

**Manuscript version: Author's Accepted Manuscript**

The version presented in WRAP is the author's accepted manuscript and may differ from the published version or Version of Record.

**Persistent WRAP URL:**

<http://wrap.warwick.ac.uk/141079>

**How to cite:**

Please refer to published version for the most recent bibliographic citation information. If a published version is known of, the repository item page linked to above, will contain details on accessing it.

**Copyright and reuse:**

The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions.

© 2020 Elsevier. Licensed under the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International <http://creativecommons.org/licenses/by-nc-nd/4.0/>.



**Publisher's statement:**

Please refer to the repository item page, publisher's statement section, for further information.

For more information, please contact the WRAP Team at: [wrap@warwick.ac.uk](mailto:wrap@warwick.ac.uk).

# ADELIC MODELS OF TENSOR-TRIANGULATED CATEGORIES

SCOTT BALCHIN AND J.P.C. GREENLEES

ABSTRACT. We show that a well behaved Noetherian, finite dimensional, stable, monoidal model category has a model built from categories of modules over completed rings in an adelic fashion. Special cases include abelian groups (the Hasse square), chromatic homotopy theory (a module theoretic chromatic fracture square), and rational torus-equivariant homotopy theory (first step to the model of [GS18]).

## CONTENTS

1. Introduction	1
2. Tensor-triangulated categories	4
3. Localization and completion in the homotopy category	6
4. Model categories	9
5. Localizing model categories at a prime	14
6. Cellularizing and completing model categories at a prime	17
7. The adelic cube	20
8. The adelic approximation theorem	23
9. The adelic model	26
10. Examples	29
References	34

## 1. INTRODUCTION

This paper is concerned with models for well behaved tensor triangulated categories  $\overline{\mathcal{C}}$ . Two motivating examples are the derived category  $D(R)$  of a commutative Noetherian ring  $R$ , and the category of rational torus-equivariant cohomology theories. Our results have new implications even for the classical example  $D(R)$ .

**1.A. Models of tensor-triangulated categories.** This introductory section will be informal, and during the rest of this paper it will be made precise. To give us an appropriate context, we assume that our tensor triangulated category  $\overline{\mathcal{C}}$  is the homotopy category of a stable, symmetric monoidal model category  $\mathcal{C}$ .

There are a number of approaches to providing models of tensor-triangulated categories  $\overline{\mathcal{C}}$ . The best known approach is Morita theory: if  $\overline{\mathcal{C}}$  has a small generator  $k$  then  $\mathcal{C}$  is equivalent

---

*Date:* July 27, 2020.

The authors are grateful to EPSRC for support from EP/P031080/1. The authors would also like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Homotopy Harnessing Higher Structures* where the first version of this paper was completed. This work was supported by EPSRC grant number EP/K032208/1.

to the category of right modules over the endomorphism ring  $\mathcal{E} := \text{End}(k)$ , in a context where this makes sense [SS03]. This is an extremely powerful technique, and very valuable for many purposes, but it has the disadvantage that the ring  $\mathcal{E}$  is typically non-commutative and usually its coefficient ring  $\pi_*(\mathcal{E})$  has infinite homological dimension. For our purposes these are serious disadvantages.

The second approach is typified by the case when  $\overline{\mathcal{C}} = \mathbf{D}(R)$  is the derived category of a commutative Noetherian ring  $R$ . It is well known that the abelian category of  $R$ -modules is equivalent to the category of quasi-coherent sheaves of modules over the topological space  $\text{Spec}(R)$ , and the corresponding statement applies at the level of derived categories. This method is also a very powerful technique: in effect it is reconstructing  $R$ -modules from the categories of modules over the local rings  $R_{\mathfrak{p}}$  for primes  $\mathfrak{p}$ . The disadvantages for us are (i) the category of modules over the local ring  $R_{\mathfrak{p}}$  is still quite complicated and (ii) the process for assembling stalks into a sheaf is intricate.

Nonetheless, this approach can be extended to general tensor-triangulated categories. We replace  $\text{Spec}(R)$  by the Balmer spectrum  $\text{Spc}^{\omega}(\overline{\mathcal{C}})$  of small objects in  $\overline{\mathcal{C}}$  (see Subsection 2.A for definitions). In the case of  $R$ -modules there is an order reversing bijection  $\text{Spec}(R) \cong \text{Spc}^{\omega}(\mathbf{D}(R))$ , so this does extend the classical case of commutative algebra.

Our work goes further, by using simpler building blocks to construct the model of  $\overline{\mathcal{C}}$ . In the commutative algebra case, we reconstruct the category of  $R$ -modules from the categories of modules over the localized completed rings  $(R_{\mathfrak{p}}^{\wedge})_{\mathfrak{p}}$ . In fact we will show that one can construct such a model rather generally, for categories  $\overline{\mathcal{C}}$  where the Balmer spectrum has the formal properties of the Zariski spectrum of a finite dimensional Noetherian commutative ring. In this case  $(R_{\mathfrak{p}}^{\wedge})_{\mathfrak{p}}$  is a ring in the underlying category  $\mathcal{C}$ , which will have a coefficient ring  $\pi_*((R_{\mathfrak{p}}^{\wedge})_{\mathfrak{p}})$ . This type of model has significant advantages. First, the pieces are genuinely simpler, and so is the assembly process. Second, and more important, we may often prove that the localized completed rings  $(R_{\mathfrak{p}}^{\wedge})_{\mathfrak{p}}$  are also commutative. In that case, under smoothness assumptions, they may be shown to be formal in the sense that the rings  $(R_{\mathfrak{p}}^{\wedge})_{\mathfrak{p}}$  in  $\mathcal{C}$  are determined by their coefficient rings. The process of assembling the model from these pieces is then also quite rigid. In this happy situation, the category  $\mathcal{C}$  is determined up to Quillen equivalence by the homotopy of the rings  $(R_{\mathfrak{p}}^{\wedge})_{\mathfrak{p}}$  and therefore equivalent to one which is algebraic in nature. This happens for rational torus-equivariant spectra [GS18].

Tensor triangulated categories, localizations, completions and support have been studied in various contexts including algebraic geometry, chromatic and equivariant homotopy theory, commutative algebra and representation theory. The terminology and standard references are not always the same in the different traditions, and different facts are viewed as being well-known, so we have tried to be thorough in documenting our sources and apologize in advance if we have failed to give proper credit in some cases. In homotopy theory, related reconstruction methods are used in the recent preprint [AMGR19] using categorical localizations of the entire category, rather than the categories of modules over rings used here.

**1.B. Abelian groups.** To bring the discussion down to earth, consider the category of abelian groups. The road to our model proceeds as follows. The first step is to recognize abelian groups as  $\mathbb{Z}$ -modules. Since these are modules in the original category, this looks perverse, but it provides a better starting point.

We then attempt to analyse the category by analysing the unit object  $\mathbb{Z}$ , and for this purpose we consider the Hasse square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \mathbb{Z}_p^\wedge & \longrightarrow & \mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge . \end{array}$$

The key to progress is that this is a pullback square. Classically, this enables number theoretic questions to be reduced to statements over  $\mathbb{Q}$  and over  $\mathbb{Z}_p^\wedge$ . We are interested in the *derived* category of abelian groups, so we will use the fact that the square is also a *homotopy* pullback. This lets us reconstruct the derived category of abelian groups from derived categories of  $\mathbb{Q}$  and  $\mathbb{Z}_p^\wedge$ . The point of this is that modules over  $\mathbb{Q}$  and  $\mathbb{Z}_p^\wedge$  are much simpler than modules over  $\mathbb{Z}$ , or even over the local rings  $\mathbb{Z}_{(p)}$ . Similarly, if we start with the derived category of a finite dimensional commutative Noetherian ring then we may reconstruct the derived category of  $R$ -modules from the derived categories of the localized completed rings  $(R_p^\wedge)_p$ .

**1.C. Outline of paper.** We work with tensor-triangulated categories  $\bar{\mathcal{C}}$  which we think of as analogues of categories of modules over a commutative ring. We will impose restrictions on these categories so that they have the same formal properties as categories of modules of finite dimensional Noetherian rings. The principal input from the category  $\bar{\mathcal{C}}$  is its Balmer spectrum, which is the categorical analogue of the Zariski spectrum of a commutative ring. We recall the necessary definitions in Section 2.

In particular, we can discuss localization and completion of objects at some prime of the Balmer spectrum. We recall the classical theory of localization and completion in derived commutative algebra in Section 3 along with the associated notions of support and cosupport.

We wish to apply these ideas more generally, in the context where the tensor-triangulated categories are homotopy categories of well-structured Quillen model categories as discussed in Section 4. In the language of model categories, localization and completion are two examples of Bousfield localizations: Section 5 discusses localization, and Section 6 discusses completion.

The fundamental result is the Adelic Approximation Theorem 8.1 which shows the monoidal unit of  $\bar{\mathcal{C}}$  is a homotopy limit in a cubical diagram of products of localized completed rings, directly analogous to the Hasse square.

From the Adelic Approximation Theorem we deduce our main result (Theorem 9.3) giving the Adelic Model for  $\bar{\mathcal{C}}$  in terms of modules over the adelic rings. The further step of giving a model in terms of modules over individual localized completed rings will be dealt with in a sequel [BaGre20].

**1.D. Notation.** We write  $\mathcal{C}$  for the model category we are studying, and  $\bar{\mathcal{C}}$  for its homotopy category. The monoidal unit is denoted  $\mathbb{1}$ . We shall use  $\mathfrak{p}$  and  $\mathfrak{q}$  when we discuss Balmer primes, we generally assume that containment corresponds to alphabetical order so that  $\mathfrak{p} \supseteq \mathfrak{q}$ . We will use this Balmer ordering in all cases, even for  $D(R)$  where it corresponds to the reverse of the commutative algebra ordering. The functor  $L_{\mathfrak{p}}$  will be the Bousfield localization at  $\mathfrak{p}$  and  $\Lambda_{\mathfrak{p}}$  the Bousfield completion at  $\mathfrak{p}$ . We write  $\mathrm{Spc}^\omega(\bar{\mathcal{C}})$  for the Balmer spectrum of the triangulated category of small objects  $\bar{\mathcal{C}}^\omega$ .

Commutative algebra	Adelic models
$\mathbf{Ch}(R\text{-mod})$	$\mathcal{C}$
$D(R)$	$\overline{\mathcal{C}}$
$R$	$\mathbb{1}$
$R_{\mathfrak{p}}$	$L_{\mathfrak{p}}\mathbb{1}$
$R_{\mathfrak{p}}^{\wedge}$	$\Lambda_{\mathfrak{p}}\mathbb{1}$
$\mathrm{Spec}(R)$	$\mathrm{Spc}^{\omega}(\overline{\mathcal{C}})$
Hasse cube	Adelic approximation cube

TABLE 1. Commutative algebra and general adelic constructions

When it comes to finiteness properties, all conventions contradict some respected source: for us ‘compact’, ‘finite’ and ‘perfect’ apply to the model category whilst ‘small’ and ‘rigid’ refer to the homotopy category. Precise definitions are given in Sections 2 and 4.

**Acknowledgements.** The authors would like to thank to Tobias Barthel for many interesting conversations and comments on this work, and the referee for supplying several useful references and observing some of our original hypotheses were unnecessary.

## 2. TENSOR-TRIANGULATED CATEGORIES

In this section we introduce the basic language at the homotopy category level. This is in the context of tensor-triangulated categories where the triangulated structure is augmented by a symmetric monoidal tensor product exact in each variable: the survey by Stevenson [Ste18] lays out the context.

**2.A. Basic terminology.** We recall some standard terminology from the study of tensor-triangulated categories and the basic definitions from [Bal05].

If  $\overline{\mathcal{C}}$  is a tensor-triangulated category, an object  $T$  is called *small* if for any set of objects  $Y_i$ , the natural map

$$\bigoplus_i [T, Y_i] \xrightarrow{\cong} [T, \bigvee_i Y_i]$$

is an isomorphism (where  $[A, B]$  denotes the  $\overline{\mathcal{C}}$ -morphisms from  $A$  to  $B$ ). We warn that these are sometimes called ‘compact’ or ‘finite’ but we are reserving those terms for model category level notions. We write  $\overline{\mathcal{C}}^{\omega}$  for the triangulated subcategory of small objects.

We say that a full subcategory  $A$  of  $\overline{\mathcal{C}}$  is *thick* if it is closed under completing triangles and taking retracts. It is *localizing* if it is closed under completing triangles and taking arbitrary coproducts (it is then automatically closed under retracts as well). We say that  $A$  is an *ideal* if it is closed under triangles and tensoring with an arbitrary element.

For a general subcategory  $B$  we write  $\mathrm{Thick}(B)$  for the thick subcategory generated by  $B$  and  $\mathrm{Thick}^{\otimes}(B)$  for the thick tensor ideal generated by  $B$ . The latter depends on the ambient category, and we will only write  $\mathrm{Thick}^{\otimes}(B)$  in the category  $\overline{\mathcal{C}}^{\omega}$  of small objects (so  $B$  is small, and only tensor products with small objects are permitted). We write  $\mathrm{Loc}(B)$  for the localizing subcategory generated by  $B$ , and  $\mathrm{Loc}^{\otimes}(B)$  for the localizing tensor ideal generated by  $B$ ; because an infinite coproduct of small objects will usually not be small, this

only makes sense for the full category  $\overline{\mathcal{C}}$  and tensor products with arbitrary objects of  $\overline{\mathcal{C}}$  are permitted.

We will generally be interested in thick and localizing tensor ideals, because without closure under tensor products the structure is hard to understand.

A triangulated category  $\overline{\mathcal{C}}$  is *small-generated* if there is a set  $\mathcal{G}$  of small objects in  $\overline{\mathcal{C}}$  such that an object  $X \in \overline{\mathcal{C}}$  is zero if and only if

$$[G, \Sigma^i X] = 0 \text{ for every } G \in \mathcal{G} \text{ and } i \in \mathbb{Z}.$$

In a small-generated tensor-triangulated category, the tensor product has a right adjoint, and therefore it is in fact a closed monoidal category, in that we have an internal hom right adjoint to  $\otimes$ , which we shall denote  $\underline{\text{hom}}(-, -) \in \overline{\mathcal{C}}$ . For every object  $X \in \overline{\mathcal{C}}$ , we define its *dual* to be the object  $DX := \underline{\text{hom}}(X, \mathbb{1})$ . We shall say that  $X$  is *rigid* (or *strongly dualizable*) if the natural map  $X \otimes DX \rightarrow \underline{\text{hom}}(X, X)$  is an equivalence.

The usual context for working with tensor-triangulated categories is as follows.

**Definition 2.1.** A *rigidly small-generated tensor-triangulated category*  $\overline{\mathcal{C}}$  is a small-generated tensor-triangulated category such that the small objects  $X \in \overline{\mathcal{C}}^\omega$  coincide with the rigid objects (in particular, the tensor product of small objects is small).

**2.B. The Balmer spectrum and support of small objects.** We may now introduce the organizational principle on which the construction is based.

**Definition 2.2.** A *prime ideal* in a tensor-triangulated category is a proper thick tensor ideal  $\mathfrak{p}$  with the property that  $a \otimes b \in \mathfrak{p}$  implies that  $a$  or  $b$  is in  $\mathfrak{p}$ .

The *Balmer spectrum* of a tensor-triangulated category  $\overline{\mathcal{C}}$  is the set of prime tensor ideals of the triangulated category of small objects:

$$\text{Spc}^\omega(\overline{\mathcal{C}}) = \{ \mathfrak{p} \subseteq \overline{\mathcal{C}}^\omega \mid \mathfrak{p} \text{ is prime} \}.$$

We may use this to define the *support* of a small object  $X$ :

$$\text{supp}(X) = \{ \mathfrak{p} \in \text{Spc}^\omega(\overline{\mathcal{C}}) \mid X \notin \mathfrak{p} \}.$$

This in turn lets us define the Zariski topology on  $\text{Spc}^\omega(\overline{\mathcal{C}})$  as generated by the closed sets  $\text{supp}(X)$  as  $X$  runs through small objects of  $\overline{\mathcal{C}}$ . We note that the set  $\text{Spc}^\omega(\overline{\mathcal{C}})$  equipped with the Zariski topology is always a spectral space in the sense of Hochster [BKS].

**Example 2.3.** The motivating example is that if  $\overline{\mathcal{C}} = \mathbf{D}(R)$  is the derived category of a commutative Noetherian ring  $R$  then there is a natural homeomorphism

$$\text{Spec}(R) \xrightarrow{\cong} \text{Spc}^\omega(\mathbf{D}(R))$$

where the classical *algebraic* prime  $\mathfrak{p}_a$  corresponds to the *Balmer* prime  $\mathfrak{p}_b = \{ M \mid M_{\mathfrak{p}_a} \simeq 0 \}$ . To avoid disorientation it is essential to emphasize that this is order-reversing, so that maximal algebraic primes correspond to minimal Balmer primes; either way these are the closed points. Even in this classical case we will use the Balmer order on primes.

We indicate the (specialization) closure operation by a bar:

$$\overline{\{ \mathfrak{p} \}} = \Lambda(\mathfrak{p}) = \{ \mathfrak{q} \mid \mathfrak{q} \subseteq \mathfrak{p} \}.$$

We say that a prime  $\mathfrak{p}$  is *visible* if its closure is of the form  $\text{supp}(M)$  a small object  $M$  of  $\overline{\mathcal{C}}$  (the term is also used for a weaker notion as in [Ste17]).

The results of this paper require all primes to be visible; if it fails, there is an additional layer of complication. The point is that if every prime is visible then the topology on  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$  is generated by the closures of points, and hence determined by the poset structure of  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$ . To explain this, recall that a topological space is said to be *Noetherian* if its open sets satisfy the ascending chain condition.

**Lemma 2.4** ([Bal05, Corollary 2.17]). *The topological space  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$  is Noetherian if and only if any closed subset of  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$  is the support of an object in  $\overline{\mathcal{C}}^\omega$ .*  $\square$

The space  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$  is Noetherian if and only if all primes are visible [BF11, Corollary 7.14].

**2.C. Classification of thick tensor ideals.** The purpose of the Balmer spectrum is to give a systematic approach to the classification of thick tensor ideals. Balmer shows [Bal05] that there is a bijection between the collection of Thomason subsets  $Y$  (i.e.,  $Y$  is a union of closed sets whose complements are compact) of  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$  and the radical thick tensor ideals of  $\overline{\mathcal{C}}^\omega$ . In the case when all small objects are rigid, thick tensor ideals are automatically radical. We shall make two uses of this fact.

**Lemma 2.5.**

- (i) *If  $K_{\mathfrak{p}}, K'_{\mathfrak{p}}$  are two small objects with support  $\Lambda(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$  then they generate the same thick tensor ideal.*
- (ii) *If  $K_i$  are small objects whose supports cover  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$  and  $\overline{\mathcal{C}}$  is small-generated then the objects  $K_i$  generate  $\overline{\mathcal{C}}$  as a localizing tensor ideal.*  $\square$

**2.D. Dimension and the partial order.** The primes in the Balmer spectrum are partially ordered by inclusion. The minimal elements are the closed points, and every prime contains a closed point ([Bal07, 2.12]). We will filter this by the *Krull dimension* [Bal05, 3.2]

$$\dim(\mathfrak{p}) = \sup\{m \mid \exists \mathfrak{p}_0 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_m = \mathfrak{p}\}.$$

We will henceforth simply refer to this as ‘dimension’.

We will restrict attention to tensor-triangulated categories which are rigidly small-generated with  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$  finite dimensional and Noetherian.

### 3. LOCALIZATION AND COMPLETION IN THE HOMOTOPY CATEGORY

We begin by recalling the standard constructions of localization at a prime, torsion at a prime and completion at a prime from commutative algebra, since our analysis is based on counterparts of them. We also explain how to decompose the category  $\overline{\mathcal{C}}$  corresponding to natural structures on  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$ , these are widely used in equivariant topology and commutative algebra (see [Har67, GM92, DG02, Bal07, Bal05, BIK12, Ste17]).

In Sections 5 and 6 we will introduce corresponding constructions for model categories and freely apply the language introduced here.

**3.A. Localization, completion and local cohomology in commutative algebra.** Our construction is based on commutative algebra, so we need to explain the localization and completion from commutative algebra that we need in a way that makes it clear how to extend it.

In this subsection we fix a commutative Noetherian ring  $R$ . We will continue to use  $\mathfrak{p}_a$  for the algebraic prime in the commutative ring  $R$  (which is a subset of  $R$ ) and  $\mathfrak{p}_b$  for the corresponding Balmer prime (which is a collection of small objects of the derived category).

For localization at  $\mathfrak{p}_a$ , we write  $L_{\mathfrak{p}_a}M = M_{\mathfrak{p}_a}$ . In the classical world, this is obtained by inverting the multiplicative set  $R \setminus \mathfrak{p}_a$ . However in terms of the Balmer spectrum, for any object  $M$  of the derived category  $\mathcal{D}(R)$ , the localization map

$$M \longrightarrow L_{\mathfrak{p}_a}M$$

is the initial map nullifying all small objects  $T$  with  $T_{\mathfrak{p}_a} \simeq 0$ . In other words it is the nullification of the Balmer prime  $\mathfrak{p}_b$ .

Next, we define the  $\mathfrak{p}_a$ -power torsion and local cohomology. For an  $R$ -module  $M$ , at the level of abelian categories the  $\mathfrak{p}_a$ -power torsion is defined by

$$\Gamma'_{\mathfrak{p}_a}M := \{x \in M \mid (\mathfrak{p}_a)^n x = 0 \text{ for } n \gg 0\}$$

(the dash is to distinguish this from the derived version, which we will use constantly throughout the rest of this paper). The functor  $\Gamma'_{\mathfrak{p}_a}$  is left exact and if  $R$  is Noetherian, the derived functors are calculated by the stable Koszul complex. To define this, suppose  $\mathfrak{p}_a = (x_1, \dots, x_n)$  and define

$$\Gamma_{\mathfrak{p}_a}M = (R \longrightarrow R[1/x_1]) \otimes_R \cdots \otimes_R (R \longrightarrow R[1/x_n]) \otimes_R M.$$

Up to equivalence this is independent of generators, and the cohomology of this complex is by definition the local cohomology; Grothendieck observed that it calculates the right derived functors of  $\Gamma'_{\mathfrak{p}_a}$ . From our point of view the important thing is that for any object  $M$  of the derived category  $\mathcal{D}(R)$ , the map

$$\Gamma_{\mathfrak{p}_a}M \longrightarrow M$$

has the universal property of  $K_{\mathfrak{p}_a}$ -cellularization where  $K_{\mathfrak{p}_a}$  is the unstable Koszul complex

$$K_R(x_1, \dots, x_n) = (R \xrightarrow{x_1} R) \otimes_R \cdots \otimes_R (R \xrightarrow{x_n} R).$$

This object  $K_{\mathfrak{p}_a}$  depends on the generators, but the cellularization only depends on the fact that  $K_{\mathfrak{p}_a}$  is a small object with support  $\overline{\{\mathfrak{p}_b\}}$ .

These two functors often occur together. We note that the composite is smashing in the sense that

$$\Gamma_{\mathfrak{p}_a}L_{\mathfrak{p}_a}M \simeq (\Gamma_{\mathfrak{p}_a}L_{\mathfrak{p}_a}R) \otimes_R M.$$

**Remark 3.1.** We warn that [BIK12, Ste18] use  $\Gamma_{\mathfrak{p}_a}$  for the composite

$$L_{\mathfrak{p}_a}\Gamma_{\mathfrak{p}_a}M = \Gamma_{\mathfrak{p}_a}L_{\mathfrak{p}_a}M,$$

which we will never do.

For  $\mathfrak{p}_a$ -completion we start from the  $\mathfrak{p}_a$ -cellularization. We use the traditional notation  $\Lambda_{\mathfrak{p}_a}M$  for the derived  $\mathfrak{p}_a$ -completion functor, so that

$$\Lambda_{\mathfrak{p}_a}M = \text{Hom}_R(\Gamma_{\mathfrak{p}_a}R, M).$$

For a Noetherian ring  $R$  and a module  $M$ , the homotopy groups of this are the left derived functors of  $\mathfrak{p}_a$ -adic completion [GM92]. For a general object  $M$  of the derived category  $\mathbf{D}(R)$ , the map

$$M \longrightarrow \Lambda_{\mathfrak{p}_a} M$$

is the Bousfield localization with respect to  $K_{\mathfrak{p}_a}$ .

Finally, we also write

$$V_{\mathfrak{p}_a} M = \mathrm{Hom}_R(L_{\mathfrak{p}_a} R, M).$$

The functors  $\Lambda_{\mathfrak{p}_a}$  and  $V_{\mathfrak{p}_a}$  often occur together, and we note that

$$\Lambda_{\mathfrak{p}_a} V_{\mathfrak{p}_a} M \simeq \mathrm{Hom}_R(\Gamma_{\mathfrak{p}_a} L_{\mathfrak{p}_a} R, M).$$

**Remark 3.2.** We note that [BIK12] uses  $\Lambda_{\mathfrak{p}_a}$  for the composite

$$V_{\mathfrak{p}_a} \Lambda_{\mathfrak{p}_a} M = \Lambda_{\mathfrak{p}_a} V_{\mathfrak{p}_a} M,$$

which we will never do.

**3.B. Support and cosupport for arbitrary objects.** We have defined support for small objects in terms of the primes, and we now extend this to general objects.

**Definition 3.3.** [BF11, BIK12] The support and cosupport of an  $R$ -module  $M$  are defined by

$$\begin{aligned} \mathrm{supp}(M) &= \{\mathfrak{p} \mid \Gamma_{\mathfrak{p}} L_{\mathfrak{p}} R \otimes_R M \neq 0\}. \\ \mathrm{cosupp}(M) &:= \{\mathfrak{p} \mid V_{\mathfrak{p}} \Lambda_{\mathfrak{p}} M \neq 0\} = \{\mathfrak{p} \mid \mathrm{Hom}_R(\Gamma_{\mathfrak{p}} R_{\mathfrak{p}}, M) \neq 0\}. \end{aligned}$$

**Remark 3.4.** When  $M$  is small, the support is

$$\{\mathfrak{p}_a \mid M_{\mathfrak{p}_a} \neq 0\} = \{\mathfrak{p}_a \mid M \notin \mathfrak{p}_b\},$$

but in general the support is a proper subset of  $\{\mathfrak{p}_a \mid M_{\mathfrak{p}_a} \neq 0\}$ .

The main fact that we shall use is that, for categories with a model, an object is trivial if it has empty support ([Ste13]) and hence if it has empty cosupport.

**3.C. Filtrations by support.** The following filtrations are well known by various names in equivariant topology, representation theory and algebraic geometry. We consider collections  $\mathcal{F}$  of primes closed under specialization (‘families’) and collections  $\mathcal{G}$  of primes closed under generization (‘cofamilies’). If  $\mathcal{F}$  is a family, we write  $\tilde{\mathcal{F}}$  for the complementary cofamily.

In particular we consider the cones above and below a fixed prime  $\mathfrak{q}$ :

$$\Lambda(\mathfrak{q}) = \{\mathfrak{p} \mid \mathfrak{p} \leq \mathfrak{q}\} \text{ and } V(\mathfrak{q}) = \{\mathfrak{p} \mid \mathfrak{q} \leq \mathfrak{p}\}.$$

The first is a family (namely the closure of  $\{\mathfrak{q}\}$ ) and the second is a cofamily.

Given a family  $\mathcal{F}$ , we may consider the set of Koszul objects for primes in  $\mathcal{F}$ . Taking the cellularization with respect to these small objects gives  $\Gamma_{\mathcal{F}} X$  (so that  $\Gamma_{\mathfrak{p}} = \Gamma_{\Lambda(\mathfrak{p})}$ ) and the nullification gives  $L_{\tilde{\mathcal{F}}} X$  (so that  $L_{\mathfrak{p}} = L_{V(\mathfrak{p})}$ ). We then have a natural cofibre sequence

$$\Gamma_{\mathcal{F}} X \longrightarrow X \longrightarrow L_{\tilde{\mathcal{F}}} X$$

with

$$\mathrm{supp}(\Gamma_{\mathcal{F}} X) = \mathrm{supp}(X) \cap \mathcal{F} \text{ and } \mathrm{supp}(L_{\tilde{\mathcal{F}}} X) = \mathrm{supp}(X) \setminus \mathcal{F}.$$

All maps  $\Gamma_{\mathcal{F}} X \longrightarrow L_{\tilde{\mathcal{F}}} X$  are trivial so this gives a semi-orthogonal decomposition of the category.

A map  $X \rightarrow Y$  is an  $\mathcal{F}$ -equivalence if  $\Gamma_{\mathcal{F}}X \rightarrow \Gamma_{\mathcal{F}}Y$  is an equivalence or equivalently if  $\text{supp}(C) \cap \mathcal{F} = \emptyset$  where  $C$  is the cofibre of  $X \rightarrow Y$ .

If  $\mathcal{F}$  is the family of primes of dimension  $\leq i$  and  $\tilde{\mathcal{F}}$  is the complementary cofamily of primes of dimension  $\geq i + 1$  we write

$$M_{\leq i} \rightarrow M \rightarrow M_{\geq i+1}$$

for the cellularization and nullification. We say that a map  $X \rightarrow Y$  is a  $(\leq i)$ -equivalence if it induces an equivalence  $X_{\leq i} \rightarrow Y_{\leq i}$ , or equivalently if it is a equivalence when tensored with any small object  $K$  with  $\text{supp}(K)$  consisting of primes of dimension  $\leq i$ .

#### 4. MODEL CATEGORIES

To start with, we recall basic terminology and constructions from model categories, and their relation to structures at the triangulated category level. The discussion of diagrams and limits of model categories in Subsection 4.C provides an essential framework for our adelic model.

**4.A. Compact, finite and small.** The terminology describing finiteness properties is in chaos, in the sense that there is no single set of conventions applying consistently to both model categories and homotopy categories. We use conventions that are necessarily non-standard, but we hope they are clear. The basic principle is that compact, finite and perfect are reserved for model category level concepts while small and rigid are exclusively for the derived category level.

Most straightforward is the analogue of a finite cell complex. For a set  $I$  of morphisms, an  $I$ -cell complex is an object constructed as a transfinite composition of pushouts of elements of  $I$  (see [Hir03, 10.5.8]), and it is a finite  $I$ -cell complex if only finitely many steps are required.

There are two counterparts of properties of compact topological spaces that are important. First (colimit-compactness) that when mapping into a sequential colimit they map into some finite part, and second (cell-compactness) that when mapping into a CW complex they map into a finite part.

**Definition 4.1** (Colimit-compactness). Let  $\mathcal{C}$  be a category, and  $\mathcal{D}$  a collection of morphisms.

- (1) If  $\gamma$  is a cardinal, then an object  $W$  in  $\mathcal{C}$  is  $\gamma$ -colimit-compact over  $\mathcal{D}$  if for all  $\gamma$ -filtered ordinals  $\lambda$  and all diagrams  $X : \lambda \rightarrow \mathcal{D}$ , we have

$$\lim_{\rightarrow \beta < \lambda} \mathcal{C}(W, X(\beta)) \xrightarrow{\cong} \mathcal{C}(W, \lim_{\rightarrow \beta < \lambda} X(\beta)).$$

- (2) An object  $W$  in  $\mathcal{C}$  is colimit-compact over  $\mathcal{D}$  if there is a cardinal  $\gamma$  for which it is  $\gamma$ -compact.

Colimit-compactness is referred to as smallness in Hovey [Hov99] and Hirschhorn [Hir03].

**Definition 4.2** (Cell-compactness). Let  $\mathcal{C}$  be a cofibrantly generated model category with generating cofibrations  $I$ .

- (1) If  $\gamma$  is a cardinal, then an object  $W$  in  $\mathcal{C}$  is  $\gamma$ -cell-compact if it is  $\gamma$ -compact over  $I$ . That is, for every relative  $I$ -cell complex  $f : X \rightarrow Y$ , every map from  $W$  to  $Y$  factors through a subcomplex of size at most  $\gamma$ .

(2) An object  $W$  in  $\mathcal{C}$  is *cell-compact* if there is a cardinal  $\gamma$  for which it is  $\gamma$ -cell-compact. Cell-compactness is referred to as compactness in Hirschhorn [Hir03].

**Definition 4.3** ([TV08, Definition 1.2.3.6]). An object  $X$  in  $\mathcal{C}$  is *perfect* if its image in  $\overline{\mathcal{C}}$  is rigid in the sense of Subsection 2.A.

**4.B. From model category to homotopy category.** We impose conditions on our model category  $\mathcal{C}$  that lead to the homotopy category being a well behaved rigidly small-generated tensor-triangulated category.

Recall that a model category is said to be *symmetric monoidal* if the underlying category is symmetric monoidal  $(\otimes, \mathbb{1})$  such that  $\otimes: \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$  is a left Quillen bifunctor and such that for every cofibrant object  $X$  and every cofibrant resolution  $\emptyset \hookrightarrow Q\mathbb{1} \rightarrow \mathbb{1}$  of the tensor unit there is an induced weak equivalence

$$Q\mathbb{1} \otimes X \rightarrow \mathbb{1} \otimes X \xrightarrow{\cong} X.$$

**Proposition 4.4.** [Hov99, Theorem 4.3.2, §7.1, Theorem 6.6.4] *Let  $\mathcal{C}$  be a stable and symmetric monoidal model category, then  $\overline{\mathcal{C}}$  is a tensor-triangulated category with the induced tensor and unit.*

**Remark 4.5.** Note that Proposition 4.4 is not stated as such in [Hov99]. At the time, there was a conjectural assumption that we needed for this to hold, however, this conjecture ([Hov99, Conjecture 5.7.5]) has been proved to hold in generality in [Cis08].

**Definition 4.6** ([Hir03, Definition 12.1.1]). A model category  $\mathcal{C}$  is said to be *cellular* if it is cofibrantly generated with generating cofibrations  $I$  and generating acyclic cofibrations  $J$  such that

- All domain and codomain objects of elements of  $I$  are cell-compact objects over  $I$ .
- The domain objects of the elements of  $J$  are colimit-compact objects over  $I$ .
- The cofibrations are effective monomorphisms. That is, a cofibration  $f: A \rightarrow B$  is the equalizer of the pair of natural inclusions  $B \rightrightarrows B \coprod_A B$ .

We now strengthen a cellular model category to a *compactly generated* category.

**Definition 4.7.** Let  $\mathcal{C}$  be a model category with a set of generating cofibrations  $I$ . We say that  $\mathcal{C}$  is *compactly generated* if

- (1) The model category  $\mathcal{C}$  is cellular.
- (2) There exists a set of generating cofibrations  $I$  and generating acyclic cofibrations  $J$  whose domains and codomains are cofibrant,  $\omega$ -cell-compact relative to  $I$  and  $\omega$ -colimit-compact with respect to the whole category  $\mathcal{C}$ .
- (3) Filtered colimits commute with finite limits in  $\mathcal{C}$ .

**Lemma 4.8.** *Suppose that  $\mathcal{C}$  is a stable and compactly generated symmetric monoidal model category with generating cofibrations  $I$ , then the tensor-triangulated category  $\overline{\mathcal{C}}$  has a set of small generators given by the set  $\mathcal{G}$  of cofibres of maps in  $I$ .*

**Proof:** The property of being compactly generated implies that the model category is *finitely generated* in the sense of [Hov99, §7.4]. We can then use [Hov99, Corollary 7.4.4] to obtain the result.  $\square$

**Theorem 4.9** ([TV08, Corollary 1.2.3.8]). *Suppose that  $\mathcal{C}$  is a stable and compactly generated symmetric monoidal model category with  $\mathbb{1}$  being  $\omega$ -cell-compact and cofibrant. We assume that the set  $I$  of morphisms of the form*

$$S^n \otimes \mathcal{G} \rightarrow \Delta^{n+1} \otimes \mathcal{G}$$

*is a set of generating cofibrations for  $\mathcal{C}$ . Here,  $\mathcal{G}$  is a set of  $\omega$ -cell-compact and  $\omega$ -colimit-compact cofibrant perfect objects. Then the rigid objects and the small objects in  $\overline{\mathcal{C}}$  coincide, and these coincide with the retracts of finite  $I$ -cell complexes.*

**Remark 4.10.** In the above theorem we are using the fact that any model category  $\mathcal{C}$  with functorial factorizations has the property that  $\mathrm{Ho}(\mathcal{C})$  has the structure of a  $\mathrm{Ho}(\mathbf{sSet})$ -module via the theory of framings [Hov99, §5]. This allows us to form the tensor products  $S^n \otimes \mathcal{G}$  and  $\Delta^{n+1} \otimes \mathcal{G}$ .

Note that by using [TV08, Proposition 1.2.3.7] it is possible to see that under the assumptions of the above theorem, the small objects form a tensor-triangulated subcategory of  $\overline{\mathcal{C}}$ . This then leads us to the following corollary.

**Corollary 4.11.** *Suppose that  $\mathcal{C}$  is as in Theorem 4.9, then  $\overline{\mathcal{C}}$  is a rigidly small-generated tensor-triangulated category with the compact generators being the objects of the set  $\mathcal{G}$ .*

**Definition 4.12.** We say that a symmetric monoidal model category  $\mathcal{C}$  is *rigidly-compactly generated* if

- (1) It is stable, proper, and compactly generated.
- (2) The monoidal unit  $\mathbb{1}$  is  $\omega$ -cell-compact and cofibrant.
- (3) There is a generating set of cofibrations of the form  $S^n \otimes \mathcal{G} \rightarrow \Delta^{n+1} \otimes \mathcal{G}$  with  $\mathcal{G}$  a set of  $\omega$ -cell-compact and  $\omega$ -small cofibrant objects whose images in  $\overline{\mathcal{C}}$  are rigid.

In light of Definition 4.12 and Corollary 4.11 we conclude

**Corollary 4.13.** *Let  $\mathcal{C}$  be a rigidly compactly generated symmetric monoidal model category so that  $\mathrm{Sp}^\omega(\overline{\mathcal{C}})$  is finite dimensional and Noetherian then  $\overline{\mathcal{C}}$  is a Noetherian tensor-triangulated category.*

**Example 4.14.** If  $R$  is a commutative Noetherian ring then the (unbounded) derived category  $\overline{\mathcal{C}} = \mathrm{D}(R)$  is a rigidly small-generated tensor-triangulated category [Ste13, Example 3.7]. It is the homotopy category of the rigidly compactly generated monoidal model category  $\mathcal{C} = \mathbf{Ch}(R)$  equipped with the projective model structure. The Balmer spectrum of  $\mathrm{D}(R)^\omega$  can be identified with  $\mathrm{Spec}(R)$  [Bal05, §5]. The category  $\mathbf{Ch}(R)$  is an Noetherian model category, with the same dimension as  $R$ .

**Example 4.15.** If  $G$  is a compact Lie group then the category of rational  $G$ -equivariant cohomology theories is a rigidly small-generated tensor-triangulated category. It is the homotopy category of the rigidly compactly generated monoidal model category of rational orthogonal  $G$ -spectra [MM02].

The Balmer spectrum can be identified with the space of conjugacy classes of closed subgroups [Gre17]. We have  $\mathfrak{p}_K \subseteq \mathfrak{p}_H$  if and only if  $K$  is cotoral in  $H$  (i.e., conjugate to a subgroup  $K'$  normal in  $H$  with  $H/K'$  a torus). The rank of the subgroup agrees with the Krull dimension on  $\mathrm{Sp}^\omega(\overline{\mathcal{C}})$ . The spectrum is Noetherian if  $G$  is a torus, but in general the closures of points do not generate the topology.

**4.C. Diagram model categories.** We will several times need to consider generalized diagram categories, and we briefly recall the construction.

**Definition 4.16.** Let  $\mathbf{D}$  be a small category, and  $\mathcal{M}$  a diagram of model categories indexed by  $\mathbf{D}$ . That is, for each  $s \in \mathbf{D}$ , we have a model category  $\mathcal{M}(s)$  and for each  $a: s \rightarrow t$  in  $\mathbf{D}$ , a left Quillen functor  $a_*: \mathcal{M}(s) \rightarrow \mathcal{M}(t)$  (with right adjoint  $a^*$ ). Then an  $\mathcal{M}$ -diagram  $X$  specifies for each object  $s$  in  $\mathbf{D}$  an object  $X(s)$  of  $\mathcal{M}(s)$  and for each morphism  $a: s \rightarrow t$  in  $\mathbf{D}$  a *base change map*  $\tilde{X}(a): a_*X(s) \rightarrow X(t)$  pseudofunctorial for composition. When  $\mathcal{M}$  is not mentioned, we refer to this as a *generalized diagram*.

**Proposition 4.17** ([GS14, Theorem 3.1, Proposition 3.3]). *Suppose given a diagram of model categories  $\mathcal{M}$  indexed on  $\mathbf{D}$ .*

(i) *If  $\mathbf{D}$  is a direct category, there is a diagram-projective model structure on the category of diagrams over  $\mathcal{M}$  with objectwise weak equivalences and fibrations.*

(ii) *If  $\mathbf{D}$  is an inverse category, there is a diagram-injective model structure on the category of diagrams over  $\mathcal{M}$  with objectwise weak equivalences and cofibrations.*

(iii) *In both the direct and the inverse case, if each model structure appearing in the diagram is cellular and proper, then so are the model structures on the category of diagrams.  $\square$*

We will write  $\mathcal{L}_{\mathbf{D}}\mathcal{M}$  for the category of  $\mathcal{M}$ -diagrams with the diagram injective model structure, since it is Bergner's *lax homotopy limit* of the diagram of model categories [Ber12]. This model structure also appears in the literature as the *diagram of left sections* in the work of Barwick [Bar10]. Under mild conditions we can also describe the corresponding *strict homotopy limit* of model categories as a certain right Bousfield localisation of this lax limit. However, it may be more helpful to first describe what will be its subcategory of fibrant and cofibrant objects as first observed in the work of Toën [Toë06].

**Definition 4.18.** The *cocartesian skeleton*  $\text{Skel}_{\mathbf{D}}X$  of the diagram  $\mathcal{M}$  of model categories is the subcategory of  $\mathcal{M}$ -diagrams which are objectwise fibrant and cofibrant and the base change maps  $a_*X(s) \rightarrow X(t)$  are weak equivalences. That is, we take the subcategory of those left sections which are *homotopy cocartesian*.

This subcategory should be thought of as the homotopical skeleton of the homotopy limit of model categories. To explain further, there are several models for the homotopy theory of homotopy theories. We are working in the category of model categories, which is not complete. However the category  $\mathbf{CSS}$  of complete Segal spaces is complete and Bergner [Ber07] shows that complete Segal spaces gives a model of the homotopy theory of homotopy theories.

Given a model category  $\mathcal{C}$  we can consider its associated complete Segal space  $L_{\mathcal{C}}\mathcal{C}$ , and hence given a diagram  $\mathcal{M}$  of model categories and left Quillen functors, we may consider the corresponding diagram  $L_{\mathcal{C}}\mathcal{M}$  of complete Segal spaces and take the homotopy limit  $\mathcal{L}im_{\mathbf{D}}(L_{\mathcal{C}}\mathcal{M}) \in \mathbf{CSS}$ . We aim to define a strict homotopy limit of model categories so that  $\mathcal{L}im_{\mathbf{D}}(L_{\mathcal{C}}\mathcal{M}) \simeq L_{\mathcal{C}}(\mathcal{L}im_{\mathbf{D}}\mathcal{M})$ . The following theorem uses the machinery of right Bousfield localisation as in [Hir03]. We shall also recap the relevant definitions in Section 6.D.

**Theorem 4.19** ([Ber12, Theorem 3.2],[Bar10, Theorem 5.25]). *Let  $\mathcal{M}$  be a diagram of shape  $\mathbf{D}$  of model categories and left Quillen functors between them. If  $\mathcal{L}_{\mathbf{D}}\mathcal{M}$  is right proper and combinatorial, then there is a model structure  $\mathcal{L}im_{\mathbf{D}}\mathcal{M}$  on the category of  $\mathcal{M}$ -diagrams so*

that there is an equivalence of complete Segal spaces (and therefore of homotopy theories)  $L_{\mathcal{C}}(\mathcal{L}im_{\mathbf{D}}\mathcal{M}) \simeq \mathcal{L}im_{\mathbf{D}}(L_{\mathcal{C}}\mathcal{M})$ . Moreover, the model structure may be described by saying that the identity is a right Quillen functor

$$\mathcal{L}_{\mathbf{D}}\mathcal{M} \longrightarrow \mathcal{L}im_{\mathbf{D}}\mathcal{M}$$

which is a right Bousfield localisation at the generalized diagrams in which the base change maps are all weak equivalences. As such, the category of fibrant and cofibrant objects of  $\mathcal{L}im_{\mathbf{D}}\mathcal{M}$  is the cocartesian skeleton  $\text{Skel}_{\mathbf{D}}\mathcal{M}$ .

**Proof:** As we have assumed each model category is combinatorial, it follows *a fortiori* each  $\mathcal{M}(s)$  appearing in  $\mathcal{M}$  is locally presentable. This property is used to find a set of objects  $\mathcal{A}(s)$  which generate each  $\mathcal{M}(s)$  under  $\lambda$ -filtered colimits for some sufficiently large cardinal  $\lambda$ . We may choose these objects such that the objects of  $\mathcal{A}(s)$  are cofibrant in  $\mathcal{M}(s)$ . Given  $X(s) \in \mathcal{A}(s)$ , and a left Quillen functor  $a_*: \mathcal{M}(s) \rightarrow \mathcal{M}(t)$ , we can consider the class of all objects  $X(t) \in \mathcal{M}(t)$  which are equipped with a weak equivalence  $a_*X(s) \rightarrow X(t)$ . We choose a cofibrant replacement of the objects  $X(t)$  in  $\mathcal{A}(t)$  if possible. However if  $X(t) \notin \mathcal{A}(t)$  then we must add it to the generating set of  $\mathcal{M}(t)$ . We repeat this process for all  $s \in \mathbf{D}$  and all  $X(s) \in \mathcal{A}(s)$ , and we end up with a potentially larger set of objects  $\mathcal{A}(s)^{(1)}$ . We repeat this process to get a diagram

$$\mathcal{A}(s) \rightarrow \mathcal{A}(s)^{(1)} \rightarrow \mathcal{A}(s)^{(2)} \rightarrow \dots$$

whose colimit we denote  $\mathcal{B}(s)$ . We then define a set of objects in  $\mathcal{M}$

$$\{(X(s), a_*: X(s) \rightarrow X(t))_{s,t} \mid X(s) \in \mathcal{B}(s), \tilde{X}(a): a_*X(s) \rightarrow X(t) \text{ weak equivalence in } \mathcal{M}(t)\}.$$

Bergner [Ber12] shows that cellularization at this set of objects gives the strict homotopy limit model structure: the weak equivalence to the complete Segal space limit is given as [Ber12, Theorem 5.1]. It is clear by construction that the category of fibrant-cofibrant objects exactly coincides with the cocartesian skeleton.  $\square$

**Remark 4.20.** Note that the above theorem uses a combinatorial hypothesis which we do not assume for our Noetherian model categories. It is sometimes still possible to construct such a strict homotopy limit without the combinatorial hypothesis by using the compact generators (see Theorem 9.3 below).

**4.D. Model structures on module categories.** We recall the necessary definitions and results from [SS00] which allow us to form module categories in monoidal model categories. Recall that a *monoid* is an object  $R \in \mathcal{C}$  together with a map  $R \otimes R \rightarrow R$  and a unit  $\mathbb{1} \rightarrow R$  which satisfy the obvious associativity and unit conditions. A *left  $R$ -module* in  $\mathcal{C}$  is an object  $N$  together with a map  $R \otimes N \rightarrow N$  satisfying associativity and unit conditions. We can then construct the category  $R\text{-mod}_{\mathcal{C}}$  of (left)  $R$ -modules in  $\mathcal{C}$ . There is a forgetful functor  $i^*: R\text{-mod}_{\mathcal{C}} \rightarrow \mathcal{C}$ .

**Proposition 4.21** ([SS00, Theorem 4.1]). *Let  $\mathcal{C}$  be a cofibrantly generated monoidal model category, and  $R$  a cofibrant commutative monoid in  $\mathcal{C}$ . Then there is a cofibrantly generated monoidal model structure on  $R\text{-mod}_{\mathcal{C}}$  where a map  $f: N \rightarrow M$  is*

- A weak equivalence if  $i^*(f): i^*(N) \rightarrow i^*(M)$  is a weak equivalence in  $\mathcal{C}$ .
- A fibration if  $i^*(f): i^*(N) \rightarrow i^*(M)$  is a fibration in  $\mathcal{C}$ .

- A cofibration if it has the LLP with respect to all acyclic fibrations.  $\square$

## 5. LOCALIZING MODEL CATEGORIES AT A PRIME

Motivated by the commutative algebra in Subsection 3.A, we construct localization  $L_{\mathfrak{p}}$  at a Balmer prime  $\mathfrak{p}$  in model categorical terms as fibrant replacement in the nullification of  $\mathfrak{p}$ , and show it is covariant in the Balmer ordering.

**5.A. Left Bousfield localization.** We recall the theory of left Bousfield localization using [Hir03] for reference, and [BR15] for some further properties. We will denote by  $\mathrm{Map}_{\mathcal{C}}(-, -) \in \mathbf{sSet}$  the homotopy function complex. The idea is to specify a set  $S$  of maps in  $\mathcal{C}$  and to construct model category by making elements of  $S$  into weak equivalences.

**Definition 5.1.** Let  $\mathcal{C}$  be a model category and  $S$  be a set of maps in  $\mathcal{C}$ .

- An object  $Z \in \mathcal{C}$  is said to be  $S$ -local if

$$\mathrm{Map}_{\mathcal{C}}(s, Z): \mathrm{Map}_{\mathcal{C}}(B, Z) \rightarrow \mathrm{Map}_{\mathcal{C}}(A, Z)$$

is a weak equivalence in  $\mathbf{sSet}$  for any  $s: A \rightarrow B$  in  $S$ .

- A map  $f: X \rightarrow Y$  in  $\mathcal{C}$  is an  $S$ -equivalence if

$$\mathrm{Map}_{\mathcal{C}}(f, Z): \mathrm{Map}_{\mathcal{C}}(Y, Z) \rightarrow \mathrm{Map}_{\mathcal{C}}(X, Z)$$

is a weak equivalence for any  $S$ -local object  $Z \in \mathcal{C}$ .

- An object  $W \in \mathcal{C}$  is  $S$ -acyclic if  $\mathrm{Map}_{\mathcal{C}}(W, Z) \simeq *$  for any  $S$ -local object  $Z \in \mathcal{C}$ .

**Definition 5.2.** The *left Bousfield localization* of a model category  $\mathcal{C}$  inverting a set of maps  $S$  (if it exists) is the model category  $L_S\mathcal{C}$  with underlying category of  $\mathcal{C}$  such that

- The weak equivalences of  $L_S\mathcal{C}$  are the  $S$ -equivalences.
- The fibrations of  $L_S\mathcal{C}$  are those maps with the RLP with respect to the cofibrations which are also  $S$ -equivalences (i.e., the acyclic cofibrations of  $L_S\mathcal{C}$ ).
- The cofibrations of  $L_S\mathcal{C}$  are the cofibrations of  $\mathcal{C}$ .

The fibrant objects in  $L_S\mathcal{C}$  are exactly the fibrant objects of  $\mathcal{C}$  which also happen to be  $S$ -local. Accordingly the identity functor  $\mathrm{id}: \mathcal{C} \rightarrow L_S\mathcal{C}$  is a left Quillen functor. Finally, for any object  $X$  of  $\mathcal{C}$  we write  $X \longrightarrow L_S X$  for a functorial fibrant replacement.

**Proposition 5.3** ([Hir03, Theorem 4.1.1]). *If  $\mathcal{C}$  is a left proper and cellular model category, then the Bousfield localization inverting any given set of maps  $S$  exists.*  $\square$

Localizations preserve certain properties as we see from [BR15]. By cofibrant replacement, we may assume that the set  $S$  consists of cofibrations between cofibrant objects.

**Proposition 5.4** ([BR15, Propositions 3.6, 4.7 and 5.4]). *Let  $\mathcal{C}$  be a proper, cellular, stable monoidal model category and  $S$  a set of cofibrations between cofibrant objects. Assume moreover that:*

- (1)  $S$  is closed under taking  $\Sigma$ ;
- (2)  $S \square I$  is contained in the class of  $S$ -equivalences, where  $I$  is the class of generating cofibrations and  $-\square-$  is the pushout-product.

Then the Bousfield localization  $L_S\mathcal{C}$  is a proper, cellular, stable monoidal model category.  $\square$

We are interested in two particular flavours of Bousfield localization. First, the nullification of a set of small objects (i.e., the set of maps inverted are maps  $N \rightarrow 0$  from the nullified objects to a point) gives the localization at a prime discussed in Subsection 5.B. Second, for monoidal model categories, classical Bousfield localization with respect to an object  $E$  (i.e., inverting all maps inducing an isomorphism in  $E$ -homology), which we will use in Section 6 to give completion at a prime.

**5.B. Localization at a prime.** Suppose  $\mathcal{C}$  is an Noetherian model category and  $\mathfrak{p}$  is a Balmer prime of  $\overline{\mathcal{C}}$ . Motivated by the commutative algebra in Subsection 3.A, we take  $L_{\mathfrak{p}}$  to be the nullification of  $\mathfrak{p}$ . More precisely, we note that  $\mathfrak{p}$  has a small skeleton and invert the set of maps  $S_{\mathfrak{p}} = \{X \rightarrow 0 \mid X \in \mathfrak{p}\}$ .

Recall that a Bousfield localization is called *smashing* if it preserves homotopy colimits.

**Lemma 5.5.** *The localizations  $L_{\mathfrak{p}}$  exist, and are monoidal, stable, and smashing.*

**Proof:** As  $\mathfrak{p}$  is a thick-tensor ideal, it is clear that the localization exists, and is stable. To see that it is monoidal, note that for an arbitrary map in  $S_{\mathfrak{p}}$ , say  $X \rightarrow 0$ , the map  $A \otimes X \rightarrow 0$  is also an  $S_{\mathfrak{p}}$  equivalence for any cofibrant object  $A$  due to  $\mathfrak{p}$  being an ideal.

To see that the localization is smashing, note, that by construction, it is a finite localization in the sense of [Mil92] (indeed the localizing subcategory of  $\mathfrak{p}$  is generated by small objects).  $\square$

**5.C. Variance in the prime.** We observe that if  $\mathfrak{q} \subseteq \mathfrak{p}$  then  $L_{\mathfrak{p}}$  nullifies more than  $L_{\mathfrak{q}}$ , so we have left Quillen functors

$$\mathcal{C} \longrightarrow L_{\mathfrak{q}}\mathcal{C} \longrightarrow L_{\mathfrak{p}}\mathcal{C},$$

where we note that this is *covariant* in the Balmer ordering. Indeed if we let  $MC_L(\mathcal{C})$  denote the category whose objects are model structures on  $\mathcal{C}$  with the same set of cofibrations as  $\mathcal{C}$ , and with morphisms the left Quillen functors then we have a functor

$$L_{\bullet}: \mathrm{Spc}^{\omega}(\overline{\mathcal{C}}) \longrightarrow MC_L(\mathcal{C}).$$

**Lemma 5.6.** *Suppose that the tensor-triangulated category  $\overline{\mathcal{C}}$  is finite dimensional Noetherian. For each object  $X$  there is a diagram*

$$L_{\bullet}X: \mathrm{Spc}^{\omega}(\overline{\mathcal{C}}) \longrightarrow \mathcal{C}$$

so that  $L_{\mathfrak{p}}X$  is a fibrant replacement of  $X$  in  $L_{\mathfrak{p}}\mathcal{C}$ . If  $X$  is a ring, then this may be taken to be a diagram of rings.

**Proof:** The idea is to choose fibrant replacements  $L_{\mathfrak{p}}X$  for all  $\mathfrak{p}$ , and then fill in the maps in adjacent layers by functoriality, starting with the top dimension. This may be made precise as follows. We have considered the diagram

$$L_{\bullet}\mathcal{C}: \mathrm{Spc}^{\omega}(\overline{\mathcal{C}}) \longrightarrow MC_L(\mathcal{C})$$

of model categories. Krull dimension, viewed as a function on  $\mathrm{Spc}^{\omega}(\overline{\mathcal{C}})$  shows that it is a direct category so Proposition 4.17 shows that the category of generalized diagrams admits

the diagram-projective model structure with weak equivalences and fibrations objectwise. Furthermore,  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$  is a poset, so since we start with a diagram of proper, cellular model categories, the resulting category of diagrams is also proper and cellular. Accordingly if we take the fibrant replacement of the constant diagram  $X$  then we obtain a diagram  $L_\bullet X$ .

To show we have a diagram of rings when  $X$  is a ring, we use a different model structure, constructed directly as a monoidal Bousfield localization. We start with the diagram of shape  $\mathrm{Spc}^\omega(\overline{\mathcal{C}})$  constant at the model category  $\mathcal{C}$ . We equip this with the injective model structure, with weak equivalences and cofibrations determined objectwise.

Now consider the class of maps of diagrams in which the component at  $\mathfrak{p}$  is required to lie in  $S_{\mathfrak{p}}$ :

$$S = \{f = (f_{\mathfrak{p}})_{\mathfrak{p}} \mid f_{\mathfrak{p}} \in S_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in \mathrm{Spc}^\omega(\overline{\mathcal{C}})\}.$$

We consider the left Bousfield localization  $L_S$  inverting  $S$ . It is easy to see that  $S$  satisfies the second condition of Proposition 5.4. Indeed, if  $A$  is a cofibrant diagram (i.e.,  $A_{\mathfrak{p}}$  is cofibrant for all  $\mathfrak{p}$ ), then by Lemma 5.5,  $A_{\mathfrak{p}} \otimes f_{\mathfrak{p}}$  is an  $S_{\mathfrak{p}}$  equivalence and  $A \otimes f$  is in  $S$ . Using this and the properness, it follows that the pushout product map lies in  $S$  as required.

It remains to observe that if  $\eta: X \rightarrow L_S X$  is a fibrant replacement in the  $S$ -localization then for each prime  $\mathfrak{p}$  the map  $X_{\mathfrak{p}} \rightarrow (L_S X)_{\mathfrak{p}}$  is also fibrant replacement in the  $S_{\mathfrak{p}}$ -localization, so that  $(L_S X)_{\mathfrak{p}} = L_{S_{\mathfrak{p}}}(X_{\mathfrak{p}})$ . This argument is quite general, and only uses the fact that the sets  $S_{\mathfrak{p}}$  increase with  $\mathfrak{p}$  (in our case this is clear since if  $\mathfrak{q} \subseteq \mathfrak{p}$  then nullifying objects of  $\mathfrak{p}$  nullifies objects in  $\mathfrak{q}$ ).

Firstly, we show that if  $f: X \rightarrow Y$  is an  $S$ -equivalence, then  $f_{\mathfrak{p}}: X_{\mathfrak{p}} \rightarrow Y_{\mathfrak{p}}$  is an  $S_{\mathfrak{p}}$ -equivalence. The right adjoint to evaluation at  $\mathfrak{p}$  is the ‘constant below  $\mathfrak{p}$ ’ functor. Indeed, for any object  $Y_{\mathfrak{p}}$  we may consider the diagram  $\Lambda(\mathfrak{p}, Y_{\mathfrak{p}})$  which is constant at  $Y_{\mathfrak{p}}$  on  $\Lambda(\mathfrak{p})$  and 0 otherwise. Then

$$\mathrm{Map}(X, \Lambda(\mathfrak{p}, Y_{\mathfrak{p}})) = \mathrm{Map}(X_{\mathfrak{p}}, Y_{\mathfrak{p}}).$$

It follows that if  $Y_{\mathfrak{p}}$  is  $S_{\mathfrak{p}}$ -local then  $\Lambda(\mathfrak{p}, Y_{\mathfrak{p}})$  is  $S$ -local. Testing  $f$  against the  $S$ -local objects  $\Lambda(\mathfrak{p}, Y_{\mathfrak{p}})$  we see that  $f_{\mathfrak{p}}$  is an  $S_{\mathfrak{p}}$ -equivalence. In particular this applies to  $f = \eta$ .

Secondly, we show that if  $Z$  is  $S$ -local then  $Z_{\mathfrak{p}}$  is  $S_{\mathfrak{p}}$ -local. The left adjoint to evaluation at  $\mathfrak{p}$  is the ‘constant above  $\mathfrak{p}$ ’ functor. Indeed, for any object  $A_{\mathfrak{p}}$  we may define  $V(\mathfrak{p}, A_{\mathfrak{p}})$  to be constant at  $A_{\mathfrak{p}}$  on  $V(\mathfrak{p})$  and zero elsewhere and then

$$\mathrm{Map}(V(\mathfrak{p}, A_{\mathfrak{p}}), X) = \mathrm{Map}(A_{\mathfrak{p}}, X_{\mathfrak{p}}).$$

If  $s_{\mathfrak{p}}: A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$  lies in  $S_{\mathfrak{p}}$  then by the hypothesis that  $S_{\mathfrak{p}}$  increases with  $\mathfrak{p}$  it follows that  $s_{\mathfrak{p}}: A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$  lies in  $S_{\hat{\mathfrak{p}}}$  whenever  $\mathfrak{p} \subseteq \hat{\mathfrak{p}}$  and hence  $V(\mathfrak{p}, s_{\mathfrak{p}}): V(\mathfrak{p}, A_{\mathfrak{p}}) \rightarrow V(\mathfrak{p}, B_{\mathfrak{p}})$  lies in  $S$ . Now testing  $Z$  against  $V(\mathfrak{p}, s_{\mathfrak{p}})$  is testing  $Z_{\mathfrak{p}}$  against  $s_{\mathfrak{p}}$ , so  $Z_{\mathfrak{p}}$  is  $S_{\mathfrak{p}}$ -local as required. This applies in particular to  $L_S X$ .  $\square$

**Remark 5.7.** As far as the formal proof is concerned, we need never have mentioned the diagram-projective model structure, but we consider it useful motivation. We note that the identity functor from the diagram-projective model structure to the  $S$ -local model structure is a left Quillen functor.

## 6. CELLULARIZING AND COMPLETING MODEL CATEGORIES AT A PRIME

Motivated by commutative algebra in Subsection 3.A, we construct completion  $\Lambda_{\mathfrak{p}}$  at a Balmer prime  $\mathfrak{p}$  in model theoretic terms and show this is contravariant in the Balmer ordering. Indeed,  $\Lambda_{\mathfrak{p}}$  is the fibrant replacement in the  $K_{\mathfrak{p}}$ -localization of  $\mathcal{C}$ . Furthermore, we can construct this fibrant replacement by cellularizing the unit to form  $\Gamma_{\mathfrak{p}}\mathbb{1}$  and then taking  $\Lambda_{\mathfrak{p}}X := \text{Hom}(\Gamma_{\mathfrak{p}}\mathbb{1}, X)$ .

**6.A. Classical Bousfield localization.** For monoidal model categories we may define homology the  $E$ -homology for an object  $E$  the  $E$ -homology by  $E_*(X) = [\mathbb{1}, E \otimes X]_*$ . The classical Bousfield  $E$ -homology localization (if it exists) is the localization which inverts the set of  $E$ -homology equivalences. In favourable circumstances, Bousfield's method [Bou79] identifies a generating set  $T_E$  of maps such that the  $T_E$ -equivalences will coincide with the  $E$ -equivalences, and establishes the existence of the localization.

Up to equivalence, we may assume that  $E$  is a cell object in  $\mathcal{C}$  (with respect to the generating cofibrations). Let  $X$  be a cell object, and denote by  $\#X$  the cardinality of the set of cells of  $X$ . We then fix an infinite cardinal  $c$  which is at least the cardinality of  $\max(\#(E \otimes G))$  for  $G \in \mathcal{G}$ , a compact generator of  $X$ . We then let  $T$  be the set of  $E$ -acyclic inclusions of subcomplexes in cell objects  $Y$  such that  $\#Y \leq c$ . We then have that  $T$  is a test set for the  $E$ -fibrations. Note that the maps in  $T$  are then cofibrations between cofibrant objects.

Using the arguments from [EKMM97, §VIII.1], we can conclude the following in the case that  $\mathcal{C}$  is a sufficiently well behaved model category.

**Corollary 6.1.** *Let  $E$  be an object of a rigidly compactly generated model category  $\mathcal{C}$ , then there is a left Bousfield localization of  $\mathcal{C}$ , denoted  $L_E\mathcal{C}$ , so that the weak equivalences are the  $E$ -equivalences and the cofibrations are the cofibrations in the original model category  $\mathcal{C}$ . This localization is stable and monoidal.*

**6.B. Completion at a prime.** We suppose that the prime  $\mathfrak{p}$  is visible, so that there is a small object  $K_{\mathfrak{p}}$  with support  $\Lambda(\mathfrak{p})$ . By Lemma 2.5, any two such small objects generate the same thick tensor ideal of small objects, so the following construction does not depend on this choice. By Lemma 2.4, the objects  $K_{\mathfrak{p}}$  exist for all primes if  $\text{Spc}^{\omega}(\overline{\mathcal{C}})$  is Noetherian.

**Definition 6.2.** If  $\mathfrak{p}$  is visible, we define *completion at  $\mathfrak{p}$*  by  $\Lambda_{\mathfrak{p}} = L_{K_{\mathfrak{p}}}$ . Accordingly we have a left Quillen functor  $\mathcal{C} \rightarrow \Lambda_{\mathfrak{p}}\mathcal{C}$  and a fibrant replacement  $X \rightarrow \Lambda_{\mathfrak{p}}X$  for any object  $X$ .

Completion at  $\mathfrak{p}$  is usually not smashing and it need not preserve small objects. Even the monoidal unit may fail to be small in  $\Lambda_{\mathfrak{p}}\mathcal{C}$ .

**6.C. Variance in the prime.** We observe that if  $\mathfrak{q} \subseteq \mathfrak{p}$  then the classification of thick categories shows that  $\text{Thick}(K_{\mathfrak{q}}) \subseteq \text{Thick}(K_{\mathfrak{p}})$  and therefore every  $K_{\mathfrak{p}}$ -equivalence is a  $K_{\mathfrak{q}}$ -equivalence. Accordingly we have left Quillen functors

$$\mathcal{C} \rightarrow \Lambda_{\mathfrak{p}}\mathcal{C} \rightarrow \Lambda_{\mathfrak{q}}\mathcal{C},$$

where we note that this is *contravariant* in the Balmer ordering. Indeed if we let  $MC_L(\mathcal{C})$  denote the category whose objects are model structures on  $\mathcal{C}$  with the same set of cofibrations as  $\mathcal{C}$ , and with morphisms the left Quillen functors then we have a functor

$$\Lambda_{\bullet} : \text{Spc}^{\omega}(\overline{\mathcal{C}})^{op} \rightarrow MC_L(\mathcal{C}).$$

**Lemma 6.3.** *Suppose that the tensor-triangulated category is finite dimensional Noetherian. For each object  $X$  there is a diagram*

$$\Lambda_{\bullet}X: \mathrm{Spc}^{\omega}(\overline{\mathcal{C}})^{op} \longrightarrow \mathcal{C}$$

*so that  $\Lambda_{\mathfrak{p}}X$  is a fibrant replacement of  $X$  in  $\Lambda_{\mathfrak{p}}\mathcal{C}$ . If  $X$  is a ring, this is a diagram of rings.*

**Proof:** The idea is to choose fibrant replacements  $\Lambda_{\mathfrak{p}}X$  for all  $\mathfrak{p}$ , and then fill in the maps in adjacent layers by functoriality, starting with the closed points. This may be made precise as follows. We have considered the diagram

$$\Lambda_{\bullet}\mathcal{C}: \mathrm{Spc}^{\omega}(\overline{\mathcal{C}})^{op} \longrightarrow MC_L(\mathcal{C})$$

of model categories. Krull dimension, viewed as a function on  $\mathrm{Spc}^{\omega}(\overline{\mathcal{C}})$  shows that it is an inverse category so its opposite is direct, and Proposition 4.17 shows that the generalized diagram category admits the diagram-projective model structure with weak equivalences and fibrations objectwise. Furthermore,  $\mathrm{Spc}^{\omega}(\overline{\mathcal{C}})$  is a poset, so that since we start with a diagram of proper, cellular model categories, the resulting category of diagrams is also proper and cellular.

This in turn means that if we take the fibrant replacement of the constant diagram  $X$  then we obtain a diagram  $\Lambda_{\bullet}X$ .

To see that this is a diagram of rings we apply the same argument as in Lemma 5.6. Noting that completion is contravariant in the prime, we need only observe that if  $S_{\mathfrak{p}}$  consists of  $K_{\mathfrak{p}}$ -equivalences, then  $S_{\mathfrak{p}}$  increases as the prime gets smaller. Indeed  $K_{\mathfrak{p}}$ -equivalences are the  $\Lambda(\mathfrak{p})$ -equivalences and if  $\mathfrak{q} \subseteq \mathfrak{p}$  then  $\Lambda(\mathfrak{q}) \subseteq \Lambda(\mathfrak{p})$ .  $\square$

**6.D. Cellularization.** We recall the definition of cellularization (or right Bousfield localization).

**Definition 6.4.** Let  $\mathcal{C}$  be a model category and  $\mathcal{K}$  a set of objects of  $\mathcal{C}$ .

- A map  $f: A \rightarrow B$  in  $\mathcal{C}$  is a  $\mathcal{K}$ -coequivalence if

$$\mathrm{Map}_{\mathcal{C}}(X, f): \mathrm{Map}_{\mathcal{C}}(X, A) \rightarrow \mathrm{Map}_{\mathcal{C}}(X, B)$$

is a weak equivalence in  $\mathbf{sSet}$  for each  $X \in \mathcal{K}$ .

- An object  $Z \in \mathcal{C}$  is  $\mathcal{K}$ -colocal if

$$\mathrm{Map}_{\mathcal{C}}(Z, f): \mathrm{Map}_{\mathcal{C}}(Z, A) \rightarrow \mathrm{Map}_{\mathcal{C}}(Z, B)$$

is a weak equivalence for any  $\mathcal{K}$ -coequivalence.

- An object  $A \in \mathcal{C}$  is  $\mathcal{K}$ -coacyclic if  $\mathrm{Map}_{\mathcal{C}}(W, A) \simeq *$  for any  $\mathcal{K}$ -colocal object  $W \in \mathcal{C}$ .

**Definition 6.5.** The  $\mathcal{K}$ -cellularization (or right Bousfield localization at  $\mathcal{K}$ ) of  $\mathcal{C}$  is the model category  $\mathrm{Cell}_{\mathcal{K}}\mathcal{C}$  with underlying category of  $\mathcal{C}$  such that

- The weak equivalences of  $\mathrm{Cell}_{\mathcal{K}}\mathcal{C}$  are the  $\mathcal{K}$ -coequivalences.
- The fibrations of  $\mathrm{Cell}_{\mathcal{K}}\mathcal{C}$  are the fibrations of  $\mathcal{C}$ .
- The cofibrations of  $\mathrm{Cell}_{\mathcal{K}}\mathcal{C}$  are those maps with the LLP with respect to the fibrations which are also  $\mathcal{K}$ -coequivalences (i.e., the acyclic fibrations of  $\mathrm{Cell}_{\mathcal{K}}\mathcal{C}$ ).

The cofibrant objects in  $\mathrm{Cell}_{\mathcal{K}}\mathcal{C}$  are exactly the cofibrant objects of  $\mathcal{C}$  which also happen to be  $\mathcal{K}$ -colocal. The identity functor  $\mathrm{id}: \mathcal{C} \rightarrow \mathrm{Cell}_{\mathcal{K}}\mathcal{C}$  is a right Quillen functor. Finally, for any object  $X$  of  $\mathcal{C}$  we write  $\mathrm{Cell}_{\mathcal{K}}X \longrightarrow X$  for a functorial cofibrant replacement.

**Proposition 6.6** ([Hir03, Theorem 5.1.1]). *If  $\mathcal{C}$  is a right proper and cellular model category, then cellularization exists at any set of objects  $\mathcal{K}$ .*  $\square$

**6.E. Cellularization at a prime.** As before, Balmer's classification shows that any two objects  $K_{\mathfrak{p}}$  with support  $\Lambda(\mathfrak{p})$  generate the same thick tensor ideal of small objects, so the following construction does not depend on this choice.

**Definition 6.7.** Given a visible prime  $\mathfrak{p}$  we write  $\Gamma_{\mathfrak{p}} = \text{Cell}_{K_{\mathfrak{p}}}$  for the  $K_{\mathfrak{p}}$ -cellularization, so that for an object  $X$  of  $\mathcal{C}$  there is a map  $\Gamma_{\mathfrak{p}}X \rightarrow X$  which is a  $K_{\mathfrak{p}}$ -coequivalence from a  $K_{\mathfrak{p}}$ -cellular object. Indeed, we have a right Quillen functor

$$\text{Cell}_{K_{\mathfrak{p}}}\mathcal{C} \rightarrow \mathcal{C}$$

and we may take  $\Gamma_{\mathfrak{p}}X$  to be the cofibrant approximation of  $X$  in  $\Gamma_{\mathfrak{p}}\mathcal{C}$ .

In our situation this is a rather concrete construction: since  $K_{\mathfrak{p}}$  is small we may construct  $\Gamma_{\mathfrak{p}}X$  up to equivalence by repeatedly attaching cells  $K_{\mathfrak{p}}$  and passing to sequential colimits.

**6.F. Variance in the prime.** We observe that if  $\mathfrak{q} \subseteq \mathfrak{p}$  then the classification of thick categories shows that  $\text{Thick}(K_{\mathfrak{q}}) \subseteq \text{Thick}(K_{\mathfrak{p}})$  and therefore every  $K_{\mathfrak{p}}$ -coequivalence is a  $K_{\mathfrak{q}}$ -coequivalence. Accordingly we have right Quillen functors

$$\Gamma_{\mathfrak{q}}\mathcal{C} \rightarrow \Gamma_{\mathfrak{p}}\mathcal{C} \rightarrow \mathcal{C},$$

where we note that this is *covariant* in the Balmer ordering. Indeed if we let  $MC_R(\mathcal{C})$  denote the category whose objects are model structures on  $\mathcal{C}$  with the same set of fibrations as  $\mathcal{C}$ , and with morphisms the right Quillen functors then we have a functor

$$\Gamma_{\bullet}: \text{Spc}^{\omega}(\overline{\mathcal{C}}) \rightarrow MC_R(\mathcal{C}).$$

**Lemma 6.8.** *Suppose that the tensor-triangulated category is Noetherian. For each object  $X$  there is a diagram*

$$\Gamma_{\bullet}X: \text{Spc}^{\omega}(\overline{\mathcal{C}}) \rightarrow \mathcal{C}$$

so that  $\Gamma_{\mathfrak{p}}X$  is a cofibrant replacement of  $X$  in  $\Gamma_{\mathfrak{p}}\mathcal{C}$ .

**Proof:** In effect, we choose cofibrant replacements  $\Gamma_{\mathfrak{p}}X$  for all  $\mathfrak{p}$ , and then fill in the maps in adjacent layers by functoriality, starting with the top dimension. This may be made precise as follows. We have considered the diagram

$$\Gamma_{\bullet}\mathcal{C}: \text{Spc}^{\omega}(\overline{\mathcal{C}}) \rightarrow MC_R(\mathcal{C})$$

of model categories. The dimension function on  $\text{Spc}^{\omega}(\overline{\mathcal{C}})$  shows that it is a inverse category so Proposition 4.17 shows that it admits a diagram-injective model structure with weak equivalences and cofibrations objectwise. Furthermore,  $\text{Spc}^{\omega}(\overline{\mathcal{C}})$  is a poset, so that since we start with a diagram of proper, cellular model categories, the resulting category of diagrams is also proper and cellular.

This in turn means that if we take the cofibrant replacement of the constant diagram  $X$  then we obtain a diagram  $\Gamma_{\bullet}X$ .  $\square$

**6.G. Completion at a prime via cellularization of the unit.** The connection between local cohomology and completion has a natural counterpart in the general context.

**Lemma 6.9.** *For an object  $X$  fibrant in  $\mathcal{C}$ , the map  $X \rightarrow \mathrm{Hom}(\Gamma_{\mathfrak{q}}\mathbb{1}, X)$  is a  $\Lambda_{\mathfrak{q}}$ -fibrant approximation. More informally*

$$\Lambda_{\mathfrak{q}}X \simeq \mathrm{Hom}(\Gamma_{\mathfrak{q}}\mathbb{1}, X). \quad \square$$

**6.H. Two extremes.** We finish by recording two different results regarding the behaviour of the completion at the maximal and minimal elements of the Balmer spectrum.

**Lemma 6.10.** *If  $\mathrm{Spc}^{\omega}(\mathcal{C})$  is irreducible (in the sense that it has a unique maximal point, the generic point  $\mathfrak{g}$ ), then  $X \xrightarrow{\simeq} \Lambda_{\mathfrak{g}}X$  is an equivalence.*

**Proof:** By definition we have  $\overline{\{\mathfrak{g}\}} = \mathrm{Spc}^{\omega}(\mathcal{C})$ , and we may take  $K_{\mathfrak{g}} = \mathbb{1}$ . The localization therefore only inverts those maps that are already weak equivalences.  $\square$

**Lemma 6.11.** *Let  $\mathfrak{m}$  be a Balmer minimal prime (closed point), then  $\Lambda_{\mathfrak{m}}X \xrightarrow{\simeq} L_{\mathfrak{m}}\Lambda_{\mathfrak{m}}X$  is an equivalence.*

**Proof:** We need to show that nullifying elements of  $\mathfrak{m}$  makes no difference to a completion. Indeed, since  $\Lambda_{\mathfrak{m}}X$  is  $K_{\mathfrak{p}}$ -local, we need only observe that if  $A \in \mathfrak{m}$  then  $A \rightarrow 0$  is a  $K_{\mathfrak{p}}$ -equivalence. However  $A \in \mathfrak{m}$  means  $\mathfrak{m} \notin \mathrm{supp}(A)$  and so  $\mathrm{supp}(A \otimes K_{\mathfrak{m}}) \subseteq \mathrm{supp}(A) \cap \mathrm{supp}(K_{\mathfrak{m}}) = \emptyset$ . Accordingly  $A \otimes K_{\mathfrak{m}} \simeq 0$  and  $A \rightarrow 0$  is a  $K_{\mathfrak{m}}$ -equivalence as required.  $\square$

## 7. THE ADELIC CUBE

In [Gre19], a certain ‘adelic cochain complex’  $C_{ad}^*(\mathfrak{X}; L, \mathcal{F})$  was constructed. Taking the relevant special case, this takes as input

- a poset  $\mathfrak{X}$  with a dimension function;
- a coefficient system  $\mathcal{F}: \mathfrak{X}^{op} \rightarrow \mathcal{A}$ ;
- a compatible system of localizations  $L: \mathfrak{X} \rightarrow [\mathcal{A}, \mathcal{A}]$ .

We will recall the definitions below, but we may take  $\mathfrak{X} = \mathrm{Spc}^{\omega}(\overline{\mathcal{C}})$ , which we assume to be of finite dimension  $r$  equipped with the Krull dimension. The motivating example has  $\overline{\mathcal{C}} = \mathrm{D}(R)$  for a finite dimensional commutative Noetherian ring,  $\mathcal{A}$  is the category of  $R$ -modules  $\mathcal{F}(\mathfrak{p}) = M_{\mathfrak{p}}^{\wedge}$  ( $\mathfrak{p}$ -adic completion of a fixed  $R$ -module  $M$ ) and  $L_{\mathfrak{p}}M = M_{\mathfrak{p}}$  (localization at  $\mathfrak{p}$ ).

The purpose of this section is to describe the counterpart at the model category level, and the reader wanting precision should immediately skip over the next two short paragraphs sketching the vision.

The idea is that an object can be assembled from information at each prime  $\mathfrak{p}$ . If we start from localized completed objects, we can hope to retain good multiplicative information. The data for an object should be thought of as being based on the completed information at closed points, with additional structure added to its localizations so as to describe the whole.

To understand the formal structure used to assemble this, one might wish to think of having a local coefficient system of rings on the simplicial complex of the Balmer spectrum poset. To convert to algebra one takes ‘locally finite sections’. The model then consists of a category of modules over this local system of rings.

**7.A. Objects of the adelic diagram.** The construction at the abelian category level in [Gre19] may be copied verbatim to our present context. The adelic diagram is based on the cube consisting of subsets of  $\Delta^r = \{0, 1, \dots, r\}$ , where  $r = \dim(\mathfrak{X})$ . The diagram is a functor on the punctured cube  $(\Delta^r)'$  of nonempty subsets. For an object  $M$  in  $\mathcal{C}$ , we will define the adelic diagram

$$M_{ad}: (\Delta^r)' \longrightarrow \mathcal{C}.$$

(in the notation of [Gre19] we have  $M_{ad} := C_{ad}^\bullet(\mathfrak{X}; L, \Lambda M)$ ). The value on a nonempty subset  $\mathbf{d} = (d_0 > d_1 > \dots > d_s)$  is obtained from the localized completed cobjects  $L_{\mathfrak{p}_s} \Lambda_{\mathfrak{p}_s} M$  with  $\dim(\mathfrak{p}_s) = d_s$  by taking iterated products and localizations as follows:

$$M_{ad}(\mathbf{d}) := \prod_{\dim(\mathfrak{p}_0)=d_0} L_{\mathfrak{p}_0} \left( \prod_{\dim(\mathfrak{p}_1)=d_1, \mathfrak{p}_1 \subset \mathfrak{p}_0} L_{\mathfrak{p}_1} \cdots \right. \\ \left. \cdots \left( \prod_{\dim(\mathfrak{p}_{s-1})=d_{s-1}, \mathfrak{p}_{s-1} \subset \mathfrak{p}_{s-2}} L_{\mathfrak{p}_{s-1}} \left( \prod_{\dim(\mathfrak{p}_s)=d_s, \mathfrak{p}_s \subset \mathfrak{p}_{s-1}} L_{\mathfrak{p}_s} \Lambda_{\mathfrak{p}_s} M \right) \right) \cdots \right).$$

In order to fill in the edges of the punctured cube and make it commutative we need  $\Lambda_\bullet M$  to be a functor as shown in Lemma 6.3, and  $L$  to consist of functors with natural transformations  $\eta: id \longrightarrow L_{\mathfrak{p}}$ . We write  $M(\mathfrak{p}) = \Lambda_{\mathfrak{p}} M$  to emphasize the (contravariant) functoriality in  $\mathfrak{p}$ .

**7.B. Morphisms in the adelic diagram.** Suppose we consider a flag  $\mathbf{d} = (d_0 > d_1 > \dots > d_s)$  and the edge of the cube  $\partial_i \mathbf{d} \xrightarrow{\delta_i} \mathbf{d}$  corresponding to the face omitting  $i$ . If  $i < s$ , we may take

$$M_{i+1}(\mathfrak{p}_i) := \prod_{\mathfrak{p}_{i+1} < \mathfrak{p}_i} L_{\mathfrak{p}_{i+1}} \prod_{\mathfrak{p}_{i+2} < \mathfrak{p}_{i+1}} L_{\mathfrak{p}_{i+2}} \cdots L_{\mathfrak{p}_{s-2}} \prod_{\mathfrak{p}_{s-1} < \mathfrak{p}_{s-2}} L_{\mathfrak{p}_{s-1}} \prod_{\mathfrak{p}_s < \mathfrak{p}_{s-1}} L_{\mathfrak{p}_s} M(\mathfrak{p}_s).$$

Then the map is simply given by taking

$$M_{i+1}(\mathfrak{p}_{i-1}) \longrightarrow M_{i+1}(\mathfrak{p}_i) \longrightarrow L_{\mathfrak{p}_i} M_{i+1}(\mathfrak{p}_i)$$

at the  $i$ th spot (where the first map uses functoriality in the prime and collapses the factors corresponding to primes  $\mathfrak{p}_{i+1}$  containing  $\mathfrak{p}_{i-1}$  but not  $\mathfrak{p}_i$ , and the second is the unit of localization) and then applying the same sequence of products and localizations to both domain and codomain.

If  $i = s$  we take the map

$$M(\mathfrak{p}_{s-1}) \longrightarrow \prod_{\mathfrak{p}_s < \mathfrak{p}_{s-1}} M(\mathfrak{p}_s) \longrightarrow \prod_{\mathfrak{p}_s < \mathfrak{p}_{s-1}} L_{\mathfrak{p}_s} M(\mathfrak{p}_s)$$

with components  $M(\mathfrak{p}_{s-1}) \longrightarrow M(\mathfrak{p}_s) \longrightarrow L_{\mathfrak{p}_s} M(\mathfrak{p}_s)$  given by the coefficient system, and then apply  $L_{\mathfrak{p}_{s-1}}$  and the same sequence of products and localizations to both domain and codomain.

To see we get a cochain complex we need only observe that the composite of two  $\delta_i$ s depends only on the dimensions omitted. More precisely, if the numbers omitted are  $0 \leq a < b \leq s$ , then we may omit  $a$  and  $b$  in either order and we need to know that  $\delta_a \delta_b = \delta_{b-1} \delta_a$ .

If  $a < b < s$  then the verification is immediate from the fact that  $L_\bullet$  is a functor on the diagram category, together with the categorical properties of the product.

If  $b = s$  there are two cases. The simplest is when  $a < s - 2$ . Then the diagram

$$\begin{array}{ccc} L_{\mathbf{p}_{a+1}} \cdots L_{\mathbf{p}_{s-1}} M(\mathbf{p}_{s-1}) & \longrightarrow & L_{\mathbf{p}_{a+1}} \cdots L_{\mathbf{p}_{s-1}} L_{\mathbf{p}_s} M(\mathbf{p}_s) \\ \downarrow & & \downarrow \\ L_{\mathbf{p}_a} L_{\mathbf{p}_{a+1}} \cdots L_{\mathbf{p}_{s-1}} M(\mathbf{p}_{s-1}) & \longrightarrow & L_{\mathbf{p}_a} L_{\mathbf{p}_{a+1}} \cdots L_{\mathbf{p}_{s-1}} L_{\mathbf{p}_s} M(\mathbf{p}_s) \end{array}$$

commutes since  $\eta : 1 \rightarrow L_{\mathbf{p}_a}$  is a natural transformation. The required commutation then follows from the categorical properties of the product.

The case  $b = s, a = s - 1$  is the most complicated. We will abbreviate  $M(\mathbf{p}_s) = M(s)$  and  $L_{\mathbf{p}_s} = L_s$  for readability. The following diagram has  $L_0 L_1 \dots L_{s-2}$  applied to it.

$$\begin{array}{ccccc} & & M(s-2) & & \\ & \swarrow & \downarrow & \searrow & \\ L_{s-1}M(s-2) & & & & L_sM(s-2) \\ & \searrow & & \swarrow & \\ & & L_{s-1}L_sM(s-1) & & \\ & & \downarrow & & \\ & & M(s-1) & & \\ & \swarrow & \downarrow & \searrow & \\ L_{s-1}M(s-1) & & & & L_sM(s-1) \\ & \searrow & & \swarrow & \\ & & L_{s-1}L_sM(s-1) & & \\ & & \downarrow & & \\ & & M(s) & & \\ & \swarrow & \downarrow & \searrow & \\ L_{s-1}M(s) & & & & L_sM(s) \\ & \searrow & & \swarrow & \\ & & L_{s-1}L_sM(s) & & \end{array}$$

The left and right faces commute since the unit for  $L_{s-1}$  is a natural transformation. The front and back faces commute since the unit for  $L_s$  is a natural transformation. The top and bottom faces commute because of the natural transformation  $id \rightarrow L_p$ . The relevant square involves  $M(s-2), L_{s-1}M(s-1), L_sM(s)$  and  $L_{s-1}L_sM(s)$ . The required commutation then follows from the categorical properties of the product.

**Remark 7.1.** (*Finite number of primes*) We observe that if there are only a finite number of primes, since the localization commutes with finite products, we have

$$M_{ad}(d_0 > \cdots > d_s) \simeq \prod_{\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_s, \dim(\mathfrak{p}_i) = d_i} L_{\mathfrak{p}_0} \Lambda_{\mathfrak{p}_s} M.$$

## 8. THE ADELIC APPROXIMATION THEOREM

We now come the main ingredient in constructing the adelic model. It states that the monoidal unit  $\mathbb{1}$  can be reconstructed from localizations of completions. We use the dimension filtration to give us an inductive approach to proving an equivalence and the finite dimensionality to show this terminates at a finite stage. The Noetherian condition is used so that we can work with a diagram without the need to consider continuity of various constructions in the prime.

**8.A. Statement of result.** Let  $\Delta^r = \{0, 1, \dots, r\}$  and write  $(\Delta^r)'$  for the poset of its non-empty subsets, which we may think of as a punctured  $(r + 1)$ -cube.

**Theorem 8.1** (Adelic Approximation). *Let  $\mathcal{C}$  be a Noetherian model category with  $r$ -dimensional Balmer spectrum. Then  $\mathbb{1} \in \mathcal{C}$  is the homotopy pullback of the punctured  $(r + 1)$ -cube  $\mathbb{1}_{ad}: (\Delta^r)' \rightarrow \mathcal{C}$  where the object at position  $(d_0 > d_1 > \cdots > d_s)$  is*

$$\mathbb{1}_{ad}(d_0 > d_1 > \cdots > d_s) = \prod_{\dim \mathfrak{p}_0 = d_0} L_{\mathfrak{p}_0} \prod_{\substack{\mathfrak{p}_1 \subset \mathfrak{p}_0 \\ \dim \mathfrak{p}_1 = d_1}} L_{\mathfrak{p}_1} \cdots \prod_{\substack{\mathfrak{p}_{s-1} \subset \mathfrak{p}_{s-2} \\ \dim \mathfrak{p}_{s-1} = d_{s-1}}} L_{\mathfrak{p}_{s-1}} \prod_{\substack{\mathfrak{p}_s \subset \mathfrak{p}_{s-1} \\ \dim \mathfrak{p}_s = d_s}} L_{\mathfrak{p}_s} \Lambda_{\mathfrak{p}_s} \mathbb{1},$$

and the maps are as described in Section 7.

**Example 8.2.** In the special case when  $\bar{\mathcal{C}}$  is the derived category of abelian groups the statement of the adelic approximation theorem is that the Hasse square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \mathbb{Z}_p^\wedge & \longrightarrow & \mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge, \end{array}$$

is a homotopy pullback.

**Example 8.3.** When the Balmer spectrum is two-dimensional and irreducible, the diagram of the adelic approximation theorem states that the cube

$$\begin{array}{ccc}
\prod_{\mathfrak{p}_1} L_{\mathfrak{p}_1} \Lambda_{\mathfrak{p}_1} \mathbb{1} & \longrightarrow & L_{\mathfrak{p}_0} \prod_{\mathfrak{p}_1 \subset \mathfrak{p}_0} L_{\mathfrak{p}_1} \Lambda_{\mathfrak{p}_1} \mathbb{1} \\
\uparrow & & \uparrow \\
\mathbb{1} & \longrightarrow & L_{\mathfrak{p}_0} \mathbb{1} \\
\downarrow & & \downarrow \\
\prod_{\mathfrak{p}_1} L_{\mathfrak{p}_1} \prod_{\mathfrak{p}_2 \subset \mathfrak{p}_1} \Lambda_{\mathfrak{p}_2} \mathbb{1} & \longrightarrow & L_{\mathfrak{p}_0} \prod_{\mathfrak{p}_1 \subset \mathfrak{p}_0} L_{\mathfrak{p}_1} \prod_{\mathfrak{p}_2 \subset \mathfrak{p}_1} \Lambda_{\mathfrak{p}_2} \mathbb{1} \\
\downarrow & & \downarrow \\
\prod_{\mathfrak{p}_2} \Lambda_{\mathfrak{p}_2} \mathbb{1} & \longrightarrow & L_{\mathfrak{p}_0} \prod_{\mathfrak{p}_2 \subset \mathfrak{p}_0} \Lambda_{\mathfrak{p}_2} \mathbb{1}
\end{array}$$

is a homotopy pullback, where  $\mathfrak{p}_i$  runs through primes of dimension  $i$ .

**8.B. Strategy.** First we recall the Cubical Reduction Principle for homotopy pullbacks. A cubical diagram  $X: C \rightarrow \mathbb{D}$  is a homotopy pullback if the initial point  $X(\emptyset)$  is the homotopy inverse limit over the punctured cube  $PC$ . It is thus clear that a 0-cube is a homotopy pullback if  $X(\emptyset) \simeq *$ . For a 1-cube  $X: \mathbb{I} \rightarrow \mathcal{D}$  (with  $\mathbb{I} = (0 \rightarrow 1)$ ), we write  $X_f = \text{fibre}(X(0) \rightarrow X(1))$  for the homotopy fibre. This diagram is a homotopy pullback if and only if the map  $X(0) \xrightarrow{\simeq} X(1)$  is an equivalence which happens if and only if  $X_f \simeq *$ .

Now suppose  $C = \mathbb{I} \times C'$ , and note that  $X: C \rightarrow \mathbb{D}$  induces a cube  $X_f^1: C' \rightarrow \mathbb{D}$  of homotopy fibres, where the 1 refers to the fact that the fibre has been taken with respect to the first coordinate. The Cubical Reduction Principle states that the diagram  $X$  is a homotopy pullback if and only if  $X_f^1$  is a homotopy pullback.

**Proof of 8.1:** For each  $n$ -dimensional prime we may consider the set  $\Lambda(\mathfrak{p})$  of primes below  $\mathfrak{p}$ , and form the  $(n+1)$ -cube indexed by subsets of  $\{0, 1, \dots, n\}$ . We consider the  $(n+1)$ -cube  $\mathbb{1}_{ad}(\mathfrak{p})$ , with the same definition as  $\mathbb{1}_{ad}$ , but the primes are restricted to  $\Lambda(\mathfrak{p})$  and hence the dimensions are restricted to  $\{0, 1, \dots, n\}$ . Evidently if  $\mathfrak{q} \subseteq \mathfrak{p}$  we have maps of diagrams

$$\mathbb{1}_{ad}(\mathfrak{q}) \longrightarrow \mathbb{1}_{ad}(\mathfrak{p}) \longrightarrow \mathbb{1}_{ad}.$$

Note that  $\mathbb{1}_{ad}$  is a homotopy pullback if and only if  $K_{\mathfrak{p}} \otimes \mathbb{1}_{ad}$  is a pullback for all  $\mathfrak{p}$  by Lemma 2.5. Since  $K_{\mathfrak{p}} \otimes \mathbb{1}_{\mathfrak{q}} \simeq 0$  unless  $\mathfrak{q} \subseteq \mathfrak{p}$  we see

$$K_{\mathfrak{p}} \otimes_R \mathbb{1}_{ad} \simeq K_{\mathfrak{p}} \otimes_R \mathbb{1}_{ad}(\mathfrak{p}),$$

so that it suffices to show  $K_{\mathfrak{p}} \otimes_R \mathbb{1}_{ad}(\mathfrak{p})$  is a pullback for all primes  $\mathfrak{p}$ .

We will prove by induction that if  $\dim(\mathfrak{p}) = n$  then  $\mathbb{1}_{ad}(\mathfrak{p})$  is a homotopy pullback in dimension  $\leq n$ . The base of the induction is the trivial case  $n = -1$ .

For the inductive step we suppose that  $\dim(\mathfrak{p}) = n$  and if  $\mathfrak{q} \subseteq \mathfrak{p}$  with  $\dim(\mathfrak{q}) = i \leq n - 1$  then  $\mathbb{1}_{ad}(\mathfrak{q})$  is a homotopy pullback in dimension  $\leq i$ . By the Cubical Reduction Principle,  $\mathbb{1}_{ad}(\mathfrak{p})$  is a homotopy pullback if and only if  $(\mathbb{1}_{ad}(\mathfrak{p}))_f^n$  is a homotopy pullback.

Since  $\mathfrak{p}$  is the only  $n$ -dimensional prime in  $\mathbb{1}_{ad}(\mathfrak{p})$ , the cubical reduction takes the fibre of localization  $L_{\mathfrak{p}} = L_{V(\mathfrak{p})}$  (see Section 3), and in view of the fibre sequence  $\Gamma_{V(\mathfrak{p})^c} \mathbb{1} \longrightarrow \mathbb{1} \longrightarrow L_{V(\mathfrak{p})} \mathbb{1}$  we have

$$\mathbb{1}_{ad}(\mathfrak{p})_f^n(d_0 > \cdots > d_s) = (\Gamma_{V(\mathfrak{p})^c} R) \otimes_R [\mathbb{1}_{ad}(\mathfrak{p})(d_0 > \cdots > d_s)].$$

Any prime  $\mathfrak{q} \subseteq \mathfrak{p}$  of dimension  $\leq n$  in  $V(\mathfrak{p})^c$  is actually of dimension  $\leq n - 1$ . Next note that

$$K_{\mathfrak{q}} \otimes \mathbb{1}_{ad}(\mathfrak{p})(d_0 > \cdots > d_s) \simeq 0$$

unless  $\dim(\mathfrak{q}) \geq d_0$ : this uses the fact that  $K_{\mathfrak{q}} \otimes \mathbb{1}_{\mathfrak{q}_0} \simeq 0$  unless  $\mathfrak{q}_0 \leq \mathfrak{q}$ , and the fact that  $K_{\mathfrak{q}}$  is small so that it passes inside the products. Accordingly,

$$K_{\mathfrak{q}} \otimes \mathbb{1}_{ad}(\mathfrak{p})_f^n \simeq K_{\mathfrak{q}} \otimes \mathbb{1}_{ad}(\mathfrak{q}),$$

which is a pullback cube by the induction hypothesis, completing the inductive step.

By induction we see that  $K_{\mathfrak{p}} \otimes_R \mathbb{1}_{ad}(\mathfrak{p})$  is a homotopy pullback for all primes of dimension  $r$ , and hence  $\mathbb{1}_{ad}$  is a homotopy pullback as required.  $\square$

**8.C. The Beilinson-Parshin cube.** The adelic approximation theorem emerged from the algebraic model for torus-equivariant rational spectra. Related constructions occur in the construction of Beilinson-Parshin adeles [Bei80, Hub91, Mor12], and there is a corresponding statement.

In fact the inductive scheme of the proof of the Adelic Approximation Theorem applies equally well to other localization systems provided  $K_{\mathfrak{p}} \otimes A_{\mathfrak{q}} \simeq 0$  unless  $\mathfrak{q} \subseteq \mathfrak{p}$ , and provided the support of the fibre of  $1 \longrightarrow A_{\mathfrak{p}}$  does not contain  $\mathfrak{p}$ .

If  $A_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} L_{\mathfrak{p}}$  as for the Beilinson-Parshin case the first condition is clear since  $K_{\mathfrak{p}}$  is small and  $K_{\mathfrak{p}} \otimes_R L_{\mathfrak{q}} R \simeq 0$  unless  $\mathfrak{q} \leq \mathfrak{p}$ . For the second condition we factor it as  $1 \longrightarrow L_{\mathfrak{p}} \longrightarrow \Lambda_{\mathfrak{p}} L_{\mathfrak{p}}$ , and it suffices to show that the fibres of both factors are supported in dimension  $\leq n - 1$ . This is true as before for the first map. For the second the fibre is of the form  $\text{Hom}(L_{\Lambda(\mathfrak{p})^c} R, L_{\mathfrak{p}} M)$ , and since  $K_{\mathfrak{p}}$  is small and  $\mathfrak{p} \notin \Lambda(\mathfrak{p})^c \cap V(\mathfrak{p})$  its tensor product with  $K_{\mathfrak{p}}$  is trivial.

Now consider the composite functors  $A_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} L_{\mathfrak{p}}$ , which is equipped with a natural transformation from the identity.

**Theorem 8.4** (Beilinson-Parshin Adelic Approximation). *Let  $\mathcal{C}$  be a Noetherian model category whose Balmer spectrum is of topological dimension  $r$ . Then  $\mathbb{1} \in \mathcal{C}$  is the homotopy pullback of the punctured  $n$ -cube  $\mathbb{1}_{BP}: (\Delta^r)' \rightarrow \mathcal{C}$  where the object at position  $(d_0 > d_1 > \cdots > d_s)$  is*

$$\mathbb{1}_{BP}(d_0 > d_1 > \cdots > d_s) = \prod_{\dim \mathfrak{p}_0 = d_0} \Lambda_{\mathfrak{p}_0} L_{\mathfrak{p}_0} \prod_{\substack{\mathfrak{p}_1 \subset \mathfrak{p}_0 \\ \dim \mathfrak{p}_1 = d_1}} \Lambda_{\mathfrak{p}_1} L_{\mathfrak{p}_1} \cdots \prod_{\substack{\mathfrak{p}_{s-1} \subset \mathfrak{p}_{s-2} \\ \dim \mathfrak{p}_{s-1} = d_{s-1}}} \Lambda_{\mathfrak{p}_{s-1}} L_{\mathfrak{p}_{s-1}} \prod_{\substack{\mathfrak{p}_s \subset \mathfrak{p}_{s-1} \\ \dim \mathfrak{p}_s = d_s}} \Lambda_{\mathfrak{p}_s} L_{\mathfrak{p}_s} \mathbb{1}$$

with maps as in Section 7.

**Remark 8.5.** If  $r = 1$  and the Balmer spectrum is irreducible, then (using Lemmas 6.10 and 6.11) the Beilinson-Parshin square coincides exactly with the standard adelic square.

**Example 8.6.** Let us rewrite Example 8.3 of the two-dimensional irreducible Balmer spectrum in the Beilinson-Parshin variant:

$$\begin{array}{ccccc}
& & \prod_{\mathfrak{p}_1} \Lambda_{\mathfrak{p}_1} L_{\mathfrak{p}_1} \mathbb{1} & \longrightarrow & L_{\mathfrak{p}_0} \prod_{\mathfrak{p}_1 \subset \mathfrak{p}_0} \Lambda_{\mathfrak{p}_1} L_{\mathfrak{p}_1} \mathbb{1} \\
& \nearrow & \downarrow & & \nearrow \\
\mathbb{1} & \longrightarrow & L_{\mathfrak{p}_0} \mathbb{1} & & \\
& \downarrow & \downarrow & & \downarrow \\
& & \prod_{\mathfrak{p}_1} \Lambda_{\mathfrak{p}_1} L_{\mathfrak{p}_1} \prod_{\mathfrak{p}_2 \subset \mathfrak{p}_1} \Lambda_{\mathfrak{p}_2} \mathbb{1} & \longrightarrow & L_{\mathfrak{p}_0} \prod_{\mathfrak{p}_1 \subset \mathfrak{p}_0} \Lambda_{\mathfrak{p}_1} L_{\mathfrak{p}_1} \prod_{\mathfrak{p}_2 \subset \mathfrak{p}_1} \Lambda_{\mathfrak{p}_2} \mathbb{1} \\
& \nearrow & \downarrow & & \nearrow \\
\prod_{\mathfrak{p}_2} \Lambda_{\mathfrak{p}_2} \mathbb{1} & \longrightarrow & L_{\mathfrak{p}_0} \prod_{\mathfrak{p}_2 \subset \mathfrak{p}_0} \Lambda_{\mathfrak{p}_2} \mathbb{1} & & 
\end{array}$$

Notice how the front face is the same as the standard version because of the properties of the completion at the maximal primes and the localizations at the minimal primes (Section 6.H).

## 9. THE ADELIC MODEL

The Adelic Approximation Theorem expresses the unit as a pullback of adelic rings. In this section we show that this in turn gives a model for a finite dimensional Noetherian model category  $\mathcal{C}$  in terms of categories of modules over these adelic rings.

**9.A. The diagram of adelic module categories.** The basic ingredient is the diagram  $\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}}$  of module categories. We use the constructions of Subsection 4.C to put model structures on the corresponding categories of generalized diagrams.

**Definition 9.1.** Let  $\mathcal{C}$  be a model category with  $r$ -dimensional Noetherian Balmer spectrum. We define the punctured  $(r+1)$ -cube of model categories by taking the value at  $(d_0 > d_1 \cdots > d_s)$  to be the module category of the corresponding adelic ring

$$[\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}}](d_0 > \cdots > d_s) = [\mathbb{1}_{ad}(d_0 > \cdots > d_s)]\text{-mod}_{\mathcal{C}}$$

equipped with the diagram injective model structure of Proposition 4.17. The morphisms in the diagram are the extension of scalars functors corresponding to the maps of rings.

Proposition 4.17 shows that the diagram-injective model structure on  $\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}}$  gives a cellular and proper model category,  $\mathcal{L}(\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}})$ , the lax limit of the diagram of module categories. This category of generalized diagrams is related to our original model category  $\mathcal{C}$  by a Quillen adjunction. Indeed an object  $X \in \mathcal{C}$  gives an object in  $\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}}$  by tensoring with the diagram  $\mathbb{1}_{ad}$ . This has right adjoint given by taking the inverse limit over the diagram.

**Proposition 9.2.** *There is a Quillen adjunction*

$$\mathbb{1}_{ad} \otimes - : \mathcal{C} \rightleftarrows \mathbb{1}_{ad}\text{-mod}_{\mathcal{C}} : \lim_{\leftarrow} . \quad \square$$

We apply the Cellularization Principle to this Quillen adjunction to obtain an equivalence. We then use the framework of Subsection 4.C to give a conceptual reformulation of the result.

**9.B.  $\mathcal{C}$  is the strict limit of the adelic diagram.** We are now ready to prove our main theorem.

**Theorem 9.3.** (The Adelic Model) *Let  $\mathcal{C}$  be a finite dimensional Noetherian model category. The adjunction of Proposition 9.2 induces a Quillen equivalence*

$$\mathcal{C} \simeq \mathcal{L}im(\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}})$$

*between  $\mathcal{C}$  and the strict homotopy limit of the diagram of adelic module categories. Any object is therefore equivalent to one in the cocartesian skeleton  $\text{Skel}(\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}})$  (i.e., with all base change maps being weak equivalences). Moreover, if  $\mathbb{1}_{ad}$  is a diagram of commutative ring objects, then the Quillen adjunction is symmetric monoidal.*

**Remark 9.4.** The same proof applied to Beilinson-Parshin approximation gives a Quillen equivalence

$$\mathcal{C} \simeq_Q \mathcal{L}im(\mathbb{1}_{BP}\text{-mod}_{\mathcal{C}}).$$

**Proof:** We start by applying the Cellularization Principle [GS13] to show that  $\mathcal{C}$  is equivalent to a cellularization of the category  $\mathcal{L}(\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}})$  of generalized diagrams. We choose a set  $\mathcal{G}$  of cofibrant small generators of  $\mathcal{C}$ . Accordingly  $\mathcal{G}\text{-cell-}\mathcal{C} \cong \mathcal{C}$  [GS13, Proposition 6.2]. To apply the Cellularization Principle, we need to show that the objects of  $\mathcal{G}$  are small in  $\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}}$  and that the derived unit is an equivalence.

It is clear that if  $X$  is small and cofibrant in  $\mathcal{C}$ , then it is small and cofibrant in each model category appearing in the diagram  $\mathbb{1}_{ad}$ . Consequently it is small and cofibrant in the diagram injective model structure on  $\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}}$  as in [GS13, Proposition 6.2]. Therefore, the images of elements of  $\mathcal{G}$  are small and cofibrant as required. The fact that the derived unit is an equivalence follows from the Adelic Approximation Theorem, 8.1. The Cellularization Principle gives a Quillen equivalence

$$\mathcal{C} \simeq_Q \mathcal{G}\text{-cell-}\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}}.$$

All that is left to show is that this cellularization with respect to  $\mathcal{G}$  is the strict homotopy limit. It is clear that every object  $g$  of  $\mathcal{G}$  gives an object  $\mathbb{1}_{ad} \otimes g$  of the cocartesian skeleton  $\text{Skel}(\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}})$ . Therefore what is left to show is that every object of the cocartesian skeleton can be built from the images of the generators. In particular, it is enough to show that  $\mathbb{1}_{ad} \otimes \mathcal{G}$  and  $\text{Skel}(\mathbb{1}_{ad}\text{-mod}_{\mathcal{C}})$  generate the same localizing subcategory. This follows from Proposition 9.5 below.

The Quillen equivalence being symmetric monoidal under the additional assumptions follows from [BGKS17, Proposition 5.1.6].  $\square$

**9.C. All homotopy cocartesian modules come from  $\mathcal{C}$ .** We shall follow the argument from [Gre08, Theorem 4.5] to prove the following result.

**Proposition 9.5.** *The thick subcategory of the homotopy category of  $\mathbb{1}_{ad}$ -modules obtained from  $\overline{\mathcal{C}}$  is precisely the category of homotopy cocartesian objects.*

**Proof:** We say that an object  $X$  of  $\mathbb{1}_{ad}$ -modules is *supported in dimension  $\leq d$*  if  $X(s) \simeq 0$  for  $s > d$ . We show by induction on  $d$  that all homotopy cocartesian objects supported in dimension  $\leq d$  are built from objects in the image of  $\mathcal{C}$ . This is clearly true if  $d = -1$  since these objects are all contractible.

Now suppose that objects supported in dimension  $\leq d - 1$  are built from objects in the image of  $\mathcal{C}$ , and suppose that  $X$  is supported in dimension  $\leq d$ . We will construct a  $\mathbb{1}_{ad}$ -module  $f_d(X(d))$  that comes from  $\mathcal{C}$  (and hence is homotopy cocartesian) and is supported in dimension  $\leq d$ , and a map

$$\eta: X \longrightarrow f_d(X(d))$$

that is an equivalence at  $d$ . It follows that the mapping cone of  $\eta$  is supported in dimension  $\leq d - 1$ , and hence built from objects in the image of  $\mathcal{C}$  by induction. Thus  $X$  is built from the image of  $\mathcal{C}$  as required. An arbitrary object is supported in dimension  $\leq r$  so after  $r$  steps we find all objects are built from the image of  $\mathcal{C}$ .

It remains to construct  $f_d(X(d))$  and the map  $\eta$ . In fact we will show that  $f_d$  is right adjoint to evaluation at  $d$  on a suitable subcategory of  $\mathbb{1}_{ad}$ -modules. We suppose given a  $\mathbb{1}_{ad}$ -module  $M$  and define a  $\mathbb{1}_{ad}$ -module  $f_d(M)$  as follows. Writing  $\mathbf{d} = (d_0 > d_1 > \cdots > d_s)$  for a flag of dimensions, define

$$f_d(M)(\mathbf{d}) = \begin{cases} \mathbb{1}_{ad}(\mathbf{d}) \otimes_{\mathbb{1}_{ad}(d)} M & \text{if } d_0 = d \\ \mathbb{1}_{ad}(d > \mathbf{d}) \otimes_{\mathbb{1}_{ad}(d)} M & \text{if } d_0 < d \\ 0 & \text{if } d_0 > d \end{cases} .$$

We note that the structure maps of  $f_d(M)$  are extensions of scalars for the maps of flags  $(d) \longrightarrow \mathbf{d}$  if  $d_0 = d$ , and we write  $\mathcal{C}_{ad}^{e(d)}$  for the category of  $\mathbb{1}_{ad}$ -modules with this property.

**Lemma 9.6.** *The functor  $f_d$  is right adjoint to evaluation  $ev_d: \mathcal{C}_{ad}^{e(d)} \longrightarrow \mathbb{1}_{ad}(d)$ -modules.*

**Proof:** We take the counit  $f_d(M)(d) = M \longrightarrow M$  to be the identity. To define the unit, we suppose  $X(d) = M$  and construct  $\eta: X \longrightarrow f_d(M)$ . Indeed, we take the identity at  $d$  and use extension of scalars to define  $\eta(\mathbf{d})$  when  $d = d_0$ , as we may do since  $X$  lies in  $\mathcal{C}_{ad}^{e(d)}$ . There is nothing to do at  $\mathbf{d}$  if  $d_0 > d$  since  $f_d(M)(\mathbf{d})$  is trivial. Finally, if  $d > d_0$  the definition is determined by the square

$$\begin{array}{ccc} X(\mathbf{d}) & \xrightarrow{\eta(\mathbf{d})} & \mathbb{1}_{ad}(d > \mathbf{d}) \otimes_{\mathbb{1}_{ad}(d)} M \\ \downarrow & & \downarrow = \\ X(d > \mathbf{d}) & \xrightarrow{\eta(d > \mathbf{d})} & \mathbb{1}_{ad}(d > \mathbf{d}) \otimes_{\mathbb{1}_{ad}(d)} M \end{array}$$

□

Applying this when  $M = X(d)$  we obtain a map  $\eta: X \longrightarrow f_d(X(d))$ . By construction  $f_d(X(d))$  is supported in dimensions  $\leq d$  and  $\eta(d)$  is an equivalence.

It remains to show that  $f_d(X(d))$  comes from  $\mathcal{C}$ . For this we note that all terms of  $f_d(\mathbb{1}_{ad}(d))$  are  $\mathbb{1}_{ad}(d)$ -modules, and

$$f_d(X(d)) = f_d(\mathbb{1}_{ad}(d)) \otimes_{\mathbb{1}_{ad}(d)} X(d).$$

The unit of the adjunction gives an obvious map  $\mathbb{1}_{ad} \longrightarrow f_d(\mathbb{1}_{ad}(d))$ . Tensoring with  $X(d)$  gives a map

$$\phi: \mathbb{1}_{ad} \otimes_{\mathbb{1}_{ad}(d)} X(d) \longrightarrow f_d(X(d)),$$

which we will show is an equivalence, thereby establishing that  $f_d(X(d))$  comes from  $\mathcal{C}$ .

It is clear that  $\phi(\mathbf{d})$  is an equivalence if  $d_0 = d$ , since the map is the identity. Since  $X(d)$  is supported in dimensions  $\leq d$ , it follows that if  $d_0 > d$  then  $\mathbb{1}_{ad}(\mathbf{d}) \otimes_{X(d)} X(d) \simeq 0$  and  $\phi(\mathbf{d})$  is an equivalence if  $d_0 > d$ .

Finally we consider  $d > d_0$ , where we have the map

$$\phi(\mathbf{d}): \mathbb{1}_{ad}(\mathbf{d}) \otimes_{\mathbb{1}_{ad}(d)} X(d) \longrightarrow \mathbb{1}_{ad}(d > \mathbf{d}) \otimes_{\mathbb{1}_{ad}(d)} X(d).$$

Cellularizing at any prime of dimension  $< d$ , both sides are contractible because  $X(d)$  is an  $\mathbb{1}_{ad}(d)$ -module. Cellularizing at any prime of dimension  $> d$ , both sides are contractible because  $X(d)$  is a torsion module. For primes of dimension  $d$  note that we are considering the map

$$\mathbb{1}_{ad}(\mathbf{d}) \longrightarrow \prod_{\dim(\mathfrak{p})=d} L_{\mathfrak{p}} e_{\mathfrak{p}} \mathbb{1}_{ad}(\mathbf{d}).$$

If we choose a prime  $\mathfrak{q}$  is of dimension  $d$  and cellularize, factors for  $\mathfrak{p} \neq \mathfrak{q}$  are trivial and we have

$$\mathbb{1}_{ad}(\mathbf{d}) \longrightarrow L_{\mathfrak{q}} e_{\mathfrak{q}} \mathbb{1}_{ad}(\mathbf{d}),$$

and this is an equivalence at  $\mathfrak{q}$  as required.  $\square$

**Remark 9.7.** The fact that the adelic model is equivalent to the strict homotopy limit relies on special properties of the adelic fracture square (essentially the stratification by dimension). Taking a commutative ring to its category of chain complexes does not preserve arbitrary finite homotopy limits. For example, we may consider the diagram of rings

$$\begin{array}{ccc} & & \mathbb{Z} \\ & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Q} \end{array}$$

whose pullback is  $\mathbb{Z}$ . The corresponding diagram of module categories is not a strict homotopy limit. Indeed, it is clear that the diagram

$$\begin{array}{ccc} & & \mathbb{Z}/p \\ & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

is in the strict limit but does not come from a  $\mathbb{Z}$ -module.

## 10. EXAMPLES

We comment on three classes of examples, from algebraic geometry, algebraic topology and representation theory.

10.A. **Algebraic geometry.** We have already seen the prototypical example coming from the theory of commutative algebra, that is, we take  $R$  a finite dimensional Noetherian ring and we can consider the projective model structure on  $\mathbf{Ch}(R\text{-mod})$  whose homotopy category is exactly the unbounded derived category of  $R$ . We also know that the Balmer spectrum  $\mathrm{Spc}^\omega(\mathbf{D}(R))$  is in bijection with  $\mathrm{Spec}(R)$ . The adelic approximation theorem then tells us that there is a natural way to reconstruct  $R$  from localized completed rings. In the case of a 2-dimensional irreducible Noetherian ring  $R$ , we would get a cube of the following form:

$$\begin{array}{ccccc}
& & \prod_{\mathfrak{p}} (R_{\mathfrak{p}}^{\wedge}) & \longrightarrow & R_{(0)} \otimes \prod_{\mathfrak{p}} (R_{\mathfrak{p}}^{\wedge}) \\
& & \downarrow & & \downarrow \\
R & \nearrow & & \longrightarrow & R_{(0)} \\
& & \downarrow & & \downarrow \\
& & \prod_{\mathfrak{p}} R_{\mathfrak{p}} \otimes \prod_{m \leq \mathfrak{p}} R_m^{\wedge} & \longrightarrow & R_{(0)} \otimes \prod_{\mathfrak{p}} R_{\mathfrak{p}} \otimes \prod_{m \leq \mathfrak{p}} R_m^{\wedge} \\
& & \downarrow & & \downarrow \\
& & \prod_m R_m^{\wedge} & \longrightarrow & R_{(0)} \otimes \prod_m R_m^{\wedge}
\end{array}$$

The category of  $R$ -modules is equivalent to the category of quasi-coherent sheaves on  $\mathrm{Spec}(R)$ . Accordingly, chain complexes of  $R$ -modules is a special case of a wider class of examples arising from algebraic geometry.

More generally, we let  $X$  be a topologically Noetherian scheme. In [Bal05, Corollary 5.6], it was shown that  $\mathrm{Spc}^\omega(\mathbf{D}(\mathbf{Qcoh}(\mathcal{O}_X)))$  can be used to recover the scheme  $X$ . That is, there is a homeomorphism  $f: X \xrightarrow{\sim} \mathrm{Spc}^\omega(\mathbf{D}(\mathbf{Qcoh}(\mathcal{O}_X)))$  with

$$f(x) = \{a \in \mathbf{Perf}(X) \mid a_x \simeq 0 \text{ in } \mathbf{Perf}(\mathcal{O}_{X,x})\} \quad \text{for all } x \in X.$$

Since  $X$  is topologically Noetherian,  $\mathbf{D}(\mathbf{Qcoh}(\mathcal{O}_X))$  is a Noetherian tensor-triangulated category which is generated by a single perfect complex  $P$  [Sta18, Tag 09IS].

Our main result therefore shows that  $\mathbf{Ch}(\mathbf{Qcoh}(X))$  with the projective model structure is Quillen equivalent to  $\mathcal{L}im(\mathbb{1}_{ad}\text{-mod})$ , where the values of  $\mathbb{1}_{ad}$  are the adelic restricted products built from the localized completed stalks  $((\mathcal{O}_X)_x^{\wedge})_x$ .

This extends to algebraic stacks. Quasi-coherent sheaves on suitable algebraic stacks carry a compatible model structure [Est15]. When  $\mathcal{X}$  is a tame stack, the Balmer spectrum of the derived category of perfect complexes of quasi-coherent  $\mathcal{O}_{\mathcal{X}}$ -modules has been computed [Hal16]. Under the tameness condition, the category of quasi-coherent sheaves on  $\mathcal{X}$  is a compactly rigidly-generated model category.

10.B. **Algebraic topology.** We describe two examples. In chromatic homotopy theory at  $p$  there is only one prime of each dimension so the adelic flavour is not expressed, but our approach gives a module-theoretic version of chromatic fracture methods. The second example is rational equivariant cohomology theories, and indeed, the principal motivation this work was to understand the underpinnings of the results of [GS18]

10.B.1. *Chromatic homotopy theory.* We pick a suitable model for the category of spectra, such as orthogonal spectra [MMSS01]  $\mathbf{Sp}^{\mathcal{O}}$ . The homotopy category is a rigidly small-generated tensor-triangulated category (with the sphere spectrum  $\mathbb{S}^0$  as small generator). The Balmer spectrum of  $\mathbf{Spectra}^{\omega}$  was identified in [HS98]. To describe this, for  $n \geq 1$  we write  $K(p, n)$  for the  $n$ th mod  $p$  Morava  $K$ -theory, extending this to let  $K(p, \infty) = HF_p$  denote mod  $p$  cohomology and  $K(p, 0) = H\mathbb{Q}$  denote rational cohomology. The primes are

$$\mathcal{P}(p, n) = \{X \mid K(p, n)_*(X) = 0\}.$$

When  $n = 0$ , this is independent of  $p$  and consists of the finite torsion spectra.

For each prime integer prime  $p > 0$  the Balmer primes are linearly ordered:

$$\mathcal{P}(p, 0) \supset \mathcal{P}(p, 1) \supset \mathcal{P}(p, 2) \supset \cdots \supset \mathcal{P}(p, \infty).$$

Corresponding to  $\mathcal{P}(p, n)$  we have a localization and a completion. The localization  $L_{\mathcal{P}(p, n)}$  is finite Bousfield localization with respect to  $K(p, 0) \vee K(p, 1) \vee \cdots \vee K(p, n)$ . If  $p$  is understood this is usually written  $L_n^f$ . When the telescope conjecture is true (it is known only for  $n = 0, 1$ ) this coincides with the ordinary Bousfield localization with respect to  $K(p, 0) \vee K(p, 1) \vee \cdots \vee K(p, n)$ . The completion corresponding to  $\mathcal{P}(p, n)$  is the Bousfield localization  $L_{K(p, n)}$  with respect to  $K(p, n)$ .

From the infinite decreasing chain we see that this Balmer spectrum is not Noetherian, and the infinite primes  $K(n, \infty)$  are not visible. For the prime  $\mathcal{P}(p, n)$  with  $n$  finite, we may take  $K_{\mathcal{P}(p, n)}$  to be any generalized Smith-Toda complex  $\mathbb{1}/p^{i_0}, v_1^{i_1}, \dots, v_n^{i_n}$ . In particular, all primes  $\mathcal{P}_{p, n}$  with  $n$  finite are visible.

However, we can take the  $L_n$  localization of  $\mathbf{Sp}^{\mathcal{O}}$  at a fixed prime  $p$  which will truncate the Balmer spectrum to a single branch (namely the  $p$ -branch) truncated to level  $n$ .

Now, let us fix a prime  $p$  consider the category  $L_n \mathbf{Sp}^{\mathcal{O}}$  of those spectra with chromatic level at most  $n$ . Then from the above discussion we know that this is a Noetherian model category where the monoidal unit coincides with small generator  $L_n \mathbb{S}^0$ . Our main result therefore shows that  $L_n \mathbf{Sp}^{\mathcal{O}}$  is Quillen equivalent to  $\mathcal{L}im(\mathbb{1}_{ad}\text{-mod})$ , where the values of  $\mathbb{1}_{ad}$  are the localizations  $L_t L_{K(s)} \mathbb{S}^0$  for  $n \geq t \geq s$ .

**Remark 10.1.** In chromatic homotopy theory, results similar to ours are well known. To start with, the one dimensional case for assembling two adjacent chromatic levels is the chromatic fracture square and [BAC14] extends this to higher cubes. However these differ from our results in two ways.

Firstly, they express the  $E(n)$ -local category in terms of the  $L_{\mathcal{P}(p, i)}$ - and  $\Lambda_{\mathcal{P}(p, i)}$ -localizations of the category for  $i \leq n$  whereas our model is in terms of *module* categories over the localized rings. Secondly, for dimensions higher than 2, the diagrams of [BAC14] are more complicated, and oriented limits are used.

10.B.2. *Rational torus-equivariant cohomology theories.* We finally return to the examples that spawned this research, namely the algebraic models of  $G$ -equivariant rational cohomology theories. The results in the present paper reproduce the first step in the construction of the algebraic models for tori. The final result requires (i) an argument to show that we may take fixed points without losing information (ii) application of Shipley's equivalence (iii) proving commutativity and (iv) a formality argument [GS18, BGKS17, Bar17, Ke17].

For a general compact Lie group  $G$ , where we recall that an inclusion  $K \subseteq H$  of subgroups is said to be *cotoral* if  $K$  is normal and  $H/K$  is a torus. As described in Subsection 4.15, as

a poset, the Balmer spectrum consists of the conjugacy classes of subgroups under cotoral inclusion.

10.B.3. *G a finite group.* The easiest case is when  $G$  is finite, when we recover the Quillen equivalence of [Bar09]. The Balmer spectrum of  $\mathbf{Sp}_{\mathbb{Q}}^G$  is a discrete space, with the points being conjugacy classes of subgroups. In particular, the Balmer spectrum is 0-dimensional. Module categories over a finite product of rings splits correspondingly [Bar09].

**Corollary 10.2.** *Let  $G$  be a finite group, then*

$$\mathbf{Sp}_{\mathbb{Q}}^G \simeq_Q \prod_{H \in \text{Sub}(G)/G} (L_H \mathbb{S})\text{-mod} \simeq_Q \prod_{H \in \text{Sub}(G)/G} \mathbf{Ch}(\mathbb{Q}[W_G H]\text{-mod}).$$

**Proof:** In the 0-dimensional case Theorem 9.3 states that  $\mathcal{C}$  is equivalent to the category of modules over the product ring  $\prod_H L_H \mathbb{S}$ . Barnes's Theorem shows this is a product of module categories of the individual rings  $L_H \mathbb{S}$ : this gives the first equivalence. Morita theory and the formality of  $[L_H \mathbb{S}, L_H \mathbb{S}]^G = \mathbb{Q}W_G(H)$  gives the second equivalence.  $\square$

10.B.4. *G =  $\mathbb{T}$ , the circle group.* Here we recover a theorem of [GS18]. The Balmer spectrum of  $\mathbf{Sp}_{\mathbb{Q}}^{\mathbb{T}}$  is

$$\text{Spc}^{\omega}(\mathbf{Sp}_{\mathbb{Q}}^{\mathbb{T}}) = \begin{array}{c} \mathbb{T} \\ \nearrow \quad \nearrow \quad \uparrow \quad \searrow \\ \cdots \quad C_4 \quad C_3 \quad C_2 \quad C_1 \end{array}$$

where  $C_i$  is the cyclic group of order  $i$ . The diagram  $\mathbb{1}_{ad}$  is then

$$\begin{array}{c} L_{\mathbb{T}} \mathbb{S} \\ \downarrow \\ \prod_n \Lambda_{C_n} \mathbb{S} \longrightarrow L_{\mathbb{T}} \prod_n \Lambda_{C_n} \mathbb{S} . \end{array}$$

which can be written as

$$\mathbb{1}_{ad} = \left( \begin{array}{ccc} & \tilde{E}\mathcal{F} & \\ & \downarrow & \\ \prod_n DE\langle n \rangle_+ & \longrightarrow & \tilde{E}\mathcal{F} \wedge \prod_n DE\langle n \rangle_+ \end{array} \right) \simeq \left( \begin{array}{ccc} & \tilde{E}\mathcal{F} & \\ & \downarrow & \\ DEF_+ & \longrightarrow & \tilde{E}\mathcal{F} \wedge DEF_+ \end{array} \right)$$

which is the usual Tate square for rational  $\mathbb{T}$ -equivariant homotopy theory [GM95].

**Corollary 10.3.** *There is a Quillen equivalence*

$$\mathbf{Sp}_{\mathbb{Q}}^{\mathbb{T}} \simeq \mathcal{L}im \left( \begin{array}{ccc} & \tilde{E}\mathcal{F}\text{-mod} & \\ & \downarrow & \\ DEF_+\text{-mod} & \longrightarrow & \tilde{E}\mathcal{F} \wedge DEF_+\text{-mod} \end{array} \right).$$

10.B.5. *Aside on invertible objects.* Continuing with rational  $\mathbb{T}$ -spectra, the adelic point of view gives an interesting perspective on spheres  $S^V$ , where  $V^{\mathbb{T}} = 0$ . Spheres are invertible objects, and as such their models are invertible at each Balmer prime. Because the rings  $\mathbb{Q}$  and  $\mathbb{Q}[c]$  are local, invertible modules are free so the module at each prime is an integer suspension of the ring. More explicitly, the model of  $S^V$  is

$$\prod_F \Sigma^{|V^F|} \mathbb{Q}[c] \longrightarrow \mathcal{E}^{-1} \prod_F \mathbb{Q}[c].$$

$$\begin{array}{c} \mathbb{Q} \\ \downarrow \\ \mathcal{E}^{-1} \prod_F \mathbb{Q}[c] \end{array}$$

All three terms are invertible modules over their respective rings and maps between them given by units in the respective rings will give isomorphisms. Because of grading, isomorphism is rather straightforward in this case.

More generally, we may have a cospan

$$\begin{array}{ccc} & B & \\ & \downarrow & \\ C & \longrightarrow & D \end{array}$$

of commutative rings from the adelic square of a 1-dimensional Balmer spectrum. We may ask for isomorphism classes of objects which are free on one generator at each Balmer prime but where the horizontal and vertical maps are non-standard. It may even happen that the  $B$ -module,  $C$ -module and  $D$ -module are  $B, C$  and  $D$  respectively. In this case the identification of the images of  $B$  and  $C$  is given by a unit in  $D$ , so that elements of

$$\text{cok}(B^\times \times C^\times \longrightarrow D^\times)$$

will give rise to exotic invertible objects.

One type of example arises when  $H_{ad}^1 = 0$  (i.e., when the maps  $B \longrightarrow D$  and  $C \longrightarrow D$  are jointly surjective). For example, by [Gre19] this occurs when we have the adelic square for the ring of integers  $A$  in a number field. In this case invertible elements correspond to elements of the class group of  $A$ , and there is a Mayer-Vietoris sequence for  $K_0(A)$ . A second class of examples arise for  $H_{ad}^1 \neq 0$ . This occurs for rational  $G$ -equivariant cohomology theories for a 1-dimensional abelian compact Lie group  $G$  (as we have just seen for  $G = \mathbb{T}$ ) where representation spheres give invertible objects. It also arises from quasi-coherent sheaves on a projective curve, where line bundles give invertible objects.

10.B.6.  *$G$  a torus.* Let  $\mathbb{T}^n$  be an  $n$ -dimensional torus, then the category of  $\mathbb{T}^n$ -equivariant rational cohomology theories is an  $n$ -dimensional Noetherian model category. For  $r = 2$ ,

[GS18] constructs the Adelic Approximation Cube

$$\begin{array}{ccccc}
& \prod_H S^{\infty V(H)} \wedge DEF/H_+ & \longrightarrow & S^{\infty V(G)} \wedge \prod_H (S^{\infty V(K_s)} \wedge DEF/K_{s+}) & \\
& \nearrow & \downarrow & \nearrow & \downarrow \\
S^0 & \xrightarrow{\quad} & S^{\infty V(G)} & & \\
& \downarrow & \downarrow & & \downarrow \\
& \prod_H S^{\infty V(H)} \wedge DEF_+ & \longrightarrow & S^{\infty V(G)} \wedge \prod_H (S^{\infty V(H)} \wedge DEF_+) & \\
& \nearrow & \downarrow & \nearrow & \\
DEF_+ & \xrightarrow{\quad} & S^{\infty V(G)} \wedge DEF_+ & & 
\end{array}$$

where  $S^{\infty V(H)} = \bigcup_{V_H=0} S^V$  and  $\mathcal{F}/H$  is the family of subgroups with finite image in  $G/H$ .

We note that for ranks  $r \geq 2$  this cube from [GS18] uses  $DEF_+$  because of its multiplicative properties. This differs from slightly different from the one used in this paper, because  $DEF/H_+$  involves a product over all finite subgroups instead of just those contained in the subgroup of the next dimension. The two cubes have the same pullback (see [Gre19]).

**10.C. Representation theory.** Our final example comes from modular representation theory. We take  $G$  to be a finite group and  $k$  a field with characteristic dividing the order of  $G$ , we then consider the category of  $kG$ -modules. Say that two morphisms  $f, g: M \rightarrow N$  are *stably homotopic* if  $f - g: M \rightarrow N$  factors through a projective module. The weak equivalences are then the homotopy equivalences. The fibrations are the epimorphisms and the cofibrations are the monomorphisms. The homotopy category of this model category is the stable module category  $\mathbf{Mod} - kG$  [Hov99, 2.2.12].

This is a rigidly small-generated tensor-triangulated category with unit object the trivial representation  $k$ . The Balmer spectrum of the small objects is the same as the space which underlies the projective scheme associated to the group cohomology ring [BCR97]:

$$\mathrm{Spc}^\omega(\mathbf{Mod} - kG) \cong \mathrm{Proj} H^*(G; k).$$

In particular, since  $H^*(G; k)$  is a Noetherian graded ring by Venkov's Theorem, the Balmer spectrum is Noetherian.

Accordingly, the adelic model for  $\mathbf{Mod} - kG$  is closely related to the the adelic model for quasi-coherent modules over  $\mathrm{Proj} H^*(G; k)$ . In fact the fibre over a homogenous prime  $\mathfrak{p}$  in the respective adelic models is  $(C^*(BG; k)_\mathfrak{p}^\wedge)_\mathfrak{p}$  in the first case and  $(H^*(BG; k)_\mathfrak{p}^\wedge)_\mathfrak{p}$  in the second case.

Usually these are different: for example, even if  $G$  is cyclic of odd prime order, one sees there are Massey products showing the cochain algebra is not formal.

## REFERENCES

[AMGR19] David Ayala, Aaron Mazel-Gee, and Nick Rozenblum. Stratified noncommutative geometry. *arXiv preprints*, 2019. arXiv:1910.14602.

- [BaGre20] S. Balchin and J. P. C. Greenlees. The separated adelic model of a tensor triangulated category, *In preparation* 2020.
- [Bal05] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.
- [Bal07] Paul Balmer. Supports and filtrations in algebraic geometry and modular representation theory. *American J. Math.*, 129 (5):1227–1250, 2007.
- [BF11] Paul Balmer and Giordano Favi. Generalized tensor idempotents and the telescope conjecture. *Proc. Lond. Math. Soc. (3)*, 102(6):1161–1185, 2011.
- [Bar09] David Barnes. Splitting monoidal stable model categories. *J. Pure Appl. Algebra*, 213(5):846–856, 2009.
- [Bar17] David Barnes. Rational  $O(2)$ -equivariant spectra. *Homology Homotopy Appl.*, 19(1):225–252, 2017.
- [BGKS17] David Barnes, J. P. C. Greenlees, Magdalena Kędziorek, and Brooke Shipley. Rational  $SO(2)$ -equivariant spectra. *Algebr. Geom. Topol.*, 17(2):983–1020, 2017.
- [BR15] David Barnes and Constanze Roitzheim. Homological localisation of model categories. *Appl. Categ. Structures*, 23(3):487–505, 2015.
- [BAC14] Tobias Barthel and Omar Antolín-Camarena. Chromatic fracture cubes. *arXiv preprints*, 2014. arXiv:1410.7271.
- [Bar10] Clark Barwick. On left and right model categories and left and right Bousfield localizations. *Homology Homotopy Appl.*, 12(2):245–320, 2010.
- [Bei80] A. A. Beilinson. Residues and adèles. *Funktsional. Anal. i Prilozhen.*, 14(1):44–45, 1980.
- [BCR97] David J. Benson, Jon F. Carlson, and Jeremy Rickard. Thick subcategories of the stable module category. *Fundam. Math.*, 153(1):59–80, 1997.
- [BIK12] David J. Benson, Srikanth B. Iyengar, and Henning Krause. Colocalizing subcategories and cosupport. *J. Reine Angew. Math.*, 673:161–207, 2012.
- [Ber07] Julia E. Bergner. Three models for the homotopy theory of homotopy theories. *Topology*, 46(4):397–436, 2007.
- [Ber12] Julia E. Bergner. Homotopy limits of model categories and more general homotopy theories. *Bull. Lond. Math. Soc.*, 44(2):311–322, 2012.
- [Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.
- [BKS] A. B. Buan, H. Krause and Ø. Solberg. Support varieties: an ideal approach. *Homology, Homotopy Appl.* 9(1):45–74, 2007.
- [Cis08] Denis-Charles Cisinski. Propriétés universelles et extensions de Kan dérivées. *Theory Appl. Categ.*, 20:No. 17, 605–649, 2008.
- [DG02] W. G. Dwyer and J. P. C. Greenlees. Complete modules and torsion modules. *Amer. J. Math.*, 124(1):199–220, 2002.
- [EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.
- [Est15] Sergio Estrada. The derived category of quasi-coherent modules on an Artin stack via model structures. *Int. Math. Res. Not. IMRN*, (15):6411–6432, 2015.
- [Gre08] J. P. C. Greenlees. Rational torus-equivariant stable homotopy I: calculating groups of stable maps *J. Pure Appl. Algebra*, 212(1):72–98, 2008.
- [Gre17] J. P. C. Greenlees. The Balmer spectrum of rational equivariant cohomology theories *J. Pure Appl. Algebra* **223** (2019) 2845–2871, arXiv:1706.07868.
- [Gre19] J. P. C. Greenlees. Adelic cohomology, 2019. arXiv:1903.02668
- [GM92] J. P. C. Greenlees and J. P. May. Derived functors of  $I$ -adic completion and local homology. *J. Algebra*, 149(2):438–453, 1992.
- [GM95] J. P. C. Greenlees and J. P. May. “Generalized Tate cohomology” *Memoirs of the American Maths. Soc.*, **543** (1995) 178pp.
- [GS13] J. P. C. Greenlees and B. Shipley. The cellularization principle for Quillen adjunctions. *Homology Homotopy Appl.*, 15(2):173–184, 2013.

- [GS14] J. P. C. Greenlees and B. Shipley. Homotopy theory of modules over diagrams of rings. *Proc. Amer. Math. Soc. Ser. B*, 1:89–104, 2014.
- [GS18] J. P. C. Greenlees and B. Shipley. An algebraic model for rational torus-equivariant spectra. *J. Topology*, **11** (2018) 666–719, arXiv:1101.2511v6.
- [Hal16] Jack Hall. The Balmer spectrum of a tame stack. *Ann. K-Theory*, 1(3):259–274, 2016.
- [Har67] Robin Hartshorne. *Local cohomology, A 1961 seminar given by A. Grothendieck, Harvard University, Fall*, Lecture Notes in Mathematics **41**, Springer-Verlag, Berlin, 1967.
- [Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
- [Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
- [HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.
- [Hub91] A. Huber. On the Parshin-Beilinson adèles for schemes. *Abh. Math. Sem. Univ. Hamburg*, 61:249–273, 1991.
- [Kę17] Magdalena Kędziorek. An algebraic model for rational  $SO(3)$ -spectra. *Algebr. Geom. Topol.*, 17(5):3095–3136, 2017.
- [MM02] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and  $S$ -modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002.
- [MMS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc. (3)*, 82(2):441–512, 2001.
- [Mil92] Haynes Miller. Finite localizations. *Bol. Soc. Mat. Mexicana (2)*, 37(1-2):383–389, 1992. Papers in honor of José Adem (Spanish).
- [Mor12] Mathew Morrow. An introduction to higher dimensional local fields. *arXiv preprints*, 2012. arXiv:1204.0586.
- [SS00] Stefan Schwede and Brooke Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.
- [SS03] Stefan Schwede and Brooke Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003.
- [Sta18] The Stacks Project Authors. *Stacks Project*. <http://stacks.math.columbia.edu>, 2018.
- [Ste13] Greg Stevenson. Support theory via actions of tensor triangulated categories. *J. Reine Angew. Math.*, 681:219–254, 2013.
- [Ste17] Greg Stevenson. The local-to-global principle for triangulated categories via dimension functions. *J. Algebra*, 473:406–429, 2017.
- [Ste18] Greg Stevenson. A tour of support theory for triangulated categories through tensor triangular geometry Building bridges between algebra and topology, Adv. Courses Math. CRM Barcelona, 63–101, Birkhäuser/Springer, Cham, (2018),
- [Toë06] Bertrand Toën. Derived Hall algebras. *Duke Math. J.*, 135(3):587–615, 2006.
- [TV08] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.

(Balchin) WARWICK MATHEMATICS INSTITUTE, ZEEMAN BUILDING, COVENTRY, CV4 7AL, UK

(Greenlees) WARWICK MATHEMATICS INSTITUTE, ZEEMAN BUILDING, COVENTRY, CV4 7AL, UK