Mathematische Zeitschrift
https://doi.org/10.1007/s00209-020-02554-0

Mathematische Zeitschrift

An algebraic model for rational naïve-commutative ring $SO(2)$-spectra and equivariant elliptic cohomology

David Barnes[1] · J. P. C. Greenlees[2] · Magdalena Kędziorek[3]

Received: 21 October 2018 / Accepted: 7 May 2020

Abstract

Equipping a non-equivariant topological E_∞-operad with the trivial G-action gives an operad in G-spaces. For a G-spectrum, being an algebra over this operad does not provide any multiplicative norm maps on homotopy groups. Algebras over this operad are called naïve-commutative ring G-spectra. In this paper we take $G = SO(2)$ and we show that commutative algebras in the algebraic model for rational $SO(2)$-spectra model rational naïve-commutative ring $SO(2)$-spectra. In particular, this applies to show that the $SO(2)$-equivariant cohomology associated to an elliptic curve C of Greenlees (Topology 44(6):1213–1279, 2005) is represented by an E_∞-ring spectrum. Moreover, the category of modules over that E_∞-ring spectrum is equivalent to the derived category of sheaves over the elliptic curve C with the Zariski torsion point topology.

Contents

✉ Magdalena Kędziorek
m.kedziorek@math.ru.nl

David Barnes
d.barnes@qub.ac.uk

J. P. C. Greenlees
john.greenlees@warwick.ac.uk

1 Mathematical Sciences Research Centre, Queen's University Belfast, Belfast, UK

2 Warwick Mathematics Institute, Coventry, UK

3 IMAPP, Radboud University Nijmegen, Nijmegen, The Netherlands

Published online: 10 August 2020

1 Introduction

Rational equivariant cohomology theories

We are interested in the category of rational G, where $G = SO(2)$ is the circle group and the indexing universe is a complete G-universe U. This is a model for the rational equivariant stable homotopy category where all G-representation spheres are invertible. Building on work of Greenlees and Shipley [15] and Barnes [2], Barnes et al. [5] gave a symmetric monoidal algebraic model for the category of rational G, when $G = SO(2)$. As a consequence, one obtains a model for rational ring G in terms of monoids in the algebraic model. However, this does not imply analogous results about strict commutative rational ring G-spectra (Comm-algebras). This is because of the well-known but surprising result that *symmetric* monoidal Quillen functors can fail to preserve *commutative* monoids (algebras for the operad Comm) in the equivariant setting.

Recent work of Blumberg and Hill [7] describes a class of commutative multiplicative structures on the equivariant stable homotopy category. These multiplicative structures are governed by G-operads called N_∞-operads. Roughly speaking, such a multiplicative structure is characterised by the set of Hill–Hopkins–Ravenel norms (see [18]) which exist on the commutative algebras corresponding to it. Originally [7] considered N_∞-operads for a finite group G, however the definition can be extended to any compact Lie group G. The present paper uses only the most trivial N_∞-operad E^1_∞ which for any group G is modelled by a non-equivariant topological E_∞-operad inflated to the equivariant world of G-topological spaces.

The work of Blumberg and Hill raises a question: which level of commutative ring $\mathbb{T}$ is modelled by commutative algebras in the algebraic model of [5]?

For G a finite group, an understanding of which levels of commutativity are visible in the algebraic models for rational G was done in [4]. The new ingredient there was an analysis of which localised model structures $L_A G\mathrm{Sp}$ can be right-lifted to the category of $\mathcal{O}$-algebras in $L_A G\mathrm{Sp}$. Here, $L_A G\mathrm{Sp}$ is a left Bousfield localisation at an object A of the positive stable model structure on G-spectra and $\mathcal{O}$ is an N_∞-operad.

The work mentioned above is related to another surprising fact in equivariant homotopy theory, namely that a left Bousfield localisation of a genuinely commutative ring G-spectrum might fail to be commutative. Historically, the first example of that phenomenon appeared in [25], where McClure showed that for a finite group G and a family $\mathcal{F}$ of proper subgroups of G, $\widetilde{E}\mathcal{F} \simeq L_{\widetilde{E}\mathcal{F}}\mathbb{S}$ does not have a strictly commutative ring spectrum model. Here $\widetilde{E}\mathcal{F}$ is the cofibre of the natural map from a universal space $E\mathcal{F}_+$ to S^0. Recently, more examples have been discussed in [17,19].

The failure of $\widetilde{E}\mathcal{F}$ to be genuinely-commutative for a finite G is related to the fact that the restriction of $\widetilde{E}\mathcal{F}$ to $H\,\mathrm{Sp}$ for any proper subgroup H of G is trivial, while $\widetilde{E}\mathcal{F}$ is not contractible in $G\mathrm{Sp}$. If $\widetilde{E}\mathcal{F}$ were commutative, then there would exist a ring map $* \simeq N^G_H \mathrm{res}^G_H \widetilde{E}\mathcal{F} \longrightarrow \widetilde{E}\mathcal{F}$, which gives a contradiction, since $\widetilde{E}\mathcal{F}$ is not equivariantly contractible. Here N^G_H denotes a norm *functor* (see [18]) which is a left adjoint to the restriction res^G_H at the level of equivariant commutative ring spectra (algebras for the operad Comm).

In the case of the circle, we will always take $\mathcal{F}$ to be the collection of all finite subgroups. Whereupon, $\widetilde{E}\mathcal{F}$ has a strictly commutative model by [12]. The different behaviour in case of a circle comes from the fact that all proper subgroups are of infinite index in $SO(2)$ and the norm maps can only link subgroups K and H if $K \leq H$ is of finite index in H. We will use this observation in Sect. 4, where we summarise the zig–zag of Quillen equivalences to the algebraic model for rational $\mathbb{T}$-spectra.

Contents of part 1 of this paper

Let $G = \mathbb{T} = SO(2)$ and write E^1_∞ for the *non-equivariant* E_∞-operad equipped with the trivial $\mathbb{T}$-action.

The main theorem of part 1 appears later in the paper as Theorem 5.8.

Theorem A *There is a zig–zag of Quillen equivalences*

$$\mathrm{E}^1_\infty\text{−alg-in−}(\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}) \simeq \mathit{Comm}\text{−alg-in−}d\mathcal{A}(\mathbb{T}),$$

where $d\mathcal{A}(\mathbb{T})$ denotes the algebraic model for rational $\mathbb{T}$-spectra from [5,10].

In view of [7] E^1_∞-operad has the least commutative structure, in particular it encodes the algebra structure without multiplicative norms.

Note that the operad Comm in the algebraic model doesn't encode any additional structure beyond that of commutative algebras in the usual sense, and thus the category of Comm-algebras in the algebraic model can only model the rational naïve-commutative ring $\mathbb{T}$-spectra.

The present work is the first step towards obtaining an algebraic model for $\mathbb{T}$-spectra that are equivariantly commutative in the strongest sense. Notice that, in the case of a finite group G, there was a clear reason why it should be expected that the model categorical methods used to obtain the monoidal algebraic model only capture the lowest level of commutativity in the equivariant setting. The methods for rational G for a finite G rely on the complete idempotent splitting using results of Barnes [1]. The splitting operation does not preserve norms, and hence won't preserve more structured commutative objects (this is related to the work of McClure [25] mentioned earlier).

For the case of $\mathbb{T}$, the additional step beyond the splitting for finite subgroups is to use an isotropy separation technique (known as the Tate square or Hasse square). For suitable families of subgroups, this technique preserves the strongest form of equivariant commutativity, so there is a good chance of obtaining an algebraic model for more structured commutative $\mathbb{T}$-ring spectra. This is work in progress.

Summary of the zig–zag of Quillen equivalences

To illustrate the zig–zag of Quillen equivalences from [5, Theorem 5.2.1] we present a diagram of key steps. At the top we have our preferred model for rational $\mathbb{T}$-spectra (namely the left Bousfield localization $L_{\mathbb{S}_\mathbb{Q}}\mathbb{T}\mathrm{Sp}$ of the category of orthogonal $\mathbb{T}$ at the rational sphere spectrum). At the bottom we have the algebraic model $d\mathcal{A}(\mathbb{T})_{dual}$ with the dualizable model structure from [2].

The reader may wish to refer to this diagram now, but the notation will be introduced as we proceed. In the diagram, left Quillen functors are placed on the left and $\mathbb{T} = SO(2)$. A lift to the level of algebras over an E^1_∞-operad is given on the left, on the right there is an indication of the ambient category.

$$
\begin{tikzcd}[row sep=large]
\mathrm{E}^1_\infty\text{--alg-in--}\mathbb{T}\mathrm{Sp}_\mathbb{Q} \arrow[d, shift right, "\mathbb{S}^{\lrcorner}\wedge-"'] & \mathbb{T}\mathrm{Sp}_\mathbb{Q} \arrow[d, shift right, "\mathbb{S}^{\lrcorner}\wedge-"'] & \text{in } \mathbb{T}\mathrm{Sp} \\
\mathrm{E}^1_\infty\text{--alg-in--}K_{\mathsf{Top}}\text{--cell--}\mathbb{S}^{\lrcorner}_{\mathsf{Top}}\text{--mod} \arrow[u, shift right, "\mathrm{pb}"'] \arrow[d, shift right, "(-)^{\mathbb{T}}"'] & K_{\mathsf{Top}}\text{--cell--}\mathbb{S}^{\lrcorner}_{\mathsf{Top}}\text{--mod} \arrow[u, shift right, "\mathrm{pb}"'] \arrow[d, shift right, "(-)^{\mathbb{T}}"'] & \text{in } \mathbb{T}\mathrm{Sp} \\
\mathrm{E}^1_\infty\text{--alg-in--}K^{\mathbb{T}}_{\mathsf{Top}}\text{--cell--}(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}\text{--mod} \arrow[u, shift right, "\zeta_\#"'] \arrow[d, shift right, "\text{change}"'] & K^{\mathbb{T}}_{\mathsf{Top}}\text{--cell--}(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}\text{--mod} \arrow[u, shift right, "\zeta_\#"'] \arrow[d, leftrightarrow, "\mathrm{Id}"'] & \text{in } \mathrm{Sp} \\
\mathsf{Comm}\text{--alg-in--}K^{\mathbb{T}}_{\mathsf{Top}}\text{--cell--}(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}\text{--mod} \arrow[u, shift right, "\text{of operads}"'] \arrow[d, leftrightarrow, "\text{zig--zag}"', "\text{of Quillen equivalences}"] & K^{\mathbb{T}}_{\mathsf{Top}}\text{--cell--}(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}\text{--mod} \arrow[d, leftrightarrow, "\text{zig--zag}"', "\text{of Quillen equivalences}"] & \text{in } \mathrm{Sp} \\
\mathsf{Comm}\text{--alg-in--}K_t\text{--cell--}\mathbb{S}^{\lrcorner}_t\text{--dmod} \arrow[d, leftrightarrow, "\text{zig--zag}"', "\text{of Quillen equivalences}"] & K_t\text{--cell--}\mathbb{S}^{\lrcorner}_t\text{--dmod} \arrow[d, leftrightarrow, "\text{zig--zag}"', "\text{of Quillen equivalences}"] & \text{in } \mathrm{Ch}(\mathbb{Q}) \\
\mathsf{Comm}\text{--alg-in--}K_a\text{--cell--}\mathbb{S}^{\lrcorner}_a\text{--dmod} \arrow[d, shift left, "\Gamma"] & K_a\text{--cell--}\mathbb{S}^{\lrcorner}_a\text{--dmod} \arrow[d, shift left, "\Gamma"] & \text{in } \mathrm{Ch}(\mathbb{Q}) \\
\mathsf{Comm}\text{--alg-in--}d\mathcal{A}(\mathbb{T})_{dual} \arrow[u, shift left, "\mu"] & d\mathcal{A}(\mathbb{T})_{dual} \arrow[u, shift left, "\mu"] & \text{in } \mathrm{Ch}(\mathbb{Q})
\end{tikzcd}
$$

In the above the subscript 'Top' indicates that the corresponding object has a topological origin, whereas the subscript 't' indicates that the object is algebraic, but has been produced by applying the results of [29] and thus usually it does not have an explicit description. The subscript 'a' indicates that the object is algebraic in nature and has a small and explicit description. The symbols $\mathbb{S}^{\lrcorner}_{(-)}$ refer to particular diagrams of rings, and the various categories $\mathbb{S}^{\lrcorner}_{(-)}$−mod are diagrams of modules over a diagram of rings, see Sect. 2.2. We use −dmod to denote differential objects in graded modules, once we are in algebra. These form model categories and are cellularized (i.e., right Bousfield localized) at the sets of objects $K_{(-)}$, which at every level of the diagram are the derived images of the usual stable generators $\mathbb{T}/H_+$ of $\mathbb{T}$, where H varies through closed subgroups of $\mathbb{T}$.

Contents of part 2 of this paper: elliptic cohomology

Given an elliptic curve C over a $\mathbb{Q}$-algebra K, together with some coordinate data [11], shows that one may construct an $SO(2)$-equivariant elliptic cohomology associated to C. Indeed, choosing some auxiliary data, one may write down an object EC_a of the algebraic model $d\mathcal{A}(\mathbb{T})$ (the construction of EC_a is recalled in Sect. 9.1). In view of the results of [10] and [5], there is an associated rational $\mathbb{T}$-spectrum EC_{Top}, and the cohomology theory is the one it represents. Moreover, $d\mathcal{A}(\mathbb{T})$ is a symmetric monoidal category and EC_a is visibly a commutative monoid.

It was asserted in [11, Theorem 11.1] that [15] would (a) establish a monoidal Quillen equivalence and (b) that would allow one to deduce that EC_{Top} is a commutative ring spectrum. It was then further stated (c) that its category of modules would be monoidally equivalent to a derived category of quasi-coherent sheaves over the elliptic curve C.

In the event, [15] did not prove a monoidal equivalence, but we may refer instead to [5]. This shows we may take EC_{Top} to be a ring spectrum. But even then, the deduction of (b) from (a) requires substantial additional work, and part 1 of this paper establishes a version of it. Finally, it is the purpose of Part 2 of the present paper to give a complete proof of (c).

As noted above, EC_a is visibly a commutative monoid, and therefore an algebra over Comm, and by Theorem 5.8, it corresponds to an E^1_∞-algebra EC_{Top} in rational $\mathbb{T}$. We can therefore consider the symmetric monoidal category of modules. This is effectively what the author of [11] had in mind when stating [11, Theorem 11.1], and this structure suffices to give a natural symmetric monoidal product on EC_{Top}-modules.

To be precise on the geometric side, we form a 2-periodic version $P\mathcal{O}_C$ of the structure sheaf $\mathcal{O}_C$ to correspond to the fact that the cohomology theory $EC^*_{\mathbb{T}}(\cdot)$ should be 2-periodic. The non-empty open sets of the usual Zariski topology on C consist of the complements of arbitrary finite sets of points, but we use the torsion point (tp) topology in which only points of finite order may be deleted.

The main theorem of part 2 of the paper appears later as Theorem 6.1.

Theorem B *There is a zig–zag of symmetric monoidal Quillen equivalences*

$$EC_{Top}\text{−mod−}\mathbb{T}\mathrm{Sp}_{\mathbb{Q}} \simeq \textit{quasi-coherent}-P\mathcal{O}_C\text{−dmod−}\textit{sheaves}/C_{tp}$$

giving an equivalence of tensor triangulated categories

$$\mathrm{Ho}(EC_{Top}\text{−mod−}\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}) \simeq D(PC_{tp}).$$

We note that our result showing that EC_{Top} is an E^1_∞-algebra falls short of showing EC_{Top} may be represented by a commutative orthogonal $\mathbb{T}$-spectrum. It is now well known that commutative ring spectra in the category of orthogonal $\mathbb{T}$ have additional structure (in particular, they admit multiplicative norms along finite index inclusions) but the author of [11] was not aware of this in 2005. We do not know whether EC_{Top} can be taken to be an E^G_∞-ring spectrum and hence admit a model as a commutative orthogonal $\mathbb{T}$-spectrum.

Notation

From now on we will write $\mathbb{T}$ for the group $SO(2)$. We also stick to the convention of drawing the left adjoint above the right one (or to the left, if drawn vertically) in any adjoint pair. We use $\mathrm{Ch}(\mathbb{Q})$ for the category of chain complexes of rational vector spaces, Sp for the category of orthogonal spectra, GSp for the category of orthogonal G, and Sp^Σ for the category of symmetric spectra. We add a subscript $\mathbb{Q}$ to indicate *rational* (G-) spectra. We will also use notation E^1_∞ for non-equivariant operad E_∞ given a trivial $\mathbb{T}$-action and considered as an operad in $\mathbb{T}$-spaces.

The authors would also like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Homotopy Harnessing Higher Structures* where work on this paper was completed. This work was supported by EPSRC Grant number EP/K032208/1.

The authors wish to thank the anonymous referee for many helpful comments that improved the exposition of this paper.

Part 1. Quillen equivalence with the algebraic model

2 Recollections

We start this section by recalling several results on lifting various model structures to the categories of algebras over certain operads in equivariant stable setting.

2.1 Model structures on algebras over operads

Definition 2.1 Let $\mathcal{O}$ be an operad in G-topological spaces, which has a trivial G-action and whose underlying non-equivariant operad is an E_∞-operad in topological spaces. We will call such an operad E^1_∞.

Example 2.2 Let U denote a G-universe, i.e. a countably infinite-dimensional real G-inner product space which contains each finite dimensional sub-representation infinitely often. The *linear isometries operad* $\mathcal{L}(U)$ is a G-operad such that $\mathcal{L}(U)(n) := \mathcal{L}(U^n, U)$, where $\mathcal{L}(U^n, U)$ denotes non-equivariant linear isometries from U^n to U. It is a $G \times \Sigma_n$ space by conjugation and diagonal action. The identity map $U \longrightarrow U$ is the distinguished element of $\mathcal{L}(U)(1)$ and the structure maps are given by composition.

If U is a G-**fixed** G-universe then $\mathcal{L}(U)$ is an example of an E^1_∞-operad.

Since the category of G-spectra is tensored over G-spaces we can consider $\mathcal{O}$-algebras in G, where $\mathcal{O}$ is a G-operad (an operad in G-spaces). In particular this applies when $\mathcal{O}$ is an E^1_∞-operad in G-spaces.

In the next proposition we consider the category of orthogonal G-spectra with the positive stable model structure of [26, Section III.5].

Proposition 2.3 *For any E^1_∞-operad $\mathcal{O}$ such that each $\mathcal{O}(n)$ has a homotopy type of a $(G \times \Sigma_n)$-CW complex there exists a right-lifted model structure on $\mathcal{O}$-algebras in orthogonal G-spectra (i.e. the weak equivalences and fibrations are created in the category of orthogonal G-spectra with the positive stable model structure).*

Proof It is shown in [7, Proposition A.1] that the positive complete model structure on G can be right-lifted to the category of $\mathcal{O}$-algebras for any E^1_∞-operad $\mathcal{O}$ such that each $\mathcal{O}(n)$ has a homotopy type of a $(G \times \Sigma_n)$-CW complex. To show that one can lift the positive stable model structure it is enough to show that the following acyclicity condition is satisfied. Let U denote the forgetful functor from $\mathcal{O}$-algebras to G-spectra. The acyclicity condition is

$$\mathsf{LLP}(U^{-1}(\mathsf{Fib}_{+\mathrm{stable}})) \subseteq U^{-1}(\mathsf{WE}).$$

First notice that $\mathsf{Fib}_{+\mathrm{complete}} \subseteq \mathsf{Fib}_{+\mathrm{stable}}$ and both positive stable and positive complete model structures have the same classes of weak equivalences, denoted by WE. Since the acyclicity condition is satisfied for the positive complete model structure

$$\mathsf{LLP}(U^{-1}(\mathsf{Fib}_{+\mathrm{complete}})) \subseteq U^{-1}(\mathsf{WE})$$

it follows that

$$\mathsf{LLP}(U^{-1}(\mathsf{Fib}_{+\mathrm{stable}}) \subseteq \mathsf{LLP}(U^{-1}(\mathsf{Fib}_{+\mathrm{complete}})) \subseteq U^{-1}(\mathsf{WE})$$

which finishes the proof. □

Lemma 2.4 *Let (L, R) be a $\mathbb{V}$-enriched strong symmetric monoidal adjunction between $\mathbb{V}$-tensored symmetric monoidal categories $\mathbb{C}$ and $\mathbb{D}$.*

Let $\mathcal{O}$ be an operad in $\mathbb{V}$, then L and R extend to functors of $\mathcal{O}$-operads in $\mathbb{C}$ and $\mathbb{D}$, and we obtain a square of adjunctions

$$\begin{tikzcd}
\mathcal{O}\text{−alg-in−}\mathbb{C} \arrow[r, shift left, "L"] \arrow[d, shift left, "U_{\mathbb{C}}"] & \mathcal{O}\text{−alg-in−}\mathbb{D} \arrow[l, shift left, "R"] \arrow[d, shift left, "U_{\mathbb{D}}"] \\
\mathbb{C} \arrow[r, shift left, "L"] \arrow[u, shift left, "F_{\mathcal{O}}"] & \mathbb{D} \arrow[l, shift left, "R"] \arrow[u, shift left, "F_{\mathcal{O}}"]
\end{tikzcd}$$

which is commutative in the sense that the square of left adjoints commutes and the square of right adjoints commutes.

Now we consider the case when the original L, R-adjunction is a Quillen equivalence and give criteria for it to lift to a Quillen equivalence of $\mathcal{O}$-algebras.

Lemma 2.5 [4, Lemma 3.6] *Suppose that L is a strong symmetric monoidal functor tensored over $\mathbb{V}$, L and R form a Quillen equivalence (at the level of categories $\mathbb{C}$ and $\mathbb{D}$) and that the categories of $\mathcal{O}$-algebras in $\mathbb{C}$ and $\mathbb{D}$ have right-lifted cofibrantly generated model structures from the ones on $\mathbb{C}$ and $\mathbb{D}$ respectively.*

If $U_{\mathbb{C}}$ preserves cofibrant objects then the lifted adjoint pair L, R at the level of $\mathcal{O}$-algebras is a Quillen equivalence:

$$\mathcal{O}\text{−alg-in−}\mathbb{C} \simeq \mathcal{O}\text{−alg-in−}\mathbb{D}.$$

We will apply this result numerous times in the rest of the paper in case where $\mathcal{O}$ is an E^1_∞-operad and categories $\mathbb{C}$ and $\mathbb{D}$ are built from equivariant orthogonal spectra, as we explain in the next section.

2.2 Diagrams of model categories

The methods used to obtain an algebraic model for rational $\mathbb{T}$-spectra are substantially different than the ones for a finite group. We recall here the basic building blocks from [5] and [15], i.e. diagrams of model categories. This idea has been studied in some detail in [23] and in [14], among others. In this section we introduce the relevant structures and leave most of the proofs to the references. We will only use one shape of a diagram, the pullback diagram $\mathscr{P}$:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet,$$

where the direction of the arrows refers to the left adjoints. Pullbacks of model categories are also considered in detail in [3].

Definition 2.6 A **$\mathscr{P}$-diagram of model categories $R^\bullet$** is a pair of Quillen pairs

$$\begin{array}{c} L : \mathcal{A} \rightleftarrows \mathcal{B} : R \\ F : \mathcal{C} \rightleftarrows \mathcal{B} : G \end{array}$$

with L and F the left adjoints. We will usually draw this as the diagram below.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "L"] & \mathcal{B} \arrow[l, shift left, "R"] \arrow[r, shift right, "G"'] & \mathcal{C} \arrow[l, shift right, "F"']
\end{tikzcd}$$

A standard example, which is of the main interest to this paper, comes from a $\mathscr{P}$-diagram of rings $R^{\lrcorner} = (R_1 \xrightarrow{f} R_2 \xleftarrow{g} R_3)$. Using the adjoint pairs of extension and restriction of scalars we obtain a $\mathscr{P}$-diagram of model categories $R^{\bullet}$ as below.

$$R_1\text{-mod} \underset{f^*}{\overset{R_2 \otimes_{R_1} -}{\rightleftarrows}} R_2\text{-mod} \underset{g^*}{\overset{R_2 \otimes_{R_3} -}{\leftrightarrows}} R_3\text{-mod}.$$

Definition 2.7 Given a $\mathscr{P}$-diagram of model categories $R^{\bullet}$ we can define a new category, $R^{\bullet}$−mod. The objects of this category are pairs of morphisms, $\alpha : La \to b$ and $\gamma : Fc \to b$ in $\mathcal{B}$. We usually abbreviate a pair $(\alpha : La \to b, \gamma : Fc \to b)$ to a quintuple $(a, \alpha, b, \gamma, c)$.

A morphism in $R^{\bullet}$−mod from $(a, \alpha, b, \gamma, c)$ to $(a', \alpha', b', \gamma', c')$ is a triple of maps $x : a \to a'$ in $\mathcal{A}$, $y : b \to b'$ in $\mathcal{B}$, $z : c \to c'$ in $\mathcal{C}$ such that we have a commuting diagram in $\mathcal{B}$

$$\begin{tikzcd}
La \arrow[r, "\alpha"] \arrow[d, "Lx"] & b \arrow[d, "y"] & Fc \arrow[l, "\gamma"'] \arrow[d, "Fz"] \\
La' \arrow[r, "\alpha'"] & b' & Fc' \arrow[l, "\gamma'"']
\end{tikzcd}$$

Note that we could also have defined an object as a sequence $(a, \bar{\alpha}, b, \bar{\gamma}, c)$. where $\bar{\alpha} : a \to Rb$ is a map in $\mathcal{A}$ and $\bar{\gamma} : c \to Gb$ is a map in $\mathcal{C}$.

Notation 2.8 *We use* $R^{\lrcorner}$−mod *to denote a category of modules over a diagram of rings* $R^{\lrcorner}$. *This is a special case of the category* $R^{\bullet}$−mod *from Definition 2.7.*

We say that a map (x, y, z) in $R^{\bullet}$−mod is an objectwise cofibration if x is a cofibration of $\mathcal{A}$, y is a cofibration of $\mathcal{B}$ and z is a cofibration of $\mathcal{C}$. We define objectwise weak equivalences similarly.

Lemma 2.9 [14, Proposition 3.3] *Consider a* $\mathscr{P}$*-diagram of model categories* $R^{\bullet}$ *as below, with each category cellular and proper.*

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{C}$$

The category $R^{\bullet}$−mod *admits a cellular proper model structure with cofibrations and weak equivalences defined objectwise. This is called the* diagram injective *model structure.*

Whilst there is also a *diagram projective* model structure, in this paper we only use the diagram injective model structure (and cellularizations thereof) on diagrams of model categories.

Now consider maps of $\mathscr{P}$-diagrams of model categories. Let $R^{\bullet}$ and $S^{\bullet}$ be two $\mathscr{P}$-diagrams, where $R^{\bullet}$ is as above and $S^{\bullet}$ is given below.

$$\mathcal{A}' \underset{R'}{\overset{L'}{\rightleftarrows}} \mathcal{B}' \underset{G'}{\overset{F'}{\leftrightarrows}} \mathcal{C}'$$

Now we assume that we have Quillen adjunctions as below such that $P_2 L$ is naturally isomorphic to $L' P_1$ and $P_2 F$ is naturally isomorphic to $F' P_3$.

$$\begin{aligned} P_1 : \mathcal{A} &\rightleftarrows \mathcal{A}' : Q_1 \\ P_2 : \mathcal{B} &\rightleftarrows \mathcal{B}' : Q_2 \\ P_3 : \mathcal{C} &\rightleftarrows \mathcal{C}' : Q_3 \end{aligned}$$

We then obtain a Quillen adjunction (P, Q) between $R^\bullet-$mod and $S^\bullet-$mod. For example, the left adjoint P takes the object $(a, \alpha, b, \gamma, c)$ to $(P_1 a, P_2\alpha, P_2 b, P_2\gamma, P_3 c)$. The natural isomorphisms $P_2 L \simeq L' P_1$ and $P_2 F \simeq F' P_3$ ensure that this is an object of $S^\bullet-$mod. It is easy to see the following

Lemma 2.10 *If the Quillen adjunctions (P_i, Q_i) are Quillen equivalences then the adjunction (P, Q) between $R^\bullet-$mod and $S^\bullet-$mod is a Quillen equivalence.*

Now we turn to monoidal considerations. There is an obvious monoidal product for $R^\bullet-$mod, provided that each of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ is monoidal and that the left adjoints L and F are strong monoidal.

$$(a, \alpha, b, \gamma, c) \wedge (a', \alpha', b', \gamma', c') := (a \wedge a', \alpha \wedge \alpha', b \wedge b', \gamma \wedge \gamma', c \wedge c')$$

Let $S_{\mathcal{A}}$ be the unit of $\mathcal{A}$, $S_{\mathcal{B}}$ be the unit of $\mathcal{B}$ and let $S_{\mathcal{C}}$ be the unit of $\mathcal{C}$. Since L and F are strong monoidal, we have maps $\eta_{\mathcal{A}} : LS_{\mathcal{A}} \to S_{\mathcal{B}}$ and $\eta_{\mathcal{C}} : FS_{\mathcal{C}} \to S_{\mathcal{B}}$. The unit of the monoidal product on $R^\bullet-$mod is $(S_{\mathcal{A}}, \eta_{\mathcal{A}}, S_{\mathcal{B}}, \eta_{\mathcal{C}}, S_{\mathcal{C}})$.

It is worth noting that this category has an internal function object when $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are closed monoidal categories and thus itself is closed.

Lemma 2.11 *Consider a $\mathscr{P}$-diagram of model categories $R^\bullet$ such that each vertex is a cellular monoidal model category. Assume further that the two adjunctions of the diagram are strong monoidal Quillen pairs. Then $R^\bullet-$mod is a monoidal model category. If each vertex also satisfies the monoid axiom, so does $R^\bullet-$mod.*

Proof Since the cofibrations and weak equivalences are defined objectwise, the pushout product and monoid axioms hold provided they do so in each model category in the diagram $R^\bullet$. □

We can also extend our monoidal considerations to maps of diagrams. Return to the setting of a map (P, Q) of $\mathscr{P}$-diagrams from $R^\bullet$ to $S^\bullet$ as described above. If we assume that each of the adjunctions (P_1, Q_1), (P_2, Q_2) and (P_3, Q_3) is a symmetric monoidal Quillen equivalence, then we see that (P, Q) is a symmetric monoidal Quillen equivalence.

Lemma 2.12 *Consider a $\mathscr{P}$-diagram of model categories $R^\bullet$ such that each vertex is tensored over a symmetric monoidal category $\mathbb{V}$ and the two left adjoints of the diagram are also tensored over $\mathbb{V}$. Then $R^\bullet-$mod is tensored over $\mathbb{V}$, using the formula*

$$v \otimes (a, \alpha, b, \gamma, c) := (v \otimes a, v \otimes \alpha, v \otimes b, v \otimes \beta, v \otimes c).$$

Lemma 2.13 *Under the assumptions from the lemma above, if $\mathcal{O}$ is an operad in $\mathbb{V}$ and both left adjoints of the diagram $R^\bullet$ are strong symmetric monoidal, then the category of $\mathcal{O}$-algebras in $R^\bullet-$mod is equivalent to the category $(\mathcal{O}-\text{alg-in}-R^\bullet)-$mod, where $\mathcal{O}-\text{alg-in}-R^\bullet$ is the following diagram*

$$\mathcal{O}-\text{alg-in}-\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{O}-\text{alg-in}-\mathcal{B} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{O}-\text{alg-in}-\mathcal{C}$$

2.3 Cellularization

A cellularization of a model category is a right Bousfield localization at a set of objects. Such a localization exists by [20, Theorem 5.1.1] whenever the model category is right proper and cellular. When we are in a stable context the results of [8] can be used.

In this subsection we recall the notion of cellularization when $\mathcal{C}$ is stable.

Definition 2.14 Let $\mathcal{C}$ be a stable model category and K a stable set of objects of $\mathcal{C}$, i.e. such that a class of K-cellular objects of $\mathcal{C}$ is closed under desuspensions.[1] We say that a map $f : A \longrightarrow B$ of $\mathcal{C}$ is a K**-cellular equivalence** if the induced map

$$[k, f]_*^{\mathcal{C}} : [k, A]_*^{\mathcal{C}} \longrightarrow [k, B]_*^{\mathcal{C}}$$

is an isomorphism of graded abelian groups for each $k \in K$. An object $Z \in \mathcal{C}$ is said to be K**-cellular** if

$$[Z, f]_*^{\mathcal{C}} : [Z, A]_*^{\mathcal{C}} \longrightarrow [Z, B]_*^{\mathcal{C}}$$

is an isomorphism of graded abelian groups for any K-cellular equivalence f.

Definition 2.15 A **right Bousfield localization** or **cellularization** of $\mathcal{C}$ with respect to a set of objects K is a model structure $K-\text{cell}-\mathcal{C}$ on $\mathcal{C}$ such that

- the weak equivalences are K-cellular equivalences
- the fibrations of $K-\text{cell}-\mathcal{C}$ are the fibrations of $\mathcal{C}$
- the cofibrations of $K-\text{cell}-\mathcal{C}$ are defined via left lifting property.

By [20, Theorem 5.1.1], if $\mathcal{C}$ is a right proper, cellular model category and K is a set of objects in $\mathcal{C}$, then the cellularization of $\mathcal{C}$ with respect to K, $K-\text{cell}-\mathcal{C}$, exists and is a right proper model category. The cofibrant objects of $K-\text{cell}-\mathcal{C}$ are called K**-cofibrant** and are precisely the K-cellular and cofibrant objects of $\mathcal{C}$. Moreover, if $\mathcal{C}$ is a stable model category and K is a stable set of cofibrant objects (see [8, Definition 4.3]), then $K-\text{cell}-\mathcal{C}$ is also cofibrantly generated, and the set of generating acyclic cofibrations is the same as for $\mathcal{C}$ (see [8, Theorem 4.9]).

3 Preliminaries

We now proceed to lifting model structures to the categories of algebras over E_∞^1-operads in this new setting of diagrams of model categories.

3.1 Lifting model structures

If $\mathbb{C}$ is a model category we want to right-lift the model structure from $\mathbb{C}$ to the category $\mathcal{O}-\text{alg-in}-\mathbb{C}$ of $\mathcal{O}$-algebras in $\mathbb{C}$ using the right adjoint U, i.e. we want the weak equivalences and fibrations to be created by U. In our case, $\mathbb{C}$ will be some category of R-modules in $\mathbb{T}\mathrm{Sp}$ (or spectra) with weak equivalences those maps which forget to rational equivalences of $\mathbb{T}\mathrm{Sp}$ (or spectra, respectively). Here R is a Comm-algebra either in spectra or in $\mathbb{T}\mathrm{Sp}$. Before we restrict attention to particular cases of R, we will establish some general results.

First we recall Kan's result for right lifting model structures in this setting.

Lemma 3.1 [20, Theorem 11.3.2] *Suppose $\mathbb{C}$ is a cofibrantly generated model category with a set of generating acyclic cofibrations J. If*

- *the free $\mathcal{O}$-algebra functor $F_{\mathcal{O}}$ preserves small objects (or the forgetful functor U preserves filtered colimits) and*
- *every transfinite composition of pushouts (cobase extensions) of elements of $F_{\mathcal{O}}(J)$ is sent to a weak equivalence of $\mathbb{C}$ by U,*

[1] Note that this class is always closed under suspensions.

then the model structure on $\mathbb{C}$ may be lifted using the right adjoint to give a cofibrantly generated model structure on $\mathcal{O}-\mathrm{alg\text{-}in}-\mathbb{C}$. The functor U then creates fibrations and weak equivalences.

The following remark tells us that cellularizations interact well with lifted model structures.

Remark 3.2 Let $\mathbb{C}$ be a cofibrantly generated model structure and $\mathcal{O}$ be an operad such that the free $\mathcal{O}$-algebra functor $F_{\mathcal{O}}$ preserves small objects and such that the model structure on $\mathbb{C}$ right lifts to $\mathcal{O}$-algebras in $\mathbb{C}$. Assume that for a set of objects K the cellularization $K-\mathrm{cell}-\mathbb{C}$ of the model structure on $\mathbb{C}$ exists. Then there exists a right-lifted model structure on $\mathcal{O}$-algebras in $\mathbb{C}$ from $K-\mathrm{cell}-\mathbb{C}$.

This follows directly from Kan's lifting lemma and the fact that the generating acyclic cofibrations for $K-\mathrm{cell}-\mathbb{C}$ are the same as the ones for $\mathbb{C}$.

Suppose R is a Comm-algebra in $\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}$. We will lift various model structures on R-modules in $\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}$ to the level of algebras over an E^1_∞-operad $\mathcal{O}$. We first consider the categorical setting. Since R is a Comm-algebra we get a distributivity law

$$F_{\mathcal{O}}(R \wedge -) \longrightarrow R \wedge F_{\mathcal{O}}(-),$$

and hence

$$\mathcal{O}-\mathrm{alg\text{-}in}-R-\mathrm{mod}-\mathbb{T}\mathrm{Sp} \cong R-\mathrm{mod}-\mathcal{O}-\mathrm{alg\text{-}in}-\mathbb{T}\mathrm{Sp}.$$

Therefore the functor $R\wedge -$ lifts to a functor at the level of $\mathcal{O}$-algebras as in the diagram below. In the following, U denotes forgetful functors and $F_{\mathcal{O},R}$ is defined by $X \mapsto \vee_{n\geq 0}\mathcal{O}(n)_+ \wedge_{\Sigma_n} X^{\wedge^n_R}$.

$$\begin{tikzcd}
\mathcal{O}-\mathrm{alg\text{-}in}-R-\mathrm{mod}-\mathbb{T}\mathrm{Sp} \arrow[r, shift right, "U"'] \arrow[d, shift left, "U"] & \mathcal{O}-\mathrm{alg\text{-}in}-\mathbb{T}\mathrm{Sp} \arrow[l, shift right, "R\wedge -"'] \arrow[d, shift left, "U"] \\
R-\mathrm{mod}-\mathbb{T}\mathrm{Sp} \arrow[r, shift right, "U"'] \arrow[u, shift left, "F_{\mathcal{O},R}"] & \mathbb{T}\mathrm{Sp} \arrow[l, shift right, "R\wedge -"'] \arrow[u, shift left, "F_{\mathcal{O}}"]
\end{tikzcd}$$

Now considering model categories, we want to lift the positive stable model structure on rational $\mathbb{T}$ to the other three corners. Lifting to R-modules is standard and lifting to $\mathcal{O}$-algebras is Proposition 2.3. Thus all that remains is the top left corner.

Since the homotopy theory of $\mathcal{O}$-algebras does not depend on the choice of the E^1_∞-operad $\mathcal{O}$ which at level n has a homotopy type of $(\mathbb{T} \times \Sigma_n)$-CW complex, from now on we will use the generic notation E^1_∞-algebras.

Lemma 3.3 *For R a commutative ring $\mathbb{T}$-spectrum there is a cofibrantly generated model structure on E^1_∞-algebras in $R-\mathrm{mod}$ where the weak equivalences are those maps which forget to rational stable equivalences of $\mathbb{T}$. Furthermore, the forgetful functor to $R-\mathrm{mod}$ preserves cofibrant objects.*

Proof We adapt the proof of [4, Theorem 4.4]. A corollary of that theorem states that there is a lifted model structure on E^1_∞-algebras in $\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}$. We extend this to E^1_∞-algebras in $R-\mathrm{mod}-\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}$. Specifically, we show that the adjunction $(F_{\mathrm{E}^1_\infty,R}, U)$ can be used to lift the model structure on $R-\mathrm{mod}-\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}$ to E^1_∞-algebras in $R-\mathrm{mod}-\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}$.

As in [4, Theorem 4.4], this result has two parts. The first part is about the interaction of pushouts, sequential colimits and h-cofibrations (also known as the Cofibration Hypothesis). This is well known and uses a standard technique of describing pushouts of E^1_∞-algebras as sequential colimits in the underlying category (in this case $R-$mod). This argument appears in several places, one of the earlier occurrences is Elmendorf et al. [9, Chapters VII and VIII] where they construct model structure of commutative algebras and localisations of commutative algebras. A more recent description is given by Harper and Hess in [16, Proposition 5.10].

The second part is that every map built using pushouts (cobase extensions) and sequential colimits from $F_{\mathrm{E}^1_\infty,R}(R \wedge J)$ (i.e. maps of the form $F_{\mathrm{E}^1_\infty,R}(R \wedge j)$, for j a generating acyclic cofibration of $\mathbb{T}\mathrm{Sp}_\mathbb{Q}$) is a rational equivalence. It can be broken into a number of steps.

Step 1 If $f : X \to Y$ is a generating acyclic cofibration of the rational model structure on R-modules then $F_{\mathrm{E}^1_\infty,R} f$ is a rational equivalence and a h-cofibration of underlying $\mathbb{T}$.
Step 2 Any pushout (cobase extension) of $F_{\mathrm{E}^1_\infty,R} f$ is a rational equivalence and a h-cofibration of underlying $\mathbb{T}$.
Step 3 Any sequential colimit of such maps is a rational equivalence.

To prove Step 1 we first identify the generating acyclic cofibrations as maps of the form $R \wedge g : R \wedge Z \to R \wedge Z'$ where $g : Z \to Z'$ is an acyclic generating cofibration of $\mathbb{T}\mathrm{Sp}_\mathbb{Q}$ and hence has cofibrant domain and codomain. We know that $F_{\mathrm{E}^1_\infty,R}(R \wedge g) \simeq R \wedge F_{\mathrm{E}^1_\infty} g$. By [4, Theorem 4.4], $F_{\mathrm{E}^1_\infty} g$ is a rational acyclic cofibration of cofibrant $\mathbb{T}$-spectra. Smashing such a map with a spectrum gives a rational acyclic h-cofibration of $\mathbb{T}$ that in $R-$mod is a cofibration of cofibrant objects. Steps 2 and 3 follow as in the classical case.

The last part of the Lemma follows from [16, Theorem 5.18], which shows that the forgetful functor

$$U : \mathrm{E}^1_\infty-\text{alg-in}-R-\text{mod}-\mathbb{T}\mathrm{Sp}_\mathbb{Q} \longrightarrow R-\text{mod}-\mathbb{T}\mathrm{Sp}_\mathbb{Q}$$

preserves cofibrant objects. □

Lemma 3.4 *Suppose there is a right-lifted model structure to $\mathcal{O}-$alg-in$-\mathbb{C}$ from each category $\mathbb{C}$ used to build the diagram category $R^\lrcorner-$mod separately and*

$$U : \mathcal{O}-\text{alg-in}-R^\lrcorner-\text{mod} \longrightarrow R^\lrcorner-\text{mod}$$

commutes with filtered colimits. Then there is a right-lifted model structure from the diagram injective model structure on $R^\lrcorner-$mod to $\mathcal{O}-$alg-in$-R^\lrcorner-$mod.

Proof If each separate lift exists, then the conditions of Kan's result are satisfied for the (generalised) diagram category, since the free $\mathcal{O}$-algebra functor, acyclic cofibrations, weak equivalences, transfinite compositions and pushouts are defined objectwise. □

As well as E^1_∞-algebras on $\mathbb{T}\mathrm{Sp}$ and Sp we will need Comm-algebras on various model categories of non-equivariant spectra. Specifically we will need to consider Comm-algebras in the rationalised positive stable model structure on orthogonal spectra, Comm-algebras in the rationalised positive stable model structure on symmetric spectra, and Comm-algebras in the positive stable model structure on $H\mathbb{Q}$-modules in symmetric spectra. We give a proof that the various lifted model structures exist for the orthogonal spectra case, the remaining cases are similar.

Lemma 3.5 *Let R be a commutative ring spectrum. If we equip orthogonal spectra with the positive rational stable model structure then there are lifted model structures on the categories of* Comm*-algebras in R-modules in* $\mathrm{Sp}_{\mathbb{Q}}$ *and* E^1_∞*-algebras in R-modules in* $\mathrm{Sp}_{\mathbb{Q}}$.

Proof The category of commutative algebras in R-modules is the category of commutative R-algebras, which in turn is the category of commutative algebras under R. Hence the model structure for Comm-algebras in $\mathrm{Sp}_{\mathbb{Q}}$ [4, Theorem A.2] gives a model structure on Comm-algebras in R-modules in $\mathrm{Sp}_{\mathbb{Q}}$.

For the E^1_∞-case we use a non-equivariant version of Lemma 3.3. □

4 An algebraic model for rational $\mathbb{T}$

We start by summarising the classification of rational $\mathbb{T}$ in terms of an algebraic model. We will only outline the strategy here. For details and definitions we refer the reader to [5,15].

Remark 4.1 We choose to work with a simplified version of the proof presented in [5]. The simplification arises from very recent work of the second author, [12], which proves that the ring $\mathbb{T}$-spectrum $\widetilde{E}\mathcal{F}$ has a genuinely commutative model.

This means that instead of working with localisations of model categories used in [5], we work with diagrams of modules over a diagram of *genuinely* commutative ring $\mathbb{T}$. This approach is simpler, and more closely follows the approach of [15].

We remark here that there are three methods to obtain an algebraic model: the one presented in this paper, the one used in [5] or the one using $\mathcal{L}(1)$-modules in $\mathbb{T}$-orthogonal spectra (where $\mathcal{L}$ is the linear isometries operad for the trivial $\mathbb{T}$ universe) suggested in [15]. Either method can be used to prove that there is a zig–zag of Quillen equivalences

$$\mathrm{E}^1_\infty\text{–alg-in–}\mathbb{T}\mathrm{Sp}_{\mathbb{Q}} \simeq \mathsf{Comm}\text{–alg-in–}d\mathcal{A}(\mathbb{T})_{dual}.$$

The Tate square for the family of finite subgroups is the homotopy pullback square

$$\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d] & S^{\infty V(\mathbb{T})} \arrow[d] \\
DE\mathcal{F}_+ \arrow[r] & S^{\infty V(\mathbb{T})} \wedge DE\mathcal{F}_+
\end{tikzcd}$$

of (genuinely) commutative ring $\mathbb{T}$ (Note that $S^{\infty V(\mathbb{T})} = \widetilde{E}\mathcal{F}$ has a commutative model in rational orthogonal $\mathbb{T}$ by [12]). We then omit the copy of $\mathbb{S}$ at the top left, to leave the fork

$$\mathbb{S}^{\lrcorner}_{\mathsf{Top}} = \left(\begin{tikzcd}
 & S^{\infty V(\mathbb{T})} \arrow[d] \\
DE\mathcal{F}_+ \arrow[r] & S^{\infty V(\mathbb{T})} \wedge DE\mathcal{F}_+
\end{tikzcd}\right).$$

Using Definition 2.7 one can consider the category $\mathbb{S}^{\lrcorner}_{\mathsf{Top}}$–mod (see also Notation 2.8). By Lemma 2.11 this is a cellular, monoidal model category. The Cellularization Principle [13, Theorem 2.1] shows that the category of $\mathbb{T}$ can then be recovered from modules over $\mathbb{S}^{\lrcorner}_{\mathsf{Top}}$ by taking a pullback construction [14, Proposition 4.1] (see also [15, Proposition 4.1], [13, Proposition 6.2]):

$$\mathbb{T}\mathrm{Sp}_{\mathbb{Q}} = \mathbb{S}\text{–mod–}\mathbb{T}\mathrm{Sp}_{\mathbb{Q}} \simeq K_{\mathsf{Top}}\text{–cell–}\mathbb{S}^{\lrcorner}_{\mathsf{Top}}\text{–mod–}\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}.$$

The cellularization here is at the derived image of the generators for $\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}$, which we denote by K_{Top}. Taking $\mathbb{T}$-fixed points gives the diagram of rings (arranged on a line for typographical convenience)

$$(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}} = \left(DE\mathcal{F}_+^{\mathbb{T}} \longrightarrow (S^{\infty V(\mathbb{T})} \wedge DE\mathcal{F}_+)^{\mathbb{T}} \longleftarrow (S^{\infty V(\mathbb{T})})^{\mathbb{T}} \right).$$

If M is a module over the ring $\mathbb{T}$-spectrum R, then $M^{\mathbb{T}}$ is a module over the (non-equivariant) ring spectrum $R^{\mathbb{T}}$. Taking $\mathbb{T}$-fixed points at every place of the diagram gives a Quillen equivalence

$$\zeta_{\#} : (\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}\text{−mod− }\mathrm{Sp}_{\mathbb{Q}} \rightleftarrows \mathbb{S}^{\lrcorner}_{\mathsf{Top}}\text{−mod−}\mathbb{T}\mathrm{Sp}_{\mathbb{Q}} : (-)^{\mathbb{T}}$$

by [5, Section 3.3] or [15, Section 7.A]. Let $K^{\mathbb{T}}_{\mathsf{Top}}$ be the derived image of the cells K_{Top} under this adjunction. Then we have a symmetric monoidal Quillen equivalence

$$\zeta_{\#} : K^{\mathbb{T}}_{\mathsf{Top}}\text{−cell−}(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}\text{−mod− }\mathrm{Sp}_{\mathbb{Q}} \rightleftarrows K_{\mathsf{Top}}\text{−cell−}\mathbb{S}^{\lrcorner}_{\mathsf{Top}}\text{−mod−}\mathbb{T}\mathrm{Sp}_{\mathbb{Q}} : (-)^{\mathbb{T}} .$$

Rational orthogonal spectra are symmetric monoidally Quillen equivalent to rational symmetric spectra in simplicial sets $\mathrm{Sp}^{\Sigma}_{\mathbb{Q}}$, which in turn are symmetric monoidally Quillen equivalent to $H\mathbb{Q}$-modules in symmetric spectra. Hence we can apply the results of [29] to obtain a diagram $\mathbb{S}^{\lrcorner}_t$ of commutative ring objects in rational chain complexes and symmetric monoidal Quillen equivalences

$$(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}\text{−mod} \simeq \mathbb{S}^{\lrcorner}_t\text{−dmod} \qquad K^{\mathbb{T}}_{\mathsf{Top}}\text{−cell−}(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}\text{−mod} \simeq K_t\text{−cell−}\mathbb{S}^{\lrcorner}_t\text{−dmod}$$

where K_t is the derived image of the cells $K^{\mathbb{T}}_{\mathsf{Top}}$, see [5, Section 3.4] or [15, Section 8]. Here, −dmod denotes differential objects in graded modules. The results of [29] give isomorphisms between the homology of these rational chain complexes and homotopy groups of the original spectra.

There is a product splitting

$$DE\mathcal{F}_+ \simeq \prod_n DE\langle n\rangle$$

where the single isotropy space $E\langle n\rangle$ is defined by the cofibre sequence

$$E[H \subset C_n]_+ \longrightarrow E[H \subseteq C_n]_+ \longrightarrow E\langle n\rangle.$$

We may then calculate

$$\pi_*(DE\langle n\rangle^{\mathbb{T}}) = \pi_*^{\mathbb{T}}(DE\langle n\rangle) \cong H^*(BS^1/C_n) = \mathbb{Q}[c], \qquad \pi_*((S^{\infty V(\mathbb{T})})^{\mathbb{T}}) = \mathbb{Q}$$

where c is in degree −2. As these rings are polynomial, we use formality arguments to see that $\mathbb{S}^{\lrcorner}_t$ is quasi-isomorphic (as diagrams of commutative rings) to the diagram

$$\mathbb{S}^{\lrcorner}_a = \left(\begin{array}{ccc} & & \mathbb{Q} \\ & & \downarrow{\scriptstyle \iota} \\ \mathcal{O}_{\mathcal{F}} & \xrightarrow{l} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \end{array} \right)$$

of graded rings. Recall ([10, 2.4.1]) that $\mathcal{O}_{\mathcal{F}} \cong \prod_n \mathbb{Q}[c]$, where c is in degree −2 and

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} = \operatorname{colim}_{V^{\mathbb{T}}=0} \Sigma^V \mathcal{O}_{\mathcal{F}}$$

where $\Sigma^V \mathcal{O}_{\mathcal{F}} = \prod_n \Sigma^{|V^{C_n}|}\mathbb{Q}[c]$. It is useful to think of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ as an $\mathcal{O}_{\mathcal{F}}$ with "Euler classes" $\mathcal{E}$ inverted. For a function $\nu : \mathcal{F} \longrightarrow \mathbb{Z}_{\geq 0}$ with finite support we can define an inhomogeneous element c^ν in $\mathcal{O}_{\mathcal{F}}$ such that $e_H c^\nu = c_H^{\nu(H)}$ where e_H is the projection onto H-factor of $\mathcal{O}_{\mathcal{F}}$ and c_H is the polynomial generator in the factor H. The primary example is the function sending a subgroup H to the dimension of V^H, for V a representation of $\mathbb{T}$ with $V^{\mathbb{T}} = 0$. We define

$$\mathcal{E} = \{c^\nu | \nu : \mathcal{F} \longrightarrow \mathbb{Z}_{\geq 0} \text{ with finite support } \}$$

and call the elements of $\mathcal{E}$ *Euler classes*.

A module over the diagram of rings $\mathbb{S}_a^{\lrcorner}$ is of the form

$$N \xrightarrow{k} P \xleftarrow{m} V$$

where k is the map of $\mathcal{O}_{\mathcal{F}}$-modules and m is a map of $\mathbb{Q}$-modules (see the second part of the Definition 2.7).

Thus formality arguments allowed us to simplify the diagram of commutative rings $\mathbb{S}_t^{\lrcorner}$ to a quasi-isomorphic diagram of commutative rings $\mathbb{S}_a^{\lrcorner}$. Another sequence of formality arguments gives a simpler set of cells K_a defined in [5, Lemma 4.2.2]. Hence we have zig–zags of symmetric monoidal Quillen equivalences

$$\mathbb{S}_a^{\lrcorner}\text{−dmod} \simeq \mathbb{S}_t^{\lrcorner}\text{−dmod} \qquad K_a\text{−cell−}\mathbb{S}_a^{\lrcorner}\text{−dmod} \simeq K_t\text{−cell−}\mathbb{S}_t^{\lrcorner}\text{−dmod}$$

see [15, Section 9].

Let $\mathcal{A}(\mathbb{T})$ be the subcategory of $\mathbb{S}_a^{\lrcorner}$-modules where the structure maps induce isomorphisms $l_* N \cong P$ and $\iota_* V \cong P$ (these are called *qce-modules*). One can use a simplified description for the category $\mathcal{A}(\mathbb{T})$. An object $X \in \mathcal{A}(\mathbb{T})$ is a triple $(N, V, \beta : N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$, where N is an $\mathcal{O}_{\mathcal{F}}$-module, V is a graded $\mathbb{Q}$ vector space and β is a map of $\mathcal{O}_{\mathcal{F}}$-modules which induces an isomorphism $\mathcal{E}^{-1}\beta : \mathcal{E}^{-1}N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$. To recover the original description one defines the map from V into the codomain of β to be the adjoint of the identity on $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$. A morphism consists of a map of $\mathcal{O}_{\mathcal{F}}$-modules and a map of graded $\mathbb{Q}$-vector spaces which make the obvious square commute. This simplified notation will be used in part 2. We use notation $d\mathcal{A}(\mathbb{T})$ for the category of objects of $\mathcal{A}(\mathbb{T})$ equipped with a differential. These categories are described in detail in [5].

We give the differential module categories of the individual rings in $\mathbb{S}_a^{\lrcorner}$ the (algebraic) projective model structures and $\mathbb{S}_a^{\lrcorner}$−dmod the diagram injective model structure. We give the category $d\mathcal{A}(\mathbb{T})$ of differential objects of $\mathcal{A}(\mathbb{T})$ the dualizable model structure of Barnes [2]. The essential fact is that this is a monoidal model structure.

The final stage in our sequence of Quillen equivalences is to remove the cellularization. There is a symmetric monoidal Quillen equivalence

$$K_a\text{−cell−}\mathbb{S}_a^{\lrcorner}\text{−dmod} \simeq d\mathcal{A}(\mathbb{T}),$$

by [15, 11.5], see also Lemma 8.1 for more details.

5 An algebraic model for rational naïve-commutative ring $\mathbb{T}$-spectra

In this section we construct our sequence of Quillen equivalences between E^1_∞-algebras in rational $\mathbb{T}$-spectra and commutative algebras in the algebraic model. The strategy is to replace E^1_∞ by Comm as soon as we have moved from equivariant spectra to (modules over $(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}$ in) non-equivariant spectra. We then apply the results of Richter and Shipley [27] to compare commutative algebras in rational spectra to commutative algebras in rational chain complexes.

Our next task is to check the compatibility of the zig–zag of Quillen equivalences from Sect. 4 with algebras over E^1_∞-operads. We first note that in this zig–zag of Quillen equivalences we can use either the stable model structures (as in [5] and [15]), or the positive stable model structures. Recall that we need to work with *positive* stable model structures to be able to lift them to algebras over E^1_∞-operads.

Remark 5.1 The model structures and Quillen equivalences of Sect. 4 relating rational $\mathbb{T}$ to the algebraic model $d\mathcal{A}(\mathbb{T})$ can all be obtained using the positive stable model structure on the various categories of spectra. Moreover, the adjunction

$$H\mathbb{Q} \wedge - : \mathrm{Sp}^{\Sigma}_{\mathbb{Q}} \rightleftarrows H\mathbb{Q}\mathrm{-mod} : U$$

is a Quillen equivalence when the left side is considered with the positive stable model structure and the right hand side is considered with the positive flat stable model structure of [28]. This model structure on $H\mathbb{Q}$−mod is the starting point for the work in [27].

Notice that all left adjoints of the zig–zag of Quillen equivalences from Sect. 4 (see also [5] and [15]) up to $H\mathbb{Q}$−mod are strong symmetric monoidal.

By Lemmas 2.5, 3.4, 3.5 and Remark 3.2 we obtain lifted model structures on the categories of E^1_∞-algebras and Quillen equivalences at the levels of E^1_∞-algebras as below.

$$\begin{tikzcd}
\mathrm{E}^1_\infty\mathrm{-alg\text{-}in-}\mathbb{T}\mathrm{Sp}_{\mathbb{Q}} \arrow[d, shift right, "\mathbb{S}^{\lrcorner}\wedge -"'] & \text{in } \mathbb{T}\mathrm{Sp} \\
\mathrm{E}^1_\infty\mathrm{-alg\text{-}in-}K_{\mathsf{Top}}\mathrm{-cell-}\mathbb{S}^{\lrcorner}_{\mathsf{Top}}\mathrm{-mod} \arrow[u, shift right, "\mathrm{pb}"'] \arrow[d, shift left, "(-)^{\mathbb{T}}"] & \text{in } \mathbb{T}\mathrm{Sp} \\
\mathrm{E}^1_\infty\mathrm{-alg\text{-}in-}K^{\mathbb{T}}_{\mathsf{Top}}\mathrm{-cell-}(\mathbb{S}^{\lrcorner}_{\mathsf{Top}})^{\mathbb{T}}\mathrm{-mod} \arrow[u, shift left, "\zeta_{\#}"] & \text{in } \mathrm{Sp}
\end{tikzcd}$$

At this point we change the operad from E^1_∞ to Comm obtaining a Quillen equivalence. This Quillen equivalence at the level of spectra induces a Quillen equivalence at the level of diagram categories and their cellularizations.

Lemma 5.2 [4, Lemma 6.2] *There is an adjunction*

$$\eta_* : \mathrm{E}^1_\infty\mathrm{-alg\text{-}in-Sp}_{\mathbb{Q}} \rightleftarrows \mathit{Comm}\mathrm{-alg\text{-}in-Sp}_{\mathbb{Q}} : \eta^*$$

induced by the map of operads $\eta : \mathrm{E}^1_\infty \longrightarrow \mathit{Comm}$ *in Top. This adjunction is a Quillen equivalence with respect to right-induced model structures from the rational positive stable model structure on* $\mathrm{Sp}_{\mathbb{Q}}$.

Lemma 5.3 *There is an adjunction*

$$\eta_* : \mathrm{E}^1_\infty\mathrm{-alg\text{-}in-}K^{\mathbb{T}}_{\mathit{Top}}\mathrm{-cell-}(\mathbb{S}^{\lrcorner}_{\mathit{Top}})^{\mathbb{T}}\mathrm{-mod} \rightleftarrows \mathit{Comm}\mathrm{-alg\text{-}in-}K^{\mathbb{T}}_{\mathit{Top}}\mathrm{-cell-}(\mathbb{S}^{\lrcorner}_{\mathit{Top}})^{\mathbb{T}}\mathrm{-mod} : \eta^*$$

induced by the map of operads $\eta : \mathrm{E}^1_\infty \longrightarrow \mathit{Comm}$ *in Top. This adjunction is a Quillen equivalence with respect to right-induced model structures from* $K^{\mathbb{T}}_{\mathit{Top}}\mathrm{-cell-}(\mathbb{S}^{\lrcorner}_{\mathit{Top}})^{\mathbb{T}}\mathrm{-mod}$ *(whose model structure is built from the positive stable model structure on* Sp*).*

Another application of Lemma 2.5 gives two more results.

Lemma 5.4 *There is a Quillen equivalence*

$$\mathbb{P}\circ|-|: \mathit{Comm}\text{−alg-in−}\mathrm{Sp}^{\Sigma}_{\mathbb{Q}} \rightleftarrows \mathit{Comm}\text{−alg-in−}\mathrm{Sp}_{\mathbb{Q}} : \mathrm{Sing}\circ\mathbb{U}$$

where both model structures are right induced from the rational positive stable model structures on Sp^{Σ} *and* Sp *respectively. Recall that* $|-|$ *is the geometric realization functor from* sSet *to* Top *and* Sing *is its right adjoint.* $\mathbb{U}$ *denotes the restriction functor from orthogonal spectra (in* Top*) to symmetric spectra (in* Top*) and* $\mathbb{P}$ *is its left adjoint.*

Proposition 5.5 *There is a Quillen equivalence*

$$H\mathbb{Q}\wedge - : \mathit{Comm}\text{−alg-in−}\mathrm{Sp}^{\Sigma}_{\mathbb{Q}} \rightleftarrows \mathit{Comm}\text{−alg-in−}(H\mathbb{Q}\text{−mod}) : U\ .$$

These results lift to the level of diagrams of modules over a fork of commutative rings used in Sect. 4 and so does the following result.

Theorem 5.6 ([27, Corollary 8.4]) *There is a zig–zag of Quillen equivalences between the model category of commutative* $H\mathbb{Q}$*-algebras (in symmetric spectra) and differential graded commutative* $\mathbb{Q}$*-algebras.*

The model structure on commutative $H\mathbb{Q}$-algebras is lifted from the positive flat stable model structure on symmetric spectra, see for example [28]. The model structure on differential graded commutative $\mathbb{Q}$-algebras has fibrations the surjections and weak equivalences the homology isomorphisms.

We can summarise the above in the next lemma, where $\mathbb{S}^{\lrcorner}_{t}$ is a diagram of commutative rings obtained from $(\mathbb{S}^{\lrcorner}_{\mathrm{Top}})^{\mathbb{T}}$ by applying the result of [29].

Lemma 5.7 *There is a zig–zag of Quillen equivalences*

$$\begin{array}{lr}
\mathrm{E}^{1}_{\infty}\text{−alg-in−}K^{\mathbb{T}}_{\mathit{Top}}\text{−cell−}(\mathbb{S}^{\lrcorner}_{\mathit{Top}})^{\mathbb{T}}\text{−mod} & \text{in Sp} \\
\Big\updownarrow \scriptstyle{\textit{zig–zag}\ \text{of Quillen equivalences}} & \\
\mathit{Comm}\text{−alg-in−}K_{t}\text{−cell−}\mathbb{S}^{\lrcorner}_{t}\text{−dmod.} & \text{in Ch}(\mathbb{Q})
\end{array}$$

Via Lemma 2.5, the arguments using formality of certain commutative rings and the removal of cellularization from Sect. 4 pass to the level of commutative algebras to produce the Quillen equivalences below.

$$\begin{array}{lr}
\mathrm{Comm}\text{−alg-in−}K_{t}\text{−cell−}\mathbb{S}^{\lrcorner}_{t}\text{−dmod} & \text{in Ch}(\mathbb{Q}) \\
\Big\updownarrow \scriptstyle{\text{zig–zag of Quillen equivalences}} & \\
\mathrm{Comm}\text{−alg-in−}K_{a}\text{−cell−}\mathbb{S}^{\lrcorner}_{a}\text{−dmod} & \text{in Ch}(\mathbb{Q}) \\
\mu\Big\uparrow\ \Big\downarrow\Gamma & \\
\mathrm{Comm}\text{−alg-in−}d\mathcal{A}(\mathbb{T})_{dual} & \text{in Ch}(\mathbb{Q})
\end{array}$$

We collect all the Quillen equivalences from this section into one result.

Theorem 5.8 *There is a zig–zag of Quillen equivalences*

$$\mathrm{E}^1_\infty\text{−alg-in−}(\mathbb{T}\mathrm{Sp}_\mathbb{Q}) \simeq \mathsf{Comm}\text{−alg-in−}d\mathcal{A}(\mathbb{T}),$$

where $d\mathcal{A}(\mathbb{T})$ denotes the algebraic model for rational $\mathbb{T}$ from [5,10].

We may extend the above results to modules over rational E^1_∞-ring $\mathbb{T}$. Rather than give the somewhat complicated definition of a module over an algebra over an operad (and the monoidal product of such modules), we use a different model category of spectra and simply talk about modules over a commutative ring object.

Recall a variant of the category of rational $\mathbb{T}$: the category of unital $\mathcal{L}(1)$-modules in orthogonal $\mathbb{T}$ from [6], where $\mathcal{L}$ is the linear isometries operad for the universe $U = \mathbb{R}^\infty$. Thus the additive universe is the complete $\mathbb{T}$-universe and the multiplicative universe is taken to be $\mathbb{R}^\infty$. This category will be denoted $\mathcal{L}(\mathbb{R}^\infty)\text{−}\mathbb{T}\mathrm{Sp}_\mathbb{Q}$. Recall further that

$$\mathsf{Comm}\text{−alg-in−}(\mathcal{L}(\mathbb{R}^\infty)\text{−}\mathbb{T}\mathrm{Sp}_\mathbb{Q}) \simeq \mathrm{E}^1_\infty\text{−alg-in−}\mathbb{T}\mathrm{Sp}_\mathbb{Q},$$

so that an E^1_∞-ring in rational orthogonal $\mathbb{T}$ has a strictly commutative model in rational unital $\mathcal{L}(1)$-modules in orthogonal $\mathbb{T}$, $\mathcal{L}(\mathbb{R}^\infty)\text{−}\mathbb{T}\mathrm{Sp}_\mathbb{Q}$.

One may rewrite our zig–zag of Quillen equivalences between $\mathrm{E}^1_\infty\text{−alg-in−}\mathbb{T}\mathrm{Sp}_\mathbb{Q}$ and commutative rings in $d\mathcal{A}(\mathbb{T})$ to start from $\mathsf{Comm}\text{−alg-in−}\mathcal{L}(\mathbb{R}^\infty)\text{−}\mathbb{T}\mathrm{Sp}_\mathbb{Q}$ and use unital $\mathcal{L}(1)$-modules in orthogonal $\mathbb{T}$ in place of $\mathbb{T}\mathrm{Sp}$, as discussed in Remark 4.1.

Corollary 5.9 *For any rational E^1_∞-ring $\mathbb{T}$-spectrum R there is a zig–zag of monoidal Quillen equivalences*

$$R\text{−mod−}(\mathcal{L}(\mathbb{R}^\infty)\text{−}\mathbb{T}\mathrm{Sp}_\mathbb{Q}) \simeq \widetilde{R}\text{−mod−}d\mathcal{A}(\mathbb{T}),$$

where $\widetilde{R}$ is the derived image of R under the zig–zag of Quillen equivalences of Theorem 5.8.

Proof Let $\bar{R}$ be a strictly commutative model for R in the category of unital $\mathcal{L}(1)$-modules in orthogonal $\mathbb{T}$ as discussed in Remark 4.1. Taking the symmetric monoidal category of $\bar{R}$-modules to be the starting point, we may lift the zig–zag of Quillen equivalences described just before the current corollary to categories of modules using Blumberg and Hill [6, Theorem 4.13] and Schwede and Shipley [31, Theorem 3.12]. All steps of the zig–zag are symmetric monoidal, since the Quillen equivalences were so. The algebraic model $d\mathcal{A}(\mathbb{T})$ is taken with the dualizable model structure from [2] as in the current paper. The end of the zig–zag is then a symmetric monoidal category $\widetilde{R}\text{−mod−}d\mathcal{A}(\mathbb{T})$, where $\widetilde{R}$ can be taken to be the derived image of R under the zig–zag of Quillen equivalences of Theorem 5.8, and hence is a commutative ring in $d\mathcal{A}(\mathbb{T})$. □

Part 2. An application to equivariant elliptic cohomology

The purpose of the second part is to apply the results of Part 1 to a case of particular interest.

6 An island of algebraic geometry in rational $\mathbb{T}$

6.1 Overview

As mentioned in the introduction, given an elliptic curve C over a $\mathbb{Q}$-algebra K, together with some coordinate data, one may construct a $\mathbb{T}$-equivariant elliptic cohomology theory

$EC^*_{\mathbb{T}}(\cdot)$ associated to C [11] as follows. First, using C and the coordinate data one may write down an object EC_a of $d\mathcal{A}(\mathbb{T})$; by [10], this corresponds to a $\mathbb{T}$-spectrum EC_{Top} which represents $EC^*_{\mathbb{T}}(\cdot)$, and by [5] this may be taken to be a ring. The construction of EC_a is recalled in Sect. 9.1. Furthermore, $d\mathcal{A}(\mathbb{T})$ is a symmetric monoidal category and EC_a is visibly a commutative monoid.

By Corollary 5.9, EC_{Top} can be chosen to be an E^1_∞-algebra in rational $\mathbb{T}$, which for this part we take to be $\mathcal{L}(\mathbb{R}^\infty)-\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}$. This is the category of unital $\mathcal{L}(1)$-modules in $\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}$, see Blumberg and Hill [6]. It therefore makes sense to discuss the monoidal model category of E^1_∞-modules over EC_{Top}, and the following theorem states that it is equivalent to a category of sheaves of quasi-coherent modules over C. The exact definitions of the following categories will be given as we proceed.

Since the cohomology theory $EC^*_{\mathbb{T}}(\cdot)$ is 2-periodic, we form a 2-periodic version $P\mathcal{O}_C$ of the structure sheaf $\mathcal{O}_C$. The non-empty open sets of the usual Zariski topology on C consist of the complements of arbitrary finite sets of points, but we use the torsion point (tp) topology in which only points of finite order may be deleted. The purpose of part 2 is to prove the following result.

Theorem 6.1 *There is a zig–zag of symmetric monoidal Quillen equivalences*

$$EC_{Top}-\mathrm{mod}-\mathbb{T}\mathrm{Sp}_{\mathbb{Q}} \simeq quasi-coherent-P\mathcal{O}_C-\mathrm{dmod}-\mathrm{sheaves}/C_{tp}$$

giving an equivalence of tensor triangulated categories

$$\mathrm{Ho}(EC_{Top}-\mathrm{mod}-\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}) \simeq D(PC_{tp}),$$

where dmod *denotes differential objects in graded modules.*

Proof We will prove the theorem by showing that both categories are formal and have the same abelian skeleton, $\mathcal{A}(PC)$. More precisely, we construct the following zig–zag of Quillen equivalences

$$EC_{\mathsf{Top}}-\mathrm{mod}-\mathbb{T}\mathrm{Sp}_{\mathbb{Q}} \overset{(1)}{\simeq} EC_a-\mathrm{mod}-d\mathcal{A}(\mathbb{T}) \overset{(2)}{\simeq} d\mathcal{A}(PC)$$
$$\overset{(3)}{\simeq} d\mathcal{A}(P\mathcal{O}_C) \overset{(4)}{\simeq} \mathrm{cell}-P\mathcal{O}^{\lrcorner}_C-\mathrm{dmod}/C_{tp} \overset{(5)}{\simeq} P\mathcal{O}_C-\mathrm{dmod}/C_{tp}$$

Equivalence (1) is an application of the monoidal equivalence of Corollary 5.9.
Equivalence (2) comes from the elementary reformulation of the abelian category given in Lemma 9.3.
Equivalence (3) arises by replacing skyscraper and constant sheaves by the associated modules as in Proposition 8.6.
Equivalence (4) is the Cellular Skeleton Theorem 8.4 in this context.
Equivalence (5) is an immediate application of the Cellularization Principle (Lemma 8.3).

□

Remark 6.2 There are similar results for Lurie's elliptic cohomology associated to a derived elliptic curve [24]. Lurie constructs the representing E^1_∞-ring spectra as sections of a structure sheaf constructed from a derived elliptic curve. At the level of derived algebraic geometry the counterpart of our theorem is built into his machinery, and over $\mathbb{Q}$ classical consequences can be inferred by theorems such as [24, Theorem 2.1.1].

Unpublished work of Gepner and the second author analyses the relationship between Lurie's theories and the theories EC_{Top}. Lurie's theories are defined and natural for all compact Lie groups, in particular the coordinate data is determined by the coordinate at the identity pullback along the power maps. Accordingly, not all of the theories EC_a are restrictions of such theories.

7 Elliptic curves

Let C be an elliptic curve over a $\mathbb{Q}$-algebra K. There are certain associated structures that we recall. For more details on the theory of elliptic curves we refer the reader to [30].

7.1 Functions, divisors and differentials

We write e for the identity of C as an abelian group, $C[n] = \ker(C \xrightarrow{n} C)$ for the points of order dividing n and $C\langle n\rangle$ for the points of exact order n. Coming to divisors, Abel's theorem states that $\Sigma_i n_i[x_i]$ is the divisor of a regular function if and only if $\Sigma_i n_i = 0$ and $\Sigma_i n_i x_i = e$.

We write $\mathcal{O}_C$ for the structure sheaf of the curve C and $\mathcal{K}$ for the (constant) sheaf of meromorphic functions with poles only at points of finite order. We then write $\mathcal{O}_{C,x}$ for the ring of functions regular at x, and $(\mathcal{O}_C)^\wedge_x$ for its completion. For a divisor D we write $\mathcal{O}_C(D) = \{f \in \mathcal{K} \mid \mathrm{div}(f) + D \geq 0\}$ for the sheaf of functions with poles bounded by D.

By the Riemann–Roch theorem $H^0(C; \mathcal{O}_C(ne))$ is n-dimensional over K if $n > 0$. It follows that there is a short exact sequence

$$0 \longrightarrow C \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $\mathrm{Pic}(C)$ is the Picard group of line bundles on C. Thus any line bundle is determined up to equivalence by a point on C and its degree. We are especially concerned with the line bundles associated to sums of divisors $C[n]$ where we have in particular $\mathcal{O}_C(C[n]) \cong \mathcal{O}_C(n^2e)$.

Since C is an abelian group, the sheaf Ω^1_C of differentials is a trivializable line bundle. We will make our theory 2-periodic by using Ω^1_C for the periodicity.

Definition 7.1 The periodic structure sheaf is the sheaf of graded rings $P\mathcal{O}_C = \bigoplus_{n\in\mathbb{Z}}(\Omega^1_C)^{\otimes n}$, where $(\Omega^1_C)^{\otimes n}$ is the degree $2n$ part. A *periodic* $\mathcal{O}_C$-module M is a graded module over $P\mathcal{O}_C$.

7.2 Derived categories of sheaves of modules

The structure sheaf $\mathcal{O}_C$ is a sheaf of rings over C and we consider complexes of sheaves of quasi-coherent $\mathcal{O}_C$-modules.

The non-empty open sets of the usual Zariski topology on C consist of the complements of arbitrary finite sets of points. We use the torsion point (tp) topology where the non-empty open sets are obtained by deleting a finite sets of points, each of which has finite order. In effect, we only permit poles at points of finite order.

Consider four classes of maps $f : \mathcal{P} \longrightarrow \mathcal{Q}$ of sheaves over C_{tp}.

- $\mathcal{F}_1 = \{f \mid f : \mathcal{P} \xrightarrow{\cong} \mathcal{Q}\}$
- $\mathcal{F}_2 = \{f \mid f(D) : \mathcal{P}(D) \xrightarrow{\cong} \mathcal{Q}(D)$ for all effective torsion point divisors D$\}$
- $\mathcal{F}_3 = \{f \mid f_x : \mathcal{P}_x \xrightarrow{\cong} \mathcal{Q}_x$ for all points x of finite order$\}$
- $\mathcal{G} = \{f \mid \Gamma(f) : \Gamma(\mathcal{P}) \xrightarrow{\cong} \Gamma(\mathcal{Q})\}$

The first three classes are in fact the same.

Lemma 7.2 *We have equalities* $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$. *We use* $\mathcal{F}_{tp}$ *to denote it and we call elements of this class* tp-isomorphisms. *We call a map* f *between complexes of sheaves a* tp-equivalence *if* $H_*(f)$ *is a* tp*-isomorphism.*

Proof It is clear that $\mathcal{F}_1 \supseteq \mathcal{F}_2$ since we can take the tensor product with the locally free sheaf $\mathcal{O}_C(D)$. It is clear that $\mathcal{F}_2 \supseteq \mathcal{F}_3$ since $\mathcal{P}_x$ is the direct limit of sections of $\mathcal{P}(D)$ over D not containing x.

The proof that $\mathcal{F}_3 \supseteq \mathcal{F}_1$ is the local to global property of sheaves. □

The derived category $D(C_{tp})$ is formed from complexes of sheaves by inverting all homology tp-isomorphisms. The method of Lemma 8.2 below shows that not every equivalence of global sections is a tp-equivalence of sheaves, but since $\mathcal{F}_{tp} \subseteq \mathcal{G}$ we obtain a localization

$$D(C_{tp}) \longrightarrow D(C_{tp})[\mathcal{G}^{-1}] =: D_{\mathcal{G}}(C).$$

7.3 Model categories of sheaves of modules

The natural way for a homotopy theorist to construct these derived categories is as homotopy categories of model categories. As seen in [22], there are numerous different model structures with the class of homology isomorphisms as weak equivalences.

One method of constructing model structures is by specifying a generating set $\mathbb{K}$ of small objects. Writing $S^n(M)$ for the complex with M in degree n and zero elsewhere and $D^{n+1}(M)$ for the mapping cone of the identity map of $S^n(M)$ as usual we take generating cofibrations and acyclic cofibrations to be

$$\begin{aligned} I(\mathbb{K}) &= \{S^{n-1}(M) \longrightarrow D^n(M) \mid n \in \mathbb{Z}, M \in \mathbb{K}\} \\ J(\mathbb{K}) &= \{0 \longrightarrow D^n(M) \mid n \in \mathbb{Z}, M \in \mathbb{K}\}. \end{aligned}$$

We then attempt to use these to generate a model structure.

We are particularly interested in monoidal model structures so we want to take $\mathbb{K}$ to consist of flat objects. We define

$$\mathbb{K}_{tp} = \{\mathcal{O}_C(D) \mid D \text{ a divisor of torsion points}\}.$$

Proposition 7.3 *The sets* $I(\mathbb{K}_{tp})$ *and* $J(\mathbb{K}_{tp})$ *generate a proper monoidal model structure on the category of differential graded sheaves of quasi-coherent* $\mathcal{O}_C$*-modules, with* tp*-equivalences as weak equivalences. We call it the* flat tp-model structure.

We will show that these sets generate a proper model structure using Quillen's argument as codified by [22, Theorem 1.7]. The proof that it is monoidal will be based on the flatness of objects in $\mathbb{K}_{tp}$. Before proceeding, we record the usual calculation of maps out of discs and spheres and categorical generators.

Lemma 7.4 *For a complex* $\mathcal{P}$ *of sheaves,*

$$\mathrm{Hom}(D^n(M), \mathcal{P}) = \mathrm{Hom}(M, \mathcal{P}_n)$$

and

$$\mathrm{Hom}(S^n(M), \mathcal{P}) = \mathrm{Hom}(M, Z_n(\mathcal{P})).$$

Lemma 7.5 *The set $\mathbb{K}_{tp}$ is a set of generators for quasi-coherent tp-sheaves.*

Proof. We must show that a sheaf $\mathcal{P}$ with only the zero map from the sheaves $\mathcal{O}_C(D)$ is zero, which we do by showing the stalks are all zero.

By definition the stalk of a sheaf $\mathcal{P}$ at x is the direct limit of $\mathcal{P}(U)$ over open sets U containing x. The complement of such an open set U is a closed set not containing x. Let $\mathcal{V}_x$ denote the collection of closed sets not containing x and $\mathcal{D}_x$ denote the collection of effective divisors not containing x.

Since $\mathcal{D}_x$ is cofinal in $\mathcal{V}_x$,

$$\mathcal{P}_x = \operatorname{colim}_{D\in\mathcal{D}_x} \mathcal{P}(X \setminus D).$$

Finally

$$\mathcal{P}(X \setminus D) = \operatorname{colim}_k \Gamma\mathcal{P}(kD) \cong \operatorname{colim}_k \operatorname{Hom}(\mathcal{O}_C(-kD), \mathcal{P}).$$

The last step follows from the enriched isomorphism

$$\operatorname{Hom}(\mathcal{O}_C(-D), \mathcal{P}) \cong \mathcal{P}(D). \qquad \square$$

Remark 7.6 The proof applies to an arbitrary quasi-projective variety with the Zariski topology.

Proof of Proposition 7.3. As usual, we define fibrations to be $J-$inj (i.e. the maps with the right lifting property with respect to all elements of J) and cofibrations to be $I-$cof (i.e. the maps with the left lifting property with respect to all elements of $I-$inj).

It remains to verify the conditions of [22, Theorem 1.7]. Firstly, $\mathbb{K}_{tp}$ is a generating set by Lemma 7.5 so we need to show that if $\mathcal{P}$ is $\mathbb{K}_{tp}$-flabby then $\operatorname{Hom}(M, \mathcal{P})$ is acyclic for all $M \in \mathbb{K}_{tp}$.

Suppose that $\mathcal{P}$ is $\mathbb{K}_{tp}$-flabby. By Lemma 7.4, this means that the map

$$\Gamma(\mathcal{P}(D)_n) = \operatorname{Hom}(D^n(\mathcal{O}_C(-D)), \mathcal{P}) \longrightarrow \operatorname{Hom}(S^{n-1}(\mathcal{O}_C(-D)), \mathcal{P}) \longrightarrow \Gamma(Z_n\mathcal{P}(D))$$

is surjective for all divisors D. Taking colimits as in Lemma 7.5, we see that

$$\mathcal{P}_n \longrightarrow Z_n\mathcal{P}$$

is an epimorphism on stalks, which is to say it is an epimorphism of sheaves. Thus every cycle is a boundary and $\mathcal{P}$ is acyclic. Tensoring with the flat sheaf $\mathcal{O}_C(D)$ we see that $\operatorname{Hom}(\mathcal{O}_C(-D), \mathcal{P})$ is acyclic as required.

For the monoidal statement we need to check the pushout product axiom. By [21, Chapter 4], we need only check that $i\Box j$ is an acyclic cofibration when $i \in I$ and $j \in J$ and that $i\Box i'$ is a cofibration for $i, i' \in I$.

If $i = (S^{m-1}(M) \longrightarrow D^m(M))$ and $j = (0 \longrightarrow D^n(N))$ this involves considering the map

$$S^{m-1}(M) \otimes D^n(N) \longrightarrow D^m(M) \otimes D^n(N).$$

This is obtained from

$$M \otimes N \longrightarrow D^1(M \otimes N)$$

by suspending and taking the mapping cones of the identity. The class $\mathbb{K}_{tp}$ is closed under tensor products, so this is a generating cofibration and the mapping cone of the identity will give another cofibration with both terms acyclic.

For $i = (S^{m-1}(M) \longrightarrow D^m(M))$, $i' = (S^{m'-1}(M') \longrightarrow D^{m'}(M'))$ the resulting map comes from the one for $\bar{i} = (S^{m-1}(\mathbb{Q}) \longrightarrow D^m(\mathbb{Q}))$, $\bar{i}' = (S^{m'-1}(\mathbb{Q}) \longrightarrow D^{m'}(\mathbb{Q}))$ in chain complexes over $\mathbb{Q}$ by tensoring with $M \otimes M'$. Since $\bar{i}\Box\bar{i}'$ is a cofibration in $\mathbb{Q}$-modules it follows that $i\Box i'$ is a cofibration. □

Remark 7.7 We will need the periodic variant, where we start from the sheaf $P\mathcal{O}_C$ of *graded* rings. We then consider differential graded sheaves of quasi-coherent $P\mathcal{O}_C$-modules in the tp-topology, and form $D(P\mathcal{O}_C)$ as before.

8 Hasse squares

8.1 The Hasse square for $\mathbb{T}$

We will give some further details of the model for rational $\mathbb{T}$ outlined in Sect. 4, because the models for quasi-coherent sheaves over an elliptic curve are constructed analogously.

In particular, recall the Quillen equivalence between a cellularization of $\mathbb{S}_a^{\lrcorner}-$mod and the algebraic model $d\mathcal{A}(\mathbb{T})$. We are replacing the cellularization of $\mathbb{S}_a^{\lrcorner}-$mod by restricting to the subcategory $\mathcal{A}(\mathbb{T})$ of qce-modules, which we think of as the essential skeleton. Since we are preparing for the analogue, we recall the outline, which is an elaboration of [13, Section 5]. The key fact is that we have an adjunction where the left adjoint i is the inclusion (clearly strong symmetric monoidal) and the right adjoint is Γ (see [10] for an explicit description of Γ).

This is a Quillen pair, and we apply the Cellularization Principle [13, Theorem 2.1] to get a Quillen equivalence. The main content of the proof is that a cellular equivalence between torsion objects is a homology isomorphism. We recall the proof from [15] partly because it is much simpler for the circle than a general torus, and partly because we will need to adapt it to sheaves in Lemma 8.4 below. In the following proofs we will make use of the special objects $e(V) = (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ for a graded vector space V and $f(T) = (T \longrightarrow 0)$ for an $\mathcal{O}_{\mathcal{F}}$-module T, such that $\mathcal{E}^{-1}T = 0$ (we say T is an $\mathcal{E}$-torsion module).

Lemma 8.1 *(Cellular Skeleton Theorem [15, 11.5]) There is a monoidal Quillen equivalence*

$$\text{cell}-\mathbb{S}_a^{\lrcorner}-\text{dmod} \simeq d\mathcal{A}(\mathbb{T}).$$

Proof We show that a cellular equivalence is a homology isomorphism. We do that by showing that if $X = (N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ in $d\mathcal{A}(\mathbb{T})$ is cellularly trivial then $X \simeq 0$.

First note

$$0 \simeq \text{Hom}(S^{-W}, X) \cong (\Sigma^W N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$$

so that

$$0 \simeq \text{colim}_{W^{\mathbb{T}}=0}\,\text{Hom}(S^{-W}, X) = (\mathcal{E}^{-1}N \xrightarrow{\cong} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V) = e(V)$$

is trivial. Hence $V \simeq 0$, $N \simeq T$ is a torsion module and $X = f(T)$. Since T is torsion, $T \cong \bigoplus_n e_n T$. Now consider the algebraic counterpart of the basic cell $\Sigma^{-1}\sigma_n = \mathbb{T} \wedge_{\mathbb{T}[n]} e_n S^{-1}$, namely $(\mathbb{Q}_n \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes 0)$, where $\mathbb{T}[n]$ are points of order n and $\mathbb{Q}_n$ denotes $\mathbb{Q}$ at degree 0 at the spot corresponding to C_n in the $\mathcal{O}_{\mathcal{F}}$-module. Then we have

$$0 \simeq \text{Hom}(\sigma_n, f(T)) = \text{Hom}_{\mathbb{Q}[c]}(\mathbb{Q}, e_n T)$$

Hence $e_n T \simeq 0$. Since this applies to all $n \geq 1$ we see $T \simeq 0$ as required. □

For comparison, it is illuminating to identify the non-trivial objects $X = (\beta : N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ of $d\mathcal{A}(\mathbb{T})$ so that $\pi_*^{\mathbb{T}}(X) = 0$.

Lemma 8.2 *If T is an $\mathcal{E}$-divisible $\mathcal{E}$-torsion module with action map $\mu : \mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} T \longrightarrow T$ then the object*

$$a(T) = (NT \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes T) \text{ with } NT = \ker(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes T \xrightarrow{\mu} T)$$

has $\pi_*^{\mathbb{T}}(a(T)) = 0$. Conversely, if X has $\pi_*^{\mathbb{T}}(X) = 0$ then $X \simeq a(T)$ where $T = \pi_*^{\mathbb{T}}(X \wedge \Sigma E\mathcal{F}_+)$.

Proof Suppose that $X = (N \xrightarrow{\beta} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ has $\pi_*^{\mathbb{T}}(X) = 0$. Since all objects are formal we may suppose that X has zero differential, and any such X splits as an even part X_{ev} and an odd part X_{od}. Since X is $\pi_*^{\mathbb{T}}$-acyclic so are its even and odd parts. We may therefore assume without loss of generality that X is even. Next, we see that β must be injective. Otherwise, if $K = \ker(\beta)$ we have a cofibre sequence

$$X \longrightarrow e(V) \longrightarrow f(T) \oplus \Sigma f(K),$$

where $T = (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)/N$.

Since $\pi_*^{\mathbb{T}}(e(V)) = V$ is in even degrees, the vanishing of $\pi_*^{\mathbb{T}}(X)$ forces $\pi_*^{\mathbb{T}}(\Sigma f(K))$ to be in even degrees. However, K is a torsion module in even degrees so $K \simeq 0$ (if $K \not\simeq 0$ then $\Sigma f(K)$ will have some odd degree homotopy).

Since X has injective structure map β, we have an injective resolution

$$0 \longrightarrow X \longrightarrow e(V) \longrightarrow f(T) \longrightarrow 0,$$

where T is $\mathcal{E}$-torsion and divisible. Now $\pi_*^{\mathbb{T}}(X) = 0$ if and only if $V = \pi_*^{\mathbb{T}}(e(V)) \longrightarrow \pi_*^{\mathbb{T}}(f(T)) = T$ is an isomorphism, so that $X \simeq a(T)$ as required. □

8.2 The Hasse square for sheaves over *C*

Returning to the elliptic curve C, just as in Sect. 4 there is a pullback square

$$\begin{tikzcd}
\mathcal{O}_C \arrow[r] \arrow[d] & \mathcal{K} \arrow[d] \\
\mathcal{O}^{\wedge}_{C(\mathcal{F})} \arrow[r] & \mathcal{K} \otimes_{\mathcal{O}_C} \mathcal{O}^{\wedge}_{C(\mathcal{F})}
\end{tikzcd}$$

of sheaves of rings over C, where $\mathcal{O}^{\wedge}_{C(\mathcal{F})} = (\mathcal{O}_C)^{\wedge}_{\mathcal{F}}$ is the $\mathcal{F}$-completion of $\mathcal{O}_C$. Recall that $\mathcal{K}$ denotes the constant sheaf of meromorphic functions with poles only at points of finite order. We note that there is a product splitting

$$\mathcal{O}^{\wedge}_{C(\mathcal{F})} \cong \prod_n (\mathcal{O}_C)^{\wedge}_{C\langle n\rangle}.$$

We write $\mathcal{O}_C^{\lrcorner}$–dmod for the category of differential modules over the diagram

$$\mathcal{O}_C^{\lrcorner} = \left(\begin{tikzcd}
 & \mathcal{K} \arrow[d, "\iota"] \\
\mathcal{O}^{\wedge}_{C(\mathcal{F})} \arrow[r, "l"] & \mathcal{K} \otimes_{\mathcal{O}_C} \mathcal{O}^{\wedge}_{C(\mathcal{F})}
\end{tikzcd} \right)$$

of sheaves of rings.

We write $\mathcal{O}_C^{\dashv}$−dmod for the category of all diagrams

$$\mathcal{N} \xrightarrow{k} \mathcal{P} \overset{m}{\leftarrow} \mathcal{V}$$

of differential modules over $\mathcal{O}_C^{\dashv}$ where k is a map of differential $\mathcal{O}^{\wedge}_{C(\mathcal{F})}$-modules and m is a map of differential $\mathcal{K}$-modules. The category $\mathcal{A}(\mathcal{O}_C)$ consists of diagrams where the maps induce isomorphisms $l_*\mathcal{N} \cong \mathcal{P}$ and $\iota_*\mathcal{V} \cong \mathcal{P}$.

Just as happened for modules over the sphere spectrum, the model category of differential graded $\mathcal{O}_C$−mod with tp-equivalences from Lemma 7.2 can be recovered from differential modules over $\mathcal{O}_C^{\dashv}$.

Lemma 8.3 *There is a Quillen equivalence*

$$\mathcal{O}_C\text{−dmod} \simeq \text{cell−}\mathcal{O}_C^{\dashv}\text{−dmod},$$

where the model structure on the left is the one from Proposition 7.3 and the one on the right is the cellularization (at the derived images of the generators $\mathcal{O}(D)$ of differential graded $\mathcal{O}_C$-modules) of the diagram injective model structure built from the tp-flat model structures on the categories of sheaves.

Proof Extension of scalars from $\mathcal{O}_C$ to $\mathcal{O}_C^{\dashv}$ is left adjoint to the pullback functor. If we use the tp-flat model structures on the categories of sheaves and the diagram injective model structure, then this is a Quillen pair.

The generators $\mathcal{O}(D)$ of differential $\mathcal{O}_C$-modules are small, and have small images under the derived left adjoint. The fact that the $\mathcal{O}_C$ is the pullback of the diagram $\mathcal{O}_C^{\dashv}$ and the fact that $\mathcal{O}_C(D)$ is locally free shows that all generators are locally free. The result then follows from the Cellularization Principle [13]. □

Similarly, we may obtain a graded abelian category if we replace $\mathcal{O}_C$ by $P\mathcal{O}_C$, and then form $d\mathcal{A}(P\mathcal{O}_C)$.

We give the categories of differential graded modules over the individual graded sheaves of rings in $P\mathcal{O}_C^{\dashv}$ the flat tp-model structures, which we have seen to be proper and monoidal. We then give the category $P\mathcal{O}_C^{\dashv}$−dmod the diagram-injective model structure.

Lemma 8.4 (Cellular Skeleton Theorem) *There is a monoidal Quillen equivalence*

$$\text{cell−}\mathcal{O}_C^{\dashv}\text{−dmod} \simeq d\mathcal{A}(\mathcal{O}_C)$$

where the left hand side is cellularised at the derived images of the generators of the differential $\mathcal{O}_C$-modules. Both of these model categories are Quillen equivalent to differential $\mathcal{O}_C$-modules with the tp-flat model structure and so all three have homotopy categories $D_{tp}(\mathcal{O}_C)$.

Remark 8.5 The corresponding result for the periodic case follows directly, so that we also have a monoidal Quillen equivalence

$$\text{cell−}P\mathcal{O}_C^{\dashv}\text{−dmod} \simeq d\mathcal{A}(P\mathcal{O}_C).$$

Proof We begin with the adjunction

$$i : d\mathcal{A}(\mathcal{O}_C) \rightleftarrows \mathcal{O}_C^{\dashv}\text{−dmod} : \Gamma\ .$$

The left adjoint is inclusion i and the right adjoint Γ is the torsion functor constructed just like in the case of $\mathcal{A}(\mathbb{T})$, see [5]. The left adjoint is clearly strong symmetric monoidal.

Passing to differential graded objects, we obtain a monoidal Quillen pair. The objects $\mathcal{O}_C(D)$ with D a torsion point divisor lie in $d\mathcal{A}(\mathcal{O}_C)$, so the cellularization Principle [13] shows that the cellularizations are Quillen equivalent.

Finally, we observe that any $\mathcal{O}_C(D)$-cellular equivalence of torsion objects for all torsion point divisors D is a homology isomorphism. The proof is the precise analogue of the one in $d\mathcal{A}(\mathbb{T})$ (Lemma 8.1). We show that if $X = (N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ is $\mathcal{O}_C(D)$-trivial for all torsion point divisors D then $X \simeq 0$.

First note

$$0 \simeq \mathrm{Hom}(\mathcal{O}(-D), X) \cong (N(D) \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$$

so that

$$0 \simeq \mathrm{colim}_D \, \mathrm{Hom}(\mathcal{O}_C(-D), X) = (\mathrm{colim}_D \, N(D) \xrightarrow{\cong} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V) = e(V)$$

is trivial. Hence $V = 0$, $N = T$ is a torsion sheaf and $X = f(T)$. It follows that its stalks at non-torsion points are trivial. Finally we use the short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(C\langle n\rangle) \longrightarrow i_*^{C\langle n\rangle} k \longrightarrow 0$$

where $i^{C\langle n\rangle} : C\langle n\rangle \longrightarrow \mathbb{T}$ denotes the inclusion. We see

$$0 \simeq \mathrm{Hom}(f(i_*^{C\langle n\rangle} k), f(T)) = \mathrm{Hom}_{C\langle n\rangle}(k, T|_{C\langle n\rangle})$$

Hence $T|_{C\langle n\rangle} \simeq 0$. Since this covers all torsion points $T \simeq 0$ as required. □

8.3 From algebra to algebraic geometry

We now wish to observe that the abelian category $\mathcal{A}(\mathcal{O}_C)$ of diagrams of sheaves is equivalent to a purely algebraic category $\mathcal{A}(C)$ of modules over the diagram of rings. All results in this section apply equally well to periodic modules, but since the results are about abelian categories we will state them for the non-periodic version for brevity.

We note that the diagram $\mathcal{O}_C^{\lrcorner}$ consists of constant sheaves and skyscraper sheaves, so it is equivalent to the diagram of rings

$$\Gamma\mathcal{O}_C^{\lrcorner} = \left(\begin{array}{ccc} & & \mathcal{K} \\ & & \downarrow \iota \\ \prod_n \Gamma\mathcal{O}^{\wedge}_{C\langle n\rangle} & \xrightarrow{l} & \mathcal{K} \otimes_{\mathcal{O}_C} \prod_n \Gamma\mathcal{O}^{\wedge}_{C\langle n\rangle} \end{array} \right)$$

(in modules over the ground ring K) formed by taking global sections. Let $\Gamma\mathcal{O}_C^{\lrcorner}-\mathrm{mod}$ denote the category of modules over $\Gamma\mathcal{O}_C^{\lrcorner}$. We let $\mathcal{A}(C)$ to be the category of these $\Gamma\mathcal{O}_C^{\lrcorner}$-modules so that the horizontal and vertical maps are extensions of scalars. We write $\Gamma\mathcal{O}_C^{\lrcorner}-\mathrm{mod}$ for the category of all diagrams

$$\mathcal{N} \longrightarrow \mathcal{P} \leftarrow \mathcal{V}$$

of modules over $\Gamma\mathcal{O}_C^{\lrcorner}$. The category $\mathcal{A}(C)$ consists of diagrams where the maps induce isomorphisms $l_*\mathcal{N} \cong \mathcal{P}$ and $\iota_*\mathcal{V} \cong \mathcal{P}$.

Lemma 8.6 *We have an equivalence of categories*

$$\mathcal{A}(\mathcal{O}_C) \simeq \mathcal{A}(C).$$

Proof For clarity, in this proof we will introduce notation to distinguish between sheaves and global sections.

At the generic point we note that the sheaf $\widetilde{\mathcal{K}}$ of meromorphic functions is constant at $\mathcal{K} = \Gamma\widetilde{\mathcal{K}}$. Similarly any quasi-coherent module $\widetilde{\mathcal{M}}$ over the sheaf $\widetilde{\mathcal{K}}$ is also constant (at $\mathcal{M} = \Gamma\widetilde{\mathcal{M}}$). This means that passage to sections gives an equivalence between sheaves of modules over $\widetilde{\mathcal{K}}$ and modules over the ring $\mathcal{K}$.

Next, if $\mathcal{F}$ is any skyscraper sheaf of rings concentrated over a finite set of points $x_1, \ldots, x_s$ passage to global sections gives an isomorphism

$$\Gamma\mathcal{F} \xrightarrow{\cong} \prod_{i=1}^{s} \mathcal{F}_{x_i}$$

between the global sections and the product of the stalks. Furthermore, any $\mathcal{F}$-module $\mathcal{N}$ is a skyscraper over $x_1, \ldots, x_s$, because $0 = 1$ in the stalk at any other point. Next, we have an isomorphism

$$\Gamma\mathcal{N} \xrightarrow{\cong} \prod_{i=1}^{s} \mathcal{N}_{x_i}.$$

It follows that there is an equivalence of categories between sheaves of $\mathcal{F}$-modules and modules over $\prod_{i=1}^{s} \mathcal{F}_{x_i}$. Finally, this is natural for maps of sheaves of rings concentrated over $x_1, \ldots, x_s$.

Next, if $\widetilde{\mathcal{C}}$ is constant at $\mathcal{C}$ and $\mathcal{N}$ is concentrated at $x_1, \ldots, x_s$ then passage to global sections gives an isomorphism

$$\Gamma : \mathrm{Hom}(\widetilde{\mathcal{C}}, \mathcal{N}) \xrightarrow{\cong} \prod_{i=1}^{s} \mathrm{Hom}(\mathcal{C}, \mathcal{N}_{x_i}).$$

Finally we turn to the question of dealing with an infinite number of points. In our case we have the sheaf, $(\mathcal{O}_C)^{\wedge}_{\mathcal{F}}$ which is itself a product: $(\mathcal{O}_C)^{\wedge}_{\mathcal{F}} = \prod_n (\mathcal{O}_C)^{\wedge}_{C\langle n\rangle}$. Since Γ is lax monoidal, passage to sections therefore gives a functor

$$\Gamma : (\mathcal{O}_C)^{\wedge}_{\mathcal{F}}\text{−mod} = \left[\prod_n (\mathcal{O}_C)^{\wedge}_{C\langle n\rangle}\right]\text{−mod} \longrightarrow \left[\prod_n \Gamma(\mathcal{O}_C)^{\wedge}_{C\langle n\rangle}\right]\text{−mod}.$$

Altogether, taking sections at all three points of the diagram gives a functor

$$\mathbf{0} : \mathcal{O}_C^{\lrcorner}\text{−mod} \longrightarrow \Gamma\mathcal{O}_C^{\lrcorner}\text{−mod}.$$

Next we define a functor in the other direction

$$\mathbf{t} : \Gamma\mathcal{O}_C^{\lrcorner}\text{−mod} \longrightarrow \mathcal{O}_C^{\lrcorner}\text{−mod}.$$

On the two right hand entries, we use the constant sheaf functor. For the bottom left, we note that if M is a module over $\prod_n \Gamma(\mathcal{O}_C)^{\wedge}_{C\langle n\rangle}$ we may define a sheaf $t\mathcal{M}$ as follows. The open sets are obtained by choosing a finite set $n_1, \ldots, n_t$ of orders and taking

$$U(n_1, \ldots, n_t) = C \setminus \coprod_j C\langle n_j\rangle.$$

Associated to the numbers $n_1, \ldots, n_t$ there is an idempotent $e(n_1, \ldots, n_t)$ in $\prod_n \Gamma(\mathcal{O}_C)^{\wedge}_{C\langle n\rangle}$ supported on the subgroups of these orders and we take

$$t(\mathcal{M})(U(n_1, \ldots, n_t)) = \mathcal{M}/e(n_1, \ldots, n_j)\mathcal{M}.$$

The map $\mathcal{M} \longrightarrow \mathcal{P}$ along the horizontal map in diagram defining $\Gamma\mathcal{O}_C^{\dashv}$ gives a map $t(\mathcal{M}) \longrightarrow \widetilde{\mathcal{P}}$, where $\widetilde{\mathcal{P}}$ is the constant sheaf. Since Γ and t are inverse equivalences, so are $\mathbf{0}$ and $\mathbf{t}$. □

Remark 8.7 It is clear that the analogous statement works for differential objects and also for $P\mathcal{O}_C$ and PC instead of $\mathcal{O}_C$ and C respectively. Thus we have an equivalence of categories

$$d\mathcal{A}(P\mathcal{O}_C) \simeq d\mathcal{A}(PC).$$

9 The algebraic model of $\mathbb{T}$-equivariant elliptic cohomology

9.1 The algebraic object of an elliptic curve with coordinates

Coordinate data on our elliptic curve C consists of functions $t_1, t_2, \ldots$ with t_n vanishing to the first order on the points of exact order n. We also let Dt denote the regular differential agreeing with dt_1 at e.

This data allows us to write down an object EC_a in the algebraic model for rational $\mathbb{T}$. Indeed we will define

$$EC_a = (NC \xrightarrow{\beta} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes VC)$$

where β is injective. We take

$$VC = P\mathcal{K}$$

to consist of the meromorphic functions made periodic, and NC to be the following module of regular functions:

$$NC = \ker \left[\mathcal{O}_{\mathcal{F}} \otimes VC \xrightarrow{p} \bigoplus_{n \geq 1} H^1_{C\langle n\rangle}(C; P\mathcal{O}_C) \right]$$

where the principal part map p has nth component

$$p(c^v \otimes f) = \overline{\left(\frac{t_n}{Dt}\right)^{v(n)} f},$$

where $v : \mathcal{F} \longrightarrow \mathbb{Z}$ is a function which is zero almost everywhere (it is checked in [11, Lemma 10.8] that this determines the map p).

Remark 9.1 For $i = 0, 1$ we have

$$EC^{\mathbb{T}}_{-i}(S^V) = H^i(C; \mathcal{O}(D(V)).$$

The complete proof is given in [11, Section 10], but the $i = 0$ part of the result is immediate from the definition. Indeed, $c^v \otimes f$ lies in the kernel of p if, for every point Q of finite order n, the pole of f at Q is of order $\leq v(n)$ (i.e., $v_Q(f) + v(n) \geq 0$):

$$\ker(p|_{c^v \otimes \mathcal{K}}) = \Gamma(\mathcal{O}(D(v))),$$

where

$$D(v) = \sum_n v(n)C\langle n\rangle.$$

9.2 Multiplicativity of EC_a

The product of two meromorphic functions is meromorphic. Indeed if f and g have poles at points only of finite order the product fg has the same property. Hence $\mathcal{K}$ is a commutative ring and $P\mathcal{K}$ and $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes P\mathcal{K}$ are commutative graded rings in even degrees.

Lemma 9.2 ([11, Theorem 11.1]) *Multiplication of meromorphic functions defines a map* $EC_a \otimes EC_a \longrightarrow EC_a$ *making* EC_a *into a commutative monoid in* $d\mathcal{A}(\mathbb{T})$.

9.3 From modules over EC_a to modules over $\Gamma\mathcal{O}_C^{\lrcorner}$.

The connection between differential EC_a-modules and $d\mathcal{A}(PC)$ is entirely elementary.

Lemma 9.3 *We have an equivalence of abelian categories*

$$EC_a\text{−mod−}\mathcal{A}(\mathbb{T}) \simeq \mathcal{A}(PC).$$

Proof This is an immediate reformulation: we just need to make it apparent how the same structures are expressed in the two categories.

We have $EC_a = (NC \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes P\mathcal{K})$ so that if $X = (N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ is a module over EC_a we have a map

$$EC_a \otimes X = (NC \otimes_{\mathcal{O}_{\mathcal{F}}} N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes P\mathcal{K} \otimes V) \longrightarrow (N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V) = X.$$

This makes V into a $P\mathcal{K}$-module, N into an NC-module, and the Euler classes e_W (induced by the map $S^0 \longrightarrow S^W$ for a representation W of $\mathbb{T}$ with $W^{\mathbb{T}} = 0$) come from EC_a (locally $e_W(n) = (t_n/Dt)^{d_n}$, where $d_n = \dim_{\mathbb{C}}(W^{\mathbb{T}[n]})$). For definition and detailed discussion of Euler classes in case of a circle group see [10, Section 4.6]. □

This lemma clearly extends to the level of differential objects in both abelian categories and thus finishes the proof of Theorem 6.1.

Acknowledgements The authors would like to thank the Mathematisches Forschungsinstitut Oberwolfach and the Centre International de Rencontres Mathématiques in Luminy for providing ideal environments to work on this project as part of Research in Pairs Programmes. The second author is grateful to EPSRC for support from EP/P031080/1. The third author was supported by a NWO Veni Grant 639.031.757 during final stages of this work.

References

1. Barnes, D.: Splitting monoidal stable model categories. J. Pure Appl. Algebra **213**(5), 846–856 (2009)

2. Barnes, D.: A monoidal algebraic model for rational $SO(2)$-spectra. Math. Proc. Camb. Philos. Soc. **161**(1), 167–192 (2016)
3. Bergner, J.E.: Homotopy fiber products of homotopy theories. Isr. J. Math. **185**, 389–411 (2011)
4. Barnes, D., Greenlees, J.P.C., Kędziorek, M.: An algebraic model for rational naïve-commutative G-equivariant ring spectra for finite G. Homol Homotopy Appl. **21**(1), 73–93 (2018)
5. Barnes, D., Greenlees, J.P.C., Kędziorek, M., Shipley, B.: Rational SO(2)-equivariant spectra. Algebr. Geom. Topol. **17**(2), 983–1020 (2017)
6. Blumberg, A.J., Hill, M.A.: G-symmetric monoidal categories of modules over equivariant commutative ring spectra. Tunisian J. Math. **2**(2), 237–286 (2020). https://doi.org/10.2140/tunis.2020.2.237
7. Blumberg, A.J., Hill, M.A.: Operadic multiplications in equivariant spectra, norms, and transfers. Adv. Math. **285**, 658–708 (2015)
8. Barnes, D., Roitzheim, C.: Stable left and right Bousfield localisations. Glasg. Math. J. **2**, 1–30 (2013)
9. Elmendorf, A.D., Kriz, I., Mandell, M.A., May, J.P.: Rings, Modules, and Algebras in Stable Homotopy Theory. Mathematical Surveys and Monographs, vol. 47. American Mathematical Society, Providence (1997)
10. Greenlees, J.P.C.: Rational S^1-equivariant stable homotopy theory. Mem. Am. Math. Soc. **661**, 138–289 (1999)
11. Greenlees, J.P.C.: Rational S^1-equivariant elliptic cohomology. Topology **44**(6), 1213–1279 (2005)
12. Greenlees, J.P.C.: Couniversal spaces which are equivariant commutative ring spectra. Homol. Homotopy Appl. **22**(1), 69–75 (2020)
13. Greenlees, J.P.C., Shipley, B.: The cellularization principle for Quillen adjunctions. Homol. Homotopy Appl. **15**(2), 173–184 (2013)
14. Greenlees, J.P.C., Shipley, B.: Homotopy theory of modules over diagrams of rings. Proc. Am. Math. Soc. Ser. B **1**, 89–104 (2014)
15. Greenlees, J.P.C., Shipley, B.: An algebraic model for rational torus-equivariant spectra. J. Topol. **11**(3), 666–719 (2018)
16. Harper, J.E., Hess, K.: Homotopy completion and topological Quillen homology of structured ring spectra. Geom. Topol. **17**(3), 1325–1416 (2013)
17. Hill, M.A., Hopkins, M.J.: Equivariant Multiplicative Closure Algebraic Topology: Applications and New Directions, vol. 620, pp. 183–199. American Mathematical Society, Providence (2014)
18. Hill, M.A., Hopkins, M.J., Ravenel, D.C.: On the nonexistence of elements of Kervaire invariant one. Ann. Math. (2) **184**(1), 1–262 (2016)
19. Hill, M.A.: Equivariant chromatic localizations and commutativity. J. Homotopy Relat. Struct. **14**, 647–662 (2019). https://doi.org/10.1007/s40062-018-0226-2
20. Hirschhorn, P.S.: Model Categories and their Localizations. Mathematical Surveys and Monographs, vol. 99. American Mathematical Society, Providence (2003)
21. Hovey, M.: Model Categories. Mathematical Surveys and Monographs, vol. 63. American Mathematical Society, Providence (1999)
22. Hovey, M.: Model category structures on chain complexes of sheaves. Trans. Am. Math. Soc. **353**(6), 2441–2457 (2001). (electronic)
23. Huettemann, T., Roendigs, O.: Twisted diagrams and homotopy sheaves. arXiv:0805.4076 (2008)
24. Lurie, J.: Elliptic cohomology I: Spectral abelian varieties. http://www.math.harvard.edu/~lurie/papers/Elliptic-I.pdf (2018)
25. McClure, J.E.: E_∞-ring structures for Tate spectra. Proc. Am. Math. Soc. **124**(6), 1917–1922 (1996)
26. Mandell, M.A., May, J.P.: Equivariant orthogonal spectra and S-modules. Mem. Am. Math. Soc. **159**(755), x+108 (2002)
27. Richter, B., Shipley, B.: An algebraic model for commutative $H\mathbb{Z}$-algebras. Algebr. Geom. Topol. **17**(4), 2013–2038 (2017)
28. Shipley, B.: A Convenient Model Category for Commutative Ring Spectra Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory, vol. 346, pp. 473–483. Ameican Mathematical Society, Providence (2004)
29. Shipley, B.: $H\mathbb{Z}$-algebra spectra are differential graded algebras. Am. J. Math. **129**(2), 351–379 (2007)
30. Silverman, J.H.: The Arithmetic of Elliptic Curves. Graduate Texts in Mathematics, vol. 106, 2nd edn. Springer, Dordrecht (2009)
31. Schwede, S., Shipley, B.: Equivalences of monoidal model categories. Algebr. Geom. Topol. **3**, 287–334 (2003). (electronic)