# GROUPS WITH FEW MAXIMAL SUM-FREE SETS

HONG LIU AND MARYAM SHARIFZADEH

Abstract. A set of integers is sum-free if it does not contain any solution for $x+y=z$. Answering a question of Cameron and Erdős, Balogh, Liu, Sharifzadeh and Treglown recently proved that the number of maximal sum-free sets in $\{1,\dots,n\}$ is $\Theta(2^{\mu(n)/2})$, where $\mu(n)$ is the size of a largest sum-free set in $\{1,\dots,n\}$. They conjectured that, in contrast to the integer setting, there are abelian groups $G$ having exponentially fewer maximal sum-free sets than $2^{\mu(G)/2}$, where $\mu(G)$ denotes the size of a largest sum-free set in $G$.

We settle this conjecture affirmatively. In particular, we show that there exists an absolute constant $c>0$ such that almost all even order abelian groups $G$ have at most $2^{(1/2-c)\mu(G)}$ maximal sum-free sets.

## 1. Introduction

A triple $\{x,y,z\}$ of natural numbers is a *Schur triple* if $x+y=z$[1]. A set $S\subseteq\mathbb{N}$ is *sum-free* if it does not contain any Schur triples, in other words, $(S+S)\cap S=\emptyset$. Sum-free set is a fundamental notion in combinatorial number theory. Its study dates back to the classical Schur's theorem [16] in Ramsey theory from 1916, which states that any finite colouring of $\mathbb{N}$ contains infinitely many monochromatic Schur triples.

If we look at sum-free sets in the first $n$ integers $[n]:=\{1,\dots,n\}$, it is easy to see that $\mu(n)$, the size of a largest sum-free subset of $[n]$, is $\lceil n/2\rceil$. Both the set of odd integers and $\{\lfloor n/2\rfloor+1,\dots n\}$ are examples of extremal sets. Denote by $f(n)$ the number of sum-free subsets of $[n]$. Since all subsets of a sum-free set are also sum-free, we have that $f(n)\geq 2^{\mu(n)}$. Cameron and Erdős [4] conjectured that this trivial lower bound in fact gives the correct order, that is, $f(n)=O(2^{\mu(n)})$. Their conjecture was only proven more than a decade later independently by Green [7] and Sapozhenko [14], both of whom proved the stronger statement that there are two constants $C_0$ and $C_1$ such that $f(n)=(C_i+o(1))2^{\mu(n)}$, for $n\equiv i \mod 2$.

Let us consider now a subcollection of "largest" sum-free sets. A sum-free set $S\subseteq[n]$ is *maximal* if it is not contained in any larger sum-free subset of $[n]$, and denote by $f_{\max}(n)$ the number of maximal sum-free subsets of $[n]$. Motivated by the fact that all the sum-free sets in the above trivial lower bound for $f(n)$ lie in two maximal ones, Cameron and Erdős [5] raised the question of enumerating maximal sum-free subsets of $[n]$. In particular, they asked whether $f_{\max}(n)$ is exponentially smaller than $f(n)$. In the same paper, they showed that $f_{\max}(n)\geq 2^{\mu(n)/2}$. Recently, Balogh, Treglown, and the authors [2] gave an exact answer to this question, showing that, there exist constants $C_i$, $i\in[4]$, such that $f_{\max}(n)=(C_i+o(1))2^{\mu(n)/2}$ for $n\equiv i \mod 4$.

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[1]Note that $x$ and $y$ are not necessarily distinct.

Another interesting direction to study is the analogous parameter for finite abelian groups. For an abelian group $G$, we can define $\mu(G)$, $f(G)$, and $f_{\max}(G)$ analogously to the integer setting. Interest in sum-free subsets of abelian groups goes back to the 1960s. Estimating $\mu(G)$ turns out to be a much more difficult task. It was not until 2005 that Green and Ruzsa [9] determined $\mu(G)$ for all finite abelian groups $G$. They also proved that, analogously to the integer setting, $f(G) = 2^{(1+o(1))\mu(G)}$. One can then ask the question similar to Cameron and Erdős': Is $f_{\max}(G)$ exponentially smaller than $f(G)$? Wolfovitz [19] showed that this is indeed the case for all even order abelian groups. This was extended to all abelian groups by Balogh, Treglown, and the authors [2], in particular,

$$f_{\max}(G) \leq 3^{(1/3+o(1))\mu(G)}. \tag{1.1}$$

Considering $f_{\max}(n) = \Theta(2^{\mu(n)/2})$, the following question was raised in [2], asking whether analogous bound holds for abelian groups.

**Question 1.1.** *Given an abelian group $G$, is it true that $f_{\max}(G) \leq 2^{(1/2+o(1))\mu(G)}$?*

This stronger bound holds ([2]) for the group $\mathbb{Z}_2^k := \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$: $f_{\max}(\mathbb{Z}_2^k) = 2^{(1/2+o(1))\mu(\mathbb{Z}_2^k)}$. It was also suspected in [2] that there is an infinite class of abelian groups for which the upper bound in Question 1.1 is far from tight.

**Conjecture 1.2.** *There exists a sequence of finite abelian groups $\{G_i\}_{i\in\mathbb{N}}$ of increasing order such that for all $i$, $f_{\max}(G_i)$ is exponentially smaller than $2^{\mu(G_i)/2}$.*

We confirm this conjecture, showing that, somewhat surprisingly for almost all even order groups $f_{\max}(G)$ is substantially smaller compared to the integer setting.

**Theorem 1.3.** *There exists a constant $c > 0$ and an integer $n_0$ such that for almost all even order groups $G$ with $|G| > n_0$,*

$$f_{\max}(G) \leq 2^{(1/2-c)\mu(G)}. \tag{1.2}$$

A more formal statement will be given in Section 3. We remark that the constant $c$ can be taken as for instance $10^{-4}$. Our result suggests that $\mathbb{Z}_2^k$ might be the only exception among all even order groups achieving the bound $2^{(1/2+o(1))\mu(G)}$. We will discuss more on this in the concluding remarks. Our proof can be extended to find more groups satisfying the bound in (1.2).

**Theorem 1.4.** *Let $G$ be $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ or $\mathbb{Z}_5 \oplus H$ with $|H|$ odd. Then $f_{\max}(G)$ is exponentially smaller than $2^{\mu(G)/2}$.*

On the other hand, we show that the bound in (1.1) cannot be improved, giving a negative answer to Question 1.1.

**Proposition 1.5.** *Let $G$ be an abelian group of order $n$, such that $9|n$. Also, there does not exist a prime $p$ such that $p \equiv 2 \pmod 3$ and $p|n$. Then*

$$f_{\max}(G) = 3^{(1/3+o(1))\mu(G)}.$$

The rest of the paper is organised as follows. In Section 2, we introduce all the tools and useful results. Then, we prove Theorem 1.3 in Section 3. The proof of Theorem 1.4 will be given in Section 4. The proof of Proposition 1.5 will be given in Section 5. Some concluding remarks and open problems will be given in Section 6.

## 2. Preliminaries

2.1. **Notation.** For a graph $\Gamma$, we write $V(\Gamma)$ and $E(\Gamma)$ for the set of vertices and edges of $\Gamma$, respectively. We allow at most one loop at each vertex. Denote $e(\Gamma) := |E(\Gamma)|$ and $v(\Gamma) := |V(\Gamma)|$. Given $x \in V(\Gamma)$, we write $N(x, \Gamma)$ for the neighbourhood of $x$ in $\Gamma$, i.e. the multi-set of vertices adjacent to $x$ in $\Gamma$. We also define $d(x, \Gamma) := |N(x, \Gamma)|$ to be the degree of $x$ in $\Gamma$. Note that a loop at $x$ contributes two to the degree of $x$. We write $\delta(\Gamma)$ for the minimum degree and $\Delta(\Gamma)$ for the maximum degree of $\Gamma$. Denote by $\Gamma[T]$ the induced subgraph of $\Gamma$ on the vertex set $T$, and $\Gamma \setminus T$ the induced subgraph of $\Gamma$ on the vertex set $V(\Gamma) \setminus T$. For $E_1 \subseteq E(\Gamma)$, define $\Gamma \setminus E_1 \subseteq \Gamma$ to be the subgraph on the same vertex set with $E(\Gamma \setminus E_1) = E(\Gamma) \setminus E_1$.

Throughout the paper, unless otherwise stated, all groups are finite and abelian and all logarithms are taken base 2. We omit floors and ceilings where the argument is unaffected.

2.2. **Number theoretic tools.**

**Definition 2.1.** *Let $G$ be an abelian group of order $n$.*

- *If $n$ is divisible by a prime $p \equiv 2$ (mod 3), then we say that $G$ is* type I($p$), *for smallest such $p$.*
- *If $n$ is not divisible by any prime $p \equiv 2$ (mod 3), but $3|n$, then we say that $G$ is* type II.
- *Otherwise, $G$ is* type III.

The following theorem was proved for type I and II groups by Diananda and Yap [6]. Later, it was proved for some special type III groups (see [13, 17, 18]), and for all type III groups in [9].

**Theorem 2.2.** *For all finite abelian groups $G$, if $G$ is type I(p) then $\mu(G) = \left(\frac{1}{3} + \frac{1}{3p}\right)|G|$. Otherwise, if $G$ is type II then $\mu(G) = \frac{|G|}{3}$. Finally, if $G$ is type III then $\mu(G) = \left(\frac{1}{3} - \frac{1}{3m}\right)|G|$, where $m$ is the exponent (largest order of any element) of $G$.*

We will use the following result by Green and Ruzsa [9] on the structure of large sum-free sets in type I group.

**Lemma 2.3.** *Suppose that $G$ is type $I(p)$ group of order $n$, and write $p = 3k+2$. Let $A \subseteq G$ be sum-free of size $|A| > \left(\frac{1}{3} + \frac{1}{3(p+1)}\right)n$. Then there exists a homomorphism $\phi : G \to \mathbb{Z}/p\mathbb{Z}$ such that $A$ is contained in $\phi^{-1}(k+1, ..., 2k+1)$.*

We will need the following simple fact about abelian groups.

**Fact 2.4.** *Let $G := \mathbb{Z}_{2^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{\alpha_r}} \oplus \mathbb{Z}_{p_1^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{\beta_t}}$ and $g \in G$. Then there are at most $2^r$ solutions in $G$ to the equation $2x = g$.*

We will also use the classical result of Hardy and Ramanujan on asymptotics of the partition function. Recall that a partition of a positive integer $n$ is a way of writing $n$ as a sum of positive integers.

**Theorem 2.5.** *The number of partitions of integer $n$ is asymptotically*

$$\frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

2.3. **Maximal independent sets in graphs.** In this subsection we collect together results on maximal independent sets in a graph. Let $\mathrm{mis}(\Gamma)$ denote the number of maximal independent sets in a graph $\Gamma$. Note that adding loops to a graph does not increase the number of maximal independent sets. Therefore, all upper bounds on the number of maximal independent sets holds even after adding loops to some of the vertices.

Moon and Moser [12] showed that for any graph $\Gamma$,

$$\mathrm{mis}(\Gamma) \leq 3^{|\Gamma|/3}, \tag{2.1}$$

and this bound is optimal for disjoint union of triangles. When a graph is almost regular and relatively dense, the bound above can be improved as follows (Equation (3) in [1]).

**Lemma 2.6.** *Let $k \geq 1$ and let $\Gamma$ be a graph on $n$ vertices. Suppose that $\Delta(\Gamma) \leq k\delta(\Gamma)$ and set $b := \sqrt{\delta(\Gamma)}$. Then*

$$\mathrm{mis}(\Gamma) \leq \sum_{0 \leq i \leq n/b} \binom{n}{i} 3^{\left(\frac{k}{k+1}\right)\frac{n}{3} + \frac{2n}{3b}}. \tag{2.2}$$

When a graph is triangle-free, Hujter and Tuza [10] obtained the following exponential improvement, which is optimal witnessed by a perfect matching. If $\Gamma$ is triangle-free, then

$$\mathrm{mis}(\Gamma) \leq 2^{|\Gamma|/2}. \tag{2.3}$$

We will also make use of the following version for 'almost triangle-free' graphs (Corollary 3.3 in [2]).

**Lemma 2.7.** *Let $n, D \in \mathbb{N}$ and $k \in \mathbb{R}$. Suppose that $\Gamma$ is a graph and $T$ is a subset of its vertex set such that $\Gamma' := \Gamma \setminus T$ is triangle-free. Suppose that $\Delta(\Gamma) \leq D$, $v(\Gamma') = n$ and $e(\Gamma') \geq n/2 + k$. Then*

$$\mathrm{mis}(\Gamma) \leq 2^{n/2 - k/(100D^2) + 2|T|}.$$

## 3. Proof of Theorem 1.3

To state Theorem 1.3 formally, we use the following proposition.

**Proposition 3.1.** *For any $\varepsilon > 0$, there exists $n_0 > 0$ such that the following holds for all $n > n_0$. Let $\mathbb{Z}_{2^{\alpha_1}} \oplus \ldots \oplus \mathbb{Z}_{2^{\alpha_r}} \oplus \mathbb{Z}_{p_1^{\beta_1}} \oplus \ldots \oplus \mathbb{Z}_{p_t^{\beta_t}}$ be the canonical decomposition of an abelian group of order $n$. Then all but $\varepsilon$-proportion of abelian groups of order $n$ satisfy $2^r \leq \varepsilon n$.*

*Proof.* Fix $\varepsilon > 0$, and let $n$ be sufficiently large. Let $\alpha, h \in \mathbb{N}$ be such that $n = 2^\alpha \cdot h$ and $2 \nmid h$. So $\alpha = \sum_{i \in [r]} \alpha_i \geq r$. We may assume that $2^\alpha = n/h > \varepsilon n$, as otherwise all order-$n$ groups have the desired property.

We first bound the number of groups with $2^r > \varepsilon n$. As $\prod_{i \in [t]} p_i^{\beta_i} = h < 1/\varepsilon$, there are only $O_\varepsilon(1)$ ways to choose the odd components $\mathbb{Z}_{p_i^{\beta_i}}$. Similarly, as

$$\prod_{i=1}^{r} 2^{\alpha_i - 1} = \frac{2^\alpha}{2^r} < \frac{1}{\varepsilon},$$

the number of possibilities for $\alpha_i \geq 2$, i.e. the non-$\mathbb{Z}_2$ even components, is at most $O_\varepsilon(1)$.

On the other hand, the number of non-isomorphic abelian groups of order $n$ is at least the number of partitions of $\alpha$, which, by Theorem 2.5, is at least $2^{\sqrt{\alpha}}$ (as $\alpha > \log(\varepsilon n)$ is sufficiently large). □

We can now restate Theorem 1.3 as follows.

**Theorem 3.2.** *There exists a constant $c > 10^{-4}$ such that the following holds. For all $0 < \varepsilon < 10^{-20}$, there is an integer $n_0$ such that for any abelian group $G = \mathbb{Z}_{2^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{\alpha_r}} \oplus \mathbb{Z}_{p_1^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{\beta_t}}$ with even order $n > n_0$ and $2^r \leq \varepsilon n$, we have*

$$f_{\max}(G) \leq 2^{(1/2-c)\mu(G)}.$$

3.1. **Connection between sum-free sets and independent sets.** In this subsection, we will reduce the problem of estimating $f_{\max}(G)$ to bounding $\mathrm{mis}(\Gamma)$, for some Cayley-like graph $\Gamma$. For subsets $B, S \subseteq V(G)$, let $L_S[B]$ be the *link graph of $S$ on $B$*, which is a simple graph with possible loops, defined as follows. The vertex set of $L_S[B]$ is $B$. The edge set of $L_S[B]$ consists of the following two types of edges:

(i) Two vertices $x$ and $y$ are adjacent if there exists an element $z \in S$ such that $\{x, y, z\}$ is a Schur triple;
(ii) There is a loop at a vertex $x$ if $\{x, x, z\}$ or $\{x, z, z'\}$ is a Schur triple for some $z, z' \in S$.

We will use the following container theorem of Green and Ruzsa (Proposition 2.1′ in [9]). See also [3] and [15] for more on container method.

**Lemma 3.3.** *Let $n$ be sufficiently large. For all finite abelian groups $G$ of order $n$, there is a family $\mathcal{F}$ of subsets of $G$ with the following properties.*

*(1) Every $F \in \mathcal{F}$ has at most $(\log n)^{-1/9} n^2$ Schur triples.*
*(2) If $S \subseteq G$ is sum-free, then $S$ is contained in some $F \in \mathcal{F}$.*
*(3) $|\mathcal{F}| \leq 2^{n(\log n)^{-1/18}}$.*

We refer to the sets in $\mathcal{F}$ as *containers*. For the rest of this section, fix an arbitrary $0 < \varepsilon < 10^{-20}$ and a group $G$ of order $n$ that satisfies the hypothesis of Theorem 3.2. Let $F \in \mathcal{F}$ be an arbitrary container. Recall that since $G$ is an even order group, $\mu(G) = n/2$. Thus, to prove Theorem 3.2, by Lemma 3.3 (2) and (3), for sufficiently large $n$, it suffices to show that $f_{\max}(F) \leq 2^{0.2499n}$, where $f_{\max}(F)$ denotes the number of maximal sum-free subsets of $G$ that lie in $F$.

By a group removal lemma of Green (Theorem 1.4 in [8], see also [11]), $F = B \cup C$, where $B$ is sum-free and $|C| = o(n)$. Notice that every maximal sum-free subset of $G$ in $F$ can be built in the following two steps:

(1) Choose a sum-free set $S$ in $C$;
(2) Extend $S$ in $B$ to a maximal one.

Since the set $C$ is small, the number of choices for the first step is negligible. We will use the following lemma from [1] to bound the number of choices in the second step.

**Lemma 3.4** ([1]). *Suppose that $B, S \subseteq G$ are both sum-free. If $I \subseteq B$ is such that $S \cup I$ is a maximal sum-free subset of $G$, then $I$ is a maximal independent set in $L_S[B]$.*

For a fixed $S$, by Lemma 3.4, the number of extensions of $S$ in $B$ in Step (2) is at most $\mathrm{mis}(L_S[B])$. Thus,

$$f_{\max}(F) \leq 2^{o(n)} \cdot \max_{\substack{S \subseteq C \\ S \text{ is sum-free}}} \mathrm{mis}(L_S[B]).$$

Therefore, it suffices to show that

$$\mathrm{mis}(L_S[B]) \leq 2^{0.2498n}. \tag{3.1}$$

3.2. **A new bound for maximal independent sets of dense graphs.** We will use the following lemma to bound the number of maximal independent sets in the link graph. This lemma can be viewed as a stability version of Moon and Moser's bound (2.1).

**Lemma 3.5.** *Let $k \in \mathbb{Z}$, $\Delta \in \mathbb{N}$, and $C = 3^{\Delta/13}$. Let $\Gamma$ be an $n$-vertex graph with $n+k$ edges and maximum degree $\Delta$, then*

$$\mathrm{mis}(\Gamma) \le C \cdot 3^{\frac{n}{3} - \frac{k}{13\Delta}}.$$

We will use the following fact. Write $N[v] := N(v) \cup \{v\}$. We partition maximal independent sets of $\Gamma$ into two parts depending on whether they contain $v$ or not. The former is exactly $\mathrm{mis}(\Gamma \setminus N[v])$. The latter is at most $\mathrm{mis}(\Gamma \setminus \{v\})$. Indeed, every maximal independent set $\Gamma$ that does not contain $v$ must be a maximal independent set of $\Gamma \setminus \{v\}$. Therefore, we have

$$\mathrm{mis}(\Gamma) \le \mathrm{mis}(\Gamma \setminus N[v]) + \mathrm{mis}(\Gamma \setminus \{v\}). \tag{3.2}$$

*Proof of Lemma 3.5.* We use induction on $n$. For the base case, suppose that $n \le 3$, then $k \le 0$. Thus, by (2.1), the lemma trivially holds. Now, let $\Gamma$ be an $n$-vertex graph that satisfies the hypothesis of the lemma. Therefore, $e(\Gamma) - n \ge 1$, and $\Delta \ge 3$. Also, if $k < \Delta^2$, by (2.1), we have

$$\mathrm{mis}(\Gamma) \le 3^{\frac{n}{3}} = 3^{\frac{\Delta}{13}} \cdot 3^{\frac{n}{3} - \frac{\Delta}{13}} \le C \cdot 3^{\frac{n}{3} - \frac{k}{13\Delta}}.$$

We may assume $k \ge \Delta^2$. Fix a vertex $v$ of degree $d$ with $3 \le d \le \Delta$. Let $\Gamma' := \Gamma \setminus \{v\}$ with $\Delta' := \Delta(\Gamma')$, $n' := v(\Gamma') = n-1$ and

$$e(\Gamma') = (n-1) + (k-d+1) =: n' + k';$$

and $\Gamma'' := \Gamma \setminus N[v]$ with $\Delta'' := \Delta(\Gamma'')$, $n'' := n-d-1$ and

$$e(\Gamma'') \ge n + k - d\Delta = (n-d-1) + (k - d\Delta + d + 1) =: n'' + k''.$$

As $k \ge \Delta^2$, we have $k', k'' > 0$. By induction hypothesis on $\Gamma'$ and $\Gamma''$, and that $\Delta', \Delta'' \le \Delta$, we get

$$\mathrm{mis}(\Gamma') \le C \cdot 3^{\frac{n-1}{3} - \frac{k-d+1}{13\Delta'}} \le C \cdot 3^{\frac{n-1}{3} - \frac{k-d+1}{13\Delta}} = C \cdot 3^{\frac{n}{3} - \frac{k}{13\Delta}} \cdot 3^{-\frac{1}{3} + \frac{d-1}{13\Delta}},$$

and

$$\begin{aligned}\mathrm{mis}(\Gamma'') &\le C \cdot 3^{\frac{n-d-1}{3} - \frac{k-d\Delta+d+1}{13\Delta''}} \le C \cdot 3^{\frac{n}{3} - \frac{d+1}{3} - \frac{k-d\Delta+d+1}{13\Delta}} \\ &= C \cdot 3^{\frac{n}{3} - \frac{k}{13\Delta}} \cdot 3^{-\frac{d+1}{3} + \frac{d}{13} - \frac{d+1}{13\Delta}}.\end{aligned}$$

This finishes the proof as $\mathrm{mis}(\Gamma) \le \mathrm{mis}(\Gamma') + \mathrm{mis}(\Gamma'')$ due to (3.2), and

$$3^{-\frac{d+1}{3} + \frac{d}{13} - \frac{d+1}{13\Delta}} + 3^{-\frac{1}{3} + \frac{d-1}{13\Delta}} \le 0.9997,$$

subject to $3 \le d \le \Delta$. □

3.3. **Proof of** (3.1)**.** We will use the following definitions and notations throughout the rest of this section. For disjoint sum-free subsets $A, A' \subseteq F$, we call an edge $xy \in E(L_A[A'])$ a *type 1 edge* if $x - y = a$, for some $a \in A \cup (-A)$. Also, if an edge can be generated in multiple ways, we do not allow parallel edges. In particular, if $x - y = a_1$ and $y - x = a_2$, for $a_1, a_2 \in A$, then we only have one edge between $x$ and $y$. For any $x$ and $y$ that are not connected by a type 1 edge, if $x + y = a$, for some $a \in A$, we call the edge $xy$ *type 2*. Notice that we do not allow an edge to be both type 1 and type 2. Denote by $E_1(L_A[A'])$ and $E_2(L_A[A'])$ the set of type 1 and 2 edges, respectively. For $i \in [2]$ and $x \in A'$, let $e_i(L_A[A']) := |E_i(L_A[A'])|$

and $d_i(x, L_A[A'])$ be the number of type $i$ edges incident to $x$ in $L_A[A']$. We will omit $L_A[A']$ from the above notations whenever clear from the context.

Let $\Gamma := L_S[B]$. Notice crucially that since $S$ is sum-free, we have

$$0_G \notin S.$$

Note that there are loop edges at vertices $x \in S \pm S$ that are neither type 1 nor type 2. However, as removing loops will not decrease the number of maximal independent sets, for the rest of this section, we consider $\Gamma$ without loops, unless mentioned otherwise. In particular, we only allow loops in the second part of the case $S = \{s\}$. In all other cases, $\Gamma$ contains only non-loop edges of type 1 and 2. Recall that $\mu(G) = n/2$ and $B$ is sum-free, therefore, $v(\Gamma) \leq n/2$. Denote by $\Gamma_1$ and $\Gamma_2$ the subgraph of $\Gamma$ consisting of type 1 and 2 edges, respectively.

We claim that we may assume $B$ is the complement of an index 2 subgroup of $G$. Indeed, if $|B| \leq 4n/9$, then by (2.1), we have

$$\text{mis}(\Gamma) \leq 3^{|B|/3} = 3^{4n/27} < 2^{0.24n}, \tag{3.3}$$

as desired. Therefore, we can assume that $|B| > 4n/9$. Thus, by Lemma 2.3 (with $k = 0$ and $p = 2$), we have that $B$ is contained in the complement of an index 2 subgroup, say $H$, of $G$. As $G \setminus H$ is sum-free, we may assume then the partition of the container $F = B \cup C$ from the removal lemma is such that $C \subseteq H$. In other words, $B$ only consists of elements with odd values in exactly one of the first $r$ coordinates, and $S \subseteq G$ is a subset of the set of elements with even values in the same coordinate. Note that adding new vertices to a graph will not decrease the number of maximal independent sets, and thus without loss of generality we can assume that

$$B = \{1, 3, 5, 7, \dots, 2^{\alpha_1} - 1\} \oplus \mathbb{Z}_{2^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{2^{\alpha_r}} \oplus \mathbb{Z}_{p_1^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{\beta_t}};$$

and the generating set is

$$S \subseteq \{0, 2, 4, 6 \dots, 2^{\alpha_1} - 2\} \oplus \mathbb{Z}_{2^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{2^{\alpha_r}} \oplus \mathbb{Z}_{p_1^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{\beta_t}} \setminus \{0_G\}.$$

We first show that (3.1) holds for large $S$. We will use the following claim, which bounds the degree of every vertex, as well as, their type 1 and type 2 degree.

**Claim 3.6.** *For all $x \in B$,*

*(i)* $d_1(x, \Gamma) = |S \cup (-S)|$,

*(ii)* $d_2(x, \Gamma) \leq |S|$ *with equality if and only if* $2x \notin S + (S \cup (-S))$ *and* $2x \notin S$*; and when* $2x \in S$*, then* $d_2(x, \Gamma) = |S| - 1$.

*Consequently,*

*(iii)* $|S \cup (-S)| \leq \delta(\Gamma) \leq \Delta(\Gamma) \leq |S \cup (-S)| + |S| \leq 2\delta(\Gamma)$.

*Proof.* Fix an arbitrary element $x \in B$. For the first part, since we do not allow parallel edges, each $s \in S \cup (-S)$ generates a unique type 1 neighbour $y = x + s \in B$.

For the second part, it is clear that $x$ is incident to exactly $|S|$ many edges of the form $(x, s - x)$, except when $2x \in S$, in which case there are exactly $|S| - 1$ many edges of this form as we removed loops from $\Gamma$. Fix an $s \in S$. It suffices to show that the edge $(x, s-x) \in E_2(L_S[B])$ if and only if $2x \notin s+(S\cup(-S))$. By definition, $(x, s-x) \in E_2(L_S[B])$, if and only if $(x, s-x) \notin E_1(L_S[B])$. This is equivalent to $s-x \neq x-s'$, for all $s' \in S\cup(-S)$. Which implies $2x \notin s + (S \cup (-S))$. □

**Claim 3.7.** *If* $|S| \geq 10^4$*, then* $\text{mis}(\Gamma) \leq 2^{0.24n}$.

*Proof.* By Claim 3.6(iii), $10^4 \leq |S| \leq \delta(\Gamma) \leq \Delta(\Gamma) \leq 2\delta(\Gamma)$. Since the right hand side of (2.2) is decreasing in $b$, we can apply Lemma 2.6 with $k = 2$ and $b = 100$. Thus, we get

$$\text{mis}(\Gamma) \leq \sum_{0 \leq i \leq n/200} \binom{n/2}{i} \cdot 3^{\frac{n}{9}+\frac{n}{300}} \leq 2 \cdot (100e)^{\frac{n}{200}} \cdot 3^{\frac{n}{9}+\frac{n}{300}} \leq 2^{0.24n}.$$

□

Thus, we may assume that $|S| < 10^4$. The next claim will bound the density of $\Gamma$ with size of $S$.

**Claim 3.8.** *We have*

$$\begin{aligned} e(\Gamma) &\geq \frac{(|S \cup (-S)| + |S|) \cdot |B|}{2} - (|S| \cdot |S \cup (-S)| + |S|) \cdot 2^r \\ &\geq \frac{\Delta(\Gamma) \cdot |B|}{2} - (|S| \cdot |S \cup (-S)| + |S|) \cdot 2^r. \end{aligned}$$

*Proof.* By Claim 3.6(i), we only need to prove

$$e_2(\Gamma) \geq \frac{|S| \cdot |B|}{2} - (|S| \cdot |S \cup (-S)| + |S|) \cdot 2^r.$$

Define

$$A := \{(x, s - x) : x \in B \text{ and } s \in S\} \quad \text{and} \quad X := \{x \in B : 2x \in S\}.$$

By definition $E_2 \subseteq A \subseteq E$, and $e_2(\Gamma) = |A| - |A \cap E_1| - |X|$. Then, it suffices to prove that $|A \cap E_1| \leq |S| \cdot |S \cup (-S)| \cdot 2^r$ and $|X| \leq |S| \cdot 2^r$. The latter follows immediately from Fact 2.4. For the former, recall that an edge $(x, s - x) \in A$ is a type 1 edge when $x - (s - x) = s'$, for some $s' \in S \cup (-S)$. Therefore, $|A \cap E_1|$ is at most the number of triples $(x, s, s')$ with $x \in B$, $s \in S$, $s' \in S \cup (-S)$, and $2x = s + s'$. By Fact 2.4, the number of such triples is at most $|S| \cdot |S \cup (-S)| \cdot 2^r$, yielding the desired bound. □

**Case 0:** $S = \emptyset$**.** The graph $\Gamma$ contains no edges. Therefore, $\text{mis}(\Gamma) = 1$.

**Case 1:** $S = \{s\}$**.** Let $\ell$ be the order of $s$. First, assume $\ell \neq 3$. Next claim shows that, in this case, $\Gamma$ can be made triangle-free by removing $\varepsilon n$ vertices.

**Claim 3.9.** *If* $\ell \neq 3$*, then there exists a subset* $B_t \subseteq B$ *with* $|B_t| \leq 2^r \leq \varepsilon n$ *that intersects all triangles in* $\Gamma$*, i.e.* $V(T) \cap B_t \neq \emptyset$*, for all triangles* $T \subseteq \Gamma$*.*

*Proof.* Let $T$ be a triangle in $\Gamma$ with $V(T) = \{x, y, z\}$. First, we will show that $E(T)$ contains exactly one type 2 edge. Indeed, if $|E(T) \cap E_2| \geq 2$, say $xy, xz \in E_2$, then $y = s - x = z$, a contradiction. Otherwise, if $|E(T) \cap E_2| = 0$, then it is not hard to check that either $x - y = s$, $y - z = s$, and $z - x = s$, in which case, we have $3s = 0$, or $x - y = s$ and $x - z = s$, in which case $y = z$, leading to a contradiction in either case.

Therefore, without loss of generality, we can assume that $xy, xz \in E_1$ and $yz \in E_2$. We also assume that $x - y = s$ (the case $y - x = s$ is almost identical). Then, we must have $z - x = s$, which, together with $y + z = s$, implies $2z = 3s$. By Fact 2.4, since $s$ is fixed, there are at most $2^r$ choices for $z$. Then $B_t := \{z : 2z = 3s\}$ is the desired set. □

Let $B_t$ be the set guaranteed by Claim 3.9. Let $\Gamma' = \Gamma \setminus B_t$. By Claims 3.6, for all $x \in B$, we have $2 \leq d(x) \leq 3$. Therefore, by Claim 3.8, $e(\Gamma) \geq |B| - 3 \cdot 2^r$, and thus,

$$e(\Gamma') \geq e(\Gamma) - 3|B_t| \geq |B| - 6 \cdot 2^r.$$

Thus by Lemma 2.7, we get

$$\text{mis}(\Gamma) \leq 2^{\frac{\mu(G)}{2} - \frac{\mu(G)}{1800} + 3\varepsilon n} \leq 2^{0.4996\mu(G)},$$

as desired. Therefore, we can assume that $\ell = 3$. In this case, $\Gamma_1$ is a disjoint union of $|B|/3$ triangles, where the vertex set of each triangle is $\{x, x+s, x+2s\}$, for some $x \in B$. For type 2 edges, we call a vertex $x \in B$ irregular if $2x \in \{0, s, 2s\}$, and regular, otherwise. By Claim 3.6, if $x \in B$ is regular then $d_2(x) = 1$, otherwise $d_2(x) = 0$. Note that, for all $x \in B$, the vertex $x + s \in B$. Therefore, for every vertex $x \in B$ with $2x \in \{0, s, 2s\}$, there are two unique vertices $x', x'' \in B$ with $\{2x, 2x', 2x''\} = \{0, s, 2s\}$. Indeed, if $2x = 0$ then $x'' = x + s$ and $x' = x'' + s$, and the other two cases are similar. Thus, if there exists a triangle in $\Gamma_1$ with one irregular vertex, then the other two vertices of the triangle are irregular as well. Therefore, by Fact 2.4, there are at most $2^r$ irregular triangles, i.e. triangles with all irregular vertices. Denote by $\mathcal{T}'$ and $\mathcal{T}$ the set of irregular and all triangles in $\Gamma$, respectively. Then it is not hard to see that there exists a partition of all triangles in $\mathcal{T} \setminus \mathcal{T}'$ into pairs, $\{x, x+s, x+2s\}$ and $\{s-x, -x, -x-s\}$, for $x \in B$, such that $\Gamma_2$ induces a perfect matching between each pair. To count the number of maximal independent sets, note that, as we showed above, every irregular triangle contains a vertex $x'$ with $2x' = s$. Therefore, $x'$ cannot be in any maximal independent edge (this is the only exception where we consider loops). Thus, the number of maximal independent sets for an irregular triangle is two. Also, the number of maximal independent sets for two disjoint triangles joined by a perfect matching is 6, we obtain that

$$\text{mis}(\Gamma) \leq 6^{\frac{|\mathcal{T} \setminus \mathcal{T}'|}{2}} \cdot 2^{|\mathcal{T}'|} \leq 6^{\frac{n/6 - |\mathcal{T}'|}{2}} \cdot 2^{|\mathcal{T}'|} \leq 6^{\frac{n}{12}} \leq 2^{0.45\mu(G)},$$

which finishes the proof of (3.1) for the case when $S$ is a singleton.

**Case 2:** $2 \leq |S| \leq 10000$ **and** $2 \leq |S \cup (-S)| \leq 20000$**.** Then, Claim 3.8 implies

$$e(\Gamma) \geq \frac{(|S \cup (-S)| + |S|) \cdot n}{4} - 10^9 \varepsilon n$$

Also, by Claim 3.6 we have $\Delta(\Gamma) \leq |S \cup (-S)| + |S| \leq 10^5$. Therefore, we can apply Lemma 3.5 , which shows that

$$\begin{aligned}
\text{mis}(\Gamma) &\leq 3^{\Delta/13} \cdot 3^{\frac{n}{6} - \frac{e(\Gamma) - n/2}{13\Delta}} \leq 3^{\Delta/13} \cdot 3^{\frac{n}{6} - \frac{(|S \cup (-S)| + |S|) \cdot n/4 - n/2}{13(|S \cup (-S)| + |S|)} + 10^9 \varepsilon n} \\
&= 3^{\Delta/13} \cdot 3^{\left(\frac{1}{6} - \frac{1}{52} + \frac{1}{26(|S \cup (-S)| + |S|)} + 10^9 \varepsilon\right) n} \\
&\leq 3^{\Delta/13} \cdot 3^{\left(\frac{1}{6} - \frac{1}{52} + \frac{1}{104} + 10^9 \varepsilon\right) n} \leq 2^{0.499\mu(G)}.
\end{aligned}$$

## 4. Other type I groups

We can extend the proof for Theorem 1.3 to some other type I groups. We streamline the proof of Theorem 1.4 in this section.

4.1. **Group $G = \mathbb{Z}_2^t \oplus \mathbb{Z}_4$.** We may again assume, using Lemma 2.3 (see the paragraph below (3.3)), that $B$ is the complement of an index 2 subgroup of $G$. We split the proof into the following two cases.

**Case 1:** $B = \mathbb{Z}_2^t \oplus \{1,3\}$ and $S \subseteq \mathbb{Z}_2^t \oplus \{0,2\}$. We may assume that $\Gamma$ does not contain any loops. Indeed, as $0 \notin S$, a loop at a vertex $x \in B$ implies that $2x = s$ for some $s \in S$. But then every vertex in $B$ has a loop, as $2B = \{(0,\dots,0,2)\}$. For all $s \in S$ and $A \subseteq S$, define $\overline{s} := (0,\dots,0,2) + s$, $\overline{A} := \cup_{s\in A}\overline{s}$, and $A^* = A \cup \overline{A}$.

**Lemma 4.1.** *The graph $\Gamma$ is $|S^*|$-regular.*

*Proof.* Note first that all elements in $S^*$ have order at most 2, so $S^* \cup (-(S^*)) = S^*$. Then Claim 3.6 implies that in $L_{S^*}[B]$ all vertices are adjacent to exactly $|S^*|$ many type 1 edges. It suffices to show that $xy \in E(\Gamma)$ if and only if $xy \in E_1(L_{S^*}[B])$. ($\Rightarrow$) If $xy \in E_1(\Gamma)$, it is trivial as $S \subseteq S^*$. Otherwise, if $xy \in E_2(\Gamma)$, then there exists an $s \in S$ such that $s - x = y$. This, together with $2x = s - \overline{s}$, implies that $x + \overline{s} = s - x = y$, i.e. $xy \in E_1(L_{S^*}[B])$. ($\Leftarrow$) Let $xy \in E_1(L_{S^*}[B])$. We may assume that $x - y = \overline{s}$ for some $\overline{s} \in \overline{S}$. Then $x + y = x - y + 2y = \overline{s} + (0,\dots,0,2) = s$, that is, $xy \in E_2(\Gamma)$ as claimed. $\square$

If $|S^*| \geq 4$, then we can apply Lemma 3.5 to get the desired bound. Suppose that $|S^*| \leq 3$. Then it must be that $S^* = \{s, \overline{s}\}$. We claim that $\Gamma$ is triangle-free, which together with $e(\Gamma) = \mu(G)$ and Lemma 2.7 yields the desired bound. Indeed, a triangle $T$ in $\Gamma$ must have $V(T) = \{x, x+s, x+\overline{s}\}$, for some $x \in B$. Then we have $(x+s) + \overline{s} = x + \overline{s}$, or $s = 0$, a contradiction as otherwise every vertex has a loop.

**Case 2:** $B = \{1\} \oplus \mathbb{Z}_2^{t-1} \oplus \mathbb{Z}_4$ and $S \subseteq \{0\} \oplus \mathbb{Z}_2^{t-1} \oplus \mathbb{Z}_4$. Define $B_0 = \{1\} \oplus \mathbb{Z}_2^{t-1} \oplus \{0,2\}$ and $B_1 = \{1\} \oplus \mathbb{Z}_2^{t-1} \oplus \{1,3\}$. We partition $S = S_0 \cup S_1$ such that $S_0 \subseteq \{0\} \oplus \mathbb{Z}_2^{t-1} \oplus \{0,2\}$ and $S_1 \subseteq \{0\} \oplus \mathbb{Z}_2^{t-1} \oplus \{1,3\}$.

We may assume that $\Gamma$ does not contain any loop. Similar to Case 1, since $2B_i = \{(0,\dots,0,2i)\}$, if there is one loop on a vertex $x \in B_i$, then every vertex in $B_i$ would have a loop, and by (2.1), we have $\mathrm{mis}(\Gamma) \leq 3^{\frac{\mu(G)/2}{3}} \leq 2^{0.27\mu(G)}$.

Note that all edges in $E(\Gamma[B_0]) \cup E(\Gamma[B_1])$ and $E(\Gamma[B_0,B_1])$ are generated by $S_0$ and $S_1$ respectively.

**Lemma 4.2.** *For all $i \in \{0,1\}$ and $x_i \in B_i$*

- $N_{\Gamma[B_0]}(x_0) = x_0 + S_0$,
- $N_{\Gamma[B_1]}(x_1) = x_1 + S_0^*$,
- $N_{\Gamma[B_0,B_1]}(x_i) = x_i + S_1^*$.

*In particular, $\Gamma[B_0]$, $\Gamma[B_1]$, and $\Gamma[B_0,B_1]$ are $|S_0|$, $|S_0^*|$, and $|S_1^*|$-regular, respectively. Furthermore, $\Gamma[B_0]$ is triangle-free.*

*Proof.* Recall that edges in $\Gamma[B_0]$ are generated by $S_0$. As all elements in $B_0 \cup S_0$ are of order 2, for any $x \in B_0$ and $s_0 \in S_0$, all three edges incident to $x$ generated by $s_0$, $\{x, x+s_0\}$, $\{x, x-s_0\}$ and $\{x, s_0 - x\}$, coincide, showing that $\Gamma[B_0]$ is $|S_0|$-regular. To see that $\Gamma[B_0]$ is triangle-free, assume to the contrary that there exists a triangle $T \subseteq \Gamma[B_0]$ with $V(T) = \{x_0, x_0+s_0, x_0+s_0'\}$, for some $s_0, s_0' \in S_0$. Then $x_0 + s_0 + s_0'' = x_0 + s_0'$ for some $s_0'' \in S_0$. This implies that $s_0 + s_0'' = s_0'$, contradicting to $S_0$ being sum-free. The proof for $\Gamma[B_1]$ being $|S_0^*|$-regular is almost identical to that of Lemma 4.1.

For the bipartite graph $\Gamma[B_0,B_1]$, all edges are generated by $S_1$. Note that there is no type 2 edges, since elements in $B_0$ have order 2 and so $\{x, s_1 - x\}$ coincides with $\{x, s_1 + x\}$ for any $x \in B_0$ and $s_1 \in S_1$. Thus, all edges are of the form $x \pm s_1$, showing that $\Gamma[B_0, B_1]$ is $|S_1^*|$-regular as $S_1^* = S_1 \cup (-S_1)$ due to $\overline{s_1} = -s_1$. $\square$

An immediate consequence is that the link graph is relatively regular: $|S| \leq \delta(\Gamma) \leq \Delta(\Gamma) \leq 2\delta(\Gamma)$. We may then assume that $|S| \leq 20000$, as otherwise it can be handled as in Claim 3.7.

Suppose that $S_1 = \emptyset$. Then $\Gamma$ is a disjoint union of $\Gamma[B_i]$, $i \in \{0, 1\}$. By (2.3) and Lemma 4.2, $\text{mis}(\Gamma[B_0]) \leq 2^{\mu(G)/4}$. It suffices to show that $\text{mis}(\Gamma[B_1])$ is exponentially smaller than $2^{\mu(G)/4}$. Recall that $\Gamma[B_1]$ is $|S_0^*|$-regular, then similar analysis as in Case 1 implies the desired bound.

We may now assume that $|S_1| \geq 1$. Furthermore, $|S_0| \geq 1$, as otherwise $\Gamma = \Gamma[B_0, B_1]$ is a $D$-regular bipartite graph with $D \geq 2$ and Lemma 2.7 implies the desired bound.

Define $d_0 := |S_0| + |S_1^*|$ and $d_1 := |S_0^*| + |S_1^*|$. By Lemma 4.2, all vertices in $B_0$ and $B_1$ have degree $d_0$ and $d_1$, respectively. Note that $d_0 \leq d_1 \leq 2d_0$. Thus, $e(\Gamma) = \frac{\mu(G)}{2}\left(\frac{d_0}{2} + \frac{d_1}{2}\right)$. Hence, Lemma 3.5, together with $\Delta(\Gamma) = d_1$, implies

$$\text{mis}(\Gamma) \leq 3^{\frac{d_1}{13}} \cdot 3^{\frac{\mu(G)}{3} - \frac{\mu(G)}{4} \cdot \frac{d_0 + d_1 - 4}{13 d_1}},$$

which, by a short calculation, is exponentially smaller than $2^{\mu(G)/2}$ when $d_0 \geq 4$. We can then assume

$$d_0 = |S_0| + |S_1 \cup (-S_1)| \leq 3.$$

As $S_0$ and $S_1$ are non-empty and elements in $S_1$ have order 4, we must have $S_0 = \{s_0\}$ and $S_1 = \{s_1\}$, in which case $\Gamma[B_0], \Gamma[B_1], \Gamma[B_0, B_1]$ are 1-, 2- and 2-regular respectively. We claim that $\Gamma$ is triangle-free. Then Lemma 2.7, together with $e(\Gamma) = 7\mu(G)/4$, implies the desired bound.

Suppose to the contrary that there exists a triangle $T$. As $\Gamma[B_0]$ is triangle-free, $V(T) \cap B_1 \neq \emptyset$. If $V(T) \subseteq B_1$, then $V(T) = \{x, x + s_0, x + \overline{s_0}\}$ and $N_{\Gamma[B_1]}(x + s_0) = \{x, x + s_0 + \overline{s_0}\}$, implying that $s_0 = 0$, a contradiction. If $V(T)$ intersects $B_1$ at two vertices, then we must have $V(T) = \{x_0, x_0 + s_1, x_0 - s_1\}$ for some $x_0 \in B_0$. This, however, implies that either $(x_0 + s_1) + s_0 = x_0 - s_1$ or $(x_0 + s_1) + \overline{s_0} = x_0 - s_1$. The former case implies that $2s_1 = s_0$; while the latter case yields $s_0 = 0$, leading to contradictions in both cases. The case when $V(T)$ intersects $B_0$ at two vertices can be handled similarly.

4.2. $G = \mathbb{Z}_5 \oplus H$ **and** $2 \nmid |H|$. In this section, we prove that $\text{mis}(\Gamma)$ is exponentially smaller than $2^{\mu(G)/2} = 2^{n/5}$. In particular, we will show that there exists a positive constant $c$,

$$\text{mis}(\Gamma) \leq 2^{(1/2 - c)\mu(G)}. \tag{4.1}$$

If $B$ is smaller than $0.37n$, then (2.1) suffices. Note that for type I(5) groups, the stability Lemma 2.3 applies only to sets of size at least $7n/18 \approx 0.389n$, nonetheless with the same proof in [9], the stability can be slightly strengthened to cover sets of size at least $11n/30 \approx 0.367n$.

**Lemma 4.3.** *Suppose that $G$ is type $I(5)$ group of order $n$ Let $A \subseteq G$ be sum-free of size $|A| > 11n/30$. Then there exists a homomorphism $\phi : G \to \mathbb{Z}/p\mathbb{Z}$ such that $A$ is contained in $\phi^{-1}(2, 3)$.*

We may then assume that $B = \{2, 3\} \oplus H$ and $S \subseteq \{0, 1, 4\} \oplus H$. For all subsets $G' \subseteq G$, denote $G'_i$ to be the set $G' \cap \{i\} \oplus H$, for all $i \in \mathbb{Z}_5$.

Similar to Lemma 4.2, the following claim on neighbourhoods of vertices in $\Gamma$ can be derived. We omit its proof.

**Claim 4.4.** *For all $i \in \{2,3\}$ and $x_i \in B_i$,*

- $d_1(x_i, \Gamma[B_i]) = |S_0 \cup (-S_0)|$;
- $d_2(x_i, \Gamma[B_i]) = |S_{2i}| - |\{s \in S_{2i} : 2x_i \in s + (S_0 \cup (-S_0))\}|$;
- $d_1(x_i, \Gamma[B_2, B_3]) = |S_4 \cup (-S_1)|$;
- $d_2(x_i, \Gamma[B_2, B_3]) = |S_0| - |\{s \in S_0 : 2x_i \in s + (S_{2i} \cup (-S_{-2i}))\}|$.

An immediate consequence is that the link graph is relatively regular: $|S|/2 \le \delta(\Gamma) \le \Delta(\Gamma) \le 2\delta(\Gamma)$. We may again assume that $|S| = O(1)$, as otherwise it can be handled as in Claim 3.7. As now $\Delta(\Gamma) = O(1)$, we can make use of the following corollary of Lemma 3.5.

**Claim 4.5.** *If $e(\Gamma) \ge (1+\alpha)|B| - O_S(1)$, and $\alpha/\Delta(\Gamma) \ge 1/4$, then $\Gamma$ satisfies (4.1).*

For the rest of the proof, without loss of generality, assume that $|S_1| \ge |S_4|$. Next, we will calculate the ratio $\alpha/\Delta(\Gamma)$ depending on size of $S$. By Claim 4.4,

$$e(\Gamma) = \frac{|B|}{4} \cdot (2|S_0 \cup (-S_0)| + |S_4| + |S_1| + 2|S_4 \cup (-S_1)| + 2|S_0|) - O_S(1),$$

and

$$\Delta(\Gamma) = |S_0 \cup (-S_0)| + |S_1| + |S_0| + |S_4 \cup (-S_1)|.$$

Therefore,

$$\frac{\alpha}{\Delta(\Gamma)} = \frac{1}{4} \cdot \frac{2|S_0 \cup (-S_0)| + |S_4| + |S_1| + 2|S_4 \cup (-S_1)| + 2|S_0| - 4}{|S_0 \cup (-S_0)| + |S_1| + |S_0| + |S_4 \cup (-S_1)|}.$$

By Claim 4.5, we may assume $\alpha/\Delta(\Gamma) < 1/4$, implying that

$$|S_0 \cup (-S_0)| + |S_0| + |S_4 \cup (-S_1)| + |S_4| \le 3. \tag{4.2}$$

In particular, we must have $|S_0| \le 1$.

Suppose that $|S_0| = 1$. As $H$ has no order-2 element, $|S_0 \cup (-S_0)| = 2$ and hence $S_1, S_4 = \emptyset$. By Claim 4.4, $\Gamma[B_2, B_3]$ is a matching and apart from $O_S(1)$ vertices, $\Gamma[B_i]$, $i \in \{2,3\}$, is a disjoint union of $\ell$-cycle, where $\ell$ is the order of $s \in S_0$. If $\ell \ne 3$, then $\Gamma$ is triangle-free. Note that $\Delta(\Gamma) = 3$ and $e(\Gamma) = 3\mu(\Gamma)/2 - O_S(1)$, then Lemma 2.7 finishes the proof of this case. If $\ell = 3$, then apart from constantly many vertices, $\Gamma$ is a disjoint union of the six-vertex graph obtained by adding a perfect matching between two triangles. Thus $\mathrm{mis}(\Gamma) \le 6^{(1/6+o(1))\mu(G)} \le 2^{0.45\mu(G)}$.

We may then assume that $S_0 = \emptyset$. Note that $S_1 \ne \emptyset$, as otherwise $S = \emptyset$. Thus $e(\Gamma) \ge 3|S_1||B|/4 \ge 3|B|/4$. We shall see that in this case $\Gamma$ can be made triangle-free by removing constantly many vertices. Then as $e(\Gamma) \ge 3|B|/4$, Lemma 2.7 finishes the proof. Recall that $H$ contains no element of order 2, thus, it suffices to show every triangle contains a vertex $y$ with $2y \in S + S - S$ as $|S| = O(1)$. Let $T$ be a triangle induced by $\{x, y, z\}$. Assume that $x \in B_2$ and $y, z \in B_3$, other cases are similar. Recall that edges in $[B_2, B_3]$ and $B_3$ are type 1 and type 2 respectively. Then $x + s' = y$, $x + s'' = z$ and $y + z = s$ for some $s, s', s'' \in S_1$, implying that $2y = s + s' - s''$ as desired.

## 5. Type II groups

*Proof of Proposition 1.5.* Upper bound follows from (1.1). For the lower bound, as $9 \,|\, |G|$, either $G = \mathbb{Z}_{3^a} \oplus G'$ with $a \ge 2$, or $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus G'$. In the former case, let $H < \mathbb{Z}_{3^a}$ be a subgroup of index three. Then, $B := (1 + H) \oplus G'$ is a sum-free subset of size $\mu(G)$. Since $a \ge 2$, we have that $3 \,|\, |H|$. Let $x \in H$ be of order three in $H$, and define $s := (x, 0_{G'})$. Note that $s$ has order three in $G$. Note that the graph $L_{\{s\}}[B]$ does not have any type 2 edges. Indeed, for all $(1+y,z)\in B$, with $y\in H$ and $z\in G'$, the element $s-(1+y,z)\in(2+H)\oplus G'$. Therefore, every vertex in $L_{\{s\}}[B]$ has degree exactly two. Since $s$ has order three, it is easy to check that $L_{\{s\}}[B]$ is a disjoint union of triangles $T$ with

$$V(T)=\{(1+y,z),(1+y+x,z),(1+y+2x,z)\},\ \text{for } y\in H \text{ and } z\in G'.$$

Suppose now $G=\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus G'$. Then, the set $B:=\{1\}\oplus\mathbb{Z}_3\oplus G'$ is a sum-free subset of size $\mu(G)$. Let $s:=(0,1,0_{G'})$, and similar to the previous case, there are no type 2 edges in $L_{\{s\}}[B]$, and also $L_{\{s\}}[B]$ is a disjoint union of triangles $T$ with

$$V(T)=\{(1,y,z),(1,y+1,z),(1,y+2,z)\},\ \text{for some } y\in\mathbb{Z}_3 \text{ and } z\in G'.$$

Thus, in either case, the link graph is a disjoint union of triangles. Note that every maximal independent set $I$ in $L_{\{s\}}[B]$ corresponds naturally to a maximal sum-free set containing $I\cup\{s\}$ in $G$, and thus, $f_{\max}(G)\geq \text{mis}(L_{\{s\}}[B])=3^{|B|/3}$, as desired. □

## 6. Concluding remarks

In this paper, we show that type II groups of order divisible by 9 have many maximal sum-free sets, $3^{(1/3+o(1))\mu(G)}$; while almost all even order groups, i.e. type I(2), have exponentially fewer than $2^{\mu(G)/2}$. This is in sharp contrast to the integer setting. Many interesting problems remain. For example, very little is known about type III groups. We conclude this paper with two further remarks.

- We establish the bound $f_{\max}(G)\leq 2^{(1/2-c)\mu(G)}$ for even order groups with sublinear number of order 2 elements. New ideas are needed to handle the remaining constant many even order groups with $\Omega(n)$ number of order 2 elements. We see in Section 4.1 that the same bound holds for the group $\mathbb{Z}_2\oplus\cdots\oplus\mathbb{Z}_2\oplus\mathbb{Z}_4$, which is in a sense the 'worst' even order group as it has the largest number of order 2 elements (other than $\mathbb{Z}_2^k$). Considering also the result on type I(5) groups $\mathbb{Z}_5\oplus H$, it is plausible that $\mathbb{Z}_2^k$ is the group with the maximum number of maximal sum-free sets among type I groups.

  **Conjecture 6.1.** *All type I groups $G$ except $\mathbb{Z}_2^k$ have exponentially fewer maximal sum-free sets than $2^{\mu(G)/2}$.*

  Apart from the even order groups with many order 2 elements, another difficulty for the above conjecture is that the stability result gets weaker for type I($p$) groups when $p$ gets larger. As a result, we might not be able to assume the ground set of the link graph is a union of cosets, which is very useful in our analysis.

- The remaining type II groups not covered by Proposition 1.5 are of the form $G=\mathbb{Z}_3\oplus_i\mathbb{Z}_{p_i^{a_i}}$ with $p_i\equiv 1 \pmod 3$. It is known [2] that

  $$2^{\mu(G)/2}\leq f_{\max}(G)\leq 3^{(1/3+o(1))\mu(G)}.$$

  It would be interesting to know which bound is closer to the truth.

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Hong Liu
Mathematics Institute
University of Warwick
UK

Maryam Sharifzadeh
School of Mathematics
University of Birmingham
UK

*E-mail addresses:* h.liu.9@warwick.ac.uk, m.sharifzadeh@gmail.com