UNRAMIFIED BRAUER GROUPS OF CONIC BUNDLE THREEFOLDS IN CHARACTERISTIC TWO

ASHER AUEL, ALESSANDRO BIGAZZI, CHRISTIAN BÖHNING, AND HANS-CHRISTIAN GRAF VON BOTHMER

Abstract. We establish a formula for computing the unramified Brauer group of tame conic bundle threefolds in characteristic 2. The formula depends on the arrangement and residue double covers of the discriminant components, the latter being governed by Artin–Schreier theory (instead of Kummer theory in characteristic not 2). We use this to give new examples of threefold conic bundles defined over $\mathbb{Z}$ that are not stably rational over $\mathbb{C}$.

1. Introduction

Motivated by the rationality problem in algebraic geometry, we compute new obstructions to the universal triviality of the Chow group of 0-cycles of smooth projective varieties in characteristic $p > 0$, related to ideas coming from crystalline cohomology (see [CL98] for a survey): the p-torsion in the unramified Brauer group. In [ABBB18], we proved that p-torsion Brauer classes do obstruct the universal triviality of the Chow group of 0-cycles; here we focus on computing these obstructions in the case of conic bundles. In particular, we provide a formula to compute the two torsion in the unramified Brauer group of conic bundle threefolds in characteristic 2. We provide some applications showing that one can obtain results with this type of obstruction that one cannot by other means: there exist conic bundles over $\mathbb{P}^2$, defined over $\mathbb{Z}$, that are smooth over $\mathbb{Q}$ and whose reduction modulo p has (1) nontrivial two torsion in the unramified Brauer group and a universally CH_0-trivial resolution for $p = 2$, and (2) irreducible discriminant, hence trivial Brauer group, for all $p > 2$. The roadmap for this paper is as follows.

In Section 2, we assemble some background on conic bundles and quadratic forms in characteristic 2 to fix notation and the basic notions.

Section 3 contains a few preliminary results about Brauer groups, in particular their p-parts in characteristic p, and then goes on to discuss residue maps, which, in our setting, are only defined on a certain subgroup of the Brauer group, the so-called tamely ramified, or tame, subgroup. We also interpret these residues geometrically for Brauer classes induced by conic bundles in characteristic 2, and show that their vanishing characterizes unramified elements.

Besides the fact that residues are only partially defined, we encounter another new phenomenon in the bad torsion setting, which we discuss in Section 4, namely, that Bloch–Ogus type complexes fail to explain which residue profiles are actually realized by Brauer classes on the base. We also investigate some local analytic normal forms of the discriminants of conic bundles in characteristic 2 that can arise or are excluded for various reasons.

In Section 5 we prove Theorem 5.1, which computes the two torsion in the unramified Brauer group of some conic bundles over surfaces in characteristic two. In the

Date: May 6, 2020.

hypotheses, we have to assume the existence of certain auxiliary conic bundles over the base with predefined residue subprofiles of the discriminant profile of the initial conic bundle. This is because of the absence of Bloch–Ogus complex methods, as discussed in Section 4.

In Section 6 we construct examples of conic bundle threefolds defined over $\mathbb{Z}$ that are not stably rational over $\mathbb{C}$, of the type described in the first paragraph of this Introduction.

Acknowledgments. We would like to thank Jean-Louis Colliot-Thélène and Burt Totaro for their interest in this work, their helpful suggestions and pointers to the literature. We would also like to thank the Simons Foundation for hosting the Conference on Birational Geometry August 2017 during which part of this work was presented and several participants gave us useful feedback. Finally, we thank the referees, who contributed useful comments on the exposition.

2. Background on conic bundles

Let K be a field. The most interesting case for us in the sequel will be when K has characteristic 2. Typically, K will not be algebraically closed, for example, the function field of some positive-dimensional algebraic variety over an algebraically closed ground field k of characteristic 2, which is the base space of certain conic bundles or, more generally, quadric fibrations.

2.a. **Quadratic forms.** As a matter of reference, let us recall here some basic notions concerning the classification of quadratic forms in characteristic 2. We refer to [EKM08, Chapters I–II].

Definition 2.1. Let V be a finite dimensional vector space over K. A *quadratic form* over V is a map $q : V \longrightarrow K$ such that:

a) $q(\lambda v) = \lambda^2 q(v)$ for each $\lambda \in K$ and $v \in V$;

b) the map $b_q : V \times V \longrightarrow K$ defined by

$$b_q(v, w) = q(v + w) - q(v) - q(w)$$

is K-bilinear.

When the characteristic of K is not 2, a quadratic form q can be completely recovered by its associated bilinear form b_q and, thus, by its associated symmetric matrix. This correspondence fails to hold when $\operatorname{char} K = 2$, due to the existence of non-zero quadratic forms with identically zero associated bilinear form; these forms are called *totally singular* and play a significant role in the decomposition of quadratic forms over such fields.

Definition 2.2. Let b be a bilinear form over V; its *radical* is the set

$$r(b) := \{v \in V \mid b(v, w) = 0 \text{ for any } w \in V\}$$

Let q be a quadratic form; the *quadratic radical* is

$$r(q) := \{v \in V \mid q(v) = 0\} \cap r(b_q)$$

In general, we have strict inclusion $r(q) \subset r(b_q)$ if $\operatorname{char} K = 2$. A form such that $r(q) = 0$ is called *regular*.

We introduce the following notation: let q be a quadratic form over V and let $U, W \subseteq V$ be vector subspaces such that $V = W \oplus U$. If U and W are orthogonal with respect to the associated bilinear form b_q (we write $U \subset W^{\perp}$ to mean this), then q decomposes as sum of its restrictions $q|_W$ and $q|_U$ and we write $q = q|_W \perp q|_V$.

We will also say that two quadratic forms q_1, q_2 defined respectively over V_1 and V_2, are *isometric* if there exists an isometry $f : V_1 \longrightarrow V_2$ of the associated bilinear forms and satisfying $q_1(v) = q_2(f(v))$. In this case, we write $q_1 \simeq q_2$. We say that q_1, q_2 are *similar* if $q_1 \simeq cq_2$ for some $c \in K^\times$.

Definition 2.3. Let $a, b \in K$. We denote by $\langle a\rangle$ the *diagonal quadratic form* on K (as K-vector space over itself) defined by $v \mapsto a\, v^2$. Also, we denote by $[a, b]$ the quadratic form on K^2 defined by $(x, y) \mapsto ax^2 + xy + by^2$.

We say that a quadratic form q is *diagonalizable* if there exists a direct sum decomposition $V = V_1 \oplus \ldots \oplus V_n$ such that each V_i has dimension 1, we have $V_i \subseteq V_j^\perp$ for every $i \neq j$ and $q|_{V_i} \simeq \langle a_i\rangle$ so that

$$q \simeq \langle a_1, \ldots, a_n\rangle := \langle a_1\rangle\perp \ldots \perp\langle a_n\rangle$$

We will also write

$$n \cdot q := \underbrace{q\perp \ldots \perp q}_{n \text{ times}}$$

If $\operatorname{char} K = 2$, then q is diagonalizable if and only if q is totally singular. This is in contrast with the well known case of $\operatorname{char} K \neq 2$, since over such fields every quadratic form is diagonalizable.

A quadratic form q is called *anisotropic* if $q(v) \neq 0$ for every $0 \neq v \in V$. Geometrically, this means that the associated quadric $Q := \{q = 0\} \subset \mathbb{P}(V)$ does not have K-rational points. We remark that the associated quadric only depends on the similarity class of the quadratic form.

A form q is called *non-degenerate* if it is regular and $\dim r(b_q) \leq 1$. Geometrically speaking, non-degeneracy means that the quadric Q is smooth over K, while regularity means that the quadric Q is regular as a scheme, equivalently, is not a cone in $\mathbb{P}(V)$ over a lower dimensional quadric. In characteristic 2, there can exist regular quadratic forms that fail to be non-degenerate.

For example, consider the subvariety X of $\mathbb{P}^2_{(u:v:w)} \times \mathbb{P}^2_{(x:y:z)}$ defined by

$$ux^2 + vy^2 + wz^2 = 0$$

over an algebraically closed field k of characteristic 2. This is a conic fibration over $\mathbb{P}^2_{(u:v:w)}$ such that the generic fiber X_K over $K = k(\mathbb{P}^2_{(u:v:w)})$ is defined by a quadratic form q that is anisotropic, but totally singular. The form q is regular, but fails to be non-degenerate. Geometrically, this means that the conic fibration has no rational section (anisotropic), has a geometric generic fiber that is a double line (totally singular), X_K is not a cone (regular), but X_K is of course not smooth over K. On the other hand, the total space X of this conic fibration is smooth over k.

One has the following structure theorem.

Theorem 2.4. *Let K be a field of characteristic 2 and let q be a quadratic form on a finite-dimensional vector V over K. Then there exist a m-dimensional vector subspace $W \subseteq r(b_q)$ and 2-dimensional vector subspaces $V_1, \ldots, V_s \subseteq V$ such that the following orthogonal decomposition is realized:*

$$q = q|_{r(q)}\perp q|_W \perp q|_{V_1}\perp \ldots \perp q|_{V_s}$$

with $q|_{V_i} \simeq [a_i, b_i]$ for some $a_i, b_i \in K$ a non-degenerate form. Moreover, $q|_W$ is anisotropic, diagonalisable and unique up to isometry. In particular,

$$q \simeq r \cdot \langle 0\rangle\perp\langle c_1, \ldots, c_m\rangle\perp[a_1, b_1]\perp \ldots \perp[a_s, b_s]$$

We now classify quadratic forms in three variables.

Corollary 2.5. *Let K be a field of characteristic 2, let q be a nonzero quadratic form in three variables over K, and let $Q \subset \mathbb{P}^2$ be the associated conic. In the following table, we give the classification of normal forms of q, up to similarity, and the corresponding geometry of Q.*

$\dim r(q)$	$\dim r(b_q)$	*normal form of q*	*geometry of Q*
0	1	$ax^2 + by^2 + xz + z^2$	*smooth conic*
0	3	$ax^2 + by^2 + z^2$	*regular conic, geom. double line*
1	1	$ax^2 + xz + z^2$	*cross of lines over K_a*
1	1	xz	*cross of lines*
1	3	$ax^2 + z^2$	*singular conic, geom. double line*
2	3	z^2	*double line*

Here, $(x : y : z)$ are homogeneous coordinates on $\mathbb{P}^2$; by cross of lines, we mean a union of two disjoint lines in $\mathbb{P}^2$; and by K_a, we mean the Artin–Schreier extension of K defined by $x^2 - x - a$.

Proof. According to the classification in Theorem 2.4, we have the following normal forms for q up to isometry over K.

$\dim r(q)$	$\dim r(b_q)$	normal form of q up to isometry	
0	1	$ax^2 + by^2 + xz + cz^2$	$a, c \in K,\ b \in K^\times$
0	3	$ax^2 + by^2 + cz^2$	$a, b, c \in K^\times$
1	1	$ax^2 + xz + cz^2$	$a, c \in K$
1	3	$ax^2 + cz^2$	$a, c \in K^\times$
2	3	cz^2	$c \in K^\times$

Here, in the cases $\dim r(q) \leq 1$ and $\dim r(b_q) = 3$, we are assuming that the associated diagonal quadratic forms $\langle a, b, c\rangle$ in 3 variables or $\langle a, c\rangle$ in 2 variables, respectively, are anisotropic. Otherwise, these cases are not necessarily distinct.

We remark that up to the change of variables $z \mapsto c^{-1}z$ and multiplication by c, the quadratic forms $[a, c]$ and $[ac, 1]$ are similar. Hence up to similarity, we can assume that $c = 1$ in the above table of normal forms up to isometry.

The fact that when q is totally singular, Q is geometrically a double line follows since any diagonal quadratic form over an algebraically closed form of characteristic 2 is the square of a linear form.

Thus, the only case requiring attention is the case $\dim r(q) = \dim r(b_q) = 1$, where we claim that if $a = \alpha^2 - \alpha$ for some $\alpha \in K$, then q is similar to xz and thus Q is a cross of lines. Indeed, after assuming that $c = 1$, as above, we change variables $z \mapsto z - \alpha x$ and $x \mapsto x - z$. In particular, when $a \in K/\wp(K)$ is nonzero, where $\wp : K \to K$ is given by $\wp(x) = x^2 - x$, then Q becomes a cross of lines over the Artin–Schreier extension L/K defined by $x^2 - x - a$. This distinguished the two cross of lines cases in the table in the statement of the corollary. ☐

2.b. **Conic bundles.** Let k be an algebraically closed field. We adopt the following definition of conic bundle.

Definition 2.6. Let X and B be projective varieties over k and let B be smooth. A *conic bundle* is a morphism $\pi : X \longrightarrow B$ such that π is flat and proper with every geometric fiber isomorphic to a plane conic and with smooth geometric generic fiber. In practice, all conic bundles will be given to us in the following form: there is a rank 3 vector bundle $\mathscr{E}$ over B and a quadratic form $q : \mathscr{E} \longrightarrow \mathscr{L}$ (with values in some line bundle $\mathscr{L}$ over B) which is not identically zero on any fiber. Suppose that q is non-degenerate on the generic fiber of $\mathscr{E}$. Then putting $X = \{q = 0\} \subseteq \mathbb{P}(\mathscr{E}) \longrightarrow B$, where the arrow is the canonical projection map to B, defines a conic bundle.

The hypothesis on the geometric generic fiber is not redundant in our context. Suppose that $\operatorname{char} k = 2$, and let $\pi : X \longrightarrow B$ be a flat, proper morphism such that every geometric fiber is isomorphic to a plane conic. Let η be the generic point of B and $K = k(B)$; note that the geometric generic fiber $\overline{X}_\eta$ is a conic in $\mathbb{P}^2_{\overline{K}}$ and it is defined by the vanishing of some quadratic form q_η. By Corollary 2.5, we conclude that then $\overline{X}_\eta$ is cut out by one of the following equations:

$$ax^2 + by^2 + cz^2 = 0 \tag{1}$$

or

$$ax^2 + by^2 + yz + cz^2 = 0 \tag{2}$$

where $(x : y : z)$ are homogeneous coordinates for $\mathbb{P}^2_{\overline{K}}$. The additional assumption on smoothness of the geometric generic fiber allows us to rule out the case of (1), which would give rise to *wild conic bundles*.

We must define discriminants of conic bundles together with their scheme-structure using a notion (often referred to as the "half-discriminant" or "semi-discriminant") discovered independently by Grothendieck and M. Kneser, see [Kn91, IV.3.1], [Co14, Proposition C.1.4]. First we discuss the discriminant of the generic conic.

Remark 2.7. Let $\mathbb{P}^2$ have homogeneous coordinates $(x : y : z)$ and $\mathbb{P} = \mathbb{P}(H^0(\mathbb{P}^2, \mathscr{O}(2)))$ the 5-dimensional projective space of all conics in $\mathbb{P}^2$. We have the universal conic over $X_{\text{univ}} \longrightarrow \mathbb{P}$ defined as the projection of the incidence $X_{\text{univ}} \subset \mathbb{P} \times \mathbb{P}^2$, which can be written as a hypersurface of bidegree $(1, 2)$ defined by the generic conic

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + a_{xz}xz + a_{yz}yz,$$

where we can consider $(a_{xx} : a_{yy} : a_{zz} : a_{xy} : a_{xz} : a_{yz})$ as a system of homogeneous coordinates on $\mathbb{P}$. In these coordinates, the equation of the so-called universal discriminant $\Delta_{\text{univ}} \subset \mathbb{P}$ parametrizing singular conics is

$$4a_{xx}a_{yy}a_{zz} + a_{xy}a_{yz}a_{xz} - a_{xz}^2 a_{yy} - a_{yz}^2 a_{xx} - a_{xy}^2 a_{zz} \tag{3}$$

if $\operatorname{char} k \neq 2$, and

$$a_{xy}a_{yz}a_{xz} + a_{xz}^2 a_{yy} + a_{yz}^2 a_{xx} + a_{xy}^2 a_{zz} \tag{4}$$

if $\operatorname{char} k = 2$. In any characteristic, $\Delta_{\text{univ}} \subset \mathbb{P}$ is a geometrically integral hypersurface parameterizing the locus of singular conics in $\mathbb{P}^2$.

If $\operatorname{char} k = 2$, then the locus of double lines in Δ_{univ}, with its reduced scheme structure, is given by $a_{xy} = a_{xz} = a_{yz} = 0$. We call this the universal subscheme of double lines.

Definition 2.8. Let $\pi \colon X \to B$ be a conic bundle as in Definition 2.6.

The discriminant Δ of the conic bundle is a subscheme of B defined in the following way. Locally around each point of B, we choose an open U on which $\mathscr{E}$ and $\mathscr{L}$ from Definition 2.6 trivialize; then there is a unique morphism $f \colon U \to \mathbb{P}$ such that $\pi|_{\pi^{-1}(U)} \colon X_{\pi^{-1}(U)} \to U$ is isomorphic to the pull-back via f of the universal conic bundle. Then we define $\Delta \cap U$ as the scheme-theoretic pullback of Δ_{univ} via f. By the uniqueness of the f these local descriptions glue to give a subscheme Δ of B.

In a similar way, if $\operatorname{char} k = 2$, we define a subscheme of Δ of double lines of the conic bundle locally as the scheme-theoretic pullback of the universal subscheme of double lines.

3. Brauer groups and partially defined residues

For a Noetherian scheme X, we denote by $\mathrm{Br}(X)$ Grothendieck's cohomological Brauer group, the torsion subgroup of $H^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$. If $X = \mathrm{Spec}(A)$ for a commutative ring A, we also write $\mathrm{Br}(A)$ for the Brauer group of $\mathrm{Spec}\, A$.

If X is a regular scheme, then every class in $H^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$ is torsion [Gro68, II, Proposition 1.4]. If X is quasi-projective (over any ring), a result of Gabber [deJ03] says that this group equals the Azumaya algebra Brauer group, defined as the group of Azumaya algebras over X up to Morita equivalence.

If X is a smooth connected projective variety over a field k, then restriction to the generic point defines an inclusion $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ by [Gro68, II, Corollaire 1.10]. In various applications, especially where one is presented with a highly singular model of X, it is convenient to work with the image of this inclusion in the Brauer group of the function field, which is the idea behind the unramified Brauer group, see [Bogo87], [CTO], [CT95]. A purity result due to Hoobler [Ho80] in the affine case together with an argument (cf. [ABBB18, Theorem 2.5]) with the Mayer–Vietoris sequence for étale cohomology shows that this image coincides with the set of all classes $\alpha \in \mathrm{Br}(k(X))$ that are in the image of the natural map $\mathrm{Br}(A_v) \to \mathrm{Br}(k(X))$ for all divisorial valuations v (trivial on k) of $k(X)$ corresponding to prime divisors on X, where $A_v \subset k(X)$ is the valuation ring. More generally, if X is any integral (possibly singular) k-scheme, the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(k(X))$ is defined to be the set of all classes $\alpha \in \mathrm{Br}(k(X))$ that are in the image of the natural map $\mathrm{Br}(A_v) \to \mathrm{Br}(k(X))$ for all divisorial valuations v (trivial on k) of $k(X)$ that correspond to a prime divisor D on some model X' of X, where X' is generically smooth along D. In our applications, we will produce explicit resolutions $\widetilde{X}$ of X, and thus we will have $\mathrm{Br}_{\mathrm{nr}}(k(X)) = \mathrm{Br}(\widetilde{X})$.

Next we want to characterize elements in $\mathrm{Br}_{\mathrm{nr}}(k(X))$ in terms of partially defined residues in the sense of Merkurjev as in [GMS03, Appendix A]; this is necessitated by the following circumstance: if one wants to give a formula for the unramified Brauer group of a conic bundle over some smooth projective rational base (see, e.g., [Pi16], [ABBP16]), for example a smooth projective rational surface, the idea of [CTO] is to produce the nonzero Brauer classes on the total space of the given conic bundle as pull-backs of Brauer classes represented by certain other conic bundles on the base whose residue profiles are a proper subset of the residue profile of the given conic bundle. Hence, one also has to understand the geometric meaning of residues because in the course of this approach it becomes necessary to decide when the residues of two conic bundles along the same divisor are equal.

Let K be a field of characteristic p. We denote the subgroup of elements in $\mathrm{Br}(K)$ whose order equals a power of p by $\mathrm{Br}(K)\{p\}$. Let v be a discrete valuation of K, K_v the completion of K with respect to the absolute value induced by v. Denote the residue field of v by $k(v)$ and by $\overline{K}_v$ an algebraic closure of K_v. One can extend v uniquely from K_v to $\overline{K}_v$, the residue field for that extended valuation on $\overline{K}_v$ will be denoted by $\overline{k(v)}$.

By [Artin67, p. 64–67], unramified subfields of $\overline{K}_v$ correspond to separable subfields of $\overline{k(v)}$, and, in particular, there is a maximal unramified extension, with residue field $k(v)^{\mathrm{s}}$ (separable closure), called the *inertia field*, and denoted by K_v^{nr} or $T = T_v$ (for *Trägheitskörper*). One also has that the Galois group $\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v)$ is isomorphic to $\mathrm{Gal}(k(v))$. Now, recall that by the Galois cohomology characterization, the Brauer group $\mathrm{Br}(K_v)\{p\}$ is isomorphic to $H^2(K_v, K_v^{\mathrm{s}\times})\{p\}$ and there is a natural map

$$H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), K_v^{\mathrm{nr}\times})\{p\} \to H^2(K_v, K_v^{\mathrm{s}\times})\{p\} \tag{5}$$

which is injective [GMS03, Appendix A, Lemma A.6, p. 153].

Definition 3.1. With the above setting, we call the image of (5) the *tame subgroup* or *tamely ramified* subgroup of $\mathrm{Br}(K_v)\{p\}$ associated to v, and denote it by $\mathrm{Br}_{\mathrm{tame},v}(K_v)\{p\}$. We denote its preimage in $\mathrm{Br}(K)\{p\}$ by $\mathrm{Br}_{\mathrm{tame},v}(K)\{p\}$ and likewise call it the tame subgroup of $\mathrm{Br}(K)\{p\}$ associated to v.

Writing again v for the unique extension of v to K_v^{nr} we have a group homomorphism

$$v\colon K_v^{\mathrm{nr}\times} \to \mathbb{Z}$$

Definition 3.2. Following [GMS03, Appendix A] one can define a map as the composition

$$r_v\colon \mathrm{Br}_{\mathrm{tame},v}(K)\{p\} \to H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), K_v^{\mathrm{nr}\times})\{p\} \to H^2(k(v),\mathbb{Z})\{p\} \simeq H^1(k(v),\mathbb{Q}/\mathbb{Z})\{p\}.$$

which we call the *residue map* with respect to the valuation v. We will say that the residue of an element $\alpha \in \mathrm{Br}(K)\{p\}$ with respect to a valuation v is *defined*, equivalently, that α is *tamely ramified* at v, if α is contained in $\mathrm{Br}_{\mathrm{tame},v}(K)\{p\}$.

Remark 3.3. If $\alpha \in \mathrm{Br}(K)[p]$ for which the residue with respect to a valuation v is defined, as in Definition 3.2, then $r_v(\alpha) \in H^1(k(v), \mathbb{Z}/p)$. By Artin–Schreier theory [GS06, Proposition 4.3.10], one has $H^1(k(v),\mathbb{Z}/p) \simeq k(v)/\wp(k(v))$ where $\wp\colon k(v) \to k(v)$, $\wp(x) = x^p - x$, is the Artin–Schreier map. This group classifies pairs, consisting of a finite $\mathbb{Z}/p$-Galois extension of $k(v)$ together with a chosen generator of the Galois group. Indeed, $\mathbb{Z}/p$-Galois extensions of $k(v)$ are Artin–Schreier extensions, i.e., generated by the roots of a polynomial $x^p - x - a$ for some $a \in k(v)$. The isomorphism class of such an Artin–Schreier extension is unique up to the substitution

$$a \mapsto \eta a + (c^p - c)$$

where $\eta \in \mathbb{F}_p^\times$ and $c \in k(v)$, see for example [Artin07, §7.2]. In particular, for $p = 2$, one may also identify $H^1(k(v), \mathbb{Z}/2)$ with $\acute{\mathrm{E}}\mathrm{t}_2(k(v))$, the set of isomorphism classes étale algebras of degree 2 over $k(v)$, cf. [EKM08, p. 402, Example 101.1]; more geometrically, if D is a prime divisor on a smooth algebraic variety over a field k and v_D the corresponding valuation, the residue can be thought of as being given by an étale double cover of an open part of D.

Remark 3.4. Keep the notation of Definition 3.2. The tame subgroup

$$\mathrm{Br}_{\mathrm{tame},v}(K_v)\{p\} = H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), K_v^{\mathrm{nr}\times})\{p\}$$

of $\mathrm{Br}(K_v)\{p\}$ has a simpler description by [GS06, Theorem 4.4.7, Definition 2.4.9]: it is nothing but the subgroup of elements of order a power of p in the relative Brauer group $\mathrm{Br}(K_v^{\mathrm{nr}}/K_v)$ of Brauer classes in $\mathrm{Br}(K_v)$ that are split by the inertia field K_v^{nr}, in other words, are in the kernel of the natural map

$$\mathrm{Br}(K_v) \to \mathrm{Br}(K_v^{\mathrm{nr}}).$$

To explain the name, one can say that the tame subgroup of $\mathrm{Br}(K_v)\{p\}$ consists of those classes that become trivial in $\mathrm{Br}(V)$, where V is the *maximal tamely ramified extension* of K_v, the ramification field (*Verzweigungskörper*) [Artin67, Chapter 4, §2], because the classes of orders a power of p split by $\mathrm{Br}(T)\{p\}$ coincide with those split by $\mathrm{Br}(V)\{p\}$ (since V is obtained from T by adjoining roots $\sqrt[m]{\pi}$ of a uniformizing element π of K_v of orders m not divisible by p and restriction followed by corestriction is multiplication by the degree of a finite extension). Since in characteristic 0 every extension of K_v is tamely ramified, one can say that in general residues are defined on the subgroup of those classes in $\mathrm{Br}(K_v)\{p\}$ that become trivial on $\mathrm{Br}(V)\{p\}$.

The terminology tame subgroup was suggested to us by Burt Totaro, who also kindly provided other references to the literature. It has the advantage of avoiding the confusing terminology "unramified subgroup", also sometimes used, for elements that can have nontrivial residues. The terminology here is consistent with [TiWa15, §6.2, Proposition 6.63], and one could also have called the tamely ramified subgroup the *inertially split part*, following that source, as the two notions coincide in our context. Our terminology is also consistent with the one in [Ka82, Theorem 3].

Remark 3.5. More generally, given a field F of characteristic $p > 0$, one can define a version of Galois cohomology "with mod p coefficients", following Kato [Ka86] or Merkurjev [GMS03, Appendix A], [Mer15]:

$$H^{n+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) := H^2(F, \mathrm{K}_n(F^{\mathrm{s}}))\{p\} \tag{6}$$

where $\mathrm{K}_n(F^{\mathrm{s}})$ is the n-th Milnor K-group of the separable closure of F, and the cohomology on the right hand side is usual Galois cohomology with coefficients in this Galois module. The coefficients group $\mathbb{Q}_p/\mathbb{Z}_p(n)$ on the left hand side is just a symbol in analogy with the case of characteristics coprime to p. Given a discrete rank 1 valuation v of F with residue field E, one can define a *tame subgroup* (or *tamely ramified subgroup*)

$$H^{n+1}_{\mathrm{tame},v}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \subset H^{n+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(n))$$

in this more general setting so that one recovers the definition given for the Brauer group in Definition 3.2, see [GMS03, p. 152, Example A.3].

Theorem 3.6. *Let k be an algebraically closed field of characteristic 2. Let X and B be projective varieties over k, let B be smooth of dimension ≥ 2 and let $\pi\colon X \to B$ be a conic bundle. Let K be the function field of B, let $\alpha \in \mathrm{Br}(K)[2]$ be the Brauer class determined by the conic bundle, and let D be a prime divisor on B. Suppose that one is in either one of the following two cases:*

a) The geometric generic fiber of

$$\pi|_{\pi^{-1}(D)}\colon X|_{\pi^{-1}(D)} \to D$$

is a smooth conic.

b) The geometric generic fiber of

$$\pi|_{\pi^{-1}(D)}\colon X|_{\pi^{-1}(D)} \to D$$

is a cross of lines (as in Corollary 2.5). In this case, we say that the conic bundle is tamely ramified *over D.*

Then in both cases, the residue of α with respect to the divisorial valuation v_D determined by D is defined, and in case a) it is zero, whereas in case b), if the multiplicity of D as a discriminant component is even, it is zero, and if this multiplicity is odd, it is the class of the double cover of D induced by the restriction of the conic bundle π over D, which is étale over an open part of D by the assumption on the type of geometric generic fiber.

We need the following auxiliary results before commencing with the proof.

Lemma 3.7. *Under all the hypotheses of a) or b) of Theorem 3.6, let $P \in D$ be a point where the fiber X_P is a smooth conic or a cross of lines, respectively. Then we can assume that Zariski locally around P the conic bundle is defined by*

$$ax^2 + by^2 + xz + z^2 = 0$$

with x, y, z fiber coordinates and a, b functions on B, both regular locally around P and b not identically zero.

Proof. Locally around P, the conic bundle is given by an equation

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + a_{xz}xz + a_{yz}yz = 0$$

with the a's regular functions locally around P. Since the fiber X_P is smooth or a cross of lines, we have that one of the coefficients of the mixed terms, without loss of generality a_{xz}, is nonzero in P. Introducing new coordinates by the substitution $x \mapsto (1/a_{xz})x$ (here and in the following we treat x, y, z as well as the a's as dynamical variables to ease notation) one gets the form

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + xz + a_{yz}yz = 0.$$

Now the substitutions $x \mapsto x + a_{yz}y, \quad y \mapsto y, \quad z \mapsto z + a_{xy}y$ transforms this into

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + xz = 0.$$

Now if one of a_{xx} or a_{zz} is nonzero in P, without loss of generality $a_{zz}(P) \neq 0$, then multiplying the equation by a_{zz}^{-1} and subsequently applying the substitution $x \mapsto a_{zz}x$ we obtain the desired normal form

$$a_{xx}x^2 + a_{yy}y^2 + xz + z^2 = 0. \tag{7}$$

But if both a_{xx} and a_{zz} vanish in P, then after applying the substitution $x \mapsto x + z$, we get that $a_{zz}(P) \neq 0$ and proceed as before. $\square$

Before stating the next auxiliary result, we recall the existence of a cup product homomorphism

$$K_i(K) \otimes H^{n+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^{n+i+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(n+i)),$$

which restricts to the tamely ramified subgroups, see [GMS03, p. 154, (A.7)].

Proposition 3.8. *Let K be a field of characteristic 2 and Q the conic defined by $ax^2 + by^2 + xz + z^2 = 0$ for $a \in K$ and $b \in K^\times$. Then the Brauer class associated to Q is the cup product $b \cup a$, via the cup product homomorphism*

$$K_1(K) \otimes H^1(K, \mathbb{Q}_2/\mathbb{Z}_2(0)) \to H^2(K, \mathbb{Q}_2/\mathbb{Z}_2(1)) = \mathrm{Br}(K)\{2\},$$

where we consider $a \in H^1(K, \mathbb{Q}_2/\mathbb{Z}_2)[2] = K/\wp(K)$ and $b \in K_1(K) = K^\times$.

Proof. The Brauer class associated to Q is the generalized quaternion algebra $[a, b)$, defined as the free associative K-algebra on generators i and j with the relations

$$i^2 - i = a, \quad j^2 = b, \quad ij = ji + j,$$

see [GS06, Chapter 1, Exercise 4]. Let L/K be the Artin–Schreier extension, which could be the split étale algebra, generated by $x^2 - x - a$ and $\chi_{L/K} : \mathrm{Gal}(K^{\mathrm{s}}/K) \to \mathbb{Z}/2\mathbb{Z}$ the canonically associated character of the absolute Galois group of K. By [GS06, Corollary 2.5.5b], the quaternion algebra $[a, b)$ is K-isomorphic to the cyclic algebra $(\chi_{L/K}, b)$, generated as a K-algebra by L and an element y with the relations

$$y^2 = b, \quad \lambda y = y\sigma(\lambda)$$

where $\lambda \in L$ and σ is the generator of $\mathrm{Gal}(L/K)$.

Letting $\delta : H^1(K, \mathbb{Z}/2) \to H^2(K, \mathbb{Z})$ be the *Bockstein homomorphism* induced from the coboundary map in Galois cohomology associated to the exact sequence of trivial Galois modules

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

then $\delta : H^1(K, \mathbb{Z}/2) \to H^2(K, \mathbb{Z})[2]$ is an isomorphism that gives sense to Merkurjev's definition $H^1(K, Z/2Z(0)) := H^2(K, K_0(F^{\mathrm{s}}))[2] = H^2(K, \mathbb{Z})[2]$. By [GS06, Proposition 4.7.3], the cup product pairing in Galois cohomology

$$H^2(K, \mathbb{Z}) \times H^0(K, K^{\mathrm{s}\times}) \to H^2(K, K^{\mathrm{s}\times}) = \mathrm{Br}(F)$$

has the property that the cup product of the class $\delta(a) \in H^2(K, \mathbb{Z})[2]$, where we consider $a \in K/\wp(K) = H^1(K, \mathbb{Z}/2)$, with the class $b \in H^0(K, K^{\mathrm{s}\times}) = K^\times$, results in the Brauer class of the cyclic algebra $(\chi_{L/K}, b) \in \mathrm{Br}(K)[2]$.

Finally, under the canonical identification $H^0(K, K^{\mathrm{s}\times}) = K^\times = K_1(K)$, the isomorphism $\delta : H^1(K, \mathbb{Z}/2) \to H^2(K, \mathbb{Z})[2]$, and the definition of the action of $K_1(K)$ on $H^1(K, \mathbb{Q}_2/\mathbb{Z}_2(0))$, the cup product pairing

$$H^1(K, \mathbb{Z}/2(0)) \times K_1(K) \to H^2(K, \mathbb{Z}/2(1)) = \mathrm{Br}(K)[2]$$

is identified with the 2-torsion part of the above cup product pairing in Galois cohomology. Since the cup product is commutative on 2-torsion classes, we get the desired formula.

We also point out that the relevant cocycle calculation in the proof of this result in [KMRT98, Proposition 30.4], though stated for K of characteristic not 2, can be generalized to the case of characteristic 2 using the machinery of flat cohomology. $\square$

Proof of Theorem 3.6. Let $v = v_D$ be the divisorial valuation associated to D. We have to check that in both cases of the Theorem, $\alpha \in \mathrm{Br}(K_v^{\mathrm{nr}}/K_v)$, see Remark 3.4, in other words, that α is split by K_v^{nr}. We have the normal form of Lemma 3.7 locally around the generic point of D. The conic bundle is then obviously split by the Galois cover of the base defined by adjoining to $k(B)$ the roots of T^2+T+a because then the quadratic form in Lemma 3.7 acquires a zero. Moreover, that Galois cover does not ramify in the generic point of D because a has no pole along D. Hence it defines an extension of K_v contained in K_v^{nr}. See also [Artin67, Example 1, p. 67].

By formula (4) we find that the discriminant of $ax^2+by^2+xz+z^2 = 0$ is given by b, hence by absorbing even powers of a local parameter for D into the fiber coordinate y, we can assume b itself is a local parameter for D, or a unit in the generic point of D. By Proposition 3.8, the Brauer class $\alpha \in \mathrm{Br}(k(B))\{2\} = H^2(k(B), \mathbb{Q}_2/\mathbb{Z}_2(1))$ associated to the conic bundle defined by the preceding formula is the cup product $\alpha = b \cup a$ of the class $b \in K_1(k(B)) = k(B)^\times$ and the class $a \in H^1(k(B), \mathbb{Q}_2/\mathbb{Z}_2) = k(B)/\wp(k(B))$. We now conclude the proof in a number of steps.

Step 1. If π is a local equation for D in $\mathscr{O}_{B,D}$, then a polynomial in a with coefficients in k can only vanish along D if a is congruent modulo π to some element in $k^\times$. If a is not congruent modulo π to an element in k, we consequently have that $k(a) \subset k(B)$ is a subfield of the valuation ring $\mathscr{O}_{B,D}$ of v. By [GMS03, p. 154, sentence before formula (A.8)], the element that a induces in $H^1(k(B), \mathbb{Q}_2/\mathbb{Z}_2)$ is in $H^1_{\mathrm{tame},v}(k(B), \mathbb{Q}_2/\mathbb{Z}_2)$, and then formula (A.8) of loc.cit. implies

$$r_v(b \cup a) = a|_D \in k(D)/\wp(k(D)) = H^1(k(D), \mathbb{Z}/2)$$

in case b vanishes along D with order 1, and $r_v(b \cup a) = 0$ if b is a unit generically along D. Since $a|_D$ is precisely the element defining the Artin–Schreier double cover induced by the conic bundle on $D = (b = 0)$, the residue is given by this geometrically defined double cover.

Step 2. If a is congruent modulo π to an element in k, and since k is algebraically closed, we can make a change in the fiber coordinate z so that a is actually a power of π times a unit in $\mathscr{O}_{B,D}$. Since $\dim B \geq 2$, we can find a unit $a' \in \mathscr{O}_{B,D}$ that is not congruent to an element in k modulo π, and write $a = (a - a') + a'$. Now applying Step 1 to $a - a'$ and a' finishes the proof since the cup product $\cup$ is bilinear and r_v is linear, so $a|_D$, the element defining the Artin–Schreier double cover induced by the conic bundle, is equal to the residue of the conic bundle along D in general. $\square$

Lemma 3.9. *Let R be a complete discrete valuation ring with field of fractions K and let K^{nr} be the maximal unramified extension of K, as before. Let R^{nr} be the integral closure of R in K^{nr}. Then $\mathrm{Br}(R) = H^2(\mathrm{Gal}(K^{\mathrm{nr}}/K), R^{\mathrm{nr}\times})$.*

Proof. This is contained in[AB68], see the proof at the top of page 289, combined with the remark in §3, and the first sentence of the proof of Theorem 3.1. $\square$

Theorem 3.10. *Let X be a smooth and projective variety over an algebraically closed field k of characteristic p. Assume $\alpha \in \mathrm{Br}(k(X))\{p\}$ is such that the residue $r_{v_D}(\alpha)$ is defined in the sense of Definition 3.2 and is trivial for all divisorial valuations v_D corresponding to prime divisors D on X. Then $\alpha \in \mathrm{Br}_{\mathrm{nr}}(k(X)) = \mathrm{Br}(X)$.*

If $Z \subset X$ is an irreducible subvariety with local ring $\mathscr{O}_{X,Z}$ and the assumption above is only required to hold for all prime divisors D passing through Z, the class α comes from $\mathrm{Br}(\mathscr{O}_{X,Z})$.

Proof. Letting $K = k(X)$ and $v = v_D$, it suffices to show that $\alpha \in \mathrm{Br}(A_v) \subset \mathrm{Br}(K)$. Keeping the notation of Definition 3.2 we have an exact sequence

$$H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), A_v^{\mathrm{nr}\times})\{p\} \longrightarrow H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), K_v^{\mathrm{nr}\times})\{p\} \xrightarrow{r_v} H^1(k(v), \mathbb{Q}/\mathbb{Z})\{p\}$$

resulting from the exact sequence of coefficients $1 \to A_v^{\mathrm{nr}\times} \to K_v^{\mathrm{nr}\times} \to \mathbb{Z} \to 1$ where $A_v^{\mathrm{nr}\times}$ is the valuation ring, inside of K_v^{nr}, of the extension of v to K_v^{nr}. Thus it suffices to show that classes in $\mathrm{Br}_{\mathrm{tame},v}(K)\{p\} \subset \mathrm{Br}(K)\{p\}$ that, under the map

$$\mathrm{Br}_{\mathrm{tame},v}(K)\{p\} \to H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), K_v^{\mathrm{nr}\times})\{p\} \subset \mathrm{Br}(K_v)\{p\},$$

land in the image of $H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), A_v^{\mathrm{nr}\times})\{p\}$ actually come from $\mathrm{Br}(\mathscr{O}_{X,\xi_D})\{p\}$ where $\mathscr{O}_{X,\xi_D}$ is the local ring of D in $k(X)$. Now, by Lemma 3.9, we have

$$H^2(\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v), A_v^{\mathrm{nr}\times}) \simeq \mathrm{Br}(A_v)$$

A class γ in $\mathrm{Br}(K)$ whose image γ_v in $\mathrm{Br}(K_v)$ is contained in $\mathrm{Br}(A_v)$ comes from the valuation ring $A = \mathscr{O}_{X,\xi_D}$ of v in K by Lemma 3.11 below, hence is unramified. $\square$

Lemma 3.11. *Let K be the function field of an algebraic variety and v a discrete rank 1 valuation of K. Let $A \subset K$ be the valuation ring, let K_v be the completion of K with respect to v, and let $A_v \subset K_v$ be the valuation ring of the extension of v to K_v. Then a Brauer class $\alpha \in \mathrm{Br}(K)$ whose image in $\mathrm{Br}(K_v)$ comes from a class $\alpha^\sharp \in \mathrm{Br}(A_v)$ is already in the image of $\mathrm{Br}(A)$.*

Proof. This is a special case of [Ha67, Lemma 4.1.3] or [CTPS12, Lemma 4.1], but we include a proof for completeness.

Suppose the class α is represented by an Azumaya algebra $\mathscr{A}$ over K, and that $\alpha^\sharp$ is represented by an Azumaya algebra $\mathscr{B}$ over A_v. By assumption, $\mathscr{A}$ and $\mathscr{B}$ become Brauer equivalent over K_v, and we can assume that they even become isomorphic over K_v by replacing $\mathscr{A}$ and $\mathscr{B}$ by matrix algebras over them so that they have the same degree. Let $\mathscr{A}_A$ be a maximal A-order of the algebra $\mathscr{A}$ in the sense of Auslander–Goldman [AG60], which means that $\mathscr{A}_A$ is a subring of $\mathscr{A}$ that is finitely generated as an A-module, spans $\mathscr{A}$ over K and is maximal with these properties.

We seek to prove that $\mathscr{A}_A$ is Azumaya. Now we know that the base change $(\mathscr{A}_A)_{A_v}$ is a maximal order, but also any Azumaya A_v-algebra is a maximal order, and by [AG60, Proposition 3.5], any two maximal orders over a rank 1 discrete valuation ring are conjugate, so in fact the base change $(\mathscr{A}_A)_{A_v}$ is Azumaya because $\mathscr{B}$ is. But then this implies that $\mathscr{A}_A$ is Azumaya since A_v is faithfully flat over A, so if the canonical algebra homomorphism $\mathscr{A}_A \otimes \mathscr{A}_A \to \mathrm{End}(\mathscr{A}_A)$ becomes an isomorphism over A_v, it is already an isomorphism over A. □

4. Discriminant profiles of conic bundles in characteristic two: an instructive example

In this Section, we work over an algebraically closed ground field k. First k may have arbitrary characteristic, later we will focus on the characteristic two case. Let X be a conic bundle over a smooth projective base B as in Definition 2.6.

Definition 4.1. We denote by $B^{(1)}$ the set of all valuations of $k(B)$ corresponding to prime divisors on B. A conic bundle $\pi\colon X \to B$ determines a Brauer class $\alpha \in \mathrm{Br}(k(B))[2]$. Moreover, we have natural maps, for $\mathrm{char}(k) \neq 2$,

$$\mathrm{Br}(k(B))[2] \xrightarrow{\oplus \partial_v} \bigoplus_{v \in B^{(1)}} H^1(k(v), \mathbb{Z}/2) \simeq \bigoplus_{v\in B^{(1)}} k(v)^\times / k(v)^{\times 2}$$

where ∂_v are the usual residue maps as in, for example, [GS06, Chaper 6], see also [Pi16, §3.1]; and for $\mathrm{char}(k) = 2$,

$$\mathrm{Br}(k(B))[2] \xrightarrow{\oplus r_v} \bigoplus_{v \in B^{(1)}} H^1(k(v), \mathbb{Z}/2) \simeq \bigoplus_{v\in B^{(1)}} k(v)/\wp(k(v))$$

where r_v is the residue map as in Definition 3.2, provided it is defined for α. In both of these case, we call the image of α in $\bigoplus_{v\in B^{(1)}} k(v)^\times / k(v)^{\times 2}$ in the first case, and in $\bigoplus_{v\in B^{(1)}} k(v)/\wp(k(v))$ in the second case, the *residue profile* of the conic bundle $\pi\colon X \to B$. Note that the valuations v for which the component in $H^1(k(v), \mathbb{Z}/2)$ of the residue profile of a conic bundle is nontrivial are a (possibly proper) subset of the divisorial valuations corresponding to the discriminant components of the conic bundle.

One main difference between characteristic not equal to 2 and equal to 2 (besides the fact that the residue profiles are governed by Kummer theory in the first case and by Artin–Schreier theory in the second case) is the following: for $\mathrm{char}(k) \neq 2$ and B, for concreteness and simplicity of exposition, a smooth projective rational surface, the residue profiles of conic bundles that can occur can be characterised as kernels of another explicit morphism, induced by further residues; more precisely, there is an exact sequence

(8)

$$0 \longrightarrow \mathrm{Br}(k(B))[2] \xrightarrow{\oplus \partial_v} \bigoplus_{v \in B^{(1)}} H^1(k(v), \mathbb{Z}/2) \xrightarrow{\oplus \partial_p} \bigoplus_{p \in B^{(2)}} \mathrm{Hom}(\mu_2, \mathbb{Z}/2)$$

where $B^{(2)}$ is the set of codimension 2 points of B, namely, the closed points when S is a surface, see [A-M72, Theorem 1], [Pi16, Proposition 3.9], but also the far-reaching generalization via Bloch–Ogus–Kato complexes in [Ka86]. The maps ∂_p are also induced by residues, more precisely, if $C \subset B$ is a curve, $p \in C$ a point in the smooth locus of C, then

$$\partial_p\colon H^1(k(C), \mathbb{Z}/2) = k(C)^\times / k(C)^{\times 2} \to \mathrm{Hom}(\mu_2, \mathbb{Z}/2)_p \simeq \mathbb{Z}/2$$

is just the valuation taking the order of zero or pole of a function in $k(C)^\times/k(C)^{\times 2}$ at p, modulo 2 (if C is not smooth at p, one has to make a slightly more refined definition involving the normalisation).

One has the fundamental result of de Jong [deJ04], [Lieb08, Theorem 4.2.2.3] (though for 2-torsion classes, it was proved earlier by Artin [Artin82, Theorem 6.2]) that for fields of transcendence degree 2 over an algebraically closed ground field k (of any characteristic), the period of a Brauer class equals the index, hence that every class in $\mathrm{Br}(k(B))[2]$ can be represented by a quaternion algebra, i.e., by a conic bundle over an open part of B.

However, in characteristic 2, we cannot expect a sequence that naïvely has similar exactness properties as the one in (8). The following example, which is essential for the proof of Theorem 6.2 since it is used to produce a nontrivial Brauer class, exhibits this phenomenon.

Example 4.2. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the conic bundle defined by an equation

$$Q = ax^2 + axz + by^2 + byz + cz^2 = 0,$$

where x, y, z are fiber coordinates in the "fiber copy" $\mathbb{P}^2$ in $\mathbb{P}^2 \times \mathbb{P}^2$, and a, b, c are general linear forms in the homogeneous coordinates u, v, w on the base $\mathbb{P}^2$. Then the discriminant is of degree 3 and consists of the three lines given by $a = 0$, $b = 0$ and $a = b$, which intersect in the point $P = (0 : 0 : 1)$. Indeed, if we want to find the points with coordinates $(u : v : w)$ on the base such that the fiber of the conic bundle over this point is singular, in other words, is such that there exist $(x : y : z)$ in $\mathbb{P}^2$ satisfying

$$Q_x = az = 0, \quad Q_y = bz = 0, \quad Q_z = by + ax = 0$$

and also $ax^2 + axz + by^2 + byz + cz^2 = 0$ (Euler's relation does not automatically imply the vanishing of the equation of the conic itself because the characteristic is two), then we have to look for those points $(u : v : w)$ where

$$\begin{pmatrix} a & b & 0 \\ 0 & 0 & a \\ 0 & 0 & b \\ \sqrt{a} & \sqrt{b} & \sqrt{c} \end{pmatrix}$$

has rank less than or equal to 2, which, on quick inspection, means $a = 0, b = 0$, or $a = b$. The rank drops to 1 only at the point P.

More precisely, the conic bundle induces Artin–Schreier double covers ramified only in P on each of those lines: For $a = 0$ we have

$$Q_{a=0} = b(y^2 + yz) + cz^2$$

which describes a nontrivial Artin–Schreier cover ramified only at $b = 0$. The same happens on the line $b = 0$ and also on the line $a = b$:

$$\begin{aligned} Q_{a=b} &= a(x^2 + xz + y^2 + yz) + cz^2 \\ &= a\big((x^2 + y^2) + (x + y)z\big) + cz^2 \\ &= a\big((x + y)^2 + (x + y)z\big) + cz^2. \end{aligned}$$

This example captures the étale local structure of conic bundles over a smooth surface in characteristic 2, namely that when multiple components of the residue profile meet in a point over which the conic has rank 1, then at least 3 components must meet (as opposed to in characteristic $\neq 2$, where two components meeting in such a point is the generic behavior).

5. Nontriviality of the unramified Brauer group of a conic bundle threefold in characteristic two

We seek to prove the following result.

Theorem 5.1. *Let k be an algebraically closed field of characteristic 2 and B a smooth projective surface over k. Let $\pi\colon X \longrightarrow B$ be a conic bundle with discriminant $\Delta = \cup_{i\in I}\Delta_i$ (as in Definition 2.8) with irreducible components Δ_i. Suppose that the conic bundle is tamely ramified over each Δ_i (as in Theorem 3.6b). Let $\alpha_i \in H^1(k(\Delta_i), \mathbb{Z}/2) = k(\Delta_i)/\wp(k(\Delta_i))$ be the element determined by the Artin–Schreier double cover induced by π over Δ_i.*

Assume that one can write $I = I_1 \sqcup I_2$ with both I_1, I_2 nonempty such that:

a) There exists a tamely ramified conic bundle $\psi\colon Y \to B$ that induces a Brauer class in $\mathrm{Br}(k(B))$ with residue profile (as in Definition 4.1) given by $(\alpha_i)_{i\in I_1} \in \bigoplus_{i\in I_1} H^1(k(\Delta_i), \mathbb{Z}/2)$, and such that for any point P in the intersection of some Δ_i and Δ_j, where $i \in I_1, j \in I_2$, the fiber Y_P is a cross of lines.

b) There exist $i_0 \in I_1$ and $j_0 \in I_2$ such that α_{i_0} and α_{j_0} are nontrivial.

Then $\mathrm{Br}_{\mathrm{nr}}(k(X))[2]$ is nontrivial.

Note that, by the discussion following Example 4.2, the assumption *a)* seems hard to replace by something more cohomological or syzygy-theoretic.

Proof. Let us start the proof with a preliminary remark. By the work of Cossart and Piltant [CP08], [CP09], resolution of singularities is known for quasiprojective threefolds in arbitrary characteristic. Then a smooth projective model $\widetilde{X}$ of X always exists and we have $\mathrm{Br}_{\mathrm{nr}}(k(X))[2] = \mathrm{Br}(\widetilde{X})[2]$. Still, in all applications, for example in Section 6, we will exhibit such a resolution explicitly.

By a result of Witt [Witt35], cf. [GS06, Theorem 5.4.1], the kernel of the natural homomorphism

$$\pi^*\colon \mathrm{Br}(k(B)) \to \mathrm{Br}(k(X))$$

is generated by the class of the conic bundle $X \to B$ itself. Denote by α that class in $\mathrm{Br}(k(B))$. Denote by β the class of $\psi\colon Y \to B$ in $\mathrm{Br}(k(B))$. We claim that $\pi^*(\beta) \in \mathrm{Br}(k(X))$ is nontrivial and unramified. It is nontrivial because $\beta \neq \alpha$ by assumption *b)*: α and β have different residues along some irreducible component Δ_{j_0} of Δ. Now to check that $\pi^*(\beta)$ is unramified, it suffices to check that for any valuation $v = v_D$ corresponding to a prime divisor D on a model $X' \to X$ which is smooth generically along D we have that $\pi^*(\beta)$ is unramified with respect to that valuation, in the sense that it is in the image of $\mathrm{Br}(\mathscr{O}_{X',D})$. Let $\Delta^{(1)} = \bigcup_{i\in I_1}\Delta_i$, $\Delta^{(2)} = \bigcup_{j\in I_2}\Delta_j$. There are two cases to distinguish:

(i) The centre Z of v on B, in other words the image of D on B, is not contained in $\Delta^{(1)} \cap \Delta^{(2)}$. In general, notice that the in general only partially defined residue map is defined for the classes β and α with respect to any divisor D' on the base B by the assumption on the geometric generic fibers of $X \to B$ over discriminant components and by Theorem 3.6. Moreover, if the centre Z is not contained in $\Delta^{(1)} \cap \Delta^{(2)}$, then β or $\beta - \alpha$ has residue zero along every divisor D' on B passing through Z. By Theorem 3.10, the class $\beta - \alpha$ comes from $\mathrm{Br}(\mathscr{O}_{B,Z})$. But $\pi^*(\beta-\alpha) = \pi^*(\beta)$, and hence $\pi^*(\beta)$ comes from $\mathrm{Br}(\mathscr{O}_{X',D})$ as desired.

(ii) The centre Z of v on B is contained in $\Delta^{(1)} \cap \Delta^{(2)}$, hence a point $Z = P$ over which the fiber Y_P is a cross of lines by the assumption in *a)* of the Theorem. Then the class $\pi^*(\beta)$ is represented by a conic bundle on X' whose residue along D is defined and trivial by Theorem 3.6. So $\pi^*(\beta)$ comes from $\mathrm{Br}(\mathscr{O}_{X',D})$ as desired by Theorem 3.10 again.

Thus $\pi^*(\beta) \in \mathrm{Br}_{\mathrm{nr}}(k(X))[2]$ is a nontrivial class. □

6. Examples of conic bundles in characteristic two with nontrivial Brauer groups

Definition 6.1. Consider the following symmetric matrix defined over $\mathbb{Z}$

$$S = \begin{pmatrix} 2uv + 4v^2 + 2uw + 2w^2 & u^2 + uw + w^2 & uv \\ u^2 + uw + w^2 & 2u^2 + 2vw + 2w^2 & u^2 + vw + w^2 \\ uv & u^2 + vw + w^2 & 2v^2 + 2uw + 2w^2 \end{pmatrix}.$$

The bihomogeneous polynomial

$$(x, y, z)S(x, y, z)^t$$

is divisible by 2. Let $X \subset \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$ be the conic bundle defined by

$$\frac{1}{2}(x, y, z)S(x, y, z)^t = 0.$$

Here we denote by $(u : v : w)$ the coordinates of the first (base) $\mathbb{P}^2_{\mathbb{Z}}$ and by $(x : y : z)$ the coordinates of the second (fiber) $\mathbb{P}^2_{\mathbb{Z}}$.

The determinant of S is divisible by 2 so

$$D = \frac{1}{2}\det S$$

is still a polynomial over $\mathbb{Z}$. Its vanishing defines the discriminant Δ of X in the sense of Definition 2.8. We denote by $X_{(p)}$ the conic bundle over $\overline{\mathbb{F}}_p$ defined by reducing the defining equation of X modulo p. It has discriminant $\Delta_{(p)}$ defined by the reduction of D modulo p.

This example was found using the computer algebra system `Macaulay2` [M2] and Jakob Krökers Macaulay2 packages `FiniteFieldExperiments` and `BlackBoxIdeals` [Kr15].

Our aim is to prove the following result whose proof will take up the remainder of this Section.

Theorem 6.2. *The conic bundle $X \to \mathbb{P}^2$ has smooth total space that is not stably rational over $\mathbb{C}$. More precisely, X has the following properties:*

a) The discriminant $\Delta_{(p)}$ is irreducible for $p \neq 2$, hence

$$\mathrm{Br}_{\mathrm{nr}}\left(\overline{\mathbb{F}}_p\left(X_{(p)}\right)\right) = 0 \quad \textit{for } p \neq 2.$$

b) The conic bundle $X_{(2)}$ satisfies the hypotheses of Theorem 5.1, hence

$$\mathrm{Br}_{\mathrm{nr}}\left(\overline{\mathbb{F}}_2\left(X_{(2)}\right)\right)[2] \neq 0.$$

c) There is a CH_0-universally trivial resolution of singularities $\sigma\colon \widetilde{X}_{(2)} \to X_{(2)}$.

Notice that the degeneration method of [CT-P16] (and [Voi15] initially, see also [To16]) shows that *b*) and *c*) imply that X is not stably rational over $\mathbb{C}$: indeed, we have $\mathrm{Br}(\widetilde{X}_{(2)}) = \mathrm{Br}_{\mathrm{nr}}\left(\overline{\mathbb{F}}_2\left(X_{(2)}\right)\right) \neq 0$, because of *b*); then [ABBB18, Theorem 1.1] yields that $\widetilde{X}_{(2)}$ is not CH_0-universally trivial. Finally, [CT-P16, Théorème 1.14] implies that X is not retract rational, in particular not stably rational, over $\overline{\mathbb{Q}}$, which is equivalent to saying it is not stably rational over any algebraically closed field of characteristic 0, see [KSC04, Proposition 3.33].

Moreover, item *a*) shows that the degeneration method, using reduction modulo $p \neq 2$ and the unramified Brauer group, cannot yield this result. This follows from work of Colliot-Thélène, see [Pi16, Theorem 3.13, Remark 3.14]; note that one only has to assume X is a threefold which is nonsingular in codimension 1 in [Pi16, Theorem 3.13]. Likewise, usage of differential forms as in [A-O16], see in particular their Theorem 1.1 and Corollary 1.2, does not imply the result either.

6.a. **Irreducibility of $\Delta_{(p)}$ for $p \neq 2$.** When we speak about irreducibility or reducibility in the following, we always mean geometric irreducibility or reducibility. Our first aim is to prove that $\Delta_{(p)}$ is irreducible for $p \neq 2$. This is easy for generic p since X is smooth over $\mathbb{Q}$ (by a straight-forward Gröbner basis computation [ABBBM2]). Since being singular is a codimension 1 condition, we expect that $\Delta_{(p)}$ is singular for a finite number of primes. So we need a more refined argument to prove irreducibility. Our idea is to prove that there is (counted with multiplicity) at most one singular point for each $p \neq 2$.

Lemma 6.3. *Let C be a reduced and reducible plane curve of degree at least 3 over an algebraically closed field. Then the length of the singular subscheme, defined by the Jacobi ideal on the curve, is at least 2.*

Proof. The only singularities of length 1 are those where étale locally two smooth branches of the curve cross transversely: if $f(x, y) = 0$ is a local equation for C with isolated singular point at the origin, then the length can only be 1 if

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

have leading terms consisting of linearly independent linear forms. This means two smooth branches cross transversely. The only reducible curve that has only one transverse intersection is the union of two lines. □

We also need the following technical lemma.

Lemma 6.4. *Let $I \subset \mathbb{Z}[u, v, w]$ be a homogeneous ideal, $B = \{l_1, \dots, l_k\}$ a $\mathbb{Z}$-Basis of the space of linear forms $I_1 \subset I$, and M the $k \times 3$ matrix of coefficients of the l_i. Let g be the minimal generator of the ideal of 2×2 minors of M in $\mathbb{Z}$.*

If a prime p does not divide g, then I defines a finite scheme of degree at most 1 in characteristic p.

Proof. If p does not divide g there is at least one 2×2 minor m with $p \nmid m$. Therefore in characteristic p this minor is invertible and the matrix has rank at least 2. It follows that I contains at least 2 independent linear forms in characteristic p and therefore the vanishing set is either empty or contains 1 reduced point. □

Remark 6.5. Notice that the condition $p \nmid g$ is sufficient, but not necessary. For example the ideal (u^2, v^2, w^2) vanishes nowhere, but still has $g = 0$ and therefore $p|g$. The condition becomes necessary if I is saturated.

Proposition 6.6. *For $p \neq 2$, $\Delta_{(p)}$ is an irreducible sextic curve.*

Proof. We apply Lemma 6.4 to the saturation of $(D, \frac{dD}{du}, \frac{dD}{dv}, \frac{dD}{dw}) \subset \mathbb{Z}[u, v, w]$. A Macaulay2 computation gives $g = 2^{10}$ [ABBBM2]. So we have at most one singular point over $p \neq 2$ and therefore $\Delta_{(p)}$ is irreducible. □

6.b. **The unramified Brauer group of $X_{(2)}$ is nontrivial.** Let us now turn to characteristic $p=2$. We will prove the nontriviality of the unramified Brauer group by verifying the conditions of Theorem 5.1.

Proposition 6.7. *We have*

$$D \equiv uw(u+w)(\gamma u + v^3) \mod 2.$$

with $\gamma = v^2+uv+vw+w^2$. Furthermore

- *γ does not vanish at $(0:0:1)$.*
- *$\gamma u + v^3 = 0$ defines a smooth elliptic curve $E \subset \mathbb{P}^2_{\overline{\mathbb{F}}_2}$.*
- *E does not contain the intersection point $(0:1:0)$ of the three lines.*
- *The intersection of E with each of the lines $w=0$ and $u+w=0$ is transverse.*
- *The line $u=0$ is an inflectional tangent to E at the point $(0:0:1)$.*

Proof. All of this is a straight forward computation. See [ABBBM2]. □

The next lemma gives us a criterion for the irreducibility of the Artin–Schreier double covers induced on the discriminant components and hence for the nontriviality of the residues of the conic bundle along these components.

Lemma 6.8. *Let $\pi\colon \overline{X} \to \mathbb{P}^2$ be a conic bundle defined over $\mathbb{F}_2$. Let $C \subset \mathbb{P}^2$ be an irreducible curve over $\mathbb{F}_2$, over which the fibers of $\overline{X}$ generically consist of two distinct lines. Let $\widetilde{C} \to C$ be the natural double cover of C induced by π. Then $\widetilde{C}$ is irreducible if the following hold:*

- *There exists an $\mathbb{F}_2$-rational point $p_1 \in C$ such that the fiber of $\overline{X}$ over p_1 splits into two lines defined over $\mathbb{F}_2$.*
- *There exists an $\mathbb{F}_2$-rational point $p_2 \in C$ such that the fiber of $\overline{X}$ over p_2 is irreducible over $\mathbb{F}_2$ but splits into two lines over $\overline{\mathbb{F}}_2$.*

Proof. Under the assumptions the double cover $\widetilde{C} \to C$ is defined over $\mathbb{F}_2$. Suppose, by contradiction, that $\widetilde{C}$ were (geometrically) reducible. Then the Frobenius morphism F would either fix each irreducible component of $\widetilde{C}$ as a set, or interchange the two irreducible components. But since C is defined over $\mathbb{F}_2$, this would mean that F either fixes each of the two lines as a set in every fiber over a $\mathbb{F}_2$-rational point of the base, or F interchanges the two lines in every fiber over a $\mathbb{F}_2$-rational point. This contradicts the existence of p_1, p_2. □

Proposition 6.9. *We consider the fibers of $X_{(2)}$ over the $\mathbb{F}_2$-rational points of the base $\mathbb{P}^2_{\overline{\mathbb{F}}_2}$ and obtain the following table:*

point	*fiber*	u	w	$u+w$	$\gamma u+v^3$
$(0:1:0)$	1 *double line*	$\times$	$\times$	$\times$	
$(0:1:1)$	2 *rational lines*	$\times$			
$(1:0:0)$	2 *rational lines*		$\times$		$\times$
$(1:0:1)$	2 *rational lines*			$\times$	
$(0:0:1)$	2 *conjugate lines*	$\times$			$\times$
$(1:1:0)$	2 *conjugate lines*		$\times$		
$(1:1:1)$	2 *conjugate lines*			$\times$	

Here if a $\mathbb{F}_2$-rational point lies on a particular component of the discriminant, we put an '$\times$' in the corresponding row and column.

Proof. All of this is again a straight forward computation. See [ABBBM2]. □

Corollary 6.10. *The conic bundle $X_{(2)}$ induces a nontrivial Artin–Schreier double cover on each component of the discriminant locus. In particular, condition b) of Theorem 5.1 is satisfied if we let I_1 index the three lines of $\Delta_{(2)}$ and I_2 the elliptic curve E.*

Proof. Use Lemma 6.8 and Proposition 6.9. □

We now want to check that the $2:1$ covers induced by $X_{(2)}$ over the three lines yield the same element in $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2)$ as the $2:1$ covers in our Example 4.2. We need this to verify condition *a)* of Theorem 5.1. We only have to check that all these covers are birational to each other over the base $\mathbb{P}^1$. For this we use:

Proposition 6.11. *Let $\overline{X} \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a divisor of bidegree $(d, 2)$, considered as a conic bundle over $\mathbb{P}^2$ via the first projection, and all defined over the ground field $k = \overline{\mathbb{F}}_2$. Assume that the discriminant of the conic bundle has a line L as an irreducible component, and assume that the subscheme of double lines in L is a reduced point R. Furthermore, assume that $\overline{X}$, L and R are defined over $\mathbb{F}_2$.*

Then either the $2:1$ cover defined over L by $\overline{X}|_L$ is trivial, or it is birational over L to the Artin–Schreier cover

$$x^2 + x + \frac{v}{u} = 0$$

where $(u : v)$ are coordinates on $L \cong \mathbb{P}^1_{\mathbb{F}_2}$ and $R = (0 : 1)$. In particular all nontrivial covers arising geometrically in the way described above yield the same element in $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2) = H^1(\overline{\mathbb{F}}_2(L), \mathbb{Z}/2)$.

Remark 6.12. The hypothesis that $\overline{X}$, L, and R must be defined over $\mathbb{F}_2$, which at first sight may seem unnatural, cannot be dropped. More generally, the proof shows, cf. also Remark 6.13, that either the cover would be trivial or would induce an element in $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2)$ of the form γ/t with $\gamma \in \overline{\mathbb{F}}_2^\times$. However, $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2)$ is nothing but the additive group of $\overline{\mathbb{F}}_2(t)$ modulo the image of the Artin–Schreier map $y \mapsto y^2 + y$, and γ/t and γ'/t, for $\gamma, \gamma' \in \overline{\mathbb{F}}_2^\times$, will define distinct elements of $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2)$ when $\gamma \neq \gamma'$. Algebraically, the group $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2)$ classifies degree 2 Galois extensions of $\overline{\mathbb{F}}_2(t)$ up to isomorphism over $\overline{\mathbb{F}}_2(t)$, cf. Remark 3.3. Geometrically, $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2)$ can be thought of as parametrizing separable (i.e., generically étale) double covers $C \to \mathbb{P}^1$, up to birational isomorphism *over* $\mathbb{P}^1$ in the sense that $C_1 \to \mathbb{P}^1$ and $C_2 \to \mathbb{P}^1$ are considered the same if there is a diagram

$$\begin{tikzcd} C_1 \arrow[rr, dashed, "\phi"] \arrow[dr] & & C_2 \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

with ϕ birational. However, if we drop the assumption that $\overline{X}$, L and R be defined over $\mathbb{F}_2$, we could only conclude in the above situation that two nontrivial covers $C_1 \to \mathbb{P}^1$ and $C_2 \to \mathbb{P}^1$ arising as in the Proposition would be related by a diagram

$$\begin{tikzcd} C_1 \arrow[r, dashed, "\phi"] \arrow[d] & C_2 \arrow[d] \\ \mathbb{P}^1 \arrow[r, dashed, "\phi'"] & \mathbb{P}^1 \end{tikzcd}$$

with ϕ, ϕ' birational, but where ϕ' is not necessarily the identity. More concisely, under the weaker assumptions we could conclude that the covers are *square-birational*, but not birational *over* $\mathbb{P}^1$. However, that would not be sufficient for our purposes, since to apply Theorem 5.1 to get a nontrivial Brauer class in $\mathrm{Br}_{\mathrm{nr}}(\overline{\mathbb{F}}_2(X_{(2)}))[2]$, we need to check that certain residues, which live in $H^1(\overline{\mathbb{F}}_2(t), \mathbb{Z}/2)$, are *the same* (for the standard "ramification kills ramification" argument to apply). We achieve this by assuming $\overline{X}$, L and R are defined over $\mathbb{F}_2$, whence γ will actually be in $\mathbb{F}_2$, hence 1 if nontrivial, as shown in the proof of Proposition 6.11 below.

Proof. Note that $\overline{X}|_L \to L$ is defined over $\mathbb{F}_2$, and defines a double cover $\pi\colon Y \to L$, where Y is the relative Grassmannian of lines in the fibers of $\overline{X}|_L \to L$. Then $Y \to L$ is also defined over $\mathbb{F}_2$ and flat over L. Hence $\mathscr{E} = \pi_*(\mathscr{O}_Y)$ is a rank 2 vector bundle on L, and Y can be naturally embedded into $\mathbb{P}(\mathscr{E})$. Then $\mathscr{E} = \mathscr{O}_L(e) \oplus \mathscr{O}_L(f)$, and Y is defined inside $\mathbb{P}(\mathscr{E})$ by an equation

$$ax^2 + bxy + cy^2 = 0$$

with a, b, c homogeneous polynomials with $\deg(a) + \deg(c) = 2\deg(b)$. Notice that $b = 0$ defines the locus of points of the base L over which the fiber is a double point. By our assumption $b = 0$ is a single reduced point. Hence $\deg(b) = 1$.

Notice that if the double cover Y is nontrivial, both a and c are nonzero, hence $\deg(a) \geq 0, \deg(c) \geq 0$ and $\deg(a) + \deg(c) = 2$.

Let $(u : v)$ be homogeneous coordinates on L. We now put $a' = a/u^{\deg(a)}, b' = b/u^{\deg(b)}, c' = c/u^{\deg c}$ and calculate over the function field of L. Apply $(x, y) \mapsto (b'x, a'y)$ to obtain

$$a'(b')^2x^2 + a'(b')^2xy + (a')^2c'y^2 = 0.$$

Divide by $a'(b')^2$ and dehomogenise via $y \mapsto 1$ to obtain the Artin–Schreier normal form

$$x^2 + x + \frac{ac}{b^2} = 0.$$

We now use the fact that we can choose the coordinates $(u : v)$ such that $b = u$. We can write $ac = \alpha u^2 + \beta uv + \gamma v^2$ with $\alpha, \beta, \gamma \in \mathbb{F}_2$:

$$x^2 + x + \alpha + \beta\frac{v}{u} + \gamma\frac{v^2}{u^2} = 0.$$

Now we use extensively the fact that we work over $\mathbb{F}_2$: firstly, either $\alpha = 0$ or $\alpha = 1$. In the second case let $\rho \in \overline{\mathbb{F}}_2$ be a root of $x^2 + x + 1$ and apply $x \mapsto x + \rho$. This gives

$$x^2 + x + \beta\frac{v}{u} + \gamma\frac{v^2}{u^2} = 0$$

in both cases. Even though the transformation was defined over $\overline{\mathbb{F}}_2$ this does not change the fact that β and γ are in $\mathbb{F}_2$.

Secondly either $\gamma = 0$ and we have

$$x^2 + x + \beta\frac{v}{u} = 0$$

or $\gamma = 1$ and we apply $x \mapsto x + \frac{v}{u}$ to obtain

$$x^2 + x + (\beta + 1)\frac{v}{u} = 0.$$

In both cases the coefficient in front of $\frac{v}{u}$ is either 0 or 1, thus the cover is either trivial or has the normal form

$$x^2 + x + \frac{v}{u} = 0.$$

□

Remark 6.13. Notice that the proof works over any field k of characteristic 2 until we have

$$x^2 + x + \beta\frac{v}{u} + \gamma\frac{v^2}{u^2} = 0.$$

Now we can eliminate γ only if it is a square in k. Even if this happens (for example if we work over $\overline{\mathbb{F}}_2$) we obtain, using $x \mapsto x + \sqrt{\gamma}(v/u)$,

$$x^2 + x + \underbrace{\left(\beta + \sqrt{\gamma}\right)}_{\beta'}\frac{v}{u} = 0.$$

So there seems to be a 1-dimensional moduli space of such covers.

Note that Propositions 6.7, 6.9, Corollary 6.10, and Proposition 6.11 together with the conic bundle exhibited in Example 4.2 show that Theorem 5.1 is applicable in the case of $X_{(2)}$, hence $\mathrm{Br}_{\mathrm{nr}}(\overline{\mathbb{F}}_2(X_{(2)}))[2] \neq 0$.

6.c. **A CH_0-universally trivial resolution of $X_{(2)}$.** Now we conclude the proof of Theorem 6.2 by showing the remaining assertion *c)*, the existence of a CH_0-universally trivial resolution of singularities $\sigma\colon \widetilde{X}_{(2)} \to X_{(2)}$.

We will use the following criterion [Pi16, Example 2.5 (1),(2),(3)] which summarizes results of [CT-P16] and [CTP16-2].

Proposition 6.14. *A sufficient condition for a projective morphism $f\colon V \to W$ of varieties over a field k to be CH_0-universally trivial is that the fiber V_ξ of f over every scheme-theoretic point ξ of W is a (possibly reducible) CH_0-universally trivial variety over the residue field $\kappa(\xi)$ of the point ξ. This sufficient condition in turn holds if X_ξ is a projective (reduced) geometrically connected variety, breaking up into irreducible components X_i such that each X_i is CH_0-universally trivial and geometrically irreducible, and such that each intersection $X_i \cap X_j$ is either empty or has a zero-cycle of degree 1 (of course the last condition is automatic if $\kappa(\xi)$ is algebraically closed).*

Moreover, a smooth projective retract rational variety Y over any field is universally CH_0-trivial. If Y is defined over an algebraically closed ground field, one can replace the smoothness assumption on Y by the requirement that Y be connected and each component of Y^{red} be a rational variety with isolated singular points.

We now study the behaviour of $X_{(2)}$ locally above a point P on the base $\mathbb{P}^2$, distinguishing several cases; for the cases when $X_{(2)}$ is singular locally above P, we exhibit an explicit blow-up scheme to desingularise it, with exceptional locus CH_0-universally trivial, so that Proposition 6.14 applies.

(i) The case when $P \notin \Delta_{(2)}$. In that case, $X_{(2)}$ is nonsingular locally above P.

(ii) The case when P is in the smooth locus of $\Delta_{(2)}$. In that case, $X_{(2)}$ is nonsingular locally above P as well. This can be seen by direct computation [ABBBM2].

(iii) The case when $P = (0:1:0)$ is the intersection point of the three lines $(u=0)$, $(w=0)$, $(u+w=0)$ in $\Delta_{(2)}$. A direct computation shows that here $X_{(2)}$ is nonsingular locally above P as well [ABBBM2].

(iv) The case when P is one of the six intersection points of $w=0$ or $u+w=0$ with E. In these cases, the intersection of the two discriminant components is transverse, and the fiber above the intersection point is a cross of lines. Then $X_{(2)}$ locally above P has a CH_0-universally trivial desingularization because we have the local normal form as in Lemma 6.15 with $n = 1$, and thus, by Proposition 6.16 and Proposition 6.17, one blow-up with exceptional divisor a smooth quadric resolves the single singular point of $X_{(2)}$ above P.

(v) The case when $P = (0 : 0 : 1)$ is the point where the components $(u = 0)$ and E of $X_{(2)}$ intersect in such a way that $(u = 0)$ is an inflectional tangent to the smooth elliptic curve E. In this case, the fiber of $X_{(2)}$ above P is a cross of two conjugate lines by Proposition 6.9. We need some auxiliary results.

Lemma 6.15. *Let $\hat{\mathbb{A}}^2_{\overline{\mathbb{F}}_2}$ be the completion of $\mathbb{A}^2_{\overline{\mathbb{F}}_2}$ with affine coordinates u, v along $(0,0)$, and let $\overline{X}$ be a conic bundle over $\hat{\mathbb{A}}^2_{\overline{\mathbb{F}}_2}$. Thus $\overline{X}$ has an equation*

$$c_{xx}x^2 + c_{xy}xy + c_{yy}y^2 + c_{xz}xz + c_{yz}yz + c_{zz}z^2 = 0$$

where the c's are formal power series in u and v with coefficients in $\overline{\mathbb{F}}_2$.

Assume that

a) locally around $(0,0)$ the discriminant of $\overline{X}$ has a local equation $u(u + v^n)$, $n \geq 1$.
b) The fiber over $(0,0)$ has the form $x^2 + xy + y^2$

Then, after a change in the fiber coordinates x, y and z, we can assume the normal form

$$x^2 + xy + c_{yy}y^2 + c_{zz}z^2 = 0$$

with c_{yy} a unit, $c_{zz} = \beta u(u + v^n)$ and β a unit.

Proof. Because of assumption (*b*) we can assume that c_{xx} is a unit. After dividing by c_{xx} we can assume that we have the form

$$x^2 + c_{xy}xy + c_{yy}y^2 + c_{xz}xz + c_{yz}yz + c_{zz}z^2 = 0$$

with c_{xy} and c_{yy} units. After the substition of $x \mapsto c_{xy}x$ we can divide the whole equation by c_{xy}^2 and can assume that we have the form

$$x^2 + xy + c_{yy}y^2 + c_{xz}xz + c_{yz}yz + c_{zz}z^2 = 0$$

with c_{yy} a unit. Now substituting $x \mapsto x + c_{yz}z$ and $y \mapsto y + c_{xz}z$ we obtain the normal form

$$x^2 + xy + c_{yy}y^2 + c_{zz}z^2 = 0$$

with c_{yy} still a unit. Now the discriminant of this conic bundle ist c_{zz}. Since the discriminant was changed at most by a unit during the normalization process above, we have $c_{zz} = \beta u(u + v^n)$ as claimed. □

Proposition 6.16. *Let Y be a hypersurface in $\hat{\mathbb{A}}^4_{\overline{\mathbb{F}}_2}$ with coordinates x, y, u, v, with equation*

$$x^2 + xy + \alpha y^2 + \beta u(u + v^n) = 0, \quad n \geq 1,$$

where α and β are units in $\overline{\mathbb{F}}_2[[u, v]]$. Then Y is singular only at the origin.

Let $\widetilde{\mathbb{A}^4}$ be the blow up of $\hat{\mathbb{A}}^4_{\overline{\mathbb{F}}_2}$ in the origin and let $\widetilde{Y} \subset \widetilde{\mathbb{A}^4}$ be the strict transform of Y. If $n = 1$, then $\widetilde{Y}$ is smooth. If $n > 1$, then $\widetilde{Y}$ is singular at only one point, which we can assume to be the origin again. Around this singular point $\widetilde{Y}$ has a local equation

$$x^2 + xy + \alpha' y^2 + \beta' u(u + v^{n-1}) = 0$$

with α' and β' units in $\overline{\mathbb{F}}_2[[u, v]]$.

Proof. In $\widetilde{\mathbb{A}^4}$, we obtain 4 charts. It will turn out that in three of them $\widetilde{Y}$ is smooth and in the fourth we obtain the local equation given above.

a) $(x, y, u, v) \mapsto (x, xy, xu, xv)$ gives

$$x^2 + x^2y + \alpha' x^2y^2 + \beta' xu(xu + x^n v^n) = 0$$

as the total transform, and

$$1 + y + \alpha' y^2 + \beta' u(u + x^{n-1}v^n) = 0$$

as the strict transform. Notice that α' and β' are power series that only involve u, v and x. Therefore the derivative with respect to y is 1 in both cases and the strict transform is smooth in this chart.

b) $(x, y, u, v) \mapsto (xy, y, yu, yv)$ gives

$$x^2y^2 + xy^2 + \alpha' y^2 + \beta' yu(yu + y^n v^n) = 0$$

as the total transform, and

$$x^2 + x + \alpha' + \beta' u(u + y^{n-1}v^n) = 0$$

as the strict transform. Notice that α' and β' are power series that only involve u, v and y. Therefore the derivative with respect to x is 1 in both cases and the strict transform is smooth in this chart.

c) $(x, y, u, v) \mapsto (xu, yu, u, uv)$ gives

$$x^2u^2 + xyu^2 + \alpha' y^2u^2 + \beta' u(u + u^n v^n) = 0$$

as the total transform, and

$$x^2 + xy + \alpha' y^2 + \beta'(1 + u^{n-1}v^n) = 0$$

as the strict transform. Notice that α' and β' are power series that only involve u, v. Therefore the derivative with respect to x and y are y and x respectively. So the singular locus lies on $x = y = 0$. Substituting this into the equation of the strict transform we get

$$\beta'(1 + u^{n-1}v^n) = 0$$

This is impossible since β' and $(1 + u^{n-1}v^n)$ are units. Therefore the strict transform is smooth in this chart.

d) $(x, y, u, v) \mapsto (xv, yv, uv, v)$ gives

$$x^2v^2 + xyv^2 + \alpha' y^2v^2 + \beta' uv(uv + v^n) = 0$$

as the total transform, and

$$x^2 + xy + \alpha' y^2 + \beta' u(u + v^{n-1}) = 0$$

as the strict transform. Notice that α' and β' are power series that only involve u, v. Therefore the derivative with respect to x and y are y and x respectively. So the singular locus lies on $x = y = 0$. Substituting this into the equation of the strict transform we get

$$\beta' u(u + v^{n-1}) = 0.$$

Let us now look at the derivative with respect to u:

$$\frac{d\alpha'}{du} y^2 + \frac{d\beta'}{du} u(u + v^{n-1}) + \beta' v^{n-1} = 0$$

Since $x = y = u(u + v^{n-1}) = 0$ on the singular locus, this equation reduces to $v^{n-1} = 0$. If $n = 1$, this shows that $\widetilde{Y}$ is smooth everywhere. If $n \geq 2$, we obtain that the strict transform is singular at most at $x = y = u = v = 0$ in this chart. To check that this is indeed a singular point we also calculate the derivative with respect to v:

$$\frac{d\alpha'}{dv}y^2 + \frac{d\beta'}{dv}u(u+v^{n-1}) + \beta'(n-1)uv^{n-2} = 0$$

which is automatically satisfied at $x = y = u = v = 0$.

This proves all claims of the proposition. □

Proposition 6.17. *Keeping the notation of Proposition 6.16, the exceptional divisor of $\widetilde{Y} \to Y$ is a quadric with at most one singular point.*

Proof. Recall that the equation of Y is

$$x^2 + xy + \alpha y^2 + \beta u(u + v^n) = 0.$$

We see immediately that the leading term around the origin is

$$x^2 + xy + \alpha_0 y^2 + \beta_0 u^2$$

for $n > 1$ with α_0, β_0 nonzero constants, and

$$x^2 + xy + \alpha_0 y^2 + \beta_0 u^2 + \beta_0 uv$$

for $n = 1$. The first is a quadric cone with an isolated singular point, the second is a smooth quadric. □

Summarizing, we see that Lemma 6.15, Proposition 6.16, and Proposition 6.17 show that, locally around the singular point lying above $P = (0 : 0 : 1)$, the conic bundle $X_{(2)}$ has a resolution of singularities with CH_0-universally trivial fibers. By Proposition 6.14, and taking into account cases (i)-(v) above, we conclude that $X_{(2)}$ has a CH_0-universally trivial resolution of singularities $\sigma\colon \widetilde{X}_{(2)} \to X_{(2)}$. This concludes the proof of Theorem 6.2.

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Asher Auel, Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755, USA

E-mail address: asher.auel@dartmouth.edu

Alessandro Bigazzi, Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

E-mail address: A.Bigazzi@warwick.ac.uk

Christian Böhning, Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

E-mail address: C.Boehning@warwick.ac.uk

Hans-Christian Graf von Bothmer, Fachbereich Mathematik der Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany

E-mail address: hans.christian.v.bothmer@uni-hamburg.de