



Parabolic eigenvarieties via overconvergent cohomology

Daniel Barrera Salazar¹ · Chris Williams²

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Abstract

Let \mathcal{G} be a connected reductive group over \mathbf{Q} such that $G = \mathcal{G}/\mathbf{Q}_p$ is quasi-split, and let $Q \subset G$ be a parabolic subgroup. We introduce parahoric overconvergent cohomology groups with respect to Q , and prove a classically theorem showing that the small slope parts of these groups coincide with those of classical cohomology. This allows the use of overconvergent cohomology at parahoric, rather than Iwahoric, level, and provides flexible lifting theorems that appear to be particularly well-adapted to arithmetic applications. When Q is a Borel, we recover the usual theory of overconvergent cohomology, and our classically theorem gives a stronger slope bound than in the existing literature. We use our theory to construct Q -parabolic eigenvarieties, which parametrise p -adic families of systems of Hecke eigenvalues that are finite slope at Q , but that allow infinite slope away from Q .

Contents

1	Introduction
1.1	Context
1.2	Parabolic families and classically
1.3	Methods and results
1.4	A note on assumptions
1.5	Comparison to the literature
1.6	Acknowledgements
2	Preliminaries and structure theory
2.1	Global notation
2.2	Local notation and root data at p
2.3	Parabolic subgroups
2.4	The Hecke algebra
2.5	Controlling operators
3	Parahoric overconvergent cohomology
3.1	Weight spaces
3.2	Parahoric distributions
3.2.1	Algebraic induction and highest weight representations

✉ Chris Williams
christopher.d.williams@warwick.ac.uk
Daniel Barrera Salazar
daniel.barrera.s@usach.cl

¹ Universidad de Santiago de Chile, Santiago, Chile

² University of Warwick, Coventry, UK

3.2.2	Analytic function spaces
3.2.3	Analytic induction modules
3.2.4	Locally analytic induction at single weights
3.2.5	Integral structures
3.2.6	Analytic functions in families
3.2.7	Distributions in families
3.3	Summary of notation
3.4	The action of Σ_Q and local systems
3.5	Compact operators and slope decompositions
4	Parahoric classicality theorems
4.1	The parahoric classicality theorem
4.2	Analytic BGG for the Borel
4.3	Theta operators on parahoric distributions
4.4	The parahoric analytic BGG resolution
4.5	Proof of Theorem 4.4
4.6	Hecke normalisations and connections to automorphic representations
5	Parabolic eigenvarieties
5.1	Local pieces of the eigenvariety
5.2	The global eigenvariety
5.2.1	Fredholm power series and hypersurfaces
5.2.2	The eigenvariety datum
5.3	Hansen's spectral sequences
5.3.1	Points in eigenvarieties and classical eigensystems
5.3.2	The dimension of components
5.4	Parabolic families of cuspidal automorphic representations
	References

1 Introduction

1.1 Context

Hida and Coleman families describe the variation of automorphic representations as their weight varies p -adically. They have become ubiquitous in many areas of number theory, and are vital tools in the study of the Langlands program and the Bloch–Kato conjectures. Their behaviour is captured geometrically in the theory of *eigenvarieties*. To construct and study an eigenvariety, one requires:

- a rigid analytic *weight space* \mathcal{W} , encoding p -adic analytic variation of weights;
- for each $\lambda \in \mathcal{W}$, a space M_λ that varies analytically in λ , and which carries an action of a suitable Hecke algebra;
- and a notion of ‘classical structure/classicality’, relating finite-slope systems of Hecke eigenvalues appearing in M_λ to those arising from p -refinements of automorphic representations of weight λ .

The eigenvariety is then a rigid analytic space \mathcal{E} , with a weight map $w : \mathcal{E} \rightarrow \mathcal{W}$, whose points lying above a weight λ parametrise finite-slope systems of Hecke eigenvalues that appear in M_λ . Via the classical structure these relate to eigensystems attached to automorphic representations.

Let \mathcal{G} be a connected reductive group over \mathbf{Q} , and suppose $G := \mathcal{G}/\mathbf{Q}_p$ is quasi-split. In this case Hansen [23] has constructed eigenvarieties for \mathcal{G} by taking M_λ to be *overconvergent cohomology groups*; his work generalises earlier constructions of Ash–Stevens and Urban [1,42]. Cohomological automorphic representations of $\mathcal{G}(\mathbf{A})$ of weight λ arise in the cohomology of locally symmetric spaces S_K for \mathcal{G} , of level K , with coefficients in an algebraic representation V_λ^\vee of weight λ . Overconvergent cohomology is defined by replacing V_λ^\vee with

an (infinite-dimensional) module \mathcal{D}_λ^G of p -adic distributions. The classical structure is then furnished by a *classicality theorem*, which says that the ‘non-critical/small slope’ parts of the overconvergent and classical cohomology coincide, so that non-critical slope systems of Hecke eigenvalues in M_λ are classical. Here the *slope* of an eigensystem is the p -adic valuation of the U eigenvalue (for an appropriate ‘controlling operator’ U). A slope 0 eigensystem is *ordinary*.

This classicality theorem was first introduced in [40] for modular forms, and is a cohomological analogue of Coleman’s classicality criterion [11]. It has, in its own right, had far-reaching arithmetic consequences: to give a brief flavour, it has been used to construct p -adic L -functions [35], to study \mathcal{L} -invariants [22], to construct Stark–Heegner points [14], and to give conjectural analogues of class field theory over real quadratic fields [15].

1.2 Parabolic families and classicality

In the usual theory, p -adic families for \mathcal{G} encode variation with respect to a Borel subgroup $B \subset G$. In particular, U is a B -controlling operator in the sense of Sect. 2.5, the natural generalisation of the U_p operator for modular forms. Then the eigenvariety encodes U -finite-slope eigensystems, and the non-critical slope bound depends on U .

All of the above is defined using the Iwahori subgroup at p . When applying this to the study of an automorphic representation π , this forces one to work at Iwahoric level, studying ‘full’ p -refinements of π . In practice, however, it is frequently more natural to work only at *parahoric* level for a parabolic subgroup $Q \subset G$, corresponding to a weaker p -refinement. In this setting, passing further to full Iwahoric level often requires stronger hypotheses and a loss of information.

In this paper, we present a refined version of overconvergent cohomology which applies to Q -parahoric level, and prove a classicality theorem for this refined theory. We vary this in p -adic families and use it to construct ‘parabolic eigenvarieties’, parametrising parabolic families of automorphic representations. This approach brings two further benefits:

- the criterion for non-critical slope is weaker, giving more control in the classicality theorem;
- the resulting parabolic families parametrise Q -finite-slope eigensystems, without requiring finite slope away from Q .

This is offset by the fact that these spaces vary over smaller-dimensional weight spaces.

A very special case of this is as follows. Suppose F is a real quadratic field in which p splits as $\mathfrak{p}\bar{\mathfrak{p}}$, and let $\mathcal{G} = \text{Res}_{F/\mathbf{Q}} \text{GL}_2$. Then $G = \text{GL}_2 \times \text{GL}_2$, and $U_p = U_{\mathfrak{p}}U_{\bar{\mathfrak{p}}}$ is a B -controlling operator. Let E/F be a modular elliptic curve with good ordinary reduction at \mathfrak{p} and bad (additive) reduction at $\bar{\mathfrak{p}}$. The attached system of Hecke eigenvalues has infinite slope for $U_{\bar{\mathfrak{p}}}$ and hence U_p , and does not appear in the (2-dimensional) Hilbert eigenvariety. However, we may take a parabolic $Q = B_2 \times \text{GL}_2 \subset G$, where B_2 is the Borel in GL_2 ; then $U_{\mathfrak{p}}$ is a Q -controlling operator, and the ordinary \mathfrak{p} -refinement of E satisfies the Q -classicality theorem, giving a 1-dimensional ‘ p -adic family’ through E . Moreover, this classicality yields a class in the \mathfrak{p} -adic overconvergent cohomology attached to E , which has been used to construct p -adic points on E [20].

1.3 Methods and results

Our parahoric overconvergent cohomology groups are defined using *parahoric*¹ *distribution modules*. Any weight λ is naturally a character on the torus $T(\mathbf{Z}_p)$; we are most interested in those that are algebraic dominant, and call these *classical*. The typical coefficient modules used in overconvergent cohomology are:

- overconvergent coefficients \mathcal{D}_λ^G , dual to the *locally analytic* induction of λ to the Iwahori subgroup of $G(\mathbf{Z}_p)$,
- and classical coefficients V_λ^\vee , dual to the *algebraic* induction of λ to $G(\mathbf{Z}_p)$.

We consider a hybrid construction, defining spaces \mathbf{D}_λ^Q by taking the algebraic induction of λ to the Levi subgroup L_Q of Q , then (locally) analytically inducing to the parahoric subgroup for Q , then taking the dual. These groups are naturally quotients of \mathcal{D}_λ^G . Moreover if we take $Q = B$ to be the Borel, we recover \mathcal{D}_λ^G ; and if we take $Q = G$ the ‘trivial’ parabolic we recover V_λ^\vee . All of this is described in Sect. 3, and summarised in Table 1.

In Sect. 4, we construct a parahoric version of Jones–Urban’s locally analytic BGG resolution. This is an analytic version of the main result of [28], and provides a tool for our main result, which is a Q -classicality theorem giving an isomorphism between the small-slope parts of cohomology with \mathbf{D}_λ^Q and V_λ^\vee coefficients. In particular, in Theorem 4.4 we prove:

Theorem A *Let $Q = P_0 \subset P_1 \subset \dots \subset P_m = G$ be a maximal chain of parabolics containing Q , and let U_Q be a Q -controlling operator which factorises as $U_Q = U_1 \dots U_m$, where each $U_i \dots U_m$ is a P_{i-1} -controlling operator. Let ϕ be a system of Hecke eigenvalues and λ a classical weight.*

There exist precise bounds $h_i \in \mathbf{Q}_{>0}$, depending on λ , such that if $v_p(\phi(U_i)) < h_i$ for each i , then the ϕ -parts of the weight λ classical and Q -overconvergent cohomology are isomorphic.

If $v_p(\phi(U_i)) < h_i$ for each i , we say ϕ has Q -non-critical slope. The notion of being a controlling operator, and the precise values of h_i , are described in terms of root data and Weyl groups, which we recap in Sect. 2. We describe this theorem in a number of explicit cases in Examples 4.1.

Remark If Q is the Borel, the most general classicality theorems for (Iwahoric) overconvergent cohomology that currently appear in the literature—for example, [42, Prop. 4.3.10]—require $v_p(\phi(U_Q)) < \min_i(h_i)$, so even in this case we give a significant improvement on the known range of non-critical slopes. Such improved ranges were known to exist in other settings (for example, see [17, Sect. 4.4]), and we believe an analogue for overconvergent cohomology was expected by experts. However, it does not appear in the literature, which we aim to rectify here.

The parahoric overconvergent cohomology groups can be naturally varied analytically in the weight, from which the construction of p -adic families and eigenvarieties—and their basic properties—is fairly standard. In particular, we construct rigid analytic spaces whose points parametrise Q -finite slope systems of eigenvalues, and coherent sheaves on these spaces that interpolate Q -finite slope eigenspaces in classical cohomology. We describe this in Sect. 5. In Sect. 5.4, we give sufficient conditions for the existence of parabolic families of cuspidal automorphic representations.

¹ Though we only consider parahoric subgroups attached to parabolics, we write ‘parahoric distributions/overconvergent cohomology’ to avoid conflict with the established definition of parabolic cohomology.

1.4 A note on assumptions

We will use results from [23,42], which work in slightly different settings to us. In [42], Urban's main applications are in the case where \mathcal{G} is quasi-split at p and satisfies the Harish–Chandra condition at infinity (i.e. $\mathcal{G}^{\text{der}}(\mathbf{R})$ admits discrete series). The Harish–Chandra condition is assumed only to control the geometry of the eigenvarieties he constructs. In particular it is not used anywhere in Sect. 2,3 of [42], which covers the results we use; in these sections Urban sets up the theory of (Iwahoric) overconvergent cohomology assuming only \mathcal{G}/\mathbf{Q}_p is quasi-split. (We indicate briefly where the Harish–Chandra condition is useful in our setting. In the notation of Definition 5.11 below, it implies that at any Q -non-critical slope cuspidal point x we have $\ell_Q(x) = 0$; and thus by Proposition 5.12, any irreducible component of the parabolic eigenvariety through x has the same dimension as the weight space. Without the Harish–Chandra condition this might not be true).

In [23], Hansen works under the assumption that \mathcal{G}/\mathbf{Q}_p is split instead of quasi-split. This appears to have been done only for convenience, since (as explained in [42, Sect. 3.1.1, Sect. 3.2]) the formalism of locally analytic distributions goes through equally well when \mathcal{G} is quasi-split, up to keeping track of a finite field extension (the field L for us). Moreover Hansen's main tools—the spectral sequences—require only formal properties of distributions that hold in the quasi-split case.

In fact, as remarked on p.1712, footnote 16 of [42], it should be possible to drop the quasi-split requirement altogether if one uses Bruhat–Tits buildings. One then replaces the parahoric subgroup with any open compact subgroup with a Bruhat–Iwahori decomposition. This approach is taken in [25,30], where there are no assumptions at all on \mathcal{G} at p . We have opted to stick to the notationally much simpler, but still very general, quasi-split setting.

Finally, we choose to use compactly supported cohomology throughout this paper as it best suits our future applications, but all of the results go through identically replacing this with singular cohomology (and, in Sect. 5, Borel–Moore homology with singular homology).

1.5 Comparison to the literature

Constructions of parabolic families/eigenvarieties have been previously given using methods different to this paper. The theory was introduced for Hida families in [24], and other papers on this subject include [30] (for unitary groups), [34] (Hida theory for Siegel modular forms), and in particular [25], which treats a very general setting using Emerton's completed cohomology. They are also related to the μ -ordinary setting of [16]. Parabolic families have important applications in arithmetic: for example, in the case of $\mathcal{G} = \text{GSp}_4$, Siegel-parabolic families are used in [32, Sect. 17], where new cases of the Bloch–Kato conjecture are proved; when \mathcal{G} is a definite unitary group, parabolic eigenvarieties were used in [10] to attach Galois representations to certain regular, polarised automorphic representations of GL_n ; and parabolic Hida families are used in upcoming work of Caraiani–Newton to answer deep questions about local–global compatibility for Galois representations.

In this spirit, the main motivation for giving a *new* version of this theory comes through arithmetic applications, for which parahoric overconvergent cohomology appears particularly well-suited; it adapts a very useful arithmetic tool (overconvergent cohomology) to a setting of increasing arithmetic interest (parahoric level/families).

This utility is illustrated in the example of GL_2 over a number field F , where special cases of the above theory have appeared repeatedly:

- In the case where F is totally real, partial p -adic families were used in [4,26], with applications to the trivial zero and parity conjectures respectively.
- For more general F , versions of Theorem A have been proved and used to construct Stark–Heegner points on elliptic curves [19,20,41], and when F is imaginary quadratic, to construct conjectural Stark–Heegner cycles attached to Bianchi modular forms [43]. It was also used in [7] to construct p -adic \mathcal{L} -invariants and prove an exceptional zero conjecture for Bianchi modular forms.
- Moreover, versions of the refined slope conditions given by Theorem A were used in [8,44] to construct p -adic L -functions attached to automorphic forms for GL_2 .

In forthcoming work with Dimitrov, we use Theorem A in the setting of GL_{2n} over totally real fields, using the parabolic Q with Levi $\mathrm{GL}_n \times \mathrm{GL}_n$, to construct p -adic L -functions attached to Q -non-critical conjugate-symplectic automorphic representations of GL_{2n} . We use the results of the present paper to give stronger non-critical-slope and growth conditions than could be achieved with Iwahoric overconvergent cohomology. We also vary this construction in Q -families.

These methods also appear well-adapted to the study of the general automorphic \mathcal{L} -invariants defined in [18], in which parabolic subgroups arise very naturally. In addition to the examples for GL_2 above, a combination of parahoric overconvergent cohomology with recent work of Gehrmann and Rosso [21] should, in nice examples (such as the setting of conjugate-symplectic GL_{2n}) yield arithmetic interpretations of automorphic \mathcal{L} -invariants. For GL_2 , such interpretations are already crucial in the construction of the Stark–Heegner points/cycles mentioned above.

Finally, we note the recent related work of Loeffler [29] on universal deformation spaces, which can be described as ‘big’ parabolic eigenvarieties. The eigenvarieties we construct are the ‘small’ automorphic eigenvarieties of Sect. 6.2 *op. cit.*; as yet there is no ‘big’ automorphic analogue.

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2 Preliminaries and structure theory

2.1 Global notation

Let F be a number field, and for each non-archimedean place v let F_v denote its completion at v , with ring of integers \mathcal{O}_v and uniformiser ϖ_v . Let \mathcal{G}' be a connected reductive group over F , and $\mathcal{G} := \mathrm{Res}_{F/\mathbf{Q}}\mathcal{G}'$ be the Weil restriction of scalars. We will be fundamentally interested in the cohomology of locally symmetric spaces attached to \mathcal{G} . Let $K \subset \mathcal{G}(\mathbf{A}_f)$ be an open compact subgroup, where \mathbf{A}_f denotes the finite adeles of \mathbf{Q} , let C_∞ (resp. Z_∞) be the maximal compact subgroup (resp. centre) of $\mathcal{G}(\mathbf{R})$, and let $K_\infty = C_\infty Z_\infty$. Then let

$$S_K := \mathcal{G}(\mathbf{Q}) \backslash \mathcal{G}(\mathbf{A}) / K K_\infty^\circ$$

be the locally symmetric space attached to K , where K_∞° is the identity component of K_∞ . If M is a right K -module such that the centre $Z(K \cap \mathcal{G}(\mathbf{Q}))$ acts trivially, then we get an associated local system on S_K given by the fibres of the projection

$$\mathcal{G}(\mathbf{Q}) \backslash [\mathcal{G}(\mathbf{A}) \times M] / KK_\infty^\circ \rightarrow S_K, \tag{2.1}$$

with action $\gamma(g, m)uk = (\gamma g u k, m|u)$.

2.2 Local notation and root data at p

Let $G = \mathcal{G}/\mathbf{Q}_p$. We assume that G is quasi-split, and splits over a (fixed) finite Galois extension E/\mathbf{Q}_p . As far as possible we will suppress E from notation. We take \mathcal{G}'/F_v and G to have (henceforth fixed) models over \mathcal{O}_v and \mathbf{Z}_p respectively. Let T be a maximal torus in G , and B a Borel subgroup containing T . Let B^- denote the opposite Borel, and N, N^- the unipotent radicals of B, B^- . Attached to all of these groups we have corresponding Lie algebras $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}, \mathfrak{b}^-, \mathfrak{n}, \mathfrak{n}^-$ over \mathbf{Q}_p . Let

$$X^\bullet(T) := \text{Hom}(T, \mathbf{G}_m), \quad X_\bullet(T) := \text{Hom}(\mathbf{G}_m, T)$$

be the lattices of algebraic characters and cocharacters of the torus, and $\langle \cdot, \cdot \rangle$ the canonical pairing on $X^\bullet(T) \otimes X_\bullet(T)$. Let $R \subset X^\bullet(T)$ denote the set of roots for (G, T) . For each root α , let $H_\alpha \in \mathfrak{t}$ and $\alpha^\vee \in X_\bullet(T)$ be the corresponding coroots, defined so that $\langle \alpha, \alpha^\vee \rangle = \alpha(H_\alpha) = 2$. We fix a basis X_α of

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : \text{ad}(t) \cdot X = \alpha(t)X \text{ for all } t \in T\}$$

normalised so that $[X_\alpha, X_{-\alpha}] = H_\alpha$ in \mathfrak{g} . Our choice of Borel fixes a set of positive roots $R^+ \subset R$ and a set $\Delta \subset R^+$ of simple roots. We say a character $\lambda \in X^\bullet(T)$ is *dominant* (with respect to B) if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta$.

Let W_G denote the Weyl group of (G, T) , generated by reflections w_α for $\alpha \in \Delta$, acting on $X^\bullet(T)$ by $\lambda^{w_\alpha} = \lambda - \lambda(H_\alpha)\alpha$. Also define the $*$ -action of W_G on $X^\bullet(T)$ by

$$w * \lambda = (\lambda + \rho)^w - \rho, \quad \lambda \in X^\bullet(T), w \in W_G,$$

where $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in X^\bullet(T) \otimes_{\mathbf{Z}} \frac{1}{2}\mathbf{Z}$ is half the sum of the positive roots. One may check (see e.g. the proof of [42, Prop. 3.2.11]) that this action is by

$$w_\alpha * \lambda = \lambda - [\langle \lambda, \alpha^\vee \rangle + 1]\alpha. \tag{2.2}$$

Example To anchor this general framework, we keep in mind the familiar example of GL_n/\mathbf{Q} . Here G is split, $\mathfrak{g} = M_n(\mathbf{Q})$, $X^\bullet(T) = \mathbf{Z}e_1 \oplus \dots \oplus \mathbf{Z}e_n$, and $X_\bullet(T) = \mathbf{Z}e_1^\vee \oplus \dots \oplus \mathbf{Z}e_n^\vee$. For B the upper-triangular Borel, $\Delta = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n\}$. We have $\alpha_i^\vee = e_i^\vee - e_{i+1}^\vee$, H_{α_i} is the $n \times n$ matrix with (i, i) entry 1, $(i + 1, i + 1)$ entry -1 and all other entries 0, and X_{α_i} is the $n \times n$ matrix with $(i, i + 1)$ entry 1 and all others 0. The Weyl group is S_n ; the standard action is by permutations of the e_i , and the $*$ action on $\lambda = (\lambda_1, \dots, \lambda_n) = \lambda_1 e_1 + \dots + \lambda_n e_n$ is

$$w_{\alpha_i} * \lambda = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_i + 1, \lambda_{i+2}, \dots, \lambda_n). \tag{2.3}$$

The dominant weights are the λ with $\lambda_m \geq \lambda_{m+1}$ for all m . In particular, if λ is dominant, then $w_{\alpha_i} * \lambda$ is *never* dominant for any i , as $\lambda_{i+1} - 1 < \lambda_i + 1$.

2.3 Parabolic subgroups

There is a well-known correspondence between the standard parabolic subgroups $B \subset Q \subset G$ and subsets of the simple roots: if $\mathfrak{q} := \text{Lie}(Q)$, we let

$$\Delta_Q := \{\alpha \in \Delta : X_{-\alpha} \in \mathfrak{q}\}. \tag{2.4}$$

The correspondence $Q \leftrightarrow \Delta_Q$ is inclusion-preserving: in particular, $\Delta_B = \emptyset$ and the maximal standard parabolics correspond to excluding a single simple root. It is convenient (if non-standard) to allow G to be the ‘trivial’ parabolic subgroup, equal to its Levi subgroup and with $\Delta_G = \Delta$.

Let L_Q denote the Levi group attached to Q , and N_Q the unipotent radical of Q , so that $Q = L_Q N_Q$. Note Δ_Q can be identified with Δ_{L_Q} . Also let Q^- and N_Q^- be the opposite groups.

Define the *parahoric subgroup at Q* to be $J_Q = \prod_{v|p} J_{Q,v}$, where

$$J_{Q,v} := \{g \in \mathcal{G}'(\mathcal{O}_v) : g \pmod{\varpi_v} \in Q(\mathbf{F}_p)\}.$$

We also define $J_Q^- := J_Q \cap N_Q^-(\mathbf{Z}_p)$. For non-trivial Q we have a parahoric decomposition

$$J_Q = J_Q^- \cdot L_Q(\mathbf{Z}_p) \cdot N_Q(\mathbf{Z}_p), \tag{2.5}$$

and for $g \in J_Q$, we write this as $g = n_g^- \cdot t_g \cdot n_g$. If the context is clear, we sometimes drop the subscript g . Note that when $Q = B$ is the Borel, J_B is the usual Iwahori subgroup and we recover the Iwahori decomposition [33, Prop. 5.3.3].

2.4 The Hecke algebra

Fix a parabolic subgroup Q , and let $K = \prod_{v \nmid p} K_v \subset \mathcal{G}(\mathbf{A}_f)$ be an open compact subgroup. We take K to be *parahoric* in that $K_p := \prod_{v|p} K_v \subset J_Q \subset G(\mathbf{Z}_p)$. To define the (Q -parahoric) Hecke algebra at p , we define

$$T^+ := \{t \in T(\mathbf{Q}_p) : t^{-1} \cdot N(\mathbf{Z}_p) \cdot t \subset N(\mathbf{Z}_p)\}. \tag{2.6}$$

Proposition 2.1 (i) *An element $t \in T(\mathbf{Q}_p)$ is in T^+ if and only if $v_p(\alpha(t)) \leq 0$ for all $\alpha \in \Delta$.*

(ii) *If $t \in T^+$, then $t^{-1} \cdot N_Q(\mathbf{Z}_p) \cdot t \subset N_Q(\mathbf{Z}_p)$ for any parabolic Q .*

(iii) *If $t \in T^+$, then $t^{-1} \cdot B(\mathbf{Z}_p) \cdot t \subset B(\mathbf{Z}_p)$.*

Proof For (i), first suppose $v_p(\alpha(t)) \leq 0$ for all α . The Lie algebra of N is $\mathfrak{n} = \bigoplus_{\beta \in R^+} \mathbf{Q}_p X_\beta \subset \mathfrak{g}$, which has a basis indexed by the positive roots R^+ . We obtain coordinates $\{x_\beta(n) \in \mathbf{Q}_p : \beta \in R^+\}$ for any $n \in N(\mathbf{Q}_p)$, with the property that for any $\beta, \beta' \in R^+$, we have

$$x_\beta(\exp(X_{\beta'})) = \begin{cases} 1 & : \beta = \beta' \\ 0 & : \beta \neq \beta' \end{cases}$$

Let B_t be the matrix of conjugation by t in this basis; it is diagonal with value $\beta^{-1}(t)$ at (β, β) . By the valuation condition, we have $v_p(\beta^{-1}(t)) \geq 0$ for all t . Now, the subgroup $N(\mathbf{Z}_p)$ is exactly the subspace of n such that $x_\beta(n) \in \mathbf{Z}_p$ for all β , and this is clearly preserved by B_t .

Conversely, if there exists $\alpha \in \Delta$ with $v_p(\alpha(t)) > 0$, then we see that $t^{-1} \exp(X_\alpha) t \notin N(\mathbf{Z}_p)$.

To see (ii), observe that we have $\mathfrak{n}_Q(\mathbf{Q}_p) = \bigoplus_{\beta \in R^+ \setminus R_Q^+} \mathbf{Q}_p X_\beta \subset \mathfrak{n}(\mathbf{Z}_p)$, where

$$R_Q^+ := \{\beta \in R^+ : \beta \text{ is a root of } L_Q\}. \tag{2.7}$$

Thus $N_Q(\mathbf{Z}_p)$ is the subgroup of $N(\mathbf{Z}_p)$ characterised by $x_\beta(n) = 0$ for $\beta \in R_Q^+$. But this space is preserved by the action of $t \in T^+$ by the arguments above. Finally (iii) is immediate since $B(\mathbf{Z}_p) = T(\mathbf{Z}_p)N(\mathbf{Z}_p)$ and T^+ commutes with $T(\mathbf{Z}_p)$. \square

Definition 2.2 • We define $\mathcal{H}_p(K_p)$ to be the commutative \mathbf{Q}_p -algebra generated by

$$U_t := [K_p t K_p], \quad t \in T^+.$$

- For the (all but finitely many) places v of F at which K_v is hyperspecial maximal compact and \mathcal{G}'/F_v is unramified, define the local Hecke algebra $\mathcal{H}_v(K_v)$ to be the commutative \mathbf{Q}_p -algebra generated by the double coset operators $T_v(\gamma) := [K_v \gamma K_v]$, for $\gamma \in \mathcal{G}'(F_v)$.
- For all other $v \nmid p\infty$, define $\mathcal{H}_v(K_v) = 1$.
- We define the Hecke algebra to be $\mathcal{H}(K) := \mathcal{H}_p(K_p) \otimes \bigotimes_{v \nmid p\infty} \mathcal{H}_v(K_v)$.

If S is a \mathbf{Q}_p -algebra, then a *system of Hecke eigenvalues over S* is a non-trivial algebra homomorphism $\phi : \mathcal{H}(K) \rightarrow S$. If M is an S -module upon which $\mathcal{H}(K)$ acts S -linearly, then we write M_ϕ for the localisation of M —as a $\mathcal{H}(K) \otimes_{\mathbf{Q}_p} S$ -module—at the ideal $\ker(\phi) \subset \mathcal{H}(K) \otimes_{\mathbf{Q}_p} S$. If S is a field and M a finite-dimensional S -vector space, this is the generalised eigenspace where $\mathcal{H}(K)$ acts as ϕ . We say ϕ *occurs in M* if $M_\phi \neq 0$.

Remark We could take other choices of ramified Hecke algebra, altering the local geometry of the eigenvariety to suit particular arithmetic applications. The construction and results we present here go through for any reasonable choice of ramified Hecke algebra.

2.5 Controlling operators

In the general theory, the role of U_p operator for modular forms is played by *controlling operators*. Let Q be a parabolic subgroup. For $s \geq 0$, let $B^s(\mathbf{Z}_p) = \{b \in B(\mathbf{Z}_p) : b \equiv 1 \pmod{p^s}\}$ and define $N_Q^s = N_Q(\mathbf{Z}_p) \cap B^s(\mathbf{Z}_p)$. If $t \in T^+$, then by Proposition 2.1 we know conjugation by t preserves $N_Q(\mathbf{Z}_p)$. We define

$$T_Q^{++} := \{t \in T^+ : t^{-1} \cdot N_Q^s \cdot t \subset N_Q^{s+1} \ \forall s \geq 0\} = \left\{t \in T^+ : \bigcap t^{-i} N_Q t^i = 1\right\}.$$

If $P \subset Q$ are two parabolics and $t \in T_P^{++}$, then $t^{-1} N_Q^s t \subset N_Q(\mathbf{Z}_p) \cap N_P^{s+1} = N_Q^{s+1}$, so $T_P^{++} \subset T_Q^{++}$.

Proposition 2.3 *Let $t \in T^+$. Then $t \in T_Q^{++}$ if and only if $v_p(\alpha(t)) < 0$ for all $\alpha \in \Delta \setminus \Delta_Q$.*

Proof Suppose $v_p(\alpha(t)) < 0$ for all $\alpha \in \Delta \setminus \Delta_Q$, and let $n \in N_Q(\mathbf{Z}_p)$. In the notation of the proof of Proposition 2.1, the set $R^+ \setminus R_Q^+$ is precisely the set of $\beta \in R^+$ whose decomposition $\beta = \sum \alpha_i$ into simple roots (in G) has at least one of the $\alpha_i \in \Delta \setminus \Delta_Q$. Then $v_p(\beta(t)) < 0$ for all $\beta \in R^+ \setminus R_Q^+$, and every entry of B_t restricted to $N_Q(\mathbf{Z}_p)$ is divisible by p . Since N_Q^s is the subgroup of n with $x_\beta(n) \equiv 0 \pmod{p^s}$ for all $\beta \in R^+ \setminus R_Q^+$, we see that B_t sends N_Q^s to N_Q^{s+1} .

Conversely, if $\alpha \in \Delta \setminus \Delta_Q$ with $v_p(\alpha(t)) = 0$, then $t^{-1} \exp(X_\alpha) t \not\equiv I \pmod{p}$. Thus $t^{-1} \cdot N_Q^0 \cdot t \not\subset N_Q^1$, so $t \notin T_Q^{++}$. \square

Definition 2.4 If $t \in T_Q^{++}$, we call U_t a Q -controlling operator.

Example Consider the case of $G = \mathrm{GL}_n$. For Q the parabolic with Levi $\mathrm{GL}_{n-r} \times \mathrm{GL}_r$, the element $t = \mathrm{diag}(1, \dots, 1, p, \dots, p)$, with p 's in the last r entries, defines a Q -controlling operator, but *not* a B -controlling operator. The element $t = \mathrm{diag}(1, p, \dots, p^{n-2}, p^{n-1})$ defines a B -controlling operator and hence a Q -controlling operator for any standard parabolic Q .

3 Parahoric overconvergent cohomology

We now introduce the coefficient modules for overconvergent cohomology, using a more flexible notion of ‘parahoric distributions’ defined relative to a parabolic Q . When $Q = B$ is the Borel, this specialises to the usual definition of locally analytic distributions; and when $Q = G$, we recover classical coefficient modules. Cohomology with coefficients in Q -parahoric distributions is more easily controlled (but varies over smaller weight spaces) as Q gets larger.

3.1 Weight spaces

Let $K \subset \mathcal{G}(\mathbb{A}_f)$ be an open compact subgroup such that $K_p \subset G(\mathbb{Z}_p)$, and let $\overline{Z(K)}$ denote the p -adic closure of $Z_G(\mathbf{Q}) \cap K$ in $T(\mathbb{Z}_p)$.

Definition 3.1 (Weights for T) Define the *weight space* of level K for G to be the \mathbf{Q}_p -rigid analytic space whose L -points, for $L \subset \mathbf{C}_p$ any sufficiently large extension of \mathbf{Q}_p , are given by

$$\mathcal{W}_K(L) = \mathrm{Hom}_{\mathrm{cts}}(T(\mathbb{Z}_p)/\overline{Z(K)}, L^\times).$$

This space has a natural group structure, and has dimension $\dim T(\mathbb{Z}_p) - \dim \overline{Z(K)}$. It is usually more convenient to identify a weight $\lambda \in \mathcal{W}_K(L)$ with the corresponding character on $T(\mathbb{Z}_p)$ that is trivial under $\overline{Z(K)}$, and we do this freely throughout. The condition that characters be trivial on $\overline{Z(K)}$ ensures the local systems we define later are well-defined, as discussed before (2.1). Since K will typically be fixed, we will henceforth mostly drop it from the notation.

Definition 3.2 Each $\lambda \in X^\bullet(T)$ induces a character on $T(\mathbb{Z}_p)$; let $X^\bullet(T)_K$ be the subspace of such λ trivial on $\overline{Z(K)}$. There is a natural inclusion $X^\bullet(T)_K \subset \mathcal{W}(L)$, and we call this the subspace of *algebraic weights*. Via Sect. 2.2, the algebraic weights carry the $*$ -action of the Weyl group and can be paired naturally, via $\langle -, - \rangle$, with $X_\bullet(T)$. A *classical weight* is a dominant algebraic weight.

When using the standard notion of distributions with respect to the Borel subgroup, it is possible to define distributions over arbitrary affinoids in \mathcal{W} (see, for example, [23, Sect. 2.2]). The additional flexibility we obtain with parahoric distributions, i.e. weaker notions of finite-slope families and non-criticality, come at the cost of less flexibility when defining distributions in families. In particular, they vary only over the following smaller weight spaces.

Definition 3.3 (Weights for Q) Let Q be a standard parabolic subgroup.

(i) For K and L as above, let $\mathcal{W}^Q(L)$ be the \mathbf{Q}_p -rigid analytic space with L -points

$$\mathcal{W}^Q(L) = \mathcal{W}_K^Q(L) = \text{Hom}_{\text{cts}}(L_Q(\mathbf{Z}_p)/\overline{Z(K)}, L^\times).$$

More precisely, \mathcal{W}^Q is the rigid generic fibre of $\text{Spf}(\mathbf{Z}_p[[L_Q^{\text{ab}}(\mathbf{Z}_p)/\overline{Z(K)}]])$, where the quotient is by the image of $\overline{Z(K)}$ in $L_Q^{\text{ab}}(\mathbf{Z}_p)$. Precomposition with $T(\mathbf{Z}_p) \hookrightarrow L_Q(\mathbf{Z}_p) \rightarrow L_Q^{\text{ab}}(\mathbf{Z}_p)$ realises \mathcal{W}^Q as a closed rigid subgroup of \mathcal{W} .

(ii) For $\lambda_0 \in \mathcal{W}(\mathbf{Q}_p)$ a fixed classical weight, define $\mathcal{W}_{\lambda_0}^Q$ to be the coset $\lambda_0\mathcal{W}^Q$ inside \mathcal{W} , which hence obtains the structure of a \mathbf{Q}_p -rigid space. We have

$$\mathcal{W}_{\lambda_0}^Q(L) := \{\lambda \in \mathcal{W}(L) : \lambda\lambda_0^{-1} \in \mathcal{W}^Q(L)\}.$$

Again, we identify these weights with characters on $L_Q(\mathbf{Z}_p)$ that are trivial under $\overline{Z(K)}$. This space has dimension $\dim(L_Q^{\text{ab}}(\mathbf{Z}_p)) - \dim(\overline{Z(K)})$, which is at most $\dim(\mathcal{W}_K)$. Whilst we encode λ_0 in the notation, the space $\mathcal{W}_{\lambda_0}^Q$ evidently only depends on λ_0 up to translation by \mathcal{W}^Q .

Example Let $G = \text{GL}_{2n}$, and Q the standard parabolic with Levi $L_Q = \text{GL}_n \times \text{GL}_n$ embedded diagonally. Then $\mathcal{W}(L)$ comprises $2n$ -tuples $\lambda = (\lambda_1, \dots, \lambda_{2n})$ of characters $\mathbf{Z}_p^\times \rightarrow L^\times$ (that are trivial on $\overline{Z(K)}$), and $\mathcal{W}^Q(L)$ is the subspace where $\lambda_1 = \dots = \lambda_n$ and $\lambda_{n+1} = \dots = \lambda_{2n}$.

3.2 Parahoric distributions

Locally analytic induction modules for a group G , as for example seen in [1,27,42], are usually defined through p -adic analytic functions on the Iwahori subgroup, and are uniquely defined by their restriction to $N(\mathbf{Z}_p)$. For $G = \text{GL}_n$, for example, this translates into functions that are locally analytic in $n(n - 1)/2$ variables, corresponding to the off-diagonal entries in $N(\mathbf{Z}_p)$.

We now define ‘partially overconvergent’ distribution modules, defined with respect to the parabolic Q , where we only allow analytic variation in some subset of the variables in $N(\mathbf{Z}_p)$ and dictate algebraic variation in the others. For this, we first algebraically induce up to the Levi L_Q , and then analytically induce to the parahoric J_Q . This is explained in explicit detail for GL_3/\mathbf{Q} in [45, Sect. 4.3]; the concrete setting *op. cit.* simplifies the concepts whilst retaining the key ideas.

We recap standard results on locally analytic induction. As G splits over E , all our coefficient modules come from representations of $\mathfrak{g}_{/E}$. For the rest of the paper, fix L/\mathbf{Q}_p finite containing E , and an L -Banach algebra R .

3.2.1 Algebraic induction and highest weight representations

Let $\lambda \in X^\bullet(T) \subset \mathcal{W}(L)$ be a classical weight for the group G . We have a finite-dimensional irreducible representation V_λ^G of highest weight λ , whose L -points can be realised as the algebraic induction

$$\begin{aligned} V_\lambda^G(L) &:= \text{Ind}_{B^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} \lambda \\ &:= \{f : G(\mathbf{Z}_p) \rightarrow L \mid f \text{ algebraic, } f(n^-tg) = \lambda(t)f(g) \forall n^- \in N^-(\mathbf{Z}_p), \\ &\quad t \in T(\mathbf{Z}_p), g \in G(\mathbf{Z}_p)\}. \end{aligned}$$

(By an *algebraic function* $f : G(\mathbf{Z}_p) \rightarrow L$, we mean a function on $G(\mathbf{Z}_p)$ induced by a global section $f \in L[G]$ of the structure sheaf of the (affine) group $G_{/L}$.) The space $V_\lambda^G(L)$ is a left $G(\mathbf{Z}_p)$ -module by right translation, and we denote this action by $\langle \cdot \rangle_\lambda$. Any $f \in V_\lambda^G(L)$ is determined by its restriction to the (open, dense) Iwahori subgroup J_B , and thus (by the transformation property and (2.5)) by its restriction to $N(\mathbf{Z}_p)$. Moreover, it is standard² that any algebraic $f : J_B \rightarrow L$ with $f(n^{-t}g) = \lambda(t)f(g)$ has a unique algebraic extension to $G(\mathbf{Z}_p)$.

3.2.2 Analytic function spaces

Let $X \subset \mathbf{Z}_p^r$ be open compact and L and R be as above. A function $f : X \rightarrow R$ is *analytic* if it can be written as a convergent power series

$$f(x_1, \dots, x_r) = \sum_{n_1, \dots, n_r} a_{n_1, \dots, n_r} (x_1 - a_1)^{n_1} \cdots (x_r - a_r)^{n_r}, \quad a_n \in R,$$

for some $(a_1, \dots, a_r) \in X$. We write the space of such functions as $\mathcal{A}_0(X, R)$; note that as the a_n converge to zero, $\mathcal{A}_0(X, R) \cong \mathcal{A}_0(X, L) \widehat{\otimes}_L R$ is the completed tensor product. We say f is *algebraic* if $a_n = 0$ for all but finitely many \mathbf{n} , and denote the subspace of such f as $V(X, R) \subset \mathcal{A}_0(X, R)$. For any integer s , we say $f : X \rightarrow R$ is *s-analytic* (resp. *s-algebraic*) if it is analytic (resp. algebraic) on each open disc of radius p^{-s} in X (inside \mathbf{Z}_p^r), and write $\mathcal{A}_s(X, R)$ for the space of s -analytic functions. Note 0-analytic is the same as analytic, so the notation is consistent. The spaces $\mathcal{A}_s(X, L)$ are Banach spaces under a suitable sup norm [42, Sect. 3.2.1], and the inclusions $\mathcal{A}_s(X, L) \subset \mathcal{A}_{s+1}(X, L)$ are compact [42, Lem. 3.2.2]. The spaces $\mathcal{A}_s(X, R) \cong \mathcal{A}_s(X, L) \widehat{\otimes}_L R$ inherit a Banach R -module structure from the completed tensor product, which can again be described in terms of sup norms (see e.g. [23, Sect. 2.2], [1, Prop. 3.6.7]). The inclusions $\mathcal{A}_s(X, R) \subset \mathcal{A}_{s+1}(X, R)$ are compact by [9, Cor. 2.9], noting the potential ONability hypothesis follows from ONability of $\mathcal{A}_s(X, L)$ (see below) and Lem. 2.8 *op. cit.* We write $\mathcal{A}(X, R) = \varinjlim_s \mathcal{A}_s(X, R)$.

If M is a finite Banach R -module, then we say a function $f : X \rightarrow M$ is *s-analytic* if it is an element of $\mathcal{A}_s(X, R) \widehat{\otimes}_R M$. We write $\mathcal{A}_s(X, M)$ for the space of such functions, which (by [39, Sect. 4]) inherits R -Banach module structure from the completed tensor product. Again, the inclusion maps $\mathcal{A}_s(X, M) \subset \mathcal{A}_{s+1}(X, M)$ are compact, and we let $\mathcal{A}(X, M) = \varinjlim_s \mathcal{A}_s(X, M)$.

Recall the definition of *orthonormalisable (ONable)* from [12, Sect. A1]. Any Banach space over L is ONable [42, Lem. 2.1.5], so the spaces $\mathcal{A}_s(X, L)$ and (when M is a finite-dimensional L -vector space) $\mathcal{A}_s(X, M)$ are ONable. When R is a contractive L -algebra (for example, if R is an L -affinoid algebra), then the completed tensor product of an ONable L -Banach space with R is an ONable R -Banach module [12, Prop. A1.3], so for such R the spaces $\mathcal{A}_s(X, R)$ are ONable. If A and B are two ONable Banach R -modules with ON bases $\{e_i\}, \{f_j\}$, then $A \widehat{\otimes}_R B$ is an ONable Banach R -module with ON basis $\{e_i \otimes f_j\}$; hence when R is contractive and M is a finite Banach R -module, the spaces $\mathcal{A}_s(X, M)$ are ONable.

For a Banach R -module A , let $\text{Hom}_R(A, R)$ denote the space of continuous R -module maps $A \rightarrow R$. This is a Banach space via [12, Sect. A1]. If $R = L$, and M is a finite-dimensional (normed) L -vector space, then we write $\mathcal{D}_s(X, M) := \text{Hom}_L(\mathcal{A}_s(X, M), L)$. In this case the maps $\mathcal{D}_{s+1}(X, M) \subset \mathcal{D}_s(X, M)$ are compact via the analogous statements for $\mathcal{A}_s(X, M)$ and [38, Lem. 16.4], and (as it is an L -Banach space) $\mathcal{D}_s(X, M)$ is ONable. Similar statements for dual spaces over R are more subtle: see Sect. 3.2.7 below.

² See e.g. [42, Sect. 3.2.9], where for $\varepsilon = 1$ this is implicit in the statement $\mathbb{V}_\lambda(\varepsilon, L) = V_\lambda(L) \cap \mathcal{A}_m(I, L)$.

3.2.3 Analytic induction modules

Let $Q = L_Q N_Q$ be a parabolic. We may identify J_Q with an open compact subset of \mathbf{Z}_p^r for some r , and thus apply the above formalism of analytic functions on J_Q . Let M be a finite Banach R -module with a left-action of $L_Q(\mathbf{Z}_p)$. We extend this action to $Q^-(\mathbf{Z}_p) \cap J_Q = (J_Q^- L_Q)(\mathbf{Z}_p)$ by dictating that J_Q^- acts trivially.

Definition 3.4 Define the s -analytic induction of M to J_Q , denoted $\text{LA}_s \text{Ind}_Q M$, to be the space of functions $f : J_Q \rightarrow M$ such that $f \in \mathcal{A}_s(J_Q, M)$ and

$$f(bg) = b \cdot f(g) \quad \text{for all } b \in Q^-(\mathbf{Z}_p) \cap J_Q \text{ and } g \in J_Q. \tag{3.1}$$

We write $\text{LAInd}_Q M$ for the space of such functions f such that $f \in \mathcal{A}(J_Q, M)$.

Note that any such function f is uniquely determined by its restriction to $N_Q(\mathbf{Z}_p)$ by (3.1) and the parahoric decomposition (2.5). Recall from (2.7) we have an explicit realisation of $N_Q(\mathbf{Z}_p)$ as an open compact subset of \mathbf{Z}_p^r via the product decomposition $N_Q(\mathbf{Z}_p) \cong \prod_{\beta \in R^+ \setminus R_Q^+} \mathbf{Z}_p X_\beta$. Note then that a function on $N_Q(\mathbf{Z}_p)$ is s -analytic if and only if it is analytic on each $N_Q^s(\mathbf{Z}_p)$ -coset.

3.2.4 Locally analytic induction at single weights

We recap the usual locally analytic modules. Here we take Q to be the Borel B , with Levi T . Let $\lambda \in \mathcal{W}(L)$ be a classical weight.

Definition 3.5 • Denote the s -analytic induction of λ by

$$\mathcal{A}_{\lambda,s}^G(L) := \text{LA}_s \text{Ind}_B \lambda,$$

realised as functions $f : J_B \rightarrow L$ with $f(n^- t g) = \lambda(t) f(g)$ for $t \in T(\mathbf{Z}_p)$, $n^- \in N^-(\mathbf{Z}_p)$.

- Let $\mathcal{A}_\lambda^G(L) := \text{LAInd}_B \lambda = \varinjlim_s \mathcal{A}_{\lambda,s}^G(L)$ be the module of locally analytic functions.
- We write $\mathcal{D}_{\lambda,s}^G(L)$ and $\mathcal{D}_\lambda^G(L)$ for the respective topological L -duals of the above spaces.

The module $\mathcal{A}_{\lambda,s}^G(L)$ can be identified with $\mathcal{A}_s(N(\mathbf{Z}_p), L)$ by restriction from J_B to $N(\mathbf{Z}_p)$, and inherits an L -Banach space structure from this space. Similarly the natural inclusions $\mathcal{A}_{\lambda,s}^G(L) \subset \mathcal{A}_{\lambda,s+1}^G(L)$ are all compact. Note also that via the restriction to J_B explained in Sect. 3.2.1, we may view $V_\lambda^G(L)$ as the subspace of algebraic functions in $\mathcal{A}_{\lambda,0}^G(L)$.

Now we work with a general Q , with Levi L_Q . Let λ be a classical weight; it is also a weight for L_Q , and we have an algebraic L_Q -representation $V_\lambda^{L_Q}(L)$ of highest weight λ via Sect. 3.2.1.

Definition 3.6 • Let $\mathbf{A}_{\lambda,s}^Q(L) := \text{LA}_s \text{Ind}_Q [V_\lambda^{L_Q}(L)]$.

- Let $\mathbf{A}_\lambda^Q(L) := \text{LAInd}_Q [V_\lambda^{L_Q}(L)] = \varinjlim_s \mathbf{A}_{\lambda,s}^Q(L)$.
- Let $\mathbf{D}_{\lambda,s}^Q(L) := \text{Hom}_L(\mathbf{A}_{\lambda,s}^Q(L), L)$ and $\mathbf{D}_\lambda^Q(L) = \text{Hom}_L(\mathbf{A}_\lambda^Q(L), L)$.

As above, all the spaces with subscript s 's are Banach spaces over L . The spaces $\mathbf{A}_\lambda^Q(L)$ and $\mathbf{D}_\lambda^Q(L)$ are Fréchet spaces, and $\mathbf{D}_\lambda^Q(L)$ is compact Fréchet in the sense of [42, Sect. 2.3.12].

Remark 3.7 As above, for any Q , the subspace of algebraic functions in $\mathbf{A}_{\lambda,0}^Q(L)$ is $V_\lambda^G(L)$, and hence $V_\lambda^{G,\vee}(L)$ is a quotient of $\mathbf{D}_{\lambda,0}^Q(L)$. At the extreme end, where we take $Q = G$, then from the definition $\mathbf{A}_{\lambda,0}^G(L) = V_\lambda^G(L)$ and $\mathbf{D}_{\lambda,0}^G(L) = V_\lambda^{G,\vee}(L)$.

3.2.5 Integral structures

All of the above Banach spaces have natural integral structures, where we replace L with \mathcal{O}_L ; in particular, as in [42, 3.2.13] we define

$$\mathbf{A}_{\lambda,s}^G(\mathcal{O}_L) := \mathbf{A}_{\lambda,s}^G(L) \cap \mathcal{A}_s(J_B, \mathcal{O}_L), \quad \mathbf{A}_{\lambda,s}^Q(\mathcal{O}_L) := \mathbf{A}_{\lambda,s}^Q(L) \cap \mathcal{A}_s(J_Q, V_\lambda^{L_Q}(\mathcal{O}_L)).$$

The dual modules $V_\lambda^{G,\vee}(\mathcal{O}_L)$, $\mathbf{D}_{\lambda,s}^G(\mathcal{O}_L)$ and $\mathbf{D}_{\lambda,s}^Q(\mathcal{O}_L)$ are then all defined via \mathcal{O}_L -duals.

3.2.6 Analytic functions in families

We now vary these spaces in families. Fix a classical weight $\lambda_0 \in \mathcal{W}(L)$, and let $\mathcal{U} \subset \mathcal{W}_{\lambda_0}^Q$ be an affinoid (which we *always* take to be admissible in the sense of [13, Def. 2.2.6], so that it is open in the Tate topology on $\mathcal{W}_{\lambda_0}^Q$). If $\lambda \in \mathcal{U}(L)$, then by definition $\lambda\lambda_0^{-1} \in \mathcal{W}^Q(L)$ is a character of $L_Q(\mathbf{Z}_p)$.

Lemma 3.8 *If $\lambda \in \mathcal{U}(L)$ is classical, then we have an isomorphism of $L_Q(\mathbf{Z}_p)$ -modules*

$$V_\lambda^{L_Q}(L) \cong V_{\lambda_0}^{L_Q}(L) \otimes_L \lambda\lambda_0^{-1}.$$

Proof The character $\lambda\lambda_0^{-1}$ can itself, as an irreducible representation of L_Q , be viewed as the highest weight representation $V_{\lambda\lambda_0^{-1}}^{L_Q}$. Then $V_\lambda^{L_Q} = V_{\lambda_0\lambda\lambda_0^{-1}}^{L_Q}$ is a subrepresentation of the tensor product; but the tensor product of an irreducible representation with a character is irreducible. \square

Crucial for variation is the fact that the underlying spaces of $V_\lambda^{L_Q}(L)$ and $V_{\lambda_0}^{L_Q}(L)$ are the same: only the $L_Q(\mathbf{Z}_p)$ -action is different. We now vary the action analytically.

As \mathcal{W} is a rigid analytic group, translation by λ_0 defines a rigid analytic automorphism of \mathcal{W} . Let $\mathcal{U}_0 := \lambda_0^{-1}\mathcal{U} \subset \mathcal{W}^Q$; this translation identifies \mathcal{U}_0 isomorphically with \mathcal{U} , so it is an affinoid defined over L . Attached to such an affinoid, there exists a tautological/universal character $\chi_{\mathcal{U}_0} : L_Q(\mathbf{Z}_p) \rightarrow \mathcal{O}(\mathcal{U}_0)^\times$ with the property that for each weight $\lambda\lambda_0^{-1} \in \mathcal{U}_0(L)$, composing $\chi_{\mathcal{U}_0}$ with evaluation $\mathcal{O}(\mathcal{U}_0) \rightarrow L$ at $\lambda\lambda_0^{-1}$ recovers the corresponding map $L_Q(\mathbf{Z}_p) \rightarrow L^\times$. Necessarily such a character must factor through the abelianisation $L_Q^{\text{ab}}(\mathbf{Z}_p)$, and L_Q^{ab} (as a commutative reductive group) is a torus. Any character of $L_Q^{\text{ab}}(\mathbf{Z}_p)$ is then locally analytic by [9, Prop. 8.3]. We deduce $\chi_{\mathcal{U}_0}$ is the composition of a locally analytic map with the analytic (even algebraic) map $L_Q \rightarrow L_Q^{\text{ab}}$, hence it is s -analytic for all s greater than some (minimal) integer $s[\mathcal{U}]$.

Definition 3.9 Define a finite free $\mathcal{O}(\mathcal{U}_0)$ -module $V_{\mathcal{U}}^{L_Q} := V_{\lambda_0}^{L_Q}(L) \otimes_L \mathcal{O}(\mathcal{U}_0)$, and a map

$$\begin{aligned} \langle \cdot \rangle_{\mathcal{U}} : L_Q(\mathbf{Z}_p) &\rightarrow \text{Aut}\left(V_{\lambda_0}^{L_Q}(L)\right) \otimes_L \mathcal{O}(\mathcal{U}_0)^\times \subset \text{Aut}\left(V_{\mathcal{U}}^{L_Q}\right) \\ \ell &\mapsto \langle \ell \rangle_{\lambda_0} \otimes \chi_{\mathcal{U}_0}(\ell). \end{aligned}$$

This makes $V_{\mathcal{U}}^{L_Q}$ into an $L_Q(\mathbf{Z}_p)$ -representation. From the definition of $\chi_{\mathcal{U}_0}$, we deduce:

Proposition 3.10 For any classical $\lambda \in \mathcal{U}(L)$, evaluation $\mathcal{O}(\mathcal{U}_0) \rightarrow L$ at $\lambda\lambda_0^{-1} \in \mathcal{U}_0$ induces a surjective map

$$\text{sp}_\lambda : V_{\mathcal{U}}^{L_Q} \longrightarrow V_{\lambda_0}^{L_Q}(L) \otimes_L \lambda\lambda_0^{-1} \cong V_\lambda^{L_Q}(L)$$

of $L_Q(\mathbf{Z}_p)$ -representations. Thus $V_{\mathcal{U}}^{L_Q}$ interpolates the representations $V_\lambda^{L_Q}(L)$ as λ varies in \mathcal{U} .

Here sp_λ is surjective since $\mathcal{O}(\mathcal{U}_0) \rightarrow L$ (evaluation at $\lambda\lambda_0^{-1}$) is surjective and $V_{\lambda_0}^{L_Q}(L)$ is L -flat.

Remark 3.11 The choice of λ_0 fixes an identification of \mathcal{U} and \mathcal{U}_0 , and hence of $\mathcal{O}(\mathcal{U})$ and $\mathcal{O}(\mathcal{U}_0)$, which is compatible with our normalisation of specialisation maps. Henceforth we work only with \mathcal{U} , and implicitly the transfer of structure is with respect to this choice of identification.

Definition 3.12 For any $s \geq s[\mathcal{U}]$, define³

$$\mathbf{A}_{\mathcal{U},s}^Q := \text{LA}_s \text{Ind}_Q V_{\mathcal{U}}^{L_Q}, \quad \text{and} \quad \mathbf{A}_{\mathcal{U}}^Q = \varinjlim_s \mathbf{A}_{\mathcal{U},s}^Q.$$

Lemma 3.13 $\mathbf{A}_{\mathcal{U},s}^Q \cong \mathcal{A}_s(N_Q(\mathbf{Z}_p), V_{\lambda_0}^{L_Q}(L)) \widehat{\otimes}_L \mathcal{O}(\mathcal{U})$ is an ONable $\mathcal{O}(\mathcal{U})$ -Banach module. If $\{e_i\}_{i \in \mathbf{N}}$ is an ON basis of $\mathcal{A}_s(N_Q(\mathbf{Z}_p), V_{\lambda_0}^{L_Q}(L))$, then $\{e_i \otimes 1\}_{i \in \mathbf{N}}$ is an ON basis of $\mathbf{A}_{\mathcal{U},s}^Q$.

Proof By (3.1), restriction to $N_Q(\mathbf{Z}_p)$ gives $\mathbf{A}_{\mathcal{U},s}^Q \cong \mathcal{A}_s(N_Q(\mathbf{Z}_p), V_{\mathcal{U}}^{L_Q})$, which from the definitions is isomorphic to $\mathcal{A}_s(N_Q(\mathbf{Z}_p), V_{\lambda_0}^{L_Q}(L)) \widehat{\otimes}_L \mathcal{O}(\mathcal{U})$. The rest now follows from [12, Prop. A1.3]. □

3.2.7 Distributions in families

Since $\mathcal{O}(\mathcal{U})$ -duals are not as well-behaved as L -duals, we have to work harder to study the distributions in this setting. See e.g. [2, Rem. 3.1] or [23, Sect. 2.2] for analogous discussions. The first natural space to study is the Banach/continuous dual

$$\mathbf{D}_{\mathcal{U},s}^Q := \text{Hom}_{\mathcal{O}(\mathcal{U})}(\mathbf{A}_{\mathcal{U},s}^Q, \mathcal{O}(\mathcal{U})).$$

The natural restriction maps $\mathbf{D}_{\mathcal{U},s+1}^Q \rightarrow \mathbf{D}_{\mathcal{U},s}^Q$ are injective (as in [23, Sect. 2.2]). However this is not obviously ONable. Since we require this for slope decompositions, we also define

$$\widetilde{\mathbf{D}}_{\mathcal{U},s}^Q := \mathcal{D}_s(N_Q(\mathbf{Z}_p), V_{\lambda_0}^{L_Q}(L)) \widehat{\otimes}_L \mathcal{O}(\mathcal{U}).$$

Since $\mathcal{O}(\mathcal{U})$ is a contractive Banach L -algebra, this space is an ONable Banach R -module [12, Prop. A1.3]; and the restriction maps $\widetilde{\mathbf{D}}_{\mathcal{U},s+1}^Q \rightarrow \widetilde{\mathbf{D}}_{\mathcal{U},s}^Q$ are compact by [9, Cor. 2.9]. By formalism of duals/tensor products there is a natural inclusion

$$r_s : \widetilde{\mathbf{D}}_{\mathcal{U},s}^Q \hookrightarrow \mathbf{D}_{\mathcal{U},s}^Q, \tag{3.2}$$

defined on pure tensors by $\mu \otimes \alpha \mapsto [(f \otimes \beta) \mapsto \mu(f)\alpha\beta]$ (using Lemma 3.13). Then (cf. [23, Sect. 2.2]):

³ Note this is only well-defined for $s \geq s[\mathcal{U}]$ since otherwise the action of $L_Q(\mathbf{Z}_p)$ is not s -analytic.

Lemma 3.14 For all s , there is a compact injective map $j_s : \mathbf{D}_{\mathcal{U},s+1}^{\mathcal{O}} \hookrightarrow \tilde{\mathbf{D}}_{\mathcal{U},s}^{\mathcal{O}}$ making the following diagram commute:

$$\begin{array}{ccc}
 \tilde{\mathbf{D}}_{\mathcal{U},s+1}^{\mathcal{O}} & \xrightarrow{\text{res}} & \tilde{\mathbf{D}}_{\mathcal{U},s}^{\mathcal{O}} \\
 \downarrow r_s & \nearrow j_s & \downarrow r_s \\
 \mathbf{D}_{\mathcal{U},s+1}^{\mathcal{O}} & \xrightarrow{\text{res}} & \mathbf{D}_{\mathcal{U},s}^{\mathcal{O}}
 \end{array} \tag{3.3}$$

Proof If the map exists, it is compact (by considering the top triangle) and injective (by considering the bottom triangle). To prove existence, let $\{e_i\}_{i \in \mathbb{N}}$ be an ON basis of $A_s(N_Q(\mathbf{Z}_p), V_{\lambda_0}^{L_Q}(L))$. For $i \in \mathbb{N}$ define distributions

$$v_i \in \mathcal{D}_s(N_Q(\mathbf{Z}_p), V_{\lambda_0}^{L_Q}(L)), \quad v_i(e_j) = \begin{cases} 1 & :i=j \\ 0 & :i \neq j, \end{cases}$$

extended continuously. Then given $\mu \in \mathbf{D}_{\mathcal{U},s+1}^{\mathcal{O}}$, we define

$$j_s(\mu) := \sum_{i \in \mathbb{N}} v_i \otimes \mu(e_i \otimes 1) \in \tilde{\mathbf{D}}_{\mathcal{U},s}^{\mathcal{O}}. \tag{3.4}$$

To see this is well-defined, note $\{e_i \otimes 1\}$ is an ON basis of $\mathbf{A}_{\mathcal{U},s}^{\mathcal{O}}$ by Lemma 3.13. As the inclusion $\mathbf{A}_{\mathcal{U},s}^{\mathcal{O}} \subset \mathbf{A}_{\mathcal{U},s+1}^{\mathcal{O}}$ is compact, the sequence $e_i \otimes 1$ tends to zero in $\mathbf{A}_{\mathcal{U},s+1}^{\mathcal{O}}$, and $\mu(e_i \otimes 1) \rightarrow 0$; hence the sum in (3.4) converges in the completed tensor product. Commutativity of (3.3) follows easily from the definitions. \square

Definition 3.15 Define the space of *parahoric locally analytic distributions over \mathcal{U}* to be

$$\mathbf{D}_{\mathcal{U}}^{\mathcal{O}} := \text{Hom}_{\mathcal{O}(\mathcal{U})}(\mathbf{A}_{\mathcal{U}}^{\mathcal{O}}, \mathcal{O}(\mathcal{U})).$$

Lemma 3.16 The space $\mathbf{D}_{\mathcal{U}}^{\mathcal{O}} \cong \varprojlim_s \mathbf{D}_{\mathcal{U},s}^{\mathcal{O}} \cong \varprojlim_s \tilde{\mathbf{D}}_{\mathcal{U},s}^{\mathcal{O}}$ is a compact Fréchet $\mathcal{O}(\mathcal{U})$ -module.

Proof The first isomorphism is standard, and the second isomorphism (between inverse limits) follows from Lemma 3.14. We conclude since $\varprojlim_s \tilde{\mathbf{D}}_{\mathcal{U},s}^{\mathcal{O}}$ is compact Fréchet by definition. \square

Remark 3.17 If $\lambda \in \mathcal{W}_{\lambda_0}^{\mathcal{O}}$ is any (possibly non-classical) weight, then we may still define an $L_Q(\mathbf{Z}_p)$ -module $V_{\lambda}^{L_Q}(L) := V_{\lambda_0}^{L_Q}(L) \otimes \lambda \lambda_0^{-1}$. Hence we can define $\mathbf{D}_{\lambda,s}^{\mathcal{O}}(L)$ and $\mathbf{D}_{\lambda}^{\mathcal{O}}(L)$ identically to Definition 3.6. Note $V_{\lambda}^{L_Q}(L)$ is independent of the choice of base weight λ_0 , since if λ'_0 is another choice, by Lemma 3.8 (in the first isomorphism) we have

$$V_{\lambda}^{L_Q}(L) := V_{\lambda_0}^{L_Q}(L) \otimes_L \lambda \lambda_0^{-1} \cong [V_{\lambda'_0}^{L_Q}(L) \otimes_L \lambda_0(\lambda'_0)^{-1}] \otimes_L \lambda \lambda_0^{-1} \cong V_{\lambda'_0}^{L_Q}(L) \otimes_L \lambda(\lambda'_0)^{-1}.$$

Hence $\mathbf{D}_{\lambda,s}^{\mathcal{O}}(L)$ and $\mathbf{D}_{\lambda}^{\mathcal{O}}(L)$ are also independent of the choice of λ_0 .

Remark 3.18 If $\mathcal{U}' \subset \mathcal{U}$ is a closed affinoid subspace, then (by definition of $\tilde{\mathbf{D}}_{\mathcal{U},s}^{\mathcal{O}}$) we have $\mathbf{D}_{\mathcal{U}}^{\mathcal{O}} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U}') \cong \mathbf{D}_{\mathcal{U}'}^{\mathcal{O}}$. If $\lambda \in \mathcal{U}(L)$ corresponds to the maximal ideal $\mathfrak{m}_{\lambda} \subset \mathcal{O}(\mathcal{U})$, we thus have

$$\mathbf{D}_{\mathcal{U}}^{\mathcal{O}} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U})/\mathfrak{m}_{\lambda} \cong \mathbf{D}_{\lambda}^{\mathcal{O}}(L),$$

and a specialisation map $\text{sp}_{\lambda} : \mathbf{D}_{\mathcal{U}}^{\mathcal{O}} \rightarrow \mathbf{D}_{\lambda}^{\mathcal{O}}(L)$. Thus $\mathbf{D}_{\mathcal{U}}^{\mathcal{O}}$ interpolates $\mathbf{D}_{\lambda}^{\mathcal{O}}(L)$ as λ varies in $\mathcal{U}(L)$.

Table 1 Modules of coefficients

Module	On unipotent	Extension	Dual	Nomenclature
V_λ^G	–	–	$V_\lambda^{G,\vee}$	Algebraic on G
$\mathcal{A}_{\lambda,s}^G$	$\mathcal{A}_s(N(\mathbf{Z}_p), L)$	$f : J_B \rightarrow L$	$\mathcal{D}_{\lambda,s}^G$	s -analytic on N
\mathcal{A}_λ^G	$\mathcal{A}(N(\mathbf{Z}_p), L)$	$f : J_B \rightarrow L$	\mathcal{D}_λ^G	locally analytic on N
$\mathbf{A}_{\lambda,s}^Q$	$\mathcal{A}_s(N_Q(\mathbf{Z}_p), V_\lambda^{LQ})$	$f : J_Q \rightarrow V_\lambda^{LQ}$	$\mathbf{D}_{\lambda,s}^Q$	s -an. on N_Q , s -alg. on L_Q
\mathbf{A}_λ^Q	$\mathcal{A}(N_Q(\mathbf{Z}_p), V_\lambda^{LQ})$	$f : J_Q \rightarrow V_\lambda^{LQ}$	\mathbf{D}_λ^Q	loc. an. on N_Q , loc. alg. on L_Q
$\mathbf{A}_\mathcal{U}^Q$	$\mathcal{A}(N_Q(\mathbf{Z}_p), V_\mathcal{U}^{LQ})$	$f : J_Q \rightarrow V_\mathcal{U}^{LQ}$	$\mathbf{D}_\mathcal{U}^Q$	loc. an. on N_Q , loc. alg. on L_Q

3.3 Summary of notation

The notation in the above is heavy. To ease notation, henceforth we will fix a coefficient field L/\mathbf{Q}_p , containing the fixed splitting field E of G , and drop it from the notation, writing $\mathbf{A}_{\lambda,s}^Q = \mathbf{A}_{\lambda,s}^Q(L)$, $V_\lambda^G = V_\lambda^G(L)$, etc.

In Table 1 we give a brief key of our notation in the language of Sect. 3.2.2. Note that all of the analytic function spaces can be characterized uniquely by their restrictions to a unipotent subgroup, valued in some Banach module, and then extended uniquely to J_B or J_Q using the weight action. For a classical weight λ and any $s \geq 0$, we get the chain of modules

$$\begin{aligned}
 \text{(Banach)} \quad & V_\lambda^G = \mathbf{A}_{\lambda,0}^G \subset \mathbf{A}_{\lambda,s}^Q \subset \mathbf{A}_{\lambda,s}^B = \mathcal{A}_{\lambda,s}^G \quad (3.5) \\
 & \qquad \qquad \qquad \cap \qquad \qquad \cap \qquad \qquad \cap \\
 \text{(Fréchet)} \quad & \mathbf{A}_\lambda^Q \subset \mathbf{A}_\lambda^B = \mathcal{A}_\lambda^G.
 \end{aligned}$$

The notation we maintain is that \mathbf{A}^Q means Q -parabolic induction and \mathcal{A}^G means full induction. Modules with subscripts s are Banach modules, and s denotes the degree of analyticity; those without a subscript s are Fréchet modules. Despite the equality $\mathbf{A}_{\lambda,s}^B = \mathcal{A}_{\lambda,s}^G$, we choose to maintain the separate notation \mathbf{A} and \mathcal{A} both for clarity and because the modules $\mathcal{A}_{\lambda,s}^{LQ}$ play a crucial role in the sequel.

3.4 The action of Σ_Q and local systems

Definition 3.19 Let Σ_Q denote the monoid in $G(\mathbf{Q}_p)$ generated by J_Q and T^+ .

Let \diamond denote either a single classical weight λ or an affinoid \mathcal{U} in $\mathcal{W}_{\lambda_0}^Q$ for a fixed classical λ_0 . The parahoric J_Q acts on itself by right multiplication, which then give rise to left actions of J_Q on $\mathbf{A}_{\diamond,s}^Q$ and \mathbf{A}_\diamond^Q and dual right actions on $\mathbf{D}_{\diamond,s}^Q$ and \mathbf{D}_\diamond^Q .

The action of T^+ is more subtle; we note that any function $f \in \mathbf{A}_{\diamond,s}^Q$ is uniquely determined by its restriction to $B(\mathbf{Z}_p)$, upon which $t \in T^+$ acts by $b \mapsto t^{-1}bt$ (by Proposition 2.1(iii)). In itself, this is not compatible with the action of J_Q above due to the left multiplication by t^{-1} . To rectify this, note that our choice of uniformisers defines a splitting

$$T(\mathbf{Q}_p) \xrightarrow{\sim} T(\mathbf{Z}_p) \times T(\mathbf{Q}_p)/T(\mathbf{Z}_p), \quad t \mapsto (\sigma(t), \zeta(t)). \quad (3.6)$$

Also write ζ for the composition $T(\mathbf{Q}_p) \xrightarrow{\zeta} T(\mathbf{Q}_p)/T(\mathbf{Z}_p) \hookrightarrow T(\mathbf{Q}_p)$. Then T^+ acts on $B(\mathbf{Z}_p)$ by

$$b * t = \zeta(t)^{-1}bt = \sigma(t)t^{-1}bt. \tag{3.7}$$

Now if $t \in T(\mathbf{Z}_p) = T(\mathbf{Q}_p) \cap J_Q$, then $\sigma(t)t^{-1} = 1$ and (3.7) coincides with right translation by t . If $f \in \mathbf{A}_{\delta,s}^Q$, define $t * f$ on $B(\mathbf{Z}_p)$ by $(t * f)(b) = f(b * t)$, and extend to J_Q via (3.1). A simple check shows $t * f \in \mathbf{A}_{\delta,s}^Q$ is well-defined, giving a left action of T^+ on $\mathbf{A}_{\delta,s}^Q$ and a right action on $\mathbf{D}_{\delta,s}^Q$.

Notation 3.20 If $g \in \Sigma_Q$, denote the action of g on $f \in \mathbf{A}_{\delta,s}^Q$ by $g * f$, and on $\mu \in \mathbf{D}_{\delta,s}^Q$ by $\mu * g$.

Lemma 3.21 The image of the map $r_s : \tilde{\mathbf{D}}_{\mathcal{U},s}^Q \hookrightarrow \mathbf{D}_{\mathcal{U},s}^Q$ from (3.2) is preserved by Σ_Q .

Proof We can argue exactly as in [2, Rem. 3.1]. Alternatively, we can directly write down an action on $\tilde{\mathbf{D}}_{\mathcal{U},s}^Q$: let $j \in J_Q$, $t \in T^+$ and $\mu \otimes \alpha \in \mathbf{D}_{\lambda_0,s}^Q(L) \widehat{\otimes}_L \mathcal{O}(\mathcal{U}_0)$, which we identify with $\tilde{\mathbf{D}}_{\mathcal{U},s}^Q$ via restriction to $N_Q(\mathbf{Z}_p)$. Write $j = j^{-\ell} j_n j$ under (2.5). On pure tensors, define

$$(\mu \otimes \alpha) * j = (\mu * j) \otimes \chi_{\mathcal{U}_0}(\ell_j)\alpha, \quad (\mu \otimes \alpha) * t = (\mu * t) \otimes \chi_{\mathcal{U}_0}(\sigma(t))\alpha,$$

extended by continuity. One may check explicitly that (3.2) is equivariant for the $*$ -actions. □

Suppose $K \subset \mathcal{G}(\mathbf{A}_f)$ is open compact with $K_p \subset J_Q$. Via projection to K_p , these spaces of locally analytic distributions are K -modules which then, via (2.1), give local systems over the locally symmetric space, which in a slight abuse of notation we denote by the same symbols.

Definition 3.22 The *parahoric overconvergent cohomology groups* (with respect to the parabolic Q) are the groups $H_c^i(S_K, \mathbf{D}_{\delta,s}^Q)$, $H_c^i(S_K, \mathbf{D}_{\delta}^Q)$ and $H_c^i(S_K, \tilde{\mathbf{D}}_{\mathcal{U},s}^Q)$.

The action of $t \in T^+$ then allows us to define Hecke operators U_t on the parahoric overconvergent cohomology groups, exactly as in [23, Sect. 2.1]. We extend this to an action of $\mathcal{H}(K)$ by letting $\mathcal{G}'(F_v)$ act trivially on $\mathbf{D}_{\delta,s}^Q$ for all $v \nmid p$.

Remark 3.23 (i) Note that more or less by definition, the $*$ -action of Σ_Q defined here preserves the integral subspaces $\mathbf{D}_{\lambda,s}^Q(\mathcal{O}_L)$ of Sect. 3.2.5.

(ii) The $*$ -action also preserves algebraic subspaces. In particular, we get a $*$ -action of Σ_Q on $V_\lambda^G(L)$ which preserves $V_\lambda^G(\mathcal{O}_L)$. But any $f \in V_\lambda^G(L)$ extends uniquely from $G(\mathbf{Z}_p)$ to $G(\mathbf{Q}_p)$, from which we get a natural ‘algebraic’ action of $G(\mathbf{Q}_p)$ defined by $(t \cdot f)(g) := f(gt)$. From the definition, we find that for $f \in V_\lambda^G$ and $t \in T^+$, we have

$$(t * f)(g) = f(\sigma(t)t^{-1}gt) = \lambda(\sigma(t)t^{-1})(t \cdot f)(g) \tag{3.8}$$

(compare [42, (15)]). The $--$ -action does *not* preserve $V_\lambda^G(\mathcal{O}_L)$, and the $*$ -action can be viewed as an ‘optimal’ integral normalisation of it.

(iii) For GL_2 , it is easy to write down the $--$ -action on $V_\lambda^{G,\vee}$ explicitly, and one easily sees that this explicit action extends to distributions; this is done, for example, in [2,8,35]. We warn the reader, however, that this does not give the $*$ -action of T^+ on distributions defined here: in particular, it does not preserve integrality (see [8, Sect. 9.1]).

For the remainder of this paper, unless explicitly stated, all actions will be the $*$ -actions.

3.5 Compact operators and slope decompositions

We now recap the (standard) arguments that show the parahoric overconvergent cohomology groups admit slope decompositions with respect to Q -controlling operators.

Lemma 3.24 *If $t \in T_Q^{++}$, then t acts compactly on $\mathbf{D}_{\lambda,s}^Q$ and \mathbf{D}_{λ}^Q , and on $\tilde{\mathbf{D}}_{\mathcal{U},s}^Q$ and $\mathbf{D}_{\mathcal{U}}^Q$.*

Proof At a single weight λ , we follow [42, Lemma 3.2.8]. Firstly, since by definition of T_Q^{++} we have $t^{-1}N_Q^s(\mathbf{Z}_p)t \subset N_Q^{s+1}(\mathbf{Z}_p)$, we see that $t \cdot \mathbf{A}_{\lambda,s+1}^Q(L) \subset \mathbf{A}_{\lambda,s}^Q(L)$. Hence on distributions, we have $\mathbf{D}_{\lambda,s}^Q(L) \cdot t \subset \mathbf{D}_{\lambda,s+1}^Q(L)$ (that is, t improves the analyticity). Thus the action of t factors through the (compact) inclusion map $\mathbf{D}_{\lambda,s+1}^Q(L) \hookrightarrow \mathbf{D}_{\lambda,s}^Q(L)$. This ensures that it acts compactly on $\mathbf{D}_{\lambda,s}^Q(L)$, and also the limit $\mathbf{D}_{\lambda}^Q(L)$ by definition. The statements for \mathcal{U} then follow combining this with [9, Lem. 2.9], the definition of $\tilde{\mathbf{D}}_{\mathcal{U},s}^Q$ (from Sect. 3.2.7), and Lemma 3.16. \square

If M is a module admitting a slope $\leq h$ decomposition with respect to an operator U (see, for example, [23, Definition 2.3.1]), we write it as

$$M = M^{U \leq h} \oplus M^{U > h}. \tag{3.9}$$

Let H_c^\bullet denote compactly supported (Betti) cohomology, dual to the Borel–Moore homology. The following adaptation of [1, Sect. 4] is the main reason we introduced the (ONable) spaces $\tilde{\mathbf{D}}_{\mathcal{U},s}^Q$.

Proposition 3.25 *Let K be an open compact subgroup of $\mathcal{G}(\mathbf{A}_f)$ with $K_p \subset J_Q$, let $\mathcal{U} \subset \mathcal{W}_{\lambda_0}^Q$ be an open affinoid, let $h \geq 0$, and let $t \in T_Q^{++}$. Then, possibly up to replacing \mathcal{U} with a smaller affinoid neighbourhood of λ :*

- (i) *The spaces $H_c^\bullet(S_K, \mathbf{D}_{\lambda,s}^Q)$ and $H_c^\bullet(S_K, \tilde{\mathbf{D}}_{\mathcal{U},s}^Q)$ admit slope $U_t \leq h$ decompositions for all s .*
- (ii) *The small slope parts $H_c^\bullet(S_K, \mathbf{D}_{\lambda,s}^Q)^{U_t \leq h}$ and $H_c^\bullet(S_K, \tilde{\mathbf{D}}_{\mathcal{U},s}^Q)^{U_t \leq h}$ are independent of s .*
- (iii) *Both $H_c^\bullet(S_K, \mathbf{D}_{\lambda}^Q)$ and $H_c^\bullet(S_K, \mathbf{D}_{\mathcal{U}}^Q)$ admit slope $U_t \leq h$ decompositions, and for any s*

$$H_c^\bullet(S_K, \mathbf{D}_{\lambda}^Q)^{U_t \leq h} \cong H_c^\bullet(S_K, \mathbf{D}_{\lambda,s}^Q)^{U_t \leq h}, \quad H_c^\bullet(S_K, \mathbf{D}_{\mathcal{U}}^Q)^{U_t \leq h} \cong H_c^\bullet(S_K, \tilde{\mathbf{D}}_{\mathcal{U},s}^Q)^{U_t \leq h}.$$

Proof These results are all standard, so we only give analogous references. The modules we have defined give rise to compactly supported chain complexes $C_c^\bullet(K, \mathbf{D}_{\lambda,s}^Q)$ and $C_c^\bullet(K, \tilde{\mathbf{D}}_{\mathcal{U},s}^Q)$, as at the end of [23, Sect. 3], and the compactness of t on distributions lifts to compactness of t on the complex. The cohomology of this complex gives rise to the compactly supported cohomology groups in which we are primarily interested. Since the $\mathbf{D}_{\lambda,s}^Q$ and $\tilde{\mathbf{D}}_{\mathcal{U},s}^Q$ are ONable, Propositions 2.3.3–2.3.5 of [23] then show part (i). Part (ii) is the parahoric analogue of Proposition 3.1.5 *op. cit.*, arguing identically using instead the parahoric chain complexes. Part (iii) follows in the inverse limit (using Lemma 3.16 for distributions over \mathcal{U}). \square

Note that, directly from the definitions, if M is a \mathbf{Q}_p -module that admits a slope decomposition with respect to an operator U , and $\beta \in \mathbf{Q}_p$, then

$$M^{(\beta U) \leq h} \cong M^{U \leq [h - v_p(\beta)]}. \tag{3.10}$$

4 Parahoric classicality theorems

We now prove our central result, a relative classicality theorem for parahoric overconvergent cohomology. This encompasses the analogous theorem for lifting from fully algebraic to fully analytic coefficients, and indeed we expect that it gives a numerically optimal slope bound for such a result. Our main tool is a parahoric version of Jones and Urban’s locally analytic Bernstein–Gelfand–Gelfand (BGG) resolution for classical weights λ (Corollary 4.16), which we develop in Sects. 4.2–4.4. This can also be considered as a locally analytic version of the main result of [28].

As in Sect. 3.3, we fix a coefficient field L/\mathbf{Q}_p , containing E splitting G , and drop it from notation.

4.1 The parahoric classicality theorem

Fix throughout this section a parabolic $Q \subset G$, an open compact $K \subset \mathcal{G}(\mathbf{A}_f)$ with $K_p \subset J_Q$ and a classical weight λ . Dualising the natural inclusion $V_\lambda^G \subset \mathbf{A}_{\lambda,0}^Q \subset \mathbf{A}_{\lambda,s}^Q$ yields a map $\mathbf{D}_{\lambda,s}^Q \rightarrow V_\lambda^{G,\vee}$, and a corresponding map on cohomology:

$$\rho_\lambda : \mathbf{H}_c^\bullet(S_K, \mathbf{D}_{\lambda,s}^Q) \longrightarrow \mathbf{H}_c^\bullet(S_K, V_\lambda^{G,\vee}). \tag{4.1}$$

Definition 4.1 Let ϕ be a system of Hecke eigenvalues (for $\mathcal{H}(K)$) occurring in $\mathbf{H}_c^\bullet(S_K, V_\lambda^{G,\vee})$. We say ϕ is *Q-non-critical* if the map ρ restricts to an isomorphism of ϕ -generalised eigenspaces

$$\mathbf{H}_c^\bullet(S_K, \mathbf{D}_{\lambda,s}^Q)_\phi \xrightarrow{\sim} \mathbf{H}_c^\bullet(S_K, V_\lambda^{G,\vee})_\phi.$$

Such systems ϕ naturally arise from ‘ p -refined’ automorphic representations $\tilde{\pi}$; see Sect. 4.6. We say such a $\tilde{\pi}$ is *Q-non-critical* if the associated ϕ is. We observe that for finite slope systems, this definition has no dependence on the radius of analyticity s , so is well-defined; and in fact we may pass to distributions that are fully locally analytic in Q :

Lemma 4.2 *Let ϕ be a Q-non-critical system of Hecke eigenvalues, and assume ϕ has Q-finite slope (i.e. $\phi(U_t) \neq 0$ for some $t \in T_Q^{++}$). Then for any $s \geq 0$, we have*

$$\mathbf{H}_c^\bullet(S_K, \mathbf{D}_\lambda^Q)_\phi \cong \mathbf{H}_c^\bullet(S_K, \mathbf{D}_{\lambda,s}^Q)_\phi \cong \mathbf{H}_c^\bullet(S_K, V_\lambda^{G,\vee})_\phi$$

Proof This follows from Proposition 3.25 applied with some $h \geq v_p(\phi(U_t))$. □

Definition 4.3 For λ a classical weight, $t \in T^+$ and $\alpha \in \Delta$, let

$$\begin{aligned} h^{\text{crit}}(t, \alpha, \lambda) &:= v_p(t^{w_\alpha * \lambda - \lambda}) \\ &= -[\langle \lambda, \alpha^\vee \rangle + 1] \cdot v_p(\alpha(t)). \end{aligned}$$

Here $t^\lambda := \lambda(t)$, and the equality is (2.2).

This provides a numerical criterion for Q -non-criticality. Define a maximal chain of parabolics

$$Q = P_0 \subset P_1 \subset \dots \subset P_m = G$$

containing Q , so that $\Delta_{P_i} = \Delta_{P_{i-1}} \cup \{\alpha_i\}$ for some simple root α_i . For each $i = 1, \dots, m$, let $t_i \in T^+$ such that $v_p(\alpha_i(t_i)) < 0$, and let $U_i = U_{t_i}$. The rest of Sect. 4 will be dedicated to proving:

Theorem 4.4 *Let ϕ be as in Definition 4.1. Suppose ϕ is Q -non-critical slope in the sense that*

$$h_i := v_p[\phi(U_i)] < h^{\text{crit}}(t_i, \alpha_i, \lambda)$$

for all $i = 1, \dots, m$. Then ϕ is Q -non-critical.

Example • Let $G = \text{GL}_n$, with root system A_{n-1} and simple roots $\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$.

We get a chain of parabolics P_i corresponding to $\emptyset \subset \{\alpha_1\} \subset \dots \subset \{\alpha_1, \dots, \alpha_{n-1}\} = \Delta$: precisely, $P_0 = B$ and if $i > 0$, then P_i is the parabolic with Levi $\text{GL}_{i+1} \times \text{GL}_1^{n-i-1}$. We may take $t_i = \text{diag}(1, \dots, 1, p, \dots, p)$, with p 's in the last $n - i$ entries; then for $\lambda = (\lambda_1, \dots, \lambda_n)$, we have $h^{\text{crit}}(t_i, \alpha_i, \lambda) = \lambda_i - \lambda_{i+1} + 1$. Thus a p -refined automorphic representation $\tilde{\pi}$ of $\text{GL}_n(\mathbf{A})$ with U_i -eigenvalues A_i is B -non-critical if $v_p(A_i) < \lambda_i - \lambda_{i+1} + 1$ for $i = 1, \dots, n - 1$. If we just use the usual U_p -operator corresponding to $\text{diag}(1, p, \dots, p^{n-2}, p^{n-1})$, with eigenvalue A , then the small slope criterion is (the much more restrictive) $v_p(A) < \min_i(\lambda_i - \lambda_{i+1} + 1)$.

- In the same set-up, if Q_j is the maximal standard parabolic with $\Delta_{Q_j} = \Delta \setminus \{\alpha_j\}$, then $\tilde{\pi}$ is Q_j -non-critical slope if $v_p(A_j) < h^{\text{crit}}(t_j, \alpha_j, \lambda) = \lambda_j - \lambda_{j+1} + 1$.
- Let $G = \text{GSp}_4$, with root system a generalised form of C_2 (with an additional basis vector e_3 for the character space; see [37, Sect. 2.3] for more details). Let \mathcal{F} be a Siegel eigenform with weight $\lambda = (k_1 + 3, k_2 + 3)$ with $k_1 \geq k_2 \geq 0$; then \mathcal{F} is cohomological. This corresponds to the character $k_1 e_1 + k_2 e_2 + 0e_3$. The simple roots are $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2 - e_3$, with coroots $\alpha_1^\vee = e_1^\vee - e_2^\vee$ and $\alpha_2^\vee = e_2^\vee$. There are thus two non-minimal parabolics:
 - The *Siegel* parabolic corresponds to $\{\alpha_1\}$. Letting $t^{\text{Sie}} := e_3^\vee(p) \in T(\mathbf{Q}_p)$, we see $v_p(\alpha_1(t^{\text{Sie}})) = 0$ and $v_p(\alpha_2(t^{\text{Sie}})) = -1$, and we get a Siegel-controlling operator $U_p^{\text{Sie}} := U_{t^{\text{Sie}}}$. Let A_p^{Sie} be the U_p^{Sie} -eigenvalue. Then \mathcal{F} is Siegel-non-critical slope if $v_p(A_p^{\text{Sie}}) < -\langle \lambda, \alpha_1^\vee \rangle \cdot -1 = k_2 + 1$.
 - The *Klingen* parabolic corresponds to $\{\alpha_2\}$. Letting $t^{\text{Kli}} := (e_2^\vee + 2e_3^\vee)(p) \in T(\mathbf{Q}_p)$, we get $v_p(\alpha_1(t^{\text{Kli}})) = -1$ and $v_p(\alpha_2(t^{\text{Kli}})) = 0$; define $U_p^{\text{Kli}} = U_{t^{\text{Kli}}}$, with \mathcal{F} -eigenvalue A_p^{Kli} . Then \mathcal{F} is Klingen-non-critical slope if $v_p(A_p^{\text{Kli}}) < -\langle \lambda, \alpha_1^\vee \rangle \cdot -1 = k_1 - k_2 + 1$.

We may identify the torus in G with a subgroup of the diagonal matrices in GL_4 , after which t^{Sie} is the matrix $\text{diag}(1, 1, p, p)$ and t^{Kli} is $\text{diag}(1, p, p, p^2)$.

Remark 4.5 Our definition of Q -non-critical uses cohomology with compact support H_c^\bullet ; to be more precise, we could call this *Q -non-critical for H_c^\bullet* . It is also common to use Betti cohomology (without support) H^\bullet , as in for example [23, 42], giving a (directly analogous) notion of *Q -non-critical for H^\bullet* . It seems natural to expect that the two notions are equivalent, but it does not *a priori* appear obvious that this is the case. However, Theorem 4.4 applies equally well to both cases: so Q -non-critical slope implies both flavours of Q -non-criticality. Henceforth, unless specified otherwise, our notion of non-critical should be clear from the underlying setting.

4.2 Analytic BGG for the Borel

We recap the usual locally analytic BGG resolution (Theorem 4.6). Recall $\mathcal{A}(J_B, L)$ is the space of locally L -analytic functions on the Iwahori J_B , and $\mathcal{A}_\lambda^G \subset \mathcal{A}(J_B, L)$. We have a

left action of J_B on $\mathcal{A}(J_B, L)$ by $l(h) \cdot f(g) = f(h^{-1}g)$. This action is L -analytic, and thus induces an analytic action of \mathfrak{g} . Explicitly, $X \in \mathfrak{g}$ acts by

$$l(X) \cdot f(g) = \frac{d}{dt} \left([l(\exp(-tX)) \cdot f](g) \right) \Big|_{t=0}.$$

This extends in a natural way to an action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$.

By [42, Prop. 3.2.11], for each simple root $\alpha \in \Delta$, the map $f \mapsto l(X_\alpha) \cdot f$ induces a map $\mathcal{A}_\lambda^G \rightarrow \mathcal{A}_{\lambda-\alpha}^G$. By (2.2), $w_\alpha * \lambda = \lambda - [\langle \lambda, \alpha^\vee \rangle + 1]\alpha$ for a classical weight λ , and we have a map

$$\begin{aligned} \Theta_\alpha : \mathcal{A}_\lambda^G &\longrightarrow \mathcal{A}_{w_\alpha * \lambda}^G \\ f &\longmapsto l(X_\alpha^{\langle \lambda, \alpha^\vee \rangle + 1}) \cdot f. \end{aligned}$$

This is J_B -equivariant and (recalling ζ from (3.6)) transforms under $t \in T^+$ as

$$\Theta_\alpha(t * f) = \alpha(\zeta(t))^{-\langle \lambda, \alpha^\vee \rangle - 1} [t * \Theta_\alpha(f)]. \tag{4.2}$$

The following describes the first few terms of the locally analytic BGG resolution. Let $V_{\lambda, \text{loc}}^G \subset \mathcal{A}_\lambda^G$ be the subspace of functions that are locally L -algebraic on J_B , that is, the union of the subspaces of s -algebraic functions over all $s \geq 0$.

Theorem 4.6 [27, Thm. 26], [42, Prop. 3.2.12] *Let λ be a classical weight. There is an exact sequence*

$$0 \rightarrow V_{\lambda, \text{loc}}^G \longrightarrow \mathcal{A}_\lambda^G \xrightarrow{\oplus \Theta_\alpha} \bigoplus_{\alpha \in \Delta} \mathcal{A}_{w_\alpha * \lambda}^G.$$

The action of \mathfrak{g} on $\mathcal{A}(J_B, L)$ preserves $\mathcal{A}_0(J_B, L)$ (as we can define it on this space directly). Hence we have maps $X_\alpha : \mathcal{A}_{\lambda, 0}^G \rightarrow \mathcal{A}_{\lambda-\alpha, 0}^G$ and $\Theta_\alpha : \mathcal{A}_{\lambda, 0}^G \rightarrow \mathcal{A}_{w_\alpha * \lambda, 0}^G$.

Corollary 4.7 *Let λ be a classical weight. There is an exact sequence*

$$0 \rightarrow V_\lambda^G \longrightarrow \mathcal{A}_{\lambda, 0}^G \xrightarrow{\oplus \Theta_\alpha} \bigoplus_{\alpha \in \Delta} \mathcal{A}_{w_\alpha * \lambda, 0}^G.$$

Proof Since $V_\lambda^G \subset V_{\lambda, \text{loc}}^G$, it is a subset of $\ker(\oplus \Theta_\alpha)$. Conversely, if $f \in \ker(\oplus \Theta_\alpha)$, then by Theorem 4.6 it lies in $V_{\lambda, \text{loc}}^G \cap \mathcal{A}_{\lambda, 0}^G = V_\lambda^G$ (see [42, Sect. 3.2.9]). \square

4.3 Theta operators on parahoric distributions

We now describe $\mathbf{A}_{\lambda, 0}^Q$ as a canonical subspace of $\mathcal{A}_{\lambda, 0}^G$. If $f \in \mathbf{A}_{\lambda, 0}^Q$ and $n \in N_Q(\mathbf{Z}_p)$, then by definition $[f(n) : L_Q(\mathbf{Z}_p) \rightarrow L] \in V_\lambda^{L_Q}$.

Proposition 4.8 *There is an injective Σ_B -equivariant map $\iota_Q : \mathbf{A}_{\lambda, 0}^Q \hookrightarrow \mathcal{A}_{\lambda, 0}^G$ defined by*

$$\iota_Q(f) : J_B \longrightarrow L, \quad \iota_Q(f)(g) := f(g)(\text{id}_{L_Q}).$$

Proof Note $\iota_Q(f)$ is analytic since f is. Let $t \in T(\mathbf{Z}_p)$ and $n^- \in J_B^-$, and write $n^- = n_Q^- \ell^-$ with $n_Q^- \in J_Q^-$ and $\ell^- \in L_Q(\mathbf{Z}_p) \cap J_B^-$. Using (3.1) for f and the L_Q -action on $V_\lambda^{L_Q}$, we have

$$\begin{aligned} \iota_Q(f)(n^- t g) &= f(n_Q^- \ell^- t g)(\text{id}_{L_Q}) = [(\ell^- t)_\lambda f(g)](\text{id}_{L_Q}) \\ &= f(g)(\ell^- t) = \lambda(t) f(g)(\text{id}_{L_Q}) = \lambda(t) \iota_Q(f)(g), \end{aligned}$$

so $\iota_Q(f)$ has the right transformation property and ι_Q is well-defined. Since f is uniquely determined by its restriction to $N_Q(\mathbf{Z}_p) \subset J_B$, the map is injective. The map is J_B -equivariant since if $j \in J_B$, then $\iota_Q(j * f)(g) = f(jg)(\text{id}_{L_Q}) = \iota_Q(f)(gj) = j * \iota_Q(f)$. If $t \in T^+$, for $b \in B(\mathbf{Z}_p)$ similarly $\iota_Q(t * f)(b) = f(\sigma(t)t^{-1}bt)(\text{id}_{L_Q}) = \iota_Q(f)(\sigma(t)t^{-1}bt) = (t * \iota_Q(f))(b)$, so ι_Q is also T^+ -equivariant and hence Σ_B -equivariant. \square

From now on, we freely identify $\mathbf{A}_{\lambda,0}^Q$ with its image $\iota_Q(\mathbf{A}_{\lambda,0}^Q)$ in $\mathcal{A}_{\lambda,0}^G$. We can give an intrinsic criterion for an element of $\mathcal{A}_{\lambda,0}^G$ to be in this subset.

Definition 4.9 Let $n \in N_Q(\mathbf{Z}_p)$. Define a map

$$\mathcal{R}_n : \mathcal{A}_{\lambda,0}^G \longrightarrow \mathcal{A}_{\lambda,0}^{L_Q},$$

where $\mathcal{R}_n(f) : L_Q(\mathbf{Z}_p) \cap J_B \rightarrow L$ is defined by $\ell \mapsto f(\ell n)$ (noting that $L_Q(\mathbf{Z}_p) \cap J_B$ is the Iwahori subgroup in $L_Q(\mathbf{Z}_p)$). Alternatively, $\mathcal{R}_n(f)$ is the restriction of $(n * f)$ to $L_Q(\mathbf{Z}_p) \cap J_B$.

Proposition 4.10 Let $f \in \mathcal{A}_{\lambda,0}^G$. Then $f \in \mathbf{A}_{\lambda,0}^Q$ if and only if $\mathcal{R}_n(f) \in V_\lambda^{L_Q}$ for all $n \in N_Q(\mathbf{Z}_p)$, that is, for all n we have

$$\begin{array}{ccc} V_\lambda^G & \subset & \mathbf{A}_{\lambda,0}^Q \subset \mathcal{A}_{\lambda,0}^G \\ & & \downarrow \mathcal{R}_n \quad \downarrow \mathcal{R}_n \\ & & V_\lambda^{L_Q} \subset \mathcal{A}_{\lambda,0}^{L_Q}. \end{array}$$

Proof If $f' \in \mathbf{A}_{\lambda,0}^Q, \ell \in L_Q(\mathbf{Z}_p)$ and $n \in N_Q(\mathbf{Z}_p)$, then $f'(\ell n)(\text{id}_{L_Q}) = [(\ell)_\lambda f'(n)](\text{id}_{L_Q}) = f'(n)(\ell)$.

Thus if $f = \iota_Q(f')$ for some $f' \in \mathbf{A}_{\lambda,0}^Q$, then $\mathcal{R}_n(\iota_Q(f')) = f'(n) \in V_\lambda^{L_Q}$. Conversely if $\mathcal{R}_n(f) \in V_\lambda^{L_Q}$ for all n , then the function $f' : N_Q(\mathbf{Z}_p) \rightarrow V_\lambda^{L_Q}$ defined by $f'(n) = \mathcal{R}_n(f)$ defines an element of $\mathbf{A}_{\lambda,0}^Q$ and satisfies $\iota_Q(f') = f$. \square

Remark 4.11 If $P \subset Q$ are two parabolics, as $L_P \subset L_Q$ and $N_Q \subset N_P$ there is an injective extension-by-zero map $V_\lambda^{L_P} \hookrightarrow V_\lambda^{L_Q}$. We deduce that $\mathbf{A}_{\lambda,0}^Q \subset \mathbf{A}_{\lambda,0}^P$.

By definition of Δ_Q , if $\alpha \in \Delta_Q$ then $X_\alpha \in \mathfrak{l}_Q = \text{Lie}(L_Q)$, so α is a simple root of L_Q and we get a well-defined map $\Theta_\alpha : \mathcal{A}_{\lambda,0}^{L_Q} \rightarrow \mathcal{A}_{w_\alpha * \lambda,0}^{L_Q}$.

Lemma 4.12 Let $n \in N_Q(\mathbf{Z}_p)$. For all $\alpha \in \Delta_Q$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{\lambda,0}^G & \xrightarrow{\Theta_\alpha} & \mathcal{A}_{w_\alpha * \lambda,0}^G \\ \downarrow \mathcal{R}_n & & \downarrow \mathcal{R}_n \\ \mathcal{A}_{\lambda,0}^{L_Q} & \xrightarrow{\Theta_\alpha} & \mathcal{A}_{w_\alpha * \lambda,0}^{L_Q}. \end{array}$$

Proof It suffices to prove that \mathcal{R}_n commutes with the action of X_α on $\mathcal{A}_0(J_B, L)$. But if $f \in \mathcal{A}_0(J_B, L)$, then for all $\ell \in L_Q(\mathbf{Z}_p) \cap J_B$, we have

$$\begin{aligned} [l(X_\alpha) \cdot \mathcal{R}_n(f)](\ell) &= \frac{d}{dt} \mathcal{R}_n(f)(\exp(-tX_\alpha)\ell)|_{t=0} \\ &= \frac{d}{dt} f(\exp(-tX_\alpha)\ell n)|_{t=0} = [l(X_\alpha) \cdot f](\ell n) = \mathcal{R}_n(l(X_\alpha) \cdot f)(\ell). \end{aligned}$$

\square

Lemma 4.13 *Suppose $\alpha \in \Delta_Q$. Then $\mathbf{A}_{\lambda,0}^Q \subset \ker(\Theta_\alpha)$.*

Proof If $f \in \mathbf{A}_{\lambda,0}^Q$, then $\mathcal{R}_n(f) \in V_\lambda^{L_Q}$ for all $n \in N_Q(\mathbf{Z}_p)$ by Proposition 4.10; thus

$$\mathcal{R}_n(\Theta_\alpha(f)) = \Theta_\alpha(\mathcal{R}_n(f)) \in V_{w_\alpha * \lambda}^{L_Q}$$

is also algebraic, the equality being Lemma 4.12. Then 4.10 again says $\Theta_\alpha(f) \in \mathbf{A}_{w_\alpha * \lambda,0}^Q$.

As α is a root for L_Q , the weight $w_\alpha * \lambda$ is not dominant for L_Q . It follows that $V_{w_\alpha * \lambda}^{L_Q} = 0$, which forces $\mathbf{A}_{w_\alpha * \lambda,0}^Q = 0$ by definition. It follows that $\mathbf{A}_{\lambda,0}^Q \subset \ker(\Theta_\alpha)$. \square

We saw if $\alpha \in \Delta_Q$, then $\Theta_\alpha(\mathbf{A}_{\lambda,0}^Q) \subset \mathbf{A}_{w_\alpha * \lambda,0}^Q$. We want to prove this for $\alpha \notin \Delta_Q$. Such an α is not a root of L_Q , so we cannot follow the same strategy. Instead, we argue directly:

Proposition 4.14 *For $\alpha \in \Delta \setminus \Delta_Q$, we have $\Theta_\alpha(\mathbf{A}_{\lambda,0}^Q) \subset \mathbf{A}_{w_\alpha * \lambda,0}^Q$.*

Proof Choose a set of co-ordinates y_i on $L_Q(\mathbf{Z}_p) \cap J_B$ that identify it as a subset of \mathbf{Z}_p^r . We also have a set of co-ordinates z_j on $N_Q(\mathbf{Z}_p)$, indexed by $j \in R^+ \setminus R_Q^+$ as in (2.7). Let $f \in \mathbf{A}_{\lambda,0}^Q$. If $g \in Q(\mathbf{Z}_p) \cap J_B$, write it as

$$g = \ell_g n_g, \quad \ell_g \in L_Q(\mathbf{Z}_p) \cap J_B, n_g \in N_Q(\mathbf{Z}_p).$$

We may write $f(g) = f(y_i(\ell_g), z_j(n_g))$ in the co-ordinates above; then by definition, f is algebraic in the y_i and analytic in the z_j .

To show the proposition, by Proposition 4.10 we must show that $\mathcal{R}_n(\Theta_\alpha(f))$ is algebraic on $L_Q(\mathbf{Z}_p) \cap J_B$ for all $n \in N_Q(\mathbf{Z}_p)$. If $\ell \in L_Q(\mathbf{Z}_p) \cap J_B$, then

$$\mathcal{R}_n(\Theta_\alpha(f))(\ell) = (n * \Theta_\alpha(f))(\ell) = \Theta_\alpha(n * f)(\ell),$$

the last equality since Θ_α respects the $*$ action of J_B . Replacing f with $n * f$, it then suffices to prove that the restriction of $\Theta_\alpha(f)$ to $L_Q(\mathbf{Z}_p) \cap J_B$ lies in $V_\lambda^{L_Q}$. By definition, this is the function

$$\ell \mapsto \frac{d}{dt} f(\exp(-tX_\alpha)\ell) \Big|_{t=0}.$$

Since $\alpha \notin \Delta_Q$, a sufficiently small neighbourhood U of 0 in $\mathbf{Q}_p X_\alpha \subset \mathfrak{n}$ is contained in $n_Q(\mathbf{Z}_p)$. For t in such a U , we have $\exp(-tX_\alpha) \in N_Q(\mathbf{Z}_p)$. This is a normal subgroup in $G(\mathbf{Z}_p)$, so in particular, for any $\ell \in L_Q(\mathbf{Z}_p) \cap J_B$ we have $\exp(-tX_\alpha)\ell = \ell e(\ell, t)$ with $e(\ell, t) = \ell^{-1} \cdot \exp(-tX_\alpha) \cdot \ell \in N_Q(\mathbf{Z}_p)$. Then we have

$$\begin{aligned} \Theta(f)(\ell) &= \frac{d}{dt} f(\ell e(\ell, t)) \Big|_{t=0} \\ &= \frac{d}{dt} f(y_i(\ell); z_j(e(\ell, t))) \Big|_{t=0}. \end{aligned}$$

The co-ordinates $z_j(e(\ell, t))$, which are linear functions in t , are algebraic in the $y_i(\ell)$ (since inverse and multiplication operations are algebraic on a reductive group). We know f is algebraic in the $y_i(\ell)$, and analytic in the $z_j(e(\ell, t))$; and by above the coefficient of the linear term in t is algebraic in the $y_i(\ell)$. We deduce that $\Theta_\alpha(f)(\ell) = \frac{d}{dt} f(\ell e(\ell, t)) \Big|_{t=0}$ is algebraic in the $y_i(\ell)$, as required. \square

4.4 The parahoric analytic BGG resolution

Proposition 4.15 *For a classical weight λ , there is an exact sequence*

$$0 \rightarrow \mathbf{A}_{\lambda,0}^Q \rightarrow \mathcal{A}_{\lambda,0}^G \xrightarrow{\oplus \Theta_\alpha} \bigoplus_{\alpha \in \Delta_Q} \mathcal{A}_{w_\alpha * \lambda, 0}^G.$$

Proof That $\mathbf{A}_{\lambda,0}^Q \subset \bigcap \ker(\Theta_\alpha)$ is an immediate consequence of Lemma 4.13. To see the converse, suppose $f \in \ker := \bigcap \ker(\Theta_\alpha)$. Then for all $n \in N_Q(\mathbf{Z}_p)$, by Lemma 4.12 we have $\Theta_\alpha(\mathcal{R}_n(f)) = \mathcal{R}_n(\Theta_\alpha(f)) = 0$ for any $\alpha \in \Delta_Q$. Thus we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker & \longrightarrow & \mathcal{A}_{\lambda,0}^G & \xrightarrow{\oplus \Theta_\alpha} & \bigoplus_{\alpha \in \Delta_Q} \mathcal{A}_{w_\alpha * \lambda, 0}^G \\ & & \downarrow \mathcal{R}_n & & \downarrow \mathcal{R}_n & & \downarrow \mathcal{R}_n \\ 0 & \longrightarrow & V_\lambda^{L_Q} & \longrightarrow & \mathcal{A}_{\lambda,0}^{L_Q} & \xrightarrow{\oplus \Theta_\alpha} & \bigoplus_{\alpha \in \Delta_Q} \mathcal{A}_{w_\alpha * \lambda, 0}^{L_Q}, \end{array}$$

where exactness of the bottom row is Corollary 4.7 for the group L_Q , noting that Δ_Q is precisely the set of simple roots for L_Q corresponding to the Borel $B \cap L_Q$. But then $\mathcal{R}_n(f) \in V_\lambda^{L_Q}$ for any n ; thus by Proposition 4.10 we have $f \in \mathbf{A}_{\lambda,0}^Q$, as required. \square

Corollary 4.16 *Let $P \subset Q$ be two standard parabolics, with $\Delta_P \cup \{\beta\} = \Delta_Q$ (that is, there is no parabolic P' with $P \subsetneq P' \subsetneq Q$). There is an exact sequence*

$$0 \rightarrow \mathbf{A}_{\lambda,0}^Q \rightarrow \mathbf{A}_{\lambda,0}^P \xrightarrow{\Theta_\beta} \mathbf{A}_{w_\beta * \lambda, 0}^P.$$

Proof We restrict the map $\oplus \Theta_\alpha$ of 4.15 from $\mathcal{A}_{\lambda,0}^G$ to $\mathbf{A}_{\lambda,0}^P$. It is clear that the kernel of this restriction is $\mathbf{A}_{\lambda,0}^Q \cap \mathbf{A}_{\lambda,0}^P = \mathbf{A}_{\lambda,0}^Q$, the equality following by Remark 4.11. If $\alpha \in \Delta_Q$ is not equal to β , then $\alpha \in \Delta_P$, so $\mathbf{A}_{\lambda,0}^P \subset \ker(\Theta_\alpha)$ by Lemma 4.13. In particular, the direct sum $\bigoplus_{\alpha \in \Delta_Q} \Theta_\alpha$ collapses, with Θ_β the only non-zero term. The image lands in $\mathbf{A}_{w_\beta * \lambda, 0}^P$ by Proposition 4.14, giving the claimed exact sequence. \square

4.5 Proof of Theorem 4.4

We can finally prove our main result. Recall from Theorem 4.4 that $Q = P_0 \subset \dots \subset P_m = G$ is a maximal chain of parabolics, $\Delta_{P_{i-1} \cup \{\alpha_i\}} = \Delta_{P_i}$, $t_i \in T^+$ with $v_p(\alpha_i(t_i)) < 0$, $U_i := U_{t_i}$ and $h_i < h_i^{\text{crit}} := h^{\text{crit}}(t_i, \alpha_i, \lambda)$.

Proof (Theorem 4.4). First we make sense of taking U_i -slope decompositions on \mathbf{D}^{P_i} -cohomology. Note that $t_Q = t_1 \dots t_m$ is in $T_Q^{++} \subset T_{P_i}^{++}$ by Proposition 2.3, hence it acts compactly on each $\mathbf{D}_{\lambda,0}^{P_i}$ by Lemma 3.24; we get a Q -controlling operator $U_{\text{aux}} = U_{t_Q}$ on $H_c^\bullet(S_K, \mathbf{D}_{\lambda,0}^{P_i})$ for each i , and we can take slope decompositions. Let $h_{\text{aux}} \gg v_p(\phi(U_{\text{aux}}))$, so that for each i , we have

$$H_c^\bullet(S_K, \mathbf{D}_{\lambda,0}^{P_i})_\phi \subset H_c^\bullet(S_K, \mathbf{D}_{\lambda,0}^{P_i})^{U_{\text{aux}} \leq h_{\text{aux}}}. \tag{4.3}$$

By the theory of slope decompositions, the right-hand side is Hecke-stable and finite-dimensional over L ; thus we may take further slope decompositions for U_i , as they always exist on finite-dimensional spaces.

Lemma 4.17 *The map ρ_λ induces an isomorphism*

$$[H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_{i-1}})^{U_{\text{aux}} \leq h_{\text{aux}}}]^{U_i \leq h_i} \xrightarrow{\sim} [H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_i})^{U_{\text{aux}} \leq h_{\text{aux}}}]^{U_i \leq h_i}.$$

Proof Consider Corollary 4.16 applied to the pair (P_{i-1}, P_i) . Dualising gives an exact sequence

$$\mathbf{D}_{w_{\alpha_i} * \lambda, 0}^{P_{i-1}} \xrightarrow{\Theta_{\alpha_i}} \mathbf{D}_{\lambda,0}^{P_{i-1}} \longrightarrow \mathbf{D}_{\lambda,0}^{P_i} \longrightarrow 0.$$

Let $D^{P_{i-1}} = \mathbf{D}_{w_{\alpha_i} * \lambda, 0}^{P_{i-1}} / \ker(\Theta_{\alpha_i})$. We get an induced long exact sequence of cohomology

$$H_c^j(S_K, D^{P_{i-1}}) \rightarrow H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_{i-1}}) \rightarrow H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_i}) \rightarrow H_c^{j+1}(S_K, D^{P_{i-1}}).$$

By (4.2), this sequence is equivariant for the operators $\alpha_i(\zeta(t_Q))^{-(\lambda, H_{\alpha_i})^{-1}} U_{\text{aux}}$ (for $D^{P_{i-1}}$ -coefficients), and U_{aux} (for $\mathbf{D}_{\lambda,0}^{P_{i-1}}, \mathbf{D}_{\lambda,0}^{P_i}$ coefficients); let $h'_{\text{aux}} = h_{\text{aux}} - h^{\text{crit}}(t_Q, \alpha_i, \lambda)$, which is still $\gg v_p(\phi(U_{\text{aux}}))$. As taking slope decompositions is exact [42, Cor. 2.3.5], after renormalising with (3.10), for each i we obtain an exact sequence

$$\begin{aligned} H_c^j(S_K, D^{P_{i-1}})^{U_{\text{aux}} \leq h'_{\text{aux}}} &\rightarrow H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_{i-1}})^{U_{\text{aux}} \leq h_{\text{aux}}} \\ &\rightarrow H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_i})^{U_{\text{aux}} \leq h_{\text{aux}}} \rightarrow H_c^{j+1}(S_K, D^{P_{i-1}})^{U_{\text{aux}} \leq h'_{\text{aux}}}. \end{aligned}$$

This sequence in turn is equivariant for the operators $\alpha_i(\zeta(t_i))^{-(\lambda, H_{\alpha_i})^{-1}} U_i$ and U_i respectively, and taking further (renormalised) slope decompositions we obtain

$$\begin{aligned} [H_c^j(S_K, D^{P_{i-1}})^{U_{\text{aux}} \leq h'_{\text{aux}}}]^{U_i \leq h_i - h_i^{\text{crit}}} &\rightarrow [H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_{i-1}})^{U_{\text{aux}} \leq h_{\text{aux}}}]^{U_i \leq h_i} \\ &\rightarrow [H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_i})^{U_{\text{aux}} \leq h_{\text{aux}}}]^{U_i \leq h_i} \rightarrow [H_c^{j+1}(S_K, D^{P_{i-1}})^{U_{\text{aux}} \leq h'_{\text{aux}}}]^{U_i \leq h_i - h_i^{\text{crit}}}. \end{aligned}$$

using that $v_p(\alpha_i(\zeta(t_i))) = v_p(\alpha_i(t_i))$. From Sect. 3.2.5 and Remark 3.23, all of the coefficient spaces admit natural Σ_{P_i} -stable integral structures which give natural U_i -stable integral structures on the cohomology (and their small slope parts for U_{aux}). As $h_i - h_i^{\text{crit}} < 0$ by assumption, it follows that the first and last terms of the exact sequence vanish by [8, Lem. 9.1]. \square

We return to the proof of Theorem 4.4. For M as above and $\mathbf{h} = (h_1, \dots, h_m)$, define

$$M^{\leq \mathbf{h}} := \bigcap_{i=1}^m (M^{U_{\text{aux}} \leq h_{\text{aux}}})^{U_i \leq h_i}.$$

Since $\phi(U_i)$ has p -adic valuation h_i , we know U_i acts with slope $\leq h_i$ on M_ϕ for any space M ; so for each i , combining with (4.3), we immediately obtain

$$H_c^\bullet(S_K, \mathbf{D}_{\lambda,0}^{P_i})_\phi \subset H_c^\bullet(S_K, \mathbf{D}_{\lambda,0}^{P_i})^{\leq \mathbf{h}}. \tag{4.4}$$

Thus it suffices to prove that the slope criteria forces

$$H_c^\bullet(S_K, \mathbf{D}_{\lambda,0}^Q)^{\leq \mathbf{h}} \cong H_c^\bullet(S_K, V_\lambda^{G,\vee})^{\leq \mathbf{h}}.$$

For each i , using the slope assumption and restricting Lemma 4.17 we obtain isomorphisms

$$H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_{i-1}})^{\leq \mathbf{h}} \cong H_c^j(S_K, \mathbf{D}_{\lambda,0}^{P_i})^{\leq \mathbf{h}}.$$

Chaining this together for $i = 1, \dots, m$, we obtained the claimed isomorphism

$$\begin{aligned} H_c^\bullet(S_K, \mathbf{D}_{\lambda,0}^Q)^{\leq \mathbf{h}} &= H_c^\bullet(S_K, \mathbf{D}_{\lambda,0}^{P_0})^{\leq \mathbf{h}} \cong \dots \\ &\cong H_c^\bullet(S_K, \mathbf{D}_{\lambda,0}^{P_m})^{\leq \mathbf{h}} = H_c^\bullet(S_K, V_\lambda^{G,\vee})^{\leq \mathbf{h}}. \end{aligned}$$

4.6 Hecke normalisations and connections to automorphic representations

We conclude this section with some remarks on applying these results in the context of automorphic representations. Recall from Remark 3.23 that there are two natural actions of Σ_Q on $V_\lambda^{G,\vee}$: a $*$ -action induced by considering $V_\lambda^{G,\vee}$ as a stable quotient of $\mathcal{D}_{\lambda,0}^G$, well-adapted for p -adic computations; and a \cdot -action coming from the algebraic action, well-adapted to automorphic computations. As explained in Remark 3.23 these actions agree on K , so give the same local system on S_K ; but they *differ* on T^+ , giving different Hecke actions on the resulting cohomology. Thus attached to $t \in T^+$ we get two Hecke operators U_t^* and U_t on $H_c^*(S_K, V_\lambda^{G,\vee})$, and by (3.8), we have

$$U_t^* = \lambda(\sigma(t)t^{-1}) \times U_t.$$

Now, let π be a cuspidal cohomological automorphic representation of $\mathcal{G}(\mathbf{A})$ admitting K -invariant vectors, and fix an eigenform $\mathcal{F} \in \pi_f^K$. We call the pair (π, \mathcal{F}) a p -refinement of π and denote it $\tilde{\pi}$, with associated eigensystem $\phi_{\tilde{\pi}}^* : \mathcal{H}_K \rightarrow \mathbf{C}$. In favourable situations, one may use Lie algebra cohomology and complex periods to construct a (typically non-canonical) Hecke eigenclass $\psi_{\tilde{\pi}} \in H_c^*(S_K, V_\lambda^{G,\vee})$ which for the \cdot -action has the same Hecke eigenvalues as $\tilde{\pi}$. In particular, if we view $\psi_{\tilde{\pi}}$ as a class in the cohomology with the $*$ -action—as we have done throughout this paper—we must instead consider slope conditions for $\phi_{\tilde{\pi}}^*(U_t) = \lambda(\sigma(t)t^{-1}) \times \phi_{\tilde{\pi}}^*(U_t)$. We summarise this in the following corollary of Theorem 4.4.

Corollary 4.18 *Let $\tilde{\pi}$ be as above. Let $Q = P_0 \subset \dots \subset P_m = G$ be a maximal chain of parabolics, and for $i = 1, \dots, m$, let U_i be as in Theorem 4.4. Let $a_i = \phi_{\tilde{\pi}}^*(U_i)$ denote the U_i eigenvalue of \mathcal{F} , and let $a_i^\circ := \lambda(\sigma(t)t_i^{-1})a_i$, an ‘integral normalisation’ of a_i . If*

$$v_p(a_i^\circ) = v_p(\lambda^{-1}(t_i)) + v_p(a_i) < h^{\text{crit}}(t_i, \alpha_i, \lambda)$$

for all $i = 1, \dots, m$, then $\tilde{\pi}$ is Q -non-critical.

(Note the modification factor is only $v_p(\lambda^{-1}(t_i))$ as $v_p(\lambda(\sigma(t_i))) = 0$). Sometimes the operator $\lambda(\sigma(t)t^{-1})U_t$ is denoted U_t° , and (in light of Remark 3.23) is called the ‘optimal integral normalisation’ of the classical automorphic Hecke operator U_t . In other places—e.g. [23]—the specialisation map is defined using the $*$ -action on distributions and \cdot -action on algebraic coefficients, and is then referred to as an ‘intertwining’ of U_t^* and $\lambda(\sigma(t)t^{-1})U_t$.

Remark 4.19 We remark finally that it there are two common sets of conventions when defining local systems. We have taken all of our modules to be *right* K -modules, as this is natural/standard in the p -adic setting. For automorphic computations is is frequently more natural to consider only *left* K -modules. This then flips every convention in this paper, so that for example the \leq and $<$ of Propositions 2.1 and 2.3 become \geq and $>$, the action of K is induced by right-translation by the inverse, and the $*$ -action is induced by $g \mapsto \sigma(t)^{-1}tgt^{-1}$. In particular, a controlling operator for GL_2 would be given by $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$ rather than $\begin{pmatrix} 1 & \\ & p \end{pmatrix}$. Let w_0 be the longest Weyl element for G , and let $\lambda^\vee = -\lambda^{w_0}$ denote the contragredient weight. Since $V_\lambda^{G,\vee} \cong V_{\lambda^\vee}^G$ when equipped with the left \cdot -actions, by mimicking the calculation of Remark 3.23, we see that in this set-up we have instead that $U_t^* = \lambda^\vee(\sigma(t)^{-1}t) \times U_t$, and we would define $a_i^\circ = \lambda^\vee(\sigma(t)^{-1}t_i)a_i$.

5 Parabolic eigenvarieties

We now construct a theory of parabolic families of automorphic representations. There are two approaches to constructing eigenvarieties from overconvergent cohomology, with differing benefits and drawbacks. We could use *total* cohomology, as in [23,42], giving more accessible general results; or a single degree of cohomology, which is often of more arithmetic use (see, for example, the ‘middle degree’ eigenvariety of [3], or the ‘parallel weight’ eigenvariety of [6]). This, however, requires the pinning down of Hecke eigenpackets in the specified degree, so typically requires more refined arguments to study. In the below, a $*$ will denote either total cohomology \bullet or a specific degree $d \in \mathbf{Z}_{\geq 0}$.

Fix throughout a parabolic subgroup Q , and a level group K with $K_p \subset J_Q$; all our eigenvarieties will depend on this K , but since it is fixed we drop it from all notation. Fix also a ‘base-weight’ $\lambda_0 \in \mathcal{W}$, giving a subspace $\mathcal{W}_{\lambda_0}^Q \subset \mathcal{W}$ as in Sect. 3.1. All other notation will be as above.

5.1 Local pieces of the eigenvariety

The eigenvarieties we consider are defined using the parahoric overconvergent cohomology groups for Q . The local pieces are defined as the rigid analytic spectra of Hecke algebras acting on these spaces.

Fix for the rest of this section a controlling operator U_t (for $t \in T_Q^{++}$); all slope decompositions will be with respect to U_t . Let $\mathcal{U} \subset \mathcal{W}_{\lambda_0}^Q$ be an affinoid. The pair (\mathcal{U}, h) is a *slope-adapted pair* if the cohomology $H_c^*(S_K, \mathbf{D}_{\mathcal{U}}^Q)$ admits a slope $\leq h$ decomposition. Recall $\mathcal{H}(K)$ from Definition 2.2.

Definition 5.1 For a slope-adapted pair (\mathcal{U}, h) , let

$$\mathbf{T}_{\mathcal{U},h}^{Q,*} := \text{image of } \mathcal{H}(K) \otimes_{\mathbf{Q}_p} \mathcal{O}(\mathcal{U}) \text{ in } \text{End}_{\mathcal{O}(\mathcal{U})}(H_c^*(S_K, \mathbf{D}_{\mathcal{U}}^Q)^{\leq h}).$$

Define the *local piece of the eigenvariety* at (\mathcal{U}, h) to be the rigid analytic space

$$\mathcal{E}_{\mathcal{U},h}^{Q,*} := \text{Sp}(\mathbf{T}_{\mathcal{U},h}^{Q,*}).$$

The natural structure map $\mathcal{O}(\mathcal{U}) \rightarrow \mathbf{T}_{\mathcal{U},h}^{Q,*}$ gives rise to a map $w : \mathcal{E}_{\mathcal{U},h}^{Q,*} \rightarrow \mathcal{U}$, which we call the *weight map*. We get the following key property essentially by definition.

Proposition 5.2 *There is a bijection between:*

- *L -points $x = x(\phi)$ of the rigid space $\mathcal{E}_{\mathcal{U},h}^{Q,*}$ with $w(x) = \lambda$, and*
- *systems of Hecke eigenvalues $\phi : \mathcal{H}(K) \rightarrow L$ that occur in the localisation*

$$H_c^*(S_K, \mathbf{D}_{\mathcal{U}}^Q)^{\leq h} := H_c^*(S_K, \mathbf{D}_{\mathcal{U}}^Q)^{\leq h} \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U})_{\mathfrak{m}_\lambda},$$

where $\mathfrak{m}_\lambda \subset \mathcal{O}(\mathcal{U})$ is the maximal ideal corresponding to λ .

Proof Such a point x corresponds to a maximal ideal in $\mathfrak{m}_x \subset \mathbf{T}_{\mathcal{U},h}^{Q,*}$ with residue field L , and we obtain a surjective algebra homomorphism

$$\phi_x : \mathcal{H}(K) \twoheadrightarrow \mathbf{T}_{\mathcal{U},h}^{Q,*} \twoheadrightarrow \mathbf{T}_{\mathcal{U},h}^{Q,*} / \mathfrak{m}_x \cong L,$$

which by definition occurs in $H_c^*(S_K, \mathbf{D}_{\mathcal{U}}^Q)^{\leq h}$. To say that $w(x) = \lambda$ means that the contraction $\mathfrak{m}_x \cap \mathcal{O}(\mathcal{U}) = \mathfrak{m}_\lambda$, and thus ϕ_x occurs in the stated localisation. \square

5.2 The global eigenvariety

These local pieces glue into a ‘global’ eigenvariety over the weight space $\mathcal{W}_{\lambda_0}^Q$. This is straightforward using the ‘eigenvariety machine’ of [23, Sect. 4.2]; although non-minimal parabolics do not feature in Hansen’s paper, the formalism of this machine carries over to this case with little (and often no) modification. As such, our treatment of the material will be terse. The key will be to identify an *eigenvariety datum*, from which we may apply Theorem 4.2.2 *op. cit.* to obtain our global eigenvariety.

5.2.1 Fredholm power series and hypersurfaces

The modules of analytic functions from previous sections give rise to Borel–Moore chain complexes $C_*^{\text{BM}}(K, \mathbf{A}_{\mathcal{U},s}^Q)$ (dual to the compactly supported complex with distributions defined previously). The proofs of Propositions 3.1.2–3.1.5 of [23] hold in our setting with no modification, showing that the (small slope) homology and cohomology of these complexes is compatible with changing the affinoid \mathcal{U} .

For each affinoid open $\mathcal{U} \subset \mathcal{W}_{\lambda_0}^Q$, and $s \geq s[\mathcal{U}]$, we define a Fredholm series

$$F_{\mathcal{U},s}^Q(X) := \det \left(1 - U_t X | C_*^{\text{BM}}(K, \mathbf{A}_{\mathcal{U},s}^Q) \right) \in \mathcal{O}(\mathcal{U})[[X]].$$

This is independent of the choice of $s \geq s[\mathcal{U}]$ (as in [23, Proposition 3.1.1]), so we simply denote it $F_{\mathcal{U}}^Q(X)$. By Tate’s acyclicity theorem, there exists a unique

$$F^Q(X) \in \mathcal{O}(\mathcal{W}_{\lambda_0}^Q)[[X]]$$

such that $F^Q(X)|_{\mathcal{U}} = F_{\mathcal{U}}^Q(X)$, and this is a Fredholm series over all of $\mathcal{W}_{\lambda_0}^Q$. In particular, this defines a Fredholm hypersurface $\mathcal{X}^Q \subset \mathcal{W}_{\lambda_0}^Q \times \mathbf{A}^1$, where \mathbf{A}^1 is affine 1-space. There is a natural map $w : \mathcal{X}^Q \rightarrow \mathcal{W}_{\lambda_0}^Q$ given by projection to the first factor, and this has open image (see [23, Proposition 4.1.3]).

Proposition 5.3 [23, Prop. 4.1.4 and preceding discussion], or [9, Sect. 4] *Let*

$$\mathcal{X}_{\mathcal{U},h}^Q = \text{Sp}(\mathcal{O}(\mathcal{U})\langle p^h X \rangle / (F_{\mathcal{U}}^Q(X))) \subset \mathcal{X}^Q.$$

The natural map $\mathcal{X}_{\mathcal{U},h}^Q \rightarrow \mathcal{U}$ is finite flat if and only if (\mathcal{U}, h) is a slope-adapted pair; we call such $\mathcal{X}_{\mathcal{U},h}^Q$ a slope-adapted affinoid. The set of slope-adapted affinoids is an admissible cover of \mathcal{X}^Q .

5.2.2 The eigenvariety datum

We use the above to define an eigenvariety datum giving rise to the parabolic eigenvarieties. Indeed, the proof of [23, Proposition 4.3.1] (and the following paragraph) shows that there is a unique coherent analytic sheaf \mathcal{M}^* on \mathcal{X}^Q such that

$$\mathcal{M}^*(\mathcal{X}_{\mathcal{U},h}^Q) = H_c^*(S_K, \mathbf{D}_{\mathcal{U}}^Q)^{\leq h}.$$

We then let $\psi : \mathcal{H}(K) \rightarrow \text{End}_{\mathcal{O}_{\mathcal{X}^Q}}(\mathcal{M}^*)$ be the obvious map giving the Hecke action on cohomology. Then $(\mathcal{W}_{\lambda_0}^Q, \mathcal{X}^Q, \mathcal{M}^*, \mathcal{H}(K), \psi)$ is an eigenvariety datum, and [23, Theorem 4.2.2] then allows us to glue the local pieces of Proposition 5.2 into the following:

Theorem 5.4 *There exists a separated rigid analytic space $\mathcal{E}_{\lambda_0}^{Q,*}$, together with a morphism $w : \mathcal{E}_{\lambda_0}^{Q,*} \rightarrow \mathcal{W}_{\lambda_0}^Q$, such that for every finite extension L of \mathbf{Q}_p , there is a bijection between:*

- *the L -points $x = x(\phi)$ of $\mathcal{E}_{\lambda_0}^{Q,*}$ lying above a weight λ , and*
- *systems of Hecke eigenvalues $\phi : \mathcal{H}(K) \rightarrow L$ with $\phi(U_t) \neq 0$ that occur in the localization $H_c^*(S_K, \mathbf{D}_{\mathcal{U}}^Q)_\lambda$, where \mathcal{U} is any sufficiently small open affinoid containing λ .*

Remark 5.5 Additionally, during the construction one obtains a canonical coherent sheaf \mathcal{M} on $\mathcal{E}_{\lambda_0}^{Q,*}$ that interpolates Q -finite slope eigenspaces in the spaces $H_c^*(S_K, \mathbf{D}_{\mathcal{U}}^Q)$.

Remark 5.6 If $K = K^p K_p$ with $K^p \subset \mathcal{G}(\mathbf{A}_f^{(p)})$, then the eigenvariety of level K depends only on the ‘tame level at Q ’, that is, K^p and $K_p \cap L_Q(\mathbf{Z}_p)$. Indeed, let $t \in T_Q^{++}$ such that $v_p(\alpha(t)) = 0$ for all $\alpha \in \Delta_Q$. This condition ensures conjugation by t^{-1} preserves $L_Q(\mathbf{Z}_p)$, whilst by Proposition 2.3, it increases (resp. decreases) the p -adic valuation of all non-trivial entries of $n^- \in N_Q^-(\mathbf{Z}_p)$ (resp. $n \in N_Q(\mathbf{Z}_p)$). Thus if $K'_p \subset K_p$ with $K'_p \cap L_Q(\mathbf{Z}_p) = K_p \cap L_Q(\mathbf{Z}_p)$, then necessarily there exists an integer r such that

$$t^r K_p t^{-r} \cap K_p \subset K'_p. \tag{5.1}$$

Let $K' = K^p K'_p$, and $\text{res}_{K'}^K$ denote restriction from level K to K' on cohomology. Then (5.1) ensures that $U'_t = [K t^r K] = [K' t^r K] \circ \text{res}_{K'}^K$, and $(U'_t)^r := [K' t^r K'] = \text{res}_{K'}^K \circ [K' t^r K]$. In particular, the Q -finite slope eigensystems at level K and K' are the same, and the eigenvarieties at level K and K' are the same. (In fact, this is further true if K and K' have the same intersection with $\mathcal{G}(\mathbf{A}_f^{(p)}) \times L_Q(\mathbf{Z}_p)^{\text{der}}$, since the centre acts trivially on the coefficient modules).

5.3 Hansen’s spectral sequences

To study the geometry of these local pieces, we use spectral sequences introduced by Hansen in [23, Sect. 3.3]. From the construction, it can be seen that the only input required to define these spectral sequences is a theory of distributions that behaves well with respect to base-change of the weight. But all of the foundational results for fully overconvergent distributions given and used in [23] hold for parahoric distributions via exactly the same proofs, so the construction of the spectral sequences also carries over, and we conclude:

Proposition 5.7 *Fix a slope adapted pair (\mathcal{U}, h) with $\mathcal{U} \subset \mathcal{W}_{\lambda_0}^Q$, and let $\Sigma \subset \mathcal{U}$ be an arbitrary Zariski-closed subspace. Then $H_c^*(S_K, \mathbf{D}_{\Sigma}^Q)$ admits a slope $\leq h$ decomposition, and there is a convergent first quadrant spectral sequence*

$$E_2^{i,j} = \text{Ext}_{\mathcal{O}(\mathcal{U})}^i \left(H_j^{\text{BM}}(S_K, \mathbf{A}_{\mathcal{U}}^Q)^{\leq h}, \mathcal{O}(\Sigma) \right) \implies H_c^{i+j}(S_K, \mathbf{D}_{\Sigma}^Q)^{\leq h},$$

and a convergent second quadrant spectral sequence

$$E_2^{i,j} = \text{Tor}_{-i}^{\mathcal{O}(\mathcal{U})} \left(H_c^j(S_K, \mathbf{D}_{\mathcal{U}}^Q)^{\leq h}, \mathcal{O}(\Sigma) \right) \implies H_c^{i+j}(S_K, \mathbf{D}_{\Sigma}^Q)^{\leq h}.$$

There are analogous spectral sequences replacing compactly supported cohomology and Borel–Moore homology with singular cohomology and singular homology, and considering the Borel–Serre compactification of S_K —with boundary cohomology and boundary homology. These spectral sequences are all equivariant for the action of $\mathcal{H}(K)$ on their E_2 pages and abutments.

Many of the general consequences Hansen and Newton obtain from studying these spectral sequences also carry over, with identical proofs, to the context of parabolic eigenvarieties. We highlight some of these, referencing only the relevant equivalent statement in [23].

5.3.1 Points in eigenvarieties and classical eigensystems

Proposition 5.8 [23, Thm. 4.3.3] *For total cohomology $* = \bullet$, there is a bijection between:*

- *L-points $x = x(\phi)$ of $\mathcal{E}_{\lambda_0}^{Q,\bullet}$ lying above a weight $\lambda \in \mathcal{W}_{\lambda_0}^Q$, and*
- *systems of Hecke eigenvalues $\phi : \mathcal{H}(K) \rightarrow L$ with $\phi(U_1) \neq 0$ that occur in $H_c^\bullet(S_K, \mathbf{D}_\lambda^Q(L))$.*

Thus if $\tilde{\pi}$ is a Q -non-critical p -refined automorphic representation of weight $\lambda \in \mathcal{W}_{\lambda_0}^Q$, with Hecke eigensystem $\phi_{\tilde{\pi}}$ with $\phi_{\tilde{\pi}}(U_1) \leq h$, then there is a point $x_{\tilde{\pi}}$ in $\mathcal{E}_{\lambda_0}^{Q,\bullet}$ corresponding to $\tilde{\pi}$.

In particular, the coherent sheaf \mathcal{M} of Remark 5.5 actually interpolates Q -finite slope eigenspaces in $H_c^\bullet(S_K, \mathbf{D}_\lambda^Q)$, and hence—via Theorem 4.4— Q -non-critical eigenspaces in classical cohomology. We also have a partial analogue of this when we work in a single degree $* = d \in \mathbf{Z}_{\geq 0}$:

Proposition 5.9 *If $x = x(\phi) \in \mathcal{E}_{\lambda_0}^{Q,d}(L)$ lies above $\lambda \in \mathcal{W}_{\lambda_0}^Q$, then ϕ occurs in $H_c^d(S_K, \mathbf{D}_\lambda^Q(L))$.*

Proof Let \mathcal{U} be a neighbourhood of λ . By Proposition 5.2, ϕ occurs in $H_c^d(S_K, \mathbf{D}_{\mathcal{U}}^Q)_\lambda$. We localise the Tor spectral sequence (with $\Sigma = \lambda$) at ϕ . Since $\phi(U_1) \neq 0$, reducing modulo \mathfrak{m}_λ we see ϕ occurs in $E_2^{0,d} = H_c^d(S_K, \mathbf{D}_{\mathcal{U}}^Q) \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{O}(\mathcal{U})/\mathfrak{m}_\lambda$. Hence it occurs in the abutment $E_\infty^{0,d} = H_c^d(S_K, \mathbf{D}_\lambda^Q(L))$. □

Remark 5.10 The converse to Proposition 5.9 is *false*. For example, let $\mathcal{G} = \mathrm{GL}_2/\mathbf{Q}$, $Q = B$, and $K = K_0(p)$. Then as explained in [36, Thm. 7.1] and [2, Thm. 3.30], there is a weight 2 critical Eisenstein series E_2^{crit} whose eigensystem ϵ occurs in $H_c^1(S_K, \mathbf{D}_{(0,0)}^B)$ but not in $M_2^\dagger(K)$, hence not in the Coleman–Mazur eigencurve (denoted $\mathcal{E}_{(0,0)}^{B,1}$ here). In particular ϵ does not occur in $H_c^1(S_K, \mathbf{D}_{\mathcal{U}}^B)$. It does appear in $H_c^2(S_K, \mathbf{D}_{\mathcal{U}}^B)$ and hence $\mathcal{E}_{(0,0)}^{B,\bullet}$, consistent with Proposition 5.8.

5.3.2 The dimension of components

Let $\tilde{\pi}$ be a p -refined automorphic representation and suppose there is an attached point $x_{\tilde{\pi}} \in \mathcal{E}_{\lambda_0}^{Q,*}$. A p -adic family through $\tilde{\pi}$ is a positive-dimensional component of $\mathcal{E}_{\lambda_0}^{Q,*}$ through $x_{\tilde{\pi}}$. It is not obvious that such a component exists. In the case of total cohomology, however, we can exhibit lower bounds on the dimensions of components.

Definition 5.11 Let $x = x(\phi)$ be a point of $\mathcal{E}_{\lambda_0}^{Q,\bullet}$, and let $t_Q(x)$ (resp. $b_Q(x)$) denote the supremum (resp. infimum) of the set $\{i \in \mathbf{Z} : H_c^i(S_K, \mathbf{D}_\lambda^Q(L))_\phi \neq 0\}$. Define the *overconvergent defect at Q* to be $\ell_Q(x) = t_Q(x) - b_Q(x)$.

The following proposition is proved exactly as in Newton’s proof of [23, Prop. B.1].

Proposition 5.12 *Any irreducible component of $\mathcal{E}_{\lambda_0}^{Q,\bullet}$ containing a given point x has dimension at least $\dim \mathcal{W}_{\lambda_0}^Q - \ell_Q(x)$.*

5.4 Parabolic families of cuspidal automorphic representations

We now consider the construction of p -adic families of cuspidal automorphic representations. Let $x = x(\phi) \in \mathcal{E}_{\lambda_0}^{Q,*}(L)$ be a point of classical (dominant) weight $\lambda = w(x)$, corresponding to a system of eigenvalues $\phi : \mathcal{H}(K) \rightarrow L$. We require some further technical conditions:

- Definition 5.13**
1. We say ϕ is *interior* if $H_{\mathfrak{g}}^*(S_K, V_{\lambda}^{\vee}(L))_{\phi} = 0$ (boundary cohomology for the Borel–Serre compactification of S_K), and *Q -strongly interior* if $H_{\mathfrak{g}}^*(S_K, \mathbf{D}_{\lambda}^Q(L))_{\phi} = 0$.
 2. We say λ is *regular* if $\langle \lambda, \alpha \rangle > 0$ for all simple roots α of G . (For GL_n , this means that $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i > \lambda_{i+1}$ for all i).
 3. We say x is a *classical cuspidal point* if there exists a cuspidal automorphic representation π_x such that ϕ occurs in π_x^K (after appropriate renormalisations as in Sect. 4.6).

Lemma 5.14 *If ϕ is Q -non-critical slope, then it is Q -strongly interior if and only if it is interior. This is true more generally if ϕ is Q -non-critical for both $H_{\mathfrak{c}}^{\bullet}$ and H^{\bullet} (Remark 4.5).*

Proof That the second statement is a generalisation of the first is Remark 4.5. To prove the second, combining specialisation ρ_{λ} with the excision exact sequence, we have a commutative and Hecke-equivariant diagram with exact rows, which we localise at ϕ :

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_{\mathfrak{c}}^{\bullet}(S_K, \mathbf{D}_{\lambda}^Q(L))_{\phi} & \xrightarrow{\iota_1} & H^{\bullet}(S_K, \mathbf{D}_{\lambda}^Q(L))_{\phi} & \longrightarrow & H_{\mathfrak{g}}^{\bullet}(S_K, \mathbf{D}_{\lambda}^Q(L))_{\phi} \longrightarrow \cdots \\
 & & \downarrow \rho_{\lambda} & & \downarrow \rho_{\lambda} & & \downarrow \rho_{\lambda} \\
 \cdots & \longrightarrow & H_{\mathfrak{c}}^{\bullet}(S_K, V_{\lambda}^{\vee}(L))_{\phi} & \xrightarrow{\iota_2} & H^{\bullet}(S_K, V_{\lambda}^{\vee}(L))_{\phi} & \longrightarrow & H_{\mathfrak{g}}^{\bullet}(S_K, V_{\lambda}^{\vee}(L))_{\phi} \longrightarrow \cdots
 \end{array}$$

Being Q -strongly interior (resp. interior) is equivalent to ι_1 (resp. ι_2) being an isomorphism in every degree. By Q -non-criticality, the first and second vertical maps are isomorphisms; thus ι_1 is an isomorphism in every degree if and only if ι_2 is, from which we conclude. \square

Let $x = x(\phi) \in \mathcal{E}_{\lambda_0}^{Q,*}(L)$ be as above, of weight $\lambda = w(x)$, and suppose that:

- (C1) x is a classical cuspidal point,
- (C2) ϕ is Q -non-critical for $H_{\mathfrak{c}}^{\bullet}$ and H^{\bullet} ,
- (C3) and that every neighbourhood of λ in $\mathcal{W}_{\lambda_0}^Q$ contains a Zariski-dense set of regular weights. (Note this is automatic if λ itself is a regular weight).

Proposition 5.15 *Let \mathcal{V} be an irreducible component of $\mathcal{E}_{\lambda_0}^{Q,*}$ passing through x . If $\dim \mathcal{V} = \dim \mathcal{W}_{\lambda_0}^Q$, then \mathcal{V} contains a Zariski-dense set \mathcal{V}^{cl} of classical cuspidal (cohomological) points.*

Proof If an open neighbourhood of x in \mathcal{V} contains a Zariski-dense set of such points, then \mathcal{V} does. Thus we work locally, and assume \mathcal{V} is a component of a local piece $\mathcal{E}_{\mathcal{U},h}^{Q,*}$ containing x .

By (C3), we may always pick a Zariski-dense subset $\mathcal{U}^{\text{cl}} \subset \mathcal{U}$ of classical (regular) weights λ' for which h is a Q -non-critical slope. Let \mathcal{V}^{cl} denote the set of $y \in \mathcal{E}_{\mathcal{U},h}^{Q,*}$ with $\lambda_y := w(y) \in \mathcal{U}^{\text{cl}}$; this set is necessarily Zariski-dense in \mathcal{V} as $\dim(\mathcal{U}) = \dim(\mathcal{V})$. If $y \in \mathcal{V}^{\text{cl}}$, then by Propositions 5.8 (for $* = \bullet$) or 5.9 (for $* = d$) it corresponds to a system of eigenvalues ϕ_y occurring in

$$H_{\mathfrak{c}}^*(S_K, \mathbf{D}_{\lambda_y}^Q(L))^{\leq h} \cong H_{\mathfrak{c}}^*(S_K, V_{\lambda_y}^{\vee}(L))^{\leq h},$$

isomorphic since h is a Q -non-critical slope for λ_y . Hence ϕ_y is a system of eigenvalues in the classical cohomology. It remains to show ϕ_y is cuspidal (i.e. occurs in the cuspidal cohomology).

Since $\tilde{\pi}$ is cuspidal, then by a theorem of Borel (see for example [31, Sect. 0]) the associated eigensystem ϕ is interior; thus it is Q -strongly interior by Lemma 5.14. Analogously to [23, Theorem 4.5.1(ii)], from the boundary Tor spectral sequence (Proposition 5.7) we deduce that $H_{\partial}^{\bullet}(S_K, \mathbf{D}_{\mathcal{U}}^Q)_{\phi} = 0$. The boundary cohomology yields a coherent sheaf on the eigenvariety, and we see that the rigid localisation of this sheaf at x —which is a faithfully flat extension of the algebraic localisation—must be zero. Thus, perhaps after shrinking the neighbourhoods \mathcal{U} and \mathcal{V} , this vanishing lifts to \mathcal{V} . Thus for any $y \in \mathcal{V}$, we have $H_{\partial}^{\bullet}(S_K, \mathbf{D}_{\mathcal{U}}^Q)_{\phi_y} = 0$; and localising the boundary spectral sequence at y , we deduce that $H_{\partial}^{\bullet}(S_K, \mathbf{D}_{\lambda_y}^Q(L))_{\phi_y} = 0$ and ϕ_y is strongly interior.

Now suppose $y \in \mathcal{V}^{\text{cl}}$. Since ϕ_y is Q -non-critical slope, by Lemma 5.14 it is interior. But for regular weights, a class is interior if and only if it is cuspidal [31, Prop. 5.2, Sect. 5.3], so ϕ_y appears in the cuspidal cohomology, as required. \square

As a special case where the dimension hypothesis on \mathcal{V} will always be satisfied, we have:

Corollary 5.16 *Suppose $\mathcal{G}^{\text{der}}(\mathbf{R})$ admits discrete series, and let $x \in \mathcal{E}_{\lambda_0}^{Q,*}$ satisfy (C1-3). Every irreducible component of $\mathcal{E}_{\lambda_0}^{Q,*}$ through x contains a Zariski-dense set of classical cuspidal points.*

Proof The conditions on \mathcal{G} and $x = x(\phi)$ ensure that ϕ appears in only one degree of classical cohomology (e.g. [31, Sect. 4-5]); and then Proposition 5.12 ensures that any such irreducible component has dimension $\dim \mathcal{W}_{\lambda_0}^Q$. We conclude by Proposition 5.15. \square

Remark 5.17 The assumptions on regular weights ensure control over the *classical* cohomology, and in situations where we have a more complete understanding of the classical cohomology—for example, the case of GL_2 —we may relax these conditions.

For B -families, every affinoid neighbourhood of a classical weight λ_0 contains a Zariski-dense set of regular classical weights. If λ_0 is not regular, this is not necessarily true in the parahoric case. For example, consider $\mathcal{G} = \text{GL}_4$, and $\lambda_0 = (0, 0, 0, 0)$, and Q with Levi $\text{GL}_2 \times \text{GL}_2$. Then every weight $\lambda = (\lambda_1, \dots, \lambda_4) \in \mathcal{W}_{\lambda_0}^Q$ has $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$, so this space contains *no* regular weights.

Remark 5.18 Suppose $\mathcal{G}^{\text{der}}(\mathbf{R})$ does not admit discrete series. Then if a point x is cuspidal Q -non-critical, then $\ell_Q(x) \geq 1$. When $Q = B$, [23, Thm. 4.5.1] says that irreducible components through such x *never* have maximal dimension (that is, dimension equal to $\dim \mathcal{W}$), and conjecturally the inequality of Proposition 5.12 is an equality. This conjecture is false in the general parahoric setting. Indeed, in [5] examples are given of Q -parabolic families of dimension $\dim \mathcal{W}_{\lambda_0}^Q$ in the setting of $\mathcal{G} = \text{Res}_{F/Q} \text{GL}_{2n}$, even though $\ell_Q(x) = n - 1$. Conceptually these families arise through transfer from GSpin_{2n+1} (where we *do* have discrete series).

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