RETAINED EARNINGS DYNAMIC, INTERNAL PROMOTIONS AND WALRASIAN EQUILIBRIUM*

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ABSTRACT

In the early stages of the process of industry evolution, firms are financially constrained and pay different wages because workers have heterogeneous expectations about the prospects for advancement offered by each firm's job ladder. This paper argues that, nevertheless, if the output market is competitive, the *positive predictions* of the perfectly competitive model are still a good description of the long run outcome. If firms maximize the discounted sum of *constrained* profits, financing expenditure out of retained earnings, profits are driven down to zero as the perfectly competitive model predicts. *Ex ante* identical firms may follow different growth paths in which workers work for a lower entry-wage in firms expected to grow more. In the steady state, however, workers performing the same job, in *ex-ante* identical firms, receive the same wage. I explain when the long run outcome is efficient, when it is not, and why firms that produce inefficiently might drive the efficient ones out of the market even when the steady state has the positive properties of a Walrasian equilibrium. To some extent, it is not technological efficiency but workers' self-fulfilling expectations about their prospects for advancement within the firm what explains which firms have lower unit costs, grow more and dominate the market.

Keywords: Industry Evolution - Market Selection Hypothesis - Production under Incomplete Markets - Retained Earnings Dynamic - Self-Fulfilling Expectations - Internal Labor Markets

JEL Classification Numbers: D21, D52, D61, D84, D92, J41

1. INTRODUCTION

Consider a market in which many firms compete to sell an homogeneous product. Economic theory predicts that, at least in the long run, profits vanish and each firm produces the quantity that maximizes profits at the market price. Although most economists agree about this description of the long run outcome of the process of industry evolution, it is not so clear what forces lead an industry to that steady state. The theory of industry equilibrium in competitive markets relies on the existence of a perfect credit market and profit maximizing firms to explain why profits are dissipated. If there is a complete set of perfectly competitive financial markets, each firm maximizes its market value, the markets for inputs are perfectly competitive, there are no turnover costs and there is either free entry or the technology displays constant return to scale, then equilibrium profits are zero and each active firm produces the profit maximizing level.

In sharp contrast with these assumptions, however, the empirical evidence suggests that new firms are financially constrained and the labor market, rather than being in a Walrasian equilibrium from the start, it is better characterized by social institutions which are not present in the theory of the firm under perfect competition.

Indeed, the problems of asymmetric information identified by authors like Stiglitz and Weiss [12] as the main explanation for the failure of the credit market, are particularly important at the early stages of the process of industry evolution. Therefore, many firms finance production reinvesting their own funds. In modern industries, financing through retained earnings is the norm rather than the exception. To quote Allen and Gale [3]:

> "Perhaps the most striking point [...] is that in all countries [US, UK, France and Germany] except Japan, retained earnings are the most important source of funds. External finance is simply not that important" (p. 76)

The lack of access to credit may prevent firms from achieving its optimal size from the start and explains why it takes time for profits to be dissipated. In addition, the presence of asymmetric information among firms about the ability of workers causes wage rates to differ from productivity and turnover costs are significant. Therefore, workers tend to be attached to the same firm for long periods, firms carry out most of the training of their employees and prefer to promote employees internally rather than recruiting new workers. Using the term made popular by Doeringer and Piore [6], firms set up an internal labor market, with rules that are different from the ones that prevail in a Walrasian market. As S. Rosen [11] writes:

> "Many features of labor markets bear little resemblance to impersonal Walrasian auction markets. Chief among them is the remarkable degree of observed worker-firm attachment [...] The typical adult male worker spends twenty years or more on a single job"

It is apparent that modern industries display many features which are not taken into account in the static model but are key to understand why industry evolution takes time and how wages evolve. Therefore, the standard description of firm and industry behavior is at best the description of a steady state of some growth dynamics. Economists like Alchian [1] and Friedman [8] recognized this long time ago. However, Nelson and Winter [10] were the first in providing a formal explanation on how such steady state can be attained even if no firm follows a profit maximization rule. The key assumption in their work is that firms that make positive profits expand, those that make zero profits do not change capacity while those that make loses contract and search for new decision rules, a dynamic that can be motivated by the use of retained earnings to finance investment. However, Blume and Easley [5] show that even though such retained earnings dynamic explains why firms that do not maximize profits are driven out, it may not converge to a Walrasian equilibrium.

The work of Nelson and Winter and Blume and Easley focuses on the role of the retained earnings dynamic as a substitute for market completeness when the labor market is perfectly competitive. In many industries, the presence of training costs and firm specific abilities not only implies that wages are not closely related to productivity but also that they exceed wages in another industry. This is typically the case for the wage of skilled intensive jobs at the top of the progression line. Because workers anticipate that they may progress through the promotion line and obtain those high wages in the future, reservation entry-wages are usually lower than in other industries. *Ceteris paribus*, the better the prospects for advancement displayed by the firm are, the lower the worker's reservation entry-wage is. Intuitively prospects for advancement must be positively related with the growth prospects of the firm. This introduces an additional self-fulfilling aspect in the process of industry evolution. Indeed, since firms rely on internal funds, *ceteris paribus*, those that are believed to have better growth potentials pay lower wages, have more revenue and end up promoting more workers, fulfilling workers' expectations. This introduces more complexity in the process of industry evolution. If *ex-ante* identical firms follow different growth paths, does the industry converge to a steady state with zero profits? Which firms pay lower wages along the transition? What are the efficiency properties of the steady state? Is there an unambiguous positive relationship between technological efficiency and growth rates? These are some of the questions addressed in this work.

This paper argues that when firms maximize the discounted sum of *constrained* profits, financing expenditure out of retained earnings and the internal labor market arises as a cost minimizing institution, due to firm specific abilities and costly training, the industry converges to a steady state where profits are dissipated. My analysis corresponds to the case in which firms do not face a shortage in the supply of skilled workers along the process of industry evolution. Therefore, adjustment costs do not play any role in this paper. Instead, I concentrate on the role of workers' expectations in shaping factor prices, an aspect that has not been addressed yet in the literature of industry evolution towards a Walrasian equilibrium. As in Waldman [13], every firm in the industry learns something about a worker's skills by considering his job assignment and can try to hire him. Therefore, the higher the training cost is or the more general the worker's skill is, the higher is the wage of promoted workers in a two tasks job ladder. If this wage exceeds the wage those workers could obtain in another industry, their *entry-wage* depends on the worker's expectations about the firm's promotion rate.

If firms are *ex-ante* identical, I show that workers who carry out equal jobs receive the same wage in the steady state, regardless of the firm that hires them, *as if* the labor market were in a Walrasian equilibrium.

However, *ex-ante* identical firms can follow different growth paths towards the steady state. *Ceteris paribus*, firms that are expected to grow faster hire workers at a lower entry-wage, which implies that technological efficiency may not hold along the transition. However, it does hold in the steady state. Allocative efficiency, instead, is satisfied in the steady state if and only if wages at the upper levels of the job ladder are identical to those in the competing industry so that entry-wages are identical across industries. Otherwise, too little is produced compared to the efficient allocation of resources. The failure of technological and allocative efficiency is due both to the absence of a perfect credit market as well as the impossibility of enforcing a wage for old workers equal to their opportunity cost in the competing industry.

I also consider the case of firms with different technologies. Although economists long time ago recognized that firms with lower costs tend to grow more, it is usually argue that cost differentials are due to technological reasons. However, this neglects the fact that, *ceteris paribus*, those firms that are believed to display better growth prospects can hire workers at a lower wage which, in turn, contributes to lower its costs. This reverse of causality implies that even firms that produce inefficiently may end up dominating the market if workers believe they display sufficiently better prospects than the efficient ones. Indeed, the workers' willingness to work for a low entry-wage can more than compensate the cost disadvantage introduced by an inefficient technology. Can this happen in a self-fulfilling equilibrium that converges to a Walrasian-like steady state? I construct an example in which even though profits vanish in the long run, worker's expectations are fulfilled and inefficient firms grow more and dominate the market in terms of market share. If at the early stages workers are optimistic enough about the prospect for advancement offered by the firms which produce inefficiently, *almost* all workers end up employed by inefficient firms in the long run. Therefore, *almost* all workers performing the same job receive the same wage, as in a Walrasian equilibrium. In contrast with Beker [4], I do not need to assume an stochastic technology to show that inefficient firms can dominate a perfectly competitive output market.

My analysis confirms the widespread intuition that in a competitive output market, profits are driven down to zero and firms do not face financial constraints in the long run. Contrary to the standard static analysis, I do not need to assume the existence of a perfect capital market or a perfectly competitive labor market. However, this paper also confirms Winter's [16, p. 88] skepticism about the efficiency of the equilibrium in a world of incomplete markets where business firms play the role of a training institution. Indeed, he writes:

> "We know how to go about proving the Pareto optimality of equilibria in theoretical systems in which prices provide the necessary coordinating information, while actors have essentially unlimited memories and computation power, and contracts are costlessly enforced. We do not know how to -and very likely it is not true- for a system in which relevant economic information is routinely transmitted by the daily newspaper, or, indeed by any one of a large number of obviously significant social institutions. The list comprises, for example, the mass media, the schools and other educational institutions, the family, business firms (in advertising, training programs, etc.)..."

In competitive output markets, the retained earnings dynamic gives an evolutionary advantage to firms with lower unit costs. However, unit costs are determined not only by technological efficiency but also by wages. In the presence of internal promotions, unlike in Walrasian markets, worker's expectations about the opportunities for advancement within the firm are key to determine wages. Therefore, the fitness of a firm depends not only on its technological efficiency but also on the self-fulfilling beliefs of the workers. I conclude that, at least in the long run, the retained earnings dynamic justifies the use of the standard static analysis of competitive markets to make positive predictions but does not always justifies its efficiency properties. Unlike in Blume and Easley's model, even the steady state of the retained earnings dynamic may fail to be efficient in the presence of internal promotions. As in Arthur [2], what happens at the origin of the industry can have a decisive role on the technology that dominates the market. However, it is not a network externality or the presence of increasing returns what drives the result in this model but the self-fulfilling beliefs of the young workers about the prospects for advancement offered by the firms.

1.1 Overview

In section 2, I describe a partial equilibrium model of industry evolution in which retained earnings determine the scale of operation, firms are long lived and every period a new generation of workers, who live for two periods, enters the labor force. The description of the labor market is strongly influenced by Waldman's formalization of the arguments in Doeringer and Piore.[1]

In section 3, I define an Industry Equilibrium (IE). In an IE, each firm and the workers it contacts play a subgame perfect Nash equilibrium (SPNE) and the output and labor markets clear. Firms may follow different strategies either because they are endowed with different technologies or because of the existence of multiple SPNE of the game played between each firm and the workers. Since training workers is costly, firms have an incentive to hire workers trained by a competitor. However, these workers are not as productive as those promoted internally. Therefore, the higher the training cost is or the more general the training is, the higher the equilibrium wage of a promoted worker is. In section 4, I show that if the wage of a promoted worker exceeds what those workers would receive in another industry, the game played by the workers and the firm has two SPNE. In one SPNE, every generation of young workers believes the next generation will accept employment at wages low enough to induce the firm to promote a large fraction of its current employees the following period. Anticipating this, they accept employment at a low entry-wage. In another SPNE, every generation of young workers believes the next generation will accept employment only at a wage so high that the firm will promote a small fraction of workers. Therefore, they accept employment only at a high entry-wage.

Instead of looking for a further refinement of the notion of rationality, I analyze how the market share of firms which face different labor market conditions evolve along time. In sections 5 and 6, I analyze the dynamic and efficiency properties of the industry equilibrium for the case of *ex-ante* identical firms and heterogeneous firms, respectively. Conclusions are in section 7. All the proofs are in the Appendix.

[1] There are some slight differences between the two models of the labor market. In Waldman's model, workers ability takes values in a continuum while in mine it can take only two values but a law of large numbers holds at the firm level. He assumes that firms are not financially constrained but instead the technology is such that they hire only one worker each period.

2. THE MODEL

At date zero, the industry adopts a new technology to produce a final good. Let q denote the firm's output level. The technology to produce this good requires only labor and the production process can be described as a function of two tasks. The level at which task 1 and task 2 are performed are denoted by q_1 and q_2, respectively, and the production function takes the following functional form:

$$q = q_1^{\alpha} \cdot q_2^{1-\alpha} \qquad 0 < \alpha < 1.$$

Task 1 requires a skill that is not industry specific. If l is the number of workers employed in task 1 then[2]

$$q_1(l) = l$$

Every worker develops a new ability while performing the first task. Ability is a random variable that takes only two values: high or low. Ability turns out to be high with probability $\lambda \in (0, 1)$. In order to be able to perform the second task, a worker needs not only to have high ability, but also to receive some additional training to develop the industry specific skill. Then a necessary condition to be able to carry out the second task is to have performed task 1 in the past. In principle, there are three different ways in which a firm can learn whether an old worker has the necessary ability to develop the industry specific skill:

1. Since ability is revealed while performing task 1, firms learn which of their employees have developed high ability. Doeringer and Piore emphasize this point [6, p. 31]:

> "The efficiency of internal recruitment and screening derives from the fact that existing employees constitute a readily accessible and knowledgeable source of supply whose skill and behavioral characteristics are well known to management. Information about internal candidates is generated as a by-product of their work history in the enterprise."

At the beginning of period $t+1$, each worker born at t who developed high ability can be trained, at a unit cost of c, to perform task 2 during $t+1$. If the firm hires those workers to perform task 2 at date $t+1$, the firm is said to promote workers internally. Let s_t^i be the number of workers promoted internally by firm i at date t. If all workers performing the second task have been hired internally then it is said that the firm has a *closed internal labor market with one entry port.*

2. Observing who are the employees that perform the second task in other firms in the industry, a firm can learn who are those that developed high ability. A firm can make an offer to any of those workers. If the worker accepts the offer, he does not need additional training to be able to perform the second task in his new job. The firm that employs him is said to hire workers externally. However, that employee is not as productive as one that also has the skill but worked in the same firm when young. In particular, I assume that e skilled workers

[2] I assume that the number of workers takes values in $\Re_+$ so it would be more appropriate to say that l is the measure of workers hired by the firm. The same applies to all other types of labor in this paper.

that change firms are equivalent to $\frac{e}{1+\theta}$, with $\theta > 0$, skilled employees who are promoted internally. Let e_t^i be the number of workers that have been trained by another firm and are hired by firm i at date t.

3. Firms could also hire a worker who performed the first task in another industry when young and screen him in order to learn whether he has high ability or not. However, as Doeringer and Piore [6, p. 31] note:

> "In contrast, potentially interested outsiders must first be located and then screened [...] The problem of identifying the variables which will completely predict a new hire's work performance, however, is generally viewed as either insoluble or soluble only at a prohibitive cost".

Accordingly, I rule out this possibility and for the rest of the paper I assume that the second task is performed either by internally promoted workers or by externally hired employees.

If s and e denote the number of the two types of workers employed to carry out the second task, then

$$q_2(s,e) = s + \frac{e}{1+\theta}$$

denotes the level of activity of the second task. The parameter θ measures the degree of firm's specificity of the training process. Greater values of θ corresponds to greater firm specificity of the skill obtained during the training process. The technology to produce q can be written as a function of labor in the following way:

$$q(l,s,e;\alpha) = q_1(l)^{\alpha} \cdot q_2(s,e)^{1-\alpha}$$

Since production takes time, a firm that employs (l,s,e) workers at date t, obtains $q(l,s,e;\alpha)$ units of output at $t+1$. Finally, the demand for the good is $D(p)$. I assume that D has standard properties.

Assumption AD: The function $D : \Re_+ \to \Re_+$ is continuous and strictly decreasing, $\lim_{p\to\infty} D(p) = 0$ and $D(p) = 0 \Rightarrow p \cdot \left(\frac{\alpha}{\overline{w}_1}\right)^{\alpha} \cdot \left(\frac{1-\alpha}{v^*+c}\right)^{1-\alpha} > r$.

where the last condition ensures that demand is zero only at prices high enough so that firms can make positive profits hiring young and old workers at wages $\overline{w}_1$ and $v^* = Max\left\{\frac{c}{\theta}, \overline{w}_2\right\}$, respectively. This assumption will ensure that the equilibrium output level is not zero.

2.1 Workers

Every period $t \geq 0$, a new generation of workers, who live for two periods, enters the labor force. Workers do not consume the good produced by this industry. They only face uncertainty about their ability and, therefore, about their wage (and consumption) when old. Workers are risk neutral and have preferences over random bundles of the numeraire that have a discounted expected utility representation with discount rate $0 < \beta < \frac{1}{r}$. A worker who does not work in this industry can work in another industry, or at home, and obtain expected lifetime utility $\overline{u} = \overline{w}_1 + \beta \cdot \overline{w}_2$, when young, and $\overline{w}_2 \geq \overline{w}_1$, when old. Without loss of generality, one can think that $\overline{w}_1$ and $\overline{w}_2$ are the expected wages of a young and old worker, respectively, in another industry. Workers cannot borrow from future wages. Therefore, each worker consumes out of his wage and decides where to work to maximize his expected utility. Each firm in this industry faces an infinite supply of *ex-ante* identical young workers.

2.2 Efficient Allocations

Since this is a partial equilibrium model, to make efficiency considerations one has to make some additional assumptions. In particular, I assume that the consumer surplus is an adequate measure of welfare and that the social opportunity cost of working in this industry when young and old, in terms of the numeraire, is given by $\overline{w}_1$ and $\overline{w}_2$, respectively, and $\frac{1}{r}$ is the socially optimal discount rate. As usual, the set of efficient allocations can be characterized as the solution to the following Social Planner's problem:

$$\max \sum_{t=0}^{\infty} \left(\tfrac{1}{r}\right)^t \cdot \left[\frac{1}{r} \cdot CS\left(q_t\right) - \overline{w}_1 \cdot l_t - \left(\overline{w}_2 + c\right) \cdot s_t\right]$$
$$s.t. \begin{cases} q_t = l_t^{\alpha} \cdot s_t^{1-\alpha} \\ s_{t+1} \leq \lambda \cdot l_t \quad l_t, s_t \geq 0 \end{cases}$$

where $CS\left(q\right) \equiv \int_0^q D^{-1}\left(x\right) dx$ is the Marshallian Consumer Surplus. At any date $t \geq 0$, there are only two relevant types of labor for the planner: the young workers who perform task 1 and the old workers who performed the first task in this industry when young.

An industry produces efficiently if more output cannot be produced using at most the same amount of every input and strictly less of one of them. Allocative efficiency holds if the aggregate surplus is maximized. Let

$$p^* \equiv r \cdot \left(\frac{\overline{w}_1}{\alpha}\right)^{\alpha} \cdot \left(\frac{\overline{w}_2 + c}{1-\alpha}\right)^{1-\alpha}$$

and $Q^* = D\left(p^*\right)$. The following lemma characterizes the set of efficient allocations for those parameters such that the second constraint in the Social Planner's problem is not binding. This set of parameters gives the appropriate benchmark because in all the equilibria I analyze later the constraint does not bind either.

Lemma 2.1 *If $\alpha > \frac{\overline{w}_1}{\overline{w}_1 + \lambda \cdot (\overline{w}_2 + c)}$ then Q^* is the efficient level of output while the efficient allocation of labor is $l_t = l^*$ and $s_t = s^*$ where:*

$$l^* = \left(\frac{\alpha}{1-\alpha} \cdot \frac{\overline{w}_2 + c}{\overline{w}_1}\right)^{1-\alpha} \cdot Q^*$$
$$s^* = \left(\frac{1-\alpha}{\alpha} \cdot \frac{\overline{w}_1}{\overline{w}_2 + c}\right)^{\alpha} \cdot Q^*$$

2.3 Firms

Firms receive a name j in the unit interval and take the output price sequence $\{p_t\}_{t=0}^{\infty}$ as given. Each firm is endowed with $a_0 > 0$ units of the numeraire and $l_{-1} \geq \frac{1-\alpha}{w_2+c} \cdot a_0$ trainees. The lower bound chosen for l_{-1} ensures that there is no shortage of skilled workers at date zero.[3] One can think that the firms have been operating for a while, perhaps using another technology based only in task 1, and know the ability of those workers that were employed before. I assume that the workers' distribution across firms is such that a law of large numbers holds at each date: if firm i employs l_t workers in task 1 at date t, exactly a fraction λ of these workers develops high ability.[4] Therefore, since training is costly, at most $\lambda \cdot l_t$ workers receive training at date t and are ready to perform task 2 at date $t+1$.

I assume firms cannot borrow in the capital market. This may be because these firms are rationed in the credit market but I do not explicitly model this phenomena. At every date $t \geq 0$, each firm chooses how much of its assets to use as financial capital to hire inputs, $0 \leq m_t \leq a_t$, and what part to invest in an alternative activity, $b_t = a_t - m_t$, with gross rate of return $r > 1$. For the rest of the paper, I take this alternative activity as lending at the interest rate r. If a firm hires (l_t, s_t, e_t) workers and invests b_t in bonds at date t, then its assets at date $t+1$ are

$$a_{t+1} = p_t \cdot q_t + r \cdot b_t$$

where $q_t \equiv q(l_t, s_t, e_t; \alpha_i)$.

At every date $t \geq 0$, each firm collects revenue and learns who are the employees that developed high ability. In that information set, the firm decides how much of its assets to allocate as financial capital and how to spend it. That is, the firm chooses how many workers to contact and what wages to offer so its expenditure does not exceed m_t. The hiring process is described in detail below. Once the hiring phase has ended, production is carried out. Figure 1 illustrates the timing of decisions.

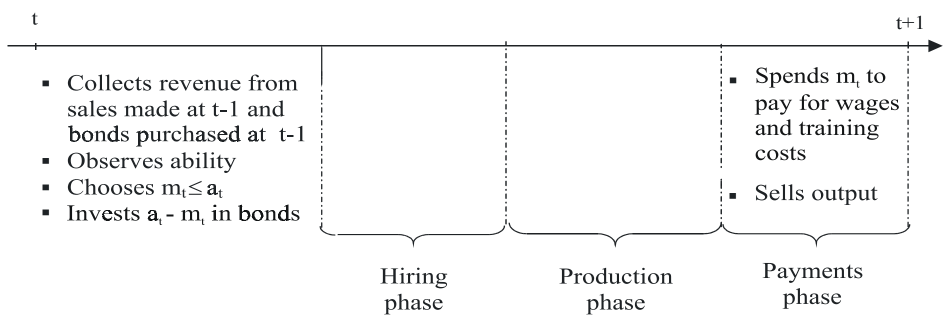

Figure 1. Timing of decisions

[3] See below for more discussion on the assumption that the internal labor market constraint is never binding.
[4] Since independence has no role in this model, the argument in Feldman and Gilles [7] implies that there exists a distribution of workers for which the law of large numbers holds in every Borel set.

Although firms are perfectly competitive in the output market, they are not so in the labor market. This is because each firm has private information about the ability of the workers that it employed the previous period. However, as in Waldman [13], when a firm makes an offer to a former employee, it realizes that other firms in the industry may learn something about that worker's ability by observing his job assignment and can try to hire him. Let $v_t^e \geq \overline{w}_2$ be the equilibrium outside value of a worker who performs task 2. Any worker who performs the second task at date t can move to another firm and obtain utility v_t^e.[5] To simplify the discussion, I do not model the game of simultaneous offers played by the firms and those workers that are promoted by its first period employer. However, I do require v_t^e to be compatible with the firms' strategies in equilibrium.

The interaction between each firm and the successive generations of workers is described as a game where firms take as given both the output price sequence as well as the outside value of a promoted worker. In principle, there is a large set of labor contracts that a firm could offer to the young workers. For example, one could imagine a contract in which a firm assigns a young worker to task 1, pays him a certain wage at date t and promises future wages contingent on being promoted or not. One could even think of a contract where the firm details the fraction of workers that it will promote at $t+1$, as in Malcomsom [9]. However, many contracts like these are not implementable because either the firm cannot commit to take actions that are not sequentially rational or the worker cannot commit to stay in the firm in case of receiving a better offer in the future. In this work, I restrict myself to spot contracts.

Assumption AC: When a firm hires a young worker, it can neither commit to a wage in the event that such worker is promoted when old nor to a promotion probability.

Each firm takes as given both the sequence of output prices $P = \{p_t\}_{t=0}^{\infty}$ as well as the reservation utility levels $V = \{v_t^e\}_{t=0}^{\infty}$. At every date, the game between the firm and the successive generations of workers has two stages:

■ $1^{\underline{st}}$ stage: Each firm decides how much of its assets (a_t) to spend as financial capital, $0 \leq m \leq a_t$. It also decides the number $(l, s, e) \in \Re_+^3$ of workers it wants to hire and makes wage offers for young and old workers, $(w, v) \in \Re_+^2$, such that its expenditure does not exceed its financial capital.

$$w \cdot l + (v + c) \cdot s + v \cdot e = m \tag{1}$$

Implicit in the financial constraint (1) is the assumption that the firm offers the same wage to all employees performing task 2, independently of their past employment history. In principle, one could allow the firm to make different wage offers to those promoted internally and those hired in the market. As I show below, since workers perform task 2 only in the last period of their life, then no firm has an incentive to pay to that worker

[5] Notice that I defined the outside value of a worker that peforms task 2 to be independent of his employment history. This seems reasonable because all high ability workers are equally productive when working in any other firm different from the one that trained them.

more than what the market would pay. Thus, given the assumption that all promoted workers are equally productive in a firm different from the one that trained them, the assumption of equal wage offers within the firm is made without loss of generality to simplify notation.

Each young worker is approached by just one firm. For simplicity, I assume that firms adopt an "up or out" promotion system: old workers who are not promoted are fired. This assumption is also made without loss of generality because, as it will become clear later in the paper, in equilibrium, no firm could make a profit by hiring an old worker to perform task one. Since the number of internal promotions cannot exceed the number of employees that developed high ability, then the firm faces the following "internal labor market" constraint:

$$0 \leq s_t \leq \lambda \cdot l_{t-1} \tag{2}$$

If $s_t < \lambda \cdot l_{t-1}$, then the firm decides at random who receives training because, from the firm's point of view, high ability workers are homogeneous. It follows that each worker hired at $t-1$ has an *ex-ante* objective probability $\frac{s_t}{l_{t-1}}$ of being promoted at t.

■ $2^{\underline{nd}}$ stage: Each young worker contacted by firm i observes the wage offer, w_t^i, and decides whether to accept (A) or reject (R) it. Those old workers that went through the training process decide whether to stay in the firm that trained them (A) or to move to another one (R) where they obtain utility v_t^e.

More formally, let $d_t = (l, s, e, m, b, w, v)$ be the quantity demanded of each factor, the financial decisions and the wages offered by a firm at date t. Let $d_t^w \in \{A, R\} \times \{A, R\}$ be the date t responses of young and old workers and let $h_t = (d_t, d_t^w)$ be the actions of the players at date $t \geq 0$.[6] Let $h^0 = (l_{-1}, a_0)$ be the history at the start of play, $h^t = (h_0, h_1, ..., h_{t-1})$ denotes the partial history of play up to date $t \geq 1$ and $h^{t-\tau}$ the partial history where the first $\tau \leq t$ elements are omitted. The set of actions that a firm can choose after history h^t is given by:

$$\mathcal{A}\left(h^t\right) = \left\{ d \in \Re_+^7 : \left| \begin{array}{c} m + b = a_t \\ w \cdot l + (v + c) \cdot s + v \cdot e = m \\ s \leq \lambda \cdot l_{t-1} \end{array} \right. \right\}$$

where

$$a_t = \begin{cases} p_{t-1} \cdot q\left(l_{t-1}, s_{t-1}, e_{t-1}; \alpha\right) + r \cdot b_{t-1} & \text{if } d_{t-1}^w = (A, A) \\ r \cdot b_{t-1} + \left(w_{t-1} \cdot l_{t-1} \cdot 1_{\sigma_{1,t-1}=R} + v_{t-1} \cdot s_{t-1} \cdot 1_{\sigma_{2,t-1}=R}\right) & \text{otherwise} \end{cases}$$

and $1_{\sigma_{k,t}=R}$ is the function that takes value 1 if $\sigma_{k,t} = R$ and zero otherwise. Therefore, the set of all histories up to date t is

$$H^t = \left\{(h_0, ..., h_{t-1}) : d_\tau \in \mathcal{A}\left(h^\tau\right) \ \& \ d_\tau^w \in \{A, R\} \times \{A, R\} \ \text{ for all } 0 \leq \tau \leq t-1\right\}$$

[6] Implicit in the description of the actions played at date t, h_t, is the assumption that all workers of the same generation take the same decision. This assumption is made without loss of generality because I only consider stationary equilibria where workers of the same generation play the same history independent strategy against a given firm.

and the set of terminal histories is $H = \left\{(h_0, h_1, ...) : (h_0, h_{1,}..., h_{t-1}) \in H^t \text{ for all } t \geq 0\right\}$.

At date t, each young worker observes history h^t and decides whether to accept or reject the wage offer he received. If he rejects, he works in another industry with lifetime utility $\overline{u}$. The payoff that a young worker obtains at date t is

$$u_1(x, w) = \begin{cases} w & \text{if } x = A \\ \overline{w}_1 & \text{if } x = R \end{cases}$$

Each old worker who worked in the firm when young and received training can stay in the firm that trained him or leave. If he stays, he obtains utility v_t. However, he can obtain utility v_t^e by leaving to another firm. It follows that the date t payoff of an old worker who underwent training is

$$u_2(x, v_t, v_t^e) = \begin{cases} v_t & \text{if } x = A \\ v_t^e & \text{if } x = R \end{cases}$$

Let $\Gamma(P, V, \alpha)$ be the extensive form game played between a firm with technology α and the infinite generations of workers. A strategy for firm j specifies the number of wage offers it makes for each task at date t, (l_t, s_t, e_t), the wages it offers, (w_t, v_t) and the financial decisions, (m_t, b_t), as a function of the history. Formally, a pure strategy for the firm is a sequence $f = \{f_t\}_{t=0}^{\infty}$ where $f_t : H^t \to \mathcal{A}\left(h^t\right)$. Let $\mathbb{F}$ be the firm's set of pure strategies.

The strategy of a worker born at date t specifies whether he accepts or rejects the offer made by a firm at date t and whether he stays or moves to another firm at $t+1$ after receiving training. That is, the strategy of a worker born at t is a pair $\sigma_t = (\sigma_{1,t}, \sigma_{2,t})$ where $\sigma_{1,t} : H^t \times \Re_+ \to \{A, R\}$ is the decision of the young worker who receives an offer to perform task 1 and $\sigma_{2,t} : H^{t+1} \times \Re_+ \to \{A, R\}$ is his response at $t+1$ after going through the training process and being offered promotion by his first period employer. Let $\mathbb{W}_t$ be the set of pure strategies for the workers born at date t. I assume that all workers of the same generation play the same strategy against a given firm. Therefore, the sequence $\sigma = \left\{(\sigma_{1,t}, \sigma_{2,t})\right\}_{t=0}^{\infty} \in \mathbb{W} \equiv \mathbb{W}_0 \times ... \times \mathbb{W}_t \times ...$ is the collection of strategies that the infinite generations of workers play against a firm. For any $(f, \sigma) \in \mathbb{F} \times \mathbb{W}$, let $\overrightarrow{h^0} = h^0$ and $\overrightarrow{h^{t+1}} = \left[\overrightarrow{h^t}, \left(f_t\left(\overrightarrow{h^t}\right), \sigma_t\left(\overrightarrow{h^t}\right)\right)\right]$ denote the actions chosen by the players before date $t \geq 0$, i.e. the path of play up to date t. Let $\overrightarrow{f}_t = \left(\overrightarrow{l}_t, \overrightarrow{s}_t, \overrightarrow{e}_t, \overrightarrow{m}_t, \overrightarrow{b}_t, \overrightarrow{w}_t, \overrightarrow{v}_t\right) = f_t\left(\overrightarrow{h^t}\right)$ and $\overrightarrow{\sigma}_t = \sigma_t\left(\overrightarrow{h^t}\right)$ be the actions chosen at date t by the firm and workers on the path of play of (f, σ).

Each worker decides whether to accept or reject an offer in order to maximize his payoff. I define the set of wages that induce workers to accept a job at date t as:

$$\Theta\left(\sigma \left| h^t\right.\right) \equiv \left\{(w, v) \in \Re_+^2 : \sigma_{1,t}\left(h^t, w\right) = \sigma_{2,t-1}\left(h^t, v\right) = A\right\}$$

Then, the payoff to the firm in the subgame that begins after history h^t is

$$\Pi\left(f,\sigma;\alpha\left|h^t\right.\right)=\sum_{k=0}^{\infty}\beta^{k+1}\cdot R\left(f_t,\sigma;\alpha\left|h^t\right.\right)\cdot a_t$$

where $f_t \in \mathcal{A}\left(h^t\right)$ and

$$R\left(f_t,\sigma;\alpha\left|h^t\right.\right)\equiv\begin{cases}\frac{p_t\cdot q\left(l_t,\frac{m_t-w_t\cdot l_t-v_t\cdot e_t}{v_t+c},e_t;\alpha\right)+r\cdot b_t}{a_t} & \text{if } (w_t,v_t)\in\Theta\left(\sigma\left|h^t\right.\right)\\ r\cdot\frac{b_t}{a_t}+\frac{w_t\cdot l_t\cdot 1_{\sigma_{1,t}=R}+v_t\cdot s_t\cdot 1_{\sigma_{2,t}=R}}{a_t} & \text{otherwise}\end{cases}$$

The payoff to the young worker born at t is

$$U\left(\sigma_t,f,\widehat{\sigma}\right|\left(h^t,w_t\right))\equiv\begin{cases}u_1\left[\sigma_{1,t},w_t\right]+\beta\cdot\left(\frac{s_{t+1}}{l_t}\cdot\left[u_2\left(\sigma_{2,t},v_{t+1},v_{t+1}^e\right)-\overline{w}_2\right]+\overline{w}_2\right) & \text{if } \widehat{\sigma}_{1,t}\left(\cdot\right)=A\\ u_1\left[\sigma_{1,t},w_t\right]+\beta\cdot\overline{w}_2 & \text{otherwise}\end{cases}$$

where the second line reflects that if a generation of workers reject working in the firm, then that firm closes.

Finally, I define the equilibrium concept for the game $\Gamma\left(P,V,\alpha\right)$. Since both young and old workers take their decisions at date t knowing only $\left(h^t,w_t\right)$ and $\left(h^t,v_t\right)$, respectively, the game $\Gamma\left(P,V,\alpha\right)$ is one of imperfect information. Therefore, subgame perfection does not exclude the possibility that workers follow a strategy that prescribes a suboptimal action on some information set out of the path of play. In particular, it does not eliminate the possibility that for some $\varepsilon>0$, old workers reject any wage offer below $v_t^e+\varepsilon$ even though they would be strictly better off accepting it. If firms make a profit by hiring workers at a wage $v_t^e+\varepsilon$, their best response would be to offer that wage to the old workers even though no other firm is willing to pay that sum. To eliminate these equilibria, I consider only those SPNE in which no worker chooses a strictly dominated action in or out of the equilibrium path.

Definition 2.1 *A $*-$Subgame Perfect Nash Equilibrium (*SPNE) of the game $\Gamma\left(P,V,\alpha\right)$ is a profile of strategies $\left(\widehat{f},\widehat{\sigma}\right)\in\mathbb{F}\times\mathbb{W}$ such that for all $t\geq 0$ and $h^t\in H^t$*

1. $u_2\left(\widehat{\sigma}_{2,t-1},v,v_t^e\left|\left(h^t,v\right)\right.\right)\geq u_2\left(x,v,v_t^e\left|\left(h^t,v\right)\right.\right)$ *for all $v\geq 0$ and $x\in\{A,R\}$*

2. $U\left(\widehat{\sigma}_t,\widehat{f},\widehat{\sigma}\left|\left(h^t,w\right)\right.\right)\geq U\left(\sigma_t,\widehat{f},\widehat{\sigma}\left|\left(h^t,w\right)\right.\right)$ *for all $w\geq 0$ and $\sigma_t\in\mathbb{W}_t$*

3. $\Pi\left(\widehat{f},\widehat{\sigma};\alpha\left|h^t\right.\right)\geq\Pi\left(f,\widehat{\sigma};\alpha\left|h^t\right.\right)$ *for all $f\in\mathbb{F}$.*

3. INDUSTRY EQUILIBRIUM

In this section, I define an Industry Equilibrium. In an Industry Equilibrium, firms take both the output prices as well as the reservation values of a skilled worker as given, the strategies of firms and workers constitute a

*SPNE of $\Gamma(P, V, \alpha)$ and all relevant markets clear when firms and workers behave according to the equilibrium path of the *SPNE they play. In section 3.1, I introduce the notion of prospects for advancement and show that in any IE, *ceteris paribus*, one firm displays better prospects for advancement than another if and only if it promotes a larger fraction of its workers than its competitor. In section 3.2, I discuss what determines the outside value of a promoted worker.

In the previous section, I described the behavior of workers and firms for exogenous sequences of the output price and the outside-value of promoted workers. This analysis is appropriate because each firm is competitive in the output market and once a worker is promoted the firm loses any monopoly power over him. However, both the output price sequence as well as the outside value of the promoted workers actually depend on the aggregate behavior of the firms through the corresponding market clearing condition. On the one hand, the output price, p_t, evolves such that the output market clears every period. On the other hand, the utility that a promoted worker can obtain by moving to another firm, v_t^e, must be consistent with the firms' actions on the equilibrium path of the *SPNE of the game $\Gamma(P, V, \alpha_i)$.

At date zero, after the firms announce their names, every worker who is contacted by a firm updates his common prior about the strategy of that firm after observing the realization of a binary *sunspot* variable that assigns probability μ^H to the strategy f^H and $1-\mu^H$ to the strategy f^L. To be more precise, the decision rule of a worker born at date t is a mapping from the set of firms, $[0,1]$, to the set $\left\{\sigma_t^L, \sigma_t^H\right\} \in \mathbb{W}_t \times \mathbb{W}_t$. In an IE, a measure $\mu^H \in (0,1)$ of the firms follow strategy f^H while the rest of the firms follow strategy f^L. At date zero, the assets in hands of those firms that follow strategies f^H and f^L are $a_0^H = \mu^H \cdot a_0$ and $a_0^L = \left(1-\mu^H\right) \cdot a_0$, respectively. If $\mu^H = 0$ or $\mu^H = 1$, then all firms follow the same strategy. For any $i \in \{L, H\}$, q_t^i denotes the output produced at date t, on the equilibrium path of the *SPNE $\left(f^i, \sigma^i\right)$, by a firm that follows strategy f^i.

Definition 3.1 *An Industry Equilibrium* (IE) *is* $(P, V) \in \Re_+^\infty \times \Re_+^\infty$ *together with strategies* $\left(f^i, \sigma^i\right) \in \mathbb{F} \times \mathbb{W}$ *for* $i \in \{L, H\}$ *and* $\mu^H \in [0,1]$ *such that* $\left\{P, V, \left(f^H, \sigma^H\right), \left(f^L, \sigma^L\right), \mu^H\right\}$ *satisfies:*

1. *For each* $i \in \{L, H\}$, $\left(f^i, \sigma^i\right)$ *is a *SPNE of* $\Gamma(P, V, \alpha_j)$ *for some* $j \in [0,1]$.
2. $q_t^L \cdot \left(1-\mu^H\right) + q_t^H \cdot \mu^H = D(p_t)$, *for all* $t \geq 0$.
3. $\overrightarrow{e}_t^i = 0$ *for all* $t \geq 0$ *and* $i \in \{L, H\}$
4. *If* $\left(f^i, \sigma^i\right)$ *is a *SPNE of* $\Gamma(P, V, \alpha_j)$ *for some* $i \in \{L, H\}$ *and* $j \in [0,1]$, *then* $\frac{\partial \Pi\left(f^i, \sigma^i; \alpha_j \middle| \overrightarrow{h^t}\right)}{\partial e_t} \leq 0$ *for all* $t \geq 0$ *(with equality for some* i *if* $v_t^e > \overline{w}_2$*.)*

If every firm follows the same strategy, I denote the IE simply by $\left\{P, V, \left(f^H, \sigma^H\right)\right\}$.

This equilibrium concept does not impose the restriction that *ex-ante* identical firms must follow the same strategy in the game $\Gamma(P, V, \alpha)$. The behavior of firms and workers may be heterogenous either because firms have different technologies or due to a coordination problem among the infinite generations of workers. For example, if for some sequences $(P, V) \in \Re_+^\infty \times \Re_+^\infty$ the game $\Gamma(P, V, \alpha)$ has multiple *SPNE equilibria, *ex-ante* identical firms may follow different growth paths. Conditions (2) - (4) refer to the quantities hired and produced by the firms on the equilibrium path of the *SPNE. This implies that unilateral deviations not only are not profitable but also they do not affect the equilibrium prices for output and promoted workers, which is consistent with the competitive hypothesis. Conditions (2) - (3) state that in an IE both the output as well as the skilled labor market clears. In the market of skilled labor, the supply is given by the sum of those workers who are offered a promotion but chose to leave to another firm. Since training is costly, in any *SPNE, those workers that are offered promotion receive a wage offer that induce them to stay with their previous employer. Hence, in an IE, the supply of externally trained workers is zero. Condition (3) says that the quantity demanded of externally trained workers, $\overrightarrow{e}_t^i$, equals the quantity supplied. Finally, the last condition guarantees that when $v_t^e > \overline{w}_2$, some firm in this industry is willing to pay v_t^e to hire a worker promoted by another firm. Notice that the derivative in condition (4) takes into account that because the firm faces a financial capital constraint, a marginal increase in e_t implies a reduction on the use of some input at date t.

Since workers perform the second task only when old, the requirement that their strategy is part of a *SPNE of $\Gamma(P, V, \alpha)$ implies that they accept any wage offer which is greater or equal than v_t^e, and reject any offer below that level. In case of indifference, I assume that an old worker prefers to stay in the firm where he worked when young. Therefore, in any *SPNE the old workers' strategy is:

$$\widehat{\sigma}_{2,t} = \begin{cases} A & \text{if } v_t \geq v_t^e \\ R & \text{if } v_t < v_t^e \end{cases}$$

and for the rest of the paper, I assume that $\{\widehat{\sigma}_{2,t}\}_{t=0}^\infty$ describes the behavior of the old workers in a *SPNE. The set of stationary, or history independent, strategies for the workers is:

$$\overline{\mathbb{W}} = \left\{\sigma \in \mathbb{W} : \forall\, k = 1, 2 \text{ and } x \geq 0,\, \sigma_{k,t}\left(h^t, x\right) = \sigma_{k,t}\left(\widetilde{h}^t, x\right)\ \forall t \geq 0 \text{ and } \forall h^t, \widetilde{h}^t \in H^t\right\}$$

3.1 Prospects for advancement

Since in the early stages of the evolution of an industry firms are financially constrained, those that pay lower wages can produce more and obtain more revenues to finance expansion. Therefore, in order to explain the outcome of industry evolution, it is important to identify what enables one firm to hire workers at lower wages than another. Insofar worker's abilities are, at least to some degree, firm specific and developed by on-the-job training, one would expect that his reservation wage depends not only on his opportunity cost and future wages, but also on other factors such as his expectations about the opportunities for promotion within the firm. For the moment, I will be rather vague and call all those relevant factors "the prospects for advancement" displayed by the firm. Although intuition suggests that prospects for advancement depends on many factors, I believe that in this model the following definition captures the main idea:

Definition 3.2 *A worker believes that firm i displays better prospect for advancement than firm j if he is willing to work in firm i at a lower wage than in firm j.*

The relevant question is: what aspects of the firms' strategies make workers believe that one firm displays better prospects for advancement than another? I show that in an IE, *ceteris paribus*, one firm displays better prospects for advancement than another at date t if and only if workers believe that the former will promote a larger fraction of its employees than the latter at $t+1$. To see why, let's consider the case of a young worker born at date t who believes that he will be promoted at date $t+1$ with probability π_t. Let $v_{t+1} > \overline{w}_2$ be the wage, or utility, he anticipates in case of being promoted. If he receives a wage offer w_t at date t and he accepts to join the firm, his lifetime expected utility is $w_t + \beta \cdot [\pi_t \cdot (v_{t+1} - \overline{w}_2) + \overline{w}_2]$. Otherwise, his lifetime utility is $\overline{w}_1 + \beta \cdot \overline{w}_2$. It is not difficult to obtain the wage offer, $w(\pi_t, v_{t+1})$, which makes the worker indifferent between accepting a job at date t or not, i.e the *reservation entry-wage*. Clearly, $w(\pi_t, v_{t+1})$ must be the unique solution to the following equation in w:

$$w + \beta \cdot [\pi_t \cdot (v_{t+1} - \overline{w}_2) + \overline{w}_2] = \overline{w}_1 + \beta \cdot \overline{w}_2$$

and it follows that the *reservation entry-wage* is:

$$w(\pi_t, v_{t+1}) = \overline{w}_1 - \beta \cdot \pi_t \cdot (v_{t+1} - \overline{w}_2)$$

As one could expect, *ceteris paribus*, workers are willing to work at a lower entry-wage in firms that are expected to promote a larger fraction of their workers. Formally,

Lemma 3.1 *Let $\{P, V, (f^H, \sigma^H), (f^L, \sigma^L), \mu^H\}$ be an IE. If $\overrightarrow{v}_{t+1}^L = \overrightarrow{v}_{t+1}^H > \overline{w}_2$, the firm that follows strategy f^H displays better prospects for advancement at date t than the firm that follows f^L does iff $\overrightarrow{\pi}_t^H > \overrightarrow{\pi}_t^L$.*

3.2 The outside value of a promoted worker

In an IE, any worker who is offered promotion is free to move to another firm where he obtains utility v_t^e. If young workers follow a strategy that do not depend on the history of play, firms have no incentive to pay to a promoted worker more than his reservation utility, v_t^e. Therefore, the reservation utility of a promoted worker is determined either by his wage in another industry or by what the firms in this industry are willing to pay to a high ability worker trained by another firm.

Since any old worker can work in another industry when old and obtain $\overline{w}_2$ then $v_t^e \geq \overline{w}_2$ for all $t \geq 0$. However, the best option of a promoted worker need not be to move out of the industry but to work for a competitor of the firm that trained him. In that case, condition (3) in the definition of an IE implies that v_t^e must be equal to the competitors' value of an externally trained worker. If those firms pay v_t^e to their internally promoted workers, and workers follow history independent strategies, the value of an internally promoted worker is at least $v_t^e + c$. Since a worker promoted internally is, roughly speaking, as productive as $1 + \theta$ workers trained by another firm then the value of the latter is at least $\frac{v_t^e + c}{1+\theta}$. Therefore, $v_t^e \geq \frac{v_t^e + c}{1+\theta}$. In the case in which $v_t^e > \overline{w}_2$, condition (4) in the definition of an IE implies that $v_t^e = \frac{v_t^e + c}{1+\theta}$ or, equivalently, $v_t^e = \frac{c}{\theta}$. One concludes that in any IE in which workers follow stationary strategies, $v_t^e = \max\left\{\frac{c}{\theta}, \overline{w}_2\right\} \equiv v^*$. Let V^* denotes the sequence with elements $v_t^e = v^*$ for all $t \geq 0$.

4. $*$-Subgame Perfect Nash Equilibrium

In this section, I consider the game which describes the interaction between a firm with technology α and the infinite generations of workers, $\Gamma(P, V^*, \alpha)$, in isolation. I divide the analysis in two cases according to the value of v^*. For each case, I restrict the analysis to a set of price sequences Σ that is the natural candidate to contain an IE price sequence and analyze the existence of a *SPNE of the game $\Gamma(P, V^*, \alpha)$ for those $P \in \Sigma$.

For the rest of the paper I assume that young workers follow a stationary strategy. Therefore, firms have no incentive to offer a promoted worker more than what its competitors would pay. As I argued in the previous section, in an IE the outside value of a promoted worker must be given by the sequence V^*. Anticipating this, the reservation entry-wage of a young worker becomes $w\left(\pi_t, v_{t+1}^e\right) = w\left(\pi_t, v^*\right)$ and depends on π_t if and only if $v^* > \overline{w}_2$. Whenever $v^* = \overline{w}_2$, the optimal strategy of the firm is the solution to a one agent problem and, therefore, easier to analyze than the case in which $v^* > \overline{w}_2$. Since $v^* = \overline{w}_2$ if and only if $\frac{c}{\theta} \leq \overline{w}_2$, I consider first the simplest case in which $\frac{c}{\theta} \leq \overline{w}_2$ and later the case $\frac{c}{\theta} > \overline{w}_2$.

4.1 Case I: $\frac{c}{\theta} \leq \overline{w}_2 = v^*$

Since young workers receive $\overline{w}_2$ when old, regardless where they work in the future, their reservation entry-wage is $\overline{w}_1$ no matter what the firm's promotion policy is. Consider the following strategy for the workers

$$\sigma_t^s = \begin{cases} A & \text{if } w_t \geq \overline{w}_1 \\ R & \text{if } w_t < \overline{w}_1 \end{cases}$$

Given the young workers' strategy, the firm has no incentive to offer its workers more than $\overline{w}_1$ when young and $\overline{w}_2$ when old, as in a Walrasian equilibrium. The determination of the optimal financial capital and the number of employees becomes a one agent problem. At every date t and partial history h^t, $\{(l_k, s_k, e_k, m_k, b_k)\}_{k=t}^{\infty}$

must solve

$$
Max \sum_{k=t}^{\infty} \beta^{k+1} \cdot R_k \cdot a_k
$$

$$
s.t. \begin{cases} \overline{w}_1 \cdot l_k + (\overline{w}_2 + c) \cdot s_k + \overline{w}_2 \cdot e_k = m_k \\ R_k = \frac{p_k \cdot q(l_k, s_k, e_k; \alpha) + r \cdot b_k}{a_k} \\ m_k + b_k = a_k, \quad a_{k+1} = R_k \cdot a_k \\ (l_k, s_k, e_k, m_k, b_k) \in \Re^5_+, \; s_k \leq \lambda \cdot l_{k-1} \end{cases}
$$

The solution to this problem depends, among other things, on the sequence P. Instead of solving the problem for each possible sequence P, I restrict myself to a set whose elements share some natural properties that makes them a candidate for an IE price sequence. Whenever profits are positive, the economic intuition suggests that firms fully reinvest earnings to hire inputs. Although the behavior of each firm in isolation does not affect the output price, the decision of fully reinvesting earnings, taken by all of them together, eventually drives the price down. If this is so, the growth rate of the aggregate financial capital is necessarily larger than one, which suggests that profits must be driven down to zero in finite time. Since the purpose of this paper is to analyze the convergence to a Walrasian-like equilibria, it seems natural to consider those price sequences in the set

$$
\Sigma = \{P \in \Re^{\infty} : \exists\, T \text{ such that } p_t < p_{t-1} \; \forall t \leq T \text{ and } p_t = p^* \; \forall t \geq T\}
$$

of decreasing price sequences that converge in finite time to p^*, the socially optimal marginal cost of the good.

From date T on, the firm can make at most zero profits. Since the firm can make zero profits by allocating all assets to the bond, it follows that the problem above has value $\frac{\beta r}{1-\beta r} \cdot a_T$ from date T on. Therefore, for any $0 \leq t \leq T-1$, $\{(l_k, s_k, e_k, m_k, b_k)\}_{k=t}^{T-1}$ must solve

$$
Max \sum_{k=t}^{T-1} \beta^{k+1} \cdot R_k \cdot a_k + \beta^{T+1} \cdot \frac{\beta r}{1-\beta r} \cdot a_T
$$

$$
s.t. \begin{cases} \overline{w}_1 \cdot l_k + (\overline{w}_2 + c) \cdot s_k + \overline{w}_2 \cdot e_k = m_k \\ R_k = \frac{p_k \cdot q(l_k, s_k, e_k; \alpha) + r \cdot b_k}{a_k} \\ m_k + b_k = a_k, \quad a_{k+1} = R_k \cdot a_k \\ (l_k, s_k, e_k, m_k, b_k) \in \Re^5_+, \; s_k \leq \lambda \cdot l_{k-1} \end{cases} \tag{3}
$$

In general, the optimal strategy of the firm depends on the history of play not only through a_t but also through l_{t-1}, making it difficult to find a closed form solution. However, the case in which $\frac{c}{\theta} = \overline{w}_2$ is easy to analyze because the cost of producing one unit of task 2 does not depend on whether the firm employs workers promoted internally or workers trained by another firm. This is because $\frac{c}{\theta} = \overline{w}_2 \Leftrightarrow \overline{w}_2 + c = \overline{w}_2 \cdot (1+\theta)$. Therefore, the firm's payoff depends only on the number of workers performing task 2, $s_t + \frac{e_t}{1+\theta}$. Hence, one can solve the problem above for $\left\{\left(l_k, s_t + \frac{e_t}{1+\theta}, m_k, b_k\right)\right\}_{k=t}^{\infty}$ ignoring the internal labor market constraint and then set s_t and e_t so that the constraint is satisfied. But once the labor market constraint is not taken into account, the problem of maximizing the intertemporal sum of discounted profits is equivalent to a sequence of one period problems. Indeed, the firm must hire workers to maximize one period profits subject to the financial capital constraint and fully allocate its assets as financial capital up to date $T-1$. One optimal strategy is $f(T, \delta; \alpha, P)$, which consists in offering wages $w_t = \overline{w}_1$ to the young workers, $v_t = \overline{w}_2$ to the old workers and allocates the firm's assets in the following way:

$$\begin{aligned} m_t &= \begin{cases} a_t & \text{if } t < T \\ \delta \cdot a_t & \text{if } t = T \\ Min\{m_T, a_t\} & \text{if } t > T \end{cases} \\ l_t &= \frac{\alpha}{\overline{w}_1} \cdot m_t \\ s_t + \frac{e_t}{1+\theta} &= \frac{1-\alpha}{\overline{w}_2 + c} \cdot m_t \text{ and } s_t = Min\left[\frac{1-\alpha}{\overline{w}_2 + c} \cdot m_t, \lambda \cdot l_{t-1}\right] \end{aligned}$$

where $\delta \in [0,1]$ and $b_t = a_t - m_t$.

Proposition 4.1 *Let $p \in \Sigma$ and $\frac{c}{\theta} = \overline{w}_2$. For any $\delta \in [0,1]$ and $T \geq T_s$, $[f(T, \delta; \alpha, P), \sigma^s]$ is a *SPNE of $\Gamma(P, V^*, \alpha)$. If $p_0 \leq \frac{p^*}{r} \frac{\alpha}{1-\alpha} \cdot \frac{\overline{w}_2 + c}{\overline{w}_1} \cdot \lambda$, then $\overrightarrow{e}_t = 0$, $\frac{\partial \Pi\left(f, \sigma^s; \alpha \middle| \overline{h}^t\right)}{\partial e_t} = 0$, $\overrightarrow{q}_t = \frac{r}{p^*} \cdot \overrightarrow{m}_t$ for all $t \geq 0$, $\overrightarrow{m}_0 = a_0$ and*

$$\overrightarrow{m}_t = \begin{cases} \frac{p_t}{p^*} \cdot r \cdot \overrightarrow{m}_{t-1} & \text{if } 1 \leq t < T \\ \delta \cdot \left(\frac{p_{T-1}}{p^*} \cdot r \cdot \overrightarrow{m}_{T-1}\right) & \text{if } t = T \\ \overrightarrow{m}_T & \text{if } t > T \end{cases} \tag{4}$$

The upper bound on the date zero price in Proposition 4.1 ensures that the internal labor market constraint is not binding at date zero. From Proposition 4.1, it is clear that the game $\Gamma(P, V^*, \alpha)$ does not have a unique *SPNE. Indeed, the game $\Gamma(P, V^*, \alpha)$ has two sources of multiplicity. However, neither of them result in an IE where identical firms follow different growth paths towards the steady state. I show in section 5 that market clearing in the output and labor market as well as the requirement that financial capital stays constant after date T helps to eliminate all but one of those *SPNE.

First, since the relative cost of a promoted worker and a worker trained by another firm equals their constant marginal rate of technical substitution, then the firm is indifferent between these two inputs. Strategy $f(T, \delta, \alpha, P)$ assumes that $s_t = Min\left[\frac{1-\alpha}{\overline{w}_2+c} \cdot m_t, \lambda \cdot l_{t-1}\right]$ but actually any $s_t \in \left(0, Min\left[\frac{1-\alpha}{\overline{w}_2+c} \cdot m_t, \lambda \cdot l_{t-1}\right]\right)$ is also a best response to the workers' strategy and implies that $e_t > 0$. However, this multiplicity is only relevant when one analyzes a single firm in isolation. In any IE, instead, market clearing implies that no firm hires an externally trained workers. Therefore, one cannot generate heterogeneous growth paths by assuming that some firms only promote internally and the rest promote both internally and externally because $\overrightarrow{e}_t^i = 0$ for every firm i. Second, there is a continuum of *SPNE of the type $[f(T, \delta; \alpha, P), \sigma^s]$ indexed by $\delta \in [0,1]$. These equilibria differ only in the financial capital at date T and arise because the scale of the firm is indeterminate once profits are driven down to zero. This feature of the model causes the existence of multiple *SPNE that differ only in the level of financial capital after date T. One could even find a *SPNE in which the firms behave according to $f(T, \delta; \alpha, P)$ up to date T but the financial capital does not stay constant afterwards. However, in all those *SPNE the behavior of the firm before date T is identical and, therefore, cannot be used to explain why *ex-ante* identical firms may follow different growth paths before profits are driven down to zero.

The hypothesis of Proposition 4.1 states that $\frac{c}{\theta} = \overline{w}_2$. If $\frac{c}{\theta} < \overline{w}_2$, instead, to produce one unit of task 2 using externally promoted workers is more expensive than employing internally promoted workers. Therefore, there are some partial histories where the internal labor market constraint is binding but, nevertheless, it is still optimal not to hire workers trained by another firm. In addition, it may be optimal to produce even if the shortage in the internal labor market drives the firm's current profits below zero, provided the firm expects to make sufficiently high positive profits in the future. Hence the optimal level of financial capital depends not only on whether profits are positive or not in the future, but also on the present discounted value of those profits and the number of workers currently available for promotion. For these reasons, it is more difficult to describe explicitly the strategy of the firm. I show that there exists a *SPNE and that it is qualitatively similar to the *SPNE found for the case $v^* = \overline{w}_2$. In any *SPNE, $\{(l_k, s_k, e_k, m_k, b_k)\}_{k=t}^{T-1}$ is the unique solution to (3) at every date $t \leq T-1$. In addition, it is also true that in any *SPNE, $\overrightarrow{e}_t = 0$, $\overrightarrow{q}_t = \frac{r}{p^*} \cdot \overrightarrow{m}_t$ for all $t \geq 0$ and that (4) holds for all $t \leq T$.

Proposition 4.2 *Let* $p \in \Sigma$ *and* $\frac{c}{\theta} < \overline{w}_2$. *The game* $\Gamma(P, V^*, \alpha)$ *has a *SPNE. If* $p_0 \leq \frac{p^*}{r} \frac{\alpha}{1-\alpha} \cdot \frac{\overline{w}_2 + c}{\overline{w}_1} \cdot \lambda$ *and* $(f, \sigma) \in \mathbb{F} \times \overline{\mathbb{W}}$ *is a *SPNE of* $\Gamma(P, V^*, \alpha)$, *then* $\sigma = \sigma^s$, $\overrightarrow{e}_t = 0$, $\frac{\partial \Pi\left(f, \sigma; \alpha \middle| \overrightarrow{h}^t\right)}{\partial e_t} < 0$, $\overrightarrow{q}_t = \frac{r}{p^*} \cdot \overrightarrow{m}_t$ *for all* $t \geq 0$ *and there exists* $\delta \in [0,1]$ *such that (4) holds for all* $t \leq T$.

In the equilibrium path of the *SPNE found in this section, firms promote workers internally, the value of these workers' marginal productivity is above their wage and once the training cost is sunk, every firm strictly prefers a worker promoted internally rather than one trained in another firm. Therefore, these *SPNE of the game describe the behavior of firms that set up a closed internal labor market with one entry port and an *up or out* promotion system.

4.2 Case II: $v^* = \frac{c}{\theta} > \overline{w}_2$

As it happens when $\frac{c}{\theta} = \overline{w}_2$, it costs the same to produce one unit of the second task either employing internally promoted workers or hiring externally trained workers. Unlike when $\frac{c}{\theta} = \overline{w}_2$, the promotion policy of the firm does affect the reservation entry-wage when $\frac{c}{\theta} > \overline{w}_2$. Although how much to invest as financial capital is part of the strategic decision that the firm makes, to simplify the exposition, it is useful to start describing a strategy for a fixed sequence of financial capital. As I show in Proposition 4.3, this is meaningful because in any *SPNE in which workers follow a stationary strategy there is a common linear relationship between the number of workers hired and the level of the financial capital of the firm. After describing this relationship, I define a set of output price sequences and I argue that it is the natural candidate to contain the equilibrium price sequences. For those sequences, I study the optimal reinvestment strategy of the firm and show that for some parameters values there exists two kinds of *SPNE of the game played by the infinite generations of workers and the firm.

Initially, I consider an exogenous strategy $\{m_t\}_{t=0}^{\infty}$ for the allocation of assets between financial capital and bonds. Suppose the firm spends a fraction α of its financial capital to hire young workers, to perform the first task, and a fraction $1-\alpha$ to hire workers, to perform the second task. This firm offers wages w_t to $l_t = \frac{\alpha}{w_t} \cdot m_t$ young workers and v^* to $s_t = \min\left\{\lambda \cdot l_{t-1}, \frac{1-\alpha}{v^*+c} \cdot m_t\right\}$ former employees and $e_t = Max\left\{0, \frac{1-\alpha}{v^*} \cdot m_t - (1+\theta) \cdot s_t\right\}$ workers trained by another firm. Therefore, the production level of the firm, per unit of financial capital, can be written as a function of the young workers' wage as follows:

$$q(w_t, \alpha) \equiv \left(\frac{\alpha}{w_t}\right)^{\alpha} \cdot \left(\frac{1-\alpha}{v^*+c}\right)^{1-\alpha}$$

Hence, for any wage w_t that induces young workers to accept employment, the firm produces $q(w_t, \alpha) \cdot m_t$.

Suppose the firm pays to its workers their reservation entry-wage, i.e. $(w_t, v_t) = \min \Theta\left(\sigma \left| h^t\right.\right)$. Since the promotion rate associated with the firm's strategy is $\frac{1-\alpha}{\alpha} \cdot \frac{w_t}{v^*+c} \cdot \frac{m_{t+1}}{m_t}$ then w_t is the reservation entry-wage if and only if it is the solution to the equation:

$$w_t = \overline{w}_1 - \beta \cdot \frac{1-\alpha}{\alpha} \cdot \frac{w_t}{v^*+c} \cdot \frac{m_{t+1}}{m_t} \cdot (v^* - \overline{w}_2) \tag{5}$$

Solving for w_t, one can write the reservation entry-wage as a function of the financial capital growth rate.

Lemma 4.1 *For any $g \geq 0$, the equation $w = \overline{w}_1 - \beta \cdot \frac{1-\alpha}{\alpha} \cdot \frac{w}{v^*+c} \cdot g \cdot (v^* - \overline{w}_2)$ has a unique solution $\omega : \Re_+ \times (0,1) \mapsto [0, \overline{w}_1]$ given by*

$$\omega(g, \alpha) = \frac{\alpha \cdot (v^*+c) \cdot \overline{w}_1}{\alpha \cdot (v^*+c) + (1-\alpha) \cdot \beta \cdot (v^* - \overline{w}_2) \cdot g}$$

The function ω is continuously differentiable and strictly decreasing in g.

There are two levels of the *reservation entry-wage* that are key in this paper: the *reservation entry-wage* associated with an stationary level of financial capital, $\omega(1, \alpha)$, and the *reservation entry-wage* associated with a growth rate of r, $\omega(r,\alpha)$. For each of these wages, one can define the output price such that the firm's rate of return is r, i.e. the price which is equal to the firm's marginal cost. This is stated in the following definition.

Definition 4.1 *Let* $p_s(\alpha) \equiv \frac{r}{q(w(1,\alpha),\alpha)}$ *and* $p_r(\alpha) \equiv \frac{r}{q(w(r,\alpha),\alpha)}$.

The following assumption ensures that for any $g \geq 0$, the promotion rate $\frac{1-\alpha}{\alpha} \cdot \frac{\omega(g,\alpha)}{v^*+c} \cdot g$ is bounded above by λ.

Assumption AW: Assume $\overline{w}_1$, $\overline{w}_2$, β, and $\frac{c}{\theta}$ are such that $w(\lambda) \leq 0 \Leftrightarrow \frac{\overline{w}_1}{\beta \cdot \left[\frac{c}{\theta} - \overline{w}_2\right]} \leq \lambda$

Assumption AW says that promotion brings about a welfare change that is large; so large that if workers believed that the promotion probability were high enough and the wage of a promoted worker were $\frac{c}{\theta}$, they would work for free in firm i when young. The assumption that *reservation entry-wages* are not bounded away from zero is sufficient to show the existence of an IE. However, as it is shown below, equilibrium entry-wages are uniformly bounded away from zero because the internal labor market constraint does not bind. I postpone further interpretations about the role of this assumption until I complete the description of the workers' and firms' strategies in an IE.

If workers follow a stationary strategy, the firm must allocate its financial capital as if it were maximizing one period profits subject to a financial constraint. That is for any $h^t \in H^t$, $(l_t, s_t, e_t, w_t, v_t)$ must solve

$$\max\ p_t \cdot l^\alpha \left(s + \frac{e}{1+\theta}\right)^{1-\alpha} \tag{6}$$
$$s.t. \begin{cases} w \cdot l + (v+c) \cdot s + v \cdot e = m_t \\ l, s, e \geq 0,\ s \leq \lambda \cdot l_{t-1} \\ (w,v) \geq \min \Theta\left(\sigma \left| h^t \right.\right) \end{cases}$$

This property of a *SPNE is proved in Proposition 4.3, where I also argue that the solution to problem (6) is to allocate the financial capital as described by (7) - (9).

Proposition 4.3 *Suppose* $(f,\sigma) \in \mathbb{F} \times \overline{\mathbb{W}}$ *is a *SPNE of* $\Gamma(P, V^*, \alpha)$. *If* $m_t > 0$, *then*

$$v_t = v^* \tag{7}$$
$$l_t = \frac{\alpha}{w_t} \cdot m_t \tag{8}$$
$$s_t \in \left[0, Min\left\{\tfrac{1-\alpha}{v^*+c} \cdot m_t, \lambda l_{t-1}\right\}\right], \quad s_t + e_t = \tfrac{1-\alpha}{v^*+c} \cdot m_t \tag{9}$$

In addition, if $s_t = Min\left\{\frac{1-\alpha}{v^*+c} \cdot m_t, \lambda \cdot l_{t-1}\right\}$ *then* $w_t = \omega\left(\frac{m_{t+1}}{m_t}, \alpha\right)$.

As in the case in which $\frac{c}{\theta} = \overline{w}_2$, those *SPNE in which $0 < s_t < Min\left\{\frac{1-\alpha}{v^*+c} \cdot m_t, \lambda l_{t-1}\right\}$ are not relevant because they imply that $\overrightarrow{e}_t > 0$, which violates condition (3) in the definition of an IE. Therefore, I restrict my attention to those *SPNE in which $s_t = Min\left\{\frac{1-\alpha}{v^*+c} \cdot m_t, \lambda \cdot l_{t-1}\right\}$. So far, I have considered an exogenous sequence of financial capital. However, how much to invest in the firm is one of the key decisions taken by the firms. Suppose the firm reinvests all earnings at date $t+1$, that is $\frac{m_{t+1}}{m_t} = p_t \cdot q(w_t, \alpha)$. Then, w_t is the reservation entry-wage if and only if w_t is a solution to the following equation in w:

$$w = \overline{w}_1 - \beta \cdot \frac{1-\alpha}{\alpha} \cdot \frac{w}{v^*+c} \cdot p_t \cdot q(w, \alpha) \cdot (v^* - \overline{w}_2) \tag{10}$$

The existence of a wage offer that solves (10) is discussed in Lemma 4.2.

Lemma 4.2 *If assumption AW holds, there exists a unique* $\omega^H : \Re_+ \times (0,1) \to [0, \overline{w}_1]$ *that solves (10). The function* ω^H *is continuous and strictly decreasing in p.*

Intuition suggests that in the presence of financial constraints and no fixed costs, the early stages of the process of industry evolution are characterized by a relatively high output price and positive profits. This induces firms to fully reinvest their earnings, driving down the price of output until profits vanish. It seems natural to think that the industry eventually converges to a steady state where all firms make zero profits and financial capital stays constant. If this intuition is correct and firms are *ex-ante* identical, Proposition 4.3 implies that in any IE the young workers' wage and the output price should converge to $\omega(1, \alpha)$ and $p_s(\alpha)$, respectively.[7] Thus, the stationary level of aggregate financial capital should be $\frac{p_s \cdot D(p_s)}{r}$, so that all active firms make zero profits at the price p_s. If firms fully reinvest earnings along the transition to the steady state, its rate of return is bounded above by $p_t \cdot q\left(\omega^H(p_t), \alpha\right)$. For profits to be positive along the transition, it is necessary that $p_t \cdot q\left(\omega^H(p_t), \alpha\right) > r$ or equivalently $p_t > p_r(\alpha)$. Thus, in order to show the existence of an IE, and with some abuse of notation, it seems natural to restrict the search to sequences of prices in the set

$$\Sigma = \{P \in \Re^\infty : \exists\, T_s \text{ such that } p_t > p_r, \forall t < T_s \text{ and } p_t = p_s, \forall t \geq T_s\}$$

and for any $P \in \Sigma$, let $P^t = \{p_\tau\}_{\tau=t}^{\infty}$.

For any $P \in \Sigma$, consider the game $\Gamma(P, V^*, \alpha)$ and the following family of cut-off strategies for the young workers, parameterized by $\delta \in [0,1]$ and $T \geq T_s$:

$$\sigma_{1,t}(T, \delta; \alpha, P) = \begin{cases} A & \text{if } w_t \geq \omega^H(p_t, \alpha) \text{ and } t < T-1 \\ A & \text{if } w_t \geq \omega^H(\delta \cdot p_{T-1}, \alpha) \text{ and } t = T-1 \\ A & \text{if } w_t \geq \omega(1, \alpha) \text{ and } t \geq T \\ R & \text{otherwise} \end{cases}$$

The cut-off wage coincides with the reservation entry-wage of a worker who assumes that the firm reinvests all its assets as financial capital up to $T-1$, it reinvest only a fraction δ of its assets at date $T \geq T_s$ and that

[7] Observe that $p_s(\alpha)$ is the firm's marginal cost when it pays $\omega(1, \alpha)$ to its young workers.

the financial capital is constant and equal to m_T thereafter. According to strategy $\sigma_{1,t}$, a young worker accepts an offer at date $t \geq 0$ if and only if it is greater or equal to his reservation entry-wage. Let $\sigma(T, \delta; \alpha, P) = \{\sigma_{1,t}(T, \delta; \alpha, P), \widehat{\sigma}_{2,t}\}_{t=0}^{\infty}$.

Abusing notation, let $f(T, \delta; \alpha, P)$ be the strategy in which (7) - (9) holds, $s_t = Min\left\{\frac{1-\alpha}{v^*+c} \cdot m_t, \lambda \cdot l_{t-1}\right\}$ and:

$$m_t = \begin{cases} a_t & \text{if } t < T \\ \delta \cdot a_T & \text{if } t = T \\ Max\{m_T, a_t\} & \text{if } t > T \end{cases} \qquad w_t = \begin{cases} \omega^H(p_t, \alpha) & \text{if } t < T-1 \\ \omega^H(\delta \cdot p_{T-1}, \alpha) & \text{if } t = T-1 \\ \omega(1, \alpha) & \text{if } t \geq T \end{cases}$$

where $\delta \in [0, 1]$ and $T \geq T_s$. According to this strategy, the firm offers a wage equal to the cutoff value of strategy $\sigma(T, \delta; \alpha, P)$ (i.e. $(w_t, v_t) \in Min\ \Theta(\sigma \left| h^t \right.)$) and spends its financial capital as if it were maximizing short run constrained profits. As the following proposition shows, these strategies constitute a *SPNE of $\Gamma(P, V^*, \alpha)$ for an open set of output price sequences in Σ.

Proposition 4.4 *Suppose $\frac{c}{\theta} > \overline{w}_2$. Let $P \in \Sigma$, $\delta \in [0, 1]$ and $T \geq T_s$. If $p_{T-1} \cdot q\left(\omega^H(\delta \cdot p_{T-1}, \alpha), \alpha\right) > r$, then $[f(T, \delta; \alpha, P), \sigma(T, \delta; \alpha, P)]$ is a *SPNE of $\Gamma(P, V^*, \alpha)$ in which $\overrightarrow{e_t} = 0$ and $\frac{\partial \Pi\left(f(\cdot), \sigma(\cdot); \alpha \left| \overrightarrow{h^t} \right.\right)}{\partial e_t} = 0$ for all $t \geq 0$.*

As in the previous section, there is a continuum of *SPNE of the type $[f(T, \delta, \alpha, P), \sigma(T, \delta; \alpha, P)]$ indexed by $\delta \in [0, 1]$. However, I show in the next section that in any IE, if $\frac{c}{\theta} > \overline{w}_2$ then the value of δ is uniquely determined by the output market clearing condition.

Now I am in a better position to explain the role of assumption AW. Along the process of industry evolution, there are potentially two limits to the growth of firms. On the one hand, firms might not achieve their optimal size because they do not have enough financial capital to finance expansion. This is represented by the financial capital constraint described by (1). On the other hand, firms may face a shortage in their internal labor market. That is, even if financial capital were available to promote workers, the internal pool might not contain as many high ability candidates as workers the firm would like to hire. This is the constraint imposed by the internal labor market and expressed by (2). In principle, any of this two constraints may be binding during the process of industry evolution. On the one hand, the possibility that a shortage in the firm's internal labor markets can be responsible for the slow growth of an industry is very realistic but, on the other hand, it complicates the analysis enormously. I introduced assumption AW to rule out this possibility along the equilibrium path of any *SPNE of the type $[f(T, \delta; \alpha, P), \sigma(T, \delta; \alpha, P)]$.

The *SPNE $[f(T, \delta; \alpha, P), \sigma(T, \delta; \alpha, P)]$ corresponds to a situation where workers are optimistic about the prospects for advancement offered by the firm. However, the game $\Gamma(P, V^*, \alpha)$ has another type of *SPNE; it corresponds to the case in which prospects for advancement within the firm are not so favorable. In this *SPNE, *reservation entry-wages* are such that the firm can just make zero profits from date 1 on. For that firm it is (weakly) optimal to reduce its financial capital along time, which justifies the workers' pessimism about the prospects for advancement offered by the firm. Let $\omega^L(p,\alpha)$ be the unique solution to the equation

$$q(w,\alpha) = \frac{r}{p}$$

For any $P \in \Sigma$, let $\underline{w} = \omega^H(p_0,\alpha)$ and $\overline{w} = Min\left\{\omega^L(p_0,\alpha), \overline{w}_1\right\}$. Let $p^{**}(\alpha) = \frac{r}{q(\overline{w}_1,\alpha)}$ be the marginal cost of a firm that is expected to close the following period, i.e. $\overline{w}_1 = \omega(0,\alpha)$. If the output price were higher than p^{**}, workers would expect negative growth from firms that follow strategy f^L but this is not feasible since financial capital is non-negative. For any $w_0 \in (\underline{w}, \overline{w})$, define the collection of young workers' strategies as $\sigma(w_0;\alpha,P) = \{\sigma_{1,t}(w_0;\alpha,P), \widehat{\sigma}_{2,t}\}_{t-0}^{\infty}$, where

$$\sigma_{1,t}(w_0;\alpha,P) = \begin{cases} A & \text{if } w_t \geq w_0 \text{ and } t = 0 \\ A & \text{if } w_t \geq \omega^L(p_t,\alpha) \text{ and } t \geq 1 \\ R & \text{otherwise} \end{cases}$$

Let G be the inverse of $\omega(g,\alpha)$. The strategy $f(w_0,a_0;\alpha,P)$, is given by (7) - (9), $s_t = Min\left\{\frac{1-\alpha}{v^*+c}\cdot m_t, \lambda\cdot l_{t-1}\right\}$ and

$$m_t = \begin{cases} a_0 & \text{if } t = 0 \\ G(w_0,\alpha)\cdot a_0 & \text{if } t = 1 \\ G\left(\omega^L(p_t,\alpha),\alpha\right)\cdot a_{t-1} & \text{if } t \geq 2 \end{cases} \qquad w_t = \begin{cases} w_0 & \text{if } t = 0 \\ \omega^L(p_t,\alpha) & \text{if } t \geq 1 \end{cases}$$

Unlike the *SPNE $[f(T,\delta;\alpha,P), \sigma(T,\delta;\alpha,P)]$, at any date $t \geq 1$ the firm is indifferent about how much to allocate as financial capital if the workers follow $\sigma(w_0;\alpha,P)$. Therefore, the financial capital of the firm at $t+1$ can be chosen to justify the young worker's reservation entry-wage at date t.

Proposition 4.5 *Suppose* $\frac{c}{\theta} > \overline{w}_2$. *Let* $P \in \Sigma$ *and* $w_0 \in (\underline{w}, \overline{w})$. *If* $p_t \leq p^{**}(\alpha)$ *for all* $t \geq 1$, *then the strategy profile* $[f(w_0,a_0;\alpha,P), \sigma(w_0;\alpha,P)]$ *is a *SPNE of* $\Gamma(P,V^*,\alpha)$ *in which* $\overrightarrow{e}_t = 0$ *and* $\frac{\partial\Pi\left(f(\cdot),\sigma(\cdot);\alpha\middle|\overrightarrow{h^t}\right)}{\partial e_t} = 0$ *for all* $t \geq 0$.

Each *SPNE in Proposition 4.5 differs from those found in Proposition 4.4 in the behavior of the firm along the transition to the steady state. This opens the possibility that *ex-ante* identical firms follow different growth paths. However, since $\omega^L(p_s,\alpha) = \omega(1,\alpha)$, in both *SPNE the worker's reservation entry-wage converges to the same level at date T. Hence, firms following strategies $f(T,\delta;\alpha,P)$ and $f(w_0,a_0;\alpha,P)$ stop growing and face the same price for labor from date T on. Any difference in their steady state size must originate during the transition towards the stationary state.

At this point, the reader may wonder whether the *SPNE discussed above describe all possible behaviors of firms and workers for a given $P \in \Sigma$. One can show that in any *SPNE of the game $\Gamma(P,V^*,\alpha)$ in which young workers follow stationary strategies and $\overrightarrow{e}_t = 0$ there exists τ and $w \in (\omega^H(p_\tau, \alpha), \overline{w}_1]$ such that (f, σ) consists in playing (7) - (9), $s_t = Min\left\{\frac{1-\alpha}{v^*+c} \cdot m_t, \lambda \cdot l_{t-1}\right\}$, $m_t = a_t$ and $w_t = \omega^H(p_t, \alpha)$ for all $t < \tau$ and according to $[f_{t-\tau}(w, a_\tau; \alpha, P^\tau), \sigma_{t-\tau}(w; \alpha, P^\tau)]$ for all $t \geq \tau$. Therefore, the two *SPNE analyzed in Propositions 4.4 and 4.5 are the special cases in which $\tau \geq T_s$ and $\tau = 0$, respectively. I prove here a weaker result that characterizes the *SPNE wages when young workers follow stationary strategies and the firm's demand for externally trained workers is zero on the equilibrium path.

Proposition 4.6 *Let $P \in \Sigma$ and $\frac{c}{\theta} > \overline{w}_2$. Suppose $(f, \sigma) \in \mathbb{F} \times \overline{\mathbb{W}}$ is a *SPNE in which $\overrightarrow{e}_t = 0$. If there exists τ such that $\overrightarrow{w}_\tau \neq \omega^H(p_\tau, \alpha)$ for the first time, then $\overrightarrow{w}_\tau \in (\omega^H(p_\tau, \alpha), \overline{w}_1]$ and $(f_t, \sigma_t) = [f_{t-\tau}(\overrightarrow{w}_\tau, a_\tau; \alpha, P^\tau), \sigma_{t-\tau}(\overrightarrow{w}_\tau; \alpha, P^\tau)]$ for all $t \geq \tau$.*

What this proposition says is that in any *SPNE the young workers' reservation entry wage can differ from $\omega^H(p_t, \alpha)$ or $\omega^L(p_t, \alpha)$ for at most one period. Moreover, once the reservation wage equals $\omega^L(p_\tau, \alpha)$ for some generation τ, the reservation entry-wage of those generations born at $t \geq \tau$ is $\omega^L(p_t, \alpha)$. That is, once a firm makes zero profits for the first time, it cannot obtain positive profits thereafter.

Like in the previous case, in the equilibrium path of these *SPNE, firms promote workers internally, the value of these workers' marginal productivity is above their wage and, once the training cost is sunk, every firm strictly prefers a worker promoted internally rather than one trained in another firm. Moreover, if $\frac{c}{\theta} > \overline{w}_2$ then the wage of old workers is always above the wage they would receive in another industry. Hence, small changes in market conditions (i.e. changes in $\overline{w}_2$) do not affect the wage of skilled workers. Therefore, each *SPNE of the game describe the behavior of firms that set up a closed internal labor market with one entry port and an *up or out* promotion system.

5. Ex ante identical firms

In this section, I consider an industry in which firms are *ex-ante* identical, i.e. $\alpha_j = \alpha$ for all $j \in [0, 1]$. First, I analyze two benchmark cases: in section 5.1, I analyze an equilibrium where firms face no financial constraint from the start and in section 5.2, I consider the case in which financial constraints are binding at date zero but $v^* = \overline{w}_2$, i.e. skills are specific enough or training cost low enough so that competition for skilled workers do not drive the market value of a promoted workers above its opportunity cost in another industry. I show there exists a unique industry equilibrium and that it converges to an allocatively efficient steady state in finite time. Section 5.3, turns into the more interesting scenario where the internal labor market is an implicit contract enforced by market competition, that is, when $v^* > \overline{w}_2$. A minimal requirement for a model of industry evolution is to have an equilibrium in which *ex-ante* identical firms behave identically. In section 5.3.1, I show that such equilibrium always exist and converges to a Walrasian-like state in finite time. I also discuss its efficiency properties. In section 5.3.2, I show that it also exists an open set of equilibria where *ex-ante* identical firms follow different growth paths. In these equilibria, productive efficiency holds but allocative efficiency fails during the transition towards the steady state.

5.1 Perfect Credit Markets

If the market for credit is perfect, entrepreneurs are not restricted to their own assets to finance production. Therefore, profits are driven down to zero from the start. Let P_s be the sequence with $p_t = p_s$ for all $t \geq 0$. Clearly, $P_s \in \Sigma$. Since firms must make zero profits, then the output per unit of financial capital must be $\frac{r}{p_s}$ every period. Hence, the workers' reservation entry-wage is $\omega(1,\alpha)$ every period and the market clears if and only if the aggregate financial capital is $\frac{p_s \cdot D(p_s)}{r}$ every period. The industry output level is $D(p_s)$. If $v^* = \overline{w}_2$, then $p_s = p^*$ and allocative efficiency holds. Otherwise, too little is produced with respect to the efficient allocation (i.e. $p_s > p^*$) but technological efficiency necessarily holds because every firm pays the same wages, $\omega(1,\alpha)$ and v^*, and maximize profits. It follows that, in the presence of a perfect credit market, *ex-ante* identical firms produce the same every period and workers that perform the same task receive the same wage regardless of the firm that employs them. To understand why too little is produced when $v^* > \overline{w}_2$, notice that the marginal cost in that case is $p_s(\alpha) = \frac{r}{q(\omega(1,\alpha),\alpha)}$ while the marginal cost in the efficient allocation is $p^* = \frac{r}{q^*}$, where $q^* = \left(\frac{\alpha}{\overline{w}_1}\right)^{\alpha} \cdot \left(\frac{1-\alpha}{\overline{w}_2+c}\right)^{1-\alpha}$ is the solution to

$$\begin{aligned} &Max\ l^{\alpha} \cdot s^{1-\alpha} \\ &s.t.\ \overline{w}_1 \cdot l + (\overline{w}_2 + c) \cdot s \leq 1 \end{aligned}$$

The two marginal costs are equal if and only if $v^* = \overline{w}_2$ so that $\omega(1,\alpha) = \overline{w}_1$. So it suffices to argue that when $v^* > \overline{w}_2$, production per unit of financial capital is smaller than in the efficient allocation, i.e. $q(\omega(1,\alpha),\alpha) < q^*$. The answer is not trivial because as v^* increases, the wage of young workers decreases. However, since workers discount the future at rate β, each unit increase in v^* leads to a less than proportional reduction in $\omega(1,\alpha)$. Indeed, let l_s and s_s be the number of young and skilled workers, respectively, hired when $v^* > \overline{w}_2$. Since $\omega(1,\alpha) = \overline{w}_1 - \beta \cdot \frac{s_s}{l_s} \cdot (v^* - \overline{w}_2)$ it follows that if $\beta \in (0,1)$ then

$$\begin{aligned} 1 &= \omega(1,\alpha) \cdot l_s + (v^* + c) \cdot s_s = \overline{w}_1 \cdot l_s + [(1-\beta) \cdot v^* + \beta \cdot \overline{w}_2 + c] \cdot s_s \\ &> \overline{w}_1 \cdot l_s + (\overline{w}_2 + c) \cdot s_s \end{aligned}$$

which implies that $q^* > q(\omega(1,\alpha),\alpha)$.

Remark 1: The argument above shows that the lack of allocative efficiency in the steady state does not depend on the assumption that firms have no access to credit. What is really important is whether skills are general enough or training is so costly that the market value of a promoted worker is above $\overline{w}_2$. Therefore, one concludes that it is the impossibility of enforcing a long term contract in which an old worker is paid $\overline{w}_2$ what is at the root of the lack of allocative efficiency.

Remark 2: Since the industry attains the steady state from the start, there is no evolution when credit markets are perfect. Therefore, one cannot talk about the growth path of a firm and how the prospects for advancement displayed by the firm affects its steady state size.

For the rest of the paper I assume that the initial financial capital falls short of the steady state level , i.e. $0 < a_0 < \frac{p_s \cdot D(p_s)}{r}$.

5.2 Case I: $\frac{c}{\theta} \leq \overline{w}_2$

This case corresponds to a situation in which workers anticipate that firms have no incentives to pay an old worker more than what firms in another industry would pay. The young worker's reservation entry-wage is independent of the firm's strategy. That is, $\omega(g, \alpha) = \overline{w}_1$ for all $g \geq 0$ and $p_s = p_r = p^*$. As was shown in Proposition 4.2, in any *SPNE of the game $\Gamma(P, V^*, \alpha)$ workers follow strategy σ^s, $\overrightarrow{e}_t = 0$, $\frac{\partial \Pi\left(f, \sigma^s; \alpha | \overrightarrow{h^t}\right)}{\partial e_t} \leq 0$ and firms produce $\frac{r}{p^*} \cdot \overrightarrow{m}_t$ at every date $t \geq 0$. To show that an IE actually exists, is suffices to find a price sequence $P \in \Sigma$ such that the output market clears at every date $t \geq 0$. If such a sequence exists, then for any $t < T$ it must be the case that p_t solves

$$\frac{r}{p^*} \cdot \overrightarrow{a}_t = D(p_t)$$

Since the function D has an inverse for any $q \in [0, D(p^*)]$, then $D^{-1}\left(\frac{r}{p^*} \cdot a\right)$ is well defined and strictly decreasing for any $0 \leq a \leq \frac{p^* \cdot D(p^*)}{r}$. Let $x_0 = a_0$ and

$$x_t = \begin{cases} D^{-1}\left(\frac{r}{p^*} \cdot x_{t-1}\right) \cdot \left(\frac{r}{p^*} \cdot x_{t-1}\right) & \text{if } \frac{r}{p^*} \cdot x_{t-1} < D(p^*) \\ \frac{p^* \cdot D(p^*)}{r} & \text{otherwise} \end{cases}$$

Since $x_{t-1} < \frac{p^* \cdot D(p^*)}{r}$ implies that $D^{-1}\left(\frac{r}{p^*} \cdot x_{t-1}\right) \cdot \frac{r}{p^*} > r$, there exists a date $T \geq 1$ such that $x_t < \frac{p^* \cdot D(p^*)}{r}$ for all $t < T$ and $x_t = \frac{p^* \cdot D(p^*)}{r}$ for all $t \geq T$. Define $P = \{\widehat{p}_t\}_{t=0}^{\infty}$ as follows:

$$\begin{aligned} p_0 &= D^{-1}\left(\frac{r}{p^*} \cdot a_0\right) \\ p_t &= \begin{cases} D^{-1}\left(p_{t-1} \cdot \ldots \cdot p_0 \cdot \left(\frac{r}{p^*}\right)^{t+1} \cdot a_0\right) & \text{if } t < T \\ p^* & \text{if } t \geq T \end{cases} \end{aligned}$$

and let $\widehat{\delta} = \frac{p^* \cdot D(p^*)}{r \cdot p_{T-1} \cdot D(p_{T-1})} \in [0, 1]$. Clearly, P is in Σ and it is strictly decreasing up to date T. It is the sequence of market clearing prices when the aggregate financial capital of the firms is given by $\{x_t\}_{t=0}^{\infty}$. As the following proposition shows, it is also an IE price sequence. The key is to show that $\{x_t\}_{t=0}^{\infty}$ describes the evolution of the firm's financial capital on the equilibrium path of $\Gamma(P, V^*, \alpha)$.

Proposition 5.1 *If $\frac{c}{\theta} \leq \overline{w}_2$ and $a_0 \geq \frac{p^*}{r} \cdot D\left(\frac{p^*}{r} \frac{\alpha}{1-\alpha} \cdot \frac{\overline{w}_2 + c}{\overline{w}_1} \cdot \lambda\right)$, then an IE exists.*

In the IE described in proposition 5.1, each firm pays wages $\overline{w}_1$ and $\overline{w}_2$ to its young and old workers, respectively. Each firm fully reinvest its revenue as financial capital, growing at rate $\frac{p_t}{p^*} \cdot r$, and promotes $\frac{1-\alpha}{\alpha} \cdot \frac{\overline{w}_2+c}{\overline{w}_1} \cdot \frac{p_t}{p^*} \cdot r$ skilled workers up to date $T-1$. The growth process stops at date T, when profits are driven down to zero. From date T on, all firms make zero profits and workers performing identical jobs receive the same wage, regardless of the firm that hires them, as in a Walrasian equilibrium. Hence, from the positive point of view, the industry attains a Walrasian-like equilibrium. Although technological efficiency holds at every date t, allocative efficiency may not be attained from the start because the financial constraint prevents the industry to produce Q^* at the early stages. Nevertheless, from date T on the industry output is Q^* and allocative efficiency is also achieved. Finally, if workers follow stationary strategies and output prices are in Σ, Corollary 5.1 shows that in any IE, *ex ante* identical firms must follow the same growth path towards the steady state. As I show in the next section, this need not be the case if $\frac{c}{\theta} > \overline{w}_2$.

Corollary 5.1 *Let $\{P, V^*, (f, \sigma^s)\}$ be the IE in Proposition 5.1. If there exists any other IE $\left\{\widehat{P}, V^*, \left(\widehat{f}, \widehat{\sigma}\right)\right\}$ in which $\widehat{\sigma} \in \overline{\mathbb{W}}$ and $\widehat{P} \in \Sigma$, then $\widehat{\sigma} = \sigma^s$, $\widehat{P} = P$ and $\overrightarrow{\widehat{f}}_t = \overrightarrow{f}_t$ for all $0 \leq t \leq T$.*

5.3 Case II: $\frac{c}{\theta} > \overline{w}_2$

This section analyzes the scenario in which firms have an incentive to pay an old worker more than what firms in another industry would pay. Hence, the young worker's reservation entry-wage depends on the strategy of the firm that contacts him. As I proved in section 4, the game between the workers and the firms has two types of *SPNE. In section 5.3.1, I show that there is a unique IE where all firms display identical prospects for advancement. It corresponds to firms and workers behaving according to one of the optimistic *SPNE described before. In section 5.3.2, I show that there is an open set of IE in which some firms display better prospects for advancement than others and follow different growth paths towards the steady state.

5.3.1 Identical Growth Paths

Suppose there is an IE where firms and workers play the *SPNE $\left(f^H, \sigma^H\right) \equiv \left[f\left(T, \widehat{\delta}; \alpha, P\right), \sigma\left(T, \widehat{\delta}; \alpha, P\right)\right]$ for some $P \in \Sigma$, $T \geq T_s$ and $\widehat{\delta} \in [0,1]$, i.e. $\mu^H = 1$. In this section, to simplify notation, I omit the parameter α in the functions q, ω and ω^H. From date T on, the wage offered by the firms and accepted by the workers is $\omega(1)$ and total supply is $q(\omega(1)) \cdot \overrightarrow{m}_T$. Since $p_t = p_s$ for all $t \geq T$, the only unknowns are the prices that clear the market along the transition towards the steady state, that is the sequence $\{p_t\}_{t=0}^{T-1}$. Market clearing at date T implies that $\overrightarrow{m}_T^H = \frac{p_s \cdot D(p_s)}{r}$. It follows that $\overrightarrow{w}_{T-1} = \omega^H\left(\widehat{\delta} \cdot p_{T-1}\right) = \omega\left(\frac{p_s \cdot D(p_s)}{r \cdot \overrightarrow{m}_{T-1}^H}\right)$. Therefore, p_{T-1} must solve:

$$q\left(\omega\left(\frac{p_s \cdot D(p_s)}{r \cdot \overrightarrow{m}_{T-1}^H}\right)\right) \cdot \overrightarrow{m}_{T-1}^H = D\left(p_{T-1}\right) \tag{11}$$

and $\widehat{\delta} = \frac{p_s \cdot D(p_s)}{r \cdot p_{T-1} \cdot D(p_{T-1})}$.

If $T = 1$, then (11) completely describes the output prices along the transition to the stationary state. If $T > 1$, however, market clearing implies that p_t must solve

$$q\left(\omega^H(p)\right) \cdot \overrightarrow{m}_t = D(p) \qquad \forall 0 \le t \le T-2 \tag{12}$$

Lemma 5.1 *Suppose assumptions AD and AW holds and* $\frac{c}{\theta} > \overline{w}_2$*. If* $m \le \frac{p_s \cdot D(p_s)}{r}$*, the equation* $q\left(\omega^H(p)\right) \cdot m = D(p)$ *has a unique solution* $\mathbb{P} : \left[0, \frac{p_s \cdot D(p_s)}{r}\right] \to \left(\frac{p_s}{r}, \infty\right)$ *and* $\mathbb{P}(m) > p_r$ *if and only if* $m < \frac{p_r \cdot D(p_r)}{r}$*.*

If $T > 1$, it follows from Lemma 5.1 that $p_0 = \mathbb{P}(a_0)$. One concludes that for any $T \ge 1$, the workers' entry-wage at date zero is uniquely determined. Indeed, $\overrightarrow{w}_0 = \omega\left(\frac{p_s \cdot D(p_s)}{r \cdot a_0}\right)$ if $T = 1$ and $\overrightarrow{w}_0 = \omega^H(\mathbb{P}(a_0))$ whenever $T > 1$. In addition, conditions (11) and (12) make it clear that at any date $t \le T-1$, prices and entry-wages depend only on the aggregate financial capital . Since $\overrightarrow{m}_t = p_{t-1} \cdot q(\overrightarrow{w}_{t-1}) \cdot \overrightarrow{m}_{t-1}$ for all $1 \le t \le T-1$, it is straightforward to see that the existence of a unique p_0 implies that if every firm plays $f\left(T, \widehat{\delta}; \alpha, P\right)$, there is at most one equilibrium sequence of prices in Σ.

In order to prove the existence of an IE, I construct a sequence $P \in \Sigma$ by iterating the map $\mathbb{P}$ until the first date that full reinvestment of revenues would make the aggregate financial capital larger than the stationary level. That date is the candidate for date $T-1$; to complete the sequence $\{p_t\}_{t=0}^{T-1}$, I choose p_{T-1} to be the solution of (11) given the value of $\overrightarrow{m}_{T-1}$.[8]

Proposition 5.2 *There is a unique IE in which* $P \in \Sigma$*,* $\mu^H = 1$ *and* $\left(f^H, \sigma^H\right) = [f(T, \delta; \alpha, P), \sigma(T, \delta; \alpha, P)]$ *for some* $T \ge T_s$ *and* $\delta \in [0, 1]$*.*

One concludes that in the unique IE in which all firms follow the same growth path, technological efficiency holds because all firms pay the same wages to workers performing identical tasks at every date. Allocative efficiency, however, fails even in the steady state because $D(p_s) < Q^*$. Since allocative efficiency would also failed if the credit market were perfect but would hold if $\frac{c}{\theta} \le \overline{w}_2$, one concludes that it is the impossibility of enforcing a wage $\overline{w}_2$ for the old workers rather than the credit constraint what causes the industry production to fall short of the efficient level in the long run. However, as the following section shows, a credit constraint is necessary to explain why *ex-ante* identical firms can follow different growth paths which result in different market shares in the long run.

5.3.2 Different growth paths

Now suppose that only a fraction $\mu^H \in (0, 1)$ of the firms follow strategy $f(T, \delta; \alpha, P)$, for some $T \ge T_s$ and $\delta \in [0, 1]$. Therefore, one can analyze how the retained earnings dynamic selects among firms in the presence of internal promotions. Proposition 5.3 shows that in any IE with $P \in \Sigma$, those firms that display the

[8] If $p_r \cdot D(p_r) \ge p_s \cdot D(p_s)$, it can be shown that the equilibrium I find is the unique IE with $P \in \sum$.

worst prospects for advancement at date zero, continue to show the worst prospects up to date T. That is, those firms that do not play $f_0(T, \delta; \alpha, P)$ at date zero must follow a strategy $f(w_0; \alpha, P)$ for some $w_0 \in (\underline{w}, \overline{w})$.

Proposition 5.3 *Suppose $\mu^H \in (0,1)$. If $w_0^L \in (\underline{w}, \overline{w})$ then in any IE with $P \in \Sigma$ and $T > 1$, $\left(f^L, \sigma^L\right) = \left[f\left(w_0^L; \alpha, P\right), \sigma\left(w_0^L; \alpha, P\right)\right]$. Hence, $\overrightarrow{w}_t^L < \overrightarrow{w}_t^H$ for all $0 \leq t \leq T-1$.*

It follows from Proposition 5.3 that, *ceteris paribus*, firms that display better prospects for advancement at date zero have a higher growth rate along the transition and a higher steady state market share than those firms that initially show worse prospects for advancement. Since in any IE $\overrightarrow{m}_{t+1}^i = G(\overrightarrow{w}_t) \cdot \overrightarrow{m}_t^i$, then there is a monotonic relationship between financial capital and prospects for advancement. One concludes that among *ex-ante* identical firms, the retained earning dynamic favors those firms that, from the workers' point of view, display better prospects for advancement at date zero.

I devote the rest of this section to characterize an IE in which *ex-ante* identical firms follow different growth paths. Afterwards, I use this characterization to build an example. Let $\left(f^H, \sigma^H\right) = \left[f(T, \delta; \alpha, P), \sigma(T, \delta; \alpha, P)\right]$ and $\left(f^L, \sigma^L\right) = \left[f(w_0, a_0; \alpha, P), \sigma(w_0; \alpha, P)\right]$.

If $T = 1$, then the aggregate financial capital at date 1 is $\overrightarrow{m}_1^H + \overrightarrow{m}_1^L = \frac{p_s \cdot D(p_s)}{r}$ or, equivalently, $G\left(\omega^H(\delta \cdot p_0)\right) \cdot a_0^H + G\left(w_0^L\right) \cdot a_0^L = \frac{p_s \cdot D(p_s)}{r}$. It follows that $\omega^H(\delta \cdot p_0) = \omega\left(\frac{p_s \cdot D(p_s) - r \cdot G(w_0^L) \cdot a_0^L}{r \cdot a_0^H}\right)$ and market clearing at date zero can be written as

$$q\left[\omega\left(\frac{p_s \cdot D(p_s) - r \cdot G(w_0^L) \cdot a_0^L}{r \cdot a_0^H}\right)\right] \cdot a_0^H + q\left(w_0^L\right) \cdot a_0^L = D(p_0) \tag{13}$$

If $\left\{P, V^*, \left(f^H, \sigma^H\right), \left(f^L, \sigma^L\right)\right\}$ is an IE in which $T = 1$, then $P = \{p_0, p_s, p_s, ...\}$, $w_0^L \in \left(\omega\left(\frac{p_s \cdot D(p_s)}{r \cdot a_0}\right), \overline{w}\right)$ and $\delta = \widehat{\delta}_0 \equiv \frac{p_s \cdot D(p_s) - r \cdot G(w_0^L) \cdot a_0^L}{r \cdot (p_0 \cdot D(p_0) - p_0 \cdot q(w_0^L) \cdot a_0^L)}$.

If $T > 1$, instead, the date zero market clearing price must satisfy

$$q\left(\omega^H(p_0)\right) \cdot a_0^H + q\left(w_0^L\right) \cdot a_0^L = D(p_0) \qquad \underline{w} < w_0^L < \overline{w} \tag{14}$$

where the left hand side is the short run industry supply function. For any initial level of aggregate financial capital, $a_0 = a_0^H + a_0^L$, the assumption that $\underline{w} < w_0^L$ implies that the industry supply shifts to the left when compared to the case where all firms display equal prospects for advancement, i.e. $\mu^H = 1$. Therefore, there is an excess of demand at $\mathbb{P}(a_0)$, the price which solves (14) when $\mu^H = 1$. Likewise, there is an excess of supply at the price $\mathbb{P}\left(a_0^H\right)$. Since the date zero supply function is strictly increasing in prices, demand is strictly decreasing and both functions are continuous, there exists a unique $\widetilde{\mathbb{P}}\left(a_0^H, a_0^L, w_0^L\right)$ that solves (14). In addition, $\mathbb{P}\left(a_0^H + a_0^L\right) < \widetilde{\mathbb{P}}\left(a_0^H, a_0^L, w_0^L\right) < \mathbb{P}\left(a_0^H\right)$. If $T > 1$, the industry financial capital at date 1 is

$$M\left(a_0^L, a_0^H, w_0^L\right) \equiv \widetilde{\mathbb{P}}(\cdot) \cdot q\left[\omega^H\left(\widetilde{\mathbb{P}}(\cdot)\right)\right] \cdot a_0^H + G\left(w_0^L\right) \cdot a_0^L$$

The function M turns out to be crucial to determine whether $T = 1$ or $T > 1$. Recall from the previous section that if there exists an IE in which $T = 1$ and $\mu^H = 1$, it must be the case that $\overrightarrow{w}_0^H = \omega\left(\frac{p_s \cdot D(p_s)}{r \cdot a_0}\right)$. For a fixed $\mu^H \in (0, 1)$, the following proposition provides the necessary and sufficient conditions for the existence of an IE where $T = 1$ and a fraction $\mu^H \in (0, 1)$ of the firms follow f^H while the others follow f^L.

Proposition 5.4 *Let $\mu^H \in (0, 1)$, $P \in \Sigma$ with $T_s \leq 1$, $(f^H, \sigma^H) = \left[f\left(1, \widehat{\delta}_0; \alpha, P\right), \sigma\left(1, \widehat{\delta}_0; \alpha, P\right)\right]$ and $(f^L, \sigma^L) = [f(w_0, a_0; \alpha, P), \sigma(w_0; \alpha, P)]$. $\{P, V^*, (f^H, \sigma^H), (f^L, \sigma^L), \mu^H\}$ is an IE iff $M\left(a_0^L, a_0^H, w_0^L\right) \geq \frac{p_s \cdot D(p_s)}{r}$, $w_0^L \in \left(\omega\left(\frac{p_s \cdot D(p_s)}{r \cdot a_0}\right), \overline{w}\right)$ and p_0 solves (13).*

Now, consider w_0^L such that $M\left(a_0^L, a_0^H, w_0^L\right) < \frac{p_s \cdot D(p_s)}{r}$; if there exists an equilibrium with $P \in \Sigma$, then it must be the case that $T > 1$. Since the equilibrium price at date zero must satisfy (14), it follows that

$$p_0 = \widetilde{\mathbb{P}}\left(a_0^H, a_0^L, w_0^L\right) \qquad \underline{w} < w_0^L < \overline{w} \tag{15}$$

Since firm L must make zero profits at any $1 \leq t \leq T - 1$ but wages are bounded above by $\overline{w}_1$, then output prices are bounded above by $p^{**} = \frac{r}{q(\overline{w}_1)}$, i.e. the marginal cost of a firm that is expected to close the following period. In addition, p_t solves

$$q\left(\omega^H(p_t)\right) \cdot \overrightarrow{m}_t^H + q\left(w^L(p_t)\right) \cdot \overrightarrow{m}_t^L = D(p_t) \qquad \text{if } 1 \leq t < T - 1$$

The left hand side of this equation is the short run industry supply function. For a given level of aggregate financial capital, $\overrightarrow{m}_t^H + \overrightarrow{m}_t^L$, it follows that the short run industry supply shifts to the left compared to the case in which all firms follow strategy f^H. That is, for any $p \leq \mathbb{P}\left(\overrightarrow{m}_t^H + \overrightarrow{m}_t^L\right)$,

$$q\left(\omega^H(p)\right) \cdot \overrightarrow{m}_t^H + q\left(w^L(p)\right) \cdot \overrightarrow{m}_t^L < q\left(\omega^H(p)\right) \cdot \left(\overrightarrow{m}_t^H + \overrightarrow{m}_t^L\right) \leq D(p)$$

Then, there is an excess of demand at any price $p \leq \mathbb{P}\left(\overrightarrow{m}_t^H + \overrightarrow{m}_t^L\right)$. Therefore, by assumption AD, for any $\overrightarrow{m}_t^H + \overrightarrow{m}_t^L < \frac{p_s \cdot D(p_s)}{r}$ there exists $\widetilde{\mathbb{P}}\left(\overrightarrow{m}_t^H, \overrightarrow{m}_t^L\right) \geq \mathbb{P}\left(\overrightarrow{m}_t^H + \overrightarrow{m}_t^L\right)$ that clears the market. However, such a price may not be the unique market clearing price. On the one hand, for a fixed level of financial capital, the supply of those firms which follow strategy f^H increases with output price. On the other hand, the supply of those firms that play f^L decreases with the output price because the young workers' reservation entry-wage adjusts so that the firm makes zero profits. As a consequence, total supply can increase or decrease with price. Some additional assumption is needed to rule out the existence of another market clearing price. Since $p \cdot q\left(\omega^H(p)\right)$ is strictly increasing and strictly convex in prices, in proposition 5.5 I assume concavity of $p \cdot D(p)$ to ensure that there is at most one market clearing price exceeding p_r. Hence, if $\overrightarrow{m}_t^H + \overrightarrow{m}_t^L < \frac{p_s \cdot D(p_s)}{r}$ then the market clearing price is

$$p_t = \widetilde{\mathbb{P}}\left(\overrightarrow{m}_t^H, \overrightarrow{m}_t^L\right) \qquad \text{for any } 1 \leq t < T - 1 \tag{16}$$

At date $T-1$, the same reasoning that motivated the necessity of condition (11) implies that the steady state aggregate financial capital must be $\frac{p_s \cdot D(p_s)}{r}$ and $\omega^H(\delta \cdot p_{T-1}) = \omega\left(\frac{p_s \cdot D(p_s) - r \cdot G(\omega^L(p_{T-1})) \cdot \overrightarrow{m}^L_{T-1}}{r \cdot \overrightarrow{m}^H_{T-1}}\right)$. Therefore, $\delta = \frac{p_s \cdot D(p_s) - r \cdot G(\omega^L(p_{T-1})) \cdot m^L_{T-1}}{r \cdot \left(p_{T-1} \cdot D(p_{T-1}) - r \cdot m^L_{T-1}\right)}$ and p_{T-1} must solve

$$q\left[\omega\left(\frac{p_s \cdot D(p_s) - r \cdot G(w^L(p_{T-1})) \cdot \overrightarrow{m}^L_{T-1}}{r \cdot \overrightarrow{m}^H_{T-1}}\right)\right] \cdot \overrightarrow{m}^H_{T-1} + q\left(\omega^L(p_{T-1})\right) \cdot \overrightarrow{m}^L_{T-1} = D(p_{T-1}) \tag{17}$$

To show that an IE with $\mu^H \in (0,1)$ and $T > 1$ actually exists, choose w^L_0 such that $M\left(a^L_0, a^H_0, w^L_0\right) < \frac{p_s \cdot D(p_s)}{r}$ and define $\rho_0 = \widetilde{\mathbb{P}}\left(a^H_0, a^L_0, w^L_0\right)$ and $x^H_0 = a^H_0$. Consider the sequence $\left\{x^H_t, x^L_t\right\}_{t=0}^{\infty}$ defined as

$$\begin{aligned}
x^L_t &= \begin{cases} G\left(w^L_0\right) \cdot a^L_0 & \text{if } t = 1 \\ G\left(\omega^L\left(\rho_{t-1}\right)\right) \cdot x^L_{t-1} & \text{otherwise} \end{cases} \\
x^H_t &= \begin{cases} G\left(\omega^H\left(\rho_{t-1}\right)\right) \cdot x^H_{t-1} & \text{if } G\left(\omega^H\left(\rho_{t-1}\right)\right) \cdot x^H_{t-1} + x^L_t < \frac{p_s \cdot D(p_s)}{r} \\ \frac{p_s \cdot D(p_s)}{r} - x^L_t & \text{otherwise} \end{cases} \\
\rho_t &= \begin{cases} Min\left\{\widetilde{\mathbb{P}}\left(x^H_t, x^L_t\right), p^{**}\right\} & \text{if } x^H_t + x^L_t < \frac{p_s \cdot D(p_s)}{r} \\ p_s & \text{otherwise} \end{cases}
\end{aligned}$$

It follows that $x^L_t \geq 0$ and $x^H_t > x^H_{t-1}$because $\rho_t \in \left(\frac{p_s}{r}, p^{**}\right]$ for all $t \geq 0$. Since $x^H_t + x^L_t < \frac{p_s \cdot D(p_s)}{r}$ implies that max $\left\{\frac{x^H_{t+1}}{x^H_t}, \frac{x^L_{t+1}}{x^L_t}\right\} \geq r$, there exists a finite T such that $x^H_T + x^L_T = \frac{p_s \cdot D(p_s)}{r}$ for the first time. The following proposition shows how to construct a candidate for a price sequence and provides a sufficient condition for the existence of an IE in which *ex-ante* identical firms follow different growth paths.

Proposition 5.5 *Suppose* $p \cdot D(p)$ *is concave,* $\mu^H \geq 1 - \frac{1}{r}$*,* $M\left(a^L_0, a^H_0, w^L_0\right) < \frac{p_s \cdot D(p_s)}{r}$ *and* $\widehat{p}$ *solves*

$$q\left[\omega\left(\frac{p_s \cdot D(p_s) - r \cdot G(w^L(p)) \cdot x^L_{T-1}}{r \cdot x^H_{T-1}}\right)\right] \cdot x^H_{T-1} + q\left(\omega^L(p)\right) \cdot x^L_{T-1} = D(p) \tag{18}$$

Let $P = \left\{\rho_0, ..., \rho_{T-2}, \widehat{p}, p_s, p_s, ...\right\}$ *and* $\widehat{\delta} = \frac{p_s \cdot D(p_s) - r \cdot G(\omega^L(\widehat{p})) \cdot x^L_{T-1}}{r \cdot \left(\widehat{p} \cdot D(\widehat{p}) - r \cdot x^L_{T-1}\right)}$*. If* $w^L_0 \in (\underline{w}, \overline{w})$*,* $p_t \in (p_r, p^{**}]$ *for all* $t \geq 1$ *and* $\widehat{\delta} \leq 1$ *then* $\left\{P, V^*, \left(f^H, \sigma^H\right), \left(f^L, \sigma^L\right), \mu^H\right\}$ *is an IE.*

The assumption that $M\left(a^L_0, a^H_0, w^L_0\right) < \frac{p_s \cdot D(p_s)}{r}$ implies that $T \geq 2$. The concavity assumption ensures that the function $\widetilde{\mathbb{P}}(\cdot)$ is the unique market clearing price at $1 \leq t \leq T-2$. Finally, $\mu^H \geq 1 - \frac{1}{r}$ implies that $\omega\left(\frac{p_s \cdot D(p_s) - r \cdot G(w^L(\widehat{p})) \cdot x^L_{T-1}}{r \cdot x^H_{T-1}}\right) \leq \overline{w}_1$. The following example illustrates how to use Propositions 5.4 and 5.5 to show that there exists an IE associated with some sequence $P \in \Sigma$.

Example 1: The demand function is $D(p) = \frac{1}{p}$, which satisfies assumption AD, and expenditure is always equal to 1. This simplifies the analysis because after one period the assets in hand of the firms exceeds the steady state financial capital. However, when firms follow different strategies convergence to the steady state may take more than one period. This is because the assets in hand of those firms that display good growth prospects may fall short of the steady state level at date 1. Suppose

$$\begin{array}{lllll} \alpha = 0.5 & \lambda \geq 0.2 & c = 1 & \overline{w}_1 = 0.2 & a_0^H = 0.14 \\ \beta = 9/10 & \frac{1}{r} = 0.9 & \theta = 0.5 & \overline{w}_2 = 8/9 & a_0^L = 0.06 \end{array}$$

For these parameters, assumption AW holds and the steady state wages are $\omega(1) = 0.15$ for the young workers and $v^* = 2 > \overline{w}_2$ for those workers who perform the second task. The steady state output price is $p_s = \frac{2}{3}\sqrt{5} > r \cdot \frac{2}{3}\sqrt{\frac{17}{5}} = p^*$. In the unique equilibrium in which all firms behave identically, the price sequence is $P = \{2\sqrt{6}, p_s, p_s, ...\}$ and $T = 1$. Young workers receive an entry-wage of 0.08 at date zero and 0.15 thereafter, while the probability of promotion for a young worker is 0.12 at date zero and 0.05 afterwards. The equilibrium levels of financial capital are $\overrightarrow{m}_0 = 0.2$ and $\overrightarrow{m}_t = 0.9$ for all $t \geq 1$.

However, there are other equilibria where, for example, $1/3$ of the firms follow strategy f^L while the rest follows f^H. I have chosen the parameters values so that in any IE either $T = 1$ or $T = 2$. The example is robust to values of μ^L in an open set of $\frac{1}{3}$. Since the equilibrium in which all firms behave identically converges to a stationary state at date 1, not surprisingly the same happens if workers believe that prospects for advancement are not very different. If firms which follow strategy f^L display different prospects for advancement, then $w_0^L > 0.08$. Hence, the date zero equilibrium price must be higher than $2\sqrt{6}$, the market clearing price when all firms display equal prospects for advancement. It follows that in any IE, $\omega^L(p_0) > \omega^L(2\sqrt{6}) > \overline{w}_1$ and one concludes that, for these parameters, $\overline{w} = \overline{w}_1$. For any $w_0^L \in (0.08, u]$, where $u \simeq 0.105$, the right hand side of Figure 2 shows that $M(a_0^H, a_0^L, w_0^L) \geq \frac{1}{r}$. Since $(0.08, u] \subset \left(\omega\left(\frac{1}{r \cdot a_0}\right), \overline{w}\right)$, by Proposition 5.4 there is an equilibrium where $w_0^L > \overrightarrow{w}_0^H$ and $T = 1$ iff p_0 solves (13). The left hand side of Figure 2 shows the unique solution to (13) for each $w_0^L \in [0.08, u]$

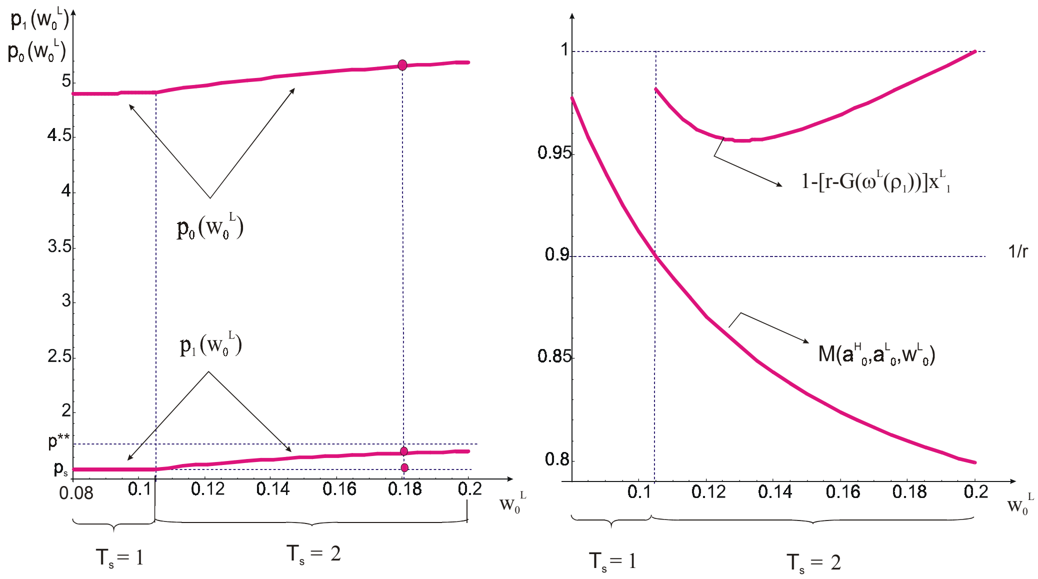

Figure 2. Equilbrium Prices and Aggregate Financial Capital

For any $w_0^L \in (u, 0.2]$, the right hand side of Figure 2 shows that $M\left(a_0^H, a_0^L, w_0^L\right) < \frac{1}{r}$. By Proposition 5.4, if an equilibrium with $w_0^L \in (u, 0.2]$ exists, it must be the case that $T \geq 2$. Since $p \cdot D(p)$ is concave and $\mu^H = \frac{2}{3} \geq 1 - \frac{1}{r} = \frac{1}{10}$, then one can use the sufficient conditions in Proposition 5.5 to check whether there exists an IE with $T = 2$ and $w_0^L \in (u, 0.2]$. First notice that if $T = 2$, then

$$\rho_1 \cdot q\left[\omega^H(\rho_1)\right] \cdot x_1^H + G\left[\omega^L(\rho_1)\right] \cdot x_1^L = 1 - \left[r - G\left(\omega^L(\rho_1)\right)\right] \cdot x_1^L \geq \frac{1}{r}$$

This implies that $\frac{1-r\cdot x_1^L}{x_1^H} \geq \frac{1-r\cdot G(\omega^L(\rho_1))\cdot x_1^L}{r\cdot x_1^H}$. Since $\omega^H(\rho_1) = \omega\left(\frac{1-r\cdot x_1^L}{x_1^H}\right)$, it follows that

$$q\left[\omega\left(\frac{1-r\cdot G(w^L(\rho_1))\cdot x_1^L}{r\cdot x_1^H}\right)\right] \cdot x_1^H + q\left(\omega^L(\rho_1)\right) \cdot x_1^L < \frac{1}{\rho_1}$$

and, by assumption AD, a solution $\widehat{p}\left(w_0^L\right) \geq \rho_1$ to (18) exists. The uniqueness of the solution follows because total revenue increases with p while expenditure is constant.

On the right hand side of figure 2, I show that for any $w_0^L \in (u, 0.2]$ it is true that $1 - \left[r - G\left(\omega^L(\rho_1)\right)\right] \cdot x_1^L \geq \frac{1}{r}$. On the left hand side, I plot $p_0 = \widetilde{\mathbb{P}}\left(0.14, 0.06, w_0^L\right)$, $p_1 = \widehat{p}\left(w_0^L\right)$ and verify that $P = \{p_0, p_1, p_s, p_s, ...\}$ satisfies $p_t \in (p_r, p^{**}]$ $\forall t \geq 1$. Finally, $\widehat{\delta} = \frac{1-r\cdot G(\omega^L(\widehat{p}))\cdot x_{T-1}^L}{r\cdot\left(1-r\cdot x_{T-1}^L\right)} \leq 1 \Leftrightarrow \omega^H\left(\widehat{\delta}\cdot\widehat{p}\right) \geq \omega^H(\widehat{p})$. Since $\rho_1 \cdot q\left(\omega^H(\rho_1)\right) \cdot x_1^H = 1 - r \cdot x_1^L = \widehat{p} \cdot q\left[\omega\left(\frac{1-r\cdot G(w^L(\widehat{p}))\cdot x_1^L}{r\cdot x_1^H}\right)\right]$ and $\widehat{p} \geq \rho_1$ it follows that $\omega^H(\rho_1) \leq \omega\left(\frac{1-r\cdot G(w^L(\rho_1))\cdot x_1^L}{r\cdot x_1^H}\right) = \omega^H\left(\widehat{\delta}\cdot\widehat{p}\right)$. Thus $\widehat{\delta} \leq 1$. By Proposition 5.5, for any $w_0^L \in (u, 0.2]$ there is an IE $\left\{P, V^*, \left(f^H, \sigma^H\right), \left(f^L, \sigma^L\right), \frac{1}{3}\right\}$.

For each $w_0^L \in (u, 0.2]$, *ex-ante* identical firms follow different growth paths. In Figure 3, I show the steady state market share of firm's L and H as a function of the young worker's entry-wage for firm L at date zero.

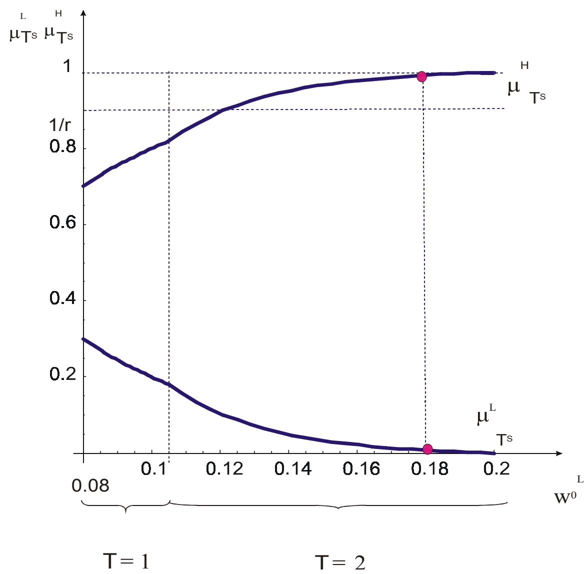

Figure 3. Steady State Market Shares

Those firms that follow f^L and display very bad prospects for advancement at date zero are almost driven out at date 2, as can be seen on the right hand side of Figure 3. For example, if at date zero the young workers' reservation entry-wage is larger than 0.18, the steady state market share of the L firms is smaller than 1%. □

This example shows that in an IE, *ex-ante* identical firms can follow different growth paths and have different sizes in the steady state. Along the transition to the steady state, firms pay different wages to their young workers. Therefore, technological efficiency only holds in the steady state. My analysis of the cases in which either the credit market is perfect or $v^* = \overline{w}_2$ implies that these results depend both on the existence of financial constraints as well as on the impossibility of enforcing a wage $\overline{w}_2$ for the skilled workers.

6. Heterogeneous Firms

In this section, I consider an example in which firms are endowed with different technologies. In particular, I assume that $\alpha_H < \alpha_L$ and a fraction $\mu_H \in (0,1)$ of the firms has technology α_H. Firms endowed with technology α_H and α_L are called firms H and L, respectively. Firm H is believed to display better prospects for advancement than firm L at date zero. I show that if the young workers born at zero are pessimistic enough about the prospects displayed by firm L, firm H may end up dominating the market even though it produces inefficiently.

With some abuse of notation, let $q(l,s,0;\alpha_i) = q(l,s;\alpha_i)$. Since the technology displays CRS, $q(l,s;\alpha_i)$ can be written as $q(l,s;\alpha_i) = l \cdot \left(\frac{s}{l}\right)^{1-\alpha_i}$ and the relationship between the two production functions can be illustrated using an arbitrary isoquant. In figure 4, I draw the isoquant corresponding to output $\overline{q}$ for each firm.

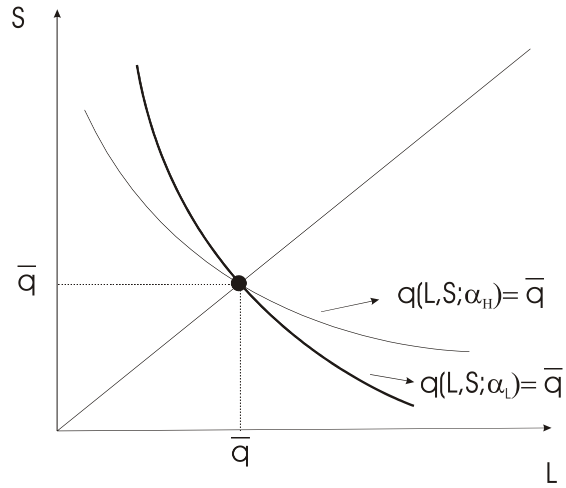

Figure 4. Isoquants of firms H and L

If in equilibrium the two firms choose an input bundle that lies below the diagonal, then firm H produces inefficiently in the sense that firm L could produce the same output using less of every input. Indeed, for any level of inputs $l > 0$ and $s > 0$ such that $\frac{s}{l} < 1$ it follows that $\alpha_H < \alpha_L$

$$\alpha_H < \alpha_L \;\Leftrightarrow q\left(l, s; \alpha_L\right) > q\left(l, s; \alpha_H\right)$$

Suppose firm H follows strategy $f\left(T, \delta; \alpha_H, P\right)$, firm L follows $f\left(w_0^L, a_0; \alpha_L, P\right)$ and young workers play $\sigma\left(T, \delta; \alpha_H, P\right)$ and $\sigma\left(w_0^L; \alpha_L, P\right)$ against firms with technologies α_H and α_L, respectively. Therefore, if the ratio of skilled to unskilled labor of both firms satisfies

$$\frac{\overrightarrow{s}_t^i}{\overrightarrow{l}_t^i} = \frac{1-\alpha_i}{\alpha_t} \cdot \frac{\overrightarrow{w}_t^i}{v^* + c} < 1$$

then firm H produces inefficiently and firm L produces efficiently every period.

Since technologies display constant returns to scale, there exists a restriction on the technologies that can coexist in steady state. Let $w_s^i = \omega\left(1, \alpha_i\right)$ denote the steady state entry-wage for the firm with technology α_i for $i \in \{L, H\}$. Since in steady state both firms must make zero profits, the technologies must satisfy:

$$q\left(w_s^H, 1\right) = q\left(w_s^L, 1\right) \Leftrightarrow \left(\frac{\alpha_H}{w_s^H}\right)^{\alpha_H} \cdot \left(\frac{1-\alpha_H}{v^*+c}\right)^{1-\alpha_H} = \left(\frac{\alpha_L}{w_s^L}\right)^{\alpha_L} \cdot \left(\frac{1-\alpha_L}{v^*+c}\right)^{1-\alpha_L} \tag{19}$$

Example 2: The demand function is $D\left(p\right) = \frac{1}{p}$, as in example 1. Suppose $\lambda = 0.95$, $\beta = \frac{1}{r} = 0.9$ and

$$\begin{array}{llll} \alpha_H = 0.1 & a_0 = 0.3 & \overline{w}_1 = \frac{50}{81} \cdot \left(\frac{125}{153}\right)^{\frac{1}{4}} \simeq 0.587 & c = 0.04 \\ \alpha_L = 0.5 & \mu^H = 0.7 & \overline{w}_2 = 0.63 & \frac{c}{\theta} = 1.33 \end{array}$$

I choose the parameters so that assumption AW and condition (19) holds. In this example, $3/10$ of the firms are endowed with technology α_L and follow strategy f^L and $7/10$ of the firms have technology α_H and play strategy f^H. The initial aggregate financial capital is 0.3 and $a_0^H = 0.21$ is in hand of the firms which follow f^H while 0.09 is owned by those following f^L. The steady state aggregate financial capital is $\frac{1}{r} = 0.9$ and the steady state entry-wages are $w_s^L \simeq 0.398$ and $w_s^H \simeq 0.113$. This means that in the steady state firm H displays better prospects for advancement than firm L does. Suppose that the young workers born at date zero also believe that firm H displays better prospects for advancement than firm L. For any $w_0^L \in [0.276, \overline{w}_1]$, there exists an IE in which $T_s = 2$, those firms endowed with technology α_L follow strategy $f^L = f\left(w_0^L, a_0; \alpha_L, P\right)$, those endowed with technology α_H follow strategy $f^H = f\left(2, \widehat{\delta}; \alpha_H, P\right)$ and workers follow $\sigma^H = \sigma\left(2, \widehat{\delta}; \alpha_H, P\right)$ and $\sigma^L = \sigma\left(w_0^L; \alpha, P\right)$ when contacted by firms H and L, respectively. First notice that there exists a price sequence P such that $\left\{P, V^*, \left(f^L, \sigma^L\right), \left(f^H, \sigma^H\right), \frac{7}{10}\right\}$ is an IE with $T_s = 2$ if and only if $p_0, p_1 \in \left(p_r, \frac{r}{q(\overline{w}_1, \alpha_L)}\right]$ solve

$$\begin{aligned} q\left(\omega^H\left(p_0, \alpha_H\right), \alpha_H\right) \cdot a_0^H + q\left(w_0^L, \alpha_L\right) \cdot a_0^L &= \frac{1}{p_0} \\ q\left(\omega\left(\frac{1 - r \cdot G\left(w^L\left(p_1, \alpha_L\right), \alpha_L\right) \cdot \overrightarrow{m}_1^L}{r \cdot \overrightarrow{m}_1^H}, \alpha_H\right), \alpha_H\right) \cdot \overrightarrow{m}_1^H + \frac{r}{p_1} \cdot \overrightarrow{m}_1^L &= \frac{1}{p_1} \end{aligned}$$

where $\overrightarrow{m}_1^L = G\left(w_0^L, \alpha_L\right) \cdot a_0^L$ and $\overrightarrow{m}_1^H = p_0 \cdot q\left(\omega^H\left(p_0, \alpha_H\right), \alpha_H\right) \cdot a_0^H$. In Figure 5, the dashed line corresponds to p_0, the solution to the first equation, while the full line corresponds to p_1, the solution to the second equation. It turns out that for each $w_0^L \in [0.276, \overline{w}_1]$ this system of equations has a unique solution that

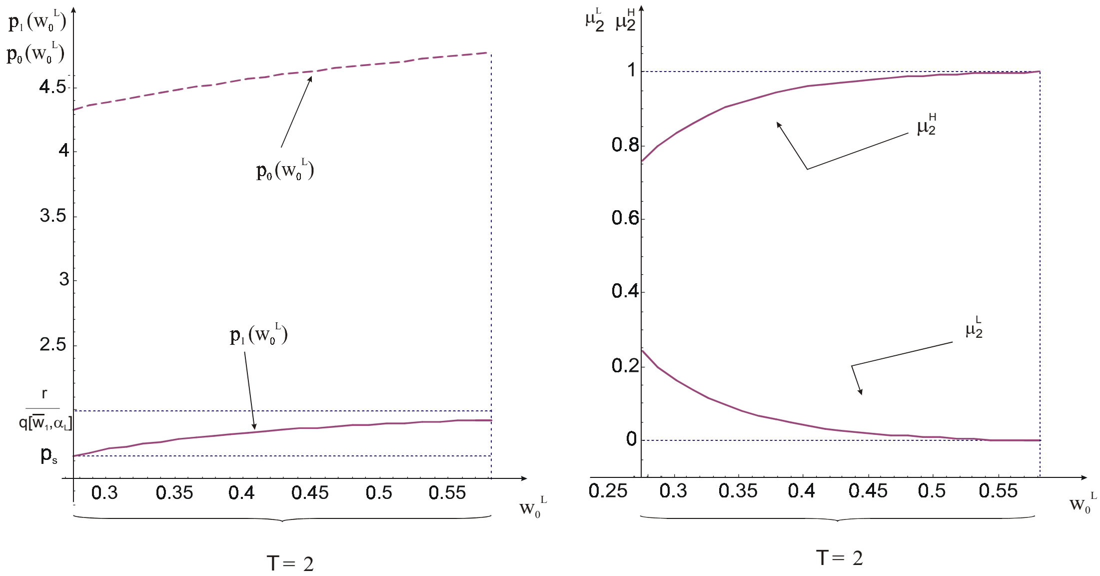

Figure 5.

satisfies $p_0, p_1 \in \left(p_r, \frac{r}{q(\overline{w}_1,\alpha_L)}\right]$, $p_1 \cdot q\left(\omega\left(\frac{1-r\cdot G(w^L(p_1,\alpha_L),\alpha_L)\cdot\overrightarrow{m}_1^L}{r\cdot\overrightarrow{m}_1^H}, \alpha_H\right), \alpha_H\right) > r$ and $w_0^L \in (\underline{w}, \overline{w})$. Clearly $\widehat{\delta} \equiv \frac{1-r\cdot G(w^L(p_1,\alpha_L))\cdot\overrightarrow{m}_1^L}{r\cdot(1-r\cdot\overrightarrow{m}_1^L)}$ satisfies $\omega^H\left(\widehat{\delta}\cdot p_1, \alpha_H\right) = \omega\left(\frac{1-r\cdot G(w^L(p_1,\alpha_L),\alpha_L)\cdot\overrightarrow{m}_1^L}{r\cdot\overrightarrow{m}_1^H}, \alpha_H\right)$ and since $p_1 \geq p_s(\alpha_L)$ it follows that $\widehat{\delta} \in [0,1]$. By propositions 4.4 and 4.5, the strategies $\left(f^H, \sigma^H\right)$ and $\left(f^L, \sigma^L\right)$ are *SPNE of $\Gamma(P, V^*, \alpha_H)$ and $\Gamma(P, V^*, \alpha_L)$, respectively, $\overrightarrow{e}_t^i = 0$ and $\frac{\partial\Pi\left(f^i, \sigma^i | \overrightarrow{h}^t\right)}{\partial e_t} = 0$ for all $t \geq 0$ and $i \in \{L, H\}$.

From date 2 on, all firms make zero profits. If date zero workers believe that firm L displays sufficiently bad prospects for advancement, then firm L is almost driven out, in terms of market share, in the steady state. For example, if $w_0^L \geq 0.5$ then the steady state market share of firm L is smaller than 0.01, which means that almost all the production in this industry is carried out by workers working in a firm with technology α_H. In these steady states not only every firm makes zero profits, is not financially constrained and maximizes profits, but also *almost* all workers who perform the same job receive the same wage, regardless of the firm that hires them, as in a Walrasian equilibrium. Although the steady state looks *almost* like a Walrasian equilibrium, these equilibria are productively inefficient in a strong sense. Notice that

$$\frac{\overrightarrow{s}_t^L}{\overrightarrow{l}_t^L} = \frac{1-\alpha_L}{\alpha_L} \cdot \frac{\overrightarrow{w}_t^L}{v^* + c} \leq \frac{\overline{w}_1}{v^* + c} < 1 \qquad \frac{\overrightarrow{s}_t^H}{\overrightarrow{l}_t^H} = \frac{1-\alpha_H}{\alpha_H} \cdot \frac{\overrightarrow{w}_t^H}{v^* + c} \leq 9 \cdot \frac{w_s^H}{v^* + c} < 1$$

If all labor were allocated to firm L, which produce efficiently, more output could be produced without altering workers' welfare. □

7. CONCLUSION

One of the most challenging tasks of economic theory is to explain how the institutions that characterize the real world influence economic phenomena. Both casual observation as well as empirical evidence suggests that at the early stages of industry evolution firms are financially constrained and social institutions, which are not present in the theory of the firm under perfect competition, characterize the inputs market. In many industries, the labor market is not in a Walrasian equilibrium from the start. Instead, firms tend to promote workers internally, creating truly labor markets inside the firms. This work gives a step in trying to incorporate such institution in the theory of industry evolution in a world where financial markets are incomplete.

In competitive output markets, the retained earnings dynamic gives an evolutionary advantage to firms with lower unit costs. However, unit costs are determined not only by technological efficiency but also by wages. In the presence of internal promotions, unlike in Walrasian markets, worker's expectations about the opportunities for advancement within the firm are key to determine wages. As a consequence, the fitness of a firm depends not only on its technological efficiency but also on the self-fulfilling beliefs of the workers. This paper suggests that, at least in the long run, the retained earnings dynamic justifies the use of the standard static analysis of competitive markets to make positive predictions but does not justify its efficiency properties. It shows that the main positive predictions of the perfectly competitive model can still be accurate in the presence of a non-Walrasian labor market and financial constraints. In addition, by taking seriously the role of internal promotions I show that phenomena such as self-fulfilling growth and the survival of inefficient firms is not necessarily incompatible with convergence to a Walrasian-like state.

The results in this paper do not depend on any strong assumption on the demand side. However, I did assume a particular CRS technology of production and homogeneity of skills among those high ability workers that receive training. It remains an interesting open question whether the process of industry evolution converges to a Walrasian-like equilibrium in a more general setup.

APPENDIX

A.1 Proofs of Section 3

Proof of Lemma 3.1: Suppose $\overrightarrow{v}^i_{t+1} = \overrightarrow{v}^j_{t+1}$. By definition, firm i displays better prospects for advancement than firm j if and only if the young worker's reservation entry wage in firm i is smaller than in firm j, i.e $w\left(\overrightarrow{\pi}^i_t, \overrightarrow{v}^i_{t+1}\right) \leq w\left(\overrightarrow{\pi}^j_t, \overrightarrow{v}^j_{t+1}\right)$. If $\overrightarrow{v}^i_{t+1} = \overrightarrow{v}^j_{t+1}$ this holds if and only if $\overrightarrow{\pi}^i_t \geq \overrightarrow{\pi}^j_t$, as desired. ■

A.2 Proofs of Section 4

Lemma 7.1 *Suppose* $\frac{c}{\theta} \geq \overline{w}_2$ *and that* $(f, \sigma) \in \mathbb{F} \times \overline{\mathbb{W}}$ *is a *SPNE of* $\Gamma(P, V^*, \alpha)$. *If* $m_t > 0$, *then* $(l_t, s_t, e_t, w_t, v_t)$ *must solve (6)*

Proof of Lemma 7.1: Let $R^*(p_t, \sigma) \cdot m_t$ be the value of problem (6). Suppose $(f, \sigma) \in \mathbb{F} \times \overline{\mathbb{W}}$ is a *SPNE but there exists $t \geq 0$ and $h^t \in H^t$ such that $m_t > 0$ and $(l_t, s_t, e_t, w_t, v_t)$ does not solve (6) for the first time. If $(w_t, v_t) \notin \Theta\left(\sigma \left| h^t\right.\right)$ and $(f, \sigma) \in \mathbb{F} \times \overline{\mathbb{W}}$ is a *SPNE, then $m_t = 0$, a contradiction. Then $(w_t, v_t) \in \Theta\left(\sigma \left| h^t\right.\right)$ and $p_t \cdot l_t^\alpha \left(s_t + \frac{e_t}{1+\theta}\right)^{1-\alpha} < R^*(p_t, \sigma) \cdot m_t$. Let $\widehat{f}$ be the strategy where

$$\widehat{m}_k = \begin{cases} m_k \text{ for all } k \leq t \\ \frac{m_k}{a_k} \cdot \widehat{a}_k \text{ for all } k > t \end{cases}$$

and $\left(\widehat{l}_k, \widehat{s}_k, \widehat{e}_k, \widehat{w}_k, \widehat{v}_k\right)$ solves (6) for all $k \geq 0$. Clearly, young and old workers would accept to work at every $k \geq 0$. Therefore, if the firm follows $\widehat{f}$, its rate of return would be to R_k at every date $k < t$ and for all $k \geq t$

$$\begin{aligned} \widehat{R}_k &= p_k \cdot \left(\frac{\alpha}{\widehat{w}_k}\right)^\alpha \left(\frac{1-\alpha}{v^* + c}\right)^{1-\alpha} \cdot \frac{\widehat{m}_k}{\widehat{a}_k} + r \cdot \frac{\widehat{b}_k}{\widehat{a}_k} \\ &= p_k \cdot \left(\frac{\alpha}{\widehat{w}_k}\right)^\alpha \left(\frac{1-\alpha}{v^* + c}\right)^{1-\alpha} \cdot \frac{m_k}{a_k} + r \cdot \frac{b_k}{a_k} \\ &\geq R_k \end{aligned}$$

with strict inequality at date t. It follows that the continuation payoff of following strategy $\widehat{f}$ is

$$\begin{aligned} \Pi\left(\widehat{f}, \sigma; \alpha \left| h^t\right.\right) &= \beta \cdot \widehat{R}_t \cdot a_t + \left(\sum_{k=t+1}^{\infty} \beta^{k+1-t} \cdot \overrightarrow{\widehat{R}}_k \cdot \ldots \cdot \overrightarrow{\widehat{R}}_t\right) \cdot a_t \\ &> \beta \cdot R_t \cdot a_t + \left(\sum_{k=t+1}^{\infty} \beta^{k+1-t} \cdot \overrightarrow{R}_k \cdot \ldots \cdot \overrightarrow{R}_t\right) \cdot a_t \\ &= \Pi\left(f, \sigma; \alpha \left| h^t\right.\right) \end{aligned}$$

a contradiction. ■

Proof of Proposition 4.1: Let $P \in \Sigma$, $\frac{c}{\theta} = \overline{w}_2$, $\delta \in [0, 1]$ and $T \geq T_s$. First, I show that $f(T, \delta, \alpha, P) \in \mathbb{F}$.

Clearly, $f_t\left(h^t\right) \in \Re_+^7$. On the one hand,

$$\begin{aligned} w_t \cdot l_t + (v_t + c) \cdot s_t + v_t \cdot e_t &= \alpha \cdot m_t + (\overline{w}_2 + c) \cdot s_t + \overline{w}_2 \cdot e_t \\ &= \alpha \cdot m_t + \overline{w}_2 \cdot (1+\theta) \cdot s_t + \overline{w}_2 \cdot e_t \\ &= \alpha \cdot m_t + \overline{w}_2 \cdot (1+\theta) \cdot \left(s_t + \frac{e_t}{1+\theta}\right) \\ &= m_t \end{aligned}$$

On the other hand, $m_t \leq a_t$ and $s_t \leq \lambda \cdot l_{t-1}$. Therefore, $f_t\left(T, \delta, \alpha, P\right) \in \mathcal{A}\left(h^t\right)$ for all h^t which proves that $f\left(T, \delta, \alpha, P\right) \in \mathbb{F}$. Consider the subgame that begins after the partial history h^t. First, I show that the firm has no profitable deviation from $f\left(T, \delta; \alpha, P\right)$.

Since $P \in \Sigma$, it follows that $p_t > p^*$ for all $0 \leq t \leq T-1$ and $p_t = p^*$ for all $t \geq T$. Therefore,

$$R^*\left(p_t, \sigma^s\right) = \frac{p_t}{p^*} \cdot r \begin{cases} > r & \text{if } t < T \\ = r & \text{if } t \geq T \end{cases}$$

where $R^*\left(p_t, \sigma^s\right)$ is defined in lemma 7.1. If the firm follows $f\left(T, \delta; \alpha, P\right)$, its continuation payoff is:

$$\begin{aligned} \Pi\left(f, \sigma^s; \alpha \left| h^t\right.\right) &= \sum_{k=t}^{\infty} \beta^{k+1-t}\left[\frac{p_k}{p^*} \cdot r \cdot \overrightarrow{m}_k + r \cdot \left(\overrightarrow{a}_k - \overrightarrow{m}_k\right)\right] \\ &= \sum_{k=t}^{\infty} \beta^{k+1-t} \cdot \frac{p_k}{p^*} \cdot r \cdot \overrightarrow{a}_k \\ &= \beta \cdot \frac{p_t}{p^*} \cdot r \cdot a_t + \left(\sum_{k=t+1}^{\infty} \beta^{k+1-t} \cdot \frac{p_k}{p^*} \cdot \ldots \cdot \frac{p_t}{p^*} \cdot r^{k+1-t}\right) \cdot a_t \end{aligned}$$

Suppose there is a profitable deviation $\widetilde{f}$, i.e. $\Pi\left(\widetilde{f}, \sigma^s; \alpha \left| h^t\right.\right) > \Pi\left(f, \sigma^s; \alpha \left| h^t\right.\right)$. Let $\widetilde{R}_t \equiv R\left(\widetilde{f}_t, \sigma^s; \alpha \left| h^t\right.\right)$. Assume $\widetilde{f}$ is such that $\overrightarrow{\widetilde{m}}_k = 0$ for all $k \geq t$. Then $\Pi\left(\widetilde{f}, \sigma^s; \alpha \left| h^t\right.\right) = \frac{\beta r}{1-\beta r} \cdot a_t \leq \Pi\left(f, \sigma^s; \alpha \left| h^t\right.\right)$, a contradiction. Therefore, $\widetilde{f}$ is such that $\overrightarrow{\widetilde{m}}_k > 0$ and $\overrightarrow{\widetilde{R}}_k > \frac{p_k}{p^*} \cdot r \geq r$ for some $k \geq t$. Hence $\left(\overrightarrow{\widetilde{w}}_k, \overrightarrow{\widetilde{v}}_k\right) \in \Theta\left(\sigma^s \left| h^k\right.\right)$, $\overrightarrow{\widetilde{m}}_k > 0$ and $p_k \cdot \frac{\overrightarrow{\widetilde{l}}_k^{\alpha} \cdot \left(\overrightarrow{\widetilde{s}}_k + \frac{\overrightarrow{\widetilde{e}}_k}{1+\theta}\right)^{1-\alpha}}{\overrightarrow{\widetilde{m}}_k} > \frac{p_k}{p^*} \cdot r$, a contradiction since $\widetilde{f}_k\left(h^k\right) \in \mathcal{A}\left(h^k\right)$ and $\left(\overrightarrow{\widetilde{w}}_k, \overrightarrow{\widetilde{v}}_k\right) \in \Theta\left(\sigma^s \left| h^k\right.\right)$ imply

$$p_k \cdot \overrightarrow{\widetilde{l}}_k^{\alpha} \cdot \left(\overrightarrow{\widetilde{s}}_k + \frac{\overrightarrow{\widetilde{e}}_k}{1+\theta}\right)^{1-\alpha} \leq R^*\left(p_k, \sigma^s\right) \cdot \overrightarrow{\widetilde{m}}_k$$

Thus, the firm has no profitable deviation at any proper subgame that begins at t.

The payoff of a young worker who accepts employment at date $k \geq t$ is $U\left(A, f, \sigma^s \left| \left(h^k, w\right)\right.\right) = w + \beta \cdot \overline{w}_2$, a strictly increasing function of the wage offer w. If he rejects the offer, he obtains a payoff $U\left(R, f, \sigma^s \left| \left(h^k, w\right)\right.\right) = \overline{w}_1 + \beta \cdot \overline{w}_2$. Therefore, σ_k^s satisfies (2) in the definition of a *SPNE.

Since $R^*\left(p_t, \sigma^s\right) = \frac{p_t}{p^*} \cdot r$, it follows that $\overrightarrow{q}_t = \frac{r}{p^*} \cdot \overrightarrow{m}_t$. Suppose $p_0 \leq \frac{p^*}{r} \frac{\alpha}{1-\alpha} \cdot \frac{\overline{w}_2 + c}{\overline{w}_1} \cdot \lambda$. If $T = 0$, then

$\overrightarrow{m}_0 = \delta \cdot a_0$ and $\overrightarrow{m}_t = \overrightarrow{m}_T$. Hence, (4) holds. If $T > 0$ then $m_0 = a_0$ and $\overrightarrow{m}_t = \frac{p_{t-1}}{p^*} \cdot r \cdot \overrightarrow{m}_{t-1}$ for all $1 \le t \le T-1$ and $\overrightarrow{m}_t = \overrightarrow{m}_T = \delta \cdot \overrightarrow{a}_T$ for all $t > T$. Thus, (4) holds. Since $\overrightarrow{m}_{t+1} \le \frac{p_{t-1}}{p^*} \cdot r \cdot \overrightarrow{m}_t$ for all $t \ge 0$ and P is a decreasing sequence,

$$\begin{aligned} \frac{1-\alpha}{\alpha} \cdot \frac{\overline{w}_1}{\overline{w}_2 + c} \cdot \frac{\overrightarrow{m}_{t+1}}{\overrightarrow{m}_t} &\le \frac{1-\alpha}{\alpha} \cdot \frac{\overline{w}_1}{\overline{w}_2 + c} \cdot \frac{p_t}{p^*} \cdot r \\ &\le \frac{1-\alpha}{\alpha} \cdot \frac{\overline{w}_1}{\overline{w}_2 + c} \cdot \frac{p_0}{p^*} \cdot r \\ &\le \lambda \end{aligned}$$

Therefore, $\overrightarrow{s}_t = \frac{1-\alpha}{\overline{w}_2+c} \cdot \overrightarrow{m}_t \le \lambda \cdot \frac{\alpha}{\overline{w}_1} \cdot \overrightarrow{m}_{t-1} = \lambda \cdot \overrightarrow{l}_{t-1}$, all $t \ge 1$. Since $\overrightarrow{s}_0 = \frac{1-\alpha}{\overline{w}_2+c} \cdot a_0 \le \lambda \cdot l_{-1}$, $\overrightarrow{e}_t = 0$ for all $t \ge 0$. Finally, $\frac{\partial \Pi\left(f,\sigma^s;\alpha\left|\overrightarrow{h^t}\right.\right)}{\partial e_t} = 0 \Leftrightarrow \frac{\partial R\left(f_t,\sigma^s;\alpha\left|\overrightarrow{h^t}\right.\right)}{\partial e_t} = 0 \Leftrightarrow v^* = \frac{c}{\theta}$. Hence, $\frac{\partial \Pi\left(f,\sigma^s;\alpha\left|\overrightarrow{h^t}\right.\right)}{\partial e_t} = 0$. ■

Proof of Proposition 4.2: Let f be the strategy in which $w_t = \overline{w}_1$, $v_t = \overline{w}_2$ for all $t \ge 0$, $\{l_t, s_t, e_t, m_t, b_t\}_{t=0}^{T-1}$ is the unique solution to problem (3) and for all $t \ge T$

$$\begin{aligned} m_t &= \begin{cases} \delta \cdot a_T & \text{if } \frac{1-\alpha}{\overline{w}_2+c} \cdot \delta a_T \le \lambda \cdot l_{t-1} \\ 0 & \text{otherwise} \end{cases} \\ l_t &= \frac{\alpha}{\overline{w}_1} \cdot m_t \\ s_t &= \frac{1-\alpha}{\overline{w}_2 + c} \cdot m_t \\ e_t &= 0 \end{aligned}$$

Clearly $f \in \mathbb{F}$. I shall show that (f, σ^s) is a *SPNE. Since the workers' strategy satisfies (1) and (2) in the definition of a *SPNE, it suffices to show that there is no history h^t such that the firm has a profitable deviation. Suppose there exists a strategy $\widetilde{f}$ and a partial history h^t such that $\Pi\left(\widetilde{f}, \sigma^s; \alpha\left|h^t\right.\right) > \Pi\left(f, \sigma^s; \alpha\left|h^t\right.\right)$. Let $\widetilde{R}_t \equiv R\left(\widetilde{f}_t, \sigma^s; \alpha\left|h^t\right.\right)$. Since $\Pi\left(f, \sigma^s; \alpha\left|h^t\right.\right) = \frac{\beta r}{1-\beta r} \cdot a_t \ge \Pi\left(\widetilde{f}, \sigma^s; \alpha\left|h^t\right.\right)$ for all $t \ge T$, then it must be the case that $t < T$ and

$$\left(\sum_{k=t}^{T-1} \beta^{k+1-t} \cdot \overrightarrow{\widetilde{R}}_k \cdot \ldots \cdot \overrightarrow{\widetilde{R}}_t\right) \cdot a_t > \left(\sum_{k=t}^{T-1} \beta^{k+1-t} \cdot \overrightarrow{R}_k \cdot \ldots \cdot \overrightarrow{R}_t\right) \cdot a_t$$

a contradiction since $\{l_t, s_t, e_t, m_t, b_t\}_{t=0}^{T-1}$ solves problem (3) and $\overrightarrow{R}_k > r$ for all $k \le T-1$.

Let $(f, \sigma) \in \mathbb{F} \times \overline{\mathbb{W}}$ be a *SPNE of $\Gamma(P, V^*, \alpha)$. Since firms pay $\overline{w}_2$ to the old workers performing task 2, then a young worker accepts employment if and only if he is offered at least $\overline{w}_1$. Therefore, $\sigma = \sigma^s$. Consider the strategy $\widetilde{f}$ where $\widetilde{w}_t = \overline{w}_1$, $\widetilde{v}_t = \overline{w}_2$ for all $t \geq 0$ and for some $\delta \in [0,1]$:

$$\begin{aligned}
\widetilde{m}_t &= \begin{cases} \widetilde{a}_t & \text{if } t < T \\ \delta \cdot \widetilde{a}_t & \text{if } t = T \\ Min\left\{\widetilde{m}_T, \widetilde{a}_t\right\} & \text{if } t > T \end{cases} \\
\widetilde{l}_t &= \frac{\alpha}{\overline{w}_1} \cdot \widetilde{m}_t \\
\widetilde{s}_t &= Min\left[\frac{1-\alpha}{\overline{w}_2 + c} \cdot \widetilde{m}_t, \lambda \cdot \widetilde{l}_{t-1}\right], \widetilde{e}_t = 0
\end{aligned}$$

and $\widetilde{b}_t = \widetilde{a}_t - \widetilde{m}_t$. Since $\widetilde{f}_t\left(h^t\right) \in \Re^7_+$, $\widetilde{s}_t \leq \lambda \cdot \widetilde{l}_{t-1}$, $\widetilde{m}_t \leq \widetilde{a}_t$ and $\widetilde{w}_t \cdot \widetilde{l}_t + (\widetilde{v}_t + c) \cdot \widetilde{s}_t + \widetilde{v}_t \cdot \widetilde{e}_t = \widetilde{m}_t$, then $\widetilde{f} \in \mathbb{F}$. Since $\left\{\widetilde{f}_t\right\}_{t=0}^{T-1}$ satisfies also the constrains in (3), it follows that $\Pi\left(\widetilde{f}, \sigma^s; \alpha \left| h^0\right.\right) \leq \Pi\left(f, \sigma^s; \alpha \left| h^0\right.\right)$. Let $\overrightarrow{R}_t = p_k \cdot \overrightarrow{l}^{\alpha}_k \cdot \left(\overrightarrow{s}_k + \frac{\overrightarrow{e}_k}{1+\theta}\right)^{1-\alpha} \cdot \frac{1}{\overrightarrow{a}_k} + r \cdot \frac{\overrightarrow{b}_k}{\overrightarrow{a}_k}$. Notice that

$$\begin{aligned}
\Pi\left(f, \sigma^s; \alpha \left| h^0\right.\right) &= \left(\sum_{k=0}^{\infty} \beta^{k+1} \cdot \overrightarrow{R}_k \cdot \ldots \cdot \overrightarrow{R}_0\right) \cdot a_0 \\
&\leq \beta \cdot \frac{p_0}{p^*} \cdot r \cdot a_0 + \left(\sum_{k=1}^{\infty} \beta^{k+1} \cdot \frac{p_k}{p^*} \cdot \ldots \cdot \frac{p_0}{p^*} \cdot r^{k+1}\right) \cdot a_0 \\
&= \Pi\left(\widetilde{f}, \sigma^s; \alpha \left| h^0\right.\right)
\end{aligned}$$

It follows that $\Pi\left(\widetilde{f}, \sigma^s; \alpha \left| h^0\right.\right) = \Pi\left(f, \sigma^s; \alpha \left| h^0\right.\right)$ and it must be the case that $\overrightarrow{R}_t = \frac{p_t}{p^*} \cdot r$. Therefore, $\overrightarrow{q}_t = \frac{r}{p^*} \cdot \overrightarrow{m}_t$, $\overrightarrow{e}_t = 0$, $\overrightarrow{m}_0 = a_0$ and

$$\overrightarrow{m}_t = \begin{cases} \frac{p_{t-1}}{p^*} \cdot r \cdot \overrightarrow{m}_{t-1} & \text{if } t < T \\ \delta \cdot \frac{p_{T-1}}{p^*} \cdot r \cdot \overrightarrow{m}_{T-1} & \text{if } t = T \end{cases}$$

for some $\delta \in [0,1]$. Finally, $\frac{\partial \Pi\left(f, \sigma^s; \alpha \left| \overrightarrow{h^t}\right.\right)}{\partial e_t} < 0 \Leftrightarrow \frac{\partial R\left(f_t, \sigma^s; \alpha \left| \overrightarrow{h^t}\right.\right)}{\partial e_t} < 0 \Leftrightarrow v^* > \frac{c}{\theta}$. Hence, $\frac{\partial \Pi\left(f, \sigma^s; \alpha \left| \overrightarrow{h^t}\right.\right)}{\partial e_t} < 0$. ■

Proof of Proposition 4.3: Let $(f, \sigma) \in \mathbb{F} \times \overline{\mathbb{W}}$ be a *SPNE of $\Gamma(P, V^*, \alpha)$. That (7) - (9) must hold whenever $m_t > 0$ follows by Lemma 7.1. Suppose $s_t = Min\left\{\frac{1-\alpha}{v^*+c} \cdot m_t, \lambda \cdot l_{t-1}\right\}$. Since $\frac{1-\alpha}{v^*+c} \cdot m_{t+1} \leq \lambda \cdot \frac{\alpha}{\omega\left(\frac{m_{t+1}}{m_t}, \alpha\right)} \cdot m_t$ for all h^t, then $\frac{s_{t+1}}{l_t} = \frac{1-\alpha}{\alpha} \cdot \frac{\omega\left(\frac{m_{t+1}}{m_t}, \alpha\right)}{v^*+c} \cdot \frac{m_{t+1}}{m_t} < \lambda$ and the young worker born at date t is better off accepting employment if and only if $w_t \geq \omega\left(\frac{m_{t+1}}{m_t}, \alpha\right)$, for any h^t. Hence, $\omega\left(\frac{m_{t+1}}{m_t}, \alpha\right) = \min \Theta\left(\sigma \left| h^t\right.\right)$. It follows by Lemma 7.1 that $w_t = \omega\left(\frac{m_{t+1}}{m_t}, \alpha\right)$. ■

Proof of Lemma 4.2 Suppose $\frac{c}{\theta} > \overline{w}_2$. Let $p \geq 0$ and $\alpha \in (0,1)$. Consider the equation

$$w = \overline{w}_1 - \beta \cdot \frac{1-\alpha}{\alpha} \cdot \frac{w}{v^* + c} \cdot p \cdot q^i(w, \alpha) \cdot (v^* - \overline{w}_2)$$

Define $B(w,p) = \overline{w}_1 - \beta \cdot \frac{1-\alpha}{\alpha} \cdot \frac{w}{v^*+c} \cdot p \cdot q^i(w, \alpha) \cdot (v^* - \overline{w}_2) - w$. A solution to (10) exists if and only if there exists w such that $B(w,p) = 0$. Notice that $B(\overline{w}_1, p) \leq 0$ and $B(0,p) = \overline{w}_1 > 0$. Since B is continuous and strictly decreasing in w, there exists a unique solution $w^H(p,\alpha)$ to the equation above. Let $0 < p_1 < p_2$. Since $B\left(w^H(p_2,\alpha),p_2\right) = 0 = B\left(w^H(p_1,\alpha),p_1\right) > B\left(w^H(p_1,\alpha),p_2\right)$ then $w^H(p_2,\alpha) < w^H(p_1,\alpha)$. Hence, w^H is strictly decreasing in p, as desired. ■

Proof of Proposition 4.4: Assume $\frac{c}{\theta} > \overline{w}_2$. Let $P \in \Sigma$, $\delta \in [0,1]$ and $T \geq T_s$. Let

$$\left(f^H,\sigma^H\right) = \left[f(T,\delta;\alpha,P), \sigma(T,\delta;\alpha,P)\right]$$

Consider the subgame that begins after partial history h^t. First, I show that the firm has no profitable deviation from $f(T,\delta;\alpha,P)$. Since $P \in \Sigma$, it follows that $p_t \cdot q\left(\omega^H(p_t,\alpha),\alpha\right) > r$ for all $0 \leq t \leq T-2$ and $p_t \cdot q(\omega(1,\alpha),\alpha) = r$ for all $t \geq T$ and by assumption $p_{T-1} \cdot q\left(\omega^H(\delta \cdot p_{T-1},\alpha),\alpha\right) > r$. Therefore,

$$R^*\left(p_t,\sigma^H\right)\begin{cases} > r & \text{if } t < T \\ = r & \text{if } t \geq T \end{cases}$$

If the firm follows the strategy $f(T,\delta;\alpha,P)$, its continuation payoff is:

$$\begin{aligned}
\Pi\left(f^H,\sigma^H;\alpha\left|h^t\right.\right) &= \sum_{k=t}^{\infty}\beta^{k+1-t}\left[R^*\left(p_k,\sigma^H\right)\cdot\overrightarrow{m}_k^H + r\cdot\left(\overrightarrow{a}_k^H - \overrightarrow{m}_k^H\right)\right] \\
&= \sum_{k=t}^{\infty}\beta^{k+1-t}\cdot R^*\left(p_k,\sigma^H\right)\cdot\overrightarrow{a}_k^H \\
&= \beta R^*\left(p_t,\sigma^H\right)\cdot a_t^H + \left(\sum_{k=t+1}^{\infty}\beta^{k+1-t}\cdot R^*\left(p_k,\sigma^H\right)\cdot\ldots\cdot R^*\left(p_t,\sigma^H\right)\right)a_t^H
\end{aligned}$$

Suppose there is a profitable deviation f, i.e. $\Pi\left(f,\sigma^H;\alpha\left|h^t\right.\right) > \Pi\left(f^H,\sigma^H;\alpha\left|h^t\right.\right)$. Let $R_t \equiv R\left(f_t,\sigma^H;\alpha\left|h^t\right.\right)$. Assume f is such that $\overrightarrow{m}_k = 0$ for all $k \geq t$. Then $\Pi\left(f,\sigma^H;\alpha\left|h^t\right.\right) = \frac{\beta r}{1-\beta r}\cdot a_t \leq \Pi\left(f^H,\sigma^H;\alpha\left|h^t\right.\right)$, a contradiction. Assume f is such that $\overrightarrow{m}_k > 0$ and $\overrightarrow{R}_k > \frac{p_k}{p^*}\cdot r \geq r$ for some $k \geq t$. Hence, $(\overrightarrow{w}_k,\overrightarrow{v}_k) \in \Theta\left(\sigma^H\left|h^k\right.\right)$, $\overrightarrow{m}_k > 0$ and $p_k \cdot \frac{\overrightarrow{l}_k^{\alpha}\cdot\left(\overrightarrow{s}_k+\frac{\overrightarrow{e}_k}{1+\theta}\right)^{1-\alpha}}{\overrightarrow{m}_k} > R^*\left(p_k,\sigma^H\right)$, a contradiction since $f_k\left(h^k\right) \in \mathcal{A}\left(h^k\right)$ and $(\overrightarrow{w}_k,\overrightarrow{v}_k) \in \Theta\left(\sigma^H\left|h^k\right.\right)$ implies

$$p_k \cdot \overrightarrow{l}_k^{\alpha}\cdot\left(\overrightarrow{s}_k + \frac{\overrightarrow{e}_k}{1+\theta}\right)^{1-\alpha} \leq R^*\left(p_k,\sigma^H\right)\cdot\overrightarrow{m}_k$$

Thus, the firm has no profitable deviation at any proper subgame that begins at h^t.

The payoff of a young worker born at date $k \geq t$ who accepts employment is

$$U\left(A,f^H,\sigma^H\left|\left(h^k,w\right)\right.\right) = w + \beta\cdot\left[\frac{1-\alpha}{\alpha}\cdot\frac{\omega\left(\frac{\overline{m}_{k+1}}{\overline{m}_k},\alpha\right)}{v^*+c}\cdot\frac{\overline{m}_{k+1}}{\overline{m}_k}\cdot\left(v^*-\overline{w}_2\right)+\overline{w}_2\right]$$

which is strictly increasing in the wage offer w. If he rejects the offer, his payoff is $U\left(R,f^H,\sigma^H\left|\left(h^k,w\right)\right.\right) = \overline{w}_1 + \beta\cdot\overline{w}_2$. Therefore, σ_k^H satisfies (2) in the definition of a *SPNE because $U\left(A,f^H,\sigma^H\left|\left(h^k,w\right)\right.\right) \geq$

$\overline{w}_1 + \beta \cdot \overline{w}_2$ for all $w \geq \omega\left(\frac{m_{k+1}}{m_k}, \alpha\right)$ and $U\left(R, f^H, \sigma^H \left|(h^k, w)\right.\right) > U\left(A, f^H, \sigma^H \left|(h^k, w)\right.\right)$ for all $0 \leq w < \omega\left(\frac{m_{k+1}}{m_k}, \alpha\right)$.

Since $s_0 = \frac{1-\alpha}{v^*+c} \cdot a_0 < l_{-1}$, it follows that $\overrightarrow{e}_0 = 0$. Notice that

$$\begin{aligned} \frac{1-\alpha}{\alpha} \cdot \frac{\overrightarrow{w}_t}{v^*+c} \cdot \frac{\overrightarrow{m}_{k+1}}{\overrightarrow{m}_k} &= \frac{1-\alpha}{\alpha} \cdot \frac{\omega\left(\frac{\overrightarrow{m}_{k+1}}{\overrightarrow{m}_k}, \alpha\right)}{v^*+c} \cdot p_t \cdot q\left(\omega\left(\frac{\overrightarrow{m}_{k+1}}{\overrightarrow{m}_k}, \alpha\right), \alpha\right) \\ &\leq \lambda \end{aligned}$$

Hence, $\overrightarrow{e}_t = 0$ for all $t \geq 0$. Finally, $\frac{\partial \Pi\left(f^H, \sigma^H \left|\overrightarrow{h}^t\right.\right)}{\partial e_t} = 0 \Leftrightarrow \frac{\partial R\left(f_t^H, \sigma^H \left|\overrightarrow{h}^t\right.\right)}{\partial e_t} = 0 \Leftrightarrow v^* = \frac{c}{\theta}$, as desired. ■

Proof of Proposition 4.5: Assume $\frac{c}{\theta} > \overline{w}_2$. Let $P \in \Sigma$, $w_0 \in (\underline{w}, \overline{w})$ and

$$\left(f^L, \sigma^L\right) = \left[f\left(w_0; \alpha, P\right), \sigma\left(w_0; \alpha, P\right)\right]$$

Consider the subgame that begins after partial history h^t. First, I show that the firm has no profitable deviation from $f^L\left(w_0; \alpha, P\right)$. Notice that

$$R^*\left(p_t, \sigma^L\right) \begin{cases} > r & \text{if } t = 0 \\ = r & \text{if } t \geq 1 \end{cases}$$

If the firm follows strategy $f^L\left(w_0; \alpha, P\right)$, its continuation payoff is:

$$\begin{aligned} \Pi\left(f^L, \sigma^L; \alpha \left|h^t\right.\right) &= \sum_{k=t}^{\infty} \beta^{k+1-t}\left[R^*\left(p_k, \sigma^L\right) \cdot \overrightarrow{m}_k^L + r \cdot \left(\overrightarrow{a}_k^L - \overrightarrow{m}_k^L\right)\right] \\ &= \sum_{k=t}^{\infty} \beta^{k+1-t} \cdot R^*\left(p_k, \sigma^L\right) \cdot \overrightarrow{a}_k^L \\ &= \left(\sum_{k=t}^{\infty} \beta^{k+1-t} \cdot r^{k-t} \cdot R^*\left(p_t, \sigma^L\right)\right) \cdot \overrightarrow{a}_t^L \end{aligned}$$

Suppose there is a profitable deviation f, i.e. $\Pi\left(f, \sigma^L; \alpha \left|h^t\right.\right) > \Pi\left(f^L, \sigma^L; \alpha \left|h^t\right.\right)$. Let $R_t \equiv R\left(f_t, \sigma^L; \alpha \left|h^t\right.\right)$. Assume f is such that $\overrightarrow{m}_k = 0$ for all $k \geq t$. Then $\Pi\left(f, \sigma^L; \alpha \left|h^t\right.\right) = \frac{\beta r}{1-\beta r} \cdot a_t \leq \Pi\left(f^L, \sigma^L; \alpha \left|h^t\right.\right)$, a contradiction. Assume f is such that $\overrightarrow{m}_k > 0$ and $\overrightarrow{R}_k > \frac{p_k}{p^*} \cdot r \geq r$ for some $k \geq t$. Hence, $(\overrightarrow{w}_k, \overrightarrow{v}_k) \in \Theta\left(\sigma^L \left|h^k\right.\right)$, $\overrightarrow{m}_k > 0$ and $p_k \cdot \frac{\overrightarrow{l}_k^{\alpha} \cdot \left(\overrightarrow{s}_k + \frac{\overrightarrow{e}_k}{1+\theta}\right)^{1-\alpha}}{\overrightarrow{m}_k} > R^*\left(p_k, \sigma^L\right)$, a contradiction since $f_k\left(h^k\right) \in \mathcal{A}\left(h^k\right)$ and $(\overrightarrow{w}_k, \overrightarrow{v}_k) \in \Theta\left(\sigma^L \left|h^k\right.\right)$ implies

$$p_k \cdot \overrightarrow{l}_k^{\alpha} \cdot \left(\overrightarrow{s}_k + \frac{\overrightarrow{e}_k}{1+\theta}\right)^{1-\alpha} \leq R^*\left(p_k, \sigma^L\right) \cdot \overrightarrow{m}_k$$

Thus, the firm has no profitable deviation at any proper subgame that begins at h^t.

Consider a young worker of generation k, for some $k \geq t$. If he accepts employment, then his payoff is

$$U\left(A, f^L, \sigma^L \left|\left(h^k, w\right)\right.\right) = w + \beta \cdot \left[\frac{1-\alpha}{\alpha} \cdot \frac{\omega^L\left(p_k, \alpha\right)}{v^* + c} \cdot G\left(\omega^L\left(p_k, \alpha\right), \alpha\right) \cdot \left(v^* - \overline{w}_2\right) + \overline{w}_2\right]$$

which is strictly increasing in the wage offer w. If he rejects the offer, his payoff is $U\left(R, f^L, \sigma^L \left|\left(h^k, w\right)\right.\right) = \overline{w}_1 + \beta \cdot \overline{w}_2$. Therefore, σ^L_k satisfies (2) in the definition of a *SPNE because $U\left(A, f^L, \sigma^L \left|\left(h^k, w\right)\right.\right) \geq \overline{w}_1 + \beta \cdot \overline{w}_2$ for all $w \geq \omega^L\left(p_k, \alpha\right)$ and $U\left(R, f^L, \sigma^L \left|\left(h^k, w\right)\right.\right) \geq U\left(A, f^L, \sigma^L \left|\left(h^k, w\right)\right.\right)$ for all $0 \leq w < \omega^L\left(p_k, \alpha\right)$.

Since $s_0 = \frac{1-\alpha}{v^*+c} \cdot a_0 < l_{-1}$, it follows that $\overrightarrow{e}_0 = 0$. Since $p_t \leq p^{**}$ and $w_0 \leq \overline{w}$ then $G\left(\overrightarrow{w}_t, \alpha\right) \geq 0$. Therefore, if $\overrightarrow{m}_t > 0$

$$\begin{aligned} \frac{1-\alpha}{\alpha} \cdot \frac{\overrightarrow{w}_t}{v^* + c} \cdot \frac{\overrightarrow{m}_{t+1}}{\overrightarrow{m}_t} &= \frac{1-\alpha}{\alpha} \cdot \frac{\overrightarrow{w}_t}{v^* + c} \cdot G\left(\overrightarrow{w}_t, \alpha\right) \\ &= \frac{1-\alpha}{\alpha} \cdot \frac{\omega\left(G\left(\overrightarrow{w}_t, \alpha\right), \alpha\right)}{v^* + c} \cdot G\left(\overrightarrow{w}_t, \alpha\right) \\ &\leq \lambda \end{aligned}$$

Hence, $\overrightarrow{e}_t = 0$ for all $t \geq 0$. Finally, $\frac{\partial \Pi\left(f^L, \sigma^L \left|\overrightarrow{h^t}\right.\right)}{\partial e_t} = 0 \Leftrightarrow \frac{\partial R\left(f^L_t, \sigma^L \left|\overrightarrow{h^t}\right.\right)}{\partial e_t} = 0 \Leftrightarrow v^* = \frac{c}{\theta}$, as desired. ■

Proof of Proposition 4.6: Let $P \in \Sigma$ and $\frac{c}{\theta} > \overline{w}_2$. Suppose $(f, \sigma) \in \mathbb{F} \times \overline{\mathbb{W}}$ is a *SPNE in which $\overrightarrow{e}_t = 0$ and τ is the first date such that $\overrightarrow{w}_\tau \neq \omega^H\left(p_\tau, \alpha\right)$. To show that $\sigma_t = \sigma_{t-\tau}\left(\overrightarrow{w}_\tau; \alpha, P^\tau\right)$ for all $t \geq \tau$, it suffices to show that $\min \Theta\left(\sigma \left|h^t\right.\right) = \min \Theta\left(\sigma\left(\overrightarrow{w}_\tau; \alpha, P^\tau\right) \left|h^{t-\tau}\right.\right)$ for all $t \geq \tau$. By proposition 4.3, $\left(w_t, v^*\right) = \min \Theta\left(\sigma \left|h^t\right.\right)$. By the stationarity of σ, $w_t = \overrightarrow{w}_t$ and $v_{t+1} = v^*$ for all $h^t \in H^t$. Therefore, $\min \Theta\left(\sigma \left|h^t\right.\right) = \left(\overrightarrow{w}_t, v^*\right)$. Suppose $\min \Theta\left(\sigma \left|h^t\right.\right) = \left(\omega^L\left(p_t, \alpha\right), v^*\right)$ for some $t \geq 1$. Then $w_t > \omega^H\left(p_t, \alpha\right)$ and $\frac{\overrightarrow{m}_{t+1}}{\overrightarrow{m}_t} < p_t \cdot q\left(\overrightarrow{w}_t, \alpha\right)$. It follows that $p_{t+1} \cdot q\left(\overrightarrow{w}_{t+1}, \alpha\right) = r$ and $\overrightarrow{w}_{t+1} = \omega^L\left(p_{t+1}, \alpha\right)$. By induction, $\min \Theta\left(\sigma \left|h^t\right.\right) = \left(\omega^L\left(p_t, \alpha\right), v^*\right)$ implies $\min \Theta\left(\sigma \left|h^\tau\right.\right) = \left(\omega^L\left(p_\tau, \alpha\right), v^*\right)$ for all $\tau \geq t+1$.

Since $p_t > p_r$ and $\overrightarrow{w}_t = \omega^H\left(p_t, \alpha\right)$ for all $t < \tau$, then $p_t \cdot q\left(\overrightarrow{w}_t, \alpha\right) > r$ for all $t < \tau$. Then, $\overrightarrow{m}_t = a_t$ for all $t < \tau$. Since $\overrightarrow{e}_{t+1} = 0$ then by proposition 4.3 $\overrightarrow{s}_{t+1} = \frac{1-\alpha}{v^*} \cdot \overrightarrow{m}_{t+1}$ and $\overrightarrow{l}_t = \frac{1-\alpha}{\omega^H(p_t,\alpha)} \cdot \overrightarrow{m}_t$ which implies that $\overrightarrow{w}_t = \omega\left(\frac{\overrightarrow{m}_{t+1}}{\overrightarrow{m}_t}, \alpha\right)$ for all $t \geq 0$. Hence, $\overrightarrow{m}_\tau = G\left(\overrightarrow{w}_{\tau-1}, \alpha\right) \cdot a_{\tau-1}$ which implies that $\overrightarrow{m}_\tau = \overrightarrow{a}_\tau$. Therefore, $\overrightarrow{w}_\tau \neq \omega^H\left(p_\tau, \alpha\right)$ implies that $\frac{\overrightarrow{m}_{\tau+1}}{\overrightarrow{m}_\tau} < p_\tau \cdot q\left(\overrightarrow{w}_\tau, \alpha\right)$. It follows that $\overrightarrow{w}_\tau \in \left(\omega^H\left(p_\tau, \alpha\right), \overline{w}_1\right]$. In addition, $\overrightarrow{m}_{\tau+1} < p_\tau \cdot q\left(\overrightarrow{w}_\tau, \alpha\right) \cdot \overrightarrow{m}_\tau$ implies that $p_{\tau+1} \cdot q\left(\overrightarrow{w}_{\tau+1}, \alpha\right) = r$. Therefore, $\overrightarrow{w}_{\tau+1} = \omega^L\left(p_{\tau+1}, \alpha\right)$ and by the argument in the first paragraph of this proof it follows that $\min \Theta\left(\sigma \left|h^t\right.\right) = \left(\omega^L\left(p_t, \alpha\right), v^*\right)$ for all $t \geq \tau + 1$. It follows that $\sigma_t = \sigma_{t-\tau}\left(\overrightarrow{w}_\tau; \alpha, P^\tau\right)$ and $f_t = f_{t-\tau}\left(\overrightarrow{w}_\tau, \overrightarrow{a}_\tau; \alpha, P^\tau\right)$ for all $t \geq \tau$. ■

A.3 Proofs of Section 5

Proof of Proposition 5.1: Suppose $\frac{c}{\theta} \leq \overline{w}_2$ and $a_0 \geq \frac{p^*}{r} \cdot D\left(\frac{p^*}{r} \frac{\alpha}{1-\alpha} \cdot \frac{\overline{w}_2 + c}{\overline{w}_1} \cdot \lambda\right)$. Since $\widehat{p}_0 = D^{-1}\left(\frac{r}{p^*} \cdot a_0\right)$, $\widehat{p}_0 \leq \frac{p^*}{r} \frac{\alpha}{1-\alpha} \cdot \frac{\overline{w}_2 + c}{\overline{w}_1} \cdot \lambda$. By propositions 4.1 and 4.2, $\Gamma\left(P, V^*, \alpha\right)$ has a *SPNE $\left(f, \sigma^s\right)$, in which $\overrightarrow{e}_t = 0$ and

$\frac{\partial \Pi\left(f,\sigma^s;\alpha|\overrightarrow{h^t}\right)}{\partial e_t} \leq 0$. Therefore, $\{P, V^*, (f, \sigma^s)\}$ satisfies conditions (3) and (4) in the definition of an IE at every date $t \geq 0$. From (4) in propositions 4.1 and 4.2, it follows that at date $0 \leq t \leq T-1$,

$$\overrightarrow{q}_t = \frac{r}{p^*} \cdot \overrightarrow{m}_t = p_{t-1} \cdot \ldots \cdot p_0 \cdot \left(\frac{r}{p^*}\right)^t \cdot a_0 = D\left(p_t\right)$$

and at any date $t \geq T$,

$$\begin{aligned} \frac{r}{p^*} \cdot \overrightarrow{m}_t &= \frac{r}{p^*} \cdot \overrightarrow{m}_T = \frac{r}{p^*} \cdot \widehat{\delta} \cdot \overrightarrow{a}_T = \frac{r}{p^*} \cdot \widehat{\delta} \cdot p_{T-1} \cdot \ldots \cdot p_0 \cdot \left(\frac{r}{p^*}\right)^T \cdot a_0 \\ &= \frac{r}{p^*} \cdot \widehat{\delta} \cdot p_{T-1} \cdot D\left(p_{T-1}\right) = D\left(p^*\right) \end{aligned}$$

It follows that the market clears at every date $t \geq 0$. Hence, there exists an IE $\{P, V^*, (f, \sigma^s)\}$. ■

Proof of Corollary 5.1: Suppose there exists another IE in which $\widehat{\sigma} \in \overline{\mathbb{W}}$ and $\widehat{P} \in \Sigma$. By Proposition 4.2, $\widehat{\sigma} = \sigma^s$, $\overrightarrow{\widehat{q}}_t = \frac{r}{p^*} \cdot \overrightarrow{m}$ and (4) holds for all $t \leq \widehat{T}$ and $h^t \in H^t$. By market clearing, $D\left(\widehat{p}_0\right) = \frac{r}{p^*} \cdot a_0 = D\left(p_0\right)$. Hence, $\widehat{p}_0 = p_0$. By induction, it follows that for any $t < \min\left\{T, \widehat{T}\right\}$,

$$D\left(\widehat{p}_t\right) = \frac{r}{p^*} \cdot a_t = \frac{r}{p^*} \cdot \widehat{p}_{t-1} \cdot \ldots \cdot \widehat{p}_0 \cdot \left(\frac{r}{p^*}\right)^t \cdot a_0 = D\left(p_t\right)$$

Hence, $\widehat{p}_t = p_t$ for all $t < \min\left\{T, \widehat{T}\right\}$. Suppose, $T > \widehat{T}$. Then

$$\frac{p^* \cdot D\left(p^*\right)}{r} = m_{\widehat{T}} \leq \widehat{p}_{\widehat{T}-1} \cdot D\left(\widehat{p}_{\widehat{T}-1}\right) = p_{\widehat{T}-1} \cdot D\left(p_{\widehat{T}-1}\right)$$

a contradiction since $p_t \cdot D\left(p_t\right) < \frac{p^* \cdot D(p^*)}{r}$ for all $t < T$. Therefore, $\widehat{T} \geq T$. Suppose $\widehat{T} > T$. Since

$$D\left(\widehat{p}_T\right) = \frac{r}{p^*} \cdot \overrightarrow{a}_T \geq \frac{r}{p^*} \cdot \overrightarrow{m}_T = D\left(p^*\right)$$

it follows that $\widehat{p}_T \leq p^*$, a contradiction since $\widehat{p}_t > p^*$ for all $t < \widehat{T}$. Thus, $\widehat{T} = T$ and $\widehat{P} = P$. Since $\left\{\widehat{l}_k, \widehat{s}_k, \widehat{e}_k, \widehat{m}_k, \widehat{b}_k\right\}_{k=t}^{T}$ must solve (3) at every date $0 \leq t \leq T$, it follows that $\overrightarrow{\widehat{f}}_t = \overrightarrow{f}_t$ for all $0 \leq t \leq T$. ■

Proof of Lemma 5.1: Suppose AD and AW hold and $m \leq \frac{p_s \cdot D(p_s)}{r}$. Consider the function $H : \Re_+ \times \left[0, \frac{p_s \cdot D(p_s)}{r}\right] \rightarrow \Re$ defined by $H\left(p, m\right) = q\left(\omega^H\left(p\right)\right) \cdot m - D\left(p\right)$. By Lemma 4.2 and assumption AD it follows that H is continuous and strictly increasing in both m and p. By AD, $\lim_{p \to \infty} H\left(p, m\right) > 0$. Since H is continuous, to show that $H\left(p, m\right) = 0$ has a solution it suffices to show that there exists p such that $H\left(p, m\right) < 0$. Notice that $\omega^H\left(\frac{p_s}{r}\right)$ is the unique solution to $w = \overline{w}_1 - \beta \cdot \frac{1-\alpha}{\alpha} \cdot \frac{w}{v^*+c} \cdot \frac{p_s}{r} \cdot q\left(w\right) \cdot \left(v^* - \overline{w}_2\right)$. Since $\omega\left(1\right)$ is also a solution to that equation, it follows that $\omega^H\left(\frac{p_s}{r}\right) = \omega\left(1\right)$. Hence $H\left(\frac{p_s}{r}, m\right) = q\left(\omega^H\left(\frac{p_s}{r}\right)\right) \cdot m - D\left(\frac{p_s}{r}\right) = q\left(\omega\left(1\right)\right) \cdot m - D\left(\frac{p_s}{r}\right) < \frac{r}{p_s} \cdot m - D\left(p_s\right) \leq 0$ where the last inequality follows from the assumption that $m \leq \frac{p_s \cdot D(p_s)}{r}$. By the intermediate value theorem there exists $p > \frac{p_s}{r}$ such that $H(p, m) = 0$. Since H is strictly increasing in its first argument, the solution is unique. Therefore there exists a function $\mathbb{P}$: $\left[0, \frac{p_s \cdot D(p_s)}{r}\right] \to \left(\frac{p_s}{r}, \infty\right)$ such that $H[\mathbb{P}(m), m] = 0$. Notice that $H[p_r, m] < H\left[p_r, \frac{p_r \cdot D(p_r)}{r}\right] = 0 = H[\mathbb{P}(m), m]$ if and only if $m < \frac{p_r \cdot D(p_r)}{r}$. Hence $\mathbb{P}(m) > p_r$ if and only if $m < \frac{p_r \cdot D(p_r)}{r}$, as desired. ■

Lemma 7.2 *Let* $\frac{c}{\theta} > \overline{w}_1$. *Suppose* $0 < m < \frac{p_s \cdot D(p_s)}{r}$ *and* $\mathbb{P}(m) \cdot D(\mathbb{P}(m)) \geq \frac{p_s \cdot D(p_s)}{r}$. *Equation (11) has a unique solution* $p_{T-1} = D^{-1}\left(q\left[\omega\left(\frac{p_s \cdot D(p_s)}{r \cdot m}\right)\right] \cdot m\right) \geq Max\{p_s, \mathbb{P}(m)\}$ *and* $p_{T-1} \cdot D(p_{T-1}) \geq \frac{p_s \cdot D(p_s)}{r}$.

Proof of Lemma 7.2: Let $\frac{c}{\theta} > \overline{w}_1$. Suppose $0 < m < \frac{p_s \cdot D(p_s)}{r}$ and $\mathbb{P}(m) \cdot D(\mathbb{P}(m)) \geq \frac{p_s \cdot D(p_s)}{r}$. Since $0 < \frac{p_s \cdot D(p_s)}{r} \leq \mathbb{P}(m) \cdot D(\mathbb{P}(m))$, it follows that

$$w_{T-1} = \omega\left(\frac{p_s \cdot D(p_s)}{r \cdot m}\right) \geq \omega\left(\frac{\mathbb{P}(m) \cdot D(\mathbb{P}(m))}{m}\right) = \omega^H(\mathbb{P}(m))$$

Hence, $\omega^H(\mathbb{P}(m)) \leq w_{T-1} \leq \omega(1)$. Since $q(w_{T-1}) \cdot m \leq q\left(\omega^H(\mathbb{P}(m))\right) \cdot m = D(\mathbb{P}(m)) \leq D\left(\frac{p_s}{r}\right)$ it follows that $p_{T-1} = D^{-1}(q(w_{T-1}) \cdot m)$ is well defined because D has an inverse on $\left[0, \frac{p_s}{r}\right]$. Clearly, p_{T-1} solves (11) and $p_{T-1} \geq \mathbb{P}(m)$. Uniqueness follows by the strict monotonicity of D in p. Since $q\left[\omega\left(\frac{p_s \cdot D(p_s)}{r \cdot m}\right)\right] \cdot m$ is strictly increasing in m, it follows that

$$D(p_{T-1}) = q\left[\omega\left(\frac{p_s \cdot D(p_s)}{r \cdot m}\right)\right] \cdot m < q[\omega(1)] \cdot \frac{p_s \cdot D(p_s)}{r} = D(p_s)$$

for all $m < \frac{p_s \cdot D(p_s)}{r}$. Therefore, $p_{T-1} > p_s$ and $p_{T-1} \geq Max\{p_s, \mathbb{P}(m)\}$.
I show that $p_{T-1} \cdot D(p_{T-1}) \geq \frac{p_s \cdot D(p_s)}{r}$ by reduction to the absurd. Suppose $p_{T-1} \cdot D(p_{T-1}) < \frac{p_s \cdot D(p_s)}{r}$. Then,

$$\begin{aligned} B\left(\omega^H(p_{T-1}), p_{T-1}\right) &= 0 = \overline{w}_1 - \frac{1-\alpha}{\alpha} \cdot \frac{w_{T-1}}{v^* + c} \cdot \frac{p_s \cdot D(p_s)}{r \cdot m} \cdot \beta \cdot (v^* - \overline{w}_2) - \omega_{T-1} \\ &< \overline{w}_1 - \frac{1-\alpha}{\alpha} \cdot \frac{w_{T-1}}{v^* + c} \cdot p_{T-1} \cdot q(w_{T-1}) \cdot \beta \cdot (v^* - \overline{w}_2) - \omega_{T-1} \\ &= B(w_{T-1}, p_{T-1}) \end{aligned}$$

and since $B(\cdot, p_{T-1})$ is decreasing in its first argument, it follows that $w_{T-1} < \omega^H(p_{T-1})$. Hence, $\omega^H(\mathbb{P}(m)) < \omega^H(p_{T-1})$ which implies that $\mathbb{P}(m) > p_{T-1}$, a contradiction since $D(p_{T-1}) = q(w_{T-1}) \cdot m \leq D(\mathbb{P}(m))$ and D is decreasing in p. It follows that $p_{T-1} \cdot D(p_{T-1}) \geq \frac{p_s \cdot D(p_s)}{r}$, as desired. ■

Proof of Proposition 5.2: First I consider the issue of existence of an IE and then I turn to its uniqueness.
EXISTENCE: There are two cases to consider depending on the value of $\mathbb{P}(a_0) \cdot D(\mathbb{P}(a_0))$.
Case (i): $\mathbb{P}(a_0) \cdot D(\mathbb{P}(a_0)) \geq \frac{p_s \cdot D(p_s)}{r}$
By Lemma 7.2, the equation

$$q\left(\omega\left(\frac{p_s \cdot D(p_s)}{r a_0}\right)\right) \cdot a_0 = D(p)$$

has a unique solution p_{T-1} and $\frac{p_s \cdot D(p_s)}{r} \leq p_{T-1} \cdot D(p_{T-1})$. Let $\widehat{\delta} = \frac{p_s \cdot D(p_s)}{r \cdot p_{T-1} \cdot D(p_{T-1})}$. Clearly, $\widehat{\delta} \in [0,1]$. Let $T_s = 1$ and define the sequence P with elements $p_0 = p_{T-1} > p_s > p_r$ and $p_t = p_s$ for all $t \geq 1$. Clearly, $P \in \Sigma$. In addition,

$$p_0 \cdot q\left[\omega^H\left(\widehat{\delta} \cdot p_0\right)\right] > p_s \cdot q(\omega(1)) = r$$

By Proposition 4.4, $\left(f^H, \sigma^H\right) \equiv \left[f\left(1, \widehat{\delta}; \alpha, P\right), \sigma\left(1, \widehat{\delta}; \alpha, P\right)\right]$ is a *SPNE of $\Gamma(P, V^*, \alpha)$ in which $\overrightarrow{e}_t = 0$ and the value of an externally trained worker is v^*. By construction, the output market clears at date zero. At any other date $t \geq 1$,

$$q(w_t) \cdot m_t = q(\omega(1)) \cdot m_1 = q(\omega(1)) \cdot \frac{p_s \cdot D(p_s)}{r} = D(p_s) = D(p_t)$$

It follows that $\left\{P, V^*, \left(f^H, \sigma^H\right)\right\}$ is an IE with $\mu^H = 1$.

Case (ii): $\mathbb{P}(a_0) \cdot D(\mathbb{P}(a_0)) < \frac{p_s \cdot D(p_s)}{r}$

Let $y_t = a_0$ and

$$y_{t+1} = \begin{cases} \mathbb{P}(y_t) \cdot D(\mathbb{P}(y_t)) & \text{if } \mathbb{P}(y_t) \cdot D(\mathbb{P}(y_t)) < \frac{p_s \cdot D(p_s)}{r} \text{ and } y_t < Min\left\{\frac{p_r \cdot D(p_r)}{r}, \frac{p_s \cdot D(p_s)}{r}\right\} \\ \frac{p_s \cdot D(p_s)}{r} & \text{otherwise} \end{cases}$$

Let τ be the first date t such that $y_t \geq \frac{p_s \cdot D(p_s)}{r}$. Clearly, $\tau \geq 1$ by the assumption that $a_0 < \frac{p_s \cdot D(p_s)}{r}$. I show that τ is finite. Suppose not. Then $y_t = \mathbb{P}(y_{t-1}) \cdot D(\mathbb{P}(y_{t-1})) = \mathbb{P}(y_{t-1}) \cdot q\left(\omega^H(\mathbb{P}(y_{t-1}))\right) \cdot y_{t-1} \geq r \cdot y_{t-1}$ because $y_{t-1} \leq \frac{p_r \cdot D(p_r)}{r}$ implies that $\mathbb{P}(y_{t-1}) \geq p_r$. It follows that $y_t \geq r^t \cdot a_0$ which implies that $y_t \to \infty$, a contradiction. Thus, τ is finite.

Let $T_s = \tau$ and P be the sequence with elements $p_t = \mathbb{P}(y_t)$ for all $t < T_s$, $p_t = p_s$ for all $t \geq T_s$ and with p_{T_s-1} as the solution to

$$\begin{aligned} q(w) \cdot y_{T_s-1} &= D(p) \\ w &= \omega\left(\frac{p_s \cdot D(p_s)}{r \cdot y_{T_s-1}}\right) \end{aligned}$$

Since $y_t < \frac{p_r \cdot D(p_r)}{r}$ for all $t < T_s$, then $p_t > p_r$ for all for all $t < T_s$. To prove that $P \in \Sigma$, I shall show that p_{T_s-1} is well defined and $p_{T_s-1} > p_r$. By definition of T_s, $y_{T_s-1} < \frac{p_s \cdot D(p_s)}{r}$. Suppose $y_{T_s-1} < \frac{p_r \cdot D(p_r)}{r}$. Then $\mathbb{P}(y_{T_s-1}) \cdot D(\mathbb{P}(y_{T_s-1})) \geq \frac{p_s \cdot D(p_s)}{r}$ and by Lemma 7.2 p_{T_s-1} is well defined, $p_{T_s-1} > p_s$ and $\widehat{\delta} = \frac{p_s \cdot D(p_s)}{r \cdot p_{T_s-1} \cdot D(p_{T_s-1})} \in [0,1]$. Therefore,

$$p_{T_s-1} \cdot q\left[\omega^H\left(\widehat{\delta} \cdot p_{T_s-1}\right)\right] = p_{T_s-1} \cdot q\left[\omega\left(\frac{p_s \cdot D(p_s)}{r \cdot y_{T_s-1}}\right)\right] > p_s \cdot q(\omega(1)) = r$$

Suppose $\frac{p_r \cdot D(p_r)}{r} \leq y_{T_s-1}$. Since $q\left[\omega\left(\frac{p_s \cdot D(p_s)}{r \cdot y_{T_s-1}}\right)\right] \cdot y_{T_s-1}$ is strictly increasing in y_{T_s-1}, it follows that

$q\left[\omega\left(\frac{p_s\cdot D(p_s)}{r\cdot y_{T_s-1}}\right)\right]\cdot y_{T_s-1} < q\left[\omega\left(1\right)\right]\cdot\frac{p_s\cdot D(p_s)}{r} = D\left(p_s\right)$ for all $y_{T_s-1} < \frac{p_s\cdot D(p_s)}{r}$. Hence, by assumption AD, there exists a unique p_{T_s-1} such that $q\left[\omega\left(\frac{p_s\cdot D(p_s)}{r\cdot y_{T_s-1}}\right)\right]\cdot y_{T_s-1} = D\left(p_{T_s-1}\right)$. In addition, $p_{T_s-1} > p_s \geq p_r$. Therefore,

$$p_{T_s-1}\cdot q\left[\omega^H\left(\widehat{\delta}\cdot p_{T_s-1}\right)\right] = p_{T_s-1}\cdot q\left[\omega\left(\frac{p_s\cdot D(p_s)}{r\cdot y_{T_s-1}}\right)\right] > p_s\cdot q\left(\omega\left(1\right)\right) = r$$

and $p_{T_s-1}\cdot D\left(p_{T_s-1}\right) = p_{T_s-1}\cdot q\left[\omega\left(\frac{p_s\cdot D(p_s)}{r\cdot y_{T_s-1}}\right)\right]\cdot y_{T_s-1} > p_r\cdot D\left(p_r\right) \geq \frac{p_s\cdot D(p_s)}{r}$, where the last inequality holds because $p_r \geq \frac{p_s}{r}$ and $D\left(p_r\right) > D\left(p_s\right)$. Hence, $\widehat{\delta} = \frac{p_s\cdot D(p_s)}{r\cdot p_{T_s-1}\cdot D(p_{T_s-1})} \in [0,1]$.

It follows that $P\in\Sigma$ and $p_{T_s-1}\cdot q\left[\omega^H\left(\widehat{\delta}\cdot p_{T_s-1}\right)\right] > r$. Let $\left(f^H,\sigma^H\right)\equiv\left[f\left(T_s,\widehat{\delta};\alpha,P\right),\sigma\left(T_s,\widehat{\delta};\alpha,P\right)\right]$. By Proposition 4.4, $\left(f^H,\sigma^H\right)$ is a *SPNE of $\Gamma\left(P,V^*,\alpha\right)$ in which $\overrightarrow{e}_t = 0$ and the value of an externally trained worker is v^*. Finally, I shall show that the output market clears at every date $t\geq 0$. Since $\overrightarrow{m}_0 = a_0 = y_0$ and $\overrightarrow{m}_t = p_{t-1}\cdot q\left(\omega^H\left(p_{t-1}\right)\right)\cdot a_{t-1} = \mathbb{P}\left(a_{t-1}\right)\cdot D\left(\mathbb{P}\left(a_{t-1}\right)\right)$, it follows that $\overrightarrow{m}_t = y_t$ for all $0\leq t\leq T_s-1$. Hence, for all $0\leq t < T_s-1$

$$q_t^H = q\left(\omega^H\left(p_t\right)\right)\cdot\overrightarrow{m}_t = q\left(\omega^H\left(p_t\right)\right)\cdot y_t = D\left(p_t\right)$$

and the output market clears. At date T_s-1,

$$q_{T_s-1}^H = q\left[\omega^H\left(\widehat{\delta}\cdot p_{T_s-1}\right)\right]\cdot\overrightarrow{m}_{T_s-1} = q\left[\omega\left(\frac{p_s\cdot D(p_s)}{r\cdot\overrightarrow{m}_{T_s-1}}\right)\right]\cdot\overrightarrow{m}_{T_s-1} = q\left[\omega\left(\frac{p_s\cdot D(p_s)}{r\cdot y_{T_s-1}}\right)\right]\cdot y_{T_s-1} = D\left(p_{T_s-1}\right)$$

Finally, at any date $t\geq T_s$,

$$q_t^H = q\left(\omega\left(1\right)\right)\cdot\overrightarrow{m}_{T_s} = q\left(\omega\left(1\right)\right)\cdot\frac{p_s\cdot D(p_s)}{r} = D\left(p_s\right)$$

as desired.

UNIQUENESS: Now I show that the equilibrium I found is the unique one in which every firm follows a strategy $\left[f\left(\widetilde{T},\widetilde{\delta};\alpha,\widetilde{P}\right),\sigma\left(\widetilde{T},\widetilde{\delta};\alpha,\widetilde{P}\right)\right]$ for some $\widetilde{T}\geq\widetilde{T}_s$ and $\widetilde{\delta}\in[0,1]$. Suppose there exists another IE $\left\{\widetilde{P},V^*,\left(\widetilde{f},\widetilde{\sigma}\right)\right\}$ where $\left(\widetilde{f},\widetilde{\sigma}\right)=\left[f\left(\widetilde{T},\widetilde{\delta};\alpha,\widetilde{P}\right),\sigma\left(\widetilde{T},\widetilde{\delta};\alpha,\widetilde{P}\right)\right]$ for some $\widetilde{\delta}\in[0,1]$ and $\widetilde{T}\geq\widetilde{T}_s$
First I show that $\widetilde{P}=P$. Suppose $\widetilde{T}>T$. Clearly, $\left(\widetilde{f}_t,\widetilde{\sigma}_t\right)=\left(f_t,\sigma_t\right)$ for all $t\leq T-2$ and $\overrightarrow{\widetilde{m}}_{T-1} = \overrightarrow{m}_{T-1} < \frac{p_sD(p_s)}{r}$. It follows that $\widetilde{p}_t = p_t$ for all $t\leq T-2$. Since $\widetilde{p}_t = p_s = p_t$ for all $t\geq\widetilde{T}$ then $\widetilde{P}\neq P$ if and only if there exists $T-1\leq t\leq\widetilde{T}-1$ such that $\widetilde{p}_t\neq p_s$ and $\widetilde{p}_t\geq p_r$. From the construction of the price equilibrium sequence P, it follows that either $\mathbb{P}\left(\overrightarrow{\widetilde{m}}_{T-1}\right)\cdot D\left(\mathbb{P}\left(\overrightarrow{\widetilde{m}}_{T-1}\right)\right)\geq\frac{p_sD(p_s)}{r}$ or $\overrightarrow{\widetilde{m}}_{T-1}\geq\frac{p_rD(p_r)}{r}$. Since $\widetilde{p}_{T-1}\geq p_r$, then $\overrightarrow{\widetilde{m}}_{T-1}\leq\frac{p_rD(p_r)}{r}$. Hence, $\overrightarrow{\widetilde{m}}_{T-1}=\frac{p_rD(p_r)}{r}$ and $\mathbb{P}\left(\overrightarrow{\widetilde{m}}_{T-1}\right)=p_r$. Since $p_r\cdot D\left(p_r\right)\geq\frac{p_sD(p_s)}{r}$ then it is always the case that $\mathbb{P}\left(\overrightarrow{\widetilde{m}}_{T-1}\right)\cdot D\left(\mathbb{P}\left(\overrightarrow{\widetilde{m}}_{T-1}\right)\right)\geq\frac{p_sD(p_s)}{r}$. Therefore, $\overrightarrow{\widetilde{m}}_t\geq\frac{p_sD(p_s)}{r}$ for all $t\geq T$. If $\overrightarrow{\widetilde{m}}_{\widetilde{T}-1}>\frac{p_sD(p_s)}{r}$, then

$$D\left(\widetilde{p}_{\widetilde{T}-1}\right) = q\left(\overrightarrow{\widetilde{w}}_{\widetilde{T}-1}\right)\cdot\overrightarrow{\widetilde{m}}_{\widetilde{T}-1} = q\left(\frac{p_sD(p_s)}{r\cdot\overrightarrow{\widetilde{m}}_{\widetilde{T}-1}}\right)\cdot\overrightarrow{\widetilde{m}}_{\widetilde{T}-1} > q\left[\omega\left(1\right)\right]\cdot\frac{p_sD(p_s)}{r} = D\left(p_s\right)$$

Then $\widetilde{p}_{\widetilde{T}-1} < p_s$ and $\widetilde{p}_{\widetilde{T}-1} \cdot q\left(\overrightarrow{\widetilde{w}}_{\widetilde{T}-1}\right) < p_s \cdot q\left[\omega\left(1\right)\right] = r$. Therefore, $\left(\widetilde{f}, \widetilde{\sigma}\right)$ is not a *SPNE of $\Gamma\left(\widetilde{P}, V^*, \alpha\right)$, a contradiction. If $\overrightarrow{\widetilde{m}}_{\widetilde{T}-1} = \frac{p_s D(p_s)}{r}$, instead, then $\overrightarrow{\widetilde{m}}_t = \frac{p_s D(p_s)}{r}$ for all $t \geq T$. Therefore, $\overrightarrow{\widetilde{w}}_t = \omega\left(1\right)$ which implies that $\widetilde{p}_t = p_s = p_t$ for all $t \geq T$. Hence $\widetilde{P} = P$. Suppose $\widetilde{T} < T$. Clearly, $\left(\widetilde{f}_t, \widetilde{\sigma}_t\right) = \left(f_t, \sigma_t\right)$ for all $t \leq \widetilde{T} - 2$ and $\overrightarrow{\widetilde{m}}_{\widetilde{T}-1} = \overrightarrow{m}_{\widetilde{T}-1} < \frac{p_s D(p_s)}{r}$. Since $\widetilde{\delta} = \frac{p_s D(p_s)}{r \cdot \widetilde{p}_{\widetilde{T}-1} \cdot D\left(\widetilde{p}_{\widetilde{T}-1}\right)}$ and $\widetilde{\delta} \leq 1$, then

$$\widetilde{p}_{\widetilde{T}-1} \cdot q\left(\overrightarrow{\widetilde{w}}_{\widetilde{T}-1}\right) = \widetilde{p}_{\widetilde{T}-1} \cdot q\left(\omega^H\left(\widetilde{\delta} \cdot \widetilde{p}_{\widetilde{T}-1}\right)\right) = \widetilde{p}_{\widetilde{T}-1} \cdot D\left(\widetilde{p}_{\widetilde{T}-1}\right) \geq \frac{p_s D(p_s)}{r}$$

It follows that

$$\mathbb{P}\left(\overrightarrow{\widetilde{m}}_{\widetilde{T}-1}\right) \cdot D\left(\mathbb{P}\left(\overrightarrow{\widetilde{m}}_{\widetilde{T}-1}\right)\right) \geq \widetilde{p}_{\widetilde{T}-1} \cdot q\left(\omega^H\left(\widetilde{\delta} \cdot \widetilde{p}_{\widetilde{T}-1}\right)\right) \geq \frac{p_s D(p_s)}{r} > \mathbb{P}\left(\overrightarrow{m}_{\widetilde{T}-1}\right) \cdot D\left(\mathbb{P}\left(\overrightarrow{m}_{\widetilde{T}-1}\right)\right)$$

a contradiction since $\overrightarrow{\widetilde{m}}_{\widetilde{T}-1} = \overrightarrow{m}_{\widetilde{T}-1}$. Thus, $\left(\widetilde{f}_t, \widetilde{\sigma}_t\right) = \left[f_t\left(T, \delta; \alpha, P\right), \sigma_t\left(T, \delta; \alpha, P\right)\right]$ for all $t \geq 0$. ■

Proof of Proposition 5.3: Let $\left\{P, V^*, \left(f^H, \sigma^H\right), \left(f^L, \sigma^L\right)\right\}$ be an IE in which $P \in \Sigma$ and $T_s > 1$. Suppose $w_0^L \in \left(\underline{w}, \overline{w}\right)$. Since $w_0^L > \omega^H\left(p_0\right)$, it follows that $G\left(w_0^L\right) < p_0 \cdot q\left(w_0^L\right)$, i.e strategy L does not reinvest all earnings at date 1. Hence, it must be the case that $p_1 \cdot q\left(\overrightarrow{w}_1^L\right) = r$ or, equivalently, $\overrightarrow{w}_1^L = w^L\left(p_1\right) < \omega^H\left(p_1\right) = \overrightarrow{w}_1^H$. By proposition 4.6, it follows that $\left(f^L, \sigma^L\right) = \left[f\left(w_0^L, a_0; \alpha, P\right), \sigma\left(w_0^L; \alpha, P\right)\right]$ and $\overrightarrow{w}_t^L = w^L\left(p_t\right) < \omega^H\left(p_t\right) = \overrightarrow{w}_t^H$ for all $1 \leq t \leq T - 1$. ■

Proof of Proposition 5.4: Suppose $M\left(a_0^L, a_0^H, w_0^L\right) \geq \frac{p_s \cdot D(p_s)}{r}$, $w_0^L \in \left(\omega\left(\frac{p_s \cdot D(p_s)}{r \cdot a_0}\right), \overline{w}\right)$ and p_0 solves (13). Let $P = \left\{p_0, p_s, p_s, ...\right\}$. To show that $\left(f^H, \sigma^H\right) = \left[f\left(1, \widehat{\delta}_0; \alpha, P\right), \sigma\left(1, \widehat{\delta}_0; \alpha, P\right)\right]$ is a *SPNE of $\Gamma\left(P, V^*, \alpha\right)$, it suffices to prove that $\widehat{\delta}_0 \in [0, 1]$ and $p_0 \cdot q\left(w_0^H\right) > r$. First I show that $\widehat{\delta}_0 \in [0, 1]$. Since $w_0^L > \omega\left(\frac{p_s \cdot D(p_s)}{r \cdot a_0}\right) > \omega\left(\frac{p_s \cdot D(p_s)}{r \cdot a_0^L}\right)$ then $G\left(w_0^L\right) < \frac{p_s \cdot D(p_s)}{r \cdot a_0^L}$. It follows that $\widehat{\delta} > 0$. Notice that $\widehat{\delta} \leq 1$ iff $\frac{p_s D(p_s) - r G(w_0^L) a_0^L}{r a_0^H} \leq p_0 \cdot q\left(w_0^H\right)$. Suppose $\widehat{\delta} > 1$. Then $\frac{p_s D(p_s)}{r} - r \cdot G\left(w_0^L\right) \cdot a_0^L > p_0 \cdot q\left(w_0^H\right) \cdot a_0^H$. Therefore,

$$p_0 \cdot q\left(w_0^H\right) \cdot a_0^H + r \cdot G\left(w_0^L\right) \cdot a_0^L < M\left(a_0^L, a_0^H, w_0^L\right) = \widetilde{\mathbb{P}}\left(\cdot\right) \cdot q\left(w^H\left(\widetilde{\mathbb{P}}\left(\cdot\right)\right)\right) \cdot a_0^H + r \cdot G\left(w_0^L\right) \cdot a_0^L$$

and since $w_0^H = \omega^H\left(\widehat{\delta} \cdot p_0\right) < \omega^H\left(p_0\right) \Rightarrow \omega^H\left(\widetilde{\mathbb{P}}\left(a_0\right)\right) < \omega^H\left(p_0\right)$. Hence, $p_0 < \widetilde{\mathbb{P}}\left(\cdot\right)$. Since $M\left(a_0^L, a_0^H, w_0^L\right) \geq \frac{p_s \cdot D(p_s)}{r}$ then $\frac{p_s \cdot D(p_s) - r G(w_0^L) a_0^L}{r \cdot a_0^H} \leq \widetilde{\mathbb{P}}\left(\cdot\right) \cdot q\left(\omega^H\left(\widetilde{\mathbb{P}}\left(\cdot\right)\right)\right)$. It follows that $w_0^H = \omega\left(\frac{p_s \cdot D(p_s) - r G(w_0^L) a_0^L}{r \cdot a_0^H}\right) \geq \omega^H\left(\widetilde{\mathbb{P}}\left(\cdot\right)\right)$ and

$$q\left(\omega\left(\frac{p_s D(p_s) - r G(w_0^L) a_0^L}{r a_0^H}\right)\right) a_0^H + q\left(w_0^L\right) a_0^L \leq q\left(\omega^H\left(\widetilde{\mathbb{P}}\left(\cdot\right)\right)\right) a_0^H + q\left(w_0^L\right) a_0^L < D\left(p\right)$$

for all $p < \widetilde{\mathbb{P}}\left(\cdot\right)$. Hence $p_0 \geq \widetilde{\mathbb{P}}\left(a_0\right)$, a contradiction. It follows that $\widehat{\delta} \in [0, 1]$ and $w_0^H \geq \omega^H\left(p_0\right) = \underline{w}$. Since $w_0^H < w_0^L \leq \omega^L\left(p_0\right)$ it follows that $p_0 \cdot q\left(w_0^H\right) > r$. By Proposition 4.4, $\left(f^H, \sigma^H\right)$ is a *SPNE of $\Gamma\left(P, V^*, \alpha\right)$

where $\overrightarrow{e_t} = 0$ and $\frac{\partial \Pi\left(f^H, \sigma^H; \alpha \middle| \overline{h^t}\right)}{\partial e_t} = 0$ for all $t \geq 0$.

By Proposition 4.5, $w_0^L \in \left(w_0^H, \overline{w}\right) \subset \left(\underline{w}, \overline{w}\right)$ implies that $\left(f^L, \sigma^L\right)$ is a *SPNE of $\Gamma\left(P, V^*, \alpha\right)$ where $\overrightarrow{e_t} = 0$ and $\frac{\partial \Pi\left(f^L, \sigma^L; \alpha \middle| \overline{h^t}\right)}{\partial e_t} = 0$ for all $t \geq 0$. Therefore $\left\{P, V^*, \left(f^H, \sigma^H\right), \left(f^L, \sigma^L\right)\right\}$ is an IE if and only if the output market clears at every date t. Since p_0 solves (13) and $\overrightarrow{m}_0^i = a_0^i$ for $i \in \{L, H\}$, the output market clears at date zero by construction. At date $t \geq 1$, $\overrightarrow{m}_t^H + \overrightarrow{m}_t^L = \overrightarrow{m}_1^H + \overrightarrow{m}_1^L = \frac{p_s D(p_s)}{r}$ and $q_t^i = \frac{r}{p_s} \cdot \overrightarrow{m}_t^i$ for $i \in \{L, H\}$. Hence, $q_t^H + q_t^L = \frac{r}{p_s} \cdot \left(\overrightarrow{m}_t^H + \overrightarrow{m}_t^L\right) = D\left(p_s\right)$ and the market clears at $t \geq 1$. It follows that $\left\{P, V^*, \left(f^H, \sigma^H\right), \left(f^L, \sigma^L\right)\right\}$ is an IE.

Now suppose $\left\{P, V^*, \left(f^H, \sigma^H\right), \left(f^L, \sigma^L\right)\right\}$ is an IE. Since $T = 1$, $G\left(\overrightarrow{w}_0^H\right) = \frac{p_s \cdot D(p_s) - rG(w_0^L) a_0^L}{r \cdot a_0^H}$. It follows that $\overrightarrow{w}_0^H = \omega\left(\frac{p_s \cdot D(p_s) - rG(w_0^L) a_0^L}{r \cdot a_0^H}\right)$ and p_0 satisfies (13) by market clearing. Since $\overrightarrow{m}_1 = G\left(\overrightarrow{w}_0^L\right) \cdot a_0^L \geq 0$, $w_0^L \leq \overline{w}_1$. If $p_0 \cdot q\left(w_0^L\right) < r$, then firm L would deviate and set $\overrightarrow{m}_1$ It follows that $w_0^L \geq \omega^L\left(p_0\right)$. Hence, $w_0^L \leq \overline{w}$. Suppose $w_0^L \leq \omega\left(\frac{p_s D(p_s)}{r a_0}\right)$. Then $G\left(w_0^L\right) \geq \frac{p_s D(p_s)}{r a_0}$. Since $G\left(\overrightarrow{w}_0^H\right) \cdot a_0^H = \frac{p_s D(p_s)}{r} - G\left(w_0^L\right) \cdot a_0^L \leq \frac{p_s D(p_s)}{r a_0} \cdot a_0^H$ then $\overrightarrow{w}_0^H \geq \omega\left(\frac{p_s D(p_s)}{r a_0}\right) \geq w_0^L$, a contradiction since I assumed that firm L displays worst prospects for advancement than H at date zero. Finally, suppose $M\left(a_0^L, a_0^H, w_0^L\right) < \frac{p_s \cdot D(p_s)}{r}$. Then $G\left(\overrightarrow{w}_0^H\right) = \frac{p_s \cdot D(p_s) - rG(w_0^L) a_0^L}{r \cdot a_0^H} > \widetilde{\mathbb{P}}\left(\cdot\right) \cdot q\left(\omega^H\left(\widetilde{\mathbb{P}}\left(\cdot\right)\right)\right) = G\left[\omega^H\left(\widetilde{\mathbb{P}}\left(\cdot\right)\right)\right]$ which implies that $\overrightarrow{w}_0^H < \omega^H\left(\widetilde{\mathbb{P}}\left(\cdot\right)\right)$. Hence, $p_0 < \widetilde{\mathbb{P}}\left(\cdot\right)$ because $D\left(p_0\right) = q\left(\overrightarrow{w}_0^H\right) \cdot a_0^H + q\left(w_0^L\right) \cdot a_0^L > D\left(\widetilde{\mathbb{P}}\left(a_0\right)\right)$. Therefore, $\omega^H\left(\widehat{\delta} \cdot p_0\right) = \overrightarrow{w}_0^H < \omega^H\left(p_0\right)$ which implies that $\widehat{\delta} > 1$. Then $\overrightarrow{m}_1 > p_0 \cdot q\left(\overrightarrow{w}_0^H\right) \cdot a_0^H = \overrightarrow{a}_1^H$, a contradiction. Thus, $M\left(a_0^L, a_0^H, w_0^L\right) \geq \frac{p_s \cdot D(p_s)}{r}$ as desired. ■

Proof of Proposition 5.5: Let $\left(m^H, m^L\right) \in \Re_+^2$ such that $m^H + m^L < \frac{p_s \cdot D(p_s)}{r}$. First, I prove that there exists a unique market clearing price $\widetilde{\mathbb{P}}\left(m^H, m^L\right)$. Consider the function $\widetilde{H}\left(p\right) = p \cdot D\left(p\right) - p \cdot q\left(\omega^H\left(p\right)\right) \cdot m^H - r \cdot m^L$. I shall show that there is a unique $p \geq p_r$ such that $\widetilde{H}\left(p\right) = 0$. Since $p \cdot D\left(p\right)$ is concave by hypothesis, to show that $\widetilde{H}\left(p\right)$ is strictly concave it suffices to show that $p \cdot q\left(\omega^H\left(p\right)\right)$ is strictly convex. Notice that $p \cdot q\left(\omega^H\left(p\right)\right) = \frac{\overline{w}_1}{A \cdot \omega^H(p)} - 1$, where $A \equiv \beta \cdot \frac{1-\alpha}{v^* + c} \cdot \left(\frac{1-\alpha}{\alpha} \frac{1}{v^* + c}\right)^{1-\alpha} \cdot \left(v^* - \overline{w}_2\right) > 0$. Hence,

$$\begin{aligned} \frac{\partial^2 p \cdot q(\omega^H(p))}{\partial p^2} &= -\frac{\overline{w}_1}{A \cdot \omega^H(p)^4} \cdot \left[\frac{\partial^2 \omega^H(p)}{\partial p^2} \cdot \omega^H\left(p\right)^2 - 2 \cdot \left(\frac{\partial \omega^H(p)}{\partial p}\right)^2 \cdot \omega^H\left(p\right)\right] \\ &= \frac{\overline{w}_1}{A \cdot \omega^H(p)^3} \cdot \left(\frac{\partial \omega^H(p)}{\partial p}\right)^2 \cdot \left[1 + \alpha + (1 - \alpha) \cdot \frac{\partial \omega^H(p)}{\partial p} \cdot \frac{p}{\omega^H\left(p\right)}\right] \end{aligned}$$

where in the second line I use the fact that

$$\frac{\partial^2 \omega^H(p)}{\partial p^2} \cdot \omega^H\left(p\right)^2 = \left(\frac{\partial \omega^H(p)}{\partial p}\right)^2 \cdot \left(1 - \alpha\right) \cdot \left[\omega^H\left(p\right) - \frac{\partial \omega^H(p)}{\partial p} \cdot p\right]$$

Therefore, $\frac{\partial^2 p \cdot q(\omega^H(p))}{\partial p^2} > 0$ because

$$\frac{\partial \omega^H(p)}{\partial p} = -\frac{A \cdot \omega^H\left(p\right)}{\omega^H\left(p\right) + \left(1 - \alpha\right) \cdot A \cdot P} \Leftrightarrow \left(1 - \alpha\right) \cdot \frac{\partial \omega^H(p)}{\partial p} \cdot \frac{p}{\omega^H\left(p\right)} > -1$$

It follows that $\widetilde{H}(p)$ is strictly concave. Notice that $\widetilde{H}(0) < 0$ and

$$\begin{aligned} \widetilde{H}(p_r) &= p_r \cdot \left[D(p_r) - q\left(\omega^H(p_r)\right) \cdot \left(m^H + m^L\right)\right] \\ &> p_r \cdot \left[D\left[\mathbb{P}\left(m^H + m^L\right)\right] - q\left[\mathbb{P}\left(m^H + m^L\right)\right] \cdot \left(m^H + m^L\right)\right] = 0 \end{aligned}$$

Hence, there exists $p' \in (0, p_r)$ such that $\widetilde{H}(p') = 0$. Since $\widetilde{H}\left[\mathbb{P}\left(m^H + m^L\right)\right] > 0$ and $\lim_{p \to \infty} p \cdot D(p) = 0$, there exists $p'' > \mathbb{P}\left(m^H + m^L\right)$ such that $\widetilde{H}(p'') = 0$. Clearly, $\widetilde{H}'(p'') < 0$. For any $p \in (p', p'')$, $\widetilde{H}(p) > 0$ by the strict concavity of $\widetilde{H}(p)$. For any $p > p''$, $\widetilde{H}(p) \leq \widetilde{H}'(p'') \cdot (p - p') < 0$. It follows that p'' is the unique market clearing price that exceeds p_r. Hence, $\widetilde{\mathbb{P}}\left(m^H, m^L\right)$ is uniquely defined for all $\left(m^H, m^L\right) \in \Re^2_+$ such that $m^H + m^L < \frac{p_s \cdot D(p_s)}{r}$ and $\{\rho_t\}_{t=0}^{\infty}$ and P are well defined.

Next, I show that $\left(f^H, \sigma^H\right) = \left[f\left(T, \widehat{\delta}; \alpha, P\right), \sigma\left(T, \widehat{\delta}; \alpha, P\right)\right]$ is a *SPNE of the game $\Gamma(P, V^*, \alpha)$. By proposition 4.4, it suffices to show that $\widehat{\delta} \in [0, 1]$. Notice that

$$\widehat{\delta} \leq 1 \Leftrightarrow x^L_{T-1} \leq \frac{r \cdot p_{T-1} \cdot D(p_{T-1}) - p_s \cdot D(p_s)}{r \cdot \left(r - G\left(\omega^L(p_{T-1})\right)\right)}$$

so it holds by definition of date T. To get a contradiction, assume $\widehat{\delta} < 0$. Then $p_s \cdot D(p_s) - rG\left(w^L(p_{T-1})\right) \cdot x^L_{T-1} \leq 0$. Hence, $x^L_{T-1} \geq \frac{p_s \cdot D(p_s)}{r \cdot G(w^L(p_{T-1}))} \geq \frac{p_s \cdot D(p_s)}{r^2}$ since $p_{T-1} > p_r$. However, $x^L_{T-1} \geq \frac{p_s \cdot D(p_s)}{r^2}$ implies that

$$\frac{x^L_{T-1}}{x^H_{T-1} + x^L_{T-1}} \geq \frac{1}{r} \Leftrightarrow \frac{G(\omega^L(p_{T-1})) \cdot \ldots \cdot G(w^L_0) \cdot \mu^L_0 \cdot a_0}{R_{T-2} \cdot \ldots \cdot R_0 \cdot \mu^H \cdot a_0 + G(\omega^L(p_{T-1})) \cdot \ldots \cdot G(w^L_0) \cdot \mu^L_0 \cdot a_0} \geq \frac{1}{r}$$

and since $R_t \geq G\left(\omega^L(p_t)\right)$ for all $0 \leq t \leq T-2$, the inequality on the right hand side implies that $\mu^L \geq \frac{1}{r}$, a contradiction. It follows that $\left(f^H, \sigma^H\right)$ is a *SPNE of the game $\Gamma(P, V^*, \alpha)$ in which $\overrightarrow{e}_t = 0$ and $\frac{\partial \Pi\left(f^H, \sigma^H | \overrightarrow{h^t}\right)}{\partial e_t} = 0$ for all $t \geq 0$. Since $w^L_0 \in (\underline{w}, \overline{w})$ and $p_t \leq p^{**}$ for all $t \geq 1$ it follows by Proposition 4.5 that $\left(f^L, \sigma^L\right) = \left[f\left(w^L_0; \alpha, P\right), \sigma\left(w^L_0; \alpha, P\right)\right]$ is a *SPNE of $\Gamma(P, V^*, \alpha)$ in which $\overrightarrow{e}_t = 0$ and $\frac{\partial \Pi\left(f^L, \sigma^L | \overrightarrow{h^t}\right)}{\partial e_t} = 0$ for all $t \geq 0$. To show that $\left\{P, V^*, \left(f^H, \sigma^H\right), \left(f^L, \sigma^L\right)\right\}$ is an IE it suffices to show that the output market clears on the equilibrium path. Since $\overrightarrow{m}^H_t = x^H_t$ and $\overrightarrow{m}^L_t = x^L_t$ then $p_0 = \widetilde{\mathbb{P}}\left(a^H_0, a^L_0, w^L_0\right)$, $p_t = \widetilde{\mathbb{P}}\left(\overrightarrow{m}^H_t, \overrightarrow{m}^L_t\right)$ for all $1 \leq t \leq T-2$, p_{T-1} that solves (11) and p_s for all $t \geq T$ are the market clearing prices if a fraction μ^H of the firms follows $\left[f\left(T, \widehat{\delta}; \alpha, P\right), \sigma\left(T, \widehat{\delta}; \alpha, P\right)\right]$ and the rest follows $\left[f\left(w^L_0; \alpha, P\right), \sigma\left(w^L_0; \alpha, P\right)\right]$. ■

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