Graph classes with linear Ramsey numbers*

Bogdan Alecu[†] Aistis Atminas[‡] Vadim Lozin[§] Viktor Zamaraev[¶]

Abstract

The Ramsey number $R_X(p, q)$ for a class of graphs X is the minimum n such that every graph in X with at least n vertices has either a clique of size p or an independent set of size q. We say that Ramsey numbers are *linear* in X if there is a constant k such that $R_X(p, q) \leq k(p + q)$ for all p, q. In the present paper we conjecture that if X is a hereditary class defined by finitely many forbidden induced subgraphs, then Ramsey numbers are linear in X if and only if X excludes a forest, a disjoint union of cliques and their complements. We prove the "only if" part of this conjecture and verify the "if" part for a variety of classes. We also apply the notion of linearity to bipartite Ramsey numbers and reveal a number of similarities and differences between the bipartite and non-bipartite case.

1 Introduction

According to Ramsey's Theorem [20], for all natural p and q there exists a minimum number $R(p, q)$ such that every graph with at least $R(p, q)$ vertices has either a clique of size p or an independent set of size q.

The exact values of Ramsey numbers are known only for small values of p and q. However, with the restriction to specific classes of graphs, Ramsey numbers can be determined for all p and q. In particular, in [23] this problem was solved for planar graphs, while in [5] it was solved for line graphs, bipartite graphs, perfect graphs, P_4-free graphs and some other classes.

These studies reveal, in particular, that different classes have different rates of growth of Ramsey numbers. In the present paper, we denote the Ramsey numbers restricted to a class X by $R_X(p, q)$ and focus on classes with a smallest speed of growth of $R_X(p, q)$. Clearly, $R_X(p, q)$ cannot be smaller than the minimum of p and q. We say that Ramsey numbers are *linear* in X if there is a constant k such that $R_X(p, q) \leq k(p + q)$ for all p, q.

All classes in this paper are *hereditary*, i.e., closed under taking induced subgraphs. It is well known that a class of graphs is hereditary if and only if it can be characterized in terms of minimal forbidden induced subgraphs. If the number of minimal forbidden induced subgraphs for a class X is finite, we say that X is *finitely defined*.

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†Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK. B.Alecu@warwick.ac.uk

‡Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, 111 Ren'ai Road, Suzhou 215123, China. Email: Aistis.Atminas@xjtlu.edu.cn

§Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK. V.Lozin@warwick.ac.uk

¶Department of Computer Science, University of Liverpool, Ashton Building, Ashton Street, Liverpool, L69 3BX, UK. Email: Viktor.Zamaraev@liverpool.ac.uk

It is not difficult to see that all classes of bounded co-chromatic number have linear Ramsey numbers, where the co-chromatic number of a graph G is the minimum k such that the vertex set of G can be partitioned into k subsets each of which is either a clique or an independent set. Unfortunately, as we show in Section 2, this is not an if and only if statement in general. We conjecture, however, that in the universe of finitely defined classes the two notions coincide.

Conjecture 1. A finitely defined hereditary class is of linear Ramsey numbers if and only if it is of bounded co-chromatic number.

In [7], it was conjectured that a finitely defined class X has bounded co-chromatic number if and only if the set of minimal forbidden induced subgraphs for X contains a P_3-free graph, the complement of a P_3-free graph, a forest (i.e., a graph without cycles) and the complement of a forest. The authors of [7] go on to show that their conjecture is equivalent to the older Gyárfás-Sumner conjecture [14, 24]. Naturally, if this conjecture is true, we expect, following Conjecture 1:

Conjecture 2. A finitely defined class X is of linear Ramsey numbers if and only if the set of minimal forbidden induced subgraphs for X contains a P_3-free graph, the complement of a P_3-free graph, a forest and the complement of a forest.

In Section 3, we prove the "only if" part of Conjecture 2. In other words, in the universe of finitely defined classes, the property of a class X having linear Ramsey numbers lies in between that of X having bounded co-chromatic number and that of X avoiding the specified induced subgraphs.

In Section 4, we focus on the "if" part of Conjecture 2 and verify it for a variety of classes defined by small forbidden induced subgraphs. Moreover, for all the considered classes we derive exact values of the Ramsey numbers.

In Section 5, we extend the notion of linearity to bipartite Ramsey numbers and show that some of the results obtained for non-bipartite numbers can be extended to the bipartite case as well. However, in general, the situation with linear bipartite Ramsey numbers seems to be more complicated and we restrict ourselves to a weaker analog of Conjecture 2, which is also verified for some classes of bipartite graphs. In the rest of the present section, we introduce basic terminology and notation.

All graphs in this paper are finite, undirected, without loops and multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $x \in V(G)$ we denote by $N(x)$ the neighbourhood of x, i.e., the set of vertices of G adjacent to x. The degree of x is $|N(x)|$. We say that x is *complete* to a subset $U \subset V(G)$ if $U \subseteq N(x)$ and *anticomplete* to U if $U \cap N(x) = \emptyset$. A subgraph of G induced by a subset of vertices $U \subseteq V(G)$ is denoted $G[U]$. By $\overline{G}$ we denote the complement of G and call it co-G.

A *clique* in a graph is a subset of pairwise adjacent vertices and an *independent set* is a subset of pairwise non-adjacent vertices. For a graph G, let $\alpha(G)$ denote the independence number of G, $\omega(G)$ the clique number, $\chi(G)$ the chromatic number and $z(G)$ the co-chromatic number.

By K_n, C_n and P_n we denote a complete graph, a chordless cycle and a chordless path with n vertices, respectively. Also, $K_{n,m}$ is a complete bipartite graph with parts of size n and m, and $K_{1,n}$ is a star. A disjoint union of two graphs G and H is denoted $G + H$. In particular, pG is a disjoint union of p copies of G.

If a graph G does not contain induced subgraphs isomorphic to a graph H, then we say that G is H-free and call H a forbidden induced subgraph for G. In case of several forbidden induced subgraphs we list them in parentheses.

A *bipartite graph* is a graph whose vertices can be partitioned into two independent sets, and a *split graph* is a graph whose vertices can be partitioned into an independent set and a clique. A graph is bipartite if and only if it is free of odd cycles, and a graph is a split graph if and only if it is $(C_4, 2K_2, C_5)$-free [12].

2 Linear Ramsey numbers and related notions

As we observed in the introduction, the notion of linear Ramsey numbers has ties with bounded co-chromatic number, and we believe that in the universe of finitely defined classes, the two notions are equivalent. In the present section, we first show that this equivalence is not valid for general hereditary classes, and then discuss the relationship between linear Ramsey numbers and some other notions that appear in the literature.

In order to show that Conjecture 1 is not valid for general hereditary classes, we consider the Kneser graph $KG_{a,b}$: it has as vertices the b-subsets of a set of size a, and two vertices are adjacent if and only if the corresponding subsets are disjoint. A well-known result due to Lovász says that, if $a \geq 2b$, then the chromatic number $\chi(KG_{a,b})$ is $a - 2b + 2$ [16].

In the following theorem, we denote by X the hereditary closure of the family of Kneser graphs $KG_{3n,n}, n \in \mathbb{N}$, i.e., $X = \{H : H$ is an induced subgraph of $KG_{3n,n}$, for some $n \in \mathbb{N}\}$.

Theorem 1. *The class X has linear Ramsey numbers and unbounded co-chromatic number.*

Proof. First, we note that by Lovász's result stated above, it follows that $\chi(KG_{3n,n}) = 3n - 2n + 2 = n + 2$. Also, it is not hard to see that the the size of the biggest clique in $KG_{3n,n}$ is 3. It follows that the co-chromatic number of $KG_{3n,n}$ is at least $\frac{n+2}{3}$. As a result, the co-chromatic number is unbounded for this class.

Now consider any induced subgraph H of $KG_{3n,n}$. We will show that $\alpha(H) \geq \frac{|V(H)|}{3}$. Indeed, the vertices of the Kneser graph in this case are n-element subsets of $\{1, 2, \ldots, 3n\}$. For each $i \in \{1, 2, \ldots, 3n\}$ let V_i be the set of vertices of H containing element i. Then, as each vertex is an n-element subset, it follows that $\sum_{n=1}^{3n} |V_i| = n \times |V(H)|$. Hence, by the Pigeonhole Principle, there is an i such that $|V_i| \geq \frac{|V(H)|}{3}$. As V_i is an independent set, it follows that $\alpha(H) \geq \frac{|V(H)|}{3}$. This implies that for any $H \in X$ we have $|V(H)| \leq 3\alpha(H) \leq 3(\alpha(H) + \omega(H))$, and hence the Ramsey numbers are linear in the class X. $\square$

We now turn to one more notion, which is closely related to the growth of Ramsey numbers. This is the notion of homogeneous subgraphs that appears in the study of the Erdős-Hajnal conjecture [11]. We will say that graphs in a class X have *linear homogeneous subgraphs* if there exists a constant $c = c(X)$ such that $\max\{\alpha(G), \omega(G)\} \geq c \cdot |V(G)|$ for every $G \in X$.

Proposition 1. *Let X be a class of graphs. Then graphs in X have linear homogeneous subgraphs if and only if Ramsey numbers are linear in X. More generally, for any $0 < \delta \leq 1$, the following two statements are equivalent:*

- *There is a constant A such that $\max\{\alpha(G), \omega(G)\} \geq A \cdot |V(G)|^{\delta}$ for every $G \in X$.*

- *There is a constant B such that $R_X(p,q) \le B(p+q)^{\frac{1}{\delta}}$.*

Proof. The second claim reduces to the first one when $\delta = 1$, so we just prove the stronger claim.

For the first implication, suppose there exists a constant A such that $\max\{\alpha(G), \omega(G)\} \ge A \cdot |V(G)|^\delta$ for all $G \in X$. Let $H \in X$, let $p, q \in \mathbb{N}$, and suppose that $|V(H)| \ge \left(\frac{p+q}{A}\right)^{\frac{1}{\delta}}$. Then $\max\{\alpha(H), \omega(H)\} \ge A \cdot |V(H)|^\delta \ge p+q$, which means that H is guaranteed to have an independent set of size p or a clique of size q, and this proves the first direction (we can put, e.g., $B = A^{-\frac{1}{\delta}}$) in the statement of the proposition.

Conversely, suppose there exists a positive constant B such that for any $p, q \in \mathbb{N}$ and $G \in X$, if $|V(G)| \ge B(p+q)^{\frac{1}{\delta}}$, then G has an independent set of size p or a clique of size q. Let H be an arbitrary graph in X and let t be the largest integer such that $|V(H)| \ge 2^{\frac{1}{\delta}} B t^{\frac{1}{\delta}} = B(t+t)^{\frac{1}{\delta}}$. By the above assumption, H has a clique or an independent set of size t, i.e., $\max\{\alpha(H), \omega(H)\} \ge t$. Notice, by definition of t, we have $|V(H)| \le 2^{\frac{1}{\delta}} B(t+1)^{\frac{1}{\delta}}$, i.e., $|V(H)|^\delta \le 2B^\delta(t+1)$. Hence if $t = 0$, then $|V(H)|^\delta \le 2B^\delta$ and therefore $\max\{\alpha(H), \omega(H)\} \ge \frac{|V(H)|^\delta}{2B^\delta} \ge \frac{|V(H)|^\delta}{4B^\delta}$. On the other hand, if $t \ge 1$, then $|V(H)|^\delta \le 2B^\delta(t+1) \le 4B^\delta t$ and therefore $\max\{\alpha(H), \omega(H)\} \ge \frac{|V(H)|^\delta}{4B^\delta}$, and putting, e.g., $A = \frac{1}{4B^\delta}$ concludes the proof. □

In particular, the Erdős-Hajnal conjecture can be stated in our terminology as follows:

Conjecture 3. (Erdős-Hajnal) Suppose X is a proper hereditary class (that is, not the class of all graphs). Then there are constants A, k such that $R_X(p,q) \le A(p+q)^k$ for every $p, q \in \mathbb{N}$, i.e., Ramsey numbers grow at most polynomially in X.

Finally, we point out the difference between the notion of Ramsey numbers *for classes* and the notion of Ramsey numbers *of graphs*. Each of them leads naturally to the notion of linear Ramsey numbers, defined differently in the present paper and, for instance, in [13]. In spite of the possible confusion, we use the terminology of Ramsey numbers, and not the terminology of homogeneous subgraphs, because most of our results deal with the exact value of $R_X(p,q)$.

3 Classes with non-linear Ramsey numbers

In this section, we prove the "only if" part of Conjecture 2.

Lemma 1. *For every fixed $k \ge 3$, the class X_k of $(C_3, C_4, \ldots, C_k)$-free graphs is not of linear Ramsey numbers.*

Proof. Assume to the contrary that Ramsey numbers for the class X_k are linear. Then, since graphs in X_k do not contain cliques of size three, there exists a constant $t = t(k)$ such that any n-vertex graph from the class has an independent set of size at least n/t.

It is well-known (see, e.g., [2]) that X_k contains n-vertex graphs with the independence number of order $O(n^{1-\epsilon} \ln n)$, where $\epsilon > 0$ depends on k, which is smaller than n/t for large n. This contradiction shows that X_k is not of linear Ramsey numbers. □

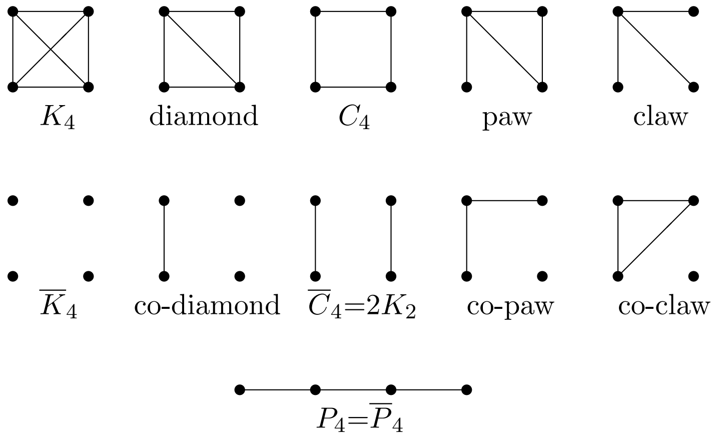

Figure 1: All 4-vertex graphs

Theorem 2. *Let X be a class of graphs defined by a finite set M of forbidden induced subgraphs. If M does not contain a graph in at least one of the following four classes, then X is not of linear Ramsey numbers: P_3-free graphs, the complements of P_3-free graphs, forests, the complements of forests.*

Proof. It is not difficult to see that a graph is P_3-free if and only if it is a disjoint union of cliques. The class of P_3-free graphs contains the graph $(q-1)K_{p-1}$ with $(q-1)(p-1)$ vertices and with no clique of size p or independent set of size q, and hence this class is not of linear Ramsey numbers. Therefore, if M contains no P_3-free graph, then X contains all P_3-free graphs and hence is not of linear Ramsey numbers. Similarly, if M contains no $\overline{P}_3$-free graph, then X is not of linear Ramsey numbers.

Now assume that M contains no forest. Therefore, every graph in M contains a cycle. Since the number of graphs in M is finite, X contains the class of $(C_3, C_4, \ldots, C_k)$-free graphs for a finite value of k and hence is not of linear Ramsey numbers by Lemma 1. Applying the same arguments to the complements of graphs in X, we conclude that if M contains no co-forest, then X is not of linear Ramsey numbers. □

4 Classes with linear Ramsey numbers

In this section, we study classes of graphs defined by forbidden induced subgraphs with 4 vertices and determine Ramsey numbers for several classes in this family that verify the "if" part of Conjecture 2. All the eleven graphs on 4 vertices are represented in Figure 1.

Below we list which of these graphs are P_3-free and which of them are forests (take the complements for $\overline{P}_3$-free graphs and for the complements of forests, respectively).

- P_3-free graphs: K_4, $\overline{K}_4$, $2K_2$, co-diamond, co-claw.
- Forests: $\overline{K}_4$, $2K_2$, P_4, co-diamond, co-paw, claw.

4.1 Claw- and co-claw-free graphs

Lemma 2. *If a (claw,co-claw)-free graph G contains a $\overline{K}_4$, then it is K_3-free.*

Proof. Assume G contains a $\overline{K}_4$ induced by $A = \{a_1, a_2, a_3, a_4\}$ and suppose by contradiction that G also contains a K_3 induced by $Z = \{x, y, z\}$.

Let first A be disjoint from Z. To avoid a co-claw, each vertex of A has a neighbour in Z and hence one of the vertices of Z is adjacent to two vertices of A, say x is adjacent to a_1 and a_2. Then, to avoid a claw, x has no other neighbours in A and y has a neighbour in $\{a_1, a_2\}$, say y is adjacent to a_1. This implies that y is adjacent to a_3 (else x, y, a_1, a_3 induce a co-claw) and similarly y is adjacent to a_4. But then y, a_1, a_3, a_4 induce a claw, a contradiction.

If A and Z are not disjoint, they have at most one vertex in common, say $a_4 = z$. Again, to avoid a co-claw, each vertex in $\{a_1, a_2, a_3\}$ has a neighbour in $\{x, y\}$ and hence, without loss of generality, x is adjacent to a_1 and a_2. But then x, a_1, a_2, a_4 induce a claw, a contradiction again. □

Lemma 3. *The maximum number of vertices in a (claw,co-claw,K_4,$\overline{K}_4$)-free graph is 9.*

Proof. Let G be a (claw,co-claw,K_4,$\overline{K}_4$)-free graph and let x be a vertex of G. Denote by A the set of neighbours and by B the set of non-neighbours of x. Clearly, A contains neither triangles nor anti-triangles, since otherwise either a K_4 or a claw arises. Therefore, A has at most 5 vertices, and similarly B has at most 5 vertices.

If $|A| = 5$, then $G[A]$ must be a C_5 induced by vertices, say, a_1, a_2, a_3, a_4, a_5 (listed along the cycle). In order to avoid a claw or K_4, each vertex of A can be adjacent to at most 2 vertices of B, which gives rise to at most 10 edges between A and B. On the other hand, to avoid a co-claw, each vertex of B must be adjacent to at least 3 vertices of A. Therefore, B contains at most 3 vertices and hence $|V(G)| \leq 9$. Similarly, if $|B| = 5$, then $|V(G)| \leq 9$.

It remains to show that there exists a (claw,co-claw,K_4,$\overline{K}_4$)-free graph with 9 vertices. The 3×3 rook's graph (also known as the Paley graph of order 9), shown in Figure 2, is a witnessing example. □

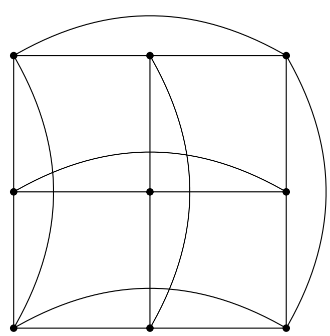

Figure 2: The 3×3 rook's graph.

Theorem 3. *For the class A of (claw,co-claw)-free graphs and all $a, b \geq 3$,*

$$R_A(a, b) = \max\left\{ \lfloor (5a-3)/2 \rfloor, \lfloor (5b-3)/2 \rfloor \right\},$$

unless $a = b = 4$ in which case $R_A(a, b) = 10$.

Proof. According to Lemma 2, the class of (claw,co-claw)-free graphs is the union of three classes:

- the class X of (claw,K_3)-free graphs,
- the class Y of (co-claw,$\overline{K}_3$)-free graphs and
- the class Z of (claw,co-claw,$K_4, \overline{K}_4$)-free graphs.

Clearly, $R_A(a,b) = \max\{R_X(a,b), R_Y(a,b), R_Z(a,b)\}$.

Since K_3 is forbidden in X, we have $R_X(a,b) = R_X(3,b)$, Also, denoting by B the class of claw-free graphs, we conclude that $R_X(3,b) = R_B(3,b)$. As was shown in [5], $R_B(3,b) = \lfloor (5b-3)/2) \rfloor$. Therefore, $R_X(a,b) = \lfloor (5b-3)/2) \rfloor$. Similarly, $R_Y(a,b) = \lfloor (5a-3)/2) \rfloor$.

In the class Z, for all $a, b \geq 4$ we have $R_Z(a,b) = 10$ by Lemma 3. Moreover, if additionally $\max\{a,b\} \geq 5$, then $R_Z(a,b) < \max\{R_X(a,b), R_Y(a,b)\}$. For $a = b = 4$, we have $R_Z(4,4) = 10 > 8 = \max\{R_X(4,4), R_Y(4,4)\}$. Finally, it is not difficult to see that $R_Z(3,b) \leq R_X(3,b)$ and $R_Z(a,3) \leq R_Y(a,3)$, and hence the result follows. □

4.2 Diamond- and co-diamond-free graphs

Lemma 4. *If a (diamond,co-diamond)-free graph G contains a $\overline{K}_4$, then it is bipartite.*

Proof. Assume G contains a $\overline{K}_4$. Let A be any maximal (with respect to inclusion) independent set containing the $\overline{K}_4$ and let $B = V(G) - A$. If B is empty, then G is edgeless (and hence bipartite). Suppose now B contains a vertex b. Then b has a neighbour a in A (else A is not maximal) and at most one non-neighbour (else a and b together with any two non-neighbours of b in A induce a co-diamond).

Assume B has two adjacent vertices, say b_1 and b_2. Since $|A| \geq 4$ and each of b_1 and b_2 has at most one non-neighbour in A, there are at least two common neighbours of b_1 and b_2 in A, say a_1, a_2. But then a_1, a_2, b_1, b_2 induce a diamond. This contradiction shows that B is independent and hence G is bipartite. □

Lemma 5. *A co-diamond-free bipartite graph containing at least one edge is either a simplex (a bipartite graph in which every vertex has at most one non-neighbour in the opposite part) or a $K_{s,t} + K_1$ for some s and t.*

Proof. Assume $G = (A, B, E)$ is a co-diamond-free bipartite graph containing at least one edge. Then G cannot have two isolated vertices, since otherwise an edge together with two isolated vertices create an induced co-diamond.

Assume G has exactly one isolated vertex, say a, and let $G' = G - a$ Then any vertex $b \in V(G')$ is adjacent to every vertex in the opposite part of G'. Indeed, if b has a non-neighbour c in the opposite part, then a, b, c together with any neighbour of b (which exists because b is not isolated) induce a co-diamond. Therefore, G' is complete bipartite and hence $G = K_{s,t} + K_1$ for some s and t.

Finally, suppose G has no isolated vertices. Then every vertex $a \in A$ has at most one non-neighbour in B, since otherwise any two non-neighbours of a in B together with a and any neighbour of a (which exists because a is not isolated) induce a co-diamond. Similarly, every vertex $b \in B$ has at most one non-neighbour in A. Therefore, G is a simplex. □

Lemma 6. *The maximum number of vertices in a (diamond,co-diamond,$K_4, \overline{K}_4$)-free graph is 9.*

Proof. Let G be a (diamond,co-diamond,$K_4, \overline{K}_4$)-free graph and x be a vertex of G. Denote by A the set of neighbours and by B the set of non-neighbours of x. Then $G[A]$ is (P_3, K_3)-free, else G contains either a diamond or a K_4. Since $G[A]$ is P_3-free, every connected component of $G[A]$ is a clique and since this graph is K_3-free, every connected component has at most 2 vertices. If at least one of the components of $G[A]$ has 2 vertices, the number of components is at most 2 (since otherwise a co-diamond arises), in which case A has at most 4 vertices. If all the components of $G[A]$ have size 1, the number of components is at most 3 (since otherwise a $\overline{K}_4$ arises), in which case A has at most 3 vertices. Similarly, B has at most 4 vertices and hence $|V(G)| \leq 9$.

To conclude the proof, we observe that the Paley graph of order $q = 3^2$ described in the proof of Lemma 3 is (diamond,co-diamond,$K_4, \overline{K}_4$)-free. □

Theorem 4. *For the class A of (diamond,co-diamond)-free graphs and $a, b \geq 3$,*

$$R_A(a, b) = \max\{2a - 1, 2b - 1\},$$

unless $a, b \in \{4, 5\}$, in which case $R_A(a, b) = 10$, and unless $a = b = 3$, in which case $R_A(a, b) = 6$.

Proof. According to Lemma 4, in order to determine the value of $R_A(a, b)$, we analyze this number in three classes:

- the class X of co-diamond-free bipartite graphs,
- the class Y of the complements of graphs in X and
- the class Z of (diamond,co-diamond,$K_4, \overline{K}_4$)-free graphs.

In the class X of co-diamond-free bipartite graphs, $R_X(a, b) = 2b - 1$, since every graph in this class with at least $2b - 1$ contains an independent set of size b, while the graph $K_{b-1,b-1}$ contains neither an independent set of size b nor a clique of size $a \geq 3$. Similarly, $R_Y(a, b) = 2a - 1$.

In the class Z of (diamond,co-diamond,$K_4, \overline{K}_4$)-free graphs, for all $a, b \geq 4$ we have $R_Z(a, b) = 10$ by Lemma 6. Moreover, if additionally $\max\{a, b\} \geq 6$, then $R_Z(a, b) < \max\{R_X(a, b), R_Y(a, b)\}$. For $a, b \in \{4, 5\}$, we have $R_Z(a, b) = 10 > \max\{R_X(a, b), R_Y(a, b)\}$. Also, $R_Z(3, 3) = 6$ (since $C_5 \in Z$) and hence $R_Z(3, 3) > \max\{R_X(3, 3), R_Y(3, 3)\}$. Finally, by direct inspection one can verify that Z contains no K_3-free graphs with more than 6 vertices and hence for $b \geq 4$ we have $R_Z(3, b) \leq R_X(3, b)$. Similarly, for $a \geq 4$ we have $R_Z(a, 3) \leq R_Y(a, 3)$. Thus for all values of $a, b \geq 3$, we have $R_A(a, b) = \max\{2a - 1, 2b - 1\}$, unless $a, b \in \{4, 5\}$, in which case $R_A(a, b) = 10$, and unless $a = b = 3$, in which case $R_A(a, b) = 6$. □

4.3 $2K_2$- and C_4-free graphs

Theorem 5. *For the class A of $(2K_2, C_4)$-free graphs and all $a, b \geq 3$,*

$$R_A(a, b) = a + b.$$

Proof. Let G be a $(2K_2, C_4)$-free graph with $a+b$ vertices. If, in addition, G is C_5-free, then the three forbidden induced subgraphs ensures that G belongs to the class of split graphs and hence it contains either a clique of size a or an independent set of size b.

If G contains a C_5, then the remaining vertices of the graph can be partitioned into a clique U, whose vertices are complete to the cycle C_5, and an independent set W, whose vertices are anticomplete to the C_5 [6]. We have $|U| + |W| = a + b - 5$ and hence either $|U| \geq a-2$ or $|W| \geq b-2$. In the first case, U together with any two adjacent vertices of the cycle C_5 create a clique of size a. In the second case, W together with any two non-adjacent vertices of the cycle create an independent set of size b. This shows that $R_A(a,b) \leq a+b$.

For the inverse inequality, we construct a graph G with $a+b-1$ vertices as follows: G consists of a cycle C_5, an independent set W of size $b-3$ anticomplete to the cycle and a clique U of size $a-3$ complete to both W and $V(C_5)$. It is not difficult to see that the size of a maximum clique in G is $a-1$ and the size of a maximum independent set in G is $b-1$. Therefore, $R_A(a,b) \geq a+b$. □

4.4 $2K_2$- and diamond-free graphs

To analyze this class, we split it into three subclasses X, Y and Z as follows:

X is the class of ($2K_2$,diamond)-free graphs containing a K_4,

Y is the class of ($2K_2$,diamond)-free graphs that do not contain a K_4 but contain a K_3,

Z is the class of ($2K_2$,diamond)-free graphs that do not contain a K_3, i.e., the class of $(2K_2, K_3)$-free graphs.

We start by characterizing graphs in the class X.

Lemma 7. *If a ($2K_2$,diamond)-free graph G contains a K_4, then G is a split graph partitionable into a clique C and an independent set I such that every vertex of I has at most one neighbour in C.*

Proof. Let G be a ($2K_2$,diamond)-free graph containing a K_4. We extend the K_4 to any maximal (with respect to inclusion) clique and denote it by C. Also, denote $I = V(G) - C$.

Assume a vertex $a \in I$ has two neighbours b, c in C. It also has a non-neighbour d in C (else C is not maximal). But then a, b, c, d induce a diamond. This contradiction shows that any vertex of I has at most one neighbour in C.

Finally, assume two vertices $a, b \in I$ are adjacent. Since each of them has at most one neighbour in C and $|C| \geq 4$, there are two vertices $c, d \in C$ adjacent neither to a nor to b. But then a, b, c, d induce a $2K_2$. This contradiction shows that I is independent and completes the proof. □

In order to characterize graphs in Z, let us say that G^* is an extended G (also known as a blow-up of G) if G^* is obtained from G by replacing the vertices of G with independent sets.

Lemma 8. *If G is a $(2K_2, K_3)$-free graph, then it is either bipartite or an extended $C_5 + K_1$.*

Proof. If G is C_5-free, then it is bipartite, because any cycle of length at least 7 contains an induced $2K_2$. Assume now that G contains a C_5 induced by a set $S = \{v_0, v_1, v_2, v_3, v_4\}$. To avoid an induced $2K_2$ or K_3, any vertex $u \notin S$ must be either anticomplete to S or have exactly two neighbours on the cycle of distance 2 from each other, i.e., $N(u) \cap S = \{v_{i-1}, v_{i+1}\}$ for some i (addition is taken modulo 5). Moreover, if $N(u) \cap S = \{v_{i-1}, v_{i+1}\}$ and $N(w) \cap S = \{v_{j-1}, v_{j+1}\}$, then

- if $i = j$ or $|i - j| > 1$, then u is not adjacent to w, since G is K_3-free.
- if $|i - j| = 1$, then u is adjacent to w, since G is $2K_2$-free.

Clearly, every vertex $u \notin S$, which is anticomplete to S, is isolated, and hence G is an extended $C_5 + K_1$. □

Now we turn to graphs G in the class Y and characterize them through a series of claims.

(1) *Any two triangles in G are vertex disjoint.* To see this, note that two triangles intersecting in two vertices induce either a K_4 or a diamond. If two triangles induced by say x_1, y_1, z and x_2, y_2, z intersect in a single vertex, there must be another edge between them, say x_1x_2, since otherwise we obtain an induced $2K_2$. But then x_1, x_2, y_1, z induce two triangles intersecting in two vertices.

(2) *For any edge xy and a triangle T containing neither x nor y, x and y each have exactly one neighbour in T.* Indeed, x and y each have at most one neighbour, since otherwise we obtain two triangles intersecting in two vertices. Moreover, if one of them does not have a neighbour, an induced $2K_2$ appears. We note that the neighbours of x and y must be distinct, since otherwise we obtain two triangles intersecting in one vertex. It follows, in particular, that the edges between two triangles form a matching.

(3) *If G has a triangle T, it does not contain an induced C_5 vertex disjoint from T.* To see this, assume that G has a triangle x, y, z and a C_5 induced by v_1, v_2, v_3, v_4, v_5. By (2), each vertex in the C_5 has exactly one neighbour in the triangle, and no two consecutive v_i (modulo 5) have the same neighbour in the triangle. It follows that up to isomorphism, the edges between the triangle and the C_5 are xv_1, yv_2, yv_4, zv_3, zv_5. But then x, v_1, v_3, v_4 induce a $2K_2$.

(4) *If G contains 3 triangles T_i, each induced by $a_i, b_i, c_i, 1 \le i \le 3$, then every other vertex in the graph is isolated. In particular, G contains at most 3 triangles.* Without loss of generality, using Claim (1) and by symmetry, the edges between the triangles are given by a_ib_j, b_ic_j, c_ia_j with $i \le j$. Suppose for a contradiction that x is a non-isolated vertex not in the T_i. Then x has exactly one neighbour in each of the triangles. Indeed, by Claim (1), it has at most one neighbour in each triangle, and if it has a neighbour anywhere in the graph, Claim (2) applies. Without loss of generality, suppose the neighbour of x in T_2 is b_2. Then x must be adjacent to exactly one of a_3 and c_1, since otherwise x, b_2, a_3, c_1 induce a $2K_2$. If x is adjacent to a_3, then x, b_2, a_2, b_3, a_3 induce a C_5 vertex disjoint from T_1, contrary to Claim (3). Similarly, if x is adjacent to c_1, then x, b_2, c_2, b_1, c_1 induce a C_5 vertex disjoint from T_3.

(5) *If G contains exactly 2 triangles T_1 and T_2 and the graph $G' = G - (T_1 \cup T_2)$ contains an edge, then G' admits a bipartition $X' \cup Y'$ such that there exist vertices $z_1 \in T_1$ and $z_2 \in T_2$ with the property that $X' \cup \{z_1, z_2\}$ and $Y' \cup \{z_1, z_2\}$ are independent sets.* Note $G' \in Z$, and by Claim (4), it is C_5-free. It follows from Lemma 8 that G' is a $2K_2$-free bipartite graph. We further split G' into G'_1 and G'_0, where G'_0 consists of the isolated vertices in G', while G'_1 contains the rest of the vertices of G'.

Note that, since G is $2K_2$-free, G'_1 is a connected graph. As this graph contains an edge, by Claim (2), every vertex of G'_1 has exactly one neighbour in each of T_1 and T_2. By standard structural results on bipartite $2K_2$-free graphs (also known as "bipartite chain graphs"), each part of G'_1 has a dominating vertex, i.e., a vertex adjacent to all the vertices in the opposite part. Write x and y for those dominating vertices, and call their respective parts X'' and Y''. Let y_1 and y_2 be the neighbours of x in T_1 and T_2 respectively, and similarly let x_1 and x_2 be the neighbours of y in those triangles. By Claim (1), $x_1 \neq y_1$, $x_2 \neq y_2$, x_1 and x_2 are not adjacent, and y_1 and y_2 are also not adjacent. Finally, write z_1, z_2 for the remaining two vertices in T_1 and T_2, respectively, and note that z_1 and z_2 are also not adjacent (otherwise z_1z_2 and xy induce a $2K_2$).

Note that any vertex in X'' must be adjacent to y_1 and to y_2: indeed, if for instance $x' \in X''$ is adjacent to $y'_1 \neq y_1$ in T_1, then y is adjacent to neither of y_1 and y'_1 (by Claim (1)), and so x, x', y, y_1, y'_1 induce a C_5 disjoint from T_2, contrary to Claim (3). Similarly, every vertex in Y'' is adjacent to x_1 and to x_2.

It remains to deal with the vertices in G'_0. Let w be a vertex in G'_0. Note that w is non-adjacent to both z_1 and z_2, since any such edge together with xy would induce a $2K_2$. Then $X'' \cup G'_0 \cup \{z_1, z_2\}$ and $Y'' \cup \{z_1, z_2\}$ are independent sets as claimed.

Theorem 6. *Let A be the class of ($2K_2$,diamond)-free graphs. Then*

- *for $a = 3$, we have $R_A(a, b) = \lfloor 2.5(b-1) \rfloor + 1$;*
- *for $a = 4$, we have $R_A(a, 3) = 7$, $R_A(a, 4) = 10$ and $R_A(a, b) = \lfloor 2.5(b-1) \rfloor + 1$ for $b \geq 5$;*
- *for $a \geq 5$, we have $R_A(a, b) = \max\{\lfloor 2.5(b-1) \rfloor + 1, a + b - 1\}$, except for $R_A(5, 4) = 10$.*

Proof. As before, we split the analysis into several subclasses of A.

For the class X of ($2K_2$,diamond)-free graphs containing a K_4 and $a \geq 5$, we have $R_X(a, b) = a + b - 1$. Indeed, every split graph with $a + b - 1$ vertices contains either a clique of size a or an independent set of size b and hence $R_X(a, b) \leq a + b - 1$. On the other hand, the split graph with a clique C of size $a - 1$ and an independent set I of size $b - 1$ with a matching between C and I belongs to X and hence $R_X(a, b) \geq a + b - 1$.

For the class Z_0 of bipartite $2K_2$-free graphs, we have $R_{Z_0}(a, b) = 2b - 1$, which is easy to see. For the class Z_1 of graphs each of which is an extended $C_5 + K_1$, we have $R_{Z_1}(a, b) = \lfloor 2.5(b-1) \rfloor + 1$. For an odd b, an extremal graph is constructed from a C_5 by replacing each vertex with an independent set of size $(b-1)/2$. This graph has $\lfloor 2.5(b-1) \rfloor$ vertices, the independence number $b - 1$ and the clique number $2 < a$. For an even b, an extremal graph is constructed from a C_5 by replacing two adjacent vertices of a C_5 with independent sets of size $b/2$ and the remaining vertices of the cycle with independent sets of size $b/2 - 1$. This again gives in total $\lfloor 2.5(b-1) \rfloor$ vertices, and the independence number $b-1$. Therefore, in the class $Z = Z_0 \cup Z_1$, we have $R_Z(a,b) = \max\{R_{Z_0}(a,b), R_{Z_1}(a,b)\} = \lfloor 2.5(b-1) \rfloor + 1$.

To compute $R_Y(a,b)$, we partition Y into Y_1, Y_2 and Y_3, where Y_s consists of the graphs in Y with s triangles. We then have:

- $R_{Y_3}(4,b) = b+6$ for $b \geq 4$. Indeed, the three triangle configuration (unique up to isomorphism) has independence number 3, and any additional vertices are isolated by Claim (4).

- $R_{Y_2}(4,b) = 2b+1$ for $b \geq 3$. To show this, let $G \in Y_2$ be a graph on $2b+1$ vertices, with triangles T_1 and T_2. As in Claim (5), $G' = G - (T_1 \cup T_2)$, G'_0 consists of the isolated vertices in G', while G'_1 contains the rest of G'.

  If G'_1 is empty (or in other words, if G' has no edges), then $G' = G'_0$ is an independent set with $2b+1-6 = 2b-5$ vertices. Provided $b \geq 5$, this number is at least b. For $b = 3$, the unique vertex in G' has at most one neighbour in each of T_1 and T_2, so in particular, it has two non-adjacent non-neighbours in the triangles, hence G has an independent set of size 3. For $b = 4$, there are 3 vertices in G'. Like before, each of them has at most one neighbour in each triangle; if their neighbourhoods do not cover the triangles, then those three vertices together with a common non-neighbour give an independent set of size 4. If their neighbourhoods do cover the triangles, then by size constraints the neighbourhoods are disjoint, and each of them is an independent set by Claim (1). In this case, any two of the vertices together with the neighbourhood of the third form an independent set of size 4.

  Now assume that G' has an edge. Then by Claim (5) G' admits a bipartition $X' \cup Y'$ such that there exist vertices $z_1 \in T_1$ and $z_2 \in T_2$ with the property that $X' \cup \{z_1, z_2\}$ and $Y' \cup \{z_1, z_2\}$ are independent sets. Given such a bipartition, it immediately follows that G has an independent set of size at least $\lceil \frac{2b-5}{2} \rceil + 2 = b$.

  Extremal counterexamples, i.e., graphs without clique of size 4 and without independent sets of size b, can be easily constructed, by making for instance G' complete bipartite with $b-3$ vertices in each part and connecting each part to the triangles appropriately.

- $R_{Y_1}(4,b) \leq 2b+1$ for $b \geq 3$. To see why, let $G \in Y_1$ be a graph on $2b+1$ vertices, write T for the triangle, and put $G' = G - T$. Like before, G' is a $2K_2$-free bipartite graph; if it has isolated vertices, one can find a bipartition of G' where one of the parts has size at least b. Otherwise, there are vertices x and y dominating each part. Those have neighbours y' and x' in T respectively; but then by Claim (1), y' is a common non-neighbour of the part containing y, and x' is a common non-neighbour of the part containing x. Since G' has one part with size at least $b-1$, this means G contains an independent set of size b.

Putting these together, we have $R_Y(4,b) = 2b+1$ for $b \geq 3$, except $R_Y(4,4) = 10$.

Combining the results for the three classes X, Y and Z, we obtain the desired conclusion of the theorem. □

4.5 The class of $(P_4, C_4,$co-claw$)$-free graphs

We start with a lemma characterizing the structure of graphs in this class, where we use the following well-known fact (see, e.g., [10]): every P_4-free graph with at least two vertices is either disconnected or the complement to a disconnected graph.

Lemma 9. *Every disconnected $(P_4, C_4,$co-claw$)$-free graph is a bipartite graph and every connected $(P_4, C_4,$co-claw$)$-free graph consists of a bipartite graph plus a number of dominating vertices, i.e., vertices adjacent to all other vertices of the graph.*

Proof. Let G be a disconnected $(P_4, C_4,$co-claw$)$-free graph. Then every connected component of G is K_3-free, since a triangle in one of them together with a vertex from any other component create an induced co-claw. Therefore, every connected component of G, and hence G itself, is a bipartite graph (since forbidding P_4 forbids every cycle of length at least 5).

Now let G be a connected graph. Since G is P_4-free, $\overline{G}$ is disconnected. Let $C^1, \ldots, C^k$ $(k \geq 2)$ be co-components of G, i.e., components in the complement of G. If at least two of them have more than 1 vertex, then an induced C_4 arises. Therefore, all co-components, except possibly one, have size 1, i.e., they are dominating vertices in G. If, say, C^1 is a co-component of size more than 1, then the subgraph of G induced by C^1 must be disconnected and hence it is a bipartite graph. $\square$

Theorem 7. *For the class A of $(P_4, C_4,$co-claw$)$-free graphs and all $a, b \geq 3$,*

$$R_A(a, b) = a + 2b - 4.$$

Proof. Let G be a graph in A with $a+2b-5$ vertices, $2b-2$ of which induce a matching (a 1-regular graph with $b-1$ edges) and the remaining $a-3$ vertices are dominating in G. Then G has neither a clique of size a nor an independent set of size b. Therefore, $R_A(a, b) \geq a+2b-4$.

Conversely, let G be a graph in A with $a + 2b - 4$ vertices. If G is disconnected, then, by Lemma 9, it is bipartite and hence at least one part in a bipartition of G has size at least b, i.e., G contains an independent set of size b. If G is connected, denote by C the set of dominating vertices in G. If $|C| \geq a - 1$, then either C itself (if $|C| \geq a$) or C together with a vertex not in C (if $|C| = a - 1$) create a clique of size a. So, assume $|C| \leq a - 2$. Then the graph $G - C$ has at least $2b - 2$ vertices and, by Lemma 9, it is bipartite. If this graph has no independent set of size b, then in any bipartition of this graph each part contains exactly $b-1$ vertices, and each vertex has a neighbour in the opposite part. But then $|C| = a-2$ and therefore C together with any two adjacent vertices in $G - C$ create a clique of size a. $\square$

4.6 The class of (co-diamond,paw,claw)-free graphs

Lemma 10. *Let G be a (co-diamond,paw,claw)-free graph.*

- *If G is connected, then it is either a path with at most 5 vertices or a cycle with at most 6 vertices or the complement of a graph of vertex degree at most 1.*
- *If G has two connected components, then either both components are complete graphs or one of the components is a single vertex and the other is the complement of a graph of vertex degree at most 1.*
- *If G has at least 3 connected components, then G is edgeless.*

Proof. Assume first that G is connected. It is known (see, e.g., [19]) that every connected paw-free graphs is either K_3-free or complete multipartite. i.e., $\overline{P}_3$-free. If G is K_3-free, then together with the claw-freeness of G this implies that G has no vertices of degree more than 2, i.e., G is either a path or a cycle. To avoid an induced co-diamond, a path cannot have more than 5 vertices and a cycle cannot have more than 6 vertices. If G is complete multipartite, then each part has size at most 2, since otherwise an induced claw arises. In other words, the complement of G is a graph of vertex degree at most 1.

Assume now that G has two connected components. If each of them contains an edge, then both components are cliques, since otherwise two non-adjacent vertices in one of the components with two adjacent vertices in the other component create an induced co-diamond. If one of the components is a single vertex, then the other is $\overline{P}_3$-free (to avoid an induced co-diamond) and hence is the complement of a graph of vertex degree at most 1 (according to the previous paragraph).

Finally, let G have at least 3 connected components. If one of them contains an edge, then this edge together with two vertices from two other components form an induced co-diamond. Therefore, every component of G consists of a single vertex, i.e., G is edgeless. □

Theorem 8. *For the class A of (co-diamond,paw,claw)-free graphs and for all $a, b \geq 3$,*

$$R_A(a, 3) = 2a - 1,$$

$$R_A(a, b) = \max\{2a, b\} \text{ for } b \geq 4,$$

except for the following four numbers $R_A(3, 3) = 6$, $R_A(3, 4) = R_A(3, 5) = R_A(3, 6) = 7$.

Proof. We start with the case $b = 3$. Since C_5 belongs to A, $R_A(3, 3) = 6$, which covers the first of the four exceptional cases.

Let $a \geq 4$. The graph $2K_{a-1}$ with $2a - 2$ vertices has neither cliques of size a nor independent sets of size 3, and hence $R_A(a, 3) \geq 2a - 1$. Conversely, let $G \in A$ be a graph with $2a - 1 \geq 7$ vertices. If G is connected, then according to Lemma 10 G is the complement of a graph of vertex degree at most 1, and hence G has a clique of size a. If G has two connected components both of which are cliques, then one of them has size at least a. If G has two connected components one of which is a single vertex, then either the second component has a couple of non-adjacent vertices, in which case an independent set of size 3 arises, or the second component is a clique of size more than a. If G has at least 3 connected components, then it contains an independent set of size more than 3. Therefore, $R_A(a, 3) \leq 2a - 1$ and hence $R_A(a, 3) = 2a - 1$ for $a \geq 4$.

From now on, $b \geq 4$. Consider the last three exceptional cases, i.e., let $a = 3$ and $4 \leq b \leq 6$. The graph C_6 that belongs to our class has neither a clique of size 3 nor an independent set of size $b \geq 4$ and hence $R_A(a, b) \geq 7$ in these cases. Conversely, let $G \in A$ be a graph with at least 7 vertices. If G is connected, then it is the complement of a graph of vertex degree at most 1 and hence contains a clique of size 3. If G has two connected components each of which is a clique, then one of them has size at least 3. If G has two components one of which is a single vertex, then the other component has at least 6 vertices and also contains a clique of size 3. If G has at least 3 connected components, then G has an independent set of size $4 \leq b \leq 6$. Therefore, $R_A(a, b) = 7$ for $a = 3$ and $4 \leq b \leq 6$.

In the rest of the proof we assume that either $a \geq 4$ or $b \geq 7$. Denote $m = \max\{2a, b\}$. If $m = 2a$, then the graph $\overline{(a-1)K_2} + K_1$ with $2a - 1$ vertices has neither cliques of size a

nor independent sets of size $b \geq 7$. If $m = b$, then the edgeless graph with $b-1$ vertices has neither cliques of size a nor independent sets of size b. Therefore, $R_A(a,b) \geq m$.

Conversely, let G be a graph with at least $m \geq 7$ vertices. If G is connected, then it is the complement of a graph of vertex degree at most 1 and hence contains a clique of size a. If G has two connected components each of which is a clique, then one of them has size at least a. If G has two components one of which is a single vertex, then the other component has at least $2a-1$ vertices and also contains a clique of size a. If G has at least 3 connected components, then G has an independent set of size b. Therefore, $R_A(a,b) = m$. □

5 Bipartite Ramsey numbers

Let $G = (A, B, E)$ be a bipartite graph given together with a bipartition $A \cup B$ of its vertex set into two independent sets. We call A and B the parts of G. The graph G is *complete bipartite*, also known as a *biclique*, if every vertex of A is adjacent to every vertex of B. A biclique with parts of size n and m is denoted by $K_{n,m}$. By $b(G)$ we denote the biclique number of G, i.e., the maximum p such that G contains $K_{p,p}$ as an induced subgraph.

Given a bipartite graph $G = (A, B, E)$, we denote by $\widetilde{G}$ the bipartite complement of G, i.e., the bipartite graph on the same vertex set in which two vertices $a \in A$ and $b \in B$ are adjacent if and only if they are not adjacent in G. We refer to the bipartite complement of a biclique as *co-biclique* and denote by $a(G)$ the maximum q such that $\widetilde{G}$ contains $K_{q,q}$ as an induced subgraph.

The notion of bipartite Ramsey numbers is an adaptation of the notion of Ramsey numbers to bipartite graphs and it can be defined as follows.

Definition 1. The bipartite Ramsey number $R^b(p,q)$ is the minimum number n such that for every bipartite graph G with at least n vertices in each of the parts, G contains $K_{p,p}$, or $\widetilde{G}$ contains $K_{q,q}$.

It is known (see, e.g., [9]) that $R^b(p,p) \geq 2^{p/2}$, and hence bipartite Ramsey numbers are generally non-linear. However, similarly to the non-bipartite case, they may become linear when restricted to a specific class X of bipartite graphs. We denote bipartite Ramsey numbers restricted to a class X by $R^b_X(p,q)$ and say that $R^b_X(p,q)$ are *linear* in X if there is a constant k such that $R^b_X(p,q) \leq k(p+q)$ for all p, q.

Similarly to the non-bipartite case, we will say that graphs in a class X of bipartite graphs have *linear bipartite homogeneous subgraphs* if there exists a constant $c = c(X)$ such that $\max\{a(G), b(G)\} \geq c \cdot |V(G)|$ for every $G \in X$. The following proposition can be proved by analogy with Proposition 1.

Proposition 2. *Let X be a class of bipartite graphs. Then graphs in X have linear bipartite homogeneous subgraphs if and only if bipartite Ramsey numbers are linear in X.*

Some classes of bipartite graphs with linear bipartite homogeneous subgraphs have been revealed recently in [4], where the authors consider bipartite graphs that do not contain a fixed bipartite graph as an induced subgraph *respecting the parts*. The subgraph containment relation respecting the parts can be thought of as the containment of *colored* bipartite graphs, where a colored bipartite graph is a bipartite graph given with a fixed bipartition of its vertices into two independent sets of black and white vertices. A colored bipartite graph H is said to be an induced subgraph of a colored bipartite graph G if there exists an isomorphism between H and an induced subgraph of G that preserves colors.

A number of related results appeared also in [15], where the authors study zero-one matrices that do not contain a fixed matrix as a submatrix. Primarily, they are interested in forbidden submatrices P that guarantee the existence of a square homogeneous submatrix of linear size in matrices avoiding P, where homogeneous means a submatrix with all its entries being equal. The problems studied in [15] can be interpreted as questions about homogeneous bipartite subgraphs in colored and (*vertex-*)*ordered* bipartite graphs which do not contain a fixed forbidden colored and ordered bipartite subgraph. In this case, the notion of graph containment must preserve not only colors but also vertex order.

In the next sections, we extend some of the results obtained in [4] using the language of Ramsey numbers.

5.1 Classes with non-linear bipartite Ramsey numbers

According to Lemma 1, classes of graphs without short cycles have non-linear Ramsey numbers. A similar result holds for bipartite graphs, which can be shown via standard probabilistic arguments. For the sake of completeness, we provide formal proofs below. We start with a result, which is an adaptation to the bipartite setting of the classical proof by Erdős of the existence of high chromatic number graphs without short cycles.

Lemma 11. *Let $k \geq 4$ and $\varepsilon > 0$. Then for any sufficiently large n, there exists a bipartite graph $G = (A, B, E)$ with n vertices in each of the parts such that G contains no cycles of length at most k, and $\widetilde{G}$ contains no $K_{s,s}$ with $s \geq \varepsilon n$.*

Proof. Let n be a natural number and let $N = 2n$. We set $\delta = \frac{1}{2k}$, $p = (2N)^{\delta-1}$, and consider the random bipartite graph $G(2N, p)$ (i.e., the probability space of bipartite graphs with two parts A and B each of size N such that every pair of vertices $a \in A, b \in B$ is connected by an edge independently with probability p).

Let Y be a random variable equal to the number of cycles of length at most k in $G(2N, p)$. The number of potential cycles of length i is at most $\frac{1}{2}(i-1)!\binom{2N}{i} \leq (2N)^i$, and each of them is present with probability p^i. Hence

$$\mathbb{E}[Y] \leq \sum_{i=4}^{k} (2N)^i p^i = \sum_{i=4}^{k} (2N)^{\delta i}.$$

Since $(2N)^{\delta i} = o(N)$ for all $i \leq k$, we conclude $\mathbb{E}[Y] = o(N)$. Hence, for every sufficiently large N, we have $\mathbb{E}[Y] < \frac{N}{2}$, and therefore, by Markov's inequality,

$$P\left[Y \geq N\right] < \frac{1}{2}.$$

Now we estimate the maximum size of a co-biclique in $G(2N, p)$, i.e., $a(G(2N, p))$. Let us set $s = \lceil \frac{3}{p} \ln N \rceil$. Then again from Markov's inequality, we have

$$P\left[a(G(2N, p)) \geq s\right] \leq \binom{N}{s}\binom{N}{s}(1-p)^{s^2} \leq N^{2s} e^{-ps^2} = e^{s(2\ln N - ps)},$$

which tends to zero as N goes to infinity. Thus again, for N sufficiently large, we have

$$P\left[a(G(2N,p)) \geq s\right] < \frac{1}{2}.$$

The above conclusions imply that there exists a graph $G = (A, B, E)$ with $Y < N$ and $a(G) < s$. Now we want to destroy all of the Y short cycles by removing one vertex from each of them. In order to guarantee that the resulting bipartite graph has many vertices in each of the parts we destroy half of the cycles by removing vertices from A, and the other half by removing vertices from B. In this way we remove at most $\frac{N}{2}$ vertices from each of A and B, and hence we obtain a graph $G' = (A', B', E')$ with at least $\frac{N}{2} = n$ vertices in each of the parts such that G' contains neither cycles of length at most k, nor the bipartite complement of $K_{s,s}$ with $s = \lceil \frac{3}{p} \ln N \rceil = \lceil 3(2N)^{1-\delta} \ln N \rceil = o(N)$. By removing some of the vertices from G' we can obtain a bipartite graph with the same properties, but with exactly n vertices in each of the parts. □

From this lemma and Proposition 2 we derive the following conclusion.

Corollary 1. *For every $k \geq 4$, bipartite Ramsey numbers are not linear in the class of bipartite graphs without cycles of length at most k.*

Theorem 9. *Let X be a class of bipartite graphs defined by a finite set M of bipartite forbidden induced subgraphs. If M does not contain a forest or the bipartite complement of a forest, then bipartite Ramsey numbers are not linear in X.*

Proof. If M does not contain a forest, then every graph in M contains a cycle. Let k be the size of a largest induced cycle in graphs in M, which is a finite number, since M is finite. Then X contains all bipartite graphs without cycles of length at most k, and hence bipartite Ramsey numbers are not linear in X by Corollary 1.

If M does not contain the bipartite complement of a forest, then bipartite Ramsey numbers are not linear in X, since they are linear in X if and only if they are linear in the class of bipartite complements of graphs in X. □

This result is half analogous to Theorem 2. Unfortunately, there is no obvious analog for the second half. In the non-bipartite case, the second half deals with P_3-free graphs and their complements. Every P_3-free graph consists of disjoint union of cliques, and the most natural analog of this class in the bipartite case is the class of P_4-free bipartite graphs, which are disjoint union of bicliques. However, bipartite Ramsey numbers are linear in this class, which is not difficult to see. In the absence of any other natural obstacles for linearity in the bipartite case, we propose the following conjecture.

Conjecture 4. Let X be a class of bipartite graphs defined by a finite set M of bipartite forbidden induced subgraphs. Then bipartite Ramsey numbers in X are linear if and only if M contains a forest and the bipartite complement of a forest.[1]

We note that an analogous conjecture, in the context of homogeneous submatrices, was proposed in [15]. In the next section, we consider some classes of bipartite graphs excluding a forest and the bipartite complement of a forest, and show that bipartite Ramsey numbers are linear for them.

[1] As pointed out by the referees, this conjecture has been solved since we first submitted this article [22].

5.2 Some classes with linear bipartite Ramsey numbers

First, we look at some classes defined by a single bipartite forbidden induced subgraph H, which is simultaneously a forest and the bipartite complement of a forest. The following theorem characterizes all graphs H of this form, where $F_{p,q}$ denotes the graph represented in Figure 3 and $S_{1,2,3}$ is a tree obtained from the claw by subdividing one of its edges ones and another edge twice (also shown in Figure 3). Implicitly, without a proof, this characterization was given in [1]. It also appeared recently in [4].

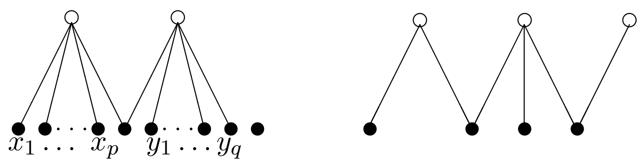

Figure 3: The graphs $F_{p,q}$ (left) and $S_{1,2,3}$ (right)

Theorem 10. *A bipartite graph H is a forest and the bipartite complement of a forest if and only if H is an induced subgraph of a P_7 or of an $S_{1,2,3}$ or of a graph $F_{p,q}$.*

The results in [4] and [15] imply that for any natural numbers p and q, $F_{p,q}$-free bipartite graphs have linear bipartite homogeneous subgraphs. Hence, by Proposition 2, bipartite Ramsey numbers are linear in the class of $F_{p,q}$-free bipartite graphs. In the next section, we prove that bipartite Ramsey numbers are linear in the class of $S_{1,2,3}$-free bipartite graphs. This leaves an intriguing open question of whether P_7-free bipartite graphs have linear Ramsey numbers or not. The structural characterization of the latter graph class from [18] may be helpful in answering this question.

We note that P_6 is symmetric with respect to bipartition and it is an induced subgraph of $S_{1,2,3}$. Therefore our result from the next section implies that colored P_6-free bipartite graphs have linear bipartite homogeneous subgraphs, which resolves one of the four open cases from [4].

5.2.1 $S_{1,2,3}$-free bipartite graphs

We start with some definitions.

- The disjoint union is the operation that creates out of G_1 and G_2 the bipartite graph $G = (A_1 \cup A_2, B_1 \cup B_2, E_1 \cup E_2)$.
- The join is the operation that creates out of G_1 and G_2 the bipartite graph which is the bipartite complement of the disjoint union of $\widetilde{G}_1$ and $\widetilde{G}_2$
- The skew join is the operation that creates out of G_1 and G_2 the bipartite graph $G = G_1 \oslash G_2 = (A_1 \cup A_2, B_1 \cup B_2, E_1 \cup E_2 \cup \{ab : a \in A_1, b \in B_2\})$. We say G is a skew join of G_1 and G_2, if either $G = G_1 \oslash G_2$ or $G = G_2 \oslash G_1$.

The three operations define a decomposition scheme, known as canonical decomposition which takes a bipartite graph G and partitions it into graphs G_1 and G_2 if G is a disjoint union, join, or skew-join of G_1 and G_2, and then the scheme applies to G_1 and G_2, recursively.

Graphs that cannot be decomposed into smaller graphs under this scheme will be called canonically indecomposable.

The following lemma from [17] characterizes $S_{1,2,3}$-free bipartite graphs containing a P_7. In the paper, the author calls a graph prime if for any two distinct vertices the neighbourhoods are also distinct.

Lemma 12. *A prime canonically indecomposable bipartite $S_{1,2,3}$-free graph G that contains a P_7 must be either a path or a cycle or the bipartite complement of either a path or a cycle.*

Theorem 11. *Let G be a canonically indecomposable $S_{1,2,3}$-free bipartite graph that contains a P_7. If G has at least $4n$ vertices in each part of the bipartition, then G contains a $K_{n,n}$ or a $\widetilde{K}_{n,n}$*

Proof. From Lemma 12 it follows that either G or its bipartite complement must be either a path or a cycle with some vertices duplicated (as we now no longer assume that G is prime). Hence, $G = (A, B, E)$ or its bipartite complement must admit a partition $A = A_1 \cup A_2 \cup \ldots \cup A_s$, $B = B_1 \cup B_2 \cup \ldots \cup B_s$ such that:

- A_i, B_i are non-empty for all $i \leq s-1$, and at most one of A_s and B_s is empty
- For any $i \leq s-1$, A_i joined with B_j if $j \in \{i, i+1\}$ and co-joined to B_j otherwise
- A_s joined to B_s and B_1 and co-joined with B_j for $j \notin \{1, s\}$

Consider first the case when there exists some i such that $|A_i| \geq n$. In this case, if $|B_i \cup B_{i+1}| \geq n$, we obtain a biclique $K_{n,n}$ induced by subsets of A_i and $B_i \cup B_{i+1}$. On the other hand, if $|B_i \cup B_{i+1}| < n$, then we obtain a $\widetilde{K}_{n,n}$ induced by subsets of A_i and $B \backslash (B_i \cup B_{i+1})$. Hence, if there exists some i such that $|A_i| \geq n$, then G contains either a $K_{n,n}$ or a $\widetilde{K}_{n,n}$. The argument when there exists some i such that $|B_i| \geq n$ is analogous.

So assume now that $|A_i| < n$ and $|B_i| < n$ for all i. Consider the smallest k such that $|A_1 \cup A_2 \cup \ldots \cup A_k| \geq n$. If $|B_{k+2} \cup B_{k+3} \cup \ldots \cup B_s| \geq n$ then we have a $\widetilde{K}_{n,n}$ induced by subsets of $A_1 \cup A_2 \cup \ldots \cup A_k$ and $B_{k+2} \cup B_{k+3} \cup \ldots \cup B_s$. Otherwise, $|B_2 \cup B_3 \cup \ldots \cup B_k| = |B| - |B_1| - |B_{k+1}| - |B_{k+2} \cup \ldots \cup B_s| > 4n - n - n - n = n$ and also $|A_{k+1} \cup A_{k+2} \cup \ldots \cup A_s| = |A| - |A_k| - |A_1 \cup A_2 \cup \ldots \cup A_{k-1}| > 4n - n - n = 2n$. Hence, we obtain a $\widetilde{K}_{n,n}$ between subsets of $A_{k+1} \cup A_{k+2} \cup \ldots \cup A_s$ and $B_2 \cup B_3 \cup \ldots \cup B_k$.

□

Theorem 12. *Let X be the class of $S_{1,2,3}$-free bipartite graphs. Then $R^b_X(p, q) \leq 6(p + q)$.*

Proof. Let $G = (A, B, E)$ be a bipartite graph in X that has $6n$ vertices in each part. If $G'_0 = G$ is canonically indecomposable, then by the previous lemma we can find $K_{n,n}$ or $\widetilde{K}_{n,n}$ in G. So assume G'_0 is a disjoint a union, a join, or a skew-join of two non-empty graphs $G_1 = (A_1, B_1, E_1)$ and $G'_1 = (A'_1, B'_1, E'_1)$. Without loss of generality we may assume that $|A_1| \leq |A'_1|$. Inductively, if G'_k for some $k \in \mathbb{N}$ is not cannonically indecomposable, then G'_k is a disjoint union, a join, or a skew join of two non-empty graphs $G_{k+1} = (A_{k+1}, B_{k+1}, E_{k+1})$ and $G'_{k+1} = (A'_{k+1}, B'_{k+1}, E'_{k+1})$. Again, without loss of generality we may assume that $|A_{k+1}| \leq |A'_{k+1}|$.

Consider first the case when the procedure stops with canonically indecomposable graph G'_k such that $|A'_k| \geq 4n$. If $|B'_k| \geq 4n$, then by the previous lemma, G contains $K_{n,n}$ or $\widetilde{K}_{n,n}$.

On the other hand, if $|B'_k| < 4n$, then we have $|B_1 \cup B_2 \cup \ldots \cup B_k| \geq 2n$ and each vertex in $B_1 \cup B_2 \cup \ldots \cup B_k$ is either joined or co-joined to the set A'_k. Hence we can find a $K_{n,n}$ or $\widetilde{K}_{n,n}$ induced by subsets of $B_1 \cup B_2 \cup \ldots \cup B_k$ and A'_k.

Now consider the case when the procedure stops with a canonically indecomposable graph G'_k such that $|A'_k| < 4n$. As $A = A_1 \cup A_2 \cup \ldots \cup A_k \cup A'_k$ and $|A| = 6n$ it follows that $|A_1 \cup A_2 \cup \ldots \cup A_k| > 2n$. Hence, in this case we can pick the smallest p such that $|A_1| + |A_2| + \ldots + |A_p| \geq 2n$. From the fact that $|A_1| + |A_2| + \ldots + |A_p| + |A'_p| = 6n$ and the definition of p it follows that $|A_p| + |A'_p| \geq 4n$. By construction $|A'_p| \geq |A_p|$, hence $|A'_p| \geq 2n$.

First, let us consider the case when $|B'_p| \geq n$. Then each vertex of $A_1 \cup A_2 \cup \ldots \cup A_p$ by construction is either joined of co-joined to B'_p. Since $|A_1 \cup A_2 \cup \ldots \cup A_p| \geq 2n$, it is clear that we will find either a $K_{n,n}$ or $\widetilde{K}_{n,n}$ in the bipartite graph induced by $A_1 \cup A_2 \cup \ldots \cup A_p$ and B'_p.

Now, let us consider the case when $|B'_p| < n$. Then each vertex of $B_1 \cup B_2 \cup \ldots B_p$ is either joined or co-joined to A'_p. Since $|A'_p| \geq 2n$ and $|B_1 \cup B_2 \cup \ldots B_p| > 5n$, we can find either a $K_{n,n}$ or $\widetilde{K}_{n,n}$ in the bipartite graph induced by $B_1 \cup B_2 \cup \ldots \cup B_p$ and A'_p.

Hence we have shown that $R^b_X(n,n) \leq 6n$ for all $n \in \mathbb{N}$. It now follows easily that for any $p, q \in \mathbb{N}$, we have $R^b_X(p,q) \leq R^b_X(\max\{p,q\}, \max\{p,q\}) \leq 6 \cdot \max\{p,q\} \leq 6(p+q)$.

$\square$

5.2.2 Exact values of bipartite Ramsey numbers for P_2+P_3-free bipartite graphs

Finding exact values of Ramsey numbers is much harder than providing bounds. Similarly, finding *tight* bounds on the size of homogeneous subgraphs is a very difficult task. In [4], such bounds have been given only for classes where the only forbidden induced subgraph has two vertices in each part of the bipartition.

In this section, we consider the class of bipartite graphs where the only forbidden induced subgraph $P_2 + P_3$ has two vertices in one of the parts and three in the other. This class is a subclass of $S_{1,2,3}$-free bipartite graphs, and hence the bipartite Ramsey numbers are linear in this class. Now we refine this conclusion by deriving *exact* values of the bipartite Ramsey numbers for the class of $P_2 + P_3$-free bipartite graphs. For this, we will use a structural characterisation of this class provided in [8].

In [8], a bipartite $P_2 + P_3$-free bipartite graph $G = (A, B, E)$ has been shown to admit a *half-graph expansion*, which means that there exists a partition $A = A_1 \cup A_2 \cup \ldots \cup A_n$, $B = B_1 \cup B_2 \cup \ldots \cup B_n$ (where some of $A_1, \ldots, A_n, B_1, \ldots, B_n$ may be empty) with the following properties:

- for $1 \leq i < j \leq n$, A_i is complete to B_j and B_i is anticomplete to A_j;
- for $1 \leq i \leq n$, either $|A_i \cup B_i| = 1$ or $G[A_i, B_i, E \cap (A_i \times B_i)]$ is an induced matching or the bipartite complement of an induced matching.

Theorem 13. *For the class X of P_2+P_3-free bipartite graphs and for all $p, q \geq 2$,* $R^b_X(p,q) = \max\{p,q\} + p + q - 2$.

Proof. To prove that $R^b_X(p,q) \geq \max\{p,q\} + p + q - 2$, assume, without loss of generality, that $q = \max\{p,q\}$ (if $p = \max\{p,q\}$, the proof is similar). Let $G = (A, B, E)$ be a $P_2 + P_3$-free bipartite graph with $|A| = |B| = 2q + p - 3$ such that $A = A_1 \cup A_2$, $B = B_2 \cup B_3$, where

$|A_1| = |B_3| = p-2$, $A_2 \cup B_2$ is an induced matching of size $2q-1$, A_1 is complete to B, while B_3 is complete to A.

Assume G contains a biclique $K_{p,p}$. Then this biclique contains at least 2 vertices in A_2 and at least two vertices in B_2. But then $A_2 \cup B_2$ is not an induced matching. This contradiction shows that G is $K_{p,p}$-free.

Assume G contains a co-biclique $\widetilde{K}_{q,q}$. This co-biclique cannot contain vertices of A_1 or B_3 (since these vertices dominate the opposite part of the graph). But then we obtain a contradiction to the assumption that the size of the matching $A_2 \cup B_2$ is $2q-1$. Therefore, G is $\widetilde{K}_{q,q}$-free. This proves the inequality $R^b_X(p,q) \geq \max\{p,q\} + p + q - 2$.

To prove the reverse inequality, consider an arbitrary $P_2 + P_3$-free bipartite graph $G = (A, B, E)$ with $|A| = |B| = \max\{p,q\}+p+q-2$. Consider a half-graph expansion of the graph G with $A = A_1 \cup \ldots \cup A_n$, $B = B_1 \cup \ldots \cup B_n$, where the partition satisfies the two half-graph expansion conditions as stated above. Fix a vertex ordering of the parts $A = \{a_1, a_2, \ldots, a_r\}$ and $B = \{b_1, b_2, \ldots, b_r\}$ (with $r = |A| = |B|$), which respects the half-graph expansion ordering of vertex subsets, i.e. such that the functions f and g defined by $a_i \in A_{f(i)}$ and $b_i \in B_{g(i)}$ are both increasing. Further, we can also assume that the ordering is consistent with matching/co-matching ordering within parts of the half-graph expansion, i.e. for each i, for which $G[A_i, B_i, E \cap (A_i \times B_i)]$ is a matching/co-matching, the bijective function h which maps each $a_k \in A_i$ to its corresponding matched/co-matched vertex $b_{h(k)} \in B_i$ is an increasing function. Following this ordering, we define the following subsets:

- $A_{[p]} = \{a_1, a_2, \ldots, a_p\}$ the set of the first p vertices of A,
- $\bar{A}_{[q]} = \{a_{r-q+1}, a_{r-q+2}, \ldots, a_r\}$ the set of the last q vertices of A,
- $B_{[q]} = \{b_1, b_2, \ldots, b_q\}$ the set of the first q vertices of B,
- $\bar{B}_{[p]} = \{b_{r-p+1}, b_{r-p+2}, \ldots, b_r\}$ the set of the last p vertices of B.

Since $p, q \geq 2$, we have that $|A_{[p]}|+|\bar{A}_{[q]}| = |B_{[q]}|+|\bar{B}_{[p]}| = p+q \leq p+q+\max\{p,q\}-2 = r$, which implies that $A_{[p]}$ and $\bar{A}_{[q]}$ are disjoint and so are $B_{[q]}$ and $\bar{B}_{[p]}$.

If $A_{[p]}$ is complete to $\bar{B}_{[p]}$ or $B_{[q]}$ is anti-complete to $\bar{A}_{[q]}$, then G contains $K_{p,p}$ or $\widetilde{K}_{q,q}$, respectively. Therefore, we assume there is a pair $a \in A_{[p]}$, $\bar{b} \in \bar{B}_{[p]}$ of non-adjacent vertices and a pair $b \in B_{[q]}$, $\bar{a} \in \bar{A}_{[q]}$ of adjacent vertices.

Let i and j be such that $a \in A_i$ and $\bar{b} \in B_j$. Since the two vertices are non-adjacent in G, it follows that $j \leq i$. Similarly, let x and y be such that $\bar{a} \in A_x$ and $b \in B_y$. As these two vertices are adjacent in G, it follows that $x \leq y$. Moreover, since a appears before $\bar{a}$ in our fixed ordering, we have $i \leq x$ and for a similar reason $y \leq j$. Hence, we have $i \leq x \leq y \leq j \leq i$ and so the equality must hold throughout. In other words, all the four vertices $a, b, \bar{a}, \bar{b}$ belong to the same "block" $G[A_i, B_i, E \cap (A_i \times B_i)] =: G_i$ for some i, which by definition of the half-graph expansion must be either a matching or a co-matching.

In what follows we assume that G_i is an induced matching (the case when G_i is a co-matching is symmetric). We consider one particular edge $b\bar{a}$ belonging to this matching. Let $s, k \leq q$ be such that $b = b_s$ and $\bar{a} = a_{r-k+1}$. Since our vertex ordering is consistent with the bijection of the matching, any vertex a_l with $l < r-k+1$ that belongs to the matching G_i has to be matched with one of the vertices b_t with $t < s$. Thus there are at most $s-1$ possible values $l < r-k+1$ for which $a_l \in G_i$. Since the vertices that appear in A_i are consecutive, in order, we conclude that the smallest value l such that a_l is contained in A_i is at least

$$r-k+1-(s-1)=r-k-s+2\geq r-2q+2=\max\{p,q\}+p-q\geq p.$$

In particular, none of $\{a_1,a_2,\ldots,a_{p-1}\}=:A_{[p-1]}$ belongs to A_i, but instead belongs to $A_1\cup A_2\cup\ldots\cup A_{i-1}$. By a similar argument, one can deduce that none of the vertices $\{b_{r-p+2},\ldots,b_{r-1},b_r\}=:\bar{B}_{[p-1]}$ belongs to B_i, but instead belongs to $B_{i+1}\cup B_{i+2}\cup\ldots\cup B_n$. Hence, $A_{[p-1]}$ and $\bar{B}_{[p-1]}$ induce a biclique $K_{p-1,p-1}$ and these sets are also complete to B_i and A_i, respectively. Hence, $A_{[p-1]}$ and $\bar{B}_{[p-1]}$ together with any extra edge of G_i, for example, $b\bar{a}$, induces a $K_{p,p}$. This finishes the proof.

□

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