On the singularity formation and long-time asymptotics in a class of nonlinear Fokker–Planck equations

by

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Declarations

Parts of the research leading to this thesis were conducted in collaboration with

- Prof. José L. Rodrigo (Chapter 3 and Chapter 8),
- Prof. José A. Carrillo (Chapter 3 and Chapter 6),
- Dr. Marie-Therese Wolfram (Chapter 6).

The problem analysed in Part I was suggested by José Carrillo, who also shared very helpful intuitions and questions motivating the analysis in Chapter 4. The problem in Part II was suggested by my advisor José Rodrigo, who accompanied me throughout my Ph.D. journey always providing most constructive feedback and supporting me through numerous stimulating discussions. The strategy of the proofs and their actual implementation has been worked out independently by myself. Furthermore, all numerical simulations have been performed by myself. The simulations and figures in the numerical study (Chapter 6) were computed in MATLAB. The graphics appearing outside Chapter 6 have been produced by myself using PGF/Ti*k*Z.

The results in Chapters 3 and 4 have been accepted for publication [26]. A manuscript containing the numerical results in Chapter 6 has been submitted for publication [27]. The results in Chapter 8 have been previously published in the form of the article [58].

I declare that this thesis is my own work except where otherwise indicated, and that the material in this thesis has not been submitted to any other university or for any other degree.

Abstract

This thesis investigates the properties and long-time behaviour of solutions to a class of Fokker–Planck-type equations with superlinear drift formally dominating the viscous term at high values of the density and potentially leading to the formation of singularities in finite time.

The first and main part of this thesis is devoted to a family of Fokker–Planck equations with superlinear drift related to condensation phenomena in quantum physics. In the drift-dominant regime, the equations have a finite critical mass above which the measure minimising the associated entropy functional displays a singular component. Our approach, which addresses the one-dimensional case, is based on a reformulation of the problem in terms of the pseudo-inverse distribution function. Motivated by the structure of the equation in the new variables, we establish a general framework for global-in-time existence, uniqueness and regularity of monotonic viscosity solutions to a class of nonlinear degenerate (resp. singular) parabolic equations, using as a key tool comparison principles and maximum arguments. We then focus on the special case of the bosonic Fokker–Planck model in 1D and study in more detail the regularity and dynamics of solutions. In particular, blow-up behaviour, formation of condensates and long-time asymptotics are investigated. We complement the rigorous analysis with numerical experiments enabling conjectures about the condensation process and long-time dynamics in the isotropic 3D Kaniadakis–Quarati model for bosons, the Fokker–Planck equation originally proposed in the physics literature. The simulations suggest that, in the L^1-supercritical regime, the bosonic Fokker–Planck problem in 1D serves as a good toy model for the Kaniadakis–Quarati model in 3D.

The second part of this thesis investigates a question related to fluid mixing and biological cell aggregation. We consider an aggregation equation with fractional (anomalous) diffusion, a generalisation of the classical parabolic-elliptic Keller–Segel system for chemotaxis, which is known to admit solutions exploding in finite time, and study the effect of an ambient incompressible flow on the system. We identify a class of stationary flows significantly enhancing dissipation in the diffusive problem and show that, provided sufficiently strong, these flows are capable of preventing the formation of singularities in our aggregation-diffusion equation and lead to a relaxation to equilibrium at an exponential rate.

Chapter 1

Introduction

1.1 Research problems

In this thesis we study the singularity formation and long-time behaviour in certain classes of nonlinear reaction-diffusion equations with mass conservation. The equations considered here describe a density $f = f(t, y) \geq 0$ evolving according to a law of the form

$$\partial_t f = \mathcal{A} f + \operatorname{div}_y(f\,V_f). \tag{1.1}$$

Here $t \geq 0$ denotes the time variable and $y \in \mathbb{R}^d$ the space variable. The linear 'diffusion' operator $\mathcal{A}$ will either be the Laplacian $\Delta = \sum_{i=1}^d \partial_{y_i}^2$ on a domain $U \subseteq \mathbb{R}^d$ or a negative, self-adjoint unbounded operator on $L^2(\mathbb{T}^d)$, where $\mathbb{T}^d$ is the flat d-dimensional torus. The 'velocity' field V_f depends on the unknown f and possibly also explicitly on the space variable.

The structure of equation (1.1) and the boundary conditions with respect to the space variable (resp. the behaviour of f as $|y| \to \infty$) will always be such that any sufficiently regular solution f of equation (1.1) has a conserved *mass* m, i.e.

$$\int f(t, y)\,\mathrm{d}y = \int f(0, y)\,\mathrm{d}y \; =: m \qquad \text{for all } t \geq 0.$$

Note that, since the function $f(t, \cdot)$ is assumed to be non-negative, its mass m agrees with its L^1-norm.

The velocity field V_f is assumed to be focusing in a suitable sense, potentially leading to singularities in finite time and counteracting the diffusive spread of the density f induced by the operator $\mathcal{A}$. It is the simultaneous presence of the (linear) diffusion and the nonlinear focusing drift which renders problem (1.1) mathematically intriguing. When terms of lower order in the density f are neglected, the equations (1.1) which we consider have an approximate scale invariance. If the corresponding scaling leaves the L^1-norm invariant, we call the problem L^1-critical. The regime where on finer scales the L^1-conservation law becomes weaker (resp. becomes more significant) will be called L^1-supercritical (resp. L^1-subcritical). In this thesis we are primarily interested in the L^1-supercritical (also referred to as drift-dominant) regime. It is the regime least understood in our applications and at the same time the most interesting one (see Subsection 1.1.1 and Chapter 2).

We consider two different problems of the form (1.1):

1.1.1 Fokker–Planck equations for Bose–Einstein particles

Part I of this thesis is concerned with a class of nonlinear Fokker–Planck equations with superlinear drift. The problem is motivated by Kaniadakis and Quarati [70] who introduced a Fokker–Planck equation with quadratic drift as a model for the relaxation to equilibrium of the velocity distribution of a spatially homogeneous system of bosons. The model is based on a direct modification of the transition probability rates governing the particle kinetics in order to account for the quantum effect. The resulting equation, the so-called Kaniadakis–Quarati model for bosons (KQ), reads

$$\begin{aligned} \partial_t f &= \Delta_v f + \nabla_v \cdot (vf(1+f)), \quad t>0,\ v \in \mathbb{R}^d, \\ f(0,\cdot) &= f_0 \geq 0. \end{aligned} \tag{KQ}$$

Here, the space variable v represents velocity. In the physically most interesting space dimension, $d=3$, equation (KQ) is L^1-supercritical, while it is critical for $d=2$ and subcritical for $d=1$. In this thesis we are interested in generalisations of KQ, where for simplicity we mainly consider the following family [11]

$$\begin{aligned} \partial_t f &= \Delta_v f + \nabla_v \cdot (vf(1+f^\gamma)), \quad t>0,\ v \in \mathbb{R}^d, \\ f(0,\cdot) &= f_0 \geq 0, \end{aligned} \tag{1.2}$$

with a parameter $\gamma > 0$. In problems with a more general superlinear drift like equation (1.2), L^1-supercriticality can also be achieved in lower space dimensions, namely by choosing $\gamma > \frac{2}{d}$. At the same time, the entropy structure of the physically motivated problem, eq. (KQ), is shared by the family of equations (1.2) (see Section 2.1). A core feature of equation (1.2), related to the entropy structure, are its equilibrium solutions or *steady states*, which for $\gamma = 1$ coincide with the classical Bose–Einstein distributions and which, in the L^1-supercritical regime, give rise to a finite *critical mass* m_c (i.e. the least upper bound for the L^1-norm of all regular steady states of the equation). In this case, when above the critical mass, minimising the entropy functional associated with eq. (1.2) leads to measures with a singular component, concentrated at the origin. Such Dirac deltas at zero are physically interpreted as *condensates*, at least in the context of equation (KQ). We will adopt this terminology for the general class of equations (1.2), which we henceforth refer to as *bosonic Fokker–Planck equations*, despite the fact that the physical description involving bosons is meaningful only if $\gamma = 1$.

In the L^1-supercritical regime the problem of understanding the long-time dynamics of the above evolutionary equations has remained largely open. The only rigorous result is due to Toscani [100], who demonstrated via contradiction that, for highly concentrated initial data or data with very large mass $m \geq \underline{m} \gg m_c$, solutions of the 3D KQ model, i.e. equation (KQ) with $d = 3$, must blow up after finite time in the sense that they cannot be extended to a global-in-time classical solution.

In this work we study the dynamics of solutions to equation (1.2) in the L^1-supercritical regime in the case $d = 1$ of a one-dimensional velocity space. We will address one of the main open questions about this problem, namely the question of whether for mass $m > m_c$ solutions eventually have condensate component, i.e. a Dirac delta at $v = 0$. A fundamental difficulty in answering this question lies in the fact that in the original formulation (1.2) measures with a singular part are not admitted. Our approach to deal with this issue takes a mass transportation perspective. Our starting point is a reformulation of the problem in terms of the pseudo-inverse of the cumulative distribution function (see Chapter 2.5), suitably rescaled, namely

$$(\partial_x u)^\gamma \partial_t u - (\partial_x u)^{\gamma-2} \partial_x^2 u + u(1 + (\partial_x u)^\gamma) = 0, \qquad (t, x) \in \Omega, \tag{1.3}$$

where $\Omega = (0, \infty) \times (0, m)$, and our wellposedness theory is based on the notion of viscosity solutions. Our concept of solution for the problem in these new variables is such that the entropy minimisers mentioned above will always be (admissible) solutions. Let us finally remark that, in higher dimensions, equation (1.2) in radial coordinates can be reformulated in a similar way. This allows us to set up a numerical scheme for equation (1.2) under radial symmetry, including 3D KQ, able to cope with singular solutions and condensates.

1.1.2 Aggregation equations with fractional diffusion

Part II of this thesis is concerned with a question arising in applications related to biological aggregation. Our point of departure is the equation

$$\partial_t \rho = -\Lambda^\gamma \rho + \nabla \cdot (\rho \nabla K * \rho), \quad t > 0,\ x \in \mathbb{T}^d. \tag{1.4}$$

Here, $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the so-called half Laplacian on the torus $\mathbb{T}^d$ (a nonlocal operator) and K denotes a singular kernel satisfying $\nabla K(x) \sim \frac{x}{|x|^{2+a}}$ near $x = 0$ for suitable $a \geq 0$. We will consider triples of parameters $d \in \mathbb{N}$, $\gamma > 0$ and $a \geq 0$ which are such that equation (1.4) is formally L^1-supercritical. In this case, solutions sufficiently concentrated in some region may explode in finite time (see Chapter 8.5.1). Let us note that a general difficulty of equation (1.4), compared to equation (1.2), is the circumstance that both diffusion and velocity field depend on the unknown in a nonlocal way.

We are now interested in the situation where aggregation takes place in an ambient fluid, and ask the question of whether the presence of an ambient flow can affect the dynamics of equation (1.4). We will show that even a stationary linear incompressible flow can qualitatively change the behaviour of solutions leading to a suppression of the formation of singularities caused by aggregation. More precisely, we identify a class of divergence-free Lipschitz vector fields u such that any local-in-time solution of the Cauchy problem

$$\begin{aligned} \partial_t \rho + u \cdot \nabla \rho &= -\Lambda^\gamma \rho + \nabla \cdot (\rho \nabla K * \rho) \quad t > 0,\ x \in \mathbb{T}^d, \\ \rho(0, \cdot) &= \rho_0, \end{aligned} \tag{1.5}$$

extends to a globally regular one if the flow is fast enough.

The mechanism behind the prevention of singularities is an enhancement of dissipation due to the mixing properties of the incompressible flows considered. Loosely speaking, in non-equilibrium states mixing leads to a transfer of energy towards higher frequencies, which, in diffusive equations, results in dissipation being amplified. One of our core references is the work by Constantin, Kiselev, Ryzhik and Zlatoš [34], which studies the effect of mixing in diffusion equations on a compact Riemannian manifold. This reference provides a sharp characterisation, in form of a spectral condition, of the incompressible flows on $\mathbb{T}^d$ which are able, in a certain sense, to significantly speed up relaxation to equilibrium in diffusion equations. We will extend this characterisation to equations involving the fractional Laplacian $-\Lambda^\gamma$ of order $\gamma < 2$, which provides us with the class flows we admit in problem (1.5). The above characterisation relies on a version of the so-called RAGE theorem from quantum mechanics (see e.g. [99, Chapter 5.2]) describing the dynamics of a quantum state in the continuous spectral subspace of the Hamiltonian, and it includes flows which are weakly mixing in the ergodic sense.

We should mention that the question of blow-up suppression through mixing was studied before by Kiselev and Xu [73] for the classical parabolic-elliptic Keller–Segel model on $\mathbb{T}^d$ for $d = 2, 3$. Our analysis provides an extension to the case of fractional diffusion and more general aggregation kernels, and applies to the Keller–Segel model in arbitrarily high dimensions.

1.2 Outline of the thesis

This thesis is divided into two parts. Part I, i.e. Chapters 2 to 7, is concerned with the study of the bosonic Fokker–Planck equations introduced in Section 1.1.1, while Part II, consisting of Chapter 8, investigates the aggregation-diffusion problem outlined in Section 1.1.2. A short summary of the content of each of these chapters is given below.

In Chapter 2 we provide relevant background information on the Kaniadakis–Quarati model (KQ) and its generalisation (1.2). We introduce the associated entropy functional and steady states, and review the existing literature related to the problem. Furthermore, in the 1D case, we introduce a transformation leading to an equation posed in mass variables, which is equivalent to the original problem as far as non-degenerate, classical solutions are concerned. This reformulation constitutes the basis of our approach towards equation (1.2) and motivates the framework in Chapter 3. In Section 2.6 we introduce some of the notations adopted in Part I.

In Chapter 3 we establish a general framework for the existence, uniqueness and regularity of viscosity solutions $u = u(t, x)$, $x \in (x_1, x_2) \Subset \mathbb{R}$, to a class of nonlinear, degenerate/singular parabolic equations

$$G(u, \partial_t u, \partial_x u, \partial_x^2 u) = 0,$$

where G is a continuous function which is non-decreasing in the first, second and last argument and satisfies an additional strict monotonicity condition in one of the first two arguments. From this framework we infer global-in-time existence, uniqueness and Lipschitz continuity of solutions $u = u(t, x)$, non-decreasing in x, to a generalised version of equation (1.3), see Theorem 3.20.

Chapter 4 is devoted to the family of L^1-supercritical 1D Fokker–Planck equations for bosons, i.e. equation (1.2) with $d = 1$ and $\gamma > 2$. Using the well-posedness and regularity results from Chapter 3, we show that the constructed viscosity solutions u of the equation in the new variables, i.e. of equation (1.3), are smooth away from the level set $\{u = 0\}$, and that the push-forward measure $u(t,\cdot)_{\#}\mathcal{L}^1_{|[0,m]} =: \mu(t) \in \mathcal{M}_b^+$, generalising the problem in the original variables, has the form

$$\mu(t) = f(t,\cdot)\mathcal{L}^1 + x_p(t)\delta_0,$$

where the map $t \mapsto x_p(t) := \mathcal{L}^1(\{u(t,\cdot) = 0\})$ is continuous and the function $f(t,\cdot) \in L^1_+$ is smooth away from the origin, where it satisfies equation (1.2) in the classical sense. Moreover, whenever the density $f(t,\cdot)$ is unbounded at the origin, its spatial blow-up profile, to leading order, is given by $c_\gamma r^{-2/\gamma}$, where $c_\gamma = (2/\gamma)^{\frac{1}{\gamma}}$. We are then able to extend entropy methods globally in time, from which we infer the long-time asymptotics of solutions. As a consequence, we obtain finite-time condensation for solutions above the critical mass as well as eventual regularity for solutions below the critical mass.

In Chapter 5 we present refinements of the theory established in Chapter 4 and discuss results providing a link to the numerical study in Chapter 6. We derive a criterion for finite-time blow-up and condensation for highly concentrated initial data (Section 5.1), and analyse the spatiotemporal behaviour during blow-up and blow-down (Section 5.2). We further provide a formula for the evolution of the condensate component and show that it is Lipschitz continuous in time (Section 5.3). Finally, we prove that (eventually) regular solutions relax to equilibrium at an exponential rate (Section 5.4) bounded below by a universal constant.

In Chapter 6 we present a time-implicit numerical scheme for the equation in the new variables, assuming radial symmetry in higher dimensions. The scheme is validated with the help of explicit solutions to 2D KQ in radial coordinates. We qualitatively replicate some of the main properties of the 1D bosonic Fokker–Planck equations proved in Chapters 4 and 5 and study numerically the condensation process in the 3D Kaniadakis–Quarati model in the isotropic case. The numerical experiments suggest that the L^1-supercritical case of the 1D model captures the main dynamical properties of 3D KQ in a qualitatively correct way.

In Chapter 7 we discuss the results obtained in the previous chapters and provide perspectives on future work.

In Chapter 8 we consider the problem outlined in Section 1.1.2. We first establish a general L^2 blow-up criterion, guaranteeing the regularity of solutions as long as their L^2 norm is controlled (Section 8.3). In Section 8.4 we introduce a specific class of flows, a generalisation of weakly mixing flows, which are relaxation enhancing with respect to fractional diffusion of order $\gamma > 0$. Using the relaxation enhancement in the diffusive equation, we then prove that, if the coupling parameter regulating the strength of the flow is large enough, the flow is able to suppress aggregation-induced singularities, leading to globally regular solutions relaxing to equilibrium at an exponential rate (Theorem 8.13). We further show how an L^p based approach allows to deduce similar results for the classical parabolic-elliptic Keller–Segel model in arbitrarily high dimensions (Theorem 8.17).

Part I

Fokker–Planck equations for Bose–Einstein particles

Chapter 2

Background on the bosonic Fokker–Planck equations

In this chapter we provide relevant background information on the family of bosonic Fokker–Planck equations (BFP) introduced in Section 1.1.1:

$$\begin{aligned} \partial_t f &= \Delta_v f + \mathrm{div}_v(vf(1+f^\gamma)), \quad t>0,\ v\in\mathbb{R}^d, \\ f(0,\cdot) &= f_0 \geq 0. \end{aligned} \tag{2.1}$$

Here $\gamma > 0$ is a fixed parameter and $f = f(t,v) \geq 0$. Let us briefly explain the origin and background of this equation. Recall that equation (2.1) with $\gamma = 1$ (i.e. eq. (KQ) in Section 1.1.1) is referred to as the *Kaniadakis–Quarati model* for bosons (KQ). It was introduced by Kaniadakis and Quarati [70] as a model for the dynamics of the velocity distribution of a homogeneous[1] system of bosons. While in quantum mechanics a system of bosons is described by a wave function, the KQ model assumes that the dynamics of the system can be well approximated by a system of interacting particles in which the transition probability rates between different states are modified in a way as to account for the quantum effect. Let us note that in the KQ model only nearest neighbour interactions are taken into account. The quantum effect observed in bosonic systems is reflected by the quadratic nonlinearity in eq. (KQ): in general, in a quantum system of identical and indistinguishable particles the presence of particles in a given energy state influences the probability of further quantum particles joining that state. (This is a consequence of the symmetry properties of the underlying wave function with respect to particle permutations.) For bosonic systems this probability is increased. In the particle model this translates into the particles obeying Bose statistics, while in the continuum KQ model the effect is encoded in

[1]Here, homogeneity means that the problem is independent of the position variable. In this case, an evolution problem in phase space reduces to an evolution law in velocity space.

the amplification factor $(1 + f)$ in the drift velocity by which it differs from the linear Fokker–Planck equation. We should mention that a rigorous derivation of eq. (KQ) from Boltzmann type equations for interacting quantum particles is not available.

For KQ the choice $d = 3$ is the physically most interesting space dimension. In this case the problem exhibits a finite critical mass m_c, above which condensates are expected to emerge in finite time. Equation (1.2) is a generalisation of KQ which has a similar entropy structure and family of steady states. This structure will be described in the following section.

2.1 Variational structure and steady states

Equation (2.1) has a natural *entropy functional*, given by

$$\mathcal{H}_\gamma(f) := \int_{\mathbb{R}^d} \left(\frac{|v|^2}{2} f + \Phi(f) \right) \mathrm{d}v,$$

where $\Phi(f) := \frac{1}{\gamma} \int_0^f \log\left(\frac{s^\gamma}{1+s^\gamma}\right) \mathrm{d}s$ and thus $\Phi''(f) = 1/h_\gamma(f)$ for $h_\gamma(s) := s(1 + s^\gamma)$. Indeed, formally, equation (2.1) can be rewritten as

$$\partial_t f = \nabla \cdot \left(h_\gamma(f) \nabla \frac{\delta \mathcal{H}_\gamma}{\delta f}(f) \right), \tag{2.2}$$

where $\frac{\delta \mathcal{H}_\gamma}{\delta f}(f)$ denotes the variational derivative of $\mathcal{H}_\gamma$ at f. Thus, for any sufficiently regular, positive (and hence mass conserving) solution $f = f(t, v)$ of eq. (2.1), one obtains the *entropy dissipation identity*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}_\gamma(f) = -\int_{\mathbb{R}^d} h_\gamma(f) \left| \nabla \frac{\delta \mathcal{H}_\gamma}{\delta f}(f) \right|^2 \mathrm{d}v. \tag{2.3}$$

Notice, however, that due to the presence of the (quantum correction) term s^γ in the definition of the mobility function $h_\gamma(s)$, equation (2.2) is not a gradient flow of the functional $\mathcal{H}_\gamma$ with respect to the classical Wasserstein metric. In contrast to the fermionic case (see Section 2.3), in which the mobility $h(s) = s(1 - s)$ enables the application of gradient flow methods based on generalised Wasserstein metrics [30], the convexity of the mobility function $h_\gamma(s)$ associated with the continuity equation (2.2) leads to issues when trying to render rigorous the gradient flow structure [40]. However, this formal gradient flow structure is a motivation for our approach to deal with condensates (see Sections 2.5 & 6.1).

We observe that, given a sufficiently regular positive function f, the right-hand side of equation (2.3) is strictly negative unless $\nabla \frac{\delta \mathcal{H}_\gamma}{\delta f}(f) = 0$. The regular positive solutions of this equation are henceforth referred to as the *steady states* associated with problem (2.1). They are explicitly given by[2]

$$f_{\infty,\theta}(v) = \left(e^{\gamma(\frac{|v|^2}{2}+\theta)} - 1\right)^{-1/\gamma}, \quad \theta \geq 0, \tag{2.4}$$

and are the natural candidates for the asymptotic behaviour of solutions $f(t,\cdot)$ as $t \to \infty$. Notice that $f_{\infty,\theta}$ is smooth and integrable for $\theta > 0$, and that the family $\{f_{\infty,\theta}\}$ is strictly ordered and approaches an unbounded 'limiting steady state' $f_c := f_{\infty,0}$ from below as $\theta \searrow 0$. Furthermore, letting $m_c := \int f_c$, the map

$$(0,\infty) \ni \theta \mapsto m_\theta := \int f_{\infty,\theta} \in (0, m_c)$$

is a bijection, and $m_c < \infty$ if and only if $\gamma > \frac{2}{d}$, i.e. if and only if the problem is L^1-supercritical. While $f_{\infty,\theta}$ is the unique minimiser of $\mathcal{H}_\gamma$ among non-negative integrable functions of mass $m = m_\theta$ [11, 28, 45], for $m > m_c$ the problem of minimising $\mathcal{H}_\gamma$ under mass constraint does not have a regular solution. Since Φ is sublinear at infinity (in the sense that $\lim_{s\to\infty} \frac{\Phi(s)}{s} = 0$), the natural extension $\widetilde{\mathcal{H}}_\gamma$ of the entropy functional $\mathcal{H}_\gamma$ to the set $\mathcal{M}_b^+(\mathbb{R}^d)$ of finite non-negative Borel measures on $\mathbb{R}^d$ is given by

$$\widetilde{\mathcal{H}}_\gamma : \quad \mu \mapsto \int_{\mathbb{R}^d} \mathrm{d}\nu,$$

where

$$\nu = \frac{|\cdot|^2}{2}\mu + \Phi(f)\mathcal{L}^d, \qquad \mu = f \cdot \mathcal{L}^d + \mu_s, \quad \mu_s \perp \mathcal{L}^d.$$

In words, f denotes the density of the component of μ which is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$. The extension $\widetilde{\mathcal{H}}_\gamma$ is convex and lower semicontinuous with respect to weak-star convergence in $\mathcal{M}$ [11, 39]. Furthermore, it is not difficult to see that sublevel sets of $\widetilde{\mathcal{H}}_\gamma$ restricted to $\{\mu \in \mathcal{M}_b^+ : \int \mu = m\}$ are tight. Hence the existence of a minimiser among measures of mass m is guaranteed by the lower semicontinuity of $\widetilde{\mathcal{H}}_\gamma$. The precise form of these minimisers and their uniqueness has been established in [11], based on explicit expansions:

Theorem 2.1 (See [11], Theorem 3.1)**.** *Let $m \in (0,\infty)$. The functional $\widetilde{\mathcal{H}}_\gamma$ restricted to the set*

$$\{\mu \in \mathcal{M}_b^+(\mathbb{R}^d) : \int \mu = m\}$$

[2]For simplicity and reasons to become clear below, we include the limiting case $\theta = 0$ in (2.4), although $f_{\infty,0}$ is not smooth.

has a unique minimiser $\mu_\infty^{(m)}$. *The minimiser is given by*

$$\mu_\infty^{(m)} = \begin{cases} f_{\infty,\theta} \cdot \mathcal{L}^d & \text{if } m \le m_c, \text{ where } \theta \ge 0 \text{ is s.t. } \int f_{\infty,\theta} = m \\ f_c \cdot \mathcal{L}^d + (m - m_c)\delta_0 & \text{if } m > m_c. \end{cases}$$

Here $f_{\infty,\theta}$ *is given by formula* (2.4), $f_c := f_{\infty,0}$ *and* $m_c = \int f_c$.

Remark 2.2 (Problem on a centred ball). In this thesis, we will also consider a slightly modified problem obtained by replacing the spatial domain $\mathbb{R}^d$ by a finite centred ball $B_R := \{|v| < R\} \subset \mathbb{R}^d$ for $R \in (1, \infty)$ and imposing a zero-flux boundary condition on ∂B_R, i.e.

$$\begin{aligned} \partial_t f &= \Delta_v f + \operatorname{div}_v(vf(1 + f^\gamma)), & t > 0,\ v \in B_R, \\ 0 &= (\nabla_v f + vf(1 + f^\gamma)) \cdot v, & t > 0,\ |v| = R, \\ f(0, v) &= f_0(v) \ge 0, & v \in B_R. \end{aligned}$$

This problem has a natural entropy functional $\widetilde{\mathcal{H}}_\gamma^{(R)} : \mathcal{M}_b^+(\bar{B}_R) \to \mathbb{R}$, given by

$$\widetilde{\mathcal{H}}_\gamma^{(R)}(\mu) = \int_{\{|v| \le R\}} \left(\frac{|v|^2}{2} \mu(\mathrm{d}v) + \Phi(f)\, \mathrm{d}v \right), \qquad \left(\mu - f \cdot \mathcal{L}^d\right) \perp \mathcal{L}^d.$$

The entropy $\widetilde{\mathcal{H}}_\gamma^{(R)}$ on $\mathcal{M}_b^+(\bar{B}_R)$ has properties completely analogous to those of the functional $\widetilde{\mathcal{H}}_\gamma$ on $\mathcal{M}_b^+(\mathbb{R}^d)$ outlined above. In particular, for a fixed positive mass $m \le \int_{B_R} f_c(v)\, \mathrm{d}v =: m_c(R)$, the unique minimiser of $\widetilde{\mathcal{H}}_\gamma^{(R)}$ on the set

$$\{\mu \in \mathcal{M}_b^+(\bar{B}_R) : \int \mu = m\} \tag{2.5}$$

is given by the absolutely continuous measure $f_{\infty,\theta} \cdot \mathcal{L}^d$, restricted to $\bar{B}_R$, where $\theta \ge 0$ is such that $\int_{B_R} f_{\infty,\theta} = m$, while in the mass-supercritical case $m > m_c(R)$ the unique minimiser of $\widetilde{\mathcal{H}}_\gamma^{(R)}$ on the set (2.5) is given by the measure $f_c \cdot \mathcal{L}^d + (m - m_c(R))\delta_0$, restricted to $\bar{B}_R$, which has a non-trivial singular part. This assertion is easily proved by following the reasoning in the constraint minimisation problem for $\widetilde{\mathcal{H}}_\gamma$, see [11, Theorem 3.1].

Let us next summarise the existing literature on the dynamical properties of the bosonic Fokker–Planck equations. So far, studies have focused on the specific choice $\gamma = 1$, i.e. on the Kaniadakis–Quarati model.

2.2 Dynamics of the Kaniadakis–Quarati model

The long-time dynamics of solutions to KQ are dimension-dependent. Recall that in both the L^1-subcritical and the L^1-critical case the limiting steady state f_c is not integrable near the origin, and for arbitrarily large mass $m \in (0, \infty)$ there exists a unique smooth and exponentially decaying steady state of mass m. The critical space dimension for KQ is $d = 2$. Loosely speaking, the space dimension determines whether all (reasonable) solutions are globally regular ($d \leq 2$) or whether there exist solutions blowing up in L^∞ in finite time ($d > 2$). More precisely, the following has been established in the literature:

1D: In the L^1-subcritical case, $d = 1$, KQ is globally wellposed in the classical sense for sufficiently regular initial data, and solutions converge to equilibrium at an exponential rate [24]. The global existence of regular solutions can be proved by a comparison principle at the level of the cumulative distribution function of the density f in a way morally similar to the proof of Proposition 5.4 below.

2D: In the L^1-critical case, $d = 2$, solutions are also globally regular and relax to equilibrium in the long-time limit [20], where the rate is exponential in the isotropic case $f(t, v) = g(t, |v|)$. The approach in [20] exploits the fact that the 2D KQ equation in isotropic coordinates can be transformed to a linear Fokker–Planck equation, which leads to explicit solutions also for the nonlinear equation. The results in [20] are valid for a large class of initial data (including isotropic finite Borel measures). Global regularity in the non-radial case is obtained upon comparison with isotropic solutions.

3D: For 3D KQ, Toscani [100] proved via contradiction, using a virial-type argument, the existence of solutions blowing up in finite time. Finite-time blow-up in this reference is obtained for any solution of sufficiently large mass m (above a technical threshold far larger than the critical mass), but also for solutions of arbitrarily small mass provided they are initially sufficiently concentrated near the origin. We use a variant of this argument in Section 5.1 to establish the existence of transient condensates for our 1D model.

Formal results on the dynamics of isotropic solutions to 3D KQ have been obtained by Sopik, Sire and Chavanis [96]. In the mass-supercritical case[3] their results suggest that near the first blow-up time, denoted by T^*, $f(t, 0) \approx$

[3]To be more precise, in [96] mass is normalised, and instead temperature is the parameter determining whether the associated equilibrium has a condensate component. Mathematically, a temperature below the critical one corresponds to a solution with supercritical mass $m > m_c$ in the notation of this thesis.

$(T^* - t)^2$ as $t \nearrow T^*$ and that, typically, the spatial blow-up profile $f(T^*, v) := \lim_{t \nearrow T^*} f(t, v)$ (is well-defined and) should satisfy

$$f(T^*, v) = f_c(v) + \frac{c(m)}{|v|} + o\left(|v|^{-1}\right) \qquad \text{as } |v| \to 0$$

for some explicit constant $c(m) > 0$ satisfying $c(m) \to 0$ as $m \searrow m_c$. Notice that this implies[4]

$$f(T^*, v)/f_c(v) = 1 + \frac{c(m)}{2}|v| + o\left(|v|\right) \qquad \text{as } |v| \to 0.$$

Our numerical simulations in Chapter 6 will qualitatively confirm some of the main findings in [96], suggesting that the dynamics depicted both in this reference and by our simulations give a good hint at the typical behaviour of solutions. We should, however, mention that the approach in [96] assumes the initial density to be sufficiently spread out, and, in fact, our numerical experiments indicate that, in general, the dynamics may display a richer variety of phenomena.

2.3 Related equations

Several equations closely related to the bosonic Fokker–Planck equations (2.1) have been considered in the literature. The most relevant equations, described below, are modifications avoiding some of the main mathematical difficulties in eq. (2.1). The study of these problems still provides valuable insights with regard to the understanding of eq. (2.1).

- *Non-diffusive case:*

  A hyperbolic version of equation (2.1) in 1D without the diffusion term was studied in [25]. The authors prove global-in-time existence and uniqueness of measure-valued solutions, which, in the large-time limit, concentrate all their mass at the origin. They further show that condensates always form in finite time and that their mass is non-decreasing in time so that, once formed, they never dissolve. The results reported in this thesis (see Chapters 5 and 6) show the genuine countereffect of linear diffusion on condensation. Indeed, the presence of diffusion leads to the possibility of non-monotonic behaviour of the size $x_p(t)$ of the point mass at the origin and to the existence of transient

[4] Let us warn that many of the quantitative findings of [96] are unlikely to hold true at the first blow-up time for solutions initially very concentrated.

condensates, as proved below in one dimension for $\gamma > 2$ (see Chapter 5.1) and conjectured in the three dimensional case for $\gamma = 1$ (see Chapter 6).

- *Sublinear diffusion & linear drift:*

  The reference [53] considers a 1D Fokker–Planck equation with sublinear diffusion and linear drift exhibiting a critical mass m_c and an entropy functional whose minimising measure of mass $m > m_c$ has a non-trivial singular component with respect to the Lebesgue measure. Exploiting the fact that the equation is the gradient flow of the entropy functional with respect to the L^2-Wasserstein distance, the authors prove global wellposedness of measure solutions relaxing to the entropy minimiser of the same mass. We anticipate that we will obtain a similar result in our problem, see Chapter 3.6 and Theorem 4.16 resp. Theorems 4.24 and 4.26. However, the fact that the drift in [53] is linear precludes the possibility of finite-time condensation for bounded initial data.

- *Fermionic case:*

  The counterpart of the bosonic Kaniadakis–Quarati model, equation (KQ) of Chapter 1, for Fermi–Dirac particles was introduced in [69, 70]. It differs from eq. (KQ) in the sign of the nonlinear part of the drift, meaning that the nonlinearity is defocusing in the fermionic case. The steady states of this equation coincide with the Fermi–Dirac distributions, which are uniformly bounded in L^∞ and can accommodate arbitrarily large mass in any space dimension. Thus, in this case there is no critical mass and, as proved in [29], solutions emanating from sufficiently regular initial data are globally regular and relax to equilibrium at an exponential rate if bounded above by one of the Fermi–Dirac distributions.

2.4 Other models for Bose–Einstein condensation

There are many other models in the literature which have been suggested in the context of Bose–Einstein condensation. Below, we review some of the most prominent examples as well as equations specifically relevant for our work.

- *Inhomogeneous (kinetic) Fokker–Planck equation for bosons:*

  A generalisation of eq. (KQ) modelling the evolution in phase space of the distribution $f(t,\cdot,\cdot) = f(t,x,v)$ of a system of bosons has been introduced in [68]. Versions of this model have been considered in [87, 89]. These studies show the stability of the smooth steady states and investigate the relaxation rates to equilibrium in the perturbative setting. The possible formation of singularities and condensates has not yet been investigated in the inhomogeneous case.

- *Boltzmann–Nordheim equation for bosons:*

  Kinetic Boltzmann-type equations for a weakly interacting gas of quantum particles have first been introduced by Nordheim [90] and Uehling & Uhlenbeck [101]. These equations are obtained by using Bose–Einstein statistics in the derivation of the Boltzmann collision operator. Most relevant in our context is the Boltzmann–Nordheim equation for bosons, also called the Boltzmann equation for Bose–Einstein particles. The spatially homogeneous and velocity isotropic Boltzmann–Nordheim equation for bosons shares with KQ its steady states, the Bose–Einstein distributions, as well as its entropy functional (up to a sign convention and a constant equal to the kinetic energy of the initial datum), see e.g. [48, 61]. In contrast to equation (2.1), the Boltzmann–Nordheim equation formally preserves the kinetic energy $\int \frac{|v|^2}{2} f \, \mathrm{d}v$.

  In the last two decades, significant progress has been made in the analysis of the Boltzmann–Nordheim equation in the homogeneous and velocity isotropic case [4, 44, 46–48, 81–85]. To roughly summarise the main results, the authors of the cited references are able to establish the existence of generalised mass- and energy-conserving solutions, which form a Bose–Einstein condensate in finite time and converge, in some sense and under certain conditions, to the entropy minimiser in the large-time limit. The question of uniqueness of the generalised solutions introduced in these references has not (yet) been investigated.

  The results in this thesis suggest that the dynamical properties of condensation in 3D KQ and its one-dimensional toy model, eq. (2.1) with $\gamma > 2$, are in some aspects similar to those observed in the homogeneous and velocity isotropic Boltzmann–Nordheim equation as described rigorously in the references [4, 47, 48, 84, 85]. We note that, regarding the nature of singularities, in the Boltzmann–Nordheim equation many questions are still open.

- *Fourth order model:*

  In [64] a degenerate fourth-order PDE has been proposed as a higher-order approximation of the spatially homogeneous and velocity isotropic Boltzmann–Nordheim equation. This PDE has recently been shown to exhibit solutions blowing up in finite time [66, 67]. In contrast to the Boltzmann–Nordheim equation and 3D KQ, this model does not possess a critical mass.

- *Kompaneets equation:*

  A model describing the momentum distribution of photons in a homogeneous plasma under the assumption that interaction with matter occurs via Compton scattering has been introduced by Kompaneets in [74]. As a special case one obtains a nonlinear Fokker–Planck-type equation on $(0,\infty)$, versions of which have been studied in the references [43] and [76]. In this model the break-down of the zero-flux boundary condition at $x=0$ is interpreted as the onset of a condensate. The phenomenon of condensation is, however, rather different from the one observed in our bosonic Fokker–Planck equations, where, in general, the condensate does interact with the density and, near the condensate, diffusion and drift are balanced to leading order. Indeed, in the Kompaneets model condensate formation is a purely hyperbolic phenomenon: near the origin, the diffusive part becomes negligible and the fraction of photons trapped in the condensate cannot decrease [76].

- *Nonlinear Schrödinger equations*:

  Other, rather different models, describing quantum effects in a gas of weakly interacting bosons at very low temperatures, involve nonlinear equations of Schrödinger type and, in particular, the Gross–Pitaevskii equation, see e.g. [5, 6, 42, 49] and references therein. In [94, 95] the authors investigate systems composed of a kinetic equation which is coupled to a nonlinear Schrödinger equation modelling the evolution of the condensate.

2.5 Equation for pseudo-inverse distribution

In this section we aim to motivate the class of equations considered in the next chapter by introducing a change of variables which constitutes the basis of our approach to eq. (2.1) in 1D in the L^1-supercritical regime.

In the following we fix some $R\in(0,\infty)$, arbitrarily large, and consider eq. (2.1) with $d=1$ and $\gamma\geq 2$, posed on a centred interval of radius R, i.e.

$$\partial_t f = \partial_r^2 f + \partial_r(rf(f^\gamma+1)), \qquad t>0,\ r\in(-R,R), \tag{2.6}$$

$$f(0,r) = f_0(r), \qquad r\in(-R,R),$$

$$0 = \partial_r f + rf(f^\gamma+1), \qquad t>0,\ r\in\{-R,R\}. \tag{2.7}$$

Notice that we have added a boundary condition, eq. (2.7), which formally ensures the conservation of mass. In equation (2.6) we have changed the order of the summands in the factor $(1+f^\gamma)$ to emphasise that on a bounded domain the linear part of the drift becomes essentially irrelevant. We use the variable r to indicate that the velocity space is one-dimensional, a property which the theory developed in Chapter 3 relies on. We would, however, like to remark that r can be negative and that our analysis does not assume symmetry in $|r|$. It will be convenient to first devise a theory for this modified problem posed on a bounded domain. In essence, the results obtained for problem (2.6)–(2.7) remain valid in the limit $R \to \infty$ for initial data f_0 satisfying a suitable decay condition at infinity (see Chapter 4.4).

Our approach to equation (2.6) is motivated by the formal Wasserstein-like gradient flow structure (2.2) and builds upon the hypothesis of mass conservation. It is based on a reformulation of the problem in terms of the pseudo-inverse cumulative distribution function

$$u(x) = \inf \left\{ r : \int_{\{r' \le r\}} f(r')\,\mathrm{d}r' \ge x \right\}, \quad x \in (0, \|f\|_{L^1}).$$

To proceed, let us first specify more precisely some of the important notations and conventions used in the next chapter. For a non-negative finite Borel measure ν on $[-R, R]$ we define the *cumulative distribution function (cdf) M* associated with ν via

$$M(r) = \nu([-R, r]), \quad r \in [-R, R]. \tag{2.8}$$

The cumulative distribution function of a function $f \in L^1(-R, R)$ is defined as the cdf associated with the measure $f \cdot \mathcal{L}^1$, where here $\mathcal{L}^1$ denotes the one-dimensional Lebesgue measure restricted to the interval $[-R, R]$.

Definition 2.3. Let $R, m > 0$. Given a strictly increasing, right-continuous function $M : [-R, R] \to [0, m]$ with $M(R) = m$, we define its *pseudo-inverse* $u : [0, m] \to [-R, R]$ via

$$u(x) = \min\{r \in [-R, R] : M(r) \ge x\}, \quad x \in [0, m].$$

The function u is well-defined and continuous, and satisfies $u(0) = -R$, $u(m) = R$ as well as $u(x) = r$ whenever $x \in [M(r-), M(r)]$, $r \in [-R, R]$.
We often use the short phrase '(pseudo-) inverse cumulative distribution function' to refer to the (pseudo-) inverse of the cumulative distribution function of a measure or density.

Notice that a Dirac mass at the origin in a measure ν translates into a jump at the origin at the level of its cumulative distribution function M as defined in formula (2.8). Figure 2.1 illustrates how such a jump for $M = M(r)$ is transformed into a flat part at the level of its pseudo-inverse u. Analytically, the function u is much better behaved than M.

Assume for the moment that $f = f(t, r)$, $t > 0$, is a strictly positive classical solution of problem (2.6)–(2.7) of mass m. Then for fixed t its cumulative distribution function $M(t, \cdot)$ satisfies the assumptions in Definition 2.3, and we can consider the pseudo-inverse $u(t.\cdot)$ of $M(t, \cdot)$, which satisfies $M(t, u(t, x)) = x$ for $x \in [0, m]$.

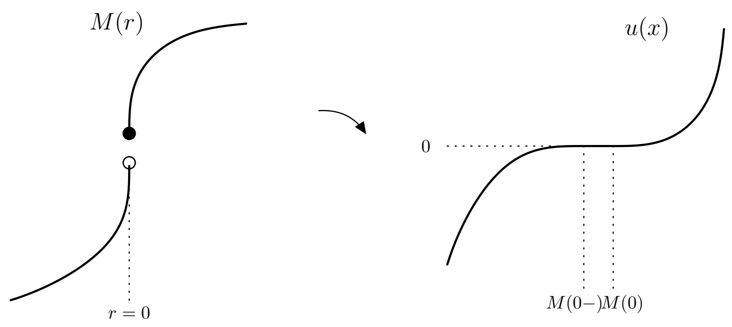


Figure 2.1: A strictly increasing, right-continuous function M with a jump discontinuity at the origin and its pseudo-inverse u, a continuous non-decreasing function with a flat part at level zero.

Under these assumptions, $\{u(t,\cdot)\}_{t>0}$ is a one-parameter family of diffeomorphisms between $(0,m)$ and $(-R,R)$. By a straightforward calculation, using in particular the relation

$$\partial_x u = \frac{1}{f(u)},$$

where we omitted the time argument, one finds that u satisfies the equation

$$\partial_t u - (\partial_x u)^{-2}\partial_x^2 u + u((\partial_x u)^{-\gamma} + 1) = 0.$$

Following an idea in [25, Section 4], we multiply the last equation by $(\partial_x u)^\gamma$ to obtain

$$(\partial_x u)^\gamma \partial_t u - (\partial_x u)^{\gamma-2}\partial_x^2 u + u(1 + (\partial_x u)^\gamma) = 0 \quad \text{in } \Omega, \tag{2.9}$$

where $\Omega := (0,T) \times (0,m)$. Observe that the choice of a time-independent domain $(0,m)$ for $u(t,\cdot)$, $t > 0$, imposes conservation of mass. Further notice that strict positivity of $f(t,\cdot)$ for $t > 0$ is a natural hypothesis in view of the uniform parabolicity (and the structure of the nonlinearity) of the problem in the original variables. This justifies supplementing equation (2.9) with the lateral boundary conditions

$$u(t,0) = -R, \quad u(t,m) = R \qquad \text{for all } t > 0, \tag{2.10}$$

satisfied by the family of diffeomorphisms. If solutions are regular up to the lateral boundary, the constant-in-time boundary conditions (2.10) can be alternatively obtained by combining the zero-flux boundary condition (2.7) for the positive density $f(t,r)$ with equation (2.9), evaluated on $(0,\infty) \times \{0,m\}$ (see Lemma 4.11 for the reverse direction). To summarise, equation (2.9) is to be complemented with the following conditions on the parabolic boundary:

$$\begin{aligned} u(0,x) &= u_0(x), && x \in (0,m), \\ u(t,0) &= -R,\ u(t,m) = R, && t > 0. \end{aligned}$$

In order to avoid technicalities at initial time, we only consider initial data satisfying the 0^{th} order compatibility conditions $u_0(0) = -R, u_0(m) = R$.

Let us remark that in our framework, based on the above change of variables, the minimisers appearing in Theorem 2.1 (resp. in Remark 2.2) will be admissible solutions while for $\theta > 0$ and $m > \|f_{\infty,\theta}\|_{L^1}$ measures of the form

$$f_{\infty,\theta} \cdot \mathcal{L}^1 + (m - \|f_{\infty,\theta}\|_{L^1})\delta_0$$

will be neither sub- nor supersolutions. In this way, the latter family is naturally ruled out as potential equilibria, which would not be the case when, for instance, considering distributional solutions of the original formulation (2.1) using test functions vanishing near the origin.

2.6 Notations and conventions (Part I)

Here, we provide a list of notations and conventions commonly adopted in Part I of this thesis. The list is non-exhaustive and further, more specific definitions will be introduced in the course of the text.

- We let $\Omega := I \times J := (0,T) \times (0,m)$, where $0 < T \leq \infty$ and $0 < m < \infty$. The parabolic boundary of Ω, denoted by $\partial_p\Omega$, is defined as the set

  $$\partial\Omega \setminus (\{T\} \times [0,m]),$$

  where $\partial\Omega$ denotes the topological boundary of Ω. This notation will be also be used for more general axis-aligned rectangles $\subset \mathbb{R} \times \mathbb{R}$. We refer to the subset $(0,T) \times \{0,m\} \subset \partial_p\Omega$ as the *lateral boundary* of Ω.
- For an interval $V \subset \mathbb{R}$, any *measure* on V is understood to be a non-negative Borel measure, and we denote by $\mathcal{M}_b^+(V)$ the set of finite measures on V.
- *Test functions* are C^1 in time and C^2 in space (meaning that the first time derivative and the second spatial derivative exist and are in $C(\Omega)$).
- In general, for a function $u = u(x_1, \dots, x_N)$ the expressions $\partial_{x_i} u$ and u_{x_i} for some $i \in \{1, \dots, N\}$ both denote the weak derivative (in the distributional

sense) of the function u in the i^{th} direction. The pointwise derivative of u with respect to x_i will be denoted by ${}^{(p)}\partial_{x_i} u$.

- For a function $u : (a, b) \subset \mathbb{R} \to \mathbb{R}$ we denote by u' its (weak) derivative.
- For $d \in \mathbb{N}$ the expression $\mathrm{Sym}(d)$ denotes the space of symmetric $d \times d$ matrices with real components.
- For $\alpha \in (0, 1]$ and $U \subset \mathbb{R}^d$ we abbreviate $[u]_{C^{0,\alpha}(U)} := \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.
- The d-dimensional Lebesgue measure on $\mathbb{R}^d$ is denoted by $\mathcal{L}^d$. We use the same symbol for its restriction to any Lebesgue measurable subset $U \subset \mathbb{R}^d$.

Apart from function spaces whose notation in the literature is mostly consistent, we use the following spaces:

- For $V \subset \mathbb{R}^2$ open, we abbreviate $C^{1,2}_{x_1,x_2}(V) = \{u \in C^1(V) : \partial^2_{x_2} u \in C(V)\}$. In this notation, x_1 will always represent the time variable.
- For $V \subset \mathbb{R}^2$ open and $\alpha \in (0, 1]$, we let $H_{2+\alpha}(\bar{V})$ denote the set of functions $u \in C^{1,2}_{t,x}(V)$ for which the quantities $\|u\|_{C^1(V)}$, $\|\partial^2_x u\|_{C(V)}$, $[\partial^2_x u]_\alpha$ and $[\partial_t u]_\alpha$ are finite, where
$$[v]_\alpha := \sup_{\substack{(t,x),(s,y)\in V\\ (t,x)\neq(s,y)}} \frac{|v(t,x) - v(s,y)|}{d_p((t,x),(s,y))^\alpha}$$
and $d_p((t,x),(s,y)) := \max\{|t-s|^{\frac{1}{2}}, |x-y|\}$.
- Unless stated otherwise, L^p spaces are to be understood with respect to the Lebesgue measure, i.e. $L^p(U) = L^p(U, \mathcal{L}^d)$ if $U \subset \mathbb{R}^d$ is Lebesgue measurable.
- $L^1_+(U) = \{f \in L^1(U) : f \geq 0 \text{ almost everywhere}\}$.
- $\mathrm{USC}(U)$ (resp. $\mathrm{LSC}(U)$) denotes the set of upper semicontinuous (resp. lower semicontinuous) functions on U (see Definition 3.4).

Chapter 3

A general framework for monotonic viscosity solutions

In this chapter we introduce a weak notion of solution for a class of equations generalising eq. (2.9) and establish an associated wellposedness theory. The equations we consider take the form

$$G(u, \partial_t u, \partial_x u, \partial_x^2 u) = 0 \quad \text{in } \Omega, \tag{3.1}$$

with $\Omega := (0, T) \times (0, m)$, where $G : \mathbb{R}^4 \to \mathbb{R}$ is a continuous function satisfying:

(A0) The function $q \mapsto G(z, \alpha, p, q)$ is non-increasing for all $z, \alpha, p \in \mathbb{R}$.

Additional structural assumptions on G will be formulated when needed the first time. We will use the 'curly font' to denote the corresponding operator, i.e. we let

$$\mathcal{G}(u) := G(u, \partial_t u, \partial_x u, \partial_x^2 u) \tag{3.2}$$

and similarly $\mathcal{F}(u) := F(u, \partial_t u, \partial_x u, \partial_x^2 u)$, where the function F is to be specified. While in the special case considered in Section 2.5 of Chapter 2 the variable x was used to represent the mass variable, we usually refer to $x \in (0, m)$ as a spatial variable provided no confusion arises with the variable v or r used for the velocity space.

In comparison to the existing literature [38, 62, 63], our approach has the following two main novelties: the first one consists in the fact that it can deal with parabolic equations which are not strictly monotonic in the time derivative, as long as G satisfies a certain strict monotonicity condition in its first argument, the second one lies in the preservation of monotonicity in x, provided the problem admits monotonic barriers.

3.1 Preliminary definitions and the notion of solution

Our concept of solution for equation (3.1) is the standard notion of a viscosity solution. In order to formulate it, we first need to introduce some additional notation.

We say that a test function ϕ *touches* the function u *from above* (resp. *from below*) at the point $\omega \in \Omega$ if $\phi(\omega) = u(\omega)$ and if there exists a neighbourhood $N \subseteq \Omega$ of ω such that $\phi \geq u$ (resp. $\phi \leq u$) in N.

Definition 3.1 (Parabolic super-/subdifferential)**.** For a function u defined on Ω and a point $\omega \in \Omega$ we let

$$\mathcal{P}^+u(\omega) = \{(\alpha, p, q) \in \mathbb{R}^3 : \ (\alpha, p, q) = (\partial_t\phi, \partial_x\phi, \partial_x^2\phi)_{|\omega} \text{ for some test function } \phi \\ \text{which touches } u \text{ from above at } \omega\}.$$

Analogously, we define

$$\mathcal{P}^-u(\omega) = \{(\alpha, p, q) \in \mathbb{R}^3 : \ (\alpha, p, q) = (\partial_t\phi, \partial_x\phi, \partial_x^2\phi)_{|\omega} \text{ for some test function } \phi \\ \text{which touches } u \text{ from below at } \omega\}.$$

We further let $\mathcal{P}u(\omega) = \mathcal{P}^+u(\omega) \cap \mathcal{P}^-u(\omega)$.

Remark. The set $\mathcal{P}u(\omega)$ is non-empty if and only if the pointwise derivatives ${}^{(p)}\partial_t u(\omega)$, ${}^{(p)}\partial_x u(\omega)$, ${}^{(p)}\partial_x^2 u(\omega)$ exist. In this case, $\mathcal{P}u(\omega) = \{({}^{(p)}\partial_t u(\omega), {}^{(p)}\partial_x u(\omega), {}^{(p)}\partial_x^2 u(\omega))\}$ is a singleton, which we will then identify with its unique element, i.e.

$$\mathcal{P}u(\omega) = ({}^{(p)}\partial_t u(\omega), {}^{(p)}\partial_x u(\omega), {}^{(p)}\partial_x^2 u(\omega)).$$

Definition 3.2. We let

$$\overline{\mathcal{P}}^{\pm}u(\omega) = \Big\{(\alpha, p, q) \in \mathbb{R}^3 : \ \exists\, \omega_n \in \Omega \text{ and } \exists(\alpha_n, p_n, q_n) \in \mathcal{P}^{\pm}u(\omega_n) \\ \text{such that } (\omega_n, u(\omega_n), \alpha_n, p_n, q_n) \to (\omega, u(\omega), \alpha, p, q)\Big\}.$$

We will also need the elliptic analogues of $\mathcal{P}$ and its versions.

Definition 3.3 (Second-order super-/subdifferential)**.** Let $d \in \mathbb{N}^+$ and $U \subset \mathbb{R}^d$ be open. For a function $v : U \to \mathbb{R}$ and $x \in U$ we define

$$\mathcal{J}^{2,+}v(x) = \Big\{(p, q) \in \mathbb{R}^d \times \mathrm{Sym}(d) : \exists\, \phi \in C^2(U) \text{ with } v - \phi \leq v(x) - \phi(x) \\ \text{such that } (p, q) = (D\phi(x), D^2\phi(x))\Big\}.$$

The sets $\mathcal{J}^{2,-}u(x), \mathcal{J}^2 u(x), \overline{\mathcal{J}}^{2,\pm}u(x)$ are then defined analogously as in the parabolic case and, if $\mathcal{J}^2 u(x)$ is non-empty, this set will be identified with its unique element $({}^{(p)}Du(x), {}^{(p)}D^2u(x))$.

We remark that $(\alpha, p, q) \in \mathcal{P}^+u(t,x)$ resp. $(\alpha, p, q) \in \mathcal{P}^-u(t,x)$ if and only if there exists a neighbourhood N of (t,x) such that as $N \ni (s,y) \to (t,x)$:

$$u(s,y) \leq u(t,x) + \alpha(s-t) + p(y-x) + \frac{q}{2}|y-x|^2 + o(|s-t| + |y-x|^2) \tag{3.3}$$

resp.

$$u(s,y) \geq u(t,x) + \alpha(s-t) + p(y-x) + \frac{q}{2}|y-x|^2 + o(|s-t| + |y-x|^2). \tag{3.4}$$

If $u(t,\cdot)$ is non-decreasing, letting $s = t$ in ineq. (3.3) resp. in ineq. (3.4) and $y \to x^+$ resp. $y \to x^-$, it follows that $p \geq 0$. In particular, for functions u which are non-decreasing in x, we have

$$\mathcal{P}^{\pm}u(\omega) \subseteq \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}$$

for all $\omega \in \Omega$.

Definition 3.4 (Semicontinuous envelopes)**.** Given $u = u(\omega)$ we define the functions

$$u^*(\omega) = \lim_{r \searrow 0} \sup\{u(\xi) : \xi \in \Omega, |\xi - \omega| \leq r\},$$
$$u_*(\omega) = \lim_{r \searrow 0} \inf\{u(\xi) : \xi \in \Omega, |\xi - \omega| \leq r\}.$$

The function u is *upper semicontinuous* (usc) if $u = u^*$, and *lower semicontinuous* (lsc) if $u = u_*$. We call u^* (resp. u_*) the *usc* (resp. *lsc*) *envelope* of u.

Notice that for any $\omega \in \Omega$ there exists a sequence $\xi_k \overset{k\to\infty}{\to} \omega$ such that $u(\xi_k) \overset{k\to\infty}{\to} u^*(\omega)$. Also note that the function u is usc if and only if $u(\omega) \geq \limsup_{k\to\infty} u(\xi_k)$ for any sequence $\xi_k \overset{k\to\infty}{\to} \omega$. Furthermore, v is lsc if and only if $-v$ is usc.

Now we are in a position to state the notion of solution we propose for eq. (3.1).

Definition 3.5 (Viscosity (sub-/super-) solution)**.** Suppose that the continuous function G satisfies property (A0), and let u be a function defined on Ω. We call u a

- *(viscosity) subsolution* of equation (3.1) in Ω if it is upper semicontinuous and if for any $\omega \in \Omega$ and any $(\alpha, p, q) \in \mathcal{P}^+u(\omega)$ we have

$$G(u(\omega), \alpha, p, q) \leq 0.$$

- *(viscosity) supersolution* of equation (3.1) in Ω if it is lower semicontinuous and if for any $\omega \in \Omega$ and any $(\alpha, p, q) \in \mathcal{P}^- u(\omega)$ we have

$$G(u(\omega), \alpha, p, q) \geq 0.$$

- *viscosity solution* of equation (3.1) in Ω if it is both a subsolution and a supersolution of equation (3.1) in Ω. (In this case u is necessarily continuous.)

In places we use the short phrase 'u is a viscosity (sub-/super-) solution of $\mathcal{G} = 0$' if it is a viscosity (sub-/super-) solution of eq. (3.1). Since we will only deal with sub- and supersolutions in the viscosity sense, we usually drop the word 'viscosity' in these cases.

Notice that, by the continuity of G, in Definition 3.5 one can replace $\mathcal{P}^{\pm} u(\omega)$ with $\overline{\mathcal{P}}^{\pm} u(\omega)$.

Remark. Of course, the mere formulation of Definition 3.5 does not require assumption (A0). However, it is this property which ensures that the definition is meaningful in the sense that it generalises the notion of a classical solution.

3.2 Stability

One advantage of the notion of viscosity solutions lies in its good stability properties. In order to demonstrate this, we reformulate [38, Proposition 4.3] (for elliptic problems) in terms of our parabolic problem.

Proposition 3.6. *Let $v \in \mathrm{USC}(\Omega)$, let $\omega \in \Omega$ and assume that $(\alpha, p, q) \in \mathcal{P}^+ v(\omega)$. Suppose that $u_n \in \mathrm{USC}(\Omega)$ is a sequence of functions satisfying*

$$\left.\begin{array}{l} (i)\ \textit{there exist } \omega_n \in \Omega \textit{ such that } (\omega_n, u_n(\omega_n)) \to (\omega, v(\omega)) \\ (ii)\ \textit{if } \xi_n \in \Omega \textit{ and } \xi_n \to \xi, \textit{ then } \limsup_{n\to\infty} u_n(\xi_n) \leq v(\xi). \end{array}\right\}$$

Then there exist $\hat{\omega}_n \in \Omega$, $(\alpha_n, p_n, q_n) \in \mathcal{P}^+ u_n(\hat{\omega}_n)$ such that

$$(\hat{\omega}_n, u_n(\hat{\omega}_n), \alpha_n, p_n, q_n) \to (\omega, v(\omega), \alpha, p, q).$$

Proof. The proof is similar to the one of [38, Proposition 4.3]. Notice that this result does not involve the equation. □

Remark 3.7 (Stability). Observe that we have the following corollaries of Proposition 3.6.

(a) The notion of viscosity solutions is stable under locally uniform convergence: let $G_n = G_n(z, \alpha, p, q)$, $n \in \mathbb{N}$, be continuous and such that $G_n \to G$ as $n \to \infty$ locally uniformly. Furthermore assume that, for each n, u_n is a viscosity solution of $\mathcal{G}_n = 0$ in Ω and that the sequence (u_n) converges locally uniformly in Ω to some function u. Then u is a viscosity solution of $\mathcal{G} = 0$ in Ω.

(b) If V is a family of subsolutions of equation (3.1) and $u := \sup_{v \in V} v$ is such that the usc envelope u^* of u satisfies $u^*(\omega) < \infty$ for all $\omega \in \Omega$, then u^* is a subsolution of equation (3.1).

3.3 Comparison

Given that our notion of solution is a rather weak one, our first concern is the question of uniqueness subject to prescribed data. The comparison principle established below is a fundamental and very powerful tool in our theory, and its range of applications goes beyond uniqueness.

Proposition 3.8 (Comparison)**.** *Suppose that, in addition to* (A0)*, the continuous function G has the following property:*

(A1) For all p, q the function $(z, \alpha) \mapsto G(z, \alpha, p, q)$ is *weakly strictly increasing* in the sense that for all $(z, \alpha), (z', \alpha') \in \mathbb{R}^2$

$$\begin{cases} [z \leq z' \text{ and } \alpha \leq \alpha'] & \Rightarrow \quad G(z, \alpha, p, q) \leq G(z', \alpha', p, q), \\ [z < z' \text{ and } \alpha < \alpha'] & \Rightarrow \quad G(z, \alpha, p, q) < G(z', \alpha', p, q). \end{cases}$$

Let $0 < T \leq \infty$ and assume that $u \in \mathrm{USC}(\Omega \cup \partial_p \Omega)$ is a subsolution and $v \in \mathrm{LSC}(\Omega \cup \partial_p \Omega)$ a supersolution of eq. (3.1) *in Ω satisfying $u \leq v$ on $\partial_p \Omega$. Then $u \leq v$ in Ω.*

Proof of Proposition 3.8. Without loss of generality we may assume that $T < \infty$ and that the upper semicontinuous $\mathbb{R}$-valued functions u and $-v$ are bounded above. (Otherwise, we apply the argument below with T replaced by $T' < T$.)

Arguing by contradiction, let us suppose that

$$\sup_{\Omega} (u - v) > 0.$$

This implies that for $\eta > 0$ sufficiently small

$$K := \sup_{(t,x) \in \Omega} \left(u(t, x) - v(t, x) - \frac{\eta}{T - t} \right) > 0.$$

Notice that the function

$$\tilde{u}(t,x) := u(t,x) - \frac{\eta}{T-t}$$

is a subsolution of eq. (3.1) which is bounded above and satisfies $\lim_{t\nearrow T} u(t,\cdot) = -\infty$ where the convergence is uniform in $x \in J$.

Due to the mere semicontinuity of the functions involved we cannot proceed using classical calculus. Also observe that we do not know whether the function $\tilde{u} - v$ is the subsolution of a suitable parabolic equation. To compensate for the lack of regularity, we use a well-known technique consisting in first doubling the independent variables and then penalising the deviation of corresponding variables. Concretely, for $\varepsilon > 0$ we consider the function

$$h_\varepsilon(t,x,s,y) := \tilde{u}(t,x) - v(s,y) - \frac{|t-s|^2}{2\varepsilon} - \frac{|x-y|^2}{2\varepsilon}.$$

Notice that whenever h_ε attains an interior maximum at some point $(\hat{t},\hat{x},\hat{s},\hat{y})$ and hence can be touched from above at that point by a constant function, by separately considering the functions $(t,x) \mapsto h_\varepsilon(t,x,\hat{s},\hat{y})$ and $(s,y) \mapsto h_\varepsilon(\hat{t},\hat{x},s,y)$ one is able to recover the first order criterion for maxima, namely the existence of first order superdifferentials of $\tilde{u}$ and of $-v$ (at the points $(\hat{t},\hat{x})$ resp. $(\hat{s},\hat{y})$) summing up to zero. Let us, however, caution the reader that this technique does not provide us with the corresponding second order information and is insufficient when dealing with second order equations. In general, it is not possible to find matrices Q_1, Q_2 (or rather elements $(P,Q_1), (-P,Q_2)$) in the second order superdifferentials $\mathcal{J}^{2,+}$ of the merely upper semicontinuous functions $\tilde{u}$ and $-v$ (at the corresponding points) whose direct sum satisfies, in the sense of quadratic forms, the inequality

$$\operatorname{diag}(Q_1,Q_2) \le \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

which in the classical case one is able to deduce from the non-positivity of the Hessian of h_ε at a maximum (see [37, Remark 5]). In the last inequality, I denotes the identity matrix in two dimensions. The fact that in an approximate sense such an inequality does hold true for the limiting superdifferentials $\overline{\mathcal{J}}^{2,+}$ is a deeper result, which lies at the heart of the classical theory of viscosity solutions for second order equations. For an introductory exposition on this issue we refer to Section 10 of Crandall's lecture notes in [7]. Here, we use the following version of this result, which is a special case of [38, Theorem 3.2]:

Theorem 3.9 ([38]). *Let $N \in \mathbb{N}$. Given open subsets $\mathcal{O}_i \subset \mathbb{R}^N, i = 1, 2$, set $\mathcal{O} := \mathcal{O}_1 \times \mathcal{O}_2$ and suppose that $u_i \in \mathrm{USC}(\mathcal{O}_i), i = 1, 2$, and $\phi \in C^2(\mathcal{O})$. Define*

$$w(\omega) = u_1(\omega_1) + u_2(\omega_2) \quad \text{for } \omega = (\omega_1, \omega_2) \in \mathcal{O}.$$

Assume that $\hat{\omega} \in \mathcal{O}$ is a local maximum of $w - \phi$. Then, for each $\delta > 0$, there exist $Q_i \in \mathrm{Sym}(N), i = 1, 2$, such that

$$(D_{\omega_i}\phi(\hat{\omega}), Q_i) \in \overline{\mathcal{J}}^{2,+} u_i(\hat{\omega}_i) \quad \text{for } i = 1, 2$$

and the block diagonal matrix $Q := \mathrm{diag}(Q_1, Q_2)$ satisfies, in the sense of quadratic forms, the inequality

$$Q \leq A + \delta A^2,$$

where $A = D^2\phi(\hat{\omega}) \in \mathrm{Sym}(2N)$.

A fairly self-contained proof of Theorem 3.9 can be found in the appendix of [38]. Let us now proceed with the proof of Proposition 3.8.

We let

$$K_\varepsilon := \sup_{(t,x),(s,y)\in\Omega} h_\varepsilon(t, x, s, y)$$

and note that $K_\varepsilon \geq K > 0$. The fact that h_ε is usc and bounded above combined with the behaviour of $\tilde{u}(t, \cdot)$ as $t \to T$ implies that for sufficiently small $\varepsilon > 0$ the supremum is attained at some point $\omega_\varepsilon := (\omega_{1,\varepsilon}, \omega_{2,\varepsilon}) := ((t_\varepsilon, x_\varepsilon), (s_\varepsilon, y_\varepsilon)) \in (\Omega \cup \partial_p\Omega) \times (\Omega \cup \partial_p\Omega)$. Moreover, $(\omega_{1,\varepsilon} - \omega_{2,\varepsilon}) \to 0$ as $\varepsilon \to 0$ and, after passing to a subsequence, $\omega_{i,\varepsilon} \to \bar{\omega}$, $i = 1, 2$, for some $\bar{\omega} \in \Omega \cup \partial_p\Omega$. First assume $\bar{\omega} \in \partial_p\Omega$. Then we obtain

$$0 < K \leq \limsup_{\varepsilon\to 0} h_\varepsilon(\omega_{1,\varepsilon}, \omega_{2,\varepsilon}) \leq \limsup_{\varepsilon\to 0}(\tilde{u}(\omega_{1,\varepsilon}) - v(\omega_{2,\varepsilon})) \leq \tilde{u}(\bar{\omega}) - v(\bar{\omega}) \leq 0,$$

a contradiction. Hence, we must have $\bar{\omega} \in \Omega$, so that for small enough ε, we have $\omega_{1,\varepsilon}, \omega_{2,\varepsilon} \in \Omega$. Now we can apply Theorem 3.9 with $N = 2$, $\mathcal{O}_i = \Omega$, $u_1 = \tilde{u}, u_2 = -v$ (which is usc), $\phi(t, x, s, y) = \frac{|t-s|^2}{2\varepsilon} + \frac{|x-y|^2}{2\varepsilon}$, and the maximiser $\hat{\omega} = (\omega_{1,\varepsilon}, \omega_{2,\varepsilon})$. Theorem 3.9 (with $\delta = 1$) guarantees the existence of $Q_{i,\varepsilon} \in \mathrm{Sym}(2)$, $i = 1, 2$, such that

$$(D_{\omega_i}\phi(\omega_\varepsilon), Q_{i,\varepsilon}) \in \overline{\mathcal{J}}^{2,+} u_i(\omega_{i,\varepsilon}) \quad \text{for } i = 1, 2$$

and

$$Q_\varepsilon := \begin{pmatrix} Q_{1,\varepsilon} & 0 \\ 0 & Q_{2,\varepsilon} \end{pmatrix} \leq A + A^2 \tag{3.5}$$

in the sense of quadratic forms, where $A = D^2\phi(\omega_\varepsilon)$. Notice that

$$D_{\omega_1}\phi(\omega_\varepsilon) = \frac{1}{\varepsilon}(t_\varepsilon - s_\varepsilon, x_\varepsilon - y_\varepsilon)^t =: (\tau_\varepsilon, p_\varepsilon)^t,$$

$$D_{\omega_2}\phi(\omega_\varepsilon) = -(\tau_\varepsilon, p_\varepsilon)^t,$$

and

$$A = D^2\phi(\omega_\varepsilon) = \frac{1}{\varepsilon}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Writing

$$Q_{i,\varepsilon} =: \begin{pmatrix} a_{i,\varepsilon} & b_{i,\varepsilon} \\ b_{i,\varepsilon} & q_{i,\varepsilon} \end{pmatrix}$$

we have for

$$\xi := (0,1,0,1)^t$$

the identity

$$\xi^t Q_\varepsilon \xi = q_{1,\varepsilon} + q_{2,\varepsilon}.$$

Hence, since $\xi \in \ker(A)$, the matrix inequality (3.5) implies

$$q_{1,\varepsilon} + q_{2,\varepsilon} \leq 0.$$

By definition, the fact that

$$(D_{\omega_1}\phi(\omega_\varepsilon), Q_{1,\varepsilon}) \in \overline{\mathcal{J}}^{2,+} u_1(\omega_{1,\varepsilon}), \quad u_1 = \tilde{u}$$

means that there exist sequences $\omega_{1,\varepsilon}^{(n)} := (t_\varepsilon^{(n)}, x_\varepsilon^{(n)})$ and

$$(\tau_\varepsilon^{(n)}, p_\varepsilon^{(n)}, Q_{1,\varepsilon}^{(n)}) \in \mathcal{J}^{2,+}\tilde{u}(\omega_{1,\varepsilon}^{(n)})$$

such that as $n \to \infty$

$$\omega_{1,\varepsilon}^{(n)} \to \omega_{1,\varepsilon}, \quad \tilde{u}(\omega_{1,\varepsilon}^{(n)}) \to \tilde{u}(\omega_{1,\varepsilon}) \text{ and } (\tau_\varepsilon^{(n)}, p_\varepsilon^{(n)}, Q_{1,\varepsilon}^{(n)}) \to (\tau_\varepsilon, p_\varepsilon, Q_{1,\varepsilon}).$$

In particular, we have as $(t,x) \to \omega_{1,\varepsilon}^{(n)}$:

$$\begin{aligned}
\tilde{u}(t,x) \le \tilde{u}(\omega_{1,\varepsilon}^{(n)}) &+ (t - t_\varepsilon^{(n)})\tau_\varepsilon^{(n)} + (x - x_\varepsilon^{(n)})p_\varepsilon^{(n)} + \frac{1}{2}(t - t_\varepsilon^{(n)})^2 a_{1,\varepsilon}^{(n)} + \frac{1}{2}(x - x_\varepsilon^{(n)})^2 q_{1,\varepsilon}^{(n)} \\
&+ (t - t_\varepsilon^{(n)})(x - x_\varepsilon^{(n)}) b_{1,\varepsilon}^{(n)} + o((t - t_\varepsilon^{(n)})^2 + (x - x_\varepsilon^{(n)})^2) \\
\le \tilde{u}(\omega_{1,\varepsilon}^{(n)}) &+ (t - t_\varepsilon^{(n)})\tau_\varepsilon^{(n)} + (x - x_\varepsilon^{(n)})p_\varepsilon^{(n)} + \frac{1}{2}(x - x_\varepsilon^{(n)})^2 q_{1,\varepsilon}^{(n)} + \frac{\sigma}{2}(x - x_\varepsilon^{(n)})^2 \\
&+ o(|t - t_\varepsilon^{(n)}| + (x - x_\varepsilon^{(n)})^2),
\end{aligned}$$

where $\sigma > 0$ can be chosen arbitrarily small. This means that for all $\sigma > 0$

$$(\tau_\varepsilon^{(n)}, p_\varepsilon^{(n)}, q_{1,\varepsilon}^{(n)} + \sigma) \in \mathcal{P}^+\tilde{u}(\omega_{1,\varepsilon}^{(n)}),$$

which, upon choosing $\sigma = \frac{1}{n}$ and letting $n \to \infty$, yields

$$(\tau_\varepsilon, p_\varepsilon, q_{1,\varepsilon}) \in \overline{\mathcal{P}}^+\tilde{u}(\omega_{1,\varepsilon})$$

or, equivalently,

$$\left(\tau_\varepsilon + \frac{\eta}{(T - t_\varepsilon)^2}, p_\varepsilon, q_{1,\varepsilon}\right) \in \overline{\mathcal{P}}^+ u(\omega_{1,\varepsilon}). \tag{3.6}$$

Starting from

$$(-\tau_\varepsilon, -p_\varepsilon, Q_{2,\varepsilon}) \in \overline{\mathcal{J}}^{2,+} u_2(\omega_{2,\varepsilon}),$$

we can argue analogously for u_2 to find

$$(-\tau_\varepsilon, -p_\varepsilon, q_{2,\varepsilon}) \in \overline{\mathcal{P}}^+ u_2(\omega_{2,\varepsilon}),$$

or, equivalently,

$$(\tau_\varepsilon, p_\varepsilon, -q_{2,\varepsilon}) \in \overline{\mathcal{P}}^- v(\omega_{2,\varepsilon}). \tag{3.7}$$

Thanks to the conclusions (3.6) and (3.7), we can make use of the fact that u (resp. v)

is a subsolution (resp. a supersolution) of equation (3.1) and obtain the inequalities

$$G(u(\omega_{1,\varepsilon}), \tilde{\tau}_\varepsilon, p_\varepsilon, q_{1,\varepsilon}) \leq 0, \tag{3.8}$$

where $\tilde{\tau}_\varepsilon = \tau_\varepsilon + \frac{\eta}{(T-t_\varepsilon)^2} > \tau_\varepsilon$, and

$$G(v(\omega_{2,\varepsilon}), \tau_\varepsilon, p_\varepsilon, -q_{2,\varepsilon}) \geq 0. \tag{3.9}$$

Subtracting ineq. (3.9) from ineq. (3.8), we infer the following contradiction

$$\begin{aligned} 0 &\geq G(u(\omega_{1,\varepsilon}), \tilde{\tau}_\varepsilon, p_\varepsilon, q_{1,\varepsilon}) - G(v(\omega_{2,\varepsilon}), \tau_\varepsilon, p_\varepsilon, -q_{2,\varepsilon}) \\ &\geq G(u(\omega_{1,\varepsilon}), \tilde{\tau}_\varepsilon, p_\varepsilon, q_{1,\varepsilon}) - G(v(\omega_{2,\varepsilon}), \tau_\varepsilon, p_\varepsilon, q_{1,\varepsilon}) > 0, \end{aligned}$$

where we used hypotheses (A0) and (A1). □

As a consequence of the proof of Proposition 3.8, viscosity solutions of $\mathcal{G} = 0$ obey an intersection comparison principle. For its precise formulation we recall the notion of the number of sign changes of a continuous function defined on an interval (see e.g. [91, Appendix F], [54] and references therein).

Definition 3.10 (Number of sign changes)**.** Let $J \subset \mathbb{R}$ be connected. Given $v \in C(J)$ define the set

$$\begin{aligned} N_v := \{k \in \mathbb{N} : \exists\, x_j \in J, j = 0, 1, \ldots, k \text{ such that } x_0 < x_1 < \cdots < x_k, \\ \text{and } v(x_{j-1}) \cdot v(x_j) < 0 \text{ for } j = 1, \ldots, k\} \end{aligned}$$

and let $Z[v] := \sup\left(N_v \cup \{0\}\right)$. We call $Z[v] \in \mathbb{N} \cup \{0, \infty\}$ the *number of sign changes* of v.

In the literature the number of sign changes is also referred to as the *zero number*. We are usually interested in the number of sign changes $Z[u_1 - u_2]$ of the difference of two functions $u_i \in C(J_i)$, $i = 1, 2$, where in general $J_1 \neq J_2$. In this case, our notation is to be understood as

$$Z[u_1 - u_2] := Z[(u_1 - u_2)_{|J'}], \quad \text{where } J' = J_1 \cap J_2.$$

We now state the intersection comparison principle in a form typically used in applications.

Corollary 3.11 (Intersection comparison)**.** *Assume that the continuous function G satisfies hypotheses* (A0) *and* (A1)*. Let $t_1 < t_2$, $x_1 < x_2$ and define $Q :=$*

$(t_1, t_2) \times (x_1, x_2)$. *Suppose that* $u, v \in C(\bar{Q})$ *are viscosity solutions of* $\mathcal{G} = 0$ *in* Q *satisfying:*

(L) *the number of connected components of* $\partial_p Q^{\pm} := \{q \in \partial_p Q : \pm(u - v)(q) > 0\}$ *does not exceed the number of connected components of* $\partial_p Q^{\pm} \cap (\{t_1\} \times [x_1, x_2])$.

Then the number of sign changes of the difference $w := u - v$ *is non-increasing in time, i.e.*

$$Z[w(t, \cdot)] \le Z[w(t_1, \cdot)] \quad \textit{for all } t \in (t_1, t_2).$$

Loosely speaking, the corollary asserts that the number of intersections of two viscosity solutions of $\mathcal{G} = 0$ is non-increasing in time provided that no intersections occur on the lateral boundary. Corollary 3.11 is a consequence of the maximum principle as it is applied in the proof of Proposition 3.8. The proof essentially follows the original approach by Sturm [97] treating linear parabolic equations (see also [54, Chapter 1]), where the application of the classical maximum principle needs to be substituted for the maximum type argument used in the proof of Proposition 3.8.

Proof of Corollary 3.11. Consider the sets $Q^{\pm} := \{q \in \bar{Q} : \pm w(q) > 0\}$ and

$$A^{\pm} := \{q \in \partial_p Q : \pm w(q) > 0\}.$$

Notice that, by the continuity of w, the number of connected components of $Q^{\pm}$ equals that of $Q^{\pm} \setminus \partial Q$.

The main ingredient in the proof is the following auxiliary result:

Lemma 3.12. *Suppose that, except for condition* (L)*, the hypotheses of Corollary 3.11 hold true. For each connected component* Q' *of* $Q^{\pm}$ *there exists a connected component* A' *of* $A^{\pm}$ *such that* $A' \subset Q'$.

Proof of Lemma 3.12. Assume that $Q' \subset Q^+$. Then the assertion follows if we can show that $\sup_{\partial Q' \cap \partial_p Q} w > 0$, where we use the convention $\sup_{\emptyset} w = -\infty$. We argue by contradiction and assume that $\sup_{\partial Q' \cap \partial_p Q} w \le 0$. By the definition of Q' and the continuity of w, we have $w = 0$ in $\partial Q' \cap Q$. Since $\sup_{Q'} w > 0$, the contradiction is now obtained as in the proof of Proposition 3.8.

If $Q' \subset Q^-$, apply the previous reasoning to $\tilde{w} = v - u$ instead of w. □

We can now conclude the proof of Corollary 3.11. Let $t \in (t_1, t_2)$ and suppose that there exist $y_j \in (x_1, x_2)$, $j = 0, \ldots, k$ such that $y_0 < y_1 < \cdots < y_k$ and $w(t, y_j) \cdot w(t, y_{j-1}) < 0$ for $j = 1, \ldots, k$. For each j let Q_j be the connected component of $Q^+ \cup Q^-$ containing (t, y_j). Using Lemma 3.12 with Q replaced by a suitable axis-aligned rectangle $\tilde{Q} \subset Q$, it is easy to see that $Q_j \neq Q_l$ whenever $j \neq l$. Applied once more, Lemma 3.12 combined with hypothesis (L) provides us with $\tilde{y}_j \in (x_1, x_2)$, $j = 0, \ldots, k$, such that $\tilde{y}_0 < \cdots < \tilde{y}_k$ and $w(t_1, \tilde{y}_j) \cdot w(t_1, \tilde{y}_{j-1}) < 0$ for $j = 1, \ldots, k$. □

3.4 Perron method

As a preparatory step towards existence we establish a Perron method for equation (3.1) for monotonic (and non-monotonic) functions, which roughly states that once a subsolution u^- and a supersolution u^+ satisfying $u^- \leq u^+$ are found, there exists an 'almost' viscosity solution squeezed between these barriers. Since in our applications we are particularly interested in functions which are non-decreasing with respect to x, we start with some preliminaries on monotonicity.

Definition 3.13 (x-monotonicity)**.** We call a function $u = u(t, x)$ *x-monotonic*, in short *x-m*, if the function $x \mapsto u(t, x)$ is non-decreasing for any t.

Fact 1. If $u = u(t, x)$ is x-monotonic, so are the semicontinuous envelopes u^* and u_* (introduced in Definition 3.4).

Let us sketch the elementary argument demonstrating the assertion for u^*, the claim for u_* can be obtained by a similar reasoning. Fix $t \geq 0$ and $x < y$. The definition of u^* implies that there exists a sequence $(t_j, x_j) \to (t, x)$ such that $u(t_j, x_j) \to u^*(t, x)$. Then, for large enough j, we have $x_j < y$ and therefore $u(t_j, x_j) \leq u(t_j, y)$. Hence

$$u^*(t, x) \leq \limsup_{j \to \infty} u(t_j, y) \leq \limsup_{j \to \infty} u^*(t_j, y) \leq u^*(t, y),$$

where the last inequality holds thanks to the semicontinuity of u^*.

Fact 2. If V is a set of functions such that all $v \in V$ are x-m, then the function u defined via $u(t, x) := \sup_{v \in V} v(t, x)$ is x-m.

While the idea of the Perron method is well-known in the literature, the assumption of monotonicity requires some non-trivial modifications. The version provided below is an adaptation of [62, Lemma 2.3.15].

Proposition 3.14. *Suppose that hypothesis* (A0) *holds true and let* $0 < T \leq \infty$*. Assume that* $u^{\pm}$ *are locally bounded x-m functions satisfying* $u^- \leq u^+$ *in* Ω *and suppose that* u^- *is a subsolution and* u^+ *a supersolution of eq.* (3.1) *in* Ω*. Then there exists an x-m function* $u : \Omega \to \mathbb{R}$ *such that* u^* *is a subsolution of eq.* (3.1) *in* Ω*,* u_* *a supersolution and* $u^- \leq u \leq u^+$*.*

The statement remains valid when the x-m property is dropped everywhere.

Proof. We confine ourselves to showing the (more interesting) assertion regarding the x-monotonic case. The proof of the second assertion is easier and can be carried out along similar lines (without the need of a distinction of cases). Consider the non-empty set

$$V = \{v : \Omega \to \mathbb{R} \mid u^- \leq v \leq u^+,\ v \text{ is } x\text{-monotonic},\ v^* \text{ is a subsolution of eq. (3.1)}\}$$

and let

$$u = \sup_{v \in V} v.$$

Then u is x-monotonic and, by Remark 3.7 (b), u^* is a subsolution of eq. (3.1) in Ω.

It remains to show that the x-m, lsc function u_* is a supersolution of eq. (3.1). We argue by contradiction and assume that there exists $\omega \in \Omega$, $(\alpha, p, q) \in \mathcal{P}^- u_*(\omega)$ and $\theta > 0$ such that

$$G(z, \alpha, p, q) \leq -\theta, \tag{3.10}$$

where $z := u_*(\omega)$. Notice that, since $u_* \leq u^+$, if $u_*(\omega) = u^+(\omega)$, then $(\alpha, p, q) \in \mathcal{P}^- u^+(\omega)$, and the fact that u^+ is a supersolution would then imply $G(z, \alpha, p, q) \geq 0$, which contradicts (3.10). Therefore

$$u_*(\omega) < u^+(\omega),$$

and, after possibly decreasing $\theta > 0$, we can assume that

$$u_*(\omega) - u^+(\omega) \leq -\theta < 0. \tag{3.11}$$

By the translation invariance of the equation with respect to the independent variable ω, we can further assume that $(0, 0) \in \Omega$ and $\omega = (0, 0)$. For small parameters $\delta, \varepsilon > 0$ to be determined later, we define

$$P(s, y) = z + \alpha s + py + \frac{1}{2} q y^2 + \delta - \varepsilon \left(|s| + \frac{1}{2} |y|^2 \right).$$

Note that for any $(s, y) \in \Omega$ and $(\alpha', p', q') \in \mathcal{P}^+ P(s, y)$ one has $|\alpha' - \alpha| \leq \varepsilon$, $p' = p + qy - \varepsilon y$ and $q' \geq q - \varepsilon$. We further let $N_r := \{(\tilde{s}, \tilde{y}) : |\tilde{s}| + |\tilde{y}|^2/2 < r\}$.

We now have to distinguish between the case in which $p > 0$ and the one in which p vanishes.

<u>Case 1:</u> $p > 0$.

In this case, P is x-monotonic in N_r for $r > 0$ small enough, and after decreasing r again and choosing $\varepsilon, \delta > 0$ sufficiently small, we have

$$G(P(s,y), \alpha', p', q') \leq -\frac{\theta}{2}$$

for any $(s,y) \in N_r$ and any $(\alpha', p', q') \in \mathcal{P}^+ P(s,y)$. Thus, P is a subsolution of eq. (3.1) in N_r.

Since, by inequality (3.11), we have $P(\omega) \leq u^+(\omega) + \delta - \theta$, the fact that P is usc and u^+ lsc ensures that, after possibly decreasing $\delta > 0$,

$$P(s,y) < u^+(s,y) \quad \text{for } (s,y) \in N_r. \tag{3.12}$$

Since $(\alpha, p, q) \in \mathcal{P}^- u_*(\omega)$, by inequality (3.4),

$$\begin{aligned} u_*(s,y) &\geq z + \alpha s + py + \frac{1}{2} q y^2 + o(|s| + |y|^2) \\ &\geq P(s,y) - \delta + \varepsilon \left(|s| + \frac{1}{2}|y|^2 \right) + o(|s| + |y|^2). \end{aligned}$$

After possibly decreasing r, we can choose $\delta = \frac{\varepsilon r}{4}$. Then for $(s,y) \in N_r \setminus N_{r/2}$

$$u_*(s,y) \geq P(s,y) - \frac{\varepsilon r}{4} + \frac{\varepsilon r}{2} + o(r) = P(s,y) + \frac{\varepsilon r}{4} + o(r)$$

and hence, for r sufficiently small,

$$u(s,y) - P(s,y) \geq \frac{\varepsilon r}{8} > 0 \quad \text{for } (s,y) \in N_r \setminus N_{r/2}.$$

Let us now define

$$U(s,y) = \begin{cases} \max\{u(s,y), P(s,y)\} & \text{if } (s,y) \in N_r, \\ u(s,y) & \text{otherwise.} \end{cases} \tag{3.13}$$

Then U is non-decreasing, U^* is a subsolution of (3.1) in Ω and $u^- \leq U \leq u^+$, where the last bound follows from ineq. (3.12). Hence $U \in V$ and thus $U \leq u$. However, by definition there exists a sequence $\xi_n \to \omega$ such that $u(\xi_n) \to u_*(\omega) = z$ and therefore

$$\liminf_{n\to\infty} (U(\xi_n) - u(\xi_n)) \geq \lim_{n\to\infty} (P(\xi_n) - u(\xi_n)) = \delta > 0.$$

This contradicts $U \leq u$.

Case 2: $p = 0$.

In this case the x-monotonicity of u_* implies that $q \leq 0$. Hence, hypothesis (A0) and inequality (3.10) imply that $G(z, \alpha, 0, 0) \leq G(z, \alpha, 0, q) \leq -\theta$.

The competitor $P = P(s,y)$ needs to be adapted since it is strictly decreasing in y for $y > 0$. We define

$$\tilde{P}(s,y) = \begin{cases} P(s,y) & \text{if } y \leq 0, \\ P(s,0) = z + \delta + s\alpha - \varepsilon|s| & \text{if } y > 0. \end{cases}$$

Notice that we can choose r, δ, ε sufficiently small such that for all $\sigma \in [-1,1]$

$$G(\tilde{P}, \alpha + \sigma\varepsilon, \partial_y \tilde{P}, \partial_y^2 \tilde{P})_{|(s,y)} = G(P, \alpha + \sigma\varepsilon, \partial_y P, \partial_y^2 P)_{|(s,y)} \leq -\frac{\theta}{2} \ \ \forall (s,y) \in N_r : \ y < 0$$

and

$$G(\tilde{P}, \alpha + \sigma\varepsilon, \partial_y \tilde{P}, \partial_y^2 \tilde{P})_{|(s,y)} = G(P(s,0), \alpha + \sigma\varepsilon, 0, 0) \leq -\frac{\theta}{2}, \quad \forall |s| < r, \ y > 0.$$

Moreover, since $\partial_y \tilde{P} \in C^0$ with $\partial_y \tilde{P}(s,0) = 0$, whenever $(\tilde{\alpha}, \tilde{p}, \tilde{q}) \in \mathcal{P}^+ \tilde{P}(s,0)$, we must have $\tilde{p} = 0$, $\tilde{q} \geq 0$, $\tilde{\alpha} = \alpha + \sigma\varepsilon$ for some $\sigma \in [-1,1]$ and therefore $G(\tilde{P}(s,0), \tilde{\alpha}, \tilde{p}, \tilde{q}) \leq -\frac{\theta}{2}$ whenever $|s| < r$. Hence, $\tilde{P}$ is a subsolution of $G = 0$ in the domain $\tilde{N}_r$ defined via

$$\tilde{N}_r := N_r \cup \{(s,y) \in \Omega : |s| < r, y \geq 0\}.$$

As in Case 1 we have $P(\omega) < u^+(\omega)$ for δ sufficiently small, so that after possibly decreasing r once more, we obtain

$$\tilde{P} < u^+ \quad \text{in } \tilde{N}_r.$$

For this conclusion we have used in particular the x-monotonicity of u^+.

Arguing as in Case 1 and letting in particular $\delta = \frac{\varepsilon r}{4}$, for r, ε sufficiently small, we can guarantee that

$$u > \tilde{P} \quad \text{in } \left(N_r \setminus N_{\frac{r}{2}}\right) \cap \{(s,y) \in \Omega : y \leq 0\}. \tag{3.14}$$

The inequality (3.14) implies that $u(s,0) > \tilde{P}(s,0)$ for $\frac{r}{2} \leq |s| < r$, and thanks to the x-monotonicity of u therefore

$$u(s,y) > \tilde{P}(s,y) \quad \text{for all } \frac{r}{2} \leq |s| < r, \ y \geq 0.$$

We now define U as in formula (3.13) with P replaced by $\tilde{P}$ and N_r replaced by $\tilde{N}_r$. Then U is x-monotonic, U^* is a subsolution of $G = 0$ in Ω, $u^- \leq u \leq U \leq u^+$ but $U \not\equiv u$, which contradicts the maximality of u. $\square$

3.5 Existence, uniqueness and Lipschitz regularity

We are now in a position to show existence and uniqueness for the Cauchy–Dirichlet problem associated with equation (3.1) conditional on the existence of appropriate barriers.

Theorem 3.15 (Existence and uniqueness)**.** *Suppose that the continuous function* G *satisfies the conditions* (A0) *and* (A1)*. Given* $0 < T \leq \infty$ *and locally bounded* x*-monotonic functions* $u^{\pm} : \Omega \cup \partial_p\Omega \to \mathbb{R}$ *such that* u^- *is a subsolution and* u^+ *a supersolution of eq.* (3.1) *in* Ω *satisfying*

(B1) $u^- \leq u^+$ in $\Omega \cup \partial_p\Omega$

(B2) $(u^-)_* = (u^+)^*$ on $\partial_p\Omega$,

there exists a unique x*-monotonic viscosity solution* $u \in C(\Omega \cup \partial_p\Omega)$ *of eq.* (3.1) *in* Ω *with the property that* $u = u^-(= u^+)$ *on* $\partial_p\Omega$*. This solution satisfies* $u^- \leq u \leq u^+$*.*

The assertion remains valid when dropping the x*-monotonicity everywhere.*

Remark. By replacing $u^{\pm}$ with $-u^{\mp}$ one obtains the same result for functions which are non-*increasing* in x.

Proof. We only consider the x-m case since the reasoning in the non-monotonic case is completely similar. From the assumptions we infer that

$$\lim_{\substack{\omega \in \Omega, \\ \omega \to \bar{\omega} \in \partial_p\Omega}} u^{\pm}(\omega) = u^-(\bar{\omega}) = u^+(\bar{\omega}) \in \mathbb{R}.$$

Thus, Proposition 3.14 guarantees the existence of an x-m function $u : \Omega \cup \partial_p\Omega \to \mathbb{R}$ satisfying $u^- \leq u \leq u^+$ such that u^* is a subsolution, u_* a supersolution of eq. (3.1) and $u_* = u^* = u^{\pm}$ on $\partial_p\Omega$. Hence, Proposition 3.8 implies that $u^* \leq u_*$, and thus $u = u^* = u_* \in C(\Omega \cup \partial_p\Omega)$ is a viscosity solution of eq. (3.1). Uniqueness subject to prescribed values on $\partial_p\Omega$ is a consequence of Proposition 3.8. □

Before providing concrete examples to Theorem 3.15, we show that if the barriers $u^{\pm}$ are Lipschitz continuous, the viscosity solution obtained in Theorem 3.15 inherits this regularity. The main ingredients in the proof are again versions of the so-called theorem on sums (see Theorem 3.9), which already was the key to proving the comparison principle (Proposition 3.8). Related approaches can be found in [63] and [62].

Proposition 3.16 (Lipschitz continuity in time)**.** *Suppose that the conditions* (A0)*,* (A1) *hold true and assume that, in addition to the hypotheses in Theorem 3.15, the* *barriers $u^{\pm}$ are locally Lipschitz continuous with respect to t in $\Omega \cup \partial_p\Omega$, i.e. for any $T' < T$ there exists $K_{T'} < \infty$ such that for all $s, t \in [0, T']$ and all $x \in \bar{J} = [0, m]$*

$$|u^{\pm}(t, x) - u^{\pm}(s, x)| \leq K_{T'}|t - s|.$$

Then for any $T' < T$ and the same constant $K_{T'}$ the associated viscosity solution u satisfies the estimate

$$|u(t, x) - u(s, x)| \leq K_{T'}|t - s| \tag{3.15}$$

for all $s, t \in [0, T']$ and all $x \in \bar{J}$.

Proof. Assume that the assertion is false. Then there exists $T' < T$ such that for $K = K_{T'}$

$$\sup_{t,s\in[0,T'],x\in J} (u(t, x) - u(s, x) - K|t - s|) > 0$$

and thus for $\eta > 0$ sufficiently small

$$M := \sup_{t,s\in[0,T'),x\in J} \left(u(t, x) - \frac{\eta}{T' - t} - \left(u(s, x) + \frac{\eta}{T' - s}\right) - K|t - s|\right) > 0.$$

With the abbreviation $u_1(t, x) := u(t, x) - \frac{\eta}{T'-t}$ and $u_2(s, x) := -\left(u(s, x) + \frac{\eta}{T'-s}\right)$ it follows that for any $\varepsilon > 0$

$$M_\varepsilon := \sup_{t,s\in[0,T'),x,y\in J} \left(u_1(t, x) + u_2(s, y) - (K|t - s| + \frac{1}{2\varepsilon}|x - y|^2)\right) \geq M > 0.$$

Let now $\varphi(t, x, s, y) := (K|t - s| + \frac{1}{2\varepsilon}|x - y|^2)$ and define $w(t, x, s, y) := u_1(t, x) + u_2(s, y) - \varphi(t, x, s, y)$. Since $u \in C([0, T'] \times \bar{J})$, the function w attains its maximum at some point $(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \in [0, T') \times \bar{J} \times [0, T') \times \bar{J}$. Notice that by the properties of $u^{\pm}$ one has $u^-(t, x) - u^-(0, x) \leq u(t, x) - u(0, x) \leq u^+(t, x) - u^+(0, x)$ and thus for all $x \in \bar{J}$ and $t \in [0, T')$

$$|u(t, x) - u(0, x)| \leq Kt,$$

which implies that $\bar{t}, \bar{s} > 0$ whenever $\varepsilon = \varepsilon(u^{\pm}(0, \cdot), M) > 0$ is sufficiently small.

We next claim that $\bar{x}, \bar{y} \notin \partial J$ for small enough $\varepsilon = \varepsilon(K) > 0$. Indeed, assuming that this is not the case, we find a sequence $\varepsilon_n \to 0$ such that $\bar{x} \in \partial J$ for all n or $\bar{y} \in \partial J$ for all n. By the boundedness of u, we must have $\bar{x} - \bar{y} \to 0$ as $n \to \infty$, and there exist $x_\infty \in \partial J$, $t_\infty, s_\infty \in [0, T')$ such that after passing to a subsequence $\bar{x}, \bar{y} \to x_\infty$, $\bar{t} \to t_\infty, \bar{s} \to s_\infty$ as $n \to \infty$. But then the continuity of u

and the fact that $u = u^{\pm}$ on $\partial_p\Omega$ lead to a contradiction to the assumption $M > 0$.

Hence $(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \in (0, T') \times J \times (0, T') \times J$. Notice also that $\bar{t} \neq \bar{s}$ for ε sufficiently small since otherwise $M_\varepsilon \to 0$ along a subsequence. This guarantees that for small enough ε, the function φ is C^2 in a neighbourhood of the maximiser of w.

We can now argue as in the proof of Proposition 3.8: by Theorem 3.9 there exist $\tau, p \in \mathbb{R}$, where $p \geq 0$, and $Q_1, Q_2 \in \mathrm{Sym}(2)$ satisfying $(\tau, p, Q_1) \in \overline{\mathcal{J}}^{2,+}(u_1)(\bar{t}, \bar{x})$, $(-\tau, -p, Q_2) \in \overline{\mathcal{J}}^{2,+}(u_2)(\bar{s}, \bar{y})$ such that for $Q = \mathrm{diag}(Q_1, Q_2)$ and $A = D^2\varphi(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ the matrix inequality $Q \leq A + A^2$ holds true. Letting $q_i := (Q_i)_{2,2}$ for $i = 1, 2$, it follows that $q_1 + q_2 \leq 0$ and, furthermore, $(\tau, p, q_1) \in \overline{P}^+ u_1(\bar{t}, \bar{x})$, $(-\tau, -p, q_2) \in \overline{P}^+ u_2(\bar{s}, \bar{y})$. By the definition of $u_i, i = 1, 2$, this means that $(\tau + \frac{\eta}{(T'-\bar{t})^2}, p, q_1) \in \overline{P}^+ u(\bar{t}, \bar{x})$, $(\tau - \frac{\eta}{(T'-\bar{s})^2}, p, -q_2) \in \overline{P}^- u(\bar{s}, \bar{y})$. A contradiction is now inferred in precisely the same way as in the proof of Proposition 3.8. □

The Lipschitz bound (3.15) implies that for all $\omega = (t, x) \in \Omega$ with $t \leq T'$ we have the implication

$$\Big[\exists\, p, q \in \mathbb{R} : \ (\tau, p, q) \in \overline{\mathcal{P}}^+ u(\omega) \ \text{ or } \ (\tau, p, q) \in \overline{\mathcal{P}}^- u(\omega)\Big] \quad \Rightarrow \quad |\tau| \leq K_{T'}. \quad (3.16)$$

Thanks to this observation, we easily obtain full Lipschitz regularity of viscosity solutions admitting barriers as in Theorem 3.15 which are Lipschitz continuous.

Proposition 3.17 (Lipschitz continuity in space)**.** *Suppose that the conditions* (A0), (A1) *hold true and assume that the barriers $u^{\pm}$ in Theorem 3.15 are in addition locally Lipschitz continuous in $\Omega \cup \partial_p\Omega$. Then for any $T' < T$ the associated viscosity solution u satisfies the estimate*

$$|u(t, x) - u(t, y)| \leq \tilde{K}_{T'}|x - y|$$

for all $t \in [0, T']$ and all $x, y \in \bar{J}$, where

$$\tilde{K}_{T'} := \max\{[u^-]_{L^\infty(0,T';C^{0,1}(\bar{J}))}, [u^+]_{L^\infty(0,T';C^{0,1}(\bar{J}))}\}.$$

Proof. Arguing by contradiction, we assume that there is $T' < T$ such that for $\tilde{K} := \tilde{K}_{T'}$

$$\sup_{t\in[0,T'], x,y\in J} \Big(u(t, x) - u(t, y) - \tilde{K}|x - y|\Big) > 0.$$

This implies that for $\eta > 0$ sufficiently small

$$M := \sup_{t\in[0,T'),\, x,y\in J} \left(u(t, x) - \frac{\eta}{T - t} - u(t, y) - \tilde{K}|x - y|\right) > 0.$$

We now define $u_1(t,x) := u(t,x) - \frac{\eta}{T-t}$, $u_2 := -u$ and $\varphi(x,y) := \tilde{K}|x-y|$, and then set $w(t,x,y) := u_1(t,x) + u_2(t,y) - \varphi(x,y)$. Since $u \in C([0,T'] \times \bar{J})$, the function w reaches its maximum M at some point $(\bar{t}, \bar{x}, \bar{y}) \in [0,T) \times [0,m] \times [0,m]$. Arguing similarly as in the proof of Proposition 3.16, we find that the maximiser $(\bar{t}, \bar{x}, \bar{y})$ of w is an interior point. Thus, in view of property (3.16) and the fact that $\bar{x} \neq \bar{y}$, the spatial version of the Theorem on Sums [38, Theorem 8.3] is applicable and yields the existence of $\tau, q_1, q_2 \in \mathbb{R}$ satisfying $q_1 + q_2 \leq 0$ and which are such that $(\tau, p, q_1) \in \overline{\mathcal{P}}^{+} u_1(\bar{t}, \bar{x})$ and $(-\tau, -p, q_2) \in \overline{\mathcal{P}}^{+} u_2(\bar{t}, \bar{y})$, where $p = \partial_x \varphi(\bar{x}, \bar{y})$. Thus

$$\left(\tau + \frac{\eta}{(T-\bar{t})^2}, p, q_1\right) \in \overline{\mathcal{P}}^{+} u(\bar{t}, \bar{x}) \quad \text{and} \quad (\tau, p, -q_2) \in \overline{\mathcal{P}}^{-} u(\bar{t}, \bar{y}).$$

Now the contradiction is obtained by using the fact that u is a sub- and a supersolution of eq. (3.1). □

As an immediate consequence of Proposition 3.16 and Proposition 3.17 we obtain

Corollary 3.18 (Lipschitz continuity)**.** *Under the hypotheses in Proposition 3.17 the corresponding viscosity solution u of equation* (3.1) *is locally Lipschitz continuous in $\Omega \cup \partial_p \Omega$ and satisfies the estimate*

$$[u]_{C^{0,1}([0,T'] \times \bar{J})} \leq \sqrt{2} \max\{K_{T'}, \tilde{K}_{T'}\},$$

where $K_{T'}$ and $\tilde{K}_{T'}$ denote the constants defined in Proposition 3.16 and Proposition 3.17.

3.6 Applications to generalised bosonic Fokker–Planck equations (GBFP)

Here we demonstrate how Theorem 3.15 can be used to derive global-in-time well-posedness for the Cauchy–Dirichlet problem associated with a class of equations generalising (2.9). In the original variables these problems correspond to a class of nonlinear Fokker–Planck equations generalising in 1D the equations (2.2) on a centred ball (cf. eq. (3.18) below). We refer to this generalised class, considered in Theorem 3.20 below, as *generalised bosonic Fokker–Planck equations* (GBFP). The equations considered are reminiscent of the setting in the reference [11] considering the stationary problem, but the precise regularity assumptions are slightly different.

Let $h \in C((0,\infty))$ be a positive function which satisfies $1/h \in L^1(1,\infty)$ and $\int_s^\infty \frac{1}{h(z)}\,\mathrm{d}z \in L^1_{\mathrm{loc}}([0,\infty))$. We then define $\Phi^{(h)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\leq 0}$ via $\Phi^{(h)}(0) = 0$,

$\Phi^{(h)'}(s) = -\int_s^\infty \frac{1}{h(z)}\,\mathrm{d}z$, $s > 0$, and consider the functional

$$\mathcal{H}^{(h,R)}(f) = \int_{-R}^{R} \left(\frac{|r|^2}{2} f + \Phi^{(h)}(f) \right) \mathrm{d}r, \qquad f \in L^1_+(-R,R), \tag{3.17}$$

where $R \in (0,\infty)$. We are interested in the equation

$$\frac{\partial f}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}r} \left(h(f) \frac{\mathrm{d}}{\mathrm{d}r} \frac{\delta \mathcal{H}^{(h,R)}}{\delta f}(f) \right), \quad t > 0,\ r \in (-R,R), \tag{3.18}$$

subject to the zero-flux boundary conditions $\frac{\mathrm{d}}{\mathrm{d}r}\frac{\delta \mathcal{H}^{(h,R)}}{\delta f}(f) = 0$ on $(0,\infty) \times \{-R, R\}$. We define the *steady states* of this conservative problem to be the positive, smooth solutions f of

$$\frac{\mathrm{d}}{\mathrm{d}r} \frac{\delta \mathcal{H}^{(h,R)}}{\delta f}(f) = 0,$$

i.e. the solutions $f^{(h)}_{\infty,\theta}$ of

$$\frac{|r|^2}{2} + \Phi^{(h)'}(f^{(h)}_{\infty,\theta}) = -\theta,$$

where θ is a constant of integration.

In the following we assume that $1/h(s)$ is not integrable near $s = 0$, which implies that $\lim_{s \to 0^+} \Phi^{(h)'}(s) = -\infty$. Since $\Phi^{(h)'}$ is strictly increasing and satisfies $\lim_{s\to\infty} \Phi^{(h)'}(s) = 0$, we can then solve the last equation for $f^{(h)}_{\infty,\theta}$ to obtain

$$f^{(h)}_{\infty,\theta}(r) = (\Phi^{(h)'})^{-1}\left(-(|r|^2/2 + \theta)\right), \quad r \in (-R,R),$$

provided that $\theta \in [0,\infty)$. Observe that here we have admitted the limiting case $\theta = 0$, despite the fact that the function $f^{(h)}_{\infty,0}$ satisfies $f^{(h)}_{\infty,0}(r) \to \infty$ as $r \to 0$. Furthermore, notice that $f^{(h)}_{\infty,\theta} \to 0$ uniformly in $[-R,R]$ as $\theta \to \infty$ and that for any $\theta < \infty$ there exists $c_\theta > 0$ such that $f^{(h)}_{\infty,\theta} \geq c_\theta$ in $[-R,R]$. Thus, letting

$$m_h^{(R,\theta)} := \int_{-R}^{R} f^{(h)}_{\infty,\theta}(r)\,\mathrm{d}r, \tag{3.19}$$

$$\theta_h^{(R,m)} := \min\{\theta \geq 0 : m_h^{(R,\theta)} \leq m\} \tag{3.20}$$

and denoting for given $m \in (0,\infty)$ and given $\theta \geq \theta_h^{(R,m)}$ by $u^{(R,m)}_{\theta,-,h} : [0,m] \to [-R,R]$ (resp. by $u^{(R,m)}_{\theta,+,h} : [0,m] \to [-R,R]$) the pseudo-inverse[1] of the cdf of $(m - m_h^{(R,\theta)})\delta_{-R} + f^{(h)}_{\infty,\theta} \cdot \mathcal{L}^1$ (resp. of $(m - m_h^{(R,\theta)})\delta_R + f^{(h)}_{\infty,\theta} \cdot \mathcal{L}^1$), we infer that $u^{(R,m)}_{\theta,\mp,h}$ are Lipschitz continuous in $[0,m]$ and that for any non-decreasing function $u_0 \in C^1([0,m])$ with

[1]See Def. 2.3 for the definition of the pseudo-inverse of an increasing, right-continuous function M.

$u_0(0) = -R$, $u_0(m) = R$ there exists $\theta < \infty$ such that

$$u_{\theta,-,h}^{(R,m)} \le u_0 \le u_{\theta,+,h}^{(R,m)}. \tag{3.21}$$

See Figure 3.1 on page 44 for an illustration of the functions $u_{\theta,\pm,h}^{(R,m)}$.

Formally, the equation for the pseudo-inverse $u(t,\cdot)$ of the cdf associated with $f(t,\cdot)$ states

$$u_t - \frac{u_{xx}}{u_x^2} + u_x h(1/u_x) u = 0 \quad \text{in } \Omega := (0,\infty) \times (0,m), \tag{3.22}$$

where m denotes the mass of the initial datum f_0, i.e. $m = \int_{-R}^{R} f_0(r)\,\mathrm{d}r$. In view of the no-flux boundary conditions for eq. (3.18), we complement eq. (3.22) with the Dirichlet conditions

$$u(t,0) = -R, \quad u(t,m) = R \tag{3.23}$$

for all $t > 0$.

We henceforth suppose that $\lim_{s\to\infty} s^3/h(s)$ exists in $[0,\infty)$ and define

$$G(z,\alpha,p,q) = \left(|p|^3 h(1/|p|)\right)^{-1} \left(|p|^2\alpha - q\right) + z, \tag{3.24}$$

with the understanding that for all $z, \alpha, q \in \mathbb{R}$

$$G(z,\alpha,0,q) := \lim_{p\to 0} G(z,\alpha,p,q),$$

which, by assumption, exists in $\mathbb{R}$. Then the function G is continuous on $\mathbb{R}^4$, satisfies the conditions (A0) and (A1), and defining $\mathcal{G}$ by formula (3.2), equation (3.22) can be reformulated as

$$\mathcal{G}(u) = 0 \quad \text{in } \Omega. \tag{3.25}$$

Notice that equations (3.22) and (3.25) are equivalent if $0 < u_x < \infty$.

Definition 3.19 (Initial data for GBFP problem)**.** For a given function h as introduced above (and G defined via (3.24)) let $\mathcal{S}_h$ denote the set of all non-decreasing functions $u_0 \in C^1([0,m])$ having the following properties:

- $u_0(0) = -R$, $u_0(m) = R$,
- $u_0'(x) > 0$ for all $x \in [0,m]$ with $|u_0(x)| > 0$,

- $u_0 \in C^2(\{x \in [0,m] : |u_0(x)| > 0\})$ and

$$C := C(u_0) := \sup_{\{|u_0|>0\}} \left|p_0 h(p_0^{-1})\mathcal{G}(u_0)\right| < \infty, \tag{3.26}$$

with $p_0 := u_0'$ and where we have used the abbreviation (3.2).

The choice of C in formula (3.26) guarantees that $u_0 \mp Ct$, $t \geq 0$, is a sub- resp. supersolution of eq. (3.25) in $\Omega := (0,\infty) \times (0,m)$. Any $u_0 \in C^2([0,m])$ with $\min_{[0,m]} u_0' > 0$ and $u_0(0) = -R, u_0(m) = R$ lies in the set $\mathcal{S}_h$, but, in general, Definition 3.19 also allows for functions which have a flat part at level zero, see Remark 3.23 for details and the meaning of the bound (3.26).

We are now in a position to show wellposedness for the problems introduced above.

Theorem 3.20 (Global existence, uniqueness and Lipschitz continuity for GBFP)**.** *Suppose that the function $h \in C((0,\infty),\mathbb{R}^+)$ satisfies $1/h \notin L^1(0,1)$, $\int_s^\infty \frac{1}{h(z)}\,\mathrm{d}z \in L^1(0,1)$ and that the limit $\lim_{s\to\infty} s^3/h(s)$ exists in $[0,\infty)$. Given $u_0 \in \mathcal{S}_h$ there exists a unique, x-monotonic viscosity solution $u \in C(\Omega \cup \partial_p\Omega)$ of problem (3.23)–(3.25) such that $u(0,\cdot) = u_0$. This solution is globally Lipschitz continuous with constant bounded above by $K = \sqrt{2}\max\{C(u_0), [u_{\theta,\pm,h}^{(R,m)}]_{C^{0,1}}\}$, where $\theta \geq 0$ is any number such that ineq. (3.21) is fulfilled.*

Proof. Choose $\theta < \infty$ such that ineq. (3.21) holds true. Then the function

$$u^-(t,x) := \max\left\{u_0(x) - Ct, u_{\theta,-,h}^{(R,m)}(x)\right\}$$

is a subsolution, while the function

$$u^+(t,x) := \min\left\{u_0(x) + Ct, u_{\theta,+,h}^{(R,m)}(x)\right\}$$

is a supersolution satisfying $u^- \leq u_0 \leq u^+$.

The functions $u^\pm$ are of class $C^{0,1}(\Omega \cup \partial_p\Omega)$ and have the desired behaviour on $\partial_p\Omega$. Thus, Theorem 3.15 yields the first claim. The Lipschitz continuity is a consequence of Corollary 3.18. □

Remark 3.21 (Critical mass $m_c(R)$). In general, the singularity of $f_{\infty,0}^{(h)}$ near the origin may not be integrable. Following [11], one finds that $m_c(R) := m_h^{(R,0)} < \infty$ if and only if

$$\int_1^\infty \frac{s}{h(s)} \left(\int_s^\infty \frac{1}{h(\sigma)}\,\mathrm{d}\sigma\right)^{-\frac{1}{2}} \mathrm{d}s < \infty.$$

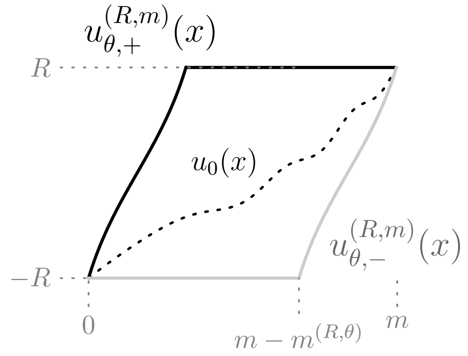


Figure 3.1: Illustration of the definition of $u_{\theta,\pm}^{(R,m)}$ for given $m, R > 0$ and $\theta > \theta^{(R,m)}$. Given an initial datum u_0 for the GBFP equation and θ satisfying (3.21), the functions $u_{\theta,\pm}^{(R,m)}$ serve as barriers enforcing the lateral boundary conditions (3.23).

Remark 3.22 (Entropy minimisers)*.* Since $\lim_{s\to\infty} \Phi^{(h)}(s)/s = 0$, we can proceed as in [11] and extend the functional $\mathcal{H}^{(h,R)}$ to the set of finite measures on $[-R, R]$ by ignoring the singular component (with respect to the Lebesgue measure) in the nonlinear term involving $\Phi^{(h)}$. Following the proof of [11, Theorem 3.1] one can show that the unique minimiser of the extended functional $\widetilde{\mathcal{H}}^{(h,R)}$ among measures $\mu \in \mathcal{M}^+([-R, R])$ of mass $m > 0$ is given by

$$\mu_\infty^{(m,R,h)} = \begin{cases} f_{\infty,\theta}^{(h)} \cdot \mathcal{L}^1 & \text{if } m < m_c(R), \text{ where } \theta = \theta_h^{(R,m)}, \\ f_c^{(h)} \cdot \mathcal{L}^1 + (m - m_c(R))\delta_0 & \text{if } m \geq m_c(R), \end{cases} \tag{3.27}$$

where $f_c^{(h)} := f_{\infty,0}^{(h)}$. Notice that for any $m > 0$ the pseudo-inverse of the cdf of $\mu_\infty^{(m,R,h)}$ is of class $C^1([0, m])$ and is a viscosity solution of eq. (3.25) while for $\theta > 0$ and $m > m_h^{(R,\theta)}$ the pseudo-inverse cdf of the measure $f_{\infty,\theta}^{(h)} \cdot \mathcal{L}^1 + (m - m_h^{(R,\theta)})\delta_0$ is neither a sub- nor a supersolution of eq. (3.25).

Remark 3.23*.* If $m_c(R) < \infty$, there exist functions $u_0 \in \mathcal{S}_h$ which have a flat part at level zero, so that there exist $0 < x_- \leq x_+ < m$ such that $u_0(x) = 0, u_0'(x) = 0$ for all $x \in [x_-, x_+]$ and $|u_0(x)| > 0$ for $x \notin [x_-, x_+]$. In this case, condition (3.26) is non-trivial and enforces that, loosely speaking, the asymptotic behaviour of $u_0(x)$ as $x \to (x_\pm)^\pm$ agrees with the corresponding behaviour of the pseudo-inverse cdf of $f_{\infty,0}^{(h)}$. For its meaning at the level of the density f_0 associated with u_0 (for a specific choice of h) see Section 4.2.

Observing that for $\gamma \geq 2$ the function $h(s) = h_\gamma(s) := s(1 + s^\gamma)$ is admissible in Theorem 3.20, we deduce wellposedness for our BFP problem in the new variables.

Corollary 3.24 (Global existence, uniqueness and Lipschitz continuity for the 1D bosonic Fokker–Planck equations in the L^1-supercritical and -critical case)**.** *Let*

$m, R \in (0,\infty)$ and abbreviate $\Omega := (0,\infty) \times (0,m)$. Suppose that $\gamma \geq 2$, let F be defined by

$$F(z,\alpha,p,q) := |p|^\gamma \alpha - |p|^{\gamma-2} q + z(1+|p|^\gamma) \tag{3.28}$$

and abbreviate $\mathcal{F}(u) := F(u, \partial_t u, \partial_x u, \partial_x^2 u)$. Given $u_0 \in \mathcal{S}_{h_\gamma}$ there exists a unique, x-monotonic viscosity solution $u \in C(\Omega \cup \partial_p \Omega)$ of the problem

$$\begin{cases} \mathcal{F}(u) = 0, & \textit{in } \Omega, \\ u(t,0) = -R, \quad u(t,m) = R, & \textit{for } t > 0, \\ u(0,x) = u_0(x), & \textit{for } x \in [0,m]. \end{cases}$$

This solution is globally Lipschitz continuous with Lipschitz constant bounded above by

$$K = K\left(C_1(u_0), [u^{(R,m)}_{\theta,\pm,h_\gamma}]_{C^{0,1}}\right) < \infty,$$

where $\theta > 0$ is any positive number[2] such that the inequalities $u^{(R,m)}_{\theta,-,h_\gamma} \leq u_0 \leq u^{(R,m)}_{\theta,+,h_\gamma}$ are satisfied.

[2]The existence of such a number θ is guaranteed by the assumption $u_0 \in C^1([0,m])$.

Chapter 4

Finite-time condensation and relaxation to equilibrium in 1D Fokker–Planck model for bosons

Given $\gamma \geq 2$, a fixed total mass $m \in (0, \infty)$, a radius $R > 0$, and an initial datum $u_0 \in C^2([0, m])$ such that $\min_{[0,m]} u_0' > 0$ and $u_0(0) = -R, u_0(m) = R$, Corollary 3.15 of Chapter 3 ensures the existence, uniqueness and Lipschitz regularity of viscosity solutions $u = u(t, x)$, non-decreasing in x, of the Cauchy–Dirichlet problem

$$\begin{cases} \mathcal{F}(u) = 0, & \text{in } \Omega := (0, \infty) \times (0, m), \\ u(t, 0) = -R, \quad u(t, m) = R, & \text{for } t > 0, \\ u(0, x) = u_0(x), & \text{for } x \in [0, m], \end{cases} \tag{4.1}$$

where $\mathcal{F}(u) := F(u, \partial_t u, \partial_x u, \partial_x^2 u)$ with

$$F(z, \alpha, p, q) := |p|^\gamma \alpha - |p|^{\gamma-2} q + z(1 + |p|^\gamma).$$

The main problems to be tackled in this chapter are as follows:

(Q1) Developing a detailed understanding of the regularity of u and analysing its implications for the problem in the original variables (see Remark 4.1).

(Q2) Establishing an entropy technique valid globally in time which enables us to identify the long-time asymptotic behaviour of solutions and allows us to prove that in the mass-supercritical case $m > m_c(R)$ singularities and condensates always emerge in finite time.

(Q3) Extending the above results to the problem corresponding to a density f defined on the whole line (i.e. corresponding formally to $R = \infty$).

Remark 4.1 (Original variables). Let $t \geq 0$ be fixed. Since the continuous function $u(t,\cdot) : [0,m] \to [-R,R]$ is non-decreasing from $u(t,0) = -R$ to $u(t,m) = R$, we can define its *generalised inverse* $M(t,\cdot) : [-R,R] \to [0,m]$ via

$$M(t,r) := \sup\{x \in [0,m] : u(t,x) \leq r\}, \qquad r \in [-R,R] \tag{4.2}$$

or, equivalently, by $M(t,r) = \max\left(u(t,\cdot)^{-1}(\{r\})\right)$. By definition $M(t,\cdot)$ is non-decreasing and satisfies $M(t,-R) \geq 0$, $M(t,R) = m$. Since $u(t,\cdot)$ is continuous, $M(t,\cdot)$ is actually strictly increasing. Indeed, the closedness of the preimages $u(t,\cdot)^{-1}([-R,r])$ implies that

$$u(t, M(t,r)) = r,$$

so that the assumption $M(t,r_1) = M(t,r_2)$ immediately yields $r_1 = r_2$. Moreover, it is easy to see that $M(t,\cdot)$ is right-continuous. Hence, there is a unique Borel measure $\mu(t) \in \mathcal{M}([-R,R])$ satisfying

$$\mu(t)([-R,r]) = M(t,r) \qquad \text{for all } r \in [-R,R], \tag{4.3}$$

see e.g. [93, Chapter 20.3]. At this stage, we know relatively little about the regularity of the family of Borel measures $\{\mu(t)\}_t$, and our first goal, formulated in problem (Q1), can be seen as a way to approach this question.

Unless otherwise stated, in the current and the subsequent chapter initial data u_0 for problem (4.1) are always assumed to be admissible in the following sense:

Definition 4.2 (Admissible initial datum for problem (4.1))**.** A function u_0 on $[0,m]$ is called an *admissible* initial datum for problem (4.1) if it satisfies $u_0 \in C^2([0,m])$ with $\min_{[0,m]} u_0' > 0$ and takes the boundary values $u_0(0) = -R, u_0(m) = R$.

Let us next briefly outline this chapter's content: we first show that our viscosity solutions are actually weak solutions (in a suitable distributional sense) satisfying a natural a priori estimate associated to the equation. The regularity derived and the equation's structure will then allow us to prove that our solutions are smooth away from the level set $\{u = 0\}$ (Section 4.1). Subsequently, we translate our results back to the original variables to obtain a finite measure $\mu(t)$, as introduced in (4.3), whose singular part with respect to the Lebesgue measure is supported at the origin and whose absolutely continuous part has a density which is smooth away from the origin. The spatial blow-up profile of the density is proved to be universal to leading order (Section 4.2.1). In Section 4.2.2 we prove that the entropy dissipation identity (at the level of $\mu(t)$) holds true globally in time, even for solutions with non-trivial singular part. This allows us to deduce the long-time asymptotics as well as the formation of a condensate in finite time provided $m > m_c(R)$ (Section 4.2.3). The outline provided here is non-exhaustive, and we refer the reader to the beginning of each individual section (or subsection) for a more detailed presentation of the results.

Finally, it will be convenient in this chapter to use the following notations.

Notations 4.3 **($\mu_\infty^{(R,m)}$ and $u_\infty^{(R,m)}$).** As above we fix $\gamma \geq 2$ and let $h(s) = s(1 + s^\gamma)$. Then for $R \in (0, \infty)$ and $\theta \geq 0$ we abbreviate $f_c = f_{\infty,0}^{(h)}$, $m^{(R,\theta)} := m_h^{(R,\theta)}$, $\theta^{(R,m)} := \theta_h^{(R,m)}$, where $m_h^{(R,\theta)}$ and $\theta_h^{(R,m)}$ are defined by (3.19) resp. (3.20). Next, for given $R, m \in (0, \infty)$ we let

$$\mu_\infty^{(R,m)} := \mu_\infty^{(R,m,h_\gamma)}, \tag{4.4}$$

where $\mu_\infty^{(R,m,h_\gamma)}$ is given by (3.27). We then denote by $u_\infty^{(R,m)}$ the pseudo-inverse (in the sense of Def. 2.3) of the cdf of $\mu_\infty^{(R,m)}$. Notice that $u_\infty^{(R,m)} \in C^1([0, m])$. Finally, given $\theta \geq \theta^{(R,m)}$ we abbreviate $u_{\theta,\pm}^{(R,m)} := u_{\theta,\pm,h}^{(R,m)}$, where $u_{\theta,\pm,h}^{(R,m)}$ has been introduced on p. 42 (see also Fig. 3.1).

4.1 Refined regularity for bosonic Fokker–Planck model

Recall that our concept of solution chosen for problem (4.1), the notion of a viscosity solution, is a rather weak one. In particular, the equation $\mathcal{F}(u) = 0$ is only satisfied (and only makes sense) at points where u is sufficiently regular. A first step towards a better understanding of the kinetics of our problem is therefore the derivation of improved regularity properties. We will now briefly motivate via formal a priori arguments the regularity results asserted in Theorem 4.4 below. The experienced reader may choose to directly move on to the statement of Theorem 4.4 and its proof. Notice that any classical solution u of the equation $\mathcal{F}(u) = 0$ in Ω, i.e. of

$$(\partial_x u)^\gamma \partial_t u - (\partial_x u)^{\gamma-2} \partial_x^2 u + u(1 + (\partial_x u)^\gamma) = 0 \qquad \text{in } \Omega$$

satisfies the a priori estimate

$$\frac{1}{\gamma - 1} \left| \frac{\mathrm{d}}{\mathrm{d}x} (\partial_x u)^{\gamma-1} \right| = \left| (\partial_x u)^{\gamma-2} \partial_x^2 u \right| \leq C(\|u\|_{C^{0,1}(\Omega)}),$$

where $\|u\|_{C^{0,1}(\Omega))}$ denotes the Lipschitz norm of $u = u(t, x)$ on Ω. Since at this stage we do not know whether our viscosity solutions are weak solutions in a distributional sense, we cannot directly manipulate our equation to extend the above estimate to viscosity solutions u. Instead, we will construct a sequence of approximate solutions (v_σ), satisfying a regularised problem, for which an estimate analogous to the above one holds true uniformly in the parameter σ. The stability and uniqueness of viscosity solutions to problem (4.1) then imply that the same estimate is valid for our viscosity solutions. We will obtain, in particular, that for each $t \geq 0$ the function $x \mapsto u(t,x)$ is of the class $C^1([0,m])$. Thus, if $\min \partial_x u(t,\cdot) > 0$, the measure $\mu(t)$ defined by formula (4.3) is absolutely continuous and its density with respect to the Lebesgue measure is a continuous function, uniquely defined via

$$f(t,u(t,x)) = \frac{1}{\partial_x u(t,x)}, \quad x \in (0,m).$$

In general, the function $u(t,\cdot)$ may, however, have critical points, giving rise to singularities at the level of $f(t,\cdot)$. In Section 4.1.2 we will prove that all critical points of the C^1 function $u(t,\cdot)$ are contained in the set $\{u(t,\cdot) = 0\}$. A formal mathematical motivation for this result is as follows: suppose that u is a classical solution of $\mathcal{F}(u) = 0$ in Ω, let $t > 0$ and assume that x_c is a critical point of $u(t,\cdot)$, i.e. $x_c \in \{\partial_x u(t,\cdot) = 0\}$. Then $\partial_x u(t,x_c) = 0$ and, since $\partial_x u(t,\cdot)$ reaches its minimum at x_c, we also have $\partial_x^2 u(t,x_c) = 0$. Hence, whenever $\gamma \geq 2$,

$$0 = \mathcal{F}(u)_{|(t,x_c)} = F(u(t,x_c), \partial_t u(t,x_c), 0, 0) = u(t,x_c).$$

Of course, in the case of viscosity solutions the rigorous argument requires more care, even when assuming the improved regularity to be derived in Section 4.1.1.

Let us now turn to the precise statement of our results and its rigorous proof.

Theorem 4.4 (Refined regularity)**.** *Suppose that $\gamma \geq 2$. Given $m, R > 0$ and an initial datum u_0 which is admissible in the sense of Definition 4.2, let $u \in C(\Omega \cup \partial_p \Omega)$ denote the unique viscosity solution of the Cauchy–Dirichlet problem* (4.1) *(see Corollary 3.24). Recall that $u \in C^{0,1}(\bar{\Omega})$ and that for each $t \geq 0$ the function $u(t,\cdot)$ is non-decreasing. The following assertions hold true:*

(R1) *We have the regularity*

$$u \in L^\infty(0,\infty; C^{1,\frac{1}{\gamma-1}}(\bar{J})),$$

where $J = (0,m)$, and u satisfies the estimate

$$\|\partial_x((\partial_x u)^{\gamma-1})\|_{L^\infty(\Omega)} \leq C([u]_{C^{0,1}(\bar{\Omega})}, R, \gamma).$$

Thus, $u \in C_b([0,\infty); C^{1,\beta}(\bar{J}))$ *with*

$$\sup_{t\geq 0} \|u(t,\cdot)\|_{C^{1,\beta}(\bar{J})} \leq C([u]_{C^{0,1}(\bar{\Omega})}, R, \gamma) \tag{4.5}$$

for $\beta \in (0, \frac{1}{\gamma-1})$.

(R2) *Defining the sets*

$$\Omega^+ := \{\omega \in \Omega :\ |u(\omega)| > 0\},$$
$$\Omega^{++} := \{\omega \in \Omega :\ \partial_x u(\omega) > 0\},$$

which, by (R1), *are open sets, the solution* u *is* C^∞ *in* Ω^{++}, *and we have*

$$\Omega^+ \subseteq \Omega^{++}.$$

In particular, in Ω^{++} *the equation* $\mathcal{F}(u) = 0$ *holds true in the classical sense.*

Remark 4.5. Observe that the regularity (R1) and our hypothesis $\inf_J u_0' > 0$ imply that there exists $t^* = t^*(u_0) > 0$ such that $\{(t,x) \in \Omega : t < t^*)\} \subset \Omega^{++}$. Thus, thanks to (R2) we deduce short-time regularity of the viscosity solution u.

For a possible extension of the regularity results to solutions of the GBFP problem considered in Theorem 3.20 see Remark 4.7.

The proof of Theorem 4.4 will be given in the following two subsections.

4.1.1 Approximate problems

Proof of Theorem 4.4 (R1). We consider a regularised version of problem (4.1) in $\Omega := (0,\infty) \times J$, $J := (0,m)$, obtained by replacing the function $F(z,\tau,p,q)$ with $F_\sigma(z,\tau,p,q) := p^\gamma \tau - (p+\sigma)^{\gamma-2} q + z(1+p^\gamma)$, $0 < \sigma \ll 1$, the lateral boundary conditions with $u(t,0) = -R_\sigma$ and $u(t,m) = R_\sigma$ for suitable $0 < R_\sigma \leq R$ with $R_\sigma \to R$ as $\sigma \to 0$ and the initial value u_0 by suitable approximations $u_{0,\sigma} \in C^2(\bar{J})$ with $\min_{\bar{J}} u_{0,\sigma}' > 0$ satisfying $u_{0,\sigma}(0) = -R_\sigma$, $u_{0,\sigma}(m) = R_\sigma$, $u_{0,\sigma} \nearrow u_0$ in $C^2(\bar{J})$. It is easy to see that such a sequence $(u_{0,\sigma})$ exists. Under these conditions the constants $C_\sigma(u_{0,\sigma})$, where

$$C_\sigma(v) := \sup_{x\in J} \left| -\frac{(p(x)+\sigma)^{\gamma-2}}{p^\gamma(x)} q(x) + v(x)(p(x)^{-\gamma} + 1) \right|, \quad p = v', q = v'', \tag{4.6}$$

are uniformly bounded in $0 < \sigma \ll 1$.

Existence and uniqueness of x-monotonic viscosity solutions are obtained by Theorem 3.15 provided appropriate barriers can be found. A possible construction of the barriers is as follows: we fix some $\theta > 0$ such that

$$u_{\theta,-} \leq u_0 \leq u_{\theta,+}$$

and define

$$\kappa(\sigma) := \sup_{x \in J : |u_\theta(x)| > 0} \left| u_\theta(x) - \frac{(p_\theta(x) + \sigma)^{\gamma-2} q_\theta(x)}{1 + p_\theta^\gamma(x)} \right|,$$

where we abbreviated $p_\theta := u_\theta'$ and $q_\theta := u_\theta''$ (which are well-defined on $\{|u_\theta| > 0\}$). We note that $\kappa \in C([0,1])$ with $\kappa(0) = 0$, and let

$$R_\sigma := R - \kappa(\sigma).$$

By construction the function

$$u_{\theta,\sigma}^- := \max\{-R_\sigma, u_{\theta,-} - \kappa(\sigma)\}$$

is a subsolution of $\mathcal{F}_\sigma = 0$, while the function

$$u_{\theta,\sigma}^+ := \min\{R_\sigma, u_{\theta,+} + \kappa(\sigma)\}$$

is a supersolution. Both functions are continuous on $\bar{J}$ and they satisfy $u_{\theta,\sigma}^\pm(0) = -R_\sigma$, $u_{\theta,\sigma}^\pm(m) = R_\sigma$. It is also clear that after possibly slightly modifying the choice of $u_{0,\sigma}$, we can assume that $u_{\theta,\sigma}^- \leq u_{0,\sigma} \leq u_{\theta,\sigma}^+$.

Letting

$$v_\sigma^-(t,x) := \max\{u_{0,\sigma}(x) - C_\sigma t, u_{\theta,\sigma}^-(x)\}$$

and

$$v_\sigma^+(t,x) := \min\{u_{0,\sigma}(x) + C_\sigma t, u_{\theta,\sigma}^+(x)\},$$

where $C_\sigma := C_\sigma(u_{0,\sigma})$ (see formula (4.6)), defines bounded x-m functions $v_\sigma^\pm \in C(\Omega \cup \partial_p \Omega)$ with the desired behaviour on $\partial_p \Omega$ such that v_σ^- is a subsolution and v_σ^+ a supersolution of $\mathcal{F}_\sigma = 0$. Thus, subject to the conditions on $\partial_p \Omega$ specified above, there exists a unique viscosity solution v_σ of $\mathcal{F}_\sigma = 0$ in $(0,\infty) \times J$, which, by Corollary 3.18, is such that the Lipschitz norm $\|v_\sigma\|_{C^{0,1}([0,\infty)\times\bar{J})}$ is uniformly bounded in $0 < \sigma \ll 1$. The Arzelà–Ascoli theorem combined with Remark 3.7 (a) and the uniqueness part of Theorem 3.15 now implies that, upon passing to a subsequence, we have $v_\sigma \to u$ locally uniformly in $\bar{\Omega}$. (Notice that the passage to a subsequence was not necessary.)

The approximate solutions v_σ are more regular: for any $\omega \in \Omega$ and any $(\tau, p, q) \in \mathcal{P}^- v_\sigma(\omega)$ we have

$$p^\gamma \tau - (p+\sigma)^{\gamma-2} q + v_\sigma(\omega)(1+p^\gamma) \geq 0$$

and therefore

$$\begin{aligned} q &\leq p^2 \tau + v_\sigma(\omega)((p+\sigma)^{2-\gamma} + p^2) \\ &\leq C([v_\sigma]_{C^{0,1}(\bar{\Omega})}) + R\left(\sigma^{2-\gamma} + C([v_\sigma]_{C^{0,1}(\bar{\Omega})})\right). \end{aligned}$$

Similarly, for any $\omega \in \Omega$ and any $(\tau, p, q) \in \mathcal{P}^+ v_\sigma(\omega)$ we have

$$p^\gamma \tau - (p+\sigma)^{\gamma-2} q + v_\sigma(\omega)(1+p^\gamma) \leq 0$$

and therefore

$$\begin{aligned} q &\geq p^2 \tau + v_\sigma(\omega)((p+\sigma)^{2-\gamma} + p^2) \\ &\geq -C([v_\sigma]_{C^{0,1}(\bar{\Omega})}) - R\left(\sigma^{2-\gamma} + C([v_\sigma]_{C^{0,1}(\bar{\Omega})})\right). \end{aligned}$$

By Proposition 4.34 (see also Definition 4.33), we conclude that for all $t > 0$ (and uniformly in t) the function $v_\sigma(t, \cdot)$ is semi-concave as well as semi-convex, which implies (see Lemma 4.35) the regularity $v_\sigma(t, \cdot) \in C^{1,1}(\bar{J})$. Then, as demonstrated in Appendix 4.A.2, the second pointwise derivative ${}^{(p)}\partial_x^2 v_\sigma$ of v_σ with respect to x exists $\mathcal{L}^2$-almost everywhere in Ω and $\partial_x v_\sigma$ has a weak derivative satisfying $\partial_x^2 v_\sigma = {}^{(p)}\partial_x^2 v_\sigma \in L^\infty(\Omega)$. Now we can relate the viscosity solution property to a more classical notion of solution. From the preceding observations and Rademacher's theorem (see e.g. [51]), it follows that $\mathcal{P} v_\sigma(\omega)$ exists for $\mathcal{L}^2$-almost every $\omega \in \Omega$ and that the function v_σ is a strong solution in the sense that the weak derivatives $\partial_t v_\sigma, \partial_x v_\sigma, \partial_x^2 v_\sigma$ exist in $L^\infty(\Omega)$ and satisfy $F_\sigma(v_\sigma, \partial_t v_\sigma, \partial_x v_\sigma, \partial_x^2 v_\sigma) = 0$ in $L^\infty(\Omega)$. In particular, in view of the inequality $\frac{1}{\gamma-1}|\partial_x((\partial_x v_\sigma)^{\gamma-1})| \leq |(\partial_x v_\sigma + \sigma)^{\gamma-2} \partial_x^2 v_\sigma|$, the equation $\mathcal{F}_\sigma(v_\sigma) = 0$ and the fact that $[v_\sigma]_{C^{0,1}(\bar{\Omega})} \leq C([u]_{C^{0,1}(\bar{\Omega})})$ yield the bound

$$\|\partial_x((\partial_x v_\sigma)^{\gamma-1})\|_{L^\infty(\Omega)} \leq C([u]_{C^{0,1}(\bar{\Omega})}, R, \gamma). \tag{4.7}$$

Hence, switching to the Bochner function perspective via Fubini's theorem, we have for any $T < \infty$

$$v_\sigma \in L^\infty(0, T; C^{1, \frac{1}{\gamma-1}}(\bar{J})), \quad \partial_t v_\sigma \in L^\infty(0, T; L^\infty(J)),$$

with norms uniformly bounded in σ (and T). Thus, thanks to the Aubin–Lions lemma (see e.g. [19, Theorem II.5.16]) and the locally uniform convergence $v_\sigma \to u$, we can pass to a subsequence satisfying for $\beta \in (0, \frac{1}{\gamma-1})$ and any $T < \infty$

$$v_\sigma \to u \quad \text{in } C([0,T]; C^{1,\beta}(\bar{J})).$$

In particular, for any compact subset $K \subset \bar{\Omega}$ we have $\partial_x v_\sigma \to \partial_x u$ in $C(K)$ and thus $(\partial_x v_\sigma)^{\gamma-1} \to (\partial_x u)^{\gamma-1}$ in $C(K)$. Now, the bound (4.7) yields

$$\|\partial_x((\partial_x u)^{\gamma-1})\|_{L^\infty(\Omega)} \leq C([u]_{C^{0,1}(\bar{\Omega})}, R, \gamma) \tag{4.8}$$

and $u \in C_b([0,\infty); C^{1,\beta}(\bar{J}))$, with

$$\sup_{t\geq 0} \|u(t,\cdot)\|_{C^{1,\beta}(\bar{J})} \leq C([u]_{C^{0,1}(\bar{\Omega})}, R, \gamma)$$

for $\beta \in (0, \frac{1}{\gamma-1})$. This completes the proof of Theorem 4.4 (R1). □

Remark 4.6. The specific form of the regularised, uniformly parabolic equation in Section 4.1.1 is not essential. For instance, we could have chosen $F_\sigma(z, \alpha, p, q) := F(z, \alpha, p + \sigma, q)$ instead.

Remark 4.7. The arguments in the proof of Theorem 4.4 (R1) can be generalised to the problem of the GBFP equation $\mathcal{G}(u) = 0$ (subject to the same Cauchy–Dirichlet conditions) whenever h satisfies the hypotheses in Theorem 3.20. Let us sketch how to argue in the general case. The family (v_σ) of approximate solutions is constructed analogously, where one can choose, for instance, as regularised problem $G_\sigma(z, \alpha, p, q) := G(z, \alpha, p + \sigma, q)$. Of course, we cannot expect to obtain the uniform bound (4.7) (as h may have rapid growth at infinity), but notice that in order to ensure compactness it is sufficient to deduce equicontinuity in x of the family $(\partial_x v_\sigma)_{\sigma\in(0,1)}$. To see the latter, define the continuous function $\kappa : [0,\infty) \to [0,\infty)$ via

$$\kappa(v) = (v^3 h(1/v))^{-1},$$

observe that κ is strictly positive for $v > 0$, and then consider the strictly increasing function

$$K(v) = \int_0^v \kappa(s)\,\mathrm{d}s, \quad v \geq 0,$$

which satisfies $K(0) = 0$. Then the equation $\mathcal{G}_\sigma(v_\sigma) = 0$ and the fact that $[v_\sigma]_{C^{0,1}} \leq$

$C_1([u]_{C^{0,1}})$ yield the bound

$$|\kappa(\partial_x v_\sigma)\partial_x^2 v_\sigma| \le \sup_{\sigma\in(0,1)} \big(\kappa(\partial_x v_\sigma + \sigma)|\partial_x^2 v_\sigma|\big)$$
$$\le C([u]_{C^{0,1}(\Omega)}, R)$$

and thus

$$\left\| \frac{\mathrm{d}}{\mathrm{d}x} K(\partial_x v_\sigma) \right\|_{L^\infty(\Omega)} \le C([u]_{C^{0,1}(\Omega)}, R) =: C_2,$$

so that $K(\partial_x v_\sigma)$ is Lipschitz continuous with respect to x uniformly in σ with constant bounded above by C_2. In the following we let $C_1 := C_1([u]_{C^{0,1}}) + 1$ and denote the inverse of $K_{|[0,C_1]} : [0, C_1] \to [0, K(C_1)]$ by K^{-1}. Then $\partial_x v_\sigma = K^{-1} \circ (K \circ \partial_x v_\sigma)$, and denoting for a uniformly continuous function a by ϑ_a its modulus of continuity, we infer that

$$\vartheta_{\partial_x v_\sigma(t,\cdot)}(\delta) \le \vartheta_{K^{-1}}(C_2\delta) \quad \text{for } \delta > 0.$$

Now compactness is obtained from the Arzelà–Ascoli theorem, so that the Aubin–Lions lemma applies as before and yields the bound

$$\left\| \frac{\mathrm{d}}{\mathrm{d}x} K(\partial_x u) \right\|_{L^\infty(\Omega)} \le C([u]_{C^{0,1}(\Omega)}, R) =: C_2$$

as well as the regularity $\partial_x u \in C(\bar{\Omega})$. Here $\frac{\mathrm{d}}{\mathrm{d}x} K(\partial_x u)$ denotes the weak derivative of $K(\partial_x u)$ with respect to x. Let us also mention that the main conclusions in Section 4.1.2 below apply to more general h. For simplicity, we only consider the case of the explicit function $h = h_\gamma$, which is in particular smooth in $(0, \infty)$.

4.1.2 The set $\Omega^+ \setminus \Omega^{++}$ is empty

Proof of Theorem 4.4 (R2). Since $u, \partial_x u \in C(\Omega)$, the sets

$$\Omega^+ = \{\omega \in \Omega :\ |u(\omega)| > 0\}$$

and

$$\Omega^{++} = \{\omega \in \Omega :\ \partial_x u(\omega) > 0\}$$

are open. From estimate (4.8) we infer that in any open rectangle $\Omega' \subset\subset \Omega^{++}$ we have $\partial_x^2 u \in L^\infty(\Omega')$. Arguing as for v_σ (see Section 4.1.1), it follows that $u_{|\Omega'}$

is a strong solution of a uniformly parabolic equation in Ω' (where the equality holds in $L^\infty(\Omega')$). This allows us to apply classical regularity theory for quasilinear parabolic equations to deduce that u is smooth in Ω^{++}: indeed, take an axis-aligned rectangle $\Omega' \subset\subset \Omega^{++}$. Then, recalling the uniqueness of (viscosity) solutions v to the Cauchy–Dirichlet problem $\mathcal{F}(v) = 0$ in Ω', $v = u$ on $\partial\Omega'$ and the fact that $u(\cdot, x)$ is Lipschitz continuous for any x and $\partial_x u(t, \cdot)$ is β-Hölder continuous for any t, as established in part (R1) of Theorem 4.4, the results [80, Theorems 8.2 & 8.3] imply local Schauder regularity of u in Ω' and, in particular, the regularity $u \in C^{1,2}_{t,x}(\Omega')$. Then, iterating the argument in the proof of [80, Lemma 14.11] (successively applied to the equation satisfied by $\partial_x^k u, k \in \mathbb{N}_0$) one deduces the regularity $u_{|\Omega'} \in C^\infty(\Omega')$.

Now define $\mathcal{N} := \Omega^+ \setminus \Omega^{++}$. Our goal is to show that $\mathcal{N}$ is empty. We proceed indirectly supposing that there exists a point $\omega = (t, x) \in \mathcal{N}$, where—by the symmetry of the equation—we may assume without loss of generality that $u(\omega) > 0$. From now on, we fix this particular time t, define $v(y) = u(t, y)$, $J' := (x_0, x]$, where $x_0 := \max\{y \in J : u(t, y) = 0\}$, and the non-empty set

$$A := J' \setminus (\Omega^{++})_t, \tag{4.9}$$

where $(\Omega^{++})_t := \{y \in J : (t, y) \in \Omega^{++}\}$ denotes the cross section of Ω^{++} at t. We call a point $y \in A$ a *left-isolated* point (of A) if there exists $\delta > 0$ such that $(y - \delta, y) \subset J' \setminus A$. Notice that in this case $(y - \delta, y) \subset (\Omega^{++})_t$, so that v is smooth in $(y - \delta, y)$.

Lemma 4.8. *Let A be defined by formula* (4.9) *and suppose that $y \in A$. Then, there cannot exist a sequence $x_n \to y$ with the property that for every n there are $(p_n, q_n) \in \mathcal{J}^{2,+}(u(t, \cdot))(x_n)$, where $p_n := \partial_x u(t, x_n)$, satisfying $q_n \leq 0$.*

Proof. We argue by contradiction and assume that such a sequence $x_n \to y$ exists. Let $z := u(t, y) > 0$ and choose $\sigma > 0$ small enough such that

$$-\sigma^\gamma K + z/2 > 0, \tag{4.10}$$

where $K := \|\partial_t u\|_{L^\infty(\Omega)}$. Next, fix some sufficiently large n such that $u(t, x_n) \geq z/2$, $\partial_x u(t, x_n) \leq \sigma$ and choose $(p_n, q_n) \in \mathcal{J}^{2,+}(u(t, \cdot))(x_n)$ such that $q_n \leq 0$. Then there exists a function $\phi \in C^2(J)$ satisfying $u(t, \cdot) - \phi \leq u(t, x_n) - \phi(x_n) = 0$ and $\phi'(x_n) = p_n, \phi''(x_n) = q_n$. After possibly replacing ϕ by $\tilde{\phi}(y) := \phi(y) + |x_n - y|^4$, we can assume that the maximum of $u(t, \cdot) - \phi$ at x_n is strict.

Now consider for some small $\delta > 0$ the function

$$w(s, y) := u(s, y) - \left(\phi(y) + \frac{1}{2\varepsilon}|s - t|^2\right) \quad \text{in} \quad Q_\delta := [t - \delta, t + \delta] \times [x_n - \delta, x_n + \delta],$$

which, by continuity, reaches its (non-negative) maximum at some point $(s_\varepsilon, y_\varepsilon)$. Notice that $s_\varepsilon \to t$ as $\varepsilon \to 0$ and, moreover, $y_\varepsilon \to x_n$. In particular, $(s_\varepsilon, y_\varepsilon) \in \mathrm{int}(Q_\delta)$ for small enough $\varepsilon > 0$, so that

$$(0,0,0) \in \mathcal{P}^+(w)(s_\varepsilon, y_\varepsilon)$$

or, equivalently,

$$\left(\frac{s_\varepsilon - t}{\varepsilon}, \phi'(y_\varepsilon), \phi''(y_\varepsilon)\right) \in \mathcal{P}^+u(s_\varepsilon, y_\varepsilon).$$

Since $|\frac{s_\varepsilon - t}{\varepsilon}| \leq K$, there exists $\bar{\tau} \in [-K, K]$ and a sequence $\varepsilon_i \to 0$ such that $\frac{s_{\varepsilon_i} - t}{\varepsilon_i} \to \bar{\tau}$. Letting $i \to \infty$, we find

$$(\bar{\tau}, p_n, q_n) \in \overline{\mathcal{P}}^+ u(t, x_n).$$

The subsolution property of u, the fact that $q_n \leq 0$ and the choice of n now imply the inequality

$$-\sigma^\gamma K + z/2 \leq 0,$$

which contradicts (4.10). □

Thanks to Lemma 4.8, we obtain the following result.

Lemma 4.9. *There cannot be any left-isolated point in the set*[1] A.

Proof. We argue again by contradiction, assuming that there exists a point $y \in A$ and $\delta > 0$ such that $(y - \delta, y) \subset J' \setminus A$. Then v' is strictly positive and smooth in $(y - \delta, y)$ and reaches its global minimum at the point y. Hence, there exists a strictly increasing sequence $(y - \delta, y) \ni \tilde{x}_n \nearrow y$, $n \geq 0$, such that $(v'(\tilde{x}_n))_n$ is strictly decreasing. Now for $n \geq 1$ let $y_n := \tilde{x}_n$ and $h_n := \tilde{x}_n - \tilde{x}_{n-1} > 0$. We then have

$$v'(y_n) - v'(y_n - h_n) = v'(\tilde{x}_n) - v'(\tilde{x}_{n-1}) < 0$$

and thus

$$\frac{v'(y_n) - v'(y_n - h_n)}{h_n} < 0$$

[1] See formula (4.9) for the definition of the set A.

for all $n \geq 1$. Since v' is absolutely continuous in $(y-\delta, y)$, we then have

$$\frac{1}{h_n}\int_{y_n-h_n}^{y_n} v''(z)\,\mathrm{d}z = \frac{v'(y_n)-v'(y_n-h_n)}{h_n} < 0.$$

Hence, there exists $x_n \in (y_n - h_n, y_n)$ such $q_n := v''(x_n) < 0$. In particular, letting $p_n := v'(x_n)$, we have $(p_n, q_n) \in \mathcal{J}^2 v(x_n)$ and by construction $x_n \to y$ as $n \to \infty$. This contradicts Lemma 4.8. □

Notice that the assumption $A = J'$ implies that $\partial_x u(t, y) = 0$ for all $y \in J'$ and thus $u(t, x) = 0$, which contradicts the definition of x. Therefore, there exists $y \in J' \setminus A$. Now let $y_1 := \min(A \cap [y, x])$, which exists since $x \in A$ and since, by the continuity of v', A is relatively closed in J'. Then $y_1 > y$, which implies that $y_1 \in A$ is left-isolated, contradicting Lemma 4.9.

We therefore conclude

$$\Omega^+ \setminus \Omega^{++} = \emptyset.$$

The proof of Theorem 4.4 (R2) is now complete. □

4.2 Relation to original equation on bounded interval

For fixed $\gamma \geq 2$, $m, R > 0$ and an initial datum u_0 admissible for problem (4.1) in the sense of Definition 4.2, we denote by u the unique global-in-time viscosity solution of the Cauchy–Dirichlet problem (4.1), restated below for the reader's convenience:

$$\begin{cases} \mathcal{F}(u) = 0, & \text{in } \Omega := (0,\infty)\times(0,m), \\ u(t,0) = -R, \quad u(t,m) = R, & \text{for } t > 0, \\ u(0,x) = u_0(x), & \text{for } x \in [0,m], \end{cases}$$

where $\mathcal{F}(u) := F(u, \partial_t u, \partial_x u, \partial_x^2 u)$ with

$$F(z,\alpha,p,q) := |p|^\gamma \alpha - |p|^{\gamma-2} q + z(1+|p|^\gamma)$$

for $z, \alpha, p, q \in \mathbb{R}$. Since the function $x \mapsto u(t, x)$ is non-decreasing for all $t \geq 0$, we could have restricted to $p \geq 0$ and dropped the absolute values in the definition F.

In the previous section we have seen that u has the improved regularity properties (R1) and (R2) of Theorem 4.4. In particular, $\partial_x u \in C([0,\infty)\times[0,m])$. In this section we investigate the conclusions which can be drawn from our theory established at the level of u for the problem in the original variables. Let us recall the definition (4.2) of the generalised inverse $M(t,\cdot)$ of $u(t,\cdot)$ as well as the definition (4.3) of the finite measure $\mu(t)$ on $[-R,R]$ associated with $M(t,\cdot)$:

$$\begin{cases} M(t,r) = \max\{x \in [0,m] : u(t,x) \le r\}, & r \in [-R,R], \\ \mu(t)([-R,r]) = M(t,r), & r \in [-R,R]. \end{cases} \tag{4.12}$$

As seen in Remark 4.1, the function $M(t,\cdot)$ is strictly increasing and right-continuous on $[-R,R]$ and satisfies $M(t,-R) \ge 0, M(t,R) = m$. In particular, the total mass of the measure $\mu(t)$ equals m for all $t \ge 0$. Thanks to Theorem 4.4, we now have a much more detailed understanding of $M(t,\cdot)$ and $\mu(t)$:

Proposition 4.10. *Let $\gamma \ge 2$, $m, R > 0$, assume that u_0 is admissible for problem* (4.1) *in the sense of Definition 4.2, let u denote the unique viscosity solution of problem* (4.1) *and define $M(t,\cdot)$ and $\mu(t)$ as in* (4.12)*. The following holds true:*

(i) *For each $t > 0$ there exist unique points $x_-(t), x_+(t) \in (0,m)$, $x_-(t) \le x_+(t)$, such that*

$$[u(t,x) = 0 \Leftrightarrow x_-(t) \le x \le x_+(t)].$$

In particular, $x_+(t) - x_-(t) = \mathcal{L}^1(\{u(t,\cdot) = 0\})$. In addition,

$$\partial_x u(t,x) > 0 \text{ for } x \in (0,m) \setminus [x_-(t), x_+(t)].$$

(ii) *For each $t > 0$ the strictly increasing and right-continuous function $M(t,\cdot)$ satisfies $M(t,-R) = 0,\ M(t,R) = m$ as well as*

$$M(t,0-) = x_-(t) \text{ and } M(t,0) = x_+(t).$$

Moreover, M is C^∞ in the set $\{(t,r) : t > 0, |r| \in (0,R)\}$.

(iii) *Letting $x_p(t) := \mathcal{L}^1(\{u(t,\cdot) = 0\})$, for each $t > 0$ there exists a unique function $f(t,\cdot) \in L^1_+(-R,R)$ such that the measure $\mu(t) \in \mathcal{M}_b^+([-R,R])$ has the decomposition*

$$\mu(t) = x_p(t)\delta_0 + f(t,\cdot)\mathcal{L}^1, \quad t \in (0,\infty). \tag{4.13}$$

Furthermore, $f(t,\cdot) \in C^\infty((-R,R) \setminus \{0\})$,

$$\begin{cases} f(t,u(t,x)) = 1/\partial_x u(t,x) & \text{for } x \in (0,m) \setminus [x_-(t), x_+(t)], \\ f(t,r) = 1/\partial_x u(t,M(t,r)) & \text{for } |r| \in (0,R), \end{cases} \tag{4.14}$$

and the function f satisfies in the classical sense the equation

$$\partial_t f - \partial_r(\partial_r f + rh_\gamma(f)) = 0, \quad t > 0, |r| \in (0, R), \tag{4.15}$$

where, as before, $h_\gamma(s) = s(1+s^\gamma)$. (Notice that eq. (4.15) *agrees with eq.* (2.6)*.)*

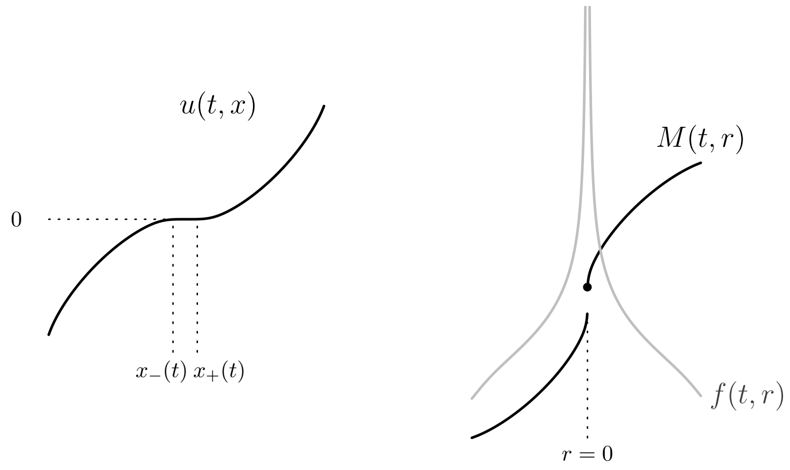


Figure 4.1: Illustration of the relation between the function $u(t, \cdot)$, its generalised inverse $M(t, \cdot)$ and the density $f(t, \cdot)$ of the absolutely continuous part of the measure $\mu(t)$ associated with $M(t, \cdot)$, as introduced in Proposition 4.10.

The proof of Proposition 4.10 is elementary. It is provided below for completeness.

Proof of Proposition 4.10. Re (i): Fix $t > 0$. The continuity and monotonicity of $u(t, \cdot)$ and the fact that $u(t, 0) = -R, u(t, m) = R$ imply that the preimage $(u(t, \cdot))^{-1}(\{0\}) \subset (0, m)$ is a closed interval. Hence there exist $x_-(t), x_+(t) \in (0, m)$ such that

$$[x_-(t), x_+(t)] = (u(t, \cdot))^{-1}(\{0\}).$$

The remaining assertions in item (i) follow from Theorem 4.4, (R2).

Re (ii): Let $J'(t) := (0, m) \setminus [x_-(t), x_+(t)]$. By (i), $u(t, \cdot)$ is strictly increasing and smooth in $J'(t)$, implying that

$$M(t, u(t, x)) = x \qquad \text{for } x \in J'(t). \tag{4.16}$$

Since $\partial_x u(t, \cdot) > 0$ in $J'(t)$, the inverse function theorem implies the smoothness of $M(t, \cdot)$ in the set $(-R, 0) \cup (0, R)$, which is exactly the image of $J'(t)$ under the function $u(t, \cdot)$. The non-degeneracy of $\partial_x u(t, M(t, r))$ for $|r| \in (0, R)$ and $t > 0$ combined with the smoothness of u in $\Omega^{++} = \{(t, x) \in \Omega : \partial_x u(t, x) > 0\}$ finally imply that M is C^∞ for $t > 0$ and $|r| \in (0, R)$. The fact that $M(t, -R) = 0$ is a consequence of identity (4.16).

Re (iii): The reasoning below uses standard results from measure theory, see e.g. Chapters 7 & 8 of the book [23]. For $r \in (-R, 0) \cup (0, R)$ let $f(t, r) = \partial_r M(t, r)$. Then $f \geq 0$ and

$$M(t, r_2) - M(t, r_1) = \int_{r_1}^{r_2} f(t, r)\, \mathrm{d}r, \qquad -R < r_1 \leq r_2 < 0.$$

Since $M(t, \cdot) \in C([-R, 0))$ with $M(t, -R) = 0$, letting $r_1 \searrow -R$, we infer

$$M(t, r_2) = \int_{-R}^{r_2} f(t, r)\, \mathrm{d}r \qquad r \in [-R, 0).$$

Thus, $M(t, \cdot)$ is absolutely continuous on $[-R, 0)$ with derivative $f(t, \cdot)$. Similarly, using also the fact that, as a consequence of the non-degeneracy (R2), we have $\lim_{r \nearrow R} M(t, r) = m$, one deduces the formula

$$M(t, r) = m - \int_{r}^{R} f(t, \rho)\, \mathrm{d}\rho \qquad r \in (0, R].$$

The two preceding representations show that the part of the measure $\mu(t)$ which is absolutely continuous with respect to the Lebesgue measure has the density $f(t, \cdot) \in L^1(-R, R)$, while the support of the singular part in the Lebesgue decomposition of the measure $\mu(t)$ is contained in $\{0\}$. Formula (4.13) now follows from the definition of $x_p(t)$. The smoothness of $f = \partial_r M$ in $(0, \infty) \times \Big((-R, 0) \cup (0, R)\Big)$ is an immediate consequence of the smoothness of M in this set. The relations (4.14) are obtained upon differentiating equation (4.16) resp. the identity $u(t, M(t, r)) = r$ at points $|r| \in (0, R)$.

From the equation $\mathcal{F}(u) = 0$, the relation $u(t, M(t, r)) = r$ and the identities (4.14), we deduce that

$$\partial_t M = \partial_r f + r h_\gamma(f) \text{ for } t > 0, |r| \in (0, R).$$

Exploiting once more the smoothness of f and M in the set $\{(t, r) : t > 0, |r| \in (0, R)\}$ and differentiating the previous equation, we infer (4.15). □

Let us also note that we have regularity up to the boundary in the following sense.

Lemma 4.11 (Regularity up to the boundary)**.** *Under the assumptions of Proposition 4.10, there exists $\sigma > 0$ only depending on the initial datum u_0 such that for all*

$t > 0$

$$\partial_x u(t,y) \geq \sigma \qquad \textit{for } y \in \{0,m\}. \tag{4.17}$$

Suppose now that, in addition,

(I1) *there exists* $\alpha > 0$ *such that* $u_0 \in C^{2,\alpha}(\bar{J})$.

(I2) u_0 *satisfies the compatibility condition* $\mathcal{F}(u_0)_{|x} = 0$ *for* $x \in \{0,m\}$.

Then for any $T < \infty$ *and* $\Omega := (0,T) \times (0,m)$ *there exists a neighbourhood* V *of* $\partial_p\Omega$ *in* $\bar{\Omega}$ *such that* u *has parabolic Schauder regularity in* V, *i.e.*

$$u_{|V} \in H_{2+\alpha}(\bar{V}) \subset C^{1,2}_{t,x}(V).$$

As a consequence, in this case $\partial_r f \in C([0,\infty) \times ([-R,R] \setminus \{0\}))$ *and*

$$\partial_r f + rh_\gamma(f) = 0 \qquad \textit{in } [0,\infty) \times \{-R,R\}. \tag{4.18}$$

Proof. Regarding the first part, we show that assertion (4.17) is satisfied on the left lateral boundary, i.e. that there exists $\sigma > 0$ such that $\partial_x u(t,0) \geq \sigma > 0$ for all t. The uniform bound $\inf_t \partial_x u(t,m) \geq \sigma' > 0$ can be deduced analogously (or by symmetry). For any $a > 0$ and $b \in (0,a]$ the time-independent function

$$u_1(x) = u_\infty^{(R,m+a)}(x+b) - u_\infty^{(R,m+a)}(b) - R, \qquad x \in [0,m]$$

is a viscosity subsolution of $\mathcal{F} = 0$ in $(0,\infty) \times (0,m)$ satisfying $u_1(0) = -R$, $u_1(m) \leq R$. It is easy to see that, by the admissibility of the initial datum u_0, $a > 0$ and $b \in (0,a]$ can be chosen such that we have the bound $u_1 \leq u_0$ *as well as* the non-degeneracy $\sigma := \partial_x u_1(0) > 0$. Hence $u_1 \leq u(t,\cdot)$ for all $t \geq 0$ and therefore $\partial_x u(t,0) \geq \sigma$.

The regularity of u, asserted under the extra assumptions (I1), (I2), is a consequence of [80, Theorems 8.2 & 8.3] and the fact that, by continuity, a lower bound of the form (4.17) (with σ replaced by some $\sigma' \in (0,\sigma)$) holds true in a neighbourhood V of $\partial_p\Omega \subset \bar{\Omega}$. The zero-flux boundary condition (4.18) is now deduced as follows: first notice that, by the non-degeneracy near the boundary, close to the boundary the equation $\mathcal{F}(u) = 0$ can be rewritten as

$$\partial_t u - (\partial_x u)^{-2}\partial_x^2 u + u((\partial_x u)^{-\gamma} + 1) = 0.$$

On the other hand, the constant-in-time lateral boundary conditions $u(\cdot,0) = -R$, $u(\cdot,m) = R$ combined with the continuity of $\partial_t u, \partial_x^2 u$ in V yield the identity $\partial_t u = 0$

on $S := (0,\infty) \times \{0, m\}$. Hence,

$$-(\partial_x u)^{-2}\partial_x^2 u + u((\partial_x u)^{-\gamma} + 1) = 0 \quad \text{on } S.$$

Reformulating the last identity in terms of f leads to equation (4.18). □

4.2.1 Spatial blow-up profile

In this subsection we aim to gain a more detailed understanding of the potential blow-up behaviour of the density $f(t,\cdot)$ introduced in Proposition 4.10 (iii). We will establish the following result.

Proposition 4.12 (Blow-up profile)**.** *Assume the hypotheses and use the notations of Proposition 4.10. Then, if $\gamma > 2$, for any $t > 0$ the following properties hold true:*

(i) *Time-uniform spatial bound: there exists a constant $C = C(R, \gamma, \|u\|_{C^{0,1}(\Omega)})$ such that for all $t > 0$ and $|r| \in (0, R)$*

$$f(t,r) \le C|r|^{-\frac{2}{\gamma}}. \tag{4.19}$$

Spatial behaviour near singularity: if $f(t,\cdot)$ is unbounded near the origin (or equivalently $\partial_x u(t, x_\pm(t)) = 0$), then

$$f(t,r) = \left(\frac{\gamma}{q(t,r)} \int_0^r sq(t,s)\,\mathrm{d}s\right)^{-\frac{1}{\gamma}}, \tag{4.20}$$

where for $|r| \in (0,R)$

$$\begin{aligned} q(t,r) &= \exp\left(\int_0^r a(t,s)\,\mathrm{d}s\right), \\ a(t,r) &= -\gamma(\tau(t,r) + r), \\ \tau(t,r) &= \partial_t u(t, M(t,r)). \end{aligned} \tag{4.21}$$

In particular, the expansion

$$f(t,r) = \left(\frac{2}{\gamma}\right)^{\frac{1}{\gamma}} |r|^{-\frac{2}{\gamma}} \left(1 + O(|r|)\right) \quad \text{as } r \to 0 \tag{4.22}$$

holds true uniformly in such t.

Furthermore,

$$\partial_x^2 u(t,\cdot) = \left(\frac{\gamma}{q(u)} \int_0^u sq(s)\,\mathrm{d}s\right)^{\frac{1}{\gamma}-1} \partial_x u \left(u - \frac{q'(u)}{(q(u))^2} \int_0^u sq(s)\,\mathrm{d}s\right), \tag{4.23}$$

where, for simplicity, we dropped the time argument on the right-hand side of eq. (4.23). *In particular, there exists a constant* $c = c(\|u\|_{C^{0,1}(\Omega)}) \in (0, R)$ *such that*

$$\partial_x^2 u(t,\cdot) \cdot \operatorname{sign}(u(t,\cdot)) > 0 \qquad \text{in } \{0 < |u(t,\cdot)| < c\}. \tag{4.24}$$

(ii) *For each* $t > 0$,

$$u(t,\cdot) \in W^{2,p}(0,m)$$

for $p < \frac{\gamma-2}{\gamma-4}$ *if* $\gamma > 4$ *and for* $p = \infty$ *if* $\gamma \in (2,4]$.

(iii) *The function* $t \mapsto x_p(t) := \mathcal{L}^1(\{u(t,\cdot) = 0\})$, *denoting the size of the condensate, is continuous.*

In the L^1*-critical case,* $\gamma = 2$, *solutions are globally regular and condensates cannot form:*

(iv) *If* $\gamma = 2$, *the density* $f(t,\cdot)$ *is bounded and smooth in* $(-R, R)$ *for all* $t \in (0,\infty)$. *In particular, in this case* $\min_{[0,m]} \partial_x u(t,\cdot) > 0$ *for all* $t > 0$, *and* f *satisfies equation* (2.6) *in the classical sense.*

Remark 4.13. In Section 5.3 we will show that the function $t \mapsto x_p(t)$ is even Lipschitz continuous.

Proof of Proposition 4.12. We fix an arbitrary time $t > 0$. For $x > x_+(t)$ we let $r = u(t,x)$, $\tau = \partial_t u(t, M(t,r))$, $p = \partial_x u(t, M(t,r))$ and $q = \partial_x^2 u(t, M(t,r))$. Notice that $r, p > 0$ and that $\tau = \tau(r)$ defines a bounded function on $(0, R)$. We have

$$p^\gamma \tau - p^{\gamma-2} q + r(1 + p^\gamma) = 0$$

and thus

$$\tau - p^{-2} q + r(p^{-\gamma} + 1) = 0. \tag{4.25}$$

In the following the fixed time argument t will be dropped. From the identity $f(u) = \frac{1}{\partial_x u}$, we deduce

$$\frac{f'(u)}{f(u)} = -\frac{\partial_x^2 u}{(\partial_x u)^2},$$

so that equation (4.25) can be rewritten as

$$\frac{f'(r)}{f(r)} + rf^{\gamma}(r) = -\tau(r) - r. \tag{4.26}$$

For later reference, we recall that in eq. (4.26) we have dropped the time argument and abbreviated $f' := \partial_r f$. We further note that $|\tau(t,r)| \leq \|u\|_{C^{0,1}(\bar{\Omega})} \leq C(u_0) < \infty$.

Letting $k(r) := f^{-\gamma}(r)$, which, by the regularity of u, is well-defined, bounded and strictly positive for $r \in (0,R)$, the last equation becomes

$$-\frac{1}{\gamma}\frac{k'(r)}{k(r)} + rk^{-1}(r) = -\tau(r) - r,$$

or, equivalently,

$$k'(r) + a(r)k(r) = \gamma r, \tag{4.27}$$

where we abbreviated $a(r) := -\gamma(\tau(r) + r)$. Introducing $q(r) := q(t,r)$, where

$$q(t,r) = \exp\left(\int_0^r a(t,s)\,\mathrm{d}s\right),$$

the left-hand side of eq. (4.27) equals $\frac{1}{q}(q \cdot k)'$. Hence, upon integration over the interval (ε, r), where $0 < \varepsilon < r$,

$$(qk)(r) = (qk)(\varepsilon) + \gamma \int_\varepsilon^r sq(s)\,\mathrm{d}s.$$

Thus,

$$k(r) = \frac{q(\varepsilon)k(\varepsilon)}{q(r)} + \frac{\gamma}{q(r)} \int_\varepsilon^r sq(s)\,\mathrm{d}s,$$

which in terms of $f = k^{-\frac{1}{\gamma}}$ becomes

$$f(r) = \left(\frac{q(\varepsilon)k(\varepsilon)}{q(r)} + \frac{\gamma}{q(r)} \int_\varepsilon^r sq(s)\,\mathrm{d}s\right)^{-\frac{1}{\gamma}}. \tag{4.28}$$

Since $\partial_x u(t,\cdot) \in C([0,m])$, the limit $f^{-\gamma}(t,0) := \lim_{\varepsilon\to 0} k(t,\varepsilon)$ exists in $[0,\infty)$. Thus, eq. (4.28) yields the identity

$$f(t,r) = \left(\frac{f^{-\gamma}(t,0)}{q(t,r)} + \frac{\gamma}{q(t,r)} \int_0^r sq(t,s)\,\mathrm{d}s\right)^{-\frac{1}{\gamma}}, \tag{4.29}$$

which implies inequality (4.19). As a side note, we observe that formula (4.29) provides an alternative means to deduce the non-degeneracy (4.17) and to quantify the lower bound σ.

Proof of assertions (i) **and** (ii)**: spatial behaviour near singularity.** Let us now suppose that the function $f(t,\cdot)$ is unbounded (from the right) near the origin, i.e. $\limsup_{r\searrow 0} f(t,r) = \infty$. Then $\lim_{\varepsilon\to 0} k(t,\varepsilon) = 0$, and thus, identity (4.29) yields

$$f(t,r) = \left(\frac{\gamma}{q(t,r)} \int_0^r sq(t,s)\,\mathrm{d}s\right)^{-\frac{1}{\gamma}}. \tag{4.30}$$

Recalling that q is given by (4.21), we find $q(t,r) = 1 - \gamma\tau(t,r)r + O(r^2)$ as $r \to 0$ with uniform control in t. Hence,

$$f(t,r) = \left(\frac{2}{\gamma}\right)^{\frac{1}{\gamma}} r^{-\frac{2}{\gamma}} (1 + O(r)) \qquad \text{as } r \to 0, \tag{4.31}$$

which again holds true uniformly in t (provided $f(t,\cdot)$ is unbounded at $r = 0$).

From now on we assume that $\gamma > 2$. By the smoothness of u in Ω^+, it is clear that the regularity of $u(t,\cdot)$ in $(0,m)$ is determined by the regularity of $u(t,\cdot)$ near $x = x_\pm(t)$. From identity (4.30) we observe

$$\partial_x u = \left(\frac{\gamma}{q(u)} \int_0^u sq(s)\,\mathrm{d}s\right)^{\frac{1}{\gamma}}, \tag{4.32}$$

from which we infer $\partial_x u = \left(\frac{\gamma}{2}\right)^{\frac{1}{\gamma}} u^{\frac{2}{\gamma}}(1 + O(u))$ as $u \searrow 0$ and, hence, as $x \searrow x_+(t)$

$$u(t,x) \approx (x - x_+(t))^{\frac{\gamma}{\gamma-2}}$$

as well as

$$\partial_x u(t,x) \approx (x - x_+(t))^{\frac{2}{\gamma-2}}.$$

Furthermore, differentiating formula (4.32) yields the identity

$$\partial_x^2 u = \left(\frac{\gamma}{q(u)} \int_0^u sq(s)\,\mathrm{d}s\right)^{\frac{1}{\gamma}-1} \partial_x u \left(u - \frac{q'(u)}{(q(u))^2} \int_0^u sq(s)\,\mathrm{d}s\right),$$

from which we observe that $\partial_x^2 u > 0$ for sufficiently small $0 < u \leq c(\|\tau\|_{L^\infty})$ and (for small enough $x > x_+(t)$)

$$\partial_x^2 u(t,x) \approx (x - x_+(t))^{-\frac{\gamma-4}{\gamma-2}},$$

where the hidden constants are independent of t. (Here $A \approx B$ for non-negative quantities A, B means that there exists a constant $1 < C < \infty$ such that $C^{-1}A \leq B \leq CA$ holds.) In particular,

$$u(t, \cdot) \in W^{2,p}(J),$$

for $p < \frac{\gamma-2}{\gamma-4}$ if $\gamma > 4$ and for $p = \infty$ if $\gamma \in (2, 4]$.

Remark 4.14. The derivation of the estimates and asymptotics established above for the region where $0 < x < x_-(t)$ is analogous. We leave it as a simple exercise for the reader.

This completes the proof of assertions (i) and (ii).

Proof of assertion (iii)**: continuity of** $x_p(t)$**.** It is now easy to see that the mass concentrated at the origin depends continuously on time. Noticing that $x_p(t) = M(t, 0) - M(t, 0-)$, we can estimate using the bound (4.19)

$$\begin{aligned}|x_p(t) - x_p(s)| &\leq |M(t, r) - M(t, 0)| + |M(s, r) - M(s, 0)| + |M(t, r) - M(s, r)| \\ &\quad + |M(t, 0-) - M(t, -r)| + |M(s, 0-) - M(s, -r)| + |M(t, -r) - M(s, -r)| \\ &\leq Cr^{1-\frac{2}{\gamma}} + |M(t, r) - M(s, r)| + |M(t, -r) - M(s, -r)|, \text{ where } 0 < r \ll R.\end{aligned}$$

Thus $\limsup_{s \to t} |x_p(t) - x_p(s)| \leq Cr^{1-\frac{2}{\gamma}}$. Since $r > 0$ can be chosen arbitrarily small, the continuity of $t \mapsto x_p(t)$ follows.

Proof of assertion (iv)**: global regularity in the** L^1**-critical case** $\gamma = 2$**.** We now suppose that $\gamma = 2$. Assuming, by contradiction, that there exists a time $T \in (0, \infty)$ such that $f(T, \cdot)$ is unbounded near the origin, identity (4.31) implies that $f(T, r) \geq r^{-1}/2$ for small enough $r > 0$. This contradicts the fact that $\|f(T, \cdot)\|_{L^1(-R,R)} \leq m$. □

4.2.2 Entropy dissipation identity

In this subsection we aim to study the time evolution of[2] $\widetilde{\mathcal{H}}^{(h_\gamma,R)}(\mu(t))$. Observe that, by (4.13), the entropy does not explicitly depend on the singular component of the measure $\mu(t)$ and thus coincides with $\mathcal{H}^{(h_\gamma,R)}(f(t, \cdot))$:

$$\widetilde{\mathcal{H}}^{(h_\gamma,R)}(\mu(t)) = \int_{-R}^{R} \left(\frac{r^2}{2} f(t, r) + \Phi(f(t, r)) \right) \mathrm{d}r.$$

[2]Here $\widetilde{\mathcal{H}}^{(h,R)}$ denotes the natural extension of $\mathcal{H}^{(h,R)}$ to $\mathcal{M}_b^+([-R, R])$ as described in Remark 3.22, where $\mathcal{H}^{(h,R)}$ is defined by formula (3.17).

Proposition 4.15 (Entropy dissipation identity)**.** *Suppose the hypotheses and use the notations of Proposition 4.10. Further assume that u_0 satisfies hypothesis* (I1) *and* (I2) *of Lemma 4.11. Then the function $t \mapsto \widetilde{\mathcal{H}}^{(h_\gamma,R)}(\mu(t)) = \mathcal{H}^{(h_\gamma,R)}(f(t,\cdot))$ is absolutely continuous, and the identity*

$$\mathcal{H}^{(h_\gamma,R)}(f(t,\cdot)) - \mathcal{H}^{(h_\gamma,R)}(f(s,\cdot)) = -\int_s^t \int_{-R}^R \frac{1}{h_\gamma(f)} |\partial_r f + r h_\gamma(f)|^2 \,\mathrm{d}r\mathrm{d}\sigma \quad (4.33)$$

holds true for all $0 \leq s \leq t < \infty$.

Proof. We will derive eq. (4.33) via approximation by a regularised problem. For convenience, our regularisations are based on the setting in Section 3.6, where the superlinearity h_γ in the drift is attenuated in such a way that is has critical growth at infinity (i.e. $h(s) \approx s^3$ as $s \to \infty$). The smoothness of the approximate solutions then follows from the theory established in Sections 4.1, 4.2.1. In order to deduce equality, we will introduce two entropy-type functionals approximating from above resp. from below the original problem. The approximation from above leads to an entropy dissipation *inequality* which is crucial for the long-time asymptotic behaviour. Here, the passage to the limit relies on the lower semicontinuity properties of the original entropy.

Let $\beta := \gamma - 2$ and take a smooth, non-decreasing function $\eta \in C^\infty(0,\infty)$ satisfying the identities

$$\eta(\sigma) = \begin{cases} \sigma^\beta & \text{if } \sigma \leq 1, \\ \left(\frac{3}{2}\right)^\beta & \text{if } \sigma \geq 2 \end{cases}$$

as well as the bound

$$\eta(\sigma) \leq \sigma^\beta \qquad \text{for all } \sigma \geq 0.$$

Then define $\eta_\varepsilon(s) = \varepsilon^{-\beta}\eta(\varepsilon s)$ and set $\varphi_\varepsilon(s) = s(1 + s^2\eta_\varepsilon(s))$. Notice that, by definition, $\varphi_\varepsilon(s) = h_\gamma(s)$ for $s \leq \frac{1}{\varepsilon}$ and $\varphi_\varepsilon \leq h_\gamma$ on $[0,\infty)$. The function $h = \varphi_\varepsilon, 0 < \varepsilon \ll 1$, satisfies the hypotheses of Theorem 3.20. Since, by assumption, our initial datum u_0 satisfies $\min u_0' > 0$, it trivially fulfils hypothesis (3.26) for any ε. Hence, Theorem 3.20 provides us with a family $\{v_\varepsilon\}$ of approximate solutions emanating from u_0, where v_ε satisfies the equation (3.25) with $h := \varphi_\varepsilon$. By the construction of the barriers $u_{\theta,\pm,h}^{(R,m)}$ (see page 42), it is obvious that for small $\varepsilon > 0$ the problem based on $h := \varphi_\varepsilon$ has barriers $u^\pm$ which are uniformly-in-ε Lipschitz continuous in space-time. Thus, Theorem 3.20 yields the bound

$$\sup_{\varepsilon} \|v_\varepsilon\|_{C^{0,1}(\Omega)} < \infty,$$

which implies that, in the limit $\varepsilon \to 0$, $\{v_\varepsilon\}$ converges locally uniformly to our viscosity solution u. Here we used the stability and uniqueness of the BFP problem at the level of u as well as the observation that $G_\varepsilon(z,\alpha,p,q) \to (1+|p|^\gamma)^{-1}F(z,\alpha,p,q)$ locally uniformly in $(z,\alpha,p,q) \in \mathbb{R}^4$, where F is defined by eq. (3.28) and

$$G_\varepsilon(z,\alpha,p,q) = \left(|p|^3\varphi_\varepsilon(1/|p|)\right)^{-1}\left(|p|^2\alpha - q\right) + z,$$

cf. (3.24). Since $\varphi_\varepsilon(s) \approx_\varepsilon s^3$ for $s \geq \frac{2}{\varepsilon}$, Sections 4.1, 4.2.1 and in particular the argument in Proposition 4.12 (iv) show that v_ε is non-degenerate and thus regular globally in time. Furthermore, by parabolic regularity, the ε-uniform bound (4.19) implies convergence of f_ε to f locally uniformly in $\{r \neq 0\}$, where $f_\varepsilon(t,\cdot)$ denotes the density of the inverse of $v_\varepsilon(t,\cdot)$. Combined with the analogue of the equation (4.26) (with h_γ replaced by φ_ε), this allows us to pass to a limit in the dissipated quantity, namely

$$\lim_{\varepsilon\to 0}\int_s^t \int_{(-R,R)} \frac{1}{\varphi_\varepsilon(f_\varepsilon)}\left|\partial_r f_\varepsilon + r\varphi_\varepsilon(f_\varepsilon)\right|^2 \,\mathrm{d}r\mathrm{d}\sigma = \int_s^t D_R(\tau)\,\mathrm{d}\tau,$$

where D_R is given by

$$D_R(\tau) = \int_{-R}^{R} \frac{1}{h_\gamma(f)}|\partial_r f + rh_\gamma(f)|^2\,\mathrm{d}r.$$

We will now define two different entropies

$$\mathcal{H}_\varepsilon(f) = \int_{(-R,R)} \left(\frac{|r|^2}{2}f(r) + \Phi_\varepsilon(f(r))\right)\,\mathrm{d}r \tag{4.34}$$

and

$$\mathcal{H}^{(\varphi_\varepsilon,R)}(f) = \int_{(-R,R)} \left(\frac{|r|^2}{2}f(r) + \Phi^{(\varphi_\varepsilon)}(f(r))\right)\,\mathrm{d}r$$

in such a way that both for $\mathcal{H} = \mathcal{H}_\varepsilon$ and for $\mathcal{H} = \mathcal{H}^{(\varphi_\varepsilon,R)}$ the density $f_\varepsilon(t,\cdot)$ of the inverse of $v_\varepsilon(t,\cdot)$ satisfies the entropy dissipation identity

$$\mathcal{H}(f_\varepsilon(t,\cdot)) - \mathcal{H}(f_\varepsilon(s,\cdot)) = \\ = -\int_s^t \int_{(-R,R)} \frac{1}{\varphi_\varepsilon(f_\varepsilon)} \left|\partial_r f_\varepsilon + r\varphi_\varepsilon(f_\varepsilon)\right|^2 \,\mathrm{d}r\mathrm{d}\sigma \tag{4.35}$$

for all $0 \le s < t < \infty$. In order to ensure (4.35), the functions $\Phi = \Phi_\varepsilon$, $\Phi = \Phi^{(\varphi_\varepsilon)}$ will be constructed in such a way that they satisfy $\Phi'' = \frac{1}{\varphi_\varepsilon}$ on $(0,\infty)$ and $\Phi(0) = 0$, i.e. they will only differ by a linear function.

The first entropy, $\mathcal{H}_\varepsilon$, will approximate the original problem from above:

$$\liminf_{\varepsilon\to 0} \mathcal{H}_\varepsilon(f_\varepsilon(t,\cdot)) \ge \mathcal{H}^{(h_\gamma,R)}(f(t,\cdot)) \qquad \text{for all } t \ge 0. \tag{4.36}$$

The second entropy, $\mathcal{H}^{(\varphi_\varepsilon,R)}$, is defined as in Section 3.6 (see eq. (3.17)) and will approximate the original problem from below:

$$\limsup_{\varepsilon\to 0} \mathcal{H}^{(\varphi_\varepsilon,R)}(f_\varepsilon(t,\cdot)) \le \mathcal{H}^{(h_\gamma,R)}(f(t,\cdot)) \qquad \text{for all } t \ge 0. \tag{4.37}$$

Since at initial time $t = 0$ we have equality in (4.36) and in (4.37) (with $\liminf$ resp. $\limsup$ replaced by $\lim$), we then infer that for all $t \ge 0$

$$\mathcal{H}^{(h_\gamma,R)}(f(t,\cdot)) = \mathcal{H}^{(h_\gamma,R)}(f(0,\cdot)) - \int_0^t \int_{-R}^R \frac{1}{h_\gamma(f)} |\partial_r f + r h_\gamma(f)|^2 \,\mathrm{d}r\,\mathrm{d}\sigma,$$

which implies the assertion (4.33).

<u>Approximation from above:</u> by construction $\varphi_\varepsilon \le h_\gamma$ and thus

$$-\int_s^\infty \frac{1}{\varphi_\varepsilon(\sigma)}\,\mathrm{d}\sigma \le -\int_s^\infty \frac{1}{h_\gamma(\sigma)}\,\mathrm{d}\sigma \qquad \text{for all } s \in (0,\infty).$$

We can therefore choose $A_\varepsilon \ge 0$ such that

$$A_\varepsilon - \int_{\frac{1}{\varepsilon}}^\infty \frac{1}{\varphi_\varepsilon(\sigma)}\,\mathrm{d}\sigma = -\int_{\frac{1}{\varepsilon}}^\infty \frac{1}{h_\gamma(\sigma)}\,\mathrm{d}\sigma.$$

We now define Φ_ε via $\Phi_\varepsilon(s) = \int_0^s \Phi_\varepsilon'(\sigma)\,\mathrm{d}\sigma$, where we let

$$\Phi_\varepsilon'(\sigma) = A_\varepsilon - \int_\sigma^\infty \frac{1}{\varphi_\varepsilon}.$$

This ensures that

$$\Phi_\varepsilon(s) = \Phi^{(h_\gamma)}(s) \qquad \text{for } s \in [0, \varepsilon^{-1}]$$

and that $\Phi_\varepsilon \geq \Phi^{(h_\gamma)}$ in $[0, \infty)$. Since $\Phi_\varepsilon'' = \frac{1}{\varphi_\varepsilon}$ in $(0, \infty)$, the functional $\mathcal{H}$ defined via (4.34) satisfies formula (4.35). The inequality (4.36) follows from the bound $\Phi_\varepsilon \geq \Phi^{(h_\gamma)}$ together with the lower semicontinuity of the extended functional $\widetilde{\mathcal{H}}^{(h_\gamma,R)}$ with respect to weak-$*$ convergence in measure [39]. We note that this inequality is sufficient to infer the long-time asymptotic behaviour in Section 4.2.3.

Approximation from below: the function $\Phi^{(\varphi_\varepsilon)}$ has been defined in Section 3.6. Observe that, since $\varphi_\varepsilon \leq h_\gamma$ on $(0, \infty)$, we have

$$\Phi^{(\varphi_\varepsilon,R)} \leq \Phi^{(h_\gamma,R)} \leq 0 \qquad \text{on } [0, \infty). \tag{4.38}$$

To see the inequality (4.37), we fix $\varepsilon_1 > 0$ small and estimate, using the non-positivity of $\Phi^{(\varphi_\varepsilon,R)}$ (and $\Phi^{(\varphi_\varepsilon,R)}(0) = 0$), mass conservation, and inequality (4.38),

$$\begin{aligned}\mathcal{H}^{(\varphi_\varepsilon,R)}(f_\varepsilon(t,\cdot)) &\leq \mathcal{H}^{(\varphi_\varepsilon,R)}(\chi_{\{|r|\geq\varepsilon_1\}} f_\varepsilon(t,\cdot)) + \frac{\varepsilon_1^2}{2} m \\ &\leq \mathcal{H}^{(h_\gamma,R)}(\chi_{\{|r|\geq\varepsilon_1\}} f_\varepsilon(t,\cdot)) + \frac{\varepsilon_1^2}{2} m.\end{aligned}$$

Hence, by the locally in $\{r \neq 0\}$ uniform convergence of f_ε to f, we infer

$$\limsup_{\varepsilon\to 0} \mathcal{H}^{(\varphi_\varepsilon,R)}(f_\varepsilon(t,\cdot)) \leq \mathcal{H}^{(h_\gamma,R)}(\chi_{\{|r|\geq\varepsilon_1\}} f(t,\cdot)) + \frac{\varepsilon_1^2}{2} m \overset{\varepsilon_1\to 0}{\to} \mathcal{H}^{(h_\gamma,R)}(f(t,\cdot)),$$

where the ε_1-limit follows from dominated convergence. □

4.2.3 Finite-time condensation and asymptotic behaviour

Thanks to Proposition 4.15, we can now show the convergence in entropy to the minimiser $\mu_\infty^{(R,m)}$ of $\widetilde{\mathcal{H}}^{(h_\gamma,R)}$ among measures of mass m. We refer to Notations 4.3 for the definition of $\theta^{(R,m)}, u_\infty^{(R,m)}$ and remind the reader of our notation $m_c(R) = \int_{-R}^{R} f_c$, where $f_c = f_{\infty,0}$.

Theorem 4.16 (Relaxation to the entropy minimiser of the given mass)**.** *Let $\gamma \geq 2$, $m, R > 0$ and assume the hypotheses and use the notations of Proposition 4.15. Then, in the long-time limit $t \to \infty$, convergence to the minimiser of the entropy holds true in the following sense:*

(C1) *Convergence in entropy:*

$$\lim_{t\to\infty} \widetilde{\mathcal{H}}^{(h_\gamma,R)}(\mu(t)) = \widetilde{\mathcal{H}}^{(h_\gamma,R)}\left(\mu_\infty^{(R,m)}\right), \tag{4.39}$$

where $\mu_\infty^{(R,m)}$ *is given by eq.* (4.4)*, i.e.*

$$\mu_\infty^{(R,m)} = \begin{cases} f_{\infty,\theta}\cdot\mathcal{L}^1 & \text{if } m \le m_c(R), \text{ where } \theta = \theta^{(R,m)}, \\ f_c\cdot\mathcal{L}^1 + (m - m_c(R))\delta_0 & \text{if } m > m_c(R). \end{cases}$$

(C2) *Uniform convergence at the level of* u*:*

$$\lim_{t\to\infty} \|u(t,\cdot) - u_\infty^{(R,m)}\|_{C([0,m])} = 0. \tag{4.40}$$

(C3) *Convergence of the Dirac mass at the origin:*

$$\lim_{t\to\infty} x_p(t) = (m - m_c(R))_+, \qquad \text{where } (m-m_c(R))_+ = \max\{0, m - m_c(R)\}.$$

Proof of Theorem 4.16. We first show assertion (C1). Define

$$D_R(t) = \int_{-R}^{R} \frac{1}{h_\gamma(f(t,r))} |\partial_r f(t,r) + r h_\gamma(f(t,r))|^2 \,\mathrm{d}r$$

and note that identity (4.33) and Theorem 2.1, together with Remark 2.2, imply $D_R \in L^1(0,\infty)$. Hence, there exists a sequence $t_k \to \infty$ such that $D_R(t_k) \to 0$. By estimate (4.5), there exists $u_\infty \in C^{1,\frac{1}{\gamma-1}}([0,m])$ such that, after transition to a subsequence,

$$u(t_k,\cdot) \to u_\infty \text{ in } C^{1,\beta}([0,m])$$

for $\beta \in (0, \frac{1}{\gamma-1})$, and

$$f(t_k,\cdot) \to f_\infty \text{ locally uniformly in } A_{0,R} \cup \{-R,R\},$$

where $A_{0,R} = (-R,R)\setminus\{0\}$ and where f_∞ is defined via $f_\infty(u_\infty) = \frac{1}{u'_\infty}$.

We now adapt an argument appearing in Step 2 of the proof of [20, Theorem 4.3]. Letting $f_k := f(t_k,\cdot)$ and $g_k := \frac{1}{f_k^{-\gamma}+1}$, we deduce

$$g_k \to g_\infty := \frac{1}{f_\infty^{-\gamma}+1} \tag{4.41}$$

locally uniformly in $A_{0,R} \cup \{-R,R\}$. We then estimate, using the Cauchy–Schwarz inequality,

$$\begin{aligned}\left(\int_{-R}^{R}|\gamma r g_k+\partial_r g_k|\mathrm{d}r\right)^2&=\gamma^2\left(\int_{-R}^{R}\left|g_k\left[r+\frac{\partial_r f_k}{f_k(1+f_k^{\gamma})}\right]\right|\mathrm{d}r\right)^2\\&\le\gamma^2\|g_k\|_{L^1}\int_{-R}^{R}g_k\left|r+\frac{\partial_r f_k}{f_k(1+f_k^{\gamma})}\right|^2\mathrm{d}r\\&\le\gamma^2\|g_k\|_{L^1}D_R(t_k)\\&\le CD_R(t_k)\to 0\quad\text{as } k\to\infty.\end{aligned}$$

Thus, we deduce that

$$\gamma r g_k+\partial_r g_k\to 0\quad\text{in } L^1(-R,R)\quad\text{as } k\to\infty,$$

which, thanks to (4.41), implies $\gamma r g_\infty+\partial_r g_\infty=0$ in $\mathcal{D}'(A_{0,R})$ and hence $\gamma r g_\infty+\partial_r g_\infty=0$ almost everywhere in $A_{0,R}$. This implies that for certain $\theta_\pm\ge 0$:

$$f_\infty=f_{\infty,\theta_-}\chi_{\{-R<r<0\}}+f_{\infty,\theta_+}\chi_{\{0<r<R\}}.$$

Since the assumption $\theta_+\ne\theta_-$ contradicts the regularity $u'_\infty\in C((0,m))$, we infer $\theta_+=\theta_-$. For the same reason, we conclude $\theta_+=\theta_-=\theta^{(R,m)}$ and thus

$$f_\infty=f_{\infty,\theta^{(R,m)}},\quad u_\infty=u_\infty^{(R,m)}.$$

By the dominated convergence theorem, we now have

$$\mathcal{H}^{(h_\gamma,R)}(f(t_k,\cdot))\to\mathcal{H}^{(h_\gamma,R)}(f_\infty)=\widetilde{\mathcal{H}}^{(h_\gamma,R)}(\mu_\infty^{(R,m)}),$$

which, combined with the monotonicity of the function $t\mapsto\mathcal{H}^{(h_\gamma,R)}(f(t,\cdot))$, implies assertion (C1).

We next prove (C2). For an arbitrary time sequence $s_n\to\infty$ we want to show that $\lim_{n\to\infty}\|u(s_n,\cdot)-u_\infty^{(R,m)}\|_{C(\bar{J})}=0$. By the global Lipschitz continuity of u (in time), we can assume without loss of generality that $|s_n-s_{n+1}|\ge\frac{2}{n}$. We now let $I_n=\{|t-s_n|\le\frac{1}{n}\}$. Then, since $D_R\in L^1(0,\infty)$, there exist n_k and $t_k\in I_{n_k}$ such that $D_R(t_k)\to 0$. Now the proof of (C1) shows that after passing to a subsequence,

$$u(t_k,\cdot)\to u_\infty^{(R,m)}\text{ uniformly in }\bar{J}.$$

Finally notice that for $K := \|\partial_t u\|_{L^\infty(\Omega)}$ we have

$$|u(s_{n_k}, x) - u_\infty^{(R,m)}(x)| \leq K \underbrace{|s_{n_k} - t_k|}_{\leq \frac{1}{n_k}} + |u(t_k, x) - u_\infty^{(R,m)}(x)|.$$

Thus the (arbitrary) sequence (s_n) has a subsequence (s_{n_k}) such that $u(s_{n_k}, \cdot) \to u_\infty^{(R,m)}$ uniformly in $\bar{J}$. This implies (4.40).

Assertion (C3) is a consequence of (C2) and the fact that the bound (4.19) holds true uniformly in time. □

Remark 4.17. In view of estimate (4.5), the convergence (C2) of $u(t, \cdot)$ to the entropy minimiser holds true in the stronger topology $C^{1,\beta}([0, m])$ for $\beta \in (0, (\gamma - 1)^{-1})$.

Corollary 4.18 ((No) Condensate after finite time)**.** *Under the hypotheses of Proposition 4.15:*

- *If $m > m_c(R)$, there exists $T < \infty$ such that $x_p(t) > 0$ for all $t > T$.*
- *If $m < m_c(R)$, there exists $T < \infty$ such that $\min_{[0,m]} \partial_x u(t, \cdot) > 0$ for all $t > T$. In particular, the condensate component is compactly supported in $(0, \infty)$, i.e. $\operatorname{supp} x_p \subset\subset (0, \infty)$, and the density $f(t, \cdot)$ is smooth for all $t > T$.*

Proof of Corollary 4.18. The assertion concerning the case $m > m_c(R)$ is an immediate consequence of Theorem 4.16 (C3). Let us now assume that $m < m_c(R)$. By identity (4.22) there exists a constant $c(m, R, u_0) > 0$ such that

$$\|u(t, \cdot) - u_\infty^{(R,m)}\|_{C([0,m])} \geq c(m, R, u_0)$$

whenever $\min_{[0,m]} \partial_x u(t, \cdot) = 0$. The assertion now follows from Theorem 4.16 (C2). □

Corollary 4.18 raises the question of whether and under which conditions finite-time blow-up and condensation may occur in the mass-subcritical case $m < m_c(R)$. As we will see in Section 5.1 finite-time blow-up and condensation does occur for any size of the mass provided the regular/smooth initial density in the original variables is sufficiently concentrated near the origin. On the other hand, there is a large class of mass-subcritical initial data for which the corresponding evolution is globally regular (see Proposition 5.4).

4.3 Higher-order comparison tools

In this section, we aim to upgrade the comparison results at the level of u in Section 3.3. In fact, we will see that the intersection comparison result for u easily yields comparison between densities, i.e. comparison at the level of f. The result may be of general interest, but will also be used explicitly in the next section.

Definition 4.19 (Translations in x)**.** Assume that $n > 0$ and let v be a function defined on $[0, n]$. For $y \in \mathbb{R}$ let

$$^{(y)}v : [y, n + y] \to [-R, R], \qquad ^{(y)}v(x) = v(x - y).$$

If $v = v(t, x)$ is time-dependent, $^{(y)}v$ is defined by $^{(y)}v(t, x) = v(t, x - y)$ for all t. Finally, given $\lambda > 0$ let

$$\mathcal{T}_\lambda[v] = \{^{(y)}v : y \in (0, \lambda)\}.$$

Proposition 4.20 (Comparison for densities)**.** *Let $\gamma \geq 2$ and $R \in (0, \infty)$. Let $g_0, \tilde{g}_0 \in C^1([-R, R])$, $g_0 \not\equiv \tilde{g}_0$, be positive functions satisfying*

$$g_0 \leq \tilde{g}_0 \qquad \text{in } [-R, R]. \tag{4.42}$$

Abbreviate $n = \|g_0\|_{L^1}$, $\tilde{n} = \|\tilde{g}_0\|_{L^1}$ and let $v_0 : [0, n] \to [-R, R]$ (resp. $\tilde{v}_0 : [0, \tilde{n}] \to [-R, R]$) be the inverse cdf of g_0 (resp. $\tilde{g}_0$). Denote by v (resp. $\tilde{v}$) the global viscosity solution of problem (4.1) *(with mass n resp. $\tilde{n}$ and initial datum v_0 resp. $\tilde{v}_0$), and let g (resp. $\tilde{g}$) denote the density of the absolutely continuous part of the measure associated with the generalised inverse of v (resp. $\tilde{v}$), as obtained in Proposition 4.10. Then*

$$g \leq \tilde{g} \text{ in } (0, \infty) \times (-R, R).$$

Proof. The assumption $g_0 \leq \tilde{g}_0, g_0 \not\equiv \tilde{g}_0$ implies that $n < \tilde{n}$. Moreover, for any $w \in \mathcal{T}_{\tilde{n}-n}[v]$ the number of sign changes (see Definition 3.10) satisfies

$$Z[\tilde{v}(0, \cdot) - w(0, \cdot)] = 1.$$

(Otherwise the fundamental theorem of calculus would lead to a contradiction with ineq. (4.42).) Since $\tilde{v}$ is non-degenerate near the lateral boundary, for any $y \in (0, \tilde{n} - n)$ and $w := {}^{(y)}v$, we have

$$w(t, y) - \tilde{v}(t, y) < 0, \qquad w(t, n + y) - \tilde{v}(t, n + y) > 0 \tag{4.43}$$

for all $t \geq 0$. Here, we used the fact that $w(t, y) = -R, w(t, n + y) = R$. Hence, by

Corollary 3.11, for all $y \in (0, \tilde{n} - n)$, $w := {}^{(y)}v$,

$$Z[[\tilde{v}(t,\cdot) - w(t,\cdot)]_{|(y,n+y)}] = 1 \text{ for all } t \geq 0. \tag{4.44}$$

Let now $(t,r) \in (0,\infty) \times ((-R,R) \setminus \{0\})$ be arbitrary. The intermediate value theorem implies the existence of $x' \in (0,\tilde{n})$ and $x'' \in (0,n)$ such that $\tilde{v}(t,x') = r$, $v(t,x'') = r$. Letting $y' = x' - x''$ and $w := {}^{(y')}v$, we infer that

$$w(t,x') = \tilde{v}(t,x') = r,$$

which, owing to properties (4.43) and (4.44), implies that

$$\partial_x w(t,x') \geq \partial_x \tilde{v}(t,x'). \tag{4.45}$$

Now, the conclusion follows by observing that, in view of eq. (4.14),

$$g(t,r) = \frac{1}{\partial_x v(t,x'-y')} = \frac{1}{\partial_x w(t,x')}$$

and

$$\tilde{g}(t,r) = \frac{1}{\partial_x \tilde{v}(t,x')},$$

where we used the convention $\frac{1}{0} = \infty$.

As a side note let us remark that if $\partial_x w(t,x') > 0$, it is possible using classical arguments for uniformly parabolic equations (see e.g. [80]) and the fact that $t > 0$ to deduce that the inequality in (4.45) is strict. □

4.4 The problem on the whole line $\mathbb{R}$

In this section we are concerned with the bosonic Fokker–Planck equations (1.2) posed on the real line, i.e. with

$$\begin{cases} \partial_t f = \partial_r^2 f + \partial_r(r\, h_\gamma(f)), & t > 0,\ r \in \mathbb{R}, \\ f(0,r) = f_0(r) > 0, & r \in \mathbb{R}, \end{cases} \tag{4.46}$$

where we suppose again that $\gamma \geq 2$ and recall that $h_\gamma(s) = s(1+s^\gamma)$. We always assume that the integrable initial density f_0 decays sufficiently fast at infinity (to be specified below) and denote by m its total mass $\|f_0\|_{L^1(\mathbb{R})}$.

As a motivation, let us first assume that $f = f(t,r)$ is a sufficiently regular, strictly positive classical solution of eq. (4.46) with finite conserved mass $m := \int f(t,\cdot)$. Defining for $t > 0$ the cumulative distribution function

$$M(t,r) = \int_{-\infty}^{r} f(t,r')\,\mathrm{d}r'$$

and letting $u(t,\cdot) : (0,m) \to \mathbb{R}$ denote the inverse of $M(t,\cdot) : \mathbb{R} \to (0,m)$, we find that u satisfies the problem

$$\begin{cases} \mathcal{F}(u) = 0, & \text{in } \Omega := (0,\infty) \times (0,m), \\ \lim_{x\searrow 0} u(t,x) = -\infty, \quad \lim_{x\nearrow m} u(t,x) = \infty, & \text{for } t > 0, \\ u(0,x) = u_0(x), & \text{for } x \in (0,m), \end{cases} \tag{4.47}$$

where, as before, $\mathcal{F}(u) := F(u, \partial_t u, \partial_x u, \partial_x^2 u)$ with

$$F(z,\alpha,p,q) := p^\gamma \alpha - p^{\gamma-2} q + z(1+p^\gamma)$$

for $p \geq 0$ and $z, \alpha, q \in \mathbb{R}$. We are primarily interested in solutions for which the limits in the second line of problem (4.47) hold true locally uniformly in time (in the sense of eq. (4.52)).

With respect to the Cauchy–Dirichlet problem (4.1) and the general framework established in Chapter 3, problem (4.47) has the added difficulty of the function u being unbounded near the lateral boundary. This is, however, mainly a technical issue, and existence, uniqueness and regularity for problem (4.47) in the spirit of Corollary 3.24 will be established below for a large class of initial data. The adaptation of the theory in Section 4.2 to eq. (4.47) will then be a fairly straightforward task and will therefore only be sketched.

Definition 4.21 (Admissible initial datum for problem (4.47))**.** We say that an initial value $u_0 \in C^2((0,m))$ is *admissible* for problem (4.47) provided it has the following properties:

(IV1) $\inf_{(0,m)} u_0' > 0$.

(IV2) The density f_0 associated with the inverse of u_0, given by $f(u_0) = \frac{1}{u_0{}'}$, satisfies

$$f_0 \geq f_{\infty,\theta} \quad \text{in } \mathbb{R} \qquad \text{for some } \theta \in (0,\infty). \tag{4.48}$$

(IV3) There exists $\varepsilon_0 > 0$ such that the function $r \mapsto |r|^{1+\varepsilon_0} f_0(r)$ lies in $L^\infty(\mathbb{R})$.

Remark 4.22. As we will see below, hypothesis (IV2) is a simple means to ensure t-uniform Lipschitz regularity, locally in $x \in (0,m)$, of the solution to be constructed. Besides, notice that hypothesis (IV2) implies the boundary behaviour $\lim_{x\to 0^+} u_0(x) = -\infty$, $\lim_{x\to m^-} u_0(x) = \infty$. It will, in fact, ensure that, for the solution to be constructed, the limits in the second line of eq. (4.47) hold true uniformly in time. The assumed boundedness of the function $r \mapsto |r|^{1+\varepsilon_0} f_0(r)$ is a technical hypothesis used to ensure that the constant $c(u_0)$ in estimate (4.55) is independent of R.

Definition 4.23. Let u_0 be admissible in the sense of Definition 4.21. Then for any $R \geq 1$ there exist points a_R and b_R satisfying $u_0(a_R) = -R$ and $u_0(b_R) = R$. Abbreviating $J_R := (a_R, b_R)$ and $\Omega_R := (0,\infty) \times J_R$, we denote by $u^{(R)}$ the unique viscosity solution of $\mathcal{F} = 0$ in Ω_R subject to the conditions $u^{(R)}(0,\cdot) = u_{0|J_R}$, $u^{(R)}(t, a_R) = -R$, $u^{(R)}(t, b_R) = R$. (See Corollary 3.24.) The measure $\mu^{(R)}(t) \in \mathcal{M}_b^+([-R, R])$ associated with the generalised inverse of $u^{(R)}(t,\cdot)$ has the form $\mu^{(R)}(t) = f^{(R)}(t,\cdot)\cdot\mathcal{L}^1 + x_p^{(R)}(t)\delta_0$, where $f^{(R)}, x_p^{(R)}(t)$ are as in Proposition 4.10.

Under the hypotheses on u_0 in Definition 4.21, we are able to construct a viscosity solution u of problem (4.47) as the limit of a sequence of solutions $\{u^{(R)}\}$ as in Definition 4.23.

We are now in a position to state the main results of this section.

Theorem 4.24 (Wellposedness)**.** *Let $\gamma \geq 2, m \in (0,\infty)$ and suppose that $u_0 \in C^2((0,m))$ is admissible for eq.* (4.47) *in the sense of Definition 4.21. Then there exists a unique x-monotonic*[3] *viscosity solution $u \in C([0,\infty)\times(0,m))$ of problem* (4.47) *with the property that*

$$\lim_{x\to 0} \sup_t u(t,x) = -\infty, \qquad \lim_{x\to m} \inf_t u(t,x) = \infty. \tag{4.49}$$

The function u satisfies the bound

$$\|u\|_{C^{0,1}([0,\infty)\times J')} \leq C_{J'} \tag{4.50}$$

for any $J' \subset\subset (0,m)$.

Definition 4.25.

(i) Given a non-decreasing, continuous function $v : (0,m) \to \mathbb{R}$ satisfying

$$\lim_{x\to 0+} v(x) = -\infty, \qquad \lim_{x\to m-} v(x) = \infty,$$

[3] Recall that $u = u(t,x)$ is called x-monotonic if $u(t,\cdot)$ is non-decreasing in x for all t (see Definition 3.13).

we define its *generalised inverse* $M_v : \mathbb{R} \to (0, m)$ via

$$M_v(r) = \sup \{x \in (0, m) : v(x) \le r\}, \quad r \in \mathbb{R}. \tag{4.51}$$

(ii) It is elementary to see that M_v in Definition 4.25 (i) is increasing, right-continuous and satisfies

$$\lim_{r\to-\infty} M_v(r) = 0, \qquad \lim_{r\to\infty} M_v(r) = m.$$

Hence, M_v is the cumulative distribution function (cdf) of a measure $\mu_v \in \mathcal{M}_b^+(\mathbb{R})$ whose mass equals m (see e.g. [93, Chapter 20.3]). The measure μ_v is uniquely determined by

$$\mu_v((-\infty, r]) = M_v(r), \quad r \in \mathbb{R}.$$

(iii) Given u as in Theorem 4.24 and $t \ge 0$ we denote by $M(t, \cdot) : \mathbb{R} \to (0, m)$ the generalised inverse of $u(t, \cdot)$, i.e.

$$M(t, \cdot) := M_{u(t,\cdot)},$$

where we used the notation (4.51). We further let $\mu(t) \in \mathcal{M}_b^+(\mathbb{R})$ denote the measure associated with the cdf $M(t, \cdot)$ as introduced in Definition 4.25 (ii), i.e. $\mu(t) = \mu_{u(t,\cdot)}$.

Theorem 4.26. *Under the hypotheses of Theorem 4.24 and with the notations in Definition 4.25, the viscosity solution u obtained in Theorem 4.24 has the following properties:*

(L1) *For all $t > 0$ there exist unique points $x_-(t), x_+(t) \in (0, m)$ such that*

$$u(t, \cdot)^{-1}(0) = [x_-(t), x_+(t)].$$

Also, $\partial_x u(t, x) > 0$ for $x \in (0, m) \setminus [x_-(t), x_+(t)]$, and away from $\{\partial_x u = 0\}$ the function u is smooth and satisfies $\mathcal{F}(u) = 0$ in the classical sense.

(L2) *For each $t > 0$ the strictly increasing and right-continuous function $M(t, \cdot)$ satisfies*

$$M(t, 0-) = x_-(t) \text{ and } M(t, 0) = x_+(t).$$

Moreover, M is C^∞ in the open set $\{(t, r) : t > 0, |r| \in (0, \infty)\}$.

(L3) *Let $x_p(t) := \mathcal{L}^1(\{u(t, \cdot) = 0\})$, $t > 0$. There exists a unique, positive function $f(t, \cdot) \in L^1(\mathbb{R})$ such that the measure $\mu(t) \in \mathcal{M}_b^+(\mathbb{R})$ associated with $M(t, \cdot)$*

has the decomposition

$$\mu(t) = f(t,\cdot)\mathcal{L}^1 + x_p(t)\delta_0, \quad t \in (0,\infty),$$

where away from $r = 0$ *the function* f *is a classical solution of eq.* (4.46)*.*

(L4) *Blow-up behaviour: if the function* $f(t,\cdot)$ *introduced in* (L3) *is unbounded near the origin (or equivalently* $\partial_x u(t, x_\pm(t)) = 0$*), then*

$$f(t,r) = \left(\frac{\gamma}{q(t,r)} \int_0^r sq(t,s)\,\mathrm{d}s\right)^{-\frac{1}{\gamma}},$$

where q *is defined as in formula* (4.21) *of Proposition 4.12. In particular, the expansion* (4.22) *holds true for small* $|r|$*. Hence, if* $\gamma = 2$*,* f *is globally regular and satisfies eq.* (4.46) *in the classical sense.*

On the whole space, an entropy dissipation identity analogous to Proposition 4.15 requires some extra control on the tails of the density. This issue has been well studied, for instance, in [29], which is why we omit the precise statements regarding the long-time asymptotics in the problem on the line. Under a suitable additional decay condition on the initial density, it should not be difficult to obtain results similar to those in Sections 4.2.2 to 4.2.3.

The rest of this section is devoted to the proofs of Theorems 4.24 and 4.26. We start by deriving uniqueness.

4.4.1 Uniqueness for unbounded monotonic viscosity solutions

In order to establish uniqueness for problem (4.47), (4.49), we first observe that the proof of the comparison principle, Proposition 3.8, shows that the assumed boundary regularity of the functions involved can be relaxed as follows:

Corollary 4.27 (Comparison, relaxed version)**.** *Let* $0 < T \leq \infty$ *and assume that the continuous function* G *satisfies* (A0) *&* (A1)*. Suppose that* $u \in \mathrm{USC}([0,T)\times(0,m))$ *is a subsolution,* $v \in \mathrm{LSC}([0,T)\times(0,m))$ *a supersolution of* $\mathcal{G} = 0$ *in* $\Omega = (0,T)\times(0,m)$ *with the boundary behaviour*

$$\limsup_{\omega\to\partial_p\Omega} (u(\omega) - v(\omega)) \leq 0.$$

Then $u \leq v$ *in* Ω*.*

Corollary 4.27 implies uniqueness for BFP on the line (at the level of u) in the following sense:

Corollary 4.28 (Uniqueness for problem (4.47))**.** *Let $T \in (0,\infty)$. Given a non-decreasing function $u_0 \in C((0,m))$, there exists at most one x-monotonic viscosity solution $u \in C([0,T) \times (0,m))$ of problem* (4.47) *with the property that*

$$\lim_{x\to 0} \sup_{t\in(0,T)} u(t,x) = -\infty, \qquad \lim_{x\to m} \inf_{t\in(0,T)} u(t,x) = \infty. \tag{4.52}$$

Proof. Suppose that u and v are x-monotonic viscosity solutions of problem (4.47) with the properties assumed in the statement of Cor. 4.28. For functions $w = w(t,x)$ and $0 < \delta \ll 1$ we denote by ${}^{(\mp\delta)}w(t,x)$ the spatially shifted function $w(t, x \pm \delta)$. The same notation will be used for time-independent functions (see Definition 4.19). We further abbreviate ${}_\delta\Omega := (0,T) \times (\delta, m-\delta)$. Then ${}^{(\delta)}u$ (resp. ${}^{(-\delta)}v$) is a viscosity subsolution (resp. supersolution) of $\mathcal{G} = 0$ in ${}_\delta\Omega$. Conditions (4.52) and the x-monotonicity ensure that

$$\limsup_{\omega\to\partial_p({}_\delta\Omega)} \left({}^{(\delta)}u(\omega) - {}^{(-\delta)}v(\omega)\right) \le 0.$$

Hence, by Corollary 4.27, ${}^{(\delta)}u \le {}^{(-\delta)}v$ in ${}_\delta\Omega$. As $\delta > 0$ can be chosen arbitrarily small, this implies, thanks to the continuity of u and v, that $u \le v$ in Ω. Since u and v are interchangeable, we infer that $u = v$. □

4.4.2 Proof of Theorems 4.24 and 4.26: Existence and Regularity

The uniqueness part of Theorem 4.24 has been established in Corollary 4.28. Now, our main task lies in establishing the existence part of Theorem 4.24 and the bound (4.50), since the assertions in Theorem 4.26 can then be deduced similarly as in the case of a bounded interval. The key is a local Lipschitz bound in space-time for $u^{(R)}$ which holds true uniformly in $R \gg 1$.

Proposition 4.29. *Let $u^{(R)}$ and Ω_R be as in Definition 4.23. Then, for any $R \ge 1$*

$$K_R := \sup_{\tilde{R}\ge R} \|u^{(\tilde{R})}\|_{C^{0,1}(\Omega_R)} < \infty. \tag{4.53}$$

Estimate (4.53) yields local compactness of our family $\{u^{(R)}\}$ of approximate solutions.

Proposition 4.29 will be proved in three steps:

In *Step 1* we establish an upper bound on the spatial Lipschitz constants of the approximate sequence $\{u^{(R)}\}$ taking the form

$$\|\partial_x u^{(\tilde{R})}\|_{L^\infty(\Omega_R)} \leq C(\theta, R), \quad \tilde{R} \geq R \geq 1, \tag{4.54}$$

where θ is the parameter in ineq. (4.48). This step relies on hypothesis (IV2) and the following bound:

Lemma 4.30. *For any $R \geq 1$ there exists $c_R < \infty$ such that for all $\tilde{R} \geq R$*

$$\sup_{t>0} \|u^{(\tilde{R})}(t, \cdot)\|_{L^\infty(J_R)} \leq c_R,$$

where $J_R = (a_R, b_R)$ are as in Definition 4.23.

Lemma 4.30 is an immediate consequence of the following estimate:

Lemma 4.31. *For all $R \geq 1$*

$$\sup_{t>0} \|u^{(R)}(t, \cdot)\|^2_{L^2(J_R)} \leq \max\{m, \|u_0\|^2_{L^2}\}.$$

Lemma 4.31 is proved in Appendix 4.A.3, where we also provide a generalisation of the estimate to L^p spaces for $p \geq 2$. Observe that the L^p norm at the level of u equals the p^{th} moment of the density f (see eq. (4.63)). In the original variables, the propagation of higher-order moments for several other (nonlinear) Fokker–Planck-type equations on $\mathbb{R}^d, d \in \mathbb{N}$, is rather well-established. See reference [29] for a proof in the case of the Kaniadakis–Quarati model for fermions.

In *Step 2* of the proof of Proposition 4.29 we derive a lower bound on $\partial_x u^{(R)}$: $\exists\, c(u_0) > 0$ such that

$$\partial_x u^{(R)} \geq c(u_0)|u^{(R)}|, \tag{4.55}$$

The constant $c(u_0)$ only depends on the mass of a symmetric, radially decreasing function $\tilde{f}_0$ lying above f_0 (see (4.57)).

Steps 1 & 2 both use the comparison principle for densities, Proposition 4.20, applied to the functions $f^{(R)}$ introduced in Definition 4.23 and a suitable reference solution.

In *Step 3* we show that, thanks to parabolic estimates, Steps 1 & 2 imply a uniform control of $|\partial_t u^{(R)}|$ on sets of the form $\{\delta < |u^{(R)}| < \delta^{-1}\}, \delta > 0$. Reasoning as in the proof of Proposition 3.16, we will then infer that an R-uniform control of the quantity $|\partial_t u^{(R)}|$ is even true on sets of the form $\{|u^{(R)}| < \delta^{-1}\}, \delta > 0$.

An alternative, in some sense more direct method to argue is sketched in Remark 4.32 below.

Let us now present the detailed arguments.

Proof of Proposition 4.29. We proceed by showing the three steps outlined above. Throughout the proof we assume that $\tilde{R} \geq R \geq 1$.
Step 1: Since $f^{(\tilde{R})}(0,\cdot) = f_0 \geq f_{\infty,\theta}$ on $[-\tilde{R},\tilde{R}]$, Proposition 4.20 yields

$$f^{(\tilde{R})}(t,\cdot) \geq f_{\infty,\theta} \quad \text{on } [-\tilde{R},\tilde{R}] \text{ for any } t \geq 0.$$

Owing to relation (4.14) and Lemma 4.30 we infer that for any $\tilde{R} \geq R$

$$\|\partial_x u^{(\tilde{R})}\|_{L^\infty(\Omega_R)} \leq (f_{\infty,\theta}(c_R))^{-1}. \tag{4.56}$$

Here we used the monotonicity of $f_{\infty,\theta}(r)$ in $|r|$. The constant $c_R < \infty$ in estimate (4.56) equals the one in Lemma 4.30. This proves estimate (4.54) and completes Step 1.

Step 2: Let $\hat{f}_0(r) = \max_{\sigma\in\{\pm1\}} f_0(\sigma r)$. Then, by (IV3), there exists $C < \infty$ such that

$$f_0(r) \leq \hat{f}_0(r) \leq C(1+|r|^2)^{-\frac{(1+\varepsilon_0)}{2}} =: \tilde{f}_0(r), \quad r \in \mathbb{R}. \tag{4.57}$$

Notice that $\tilde{f}_0$ is even, non-increasing in $|r|$, and, moreover, $\tilde{f}_0 \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$. For $R \geq 1$ consider the solutions $\tilde{u}^{(R)}$ and $u^{(R)}$ emanating from the inverse cdf of $\tilde{f}_{0|[-R,R]}$ and $f_{0|[-R,R]}$ and denote the corresponding densities, defined on $(0,\infty)\times(-R,R)$, by $\tilde{f}^{(R)}$ and $f^{(R)}$. Then, by Proposition 4.20, for all $t \geq 0$

$$f^{(R)}(t,r) \leq \tilde{f}^{(R)}(t,r), \quad r \in [-R,R].$$

By uniqueness and the equation's symmetry, $\tilde{u}^{(R)}(t.\cdot)$ is symmetric for any $t \geq 0$. Moreover, letting $\tilde{m}_R = \|\tilde{f}_0\|_{L^1(-R,R)}$, the function $\tilde{u}^{(R)}(t.\cdot)_{|(\frac{\tilde{m}_R}{2},\tilde{m}_R)}$ is convex as a consequence of a classical minimum argument combined with inequality (4.24), which controls the delicate region near the origin. (Strictly speaking, this argument requires an additional regularity hypothesis on the initial datum near the lateral boundary, which can easily be removed by approximation.) Hence, $\tilde{f}^{(R)}(t,\cdot)$ is non-increasing in $|r|$, implying that $\tilde{f}^{(R)}(t,r) \leq \frac{\tilde{m}}{2|r|}$ for $t \geq 0$, $r \in (-R,R)\setminus\{0\}$, where $\tilde{m} := \|\tilde{f}_0\|_{L^1(\mathbb{R})}$. This yields

$$\partial_x u^{(R)} \geq \frac{2|u^{(R)}|}{\tilde{m}}, \tag{4.58}$$

which concludes Step 2. In Remark 4.32 below, we sketch an alternative way to deduce estimate (4.58). The underlying method, combined with Step 1, also provides a quick means to deduce the bound (4.53).

Step 3: Thanks to hypothesis (4.48) there exist time-independent x-monotonic functions

$$u_+(t,\cdot) \equiv u_+ : (0,m) \to (\infty,\infty], \quad u_-(t,\cdot) \equiv u_- : (0,m) \to [-\infty,\infty)$$

with the following properties:

1. $u_+ \in C(\Omega \cap \{u_+ < \infty\})$ is a supersolution, $u_- \in C(\Omega \cap \{u_- > -\infty\})$ a subsolution of $\mathcal{F} = 0$ in $\Omega \cap \{u_+ < \infty\}$ resp. in $\Omega \cap \{u_- > -\infty\}$
2. $u_-(x) \leq u_0(x) \leq u_+(x)$ for all $x \in (0,m)$
3. $\lim_{x\to 0} u_+(x) = -\infty$, $\lim_{x\to m} u_-(x) = \infty$.

Thus, by comparison, for any $\tilde{R} \in [1,\infty)$

$$u_-(x) \leq u^{(\tilde{R})}(t,x) \leq u_+(x) \quad \text{for all } x \in J_{\tilde{R}},\, t \geq 0. \tag{4.59}$$

Hence, owing to bound (4.58), we infer the existence of $\underline{R} \in [1,\infty)$ and $c_1 = c_1(u_0) > 0$ such that for any $\tilde{R} \geq \underline{R}$ the inequality $\partial_x u^{(\tilde{R})}(t,\cdot) \geq c_1 > 0$ holds true in $(a_{\tilde{R}}, a_{\underline{R}}) \cup (b_{\underline{R}}, b_{\tilde{R}})$. Now, for $R \geq \underline{R}$ we can apply classical parabolic estimates (see [75, Theorem V.5.1]) to the equation for $u^{(\tilde{R})}$, $\tilde{R} \geq R+1$, in $(0,\infty)\times I_{\eta,R}$, where for $0 < \eta \ll 1$ we denote $I_{\eta,R} := (a_R, a_R+\eta) \cup (b_R - \eta, b_R)$ and, for small $\varepsilon > 0$, $I_{\eta,R,\varepsilon} := \{x \in (0,m) : \operatorname{dist}(x, I_{\eta,R}) < \varepsilon\}$. In particular, one has the bound

$$\|\partial_t u^{(\tilde{R})}\|_{L^\infty((0,\infty)\times I_{\eta,R})} \leq C\left(\varepsilon, R, \|u^{(\tilde{R})}\|_{L^\infty((0,\infty),C^1(\bar{I}_{\eta,R,\varepsilon}))}, \|u_0\|_{C^2(\bar{I}_{\eta,R,\varepsilon})}, c_1, \theta\right)$$

for any $\tilde{R} > R+1$. Arguing as in Proposition 3.16 we deduce, also owing to Lemma 4.30,

$$\|\partial_t u^{(\tilde{R})}\|_{L^\infty(\Omega_R)} \leq C(R, u_0). \tag{4.60}$$

Combining estimates (4.56) and (4.60) we obtain the bound (4.53). □

Remark 4.32. If we suppose, in addition to the hypotheses in Definition 4.2, that the initial density f_0 satisfies

$$\sup_{r\in\mathbb{R}} |\partial_r f_0(r) + r h_\gamma(f_0(r))| < \infty,$$

it is possible to simplify Steps 2 and 3 by using the uniform control of the quantity $\partial_r f^{(R)} + r h_\gamma(f^{(R)})$ obtained via an alternative approximation and comparison principle for regular solutions in the original variables. (See Section 5.3 and in particular (5.22).) Indeed, suppose that

$$B := \sup_R \|\partial_r f^{(R)} + r h_\gamma(f^{(R)})\|_{L^\infty((0,\infty)\times(-R,R))} < \infty. \tag{4.61}$$

By construction, $\partial_r f^{(R)} + u^{(R)} h_\gamma(f^{(R)}) = -\frac{\partial_t u^{(R)}}{\partial_x u^{(R)}}$, where the functions involving $f^{(R)}$ are to be evaluated at $u^{(R)}$. Then, by Step 1, for any $\omega \in \Omega_R$

$$|\partial_t u^{(\tilde{R})}(\omega)| \le B|\partial_x u^{(\tilde{R})}(\omega)| \le C(\theta, R)B \qquad \text{for all } \tilde{R} \ge R.$$

Combined with Lemma 4.30, this yields estimate (4.53).

Let us also note that estimate (4.61) and mass control imply an L^∞ bound for $f^{(R)}$ away from the origin, namely

$$|r| f^{(R)}(t,r) \le C(B,m),$$

which, up to the size of the constant, is equivalent to estimate (4.55) of Step 2.

We are now in a position to prove Theorem 4.24.

Proof of Theorem 4.24. We argue similarly to Section 4.1.1. The bound (4.53) and the equation satisfied by $u^{(\tilde{R})}$ yield

$$\sup_{\tilde{R}>R} \|\partial_x((\partial_x u^{(\tilde{R})})^{\gamma-1})\|_{L^\infty(\Omega_R)} \le C(R).$$

Thus, we find $\beta_0 > 0$, $u \in C([0,\infty); C^{1,\beta_0}_{\mathrm{loc}}((0,m))) \cap C^{0,1}_{\mathrm{loc}}([0,\infty) \times (0,m))$ and a sequence $\tilde{R} \to \infty$ such that for any $T > 0$ and any $R > 0$:

$$u^{(\tilde{R})} \overset{\tilde{R}\to\infty}{\longrightarrow} u \quad \text{in } C([0,T]; C^{1,\beta_0}(\bar{J}_R)).$$

By Remark 3.7 (a) the limit u is itself a viscosity solution of eq. (3.1), and, by construction, $u(0,\cdot) = u_0$. Owing to inequalities (4.59), we have

$$\lim_{x\to 0+} \sup_t u(t,x) \le \lim_{x\to 0+} u_+(x) = -\infty, \qquad \lim_{x\to m-} \inf_t u(t,x) \ge \lim_{x\to m-} u_-(x) = \infty.$$

Estimate (4.50) is an immediate consequence of (4.53) and the locally uniform convergence of the subsequence $\{u^{(\tilde{R})}\}$. This establishes Theorem 4.24. □

4.A Appendix

4.A.1 Semi-convexity

Definition 4.33 (Semi-convexity and -concavity)**.** Let $U \subset \mathbb{R}^d$ be convex. A function $v : U \to \mathbb{R}$ is called *semi-convex* (resp. *semi-concave*) if there exists a constant $C \in \mathbb{R}$ such that the function $x \mapsto v(x) + \frac{C}{2}|x|^2$ is convex (resp. such that $v(x) - \frac{C}{2}|x|^2$ is concave).

Proposition 4.34. *Let $u : \Omega \to \mathbb{R}$ be continuous. Suppose that there exists a constant $C < \infty$ such that for all $\omega \in \Omega$ and all $(\tau, p, q) \in \mathcal{P}^+ u(\omega)$ (resp. all $(\tau, p, q) \in \mathcal{P}^- u(\omega)$) the bound $q \geq -C$ (resp. $q \leq C$) holds true. Then, for all $t > 0$ the function $u(t, \cdot)$ is semi-convex (semi-concave) in J with constant bounded above by C.*

Proof. By symmetry, it suffices to prove the statement asserting semi-convexity. Thanks to [1, Lemma 1], it is enough to show that for all $t \in (0, \infty)$ and all $x \in J$

$$(p, q) \in \mathcal{J}^{2,+}(u(t, \cdot))(x) \quad \Rightarrow \quad q \geq -C. \tag{4.62}$$

The implication (4.62) is a consequence of the following general argument. A similar reasoning can be found in [62].

In order to see implication (4.62), we fix $t \in (0, \infty)$ and $x \in J$ and assume that $(p, q) \in \mathcal{J}^{2,+}(u(t, \cdot))(x)$. By definition (and the local boundedness of u), there exists $\phi \in C^2(J)$ such that $0 \geq u(t, y) - \phi(y)$, $0 = u(t, x) - \phi(x)$ and $p = \phi'(x)$, $q = \phi''(x)$. In particular, $u(t, \cdot) - \phi$ reaches a maximum at x. After possibly replacing ϕ with $\phi(y) + |x - y|^4$, we may assume that the maximum is strict. Now consider for suitably small $0 < \delta \ll 1$ the function

$$w(s, y) := u(s, y) - \left(\phi(y) + \frac{1}{2\varepsilon}|s - t|^2 \right) \qquad \text{in } Q_\delta := [t - \delta, t + \delta] \times [x - \delta, x + \delta].$$

By continuity, w reaches its (non-negative) maximum at some point $(s_\varepsilon, y_\varepsilon) \in Q_\delta$ and as $\varepsilon \to 0$, we must have $s_\varepsilon \to t$. Moreover, $y_\varepsilon \to x$ since if this was not the case, then along a subsequence $(s_\varepsilon, y_\varepsilon) \to (t, \tilde{x})$ for some $\tilde{x} \neq x$ and therefore $0 \leq w(s_\varepsilon, y_\varepsilon) \leq u(s_\varepsilon, y_\varepsilon) - \phi(y_\varepsilon) \to u(t, \tilde{x}) - \phi(\tilde{x}) < 0$ by the strictness of the maximum, a contradiction.

Hence for small enough $\varepsilon > 0$

$$(0, 0, 0) \in \mathcal{P}^+ w(s_\varepsilon, y_\varepsilon)$$

or, equivalently,

$$\left(\frac{s_\varepsilon - t}{\varepsilon}, \phi'(y_\varepsilon), \phi''(y_\varepsilon)\right) \in \mathcal{P}^+ u(s_\varepsilon, y_\varepsilon).$$

Hence $\phi''(y_\varepsilon) \geq -C$ and, letting $\varepsilon \to 0$, we conclude

$$q = \phi''(x) \geq -C.$$

□

Lemma 4.35. *Suppose the function $v : J \to \mathbb{R}$ is semi-convex and semi-concave with constant $C < \infty$. Then $v \in C^{1,1}(\bar{J})$ and $[v']_{C^{0,1}(\bar{J})} \leq C$.*

Proof. The fact that v is semi-convex and semi-concave implies that v is differentiable at every point (since the first order sub- and superdifferential exist everywhere). Thus, since $v(x) + \frac{C}{2}|x|^2$ is convex and $v(x) - \frac{C}{2}|x|^2$ concave, we deduce $v'(x) + Cx \leq v'(y) + Cy$ and $v'(x) - Cx \geq v'(y) - Cy$ whenever $x \leq y$. In combination, this yields

$$|v'(x) - v'(y)| \leq C|x - y|.$$

□

4.A.2 $\mathcal{L}^2$-measurability

Lemma 4.36. *Using the notation in Section 4.1.1, the second order pointwise derivative ${}^{(p)}\partial_x^2 v_\sigma$ of v_σ with respect to x exists $\mathcal{L}^2$-almost everywhere in Ω and the function $\partial_x v_\sigma$ has a weak derivative in x-direction satisfying*

$$\partial_x^2 v_\sigma = {}^{(p)}\partial_x^2 v_\sigma \text{ in } L^\infty(\Omega).$$

Proof. Throughout the proof we abbreviate $u := v_\sigma$. Recall that for fixed time this function is semi-convex, semi-concave (uniformly in t) and, thus, by Lemma 4.35, of the class $C^{1,1}(\bar{J})$ (uniformly in t). For any $t > 0$ we denote by N_t the subset of points in J where the second pointwise derivative of $u(t, \cdot)$ does not exist. Then the set N_t is an $\mathcal{L}^1$-null set, and our goal is to show that the set $\cup_t \{t\} \times N_t \subset \Omega$ is $\mathcal{L}^2$-measurable.

We choose C large enough such that the function $\tilde{u}(t, x) = u(t, x) + \frac{C}{2}|x|^2$ is convex for all t and define $v(t, x) := \partial_x \tilde{u}(t, x)$. Then $v(t, \cdot)$ is non-decreasing and $v(t, \cdot) \in C^{0,1}(\bar{J})$. Moreover, v lies in $L^\infty(\Omega)$ and is thus $\mathcal{L}^2$-measurable. Now define

$$\overline{\partial} v := \limsup_{h \to 0} \partial^h v$$

and

$$\underline{\partial} v := \liminf_{h\to 0} \partial^h v,$$

where the function $\partial^h v(t,x) := \frac{v(t,x+h)-v(t,x)}{h}$ is bounded. In view of the monotonicity and the continuity of $v(t,\cdot)$, it is clear that when taking the lim sup resp. the lim inf one can restrict to $h = \frac{1}{n}$, $n \in \mathbb{Z}$. Since $w_n := \partial^{\frac{1}{n}} v$ is $\mathcal{L}^2$-measurable, the pointwise lim sup resp. lim inf of this countable family $\{w_n\}$ must itself be $\mathcal{L}^2$-measurable. Therefore the set

$$G := \{\omega \in \Omega : \ \overline{\partial} v(\omega) - \underline{\partial} v(\omega) = 0\},$$

which is exactly the set where ${}^{(p)}\partial_x^2 u$ exists, is $\mathcal{L}^2$-measurable. Hence its complement $\Omega \setminus G = \cup_t (\{t\} \times N_t)$ is $\mathcal{L}^2$-measurable and thus, by Fubini's theorem, an $\mathcal{L}^2$-null set. Extending the function ${}^{(p)}\partial_x^2 u$ defined on G to Ω, e.g. by setting ${}^{(p)}\partial_x^2 u(\omega) = 0$ for all $\omega \in \Omega \setminus G$, the fact that ${}^{(p)}\partial_x^2 u(\omega) = \overline{\partial} v(\omega)$ for any $\omega \in G$ implies that ${}^{(p)}\partial_x^2 u$ is $\mathcal{L}^2$-measurable, so that, thanks to the boundedness of $\overline{\partial} v$, ${}^{(p)}\partial_x^2 u \in L^\infty(\Omega)$. Fubini's theorem finally yields that the identity ${}^{(p)}\partial_x^2 u = \partial_x^2 u$ holds true $\mathcal{L}^2$-almost everywhere in Ω. □

4.A.3 Propagation of moments

Proof of Lemma 4.31. For the proof we abbreviate $u := u^{(R)}$, $J := J_R = (a_R, b_R)$ and $a := a_R, b := b_R$. We first gather several observations on the regularity of the functions involved, which will justify our computations. The fact that the function $t \mapsto u(t,x)$ is Lipschitz continuous uniformly in x combined with the results in Proposition 4.12 implies that for each x the map $t \mapsto u^2(t,x)$ is differentiable with bounded derivative. Furthermore, in $\{|u| > 0\}$ we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}u^2 = u\partial_t u &= u(\partial_x u)^{-2}\partial_x^2 u - u^2(\partial_x u)^{-\gamma} - u^2 \\
&\le -u\frac{\mathrm{d}}{\mathrm{d}x}\left((\partial_x u)^{-1}\right) - u^2 \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\left(u(\partial_x u)^{-1}\right) + 1 - u^2.
\end{aligned}$$

Finally notice that, again thanks to Proposition 4.12, for every $t > 0$ the function $-\frac{\mathrm{d}}{\mathrm{d}x}\left(u(\partial_x u)^{-1}\right) = u(\partial_x u)^{-2}\partial_x^2 u$ is integrable in $\{|u(t,\cdot)| > 0\}$ and its integral satisfies

$$\begin{aligned} -\int_{(a,b)\cap\{|u(t,\cdot)|>0\}} \frac{\mathrm{d}}{\mathrm{d}x}\left(u(\partial_x u)^{-1}\right)\,\mathrm{d}x &= -\lim_{\varepsilon\to 0}\int_{(a+\varepsilon,b-\varepsilon)\cap\{|u(t,\cdot)|>0\}} \frac{\mathrm{d}}{\mathrm{d}x}\left(u(\partial_x u)^{-1}\right)\,\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0}\left[u(\partial_x u)^{-1}\right]_{a+\varepsilon}^{b-\varepsilon} \\ &= -\frac{R}{\partial_x u(t,b)} - \frac{R}{\partial_x u(t,a)}, \end{aligned}$$

where in the second step we used again Proposition 4.12 to deduce that

$$\lim_{y\to(x_\pm(t))^\pm}\left(\frac{u(t,y)}{\partial_x u(t,y)}\right) = 0.$$

Hence, the function $t \mapsto \|u(t,\cdot)\|_{L^2(a,b)}^2$ is absolutely continuous and its derivative satisfies

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|_{L^2(a,b)}^2 &= \int_{\{|u(t,\cdot)|>0\}} u(t,x)\partial_t u(t,x)\,\mathrm{d}x \\ &\le \mathcal{L}^1(\{|u(t,\cdot)| > 0\}) - \|u(t,\cdot)\|_{L^2(a,b)}^2. \end{aligned}$$

Recalling the fact that, by construction, $(a,b) = (a_R, b_R) \subset (0,m)$ and $u = u^{(R)}$ with $u^{(R)}(0,\cdot) = u_0$ in (a_R, b_R), we infer the bound

$$\begin{aligned} \|u^{(R)}(t,\cdot)\|_{L^2(a,b)}^2 &\le \max\{m, \|u^{(R)}(0,\cdot)\|_{L^2(a,b)}^2\} \\ &\le \max\{m, \|u_0\|_{L^2(0,m)}^2\} \end{aligned}$$

for all $t \ge 0$. $\square$

Using induction, Lemma 4.31 can easily be generalised to L^p for $p \in [2,\infty)$ as long as

$$\|u_0\|_{L^p(0,m)}^p = \int_{\mathbb{R}} |r|^p f_0(r)\,\mathrm{d}r < \infty. \tag{4.63}$$

More precisely, we have the bound

$$\sup_{R\ge 1}\|u^{(R)}(t,\cdot)\|_{L^q(J_R)}^q \le C(K_{2\lfloor q/2\rfloor}, m, q, \|u_0\|_{L^q(0,m)}^q) \text{ for all } t \ge 0, \tag{4.64}$$

where K_p is recursively defined via $K_0 = m$ and, for $p \in 2\mathbb{N}^+$,

$$K_p = \max\left\{(p-1)K_{p-2}, \|u_0\|_{L^p(0,m)}^p\right\}.$$

A similar inductive argument in the original variables can be found in [29].

In essence, the proof of estimate (4.64) is analogous to the proof of Lemma 4.31. Since $p > 2$, the regularity near $\{\partial_x u(t,\cdot) = 0\}$ of the functions involved is even somewhat better. Below, we therefore only provide the formal argument, where we drop for simplicity the indices involving R. We first prove for $q \in 2\mathbb{N}^+$ the bound

$$\sup_{t\geq 0} \|u^{(R)}(t,\cdot)\|^q_{L^q(J_R)} \leq K_q. \tag{4.65}$$

We argue by induction. Suppose that $p \geq 4$ and that ineq. (4.65) holds true for $q = p-2$. Then

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b |u(t,x)|^p \,\mathrm{d}x &= \int_a^b |u|^{p-2}u\partial_t u \,\mathrm{d}x = \int_{(a,b)\cap\{|u(t,\cdot)|>0\}} |u|^{p-2}u\partial_t u \,\mathrm{d}x \\
&\leq -\int_{(a,b)\cap\{|u(t,\cdot)|>0\}} \frac{\mathrm{d}}{\mathrm{d}x}\left(|u|^{p-2}u(\partial_x u)^{-1}\right)\mathrm{d}x \\
&\qquad\qquad + (p-1)\|u(t,\cdot)\|^{p-2}_{L^{p-2}} - \|u(t,\cdot)\|^p_{L^p} \\
&\leq (p-1)\|u(t,\cdot)\|^{p-2}_{L^{p-2}} - \|u(t,\cdot)\|^p_{L^p} \\
&\leq (p-1)K_{p-2} - \|u(t,\cdot)\|^p_{L^p}.
\end{aligned}$$

This implies ineq. (4.65) for $q = p$.

For $p > 2, p \notin 2\mathbb{N}$, a bound of the form (4.64) is obtained using Hölder's inequality

$$\|u(t,\cdot)\|_{L^{p-2}(a,b)} \leq |b-a|^{\frac{1}{p-2}-\frac{1}{\tilde{p}}}\|u(t,\cdot)\|_{L^{\tilde{p}}(a,b)}, \quad \tilde{p} = 2\left\lfloor\frac{p}{2}\right\rfloor,$$

in the penultimate line of the last chain of estimates and the fact that $|b-a| \leq m$.

Chapter 5

Refined dynamical properties of the 1D Fokker–Planck model for bosons

This chapter consists of a collection of results providing further insights into the dynamics of the bosonic Fokker–Planck equations in the L^1-supercritical case $\gamma > 2$. Some of the properties discussed in this chapter will be taken up and verified numerically in Chapter 6. The analysis presented builds on the results established in Chapter 4.

Throughout this chapter we use the notations and assume the hypotheses of Proposition 4.10. In particular, u denotes the viscosity solution of the bosonic Fokker–Planck problem (4.1) in the new variables. As in Proposition 4.10, for each t we let $f(t,\cdot)$ denote the density associated with the generalised inverse of $u(t,\cdot)$. Then, away from $r = 0$, the density f satisfies the first line of (5.1) in the classical sense:

$$\begin{cases} \partial_t f = \partial_r^2 f + \partial_r(rf(f^\gamma + 1)), & t > 0,\ r \in (-R, R), \\ f(0, r) = f_0(r), & r \in (-R, R), \\ 0 = \partial_r f + rf(f^\gamma + 1), & t > 0,\ r \in \{-R, R\}. \end{cases} \tag{5.1}$$

We remark that, with similar arguments, results analogous to those established in this chapter can be shown to hold true for the problem on the whole line $\mathbb{R}$ (see eq. (4.47), (4.52)).

5.1 Transient condensates and global regularity

By Corollary 4.18, under the stated hypotheses, mass-subcritical solutions will eventually be smooth, while solutions above the critical mass will eventually have a non-trivial condensate component. Here, we establish a criterion of a more local nature showing that singularities and condensates can occur for arbitrarily small mass $m > 0$. For completeness, we also provide a criterion ensuring global-in-time regularity. Below, integrals of the form $\int \ldots \,\mathrm{d}v$ are to be understood as Lebesgue integrals over the interval $(-R, R)$ although similar statements apply to the problem on the whole line.

Proposition 5.1. *In addition to the hypotheses of Proposition 4.10, suppose that $\gamma > 2$. There exists a constant $B_\gamma > 0$ only depending on γ such that if for some $\delta > 0$ the inequality*

$$m - B_\gamma \frac{m^{\frac{3\gamma}{2}}}{\left(\int |v|^2 f_0(v)\,\mathrm{d}v\right)^{\frac{\gamma-2}{2}}} \leq -\delta \tag{5.2}$$

holds true, then the function $t \mapsto x_p(t)$ cannot be identically zero.

Note that, for any fixed mass $m > 0$, inequality (5.2) is satisfied for initial data sufficiently concentrated near the origin.

Theorem 4.16, Corollary 4.18 and Proposition 5.1 show that in general the condensate does interact with the regular part of the solution and may partially or fully dissolve. This phenomenon is due to the linear diffusion and cannot occur in the hyperbolic case considered in [25]. In particular, we have the following result.

Corollary 5.2 (Existence of transient condensates)**.** *In addition to the hypotheses of Proposition 4.15, suppose that inequality* (5.2) *is satisfied for some $\delta > 0$. Then, if $m < m_c$, the point mass at velocity origin satisfies*

$$\operatorname{supp} x_p \subset\subset (0, \infty) \quad \textit{and} \quad x_p \not\equiv 0.$$

The proof of Proposition 5.1 is an adaptation of the finite-time blow-up argument in [100] combined with the bounds (4.22), (4.26). It makes use of the following inequality, established in [100].

Proposition 5.3 (Ref. [100], Lemma 2)**.** *Let $d = 1$. For any $\gamma > 2$ there exists a constant $B_\gamma \in (0, \infty)$ such that (for all sufficiently regular functions $f \not\equiv 0$)*

$$\int |v|^2 f^{\gamma+1}(v)\,\mathrm{d}v \geq B_\gamma \frac{\left(\int f(v)\,\mathrm{d}v\right)^{\frac{3\gamma}{2}}}{\left(\int |v|^2 f(v)\,\mathrm{d}v\right)^{\frac{\gamma}{2}-1}}. \tag{5.3}$$

Proposition 5.3 was originally stated for functions on the whole space. Its validity for functions on $(-R, R)$ follows via extending the functions by zero outside $(-R, R)$.

Proof of Proposition 5.1. Heuristically, the idea is to keep track of, or estimate from below, the flux of mass into the origin. For this purpose we use a virial type argument and consider the evolution of the kinetic energy

$$E(t) := \frac{1}{2}\int_{(-R,R)} |v|^2 f(t,v)\,\mathrm{d}v = \frac{1}{2}\int_{(0,m)} |u(t,x)|^2\,\mathrm{d}x.$$

The following computations, performed at the level u, can be justified in a similar way as in the proof of Lemma 4.31 (see Appendix 4.A.3). We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= -\int_{\{|u|>0\}} |u|^2 u_x^{-\gamma}\,\mathrm{d}x - \int_{\{|u|>0\}} u\frac{\mathrm{d}}{\mathrm{d}x}\left(u_x^{-1}\right)\,\mathrm{d}x - 2E(t)\\ &\leq m - \int_{\{|u|>0\}} |u|^2 u_x^{-\gamma}\,\mathrm{d}x.\end{aligned}$$

Observe that the last integral equals the left-hand side of ineq. (5.3). Hence, Proposition 5.3 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq m - B_\gamma \frac{(m - x_p(t))^{\frac{3\gamma}{2}}}{(2E(t))^{\frac{\gamma-2}{2}}}.$$

Thus, if $x_p(t) \equiv 0$, we find that whenever the bound (5.2) holds true for some $\delta > 0$, $E(t)$ would have to become negative after some time $T \leq \frac{E(0)}{\delta}$, which is impossible. □

On the other hand, there is a large class of globally bounded mass-subcritical solutions. We confine ourselves to providing a rather simple criterion. Since blow-up cannot occur in the case $\gamma = 2$ (see Proposition 4.12 (iv)), it suffices to consider the case $\gamma > 2$.

Proposition 5.4 (A criterion for global regularity)**.** *Assume that $R > 0$, $\gamma > 2$ and let $f_0 \in C^1([-R, R])$ be strictly positive. Suppose that there exists $\theta > 0$ such that the function $\tilde{f}_0(r) = \max_{\sigma\in\{\pm1\}} f_0(\sigma r)$ satisfies*

$$\Big|\int_0^r \tilde{f}_0(\rho)\,\mathrm{d}\rho\Big| \leq \Big|\int_0^r f_{\infty,\theta}(\rho)\,\mathrm{d}\rho\Big| \qquad \textit{for } r \in [-R, R]. \tag{5.4}$$

Let $m = \|f_0\|_{L^1}$ and denote by $u_0 : [0, m] \to [-R, R]$ the inverse of the cumulative distribution function of f_0. Then the corresponding viscosity solution u of (4.1) *satisfies* $\min_{[0,m]} \partial_x u(t,\cdot) > 0$ *for all* $t \geq 0$ *and, more specifically, we have*

$$\Big|\int_0^r f(t,\rho)\,\mathrm{d}\rho\Big| \leq \Big|\int_0^r f_{\infty,\theta}(\rho)\,\mathrm{d}\rho\Big| \qquad \textit{for } r \in [-R,R], \tag{5.5}$$

where $f(t,\cdot)$ *denotes the density associated with the generalised inverse of* $u(t,\cdot)$*.*

Remark 5.5. Notice that condition (5.4) implies that $\int_{-R}^{R} f_0 < m_c(R)$. Conversely, for any $m \in (0, m_c(R))$ and $f_0 \in (C^1 \cap L^1)(\mathbb{R})$ even and of mass m there exists $\lambda^* = \lambda^*(f_0) \in (0,\infty)$ such that $f_{0,\lambda}(\rho) := \lambda^{-1} f_0(\lambda^{-1}\rho)$ satisfies condition (5.4) for $r \in \mathbb{R}$ whenever $\lambda \geq \lambda^*$.

Proof of Proposition 5.4. Inequality (5.5) can be obtained from a comparison principle at the level of the cumulative distribution functions. Here, for consistency, we follow the approach pursued so far, based on the pseudo-inverse. Arguments similar to the one presented below have been used in previous parts of this thesis. The details in the current situation are provided for completeness.

We proceed in two steps:

Step 1: proof for f_0 even.

Then, for $x_0 := (m^{(R,\theta)} - m)/2$ the function

$$u_{\mathrm{barr}}(x) := u_\infty^{(R, m^{(R,\theta)})}(x + x_0)$$

is a global barrier for u ensuring that

$$\begin{cases} u(t,\cdot) \leq u_{\mathrm{barr}} \text{ in } [0, m/2], \\ u(t,\cdot) \geq u_{\mathrm{barr}} \text{ in } [m/2, m]. \end{cases} \tag{5.6}$$

Here, we have used the fact that $u(t,\cdot)$ is odd with respect to the point $m/2$. This is a consequence of the assumed point symmetry of u_0 and the uniqueness of viscosity solutions to the Cauchy–Dirichlet problem.

The bounds (5.6) combined with the non-degeneracy of u_{barr} (at height zero) imply that $\partial_x u(t,\cdot)$ is strictly positive on $[0,m]$ for any $t \in [0,\infty)$. Hence, by Theorem 4.4 (R2), the density $f(t,\cdot)$ associated with the inverse of $u(t,\cdot)$ is globally regular. The bound (5.6) then implies (5.5).

Step 2: general case.

Consider the continuous function $\widetilde{f}_0(r) = \max_{\sigma \in \{\pm 1\}} f_0(\sigma r)$, $r \in [-R,R]$, and pick a sequence $\tilde{f}_0^{(n)} \in C^1([-R,R])$ such that $\tilde{f}_0^{(n)} \geq \tilde{f}_0$ and $\|\tilde{f}_0^{(n)} - \tilde{f}_0\|_{C([0,m])} \to 0$. Then, for any $\theta' \in (0,\theta)$ there exists n sufficiently large such that bound (5.4) is satisfied with $\tilde{f}_0$ replaced by $\tilde{f}_0^{(n)}$ and $f_{\infty,\theta}$ replaced by $f_{\infty,\theta'}$. Step 1 thus yields inequality (5.5) with f replaced by the density $\tilde{f}^{(n)}$ of the generalised inverse of the viscosity solution $\tilde{u}^{(n)}$ emanating from the (pseudo-)inverse of $\tilde{f}_0^{(n)}$ and $f_{\infty,\theta}$ replaced by $f_{\infty,\theta'}$. At the same time, by comparison at the level of the densities, $f \leq \tilde{f}^{(n)}$ for all n. Thus, letting $\theta' \nearrow \theta$ we deduce (5.5). □

5.2 Type II dynamics of blow-up and blow-down

In Chapter 4 and Section 5.1 we have established different conditions ($m > m_c$ or kinetic energy $\ll m$) under which solutions f of our 1D Fokker–Planck model for bosons explode in finite time. Combined with Corollary 4.18, Proposition 5.1 further tells us that there exist solutions blowing up in finite time which will regularise or 'blow down' after some subsequent time.

In this section we are interested in the dynamics of finite-time blow-up and regularisation. We will see that in *similarity variables*, determined by the scaling properties of our equation at high values of the density (i.e. when neglecting the linear term of the drift), the profiles of blow-up and blow-down are universal and in both cases given by the power law $c_\gamma r^{-\frac{2}{\gamma}}$ with $c_\gamma = (2/\gamma)^{1/\gamma}$. In particular, blow-up and blow-down are of type II. As we will see below, these properties are a consequence of the cancellation encoded in equation (4.26). The scaling methods presented below are well-known in the literature and have been extensively used in the study of other nonlinear parabolic equations. We recommend [88] and references therein for an introduction to the technique in the context of the Fujita equation.

In order to formulate our main results, we first introduce the similarity variables: for fixed $T \in (0, \infty)$ we define

$$V_{T,+} = \{(s, y) : s > -\log(T), |y| < \exp(s/2)R\},$$
$$V_{T,-} = \{(s, y) : s \in \mathbb{R}, |y| < \exp(-s/2)R\}.$$

For simplicity we henceforth assume that $R > 2$. Given the density f associated to the generalised inverse of a global-in-time viscosity solution of eq. (4.1), we let

$$g_{T,+}(s, y) = (T - t)^{\frac{1}{\gamma}} f(t, (T - t)^{\frac{1}{2}} y),\ s = -\log(T - t), \qquad \text{for } (s, y) \in V_{T,+},$$
$$g_{T,-}(s, y) = (t - T)^{\frac{1}{\gamma}} f(t, (t - T)^{\frac{1}{2}} y),\ s = \log(t - T), \qquad \text{for } (s, y) \in V_{T,-}.$$

Notice that, by definition,

$$\partial_y^j g_{T,+}(s, y) = (T - t)^{\frac{1}{\gamma}+\frac{j}{2}} \partial_r^j f(t, r), \qquad s = -\log(T - t),\ y = (T - t)^{-\frac{1}{2}} r,$$
$$\partial_y^j g_{T,-}(s, y) = (t - T)^{\frac{1}{\gamma}+\frac{j}{2}} \partial_r^j f(t, r), \qquad s = \log(t - T),\ y = (t - T)^{-\frac{1}{2}} r.$$

In the following, the relation between (s, y) and (t, r) will always be as in the corresponding case of the previous two lines.

Proposition 5.6 (Profile of blow-up and blow-down in similarity variables)**.** *Suppose that $T \in (0,\infty)$ is such that $f(T,\cdot)$ is unbounded near $r=0$. Then*

$$g_{T,\pm}(s,y) \to f^*(y) \qquad \text{as } s \to \pm\infty,$$

locally uniformly in $\{y \neq 0\}$. Here $f^*(y) := c_\gamma |y|^{-\frac{2}{\gamma}}$ with $c_\gamma = (2/\gamma)^{1/\gamma}$.

The fact that the local blow-up profile f^* is unbounded at the origin implies that blow-up is of type II or rather the slightly stronger property:

Corollary 5.7 (Type II blow-up and blow-down)**.** *Whenever $f(T,\cdot)$ is unbounded near $r=0$, we have*

$$\lim_{t\nearrow T}(T-t)^{\frac{1}{\gamma}}\|f(t,\cdot)\|_{L^\infty} = +\infty$$

and

$$\lim_{t\searrow T}(t-T)^{\frac{1}{\gamma}}\|f(t,\cdot)\|_{L^\infty} = +\infty.$$

In preparation of the proof of Proposition 5.6 we gather several auxiliary estimates. By inequality (4.19) and identity (4.26) there exist finite constants c_1, c_2 only depending on R, γ and the Lipschitz norm $\|u\|_{C^{0,1}(\Omega)}$ of our viscosity solution such that

$$|g_{T,\pm}(s,y)| \leq c_1 |y|^{-\frac{2}{\gamma}}, \qquad (s,y) \in V_{T,\pm}, \tag{5.7}$$

$$|\partial_y g_{T,\pm}(s,y)| \leq c_2 |y|^{-\frac{2}{\gamma}-1} + \tilde{c}_2 |y|^{-\frac{2}{\gamma}}, \qquad (s,y) \in V_{T,\pm},\ \ s \geq 0. \tag{5.8}$$

Hence $g_{T,+}$ and $g_{T,-}$ are locally bounded in $V_{T,\pm} \setminus \{(s,y) : y \neq 0\}$ (uniformly in s) and satisfy

$$\begin{aligned}
\partial_s g_{T,+} &= \partial_y^2 g_{T,+} - \frac{1}{2} y \cdot \partial_y g_{T,+} - \frac{g_{T,+}}{\gamma} + \partial_y(y g_{T,+}^{\gamma+1})) + \mathrm{e}^{-s}\partial_y(y g_{T,+}),\\
\partial_s g_{T,-} &= \partial_y^2 g_{T,-} + \frac{1}{2} y \cdot \partial_y g_{T,-} + \frac{g_{T,-}}{\gamma} + \partial_y(y g_{T,-}^{\gamma+1})) + \mathrm{e}^{+s}\partial_y(y g_{T,-}).
\end{aligned}$$

Observe that the coefficient $\mathrm{e}^{\mp s}$ in the equation for $g_{T,\pm}$ is uniformly bounded for $(s,y) \in V_{T,\pm} \cap \{\pm s \geq 0\}$. Thus, by parabolic regularity estimates and inequalities (5.7) and (5.8), we have

$$\begin{aligned}
|\partial_y^j g_{T,+}(s,y)| &\leq C_{1,j}, \quad (s,y) \in V_{T,+}, \quad 1 \leq |y| \leq 2,\ s \geq 0, \qquad (5.9)\\
|\partial_y^j g_{T,-}(s,y)| &\leq C_{1,j}, \quad (s,y) \in V_{T,-}, \quad 1 \leq |y| \leq 2,\ s \leq 0,
\end{aligned}$$

where $j \in \mathbb{N}$ and $C_{1,j} \in (0,\infty)$ is a constant independent of T.

For given $(t,r) \in (0,\infty) \times (-1,1)$ satisfying $r \neq 0$, let $T_1 := t + |r|^2$ and $s_1 := -\log(T_1 - t)$, $y_1 := (T_1 - t)^{-\frac{1}{2}} r$. Then $s_1 \geq 0, |y_1| = 1$ and, by ineq. (5.9),

$$\begin{aligned}|\partial_r^j f(t,r)| &\leq (T_1 - t)^{-\frac{1}{\gamma} - \frac{j}{2}} |\partial_y^j g_{T_1,+}(s_1, y_1)| \\ &\leq C_{1,j} (T_1 - t)^{-\frac{1}{\gamma} - \frac{j}{2}} \\ &\leq C_{1,j} |r|^{-\frac{2}{\gamma} - j}.\end{aligned}$$

For $r \in (1, \tilde{R})$, $\tilde{R} := R - \frac{1}{2}$, the corresponding interior estimate, $|\partial_r^j f(t,r)| \leq C_{2,j}$, follows from the uniform bound

$$\sup_{t \geq 0,\, |r| \in (1,R)} f(t,r) \leq C(R, \gamma, \|u\|_{C^{0,1}(\Omega)})$$

and classical parabolic regularity [75]. Since $t \in (0,\infty)$ was arbitrary, we deduce

$$|\partial_r^j f(t,r)| \leq C(j) |r|^{-\frac{2}{\gamma} - j}, \quad t > 0,\ |r| \in (0, \tilde{R}). \tag{5.10}$$

The equation for f (see eq. (5.1)) then implies the rough bound

$$|\partial_t f(t,r)| \leq C |r|^{-\frac{2}{\gamma} - 2}, \quad t > 0,\ |r| \in (0, \tilde{R}). \tag{5.11}$$

In the next step, we aim to improve the control in (5.11) via interpolation.

Given $t > 0$ we define $\psi(r) := \partial_r f(t,r) + r f^{\gamma+1}(t,r)$, $|r| \in (0,R)$. By eq. (4.26) and ineq. (4.19), we have

$$|\psi(r)| \leq C_1 r^{-\frac{2}{\gamma}}, \tag{5.12}$$

while the bound (5.10) implies

$$|\psi''(r)| \leq C_2 r^{-3-\frac{2}{\gamma}}, \quad t > 0,\ |r| \in (0, \tilde{R}), \tag{5.13}$$

where $C_1, C_2 \in (0,\infty)$ are time-independent.

We will need the following simple interpolation estimate.

Lemma 5.A.1 (Interpolation)**.** *Let $I = (a,b)$ be a bounded interval and $\delta \in (0, b-a)$ be a fixed number. There exists a constant $C_\delta \in (0,\infty)$ only depending on δ such that for all $\psi \in C^2(\bar{I})$*

$$\|\psi'\|_{C((a,b-\delta))} \leq 2 \|\psi\|_{C(I)}^{\frac{1}{2}} \|\psi''\|_{C(I)}^{\frac{1}{2}} + C_\delta \|\psi\|_{C(I)}.$$

An elementary proof of Lemma 5.A.1 is given in Appendix 5.A. We now apply Lemma 5.A.1 with ψ as above, $\delta = \frac{1}{2}$ and $I = (r, \tilde{R})$ for $0 < r \ll \tilde{R} = R - \frac{1}{2}$, and use the bounds (5.12) and (5.13) to infer

$$\|\psi'\|_{C([r,R-1])} \lesssim r^{-\frac{2}{\gamma}-\frac{3}{2}} + 1$$

and thus for $r \in (0, R-1)$

$$|\partial_t f(t,r)| \lesssim r^{-\frac{2}{\gamma}-\frac{3}{2}} + 1. \tag{5.14}$$

The proof of the main result of this section is now straightforward:

Proof of Proposition 5.6. We only show the statement concerning $g_{T,+}$, the assertion involving $g_{T,-}$ follows along similar lines.

For $s > 0$ (or equivalently $t \in (T-1, T)$) and $|y| \leq \exp(s/2)(R-1)$ we compute, recalling our notation $r = (T-t)^{\frac{1}{2}} y$,

$$\begin{aligned} |g_{T,+}(s,y) - c_\gamma |y|^{-\frac{2}{\gamma}}| &\leq (T-t)^{\frac{1}{\gamma}} |f(t,r) - f(T,r)| + (T-t)^{\frac{1}{\gamma}} |f(T,r) - c_\gamma |r|^{-\frac{2}{\gamma}}| \\ &\lesssim (T-t)^{1+\frac{1}{\gamma}} (|r|^{-\frac{2}{\gamma}-\frac{3}{2}} + 1) + (T-t)^{\frac{1}{\gamma}} |r|^{1-\frac{2}{\gamma}} \\ &\lesssim (T-t)^{1-\frac{3}{4}} |y|^{-\frac{2}{\gamma}-\frac{3}{2}} + (T-t)^{\frac{1}{\gamma}}. \end{aligned}$$

In the second step, we used the mean value theorem applied to $t' \mapsto f(t', r)$ as well as the bound (5.14). Hence, as $s \to \infty$,

$$g_{T,+}(s,\cdot) \to f^* \qquad \text{locally uniformly in } \{y \neq 0\}.$$

□

5.3 Time evolution of the condensate and regularity by approximation in the original variables

In Chapter 4 we have seen that the size of the condensate component

$$t \mapsto x_p(t) = \mathcal{L}^1(\{u(t,\cdot) = 0\})$$

is a continuous function of time (see Proposition 4.12 (iii)). Here, we derive a formula for the evolution of the point mass, and provide a sketch proof showing that x_p is Lipschitz continuous. Along the way, we will see that regularisations in the original variables which preserve the Fokker–Planck-type structure lead to limiting measures $\{\bar{\mu}(t)\}$ which coincide with the measures $\{\mu(t)\}$ reconstructed from our viscosity solution. In other words, the corresponding limit in the new variables coincides with the unique viscosity solution constructed in Chapter 3.

In the following computations, we use the notations and assume the hypotheses in Proposition 4.10. In addition, we assume that the initial datum u_0 satisfies items (I1) and (I2) guaranteeing that $f(t,\cdot)$ satisfies the no-flux boundary condition (4.18) in the classical sense. Then, for any $s,t>0$, by mass conservation,

$$x_p(t)+\int_{-R}^{R} f(t,r)\,\mathrm{d}r = m = x_p(s)+\int_{-R}^{R} f(s,r)\,\mathrm{d}r.$$

Furthermore, for $\varepsilon,\delta\in(0,R)$

$$\begin{aligned}\int_{\varepsilon}^{R}(f(t,r)-f(s,r))\,\mathrm{d}r &= \int_{\varepsilon}^{R}\int_{s}^{t}\partial_t f(\sigma,r)\,\mathrm{d}\sigma\mathrm{d}r \\ &= \int_{\varepsilon}^{R}\int_{s}^{t}\partial_r(\partial_r f + rf^{\gamma+1}+rf)\,\mathrm{d}\sigma\mathrm{d}r \\ &= -\int_{s}^{t}\left[\partial_r f(\sigma,\varepsilon)+\varepsilon f^{\gamma+1}(\sigma,\varepsilon)+\varepsilon f(\sigma,\varepsilon)\right]\mathrm{d}\sigma\end{aligned} \tag{5.15}$$

and

$$\int_{-R}^{-\delta}(f(t,r)-f(s,r))\,\mathrm{d}r = \int_{s}^{t}\left[\partial_r f(\sigma,-\delta)-\delta f^{\gamma+1}(\sigma,-\delta)-\delta f(\sigma,-\delta)\right]\mathrm{d}\sigma. \tag{5.16}$$

Observing that the integral on the left-hand side of eq. (5.15) (resp. of eq. (5.16)) extends continuously to $\varepsilon=0$ (resp. $\delta=0$), we obtain

$$\begin{aligned}x_p(t)-x_p(s) &= \lim_{\varepsilon\to 0}\int_{s}^{t}\left[\partial_r f(\sigma,\varepsilon)+\varepsilon f^{\gamma+1}(\sigma,\varepsilon)\right]\mathrm{d}\sigma \\ &\quad -\lim_{\delta\to 0}\int_{s}^{t}\left[\partial_r f(\sigma,-\delta)-\delta f^{\gamma+1}(\sigma,-\delta)\right]\mathrm{d}\sigma,\end{aligned}$$

where we have used estimate (4.19). Since it is not clear whether the limits $\lim_{r\to 0^\pm}\big(\partial_r f(\sigma,r)+rf^{\gamma+1}(\sigma,r)\big)$ exist, the last formula cannot be further simplified at this stage. With the help of another approximation procedure (alternative to Section 4.1.1) it is, however, possible to show that x_p is Lipschitz continuous. Below, we outline the main steps of the proof.

- *Regularised problem:* take a smooth, non-decreasing function $\eta\in C^\infty(0,\infty)$ satisfying

$$\eta(s)=\begin{cases} s^\gamma & \text{if } s\le 1,\\ 2^\gamma & \text{if } s\ge 2,\end{cases}$$

  and let $\eta_\varepsilon(s)=\varepsilon^{-\gamma}\eta(\varepsilon s)$. Then define $\psi_\varepsilon(s)=1+\eta_\varepsilon(s)$ and let $\varphi_\varepsilon(s)=s\psi_\varepsilon(s)$, so that for $s \leq \frac{1}{\varepsilon}$ the function $\varphi_\varepsilon(s)$ coincides with the nonlinearity $h_\gamma(s) = s(1+s^\gamma)$.

For $0 < \varepsilon \ll 1$ we now consider the regularised problem

$$\begin{cases} \partial_t f_\varepsilon = \partial_r(\partial_r f_\varepsilon + r\varphi_\varepsilon(f_\varepsilon)), & t > 0,\, r \in (-R, R), \\ 0 = \partial_r f_\varepsilon + r\varphi_\varepsilon(f_\varepsilon), & \text{on } (0,\infty) \times \{\pm R\} \end{cases} \tag{5.17}$$

subject to the same initial condition $f_\varepsilon(0, r) = f_0(r)$, where $f_0 \in C^1([-R, R])$ is assumed to be strictly positive and of mass m. (This is equivalent to requiring that the inverse cdf u_0 of f_0 is admissible in the sense of Definition 4.2.)

- *Comparison for cumulative distribution function:* the advantage of the regularisation (5.17) lies in the fact that it enjoys a comparison principle at the level of the cumulative distribution function $M_\varepsilon(t, r) = \int_{-R}^r f_\varepsilon(t, \rho)\, \mathrm{d}\rho$. Indeed, the equation for M_ε corresponding to problem (5.17) states

$$\begin{cases} \partial_t M_\varepsilon = \partial_r^2 M_\varepsilon + r\varphi_\varepsilon(\partial_r M_\varepsilon), & t > 0,\, r \in (-R, R), \\ M_\varepsilon(t, -R) = 0, \quad M_\varepsilon(t, R) = m & \text{for all } t > 0. \end{cases} \tag{5.18}$$

Arguing as in the proof of Proposition 3.16 (without doubling the variables), it is easy to see that, given a family of classical solutions M_ε of problem (5.18) which are continuous up to the boundary, there exists a constant $K \in (0, \infty)$ such that

$$\sup_\varepsilon |\partial_t M_\varepsilon| \leq K. \tag{5.19}$$

Here, one also uses the fact that, by hypothesis, $f_0 \in C^1([-R, R])$.

In the limit $\varepsilon \to 0$ (see the next item), estimate (5.19) improves the bound on $\partial_t u$ near $\{\partial_x u = 0\}$.

Let us note that the existence of global-in-time regular solutions of the above problem can be obtained, for instance, by adapting the approach in Chapter 3. The comparison principle for the equation for M_ε, eq. (5.18), is a consequence of [38, Theorem 8.2], which exploits the fact that $\varphi_\varepsilon(s)$ is linear for s large enough. In order to obtain monotonic solutions in the Perron method, one uses the fact that the function 'G' determining the equation is monotonic in the space variable 'r'. Lipschitz continuity in time is obtained by following the proof of Proposition 3.16 (see also [38, Theorem 8.2]), while Lipschitz continuity in r can be deduced in a similar way as in Proposition 3.17. In both cases one uses the fact that $\varphi_\varepsilon(s)$ is linear for large s. Due to the 'r'-dependence of eq. (5.18), the spatial Lipschitz constant will, however, depend on ε. Higher regularity then follows from classical arguments (see e.g. the reasoning in the proof of Theorem 4.4).

If $R = \infty$ another convenient method to obtain global-in-time regular solutions would be to first construct local-in-time mild solutions of eq. (5.17) via a fixed point argument, and then to show that such solutions have a global extension.

- *Passage to limit:* for a strictly positive smooth function $0 < f_\varepsilon < \infty$, eq. (5.17) is equivalent to the problem for the inverse $u_\varepsilon(t,\cdot)$ of the cumulative distribution function $M_\varepsilon(t,\cdot)$ of $f_\varepsilon(t,\cdot)$:

$$\left(\psi_\varepsilon\left(\frac{1}{\partial_x u_\varepsilon}\right)\right)^{-1}\left((\partial_x u_\varepsilon)^2\partial_t u_\varepsilon - \partial_x^2 u_\varepsilon\right) + u_\varepsilon(\partial_x u_\varepsilon)^2 = 0.$$

It is possible to show that the family $\{u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is equicontinuous — for instance, by adapting the arguments leading to Corollary 3.18. Hence, along a subsequence $\varepsilon \to 0$, $\{u_\varepsilon\}$ converges locally uniformly to a viscosity solution $\bar{u}$ of the problem

$$(\partial_x\bar{u})^2 \cdot \Big((\partial_x\bar{u})^\gamma\partial_t\bar{u} - (\partial_x\bar{u})^{\gamma-2}\,\partial_x^2\bar{u} + \bar{u}(1+(\partial_x\bar{u})^\gamma)\Big) = 0. \qquad (5.20)$$

On the other hand, using the equation

$$\partial_t M_\varepsilon = \partial_r f_\varepsilon + r\varphi_\varepsilon(f_\varepsilon), \qquad (5.21)$$

it is elementary to show that the uniform bound (5.19) combined with mass conservation implies control of the term $|rf_\varepsilon(t,r)|$, uniformly in ε, t, r, which, by parabolic regularity, provides Hölder (and thus higher-order) control of the family $\{f_\varepsilon\}$, locally in $\{r \neq 0\}$. This suffices to pass to a limit in equation (5.21), possibly along another subsequence, and to deduce that there exists $b \in L^\infty((0,\infty)\times(-R,R))$ such that

$$b = \partial_r\bar{f} + rh_\gamma(\bar{f}) \quad \text{ in } (0,\infty)\times\big((-R,R)\setminus\{0\}\big), \qquad (5.22)$$

where, as before, $h_\gamma(s) = s(1+s^\gamma)$ and where $\bar{f}$ denotes the locally uniform limit of the subsequence $\{f_\varepsilon\}$ in $\{r \neq 0\}$. This, in turn, implies that $\partial_x\bar{u}$ exists everywhere in $\{\bar{u} \neq 0\}$ and satisfies $\partial_x\bar{u}(t,x) = \big(\bar{f}(t,\bar{u}(t,x))\big)^{-1} > 0$ whenever $(t,x) \in \{\bar{u} \neq 0\}$. Hence,

$$\{\partial_x\bar{u} = 0\} \subset \{\bar{u} = 0\}. \qquad (5.23)$$

Thanks to the non-degeneracy (5.23) and equation (5.20), it is now possible to use comparison in order to deduce that

$$\bar{u} = u. \tag{5.24}$$

This implies that the density f associated with the generalised inverse of u equals $\bar{f}$ and hence satisfies (5.22), which is a sharpened version of eq. (4.26) and yields the bound

$$|x_p(t) - x_p(s)| \leq 2\|b\|_{L^\infty}|t-s|.$$

The uniqueness (5.24) is interesting in its own right and provides another justification for our approach to the 1D bosonic Fokker–Planck problem.

As a side note, the above reasoning can equally be applied to the approximation introduced in the proof of Proposition 4.15.

Remark 5.8. Inserting the improved control, eq. (5.22), into formula (4.20) yields a sharper bound for the error terms in the spatial blow-up profile in Proposition 4.12 (i), namely

$$f(t,r) = c_\gamma |r|^{-\frac{2}{\gamma}} + b_1(t,r)|r|, \tag{5.25}$$

where $b_1 \in L^\infty_{t,r}$. Our numerical simulations in Chapter 6 confirm the error control (5.25), see Figures 6.1d and 6.2d. Furthermore, they suggest that, typically, the function b in eq. (5.22) has well-defined one-sided limits $\lim_{r\to 0^\pm} b(t,r)$. Notice that the existence of these limits would ensure the evolution of $x_p(t)$ to be governed by the differential equation

$$x_p'(t) = b(t,0+) - b(t,0-).$$

5.4 Rate of relaxation to equilibrium

By equation (4.39) of Chapter 4, under the stated hypotheses, in the long-time limit the global-in-time viscosity solution to our 1D Fokker–Planck model converges in entropy to the unique minimiser of the entropy of the given mass. In this short section, we show that in the mass-subcritical case the rate of convergence is (eventually) exponential. Our method exploits the fact that the entropy functional of the bosonic Fokker–Planck equation in 1D coincides with that of a nonlinear diffusion equation with linear drift, which satisfies a generalised log-Sobolev inequality [28]. An analogous idea was used in [24] for 1D KQ. The rate of convergence in the mass-supercritical case is still open. Our numerical simulations in Chapter 6 suggest a rate which is eventually exponential (see Figures 6.1b and 6.2b).

Proposition 5.9. *Let* $t_{\text{in}} \in \mathbb{R}$ *and suppose that* $f \in C([t_{\text{in}}, \infty) \times [-R, R])$ *is a classical solution of equation* (2.6) *in* $(t_{\text{in}}, \infty) \times (-R, R)$ *satisfying in the classical sense the boundary condition* (2.7)*. Further, abbreviate* $m := \|f(t_{\text{in}}, \cdot)\|_{L^1}$ *and assume*[1] *that* $m \leq m_c$*. Then, for all* $t \geq t_{\text{in}}$*,*

$$\mathcal{H}_{\text{rel}}(f(t,\cdot)|f_{\infty,\theta^{(R,m)}}) \leq \mathcal{H}_{\text{rel}}(f(t_{\text{in}},\cdot)|f_{\infty,\theta^{(R,m)}}) \exp(-2t), \tag{5.26}$$

where $\theta^{(R,m)}$ *is such that* $\|f_{\infty,\theta^{(R,m)}}\|_{L^1(-R,R)} = m$ *(see Notations 4.3). Here* $\mathcal{H}_{\text{rel}}(f_1|f_2) := \mathcal{H}_\gamma^{(R)}(f_1) - \mathcal{H}_\gamma^{(R)}(f_2)$*, where* $\mathcal{H}_\gamma^{(R)} = \mathcal{H}^{(h_\gamma,R)}$ *(see eq.* (3.17)*) with* $h(s) = h_\gamma(s) = s(1+s^\gamma)$*.*

Proof. In the proof we abbreviate $\mathcal{H} := \mathcal{H}_\gamma^{(R)}$. Let Φ be the functional defined in Section 2.1, i.e.

$$\Phi(s) = \int_0^s \psi(\sigma)\,\mathrm{d}\sigma,$$

where

$$\psi(s) = -\int_s^\infty \frac{1}{h_\gamma(\sigma)}\,\mathrm{d}\sigma = \frac{1}{\gamma}\log\left(\frac{s^\gamma}{1+s^\gamma}\right).$$

The following nonlinear diffusion equation in 1D

$$\partial_t g = \partial_v\left(g \cdot \partial_v\left[\frac{|v|^2}{2} + \Phi'(g)\right]\right) \tag{5.27}$$

has the same entropy functional as eq. (2.1) in 1D, namely

$$\mathcal{H}(g) = \int \left(\frac{|v|^2}{2} g + \Phi(g)\right)\mathrm{d}v.$$

Formally, positive solutions g of eq. (5.27) satisfy the entropy dissipation formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(g) = -\int g\left|\partial_v\left[\frac{|v|^2}{2} + \Phi'(g)\right]\right|^2\mathrm{d}v =: -\mathcal{D}^{(0)}(g),$$

while the entropy dissipation identity for positive solutions f of eq. (2.1) in 1D takes

[1] For the solutions considered in Chapter 4, the eventual regularity of $f(t,\cdot)$, assumed in Proposition 5.9, enforces the property $m \leq m_c$.

the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f) = -\int h(g)\left|\partial_v\left[\frac{|v|^2}{2} + \Phi'(f)\right]\right|^2 \mathrm{d}v =: -\mathcal{D}(f). \tag{5.28}$$

Observe that

$$\mathcal{D}^{(0)}(f) \le \mathcal{D}(f). \tag{5.29}$$

The reader may verify that for $f \in L^1_+(-R,R)$ with $\int f(v)\,\mathrm{d}v =: m \le m_c$ the entropy $\mathcal{H}(f)$ and entropy-dissipation $\mathcal{D}^{(0)}(f)$ associated with problem (5.27) satisfy the assumptions of the generalised log-Sobolev inequality [28, Eq. (72) in Corollary 1]:

$$\mathcal{H}_{\mathrm{rel}}(f|f_{\infty,\theta^{(R,m)}}) \le \frac{1}{2}\mathcal{D}^{(0)}(f), \tag{5.30}$$

where $\theta^{(R,m)} > 0$ is such that $m = \int_{(-R,R)} f_{\infty,\theta^{(R,m)}}(v)\,\mathrm{d}v$. Since f is regular, we have $\int f(t,\cdot) = m$ for all $t \ge t_{\mathrm{in}}$. Thus, the entropy dissipation formula (5.28) combined with inequalities (5.29) and (5.30) implies

$$\begin{aligned}
\mathcal{H}_{\mathrm{rel}}(f(t,\cdot)|f_{\infty,\theta^{(R,m)}}) &= \mathcal{H}_{\mathrm{rel}}(f(t_{\mathrm{in}},\cdot)|f_{\infty,\theta^{(R,m)}}) - \int_{t_{\mathrm{in}}}^t \mathcal{D}(f(s,\cdot))\,\mathrm{d}s \\
&\le \mathcal{H}_{\mathrm{rel}}(f(t_{\mathrm{in}},\cdot)|f_{\infty,\theta^{(R,m)}}) - 2\int_{t_{\mathrm{in}}}^t \mathcal{H}_{\mathrm{rel}}(f(s,\cdot)|f_{\infty,\theta^{(R,m)}})\,\mathrm{d}s.
\end{aligned}$$

Comparison with the solution of the corresponding differential equation yields the asserted bound (5.26). □

Remark 5.10. The decay formula (5.26) remains valid for the problem on the whole line $\mathbb{R}$. The logarithmic Sobolev-type inequality on $\mathbb{R}$, required in this case, has been established in [28, Theorem 17].

5.A Appendix

Below, we prove the interpolation inequality originally stated on page 96:

Lemma 5.A.1 (Interpolation)**.** *Let $I = (a,b)$ be a bounded interval and $\delta \in (0, b-a)$ be a fixed number. There exists a constant $C_\delta \in (0,\infty)$ only depending on δ such that for all $\psi \in C^2(\bar{I})$*

$$\|\psi'\|_{C((a,b-\delta))} \le 2\|\psi\|^{\frac{1}{2}}_{C(I)}\|\psi''\|^{\frac{1}{2}}_{C(I)} + C_\delta\|\psi\|_{C(I)}.$$

Proof. Let $r \in (a, b-\delta)$. Then, by Taylor's theorem, for any $\varepsilon \in (0, \delta)$ there exists $r_\varepsilon \in [r, r+\varepsilon]$ such that

$$\psi(r+\varepsilon) - \psi(r) - \varepsilon\psi'(r) - \frac{\varepsilon^2}{2}\psi''(r_\varepsilon) = 0.$$

Hence

$$|\psi'(r)| \leq \frac{2}{\varepsilon}\|\psi\|_{C^0(I)} + \frac{\varepsilon}{2}\|\psi''\|_{C^0(I)}.$$

<u>Case I:</u> $\left(\frac{\|\psi\|_{C^0(I)}}{\|\psi''\|_{C^0(I)}}\right)^{\frac{1}{2}} < \frac{\delta}{2}$.

In this case, choosing $\varepsilon = 2\left(\frac{\|\psi\|_{C^0(I)}}{\|\psi''\|_{C^0(I)}}\right)^{\frac{1}{2}}$, we deduce

$$|\psi'(r)| \leq 2\|\psi\|^{\frac{1}{2}}_{C^0(I)}\|\psi''\|^{\frac{1}{2}}_{C^0(I)}.$$

<u>Case II:</u> $\left(\frac{\|\psi\|_{C^0(I)}}{\|\psi''\|_{C^0(I)}}\right)^{\frac{1}{2}} \geq \frac{\delta}{2}$.

In this case, choosing $\varepsilon = 2\delta$, we obtain

$$\|\psi'\|_{C^0((a,b-\delta))} \leq 2\|\psi\|^{\frac{1}{2}}_{C^0(I)}\|\psi''\|^{\frac{1}{2}}_{C^0(I)} + \frac{1}{\delta}\|\psi\|_{C^0(I)}.$$

The lemma is proved. □

Chapter 6

Numerical study of Bose–Einstein condensation in the Kaniadakis–Quarati model

In this chapter, we present a numerical scheme for our bosonic Fokker–Planck equations (1.2), where in dimension $d > 1$ we consider rotationally symmetric solutions. The scheme is based on a generalisation of the change of variables leading to equation (2.9) for the pseudo-inverse distribution function, and is able to cope with singularities and Dirac measures at the origin. We use this scheme to illustrate and complement the rigorous analysis presented in Chapters 3, 4 and 5 and to study numerically the 3D Kaniadakis–Quarati (KQ) model, the equation most interesting from a physics point of view.

6.1 Overview

The main purpose of this chapter is to provide strong numerical evidence for the existence of solutions to the 3D KQ model forming a Bose–Einstein condensate in finite time. Our numerical results in higher dimensions (we focus on dimension $d = 3$) concern isotropic solutions and suggest that rotationally symmetric solutions of 3D KQ with supercritical mass $m > m_c$ will eventually have a non-trivial condensate component (see Section 6.5). From our simulations a rather clear picture of the dynamical properties of 3D KQ in the isotropic case will emerge: the long-time asymptotics will be identified, which the numerical solution converges to in entropy at an exponential rate. Numerical evidence is provided for the possibility of the condensed part failing to be monotonic in time and for even dissolving completely. The ad hoc scheme for rotationally symmetric solutions of KQ in dimension $d > 1$ is validated in dimension $d = 2$, where explicit solutions are available (see Section 6.4). Before investigating KQ in 3D, we will apply the numerical scheme to the caricature of the L^1-supercritical case in 1D, i.e. problem (2.6), or rather (2.9), with $\gamma > 2$, in order to numerically reproduce the analytical results established in Chapters 4 and 5 (see Section 6.3). Our numerical experiments in 1D further indicate that the decay of the entropy is exponential. Since for $d = 1$ non-stationary explicit solutions are not available, the 1D scheme will be validated by numerically analysing the convergence behaviour under mesh refinement with respect to a reference solution on a very fine mesh.

The proposed numerical scheme is based on the variational formulation of equation (2.1) using a mass transportation Lagrangian approach. It is motivated by the approach in [18, 32], where the gradient flow with respect to the Wasserstein distance is expressed in terms of the inverse of the cumulative distribution functions. Regarding the numerical study of mass concentration phenomena, advantages inherent in the approach based on pseudo-inverse distribution functions include mass conservation and automatic mesh refinement in regions of high concentration. A potential difficulty in our situation lies in the circumstance that we do not have the Wasserstein gradient flow structure in a rigorous sense. We will, however, see that this precise structure is not required and our proposed scheme will be shown to preserve in particular the entropy decay property (rigorously in 1D and 2D for the semidiscrete case, see Section 6.2.2). In contrast to the problems considered in [18, 32], where simulations break down at the first blow-up time (i.e. when the L^∞ norm of the density explodes), the scheme for the equations considered here, if properly formulated, allows for simulations for arbitrarily long time. In particular, our scheme allows to explore the qualitative behaviour after blow-up: condensation dynamics, spatial blow-up profile and entropy decay. These good numerical properties, consistent in 1D with the theory established in Chapters 3 to 5, corroborate our numerical findings in Section 6.5 concerning the 3D isotropic case.

6.2 Numerical method

We follow and generalise the ansatz in Section 2.5 considering the equation satisfied by the pseudo-inverse cumulative distribution function of $f(t, \cdot)$. In higher dimensions, $d > 1$, we confine ourselves to isotropic solutions and consider the pseudo-inverse of an appropriately normalised version of the *radial cdf* of $f(t, \cdot)$ returning the mass of $f(t, \cdot)$ on centred balls. At the end of Section 6.2.1, we will briefly comment on the anisotropic case.

6.2.1 Change of variables

One-dimensional case

Here, we consider the case $d = 1$ and assume that $\gamma > 2$, which determines the L^1-supercritical regime. Let us first recall from Section 2.5 that the equation satisfied by the inverse $u(t, \cdot)$ of the cumulative distribution function

$$M(t, v) = \int_{\{w \leq v\}} f(t, w) \, \mathrm{d}w$$

of $f(t, \cdot)$ formally takes the form

$$\partial_t u = (\partial_x u)^{-2} \partial_x^2 u - u(1 + (u_x)^{-\gamma}).$$

Upon multiplying by the factor $(\partial_x u)^\gamma$, it can be rewritten as

$$(\partial_x u)^\gamma \partial_t u - \frac{1}{\gamma - 1} \partial_x \left((\partial_x u)^{\gamma - 1} \right) + u((\partial_x u)^\gamma + 1) = 0. \tag{6.1}$$

Observe that the function $u \equiv 0$, which at the level of the density f corresponds to a Dirac delta at the origin, satisfies equation (6.1).

Boundary conditions. To determine the appropriate domain and boundary conditions for problem (6.1), notice that, for smooth positive densities $f(t, \cdot)$ on $(-R_1, R_1)$, the inverse cumulative distribution function $u(t, \cdot)$ maps the interval $(0, \|f(t, \cdot)\|_{L^1})$ diffeomorphically onto $(-R_1, R_1)$. Since we intend to impose mass conservation and want to consider the original problem for the density on a stationary domain $(-R_1, R_1)$, the function $u(t, \cdot)$ is understood to live on a fixed interval $(0, m)$ and assumed to take the Dirichlet boundary conditions

$$u(t, 0) \equiv -R_1, \qquad u(t, m) \equiv R_1.$$

Notice that this condition tacitly supposes strict positivity of the density or, more generally, full support of the measure in the original variables. Since, in the original variables, we are dealing with a uniformly parabolic equation without absorption, this hypothesis is, however, well-justified. To avoid regularity issues close to initial time, our initial data u_0 are chosen in such a way that they satisfy the 0^{th} order compatibility conditions $u_0(0) = -R_1, u_0(m) = R_1$.

Notations. As explained in Remark 2.2, given a radius R_1 and a mass $m = \|f_0\|_{L^1(-R_1,R_1)}$ there exists a unique measure $\mu_\infty^{(R_1,m)} \in \mathcal{M}_b^+([-R_1, R_1])$ of mass m

which minimises the entropy $\widetilde{\mathcal{H}} := \widetilde{\mathcal{H}}_\gamma^{(R_1)}$ among such measures. At the level of u, we denote this minimiser by $u_\infty := u_\infty^{(R_1,m)}$ (see Notations 4.3). We further let $H(u) := \mathcal{H}(f)$ resp. $\widetilde{\mathcal{H}}(\mu)$, where $\mu = u_\# \mathcal{L}^1$ is the push-forward measure of the Lebesgue measure on $[0,m]$ under the map u and will, in places, abbreviate $H_\infty := H(u_\infty) = \widetilde{\mathcal{H}}(\mu_\infty)$. The dependence of u_∞ on R_1 and m will be omitted. We occasionally abuse notation and write $H(t) := H(u(t))$. For later reference, let us observe that $H(u)$ is formally given by

$$H(u) = \int_{(0,m)} \left(\frac{|u|^2}{2} + \Psi(u_x) \right) \mathrm{d}x, \tag{6.2}$$

where the function

$$\Psi(s) := s\Phi(1/s) \quad \text{is convex with} \quad \Psi''(s) = s^{-3}\Phi''(1/s) = \frac{1}{s^3 h(1/s)}. \tag{6.3}$$

Higher dimensions – isotropic case

For isotropic solutions $f(t,v) = g(t,|v|)$, $v \in \mathbb{R}^d$, we can perform a similar transformation in higher dimensions: in radial form, equation (1.2) reads

$$\partial_t g = r^{1-d} \partial_r \left(r^{d-1} \partial_r g + r^d g(1+g^\gamma) \right), \ t, r > 0. \tag{6.4}$$

As a first ansatz one might try to consider the equation for the (pseudo-) inverse $R(t,z)$ of the *radial cdf* $\bar{M}(t,r) = \int_0^r g(t,s) s^{d-1} \, \mathrm{d}s$. However, for bounded densities f the function $\bar{M}$ is of class $O(r^d)$ as $r \to 0$, implying that $R(t,\cdot)$ is at most $1/d$-Hölder near $z = 0$ and $\partial_z R \gtrsim z^{1/d-1} \to \infty$ as $z \searrow 0$, whenever $d > 1$. We therefore consider the normalised version $N(t,s) = \bar{M}(t, s^{1/d})$ or, equivalently,

$$N(t,s) = \frac{1}{d} \int_0^s g(t, \sigma^{1/d}) \, \mathrm{d}\sigma,$$

which satisfies $\partial_s N(t,s) = \frac{1}{d} g(t, s^{1/d})$, and let $S(t,\cdot)$ denote the pseudo-inverse of $N(t,\cdot)$, so that $S = R^d$. From the formal relation $N(t, S(t,z)) = z$ we deduce (omitting the time argument)

$$\partial_z S = \frac{d}{g(R)}. \tag{6.5}$$

Then, the equation (6.4) for g leads to the following equation for S:

$$\tfrac{1}{d} \partial_t S - d \frac{S^{2-2/d}}{(\partial_z S)^2} \partial_z^2 S + S(1 + d^\gamma (\partial_z S)^{-\gamma}) = 0.$$

Since we want our scheme to be able to deal with condensates, i.e. $S(t,\cdot) \equiv 0$ on some subinterval $(0, z(t))$, we multiply this equation by $(\partial_z S)^\gamma$ to obtain

$$(\partial_z S)^\gamma \tfrac{1}{d} \partial_t S - d \cdot S^{2-2/d} (\partial_z S)^{\gamma-2} \partial_z^2 S + S((\partial_z S)^\gamma + d^\gamma) = 0. \tag{6.6}$$

Notice that if $\gamma \in [1,2)$, the viscosity term has a factor which becomes unbounded when S forms a condensate. We therefore consider for a small parameter $0 < \varepsilon \ll 1$ the following regularisation

$$(\partial_z S)^\gamma \tfrac{1}{d} \partial_t S - d \cdot S^{2-2/d} (\partial_z S + \varepsilon)^{\gamma-2} \partial_z^2 S + S((\partial_z S)^\gamma + d^\gamma) = 0$$

or, equivalently,

$$\begin{cases} (\partial_z S)^\gamma \frac{1}{d} \partial_t S - \frac{d}{\gamma-1} \cdot S^{2-2/d} \frac{\mathrm{d}}{\mathrm{d}z} (\partial_z S + \varepsilon)^{\gamma-1} + S((\partial_z S)^\gamma + d^\gamma) = 0, & \text{if } \gamma > 1, \\ (\partial_z S)^\gamma \frac{1}{d} \partial_t S - d \cdot S^{2-2/d} \frac{\mathrm{d}}{\mathrm{d}z} \log(\partial_z S + \varepsilon) + S((\partial_z S)^\gamma + d^\gamma) = 0, & \text{if } \gamma = 1. \end{cases}$$

We are mostly interested in the KQ model (where $\gamma = 1$) and will thus focus on the equation

$$d^{-1} \partial_z S \partial_t S - d S^{2-2/d} \frac{\mathrm{d}}{\mathrm{d}z} \log(\partial_z S + \varepsilon) + S(\partial_z S + d) = 0,$$

where $d = 2, 3$. Notice that a positive ε decreases the strength of diffusion significantly when $\partial_z S \lesssim \varepsilon$. In order to counterbalance this effect, which may potentially lead to numerical artefacts when investigating the expected phenomenon of condensation, we propose an artificial viscosity type regularisation of the form

$$d^{-1} \partial_z S \partial_t S - d(S+\delta)^{2-2/d} \frac{\mathrm{d}}{\mathrm{d}z} \log(\partial_z S + \varepsilon) + S(\partial_z S + d) = 0, \tag{6.7}$$

where $0 < \delta \ll 1$ is a small parameter. Below, $\bar{m}$ (resp. $\bar{m}_c$) denotes the total mass of the initial datum f_0 (resp. of f_c) on $B(0, R_1)$ *multiplied by the factor* $\frac{1}{|\partial B(0,1)|}$. Then, as in the 1D case, the appropriate boundary conditions for equation (6.7) are

$$S(t, 0) \equiv 0 \quad \text{and} \quad S(t, \bar{m}) \equiv R_1^d.$$

Notations. We denote by $S_\infty = S_\infty^{(R_1, \bar{m})}$ the pseudo-inverse normalised radial cdf of the unique (isotropic) minimising measure in $\mathcal{M}_b^+(\overline{B}(0, R_1))$ corresponding to the choice (R_1, m) of parameters, and generally let $H_d(S) := \widetilde{\mathcal{H}}(\mu)$, where μ is the unique isotropic measure in $\mathcal{M}_b^+(\overline{B}(0, R_1))$ satisfying $\mu(\overline{B}(0, r)) = \nu([0, r^d]) \cdot |\partial B(0,1)|$ and ν denotes the measure associated with the generalised inverse of S. We also abbreviate

$H_\infty := H(S_\infty)$ and $H(t) := H_d(S(t))$.

Higher dimensions – anisotropic case

Let us briefly discuss that one can perform a related change of variables in higher dimensions without radial symmetry. In this case, one needs to consider vector-valued transformations $u(t,\cdot) : U \to V$ between domains $U, V \subset \mathbb{R}^d$ which are formally related to the original density f via

$$\det \nabla u(t,x) \cdot f(t,u) = 1.$$

Here $\nabla u = \nabla_x u$ denotes the gradient of u with respect to $x \in U$. Similarly to [31, 52] one finds that the system governing the evolution of $u = (u^1, \ldots, u^d)^T$ can formally be written as

$$\left[(\det \nabla u)^2 \Psi''(\det \nabla u)\right] \partial_t u^i - \partial_{x_k} \left(\Psi'(\det \nabla u)(\operatorname{cof}(\nabla u))^i_k\right) + u^i = 0 \qquad (6.8)$$

for $i = 1, \ldots, d$, where Ψ is defined as in (6.3). The entropy $H_{\mathrm{ani},d}(u)$ in the new variables takes the form

$$H_{\mathrm{ani},d}(u) = \int_U \left(\tfrac{1}{2}|u|^2 + \Psi(\det \nabla u)\right) \mathrm{d}x.$$

Observe that in the vectorial case $H_{\mathrm{ani},d}(u)$ is no longer convex but merely polyconvex in ∇u. This route could potentially allow to numerically analyse concentrations without radial symmetry in higher dimensions, as it is the case in 2D for aggregation and Keller–Segel type problems close to the blow-up time [32]. While this method deserves further exploration, we focus here on the isotropic case to capture the direct generalisation of the 1D behaviour in the 3D realistic setting.

6.2.2 The semidiscrete scheme

The scalar equations (6.1) and (6.7) are discretised fully implicitly in time. We let τ be the discrete time step and denote by $\{u^n\}_{n\in\mathbb{N}}$ the time-discrete solution of the implicit Euler discretisation of equation (6.1). More precisely, given a non-decreasing function u^n satisfying $u^n(0) = -R_1$ and $u^n(m) = R_1$, the problem for $u = u^{n+1}$ reads

$$(\partial_x u)^\gamma \frac{u - u^n}{\tau} - \tfrac{1}{\gamma-1} \partial_x \left((\partial_x u)^{\gamma-1}\right) + u((\partial_x u)^\gamma + 1) = 0 \qquad (6.9)$$

subject to the Dirichlet boundary conditions $u^{n+1}(0) = -R_1, u^{n+1}(m) = R_1$.

Let us here make a short digression to explain the main difference and potential difficulty of the present problem with respect to the Wasserstein gradient flows treated in [18, 32]. Those works are based on the idea that the Wasserstein gradient flow of the entropy/free energy in the original variables is equivalent to an L^2 gradient flow for the problem in the u-variables. Loosely speaking, the semidiscrete L^2 gradient flow for $H(u)$ reads as follows: given $\tilde{u}^n$ formally define $\tilde{u}^{n+1}$ as a solution of the problem

$$\tilde{u}^{n+1} \in \arg\inf_{\tilde{u}} \left\{ \frac{1}{2\tau} \|\tilde{u} - \tilde{u}^n\|_{L^2}^2 + H(\tilde{u}) \right\}.$$

The associated Euler–Lagrange equations read

$$\frac{\tilde{u} - \tilde{u}^n}{\tau} = -[-\partial_x(\Psi'(\tilde{u}_x)) + \tilde{u}].$$

To compare this with our problem, we write eq. (6.9) in the more concise equivalent form

$$u_x^2 \Psi''(u_x) \frac{u - u^n}{\tau} = -[-\partial_x(\Psi'(u_x)) + u],$$

which suggests that in some sense a gradient flow structure is kept. At least, as will be shown below, we keep an important property in the semidiscrete numerical scheme, namely the monotonicity of the entropy. Recall that in 1D the entropy $H(u)$ in the u-variables (see (6.2)) is convex in the classical sense, and it is well-known that the implicit Euler scheme applied to a gradient flow of a convex functional satisfies the semidiscrete entropy inequality $H(\tilde{u}^{n+1}) \le H(\tilde{u}^n)$ for all n. In our situation, thanks to the convexity of the integrand of H, the entropy decay along the sequence $\{u^n\}$ can be recovered by a simple estimate:

$$\begin{aligned} H(u) - H(u^n) &\le \int_{(0,m)} (u(u - u^n) + \Psi'(u_x)(u - u^n)_x) \, \mathrm{d}x \\ &= \int_{(0,m)} (u - \partial_x(\Psi'(u_x)))(u - u^n) \, \mathrm{d}x \\ &= -\tau \int_{(0,m)} u_x^2 \Psi''(u_x) \left| \frac{u - u^n}{\tau} \right|^2 \mathrm{d}x \le 0. \end{aligned}$$

Here, we used the fact that in the above integration by parts the boundary terms vanish since, by construction, $u = u^n$ on $\partial(0, m)$. This shows the entropy decay property of the semidiscrete scheme (6.9): $H(u^{n+1}) \le H(u^n)$ for all n. We note that similar properties with a similar strategy of proof are found for related problems with a formal entropy structure, see in particular [65, Chapter 5] and references therein.

Remark 6.1 (Higher dimensions, isotropic case). In higher dimensions the entropy $H_d(S)$, introduced in Section 6.2.1, takes the form (see also (6.5))

$$H_d(S) = \int \left(\tfrac{1}{2} S^{\frac{2}{d}} + \Psi_d(\partial_z S) \right) \mathrm{d}z,$$

where $\Psi_d(s) = \Psi(\frac{s}{d})$ is again convex. If $d = 2$, thanks to convexity, the implicit Euler discretisation of eq. (6.6) can be shown to keep the entropy decay by arguing as in the 1D case. In higher dimensions, $d > 2$, this argument breaks down due to the kinetic part of the entropy failing to be a convex function of S. Notice, however, that the convexity in the highest order term, $\partial_z S$, is maintained.

6.2.3 The fully discrete scheme

The semidiscrete nonlinear system (6.9) is discretised using finite differences and solved by the Newton–Raphson method. In the one dimensional case, the finite difference approximation in space is chosen in such a way as to preserve the equation's symmetry, namely

$$\begin{aligned} (u_{i+1}^n - u_{i-1}^n)^\gamma (2h)^{-\gamma} \frac{u_i^n - u_i^{n-1}}{\tau} - ((u_{i+1}^n - u_i^n)^{\gamma-1} - (u_i^n - u_{i-1}^n)^{\gamma-1}) h^{-\gamma} (\gamma - 1)^{-1} \\ + u_i^n ((u_{i+1}^n - u_{i-1}^n)^\gamma (2h)^{-\gamma} + 1) = 0, \quad (6.10) \end{aligned}$$

for $i = 1, \ldots, N-1$, complemented with the boundary conditions $u_0^n = u_0^0 = -R_1$ and $u_N^n = u_N^0 = R_1$. We use a similar discretisation for eq. (6.7), namely

$$\begin{aligned} &(S_{i+1}^n - S_{i-1}^n)(2hd\tau)^{-1}(S_i^n - S_i^{n-1}) \\ &\quad - d(S_i^n + \delta)^{2-2/d} (\log((S_{i+1}^n - S_i^n)/h + \varepsilon) - \log((S_i^n - S_{i-1}^n)/h + \varepsilon))/h \\ &\quad + S_i^n ((S_{i+1}^n - S_{i-1}^n)/(2h) + d) = 0 \quad (6.11) \end{aligned}$$

for $i = 1, \ldots, N$, where the boundary conditions are given by $S_0^n = S_0^0 = 0$ and $S_N^n = S_N^0 = R_1^d$.

Algorithm. Given u^{n-1} the discrete approximation u^n at the subsequent time point is computed using a Newton–Raphson iteration. The iteration is stopped as soon as the smallness condition $\|F_{\mathrm{NR}}(u^n, u^{n-1}, h, \tau)\|_{l^2} < 10^{-8}$ is satisfied, where $F_{\mathrm{NR}}(u^n, u^{n-1}, h, \tau)_i$ is given by the left-hand side of equation (6.10) multiplied by h^γ. For S we proceed similarly.

Remark 6.2. In the simulations exhibiting the numerically somewhat delicate condensation phenomenon, the discrete approximate solution becomes slightly non-monotonic during the Newton–Raphson iteration, which leads to very small imaginary parts in the above scheme and of the solution at the subsequent time step. In our actual code we therefore rearrange the approximation in each Newton–Raphson iteration to ensure monotonicity. Alternatively, one can replace the first derivatives u_x by their absolute values $|u_x|$ and discretise and simulate this equation. In practice, the differences between the results using the first and the second option are negligible. A similar statement applies to the higher-dimensional case, where we choose again the option of the monotonic rearrangement.

6.3 Bosonic Fokker–Planck model in 1D: simulations replicating the theory

In this section we aim to demonstrate the reliability of the proposed numerical scheme for the L^1-supercritical bosonic Fokker–Planck equations by numerically reproducing the features of the continuous problem in 1D established in Chapters 4 and 5. In addition, we use the scheme to predict that even after the formation of a condensate the entropy decays at an exponential rate.

If not stated otherwise, we choose $\gamma = 2.9$ and use a centred Gaussian as initial datum, namely

$$f_0(v) = A\mathrm{e}^{-\frac{|v|^2}{2\sigma^2}} \tag{6.12}$$

for fixed positive constants A and σ. Moreover, we always set $R_1 = 1$. We remark that for $d = 1$ and the above choice of γ and R_1 the critical mass m_c takes the numerical value $m_c \approx 5.37$.

6.3.1 Validation in 1D

We begin with validating the 1D scheme (6.10) by comparing the solution for a given mesh with a numerical reference solution calculated on a fixed and much finer mesh. We set $\sigma = 0.7$, $A = 4.5$ in (6.12) as well as $T = 0.025$. For simplicity, the mass variable $x \in [0, m]$ is often referred to as the *spatial* variable. The numerical reference solution is computed on a grid of 12801 (equidistant) spatial mesh points and a total number of 1000 (equidistant) time points. Notice that the values of the parameters A and σ coincide with those in (P1) below and observe that, in the simulations based on (P1), well before the final time $T = 0.025$ chosen for our validation, a significant amount of mass has accumulated at the origin (cf. Figures 6.1a and 6.1c). Therefore, our validation covers the case in which condensation occurs.

timesteps	meshsize	L_z^2 error	rate
1000	50	7.3825e-3	-
1000	100	2.1290e-3	1.7939
1000	200	5.6056e-4	1.9253
1000	400	1.4222e-4	1.9788
1000	800	3.5598e-5	1.9982
1000	1600	8.8061e-6	2.0152
1000	3200	2.0991e-6	2.0687

Table 6.1: Convergence to reference solution at time $T = 0.025$.

timesteps	meshsize	$L_{t,z}^2$ error	rate
10	50	6.1372e-3	-
20	100	3.1393e-3	0.9671
40	200	1.5817e-3	0.9890
80	400	7.8542e-4	1.0099
160	800	3.8200e-4	1.0399
320	1600	1.7877e-4	1.0955
640	3200	7.6728e-5	1.2203

Table 6.2: Convergence to reference solution (on space-time grid).

Table 6.1 displays the discrete L_x^2 error of the solution on the coarser mesh with respect to the reference solution, evaluated at the final time T, while Table 6.2 indicates the L^2 space-time error between computed and reference solution. The results suggest a second order dependence of the error on the spatial increment and a first-order dependence on the temporal increment. As long as the solution is not degenerate, this can be explained by the fact that we use an implicit Euler scheme in time (which is first-order accurate), a central finite difference discretisation in space (whose truncation error is of second order) and by the fact that we have chosen a high resolution in time for the test using purely spatial refinement, which makes the temporal error negligible in this test. Notice, however, that the degenerate case requires more care and that, in this work, we do not provide a rigorous numerical analysis of the scheme.

6.3.2 Comparing simulations and theoretical results

In order to numerically confirm the dynamical properties of eq. (2.1) in 1D established in Chapters 4 and 5, we run our scheme with the following four sets of parameters covering the mass-super resp. -subcritical, the asymmetric case as well as the case of the initial datum being highly concentrated near the origin $v = 0$:

(P1) $m > m_c : \sigma = 0.7,\ A = 4.5,\ T = 0.4,\ \tau = 0.001,\ n = 2001$ ($n :=$ number of spatial grid points).

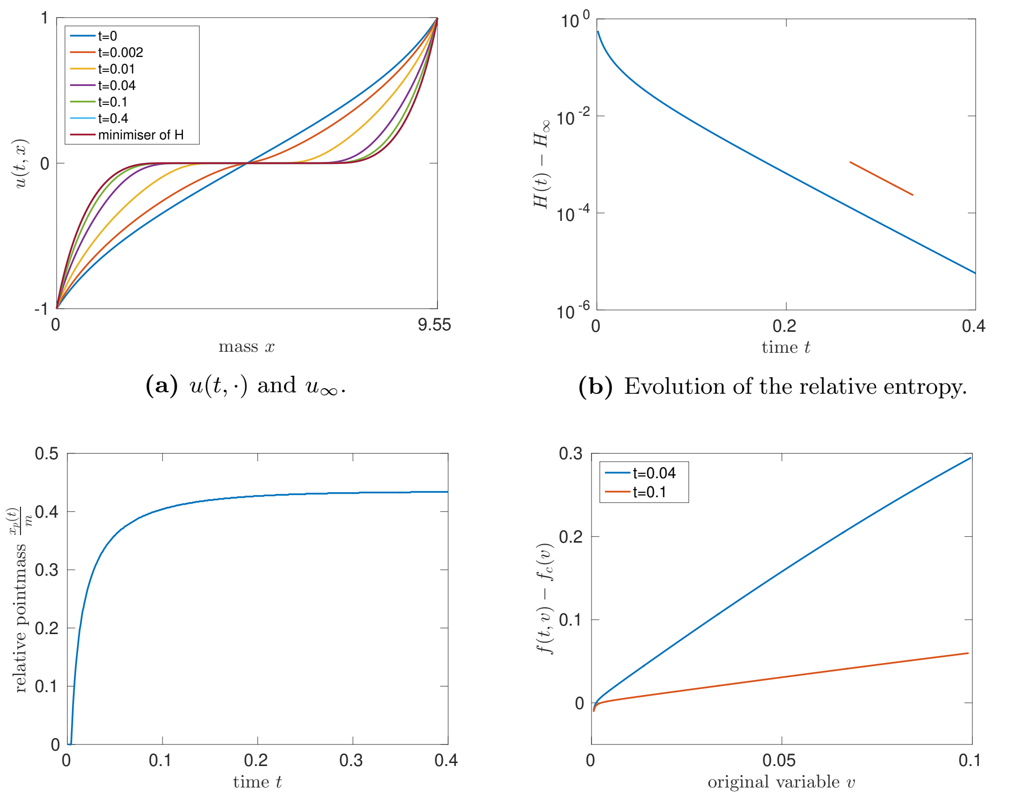
(P2) Asymmetric & $m > m_c$: translated Gaussian $f_0(v) = A\mathrm{e}^{-|v-v_0|^2/(2\sigma^2)} + 0.1$ chosen as initial datum using the parameters $v_0 = -1$, $\sigma = 0.7$ and $A = 4.5$. Moreover, $T = 0.4$, $\tau = 0.001$, $n = 2001$. The shift by $+0.1$ ensures that the cdf of f_0 is numerically still well invertible close to $v = R_1$.

(P3) $m < m_c : \sigma = 0.7,\ A = 1.5,\ T = 0.4,\ \tau = 0.001,\ n = 2001$.

(P4) Concentrated & $m < m_c$: $\sigma = 0.1$, $A = 1.5$, $T = 0.4$, $\tau = 10^{-6}$, $n = 10001$.

The approximate total mass for each of these simulations is indicated in part (a) of the corresponding figure: it is the maximal value of the part of the horizontal axis which is displayed.

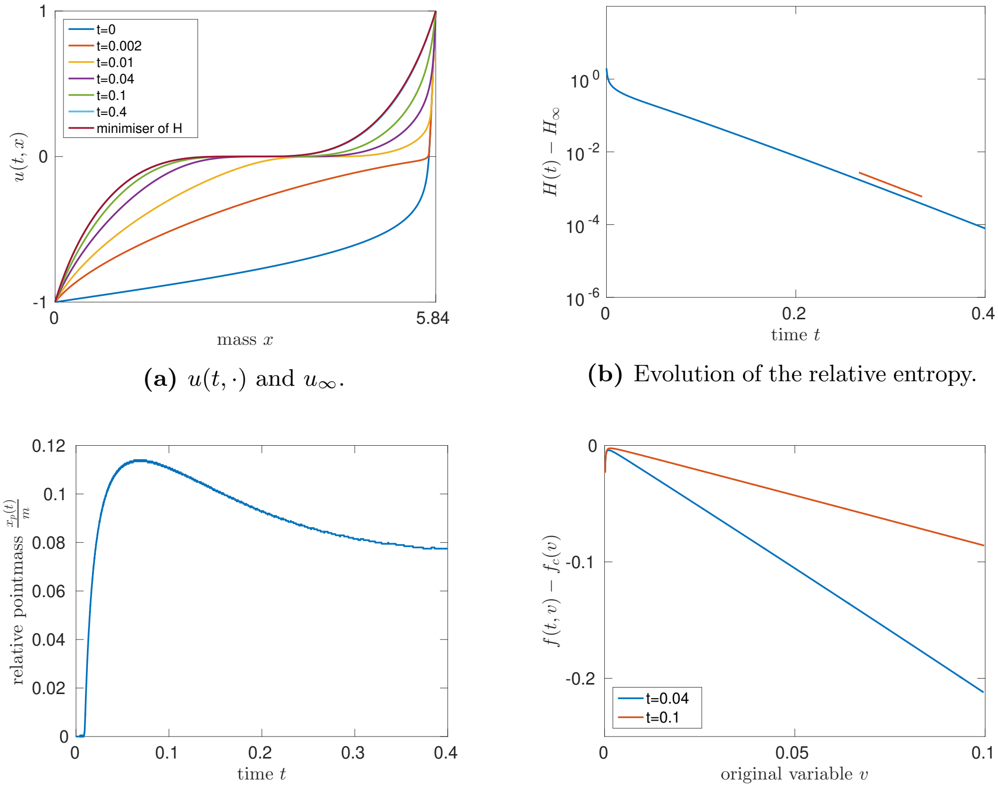
Entropy decay. The convergence to the minimiser of the entropy can be clearly observed in Figures 6.1a and 6.2a. Beyond, Figures 6.1b, 6.2b, 6.3c and 6.3d, which show the evolution of the relative entropy $H(u(t)) - H_\infty$, indicate an exponential decay of the entropy. In the mass-subcritical case (Figures 6.3c and 6.3d), the exponential decay qualitatively confirms the result in Proposition 5.9. In the mass-supercritical case, however, where solutions eventually have a condensate component, no theoretical results have been established regarding the decay rate of the entropy. The red slopes in Figures 6.1b, 6.2b, 6.3c and 6.3d indicate the approximate slopes of the graphs averaged over the intervals where they are plotted. The computed slopes imply quantitative decay rates for the entropy of the form $e^{-\alpha t}$ with the following numerical values for α: $\alpha \approx 23.7$ for (P1), $\alpha \approx 23.8$ for (P3), $\alpha \approx 23.1$ for (P4), and $\alpha \approx 23.0$ for (P2).



(a) $u(t,\cdot)$ and u_∞. **(b)** Evolution of the relative entropy.

(c) Evolution of the Dirac part. **(d)** Behaviour near singularity.

Figure 6.1: Long-time behaviour in the mass-supercritical case (P1) ($d = 1, \gamma = 2.9$).



(a) $u(t,\cdot)$ and u_∞.

(b) Evolution of the relative entropy.

(c) Evolution of the Dirac part.

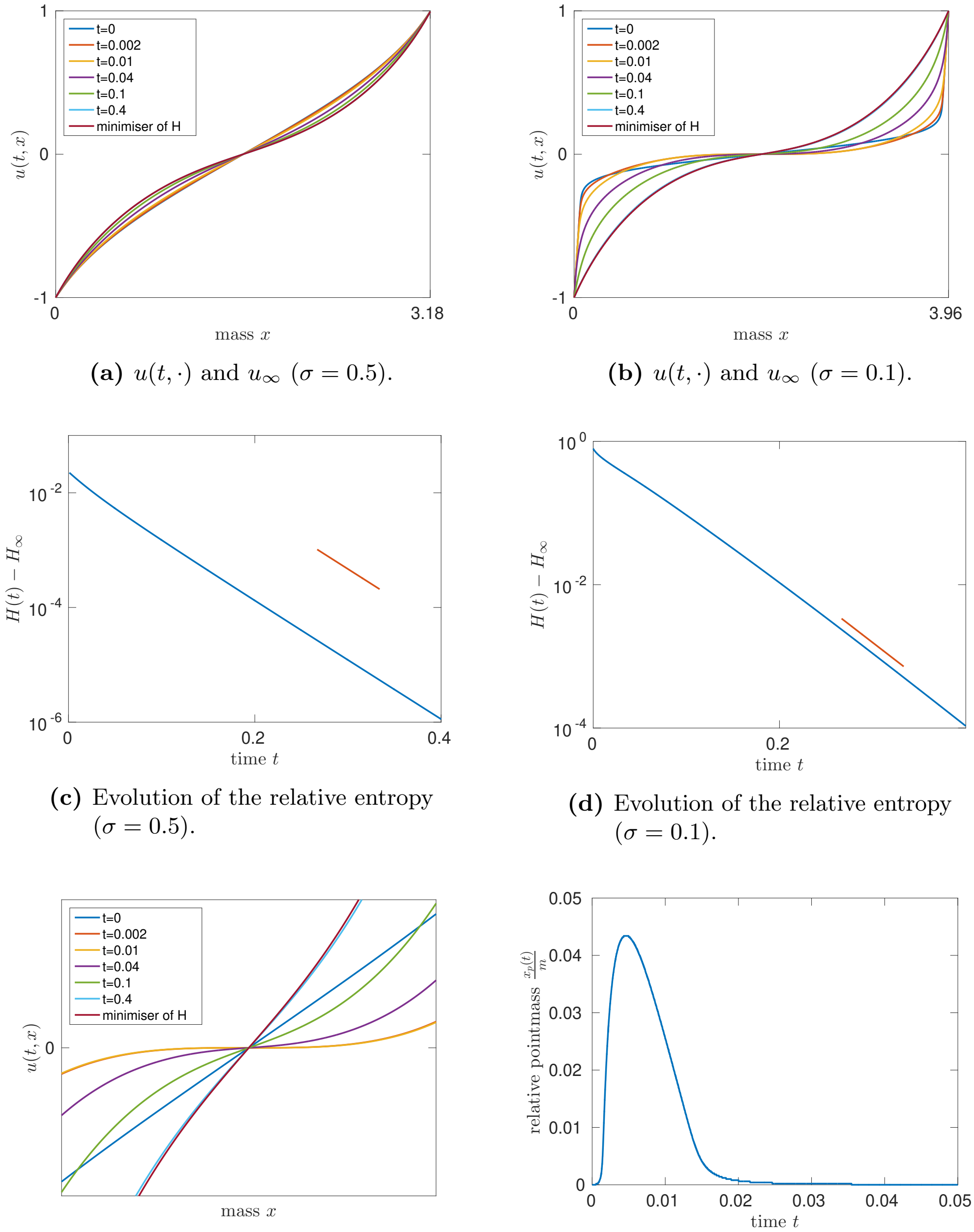
(d) Behaviour near singularity.

Figure 6.2: Long-time behaviour for asymmetric mass-supercritical datum (P2) ($d = 1, \gamma = 2.9$).

Finite-time condensation for $m > m_c$. The finite-time condensation in the mass-supercritical case is well confirmed by simulations (P1)&(P2). Recall that the condensate corresponds to the zero level set of $u(t,\cdot)$, which we numerically determine by the criterion $|u(t,\cdot)| < 10^{-6}$. Figure 6.1c shows the time evolution of the condensed part relative to the (conserved) total mass. It clearly shows the onset of a condensate after some time $0 < t \ll 0.025$. Further figures illustrating the formation of condensates are Fig. 6.1a, 6.2a and 6.2c. Interestingly, in Figure 6.2c the fraction of mass in the condensate is not monotonic, illustrating that, even when above the critical mass, a previously formed condensate may partially dissolve.

Blow-up profile. Figures 6.1d and 6.2d show the behaviour of $f(t,v) - f_c(v)$ for $0 < v \ll R_1$ at the times $t = 0.04$ and $t = 0.1$. The figures indicate an error of the form

$$f(t,v) - f_c(v) = c_\pm(t)|v| + o(|v|) \quad \text{as } v \to 0\pm \tag{6.13}$$



(a) $u(t,\cdot)$ and u_∞ ($\sigma = 0.5$).

(b) $u(t,\cdot)$ and u_∞ ($\sigma = 0.1$).

(c) Evolution of the relative entropy ($\sigma = 0.5$).

(d) Evolution of the relative entropy ($\sigma = 0.1$).

(e) Zoomed-in view of Fig. 6.3b.

(f) Dirac mass ($\sigma = 0.1$).

Figure 6.3: The mass-subcritical cases (P3) and (P4), $d = 1, \gamma = 2.9, A = 1.5$.

for suitable constants $c_+(t), c_-(t) \in \mathbb{R}$, which, for asymmetric solutions, need not necessarily coincide. The asymptotic behaviour in equation (6.13) not only confirms the leading order spatial profile obtained rigorously in Proposition 4.12 (see eq. (4.31)), but also corroborates the improved control (5.25) of the error with respect to f_c, established in Section 5.3. Let us also mention that in both figures the solution $u(t,\cdot)$ is not uniformly close to u_∞, so that the asymptotic behaviour of the density near the origin at the chosen times cannot merely be due to the fact that the long-time limit of the density equals f_c.

Transient condensates. In Figure 6.3 the behaviour of a mass-subcritical, but initially very concentrated solution is compared to the solution emanating from a more spread out datum. In both cases the entropy decays exponentially. Observe that in the case of high concentration, the solution forms a condensate in finite time which eventually vanishes again. We refer to this phenomenon, rigorously observed in Proposition 5.1 and Corollary 5.2, as a *transient condensate*. The simulations based on (P4) illustrate very explicitly how, after some finite time, the function $u(t,\cdot)$ begins to form a flat part at the horizontal axis, which eventually disappears again as the solution converges to the smooth, non-degenerate equilibrium (cf. Figure 6.3e).

6.4 Validating KQ by means of explicit solutions in 2D

As reviewed in Section 2.2, for $d = 2$ the KQ model is L^1-critical, and solutions at any level of mass are globally regular. Furthermore, KQ in its isotropic form can be transformed in an explicit way to a linear Fokker–Planck equation, whose solutions are explicit by means of the fundamental solution for this problem in $\mathbb{R}^2$ [20]. Here, we will use these explicit solutions to validate the proposed numerical scheme for KQ. Since all simulations are performed on a finite domain with zero flux boundary condition, the solutions to KQ obtained upon this transformation are only approximations of the exact solutions to our problem. However, we obtain a good approximation of the solutions in $B(0, R_1) \subset \mathbb{R}^2$ with zero flux provided R_1 is chosen sufficiently large. This is due to the fact that the exact solutions in $\mathbb{R}^2$ emanating from the chosen initial data (Gaussians) have exponential decay in $|v|$. The same is true for their derivative with respect to v, implying that on the boundary $\partial B(0, R_1)$ of a centred ball of large enough radius $R_1 \gg 1$ the flux is negligible. Hence, the exact solutions on $\mathbb{R}^2$ restricted to $B(0, R_1)$ are close to the exact solutions on $B(0, R_1)$ with zero flux.

Let us recall the transformation leading to the explicit formula of solutions on the whole space, as observed in [20]: the solutions of the linear Fokker–Planck equation

$$\begin{aligned} \partial_t h &= \Delta h + \operatorname{div}(vh), \quad t > 0, v \in \mathbb{R}^2, \\ h(0,\cdot) &= h_0 \end{aligned} \tag{6.14}$$

are given by means of the fundamental solution

$$F(t,v,w) = a(t)^{-1} K_{b(t)}(a(t)^{-1/2} v - w),$$

where $a(t) = \mathrm{e}^{-2t}$, $b(t) = \mathrm{e}^{2t} - 1$, and $K_b(z) = (2\pi b)^{-1} \mathrm{e}^{-|z|^2/2b}$. More precisely, (for sufficiently regular data h_0) the solution of equation (6.14) takes the form

$$h(t,v) = \int_{\mathbb{R}^2} F(t,v,w) h_0(w) \, \mathrm{d}w. \tag{6.15}$$

The relation between non-negative, isotropic solutions f of 2D KQ and non-negative, isotropic solutions h of eq. (6.14) is given by

$$f(t,v) = \frac{h(t,v)}{1 + \bar{M}_h(t,|v|)} \quad \text{resp.} \quad h(t,v) = f(t,v) \mathrm{e}^{\bar{M}_f(t,|v|)}, \tag{6.16}$$

where

$$\bar{M}_f(t,\rho) = \frac{1}{2\pi} \int_{\{|v| \leq \rho\}} f(t,w) \, \mathrm{d}w = \int_0^\rho g(t,r) r \, \mathrm{d}r.$$

We initialise our tests again with a centred Gaussian of the form

$$f_0(v) = A \mathrm{e}^{-\frac{|v|^2}{2\sigma^2}}$$

for fixed positive constants A and σ. Then the initial datum h_0 corresponding to f_0 via the transformation (6.16) is given by

$$h_0(v) = A \mathrm{e}^{-\frac{|v|^2}{2\sigma^2}} \mathrm{e}^{A\sigma^2 \left(1 - \mathrm{e}^{-\frac{|v|^2}{2\sigma^2}}\right)},$$

and from formula (6.15) and relation (6.16) we infer an expression for the solution f, which shows, in particular, that $f(T,\cdot)$ has exponential decay for any positive time T.

Details on the tests. We choose $R_1 > 0$ to be the smallest radius satisfying $f_c(v) \leq 10^{-4}$ for $|v| \geq R_1$. This guarantees that for any not too large $\sigma > 0$, the function $f(t,\cdot)$ is small outside $B(0,R_1)$.

Two different tests are performed using the following common set of parameters: $A = 4$, $\sigma = 0.9$, final time $T = 0.04$ and size of the coarsest mesh equal to $n_0 = 25$. Since the solution to the exact problem remains bounded, the tests are performed with $\varepsilon = \delta = 0$.

In the first test the dependence of the L^2 distance at time T between exact and computed solution for different spatial resolutions is analysed. More precisely, for $j = 0, \dots, N = 5$ we compute the error

$$E_j = \|S^{(j)}(T, \cdot) - S^{(j)}_{\text{exact}}(T, \cdot)\|_{l^2(J_j)} \cdot 2^{-j},$$

where J_j denotes the discrete mesh using a total number of $2^j n_0 + 1$ mesh points intersected with the interval $[0, m/2]$, $S^{(j)}_{\text{exact}}$ denotes the exact solution restricted to the spatial mesh J_j and $S^{(j)}$ the discrete solution computed on the mesh J_j using a total number of 400 time steps. Since we expect a polynomial dependence of the error on the spatial increment, we then let $\text{rate}(j) = \log_2(E_j/E_{j+1})$. The results of the test can be found in Table 6.3. Theoretically, since in the present case of two space dimensions the original density f remains uniformly bounded in time, which implies that $\partial_z S$ stays away from zero, the spatial discretisation based on central differences should guarantee a quadratic dependence of the truncation error on the spatial increment. The rates displayed in Table 6.3 are somewhat worse, possibly due to the fact that the mesh size has not been chosen sufficiently large to capture the asymptotic behaviour well enough.

In the second test we analyse the dependence of the L^2 space-time distance between exact and computed solution on the number of spatial and temporal grid points. The procedure is analogous to the first test except that the j-th mesh is obtained by using $2^j n_0 + 1$ spatial and $2^j m_0$ temporal grid points, where $m_0 = 4$, and that now the error is given by

$$E_j = \|S^{(j)} - S^{(j)}_{\text{exact}}\|_{l^2(I_j \times J_j)} \cdot 2^{-2j},$$

where I_j denotes the discrete temporal mesh consisting of $2^j m_0$ time points. The results are displayed in Table 6.4 and suggest a linear rate of convergence. This is in line with the backward Euler scheme used for the time stepping.

timesteps	meshsize	L^2_z error	rate
4000	25	6.2783e-3	-
4000	50	2.2323e-3	1.4919
4000	100	7.9661e-4	1.4866
4000	200	2.6080e-4	1.6109
4000	400	7.7921e-5	1.7428
4000	800	1.9283e-5	2.0147

Table 6.3: Convergence to exact solution at time $T = 0.04$.

timesteps	meshsize	$L^2_{t,z}$ error	rate
4	25	8.3850e-4	-
8	50	4.1295e-4	1.0218
16	100	2.0813e-4	0.9885
32	200	1.0427e-4	0.9971
64	400	5.1996e-5	1.0039
128	800	2.5774e-5	1.0125

Table 6.4: Convergence to reference solution (on space-time grid).

Remark 6.3 (Validation of regularisation). For completeness, we also tested the dependence of the computed solution on the regularisation parameters ε and δ, even though this is not necessary for 2D KQ since the density is theoretically known to remain bounded. We obtained a polynomial decrease of the error.

6.5 Simulations of 3D KQ in radial coordinates

Here, we simulate equation (6.7) with $d = 3$ for suitable choices of ε, δ, $0 < \varepsilon, \delta \ll 1$, where we choose $R_1 = 1$. We recall our notation $\bar{m}_c = \frac{1}{|\partial B(0,1)|} \int_{B(0,R_1)} f_c(v)\, \mathrm{d}v$, where now $|\partial B(0,1)| = 4\pi$ denotes the area of the 2-sphere, and remark that the numerical value of $\bar{m}_c$ is approximately given by $\bar{m}_c \approx 1.84$. We perform three simulations with a mass-supercritical, a mass-subcritical and a highly concentrated initial datum, respectively. More precisely, choosing as initial data again Gaussians of the form $f_0(v) = A\mathrm{e}^{-|v|^2/(2\sigma)}$, we run our scheme with the following three sets of parameters:

(P5) $m < m_c : \sigma = 0.3$, $A = 3$, $T = 0.2$, $\tau = 0.001$, $n = 2001$, $\varepsilon = 0$, $\delta = 0$.

(P6) $m > m_c : \sigma = 0.9$, $A = 10$, $T = 0.25$, $\tau = 5 \cdot 10^{-6}$, $n = 50001$, $\varepsilon = 10^{-12}$, $\delta = 0$.

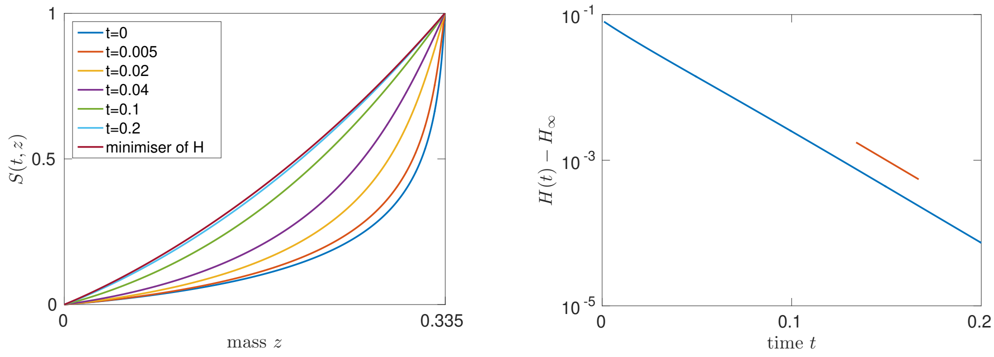
(P7) $m < m_c : \sigma = 0.15$, $A = 50$, $T = 0.25$, $\tau = 5 \cdot 10^{-5}$, $n = 2001$, $\varepsilon = 10^{-10}$, $\delta = 10^{-10}$.

The quantity $\bar{m} := m/|\partial B(0,1)|$ associated with the above choice of parameters takes the value $\bar{m} \approx 0.335$ for (P5), $\bar{m} \approx 2.59$ for (P6), and $\bar{m} \approx 1.41$ for (P7) (see Figures 6.4a, 6.5a and 6.6a).

The size of the condensate divided by $|\partial B(0,1)|$, i.e. $\bar{x}_p(t) := \mathcal{L}^1(\{S(t,\cdot) = 0\})$, is numerically determined by replacing the condition $S(t,\cdot) = 0$ with the smallness criterion $S(t,\cdot) < 10^{-10}$.

Remark 6.4. The choice of the comparatively fine mesh in (P6) was made in order to ensure a sufficiently good approximation of the evolution of the entropy. See Fig. 6.5b, which suggests an exponential decay.

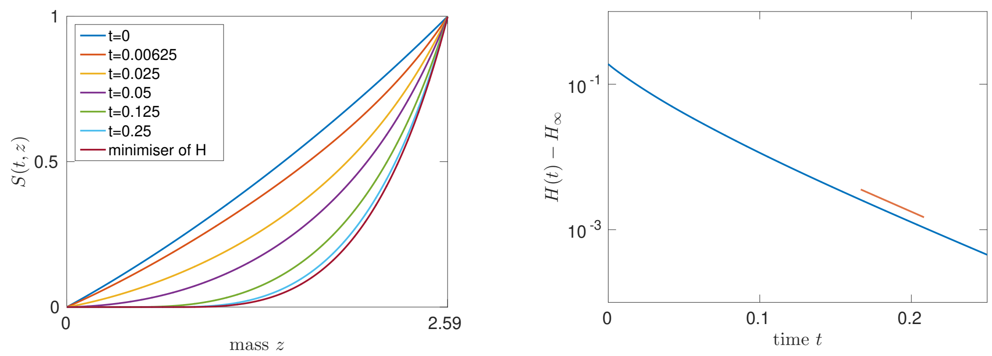
Long-time behaviour. Our simulations suggest that 3D KQ has properties which are qualitatively similar to the bosonic Fokker–Planck equations in 1D in the L^1-supercritical regime. Figures 6.4a, 6.5a and 6.6a suggest that in the long-time limit the numerical solution $S(t,\cdot)$ approximates the minimiser S_∞ of the entropy (at the level of S). Next, the decay of the relative entropy appears to be exponential in all three cases (P5)–(P7), see Figures 6.4b, 6.5b and 6.6c. In each of these plots, the red slope indicates the approximate slope of the graph averaged over the interval where it is plotted. Numerically, the relative entropy $H(t) - H_\infty$ appears to decay to zero like $e^{-\alpha t}$, where $\alpha \approx 35.3$ for (P5), $\alpha \approx 21.1$ for (P6), and $\alpha \approx 21.7$ for (P7).



(a) $S(t,\cdot)$ and S_∞.

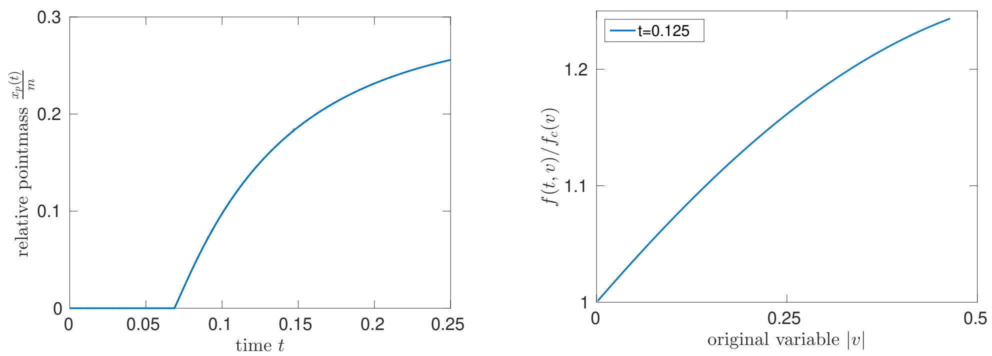
(b) Evolution of the relative entropy.

Figure 6.4: Long-time behaviour in mass-subcritical case (P5) ($\gamma = 1, d = 3$).



(a) $S(t,\cdot)$ and S_∞.

(b) Evolution of the relative entropy.



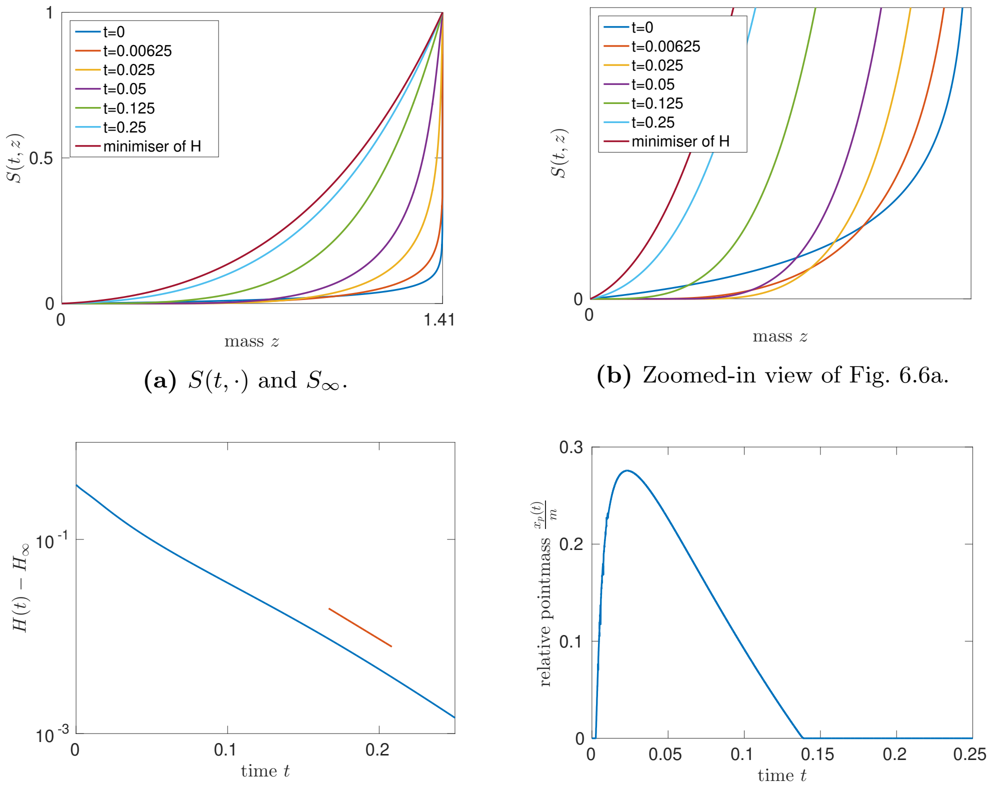
(c) Evolution of the Dirac part.

(d) Behaviour near singularity.

Figure 6.5: Long-time behaviour in the mass-supercritical case (P6) ($d = 3, \gamma = 1, \varepsilon = 10^{-12}, \delta = 0$).

Condensation. In both the mass-supercritical case (P6) and the case of high concentration near the origin (P7) we observe the onset of a flat part at the level of $S(t,\cdot)$ at height zero after some finite time, see Fig. 6.5c and 6.6d. In the original variables this means that mass is gradually absorbed by the origin. Furthermore,

Fig. 6.6d shows that, similarly to the observations in 1D (see Section 6.3), it is possible for mass previously concentrated at velocity zero to escape. In fact, the condensate component may even dissolve completely. Thus, at least in our numerical simulations, the fraction of particles in the condensate is, in general, not monotonic in time for the 3D Kaniadakis–Quarati model.



(a) $S(t,\cdot)$ and S_∞.

(b) Zoomed-in view of Fig. 6.6a.

(c) Evolution of the relative entropy.

(d) Evolution of the Dirac part.

Figure 6.6: Transient condensate in the mass-subcritical case (P7) ($d = 3, \gamma = 1, \varepsilon = \delta = 10^{-10}$).

Blow-up profile. At times where the solution has a non-trivial condensate component, we were interested in the spatial behaviour of $S(t,\cdot)$ close to $\{S(t,\cdot) = 0\}$. Owing to the results on the 1D model, one may expect the function $f(t,\cdot)$ to behave to leading order like the limiting steady state f_c, i.e. like $2|v|^{-2}$. Furthermore, the formal expansions in [96, Section III.C] suggest that for isotropic solutions of 3D KQ the error by which $f(t,\cdot)$ deviates from f_c has the form

$$f(t,v) - f_c(v) = c(t)|v|^{-1} + o(|v|^{-1}) \qquad (6.17)$$

for some constant $c(t) \in \mathbb{R}$. Our experiments corroborate formula (6.17). Indeed, Figures 6.5d and 6.7 displaying the quantity $f(t,v)/f_c(v)$ at times where $f(t,\cdot)$ is unbounded at the origin show that numerically it behaves like $1 + \tilde{c}(t)|v| + o(|v|)$ as $|v| \to 0$. Notice that in these figures the magnitude and sign of $\tilde{c}(t)$ is linked to the slope of $x_p(t)$ (see Fig. 6.5d & Fig. 6.5c and Fig. 6.7 & Fig.6.6d).

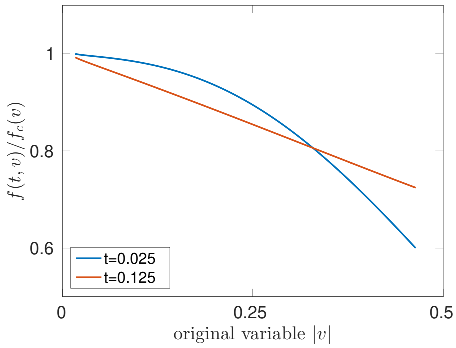


Figure 6.7: Spatial blow-up profile in (P7).

Remark 6.5. In order to produce the transient condensate in Figure 6.6, it was necessary to choose the parameter δ appearing in equations (6.7) and (6.11) strictly positive. The same simulation for $\delta = 0$ results in the flat part being trapped at height zero once it has formed. As explained in Section 6.2.1 and also in view of our results for the 1D model, this 'stickiness' appears to be a numerical artefact resulting from the circumstance that a regularisation based on a positive ε but vanishing δ is imbalanced and favours condensation.

Chapter 7

Conclusion (Part I)

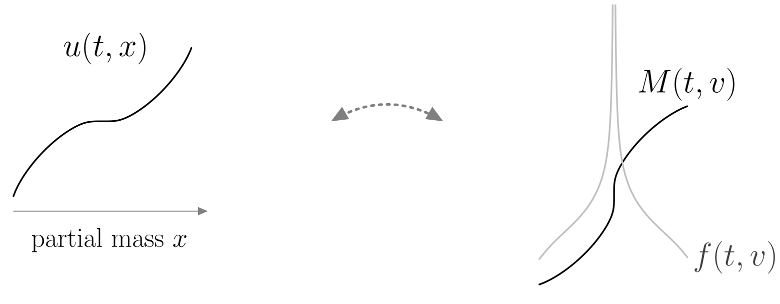
Part I of this thesis establishes a framework able to deal with singularities and Dirac measures at the origin in the one-dimensional case of the family of L^1-supercritical bosonic Fokker–Planck equations

$$\begin{aligned} \partial_t f &= \Delta_v f + \mathrm{div}_v(vf(1+f^\gamma)), \quad t>0,\ v\in\mathbb{R}^d, \qquad (\gamma > 2/d) \\ f(0,\cdot) &= f_0 \geq 0. \end{aligned} \tag{7.1}$$

The approach is based on the following reformulation of the 1D equations in terms of the pseudo-inverse distribution function u:

$$\begin{aligned} (\partial_x u)^\gamma \partial_t u - (\partial_x u)^{\gamma-2}\partial_x^2 u + u(1+(\partial_x u)^\gamma) = 0, \quad t>0,\ x\in(0,m), \\ u(0,\cdot) = u_0 \qquad (\partial_x u_0 \geq 0). \end{aligned} \tag{7.2}$$

This reformulation is motivated by the formal gradient flow structure of eq. (7.1) described in Section 2.1. The relation between the function $u(t,\cdot)$, its generalised inverse $M(t,\cdot)$ and the density $f(t,\cdot)$ of the absolutely continuous part of the measure associated with $M(t,\cdot)$ is sketched in the following graphics



indicating the advantage of formulation (7.2) over equation (7.1) when trying to make sense of singular solutions and Dirac measures (at the level of f).

A core ingredient in the framework established is a comparison principle for equations of the form (7.2). It applies to the general class of equations (3.1) under the monotonicity assumptions (A0) and (A1) and is derived using a maximum principle for semicontinuous functions developed in the literature on viscosity solutions for 2^{nd} order equations. Apart from providing uniqueness and ensuring the existence of (*continuous*) viscosity solutions, the comparison principle and its versions and consequences are essential in several arguments regarding regularity and control of solutions. In these arguments, comparison tools are typically applied in conjunction with another important feature of equation (7.2): the availability of a large class of sufficiently regular time-independent (sub- and super-) solutions. Acting as barriers, the latter naturally provide a family of a priori bounds. As we have seen in Section 4.3, such estimates are in general not restricted to 0^{th} order quantities.

Discussions

One of the main reasons for our choice to develop the wellposedness theory in the new variables lies in the fact that the entropy minimiser is a viscosity solution of equation (7.2), whereas in the original variables for mass larger than the critical one it is a measure with non-trivial singular component, not admitted in formulation (7.1). This provides a natural justification for our change of variables. Furthermore, as seen in Section 5.3, 'solutions' obtained by an approximation procedure in the original variables preserving the Fokker–Planck structure can typically be shown to be viscosity solutions themselves and thus, by uniqueness, must coincide with the solutions proposed in the thesis. We should mention that, while the regularity of the constructed viscosity solutions enables us to deduce that the corresponding measure in the original variables is regular in $\{r \neq 0\}$, where its density satisfies the PDE (7.1) in the classical sense, this thesis does not provide a comprehensive investigation of the question of wellposedness of the evolutionary problem for the reconstructed measure. The law governing the evolution of the point mass at the origin, assuming mass conservation and knowledge of the density, is described in Section 5.3. Disregarding regularity issues, it is a differential equation determining the growth of the point mass by the flux of mass (positive or negative) of the density into the origin. At the same time, the presence of a positive point mass at the origin precludes instantaneous regularisation of the density, and thus, owing to the profile in Proposition 4.12 (i), acts, to some extent, as a boundary condition for the density at $r = 0$. This informally describes the coupling between the evolution of the regular part of the reconstructed measure and the singular component.

One of the leading questions motivating our study of the nonlinear Fokker–

Planck equations (7.1) concerns the long-time asymptotics in the mass-supercritical case as well as the possibility and nature of finite-time singularities. Generally speaking, the results in Sections 4.2 to 4.4 and Chapter 5 provide a fairly comprehensive understanding of the long-time behaviour and singularities of the solutions considered in this thesis. In particular, the short-time regularity of solutions emanating from admissible initial data of mass larger than critical always breaks down in finite time, meaning that there is some positive time where solutions (in the original variables) blow up in L^∞. Theorem 4.16 and Corollary 4.18 further tell us that solutions relax to the entropy minimiser, and that for $m > m_c$ the reconstructed measure eventually has a singular part concentrated at the origin. While for $m < m_c$ relaxation implies that solutions will eventually inherit the smoothness of the associated minimiser, such solutions are still able to display transient singularities and condensates (see Proposition 5.1, Corollary 5.2).

Perspectives

We would finally like to point out different directions of research which have been provoked by or could build on the ideas in Part I of this thesis.

3D Kaniadakis–Quarati model. As shown in Section 6.2.1, within the framework of isotropic solutions, the change of variables from density to inverse distribution function can be generalised to higher dimensions. In the physically most interesting case of equation (7.1), the 3D Kaniadakis–Quarati model for bosons, this ad hoc reformulation formally takes the form

$$\partial_z S \frac{1}{d}\partial_t S - d \cdot S^{2-2/d}(\partial_z S)^{-1}\partial_z^2 S + S(\partial_z S + d) = 0 \qquad (d = 3) \tag{7.3}$$

for the unknown non-negative and non-decreasing function $S(t,\cdot)$. Recall that the method for deriving comparison for the 1D problem in Section 3.3 is based on vertical displacements of the functions involved. For the 3D problem, eq. (7.3), or for the equation in any other dimension larger than $d = 1$, however, this technique fails due to the explicit dependence of the diffusion coefficient on S, which leads to a lack of monotonicity in S of the function $\widetilde{F}(S, \partial_t S, \partial_z S, \partial_z^2 S)$ defining the equation. Thus, a viscosity solution theory for equation (7.3) in dimensions $d > 1$ would require devising a different or at least modified technique to establish comparison.

Anisotropic situation. Our approach to cope with Dirac measures in the L^1-supercritical bosonic Fokker–Planck equations certainly relies on the fact that the problem is effectively one-dimensional, and new ideas are required for the general

3D Kaniadakis–Quarati model. A careful numerical study of the vectorial generalisation (6.8) of equation (7.2) may provide first insights in the dynamics of anisotropic solutions. As regards global-in-time *existence* of measure-valued solutions one could try to elaborate the strategy outlined in Section 5.3, which is based on a family of regularisations in the original variables preserving the Fokker–Planck structure. This approach could work provided the change of mass in small balls can be appropriately controlled in a way which is uniform in the regularisation parameter. In the isotropic case, this control is a consequence of comparison at the level of the cumulative distribution function. Related approaches have been used to continue solutions beyond singularity formation in other equations like the parabolic–elliptic Keller–Segel model [41, 86, 102, 103], but also in the physically more closely related Boltzmann–Nordheim/Uehling–Uhlenbeck equation in its isotropic form (see page 16). Notice, however, that this method does not tackle the question of uniqueness, and different ways of regularisation could, in principle, give rise to different limiting solutions, as it is the case for Keller–Segel [41, 86]. On the other hand, the biological motivation of Keller–Segel does not appear to provide a good reason to expect uniqueness of continuation in this model. In our Fokker–Planck model for Bose–Einstein particles (and also in the Boltzmann–Nordheim equation) the question of continuation is motivated by the structure of the entropy minimisers and the link to quantum physics. Owing to the uniqueness results obtained for our 1D toy model, the availability of a unique continuation in the 3D Kaniadakis–Quarati model for bosons (under certain physical constraints) would be interesting to study. Let us mention that for the Boltzmann–Nordheim equation the question of uniqueness is still open.

Inhomogeneous problem in phase space. Another interesting direction of research could be the study of problem (7.1) generalised to the case of a system of Bose particles which is not homogeneous in the position variables. The resulting equation is a nonlinear kinetic Fokker–Planck-type equation describing the evolution of the particle density function $f(t, \cdot, \cdot) = f(t, v, x)$ in phase space. Existing literature on this equation is confined to a stability analysis of the *smooth* steady states [87, 89], thus a priori disregarding the case in which condensates may be expected to form.

Part II

Aggregation equations with fractional diffusion

Chapter 8

Aggregation equations with fractional diffusion: preventing explosions by mixing

In this chapter we investigate a class of aggregation-diffusion equations on the torus $\mathbb{T}^d$ with singular kernels and fractional (anomalous) dissipation in the presence of an incompressible stationary flow. Without the flow the equations are L^1-supercritical, and solutions emanating from large initial data may explode in finite time. We will show that under certain spectral conditions on the flow, which guarantee good mixing properties, the corresponding initial value problem has globally regular solutions if the coupling parameter regulating the strength of the flow is sufficiently large. We will further see that for fast enough flows the global solutions approach exponentially fast, at arbitrarily large rate, their trivial equilibrium state on $\mathbb{T}^d$.

8.1 Introduction

We are interested in the question of how the presence of a (prescribed, steady) incompressible flow may alter the long-time dynamics of solutions of a class of aggregation equations with singular kernels and fractional dissipation. More specifically, our starting point is the evolutionary problem

$$\partial_t \rho = -\Lambda^\gamma \rho + \nabla \cdot (\rho \nabla K * \rho) \quad \text{in} \quad (0, \infty) \times \mathbb{T}^d \tag{8.1}$$

subject to an initial condition $\rho(0) = \rho_0$ for some suitably regular density $\rho_0 \geq 0$. Here Λ denotes the half-Laplacian on $\mathbb{T}^d$ (see (8.6)), where $\mathbb{T}^d$ is the flat d-torus — henceforth identified with $[-\frac{1}{2}, \frac{1}{2})^d$ subject to periodic boundary conditions. To avoid short-time regularity issues (which will become clear later), the positive parameter

$\gamma > 0$ is for simplicity usually assumed to be larger than 1. The periodic convolution kernel K is assumed to have the following properties:

- Smoothness away from the origin.
- $\nabla K(x) \sim \frac{x}{|x|^{2+a}}$ near $x = 0$ for some $a \geq 0$. This is the case if $-K \sim |x|^{-a}$ in some neighbourhood of the origin (with the understanding $K \sim \log |x|$ if $a = 0$). For simplicity, we will assume that there exists $0 < \varepsilon \ll 1$ such that

$$\nabla K(x) = \frac{x}{|x|^{2+a}} \text{ on } B_\varepsilon(0).$$

The behaviour of the kernel near its singularity at the origin (including its sign) determines the short-range interaction modelled by the nonlinear term in (8.1). Our choice of the sign guarantees a predominantly attractive interaction and is essential for the construction of exploding solutions. Next, notice that for $a = d - 2$ the kernel K has the same singularity at the origin as the fundamental solution of the Laplacian on $\mathbb{T}^d$ so that, informally speaking, in this case equation (8.1) becomes a version of the fractional (or classical if $\gamma = 2$) parabolic-elliptic Keller–Segel system, which is one of the fundamental models for aggregation in several physical and biological systems, and in particular for chemotaxis, see e.g. [17, 57, 59, 60]. In this sense our model is a generalisation of Keller–Segel and, indeed, virtually the same analysis as in this paper can be used to give a direct derivation of the corresponding results for Keller–Segel. Let us also point out that for $a = 0$ we essentially recover a version of the so-called *modified* Keller–Segel model [18, 22].

The motive to allow for fractional diffusion in our model is two-fold: besides experimental evidence suggesting that in certain applications the repulsive forces may be better described by fractional rather than standard diffusion (see e.g. [2, 10, 56] and references therein), another reason to consider the more general case of fractional diffusion is the quest for a better understanding of how the equation's dynamics depends on the nature of diffusion. The mathematical literature on models for aggregation with fractional dissipation is large, see [10, 14–16, 50, 77–79] for a small selection.

One reason for our choice of periodic boundary conditions lies in the fact that in this setting chaotic dynamics generated by a time-independent flow are possible already in the physically particularly relevant case of two spatial dimensions (see Section 8.4 and Appendix 8.5.4 for more details). Let us, however, also mention that time-independence of the flow is not an essential hypothesis in our estimates.

In order to describe our results, we first need to introduce some fundamental properties of equation (8.1).

Conservation of mean. Formally, for any solution to equation (8.1) the mean value is conserved in time:

$$\int_{\mathbb{T}^d} \rho(t,x)\,\mathrm{d}x = \int_{\mathbb{T}^d} \rho_0(x)\,\mathrm{d}x.$$

All evolution equations which we shall consider here enjoy this property, and in this context we will abbreviate $\bar{\rho} = \int_{\mathbb{T}^d} \rho_0$. In applications ρ usually describes a density, and for the sake of exposition, we will henceforth assume that $\rho_0 \geq 0$, a property, which by the maximum principle (see e.g. [79] for a proof in a related setting) is preserved in time for any sufficiently regular solution to (8.1). It will, however, be obvious that (apart from the blow-up proof in Appendix 8.5.1) our results remain valid without the assumption of positivity.

Scaling. Let us for the moment replace $\mathbb{T}^d$ by $\mathbb{R}^d$ and consider the scaling properties of the equation obtained by substituting in (8.1) the kernel ∇K for its homogeneous approximation near the origin, i.e. $\frac{x}{|x|^{2+a}}$. This equation is invariant under the scaling

$$\rho_\lambda(t,x) = \lambda^{\gamma-2+d-a}\rho(\lambda^\gamma t, \lambda x), \quad \lambda > 0. \tag{8.2}$$

Moreover, by preservation of mean, non-negative solutions have conserved L^1_x-norm. Thus, the exponent $\gamma = \gamma_c$ which leaves the L^1_x-norm of the rescaled solutions ρ_λ invariant in the sense that $\|\rho_\lambda(t,\cdot)\|_{L^1} = \|\rho(\lambda^\gamma t,\cdot)\|_{L^1}$ plays a distinguished role and is generally referred to as the L^1-*critical* exponent. From (8.2) we obtain

$$\gamma_c = 2 + a.$$

Consistent with the terminology introduced in Section 1.1 (page 2), for $\gamma < \gamma_c$ (resp. $\gamma > \gamma_c$) equation (8.1) is called L^1-supercritical (resp. L^1-subcritical). In the case $a = 0$ and $\gamma \in (1,2]$ (which implies $\gamma \leq \gamma_c$) it is not difficult to produce solutions exploding in finite time using a virial type argument similar to the strategy in [73, Appendix I]. This reflects the above scaling heuristics: simplistically speaking, in the L^1-supercritical regime, the regularising effect of diffusion should be too weak to be generically able to compete with the aggregation effects induced by the quadratic drift term in (8.1) with velocity $-\nabla K * \rho$. One would therefore also expect the existence of exploding solutions for more singular kernels ($a > 0$), as proved in the case of the whole space [14]. On $\mathbb{T}^d$ this may require choosing a modified weight since in the standard virial argument, based on a (localised) moment, the arising ‘perturbation’ terms

$$\int\int \frac{x-y}{|x-y|^{a+2}} \cdot \Psi(x,y)\rho(x)\rho(y)\,\mathrm{d}y\mathrm{d}x$$

(with Ψ being some smooth cut-off which, in general, does not vanish along the diagonal) can no longer be controlled only in terms of the (conserved) mass $\int \rho$.

Background and results. One of our main goals (cf. Theorem 8.13) is to show that there exists an exponent $\gamma_0 < \gamma_c$ such that local explosions of the density can be suppressed through the action of a suitable fast flow with good mixing properties whenever $\gamma \in (\gamma_0, \gamma_c]$. This question is motivated by the work of Kiselev and Xu [73], where the authors prove a similar statement for the two- and three-dimensional parabolic-elliptic Keller–Segel model. Let us stress that in the arguably more realistic setting of a coupled chemotaxis-fluid system there does not appear to be any result in the literature proving global-in-time regularity for a model in which the existence of exploding solutions in the absence of the fluid would be known.

The class of flows we focus on is a generalisation of weakly mixing flows in the ergodic sense, and a natural adaptation of the class of *relaxation enhancing* flows considered in [73] to the case of fractional dissipation. The notion of relaxation enhancing flows was introduced in the work [34] by Constantin, Kiselev, Ryzhik and Zlatoš, which constitutes a core reference for our approach. We refer to Section 1.1.2 for an informal description of the mixing effect and enhanced dissipation in diffusive equations. For more background on fluid mixing and its possibly regularising effects in the context of reaction-diffusion equations, we refer to [73] and references therein. Let us also point out another interesting work [8, 9], which demonstrates that chemotactic singularity formation can also be prevented by mixing due to a fast shear flow. The underlying mixing mechanism is, however, rather different from the one considered here and is not able to suppress more than one dimension (of the Keller–Segel model, which is L^1-critical for $d = 2$ and L^1-supercritical in higher dimensions). In Theorem 8.17 we will show that the suppression mechanism by ergodic type mixing has a much weaker dimensional dependence in the sense that it applies to the Keller–Segel model in arbitrarily high dimension.

We finish this section by introducing two technical assumptions on the kernel K needed in large parts of our analysis, commenting on local properties of solutions to (8.1), fixing basic notations and indicating the organisation of the rest of this chapter.

Further assumptions on K. For fixed $\varepsilon > 0$ and $p_0 > 1$ we note that

$$\int_{B_\varepsilon(0)} \frac{1}{|x|^{(1+a)p_0}}\,\mathrm{d}x = c_d \int_0^\varepsilon r^{d-1-p_0(1+a)}\,\mathrm{d}r,$$

which shows that $\nabla K \in L^{p_0}(\mathbb{T}^d)$ if and only if

$$p_0 < \frac{d}{1+a}. \tag{8.3}$$

In the following we will therefore assume that the parameters $d \geq 2$ (integer) and $a \geq 0$ are such that $\frac{d}{1+a} > 1$, so that in particular there always exists $p_0 > 1$ satisfying inequality (8.3).

Moreover, since we focus on L^2-methods in our first main result (cf. Footnote 1), we will assume for this part that $2 + a - \frac{d}{2} < 2$, or equivalently,

$$\frac{d}{2a} > 1. \tag{8.4}$$

This condition ensures that the lower bound $\gamma_0 = 2 + a - \frac{d}{2}$ on γ, which makes the L^2-norm formally a subcritical quantity for (8.1), is less than 2.

LWP and smoothing. If $\gamma > 1$, problem (8.1) is locally well-posed in $H^s(\mathbb{T}^d)$ for sufficiently large $s \geq s_0(d)$. More specifically, if[1]

$$\gamma > \max\left\{2 + a - d\left(1 - \frac{1}{p}\right), 1\right\}, \tag{8.5}$$

then local existence and uniqueness already hold in $L^p(\mathbb{T}^d)$. This can be shown using semigroup estimates for $-\Lambda^\gamma$ and a fixed point argument similar to [72] and [14].

Throughout this chapter we will, for simplicity, formulate auxiliary results under the assumption of a smooth initial datum ρ_0 (resp. a smooth solution). This assumption can be removed by standard arguments exploiting the fact that, as soon as condition (8.5) holds true, the smoothing effect induced by $-\Lambda^\gamma$ is strong enough to instantaneously regularise the (local) solution emanating from an L^p datum.

[1]Notice that for $\gamma = 2 + a - d\left(1 - \frac{1}{p}\right)$ the scaling (8.2) preserves the L^p_x-norm in the sense that $\|\rho_\lambda(t,\cdot)\|_{L^p} = \|\rho(\lambda^\gamma t,\cdot)\|_{L^p}$ so that the required strength of diffusion for making the L^p norm heuristically subcritical decreases with increasing p. Thus, one may expect to obtain improved lower bounds on γ by working in L^p spaces of higher integrability. In Theorem 8.17 we will illustrate that this is indeed the case using the example of the standard Keller–Segel model. In two space dimensions, for Keller–Segel type singularities ($a = d - 2$) L^2 methods work for any $\gamma > 1$, which is why we first focus on the case $p = 2$. See also the discussion in Section 8.4 (page 150) for difficulties arising in L^p.

Notations. For smooth periodic functions $f(x) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)\mathrm{e}^{2\pi\mathrm{i}x\cdot k}$ and $\sigma \in \mathbb{R}$ we define

$$\|f\|_{\dot{H}^\sigma}^2 = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{2\sigma}|\hat{f}(k)|^2$$

and

$$\|f\|_{H^\sigma}^2 = \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2\sigma}|\hat{f}(k)|^2,$$

where $\langle k\rangle = (1+|k|^2)^{\frac{1}{2}}$. The space $H^\sigma(\mathbb{T}^d)$ is defined as the completion of $C^\infty(\mathbb{T}^d)$ under the norm $\|\cdot\|_{H^\sigma}$. We next define the fractional derivative Λ^σ via

$$\Lambda^\sigma f(x) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^\sigma \hat{f}(k)\mathrm{e}^{2\pi\mathrm{i}k\cdot x}. \tag{8.6}$$

For sufficiently regular periodic functions f, g the following identities are immediate

$$\|f\|_{\dot{H}^\sigma} = \|\Lambda^\sigma f\|_{L^2},$$
$$\Lambda^\sigma(f * g) = f * \Lambda^\sigma g.$$

Moreover,

$$\int_{\mathbb{T}^d} f\Lambda^\sigma g = \int_{\mathbb{T}^d} (\Lambda^\sigma f)\, g,$$
$$\Lambda^{\sigma_1}\Lambda^{\sigma_2} f = \Lambda^{\sigma_1+\sigma_2} f.$$

Constants C or $C(\dots)$ may change from line to line and unless explicitly indicated otherwise, they are continuous and non-decreasing functions of their (non-negative) arguments. Their possible dependence on the parameters γ, a and d will usually not be indicated explicitly. For quantities $A, B \geq 0$ the notation $A \lesssim B$ means that there exists a constant $0 < C < \infty$ (which may depend on fixed parameters) such that $A \leq CB$. Furthermore, $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$. If it is appropriate to indicate the dependence of the hidden constant in ' $\lesssim$ ' on certain parameters $p_1, \dots$, this will be done through $\lesssim_{p_1,\dots}$.

Outline. The rest of this chapter is structured as follows. In the next section we recall several well-known estimates needed for the subsequent analysis. Section 8.3 is devoted to the derivation of L^2 a priori estimates required for our first blow-up suppression result. In Section 8.4, we introduce further concepts related to mixing and dissipation enhancement in order to determine the flows leading to the specific prevention of concentration mechanism which we here focus on. We then turn to the

proof of our main results, Theorems 8.13 and 8.17.
In a supplementary section (Section 8.5.1) the existence of exploding solutions to equation (8.1) is proved in the case $a = 0$, $\gamma \in (1, 2]$. This appendix further contains two extensions of results in the literature which we require for our main argument in Section 8.4 (see Sections 8.5.2 and 8.5.3). Finally, in Section 8.5.4 we construct examples of incompressible flows, which provide a justification for our Definition 8.7 of γ-relaxation enhancing flows.

8.2 Auxiliary tools

Here we collect some standard inequalities, which will be used throughout the text.

Lemma 8.1 (Interpolation). *Let $\sigma, \mu > 0$. Then for all $f \in C^\infty(\mathbb{T}^d)$*

$$\|f\|_{\dot{H}^\sigma} \lesssim \|f\|_{L^2}^{1-b} \|f\|_{\dot{H}^{\sigma+\mu}}^b,$$

where $b = \frac{\sigma}{\sigma+\mu}$.

Proof. We compute using Plancherel's identity and Hölder inequality with $p = \frac{1}{1-b}$

$$\begin{aligned}\|f\|_{\dot{H}^\sigma}^2 &= \int |\Lambda^\sigma f|^2 \,\mathrm{d}x \approx \sum_k |k|^{2\sigma} |\hat{f}(k)|^2 = \sum_k |\hat{f}(k)|^{2(1-b)} |k|^{2\sigma} |\hat{f}(k)|^{2b} \\ &\leq \left(\sum_k |\hat{f}(k)|^2\right)^{(1-b)} \left(\sum_k |k|^{2(\sigma+\mu)} |\hat{f}(k)|^2\right)^b,\end{aligned}$$

where in the last step we used $\frac{\sigma}{b} = \sigma + \mu$. □

The following result is an immediate consequence of Plancherel's identity and Cauchy–Schwarz.

Lemma 8.2 (Duality). *Let $f, g \in C^\infty(\mathbb{T}^d)$ satisfy $\hat{f}(0)\hat{g}(0) = 0$. Then for $\sigma \in \mathbb{R}$*

$$\int_{\mathbb{T}^d} f(x) g(x) \,\mathrm{d}x \leq \|f\|_{\dot{H}^\sigma} \|g\|_{\dot{H}^{-\sigma}}.$$

In our analysis we will frequently use the following product rule estimate (also known as Kato–Ponce inequality) combined with the subsequently stated Sobolev embedding for fractional derivatives.

Lemma 8.3 (Fractional product rule estimate). *Let $\sigma \geq 0$ be given. Then for all $p_i, q_i \in (2, \infty)$ with $\frac{1}{2} = \frac{1}{p_i} + \frac{1}{q_i}$, $i = 1, 2$ the bound*

$$\|\Lambda^\sigma (fg)\|_{L^2} \lesssim \|\Lambda^\sigma f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^\sigma g\|_{L^{q_2}}$$

holds true.

Proof. For the whole space this is a special case of e.g. [55]. In the case of the torus, we refer to [33] and references therein. □

Lemma 8.4 (Homogeneous Sobolev embedding)**.** *Assume* $0 < \frac{\sigma}{d} < \frac{1}{p} < 1$ *and define* $q \in (p, \infty)$ *via*

$$\frac{\sigma}{d} = \frac{1}{p} - \frac{1}{q}.$$

Then for all $f \in C^\infty(\mathbb{T}^d)$ *with zero mean*

$$\|f\|_{L^q(\mathbb{T}^d)} \lesssim \|\Lambda^\sigma f\|_{L^p(\mathbb{T}^d)}.$$

Proof. See [12] for a direct Fourier analytic proof on the torus. □

8.3 L^2 a priori estimates

In this section we will establish L^2 a priori estimates for the evolution equation

$$\begin{aligned} \partial_t \rho + u \cdot \nabla \rho &= -\Lambda^\gamma \rho + \nabla \cdot (\rho \nabla K * \rho) \;\text{ in }\; (0,\infty) \times \mathbb{T}^d, \\ \rho(0) &= \rho_0, \end{aligned} \tag{8.7}$$

where $u = u(x)$ is a given smooth divergence-free vector field and ρ_0 a non-negative initial datum. Clearly, the conservation of mean property, preservation of positivity, LWP and the smoothing effects for the local solution mentioned in the introduction remain valid for problem (8.7). The results and estimates derived in this part will be used explicitly in and will facilitate the presentation of the proof of our first 'blow-up prevention theorem' (Theorem 8.13).

To simplify the exposition, we will prove the following results only in the (more interesting) cases $\gamma \le 2$ and $2 + a - \frac{d}{2} \ge 1$. At the end of the proofs we sketch the modifications necessary to treat the remaining cases.

8.3.1 A blow-up criterion

Here we illustrate by a formal derivation that a form of the standard blow-up resp. continuation criteria for several classical aggregation equations (including the Keller–Segel model[2]) is also valid for our problem.

[2]Counterparts of Lemmas 8.5 & 8.6 in the case of the parabolic-elliptic Keller–Segel model can be found in [73, Theorem 2.1 & Proposition 3.1].

Lemma 8.5 (L^2-control suffices)**.** *Assume that* $\gamma > \max\{2 + a - \frac{d}{2}, 1\}$ *and let*[3] $\rho_0 \in C^\infty(\mathbb{T}^d)$*. Then the following criterion holds: either the local solution* ρ *to* (8.7) *extends to a global smooth solution or there exists* $T^* \in (0, \infty)$ *and* $1 \le r < \infty$ *such that*

$$\int_0^t \|\rho(\tau) - \bar{\rho}\|_{L^2}^r \, d\tau \to \infty \text{ as } t \nearrow T^*.$$

Proof of Lemma 8.5 for $\gamma \le 2,\ 2 + a - \frac{d}{2} \ge 1$*.* It suffices to derive a priori bounds on higher-order derivatives in terms of L^2, the rest of the argument then follows as in [72, Appendix I]. Let $s \ge s_0(d)$ be a sufficiently large integer. Then we estimate as in the proof of [73, Theorem 2.1]

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{\dot{H}^s}^2 \le -\|\rho\|_{\dot{H}^{s+\frac{\gamma}{2}}}^2 + C\|u\|_{C^s}\|\rho\|_{\dot{H}^s}^2 + \left|\int \nabla \cdot (\rho \nabla K * \rho)(-\Delta)^s \rho\right|. \tag{8.8}$$

The last term on the right-hand side is estimated using Lemmas 8.2 and 8.3

$$\begin{aligned}\left|\int \Lambda^s(\rho\nabla K * \rho) \cdot \nabla \Lambda^s \rho \, \mathrm{d}x\right| &\lesssim \|\Lambda^s(\rho \nabla K * \rho)\|_{\dot{H}^{1-\frac{\gamma}{2}}} \|\nabla \Lambda^s \rho\|_{\dot{H}^{-1+\frac{\gamma}{2}}} \\ &\lesssim \left(\|\Lambda^{s+1-\frac{\gamma}{2}}\rho\|_{L^{p_1}}\|\nabla K * \rho\|_{L^{q_1}} + \|\rho\|_{L^{p_2}}\|\nabla K * \Lambda^{s+1-\frac{\gamma}{2}}\rho\|_{L^{q_2}}\right)\|\rho\|_{\dot{H}^{s+\frac{\gamma}{2}}}.\end{aligned} \tag{8.9}$$

This is valid for $p_i, q_i \in (2, \infty)$ whenever $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{2}$ for $i = 1, 2$. In the following we estimate the terms on the right-hand side of (8.9). We first choose $p_1 = 2 + \varepsilon$ for $\varepsilon > 0$ sufficiently small such that for $\sigma_1 = \left(\frac{1}{2} - \frac{1}{p_1}\right) d$ we have $b_1 := \frac{\sigma_1 + s + 1 - \frac{\gamma}{2}}{s + \frac{\gamma}{2}} < 1$. This is possible since $\gamma > 1$. Thus, using Lemmas 8.4 and 8.1, we find

$$\|\Lambda^{s+1-\frac{\gamma}{2}}\rho\|_{L^{p_1}} \le C\|\rho\|_{\dot{H}^{\sigma_1+s+1-\frac{\gamma}{2}}} \le C\|\rho - \bar{\rho}\|_{L^2}^{1-b_1}\|\rho\|_{\dot{H}^{s+\frac{\gamma}{2}}}^{b_1}.$$

Next, we apply Young's convolution inequality with suitable exponents $p_0, q_3 \in (1, \infty)$ satisfying $1 + \frac{1}{q_1} = \frac{1}{p_0} + \frac{1}{q_3}$. More precisely, we choose $p_0 = \frac{d}{1+a}(1 - \delta)$ for $\delta > 0$ small and note that if $2 + a - \frac{d}{2} \ge 1$, then $\frac{d}{1+a} \le 2$, thus implying $p_0 < 2$. Hence, for $\varepsilon > 0$ sufficiently small (which enforces q_1 to be sufficiently large) we have $q_3 \ge 2$. And clearly, for $s \ge s_0(d)$ sufficiently large we have $b_2 := \frac{(\frac{1}{2} - \frac{1}{q_3})d}{s + \frac{\gamma}{2}} < 1$. Thus,

$$\begin{aligned}\|\nabla K * \rho\|_{L^{q_1}} = \|\nabla K * (\rho - \bar{\rho})\|_{L^{q_1}} &\le \|\nabla K\|_{L^{p_0}}\|\rho - \bar{\rho}\|_{L^{q_3}} \\ &\le C\|\nabla K\|_{L^{p_0}}\|\rho - \bar{\rho}\|_{L^2}^{1-b_2}\|\rho\|_{\dot{H}^{s+\frac{\gamma}{2}}}^{b_2},\end{aligned}$$

[3] Recall that thanks to the assumed lower bound on γ, by the smoothing properties of (8.7), the assumption of smooth initial data can be removed, and the statement, mutatis mutandis, is valid for L^2 data.

where the first identity holds since $\partial_{x_i} K$ has zero mean for all i. We note that

$$\begin{aligned} b_1 + b_2 &= \frac{\left(\frac{1}{2} - \frac{1}{p_1}\right) d + s + 1 - \frac{\gamma}{2} + (\frac{1}{2} - \frac{1}{q_3})d}{s + \frac{\gamma}{2}} \\ &= \frac{\left(\frac{1}{p_0} - \frac{1}{2}\right) d + s + 1 - \frac{\gamma}{2}}{s + \frac{\gamma}{2}} \\ &= \frac{s + \frac{\gamma}{2} - \gamma - \frac{d}{2} + \frac{d}{p_0} + 1}{s + \frac{\gamma}{2}}. \end{aligned} \tag{8.10}$$

Since $\gamma > 2 + a - \frac{d}{2}$, the term $-\gamma - \frac{d}{2} + 1 + \frac{d}{p_0}$ is strictly negative if $\delta > 0$ is chosen sufficiently small. Then the strict inequality $b_1 + b_2 < 1$ holds.

The terms $\|\rho\|_{L^{p_2}}$ and $\|\nabla K * \Lambda^{s+1-\frac{\gamma}{2}} \rho\|_{L^{q_2}}$ on the right-hand side of (8.9) are treated similarly and yield bounds with only minor differences (see the proof of Lemma 8.6).

Inserting the derived bounds into (8.8), applying Young's inequality twice – once with the exponent $\frac{2}{b_1+b_2+1}$ (> 1) applied to the factor involving the highest power of $\|\rho\|_{\dot{H}^{s+\frac{\gamma}{2}}}$ – we obtain, after absorption, a bound of the form

$$\frac{1}{2}\frac{d}{dt}\|\rho\|^2_{\dot{H}^s} \leq -\frac{1}{2}\|\rho\|^2_{\dot{H}^{s+\frac{\gamma}{2}}} + C\|u\|_{C^s}\|\rho\|^2_{\dot{H}^s} + C\|\rho - \bar{\rho}\|^r_{L^2} + C(\bar{\rho})$$

for some possibly large $r \in (1, \infty)$, $r = r(a, d, \gamma, s)$. From this estimate the conclusion can easily be deduced.

Let us briefly comment on how to adapt the proof in order to obtain the result in the remaining cases where $2 + a - \frac{d}{2} < 1$ or $\gamma > 2$. If $2 + a - \frac{d}{2} < 1$ and $\gamma \leq 2$ the main difference lies in the fact that $q_3 < 2$ (using the same notation as in the above proof), and hence the estimate of the term $\|\nabla K * \rho\|_{L^{q_1}}$ simplifies to

$$\|\nabla K * \rho\|_{L^{q_1}} \leq \|\nabla K\|_{L^{p_0}}\|\rho - \bar{\rho}\|_{L^{q_3}} \leq \|\nabla K\|_{L^{p_0}}\|\rho - \bar{\rho}\|_{L^2}.$$

In consequence, when estimating the right-hand side of (8.9), the factor $\|\rho\|_{\dot{H}^{s+\frac{\gamma}{2}}}$ appears with a power of $1 + b_1$ (instead of $1 + b_1 + b_2$). Since $1 + b_1 < 2$, one then argues as before.

In the case $\gamma > 2$ first note that assumption (8.4) guarantees $2 > 2 + a - \frac{d}{2}$. Next note that

$$\|\rho\|_{\dot{H}^r} \leq \|\rho\|_{\dot{H}^{r'}}$$

whenever $r' \geq r$. Therefore the exponent γ can be replaced by 2 in all estimates, which reduces the problem to the previous cases. □

8.3.2 Local control

We now prove that solutions are locally controlled in $L^2(\mathbb{T}^d)$ for some time which only depends on the L^2-distance of the solution to the mean, the mean value and model parameters.

Lemma 8.6 (Local L^2-control)**.** *Suppose $\gamma > \max\{2 + a - \frac{d}{2}, 1\}$ and let $\rho \geq 0$ be a smooth (local) solution to* (8.7)*. Assume that $\|\rho(t_0) - \bar{\rho}\|_{L^2} = B > 0$ for some $t_0 \geq 0$. Then*

$$\|\rho(t_0 + \tau) - \bar{\rho}\|_{L^2} \leq 2B \text{ for all } 0 \leq \tau \leq \tau_0,$$

where

$$\tau_0 = C_1(\|\nabla K\|_{L^{p_0}})^{-1} \min\left\{B^{-r_1}, \bar{\rho}^{-r_2}\right\} > 0 \tag{8.11}$$

for some[4] sufficiently large $1 < p_0 < \frac{d}{1+a}$, a non-decreasing function $C_1(\dots) > 0$ and positive (possibly large) constants $r_i > 0$, $i = 1, 2$, which only depend on γ, d, a and the choice of p_0.

Proof of Lemma 8.6 for $\gamma \leq 2$, $2 + a - \frac{d}{2} \geq 1$. By multiplying (8.7) with $\rho - \bar{\rho}$ and integrating in space, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho - \bar{\rho}\|_{L^2}^2 &= -\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^2 - \int \rho \nabla K * \rho \cdot \nabla(\rho - \bar{\rho})\,\mathrm{d}x \\ &\leq -\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^2 + \|\Lambda^{1-\frac{\gamma}{2}}(\rho \nabla K * \rho)\|_{L^2}\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}.\end{aligned} \tag{8.12}$$

Here we used the incompressibility of the flow. By Lemma 8.3, for $p_i, q_i \in (2, \infty)$ with

$$p_i^{-1} + q_i^{-1} = 2^{-1}, \quad i = 1, 2, \tag{8.13}$$

we have

$$\begin{aligned}\|\Lambda^{1-\frac{\gamma}{2}}(\rho \nabla K * \rho)\|_{L^2} \leq C\Big(&\|\Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{p_1}}\|\nabla K * (\rho - \bar{\rho})\|_{L^{q_1}} \\ &+ \|\rho\|_{L^{p_2}}\|\nabla K * \Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{q_2}}\Big),\end{aligned} \tag{8.14}$$

which means that the last term on the right-hand side of (8.12) can be bounded above by

$$C\left(\|\Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{p_1}}\|\nabla K * (\rho - \bar{\rho})\|_{L^{q_1}} + \|\rho\|_{L^{p_2}}\|\nabla K * \Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{q_2}}\right)\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}. \tag{8.15}$$

[4] Recall that hypothesis (8.3) ensures $1 < \frac{d}{a+1}$.

We now claim that thanks to Young's convolution inequality and Gagliardo–Nirenberg–Sobolev estimates (see Lemma 8.4 and 8.1), term (8.15) is controlled by

$$C_{\ddagger}\|\nabla K\|_{L^{p_0}}\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}(I_1+I_2), \tag{8.16}$$

where $C_{\ddagger}$ is a fixed positive constant (depending only on γ, a and d) and

$$\begin{aligned} I_1 &= \|\rho-\bar{\rho}\|_{L^2}^{2-(b_1+b_2)}\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^{b_1+b_2}, \\ I_2 &= (\bar{\rho}+\|\rho-\bar{\rho}\|_{L^2}^{1-b_3}\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^{b_3})\|\rho-\bar{\rho}\|_{L^2}^{1-b_4}\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^{b_4}. \end{aligned}$$

Here $b_1, b_2 \in [0,1)$ are obtained as in the proof of Lemma 8.5 and satisfy $b_1+b_2<1$ (we choose again $p_0=\frac{d}{a+1}(1-\delta)$ with $\delta=\delta(a,d,\gamma)>0$ (at least) as small as in Lemma 8.5). The value of b_1+b_2 is precisely given by setting $s=0$ in (8.10), i.e.

$$b_1+b_2=\frac{-\frac{d}{2}+\frac{d}{p_0}+1}{\frac{\gamma}{2}}-1. \tag{8.17}$$

To see how the expression for I_2 and the exponents $b_3, b_4 \in [0,1)$ arise, we proceed similarly to the proof of Lemma 8.5: since $2+a-\frac{d}{2}\geq 1$ (which implies $p_0<\frac{d}{a+1}\leq 2$), we can choose $p_2>2$ sufficiently close to 2 (thus enforcing q_2 defined via (8.13) to be arbitrarily large) such that q_4 defined via

$$1+\frac{1}{q_2}=\frac{1}{p_0}+\frac{1}{q_4}$$

satisfies $q_4\geq 2$. We now apply Young's convolution inequality to the second convolution term in (8.15) estimating ∇K in L^{p_0} and use in a subsequent step Lemma 8.4 (twice) for the arising ρ-terms $\|\rho\|_{L^{p_2}}$ and $\|\Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{q_4}}$ with

$$\begin{aligned} \sigma_3 &= \left(\frac{1}{2}-\frac{1}{p_2}\right)d, \\ \sigma_4 &= \left(\frac{1}{2}-\frac{1}{q_4}\right)d \end{aligned}$$

and then Lemma 8.1 (twice) with

$$\begin{aligned} b_3 &= \frac{\sigma_3}{\gamma/2}, \\ b_4 &= \frac{\sigma_4+1-\gamma/2}{\gamma/2} \end{aligned} \tag{8.18}$$

to obtain the I_2-part of (8.16). Notice that

$$\begin{aligned} b_3+b_4 &= \frac{\sigma_3+\sigma_4+1-\gamma/2}{\gamma/2} \\ &= \frac{(1-(p_2^{-1}+q_4^{-1}))d+1}{\gamma/2}-1 \\ &= \frac{(p_0^{-1}-2^{-1})d+1}{\gamma/2}-1 \end{aligned} \tag{8.19}$$

and that the assumption $\gamma > 2+a-\frac{d}{2}$ implies that for $p_0 < \frac{d}{1+a}$ sufficiently large the strict bound $\frac{(p_0^{-1}-2^{-1})d+1}{\gamma/2}-1<1$ holds true. Hence

$$b_3+b_4<1.$$

(Since $b_i \geq 0$, this justifies in particular the application of Lemma 8.1 above.) Note that comparison of (8.17) with (8.19) shows $b_1+b_2=b_3+b_4$.

Abbreviating $b := b_3+b_4+1<2$, we thus obtain the bound

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\rho-\bar\rho\|_{L^2}^2 \leq -\|\rho\|_{\dot H^{\frac{\gamma}{2}}}^2 + C_\ddagger\|\nabla K\|_{L^{p_0}}\Big(&\|\rho-\bar\rho\|_{L^2}^{3-b}\|\rho\|_{\dot H^{\frac{\gamma}{2}}}^b \\ &+\bar\rho\|\rho-\bar\rho\|_{L^2}^{1-b_4}\|\rho\|_{\dot H^{\frac{\gamma}{2}}}^{1+b_4}\Big). \end{aligned} \tag{8.20}$$

For later use, we remark that from (8.14) and the subsequent estimates up to (8.20), we immediately deduce

$$\|\Lambda^{1-\frac{\gamma}{2}}(\rho\nabla K*\rho)\|_{L^2} \leq C_\ddagger\|\nabla K\|_{L^{p_0}}\left(\|\rho-\bar\rho\|_{L^2}^{3-b}\|\rho\|_{\dot H^{\frac{\gamma}{2}}}^{b-1}+\bar\rho\|\rho-\bar\rho\|_{L^2}^{1-b_4}\|\rho\|_{\dot H^{\frac{\gamma}{2}}}^{b_4}\right). \tag{8.21}$$

We now define

$$c_1=\left(1-\frac{b}{2}\right)^{-1}(3-b)=2\left(1+\frac{1}{2-b}\right)$$

and note that

$$\left(1-\frac{1+b_4}{2}\right)^{-1}(1-b_4)=2.$$

Applying a standard absorption argument to (8.20), we then find

$$\frac{d}{dt}\|\rho-\bar\rho\|_{L^2}^2 \leq -\|\rho\|_{\dot H^{\frac{\gamma}{2}}}^2 + C_\star(\|\nabla K\|_{L^{p_0}})\left(\|\rho-\bar\rho\|_{L^2}^{c_1}+\bar\rho^{\frac{2}{1-b_4}}\|\rho-\bar\rho\|_{L^2}^2\right). \tag{8.22}$$

Once more for later use, we note that Young's multiplication inequality applied to the right-hand side of (8.21) yields

$$\|\Lambda^{1-\frac{\gamma}{2}}(\rho\nabla K*\rho)\|_{L^2}^2 \le \frac{1}{2}\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^2 + C_\star(\|\nabla K\|_{L^{p_0}})\left(\|\rho-\bar{\rho}\|_{L^2}^{c_1} + \bar{\rho}^{\frac{2}{1-b_4}}\|\rho-\bar{\rho}\|_{L^2}^2\right) \tag{8.23}$$

with the same constants c_1 and $C_\star$ as in (8.22).

Now note that $c_1 > 2$ and that, by (8.22), the function $f(t) = \|\rho(t)-\bar{\rho}\|_{L^2}^2$ satisfies

$$f' \le C_0 f^{c_1/2} + C_0\bar{\rho}^{\frac{2}{1-b_4}} f, \qquad f(t_0) = B^2$$

where $C_0 = C_\star(\|\nabla K\|_{L^{p_0}})$. Comparison with the explicit solution $\tilde{f}$ to

$$\tilde{f}' = C_0\tilde{f}^{c_1/2} + C_0\bar{\rho}^{\frac{2}{1-b_4}}\tilde{f}, \qquad \tilde{f}(t_0) = B^2,$$

which is given by

$$\tilde{f}(t_0+t) = R^{\frac{1}{q}}\exp(C_0Rt)B^2\left(R - B^{2q}\left[\exp(C_0Rqt)-1\right]\right)^{-\frac{1}{q}}$$

with $q = \frac{c_1-2}{2}$ and $R = \bar{\rho}^{\frac{2}{1-b_4}}$, shows that

$$f(t_0+\tau) \le 4B^2, \quad \text{whenever } 0 \le \tau \le \tau_0 := \delta_0 C_0^{-1}\min\left\{\frac{2}{c_1-2}B^{-(c_1-2)}, \bar{\rho}^{-\frac{2}{1-b_4}}\right\}.$$

Here $\delta_0 > 0$ is a universal constant. Thus, the assertion of Lemma 8.6 is obtained by choosing $r_1 = c_1 - 2$ and $r_2 = \frac{2}{1-b_4}$.

The case where $2 + a - \frac{d}{2} < 1$ or $\gamma > 2$ is treated similarly to the sketch at the end of the proof of Lemma 8.5. □

8.4 Enhanced relaxation and blow-up prevention

We now introduce the mixing-type flows capable of speeding up relaxation to equilibrium in equations with anomalous diffusion induced by the operator $-\Lambda^\gamma$. While any weakly mixing flow is admissible, we aim to provide a sharp characterisation. Let us recall that a divergence-free Lipschitz vector field u on $\mathbb{T}^d$ gives rise to a flow map $\Phi : \mathbb{R}\times\mathbb{T}^d \to \mathbb{T}^d$, $(t,x)\mapsto \Phi_t(x)$ via

$$\frac{d}{dt}\Phi_t(x) = u(\Phi_t(x)),$$
$$\Phi_0 = \mathrm{Id}_{\mathbb{T}^d},$$

where the transformations Φ_t are measure-preserving bi-Lipschitz mappings. Thus, we obtain a one-parameter group of unitary operators $U^t f(x) = f(\Phi_t^{-1}(x))$ on $L^2(\mathbb{T}^d)$.

Definition 8.7. Let $\gamma \geq 1$. We call a divergence-free Lipschitz vector field $u = u(x)$ *γ-relaxation enhancing* (γ-RE) if the corresponding unitary operator U^1 does not have any non-constant eigenfunctions in $H^{\frac{\gamma}{2}}(\mathbb{T}^d)$.

The precise meaning in which relaxation is accelerated is described in Theorem 8.9 below.

Remark 8.8.

(i) The notion 'relaxation enhancing' was first introduced in [34] in a more general context. The notion used in [73] corresponds in our definition to 2-RE. Any flow which is weakly mixing in the ergodic sense (so that U^1 does not have any non-constant eigenfunctions in L^2) is also γ-RE for any γ as above. The existence of weakly mixing flows on $\mathbb{T}^d$ for any $d \geq 2$ is classical and can be shown by considering suitable time changes of appropriate irrational translations on $\mathbb{T}^d$ (see [34, Section 6] and references therein). A concrete example for a 2-RE flow which is not weakly mixing can also be found in [34, Section 6].

(ii) In Appendix 8.5.4 we provide a sketch proof showing that for any given $1 \leq \gamma_1 < \gamma_2$ there exists a smooth, incompressible flow on $\mathbb{T}^2$ which is γ_2-RE but not γ_1-RE.

We now consider for a parameter $A \gg 1$ the initial value problem

$$\begin{aligned} \partial_t \rho^A + Au \cdot \nabla \rho^A &= -\Lambda^\gamma \rho^A + \nabla \cdot (\rho^A \nabla K * \rho^A) \quad \text{in} \ (0,\infty) \times \mathbb{T}^d, \\ \rho^A(0) &= \rho_0, \end{aligned} \tag{8.24}$$

where the kernel K satisfies the conditions described in the introduction (Section 8.1) and $d \geq 2$. The crucial ingredient in the proof of our first result on suppression of singularities (Theorem 8.13) is the following result (cf. [34]):

Theorem 8.9 (Enhanced relaxation)**.** *Let $\gamma \geq 1$ and let u be a smooth divergence-free vector field on $\mathbb{T}^d$. Then u is γ-relaxation enhancing if and only if for every $\tau > 0$, $\varepsilon > 0$ there exists a positive constant $A_0 = A_0(\tau, \varepsilon)$ such that for any $A \geq A_0$ and for any $\mu_0 \in L^2(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} \mu_0 = 0$ the solution μ^A to*

$$\begin{aligned} \partial_t \mu^A + Au \cdot \nabla \mu^A &= -\Lambda^\gamma \mu^A \quad \text{in} \ (0,\infty) \times \mathbb{T}^d, \\ \mu^A(0) &= \mu_0 \end{aligned} \tag{8.25}$$

satisfies $\|\mu^A(t)\|_{L^2} \leq \varepsilon \|\mu_0\|_{L^2}$ for all $t \geq \tau$.

Remark 8.10.

(i) If restricting to initial data in $H^{\frac{1}{2}}$ (instead of general L^2 data), one is still able to obtain enhanced relaxation for $\gamma \in (0,1)$ if the unitary evolution (cf. U^1 in Definition 8.7) does not have any non-constant eigenfunctions in $H^{\frac{\gamma}{2}}$.

(ii) Theorem 8.9 (at least with $\gamma = 2$) remains true when L^2 is replaced by L^p for any $p \in [1,\infty]$, see [34, Theorem 5.5].

In the case $\gamma \geq 2$ Theorem 8.9 is a consequence of the abstract criterion in [34] (combined with Proposition 8.11). We will sketch the extension to arbitrary $\gamma \geq 1$ in Appendix 8.5.2. In any case, an important ingredient in the proof is the boundedness of the linear transport evolution in $H^{\frac{\gamma}{2}}$ for sufficiently regular vector fields:

Proposition 8.11 (Estimate for transport equation)**.** *Let $v = v(x)$ be a divergence-free smooth vector field and assume that $\gamma > 0$. Then any sufficiently regular solution η to*

$$\begin{aligned} \partial_t \eta + v \cdot \nabla \eta = 0 \quad &\textit{in} \ \ (0,\infty) \times \mathbb{T}^d, \\ \eta(0) = \eta_0 \quad & \end{aligned} \tag{8.26}$$

satisfies the bound

$$\|\eta(t)\|_{\dot{H}^{\frac{\gamma}{2}}(\mathbb{T}^d)} \lesssim \exp(C(v)t)\|\eta_0\|_{\dot{H}^{\frac{\gamma}{2}}(\mathbb{T}^d)}, \tag{8.27}$$

where $C(v) \lesssim_{\gamma,d} \|\Lambda^{\gamma+\frac{d}{2}+1} v\|_{L^2}$.

Remark 8.12.

(i) Our proof of the above estimate, provided in Appendix 8.5.3, is based on a Littlewood–Paley decomposition and relies on Sobolev-like (namely Bernstein) inequalities, thus leading to suboptimal regularity requirements on v. Using pointwise estimates and the L^2-type modulus of continuity representation of the homogeneous fractional Sobolev norm of order $\gamma \in (0,1)$ [12] allows one to by-pass the usage of Sobolev embeddings. See the recent preprint [36, Section 3.2] for a sketch of the underlying argument requiring only Lipschitz continuity of the vector field v, i.e. $C(v) = c_{\gamma,d}\|\nabla v\|_{L^\infty}$.

(ii) The assumption $\nabla \cdot v = 0$ is not necessary for the boundedness of the evolution (8.26) with respect to $\|\cdot\|_{\dot{H}^{\frac{\gamma}{2}}}$. See [3] for a proof in the case of the whole space.

We are now in a position to turn to our first main result. From now on we let $p_0 = p_0(\gamma, a, d) \in (1, \frac{d}{a+1})$ be an exponent for which both Lemma 8.5 and Lemma 8.6

are valid. Also recall that by assumption (8.3) we have $\|\nabla K\|_{L^{p_0}} < \infty$. For simplicity, any dependence of constants on γ, a and d will, as before, be omitted.

Theorem 8.13 (Prevention of blow-up for model with fractional dissipation)**.** *Let $\gamma > \max\{2 + a - \frac{d}{2}, 1\}$. Suppose that the divergence-free smooth vector field $u(x)$ is γ-relaxation enhancing. Then for any $\rho_0 \in L^2(\mathbb{T}^d)$ there exists an amplitude $A_0(\|\rho_0 - \bar{\rho}\|_{L^2}, \bar{\rho}, u, \|\nabla K\|_{L^{p_0}})$ such that, whenever $A \geq A_0$, problem (8.24) has a global solution $\rho^A \in C_b([0,\infty), L^2) \cap C^\infty((0,\infty) \times \mathbb{T}^d)$.*

Remark 8.14. Prevention of blow-up in the sense of Theorem 8.13 cannot be expected to hold for a threshold amplitude A_0 independent of the initial datum. This is essentially due to a scaling obstruction. See also Appendix 8.5.1.

The rough idea of the proof of Theorem 8.13 can be described as follows. Oversimplistically speaking, our aggregation equations with fractional diffusion are essentially driven by two competing, in general nonlocal forces: the tendency to concentrate due to aggregation versus the tendency to uniformly distribute the initial mass in space thanks to diffusion. As long as diffusion dominates, the solution should not be able to concentrate too much and thus should not blow up. In the delicate case of small dissipation (when the $H^{\frac{\gamma}{2}}$ norm is not large enough compared to L^2) the γ-RE flow – if sufficiently strong – takes care of the low frequencies by quickly stirring the density[5]. This increases spatial gradients, thus enhancing dissipation, and eventually prevents blow-up.

Proof of Theorem 8.13 for $\gamma \leq 2$ and $2 + a - \frac{d}{2} \geq 1$. Without loss of generality we may assume that ρ_0 is not constant, i.e. $\rho_0 \not\equiv \bar{\rho}$ and $\rho \in C^\infty$ (cf. page 134 (LWP and Smoothing)). By Lemma 8.5, it suffices to prove global control in $L^2(\mathbb{T}^d)$. For this purpose we first introduce the following parameters:

- Denote $B := \|\rho_0 - \bar{\rho}\|_{L^2} > 0$.
- Let $p_0 \in \left(1, \frac{d}{a+1}\right)$, $c_1 > 2$, b_4 (defined in (8.18)) and $C_\star(\|\nabla K\|_{L^{p_0}})$ be the constants introduced in the proof of Lemma 8.6. We recall that these quantities only depend on γ, a and d. Furthermore denote by $\tau_0 = \tau_0(B, \bar{\rho}, \|\nabla K\|_{L^{p_0}})$ the (possibly small) positive time span (8.11) in Lemma 8.6.
- Define now $\tau_1 = \min\left\{\frac{1}{16}\left\{4C_\star(\|\nabla K\|_{L^{p_0}})\left((2B)^{c_1-2} + \bar{\rho}^{\frac{2}{1-b_4}}\right)\right\}^{-1}, \tau_0\right\}$.

[5] Strictly speaking, this mechanism of stirring only fully applies if $\rho^A(t)$ lies in the continuous spectral subspace corresponding to U^1. In the case of a non-trivial component in the L^2-closure of the subspace spanned by all (rough) eigenfunctions the mechanism by which gradients are increased is somewhat more technical. The interested reader is referred to [34, Lemma 3.3].

- Let $A_0 = A_0(\tau_1)$ be such that for any $A \geq A_0$ and any mean-zero $\mu_0 \in L^2(\mathbb{T}^d)$ the solution $\tilde{\mu}^A$ to equation (8.25) with initial value $\tilde{\mu}^A(0) = \mu_0$ satisfies the bound
$$\|\tilde{\mu}^A(\tau_1)\|_{L^2} \leq \frac{1}{8}\|\mu_0\|_{L^2}.$$
The existence of such an A_0 is guaranteed by Theorem 8.9. Obviously, A_0 can be chosen to be non-increasing on $\mathbb{R}^+$ and it will necessarily become unbounded near $\tau_1 = 0$.

Now define $t_0 = \inf\{t > 0 : \|\rho^A(t) - \bar{\rho}\|_{L^2} \geq B\}$. If $t_0 = \infty$, there is nothing to prove. We therefore assume $t_0 < \infty$ so that by continuity $\|\rho^A(t_0) - \bar{\rho}\|_{L^2} = B$. Since $\nabla \cdot (Au) = 0$ the statement of Lemma 8.6 applies to $\rho = \rho^A$, and recalling $\tau_1 \leq \tau_0$, we deduce the bound

$$\|\rho^A(t_0 + \tau) - \bar{\rho}\|_{L^2} \leq 2B \qquad \text{for all } \tau \in [0, \tau_1]. \tag{8.28}$$

In the following we will show that the above choice of A_0 implies the bound $\|\rho^A(t_0 + \tau_1) - \bar{\rho}\|_{L^2} \leq B$. The claim then follows by iterating the argument: define $t_1 = \inf\{t > t_0 + \tau_1 : \|\rho^A(t) - \bar{\rho}\|_{L^2} \geq B\}$ and proceed as before with t_0 replaced by t_1 etc. This then results in the global bound $\|\rho^A(t) - \bar{\rho}\|_{L^2} \leq 2B$ for all $t > 0$.

Denote $R(\tau) = \int_{t_0}^{t_0+\tau} \|\rho^A\|^2_{\dot{H}^{\frac{\gamma}{2}}}$. We distinguish the following cases, which reflect the idea described above.

Case I: $R(\tau_1) > B^2$.

Here we apply estimate (8.22) (with ρ replaced by ρ^A), which is possible since Au is divergence-free. Hence on the time interval $[t_0, t_0 + \tau_1]$, we have

$$\begin{aligned}\frac{d}{dt}\|\rho^A - \bar{\rho}\|^2_{L^2} &\leq -\|\rho^A\|^2_{\dot{H}^{\frac{\gamma}{2}}} + C_\star(\|\nabla K\|_{L^{p_0}}) \left(\|\rho^A - \bar{\rho}\|^{c_1}_{L^2} + \bar{\rho}^{\frac{2}{1-b_4}}\|\rho^A - \bar{\rho}\|^2_{L^2}\right) \\ &\leq -\|\rho^A\|^2_{\dot{H}^{\frac{\gamma}{2}}} + 4C_\star(\|\nabla K\|_{L^{p_0}}) \left((2B)^{c_1-2} + \bar{\rho}^{\frac{2}{1-b_4}}\right) B^2,\end{aligned}$$

where we used (8.28) in the second step. We now integrate in time from t_0 to $t_0 + \tau_1$ to obtain

$$\begin{aligned}\|\rho^A - \bar{\rho}\|^2_{L^2}(t_0 + \tau_1) &\leq B^2 - B^2 + \tau_1 \cdot 4C_\star(\|\nabla K\|_{L^{p_0}}) \left((2B)^{c_1-2} + \bar{\rho}^{\frac{2}{1-b_4}}\right) B^2 \\ &\leq \frac{1}{16}B^2.\end{aligned}$$

Here we used the hypothesis (of Case I) and, in the second step, the choice of τ_1.

Case II: $R(\tau_1) \leq B^2$.

In this case we need to approximate $\rho^A(t_0 + t)$ by the solution $\mu^A(t_0 + t)$ to equation (8.25) with datum $\mu^A(t_0) = \rho^A(t_0)$. We estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho^A - \mu^A\|_{L^2}^2 + \|\rho^A - \mu^A\|_{\dot{H}^{\frac{\gamma}{2}}}^2 &= -\int \rho^A \nabla K * \rho^A \cdot \nabla(\rho^A - \mu^A) \\ &\leq \frac{1}{2}\|\rho^A \nabla K * \rho^A\|_{\dot{H}^{1-\frac{\gamma}{2}}}^2 + \frac{1}{2}\|\rho^A - \mu^A\|_{\dot{H}^{\frac{\gamma}{2}}}^2.\end{aligned}$$

Absorption yields

$$\frac{1}{2}\frac{d}{dt}\|\rho^A - \mu^A\|_{L^2}^2 + \frac{1}{2}\|\rho^A - \mu^A\|_{\dot{H}^{\frac{\gamma}{2}}}^2 \leq \frac{1}{2}\|\rho^A \nabla K * \rho^A\|_{\dot{H}^{1-\frac{\gamma}{2}}}^2. \tag{8.29}$$

Thanks to estimate (8.23), the right-hand side of (8.29) is bounded above by

$$\frac{1}{2}\left\{\frac{1}{2}\|\rho^A\|_{\dot{H}^{\frac{\gamma}{2}}}^2 + C_\star(\|\nabla K\|_{L^{p_0}})\left(\|\rho^A - \bar{\rho}\|_{L^2}^{c_1} + \bar{\rho}^{\frac{2}{1-b_4}}\|\rho^A - \bar{\rho}\|_{L^2}^2\right)\right\}.$$

Combination with (8.28) implies on the time interval $[t_0, t_0 + \tau_1]$

$$\begin{aligned}\frac{d}{dt}\|\rho^A - \mu^A\|_{L^2}^2 + \|\rho^A - \mu^A\|_{\dot{H}^{\frac{\gamma}{2}}}^2 \leq& \frac{1}{2}\|\rho^A\|_{\dot{H}^{\frac{\gamma}{2}}}^2 \\ &+ 4C_\star(\|\nabla K\|_{L^{p_0}})\left((2B)^{c_1-2} + \bar{\rho}^{\frac{2}{1-b_4}}\right)B^2.\end{aligned}$$

We now integrate from t_0 to $t_0 + \tau_1$ to conclude using also the hypothesis (of Case II)

$$\begin{aligned}\|\rho^A - \mu^A\|_{L^2}^2(t_0 + \tau_1) &\leq \frac{1}{2}B^2 + \tau_1 \cdot 4C_\star(\|\nabla K\|_{L^{p_0}})\left((2B)^{c_1-2} + \bar{\rho}^{\frac{2}{1-b_4}}\right)B^2 \\ &\leq \frac{1}{2}B^2 + \frac{1}{16}B^2 \\ &= \frac{9}{16}B^2.\end{aligned}$$

In the second step of the last estimate, we used the choice of τ_1.

Note that since $\mu^A(t_0) - \bar{\rho} = \rho^A(t_0) - \bar{\rho}$ (whose L^2-norm equals B), by the choice of A_0 and since $A \geq A_0$, the bound

$$\|\mu^A - \bar{\rho}\|_{L^2}(t_0 + \tau_1) \leq \frac{1}{8}B$$

holds true. We therefore obtain

$$\begin{aligned}\|\rho^A - \bar{\rho}\|_{L^2}(t_0 + \tau_1) &\leq \|\rho^A - \mu^A\|_{L^2}(t_0 + \tau_1) + \|\mu^A - \bar{\rho}\|_{L^2}(t_0 + \tau_1) \\ &\leq \frac{7}{8}B.\end{aligned}$$

In any case we have

$$\|\rho^A - \bar{\rho}\|_{L^2}(t_0 + \tau_1) \leq \frac{7}{8}B \leq B,$$

which completes the proof in the case $\gamma \leq 2$ and $2 + a - \frac{d}{2} \geq 1$.

To ensure the validity of the assertion in the remaining cases, one needs to make sure that estimates analogous to (8.22) and (8.23) hold true in these cases. This can be verified by following the ideas explained at the end of the proof of Lemma 8.6 and Lemma 8.5. □

Remark 8.15 (Long-time asymptotics). Theorem 8.13 can be refined in such a way as to obtain exponential convergence of the solution to the mean as $t \to \infty$. In fact, under the assumptions of Theorem 8.13, it follows that for any $\rho_0 \in L^2(\mathbb{T}^d)$ and any $\kappa \in (0, \infty)$ there exists $A_0(\|\rho_0 - \bar{\rho}\|_{L^2}, \bar{\rho}, u, \|\nabla K\|_{L^{p_0}}, \kappa)$ such that, whenever $A \geq A_0$, problem (8.24) has a global, regular solution ρ^A which satisfies

$$\|\rho^A(t) - \bar{\rho}\|_{L^2} \leq C \exp(-\kappa t)\|\rho_0 - \bar{\rho}\|_{L^2}, \tag{8.30}$$

where C is a universal constant (in particular independent of κ).

Let us briefly sketch how this result is obtained by adapting the proof of Theorem 8.13. Given $\kappa \in (0, \infty)$ define $\tau(\kappa) = \frac{-\ln\theta}{\kappa}$, where $\theta = \frac{7}{8}$. Then define $\tau := \min\{\tau_1, \tau(\kappa)\}$, where τ_1 and the quantities introduced before its definition are the same as in the proof of Theorem 8.13. As threshold amplitude choose $A_0 = A_0(\tau)$ satisfying the same identity as $A_0(\tau_1)$ but with the possibly smaller time τ. Now start the iteration at time $t = 0$ instead of t_0. By Lemma 8.6 the bound $\|\rho^A(t) - \bar{\rho}\|_{L^2} \leq 2B$ holds for all $t \in [0, \tau]$. Then, repeating the arguments in the two cases of the proof of Theorem 8.13, we can conclude

$$\|\rho^A(\tau) - \bar{\rho}\|_{L^2} \leq \theta B.$$

Let us now define $\rho_n = \rho^A(n\tau)$ for $n \in \mathbb{N}$ and $B_n = \|\rho_n - \bar{\rho}\|_{L^2}$. Then in the n-th iteration step one distinguishes the cases where $R_n := \int_{n\tau}^{(n+1)\tau} \|\rho^A(t)\|^2_{\dot{H}^{\frac{\gamma}{2}}} dt$ is less than B_n^2, resp. greater than or equal to B_n^2. Since, by definition, τ_0, τ_1 are non-increasing in their argument 'B', and since $\theta \in (0, 1)$, we can again argue as in the proof of Theorem 8.13 (with B replaced by B_n) and inductively obtain

$$\|\rho_n - \bar{\rho}\|_{L^2} \leq \theta^n B \leq \exp(-\kappa(n\tau))\|\rho_0 - \bar{\rho}\|_{L^2}.$$

The decay (8.30) is now easily obtained.

Remark 8.16. Note that for $d = 2$ and $a = 0$ the kernel ∇K has the same singularity at the origin as ∇N, where N denotes the two-dimensional Newton kernel. Although on the torus N is not a proper convolution kernel, an analysis almost completely analogous to the one established here shows that the statement of Theorem 8.13 also applies to the two-dimensional parabolic-elliptic Keller–Segel model with fractional diffusion $-\Lambda^\gamma$ whenever $\gamma > 1$. Similarly, for the three-dimensional parabolic-elliptic Keller–Segel model with fractional diffusion, we have blow-up prevention for L^2 data whenever $\gamma > \frac{3}{2}$.

Note that for dimension $d \geq 4$ Theorem 8.13 no longer includes the Keller–Segel case since the lower bound $\gamma_0 = d/2$ would enforce diffusion to be stronger than classical (more concretely, it is the fact that the assumption $\frac{d}{2(d-2)} > 1$ (cf. (8.4)) is violated which makes our arguments break down). As alluded to in the introduction, the reason for this failure is the fact that the L^2-norm is no longer subcritical for Keller–Segel in $d \geq 4$.

Scaling suggests that by working in L^p spaces of higher integrability ($p > 2$) smaller lower bounds on γ may be achieved, namely

$$\gamma > 2 + a - d\left(1 - \frac{1}{p}\right) \tag{8.31}$$

(as long as γ is large enough so that the nonlinear equation is locally well-posed in a suitable Lebesgue (or Sobolev) space and for data, for which Theorem 8.9 is valid for this γ — the additional condition $\gamma > 1$, for instance, would ensure these last two properties). For the Keller–Segel type (Newton kernel) singularity inequality (8.31) becomes $\gamma > \frac{d}{p}$. This may lead to the expectation that also in the higher-dimensional Keller–Segel model with fractional dissipation the mixing mechanism is able to prevent blow-up for any $\gamma > 1$ when confining to e.g. $L^\infty(\mathbb{T}^d)$ initial data. However, when trying to prove suppression using L^p- instead of L^2-estimates the following issue arises: following the notation in the proof of Theorem 8.13, it appears that in L^p, $p > 2$, the approximation of ρ^A by μ^A requires an estimate of the form

$$\|\Lambda^{1-\frac{\gamma}{2}} f\|_{L^{p_1}} \lesssim \|\Lambda^{\frac{\gamma}{2}}(|f|^{\frac{p}{2}})\|_{L^2}^{2/p} \tag{8.32}$$

for some $p_1 > 2$. Certainly such an estimate cannot hold unless $\gamma > \left(\frac{1}{p} + \frac{1}{2}\right)^{-1}$, a lower bound which is strictly larger than 1 if $p > 2$. Thus, new techniques appear to be necessary to tackle the general case. In the case $\gamma = 2$, however, estimate (8.32) becomes trivial, and indeed, in this case by working in L^p instead of L^2 the suppression mechanism can be extended as to include in particular the classical Keller–Segel model ($\gamma = 2$) in any dimension $d \geq 2$, as we will show in the following.

Let us consider the Keller–Segel model – in its precise form for clarity's sake – under the influence of a strong incompressible flow

$$\partial_t \rho^A + Au \cdot \nabla \rho^A = \Delta \rho^A + \nabla \cdot (\rho^A \nabla \Delta^{-1}(\rho^A - \bar{\rho})) \text{ in } (0,\infty) \times \mathbb{T}^d \tag{8.33}$$

with $d \geq 4$. The higher-dimensional Keller–Segel model with standard diffusion (i.e. equation (8.33) with $A = 0$) is $L^{\frac{d}{2}}$-critical and L^1-supercritical (choose $\gamma = 2$, $a = d - 2$ in (8.2)). For $p > \frac{d}{2}$ local well-posedness in L^p and regularity for positive times are well-established (see e.g. [13] for results on bounded domains and [21] for results on the whole space assuming sufficient decay at infinity), and at any (positive) level of mass (= L^1-norm for non-negative solutions) there exist smooth solutions which blow up in finite time [13, 14, 21]. Moreover, for global regularity it suffices to globally control the L^p-norm of the solution, and statements analogous to those established in Section 8.3 hold true whenever $p > \frac{d}{2}$. We will therefore directly proceed to the proof of global regularity for (8.33) whenever A is sufficiently large.

Theorem 8.17 (Prevention of blow-up for Keller–Segel model in higher dimensions)**.** *Assume $d \geq 6$ and let $p > \frac{d}{2}$. Suppose that the divergence-free smooth vector field $u(x)$ is 2-relaxation enhancing. Then for any initial datum $\rho_0 \in L^p(\mathbb{T}^d)$ there exists an amplitude $A_0(\|\rho_0 - \bar{\rho}\|_{L^p}, \bar{\rho}, u, p)$ such that, whenever $A \geq A_0$, equation* (8.33) *has a global solution $\rho^A \in C_b([0,\infty), L^p) \cap C^\infty((0,\infty) \times \mathbb{T}^d)$ with initial value $\rho^A(0) = \rho_0$. For $d = 4, 5$ the statement holds true under the stronger condition $p > \frac{4d}{d+2}$.*

Remark 8.18. For $d \geq 6$ Theorem 8.17 is optimal in terms of the regularity required for the initial data in the sense that equation (8.33) with $A = 0$ is $L^{\frac{d}{2}}$-critical.

Proof of Theorem 8.17. The result follows from arguments similar to Theorem 8.13 with L^2 replaced by L^p. In contrast to the proof of Theorem 8.13, here we do not (need to) distinguish the cases of small and large diffusion: for any time $t_0 \geq 0$ – even if diffusion is large – the local solution $\rho^A(t_0 + \tau)$ to (8.33) can be approximated sufficiently well by the solution $\mu^A(t_0 + \tau)$ to equation (8.25) with datum $\mu^A(t_0) = \rho^A(t_0)$ for small enough times $\tau > 0$, as will be shown in the following.

We first prove the case $d \geq 6$. Without loss of generality we may assume $p < d$. Note that since $p < d$ we can define $q \in (p, \infty)$ via

$$\left(\frac{1}{p} - \frac{1}{q}\right) d = 1. \tag{8.34}$$

Since $d \geq 6$ and $p > \frac{d}{2}$, we have

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{p} - \frac{1}{d} < \frac{3}{d} \leq \frac{1}{2}$$

so that there exists $r \in (2, \infty)$ satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{2}.$$

We now let $h = |\rho^A - \mu^A|^{p/2}$ and estimate using equation (8.33) and $\nabla \cdot u = 0$

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|\rho^A - \mu^A\|_{L^p}^p + \frac{4(p-1)}{p^2}\|\nabla h\|_{L^2}^2 \\
&\leq -\int \rho^A \nabla \Delta^{-1}(\rho^A - \bar{\rho}) \cdot \nabla \left((\rho^A - \mu^A)|\rho^A - \mu^A|^{p-2}\right) \\
&\leq C\|\rho^A\|_{L^p}\|\nabla \Delta^{-1}(\rho^A - \bar{\rho})\|_{L^q}\|\rho^A - \mu^A\|_{L^{r(p/2-1)}}^{p/2-1}\|\nabla h\|_{L^2} \\
&\leq C\|\rho^A\|_{L^p}\|\rho^A - \bar{\rho}\|_{L^p}\|h\|_{L^{r_1}}^{(p-2)/p}\|\nabla h\|_{L^2},
\end{aligned}$$

where r_1 is defined via

$$r_1 \cdot p/2 = r(p/2 - 1).$$

In the last estimate we used Lemma 8.4 (exploiting our choice of q) and the boundedness of the Riesz transform on L^p, $p \in (1, \infty)$. For $p \in (\frac{d}{2}, d)$ and $d \geq 6$ an elementary check yields $r_1 > 2$. (Of course, $r_1 \in [1, 2]$ would even be easier.) Now note that, by Lemmas 8.4 and 8.1, for $\sigma = \left(\frac{1}{2} - \frac{1}{r_1}\right) d$ we have

$$\|h\|_{L^{r_1}} \lesssim \|\Lambda^\sigma h\|_{L^2} \lesssim \|\nabla h\|_{L^2}^\sigma \|h\|_{L^2}^{1-\sigma}.$$

Hence we obtain

$$\begin{aligned}
\frac{d}{dt}\|\rho^A - \mu^A\|_{L^p}^p + \|\nabla h\|_{L^2}^2 \\
&\leq C(\|\rho^A - \bar{\rho}\|_{L^p} + \bar{\rho})\|\rho^A - \bar{\rho}\|_{L^p}\|h\|_{L^2}^{(1-\sigma)(p-2)/p}\|\nabla h\|_{L^2}^{1+\sigma(p-2)/p}.
\end{aligned}$$

It is elementary to verify that $p > \frac{d}{2}$ guarantees $\sigma(p-2)/p < 1$. Thus, an absorption argument yields

$$\frac{1}{p}\frac{d}{dt}\|\rho^A - \mu^A\|_{L^p}^p \leq C(\|\rho^A - \bar{\rho}\|_{L^p} + \bar{\rho})^{c_3}\|\rho^A - \bar{\rho}\|_{L^p}^{c_3}\|h\|_{L^2}^{c_4}$$

with $c_i = c_i(\sigma, p), i = 3, 4$, suitable positive exponents. Similarly to Lemma 8.6, for $B := \max\{\|\rho^A(t_0) - \bar{\rho}\|_{L^p}, 1\}$ one can show[6] that $\|\rho^A - \bar{\rho}\|_{L^p} \leq 2B$ on some small time interval $[t_0, t_0 + \tau_0]$ where $\tau_0 > 0$ only depends on $B, \bar{\rho}$ and fixed parameters. Also notice that on $[t_0, t_0 + \tau_0]$ we then have $\|h\|_{L^2} = \|\rho^A - \mu^A\|_{L^p}^{p/2}$ and $\|\rho^A - \mu^A\|_{L^p} \leq$

[6]Since for the Keller–Segel model this is a well-known result, its proof is omitted here. Of course, the condition $p > \frac{d}{2}$ is crucial for its validity.

$\|\rho^A - \bar\rho\|_{L^p} + \|\mu^A - \bar\rho\|_{L^p} \leq 3B$, where in the last bound we used the fact that $\|\mu^A - \bar\rho\|_{L^p}$ is non-increasing on $[t_0, \infty)$. The rest of the argument is similar to the reasoning in Case II of the proof of Theorem 8.13 except that here we need to use Remark 8.10 ((ii)) instead of Theorem 8.9.

If $d = 4, 5$, we assume again without loss of generality $p < d$ and define q via (8.34). The condition $p > \frac{4d}{d+2}$ ensures that $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. The rest of the proof then follows as before. □

8.5 Supplementary material

8.5.1 Blow-up in the absence of advection

In this section, we aim to show that in the case $a = 0$ and in the absence of strong advection there exist smooth initial data which lead to blow-up in finite time. We stress that blow-up can also be produced in the presence of the advective term if one *first* fixes the flow Au (including its amplitude) and choses appropriate data *afterwards*.

We consider the equation

$$\partial_t \rho = -\Lambda^\gamma \rho + \nabla \cdot (\rho \nabla K * \rho) \quad \text{in} \quad (0, \infty) \times \mathbb{T}^d, \tag{8.35}$$

where $\nabla K(x) \sim \frac{x}{|x|^2}$ near $x = 0$, $d \geq 2$ and $\gamma \in (1, 2]$. In this case, blow-up can be produced by a construction very similar to the one in [73]. We therefore confine ourselves to sketching the main argument and indicating the steps which deviate from [73]. Let us introduce the following parameters and auxiliary functions:

- $0 < 2a < b < \frac{1}{8}$ (sufficiently small).
- $\rho_0 \in C^\infty(\mathbb{T}^d)$ non-negative with $\operatorname{supp} \rho_0 \subset B_a(0)$ and mass $M \geq 1$ (sufficiently large).
- ϕ a smooth cut-off at scale b: Fix $\phi_0 \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \phi_0 \subset B_1$, $\phi_0 \equiv 1$ on $B_{\frac{1}{2}}$, $0 \leq \phi_0 \leq 1$. Then $\phi(x) := \phi_0(\frac{x}{b})$ can be considered as a function on the periodic box $\mathbb{T}^d$.

For simplicity we assume equality $\nabla K(x) = \frac{x}{|x|^2}$ on $B_{\frac{1}{4}}$. The parameters a, b, M will be fixed later. As long as the solution ρ stays regular, it preserves positivity and mass.

The main ingredient in the blow-up proof is a virial argument, which can be exploited when considering the evolution of the second moment. This is a standard technique for proving blow-up of the two- and higher-dimensional Keller–Segel model in bounded domains and the whole space.

Lemma 8.19 (Decrease of 2^{nd} moment)**.** *Let $T > 0$ and assume that problem* (8.35) *subject to initial condition $\rho(0) = \rho_0$ has a regular solution ρ on $[0, T]$. Then for all $t \in [0, T]$*

$$\frac{d}{dt}\int_{\mathbb{T}^d} |x|^2 \rho(t,x)\phi(x)\,\mathrm{d}x \leq -\left(\int \rho(t,x)\phi(x)\,\mathrm{d}x\right)^2 + C_2 M \|\rho(t,\cdot)\|_{L^1(\mathbb{T}^d \setminus B_b)} + C_3 b M^2 + C_4 M.$$

Remark 8.20. Note that since $\operatorname{supp}\phi \subset (-\frac{1}{2}, \frac{1}{2})^d$ the integrand on the left-hand side is well-defined and smooth on the periodic box $\mathbb{T}^d$.

Proof of Lemma 8.19. We compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d} |x|^2 \rho(t,x)\phi(x)\,\mathrm{d}x &= -\int_{\mathbb{T}^d} \rho(t,x)\Lambda^\gamma(|x|^2\phi(x))\,\mathrm{d}x \\ &\quad -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \nabla(|x|^2\phi(x))\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,\mathrm{d}y\mathrm{d}x \\ &=: (i) + (ii).\end{aligned}$$

In order to estimate the first term on the right-hand side, let us recall that for $\gamma \in (0, 2)$ the fractional Laplacian has the following representation (see e.g. [35] or [92]):

$$\Lambda^\gamma f(x) = \text{p.v.}\int_{\mathbb{T}^d} (f(x) - f(y))G_{\gamma,d}(x-y)\,\mathrm{d}y,$$

where

$$G_{\gamma,d}(z) = c_{\gamma,d}\sum_{\alpha\in\mathbb{Z}^d} \frac{1}{|z-\alpha|^{d+\gamma}}, \quad z \neq 0,$$

and $c_{\gamma,d}$ is a normalisation constant. Using the above formula and the smoothness of ϕ_0, it is easy to see that there exists a positive constant $C_{\phi_0} < \infty$ such that for all $b \in (0, 1]$

$$\left\|\Lambda^\gamma\left(|x|^2\phi_0\left(\frac{x}{b}\right)\right)\right\|_{L^\infty(\mathbb{T}^d)} \leq C_{\phi_0} b^{2-\gamma}.$$

Recalling $\phi(x) = \phi_0(\frac{x}{b})$, we conclude that $(i) \leq CMb^{2-\gamma}$.

To estimate the second term, we introduce the splitting

$$\begin{aligned}
(ii) &= -2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,\mathrm{d}y\mathrm{d}x\\
&\quad-\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}|x|^2\nabla\phi(x)\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,\mathrm{d}y\mathrm{d}x\\
&= -2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\phi(y)\,\mathrm{d}y\mathrm{d}x\\
&\quad-2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,\mathrm{d}y\mathrm{d}x\\
&\quad-\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}|x|^2\nabla\phi(x)\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,\mathrm{d}y\mathrm{d}x\\
&=: (iii)+(iv)+(v).
\end{aligned}$$

On $\{x-y : x,y\in\operatorname{supp}\phi\}$ we have $\nabla K(z)=\frac{z}{|z|^2}$. Thus, upon symmetrisation,

$$\begin{aligned}
(iii) &= -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|x|^2-2x\cdot y+|y|^2}{|x-y|^2}\phi(x)\rho(t,y)\rho(t,x)\phi(y)\,\mathrm{d}y\mathrm{d}x\\
&= -\left(\int_{\mathbb{T}^d}\rho(t,x)\phi(x)\,\mathrm{d}x\right)^2.
\end{aligned}$$

Next, we note

$$\begin{aligned}
(iv) &= -2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,\mathrm{d}y\mathrm{d}x\\
&= -2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\frac{x-y}{|x-y|^2}\,\chi_{B_{\frac14}}(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,\mathrm{d}y\mathrm{d}x\\
&\quad+2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\chi_{\mathbb{T}^d\setminus B_{\frac14}}(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,\mathrm{d}y\mathrm{d}x\\
&= -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}[\phi(x)(1-\phi(y))x-\phi(y)(1-\phi(x))y]\cdot\frac{x-y}{|x-y|^2}\,\cdot\\
&\qquad\qquad\cdot\chi_{B_{\frac14}}(x-y)\rho(t,y)\rho(t,x)\,\mathrm{d}y\mathrm{d}x\\
&\quad+\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\chi_{\mathbb{T}^d\setminus B_{\frac14}}(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,\mathrm{d}y\mathrm{d}x\\
&\le CM\|\rho(t)\|_{L^1(\mathbb{T}^d\setminus B_{\frac{b}{2}})}+CbM^2.
\end{aligned}$$

In the last step we used

$$|\,[\phi(x)(1-\phi(y))x-\phi(y)(1-\phi(x))y]\,|\le C\chi_{\mathbb{T}^d\times\mathbb{T}^d\setminus B_{\frac{b}{2}}\times B_{\frac{b}{2}}}(x,y).$$

Similar arguments yield

$$
\begin{aligned}
(v) &= -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} |x|^2 \nabla\phi(x)\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,\mathrm{d}y\mathrm{d}x \\
&\le CM\|\rho(t)\|_{L^1(\mathbb{T}^d\setminus B_{\frac{b}{2}})} + CbM^2.
\end{aligned}
$$

(In both estimates, and thus also in the asserted estimate, the term CbM^2 can actually be dropped.)

Using all these estimates, we conclude

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}^d} |x|^2\rho(t,x)\phi(x)\,\mathrm{d}x \le -\left(\int_{\mathbb{T}^d}\rho(t,x)\phi(x)\,\mathrm{d}x\right)^2 &+ CM\|\rho(t)\|_{L^1(\mathbb{T}^d\setminus B_{\frac{b}{2}})} \\
&+ CM^2 b + CMb^{2-\gamma}.
\end{aligned}
$$

Since $\gamma \le 2$, the claimed bound follows. □

Next, we need to ensure that the mass – initially localised near the origin – cannot escape too fast. The statement and proof are analogous to [73, Lemma 8.3], where the extension to $\gamma \in (1,2]$ follows as in the previous lemma.

The existence of exploding solutions is shown completely analogously to [73, Proof of Theorem 8.1].

8.5.2 Transport-diffusion equation

In this section we will prove Theorem 8.9 in the remaining case $\gamma \in [1,2)$. The proof of this theorem follows along the lines of the proof of [34, Theorem 1.4], and we therefore only point out the differences. First of all, if $\gamma < 2$, condition (2.1) in [34] is no longer satisfied. We have the following replacement for [34, Theorem 2.1].

Theorem 8.21 (Wellposedness)**.** *Assume $\gamma \in (1,2)$ and let $v = v(x)$ be a smooth divergence-free vector field. For any $T > 0$ and $\mu_0 \in H^{\frac{\gamma}{2}}(\mathbb{T}^d)$ there exists a unique solution*

$$\mu \in L^2(0,T;H^\gamma)\cap C([0,T];H^{\frac{\gamma}{2}}) \text{ with } \partial_t\mu \in L^2(0,T;L^2)$$

of the Cauchy problem

$$
\begin{aligned}
\partial_t\mu + v\cdot\nabla\mu &= -\Lambda^\gamma\mu \quad \text{in } (0,T)\times\mathbb{T}^d, \qquad (8.36)\\
\mu(0) &= \mu_0.
\end{aligned}
$$

Proof. The existence of weak solutions

$$\mu \in L^2(0,T;H^{\frac{\gamma}{2}})\cap C([0,T];L^2) \text{ with } \partial_t\mu \in L^2(0,T;H^{-(1-\frac{\gamma}{2})}) \qquad (8.37)$$

to initial datum $\mu_0 \in L^2(\mathbb{T}^d)$ can be shown via a simple Galerkin scheme. Since $\gamma > 1$, regularity and uniqueness are straightforward as well. $\square$

Remark. If $\gamma \in (0,1]$, local existence and uniqueness of a weak solution $\mu \in C([0,T]; H^{\frac{1}{2}})$ with $\partial_t \mu \in C([0,T]; H^{-\frac{1}{2}})$ to the Cauchy problem (8.36) with initial datum in $H^{\frac{1}{2}}$ can still be established: the existence of rough solutions is again obtained via a Galerkin method. To prove the asserted regularity and uniqueness, one first notes that the constructed weak solution μ satisfies the pointwise equality

$$\partial_t S_k \mu + \nabla \cdot S_k(v\mu) = -\Lambda^\gamma S_k \mu,$$

where S_k are the LP-projections introduced in Appendix 8.5.3, and then proceeds as in the proof of Proposition 8.23.

Owing to the worse regularity, more care has to be taken when approximating the advection-diffusion equation by the pure transport equation. Our replacement for [34, Lemma 2.4] is the following

Lemma 8.22 (Approximation by pure transport)**.** *Let $v = v(x)$ be a smooth divergence-free vector field. Assume $\gamma \in [1,2)$ and let $\eta_0 \in H^{\frac{\gamma}{2}}(\mathbb{T}^d)$. Let $\eta^0 \in C([0,\infty); H^{\frac{\gamma}{2}})$ be a weak solution of the transport problem* (8.26) *and let $\eta^\varepsilon = \mu$ solve* (8.36) *with $-\Lambda^\gamma$ replaced* [7] *by $-\varepsilon\Lambda^\gamma$ and initial datum η_0. Then*

$$\frac{d}{dt}\|\eta^\varepsilon(t) - \eta^0(t)\|_{L^2}^2 \le \frac{\varepsilon}{2}\|\eta^0(t)\|_{\dot{H}^{\gamma/2}}^2 \le \frac{\varepsilon}{2}\exp(C(v)t)\|\eta_0\|_{\dot{H}^{\gamma/2}}^2, \tag{8.38}$$

where $C(v)$ is the constant from Proposition 8.11.

Proof. The difference $\eta^\varepsilon - \eta^0$ satisfies

$$\partial_t(\eta^\varepsilon - \eta^0) + u \cdot \nabla(\eta^\varepsilon - \eta^0) = -\varepsilon\Lambda^\gamma \eta^\varepsilon, \tag{8.39}$$

where for fixed time t the equality is to be understood in $H^{\frac{\gamma}{2}-1} \subseteq H^{-\frac{\gamma}{2}}$. We can therefore take the dual pairing $\dot{H}^{-\frac{\gamma}{2}} \times \dot{H}^{\frac{\gamma}{2}}$ of the equation with $(\eta^\varepsilon - \eta^0)(t) \in H^{\frac{\gamma}{2}}$ to obtain after an absorption argument the first inequality in (8.38). (Here we also used the incompressibility and the smoothness of the flow which guarantee that $B(f,g) := \langle u \cdot \nabla f, g\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}$ satisfies $B(f,f) = 0$ for all $f \in C^\infty$ and extends uniquely to a bounded bilinear form on $H^{\frac{1}{2}} \times H^{\frac{1}{2}}$.) The second inequality in (8.38) is just the boundedness of the transport evolution with respect to $\|\cdot\|_{\dot{H}^{\frac{\gamma}{2}}}$ (cf. equation (8.27)). $\square$

[7] In order to facilitate the comparison with [34], we adopt the rescaling to 'small diffusion on long time scales' as introduced in [34].

Remark. The statement of Lemma 8.22 remains true for $\gamma \in (0,1)$ if restricting to initial data in $H^{\frac{1}{2}}$. Indeed, in this case one only needs to notice that (for fixed time) the equation (8.39) holds in $H^{-\frac{1}{2}}$ and that $(\eta^\varepsilon - \eta^0)(t) \in H^{\frac{1}{2}}$.

The remaining lemmas used in the proof of [34, Theorem 1.4] can either be shown by similar arguments as in Lemma 8.22 (where for mere L^2 data the regularity (8.37) has to be used) or require only a formal adaptation (such as replacing the 'diffusion operator' $-\Gamma$ by $-\Lambda^\gamma$).

8.5.3 Transport equation in $H^\sigma(\mathbb{T}^d)$

Here we are concerned with the linear transport equation with a (prescribed) divergence-free smooth velocity field $v = v(x)$:

$$\begin{aligned} \partial_t \eta + v \cdot \nabla \eta &= 0 \quad \text{in} \ \ (0,\infty) \times \mathbb{T}^d, \\ \eta(0) &= \eta_0. \end{aligned} \tag{8.40}$$

Our aim is to prove Proposition 8.11, i.e. the boundedness of the associated evolution in fractional Hilbert spaces $H^\sigma(\mathbb{T}^d), \sigma > 0$, where we do not aim for optimal regularity with respect to v. In the whole space case fairly general a priori estimates in Besov spaces can be found in [3]. As in [3] we will make use of a standard tool from harmonic analysis, which we shall introduce in the following.

Preliminaries

We consider a Littlewood-Paley decomposition: let $\phi_0 \in C_c^\infty(\mathbb{R}^d)$ be a radial bump function with $\operatorname{supp} \phi_0 \subset B_{11/10}(0)$ which is equal to 1 on $B_1(0)$ and satisfies $0 \le \phi_0 \le 1$. Denoting $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$, we then have

$$\phi_0(2\xi) + \sum_{k \ge 0} \phi(2^{-k}\xi) = 1, \qquad \xi \in \mathbb{R}^d.$$

For smooth functions η on $\mathbb{T}^d$ we then define the operators

$$S_{-1}\eta(x) = \sum_{\alpha \in \mathbb{Z}^d} \phi_0(2\alpha)\hat{\eta}(\alpha) \mathrm{e}^{2\pi \mathrm{i} x \cdot \alpha} = \phi_0(0)\hat{\eta}(0)$$

and for $k \ge 0$

$$S_k \eta(x) = \sum_{\alpha \in \mathbb{Z}^d} \phi(2^{-k}\alpha)\hat{\eta}(\alpha) \mathrm{e}^{2\pi \mathrm{i} x \cdot \alpha}.$$

Note that S_k localises to frequency $\sim 2^k$, i.e. $\operatorname{supp} \widehat{S_k \eta} \subset \{\alpha \in \mathbb{Z}^d : |\alpha| \approx 2^k\}$ and we have equivalence of (semi-) norms

$$\|\eta\|_{\dot{H}^\sigma}^2 \sim \sum_{k \geq 0} 2^{2\sigma k} \|S_k \eta\|_{L^2}^2. \tag{8.41}$$

We will at times also use the notation $S_{\leq N}, S_{M<\cdots<N}$ and $S_{\geq N}$ to denote the sums of operators corresponding to $\sum_{-1 \leq k \leq N} S_k, \sum_{M<k<N} S_k$ and $\sum_{k \geq N} S_k$.

Boundedness of evolution

We will now provide a proof of the transport estimate:

Proposition 8.23. *Assume $\sigma > 0$. Any sufficiently regular solution η of* (8.40) *satisfies*

$$\|\eta(t)\|_{\dot{H}^\sigma}^2 \leq \exp(C(v)t) \|\eta_0\|_{\dot{H}^\sigma}^2, \quad t \geq 0,$$

where the positive constant $C(v)$ satisfies the bound

$$C(v) \lesssim_{\sigma,d} \|\Lambda^{\sigma + \frac{d}{2} + 1} v\|_{L^2}.$$

The proof exploits the following gain at level k for the commutator involving an LP projection S_k for $k \gg 1$.

Lemma 8.24. *For smooth functions f, g on the torus the following commutator estimate holds true:*

$$\|[S_k, g]f\|_{L^2(\mathbb{T}^d)} \leq 2^{-k} \|\nabla \phi\|_{L^\infty} \|\hat{g}(\beta)\beta\|_{l^1_\beta} \|f\|_{L^2(\mathbb{T}^d)}.$$

Proof of Lemma 8.24. We first note

$$\|[S_k, g]f\|_{L^2(\mathbb{T}^d)} = \|\widehat{[S_k, g]f}\|_{l^2(\mathbb{Z}^d)}$$

and therefore consider

$$\begin{aligned}
\widehat{[S_k, g]}f(\alpha) &= \widehat{S_k(gf)}(\alpha) - \widehat{g} * \widehat{S_k f}(\alpha) \\
&= \sum_{\beta \in \mathbb{Z}^d} [\phi(2^{-k}\alpha) - \phi(2^{-k}(\alpha - \beta))] \hat{g}(\beta) \hat{f}(\alpha - \beta) \\
&= \sum_{\beta} 2^{-k} \int_0^1 \nabla \phi(2^{-k}(\alpha - (1-s)\beta)) ds \cdot \beta \hat{g}(\beta) \hat{f}(\alpha - \beta).
\end{aligned}$$

Hence

$$|\widehat{[S_k, g]f}(\alpha)| \leq 2^{-k}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)} \sum_\beta |\beta\hat{g}(\beta)||\hat{f}(\alpha-\beta)|.$$

Young's convolution inequality then yields the claim

$$\|\widehat{[S_k, g]f}\|_{l^2(\mathbb{Z}^d)} \leq 2^{-k}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)}\|\beta\hat{g}(\beta)\|_{l^1_\beta}\|f\|_{L^2},$$

where we used $\|\hat{f}\|_{l^2} = \|f\|_{L^2}$. □

We are now in a position to show the boundedness of the evolution (8.40) in $\dot{H}^\sigma(\mathbb{T}^d)$.

Sketch proof of Proposition 8.23. Without loss of generality we can assume $\hat{\eta}(0) = 0$. In the following we will omit any possible dependence of constants on σ and d. Now let $k \geq 0$ be a fixed but arbitrary integer. The equation implies

$$\partial_t S_k\eta = -\nabla \cdot S_k(v\eta)$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|S_k\eta\|^2_{L^2(\mathbb{T}^d)} = -\int \nabla \cdot S_k(v\eta)\, S_k\eta.$$

Since by incompressibility

$$\int \nabla \cdot (v\, S_k\eta)\, S_k\eta = -\frac{1}{2}\int v \cdot \nabla|S_k\eta|^2 = 0,$$

it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|S_k\eta\|^2_{L^2(\mathbb{T}^d)} &= \int \nabla \cdot [v, S_k]\eta\, S_k\eta \\ &= \int \nabla \cdot \tilde{S}_k[v, S_k]\eta\, S_k\eta \qquad\qquad (\tilde{S}_k\, S_k = S_k) \\ &\leq \|\nabla \cdot \tilde{S}_k[v, S_k]\eta\|_{L^2}\|S_k\eta\|_{L^2} \\ &\lesssim 2^k\|\tilde{S}_k[v, S_k]\eta\|_{L^2}\|S_k\eta\|_{L^2}, \end{aligned} \tag{8.42}$$

where $\tilde{S}_k$ denotes a suitable Fourier multiplier localising to frequency $\sim 2^k$ whose symbol is equal to 1 on $\operatorname{supp}\phi(2^{-k}\cdot)$. We now assume $k \gg 1$ and split

$$v = S_{\leq k-4}v + S_{>k-4}v$$

and consider

$$\tilde{S}_k[v, S_k]\eta = \tilde{S}_k[S_{\le k-4}v, S_k]\eta + \tilde{S}_k[S_{>k-4}v, S_k]\eta. \tag{8.43}$$

With regard to the regularity of η, the first term is the delicate one. It can be estimated using Lemma 8.24, as we will show now. Note that there exists a multiplier S'_k localising to frequency $\sim 2^k$ such that

$$\tilde{S}_k[S_{\le k-4}v, S_k]\eta = \tilde{S}_k[S_{\le k-4}v, S_k]S'_k\eta.$$

Now Lemma 8.24 applied to $g = S_{\le k-4}v, f = S'_k\eta$ yields

$$\begin{aligned} \|\tilde{S}_k[S_{\le k-4}v, S_k]S'_k\eta\|_{L^2} &\le \|[S_{\le k-4}v, S_k]S'_k\eta\|_{L^2} \\ &\le C2^{-k}\|\widehat{S_{\le k-4}}v(\alpha)\alpha\|_{l^1_\alpha}\|S'_k\eta\|_{L^2} \\ &\le C2^{-k}\|\hat{v}(\alpha)\alpha\|_{l^1_\alpha}\|S'_k\eta\|_{L^2}, \end{aligned}$$

where in the last step we used

$$\|\widehat{S_{\le k-4}}v(\alpha)\alpha\|_{l^1_\alpha} = \sum_\alpha |\sum_{j\le k-4} \phi(2^{-j}\alpha)\hat{v}(\alpha)\alpha| \le \sum_\alpha |\hat{v}(\alpha)\alpha|.$$

Finally notice that by the equivalence of norms (8.41)

$$\sum_{k\gg 1} 2^{2k\sigma}2^k \left(2^{-k}\|\hat{v}(\alpha)\alpha\|_{l^1_\alpha}\|S'_k\eta\|_{L^2}\right)\|S_k\eta\|_{L^2} \lesssim \|\hat{v}(\alpha)\alpha\|_{l^1_\alpha}\|\eta(t)\|^2_{\dot{H}^\sigma(\mathbb{T}^d)}.$$

Estimating the second term in (8.43) is straightforward if one is not interested in optimal regularity results for v. For a rough estimate, we note that the part of this term which requires the highest regularity of v is

$$S_k(S_{k-4<\cdots<k+4}v\,\eta)$$

as it may involve low frequencies of η. We first estimate using a Bernstein inequality (see e.g. [3, Lemma 2.1])

$$\begin{aligned} \|S_k(S_{k-4<\cdots<k+4}v\,\eta)\|_{L^2} &\lesssim 2^{\frac{kd}{2}}\|S_k(S_{k-4<\cdots<k+4}v\,\eta)\|_{L^1} \\ &\lesssim 2^{\frac{kd}{2}}\|S_{k-4<\cdots<k+4}v\|_{L^2}\|\eta\|_{L^2} \end{aligned}$$

and note that thanks to Cauchy-Schwarz and $\hat{\eta}(0) = 0$

$$\begin{aligned}\sum_{k\gg 1} 2^{2k\sigma}2^k \left(2^{\frac{kd}{2}}\|S_{k-4<\cdots<k+4}v\|_{L^2}\|\eta\|_{L^2}\right)\|S_k\eta\|_{L^2} \\ &\lesssim \sum_{k\gg 1}\|S_{k-4<\cdots<k+4}(\Lambda^{\sigma+\frac{d}{2}+1}v)\|_{L^2}2^{k\sigma}\|S_k\eta\|_{L^2}\|\eta\|_{L^2} \\ &\lesssim \|\Lambda^{\sigma+\frac{d}{2}+1}v\|_{L^2}\|\eta\|^2_{\dot{H}^\sigma(\mathbb{T}^d)}.\end{aligned}$$

For the low frequencies $k \le k_0$ (k_0 being a suitable fixed positive integer), we estimate using (8.42) and omitting the k_0 dependence

$$\begin{aligned}\frac{d}{dt}\sum_{0\le k\le k_0} 2^{2\sigma k}\|S_k\eta\|^2_{L^2(\mathbb{T}^d)} &\lesssim \sum_{0\le k\le k_0}\|[v,S_k]\eta\|_{L^2}\|S_k\eta\|_{L^2} \\ &\lesssim \|\hat{v}(\alpha)\alpha\|_{l^1_\alpha}\|\eta(t)\|^2_{\dot{H}^\sigma(\mathbb{T}^d)}.\end{aligned}$$

In the second step, we used Lemma 8.24 (mainly in order to illustrate that the estimate is independent of $\hat{v}(0)$).

We now recall (8.42) and combine our estimates for high and low frequencies to conclude

$$\frac{d}{dt}\|\eta(t)\|^2_{\dot{H}^\sigma(\mathbb{T}^d)} \lesssim \left(\|\Lambda^{\sigma+\frac{d}{2}+1}v\|_{L^2} + \|\hat{v}(\alpha)\alpha\|_{l^1_\alpha}\right)\|\eta(t)\|^2_{\dot{H}^\sigma(\mathbb{T}^d)}. \tag{8.44}$$

Finally note that since $\sigma > 0$

$$\sum_{\alpha\neq 0}|\hat{v}(\alpha)\alpha| \le \left(\sum_{\alpha\neq 0}|\hat{v}(\alpha)|^2|\alpha|^{2(1+\frac{d}{2}+\sigma)}\right)^{\frac{1}{2}}\left(\sum_{\alpha\neq 0}|\alpha|^{-d-2\sigma}\right)^{\frac{1}{2}} \lesssim \|\Lambda^{\sigma+\frac{d}{2}+1}v\|_{L^2}.$$

Hence, Gronwall's inequality applied to (8.44) yields the claim. □

8.5.4 Examples of γ-RE flows

In this section, we provide examples which show that, in general, the classes of γ-relaxation enhancing flows introduced in Definition 8.7 are different for different γ. Our construction is an adaptation of [34, Proposition 6.2].

Proposition 8.25. *For any $\gamma > \frac{1}{2}$ and any $\varepsilon > 0$ there exists a smooth, divergence-free vector field $u(x)$ on $\mathbb{T}^2$ such that the induced unitary evolution U on $L^2(\mathbb{T}^2)$ has discrete spectrum and all non-constant eigenfunctions lie in $H^{\gamma-\varepsilon} \setminus H^{\gamma+\varepsilon}$. In particular, u is $2(\gamma+\varepsilon)$-RE but not $2(\gamma-\varepsilon)$-RE.*

Sketch proof. The proof adapts the construction in [34, Proposition 6.2]. We therefore only point out the necessary modifications. Recall that a real number r is called

ι-Diophantine if there exists a constant $C > 0$ such that for all $k \in \mathbb{Z} \setminus \{0\}$:

$$\inf_{p\in\mathbb{Z}} |r \cdot k + p| \geq \frac{C}{|k|^{1+\iota}}.$$

A number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is called Liouvillean if it is not ι-Diophantine for any $\iota \in (0, \infty)$. In the following we let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be a positive Liouvillean number. Then, by [34, Proposition 6.3] (see also the original statement [71, Theorem 4.5]), there exists a smooth function $h \in C^\infty(\mathbb{T}^1)$ and a nowhere continuous, integrable function $\tilde{R}$ on $\mathbb{T}^1$ such that

$$\tilde{R}(\xi + \alpha) - \tilde{R}(\xi) = h(\xi) \text{ for all } \xi \in \mathbb{T}^1 . \tag{8.45}$$

Observe that h has zero mean and that we may assume without loss of generality $\tilde{R}$ to be mean-free as well. Since $\tilde{R} \in L^1(\mathbb{T}^1)$, it can naturally be identified with an element in $H^\sigma(\mathbb{T}^1)$ for sufficiently small $\sigma \in \mathbb{R}$. Thus, we can define

$$r := \inf\{s \in \mathbb{R} : \Lambda^{-s}\tilde{R} \in H^\gamma\}.$$

The discontinuity of $\tilde{R}$ and $\gamma > \frac{1}{2}$ imply that $r \in (0, \infty)$. We now set $R := \Lambda^{-r}\tilde{R}$ and $Q := \Lambda^{-r}h + 1$. Let further $\varepsilon > 0$ be small enough such that $\gamma - \varepsilon > \frac{1}{2}$. Clearly

$$R \in H^{\gamma-\varepsilon}(\mathbb{T}^1) \setminus H^{\gamma+\varepsilon}(\mathbb{T}^1), \tag{8.46}$$

and thanks to the Sobolev embedding into Hölder spaces, we may henceforth identify R with its Hölder continuous representative. Furthermore,

$$Q \in C^\infty(\mathbb{T}^1) \ \text{ with } \ \int_{\mathbb{T}^1} Q = 1,$$

and from (8.45) we deduce

$$R(\xi + \alpha) - R(\xi) = Q(\xi) - 1 \text{ for all } \xi \in \mathbb{T}^1 . \tag{8.47}$$

Thanks to equation (8.47) and the smoothness of Q, we may now proceed as in the proof of [34, Proposition 6.2]. Our arguments only deviate when it comes to determining the regularity of the eigenfunctions $\psi^w_{nl} \in L^2(\mathbb{T}^2)$, where we use the same notation as in [34]. For this part, let us recall (cf. [34, equation (6.2)]) that the eigenfunctions have the form

$$\psi(x, y) := \psi^w_{nl}(x, y) = \zeta(x, y)\mathrm{e}^{2\pi\mathrm{i}(n\alpha+l)R(x-\alpha y)},$$

where $n, l \in \mathbb{Z}$. Here $\zeta(x, y)$ is a smooth complex-valued function with $|\zeta| = 1$, which is not periodic in y. To complete the proof, it remains to show that the regularity of R implies the asserted regularity of ψ. The remaining steps are then exactly the same as in [34].

Regarding the regularity of ψ, we may henceforth assume $(n, l) \neq (0, 0)$ since otherwise the explicit form of ζ in [34, equation (6.2)] implies that ψ is constant. Since R is Hölder continuous and bounded, the regularity (8.46) implies that for any $\lambda \in \mathbb{R}^*$

$$R_\lambda(\xi) := \mathrm{e}^{\mathrm{i}\lambda R(\xi)} \in H^{\gamma-\varepsilon}(\mathbb{T}^1) \setminus H^{\gamma+\varepsilon}(\mathbb{T}^1). \tag{8.48}$$

This can easily be seen by noting that $\mathrm{e}^{\mathrm{i}\lambda\,\cdot} : \mathbb{R} \to \mathbb{S}^1$ is a local C^∞ diffeomorphism and by using standard fractional chain rule/Moser type estimates (see e.g. [98, Chapter 3]).

Let us next fix $\lambda = 2\pi(n\alpha + l)$, which is different from 0, and consider the function

$$\Theta_\lambda(x, y) := R_\lambda(x - \alpha y) : \mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}} \to \mathbb{S}^1,$$

where $\mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}}$ denotes the periodic box $[0, 1) \times [0, \alpha^{-1})$. By using the explicit definition of $\|\cdot\|_{\dot{H}^s}$ (in terms of Fourier coefficients), one quickly finds

$$\|\Theta_\lambda\|_{\dot{H}^s(\mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}})} = C_{s,\alpha} \|R_\lambda\|_{\dot{H}^s(\mathbb{T}^1)}$$

for some positive constant $C_{s,\alpha} > 0$. Thus, (8.48) yields

$$\Theta_\lambda \in (H^{\gamma-\varepsilon} \setminus H^{\gamma+\varepsilon})(\mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}}). \tag{8.49}$$

To conclude the regularity

$$\psi \in (H^{\gamma-\varepsilon} \setminus H^{\gamma+\varepsilon})(\mathbb{T}^1 \times \mathbb{T}^1)$$

one can use a smooth partition of unity of $\mathbb{T}^2$ in y-direction corresponding to a finite number of overlapping cylinders of height $\frac{1}{2}\alpha^{-1}$ (if $\alpha > 1$). This allows us to split ψ into a finite sum of functions, which may be considered (by first (smoothly) extending by zero to $\mathbb{T}^1 \times \mathbb{R}^1$ and then suitably periodising) as being defined on $\mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}}$. Each of these summands is the product of a smooth function with Θ_λ so that (8.49) implies $\psi \in H^{\gamma-\varepsilon}$. In order to see $\psi \notin H^{\gamma+\varepsilon}$ one can use similar arguments together with the fact that $|\zeta| = 1$ everywhere. □

Abbreviations

γ-RE	γ-relaxation enhancing
kD for $k \in \mathbb{N}^+$	k-dimensional (mainly used for $k \in \{1,2,3\}$)
x-m	x-monotonic, non-decreasing in the x variable
BFP	bosonic Fokker–Planck equations
cdf	cumulative distribution function
GBFP	generalised bosonic Fokker–Planck equations
KQ	Kaniadakis–Quarati model for bosons
LP	Littlewood–Paley
lsc	lower semicontinuous
LWP	local wellposedness
PDE	partial differential equation
usc	upper semicontinuous

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