NETWORKS AND FARSIGHTED STABILITY

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Abstract

The main contribution of this paper is to provide a framework in which the notion of farsighted stability for games, introduced by Chwe (1994), can be applied to *directed* networks. In particular, we introduce the notion of a *supernetwork*. A supernetwork is made up of a collection of directed networks (the nodes) and uniquely represents (via the arcs connecting the nodes) agent preferences and the rules governing network formation. By reformulating Chwe's basic result on the nonemptiness of farsightedly stable sets, we show that for any supernetwork (i.e., for any collection of directed networks and any collection of rules governing network formation), there exists a farsightedly stable directed network. We also introduce the notion of a Nash network relative to a given supernetwork, as well as the notions of symmetric, nonsimultaneous, and decomposable supernetworks. To illustrate the utility of our framework, we present several examples of supernetworks, compute the farsightedly stable networks, and the Nash networks.

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1 Introduction

Overview

Since the seminal paper by Jackson and Wolinsky (1996) there has been a rapidly growing literature on social and economic networks and their stability and efficiency properties (e.g., see Jackson (2001) and Jackson and van den Nouweland (2001)). As noted by Jackson (2001), an important issue that has not yet been addressed in the literature on networks and network formation is the issue of *farsighted stability* (see Jackson (2001), p.21 and p.35)). This issue is the focus of our paper. Our main contribution is to construct a framework in which the notion of farsighted stability for games, introduced by Chwe (1994), can be applied to collections of *directed networks*. Our construction proceeds in two steps. First, we extend the definition of a directed network found in the literature (e.g., see Rockafellar (1984)). Second, using our extended definition we introduce the notion a *network formation network*. We call such a network a *supernetwork*. All directed networks are composed of nodes and arcs. In most economic applications, nodes represent economic agents, while arcs represent connections or interactions between agents. In a supernetwork, nodes represent the networks in a given collection, while arcs represent coalition moves and coalitional preferences over the networks in the collection. Given any collection of directed networks and any profile of agent preferences over the collection, a supernetwork uniquely represents all the coalitional preferences and all the coalitional moves allowed by the rules governing network formation (i.e., the rules governing the addition, subtraction, or replacement of arcs).

Given the rules governing network formation as represented via the supernetwork, *a directed network (i.e., a particular node in the supernetwork) is said to be farsightedly stable if no agent or coalition of agents is willing to alter the network (via the addition, subtraction, or replacement of arcs) for fear that such an alteration might induce further network alterations by other agents or coalitions that in the end leave the initially deviating agent or coalition no better off - and possibly worse off.* By reformulating Chwe's basic result on the nonemptiness of farsightedly stable sets, we show that for any supernetwork corresponding to a given collection of directed networks, the set of farsightedly stable networks is nonempty.

Our second contribution concerns noncooperative network formation. In particular, we use our framework to define the notion of a *Nash network relative to a given supernetwork*, and to introduce the notions of *symmetric, nonsimultaneous, and decomposable supernetworks*. Extending Chwe's (1994) results on Nash equilibrium and farsighted stability, we show that any strict Nash network relative to a given symmetric, nonsimultaneous supernetwork is contained in the set of farsightedly stable networks. We also show that any strict, strong Nash network relative to a given symmetric, decomposable supernetwork is contained in the farsightedly stable set.

In order to illustrate the power and utility of our framework, we present several examples. In our first series of examples, we examine, within the context of strategic information sharing, the relationship between the rules governing the formation of information sharing networks and the resulting supernetwork. In order to illustrate the way in which these rules - as represented via the supernetwork - affect the resulting set of farsightedly stable networks as well as the set of Nash networks, we compute the Nash and farsightedly stable networks corresponding to several supernetworks representing various rules. Next, we expand our example of strategic information sharing by adding to the number of possible information sharing networks. Assuming a particular set of network formation rules, we again compute the Nash and farsightedly stable networks relative the supernetwork representing these rules for our expanded collection of information sharing networks. In our expanded example, there is a unique Nash network relative to the given supernetwork, and this unique Nash network is *not* contained in the set of farsightedly stable networks. Moreover, in our expanded example the set of farsightedly stable networks is equal to the set of Pareto efficient networks. Thus, not only is the unique Nash network not farsightedly stable, it is not efficient.

In our final example we consider the problem of strategic pollution. In particular, we construct the supernetwork corresponding to a discrete, 3-agent version of the Shapley-Shubik garbage game (Shapley and Shubik (1969)). In the garbage game, each agent has a bag of garbage that can be kept by the agent or dumped onto the property of another agent. The game is discrete in that each agent's bag of garbage cannot be divided up - each agent either keeps his bag or dumps the entire bag onto the property of one other agent. For the garbage game supernetwork, we show that the set of farsightedly stable garbage networks and the set of Nash garbage networks are not equal - but have a nonempty intersection. Moreover, we show that for *all* farsightedly stable garbage networks, each agent ends up with one bag garbage. Thus, the garbage network in which each agent keeps his own bag of garbage, and therefore chooses not to pollute his neighbor, is farsightedly stable. This is not the case for the set of Nash garbage networks. In particular, there are Nash garbage networks in which agents choose to pollute others (i.e., pollution is a Nash equilibrium).

Directed Networks vs Linking Networks

In a directed network, each arc possesses an orientation or direction: arc j connecting nodes i and i' must either go from node i to node i' or must go from node i' to node i.[1] In an undirected (or linking) network, arc j would have no orientation and would simply indicate a connection or link between nodes i and i'. Under our extended definition of directed networks, nodes are allowed to be connected by multiple arcs. For example, nodes i and i' might be connected by arcs j and j', with arc j running from node i to i' and arc j' running in the opposite direction (i.e., from node i' to node i).[2] Thus, if node i represents a seller and node i' a buyer, then arc j might represent a contract offer by the seller to the buyer, while arc j' might represent the acceptance or rejection of that contract offer. Also, under our extended definition arcs are allowed to be used multiple times in a given network. For example, arc j might be used to connect nodes i and i' as well as nodes i' and i''. However, we do not allow arc j to go from node i to node i' multiple times in the same direction. By allowing arcs to possess direction and be used multiple times and

[1] We denote arc j going from node i to node i' via the ordered pair $(j,(i,i'))$, where (i,i') is also an ordered pair. Alternatively, if arc j goes from node i' to node i, we write $(j,(i',i))$.

[2] Under our extended definition, arc j' might also run in the same direction as arc j.

by allowing nodes to be connected by multiple arcs, our extended definition makes possible the application of networks to a richer set of economic environments. Until now, most of the economic literature on networks has focused on linking networks (see for example, Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997)).

Given a particular directed network, an agent or a coalition of agents can change the network to another network by simply adding, subtracting, or replacing arcs from the existing network in accordance with certain rules represented via the supernetwork.[3] For example, if the nodes in a network represent agents, then the rule for adding an arc j from node i to node i' might require that both agents i and i' agree to add arc j. Whereas the rule for subtracting arc j, from node i to node i', might require that only agent i or agent i' agree to dissolve arc j.

In current research, we are analyzing the efficiency properties of farsightedly stable networks. While here we focus on directed networks, the same methodology can be used to deduce the existence of farsightedly stable undirected networks (i.e., linking networks - such as the networks considered by Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997)). An excellent paper on stability *and* efficiency in linking networks is Jackson (2001) (see also, Jackson and Watts (1998), Jackson and van den Nouweland (2000), Skyrms and Pemantle (2000), Watts (2001), and Slikker and van den Nouweland (2001)). In future research, we will focus on the dynamics of network formation - along the lines of Konishi and Ray (2001).

[3] Put differently, agents can change one network to another network by adding, subtracting, or replacing ordered pairs, $(j, (i, i'))$, in accordance with certain rules.

2 Directed Networks

We begin by giving a formal definition of the class of directed networks we shall consider. Let N be a finite set of nodes, with typical element denoted by i, and let A be a finite set of arcs, with typical element denoted by j. Arcs represent potential connections between nodes, and depending on the application, nodes can represent economic agents or economic objects such as markets or firms.[4]

Definition 1 *(Directed Networks)*
Given node set N and arc set A, a directed network, G, is a subset of $A \times (N \times N)$. We shall denote by $\mathbb{N}(N, A)$ the collection of all directed networks given N and A.

A directed network $G \in \mathbb{N}(N, A)$ specifies how the nodes in N are connected via the arcs in A. Note that in a directed network order matters. In particular, if $(j, (i, i^{'})) \in G$, this means that arc j *goes from* node i *to* node $i^{'}$. Also, note that under our definition of a directed network, loops are allowed - that is, we allow an arc to go from a given node back to that given node. Finally, note that under our definition an arc can be used multiple times in a given network and multiple arcs can go from one node to another. However, our definition does not allow an arc j to go from a node i to a node i' multiple times.

The following notation is useful in describing networks. Given directed network $G \subseteq A \times (N \times N)$, let

$$\left.\begin{array}{c} G(j) := \left\{(i, i^{'}) \in N \times N : (j, (i, i^{'})) \in G\right\}, \\ G(i) := \left\{j \in A : (j, (i, i^{'})) \in G \text{ or } (j, (i', i)) \in G\ \right\} \\ G(i, i^{'}) := \left\{j \in A : (j, (i, i^{'})) \in G\right\}, \\ G(j, i) := \left\{i^{'} \in N : (j, (i, i^{'})) \in G\right\}. \end{array}\right\} \tag{1}$$

Thus,

$G(j)$ is the *set of node pairs* connected by arc j in network G,
$G(i)$ is the *set of arcs* going from node i or coming to node i in network G,
$G(i, i^{'})$ is the *set of arcs* going from node i to node i' in network G,
and
$G(j, i)$ is the *set of nodes* which can be reached by arc j from node i in network G.

Note that if for some arc $j \in A$, $G(j)$ is empty, then arc j is not used in network G. Moreover, if for some node $i \in N$, $G(i)$ is empty then node i is not used in network G, and node i is said to be isolated relative to network G.

[4] Of course in a supernetwork, nodes represent networks.

If in our definition of a directed network, we had required that $G(j)$ be single-valued and nonempty for all arcs $j \in A$, then our definition would have been the same as that given by Rockafellar (1984).

Suppose that the node set N is given by $N = \{i_1, i_2, \ldots, i_5\}$, while the arc set A is given by $A = \{j_1, j_2, \ldots, j_5, j_6, j_7\}$. Consider the network, G, depicted in Figure 1.

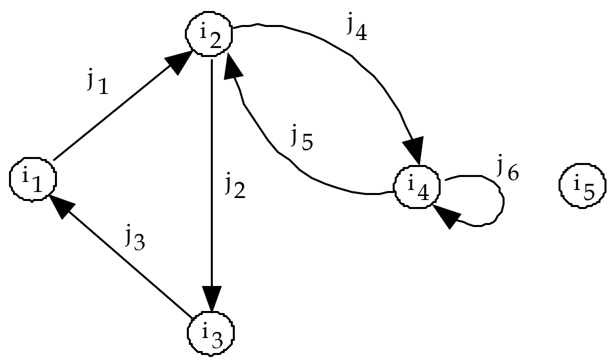


Figure 1: Network G

In network G, $G(j_6) = \{(i_4, i_4)\}$. Thus, $(j_6, (i_4, i_4)) \in G$ is a loop. Also, in network G, arc j_7 is not used. Thus, $G(j_7) = \emptyset$.[5] Finally, note that $G(i_4) = \{j_4, j_5, j_6\}$, while $G(i_5) = \emptyset$. Thus, node i_5 is *isolated* relative to G.[6]

Consider the new network, $G' \in \mathbb{N}(N, A)$ depicted in Figure 2.

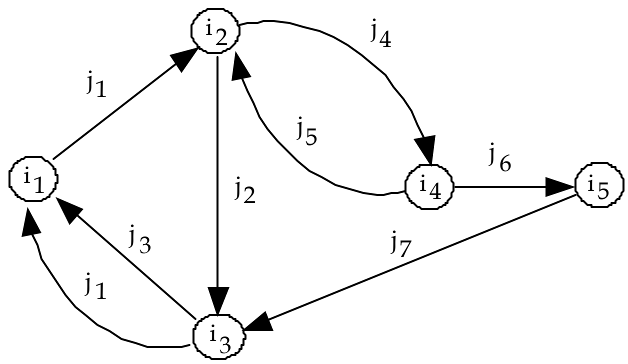


Figure 2: Network G'

[5] The fact that arc j_7 is not used in network G can also be denoted by writing

$$j_7 \notin proj_A G,$$

where $proj_A G$ denotes the projection onto A of the subset

$$G \subseteq A \times (N \times N)$$

representing the network.

[6] If the loop $(j_7, (i_5, i_5))$ were part of network G in Figure 1, then node i_5 would no longer be considered isolated under our definition. Moreover, we would have $G(i_5) = \{j_7\}$.

In network G', $G'(j_1) = \{(i_1, i_2), (i_3, i_1)\}$. Thus, $(j_1, (i_1, i_2)) \in G'$ and $(j_1, (i_3, i_1)) \in G'$. Note that in network G', node i_5 is no longer isolated. In particular, $G'(i_5) = \{j_6, j_7\}$. Also, note that nodes i_2 and i_4 are connected by two different arcs pointed in opposite directions. Under our definition of a directed network it is possible to alter network G' by replacing arc j_5 from i_4 to i_2 with arc j_4 *from* i_4 *to* i_2. However, it is *not* possible under our definition to replace arc j_5 from i_4 to i_2 with arc j_4 from i_2 to i_4 - because our definition does not allow j_4 to go *from* i_2 *to* i_4 multiple times. Finally, note that nodes i_1 and i_3 are also connected by two different arcs, but arcs pointed in the same direction. In particular, $G(i_3, i_1) = \{j_1, j_3\}$.

3 Supernetworks

3.1 Definition

Let D denote a finite set of agents (or economic decision making units) with typical element denoted by d, and let 2^D denote the collection of all nonempty subsets (or coalitions) of D with typical element denoted by S. We shall denote by $|S|$ the number agents in coalition S.

Given collection of directed networks $\mathbb{G} \subseteq \mathbb{N}(N, A)$, we shall assume that each agent's preferences over networks in $\mathbb{G}$ are specified via a network payoff function,

$$v_d(\cdot) : \mathbb{G} \to \mathbb{R}.$$

For each agent $d \in D$ and each directed network $G \in \mathbb{G}$, $v_d(G)$ is the payoff to agent d in network G. Agent d then prefers network G' to network G if and only if

$$v_d(G') > v_d(G).$$

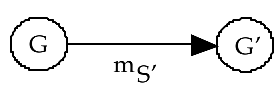
Moreover, coalition $S' \in 2^D$ prefers network G' to network G if and only if

$$v_d(G') > v_d(G) \text{ for all } d \in S'.$$

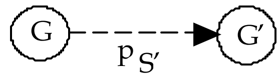
By viewing each network G in a given collection of directed networks $\mathbb{G} \subseteq \mathbb{N}(N, A)$ as a node in a larger network, we can give a precise network representation of the rules governing network formation as well as agents' preferences. To begin, let

$$\mathbb{M} := \left\{m_S : S \in 2^D\right\} \text{ denote the set of move arcs (or } m\text{-arcs for short)},$$
$$\mathbb{P} := \left\{p_S : S \in 2^D\right\} \text{ denote the set of preference arcs (or } p\text{-arcs for short)},$$
and
$$\mathbb{A} := \mathbb{M} \cup \mathbb{P}.$$

Given networks G and G' in $\mathbb{G}$, we shall denote by



(i.e., by an m-arc, belonging to coalition S', going from node G to node G') the fact that coalition $S' \in 2^D$ can change network G to network G' by adding, subtracting, or replacing arcs in network G. Moreover, we shall denote by



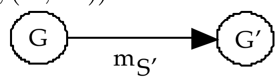
(i.e., by a p-arc, belonging to coalition S', going from node G to node G') the fact that each agent in coalition $S' \in 2^D$ prefers network G' to network G.

Definition 2 *(Supernetworks)*
Given directed networks $\mathbb{G} \subseteq \mathbb{N}(N, A)$*, agent payoff functions* $\{v_d(\cdot) : d \in D\}$*, and arc set* $\mathbb{A} := \mathbb{M} \cup \mathbb{P}$*, a supernetwork,* $\mathbf{G}$*, is a subset of* $\mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ *such that for all networks* G *and* G' *in* $\mathbb{G}$ *and for all coalitions* $S' \in 2^D$*,*

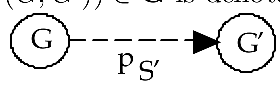
$(m_{S'}, (G, G')) \in \mathbf{G}$ *if and only if coalition* S' *can change network* G *to network*G'
by adding, subtracting, or replacing arcs in network G*,*
and
$(p_{S'}, (G, G')) \in \mathbf{G}$ *if and only if* $v_d(G') > v_d(G)$ *for all* $d \in S'$*.*

For each coalition $S' \in 2^D$*, m-arc* $m_{S'} \in \mathbb{M}$*, and p-arc* $p_{S'} \in \mathbb{P}$

$(m_{S'}, (G, G')) \in \mathbf{G}$ is denoted by



while
$(p_{S'}, (G, G')) \in \mathbf{G}$ is denoted by



Thus, supernetwork $\mathbf{G}$ specifies how the networks in $\mathbb{G}$ are connected via coalitional moves and coalitional preferences - and thus provides a *network representation* of preferences and the rules of network formation.

Remarks:

(1)Under our definition of a supernetwork, m-arc loops and p-arc loops are ruled out. Thus, for any network G and coalition S',

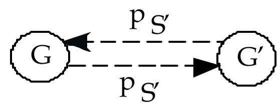
$$(m_{S'}, (G, G)) \notin \mathbf{G} \text{ and } (p_{S'}, (G, G)) \notin \mathbf{G}.$$

While m-arc loops are ruled out by definition, the absence of p-arc loops in supernetworks is due to the fact that each agent's preferences over networks are irreflexive. In particular, for each agent $d \in D$ and each network $G \in \mathbb{G}$, $v_d(G) > v_d(G)$ is not possible. Thus, $(p_{\{d\}}, (G, G)) \notin \mathbf{G}$.

(2) The definition of agent preferences via the network payoff functions,

$$\{v_d(\cdot) : d \in D\},$$

also rules out the following types of p-arc connections:



.

Thus, for all coalitions $S' \in 2^D$ and networks G and G' contained in $\mathbb{G}$,

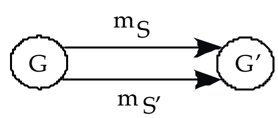
$$\text{if } (p_{S'}, (G, G')) \in \mathbf{G}, \text{ then } (p_{S'}, (G', G)) \notin \mathbf{G}.$$

(3) For all coalition $S' \in 2^D$ and networks G and G' contained in $\mathbb{G}$, if $(p_{S'}, (G, G')) \in \mathbf{G}$, then

$$(p_S, (G, G')) \in \mathbf{G} \text{ for all subcoalitions } S \text{ of } S'.$$

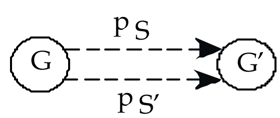
(4) Under our definition of a supernetwork, multiple m-arcs, as well as multiple p-arcs, connecting networks G and G' in supernetwork $\mathbf{G}$ are allowed. Thus, in supernetwork $\mathbf{G}$ the following types of m-arc and p-arc connections are possible:

For coalitions S and S', with $S \neq S'$



,

and



.

However, multiple m-arcs, or multiple p-arcs, from network $G \in \mathbf{G}$ to network $G' \in \mathbf{G}$ belonging to the *same* coalition are not allowed - and moreover, are unnecessary. Allowing multiple arcs can be very useful in many applications. For example, multiple m-arcs (not belonging to the same coalition) connecting networks G and G' in a given supernetwork $\mathbf{G}$ denote the fact that in supernetwork $\mathbf{G}$ there is more than one way to get from network G to network G' - or put differently, there is more than one way to change network G to network G'.

(5) In many economic applications, the set of nodes, N, used in defining the networks in the collection $\mathbb{G}$, and the set of economic agents D are one and the same (i.e., in many applications $N = D$).

The following notation, analogous to the notation introduced in expression (1) for directed networks, is useful in describing the m-arc connections in supernetworks.

Given supernetwork $\mathbf{G} \subseteq \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$

$$\left.\begin{aligned} \mathbf{G}(m_S) &:= \left\{(G, G') \in \mathbb{G} \times \mathbb{G} : (m_S, (G, G')) \in \mathbf{G}\right\} \\ \mathbf{G}(G) &:= \left\{m_S \in \mathbb{M} : (m_S, (G, G')) \in \mathbf{G} \text{ or } (m_S, (G', G)) \in \mathbf{G}\right\} \\ \mathbf{G}(G, G') &:= \left\{m_S \in \mathbb{M} : (m_S, (G, G')) \in \mathbf{G}\right\}, \\ \mathbf{G}(m_S, G) &:= \left\{G' \in \mathbb{G} : (m_S, (G, G')) \in \mathbf{G}\right\}. \end{aligned}\right\} \tag{2}$$

Similar notation can be introduced to describe p-arc connections. Thus,

$\mathbf{G}(m_S)$ is the *set of network pairs* connected by move arc m_S in supernetwork $\mathbf{G}$,
$\mathbf{G}(G)$ is the *set of move arcs* going from network G or coming to network G in supernetwork G,
$\mathbf{G}(G, G')$ is the *set of move arcs* going from network G to network G' in supernetwork $\mathbf{G}$,
and
$\mathbf{G}(m_S, G)$ is the *set of networks* which can be reached by move arc m_S from network G in supernetwork $\mathbf{G}$.

Note that if $\mathbf{G}(m_S)$ is empty for some move arc $m_S \in \mathbb{M}$, then in supernetwork $\mathbf{G}$ coalition S is not permitted to make any changes in any network, and thus in supernetwork $\mathbf{G}$, coalition S is isolated and has no power. Also, note that if for some network $G \in \mathbb{G}$, $\mathbf{G}(G)$ is empty, then network G is isolated relative to supernetwork $\mathbf{G}$. Finally, note that the set $\mathbf{G}(m_S, G)$ is simply the set of all the networks in $\mathbb{G}$ to which coalition S can deviate in supernetwork $\mathbf{G}$ starting from network G.

3.2 Supernetwork Classifications

A supernetwork $\mathbf{G}$ is *symmetric* if any move from one network to another brought about by some coalition can be reversed by that coalition. $\mathbf{G}$ is *closely connected* if it is possible to get from one non-isolated network to another non-isolated in one move via some m-arc. $\mathbf{G}$ is *simple* if there is one and only one way to get from one network to another via an m-arc.[7] $\mathbf{G}$ is *nonsimultaneous* if any move from one network to another can only be brought about by a single agent (i.e., by one agent acting alone). Finally, $\mathbf{G}$ is *decomposable* if any move from one network G to another network G' brought about by some coalition S consisting of more than one agent can

[7] Thus, if supernetwork $\mathbf{G}$ is simple, then

$$(m_S, (G, G')) \in \mathbf{G}$$

implies that

$$(m_{S'}, (G, G')) \notin \mathbf{G}$$

for all $S' \neq S$.

be decomposed for each agent $d \in S$ into two moves, a move from network G to some network G'' by coalition $S \backslash \{d\}$, and then a move from network G'' to network G' by agent d. Formally, we have the following definitions.

Definition 3 *(Supernetwork Classifications)*

(1) A supernetwork $\mathbf{G} \subseteq \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ *is said to be symmetric if for all networks* G *and* G' *in* $\mathbb{G}$,

$$\mathbf{G}(G, G') = \mathbf{G}(G', G).$$

(2) A supernetwork $\mathbf{G} \subseteq \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ *is said to be closely connected if for all networks* G *and* G' *in* $\mathbb{G}$*, not isolated relative to* $\mathbf{G}$,

$$\mathbf{G}(G, G') \text{ is nonempty.}$$

(3) A supernetwork $\mathbf{G} \subseteq \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ *is said to be simple if for all networks* G *and* G' *in* $\mathbb{G}$

$$\mathbf{G}(G, G') \text{ is either empty or consists of a single arc.}$$

(4) A supernetwork $\mathbf{G} \subseteq \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ *is said to be nonsimultaneous if for all networks* G *and* G' *in* $\mathbb{G}$,

$$m_S \in \mathbf{G}(G, G') \text{ implies that } |S| = 1.$$

(5) A supernetwork $\mathbf{G} \subseteq \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ *is said to be decomposable if for all*

$$(m_S, (G, G')) \in \mathbf{G} \text{ with } |S| > 1,$$

there exists for each agent $d \in S$ *a network* $G'' \in \mathbf{G}(m_{S \backslash \{d\}}, G)$ *such that* $G' \in \mathbf{G}(m_{\{d\}}, G'')$.

Figure 3 graphically depicts the notion of decomposability.

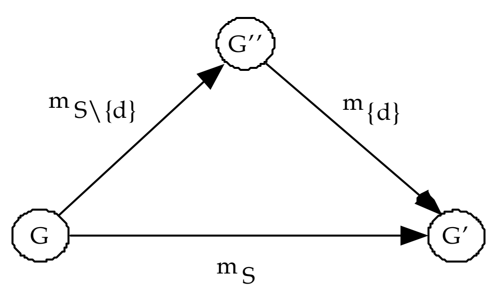


Figure 3: Decomposability

In supernetwork $\mathbf{G}$, the move from network G to network G' by coalition S, $|S| > 1$, is decomposable for each agent $d \in S$ into a move from network G to some network G'' by coalition $S \backslash \{d\}$ followed by a move from network G'' to network G' by agent d.

If supernetwork $\mathbf{G}$ is symmetric, then for all coalitions $S \in 2^D$ and all pairs of networks $(G, G') \in \mathbb{G} \times \mathbb{G}$,

$$(m_S, (G, G')) \in \mathbf{G} \text{ if and only if } (m_S, (G', G)) \in \mathbf{G}.$$

Moreover, if supernetwork $\mathbf{G}$ is simple, then for all coalitions $S \in 2^D$ and all pairs of networks $(G, G') \in \mathbb{G} \times \mathbb{G}$,

$$(m_S, (G, G')) \in \mathbf{G} \text{ implies that } (m_{S'}, (G, G')) \notin \mathbf{G} \text{ for all } S' \neq S.$$

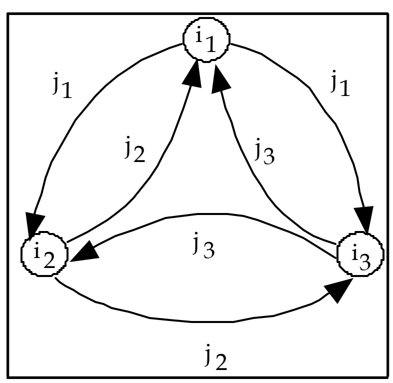
Thus, if supernetwork $\mathbf{G}$ is simple, then there is one and only one way to get from one network to another via an m-arc.

It is important to note that under our general definition of supernetworks, supernetworks are not required to be symmetry, closely connected, simple, nonsimultaneous, or decomposable.

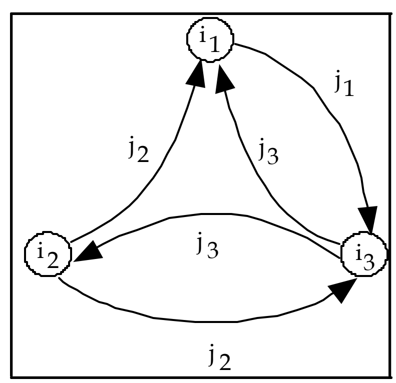
3.3 Network Formation Rules and Supernetwork Classifications

In this subsection we shall present several examples illustrating the connections between the rules governing network formation and supernetworks. For concreteness, we shall focus on the problem of strategic information sharing.

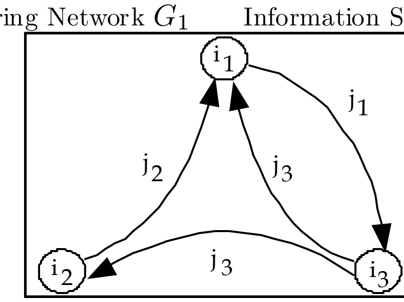
To begin, consider a situation in which three individuals, i_1, i_2, and i_3, have private information, and assume that only three configurations of information sharing arrangements are possible. In Figure 4 we have represented these configurations as an information sharing networks.



Information Sharing Network G_1



Information Sharing Network G_2



Information Sharing Network G_3

Figure 4: Three Possible Information Sharing Networks

For example in network G_1, the arc j_1 running from agent i_1 to agent i_2 indicates that agent i_1 observes the private information of agent i_2. In fact, in network G_1 there is full information sharing: each agent observes the private information of the other agents. Note that in this example the set of agents $D = \{i_1, i_2, i_3\}$ and the set of nodes N are one and the same.

In order to construct a supernetwork over our collection

$$\mathbb{G} = \{G_1, G_2, G_3\},$$

we must first specify the rules governing network formation - and therefore, we must specify the rules governing the establishment or termination of information sharing arrangements. We shall consider several possibilities.

An Example without Symmetry: We begin with an example in which the rules governing network formation are such that the resulting supernetwork is not symmetric - nor is it simple, nonsimultaneous, or decomposable, but it is closely connected.

Rules 0 - Asymmetric Network Formation Rules (arc addition is bilateral - arc subtraction is unilateral): In order for an agent to observe another agent's private information (for example, in order for agent i_1 to observe agent $i_2's$ private information, and therefore, in order to establish an arc j_1 from i_1 to i_2), both agents must agree to this arrangement. Thus, adding an arc requires a *bilateral* agreement (i.e., arc addition is bilateral). However, either agent (or both agents acting together) can terminate such an arrangement (i.e., either agent or both agents can remove the arc j_1 from i_1 to i_2). Thus, subtracting an arc can be accomplished by a *unilateral* decision (i.e., arc subtraction can be unilateral).

Figure 5 depicts the m-arc connections between networks, G_1, G_2 and G_3 in the supernetwork $\mathbf{G}_0$ corresponding to these rules.

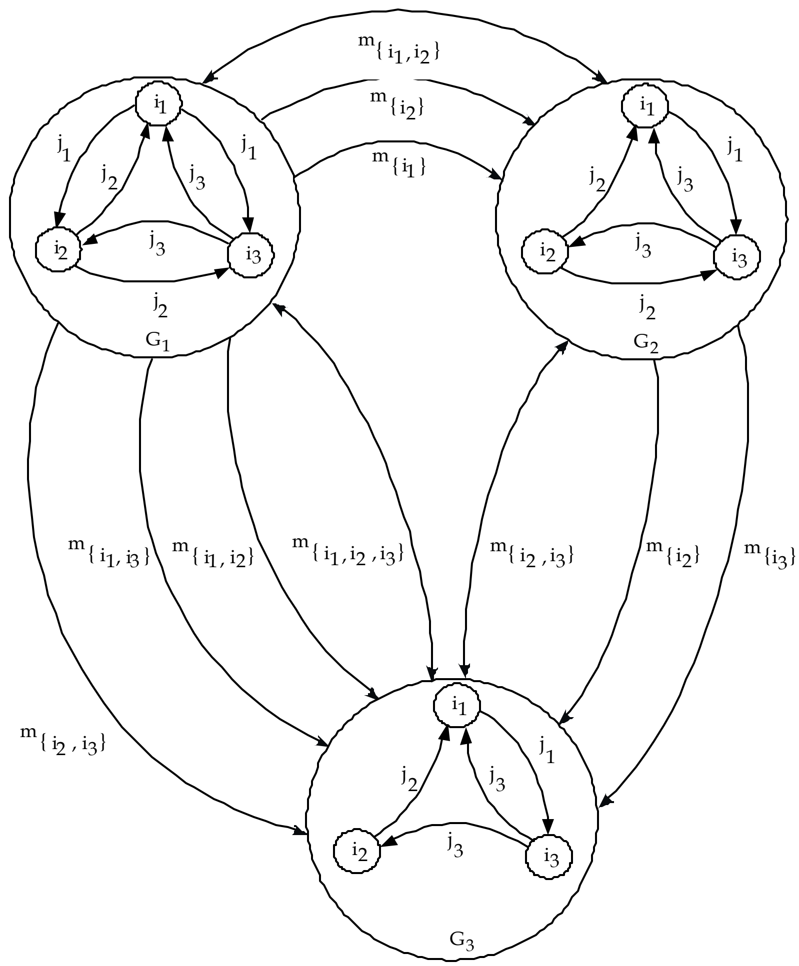


Figure 5: The m-Arcs in Asymmetric Supernetwork $\mathbf{G}_0$

Because the rules of governing network formation are a mix of bilateral and unilateral rules, the m-arc connections in supernetwork $\mathbf{G}_0$ are *asymmetric*. For example, in Figure 5, moving from network G_1 to network G_2 requires the removal of arc j_1 from i_1 to i_2 (indicating the termination of an arrangement whereby agent i_1 observes the private information of agent i_2). According to the rules, the removal of this arc can be accomplished by agent i_1, by agent i_2, or by both agents acting together. Thus, in Figure 5 there are three m-arcs, $m_{\{i_1\}}$, $m_{\{i_2\}}$, and $m_{\{i_1,i_2\}}$, running from network G_1 to network G_2. However, because the move from network G_2 to network G_1 requires the addition of arc j_1 from i_1 to i_2 (indicating the establishment of an arrangement whereby agent i_1 observes the private information of agent i_2), the rules require that both agents i_1 and i_2 agree to the addition of this arc. Hence, there is only one m-arc, $m_{\{i_1,i_2\}}$, running from network G_2 to network G_1. Because there is an $m_{\{i_1,i_2\}}$-arc running from network G_1 to network G_2, *and* an $m_{\{i_1,i_2\}}$-arc running from network

G_2 to network G_1, the $m_{\{i_1,i_2\}}$-arc connecting networks G_1 and G_2 depicted in Figure 5 has arrow heads at both ends.

Two Examples with Symmetry: In the next two examples the rules governing network formation are such that the corresponding supernetworks are symmetric, closely connected, and simple, but are not nonsimultaneous and not decomposable.

Symmetric Example 1 Suppose the rules governing the establishment or termination of information sharing arrangements are as follows:

Rules 1 - Symmetric Network Formation Rules 1 (purely bilateral): In order for an agent to observe another agent's private information (for example, in order for agent i_1 to observe agent $i_2's$ private information, and therefore, in order to establish an arc j_1 from i_1 to i_2), both agents must agree to this arrangement. Moreover, in order to terminate such an arrangement, both agents must agree (i.e., both agents must agree to remove the arc j_1 from i_1 to i_2).

As depicted in Figure 6, under these purely bilateral rules, the resulting supernetwork $\mathbf{G}_1$ is symmetric and simple. In general, if the rules governing the establishment or termination of information sharing arrangements are purely bilateral or purely unilateral, then the resulting supernetwork will be symmetric and simple. It is also easy to see from Figure 6 that supernetwork $\mathbf{G}_1$ is closely connected, but is not nonsimultaneous or decomposable.

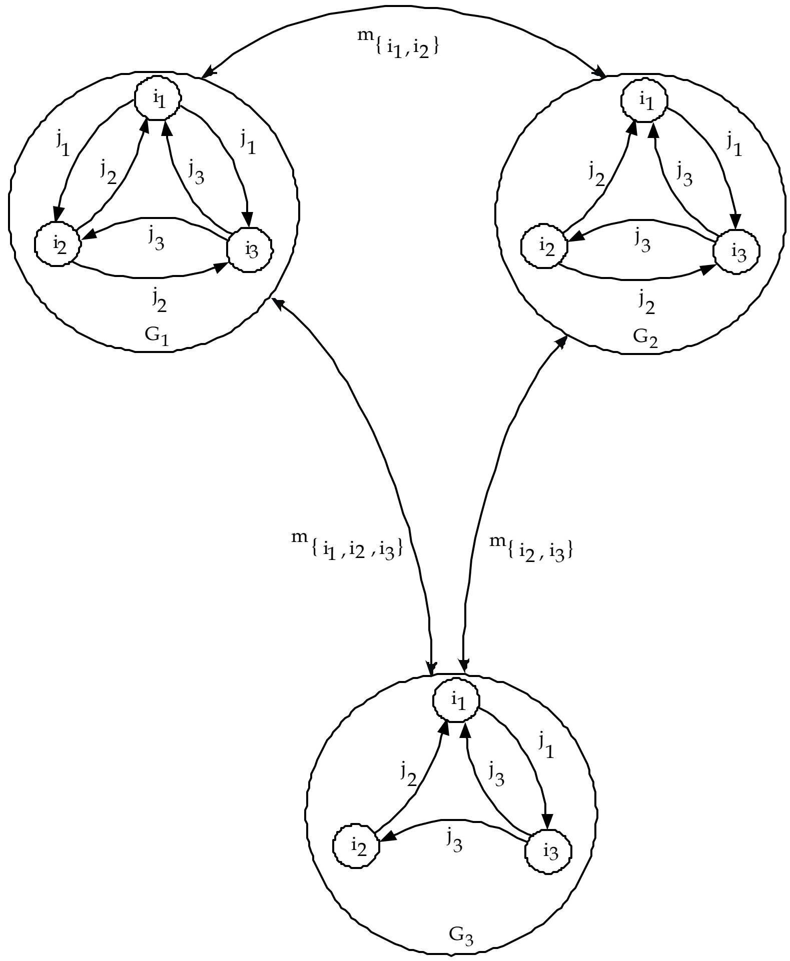


Figure 6: The m-Arcs in Symmetric Supernetwork $\mathbf{G}_1$

Symmetric Example 2 Next, let us consider rules which are purely unilateral:

Rules 2 - Symmetric Network Formation Rules 2 (purely unilateral): In order for an agent to observe another agent's private information (for example, in order for agent i_1 to observe agent $i_2's$ private information, and therefore, in order to establish an arc j_1 from i_1 to i_2), all that is required is that the agent *unilaterally initiate* the arrangement. Moreover, in order to terminate this arrangement, all that is required is that *the agent who initiated the arrangement, unilaterally terminate* the arrangement. (i.e., all that is required is that agent i_1 remove arc j_1 from i_1 to i_2).

As depicted in Figure 7, under purely unilateral rules the resulting supernetwork $\mathbf{G}_2$ is symmetric and simple. $\mathbf{G}_2$ is also closely connected, but it is not nonsimultaneous or decomposable.

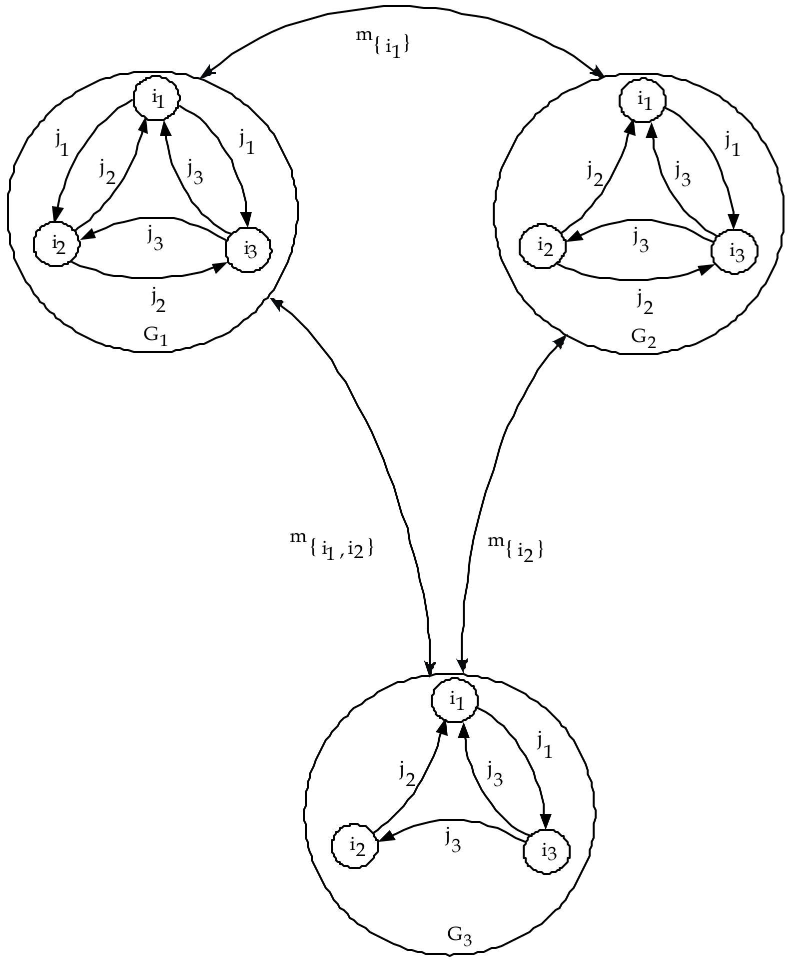


Figure 7: The m-Arcs in Symmetric Supernetwork $\mathbf{G}_2$

Note that in moving from network G_1 to network G_3 *both* agents i_1 and i_2 must act simultaneously and unilaterally to remove an arc, and in moving from network G_3 to network G_1 *both* agents i_1 and i_2 must act simultaneously and unilaterally to add an arc. Thus, in Figure 7 there is an $m_{\{i_1,i_2\}}$-arc connecting networks G_1 and G_3 with arrow heads at both ends.

An Example of a Symmetric, Simple and Nonsimultaneous Supernetwork: Suppose we modify Rules 2 as follows:

Rules 3 - Symmetric Network Formation Rules 3 (purely unilateral and nonsimultaneous): In order for an agent to observe another agent's private information (for example, in order for agent i_1 to observe agent $i_2's$ private information, and therefore, in order to establish an arc j_1 from i_1 to i_2), all that is required is that the agent *unilaterally initiate* the arrangement. Moreover, in order to terminate this arrangement, all that is required is that *the agent who initiated the* *arrangement, unilaterally terminate* the arrangement. (i.e., all that is required is that agent i_1 remove arc j_1 from i_1 to i_2). Finally, starting from any given configuration of information sharing arrangements (as represented by a given information sharing network), only one agent at a time can change the existing configuration.

As depicted in Figure 8, under Rules 3 the resulting supernetwork $\mathbf{G}_3$ is symmetric, simple, nonsimultaneous, and decomposable. However, $\mathbf{G}_3$ is not closely connected.

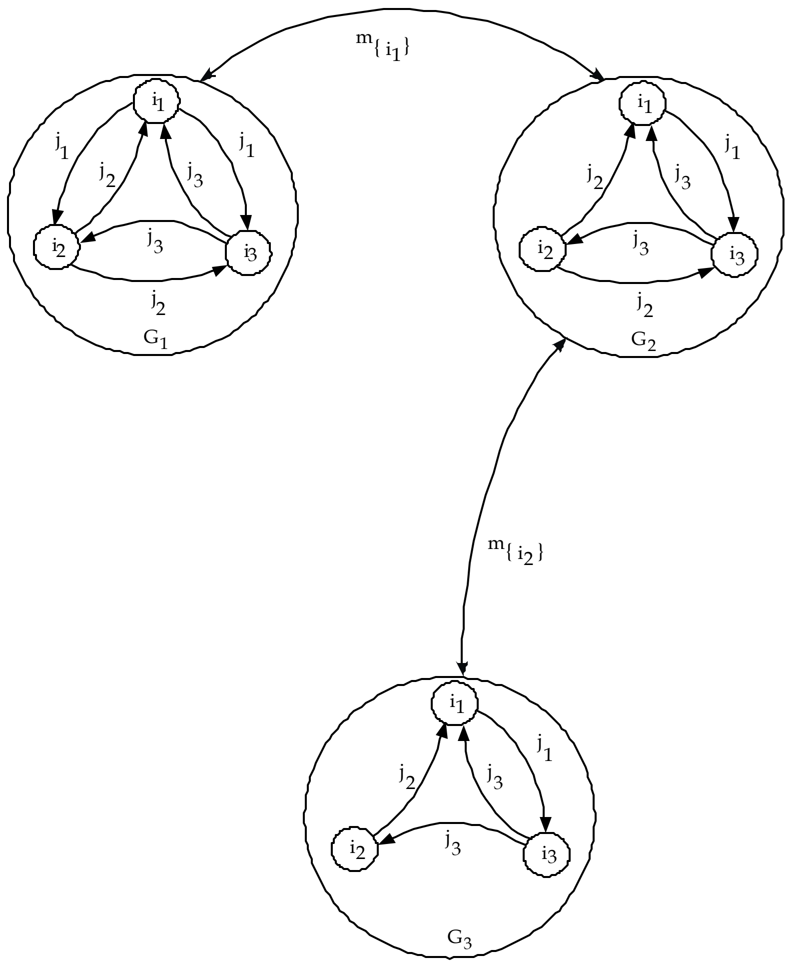


Figure 8: The m-Arcs in Symmetric Supernetwork $\mathbf{G}_3$

4 Farsightedly Stable Networks

4.1 Farsighted Dominance and Farsighted Stability

Given supernetwork $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$, we say that network $G' \in \mathbb{G}$ farsightedly dominates network $G \in \mathbb{G}$ if there is a finite sequence of networks,

$$G_0, G_1, \ldots, G_h,$$

with $G = G_0$, $G' = G_h$, and $G_k \in \mathbb{G}$ for $k = 0, 1, \ldots, h$, and a corresponding sequence of coalitions,

$$S_1, S_2, \ldots, S_h,$$

such that for $k = 1, 2, \ldots, h$

$$(m_{S_k}, (G_{k-1}G_k)) \in \mathbf{G},$$
$$\text{and}$$
$$(p_{S_k}, (G_{k-1}G_h)) \in \mathbf{G}.$$

We shall denote by $G \lhd\lhd G'$ the fact that network $G' \in \mathbb{G}$ farsightedly dominates network $G \in \mathbb{G}$. Figure 9 below provides a network representation of the farsighted dominance relation in terms of m-arcs and p-arcs. In Figure 9, network G_3 farsightedly dominates network G_0.

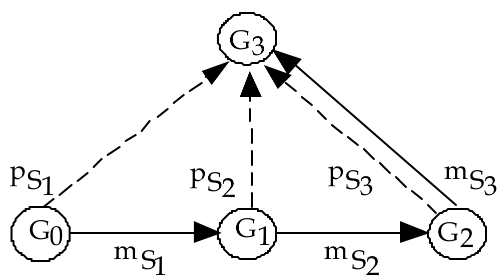


Figure 9: G_3 farsightedly dominates G_0

Definition 4 *(Farsightedly Stable Networks)*
Let $\mathbb{G} \subseteq \mathbb{N}(N, A)$ be a collection of directed networks and let $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ be a supernetwork, and let $\mathbb{F}_{\mathbf{G}}$ be a subset of $\mathbb{G}$. The subset of directed networks, $\mathbb{F}_{\mathbf{G}}$, is said to be farsightedly stable if

$$\text{for all } G_0 \in \mathbb{F}_{\mathbf{G}} \text{ and } (m_{S_1}, (G_0, G_1)) \in \mathbf{G},$$
$$\text{there exists } G_2 \in \mathbb{F}_{\mathbf{G}}$$
$$\text{with } G_2 = G_1 \text{ or } G_2 \rhd\rhd G_1 \text{ such that,}$$
$$(p_{S_1}, (G_0, G_2)) \notin \mathbf{G}.$$

Thus, a subset of directed networks $\mathbb{F}_{\mathbf{G}}$ is farsightedly stable if given any network $G_0 \in \mathbb{F}_{\mathbf{G}}$ and any m_{S_1}-deviation to network $G_1 \in \mathbb{G}$ by coalition S_1 (via adding, subtracting, or replacing arcs) there exists further deviations leading to some network $G_2 \in \mathbb{F}_{\mathbf{G}}$ where the initially deviating coalition S_1 is not better off - and possibly worse off.

There can be many farsightedly stable sets. We shall denote by $\mathbb{F}^{\star}_{\mathbf{G}}$ the largest farsightedly stable set. Thus, if $\mathbb{F}_{\mathbf{G}}$ is a farsightedly stable set, then $\mathbb{F}_{\mathbf{G}} \subset \mathbb{F}^{\star}_{\mathbf{G}}$.

4.2 Nonemptiness of the Largest Farsightedly Stable Set

By reformulating Chwe's existence and nonemptiness results for the supernetwork framework, we are able to conclude that any supernetwork contains a nonempty set of farsightedly stable networks.

Theorem 1 *($\mathbb{F}^{\star}_{\mathbf{G}} \neq \emptyset$)*

Let $\mathbb{G} \subseteq \mathbb{N}(N, A)$ be a collection of directed networks. Given any supernetwork $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$, there exists a unique, nonempty, largest farsightedly stable set $\mathbb{F}^{\star}_{\mathbf{G}}$. Moreover, $\mathbb{F}^{\star}_{\mathbf{G}}$ is externally stable with respect to farsighted dominance, that is, if network G is contained in $\mathbb{G}\backslash\mathbb{F}^{\star}_{\mathbf{G}}$, then there exists a network G' contained in $\mathbb{F}^{\star}_{\mathbf{G}}$ that farsightedly dominants G (i.e., $G' \rhd\rhd G$).

Proof. The existence of a unique, largest farsightedly stable set, $\mathbb{F}^{\star}_{\mathbf{G}}$, follows from Proposition 1 in Chwe (1994). Moreover, since the set of networks, $\mathbb{G}$, is finite and since each agent's preferences over networks are irreflexive, nonemptiness follows from the Corollary to Proposition 2 in Chwe (1994). ■

5 Nash Networks

5.1 Definitions

Within the framework developed above, we can define various Nash-type notions for networks relative to a given supernetwork.[8]

Definition 5 *(Nash Networks, Strict Nash Networks, and Strict Strong Nash Networks)*

Given directed networks $\mathbb{G} \subseteq \mathbb{N}(N, A)$, agent preferences $\{v_d(\cdot) : d \in D\}$, and arc set $\mathbb{A} := \mathbb{M} \cup \mathbb{P}$, let $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ be a supernetwork.

(1) A network $G^* \in \mathbb{G}$ is said to be a Nash network relative to supernetwork $\mathbf{G}$ if for all agents $d \in D$

$$v_d(G^*) \geq v_d(G) \text{ for all } G \in \mathbf{G}(m_{\{d\}}, G^*).$$

(2) A network $G^* \in \mathbb{G}$ is said to be a strict Nash network relative to supernetwork $\mathbf{G}$ if for all agents $d \in D$

$$v_d(G^*) > v_d(G) \text{ for all } G \in \mathbf{G}(m_{\{d\}}, G^*).$$

(3) A network $G^* \in \mathbb{G}$ is said to be a strict strong Nash network relative to supernetwork $\mathbf{G}$ if for all coalitions $S \in 2^D$ and all networks $G \in \mathbf{G}(m_S, G^*)$

$$v_{d'}(G^*) > v_{d'}(G) \text{ for some agent } d' \in S.$$

[8] In their interesting paper on noncooperative network formation, Bala and Goyal (2000) define the notion of a Nash network for one and two-way linking networks, for a particular set of noncooperative network formation rules. For example, in their framework the rules of network formation allow each agent to form links (one-way or two-way depending on the context) with any other agent. This contrasts with our framework which allows arbitrary restrictions to be placed on link formation. Moreover, in our framework, the notions of Nash and farsightedly stable networks are defined relative to any given set of network formation rules (rules represented via a supernetwork).

5.2 The Relationship Between Nash and Farsightedly Stable Networks

The following results, which relate strict Nash and strict strong Nash networks to farsightedly stable networks, extend Propositions 5 and 6 in Chwe (1994) to the supernetwork setting developed here.

Theorem 2 *(Strict and Strict Strong Nash Networks Are Farsightedly Stable)*
Let $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ be a given supernetwork.
(1) Let G^* be a strict Nash network relative to supernetwork $\mathbf{G}$. If $\mathbf{G}$ is symmetric and nonsimultaneous, then G^* is a farsightedly stable network relative to $\mathbf{G}$.
(2) Let G^* be a strict strong Nash network relative to supernetwork $\mathbf{G}$. If $\mathbf{G}$ is symmetric and decomposable, then G^* is a farsightedly stable network relative to $\mathbf{G}$.

Proof. Part (1) follows immediately from Proposition 5 in Chwe (1994). To prove part (2) we begin by defining the following mapping:

Let $P(\mathbb{G})$ denote the collection of *all* subsets (i.e., subcollections) of $\mathbb{G}$ (including the empty set or the empty subcollection), and define the mapping

$$\Lambda_{\mathbb{G}}(\cdot) : P(\mathbb{G}) \rightarrow P(\mathbb{G}),$$

as follows:

$$\begin{gathered}
\text{for subcollection of networks } \mathbb{H} \in P(\mathbb{G}) \text{ and network } G_0 \in \mathbb{G}\,, \\
G_0 \text{ is contained in } \Lambda_{\mathbb{G}}(\mathbb{H}) \\
\text{if and only if} \\
\forall\, G_1 \in \mathbb{G} \text{ such that } (m_S, (G_0, G_1)) \in \mathbf{G} \text{ for some coalition } S \in 2^D \\
\exists \text{ a network } G_2 \in \mathbb{H} \text{ such that} \\
\text{(i) } G_2 = G_1 \text{ or } G_2 \triangleright\triangleright G_1, \text{ and} \\
\text{(ii) } (p_S, (G_0, G_2)) \notin \mathbf{G}, \text{ that is, } v_d(G_2) \leq v_d(G_0) \text{ for some } d \in S.
\end{gathered}$$

By Proposition 1 in Chwe (1994), a subcollection $\mathbb{F}^{\star}_{\mathbf{G}}$ of $\mathbb{G}$ is the unique, largest farsightedly stable set if and only if $\mathbb{F}^{\star}_{\mathbf{G}}$ is a fixed point of the mapping $\Lambda_{\mathbb{G}}(\cdot)$ (i.e., if and only if $\mathbb{F}^{\star}_{\mathbf{G}} = \Lambda_{\mathbb{G}}(\mathbb{F}^{\star}_{\mathbf{G}})$). Moreover, because $\Lambda_{\mathbb{G}}(\cdot)$ is isotonic, that is, because $\mathbb{H} \subseteq \mathbb{H}'$ implies $\Lambda_{\mathbb{G}}(\mathbb{H}) \subseteq \Lambda_{\mathbb{G}}(\mathbb{H}')$, the mapping $\Lambda_{\mathbb{G}}(\cdot)$ has a fixed point - but it may be empty. However, if there exists a *nonempty* subcollection of networks $\mathbb{H}$ such that

$$\mathbb{H} \subseteq \Lambda_{\mathbb{G}}(\mathbb{H})$$

then $\Lambda_{\mathbb{G}}(\cdot)$ has a *nonempty* fixed point, $\mathbb{F}^{\star}_{\mathbf{G}}$, and in particular, $\mathbb{H} \subseteq \mathbb{F}^{\star}_{\mathbf{G}}$. We show here that for $\mathbf{G}$ a symmetric and decomposable supernetwork, if G^* is a strict strong Nash network relative to $\mathbf{G}$, then

$$\{G^*\} \subseteq \Lambda_{\mathbb{G}}(\{G^*\}).$$

In fact, it suffices to show that if G^* is a strict strong Nash network relative to a symmetric, decomposable supernetwork $\mathbf{G}$, then

$$G^* \triangleright\triangleright G' \text{ for all networks } G' \in \mathbf{G}(m_{S'}, G^*) \text{ for some coalition } S' \in 2^D. \;(*)$$

First, let $G' \in \mathbf{G}(m_{\{d\}}, G^*)$ for some agent $d \in D$. By symmetry, $G^* \in \mathbf{G}(m_{\{d\}}, G')$. Because G^* is a strict strong Nash network, $v_d(G^*) > v_d(G')$. Thus, $G^* \rhd\rhd G'$.

Now suppose statement (*) holds for all coalitions S' of size k (i.e., $|S'| = k$). We will show that under this induction hypothesis, statement (*) continues to hold for all coalitions S' of size $k+1$. Let $G' \in \mathbf{G}(m_{S'}, G^*)$ for some coalition S' with $|S'| = k+1$. Because G^* is a strict strong Nash network, $v_{d'}(G^*) > v_{d'}(G')$ for some agent $d' \in S'$. By decomposability, there exists some network $G'' \in \mathbf{G}(m_{S' \backslash \{d'\}}, G^*)$ such that $G' \in \mathbf{G}(m_{\{d'\}}, G'')$. By symmetry, $G'' \in \mathbf{G}(m_{\{d'\}}, G')$ and $G^* \in \mathbf{G}(m_{S' \backslash \{d'\}}, G'')$. Since $|S' \backslash \{d'\}| = k$, by the induction hypothesis $G^* \rhd\rhd G''$. Moreover, since $G'' \in \mathbf{G}(m_{\{d'\}}, G')$ and $v_d(G^*) > v_d(G')$, it follows from the definition of farsighted dominance that $G^* \rhd\rhd G'$. ■

A strict Nash network in a symmetric supernetwork **G** that is not nonsimultaneous may or may not be farsightedly stable. Moreover, a strict, strong Nash network in a symmetric supernetwork **G** that is not decomposable may or may not be farsightedly stable.

6 Computational Examples

In this section, we compute the Nash and farsightedly stable networks for several examples. All of our computations are carried out using a *Mathematica* package developed by Kamat and Page (2001).

6.1 The Nash and Farsightedly Stable Networks Relative to Information Sharing Supernetworks G_0,G_1,G_2, and G_3

In this subsection, we again focus on our collection,

$$\mathbb{G} = \{G_1, G_2, G_3\},$$

of information sharing networks. In particular, we compute the Nash and farsightedly stable networks corresponding to the supernetworks $\mathbf{G}_0$, $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{G}_3$ presented in section 3.3 above. In computing the Nash and farsightedly stable networks we shall assume that agents' network payoff functions,

$$v_d(\cdot) : \mathbb{H} \to \mathbb{R}, \ d = i_1, i_2, i_3$$

are given by the following table:

	i_1	i_2	i_3
G_1	$v_{i_1}(G_1)=3$	$v_{i_2}(G_1)=2$	$v_{i_3}(G_1)=3$
G_2	$v_{i_1}(G_2)=2$	$v_{i_2}(G_2)=3$	$v_{i_3}(G_2)=2$
G_3	$v_{i_1}(G_3)=2.5$	$v_{i_2}(G_3)=2.1$	$v_{i_3}(G_3)=2.5$

Table 1: Payoffs to Information Sharing Networks

Reading across the first row of payoffs, agent $i_1's$ payoff in network G_1 is $v_{i_1}(G_1) = 3$, agent $i_2's$ payoff is $v_{i_2}(G_1) = 2$, and agent $i_3's$ payoff is $v_{i_3}(G_3) = 3$.

The Supernetwork $\mathbf{G}_0$ For asymmetric supernetwork $\mathbf{G}_0$, with m-arcs depicted in Figure 6, the farsightedly stable set of networks, $\mathbb{F}^{\star}_{\mathbf{G}_0}$, is given by

$$\mathbb{F}^{\star}_{\mathbf{G}_0} = \{G_3\},$$

and the set of Nash networks, $\mathbb{N}_{\mathbf{G}_0}$, is also given by

$$\mathbb{N}_{\mathbf{G}_0} = \{G_3\}.$$

Thus, for supernetwork $\mathbf{G}_0$

$$\mathbb{F}^{\star}_{\mathbf{G}_0} = \mathbb{N}_{\mathbf{G}_0} = \{G_3\}.$$

In this example, asymmetries in the network formation rules favor the termination of information sharing arrangements. As a result, given agents' network payoffs, in the only farsightedly stable network information sharing is the network in which there is the least information sharing.

The Supernetwork $\mathbf{G}_1$ For the symmetric supernetwork $\mathbf{G}_1$, with m-arcs depicted in Figure 7, $\mathbb{F}^{\star}_{\mathbf{G}_1}$ is given by

$$\mathbb{F}^{\star}_{\mathbf{G}_1} = \{G_1, G_2, G_3\},$$

and the set of Nash networks, $\mathbb{N}_{\mathbf{G}_1}$, is also given by

$$\mathbb{N}_{\mathbf{G}_1} = \{G_1, G_2, G_3\}.$$

Thus, for supernetwork $\mathbf{G}_1$

$$\mathbb{F}^{\star}_{\mathbf{G}_1} = \mathbb{N}_{\mathbf{G}_1} = \{G_1, G_2, G_3\}.$$

In this example, the network formation rules make it equally difficult to establish or terminate information sharing arrangements. As a result, all the information sharing networks in the collection $\mathbb{G}$ are farsightedly stable as well as Nash. In this particular example, all networks are Nash under our definition because no single agent defections from a given network are possible.

The Supernetwork $\mathbf{G}_2$ For the symmetric and decomposable supernetwork $\mathbf{G}_2$, with m-arcs depicted in Figure 8, $\mathbb{F}^{\star}_{\mathbf{G}_2}$ is given by

$$\mathbb{F}^{\star}_{\mathbf{G}_2} = \{G_1, G_3\},$$

while the set of Nash networks, $\mathbb{N}_{\mathbf{G}_2}$, is given by

$$\mathbb{N}_{\mathbf{G}_2} = \{G_1\}.$$

Here, the network formation rules are purely unilateral, making it easy for each agent to establish or terminate information sharing arrangements. However, the payoffs in Table 1 are such that the extremes of maximal and minimal information sharing emerge as the farsightedly stable networks. Thus, for supernetwork $\mathbf{G}_2$

$$\mathbb{N}_{\mathbf{G}_2} = \{G_1\} \subset \{G_1, G_3\} = \mathbb{F}^{\star}_{\mathbf{G}_2}.$$

The Supernetwork $\mathbf{G}_3$ For the symmetric and nonsimultaneous supernetwork $\mathbf{G}_3$, with m-arcs depicted in Figure 9, $\mathbb{F}^{\star}_{\mathbf{G}_3}$ is given by

$$\mathbb{F}^{\star}_{\mathbf{G}_3} = \{G_1, G_2\},$$

while the set of Nash networks, $\mathbb{N}_{\mathbf{G}_3}$, is given by

$$\mathbb{N}_{\mathbf{G}_2} = \{G_1\}.$$

In this example, the network formation rules are purely unilateral and simultaneous moves are not allowed. Note that with simultaneity eliminated, under the payoffs in Table 1 maximal and intermediate information sharing emerge as the farsightedly stable networks. Thus, for supernetwork $\mathbf{G}_3$

$$\mathbb{N}_{\mathbf{G}_3} = \{G_1\} \subset \{G_1, G_2\} = \mathbb{F}^{\star}_{\mathbf{G}_3}.$$

The above series of examples illustrates how the rules governing network formation, as represented via supernetworks, affect the set of farsightedly stable networks as well as the set of Nash networks. In all of the examples above the set of Nash networks is equal to or contained in the set of farsightedly stable networks. In the next subsection, we expand upon on our information sharing examples and we show that, in general, it is not the case that the Nash networks are contained in the farsightedly stable set.

6.2 Strategic Information Sharing: An Expanded Example

6.2.1 Information Sharing Networks

Again consider the problem of strategic information sharing for the case in which three individuals, i_1, i_2, and i_3, have private information, but now consider the collection of all possible information sharing networks, *assuming that each agent observes the private information of at least one other agent.* Denote this collection by $\mathbb{G}_I$. Table

2 lists all the information sharing networks contained in the collection.

$((j_3,(i_3,i_1))$			
	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_3))$	$((j_2,(i_2,i_1)),(j_2,(i_2,i_3)))$
$(j_1,(i_1,i_2))$	G_1	G_2	G_3
$(j_1,(i_1,i_3))$	G_4	G_5	G_6
$((j_1,(i_1,i_2)),(j_1,(i_1,i_3)))$	G_7	G_8	G_9

$((j_3,(i_3,i_2))$			
	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_3))$	$((j_2,(i_2,i_1)),(j_2,(i_2,i_3)))$
$(j_1,(i_1,i_2))$	G_{10}	G_{11}	G_{12}
$(j_1,(i_1,i_3))$	G_{13}	G_{14}	G_{15}
$((j_1,(i_1,i_2)),(j_1,(i_1,i_3)))$	G_{16}	G_{17}	G_{18}

$((j_3,(i_3,i_1)),(j_3,(i_3,i_2)))$			
	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_3))$	$((j_2,(i_2,i_1)),(j_2,(i_2,i_3)))$
$(j_1,(i_1,i_2))$	G_{19}	G_{20}	G_{21}
$(j_1,(i_1,i_3))$	G_{22}	G_{23}	G_{24}
$((j_1,(i_1,i_2)),(j_1,(i_1,i_3)))$	G_{25}	G_{26}	G_{27}

Table 2: The Collection $\mathbb{G}_I$ of Information Sharing Networks

Note that network G_{27} in Table 2 corresponds to network G_1 in our previous information sharing examples - while network G_{24} in Table 2 corresponds to network G_2, and network G_{22} in Table 2 corresponds to network G_3. Also, note that our collection of networks $\mathbb{G}_I$ listed in Table 2 is a *proper* subset of the collection of all possible networks $\mathbb{N}(N,A)$ given node set $N=\{i_1,i_2,i_3\}$ and arc set $A=\{j_1,j_2,j_3\}$.

In this example, the set of nodes, $N=\{i_1,i_2,i_3\}$, and the set of agents D are one and the same, and the subscript on each arc denotes the agent to whom the arc belongs, and thus identifies the node from which the arc must emanate. In constructing the supernetwork over collection $\mathbb{G}_I$ we shall assume that the rules governing network formation are purely unilateral (see *Rules 2* above). Thus, the resulting supernetwork, $\mathbf{G}_I$, is symmetric, closely connected, simple, and decomposable, but not nonsimultaneous. Moreover, in supernetwork $\mathbf{G}_I$ no network in collection $\mathbb{G}_I$ is isolated.

6.2.2 Payoffs

Suppose now that the network payoff functions, $\{v_d(\cdot) : d \in D\}$, defined on $\mathbb{G}_I$ are specified as follows:

Given any network $G \in \mathbb{G}_I$ and any agent $d \in D = \{i_1,i_2,i_3\}$,

if d observes directly another agent d', d then receives a payoff of 2 ;

if d observes directly another agent d' who is, in turn, observing directly another agent d'', $d \neq d''$, then d receives a payoff of 3 ;

if d is observed directly by another agent d', then d pays a cost of 3;

if d is observed directly by another agent d' who is, in turn, being observed directly by another agent d'', $d \neq d''$, then d pays a cost of 6 .

Table 3 below gives the net payoffs to agents corresponding to the networks in collection $\mathbb{G}_I$ under the payoff rules listed above.

$((j_3,(i_3,i_1))$			
	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_3))$	$((j_2,(i_2,i_1)),(j_2,(i_2,i_3)))$
$(j_1,(i_1,i_2))$	$(-4,-4,3)$	$(-3,-3,-3)$	$(-6,-1,-3)$
$(j_1,(i_1,i_3))$	$(-4,3,-4)$	$(-4,3,-4)$	$(-7,6,-7)$
$((j_1,(i_1,i_2)),(j_1,(i_1,i_3)))$	$(-2,-3,-3)$	$(-1,-3,-6)$	$(-4,0,-9)$

$((j_3,(i_3,i_2))$			
	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_3))$	$((j_2,(i_2,i_1)),(j_2,(i_2,i_3)))$
$(j_1,(i_1,i_2))$	$(-4,-4,3)$	$(3,-4,-4)$	$(-3,-2,-3)$
$(j_1,(i_1,i_3))$	$(-3,-3,-3)$	$(3,-4,-4)$	$(-3,-1,-6)$
$((j_1,(i_1,i_2)),(j_1,(i_1,i_3)))$	$(-1,-6,-3)$	$(6,-7,-7)$	$(0,-4,-9)$

$((j_3,(i_3,i_1)),(j_3,(i_3,i_2)))$			
	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_3))$	$((j_2,(i_2,i_1)),(j_2,(i_2,i_3)))$
$(j_1,(i_1,i_2))$	$(-7,-7,6)$	$(-3,-6,-1)$	$(-9,-4,0)$
$(j_1,(i_1,i_3))$	$(-6,-3,-1)$	$(-3,-3,-2)$	$(-9,0,-4)$
$((j_1,(i_1,i_2)),(j_1,(i_1,i_3)))$	$(-4,-9,0)$	$(0,-9,-4)$	$(-6,-6,-6)$

Table 3: The Payoffs to the Information Sharing Networks in $\mathbb{G}_I$

6.2.3 Nash and Farsightedly Stable Networks Relative to Supernetwork $\mathbf{G}_I$

Under purely unilateral network formation rules with resulting information sharing supernetwork $\mathbf{G}_I$, the set of farsightedly stable networks, $\mathbb{F}^{\star}_{\mathbf{G}_I}$, is given by

$$\mathbb{F}^{\star}_{\mathbf{G}_I} = \{G_3, G_7, G_8, G_{12}, G_{15}, G_{16}, G_{20}, G_{22}, G_{23}\}\,.$$

Figure 10 depicts the information sharing networks contained in $\mathbb{F}^{\star}_{\mathbf{G}_I}$.

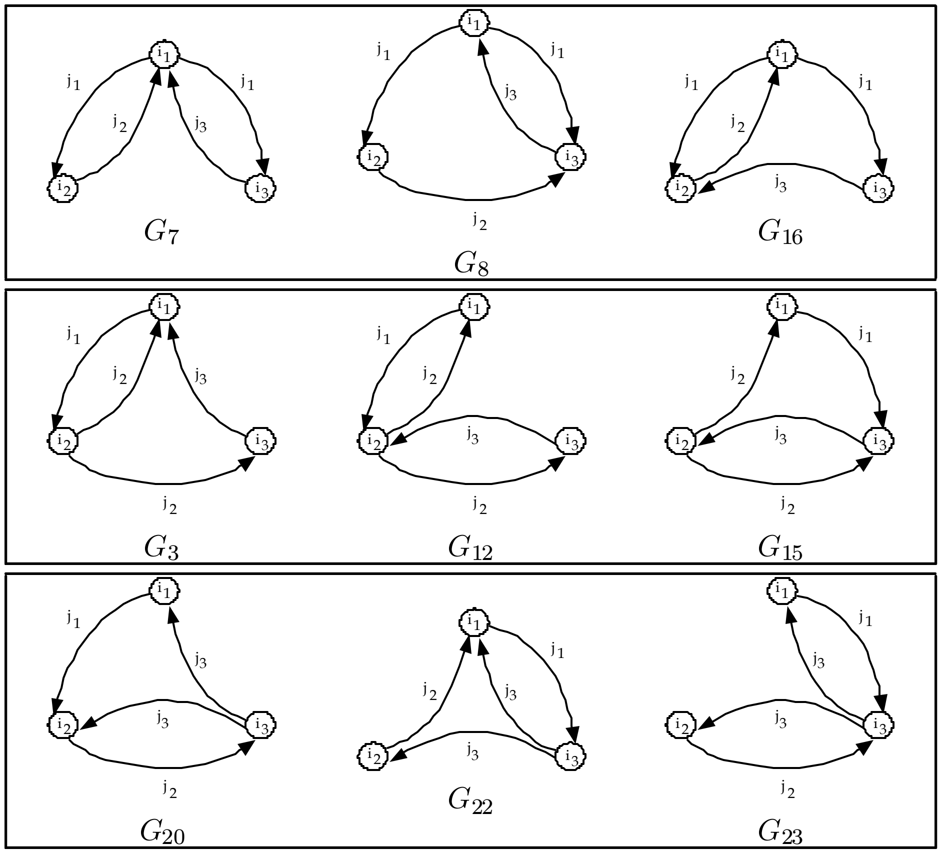


Figure 10: Farsightedly Stable Networks Relative to Information Sharing Supernetwork G_I

An interesting pattern emerges: in each of the farsighted stable networks in supernetwork $\mathbf{G}_I$, one agent observes each of the other two agents directly, while each of the two remaining agents observes one agent directly and one agent indirectly. For example, in farsightedly stable network G_{12} agent i_2 observes agents i_1 and i_3 directly, while agent i_1 observes i_2 directly and agent i_3 indirectly, and agent i_3 observes i_2 directly and agent i_1 indirectly. Note that in the other two networks, G_3 and G_{15}, in the same row with network G_{12}, agent i_2 observes agents i_1 and i_3 directly.

Relative to supernetwork G_I there is only one Nash network, network G_{27}. Figure

11 depicts network G_{27}.

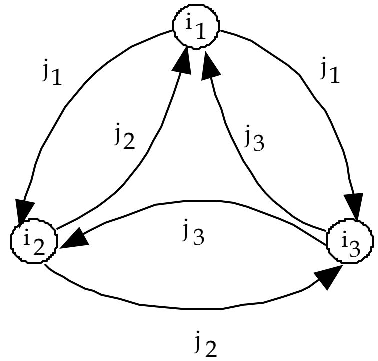


Figure 11: The Nash Network G_{27} Relative to Information Sharing Supernetwork G_I

Thus, in this example,

$$\mathbb{N}_{\mathbf{G}_I} = \{G_{27}\},$$

$$\mathbb{F}^{\star}_{\mathbf{G}_I} = \{G_3, G_7, G_8, G_{12}, G_{15}, G_{16}, G_{20}, G_{22}, G_{23}\},$$
and
$$\mathbb{N}_{\mathbf{G}_I} \cap \mathbb{F}^{\star}_{\mathbf{G}_I} = \emptyset.$$

Moreover, since the farsightedly stable set $\mathbb{F}^{\star}_{\mathbf{G}_I}$ is externally stable (see Theorem 1 above), there exists some farsightedly stable network $G \in \mathbb{F}^{\star}_{\mathbf{G}_I}$ that farsightedly dominates the Nash network G_{27}. Thus, $G \rhd\rhd G_{27}$ for some $G \in \mathbb{F}^{\star}_{\mathbf{G}_I}$.

Before moving to our last example, let us consider the issue of Pareto efficiency. Define the set of Pareto efficient networks relative to the collection $\mathbb{G}_I$ as follows:

$$\mathbb{E}_I := \left\{G \in \mathbb{G} : \text{there does not exist } G' \text{ such that } v_d(G') > v_d(G) \text{ for all } d \in D\right\}.$$

Given the payoffs in Table 3, the set of Pareto efficient networks, $\mathbb{E}_I$, is equal to the set of farsightedly stable networks. Thus, not only is the Nash network G_{27} farsightedly dominated by some network in $\mathbb{F}^{\star}_{\mathbf{G}_I}$, it is also Pareto dominated by some network in $\mathbb{F}^{\star}_{\mathbf{G}_I}$. Unfortunately, equality of the set of farsightedly stable networks and the set of efficient networks is not a general property of farsightedly stable sets in supernetworks. In fact, it is possible to construct a supernetwork example in which the set of efficient networks and the set of farsightedly stable networks are non-intersecting.

6.3 Strategic Pollution: The Garbage Game

In this our last example, we construct the supernetwork corresponding to a discrete, 3-agent version of the Shapley-Shubik garbage game (Shapley and Shubik (1969)). In the garbage game, each agent, i_1, i_2, and i_3, has a bag of garbage which can be kept by the agent or dumped onto the property of another agent. The game is discrete in that each agent's bag of garbage cannot be divided up - each agent either keeps the bag or dumps it on the property of one other agent. In the network representation of the garbage game, each agent has under his control one arc which must begin at the node representing that agent. The ending node of the arc indicates where the agent has chosen to dump his garbage. For example, if arc j_1 (the arc under the control of agent 1) runs from node i_1 to node i_3 this means that agent i_1 has chosen to dump his bag of garbage on the property of agent i_3. Alternatively, if agent i_1 decides to keep his bag of garbage, this is denoted by the arc j_1 running from node i_1 to node i_1 (i.e., by a loop). Thus, the rules for adding or subtracting arcs are *purely unilateral* (see *Rules 2* above). Figure 12 depicts the garbage network corresponding to a particular configuration of dumping strategies.

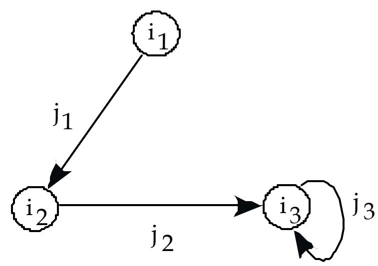


Figure 12: A Garbage Network G

In the garbage network depicted in Figure 12, agent i_1 dumps his garbage onto agent i_2, agent i_2 dumps his garbage onto agent i_3, and agent i_3 keeps his garbage. We shall assume that the payoff to each agent in any given garbage network is given by $-b$ where $b = 0, 1, 2, 3$ is number of bags of garbage the agent ends up with in that network. Thus, the payoff to each agent in the garbage network, G, depicted in Figure 12 is given by

$$v_{i_1}(G) = 0, \; v_{i_2}(G) = -1, \text{ and } v_{i_3}(G) = -2.$$

Table 4 lists all the possible networks. We shall denote by $\mathbb{G}_P$ the collection of networks in Table 4.

$((j_3,(i_3,i_1))$

	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_2))$	$(j_2,(i_2,i_3))$
$(j_1,(i_1,i_1))$	G_1	G_2	G_3
$(j_1,(i_1,i_2))$	G_4	G_5	G_6
$(j_1,(i_1,i_3))$	G_7	G_8	G_9

$((j_3,(i_3,i_2))$

	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_2))$	$(j_2,(i_2,i_3))$
$(j_1,(i_1,i_1))$	G_{10}	G_{11}	G_{12}
$(j_1,(i_1,i_2))$	G_{13}	G_{14}	G_{15}
$(j_1,(i_1,i_3))$	G_{16}	G_{17}	G_{18}

$(j_3,(i_3,i_3))$

	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_2))$	$(j_2,(i_2,i_3))$
$(j_1,(i_1,i_1))$	G_{19}	G_{20}	G_{21}
$(j_1,(i_1,i_2))$	G_{22}	G_{23}	G_{24}
$(j_1,(i_1,i_3))$	G_{25}	G_{26}	G_{27}

Table 4: All Possible Garbage Networks

Note that the network depicted in Figure 12 corresponds to network G_{24} in Table 4.

Under purely unilateral network formation rules (*Rules 2*), the resulting supernetwork, $\mathbf{G}_P$, is symmetric, closely connected, simple, and decomposable, but not nonsimultaneous. Moreover, in supernetwork $\mathbf{G}_P$, no network in collection $\mathbb{G}_P$ is isolated

Table 5 lists the payoffs corresponding to the networks in collection $\mathbb{G}_P$.

$((j_3,(i_3,i_1))$

	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_2))$	$(j_2,(i_2,i_3))$
$(j_1,(i_1,i_1))$	$(-3,0,0)$	$(-2,-1,0)$	$(-2,0,-1)$
$(j_1,(i_1,i_2))$	$(-2,-1,0)$	$(-1,-2,0)$	$(-1,-1,-1)$
$(j_1,(i_1,i_3))$	$(-2,0,-1)$	$(-1,-1,-1)$	$(-1,0,-2)$

$((j_3,(i_3,i_2))$

	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_2))$	$(j_2,(i_2,i_3))$
$(j_1,(i_1,i_1))$	$(-2,-1,0)$	$(-1,-2,0)$	$(-1,-1,-1)$
$(j_1,(i_1,i_2))$	$(-1,-2,0)$	$(0,-3,0)$	$(0,-2,-1)$
$(j_1,(i_1,i_3))$	$(-1,-1,-1)$	$(0,-2,-1)$	$(0,-1,-2)$

$(j_3,(i_3,i_3))$

	$(j_2,(i_2,i_1))$	$(j_2,(i_2,i_2))$	$(j_2,(i_2,i_3))$
$(j_1,(i_1,i_1))$	$(-2,0,-1)$	$(-1,-1,-1)$	$(-1,0,-2)$
$(j_1,(i_1,i_2))$	$(-1,-1,-1)$	$(0,-2,-1)$	$(0,-1,-2)$
$(j_1,(i_1,i_3))$	$(-1,0,-2)$	$(0,-1,-2)$	$(0,0,-3)$

Table 5: Payoffs to Garbage Networks

6.3.1 Nash and Farsightedly Stable Networks Relative to Supernetwork $\mathbf{G}_P$

For the garbage game supernetwork $\mathbf{G}_P$, $\mathbb{F}^{\star}_{\mathbf{G}_P}$ is given by

$$\mathbb{F}^{\star}_{\mathbf{G}_P} = \{G_6, G_8, G_{12}, G_{16}, G_{20}, G_{22}\}.$$

Figure 13 depicts the garbage networks contained in $\mathbb{F}^{\star}_{\mathbf{G}_P}$.

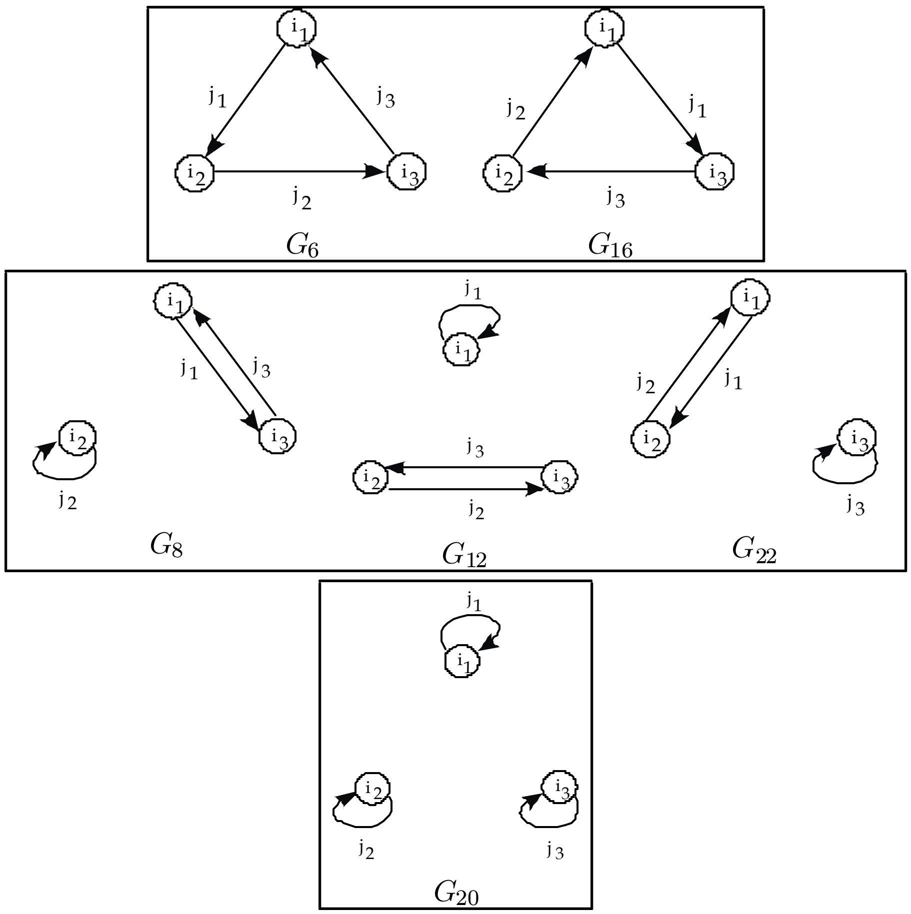


Figure 13: Farsightedly Stable Networks in the Garbage Game Supernetwork G_P

Again, an interesting pattern emerges: in each of the farsighted stable networks in supernetwork $\mathbf{G}_P$, each agent ends up with one bag of garbage - an outcome equivalent, in terms of payoff, to each agent keeping his bag of garbage. By reasoning farsightedly, agents always end up with an equitable outcome.

Relative to supernetwork $\mathbf{G}_P$, the set of Nash garbage networks is given by

$$\mathbb{N}_{\mathbf{G}_P} = \{G_4, G_6, G_7, G_9, G_{13}, G_{15}, G_{16}, G_{18}\}.$$

Figure 14 depicts the Nash garbage networks.

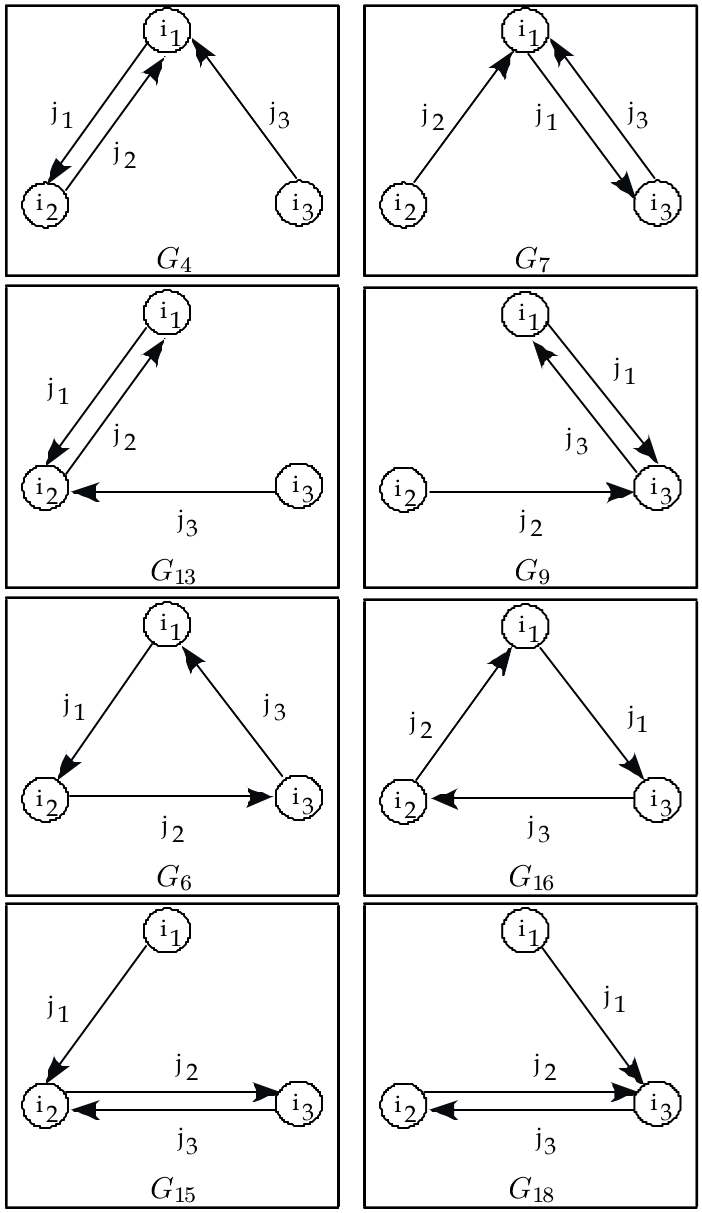


Figure 14: Nash Networks in the Garbage Game Supernetwork

Thus, in this example,

$$\mathbb{N}_{\mathbf{G}_P} = \{G_4, G_6, G_7, G_9, G_{13}, G_{15}, G_{16}, G_{18}\},$$

$$\mathbb{F}^{\star}_{\mathbf{G}_P} = \{G_6, G_8, G_{12}, G_{16}, G_{20}, G_{22}\},$$

and

$$\mathbb{N}_{\mathbf{G}_P} \cap \mathbb{F}^{\star}_{\mathbf{G}_P} = \{G_6, G_{16}\}.$$

Note that in none of the Nash equilibrium garbage networks does an agent keep his own bag of garbage (recall that in farsightedly stable garbage network G_{20} each agent keeps his bag of garbage). Moreover, except in Nash networks G_6 and G_{16}, payoffs are not equitable. For example, in Nash garbage network G_7, $v_{i_1}(G_7) = -2$, $v_{i_2}(G_7) = 0$, and $v_{i_3}(G_7) = -1$.

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