WARWICK
THE UNIVERSITY OF WARWICK

Manuscript version: Author's Accepted Manuscript
The version presented in WRAP is the author's accepted manuscript and may differ from the published version or Version of Record.

Persistent WRAP URL:
http://wrap.warwick.ac.uk/155972

How to cite:
Please refer to published version for the most recent bibliographic citation information. If a published version is known of, the repository item page linked to above, will contain details on accessing it.

Copyright and reuse:
The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions.

© 2021 Elsevier. Licensed under the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International http://creativecommons.org/licenses/by-nc-nd/4.0/.

Publisher's statement:
Please refer to the repository item page, publisher's statement section, for further information.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk.

warwick.ac.uk/lib-publications

Separating equilibria, underpricing and security design *

Dan Bernhardt [1], Kostas Koufopoulos [2], and Giulio Trigilia [3]

[1] *University of Illinois* and *University of Warwick*
[2] *University of York*
[3] *University of Rochester*

July 20, 2021

Abstract

Classical security design papers equate competitive capital markets to securities being fairly priced in expectation. We revisit Nachman and Noe (1994)'s adverse-selection setting, modeling capital-market competition as free entry of investors and letting firms propose prices for their securities, as happens in private securities placements and bank lending. We identify equilibria in which high types issue *underpriced* debt, which yields positive expected profits to uninformed lenders, while low types issue steeper securities, such as equity. In addition, pooling equilibria exist in which firms issue underpriced debt. Introducing pre-existing capital structures provides further foundations for pecking-order theories of external finance.

Keywords: Adverse selection, strictly positive profits, security design

JEL classification: G32

*We thank Ron Kaniel, Ilan Kremer, Tingjun Liu and Uday Rajan for useful comments. There are no conflicts of interest to disclose.

1 Introduction

Which securities should be issued to finance an investment project by a firm possessing private information that investors lack? And when will these securities be fairly priced? Seminal papers by Myers (1984) and Myers and Majluf (1984) posed these questions, developing a 'pecking-order theory' of external financing, according to which debt should be issued as it minimizes the mispricing of the security issued by the best firms. Nachman and Noe (1994) (henceforth, NN) provides theoretical foundations for this theory, identifying conditions under which, within the class of monotonic securities that satisfy limited liability *and* zero investor profits, issuing risky debt is uniquely optimal.[1]

This result has two caveats. First, NN predict that *all* firms would pool on the same debt contract. Thus, as Leary and Roberts (2010) observe, 'strictly speaking, the pecking order does not allow for any savings behavior or equity issuances'. That is, NN cannot be used to micro-found empirical tests based on firms' heterogenous financing choices.[2] Second, this pooling debt contract is fairly priced for the average firm type. This conflicts with the evidence that many securities issued are underpriced.[3]

Our contribution provides theoretical foundations for (1) the pecking-order theory, and (2) underpricing of securities, despite free and costless entry of investors. We prove that

[1]Specifically, NN prove that issuing debt is (uniquely) optimal if cash flow distributions across types are ordered according to (strict) conditional stochastic dominance. Noe (1988) discusses counter-examples in which conditional stochastic dominance fails, while first-order stochastic dominance holds, and debt is suboptimal.

[2]Common empirical tests of the pecking-order theory ask whether the set of 'better firms' in a sample relies more on debt than the set of 'worse firms' (e.g., Frank and Goyal (2003) and Fama and French (2005)). These tests are inconsistent with an equilibrium in which all firms pool on debt, as in NN. As Leary and Roberts (2010) put it, they test a 'modified pecking-order theory' that lacks solid foundations.

[3]See, for example, the literature on the credit spreads puzzle (e.g., Bai, Goldstein and Yang (2020)), and the recent evidence on the cost of loans in Chodorow-Reich, Darmouni, Luck and Plosser (2021).

there exist equilibria in NN's environment in which better types issue underpriced debt, while the worst type issues an alternative, fairly-priced security (e.g., equity) that is steeper than debt (in the sense of DeMarzo, Kremer and Skrzypacz (2005)). In these equilibria, competitive investors make strictly positive profits on debt because it is priced so that investors would make zero profits *if it were issued by the worst firm type.*[4] We then prove that pooling equilibria exist in which all firms issue underpriced debt, i.e., debt with a price below its full-information value when issued by the average firm type. Our characterization of all equilibria delivers clear testable predictions: worse firms issue less debt; better firms issue underpriced debt on which investors make strictly positive profits.

The contrast with NN reflects two key modeling differences. First, unlike NN, we do *not* exogenously equate competition among investors with zero expected profits for investors in equilibrium. Rather, we only assume that investors are individually rational, which just requires that they expect non-negative profits. Thus, equilibrium profits are an outcome of the model, rather than an ad hoc assumption. Second, NN do not let firms offer *prices* for their securities—for instance, firms cannot propose to raise a fixed amount with a loan at a given interest rate from banks, or seek to privately place shares with an investor at a proposed price. In contrast, we let firms choose whether or not to propose prices.

What sustains a partial-pooling equilibrium with positive profits, despite free entry of investors? In classic security design settings such as NN, three properties must hold in any partial-pooling equilibrium that satisfies standard refinements: (i) low-type firms must be indifferent between mimicking or not the equilibrium security choice of higher types—i.e.,

[4]This marks an important qualitative difference between our work and that on underpricing in IPOs, where underpricing is a costly signal that enables future refinancing at cheaper terms (Welch (1989), Allen and Faulhaber (1989)). Our model has no future refinancing periods.

their incentive constraint binds; (ii) securities issued by low-type firms cannot be underpriced in equilibrium; and (iii) zero-profit curves corresponding to different types do not intersect. Together these properties imply that investors must make profits on high-type firms in any partial-pooling equilibrium. Therefore, equilibria in which investors expect zero profits necessarily involve full pooling (Innes (1993), Nachman and Noe (1994)). Unlike these papers, we do not impose zero investor profits, and we uncover that partial-pooling equilibria with strictly positive profits exist and satisfy standard refinements.

To clarify why 'reasonable' equilibria with strictly positive investor profits exist, we first consider the case of two types. We show that there exist separating equilibria in which high-type firms issue just enough debt to cover their external financing needs, and low types issue equity that also just meets their financing needs. The properties above imply that investors must break even on the debt were it issued by *low-type* firms. In such an equilibrium, any deviation that could benefit high-type firms would also appeal to low-type firms, because high-type firms' equilibrium security (straight debt) is the *flattest* feasible security (in the sense of DeMarzo et al. (2005)), meaning that it minimizes the mispricing for high types among all feasible securities with a given expected value. This includes offering off-equilibrium a public bond—i.e., issuing debt without a proposed price—as in NN. Therefore, pessimistic off-equilibrium beliefs sustain 'reasonable' separating equilibria with positive investor profits.[5]

Turning to pooling equilibria, we first confirm that an equilibrium exists in which all firms pool on issuing debt which is fairly priced on the average type, as in NN, and in-

[5]Specifically, the standard D1 refinement of Cho and Kreps (1987), also used in NN, has no bite; therefore, such pessimistic beliefs are 'reasonable' for investors to hold off-equilibrium.

vestors break even in expectation. However, we then show that this is the sole equilibrium allocation that features zero investor profits. In particular, there also exists a continuum of equilibria in which firms pool on underpriced debt. The most severe underpricing corresponds to the case in which debt is priced so that investor profits would be zero were it issued by a low-type firm, as in partial-pooling equilibria. Thus, unlike in other security design settings, in our model *all firms are worse off (generically) under partial-pooling than pooling.*

Comparing these two types of equilibria reveals a fundamental contrast between our framework and costly signaling models. Consider, for instance, the literature on underpricing in IPOs (e.g., Welch (1989), Grinblatt and Hwang (1989) or Allen and Faulhaber (1989)). In these settings, high types credibly signal their identity by leaving money on the table in the IPO market, in order to enjoy reduced financing costs in future SEOs (Seasoned Equity Offerings). Thus, accounting for SEOs, high types *always* benefit from separating. In contrast, high types in our model *never* benefit from separation, because they issue underpriced debt once, and there are no subsequent financing rounds.

To investigate the driving forces behind our partial-pooling equilibria, we extend the analysis so that firms start with an *exogenous* amount of outstanding debt and equity. Brennan and Kraus (1987) show in this setting that if firms have enough shares to repurchase, then zero-profit curves intersect across types, unlike in NN. Thus, full separation of types without strictly positive investor profits is possible. In contrast, we focus on the intermediate case—between those of Brennan and Kraus and NN—in which firms have a positive but limited amount of securities that they can repurchase. We identify *fully separating equilibria with strictly monotonic leverage and positive investor profits*,

in a three-type setting. In these equilibria, high-type firms issue debt and repurchase the available shares. This debt is priced so that it would break even if issued by medium types, and so it yields strictly positive profits to investors. Medium types, instead, repurchase strictly less shares and issue debt that is priced at the full-information rate. Finally, low types issue fairly-priced equity (or other securities that are steeper than debt). Therefore, equilibrium leverage is strictly monotonic, providing a micro-foundation to the net leverage regressions typically run by empirical studies.

From a theoretical standpoint, our paper contributes to the vast security design literature under adverse selection. Seminal contributions include Innes (1993), Nachman and Noe (1994), DeMarzo and Duffie (1999) and DeMarzo, Kremer and Skrzypacz (2005). The crucial difference between these papers and ours is that we do not mandate that competition among investors must lead to zero profits. Instead, we allow for non-negative profits and explicitly model competition as free and costless entry. We show that the zero-profit restriction fundamentally alters the set of securities that arise in equilibrium.

Our partial-pooling equilibria with positive investor profits are robust to prior heterogeneity over the population distribution of firm types. Each security offered is priced such that investor profits would be zero were it issued by low types. Therefore, these equilibria are robust to ambiguity aversion over the fraction of types, or the cash flow distributions of different types. This is relevant for the growing literature on security design under ambiguity aversion (Carroll (2015), Lee and Rajan (2018) and Malenko and Tsoy (2020)).

Finally, our extension of NN's setting is empirically relevant. NN's mechanism in which firms choose securities and then investors compete in an auction can describe public bond

issuance by large corporations, or even IPOs and SEOs. However, it does not match the widespread phenomena of private securities placements, such as the sale of private debt and equity, or bank lending and syndication. Typical examples are raising a fixed amount with a loan at a given interest rate from banks, or offering shares of a firm's stock at a price to private equity investors. While NN exogenously rule out such securities, private financing instruments represent a common funding channel for small and medium enterprises or individuals that invest under severe informational asymmetries. In contrast, public issuances of debt or equity typically come from larger, established corporations—entities for which informational asymmetries are far lower.

Our setting is particularly relevant for bank lending, where a large literature relates bank profits to an informational advantage of banks (e.g., Dell'Ariccia and Marquez (2004)), which contrasts with recent evidence that even fintech lenders mostly rely on credit scoring and hard information (e.g., Di Maggio and Yao (2020)). In our model, lender profits are solely driven by incentive compatibility considerations—they are required to discourage low-type borrowers from mimicking high types, and issue equity instead. Because any deviation to lower interest rates would attract all firm types, pessimistic off-equilibrium beliefs sustain bank profits in equilibrium, even though banks lack informational advantages over the market on their borrowers. As a result, our generalization of NN's security design problem extends it to new economic environments.

Before proceeding to the analysis, we highlight the robustness of our main empirical predictions—which are that (i) better types issue relatively more debt, and (ii) high types issue underpriced debt securities. The first prediction holds strictly in *any* partial-pooling equilibrium, and it holds weakly under full pooling. Our second, underpricing prediction

is a *generic* feature of equilibrium—every equilibrium *except* NN's features strictly positive investor profits. Thus, our paper clarifies that the zero-profit pooling equilibrium on which the literature focuses is knife-edged in important ways.

The paper unfolds as follows. Section 2 presents a simple example that conveys the main driving forces behind our results. Section 3 introduces our baseline, two-type model. Section 4 characterizes all equilibria of the two-type model. Section 5 extends the characterization to $N > 2$ types. Section 6 introduces the possibility that firms have an outstanding amount of debt and equity that can be repurchased, showing that there exist fully separating equilibria strictly monotonic leverage across types. Section 7 concludes.

2 Example

To clarify the economic intuition, consider a simple example. A firm needs \$1 to finance a project. A high-type firm's project yields x_H with probability f_H, and zero otherwise. A low-type firm's project yields x_L with probability f_L, and zero otherwise. Assume that $x_H > x_L > 0$ and $1 > f_H > f_L > 0$. The fraction of type i in the population is $p_i \in (0, 1)$ and $\sum_i p_i = 1$. There is free entry of risk-neutral investors and the risk-free rate is zero. Firms can only issue debt, equity, or a mix of the two.[6] Debt offers bondholders a payment of $\min\{x, D\}$ for $x \in \{0, x_L, x_H\}$, where D is the face value of debt. Equity, which is junior to debt, gives shareholders a dividend of $\alpha(x - D)^+ = \max\{\alpha(x - D), 0\}$, where $\alpha \in [0, 1]$ is the fraction of shares sold. Thus, a mix of debt and equity is worth $f_i[\alpha(x_i - D)^+ + \min\{x_i, D\}]$ if issued by type $i \in \{L, H\}$. It is worth the weighted average of these

[6] In this example, one can trivially separate the types by exploiting the non-overlapping support. Our main model assumes a common support and shows that the economics is unaffected.

two expressions (with weights p_i) if all firms pool on the same mix of debt and equity.

Figure 1: Equilibria in the example

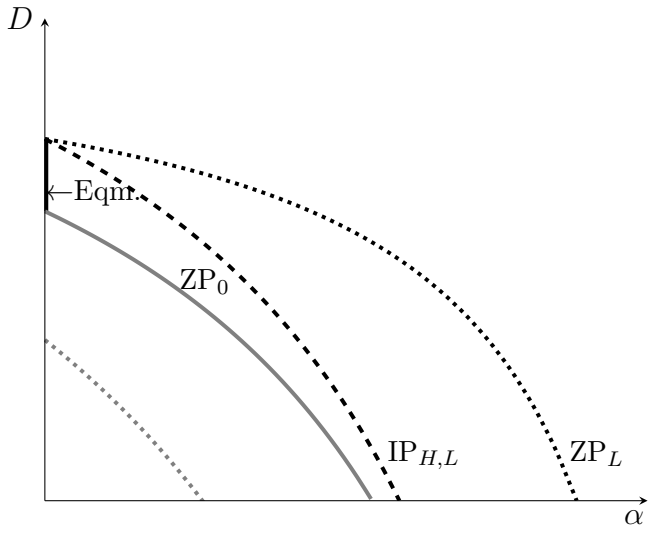


Figure 1 plots the four relevant iso-profit curves in (α, D) space given parameter choices $x_H = \$6$, $x_L = \$4$, $f_H = \frac{3}{4}$ $f_L = \frac{1}{3}$ and $p_L = 0.5$. The ZP$_i$ curves for $i \in \{L, H\}$ describe all combinations of debt and equity such that investors make zero profits on type i firms, solving $f_i\left[\alpha(x_i - D)^+ + \min\{x_i, D\}\right] = 1$. ZP$_0$ is the zero-profit curve for a pool of both types: $p_H \frac{3}{4}\left[\alpha(6 - D)^+ + \min\{6, D\}\right] + p_L \frac{1}{3}\left[\alpha(4 - D)^+ + \min\{4, D\}\right] = 1$. Finally, IP$_{H,L}$ is the iso-profit curve for high types that offer only debt ($\alpha = 0$) on ZP$_L$, and so is parallel to ZP$_H$. To derive IP$_{H,L}$, solve ZP$_L$ for D when $\alpha = 0$ to obtain $D = 3$. Next, compute the value of $(\alpha, D) = (0, 3)$ if issued by high types by substituting into the left-hand side of ZP$_H$, to get 2.25. Thus, IP$_{H,L}$ is characterized by $\frac{1}{2}[\alpha(6 - D)^+ + \min\{6, D\}] = 2.25$.

NN show that if investors must make zero profits, then, in the unique equilibrium satisfying the D1 refinement, both types pool on issuing debt, fairly priced for the average type. To derive their result, start from any mix of debt and equity (α, D). First, because ZP$_H$ < ZP$_0$ < ZP$_L$, investors cannot make zero profits on all firms in a separating

equilibrium.[7] Thus, given the zero-profit restriction, the equilibrium contract must be on the zero-profit pooling line, ZP_0. Second, if firms pool on anything other than debt, then, because ZP_H is steeper than ZP_L, high types could deviate to issuing debt that is slightly underpriced from the average firm type's perspective. The deviation would not be profitable for bad types, but it is strictly profitable for high types if accepted by investors. Therefore, issuing equity cannot be part of a 'reasonable' equilibrium.

Turning to our positive-profits equilibria, suppose high types offer only debt on ZP_L, while low types offer only equity on ZP_L. Then, debt is underpriced because it is issued by high types, but equity is fairly priced. The high type's iso-profit line to which this contract belongs is $\mathrm{IP}_{H,L}$, which lies strictly below ZP_L whenever $\alpha > 0$. Therefore, deviating to any contract that could benefit high types, also benefits low types. Thus, common refinements including D1 have no bite: with sufficiently pessimistic beliefs, this deviation would be unprofitable, sustaining the positive-profits equilibrium. Indeed, separating equilibria exist in which low types issue any mix of debt and equity such that $\alpha > 0$ on ZP_L, while high types only issue debt on ZP_L. In addition, pooling equilibria in which all firms issue underpriced debt exist: all debt contracts along the y-axis in Figure 1, between ZP_0 and ZP_L—as highlighted by the solid segment—are equilibrium securities.

Summing up, the example delivers clear testable empirical implications: (i) *high types issue relatively more debt than low types*; and (ii) *investors make strictly positive expected profits on high types' equilibrium debt securities.* We now show that this intuition extends to a standard NN setting with general contracts and cash flows distributions.

[7] Abusing notation, we write $\mathrm{ZP}_H < \mathrm{ZP}_0$ to indicate that the cost of financing under full-information for high types is strictly less than that for a fifty-fifty pool of types. Thus, the iso-profit curve for high types has a strictly lower α for any given D, relative to the pooling iso-profit curve.

3 Baseline model with two types

Our two date ($t = 0, 1$) model features two types of risk-neutral firms that seek external financing for a risky project. Firms can raise funds from a continuum of competitive, risk-neutral investors.[8] We normalize the risk-free rate to zero. Projects require investment of \$1 at date $t = 0$ to generate random cash flows $\tilde{x}$ at $t = 1$. A firm's type $\theta \in \{\theta_H, \theta_L\}$ corresponds to its expected cash flows, where $\theta_H > \theta_L > 1$. Thus, both firm types have strictly positive NPV projects and it is efficient to fund them. A fraction $p \in (0, 1)$ of firms are high types with $\theta = \theta_H$. The mean firm type is $\theta_0 := p\theta_H + (1 - p)\theta_L$.

We denote a realization of $\tilde{x}$ by x, and the probability density function over x for a type-θ firm by $f_\theta(x)$, which we assume is strictly positive on a common, finite support $[0, \bar{x}]$. The random cash flows associated with the two firm types are ordered by the strict Monotone Likelihood Ratio Property (MLRP): $\frac{\partial}{\partial x} \frac{f_H(x)}{f_L(x)} > 0$ for every $x \in [0, \bar{x}]$.[9] That is, the probability that a cash flow x comes from high types increases in x.

Contracts. As in NN, firms can sell *securities* $s(x) : [0, \bar{x}] \to \mathbb{R}$ that specify payouts to investors as a function of the realized cash flows x. NN impose the condition that the price $Q(s)$ of an equilibrium security equals its expected value given investor beliefs about the firm type offering it, imposing $Q(s) = \mathbb{E}[s|s \text{ has been offered}]$. We enlarge the contract space by allowing firms to propose selling prices, as often occurs in private security place-

[8] If instead of a continuum of investors we had any number $N \geq 2$, our results would hold a fortiori, as positive-profits equilibria are easier to sustain with Betrand competition than free entry.

[9] NN impose Conditional Stochastic Dominance, a weaker condition than MLRP. The literature mostly works with MLRP (e.g., DeMarzo et al. (2005)), as it is more tractable and does not alter economic intuition.

ments. Thus, a contract $c := (s(x), Q(s))$ consists of a security $s(x)$ and a selling price $Q(s)$. Competitive investors can accept or reject purchasing the security at price $Q(s)$.[10] With this formulation, the option *not* to propose a price (as in NN) is redundant—a firm can achieve the same allocation by proposing a price that equals the expected security auction price. NN's equilibrium in which $c^*_\theta = (s^*_\theta)$ and $Q(s^*_\theta) = \mathbb{E}[s^*_\theta | s^*_\theta \text{ has been offered}]$ corresponds in our model to the case in which $c^*_\theta = (s^*_\theta, Q^*_\theta(s^*_\theta))$, where $Q^*_\theta(s^*_\theta) = Q(s^*_\theta)$.

Our formulation has two advantages relative to NN's. First, it enlarges the contract space from which firms can choose. Second, it captures private securities placements, in which the amount that firms want to raise is often fixed and equal to their external financing needs for a budgeted project. For example, a bank loan of \$1 is a security in which $s = \min\{x, D\}$ for some $D \in [0, \bar{x}]$ and $Q(s) = \$1$. A private equity issuance for \$1 of capital is a security in which $s = \alpha x$ for some $\alpha \in [0, 1]$ and $Q(s) = \$1$. In all cases, the value of the security for investors need not be \$1, and whether or not competition pushes the price to \$1 is an outcome of the analysis, rather than an assumption.

As is standard in the literature, and also imposed by NN, we assume that a feasible security satisfies two-sided limited liability and monotonicity:[11]

Definition 1. *A contract c is* feasible *if it offers investors a feasible security s. A security s is* feasible *if it satisfies:*

(LL) Limited Liability: $s(x) \in [0, x]$ for every x.

(M) Monotonicity: for every $(x, x' < x) \in [0, \bar{x}]^2$, $s(x) \geq s(x')$ and $x - s(x) \geq x' - s(x')$.

[10] With many bidders, a lottery allocates the security to one bidder, for example, with equal probability.

[11] Limited liability guarantees that an equilibrium exists, and is needed to derive most results in the security design literature, including ours. Monotonicity ensures that the optimal security for better types is debt, but is not key for the existence of separating equilibria. We maintain this assumption to simplify exposition and comparisons with NN. Koufopoulos, Kozhan and Trigilia (2019) derive conditions under which monotonicity can be micro-founded by appealing to the possibility of window dressing.

Timing. At stage 1, a type-θ firm offers a feasible contract $c_\theta = (s_\theta, Q_\theta)$, which consists of a feasible security s_θ and a price $Q_\theta(s_\theta)$. After a contract c is issued, investors form beliefs $p_i'(c) := \Pr.[\theta = \theta_i|c]$ about the firm type offering it. We denote the vector of posterior probabilities by $\mathbf{p}'$. At stage 2, given these beliefs, investors either accept or reject the contract. We denote the acceptance decision by $r \in \{0, 1\}$, where $r = 1$ if investors accept.[12] If a price $Q \geq 1$ is offered, then (observable and verifiable) investment occurs. Otherwise, the firm is worth zero. If $Q > 1$, we follow NN and assume, without loss of generality, that the difference $Q - 1$ is paid to the firm's shareholders as a dividend at $t = 0$. It is useful to define $\mathbb{E}[s(\tau)] := \mathbb{E}[s(x)|\theta = \tau]$, which denotes the full-information value of security s when issued by type τ. Then, the expected payoff to a type-θ firm that offers a contract $c = (s, Q(s))$ when the acceptance decision is r is

$$U_\theta(c) := r(Q(s) - 1 + \theta - \mathbb{E}[s(\theta)]). \tag{1}$$

We focus on 'reasonable' pure strategy Perfect Bayesian equilibria, i.e., Perfect Bayesian equilibria that satisfy the D1 refinement (as defined in Cho and Kreps (1987), Section 5):

Definition 2. *A* 'reasonable' pure strategy PBE *consists of feasible contracts* $(c_L^*,\ c_H^*)$ *and posterior beliefs* $\mathbf{p}'(c)$ *of investors that satisfy:*

SR Sequential rationality: firms propose contracts optimally given the beliefs $\mathbf{p}'(c)$ *of investors and their associated optimal acceptance decision* $r^*(\mathbf{p}'(c))$*;*

BC Belief consistency: $\mathbf{p}'(\mathbf{c}^*)$ *are derived from Bayes' Rule.*

[12] We follow NN, as well as the rest of the security design literature, and only consider pure strategy equilibria. As is well known (see, e.g., Dasgupta and Maskin (1986)) the equilibrium set would expand dramatically if one allowed for mixed strategy equilibria.

D1 Off-equilibrium-beliefs refinement: there does not exist a pair $(\theta, \theta') \in \{\theta_1, ..., \theta_N\}^2$*, with* $\theta \neq \theta'$*, and an off-equilibrium contract* c *such that:* $\{r(\mathbf{p}'(c)) | U_{\theta'}(c|\mathbf{p}'(c)) \geq U_{\theta'}(c^*_{\theta'})\} \subset \{r(\mathbf{p}'(c)) | U_\theta(c|\mathbf{p}'(c)) > U_\theta(c^*_\theta)\}$.[13]

The D1 refinement is standard in this literature. It rules out equilibria sustained by off-equilibrium beliefs that assign positive probability to types that weakly benefit from the deviation 'in strictly less cases' than other types, in a set-inclusion sense. D1 forces off-equilibrium beliefs to assign probability one to the firm type that benefits 'in strictly more cases' than any other type. Henceforth, we dub a *'reasonable' equilibrium* any pure strategy PBE that satisfies the conditions detailed in Definition 2.

Ranking securities. We follow DeMarzo et al. (2005) and define an *ordered set of securities* as a function $s(z, x)$, where $z \in [z_0, z_1]$ is an index and $s(z, \cdot)$ denotes the payment from the firm to the investors across cash-flow states x. As before, we define $\mathbb{E}[s(z, \tau)] := \mathbb{E}[s(z, x)|\theta = \tau]$. We require the ordering to be *full* in the sense that $\mathbb{E}[s(z_0, \theta_1)] \leq \theta_1$ and $\mathbb{E}[s(z_1, \theta_N)] \geq \theta_N$, and we also require that $\mathbb{E}[s(z, x)]$ is increasing in the index z, for any probability distribution over types used to compute the expectation.

To rank ordered sets of securities, we use the notion of 'steepness' proposed in DeMarzo et al. (2005) to measure the information sensitivity of a security design.

Definition 3. *Security* $s(x)$ crosses from below *security* $s'(x)$ *if* $\mathbb{E}[s(\theta_i)] = \mathbb{E}[s'(\theta_i)]$ *implies that* $\mathbb{E}[s(\theta_{i+1})] > \mathbb{E}[s'(\theta_{i+1})]$. *An ordered set of securities* $s(z, x)$ *is* steeper *than*

[13] We use the standard definition of the D1 refinement (see Section 5 of Cho and Kreps (1987)), which ranks sets of off-equilibrium posteriors. NN's alternative definition ranks off-equilibrium prices. The definitions are equivalent, because to any price $Q \geq 1$ offered off-equilibrium there is a corresponding off-equilibrium posterior, and vice versa.

another ordered set of securities $s'(z,x)$ if, for all $s(x) \in s(z,x)$ and $s'(x) \in s'(z,x)$, $s(x)$ crosses $s'(x)$ from below.

This definition says that a security s crosses from below another security s' if there exists an $\hat{x}$ such that $s(x) > s'(x) \iff x > \hat{x}$ (DeMarzo et al. (2005), Lemma 5). For instance, equity crosses debt from below; while equity and call options cross straight equity from below. We are now equipped to solve for the set of equilibria of the financing game.

4 Equilibria with two types

4.1 Separating Equilibria

We first analyze separating equilibria. Lemma 1 details the well-known result that separating equilibria cannot exist *if investors must make zero profits* on every security offered in equilibrium.

Lemma 1. *There is no separating equilibrium in which all securities issued are fairly priced. That is, there is no equilibrium with $s_H \neq s_L$ and $\mathbb{E}_\theta[s_\theta(x)] = Q(s_\theta)$, $\forall \theta$.*

Proof. All omitted proofs are in the Appendix. □

Intuitively, if all securities were fairly priced, then regardless of the security design chosen by high types, low types would always benefit from mimicking high types. It follows directly that, if a separating equilibrium exists, it must feature strictly positive profits for competitive investors on at least one firm type. We now highlight properties of separating equilibria with positive investor profits that have been overlooked by papers imposing zero

investor profits. First, in any 'reasonable' separating equilibrium, low-type firms must be indifferent between offering their equilibrium contract c_L and mimicking high-type firms by offering $c_H \neq c_L$. Otherwise, high-type firms would have deviations that would not attract low types, but would make high types better off, violating D1. Thus,

Lemma 2. *A separating equilibrium is 'reasonable'—i.e., it satisfies D1—only if the incentive constraint for low-type firms binds:* $U_L(c_L) = U_L(c_H)$.

While the intuition behind Lemma 2 is straightforward, the lemma has the important implication that the incentive constraint for high-type firms can never bind in separating equilibria. To see this, start from a pair (s_L, s_H) with $s_H \neq s_L$ such that $\mathbb{E}_L[s_L] = \mathbb{E}_L[s_H]$. Consider the effect of a change of measure on both sides of the equation, from F_L to F_H. The effect must be heterogeneous, as long as the securities are different. Moreover, the incentive constraint must hold. Thus, high-type firms must issue *flatter securities* (in the sense of Definition 3) than low-type firms at a separating equilibrium. Indeed, the cornerstone of both NN's arguments and ours is the fact that the flattest feasible security is debt:

Lemma 3. *Debt crosses all feasible securities from below: the flattest ordered set of feasible securities is* $s^D(d, x) := \min\{x, d\}$ *for* $d \in [0, \bar{x}]$.

Lemma 3 allows us to conclude that low-type firms must issue steeper securities than debt in any separating equilibrium:

Lemma 4. *A separating equilibrium is 'reasonable' only if the incentive constraint for high-type firms is slack:* $U_H(c_H) > U_H(c_L)$. *Thus, the security issued by low-type firms,* s_L, *must cross from below that issued by high-type firms,* s_H.

Recall that Lemma 1 rules out separating equilibria with zero expected profits on all securities. We next establish that in any separating equilibrium, competitive investors never make strictly positive profits on low-type firms. Therefore, investors must make strictly positive expected profits on high types in any separating equilibrium.

Lemma 5. *A separating equilibrium is 'reasonable' only if investors do not make positive profits on low-type firms:* $\mathbb{E}_L[s_L] = Q(s_L)$.

Our last preliminary lemma establishes that in any 'reasonable' separating equilibrium, high-type firms must offer a security that exactly covers their financing needs. Intuitively, all firms value cash in the same way, but high types expect to pay more to lenders for any security, so borrowing less cash and issuing less securities appeals to high types.

Lemma 6. *A separating equilibrium is 'reasonable' only if high-type firms offer a security that raises the exact amount needed by the project:* $Q(s_H) = 1$.

Proposition 1 builds on these lemmas to establish that separating equilibria exist, and that they all share the same qualitative structure. Specifically, in every separating equilibrium, high types issue debt, while low types issue steeper securities such as equity, or a non-degenerate combination of debt and equity.

Proposition 1. *There exists a continuum of 'reasonable' separating equilibria. Every such equilibrium features the following common properties:*

1. *High-type firms issue a standard debt security:* $s_H = \min\{x, D_H\}$;
2. *The security* s_H *is underpriced: the face value* D_H *sets* $\mathbb{E}_L[\min\{x, D_H\}] = 1$;
3. *Investors make strictly positive profits on high-type firms:* $\mathbb{E}_H[\min\{x, D_H\}] > 1$.

4. *Low-type firms issue any security different from debt (i.e.,* $s_L \neq \min\{x, D\}$ *for any* $D \in [0, \bar{x}]$*), such that* $\mathbb{E}_L[s_L] = Q(s_L) \geq 1$.

Proposition 1 is novel in the context of the security design literature, providing insights into both the theoretical and empirical literature on capital structure and security design. Theoretically, our results show that, given the standard assumption of monotonic securities, there exist equilibria in which investors make strictly positive profits on high-type firms, and at which the securities issued by different types can be ranked according to their steepness, or information sensitivity. These equilibria are typically ruled out in existing papers by the imposition of zero-profits constraints on investors in equilibrium (see, e.g., the 'competitive rationality' condition for equilibrium in NN). With this added constraint, consistent with our characterization, the unique equilibrium that satisfies D1 has all firms pool on the same debt contract, which is fairly priced for the average firm type.

Empirically, our results provide theoretical foundations to empirical tests of the 'pecking-order' hypothesis. Such tests typically ask whether in some sample of firms, the set of 'better firms' (high types) relies more on debt than the set of 'worse firms' (low types). But as we just argued, while the idea behind the pecking order hypothesis is that worse firms may not *be able* to issue debt, security design under asymmetric information *and* zero profits necessarily delivers a pooling equilibrium in which every firm relies on debt, contrary to the pecking-order premise. A contribution of our paper is to show that when one allows for the possibility of positive investor profits, then a foundation for these tests of the pecking-order obtains. That is, in our separating equilibrium, high types issue *only* debt, while low-types must issue other securities as well, such as equity, to make their

payout to investors strictly more informationally sensitive. Observe that our argument does *not* rely on the concept of a firm's *debt capacity*, which is not micro-founded in this adverse selection setting (Leary and Roberts (2010)). Instead, in our model, low types have an intrinsic preference relative to high types for steeper securities (e.g., equity rather than debt). That is, in the context of our example, the iso-profit curve of low types is flatter—in the equity/debt space—than that of high types (see Figure 1).[14]

4.2 Pooling Equilibria

We next characterize all pooling equilibria in which $s_L = s_H \equiv s_P$ for some feasible s_P. Under the exogenous imposition of zero profits for investors, NN show that a unique pooling equilibrium satisfying D1 exists (see their Section 3). In this equilibrium all firm types pool on a debt contract that is fairly priced for the average firm type. We relax this restriction, allowing investors to make non-negative profits in equilibrium.

Investors make non-negative profits in a pooling equilibrium if and only if $Q(s_P) \leq \mathbb{E}_0[s_P] := p_H \mathbb{E}_H[s_p] + p_L \mathbb{E}_L[s_p]$. We first establish that in any pooling equilibrium with non-negative profits, the security s_P on which the two types pool must cross all feasible securities from below. Thus, within monotonic securities, all firm types issue straight debt.

Lemma 7. *In any 'reasonable' pooling equilibrium, all firm types issue debt:* $s_L = s_H = \min\{x, D_P\}$ *for some* $D_P \in [0, \bar{x}]$.

We next establish that in any pooling equilibrium, all firms raise the minimum amount

[14] There is also a sense in which these separating equilibria require less information than pooling equilibria, as they only demand that investors know the possible types of firms, but not necessarily the proportion of each type within the economy.

needed for investing. NN (Theorem 8) prove that this property must hold in the unique 'reasonable' equilibrium in which firms issue debt without proposing a selling price for it to investors. This result extends when firms can propose prices for the securities they issue:

Lemma 8. *In any 'reasonable' pooling equilibrium, all firm types raise exactly the capital required to invest in the project:* $Q(s_p) = 1$.

The characterization of all pooling equilibria of our game in Proposition 2 below follows directly from Lemmas 7-8. One equilibrium is the zero-profit debt equilibrium identified by the literature. The other equilibria involve all firm types issuing debt that is *underpriced* relative to the average type, with a degree of underpricing that spans from zero to the other extreme in which debt is priced so that it would make zero profits for investors *if* it were only issued by low-type firms.

Proposition 2. *There exists a continuum of 'reasonable' pooling equilibria. In any such equilibrium, all types raise exactly \$1 of capital by issuing debt with a face value* $D_P \in [D_0, D_H]$, *where* D_H *solves* $\mathbb{E}_L[\min\{x, D_H\}] = 1$ *and* D_0 *solves* $\mathbb{E}_0[\min\{x, D_0\}] = 1$.

Importantly, high-type firms are strictly better off in all pooling equilibria that have debt contracts with face value $D_P \in [D_0, D_H)$ than they are in all separating equilibria, *even though* high types cross-subsidize low types in these pooling equilibria. This reflects the fact that the debt security issued by high types in any separating equilibrium would break even for investors if it were only issued by low types. This is the opposite of what happens in standard settings in which better firms choose to separate by means of costly signals, in order to minimize the future mispricing of their securities.

5 Extension to N > 2 types

Our first extension considers the case of more than two types. Our primary objective is to establish that the key conclusions drawn from the two-type case generalize: (i) there exist equilibria in which low types issue equity while high types issue debt (but not vice versa); (ii) investor profits are strictly positive in equilibrium (generically, and in contrast to NN's pooling equilibrium, which is the sole exception); (iii) the degree of underpricing of high types' securities only depends on the technology of the worst type, and not on the number of types or their share in the population.

We now denote a firm's type by $\theta \in \{\theta_1, ..., \theta_N\}$, where types are ordered so that $1 < \theta_1 < \theta_2 < ... < \theta_N$. Thus, all firms have positive NPV projects and it is efficient to fund them. Let $p_i := \Pr.[\theta = \theta_i]$. Denote the mean firm type by $\theta_0 := \sum_{i=1}^{N} p_i \theta_i$. Strict MLRP requires that $\frac{\partial}{\partial x} \frac{f_{i+1}(x)}{f_i(x)} > 0$ for every $x \in [0, \bar{x})$ and for $i \in \{1, ..., N-1\}$.

A first useful observation is that, as in NN, if in an equilibrium a type θ issues a security that is underpriced relative to the full-information case, while another type θ' issues a security that is not underpriced, then we must have that $\theta > \theta'$. Namely, it must the better type that issues underpriced securities in equilibrium:

Lemma 9. *If there is an equilibrium security $s = s^*(\theta)$ such that $\mathbb{E}[s(\theta)] \leq Q^*(s)$, and another security $s' = s^*(\theta')$ such that $\mathbb{E}[s(\theta')] \geq Q^*(s')$, then:*

$$\min\{\theta | s^*_\theta \textit{ is underpriced relative to full information}\} >$$

$$\max\{\theta | s^*_\theta \textit{ is not underpriced relative to full information}\}.$$

Lemma 9 implies that if there is a pool of types issuing securities that are underpriced relative to the full-information case, while another pool of types issues non-underpriced securities, then the lowest type that faces underpricing must have a higher net present value than the highest type issuing a security that is not underpriced.

In addition, as in the two-type case, incentive constraints prevent two different equilibrium securities from being both priced as they would have been under full information.

Lemma 10. *There is no equilibrium such that: (i)* $s^*_{\theta'} \neq s^*_{\theta''}$*, for a pair of types* θ' *and* $\theta'' \neq \theta$*, (ii)* $\mathbb{E}[s^*_\theta(\theta)] = Q^*(s^*_\theta)$ *for* $\theta \in \{\theta', \theta''\}$*.*

Combining these results, it follows that if some type issues securities that are priced as they would have been under full information, then it must be the lowest type.

Corollary 1. *The only type that can issue a fairly-priced security in equilibrium is the lowest productivity type* $\theta = \theta_1$*.*

Proof. Immediate from Lemmas 3 and 9. □

Another implication of this analysis is that the highest type must issue underpriced securities in equilibrium, relative to the full-information benchmark. Therefore, this type can never benefit from raising more than the \$1 it needs for investment purposes.

Corollary 2. *In every equilibrium,* $Q^*(s^*_{\theta_N}) = 1$*.*

Proof. Immediate from Lemmas 3 and 9. □

To characterize equilibria, observe that there are two mutually exclusive and exhaustive types of equilibria: those in which the low type issues fairly-priced securities, and those in

which the low type issues over-priced securities—that is, securities that have a higher price relative to those that the low type would issue under full information. This is because the lowest type always has a positive net present value project, so it can always finance itself at the full-information rate. If one also imposes that investors must make zero profits, as in NN, then one can prove that fairly-priced securities cannot be issued in equilibrium, and there must be pooling. However, this is not the case with the more general contract space that we consider. To clarify intuition, we analyze these two cases separately.

5.0.1 Partial pooling

We begin with the case in which the lowest type issues fairly priced securities, that is, securities such that $\mathbb{E}[s^*_{\theta_1}(\theta_1)] = Q(s^*_{\theta_1})$. In this case the characterization of a continuum of partial-pooling equilibria obtains straightforwardly from incentive compatibility considerations, for similar reasons as those highlighted in the two-type case. All of these equilibria have the feature that investor profits are strictly positive, and that low types must issue a security that crosses debt from below, such as equity.

Proposition 3. *There exists a continuum of 'reasonable' partial-pooling equilibria. In every such equilibrium, the following properties hold:*

1. *The low type $\theta = \theta_1$ issues a security $s^*_{\theta_1}$ such that $\mathbb{E}[s^*_{\theta_1}] = Q(s^*_{\theta_1})$ that crosses debt from below (e.g., equity).*
2. *All other types issue a debt security $s^*_{-\theta_1}$ such that $\mathbb{E}[s^*_{-\theta_1}(\theta_1)] = Q(s^*_{-\theta_1})$. That is, investors would break even on $s^*_{-\theta_1}$ were it issued by type $\theta = \theta_1$ instead.*
3. *Expected investor profits are strictly positive, despite free entry.*

Proposition 3 highlights the importance of modeling competition explicitly, without imposing zero profits exogenously. By imposing zero profits, the literature has missed the existence of a continuum of equilibria in which low types issue equity, and higher types issue debt which is so underpriced not to be an attractive deviation for low types. At this debt contract, investor profits must be strictly positive, as otherwise incentive constraints would be violated, but competition among investors is not able to erode the profits, as it faces the threat of attracting low types as well. The intuition for why partial-pooling equilibria with severe underpricing exist is that when low types are expected to issue equity in equilibrium, any other security offered that would benefit higher types would also dominate equity issuance for low types, making it not incentive compatible on-the-equilibrium path, and unprofitable off-equilibrium path (if investors are sufficiently pessimistic).

5.0.2 Pooling equilibria

It remains to consider the possibility that the lowest type issues overpriced securities in equilibrium. In this case, an investor would not accept to purchase the security unless the low type was pooling with some higher-NPV type τ. Using arguments that mirror those in NN, one can show that this pooling must be on debt, as it is the flattest ordered set of securities. This logic allows one to conclude that all types must be in the same pool that issues debt. However, there still is an important contrast relative to NN, because in our model there is no reason for this debt to be fairly priced relative to the average type. Indeed, we prove that there exists a continuum of pooling equilibria, and that these equilibria span all possible degrees of underpricing, from NN's equilibrium with fair pricing on the average type, all the way to the extreme underpricing that occurs at partial-pooling equilibria.

Proposition 4. *There exists a continuum of 'reasonable' pooling aquilibria. At every such equilibrium, the following properties hold:*

1. *All types issue a debt security s_p^* such that $\mathbb{E}[s_p^*] \in [\$1, \mathbb{E}[s_p^*(\theta_1)]]$.*
2. *Investor profits are strictly positive at every pooling equilibrium, except that which implements NN's allocation—that is, they are strictly positive whenever $\mathbb{E}[s_p^*] > \$1$.*

Proposition 4 highlights a second key implication of NN's zero-profit assumption: it pins down a non-generic pooling equilibrium in which debt is fairly priced relative to the average type, so that investor profits are zero. While NN's prediction that full pooling must involve the issuance of debt securities is robust, we also find that investor profits are strictly positive at every equilibrium allocation *but* NN's, so positive expected profits are expected in settings where firms seek financing under ex-ante informational asymmetries, as in NN.

6 Pecking-order with pre-existing capital structures

As we stressed in the introduction, the motivation behind our theoretical exercise is twofold. First, we want to establish what drives the results in NN, and to characterize the set of equilibria when one drops NN's assumptions that (i) competition among investors implies that profits must be zero in equilibrium; and (ii) firms cannot propose prices for their securities. Second, we seek to provide a micro-foundation for empirical tests of pecking-order theories of external financing, which are inconsistent with a pooling equilibrium in which all firms issue debt and therefore have exactly the same capital structure.

As a preliminary step in micro-founding firms' heterogenous capital structure, we established that equity issuance by a low type is possible in equilibrium, while high types must

issue debt. Thus, capital structure heterogeneity emerges in our baseline model. However, the partial-pooling equilibrium that we derive masks further heterogeneity that is relevant in empirical applications, because it consists of a corner solution in which higher-type firms must pool. To clarify this point, we now relax the strong assumption that NN make that firms have no pre-existing capital structure, and we micro-found the cross-sectional leverage regressions that empirical papers testing the pecking-order theory typically run.

In empirical practice, firms choose how to obtain external financing when they already have a capital structure in place. This adds a dimension to a firm's optimization problem, as now the firm chooses not only what securities to issue, but also what fractions of pre-existing securities to repurchase. For concreteness, we modify our example of Section 2. First, we introduce a medium type whose project would yield $x_M \in (x_L, x_H)$ with probability $f_M \in (f_L, f_H)$, and zero otherwise. Second, we now assume that a firm's manager inherits a capital structure from the past that consists of (1) a face value of liabilities of $\Delta \geq 0$, and (2) a fraction of shares sold in the past to outside investors of $\beta \in [0, 1]$. All claims are contingent on the final realization of cash flows x.

The assumption that projects are positive net present value for the firm's manager translates into the condition $f_i(1-\beta)(x_i - \Delta)^+ \geq 1$ for $i \in \{L, M, H\}$. Since f_i and x_i strictly increase with i, it follows that a sufficient condition for all projects to have positive net present value for the firm under full information is $f_L(1-\beta)(x_L - \Delta)^+ \geq 1$. We assume that this inequality holds. Moreover, to ensure monotonicity across types (which is immediate when $\beta = \Delta = 0$, and corresponds graphically to a condition on the ordering of the crossing points of zero profit curves) we assume that the types are sufficiently different that $\frac{f_H - f_M}{f_H f_M x_H - f_H f_M x_M} > \frac{f_L - f_M}{f_L f_M x_L - f_L f_M x_M}$.

Figure 2: Extension of Figure 1 to pre-existing capital structures

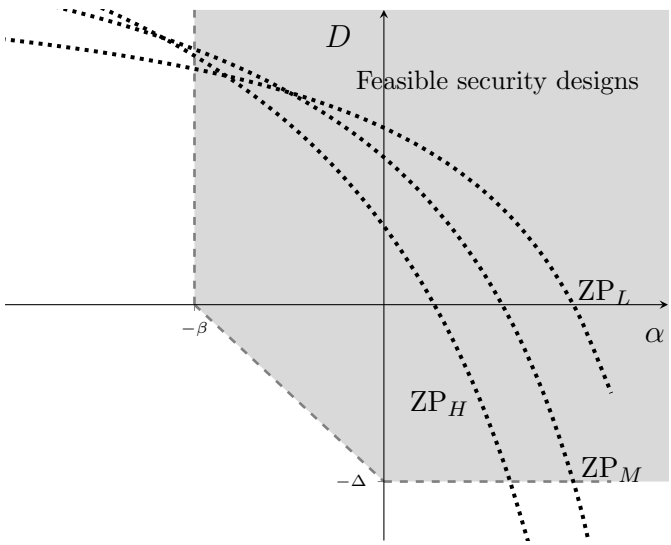


Figure 2 plots the zero-profit curves in the (α, D) space when model parameters are: $\beta = 0.45$, $x_H = \$10$, $x_M = \$8$, $x_L = \$6$, $f_H = \frac{3}{4}$, $f_M = \frac{1}{2}$, $f_L = \frac{1}{3}$ and $p_H = p_M = \frac{1}{3}$. Note that the set of feasible securities is no longer bounded below and to the left by zero, as it was in the example analyzed in Figure 1. Now, a firm can issue up to $-\beta$ shares and up to $-\Delta$ worth of bonds. If a firm has both a combination of debt and equity, then the maximum amount of shares that are available to repurchase when $f_L[(1-\beta)(x_L-\Delta)^+ \geq 1$ is $1 - \frac{1}{p_L(x_L-\Delta)^+}$, as depicted in the negative orthant of Figure 2.

As Figure 2 shows, there is never a gain from repurchasing debt in this model. The only type that could ever repurchase debt is the low type, at an equilibrium in which it separates from other types and thus can repurchase the debt at its full-information price. The nature of equilibria in this setting depend critically on the shares outstanding β.

Brennan and Kraus (1987) analyze in a related setting, the case in which full separation

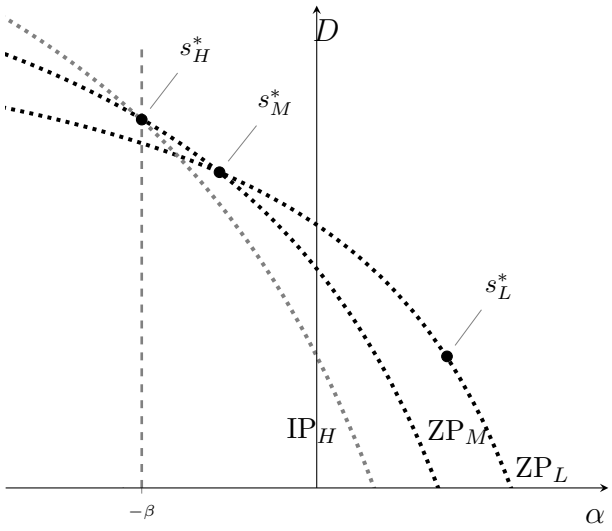
without investor profits is possible, which requires sufficient shares outstanding. In contrast, we focus on settings with fewer shares outstanding, so that $\beta \leq \overline{\beta} := \left|\frac{f_M - f_H}{f_M f_H (x_H - x_M)}\right|$, where $\overline{\beta}$ is the unique level of shares α such that ZP_H intersects ZP_M.[15] When high types are not able to repurchase $\bar{\beta}$ shares, full separation with zero investor profits on all equilibrium contracts is no longer feasible. However, in line with our earlier analysis, separating equilibria with positive investor profits continue to exist.

Figure 3 shows an example of the type of fully separating equilibria that can obtain. In this equilibrium, as with no pre-existing capital structures, low types can issue a combination of debt and equity, without repurchasing any security, such as s_L^*. Medium types issue debt and repurchase the minimal amount of shares that guarantees separation from low types, and offer s_M^*, at the intersection of ZP_M and ZP_L. Finally, high types repurchase strictly more shares than medium types, all the way to the corner $-\beta$, issuing s_H^*.

To see why this is an equilibrium, consider high types first. At s_H^* they are leaving strictly positive profits to investors (because IP_H is shifted to the right relative to ZP_H), but there is no other feasible mixture of debt, equity and repurchases that can make them better off: any feasible deviation lies below ZP_M, and thus would attract medium types and be loss making in expectation for investors off-equilibrium (if sufficiently pessimistic). Similarly, medium types do not have a profitable deviation, because any feasible deviation that would make them better off if accepted by investors, could either attract low types and be loss making, or if it does not attract low types it is loss-making on medium types for investors, and hence would be rejected.

[15] To obtain this expression, first set the right-hand side of ZP_H equal to the right-hand side of ZP_M and solve for α. Then, solve for D the equation ZP_H, and plug in the expression for α that was obtained, and solve for α again to get our expression.

Figure 3: A fully separating equilibrium with strictly monotonic leverage and positive investor profits, when $\beta \leq \bar{\beta}$



We emphasize a few features of this equilibrium. First, our main conclusions from the setting with no pre-existing securities extend: (i) high types issue underpriced debt on which competitive investors make strictly positive expected profits; (ii) medium types also issue debt, but they issue *strictly less* debt than high types, because they optimally choose to repurchase fewer shares; (iii) low types issue *strictly less debt* than medium types, and possibly choose steeper securities than debt, such as equity. However, unlike the case without pre-existing capital structure, this fully separating equilibrium features *strictly monotonic leverage* across types—the higher the type, the higher the equilibrium leverage—for a set of firms that started off with an identical inherited capital structure (β, Δ). This provides a direct foundation for the net-leverage regressions used to test the pecking-order theory, and clarifies that the partial pooling equilibrium is driven by the hidden assumption $\beta = 0$, without which there exists a continuum of fully separating equilibria *all* featuring underpricing at the top and strictly monotonic leverage across firm types.

Figure 4: Pooling equilibria with pre-existing securities (first case)

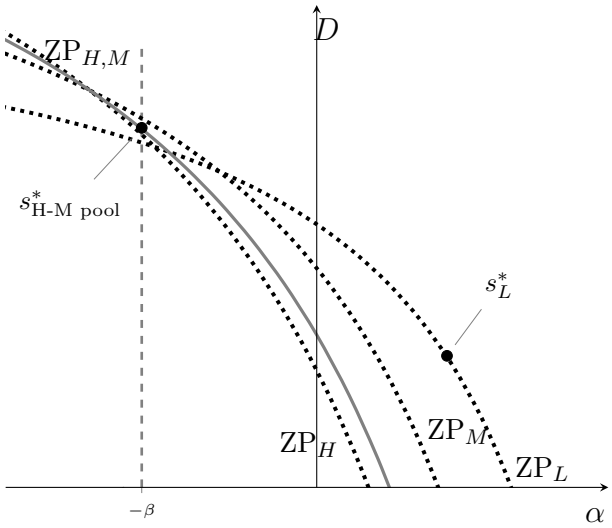
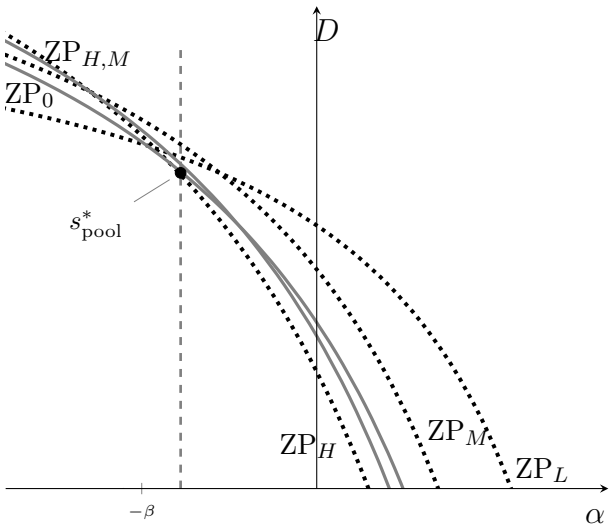


Figure 5: Pooling equilibria with pre-existing securities (second case)



To conclude the analysis, observe that such fully separating equilibria are not the only type of equilibrium that obtains. First, there is a trivial partial pooling equilibrium at which low and medium types pool and offer s^*_M, while high types issue s^*_H. Second, there exists an equilibrium in which investor profits are zero, as in NN. This equilibrium is unique, and it features either pooling of the high and medium type, while the low type

can offer something like s_L^* on ZP_L, or it features full pooling of all types. Which of these cases arises depends on the amount of outstanding shares β. Figure 4 illustrates the case where enough shares are outstanding that low types do not want to mimic a pool of high and medium types. This is because these types are repurchasing their equity at a price that low types find too expensive. Thus, the unique zero-profit equilibrium features pooling of high and medium types, both of whom issue s_{pool}^*. In contrast, in Figure 5, the available shares are not enough for a pool of high and medium types to separate from low types. Thus, the unique zero-profit equilibrium must feature full pooling, as in NN. To conclude the characterization, note that every partial pooling (or fully pooling equilibrium where high and medium types issue a convex combination of s_{pool}^* and s_H^*) is also an equilibrium, for the same reasons as when there are no pre-existing securities, and these equilibria all feature strictly positive investor profits.

7 Conclusion

Our paper reconsiders the classical security design problem of a firm that seeks finance from competitive investors for a project whose quality the firm alone knows. Nachman and Noe (1994) show that in this setting if investors must earn zero profits, under certain conditions it follow that all firm types pool on offering a debt contract that breaks even in expectation. However, there is a tension between this prediction and empirical evidence suggesting that many securities are under-priced and that there is a pecking-order relationship in which better firms issue relatively more debt.

Our paper highlights hidden assumptions that drive NN's result, and it shows that

relaxing these assumptions yields foundations for these empirical findings. In addition to imposing a zero profit requirement for investors, NN *restrict the contract space*, not allowing firms to propose *prices* for their securities, which investors can then accept or reject. In practice, prices are often proposed in private placements or bank lending, where an issuer shops around, negotiates with different potential investors, and then selects the best deal.

In this paper, we allow firms to propose prices for their securities to competitive investors, who are willing to provide capital as long as they can earn non-negative profits in expectation. We show that restrictions to zero profits are not driven by some un-modeled Bertrand competition, but rather by the exclusion of securities with proposed prices—and that this restriction has bite. In addition to a fairly-priced pooling debt equilibrium that is the analogue to NN's unique equilibrium, we identify 'reasonable' equilibria in which low types issue steeper securities than debt, while higher types issue debt that is fairly priced for low types, and hence underpriced relative to its true value given the pool of types issuing it. In such equilibria, investor profits on high types are strictly positive. We also identify a range of pooling equilibria in which all types issue debt that is underpriced relative to its true value given that it issued by the average firm type, and so also featuring positive investor profits.

We then consider firms with pre-existing capital structures and show that, because higher types have greater incentives to repurchase equity, fully separating equilibria obtain in which leverage is strictly monotonic across types, high types issue underpriced debt and low types issue steeper securities such as equity. Thus, our model provides theoretical foundations for empirical tests of the pecking-order theory of external finance and for the profitability of bank loans, syndicated loans and privately-placed debt, in a

setting in which lenders do *not* have any informational advantage relative to other investors, and there is no moral hazard in the model to justify their rents. In our setting, banks make strictly positive profits even though they are competitive and do not possess 'soft' information about firms. These lender profits are driven by incentive compatibility considerations and the need to incentivize low types to issue steeper securities.

References

Allen, F. and G. R. Faulhaber, "Signaling by underpricing in the IPO market," *Journal of Financial Economics*, 1989, *23* (2), 303–323.

Bai, J., R. S Goldstein, and F. Yang, "Is the credit spread puzzle a myth?," *Journal of Financial Economics*, 2020, *137* (2), 297–319.

Brennan, M. and A. Kraus, "Efficient financing under asymmetric information," *The Journal of Finance*, 1987, *42* (5), 1225–1243.

Carroll, G., "Robustness and linear contracts," *American Economic Review*, 2015, *105* (2), 536–563.

Cho, I. and D. M. Kreps, "Signaling games and stable equilibria," *The Quarterly Journal of Economics*, 1987, *102*, 179–221.

Chodorow-Reich, G., O. Darmouni, S. Luck, and M. Plosser, "Bank liquidity provision across the firm size distribution," *Journal of Financial Economics*, 2021.

Dasgupta, P. and E. Maskin, "The existence of equilibrium in discontinuous economic games, I: Theory," *The Review of economic studies*, 1986, *53* (1), 1–26.

Dell'Ariccia, G. and R. Marquez, "Information and bank credit allocation," *Journal of Financial Economics*, 2004, *72* (1), 185–214.

DeMarzo, P. and D. Duffie, "A liquidity-based model of security design," *Econometrica*, 1999, *67* (1), 65–99.

_ , I. Kremer, and A. Skrzypacz, "Bidding with Securities: Auctions and Security Design," *American Economic Review*, 2005, *95* (4), 936–959.

Fama, E. F. and K. R. French, "Financing decisions: who issues stock?," *Journal of financial economics*, 2005, *76* (3), 549–582.

Frank, M. Z. and V. K. Goyal, "Testing the pecking order theory of capital structure," *Journal of financial economics*, 2003, *67* (2), 217–248.

Grinblatt, M. and C. Y. Hwang, "Signalling and the pricing of new issues," *The Journal of Finance*, 1989, *44* (2), 393–420.

Innes, R., "Financial contracting under risk neutrality, limited liability and ex ante asymmetric information," *Economica*, 1993, *60* (237), 27–40.

Koufopoulos, K., R. Kozhan, and G. Trigilia, "Optimal Security Design under Asymmetric Information and Profit Manipulation," *The Review of Corporate Finance Studies*, 2019, p. cfy008.

Leary, M. T. and M. R. Roberts, "The pecking order, debt capacity, and information asymmetry," *Journal of financial economics*, 2010, *95* (3), 332–355.

Lee, S. and U. Rajan, "Robust security design," *Ross School of Business Paper*, 2018, (1338).

Maggio, M. Di and V. Yao, "Fintech borrowers: Lax-screening or cream-skimming," *Review of Financial Studies, forthcoming*, 2020.

Malenko, A. and A. Tsoy, "Asymmetric Information and Security Design under Knightian Uncertainty," *Working paper*, 2020.

Myers, S. C., "The capital structure puzzle," *The journal of finance*, 1984, *39* (3), 574–592.

_ **and N. S. Majluf**, "Corporate financing and investment decisions when firms have information that investors do not have," *Journal of Financial Economics*, 1984, *13* (2), 187–221.

Nachman, D. C. and T. H. Noe, "Optimal security design," *Review of Financial Studies*, 1994, *7* (1), 1–44.

Noe, T. H., "Capital structure and signaling game equilibria," *Review of Financial Studies*, 1988, *1* (4), 331–355.

Welch, I., "Seasoned offerings, imitation costs, and the underpricing of initial public offerings," *The Journal of Finance*, 1989, *44* (2), 421–449.

Appendix

Proof of Lemma 1

Proof. If $\mathbb{E}_\theta[s_\theta(x)] = Q(s_\theta)$, $\forall\theta$, then $U_\theta = \theta - 1$ for every θ in equilibrium. The incentive constraint for low-type firms not to mimic high-types reads: $\theta_L - 1 \geq Q(s_H) - 1 + \theta_L - \mathbb{E}_L[s_H(x)] \iff \mathbb{E}_L[s_H] \geq Q(s_H)$. Because $\mathbb{E}_H[s_H] = Q(s_H)$, incentive compatibility requires $\mathbb{E}_H[s_H] \leq \mathbb{E}_L[s_H] \iff \int_0^{\bar{x}} s_H(f_H - f_L)dx \leq 0$. Integrating by parts yields $\int_0^{\bar{x}} \frac{ds_H}{dx}(F_L - F_H)dx \leq 0$. Monotonicity of security payments in cash flows (Condition M), yields $ds_H \geq 0$ and since s_H is a continuous function, it is differentiable almost everywhere. By continuity and because investment is risky, we have $ds_H > 0$ for a positive measure of cash flows. Strict MLRP implies that $F_H(x) < F_L(x)$, for every $x \in [0, \bar{x})$. Therefore, the incentive compatibility constraint cannot hold. □

Proof of Lemma 2

Proof. Suppose, by contradiction that a separating equilibrium satisfying D1 exists at which $U_L(c_L) > U_L(c_H) \iff \mathbb{E}_L[s_H] - Q(s_H) > \mathbb{E}_L[s_L] - Q(s_L)$. Investors are individually rational so $\mathbb{E}_L[s_L] - Q(s_L) \geq 0$. Thus, $\mathbb{E}_L[s_H] - Q(s_H) > 0$. We have already shown that, by strict MLRP, $\mathbb{E}_L[s] < \mathbb{E}_H[s]$ for any strictly monotonic security s. Therefore, $\mathbb{E}_H[s_H] - Q(s_H) > \mathbb{E}_L[s_H] - Q(s_H) > 0$: the participation constraint for investors upon observing the high-type equilibrium security s_H is slack.

Consider the deviation for high types to a security $s'_H = \alpha s_H$, for some α such that $\alpha \in (\frac{Q(s_H)-Q(s_L)+\mathbb{E}_L[s_L]}{\mathbb{E}_L[s_H]}, 1)$, such that $Q(s'_H) = Q(s_H)$. To see that s'_H is a feasible security, and hence the interval is non-empty, observe that $\frac{Q(s_H)-Q(s_L)+\mathbb{E}_L[s_L]}{\mathbb{E}_L[s_H]} < 1$, because $\mathbb{E}_L[s_H] - Q(s_H) > \mathbb{E}_L[s_L] - Q(s_L)$, but positive because $\mathbb{E}_L[s_L] - Q(s_L) \geq 0$.

Deviating to s'_H would be not be profitable for low types, regardless of investor beliefs. To see this, notice that at the lowest possible α, where $\alpha = \frac{Q(s_H)-Q(s_L)+\mathbb{E}_L[s_L]}{\mathbb{E}_L[s_H]}$, incentive compatibility for low types requires $\mathbb{E}_L[s_L] - Q(s_L) \leq \mathbb{E}_L[s'_H] - Q(s'_H) = \mathbb{E}_L[\alpha s_H] - Q(s_H) = \mathbb{E}_L[s_L] - Q(s_L)$. In contrast, the deviation is profitable for high-type firms because (i) it costs less, $\mathbb{E}_H[s'_H] < \mathbb{E}_H[s_H]$; and (ii) it would be accepted by investors. Thus, a contradiction obtains. □

Proof of Lemma 3

Proof. In order to prove the Lemma, it is useful to define $r_{L,H} := \frac{1-F_H(x)}{1-F_L(x)}$. Given our assumption that strict MLRP holds, we have that $r_{L,H}$ is strictly increasing in x, for $x \in [0, \bar{x})$. To see this, note that for $x \in [0, \bar{x} - y]$ and $y \in (0, \bar{x} - x]$, we have that $r_{L,H}(x+y) - r_{L,H}(x) > 0 \iff (1-F_H(x+y))(1-F_L(x)) - (1-F_H(x))(1-F_L(x+y)) > 0$. Adding and subtracting $(1 - F_H(x))(1 - F_L(x)))$ to this expression, we get that $r_{L,H}(x+$

$y) - r_{L,H}(x) > 0 \iff (1-F_L(x))[F_H(x)-F_H(x+y)]-(1-F_H(x))[F_L(x)-F_L(x+y)] > 0$. Dividing through by $(1-F_H(x))(1-F_L(x)))$ yields that $r_{L,H}(x+y) - r_{L,H}(x) > 0 \iff \frac{F_H(x)-F_H(x+y)}{1-F_H(x)} - \frac{F_L(x)-F_L(x+y)}{1-F_L(x)} = F_L(y|x) - F_H(y|x)$. Because of strict MLRP, we know that $F_L(y|x) > F_H(y|x)$, concluding the proof that $r_{L,H}$ is strictly increasing.

First, note that $s^D(d,x)$ satisfies the definition of a full ordered set of securities, where d corresponds to the index z. Integration by parts implies that, for every possible cash-flow distribution f_i and feasible security s: $\int_0^{\bar{x}} s(x)dF_i(x) = \int_0^{\bar{x}} \frac{\partial s}{\partial x}(1-F_i(x))dx$. Moreover, for debt securities we have that $\frac{\partial s^D}{\partial x} = 1$ for $0 \leq x \leq z$, and $\frac{\partial s^D}{\partial x} = 0$ for $x > z$. It follows that $\int_0^{\bar{x}}(s(x)-s^D(x))dF_i(x) = \int_0^{\bar{x}}(\frac{\partial s}{\partial x} - \frac{\partial s^D}{\partial x})(1-F_i(x))dx = \int_0^z(\frac{\partial s}{\partial x}-1)(1-F_i(x))dx + \int_z^{\bar{x}} \frac{\partial s}{\partial x}(1-F_i(x))dx$. As we just proved that $r_{L,H}(x)$ is strictly increasing in x over the interval $[0,\bar{x})$, it follows that for $0 \leq x \leq z \leq x'$, we have $r_{L,H}(x) \leq r_{L,H}(z) \leq r_{L,H}(x')$. Generically—that is, whenever $x \neq z$—we also have that $\frac{\partial s}{\partial x}\big|_{x'} r_{L,H}(x') \geq \frac{\partial s}{\partial x}\big|_{x'} r_{L,H}(z)$ and $\left(\frac{\partial s}{\partial x}\big|_x - 1\right) r_{L,H}(x) \geq \left(\frac{\partial s}{\partial x}\big|_x - 1\right) r_{L,H}(z)$. We can now re-write:

$$\begin{aligned}
\int_0^z \left(\frac{\partial s}{\partial x} - 1\right)(1-F_L(x))dx + \int_z^{\bar{x}} \frac{\partial s}{\partial x}(1-F_L(x))dx &= \int_0^z \left(\frac{\partial s}{\partial x} - 1\right) r_{L,H}(x)(1-F_H(x))dx \\
&+ \int_z^{\bar{x}} \frac{\partial s}{\partial x} r_{L,H}(x)(1-F_H(x))dx \\
&> r_{L,H}(z)\Bigg(\int_0^z \left(\frac{\partial s}{\partial x} - 1\right)(1-F_H(x))dx \\
&+ \int_z^{\bar{x}} \frac{\partial s}{\partial x}(1-F_H(x))dx\Bigg) \\
&= r_{L,H}(z)\left(\int_0^{\bar{x}} (s(x)-s^D(x))dF_H(x)\right).
\end{aligned}$$

As a result, whenever $\mathbb{E}[s(\theta_H)] \leq \mathbb{E}[s^D(\theta_H)]$, then $\mathbb{E}[s(\theta_L)] < \mathbb{E}[s^D(\theta_L)]$. □

Proof of Lemma 4

Proof. Suppose, by contradiction, there exist s_H, s_L with $s_L \neq s_H$ and $U_H(c_H) = U_H(c_L)$. Then $U_L(c_H) = U_L(c_L)$ from Lemma 2. Combining these equations yields $\mathbb{E}_L[s_L] - \mathbb{E}_H[s_L] = \mathbb{E}_L[s_H] - \mathbb{E}_H[s_H]$, or, equivalently: $\int_0^{\bar{x}}(s_L - s_H)d(F_L(x) - F_H(x)) = 0$. From strict MLRP, this can hold only if $s_L = s_H$, a contradiction. Lemma 2 then implies that $\mathbb{E}_H[s_H] - \mathbb{E}_L[s_H] < \mathbb{E}_H[s_L] - \mathbb{E}_L[s_L]$. Lemma 2 implies that we need to consider the family of securities with the same expected value under the measure that puts probability one on low-type firms. Strict MRLP implies that, within this set of securities, high-type firms strictly prefer the flatter one, so s_H must be flatter than s_L. □

Proof of Lemma 5

Proof. Suppose, instead, that a separating equilibrium exists with $\mathbb{E}_L[s_L] > Q(s_L)$. From Lemma 4, low-type firms can deviate to a security $s'_L = \beta s_L$ for $\beta \in (\frac{Q(s_L)-Q(s_H)+\mathbb{E}_H[s_H]}{\mathbb{E}_H[s_L]}, 1)$ at which all incentive constraints hold: the security is constructed so that it is never a profitable deviation for high types. The proof logic from Lemma 2 guarantees that s'_L is feasible. Moreover, for $\beta \in (\frac{Q(s_L)}{\mathbb{E}_L[s_L]}, Q(s_L))$, which is a non-empty interval given $\mathbb{E}_L[s_L] > Q(s_L)$, the participation constraint for investors still holds. Therefore, deviating to s'_L is profitable when $\theta = \theta_L$, but not when $\theta = \theta_H$, violating D1. □

Proof of Lemma 6

Proof. Suppose, instead, there is a 'reasonable' PBE with $Q(s_H) > 1$. $\mathbb{E}_L[s_H] - Q(s_H) = \mathbb{E}_L[s_L] - Q(s_L) = 0$ from Lemmas 2 and 5; and $\mathbb{E}_H[s_H] - Q(s_H) < \mathbb{E}_H[s_L] - Q(s_L)$ from Lemma 4. Consider the deviation for high-type firms to another security $(s'_H, Q(s'_H))$ such that $Q(s'_H) = Q(s_H) - \epsilon s'_H = \alpha s_H$ for some $\alpha \in (0, 1 - \frac{\epsilon}{\mathbb{E}_H[s_H]})$. The proof logic from Lemma 2 guarantees that s'_H is feasible. If accepted, s'_H makes high types strictly better off relative to s_H. Further, low types do not mimic since $\mathbb{E}_L[s'_H] - Q(s'_H) > 0 \iff \alpha\mathbb{E}_L[s_H] - Q(s_H) + \epsilon > 0 \iff (1-\alpha)\mathbb{E}_L[s_H] - \epsilon < 0 \iff \left(\frac{\mathbb{E}_L[s_H]}{\mathbb{E}_H[s_H]} - 1\right)\epsilon < 0 \iff \mathbb{E}_L[s_H] < \mathbb{E}_H[s_H]$, where to proceed from $(1-\alpha)\mathbb{E}_L[s_H] - \epsilon$ to $\left(\frac{\mathbb{E}_L[s_H]}{\mathbb{E}_H[s_H]} - 1\right)\epsilon$ simply plug the highest possible value for α inside the first expression. $\mathbb{E}_L[s_H] < \mathbb{E}_H[s_H]$ by strict MLRP, so the deviation is profitable. Therefore, the posited equilibrium violates D1. □

Proof of Proposition 1

Proof. Lemma 4 ensures that low types do not issue debt. Suppose first, by contradiction, that property (1) fails, and both firm types issue securities different from debt. That is, $s_\theta \neq \min\{x, D_\theta\}$ for $D_\theta \in [0, \bar{x}]$, for each θ. Consider the deviation by high-type firms to a debt security $s'_H = \min\{x, D_H\}$ for some $D_H \in [0, \bar{x}]$, where D_H is such that $\mathbb{E}_L[\min\{x, D_H\}] = 1 + \epsilon$, for a small $\epsilon > 0$. By Lemma 2, this deviation is unprofitable for low-type firms, because low types are indifferent between mimicking or not when $\epsilon = 0$. High-type firms compare $\mathbb{E}_H[\min\{x, D_H\}]$ with the expected payout at the posited equilibrium $\mathbb{E}_H[s_H]$. Strict MLRP implies that if D is the face value of debt such that $\mathbb{E}_L[\min\{x, D\}] = 1$, then $\mathbb{E}_H[\min\{x, D\}] < \mathbb{E}_H[s_H]$, where the inequality follows from the fact that any feasible (monotonic) security other than debt crosses the payoff of debt from below. Thus, one can increase slightly the face value of debt from D to D_H so that $\mathbb{E}_H[\min\{x, D_H\}] < \mathbb{E}_H[s_H]$, and $\mathbb{E}_L[\min\{x, D_H\}] = 1 + \epsilon$.

Now, consider the case where high-type firms issue debt: $s_H = \min\{x, D_H\}$ for the D_H such that $\mathbb{E}_L[\min\{x, D_H\}] = 1$. The only deviation that could lower the payout for high types is to a security s' such that $\mathbb{E}_L[s'] < 1$. However, this would be profitable for low

types whenever it is profitable for high types (i.e., when the investors' off-equilibrium belief is such that the offer is accepted). Thus, D1 has no bite in this case.

Property (4) follows immediately from the previous analysis, because if low-type firms issue debt, then the high-type firms cannot find a security design that separates them without attracting the low types. Thus, if s_L is debt, there can only be pooling. Property (2) follows immediately from Lemmas 2 and 5, which imply that $\mathbb{E}_L[\min\{x, D_H\}] = \mathbb{E}_L[s_L] = 1$. Property (3) follows from the strict MLRP assumption. □

Proof of Lemma 7

Proof. Suppose, instead, that there is a pooling equilibrium where $s_P \neq \min\{x, D_P\}$ for some $D_P \in [0, \bar{x}]$. Consider the deviation to the debt security $s' = \min\{x, D'\}$ with a face value D' such that $\mathbb{E}_0[\min\{x, D'\}] = \mathbb{E}_0[s_P]$. Evidently, s' is flatter than s_P (see Definition 3). Thus, low-type firms are worse off at s' than at s_P, while high-type firms are better off, conditional on the offer being accepted by investors. Thus, D1 implies that investors hold the off-equilibrium belief $p'_H(s') = 1$, which makes the deviation profitable. □

Proof of Lemma 8

Proof. Suppose, instead, that a 'reasonable' pooling equilibrium exists in which $Q(s_p) \neq 1$. We only need to consider $Q(s_p) > 1$ because $Q(s_p) < 1$ would mean a firm cannot invest, so no capital would be provided, as investors would expect to lose money on it. Suppose now that the high type deviates to a security s' such that $Q(s') = 1$. Because of cross-subsidization, we have $\int_0^{\bar{x}} s_p(x)dF_H(x) > Q(s_p) > \int_0^{\bar{x}} s_p(x)dF_L(x)$. Therefore, the deviation is only profitable for high types, if accepted. It is never profitable for low types. □

Proof of Proposition 2

Proof. Lemma 7 implies that we can restrict attention to debt contracts being offered by all types on-the-equilibrium path. Lemma 8 implies that we can restrict attention to contracts that raise exactly \$1 of capital. Any pooling debt contract with a face value $D > D_H$ would violate the incentive constraint for low types, and so it cannot be part of an equilibrium. Moreover, investors would reject any pooling debt contract with face value $D < D_0$, as they would expect to lose money on it. Consider now a candidate equilibrium in which debt is issued with a face value $D_P \in [D_0, D_H]$. Then $\mathbb{E}_0[\min\{x, D_P\}] \geq 1$, so investors would accept such a debt contract. We must show that it satisfies D1, regardless of the expected profits that accrue to investors. To see this, observe that, starting from a debt contract, for some security s' to benefit high-type firms, it must be such that $\mathbb{E}_L[s'] < \mathbb{E}_L[\min\{x, D_P\}]$. Thus, whenever the off-equilibrium belief held by investors leads to acceptance, both types benefit strictly. It follows that

D1 has no bite regardless of the level of profits made by the investors in equilibrium. □

Proof of Lemma 9

Proof. We first prove the following instrumental claim: *If there is an equilibrium security $s = s^*(\theta)$ such that $\mathbb{E}[s(\theta)] \leq Q^*(s)$, and another security $s' = s^*(\theta')$ such that $\mathbb{E}[s(\theta')] \geq Q^*(s')$, then the we must have that $\theta' > \theta$.*

Suppose that $\theta' < \theta$ instead. From MLRP and monotonicity we know that, for any feasible security s, $\mathbb{E}[s(\theta')] < \mathbb{E}[s(\theta)]$. To see this, integrate by parts the above inequality to obtain $\int_0^{\bar{x}} \frac{ds}{dx}(F_{\theta'} - F_\theta)dx > 0$. Monotonicity of security payments in cash flows (Condition M), yields $ds \geq 0$ and since s must be a continuous function, it is differentiable almost everywhere. By continuity and because investment is risky, we also must have $ds > 0$ for a positive measure of cash flows. Strict MLRP implies that $F_\theta(x) < F_{\theta'}(x)$, for every $x \in [0, \bar{x})$. Therefore, the inequality must hold. It follows from the inequality that we have: $Q^*(s) - \mathbb{E}[s(\theta')] \geq Q^*(s) - \mathbb{E}[s(\theta)] \geq 0 \geq Q^*(s') - \mathbb{E}[s'(\theta')]$, with at least one strict inequality. Thus, type θ' would have an incentive to mimic type θ at this candidate equilibrium, and so incentive compatibility is violated. We conclude that we must have $\theta' > \theta$.

Now observe that this logic extends trivially to pool of types, as stated in the Lemma. □

Proof of Lemma 10

Proof. It follows from the three conditions above that $U_\theta = \theta - 1$ and $U_{\theta'} = \theta' - 1$ in this equilibrium. The incentive constraint for a type-θ firm not to mimic a higher type $\theta' > \theta$ reads: $\theta - 1 \geq Q^*(s^*_{\theta'}) - 1 + \theta - \mathbb{E}[s^*_{\theta'}(\theta)] \iff \mathbb{E}[s^*_{\theta'}(\theta)] \geq Q^*(s_{\theta'})$. Because $\mathbb{E}[s^*_{\theta'}(\theta')] = Q^*(s^*_{\theta'})$, incentive compatibility requires $\mathbb{E}[s^*_{\theta'}(\theta')] \leq \mathbb{E}[s^*_{\theta'}(\theta)]$. However, this inequality contradicts the fact that $\theta' > \theta$ (see the proof of Lemma 3). □

Proof of Proposition 3

Proof. Suppose that the low type $\theta = \theta_1$ issues an equilibrium security that is fairly priced—that is $\mathbb{E}[s^*_{\theta_1}(\theta_1)] = Q^*(s^*_{\theta_1})$. In this case, type θ_1 gains nothing from raising more than \$1 and therefore $Q^*(s^*_{\theta_1}) = 1$. Incentive compatibility for the pair θ_1, θ_N requires that $Q^*(s^*_{\theta_1}) - \mathbb{E}[s^*_{\theta_1}(\theta_1)] \geq Q^*(s^*_{\theta_N}) - \mathbb{E}[s^*_{\theta_N}(\theta_1)]$ for every θ. Given that also $Q^*(s^*_{\theta_N}) = 1$, incentive compatibility requires that $\mathbb{E}[s^*_{\theta_1}(\theta_1)] \leq \mathbb{E}[s^*_{\theta_N}(\theta_1)]$. In equilibrium, this incentive constraint must bind, as otherwise a high type could always deviate to a security such that the constraint is an ϵ closer to be binding and financiers would accept. Thus, we obtain that $\mathbb{E}[s^*_{\theta_N}(\theta_1)] = \mathbb{E}[s^*_{\theta_N}(\theta_1)]$. That is, in any such an equilibrium the high type must be issuing a security that is so underpriced, that investors would break even on it *if* it were issued by a type θ_1 for sure. Moreover, the same logic iterates now to type θ_{N-1}, as now

we know that this type also must issue underpriced securities and therefore raises exactly \$1. As a result, we conclude that $\mathbb{E}[s^*_\theta(\theta_1)] = \mathbb{E}[s^*_{\theta'}(\theta_1)]$ for every pair θ, θ'; all incentive constraints are binding, and therefore all types raise \$1. In addition, we know that among all securities such that $\mathbb{E}[s^*_\theta(\theta_1)]$ is constant—that is, among all the feasible securities such investors make zero profits on a low type, types $\theta > \theta_1$ the equilibrium security satisfies D1 only if it crosses all other securities from below, and therefore it must be debt. To see this, suppose that the security issued by some type $\theta > \theta_1$ is not debt. Then, there exists another security that crosses debt s^*_θ from above. Thus, one can find a pair s, Q such that the deviation would be profitable for type θ, but not for type θ_1. This follows immediately from the definition of crossing from below. Thus, we must have that $s^*_{\theta>\theta_1} = \min\{x, d\}$ for d such that $\mathbb{E}[\min\{x, d\}|\theta = \theta_1] = 1$. If there are multiple solutions to this equation, one selects the smallest feasible solution. As for s^*_1, this can be either debt, or another security which crosses debt from below, such as equity, or a mixture of the two. Evidently, any such equilibrium would satisfy D1 as the steepest ordered set of securities is debt (Lemma 3). Thus, any deviation that would benefit a type $\theta > \theta_1$ would necessarily also benefit θ_1. □

Proof of Proposition 4

Proof. Suppose, by contradiction, that the low type $\theta = \theta_1$ and this higher type who is pooling with it (i.e., type τ) pool on a feasible security s that is not debt. Then, we know that there always exists a steeper ordered set of securities s' such that one can set $Q(s') = Q(s)$ and have $\mathbb{E}[s'(\theta_1)] > \mathbb{E}[s(\theta_1)]$, while $\mathbb{E}[s'(\tau)] < \mathbb{E}[s(\tau)]$. It follows from D1 that type τ would deviate and choose s' off-equilibrium, as investors would accept the offer. Thus, the low type must be pooling on a debt contract. A similar argument implies that any pool of types must involve the issuance of debt, and as a result, by incentive compatibility, there can be at most one pool of types. Further, incentive compatibility requires that all types $\theta > \theta_1$ who are not pooling with type θ_1 to offer a security s'' such that $Q(s'') - \mathbb{E}[s''(\theta_1)] \leq Q(s) - \mathbb{E}[s(\theta_1)]$. Among all such securities, debt minimizes the difference $\mathbb{E}[s(\theta_1)] - \mathbb{E}[s''(\theta_1)]$, and hence it is optimal. We therefore conclude that there must be a single pool of types issuing debt. Moreover, underpricing of the debt issued by high types implies that all firms raise just \$1. One possibility is that this debt is priced so that investors break even, as in NN. Another possibility is that debt is underpriced *relative to the average type.* Any degree of such underpricing can be sustained in equilibrium, as it would satisfy D1 for the same reason as the partial pooling equilibria sustain. Thus, we conclude that, for every θ, in equilibrium $s^*_\theta = \min\{x, d\}$ for d such that $\mathbb{E}[\min\{x, d\}|$ all types pool$] \geq 1$. □