The McKay Correspondence and Orbifold Riemann–Roch

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Declaration

Chapter 1 is purely expository. Otherwise I declare that to the best of my knowledge, the material contained in this thesis is my original work except when explicitly stated otherwise.
Introduction

The classical statement of the McKay correspondence relates the irreducible representations of a finite subgroup $G \subset \text{SL}(2, \mathbb{C})$ with a basis for the cohomology of the unique minimal resolution $Y$ of the orbifold $\mathbb{C}^2/G$. Gonzalez-Sprinberg and Verdier [GSV83] construct this correspondence as an isomorphism on $K$-theory. This is because the $G$-equivariant $K$-theory of $\mathbb{C}^2$ is equal to the representation ring of $G$.

The exceptional locus of $Y$ consists of a chain of $-2$ curves intersecting in the way shown by the Dynkin diagram for $G$. Ito and Nakamura [IN99] show that $Y = G$-Hilb $\mathbb{C}^2$, the fine moduli space parametrising $G$-clusters, and compute the clusters along the exceptional locus of $Y$. In Chapter 2, I compute explicitly an affine cover of $G$-Hilb $\mathbb{C}^2$ when $G$ is a binary dihedral group which shows that $G$-Hilb $\mathbb{C}^2$ is a nonsingular rational variety. Unlike the case when $G$ is an Abelian group the natural cover I construct does not consist of copies of $\mathbb{C}^2$ but smooth surfaces in $\mathbb{C}^4$.

When $G$ is a finite subgroup of $\text{SL}(3, \mathbb{C})$, Bridgeland, King and Reid [BKR01] prove that $G$-Hilb $\mathbb{C}^3$ is a crepant resolution of the orbifold $\mathbb{C}^3/G$. In Chapter 3, I consider the case when $G$ is a trihedral group. I begin by classifying the possibilities for the maximal normal Abelian subgroup of $G$ and show that it must be of the form $\mathbb{Z}_r \times \mathbb{Z}_R$ where $r \mid R, \mathbb{Z}_r$ has generator $\frac{1}{r}(1,-1,0)$ and $\mathbb{Z}_R$ has generator $\frac{1}{R}(a,b,c)$. 

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Introduction

I then compute explicitly an affine cover of $G$-Hilb $\mathbb{C}^3$ using the same methods as in Chapter 2. Again $G$-Hilb $\mathbb{C}^3$ is nonsingular and rational, but the natural affine cover we construct does not consist of copies of $\mathbb{C}^3$.

In Chapter 4, I state and prove a Riemann–Roch style formula for an orbifold having only isolated quotient singularities. I begin by giving an explicit proof of Atiyah and Singer’s equivariant Riemann–Roch theorem for invertible $G$-sheaves in dimensions 1 and 2.

**Theorem 0.1 (Equivariant Riemann–Roch [AS68])** Let $Z$ be a non-

singular projective variety, $V$ a holomorphic vector bundle on $Z$ and $G$ a finite group of automorphisms of the pair $(Z, V)$. For $g \in G$, let $Z^g$ denote the fixed locus of $g$ on $Z$. Then

$$
\text{Tr} \left( g: \bigoplus_{p \geq 0} (-1)^p H^p (Z, \mathcal{O}(V)) \right) = \int_{Z^g} \frac{\text{ch}(V_{Z^g})(g) \cdot \text{td}(Z^g)}{\text{ch}_1 \left( (\mathcal{N}^g)^* \right)(g)}.
$$

I show that this theorem is an elementary consequence of Hirzebruch

Riemann–Roch and an eigensheaf decomposition of sheaves on the quotient

variety. I then use this formula to generate a Riemann–Roch type expression

for an orbifold having only isolated cyclic quotient singularities.

The structure of the proof is as follows. First prove the theorem when $Z$

is a curve. This is done by constructing eigensheaves $\mathcal{L}_i = \text{Hom}(V_i, \pi_* \mathcal{F})^G$

where $\pi: Z \to X = Z/\langle g \rangle$ is the quotient morphism. The theorem is then

proved by computing their degrees. When $Z$ is a surface, choose a resolution

$f: Y \to X$ and find locally free sheaves $\mathcal{M}_i$ on $Y$ such that $f_* \mathcal{M}_i = \mathcal{L}_i$.

Then compute $\chi(Y, \mathcal{M}_i)$.

For any smooth $n$ dimensional projective variety $Z$, we show that the

contribution an isolated fixed point, $P$, makes to Equation 1 depends only

on how $g$ acts on a local analytic neighbourhood $P$. Thus the contribution

x
$P$ would make to

$$\text{Tr} \left( g : \bigoplus_i (-1)^i H^i (Z, \mathcal{F}) \right)$$

is the same as the contribution a point $P' \in C_1 \times \cdots \times C_n$ makes to

$$\text{Tr} \left( g' : \bigoplus_i (-1)^i H^i (C_1 \times \cdots \times C_n, \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_n) \right)$$

if a local analytic neighbourhood of $P'$ in $C_1 \times \cdots \times C_n$ is isomorphic to a local analytic neighbourhood of $P$ in $Z$.

We can then use this theorem to give an expression for $\chi(X, \mathcal{L})$ when $X = Z/\langle g \rangle$ and $L = (\pi_* \mathcal{F})^{\langle g \rangle}$ for $\mathcal{F}$ an invertible $G$-sheaf on $Z$. The contribution that a isolated singularity $P \in X$ makes to this theorem depends only on the local analytic neighbourhood of $P$. Thus we can use this to give an orbifold Riemann–Roch style formula for $\chi(X, \mathcal{L})$ when $X$ is only locally a quotient of a group, that is, an orbifold.
Chapter 1

Preliminaries

1.1 Background to McKay correspondence

Throughout this thesis we will use the notation $g = \frac{1}{r}(a_1, \ldots, a_n)$ to mean that $g = \text{diag}(\varepsilon^{a_1}, \ldots, \varepsilon^{a_n}) \in \text{GL}(n, \mathbb{C})$, where $\varepsilon$ is a primitive $r$th root of unity.

Many papers have been written on the McKay correspondence over the last few years. Good introductions include the first two chapters of Alastair Craw’s thesis [Cra01] or Miles Reid’s Bourbaki talk [Rei02].

Several people have worked on adapting the classical statement of the McKay correspondence to a more general setting. Let $M$ be a smooth quasiprojective variety of dimension $n$ and $G$ a finite group of automorphisms of $M$ such that the stabiliser subgroup acts on the tangent space $T_PK$ as a subgroup of $\text{SL}(T_PK)$ for every point $P \in M$. Let $\pi: M \to X = M/G$ be the finite quotient morphism and $f: Y \to X$ a crepant resolution (if one
exists).

\[
\begin{array}{c}
\pi \\
\downarrow \\
Y \xrightarrow{f} X
\end{array}
\]

A generalisation of the McKay correspondence is that there should be a natural isomorphism between the $G$-equivariant $K$-theory of $M$ and the ordinary $K$-theory of $Y$. See [BKR01].

In the case $M = \mathbb{C}^3$ Nakamura [Nak01] conjectured, and proved when $G$ is Abelian, that $G$-Hilb $\mathbb{C}^3$, the Hilbert scheme parametrising $G$-clusters, is a crepant resolution of $\mathbb{C}^3/G$. Bridgeland, King and Reid [BKR01] prove that $G$-Hilb $\mathbb{C}^n$ is irreducible and a crepant resolution of $\mathbb{C}^n/G$ when $n \leq 3$.

Let $\text{Irr} G$ be the set of irreducible representations of $G$. The regular representation, $\mathbb{C}[G]$, is isomorphic to $\bigoplus_{\rho \in \text{Irr} G} \rho^{\otimes \deg \rho}$. Recall the following definition.

**Definition 1.1** A $G$-cluster is a subscheme $Z \subset \mathbb{C}^n$ with defining ideal $I_Z \subset \mathbb{C}[x_1, \ldots, x_n]$, and structure sheaf $\mathcal{O}_Z$ such that

- $Z$ is zero dimensional, that is a cluster,

- $Z$ is $G$-invariant and

- $\mathcal{O}_Z = \mathbb{C}[x_1, \ldots, x_n] / I_Z \cong \mathbb{C}[G]$ as $G$-modules.

The fine moduli space parametrising $G$-clusters is called $G$-Hilb $\mathbb{C}^n$.

Thus a point in $G$-Hilb $\mathbb{C}^n$ is an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ such that the quotient $\mathbb{C}[x_1, \ldots, x_n] / I \cong \mathbb{C}[G]$.

**Definition 1.2** Let $G$ be a finite diagonal Abelian subgroup of $\text{SL}(n, \mathbb{C})$. A $G$-graph is a set of monomials $\Gamma \subset \mathbb{C}[x_1, \ldots, x_n]$ satisfying the following two conditions.
1.1 Background to McKay correspondence

1. There exists one monomial in each representation space of $G$

2. If $x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \Gamma$ then $x_1^{i_1} \cdots x_n^{i_n} \in \Gamma$ for all $0 \leq i_j \leq \lambda_j$.

This is the same as Nakamura’s definition of a $G$-graph [Nak01].

Let $G$ be a finite diagonal Abelian subgroup of $\text{SL}(n, \mathbb{C})$ and $I$ an ideal defining a $G$-cluster. Now $\mathbb{C}[x_1, \ldots, x_n]/I \cong \mathbb{C}[G] \cong \bigoplus V_i$ where each $V_i$ is a 1 dimensional representation. Since $G$ is diagonal each monomial belongs to a well defined representation space. Therefore we can choose a set of monomials $\Gamma(I)$ with one monomial belonging to each $V_i$ which forms a basis for the complex vector space $\mathbb{C}[x_1, \ldots, x_n]/I$. If a monomial $x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \Gamma(I)$ then $x_1^{i_1} \cdots x_n^{i_n} \notin I$. Therefore $x_1^{i_1} \cdots x_n^{i_n} \notin I$ for all $0 \leq i_j \leq \lambda_j$ so we may assume that $x_1^{i_1} \cdots x_n^{i_n} \in \Gamma(I)$. Thus we may choose a basis $\Gamma(I)$ for $\mathbb{C}[x_1, \ldots, x_n]/I$ which is a $G$-graph. Clearly this basis will rarely be unique.

**Example 1.3** Let $G \subset \text{SL}(2, \mathbb{C})$ be generated by $\frac{1}{r}(1,-1)$ then every ideal defining a $G$-cluster has a $G$-graph of the form

\[
\begin{array}{c}
[-] \\
y^{r-i-1} \\
\vdots \\
y \\
1 \\
x & \cdots & x^i & [-]
\end{array}
\]

for some $i = 0, \ldots, r-1$, where $[-]$ means we do not take this monomial. Then polynomials of the form

\[
A := x^{i+1} - ax^{r-i-1}, \quad B := y^{r-i} - bx^i, \quad C := xy - c
\]

belong to any ideal $I$ which defines a $G$-cluster and has this $G$-graph, for
Preliminaries

some $a, b, c \in \mathbb{C}$. Notice that

$$yA - x^iC + aB = abx^i - cx^i$$

so $(ab - c)x^i \in I$. Since $x^i \notin I$ it follows that $ab = c$. Therefore $a$ and $b$
are the algebraic coordinates on this affine piece of $G$-Hilb $\mathbb{C}^2$. The relation
$yA - x^iC + aB = 0$ is called a syzygy. See for example [Rei97] for more
details.

Let $I'$ be the ideal generated by $A$, $B$ and $C$ such that $ab = c$. These poly-
nomials express every monomial in $\mathbb{C}[x, y]$ as a linear combination of mono-
nomials in $\Gamma$. They imply no relations between the monomials in $\Gamma$. Therefore
\( \Gamma \) forms a basis for $\mathbb{C}[x, y]/I'$ and so $I'$ defines a $G$-cluster.

Several explicit descriptions of $G$-Hilb $\mathbb{C}^3$ have been given when $G$ is
Abelian. Craw and Reid [CR99] calculated $G$-Hilb $\mathbb{C}^3$ explicitly by choosing
a triangulation of the junior simplex $\Delta_1 \subset \mathbb{R}^3$ in a lattice $L$. Each triangle
corresponds to an affine piece of $G$-Hilb $\mathbb{C}^3$ whose defining ideals can have
the same $G$-graph. They prove that the ideals have at most seven generators
and the affine piece requires exactly three algebraic parameters. Thus they
show that $G$-Hilb $\mathbb{C}^3$ is a smooth rational variety. Craw [Cra01] then goes on
to show how to construct a bijection between the irreducible representations
of $G$ and a basis for the cohomology of $G$-Hilb $\mathbb{C}^3$.

Let $G$ be a cyclic subgroup of $\text{SL}(3, \mathbb{C})$ generated by $\frac{1}{r}(a, b, c)$. Then
Craw and Reid define the lattice $L$ to be $L := \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(a, b, c)$ and the
junior simplex to be $\Delta_1 := \{(r_1, r_2, r_3) \in L \otimes \mathbb{R} \mid r_i \geq 0, \sum r_i = 1\}$. Any
triangulation of $\Delta_1$ into basic triangles will give a crepant resolution of $\mathbb{C}^3/G$.

In the next example we give an example of the triangulation corresponding
to $G$-Hilb $\mathbb{C}^3$.

Example 1.4 Let $G$ be generated by $\frac{1}{31}(1, 5, 25)$. The triangulation of $\Delta_1$
1.1 Background to McKay correspondence

corresponding to $G$-Hilb $\mathbb{C}^3$ is shown in Figure 1.1. Each triangle corresponds to an affine piece of $G$-Hilb $\mathbb{C}^3$. Given one such affine piece $U$ of $G$-Hilb $\mathbb{C}^3$ for any point $P \in U$ it is possible to choose the same $G$-graph as a basis for $\mathbb{C}[x, y, z]/I_P$ where $I_P$ is the defining ideal for $P$.

The triangle marked $A_1$ corresponds to ideals having the $G$-graph

$$\Gamma_{A_1} = 1 \ x \ \cdots \ x^{30} \ [\cdot].$$

and so the ideals are of the form

$$I_{\xi, \eta, \zeta} = \langle x^{31} - \xi, y - \eta x^5, z - \zeta x^{25} \rangle.$$

The coordinates $\xi, \eta, \zeta$ are the algebraic coordinates on this affine piece of $G$-Hilb $\mathbb{C}^3$. 

Figure 1.1: The triangulation of the junior simplex
Preliminaries

The triangle marked with $B$ corresponds to ideals having the $G$-graph

$$
\Gamma_B = \begin{bmatrix}
[\cdot] \\
y^6 & [\cdot] \\
y^5 & xy^5 & \cdots & x^4y^5 \\
\vdots & \vdots & \vdots \\
y & xy & \cdots & x^4y \\
1 & x & \cdots & x^4 & [\cdot]
\end{bmatrix}
$$

and so the ideals are of the form

$$I = \langle A := x^5 - \xi y, B := y^7 - \eta x^4, C := z - \zeta y^5, D := xy^6 - d \rangle.$$

Consider the relation or syzygy

$$xB - yD + \eta A = -\xi \eta y + dy.$$ 

This syzygy shows that $d = \xi \eta$ and so $\xi, \eta, \zeta$ are the algebraic coordinates on this affine piece of $G$-Hilb $\mathbb{C}^3$.

**Definition 1.5** Let $M$ be a module over a local ring $(A, m)$ then the **socle** is the annihilator in $M$ of the maximal ideal $m$. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ define a $G$-cluster then write $\text{Soc}(I)$ for the socle of $\mathbb{C}[x_1, \ldots, x_n]/I$ viewed as a module over $\mathbb{C}[x_1, \ldots, x_n]$ localised at the origin.

It follows immediately that $\text{Soc}(I)$ is zero if and only if $I$ supported at the origin.

Let $G$ be an Abelian subgroup of $SL(3, \mathbb{C})$ then [Nak01] and [CR99] prove that if an ideal $I$ defines a point in $G$-Hilb $\mathbb{C}^3$ then $I$ has a $G$-graph of the shape shown in Figure 1.2. This shows that the ideal has at most seven generators and $\text{Soc}(I) \leq 6$. 

6
1.2 Introduction to orbifolds

Figure 1.2: The shape of a $G$-graph when $G$ is an Abelian subgroup of $\text{SL}(3, \mathbb{C})$

**Definition 1.6** A *triheiral group* is a finite subgroup of $\text{SL}(3, \mathbb{C})$ of the form $G = A \times \mathbb{Z}_3$, where $A$ is a diagonal Abelian subgroup and $\mathbb{Z}_3$ is generated by

$$
\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Ito [Ito95] shows that a crepant resolution of $\mathbb{C}^3/G$ exists when $G$ is a triheiral group.

### 1.2 Introduction to orbifolds

**Definition 1.7** By *orbifold* I mean a normal projective variety having at worst quotient singularities. I say $X$ is a *cyclic orbifold* if for every singular point $P \in X$ there exists a neighbourhood $U$ of $P$ such that $U \cong V/\mathbb{Z}_r$ where $V$ is smooth and $\mathbb{Z}_r$ acts on $V$.

**Definition 1.8** Let $X$ be a normal quasiprojective variety and $\mathcal{L}$ a coherent
sheaf of $\mathcal{O}_X$ modules. Then $\mathcal{L}$ is said to be \textit{divisorial} if $\mathcal{L} = j_* \mathcal{L}^0$ where $X^0 = \text{NonSing } X \subset X$, $\mathcal{L}^0 = \mathcal{L}|_{X^0}$ is invertible and $j$ denotes the inclusion.

The following results are both taken from [Bou65] see also [Rei80, Appendix 1]. Let $X$ be a normal quasiprojective variety.

\textbf{Proposition 1.9 (VII.4 Thm. 2 [Bou65])} Let $\mathcal{L}$ be a coherent sheaf of $\mathcal{O}_X$-modules which is torsion free and of rank one. Then $\mathcal{L}$ is divisorial if and only if for every inclusion $\mathcal{L} \subset \mathcal{M}$, where $\mathcal{M}$ is a torsion free $\mathcal{O}_X$-module and $\text{Supp}(\mathcal{M}/\mathcal{L})$ has codimension $\geq 2$, $\mathcal{L} = \mathcal{M}$.

\textbf{Theorem 1.10 (VII.1 Thm. 2 [Bou65])} The map $D \mapsto \mathcal{O}_X(D)$ defines a bijection

$$\left\{ \begin{array}{c} \text{Weil divisors} \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{divisorial subsheaves} \\ \mathcal{L} \in k(X) \end{array} \right\} / \Gamma(X, \mathcal{O}_X)^*.$$

\textbf{Definition 1.11} A variety $X$ is said to be $\mathbb{Q}$-factorial if for every $\mathbb{Q}$-divisor $D = \sum a_i D_i$ for $a_i \in \mathbb{Q}$ and $D_i$ prime divisors there exists $0 \neq \lambda \in \mathbb{Z}$ such that $\lambda D$ is Cartier.

The remaining material of this section is taken from Kollár and Mori [KM98, Section 5.1].

\textbf{Definition 1.12} A variety $X$ is said to have \textit{rational singularities} if every resolution of singularities $f : Y \to X$ satisfies the following properties:

\begin{itemize}
  \item $f_* \mathcal{O}_Y = \mathcal{O}_X$ (equivalently $X$ is normal);
  \item $\mathcal{R}^i f_* (\mathcal{O}_Y) = 0$ for every $i > 0$.
\end{itemize}

\textbf{Proposition 1.13 (Prop 5.15 [KM98])} If $X$ is an orbifold then $X$ has rational singularities and $X$ is $\mathbb{Q}$-factorial.
1.3 An introduction to $G$-sheaves

**Definition 1.14** Let $Y$ be a smooth variety. A $\mathbb{Q}$-Cartier divisor $D = \sum a_i D_i$ is called a *simple normal crossing* divisor (often abbreviated to snc) if each $D_i$ is smooth and they intersect everywhere transversely. Let $X$ be a variety and $D = \sum a_i D_i$ a $\mathbb{Q}$-Cartier divisor on $X$. A *log resolution* of the pair $(X, D)$ is a proper birational morphism $f: Y \to X$ such that $Y$ is smooth, the exceptional locus $\text{Exc}(f)$ is a Cartier divisor and $\text{Exc}(f) \cup f^* \text{Supp}(D)$ is a snc divisor.

For any variety $X$ over a field of characteristic zero the fact that there exists a log resolution of $X$ follows from Hironaka’s proof of resolution of singularities. See [KM98].

1.3 An introduction to $G$-sheaves

**Definition 1.15** Let $\mathcal{F}$ be a coherent sheaf together with a homomorphism $\lambda^g : \mathcal{F} \to g^* \mathcal{F}$ for each $g \in G$. If $\lambda^g = \text{id}_\mathcal{F}$ and $\lambda^g = g^*(\lambda^F) \lambda^g$ then $\mathcal{F}$ is a $G$-sheaf. These compatibility conditions imply that $\lambda^F$ is an isomorphism.

If $P \in Z^g$ this isomorphism induces an isomorphism $\lambda_g$ on the stalk $\mathcal{F}_P \to \mathcal{F}_P$ and on the fibre $\mathcal{F}_P \to \mathcal{F}_P$. Since $G$ acts on $Z$ it follows that $G$ induces an action on the constant sheaf $k(\mathcal{F})$. Specifying that $\mathcal{F}$ is a subsheaf of $k(\mathcal{F})$ determines an action of $G$ on $\mathcal{F}$. Two isomorphic sheaves may have different actions of $G$ because they have different embeddings into $k(\mathcal{F})$.

**Example 1.16** Let $Z = \mathbb{P}^1$ and $G = \mathbb{Z}_r$ generated by $\frac{1}{r}(0,1)$. Then $1$ generates the global sections of $\mathcal{O}_Z$ and $g(1) = 1$. If $\mathcal{F} = \mathcal{O}(P_0 - P_1)$ then $x_0/x_1$ generates the global sections and $g(x_0/x_1) = \varepsilon^{-1}(x_0/x_1)$. Now $\mathcal{F} \cong \mathcal{O}_Z$ but they have different actions of $G$.

**Definition 1.17** Let $G$ be a finite group acting on a nonsingular variety $Z$. 
Preliminaries

The trivial $G$-sheaf is the sheaf $\mathcal{O}_Z$, where for every $g \in G$ the homomorphisms $\lambda^Z_g : \mathcal{O}_Z \to g^* \mathcal{O}_Z$ are induced from the action of $G$ on $k(Z)$.

A variety $X$ is a trivial $G$-variety if $gx = x$ for all $x \in X$ and $g \in G$. The following proposition comes from Atiyah’s paper on $K$-theory.

**Proposition 1.18 (Prop 1.6.2 [Ati67])** Let $X$ be a trivial $G$-variety and $V_1, \ldots, V_n$ a complete set of irreducible representations. Then every locally free $G$-sheaf $\mathcal{F}$ on $X$ is isomorphic to a direct sum $\bigoplus_i \mathcal{F}_i \otimes V_i$, where $G$ acts trivially on $\mathcal{F}_i$.

This is proved by letting $\mathcal{F}_i = \mathcal{H}om_{\mathcal{O}_X}(V_i, \mathcal{F})^G$. If $X$ is a trivial $G$-variety then by the above proposition $K_G(X) = K(X) \otimes R(G)$, where $R(G)$ is the representation ring of $G$. Therefore $a \in K_G(X)$ can be written as $a = \sum u_i \otimes V_i$ where $u_i \in K(X)$. The Chern character $\text{ch}(a)(g)$ of $a$ is defined by the formula

$$\text{ch}(a)(g) = \sum_i \text{ch}(u_i) \chi_i(g)$$

where $\text{ch}(u_i)$ is the ordinary Chern character of the sheaf $u_i$ and $\chi_i$ is the character of the representation $V_i$. 
Chapter 2

Explicit construction of
$G$-Hilb $\mathbb{C}^2$ for binary dihedral groups.

2.1 Representation theory

Throughout this chapter $G = \text{BD}_{4n} \subset \text{SL}(2, \mathbb{C})$ is the binary dihedral group of order $4n$ generated by

\[
g = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

for $\varepsilon$ a primitive $2n$th root of unity. The group has the presentation

\[
\langle g, \tau \mid g^{2n} = \tau^4 = 1, \ g^n = \tau^2, \ \tau g \tau^{-1} = g^{-1} \rangle.
\]

It is well known that the ring of invariants is generated by

\[
u = xy(x^{2n} - y^{2n}), \quad v = x^{2n} + y^{2n} \quad \text{and} \quad w = (xy)^2
\]
Explicit construction of $G$-Hilb $\mathbb{C}^2$ for binary dihedral groups.

where $x, y$ are the coordinates on $\mathbb{C}^2$. See [Rei02] for example. The quotient $X = \mathbb{C}^2 / G$ is the hypersurface singularity

$$X : (u^2 - v^2 w + 4w^{n+1} = 0) \subset \mathbb{C}^3.$$ 

This has a unique minimal resolution $Y \to X$ with exceptional locus given by a chain of $\mathbb{P}^1$'s shown by the Dynkin diagram $D_{n+2}$.

Let $\text{Irr} \ G$ be the set of distinct irreducible representations of $G$. These are given as follows. If $\rho$ is a 1 dimensional representation of the group then $\rho(g)^2 = 1$ and $\rho(\tau)^2 = \rho(g)^n$. Thus there are four 1 dimensional representations given by

$$L_\pm(g) = 1, \quad L_\pm(\tau) = \pm 1 \quad \text{and} \quad M_\pm(g) = -1, \quad M_\pm(\tau) = \pm i^n. \quad (2.1)$$

Clearly $L_+$ is the trivial representation.

For $j = 1, \ldots, n - 1$ define a 2 dimensional representation $V_j \cong \mathbb{C}^2$ as follows.

$$V_j(g) = \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix} \quad \text{and} \quad V_j(\tau) = \begin{pmatrix} 0 & 1 \\ (-1)^j & 0 \end{pmatrix} \quad (2.2)$$

Each of these representations is irreducible and $V_i \not\cong V_j$ for $i \neq j$. Since $4n = |G| = \sum_{\rho \in \text{Irr} \ G} (\deg \rho)^2 = 4 \cdot 1 + (n - 1) \cdot 4$ these are all the irreducible representations and the regular representation of $BD_{4n}$ is

$$\mathbb{C}[G] \cong L_+ \oplus L_- \oplus M_+ \oplus M_- \oplus \bigoplus_{j=1}^{n-1} V_j^{\oplus 2}.$$

2.2 $G$-graphs

**Definition 2.1** A polynomial or pair of polynomials is said to belong to one of the irreducible representations of $G$ if one of the following conditions hold.

- A polynomial $f \in \mathbb{C}[x, y]$ belongs to $L_\pm$ if $g(f) = f$ and $\tau(f) = \pm f$. 

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2.2 G-graphs

- A polynomial \( f \in \mathbb{C}[x, y] \) belongs to \( M_{\pm} \) if \( g(f) = -f \) and \( \tau(f) = \pm \bar{\tau}f \).

- For a polynomial \( f \in \mathbb{C}[x, y] \), the pair of polynomials \((f, \tau(f))\) belong to \( V_j \) if \( g(f, \tau(f)) = (\varepsilon^j f, \varepsilon^{-j} \tau(f)) \).

In the next section we will be interested in the following polynomials that belong to these representations.

<table>
<thead>
<tr>
<th></th>
<th>( L_+ )</th>
<th>( L_- )</th>
<th>( M_+ )</th>
<th>( M_- )</th>
<th>( V_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 1, x^2 y^2, x^{2n} + y^{2n} )</td>
<td>( xy, x^{2n} - y^{2n} )</td>
<td>( x^n - i^n y^n, xy(x^n + i^n y^n) )</td>
<td>( x^n + i^n y^n, xy(x^n - i^n y^n) )</td>
<td>( (x^j, y^j), (x^{j+1} y, -x y^{j+1}) ), ( (y^{2n-j}, (-1)^j x^{2n-j}) )</td>
</tr>
</tbody>
</table>

We may sometimes abuse this notation and say \((f, g) \in V_j\) when we mean \((f, -g) \in V_j\). For example we say \((x^{j+1} y, x y^{j+1}) \in V_j\) where really \((x^{j+1} y, -x y^{j+1}) \in V_j\) or \((y^{2n-j}, x^{2n-j}) \in V_j\) where really \((y^{2n-j}, (-1)^j x^{2n-j}) \in V_j\). Notice that \((y(x^n \pm i^n y^n), x(x^n \pm i^n y^n)) \in V_{n-1}\).

**Definition 2.2** A subset \( \Gamma \subset \mathbb{C}[x, y] \) is called a G-graph if it contains precisely one polynomial belonging to each 1 dimensional representation \( L_{\pm} \) and \( M_{\pm} \) and two pairs of polynomials belonging to each of the irreducible 2 dimensional representations \( V_i \).

An ideal \( I \) in \( \mathbb{C}[x, y] \) defines a G-cluster if and only if \( \mathbb{C}[x, y] / I \cong \mathbb{C}[G] \). Therefore we can choose a basis for the complex vector space \( \mathbb{C}[x, y] / I \) which is a G-graph. We note that there will never be a unique choice of G-graph.
Explicit construction of $G$-Hilb $\mathbb{C}^2$ for binary dihedral groups.

**Definition 2.3** Let $I$ be an ideal that defines a $G$-cluster. $\Gamma(I)$ is defined to be the set of $G$-graphs which form a basis for the complex vector space $\mathbb{C}[x, y]/I$.

Recall that $\text{Soc}(I)$ is defined as the socle of $\mathbb{C}[x, y]/I$ viewed as a module over $\mathbb{C}[x, y]$ localised at the origin.

**Theorem 2.4** Let $G = \text{BD}_{4n}$.

1. If an ideal $I$ defines a $G$-cluster then one of the $G$-graphs given in the second list in Theorem 2.10 belongs to $\Gamma(I)$.

2. $\text{Soc}(I)$ contains at most two irreducible representations of $G$.

3. $G$-Hilb $\mathbb{C}^2$ is a smooth rational variety.

To show that $G$-Hilb $\mathbb{C}^2$ is a smooth rational variety I find a cover of $G$-Hilb $\mathbb{C}^2$ by smooth affines. In general these affine sets are smooth surfaces in some $\mathbb{C}^n$. However we find that $G$-Hilb $\mathbb{C}^2$ contains an open set isomorphic to an open subset of $\mathbb{C}^2$ therefore $G$-Hilb $\mathbb{C}^2$ is rational.

### 2.3 Preliminaries on $G$-clusters and $G$-graphs

Let $I$ be an ideal that defines a $G$-cluster. Any 1 dimensional representation has multiplicity 1 in $\mathbb{C}[G]$. Thus for any polynomials $g$ and $h$ that belong to the same 1 dimensional representation there exists a polynomial

$$\alpha g - \beta h$$

in $I$ for $\alpha, \beta \in \mathbb{C}$. If $g \in \Gamma$ then $\beta \neq 0$.

Let $h$ be a polynomial belonging to a 1 dimensional representation. Provided that $h$ is not zero in $\mathbb{C}[x, y]/I$ it can be chosen to be basic. In other
2.3 Preliminaries on $G$-clusters and $G$-graphs

words the only reason we cannot choose $h$ in $\Gamma$ is if $h \in I$. In particular if $h = 1$ then 1 can always be chosen to be in $\Gamma$. This means that an ideal that
defines a $G$-cluster always contains polynomials of the form

$$A := x^{2n} + y^{2n} - a \quad Q := x^2 y^2 - q$$

for some $a, q \in \mathbb{C}$. For any $r \in \mathbb{Z}$ let

$$q_r := (x^2 y^2)^{r-1} + q (x^2 y^2)^{r-2} + \cdots + q^{r-2} (x^2 y^2) + q^{r-1}. \quad (2.3)$$

Then

$$q_r Q = (x^2 y^2)^r - q^r.$$

**Lemma 2.5** The submodule of $\mathbb{C}[x, y]$ consisting of polynomials belonging to $L_\omega$ is generated by $xy$ and $x^{2n} - y^{2n}$ as a $\mathbb{C}[x, y]^G$-module.

**Proof.** First note that this submodule is contained in $\mathbb{C}[x, y]^A$ where $A \cong \mathbb{Z}_{2n}$ is the normal subgroup generated by $g$. The ring $\mathbb{C}[x, y]_A^A$ is generated by $x^{2n}, xy, y^{2n}$. If $f(x, y) \in L_\omega$ and $xy \mid f(x, y)$ then $f(x, y) = xy f'(x, y)$ for some $f'(x, y) \in \mathbb{C}[x, y]^G$.

Therefore we may assume that $f(x, y)$ has no monomial term which is divisible by $xy$ and so $f(x, y) = P_1(x^{2n}) + P_2(y^{2n})$ where the $P_i$ are polynomials in one variable. Therefore $\tau(f) = P_1(y^{2n}) + P_2(x^{2n})$. Since $f \in L_\omega$ it must be that $\tau(f) = -f$ so $P_1 = -P_2$ and $f = P_1(x^{2n}) - P_1(y^{2n})$. This implies that $(x^{2n} - y^{2n}) \mid f$ and so $f = (x^{2n} - y^{2n}) f'(x, y)$ for some $f'(x, y) \in \mathbb{C}[x, y]^G$. \qed

**Corollary 2.6** If neither $xy$ nor $x^{2n} - y^{2n}$ can be chosen to be basic then $\mathbb{C}[x, y]/I$ contains no copy of $L_\omega$.

**Proof.** If neither $xy$ nor $x^{2n} - y^{2n}$ can be chosen to be basic then

$$xy, x^{2n} - y^{2n} \in I.$$

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Explicit construction of $G$-Hilb $\mathbb{C}^G$ for binary dihedral groups.

Let $f$ be any polynomial belonging to $L_-$. By Lemma 2.5

$$f = xyf' + (x^{2n} - y^{2n}) f''$$

for $f', f'' \in \mathbb{C}[x, y]^G$. Therefore $f \in I$.  

\textbf{Lemma 2.7} Let $I$ be an ideal containing the polynomials $A$ and $Q$. If neither $x^n + i^n y^n$ nor $xy(x^n - i^n y^n)$ is basic then $\mathbb{C}[x, y] / I$ contains no copy of $M_-$. If neither $x^n - i^n y^n$ nor $xy(x^n + i^n y^n)$ is basic then $\mathbb{C}[x, y] / I$ contains no copy of $M_+$.  

\textbf{Proof}. We prove the first assertion but the second is analogous. If $x^n + i^n y^n$ cannot be basic then $x^n + i^n y^n \in I$ and so

$$x^2(x^n + i^n y^n) - i^n y^{n-2}Q = x^{n+2} - i^n q y^{n-2} \in I.$$ 

Therefore only polynomials that have monomial terms of the form $x^\alpha y^\beta$ can be basic where $\alpha < n + 2$ and $\beta < 2$ or $\beta < n + 2$ and $\alpha < 2$. The only polynomials of this form that belong to $M_-$ are $x^n + i^n y^n$ and $xy(x^n - i^n y^n)$. These polynomials are both contained in $I$ so $\mathbb{C}[x, y] / I$ contains no copy of $M_-$.  

Suppose $(f_1, f_2), (g_1, g_2)$ and $(h_1, h_2)$ are pairs of polynomials that belong to some irreducible 2 dimensional representation $V_i$. Since $V_i$ has multiplicity 2 in $\mathbb{C}[G]$ these three pairs cannot be linearly independent in $\mathbb{C}[x, y] / I \cong \mathbb{C}[G]$. Therefore there exist polynomials of the form

$$\alpha f_1 + \beta g_1 + \gamma h_1, \quad \alpha f_2 + \beta g_2 + \gamma h_2$$

in $I$ for $\alpha, \beta, \gamma \in \mathbb{C}$. If $(f_1, f_2)$ and $(g_1, g_2)$ are chosen in a $G$-graph for $I$ then $\gamma$ is nonzero. 

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2.3 Preliminaries on $G$-clusters and $G$-graphs

Lemma 2.8 Let $I$ be an ideal that defines a $G$-cluster. Then there exists a $G$-graph $\Gamma \in \Gamma(I)$ containing the set \( \{ 1, (x, y), \ldots, (x^{n-1}, y^{n-1}) \} \).

Proof. The $G$-graph $\Gamma$ must contain $4n$ elements. If one of the pairs $(x^i, y^i)$ for $0 < i < n$ cannot be chosen to be in $\Gamma$ then both $x^i$ and $y^i$ belong to $I$. Therefore $x^j, y^j \in I$ for all $j \geq i$ and so $\Gamma$ is contained in the set
\[
\{ 1, x, \ldots, x^{i-1}, y, \ldots, y^{i-1}, xy, \ldots, x^{i-1}y, xy^2, \ldots, xy^{i-1} \}.
\]
This has only $4(i - 1) < 4n$ elements. Therefore the pairs $(x^i, y^i)$ can always be chosen to be in $\Gamma$ for $i = 1, \ldots, n - 1$. \qed

We therefore consider only $G$-graphs containing these monomials.

Suppose \( \{ 1, (x, y), \ldots, (x^{n-1}, y^{n-1}) \} \subset \Gamma \) and $(h_1, h_2) \in V_k$. The only reason that this pair cannot be chosen to be in $\Gamma$ is if there exist polynomials of the form
\[
h_1 = \alpha x^i, \quad h_2 = \alpha y^i
\]
in $I$.

Lemma 2.9 Let $I$ be an ideal containing the polynomials $A$ and $Q$. Suppose neither of the pairs $(x^{k+1}, xy^{k+1})$ nor $(y^{2n-k}, x^{2n-k})$ can be chosen as part of the basis of $\mathbb{C}[x, y]/I$. Then $\mathbb{C}[x, y]/I$ contains at most one copy of $V_k$.

Proof. If neither of the pairs $(x^{k+1}, xy^{k+1})$ nor $(y^{2n-k}, x^{2n-k})$ can be chosen as part of the basis of $\mathbb{C}[x, y]/I$ then the four polynomials
\[
C := x^{k+1}y - cx^k, \quad D := xy^{k+1} + cy^k, \\
E := y^{2n-k} - dx^k, \quad F := x^{2n-k} - (-1)^k dy^k
\]
belong to $I$.

Suppose that there exists a pair $(x^\alpha y^\beta, x^\beta y^\alpha)$ in $\mathbb{C}[x, y]/I$ which belongs to $V_k$. We want to show that this pair cannot be linearly independent of
Explicit construction of $G$-Hilb $\mathbb{C}^q$ for binary dihedral groups.

$(x^k, y^k)$ in $\mathbb{C}[x, y]/I$. In other words that there exist polynomials of the form

$$x^\alpha y^\beta - \gamma x^k, \quad x^\beta y^\alpha - (-1)^\beta y^k$$

in $I$.

Suppose $\alpha \geq 2$ and $\beta \geq 2$. The polynomial $Q \in I$ and so $x^{\alpha-2}y^{\beta-2}Q \in I$ and

$$x^{\alpha-2}y^{\beta-2}Q = x^\alpha y^\beta - qx^{\alpha-2}y^{\beta-2} \in I.$$ 

So $x^\alpha y^\beta$ is equal to a complex multiple of $x^{\alpha-2}y^{\beta-2}$ in $\mathbb{C}[x, y]/I$ so we may replace $x^\alpha y^\beta$ with $x^{\alpha-2}y^{\beta-2}$ in the basis. Continuing this way we may assume that either $\alpha < 2$ or $\beta < 2$. Assume that $\beta < 2$.

Again if $\alpha > 2n+1$ then the polynomial $A \in I$ can be used to replace $x^\alpha y^\beta$ with a polynomial having monomial terms of the form $x^\gamma y^\delta$ where $\gamma \leq 2n+1$ and $\delta < 2$ or $\gamma < 2$ and $\delta \leq 2n+1$.

More precisely

$$x^{\alpha-2n}y^\beta A = x^\alpha y^\beta + x^{\alpha-2n}y^{\beta+2n} - ax^{\alpha-2n}y^\beta \in I.$$ 

If $\alpha - 2n \geq 2$ then repeated use of the polynomial $Q$ replaces $x^{\alpha-2n}y^{\beta+2n}$ with a monomial of the form $x^\gamma y^\delta$ where $\gamma < 2$ or $\delta < 2$. Therefore we may assume that $\alpha \leq 2n+1$ and $\beta < 2$ or $\alpha < 2$ and $\beta \leq 2n+1$.

However the only pairs $(x^\alpha y^\beta, y^\alpha x^\beta)$ in $V_k$ where $\alpha \leq 2n+1$ and $\beta < 2$ or $\alpha < 2$ and $\beta \leq 2n+1$ are

$$(x^k, y^k), \quad (x^{k+1}, y^{k+1}) , \quad (y^{2n-k}, x^{2n-k}) , \quad (xy^{2n-k+1}, x^{2n-k}y), \quad (x^{2n+1}, y^{2n+1}) \in V_1.$$ 

Now we are assuming that $(x^{k+1}, y^{k+1})$ and $(y^{2n-k}, x^{2n-k})$ are not basic so the only other possibilities are $(xy^{2n-k+1}, x^{2n-k+1}y)$ and $(x^{2n+1}, y^{2n+1})$. 

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2.4 List of possible $G$-graphs

However

\[
\begin{align*}
xyE + dC &= xy^{2n-k+1} - cdx^k, \\
xyF + (-1)^idD &= x^{2n-k+1}y + (-1)^i cdy^k
\end{align*}
\]

so \((xy^{2n-k+1}, x^{2n-k+1}y)\) cannot be basic. If \(k=1\) then

\[
\begin{align*}
x^2F - dC &= x^{2n+1} + cdx, \\
y^2E + dD &= y^{2n+1} + cdy
\end{align*}
\]

so \((x^{2n+1}, y^{2n+1})\) cannot be basic either. Therefore \(\mathbb{C}[x, y]/I\) contains at most one copy of \(V_k\). \(\square\)

2.4 List of possible $G$-graphs

In this section we construct a list of $G$-graphs and show that every ideal that defines a $G$-cluster has a $G$-graph belonging to this list. We do this systematically by considering each representation and choosing a polynomial or pair of polynomials belonging to this representation.

Let \(I\) be an ideal that defines a $G$-cluster and \(\Gamma\) a $G$-graph in \(\Gamma(I)\). Begin by considering \(L_+\). By Lemma 2.8 we may always assume that \(1 \in L_+\) belongs to \(\Gamma\). Now \(\mathbb{C}[x, y]/I\) contains only one copy of the trivial representation so \(I\) contains the polynomials

\[
A := x^{2n} + y^{2n} - a, \quad Q := x^2 y^2 - q.
\]

By Lemma 2.8 we may assume that \((x, y), \ldots, (x^{n-1}, y^{n-1}) \in \Gamma\).

2.4.1 Case A

Next we consider \(L_-\). If \(xy \in I\) then \(xy \notin \Gamma\). By Corollary 2.6 \(x^{2n} - y^{2n} \notin I\) therefore take \(x^{2n} - y^{2n} \in \Gamma\). Since \(xy \in I\) all of the pairs \((x^{k+1}y, xy^{k+1})\)
Explicit construction of G-Hilb \( \mathbb{C}^2 \) for binary dihedral groups.

belong to \( I \) and \( xy(x^{n} \pm iy^{n}) \) belongs to \( I \) these cannot be in \( \Gamma \). By Lemma 2.9 there exist no polynomials in \( I \) of the form

\[
y^{2n-k} - cx^k, \quad x^{2n-k} - (-1)^k cy^k
\]

for \( 1 \leq k \leq n-1 \) and so \( (y^{2n-k}, x^{2n-k}) \in \Gamma \). By Lemma 2.9 \( x^n + i^n y^n \notin I \) and \( x^n - i^n y^n \notin I \) and so we may take these in \( \Gamma \).

Therefore \( \Gamma_A \in \Gamma(I) \) where

\[
\begin{bmatrix}
[y] \\
y^{2n} \\
\vdots \\
y^2 \\
y \\
\end{bmatrix}
\]

\[
\Gamma_A = 
\begin{bmatrix}
y & [\cdot] \\
1 & x & x^2 & \cdots & x^{2n} & [\cdot].
\end{bmatrix}
\]

In this \( G \)-graph we take \( x^{2n} - y^{2n} \) but not \( x^{2n} + y^{2n} \) and take \( x^n + i^n y^n \) and \( x^n - i^n y^n \) rather than \( x^n \) and \( y^n \). Clearly this satisfies the definition of a \( G \)-graph.

2.4.2 Case B

From now on we assume that \( xy \notin I \) and so we can take \( xy \in \Gamma \). Since \( (x,y) \in \Gamma \) it remains to consider the other pair in \( V_1 \). If

\[
x^2y - cx, \quad xy^2 + cy
\]

belong to \( I \) then the polynomials

\[
x^{k+1}y - cx^{k}, \quad xy^{k+1} + cy^k
\]

belong to \( I \) so \( (x^{k+1}y, xy^{k+1}) \notin \Gamma \) for all \( k \geq 1 \). By Lemma 2.9 the pair \( (y^{2n-k}, x^{2n-k}) \in \Gamma \) for \( 1 \leq k \leq n-1 \).
2.4 List of possible $G$-graphs

Also the polynomials

$$xy(x^n \pm iy^n) - c(x^n \mp iy^n) \in I.$$  

If $x^n + iy^n$ or $x^n - iy^n$ is in $I$ then $xy(x^n - iy^n)$ or $xy(x^n + iy^n)$ belong to $I$. By Lemma 2.7 this cannot happen so we may assume $x^n + iy^n$ and $x^n - iy^n$ can be chosen in $\Gamma$.

Therefore $\Gamma_B \in \Gamma(I)$ where

$$\Gamma_B := \begin{bmatrix} [\cdot] \\ y^{2n-1} \\ \vdots \\ y^2 [\cdot] \\ y \ xy [\cdot] \\ 1 \ x \ x^2 \ \cdots \ x^{2n-1} [\cdot]. \end{bmatrix}.$$  

Again $x^n + iy^n$ and $x^n - iy^n$ belong to $\Gamma_B$ rather than $x^n$ and $y^n$.

2.4.3 Case $C_k$

Fix $k \in [2, n - 1]$ and assume that $k$ is the smallest integer so that we have the polynomials of the form

$$x^{k+1}y - cx^k, \ xy^{k+1} + cy^k \in I.$$  

Then we can take $(x^{j+1}y, xy^{j+1}) \in \Gamma$ for all $j < k$. Since $(x^k, y^k) \in \Gamma$ it remains to consider the other pair in $V_k$. The polynomials

$$x^{i+1}y - cx^i, \ xy^{i+1} + cy^i \in I$$

for $i \geq k$. Therefore $(x^{i+1}y, xy^{i+1}) \notin \Gamma$ for $i \geq k$ and so we can take $(y^{2n-i}, x^{2n-i}) \in \Gamma$ for $k \leq i < n$. As in Case B $x^n + iy^n$ and $x^n - iy^n$ do not belong to $I$ so can be taken in $\Gamma$.  

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Thus $\Gamma_{C_h} \in \Gamma(I)$ where

$$[\cdot] \quad y^{2n-k} \quad \vdots \quad y^{k+1} \quad [\cdot] \quad \Gamma_{C_h} := \begin{array}{cccc}
y^k & xy^k \\
\vdots & \vdots \\
y^2 & xy^2 & [\cdot] \\
y & xy & x^2y & \cdots & x^ky & [\cdot] \\
1 & x & x^2 & \cdots & x^k & x^{k+1} & \cdots & x^{2n-k} & [\cdot].
\end{array}$$

Again $x^n + iny^n$ and $x^n - iny^n$ belong to $\Gamma_{C_h}$ rather than $x^n$ and $y^n$.

2.4.4 Case D

In each of the previous cases we were assuming that there exists some $j < n$ such that the polynomials

$$x^{j+1}y - cx^j, \quad x^{j+1}y + cy^j$$

belong to $I$. This implies that

$$xy(x^n - iny^n) - c(x^n + iny^n), \quad xy(x^n - iny^n) - c(x^n + iny^n)$$

belong to $I$. If $x^n \pm iny^n \in I$ then $xy(x^n \pm iny^n) \in I$ and by Lemma 2.7 $\mathbb{C}[x,y]/I$ contains no copy of $M_\pm$. For this reason in each of the previous cases we took $x^n \pm iny^n \in \Gamma$.

From now on we assume that $xy, (x^j, y^j)$ and $(x^{j+1}y, xy^{j+1})$ belong to $\Gamma$ for all $1 \leq j \leq n-1$. Consider the representation $M_+$. If $x^n - iny^n \in I$ then by Lemma 2.7 $xy(x^n + iny^n) \notin I$ and so $x^n + iny^n \notin I$. Therefore these cases can be
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taken in $\Gamma$. Since $x^n - i^n y^n \in I$ it follows that $y(x^n - i^n y^n), x(x^n - i^n y^n) \in I$. Therefore we can choose $(y(x^n + i^n y^n), x(x^n + i^n y^n))$ to be basic instead of $(x^n y, xy^n)$.

In this case $\Gamma_{D+} \in \Gamma(I)$ where

\[
\begin{array}{cc}
[\cdot] \\
y^{n+1} & xy^{n+1} \\
y^n & xy^n \\
\vdots & \vdots \\
y^2 & xy^2 & [\cdot] \\
y & xy & x^2 y & \cdots & x^{n-1} y & x^n y & x^{n+1} y \\
1 & x & x^2 & \cdots & x^n & x^{n+1} & [\cdot].
\end{array}
\]

This time $x^n + i^n y^n$, $x(x^n + i^n y^n)$, $y(x^n + i^n y^n)$ and $xy(x^n + i^n y^n)$ are in $\Gamma_{D+}$ but $x^n - i^n y^n$, $x(x^n - i^n y^n)$, $y(x^n - i^n y^n)$ and $xy(x^n - i^n y^n)$ are not.

Similarly consider the representation $M_-$. If $x^n + i^n y^n \in I$ then by Lemma 2.7 $xy(x^n - i^n y^n) \notin I$ and so $x^n - i^n y^n \notin I$. Therefore these can be taken in $\Gamma$. Again the pair $(x(x^n - i^n y^n), y(x^n - i^n y^n))$ can be chosen to be basic rather than $(x^n y, xy^n)$.

In this case $\Gamma_{D-} \in \Gamma(I)$ where $\Gamma_{D-}$ has the same picture as $\Gamma_{D+}$. This time the polynomials $x^n - i^n y^n$, $x(x^n - i^n y^n)$, $y(x^n - i^n y^n)$ and $xy(x^n - i^n y^n)$ are in $\Gamma_{D-}$ but $x^n + i^n y^n$, $x(x^n + i^n y^n)$, $y(x^n + i^n y^n)$ and $xy(x^n + i^n y^n)$ are not.
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2.4.5 Case E

Finally we may assume \( xy, x^n + iy^n, x^n - iy^n, (x^j, y^j) \) and \((x^{j+1}, xy^{j+1})\) belong to \( \Gamma \) for \( 1 \leq j \leq n - 1 \). Then \( \Gamma_{E_0} \in \Gamma(I) \) where

\[
\begin{bmatrix}
[ \cdot ] \\
y^n & xy^n \\
\vdots & \vdots \\
y^2 & xy^2 & [ \cdot ] \\
y & xy & x^2y & \cdots & x^ny \\
1 & x & x^2 & \cdots & x^n & [ \cdot ]
\end{bmatrix}
\]

As in Case D, we may assume either the pair \((y(x^n + i^n y^n), x(x^n + i^n y^n))\) or \((y(x^n - i^n y^n), y(x^n - i^n y^n))\) is basic. If \((x(x^n + i^n y^n), y(x^n + i^n y^n))\) is basic then \( \Gamma_{E_+} \in \Gamma(I) \) where

\[
\begin{bmatrix}
[ \cdot ] \\
y^{n+1} & [ \cdot ] \\
y^n & xy^n \\
\vdots & \vdots \\
y^2 & xy^2 & [ \cdot ] \\
y & xy & x^2y & \cdots & x^ny & [ \cdot ] \\
1 & x & x^2 & \cdots & x^n & x^{n+1} & [ \cdot ]
\end{bmatrix}
\]

Here \( x^n + iy^n, x^n - iy^n, y(x^n + iy^n) \) and \( x(x^n + iy^n) \) are in \( \Gamma_{E_+} \) but \( y(x^n - iy^n) \) and \( x(x^n - iy^n) \) are not.

If \((x(x^n - i^n y^n), y(x^n - i^n y^n))\) is basic then \( \Gamma_{E_-} \in \Gamma(I) \) where \( \Gamma_{E_-} \) has the same picture as \( \Gamma_{E_+} \) but \( x^n + iy^n, x^n - iy^n, y(x^n - iy^n) \) and \( x(x^n - iy^n) \) are in \( \Gamma_{E_-} \) but \( y(x^n + iy^n) \) and \( x(x^n + iy^n) \) are not.

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2.4.6 Conclusion

**Theorem 2.10** Let $I$ be an ideal which defines a $G$-cluster then there exist $G$-graphs $\Gamma_1$ and $\Gamma_2$ such that

\[
\Gamma_1 \in \{\Gamma_A, \Gamma_B, \Gamma_C, \ldots, \Gamma_{C_{n-1}}, \Gamma_{D_+}, \Gamma_{D_-}, \Gamma_{E_0}\}
\]

\[
\Gamma_2 \in \{\Gamma_A, \Gamma_B, \Gamma_C, \ldots, \Gamma_{C_{n-2}}, \Gamma_{D_+}, \Gamma_{D_-}, \Gamma_{E_0}, \Gamma_{E_-}\}
\]

and $\Gamma_1, \Gamma_2 \in \Gamma(I)$.

**Proof.** We have shown that any ideal $I$ which defines a $G$-cluster has $\Gamma \in \Gamma(I)$ where $\Gamma \in \{\Gamma_A, \Gamma_B, \Gamma_C, \ldots, \Gamma_{C_{n-1}}, \Gamma_{D_+}, \Gamma_{D_-}, \Gamma_{E_0}\}$. To prove the theorem it remains to show that if $\Gamma_{C_{n-1}} \in \Gamma(I)$ or $\Gamma_{E_0} \in \Gamma(I)$ then $\Gamma_{E_+} \in \Gamma(I)$ or $\Gamma_{E_-} \in \Gamma(I)$.

We prove that if $\Gamma_{C_{n-1}} \in \Gamma(I)$ then either $\Gamma_{E_+}$ or $\Gamma_{E_-}$ is in $\Gamma(I)$. The proof of the other case is identical. Let $I$ be an ideal that defines a $G$-cluster which has $\Gamma_{C_{n-1}} \in \Gamma(I)$. If neither $(y(x^n + i^n y^n), x(x^n + i^n y^n))$ nor $(y(x^n - i^n y^n), x(x^n - i^n y^n))$ can be basic in $\mathbb{C}[x, y]/I$ then there exist polynomials of the form

\[
C := y(x^n + i^n y^n) - cx^{n-1}, \quad D := x(x^n + i^n y^n) + i^n cy^{n-1}
\]

\[
E := y(x^n - i^n y^n) - dx^{n-1}, \quad F := x(x^n - i^n y^n) - i^n dy^{n-1}
\]

in $I$. However

\[
C - E = 2i^n y^{n+1} - (c - d)x^{n-1}
\]

\[
D + F = 2x^{n+1} + i^n(c - d)y^{n-1}
\]

Therefore $(y^{n+1}, x^{n+1})$ cannot form part of a basis for $\mathbb{C}[x, y]/I$ which contradicts the assumption that $\Gamma_{C_{n-1}} \in I(\Gamma)$. \qed
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2.5 Construction of open sets contained in $G$-Hilb $\mathbb{C}^2$

In this section we take a $G$-graph given in the previous section and list generators for any ideal $I$ that defines a $G$-cluster and has this $G$-graph. This will give an open subset in $G$-Hilb $\mathbb{C}^2$ of points whose defining ideal $I$ has $\Gamma \in \Gamma(I)$.

2.5.1 Case A

Let $I$ be any ideal that defines a $G$-cluster such that $\Gamma_A \in \Gamma(I)$. Then $I$ contains the polynomials

\[
\begin{align*}
A &:= x^{2n} + y^{2n} - a, & B &:= xy - b(x^{2n} - y^{2n}), \\
C &:= x^{2n+1} - cx - dy^{2n-1}, & D &:= y^{2n+1} - cy + dx^{2n-1}, \\
E &:= xy^{2n} - cx - fy^{2n-1}, & F &:= x^{2n}y - ey + fx^{2n-1}, \\
Q &:= x^2y^2 - q
\end{align*}
\]

for some $a, \ldots, f, q \in \mathbb{C}$.

**Lemma 2.11** Using only these polynomials every monomial in $\mathbb{C}[x, y]$ can be written as a linear combination of elements in $\Gamma_A$. In other words $\Gamma_A$ spans $\mathbb{C}[x, y] / \langle A, B, C, D, E, F, Q \rangle$.

**Proof.** Consider a monomial $x^\alpha y^\beta$. We want to show that this can be written as a linear combination of elements in $\Gamma_A$.

If $\alpha \geq 2$ and $\beta \geq 2$ then as in Lemma 2.9 we can use $Q$ to write

\[x^\alpha y^\beta = qx^{\alpha-2}y^{\beta-2}.
\]

Therefore it remains to prove the lemma for monomials of the form $x^\alpha$, $x^\alpha y$, $y^\alpha$ and $xy^\beta$. 

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We show that $x^\alpha$ can be written as a linear combination of elements in $\Gamma_A$. If $\alpha \leq 2n$ then either $x^\alpha \in \Gamma_A$ or $\alpha = n, 2n$. However $x^n = ((x^n + i^n y^n) + (x^n - i^n y^n))/2$ and $(x^n - i^n y^n) \in \Gamma_A$. Now $x^{2n} = ((x^{2n} + y^{2n}) + (x^{2n} - y^{2n}))/2$ and $(x^{2n} - y^{2n}) \in \Gamma_A$ but $(x^{2n} + y^{2n}) \notin \Gamma_A$.

The polynomial $A$ allows us to replace $(x^{2n} + y^{2n})$ with $a \cdot 1$. So $x^{2n} = (a \cdot 1 + (x^{2n} - y^{2n}))/2$. Therefore the lemma holds for $x^\alpha$ when $\alpha \leq 2n$.

To prove the lemma holds for $x^\alpha$ when $\alpha > 2n$ we use induction on $\alpha$. It holds for $\alpha = 2n + 1$ by the polynomial $C$.

Write $\alpha = \lambda 2n + \mu$ where $\lambda \geq 1$ and $0 \leq \mu \leq 2n - 1$. Assume that

$$x^\alpha = sx^\mu + ty^{2n-\mu}.$$  

If $\mu \neq 0, n, 2n$ then $x^\mu, y^{2n-\mu} \in \Gamma_A$. If $\mu = 0, n$ or $2n$ then as shown above $x^n, y^n, x^{2n}$ and $y^{2n}$ can be written as a linear combinations of elements in $\Gamma_A$.

Then

$$x^{\alpha+1} = sx^{\mu+1} + txy^{2n-\mu}. \quad (2.4)$$

If $\mu < 2n - 1$ then $x^{\mu+1} \in \Gamma_A$. If $\mu = 2n - 1$ then as remarked previously $x^{\mu+1}$ can be written as linear combinations of elements in $\Gamma_A$. Consider the monomial $xy^{2n-\mu} \notin \Gamma_A$. Now

$$y^{2n-\mu-1}B = xy^{2n-\mu} - bx^{2n}y^{2n-\mu-1} + by^{4n-\mu-1}$$
$$y^{2n-\mu-1}A = y^{4n-\mu-1} + x^{2n}y^{2n-\mu-1} - ay^{2n-\mu-1}$$

Again $y^{2n-\mu-1} \in \Gamma_A$.

If $\mu$ is odd then

$$x^{\mu+1}q_{(2n-\mu-1)/2}Q = x^{2n}y^{2n-\mu-1} - q^{(2n-\mu-1)/2}x^{\mu+1}.$$  

If $\mu$ is even then

$$y^{2n-\mu-2}F = x^{2n}y^{2n-\mu-1} - ey^{2n-\mu-1} + fx^{2n-1}y^{2n-\mu-2}$$
$$x^{\mu+1}q_{(2n-\mu-2)/2}Q = x^{2n-1}y^{2n-\mu-2} - q^{(2n-\mu-2)/2}x^{\mu+1}.$$  

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Substituting these values back into Equation 2.4 we have

$$x^{\alpha+1} = s'x^{\nu+1} + t'y^{2n-\nu-1}$$

where $x^{\nu+1}, y^{2n-\nu-1} \in \Gamma_A$ or can be written as linear combinations of elements in $\Gamma_A$. Thus the lemma is proved for $x^\alpha$. The proof in the other three cases is identical. \qed

Consider a linear combination $L$ of the polynomials $A, \ldots, F, Q$ with coefficients in $\mathbb{C}[x, y, a, b, c, d, e, f, q]$ which is equal to an expression $\sum a_i f_i$, where $f_i \in \Gamma_A$ and $a_i \in \mathbb{C}[a, b, c, d, e, f, q]$. Because $A, \ldots, F, Q$ belong to $I$ it follows that $\sum a_i f_i \in I$. The elements of $\Gamma_A$ are linearly independent in $\mathbb{C}[x, y]/I$ so we know that each of the coefficients $a_i$ equals zero. Therefore $L = 0$ and the relation $L$ is a syzygy between the generators. The syzygies correspond to the fact that the parameters $a, \ldots, f, q$ are not independent.

**Lemma 2.12** If $\Gamma_A$ is linearly independent in $\mathbb{C}[x, y]/\langle A, B, C, D, E, F, Q \rangle$ then the parameters $a, \ldots, f, q$ must satisfy the following relations.

$$e = a - c, \quad f = -d, \quad q = abd, \quad d = 2bc - ab, \quad c = a + 2bfq^{n-1} \quad (2.5)$$

**Proof.** Consider the relation

$$xyB - Q = -bxy (x^{2n} - y^{2n}) + q \quad (2.6)$$

in $I$. Now $xy(x^{2n} - y^{2n}) \in L_+$ and $1$ is our element in $\Gamma_A$ belonging to the representation $L_+$ so the polynomial $xy(x^{2n} - y^{2n})$ can be written as a multiple of $1$. Since

$$yC - xD = xy (x^{2n} - y^{2n}) - d (y^{2n} + x^{2n})$$
and the polynomial $A$ writes $(x^{2n} + y^{2n})$ as a multiple of 1 we have

$$xyB - Q + byC - bxD + bdA = q - abd \in I.$$  

This relation implies that $(q - abd)1 \in I$. Now $1 \notin I$ if $\Gamma_A$ are linearly independent in $\mathbb{C}[x,y]/I$ so $q = abd$. In fact this syzygy implies that $Q$ is not needed as a generator of the ideal as $Q = xyB + byC - bxD + bdA$. However we continue to include it since it will simplify the other syzygies.

Next consider the relations

$$xA - C - E = (c - a + e)x + (d + f)y^{2n-1} \quad (2.7)$$

$$yA - D - F = (c - a + e)y - (d + f)y^{2n-1}$$

in $I$. If $x$ and $y^{2n-1}$ are linearly independent in $\mathbb{C}[x,y]/I$ then the coefficient of $x$ and of $y^{2n-1}$ in Equation 2.7 must be zero. Therefore we have the relations $c = a - e$ and $d = -f$. Again these syzygies imply that we do not need the generators $E$ and $F$ since $E = xA - C$ and $F = yA - D$.

Again

$$xyA - yC - xD = -xya + 2cxy - d(x^{2n} - y^{2n})$$

belongs to $I$. Now $x^{2n} - y^{2n} \in L_{\lambda}$ belongs to $\Gamma_A$ but $xy \notin \Gamma_A$. The polynomial $B$ allows $xy$ to be replaced with $x^{2n} - y^{2n}$ and so

$$xyA - yC - xD + (a - 2c)B = (-d - ab + 2bc)(x^{2n} - y^{2n}).$$

Again since $x^{2n} - y^{2n} \notin I$ it must be that $d = -ab + 2bc$.

Finally

$$xA - y^{2n-1}B - C = -ax + bx^{2n}y^{2n-1} - by^{4n} + cx + dy^{2n-1}.$$  

Now

$$y^{2n-1}A = x^{2n}y^{2n-1} + y^{4n-1} - ay^{2n-1},$$

$$y^{2n-2}F = x^{2n}y^{2n-1} - cy^{2n-1} + fx^{2n-1}y^{2n-2},$$

$$xq_{n-1}Q = x^{2n-1}y^{2n-2} - q^{n-1}x.$$  

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Therefore

$$xA - y^{2n-1}B - C + by^{2n-1}A - 2by^{2n-2}F + 2bfq_{n-1}xQ =$$

$$(-a + c - 2bfq^{n-1})x + (d - ab + 2be)y^{2n-1}.$$

In the same way the final syzygy we will consider

$$yA - x^{2n-1}B - D - bx^{2n-1}A + 2bx^{2n-2}E + 2bfyq_{n-1}Q =$$

$$(-a + c - 2bfq^{n-1})y + (-d + ab - 2be)x^{2n-1}.$$

These syzygies can be written in a matrix $M$ as follows.

$$
\begin{pmatrix}
bd & xy & by & -bx & 0 & 0 & -1 \\
x & 0 & -1 & 0 & -1 & 0 & 0 \\
y & 0 & 0 & -1 & 0 & -1 & 0 \\
xy & a - 2c & -y & -x & 0 & 0 & 0 \\
x + by^{2n-1} & -y^{2n-1} & -1 & 0 & 0 & -2by^{2n-2} & 2bfq_{n-1}x \\
y - bx^{2n-1} & -x^{2n-1} & 0 & -1 & 2bx^{2n-2} & 0 & 2bfq_{n-1}y
\end{pmatrix}
$$

This is called the matrix of syzygies and means that

$$M \begin{pmatrix} A & B & C & D & E & F & Q \end{pmatrix}^t = 0.$$

Together these syzygies correspond to the following relations between the parameters.

$$e = a - c, \quad f = -d, \quad q = abd, \quad d = 2bc - ab, \quad c = a + 2bfq^{n-1}$$

Proposition 2.13 Let $I$ be an ideal generated by $A, \ldots, F, Q$ where the parameters $a, \ldots, f, q$ are subject to the relations given in 2.5. Then $\Gamma_A$ forms a basis for $\mathbb{C}[x, y]/I$ and so $I$ defines a $G$-cluster.
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**Proof.** By Lemma 2.11 $\Gamma_A$ spans $\mathbb{C}[x, y]/I$. It remains to show that $\Gamma_A$ is linearly independent in $\mathbb{C}[x, y]/I$. Now suppose that $\Gamma_A$ is not linearly independent. Then there exists a $G$-equivariant linear combination of the elements in $\Gamma_A$ which belongs to $I$.

Suppose $x^{2n} - y^{2n}$ is not basic then $x^{2n} - y^{2n}, xy \in I$. Now $x E \in I$ which implies $ex^2 \in I$. If $e = 0$ then $a = 0$ or $b = 0$ by Relations 2.5. If $a = 0$ then

$$x^{2n} + y^{2n}, \quad xy - b(x^{2n} - y^{2n}), \quad x^{2n+1}, \quad y^{2n+1}$$

are generators for $I$. If $b = 0$ then

$$x^{2n} + y^{2n} - a, \quad xy$$

are generators for $I$. In each case only one term in the generators divides $x^{2n} - y^{2n}$ and so these generators do not imply that $x^{2n} - y^{2n} \in I$. Therefore we can take $x^{2n} - y^{2n}$ in the basis.

Otherwise $x^2 \in I$ which implies $x^{2n} + y^{2n} \in I$ and so $a \in I$. We have already shown that if $a = 0$ then $x^{2n} - y^{2n} \notin I$. If $a \neq 0$ then $1 \in I$. Again only one term in the generators $A, B, C, D$ divides 1 and so the generators do not imply that $1 \in I$.

Suppose that $(y^{2n-i}, x^{2n-i})$ and $(x^i, y^i)$ are not linearly independent. Now $y^{2n-i}, x^{2n-i}, x^i, y^i \notin I$ because $x^{2n} - y^{2n} \notin I$. Therefore there exist polynomials of the form

$$y^{2n-i} - \alpha x^i, \quad x^{2n-i} - (-1)^i \alpha y^i$$

in $I$ where $\alpha \neq 0$. If $i$ is even then these polynomials imply that $x^{2n} - y^{2n} \in I$. We have already shown this is not possible. If $i$ is odd then as in the proof of Lemma 2.11

$$xy^{2n-i} - \alpha x^{i+1} = -aby^{2n-i-1} - (\alpha - 2bq^{(2n-i-1)/2}) x^{i+1}.$$
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However we have shown that there exists no such relation when \( i \) is even so
\[
ab = \alpha - 2bq^{(2n-i-1)/2} = 0.
\]
If \( ab = 0 \) then \( q = 0 \) so \( ab = \alpha - 2bq^{(2n-i-1)/2} = 0 \)
is impossible since \( \alpha \neq 0 \).

Therefore \( \Gamma_A \) forms a basis for \( \mathbb{C}[x,y]/I \).

Therefore this open subset of \( G \)-Hilb \( \mathbb{C}^2 \) is parametrised by \( a, b, c \) subject to the equation \( a - c + 2a^{n-1}b^2n(2c-a)^n \). In other words this open subset of \( G \)-Hilb \( \mathbb{C}^2 \) is isomorphic to the smooth surface in \( \mathbb{C}^3_{a,b,c} \) given by the equation
\[
a - c + 2a^{n-1}b^2n(2c-a)^n.
\]

**Proposition 2.14** \( \text{Soc}(I) \) contains at most the submodule of \( \mathbb{C}[x,y]/I \) generated by \( \langle x^{2n} - y^{2n} \rangle \cong L_\pm \). Moreover \( \langle x^{2n} - y^{2n} \rangle \subset \text{Soc}(I) \) if and only if \( a = 0 \).

**Proof.** Since \( \text{Soc}(I) \) is isomorphic to a direct sum of \( G \)-representations we only need to check if submodules of \( \mathbb{C}[x,y]/I \) isomorphic to any of the irreducible representations belong to \( \text{Soc}(I) \). Clearly the submodules generated by \( 1, x^n + y^n, x^n - y^n \) do not belong to \( \text{Soc}(I) \). Consider the submodule generated by \( x^j + \lambda y^{2n-j}, y^j + (-1)^j \lambda x^{2n-j} \). For this to be contained in \( \text{Soc}(I) \) it must be that \( x(x^j + \lambda y^{2n-j}) = 0 \). Now
\[
x(x^j + \lambda y^{2n-j}) = x^{j+1} + \lambda xy^{2n-j}.
\]
If \( j \) is odd then as in the proof of Lemma 2.11
\[
y^{2n-j-1}B - by^{2n-j-1}A + 2bx^{j+1}q_{(2n-j-1)/2}Q
= xy^{2n-j} - 2bq^{(2n-j-1)/2}x^{j+1} + aby^{2n-j-1}
\]
and if \( j \) is even then
\[
y^{2n-j-1}B - by^{2n-j-1}A + 2by^{2n-j-2}F - 2bf x^{j+1}q_{(2n-j-2)/2}Q
= xy^{2n-j} + 2bf q^{(2n-j-2)/2}x^{j+1} + (-2be + ab)y^{2n-j-1}.
\]

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Therefore if $j$ is odd

$$x(x^j + \lambda y^{2n-j}) = x^{j+1} + 2bq^{(2n-j-1)/2}\lambda x^{j+1} - ab\lambda y^{2n-j-1}.$$  

This is zero if and only if $\lambda ab = 0$ and $1 + 2bq^{(2n-j-1)/2}\lambda = 0$. However if $ab = 0$ then $q = 0$ and so $1 + 2bq^{(2n-j-1)/2} \neq 0$.

Similarly if $j$ is even

$$x(x^j + \lambda y^{2n-j}) = x^{j+1} - 2bfq^{(2n-j-2)/2}\lambda x^{j+1} - (-2be + ab)\lambda y^{2n-j-1}.$$  

For this to be zero $1 - 2bfq^{(2n-j-3)/2} = 0$ and $d = (-2be + ab) = 0$. If $d = 0$ then $q = 0$ and $1 + 2bq^{(2n-j-1)/2} \neq 0$. Therefore

$$\langle x^j + \lambda y^{2n-j}, y^j + (-1)^j\lambda x^{2n-j} \rangle \not\subset \text{Soc}(I).$$

It only remains to see when $x^{2n} - y^{2n} \in \text{Soc}(I)$. Using the relations $C$ and $E$

$$x(x^{2n} - y^{2n}) = (c + e)x + (d + f)y^{2n-1}.$$  

From Equation 2.5 $d + f = 0$ and $a = c + e$ and so $x^{2n} - y^{2n} \in \text{Soc}(I)$ if and only if $a = 0$.  

Notice that $u = xy(x^{2n} - y^{2n}) = d(x^{2n} + y^{2n}) = ad$, $v = x^{2n} + y^{2n} = a$ and $w = x^2y^2 = abd$. The curve $a = 0$ defines the exceptional locus $u = v = w = 0$ in this affine subset. If $a = 0$ then $c = 0$ so $b$ is the coordinate along this exceptional curve. Setting $a = 0$ and letting $b \to \infty$ we obtain the ideal which is described in the next section when the new parameters are all zero.
Explicit construction of $G$-Hilb $\mathbb{C}^9$ for binary dihedral groups.

2.5.2 Case B

Let $I$ be an ideal that defines a $G$-cluster such that $\Gamma_B \in \Gamma(I)$. Then $I$ contains the polynomials

\[
A := x^{2n} + y^{2n} - a, \quad B := x^{2n} - y^{2n} - bxy,
\]

\[
C := xy^2 - cx^{2n-1} - dy, \quad D := yx^2 - cy^{2n-1} + dx,
\]

\[
E := x y^3 - ey^{2n-2} - f y^2, \quad F := x^3 y + ey^{2n-2} + f x^2
\]

\[
Q := x^2 y^2 - q.
\]

As in Case A we can show that using only these polynomials every monomial in $\mathbb{C}[x, y]$ can be written as a linear combination of elements in $\Gamma_B$.

To find the relations between the parameters we use syzygies. First consider

\[
xC + yD + cA - 2Q = -ac + 2q \in I.
\]

Again $1 \notin I$ so $q = ac/2$ and $Q$ is not required as a generator.

Other syzygies are

\[
x(-xC + yD + cB = -(cb + 2d)xy,
\]

\[
yC - E + cx^{2n-4} F - cey^2 q_{n-2} Q
\]

\[
= (f - d - ceq^{n-2})y^2 + (e + cf)x^{2n-2},
\]

\[
xD - F + cy^{2n-4} E + ey^2 q_{n-2} Q
\]

\[
= -(f - d - ceq^{n-2})x^2 - (e + cf)y^{2n-2},
\]

\[
yA + yB - 2x^{2n-2} D - 2cyq_{n-1} Q + bC
\]

\[
= (-a + 2cq^{n-1} - bd) y - (bc - 2d)x^{2n-1},
\]

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$$xA - xB - 2y^{2n-2}C - bD - 2cxq_{n-1}Q$$

$$= (-a + bd + 2cq^{-1})x + (2d + bc)y^{2n-1}.$$ 

The matrix of syzygies is

$$
\begin{pmatrix}
  c & 0 & x & y & 0 & 0 & -2 \\
  0 & c & x & -y & 0 & 0 & 0 \\
  0 & 0 & y & 0 & -1 & cx^{2n-4} - cy^2q_{n-2} & 0 \\
  0 & 0 & 0 & x & cy^{2n-4} & -1 & cx^2q_{n-2} \\
  y & y & b & -2x^{2n-2} & 0 & 0 & -2cq_{n-1}y \\
 x & -x & -2y^{2n-2} & -b & 0 & 0 & -2cq_{n-1}x
\end{pmatrix}
$$

and these correspond to the following relations between the parameters

$$2q = ac, \quad 2d = -bc, \quad e = -cf, \quad f - d - ceq^{n-2} = 0, \quad 0 = a + bd - 2cq^{n-1}. \quad (2.8)$$

**Proposition 2.15** Let $I$ be an ideal generated by $A, \ldots, F, Q$ where the parameters $a, \ldots, f, q$ are subject to the relations given in 2.8. Then $\Gamma_B$ forms a basis for $\mathbb{C}[x, y]/I$ and so $I$ defines a $G$-cluster.

**Proof.** The proof that $\Gamma_B$ spans $\mathbb{C}[x, y]/I$ is the same as the proof of Lemma 2.11. That $\Gamma_B$ is linearly independent in $\mathbb{C}[x, y]/I$ is proved using the techniques in the proofs of Lemma 2.11 and Proposition 2.13.

Suppose that $(x, y)$ and $(y^{2n-1}, x^{2n-1})$ are not linearly independent in $\mathbb{C}[x, y]/I$. Then there exist polynomials of the form

$$\lambda x - \mu y^{2n-1}, \quad \lambda y + \mu x^{2n-1}$$

in $I$. Therefore $\mu(x^{2n} - y^{2n}) \in I$. If $\mu = 0$ then $x, y \in I$ and so $a = 0$ or $1 \in I$. As in Case A the generators $A, \ldots, F, Q$ do not imply $x, y \in I$ when $a = 0$ and do not imply $1 \in I$ in any case.

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Otherwise $x^{2n} - y^{2n} \in I$ and so $xy \in I$ or $b = 0$. As in case A the generators $A, \ldots, F, Q$ do not imply a relation between $x$ and $y^{2n-1}$ when $b = 0$. If $xy \in I$ then relations $C$ and $D$ imply that $dx^2, dy^2 \in I$ and so either $d = 0$ or $x^2, y^2 \in I$. As above we can show that neither case is possible.

Finally we show by induction that $(x^i, y^j)$ and $(y^{2n-i}, x^{2n-i})$ are linearly independent in $\mathbb{C}[x, y]/I$ for $1 \leq i \leq n$. We have shown that it is true when $i = 1$. Assume that it is not true for some $1 < i < n$. Then there exist polynomials of the form

$$y^{2n-i} - \alpha x^i, \quad x^{2n-i} - (-1)^i \alpha y^i$$

in $I$ and $\alpha \neq 0$ by induction. These polynomials imply that

$$y^{2n-i+1} - \alpha x^i y \in I.$$  (2.9)

If $i$ is even then

$$x^i y = cx^i y^{2n-1} = dx^i - \alpha q^{i-1} y^{2n-i+1} - dy^i.$$  

Substituting this back into Equation 2.9 implies

$$\alpha dx^i - (-\alpha q^{i-1}/2 + 1) y^{2n-i+1} \in I.$$  

By induction there exists no relation between $x^i$ and $y^{2n-i+1}$ in $I$ so $\alpha d = \alpha q^{i-1}/2 - 1 = 0$. This implies that $d = 0$. However if $d = 0$ then we can see that the ideal contains no polynomials of the form $y^{2n-i} - \alpha x^i$ and $x^{2n-i} - (-1)^i \alpha y^i$.

If $i$ is odd then

$$x^i y = cx^i y^{2n-2} - fx^i - \alpha q^{i-1}/2 y^{2n-i+1} - f x^i - 1.$$  

As above we can show that this leads to a contradiction. \qed
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Therefore these parameters define a smooth surface in \( \mathbb{C}_{a,b,c,f}^4 \) with equations

\[
a - b^2 c/2 - a^{n-1} c^n / 2^{n-2} = f + bc/2 + a^{n-2} c^n f / 2^{n-1} = 0
\]

**Proposition 2.16** \( \text{Soc}(I) \) contains at most the submodules of \( \mathbb{C}[x,y]/I \) generated by \( x^{2n} - y^{2n} \) and \( \langle y^{2n-1}, x^{2n-1} \rangle \). Moreover \( \langle x^{2n} - y^{2n} \rangle \cong L_\gamma \subset \text{Soc}(I) \) if and only if \( c = 0 \) and \( \langle y^{2n-1}, x^{2n-1} \rangle \subset V_1 \in \text{Soc}(I) \) if and only if \( a = b = 0 \).

**Proof.** The proof is similar to the proof of Proposition 2.14 \( \square \)

Notice that \( a = b = 0 \) is a Cartier divisor since on the open set given by \( c \neq 0 \) it has the equation \( a = 0 \) and on the open set given by \( 1-a^{n-2} c^n / 2^{n-2} \neq 0 \) it is given by the equation \( b = 0 \).

Again \( u = abc/2, \ v = a \) and \( w = ac/2 \). Therefore \( a = 0 \) defines the exceptional locus. This decomposes into two prime divisors \( a = b = 0 \) and \( c = 0 \). If \( c = 0 \) then \( b \) is the coordinate along this exceptional curve. Setting \( c = 0 \) and letting \( b \to \infty \) we obtain the ideal that was described in the previous section when the parameters were zero. If \( a = b = 0 \) then \( c \) is the coordinate along this exceptional curve. Setting \( a = b = 0 \) and letting \( c \to \infty \) then we obtain the ideal which is described in the next section when \( k = 2 \) and the new parameters are all zero.
Explicit construction of $G$-Hilb $\mathbb{C}^2$ for binary dihedral groups.

2.5.3 Case $C_k$

Fix $k \in [2, n - 2]$. Let $I$ be an ideal that defines a $G$-cluster such that $\Gamma_{C_k} \in \Gamma(I)$. Then $I$ contains the polynomials

\[
B := x^{2n-k+1} - bxy^k - cy^{k-1},
C := y^{2n-k+1} - (-1)^kbx^ky - (-1)^{j-1}cx^{k-1},
D := xy^{k+1} - dx^{2n-k} - cy^k,
E := x^{k+1}y - (-1)^{k+1}dy^{2n-k} + ex^k,
F := xy^{k+2} - f x^{2n-k-1} - gy^{k+1},
G := x^{k+2}y - (-1)^kf y^{2n-k-1} + gx^{k+1},
Q := x^2y^2 - q.
\]

Again we can show that using only these polynomial every monomial in $\mathbb{C}[x, y]$ can be written as a linear combination of elements in $\Gamma_{C_k}$.

As in Case A we can construct a matrix of syzygies and so generate the relations between these parameters. The matrix of syzygies is

\[
\begin{pmatrix}
 d & 0 & x & 0 & 0 & 0 & -y^{k-1} \\
0 & (-1)^kd & 0 & y & 0 & 0 & -x^k \\
0 & 0 & 0 & -x & (-1)^kd y^{2n-2k-2} & 1 & (-1)^kd y_{n-i-1}x^{k+1} \\
0 & 0 & y & 0 & 1 & dx^{2n-2k-2} & (-1)^kd y_{n-i-1}y^{k+1} \\
y & 0 & b & -x^{2n-2k} & 0 & 0 & (-1)^kd y_{n-k}y^k \\
0 & x & -y^{2n-2k} & (-1)^kb & 0 & 0 & -dy_{n-k}x^k
\end{pmatrix}
\]

and so the relations are

\[
e = -bd, \quad q = cd, \quad f = -dg, \quad g = e + (-1)^kd q^{n-k}, \quad 0 = (-1)^kd q^{n-k} + c + be.
\]

(2.10)

**Proposition 2.17** Let $I$ be an ideal generated by $B, \ldots, G, Q$ where the parameters $b, \ldots, g, q$ are subject to the relations given in 2.10. Then $\Gamma_{C_k}$ forms a basis for $\mathbb{C}[x, y]/I$ and so $I$ defines a $G$-cluster.
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**Proof.** This is proved using the same techniques as the proof of Proposition 2.15. $\Box$

Again we have that this affine open set is isomorphic to a smooth surface in $\mathbb{C}^4_{b,c,d,g}$ given by the equations

$$c - b^2d + (-1)^kd^{n-k+1}c^{n-k} = g + bd + (-1)^kgc^{n-k}d^{n-k+2} = 0.$$

**Proposition 2.18** $\text{Soc}(I)$ contains at most the submodules $\langle x^ky, xy^k \rangle$ and $\langle y^{2n-k}, x^{2n-k} \rangle$. Moreover $\langle x^ky, xy^k \rangle \cong V_{k-1} \subset \text{Soc}(I)$ if and only if $kd = 0$ and $\langle y^{2n-k}, x^{2n-k} \rangle \cong V_k \subset \text{Soc}(I)$ if and only if $b = c = 0$.

**Proof.** The proof is similar to the proof of Proposition 2.14. $\Box$

If $k$ is odd then $u = 2b(cd)^{(k+1)/2}$, $v = 2c(cd)^{(k-1)/2}$ and $w = cd$. If $k$ is even then $u = 2c(cd)^{k/2}$, $v = 2b(cd)^{k/2}$ and $w = cd$. Therefore $cd = 0$ defines the exceptional locus on this affine subset. This decomposes as the two prime divisors $d = 0$ and $b = c = 0$.

If $d = 0$ then $b$ is the coordinate along this exceptional curve. Setting $d = 0$ and $b \to \infty$ gives the ideal that is described in the previous case, Case $C_{k-1}$ or Case B, when all the parameters are zero. If $b = c = 0$ then $d$ is the coordinate along this exceptional curve. Setting $b = c = 0$ and $d \to \infty$ gives the ideal which is described in the next case, Case $C_{k+1}$, when all the parameters are zero.
Explicit construction of $G$-Hilb $\mathbb{C}^2$ for binary dihedral groups.

2.5.4 Case D

An ideal $I$ which defines a $G$-cluster and $\Gamma_{D_+} \in \Gamma(I)$, is generated by the polynomials

\begin{align*}
B &:= x^{n+2} - bxy^{n-1} - cy^{n-2}, \\
C &:= y^{n+2} - (-1)^{n+1} byx^{n-1} - (-1)^n cx^{n-2}, \\
D &:= (x^n - i^ny^n) - dxy(x^n + iy^n), \\
E &:= x(x^n - i^ny^n) - ey^{n-1} - f x(x^n + i^ny^n), \\
F &:= y(x^n - i^ny^n) - e(-i)^n x^{n-1} + fy(x^n + i^ny^n), \\
Q &:= x^2 y^2 - q
\end{align*}

and the syzygy matrix is

\[
\begin{pmatrix}
-dy & 0 & x & \alpha & 0 & i^n dy^{n-1} \\
0 & i^nx & x & 0 & \alpha & dx^{n-1} \\
-1 + dxy & 0 & x^2 & 0 & 0 & (bd + i^n)y^{n-2} + i^n dxy^{n-1} \\
0 & i^n (1 + xyd) & y^2 & 0 & 0 & -((-i)^n bd + 1)x^{n-2} + dxy^{n-1} \\
0 & 0 & 0 & -(-i)^n e y & x & 0
\end{pmatrix}
\]

where $\alpha = -1 - (-i)^n bd/2$. This corresponds to the relations

\begin{align*}
c &= i^n q + bdq, \\
f &= (-i)^{n} ed/2, \\
b &= dc + i^n dq, \\
0 &= e - cd + (-i)^n bde/2 - i^n dq
\end{align*}

(2.11)

between the parameters.

**Proposition 2.19** Let $I$ be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.11. Then $\Gamma_{D_+}$ forms a basis for $\mathbb{C}[x, y]/I$ and so $I$ defines a $G$-cluster.

**Proof.** This is proved using the same techniques as the proof of Proposition 2.15. \qed
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Thus this affine subset is the smooth surface in $\mathbb{C}_{k,d,e,q}^i$ given by the equations

$$b - 2i^n dq - bd^2 q = e - bd^2 q + (-i)^n bde/2 - 2i^n dq = 0.$$  

**Proposition 2.20**  
Soc$(I)$ contains at most the submodule $\langle xy(x^n + i^n y^n) \rangle$.  
Moreover $\langle xy(x^n + i^n y^n) \rangle \cong M_+ \subset$ Soc$(I)$ if and only if $q = 0$.

**Proof.** The proof is similar to the proof of Proposition 2.14.  

If $k$ is odd then $u = 2(i^n + b)q^{(n+1)/2}$, $v = 2bq^{(n-1)/2}$ and $w = q$. If $k$ is even then $u = 2bq^{n/2}$, $v = 2(i^n + bd)q^{n/2}$ and $w = q$. Therefore $q = 0$ defines the exceptional locus on this affine subset.

If $q = 0$ then $d$ is the coordinate along this exceptional curve. Setting $q = 0$ and $d \to \infty$ gives the ideal that is described in Case $E_+$ when all the parameters are zero.

An ideal $I$ that defines a $G$-cluster and $\Gamma_{D_+} \in \Gamma(I)$, is generated by the polynomials

$$B := x^{n+2} - bxy^{n-1} - cy^{n-2},$$
$$C := y^{n+2} - (-1)^{n-1} bx^{n-1} y(-1)^n - c(-1)^n x^{n-2},$$
$$D := x^n + i^n y^n - dxy (x^n - i^n y^n),$$
$$E := x (x^n + i^n y^n) - c y^{n-1} - f x (x^n - i^n y^n),$$
$$F := y (x^n + i^n y^n) + e(-i)^n x^{n-1} + f y (x^n - i^n y^n),$$
$$Q := x^2 y^2 - q,$$

the syzygy matrix is

$$
\begin{pmatrix}
   dy & 0 & x & \alpha & 0 & \bar{v}^n dy^{n-1} \\
   0 & -i^n dx & y & \alpha & 0 & dx^{n-1} \\
   -1 + dxy & 0 & x^2 & 0 & 0 & (bd - i^n) y^{n-2} - i^n dxy^{n-1} \\
   0 & -i^n (1 + dxy) & y^2 & 0 & 0 & ((-i)^n bd - 1) x^{n-2} + dxy^{n-1} \\
   0 & 0 & (-i)^n e y & -x & 0 & 0
\end{pmatrix}
$$
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where $\alpha = -1 + (-i)^n bd / 2$. The relations between the parameters are

\[
\begin{align*}
    c &= -i^n q + bdq, \\
    b &= dc - i^n dq, \\
    f &= -(-i)^n de/2 \quad 0 = e - dc - (-i)^n bde/2 + i^n dq.
\end{align*}
\]

(2.12)

**Proposition 2.21** Let $I$ be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.12. Then $\Gamma_{P_\pm}$ forms a basis for $\mathbb{C}[x, y]/I$ and so $I$ defines a $G$-cluster.

**Proof.** This is proved using the same techniques as the proof of Proposition 2.15. \qed

Thus this affine subset is the smooth surface in $\mathbb{C}^4_{b,d,e,q}$ given by the equations

\[b + 2i^n dq - bd^2 q = e - bd^2 q - (-i)^n bde/2 + 2i^n dq = 0.\]

**Proposition 2.22** $\text{Soc}(I)$ contains at most the submodule $\langle xy(x^n - i^n y^n) \rangle$. Moreover $\langle xy(x^n - i^n y^n) \rangle \cong M_\pm \subset \text{Soc}(I)$ if and only if $q = 0$.

**Proof.** The proof is similar to the proof of Proposition 2.14. \qed

If $k$ is odd then $u = 2(-i^n + bd)q^{(n+1)/2}$, $v = 2b(q)^{(n-1)/2}$ and $w = q$. If $k$ is even then $u = 2b(q)^{n/2}$, $v = 2(-i^n + bd)q^{n/2}$ and $w = q$. Therefore $q = 0$ defines the exceptional locus on this affine subset.

If $q = 0$ then $d$ is the coordinate along this exceptional curve. Setting $q = 0$ and $d \to \infty$ gives the ideal that is described in Case $E_\pm$ when all the parameters are zero.
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2.5.5 Case E

If \(I\) is an ideal that defines a \(G\)-cluster and \(\Gamma_{E+} \in \Gamma(I)\) then \(I\) is generated by the polynomials of the form

\[
B := x (x^n - i^n y^n) - by^{n-1} - cx (x^n + i^n y^n),
\]
\[
C := y (x^n - i^n y^n) - (-i)^n bx^{n-1} + cy (x^n + i^n y^n),
\]
\[
D := xy (x^n + i^n y^n) - d (x^n - i^n y^n),
\]
\[
E := x^{n+2} - ey^{n-2} - f xy^{n-1},
\]
\[
F := y^{n+2} - (-1)^n cx^{n-2} - (1)^n^{-1} x^{n-1} y,
\]
\[
Q := x^2 y^2 - q.
\]

The syzygy matrix is

\[
\begin{pmatrix}
 y & -x & 2c & 0 & 0 & 0 \\
 y^2 & xy + 2d & 2y & 0 & 0 & -4x^{n-1} \\
 0 & 0 & (1)^n f & y^2 & -i^n x^2 & -(x^n - i^n y^n) \\
 0 & y & 0 & 0 & i^n (1 - c) & -(1 + c)x^{n-2} \\
x & 0 & 0 & -(1 - c) & 0 & i^n (1 + c)y^{n-2}
\end{pmatrix}
\]

and corresponds to the relations

\[
b = 2i^n cd, \quad q = cd^2
\]
\[
0 = -e - i^n q + df \quad 0 = -f + b + ef
\]

between the parameters.

**Proposition 2.23** Let \(I\) be an ideal generated by \(B, \ldots, F, Q\) where the parameters \(b, \ldots, f, q\) are subject to the relations given in 2.13. Then \(\Gamma_{E+}\) forms a basis for \(\mathbb{C}[x, y]/I\) and so \(I\) defines a \(G\)-cluster.

**Proof.** This is proved using the same techniques as the proof of Proposition 2.15. \(\square\)
Explicit construction of $G$-Hilb $\mathbb{C}^2$ for binary dihedral groups.

This affine open set is isomorphic to a smooth surface in $\mathbb{C}_{c,d,f}^3$ given by the equation

$$-f(1-c) + 2i^n cd = 0.$$  

Notice that on the open set given by $c \neq 1$ then $f$ can be expressed as a regular function of $c$ and $d$. Therefore $G$-Hilb $\mathbb{C}^2$ contains the open set given by $c \neq 1$ in $\mathbb{C}_{c,d}^5$. Thus $G$-Hilb $\mathbb{C}^2$ is rational.

**Proposition 2.24** Soc($I$) contains at most the submodules isomorphic to $V_{n-2}$, $V_{n-1}$ and $M_+$. Moreover $V_{n-2} \subset$ Soc($I$) if and only if $c = 1$, $V_{n-1} \subset$ Soc($I$) if and only if $d = 0$ and $M_+ \subset$ Soc($I$) if and only if $c = 0$.

**Proof.** The proof that Soc($I$) contains at most the submodules isomorphic to $V_{n-2}$, $V_{n-1}$ and $M_+$ is similar to the proof of Proposition 2.14.

If $d = 0$ then it follows that $x^{n+2}(1+c) = y^{n+2}(1+c) = 0$ and so if $c \neq -1$ the submodule $\langle y(x^n + iv^n), x(x^n + iv^n) \rangle \cong V_{n-1}$ is in Soc($I$). If $c = -1$ then we have the polynomials $2x^{n+1} - by^{n-1}$ and $2y^{n+1} - b(-1)^nx^{n-1}$ in $I$ and when $d = 0$ the submodule $\langle yx^n, xy^n \rangle \cong V_{n-1}$ is in Soc($I$). The point $(c, d) = (-1, 0)$ is the ideal in Case $E_0$ given by setting every polynomial that is not in $\Gamma_{E_0}$ equal to zero. Therefore if $d = 0$ then $V_{n-1}$ is in Soc($I$).

If $c = 0$ then the submodule $\langle x^n - iv^n \rangle \cong M_+$ is in Soc($I$).

If $c = 1$ then $d = b = q = 0$. Therefore $x^2y^{n-1} = x^{n-1}y^2 = xy^n = x^ny = 0$ and so $\langle x^{n-1}y, xy^{n-1} \rangle \cong V_{n-2} \in$ Soc($I$). \hfill $\Box$

If $n$ is even then $u = 2c^{n/2}d^n e$, $v = 2c^{(n-2)/2}d^{n-2}e$ and $w = cd^2$ and if $n$ is odd $u = 2c^{(n-1)/2}d^{n-1}e$, $v = 2c^{(n-1)/2}d^{n-1}e$ and $w = cd^2$. Either way the exceptional locus on this affine subset is defined by $cd^2$. This is the union of the three prime divisors $c = 0$, $d = 0$ and $c = 1$.

If $c = 0$ then $d$ is the coordinate along this exceptional curve. Setting $c = 0$ and letting $d \to \infty$ gives the ideal in Case $D_+$ where all the parameters are
2.5 Construction of open sets contained in $G$-Hilb $\mathbb{C}^2$

zero. If $d = 0$ then $c$ is the coordinate along this exceptional curve. Setting $d = 0$ and letting $c \to \infty$ gives the ideal in Case $E_-$ when all of the parameters are zero. If $c = 1$ then $f$ is the coordinate along this exceptional curve and letting $f \to \infty$ gives the ideal in Case $C_{n-2}$ when all of the parameters are zero.

An ideal $I$ which defines a $G$-cluster and $\Gamma_{E_-} \in \Gamma(I)$, is generated by the polynomials

$$
B := x \left(x^n + iy^n\right) - by^{n-1} - c x \left(x^n - iy^n\right),
$$
$$
C := y \left(x^n + iy^n\right) + (-1)^n bx^{n-1} + cy \left(x^n - iy^n\right),
$$
$$
D := xy \left(x^n - iy^n\right) - d \left(x^n + iy^n\right),
$$
$$
E := x^{n+2} - cy^{n-2} - fxy^{n-1},
$$
$$
F := y^{n+2} - (-1)^n cxy^{n-2} - (-1)^n x^{n-1}y,
$$
$$
Q := x^2y^2 - q.
$$

The syzygy matrix is

$$
\begin{pmatrix}
y & -x & 2c & 0 & 0 & 0 \\
y^2 & xy + 2d & 0 & 0 & 2y & -4x^{n-1} \\
0 & 0 & -(-i)^n f & y^2 & i^n x^2 & -(x^n + iy^n) \\
0 & y & 0 & 0 & -i^n(1 - c) & (1 + c)x^{n-2} \\
x & 0 & 0 & -(1 - c) & 0 & -(1 + c)i^n y^{n-2}
\end{pmatrix}
$$

and corresponds to the relations

$$
b = -2i^n cd, \quad q = cd^2
$$
$$
e = i^n q + fd \quad 0 = -f + b + cf.
$$

(2.14)

**Proposition 2.25** Let $I$ be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.14. Then $\Gamma_{E_-}$ forms a basis for $\mathbb{C}[x, y]/I$ and so $I$ defines a $G$-cluster.
Explicit construction of $G$-Hilb $\mathbb{C}^2$ for binary dihedral groups.

Proof. This is proved using the same techniques as the proof of Proposition 2.15.

This affine open set is a smooth surface in $\mathbb{C}^3_{c,d,f}$ given by the equation

$$-f - 2i^n cd + cf = 0.$$ 

Notice that on the open set given by $c \neq 1$ $f$ can be expressed as a regular function of $c$ and $d$.

Proposition 2.26 Soc$(I)$ contains at most the submodules isomorphic to $V_{n-2}$, $V_{n-1}$ and $M_-$. Moreover $V_{n-2} \subset \text{Soc}(I)$ if and only if $c = 1$, $V_{n-1} \subset \text{Soc}(I)$ if and only if $d = 0$ and $M_- \subset \text{Soc}(I)$ if and only if $c = 0$.

Proof. The proof is similar to the proof of Proposition 2.24.

If $n$ is even then $u = 2c^{n/2}d^n e$, $v = 2c^{(n-2)/2}d^{n-2}e$ and $w = cd^2$ and if $n$ is odd $u = 2c^{(n-1)/2}d^{n-1}e$, $v = 2c^{(n-1)/2}d^{n-1}$ and $w = cd^2$. Either way the exceptional locus is given by $cd^2 = 0$ this is the union of the three prime divisors $c = 0$, $d = 0$ or $c = 1$.

If $c = 0$ then $d$ is the coordinate along this exceptional curve. Setting $c = 0$ and letting $d \to \infty$ gives the ideal in Case D$_-$ where all the parameters are zero. If $d = 0$ then $c$ is the coordinate along this exceptional curve. Setting $d = 0$ and letting $c \to \infty$ gives the ideal in Case E$_+$ when all of the parameters are zero. If $c = 1$ then $f$ is the coordinate along this exceptional curve and letting $f \to \infty$ gives the ideal in Case C$_{n-2}$ when all of the parameters are zero.

2.5.6 Conclusion

We have constructed an affine cover of $G$-Hilb $\mathbb{C}^2$ where every open set is a smooth surface in $\mathbb{C}^4$. Two of the open sets contain open subsets of $\mathbb{C}^2$
2.5 Construction of open sets contained in $G$-Hilb $\mathbb{C}^2$

and so $G$-Hilb $\mathbb{C}^2$ is rational. For any ideal $I$ that defines a $G$-cluster $\text{Soc}(I)$ contains at most two irreducible representations of $G$. Therefore we have proved Parts 2 and 3 of Theorem 2.4. Theorem 2.10 proves Part 1 and so Theorem 2.4 is proved.
Chapter 3

Explicit construction of $G$-Hilb $\mathbb{C}^3$ for trihedral groups

3.1 Classification of trihedral groups

Recall that a trihedral group is a finite subgroup of SL(3, $\mathbb{C}$) of the form $G = A \times \mathbb{Z}_3$, where $A$ is a diagonal Abelian subgroup and $\mathbb{Z}_3$ is generated by

$$
\tau = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
$$

We begin by classifying the possibilities for $A$.

**Proposition 3.1** If $A$ is finite Abelian subgroup of SL(3, $\mathbb{C}$). Then $A$ can be written in the form $\mathbb{Z}_r \times \mathbb{Z}_R$ where $r \mid R$.

**Proof.** Suppose $A$ contains a subgroup $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{a_3}$. If $a_i$ and $a_j$ are coprime then $\mathbb{Z}_{a_i} \times \mathbb{Z}_{a_j} = \mathbb{Z}_{a_i a_j}$. So assume $\text{hcf}(a_i, a_j) > 1$ for each pair $i, j$ and write $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{a_3} \cong \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c$ where $a = \text{lcm}(a_1, a_2)$, $b' = \text{hcf}(a_1, a_2)$, $b = \text{lcm}(b', a_3)$ and $c = \text{hcf}(b', a_3)$. If $c = 1$ then $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{a_3} \cong \mathbb{Z}_{ab}$. If $c > 1$ then $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{a_3} \cong \mathbb{Z}_{ab} \times \mathbb{Z}_c$. The case $c = 0$ is not possible since then $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{a_3}$ would be finite and Abelian, which cannot happen.
3.1 Classification of trihedral groups

\[ \mathbb{Z}_{b_1} \times \mathbb{Z}_{b_2}, \text{ where } b_1 = \text{lcm}(a, b) \text{ and } b_2 = \text{hcf}(a, b). \] Otherwise \( A \) has a nontrivial subgroup of the form \( \mathbb{Z}_c \times \mathbb{Z}_c \times \mathbb{Z}_c \).

Now compare the number of elements in \( A \) of order exactly \( c \) with the number of elements in the subset \( D \subseteq \text{SL}(3, \mathbb{C}) \) consisting of all diagonal elements of \( \text{SL}(3, \mathbb{C}) \) having order exactly \( c \).

An element \( g \in \text{SL}(3, \mathbb{C}) \) belongs to \( D \) if \( g = \frac{1}{c}(\alpha, \beta, \gamma) \) where
\[ \text{hcf}(\alpha, \beta, c) = 1 \]
and \( \gamma = c - \alpha - \beta \).

Let \( g \in \mathbb{Z}_a \), respectively \( h \in \mathbb{Z}_b \), be a generator of a subgroup of \( \mathbb{Z}_a \), respectively \( \mathbb{Z}_b \), of order \( c \) and let \( k \in \mathbb{Z}_c \) be a generator of the group. An element of \( A \) is of order \( c \) if it is of the form \( g^p h^q k^s \) where
\[ \text{hcf}(p, q, s, c) = 1, \]

Every element \( \frac{1}{c}(\alpha, \beta, \gamma) \in D \) gives a pair \( (\alpha, \beta) \) satisfying Equation 3.1. Every pair \( (\alpha, \beta) \) satisfying Equation 3.1 gives \( c \) distinct triples \( (\alpha, \beta, i) \) for \( i = 0, \ldots, r - 1 \) satisfying Equation 3.2 which correspond to \( c \) elements \( g^\alpha h^\beta k^i \in \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c \). Hence there are not enough elements in \( \text{SL}(3, \mathbb{C}) \) for \( A \) to have a subgroup of the form \( \mathbb{Z}_c \times \mathbb{Z}_c \times \mathbb{Z}_c \). Therefore \( \text{hcf}(a_i, a_j) = 1 \) for at least one pair \( (a_i, a_j) \) and \( A = \mathbb{Z}_{a_i a_j} \times \mathbb{Z}_{a_k} \). \( \square \)

**Proposition 3.2** Let \( A = \mathbb{Z}_R \times \mathbb{Z}_r \) be a finite Abelian subgroup of \( \text{SL}(3, \mathbb{C}) \) such that \( A \times \langle \tau \rangle \) is a trihedral group. Then \( A \) has a subgroup of the form \( \mathbb{Z}_r \times \mathbb{Z}_r \) which can be generated by \( \frac{1}{r}(0, 1, -1) \) and \( \frac{1}{r}(1, -1, 0) \). It maybe that \( r = 1 \).

**Proof.** By the previous proposition we may assume \( r \mid R \). Thus \( A \) has a subgroup of the form \( B = \mathbb{Z}_r \times \mathbb{Z}_r \). Let \( g_1 = \frac{1}{r}(a, b, c) \) and \( g_2 = \frac{1}{r}(d, e, f) \)
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be generators of the two groups $\mathbb{Z}_r$, and show that $h_1 = \frac{1}{r}(0, 1, -1)$ and $h_2 = \frac{1}{r}(1, -1, 0) \in \mathbb{Z}_r \times \mathbb{Z}_r$.

Claim 3.3 shows that $ae - bd$ and $r$ are coprime and so $ae - bd$ has a multiplicative inverse modulo $r$. Hence

$$
\lambda a - \mu d \equiv 1 \pmod{r}
$$

$$
\lambda b - \mu e \equiv r - 1 \pmod{r}
$$

$$
\lambda c - \mu f \equiv 0 \pmod{r}
$$

has a solution. The final equation is implied by the first two. Similarly there is a solution to

$$
\lambda a - \mu d \equiv 0 \pmod{r}
$$

$$
\lambda b - \mu e \equiv 1 \pmod{r}
$$

$$
\lambda c - \mu f \equiv r - 1 \pmod{r}.
$$

Therefore $\langle h_1, h_2 \rangle \subset \langle g_1, g_2 \rangle$. However both groups are of order $r^2$ so must be equal. □

**Claim 3.3** If $\langle \frac{1}{r}(a, b, c) \rangle = B_1$ and $\langle \frac{1}{r}(d, e, f) \rangle = B_2$ such that $B_1 \times B_2 \cong \mathbb{Z}_r \times \mathbb{Z}_r$ then $ae - bd$ and $r$ are coprime.

**Proof.** Let $g_1 = \frac{1}{r}(a, b, c)$ and $g_2 = \frac{1}{r}(d, e, f)$ then the only solution to $g_1^\lambda = g_2^\mu$ is $r \mid \lambda$ and $r \mid \mu$ because $B_1 \cap B_2 = 1$. Thus the only solution to the following set of equations is $\lambda \equiv \mu \equiv 0 \pmod{r}$.

$$
\lambda a - \mu d \equiv 0 \pmod{r}
$$

$$
\lambda b - \mu e \equiv 0 \pmod{r}
$$

$$
\lambda c - \mu f \equiv 0 \pmod{r}
$$

(3.3)

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If \( \text{hcf}(ae - bd, r) = s > 1 \) then there exist \( \xi \) and \( \eta \) such that \( \xi s = ad - be \) and \( \eta s = r \). This implies that \( \lambda = \eta d \) and \( \mu = \eta a \) is a solution to Equation 3.3 and so is \( \lambda = \eta e \) and \( \mu = \eta b \). Thus \( r \) divides each of \( \eta a, \eta b, \eta e \) and \( \eta d \) and also \( \eta c \) and \( \eta f \). If \( r \mid \eta \) then \( r = \eta \) which contradicts the assumption that \( s > 1 \). So \( r = pq \) where \( p \mid \eta \) and \( q \) divides \( a, b \) and \( c \) thus \( \frac{1}{s}(a, b, c) = \frac{1}{s}(a/q, b/q, c/q) \) which contradicts the assumption that \( \frac{1}{s}(a, b, c) \) generates a group of order \( r \).

\[ \square \]

**Proposition 3.4** Let \( G \) be a trihedral group with a cyclic normal subgroup \( \mathbb{Z}_s \). Then \( \mathbb{Z}_s \) is generated by \( \frac{1}{s}(1, k, k^2) \) where \( k^2 \) means take the smallest residue of \( k^2 \) (mod \( s \)) and \( s \mid 1 + k + k^2 \).

**Proof.** Let \( \frac{1}{s}(a, b, c) \) be the generator of \( \mathbb{Z}_s \). First show that \( a, b \) and \( c \) are coprime to \( s \). Now \( \mathbb{Z}_s < G \) so \( \frac{1}{s}(b, c, a), \frac{1}{s}(c, a, b) \in \mathbb{Z}_s \). That is, there exists \( k \) such that \( ak \equiv b, bk \equiv c \) and \( ck \equiv a \) (mod \( s \)).

Now suppose that \( \text{hcf}(a, s) = t > 1 \). Then \( t \mid ak = b + ms \) so \( t \mid b \) and similarly \( t \mid c \). Therefore \( \frac{1}{s}(a, b, c) = \frac{1}{s/t}(a/t, b/t, c/t) \) and the group is of order \( s/t \). So \( a, b \) and \( c \) must be coprime to \( s \) and \( \lambda a \equiv 1 \) (mod \( s \)) has a solution. Therefore \( \frac{1}{s}(1, k, c') \) is a generator of \( \mathbb{Z}_s \) where \( k \equiv \lambda b \) and \( c' \equiv \lambda c \equiv k^2 \) (mod \( s \)). Finally \( s = 1 + k + c' \equiv 1 + k + k^2 \) (mod \( s \)) and so \( s \mid 1 + k + k^2 \).

\[ \square \]

**Proposition 3.5** The order of \( A \), the maximal Abelian normal subgroup of \( G \), is not equal to \( 2 \) modulo \( 3 \).

**Proof.** Compare with [Ito95, Prop 3.1]. The centre of \( G \), denoted by \( Z(G) \), is the subgroup \( \langle \frac{1}{s}(1, 1, 1) \rangle \cap A \) of \( A \). It has order either 1 or 3. If an element \( g \in A \) is not in the centre of \( G \) then it belongs to a conjugacy class of \( G \)
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containing precisely three distinct elements $g, \tau g \tau^{-1}, \tau^2 g \tau^{-2}$ of $A$. Therefore $|A| - |Z(G)|$ is divisible by 3 and so $|A| \equiv |Z(G)| \pmod{3}$. 

\[\square\]

**Remark 3.6** This means that if $p \in \mathbb{Z}$ divides $1+k+k^2$ then $p \not\equiv 2 \pmod{3}$. This is because if $p \mid 1+k+k^2$ then $\left(\frac{1}{p}(1, \tau, k^2)\right) \times \mathbb{Z}_3$ is a trihedral group. By this proposition $p \not\equiv 2 \pmod{3}$.

### 3.1.1 Representation theory of $G$

First fix the following notation. Let $A = \mathbb{Z}_R \times \mathbb{Z}_r$ where $r \mid R$, and the possibilities that $r = R$ and $r = 1$ are allowed. Let $g = \frac{1}{R}(a, b, c)$ be the generator of $\mathbb{Z}_R$ and $h = \frac{1}{r}(1, -1, 0)$ the generator of $\mathbb{Z}_r$. Let $\varepsilon$ be a primitive $R$th root of unity, $\xi$ a primitive $r$th root of unity and $\omega$ a primitive 3rd root of unity.

If $|A| \equiv 1 \pmod{3}$, then $G$ has three 1 dimensional representations and $(|A| - 1)/3$ irreducible 3 dimensional representations. In this case only the identity is in the centre of $G$. The remaining elements of $A$ can be decomposed into conjugacy classes of $G$

\[
\left( g^ih^j = \frac{1}{tr}(iar + jtr, ibr - jtr, icr), \tau g^ih^j \tau^{-1}, \tau^2 g^ih^j \tau^{-2} \right).
\]

For each distinct conjugacy class define a 3 dimensional representation $V_{i,j}$ by the following formula

\[
V_{i,j}(g) = \begin{pmatrix}
\varepsilon^{ia} \xi^j & 0 & 0 \\
0 & \varepsilon^{ib} \xi^{-j} & 0 \\
0 & 0 & \varepsilon^{ic}
\end{pmatrix}
\quad \text{and} \quad
V_{i,j}(\tau) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

The three 1 dimensional representations $L_i$ are given by

\[
L_i(g) = 1 \quad \text{and} \quad L_i(\tau) = \omega^j.
\]
If \(|A| \equiv 0 \pmod{3}\), then the centre of \(G\) is \(Z(G) = \langle \frac{1}{3}(1, 1, 1) \rangle\). There are nine 1 dimensional representations and \((|A| - 3)/3\) irreducible 3 dimensional representations. As above the conjugacy class of each element of \(A\) not in the centre defines a 3 dimensional representation \(V_{i,j}\). The nine 1 dimensional representations are given by

\[
L_{i,j}(g) = \omega^i \quad \text{and} \quad L_{i,j}(\tau) = \omega^j.
\]

### 3.2 G-graphs

**Definition 3.7** A polynomial or triple of polynomials is said to belong to one of the irreducible representations of \(G\) if one of the following conditions hold.

- A polynomial \(f \in \mathbb{C}[x, y, z]\) belongs to \(L_{i,j}\) if \(g(f) = \omega^i f\) and \(\tau(f) = \omega^j f\).
- For a polynomial \(f \in \mathbb{C}[x, y]\), the triple of polynomials \((f, \tau(f), \tau^2(f))\) belongs to \(V_{i,j}\) if \(g(f, \tau(f), \tau^2(f)) = (\varepsilon^m \xi^j f, \varepsilon^{2m} \xi^{-j} \tau(f), \varepsilon^{4m} \tau^2(f))\).

Let \(h \in \mathbb{C}[x, y, z]\) be a monomial such that \(g(h) = \omega^i h\), \(\tau(h) \neq h\) then \(f = h + \omega^j \tau(h) + \omega^{2j} \tau^2(h) \in L_{i,j}\). If \(j = 0\) then we do not need the condition that \(\tau(h) \neq h\). In particular

\[
xyz \in L_{0,0},
\]
\[
x^R + \omega y^R + \omega z^R \in L_{0,j},
\]
\[
x^{R/3} + \omega^j y^{R/3} + \omega^{2j} z^{R/3} \in L_{1,j},
\]
\[
x^{2R/3} + \omega^j y^{2R/3} + \omega^{2j} z^{2R/3} \in L_{2,j}.
\]

**Definition 3.8** A subset \(\Gamma \subset \mathbb{C}[x, y, z]\) is called a \(G\)-graph if it contains precisely one polynomial belonging to each 1 dimensional representation and
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three triples of polynomials belonging to each of the irreducible 3 dimensional representations.

If an ideal $I$ in $\mathbb{C}[x,y,z]$ defines a $G$-cluster then $\mathbb{C}[x,y,z]/I \cong \mathbb{C}[G]$. Therefore we can choose a basis for the complex vector space $\mathbb{C}[x,y,z]/I$ which is a $G$-graph. We note that there will never be a unique choice of $G$-graph.

Definition 3.9 Let $I$ be an ideal which defines a $G$-cluster. Denote by $\Gamma(I)$ the set of $G$-graphs which form a basis for the complex vector space $\mathbb{C}[x,y,z]/I$.

Suppose $(f_1,f_2,f_3)$, $(g_1,g_2,g_3)$, $(h_1,h_2,h_3)$ and $(k_1,k_2,k_3)$ are distinct triples of polynomials which belong to some representation $V_{i,j}$. Since $\mathbb{C}[G]$ contains $V_{i,j}$ with multiplicity 3 any ideal $I$ which defines a $G$-cluster must contain three polynomials

$$\alpha f_1 + \beta g_1 + \gamma h_1 + \delta k_1, \quad \alpha f_2 + \beta g_2 + \gamma h_2 + \delta k_2, \quad \alpha f_3 + \beta g_3 + \gamma h_3 + \delta k_3$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. If $(f_1,f_2,f_3)$, $(g_1,g_2,g_3)$ and $(h_1,h_2,h_3)$ belong to $\Gamma$ then $\delta \neq 0$. Similarly for any polynomials $g$ and $h$ belonging to the same 1 dimensional representation there exists a polynomial of the form

$$\alpha g + \beta h$$

in $I$ where $\alpha, \beta \in \mathbb{C}$. If $g \in \Gamma$ then $\beta \neq 0$.

Let $h$ be a polynomial belonging to a 1 dimensional representation. Only if $h$ is zero in $\mathbb{C}[x,y,z]/I$ can $h$ not be chosen to be basic. In other words the only reason we cannot choose $h$ in $\Gamma$ is if $h \in I$. In particular this implies the following claim.

Claim 3.10 For every ideal $I$ which defines a $G$-cluster there exists a $G$-graph $\Gamma \in \Gamma(I)$ such that $1 \in \Gamma$. 

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Therefore every ideal $I$ defining a $G$-cluster must contain polynomials of the form

$$A := x^{\lfloor |A| \rfloor} + y^{\lfloor |A| \rfloor} + z^{\lfloor |A| \rfloor} - a, \quad Q := xyz - q$$

for some $a, q \in \mathbb{C}$.

**Claim 3.11** For every ideal $I$ which defines a $G$-cluster there exists a $G$-graph $\Gamma \in \Gamma(I)$ that contains only polynomials which have monomial terms of the form $x^\lambda y^\mu z^\nu \in \Gamma$ where $\lambda \mu \nu = 0$.

**Proof.** Assume $\lambda \geq \mu \geq \nu > 0$ then repeated use of the polynomial $Q$ shows that the polynomial $x^\lambda y^\mu z^\nu - q^\nu x^\lambda y^\mu z^\nu \in I$. Therefore we may take $x^\lambda y^\mu z^\nu$ to be basic instead of $x^\lambda y^\mu z^\nu$. $\square$

**Claim 3.12** Let a polynomial $h$ belong to some irreducible 3 dimensional representation. For every ideal $I$ which defines a $G$-cluster there exists a $G$-graph $\Gamma \in \Gamma(I)$ such that if $h \in \Gamma$ then $\tau(h)$ and $\tau^2(h) \in \Gamma$.

**Proof.** If there is a polynomial

$$\tau(h) = \alpha_1h_1 - \alpha_2h_2 - \alpha_3h_3 \quad (3.4)$$

where $h_1, h_2, h_3 \in \Gamma$. This implies there exists a polynomial $h - \alpha_1\tau^2(h_1) - \alpha_2\tau^2(h_2) - \alpha_3\tau^2(h_3) \in I$. Now $h$ is basic and so one of the $\alpha_i$ is nonzero. Multiplying Equation 3.4 by $\alpha_i^{-1}$ means we can take $\tau(h)$ to be basic rather than $h_i$. $\square$

**Claim 3.13** For every ideal $I$ which defines a $G$-cluster there exists a $G$-graph $\Gamma \in \Gamma(I)$ such that the triple of polynomials belonging to a three dimensional representation in $\Gamma$ is a triple of monomials.
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**Proof.** Let $(f_1, \tau(f_1), \tau^2(f_1))$, $(f_2, \tau(f_2), \tau^2(f_2))$ and $(f_3, \tau(f_3), \tau^2(f_3))$ be the three copies of $V_{i,j}$ in some $G$-graph $\Gamma$. Consider a monomial term $\alpha_1 x^{\lambda_1} y^{\mu_1}$ of $f_1$. Only if $\alpha_1 x^{\lambda_1} y^{\mu_1} + a_2 f_2 + a_3 f_3 \in I$ can $(f_1, \tau(f_1), \tau^2(f_1))$ not be replaced by $(x^{\lambda_1} y^{\mu_1}, y^{\lambda_1} z^{\mu_1}, z^{\lambda_1} x^{\mu_1})$ in the basis. If this is so then there exists no polynomial of the form $b_1 (f_1 - \alpha_1 x^\beta) + b_2 f_2 + b_3 f_3$ in $I$. Continuing this way we will find a monomial term $\alpha_n x^{\lambda_n} y^{\mu_n}$ of $f_1$ such that $(x^{\lambda_n} y^{\mu_n}, y^{\lambda_n} z^{\mu_n}, z^{\lambda_n} x^{\mu_n})$, $(f_2, \tau(f_2), \tau^2(f_2))$ and $(f_3, \tau(f_3), \tau^2(f_3))$ are linearly independent in $\mathbb{C}[x, y, z] / I$. □

In fact as we will see later we will not always choose triples of monomials in the $G$-graph.

**Claim 3.14** For every ideal $I$ which defines a G-cluster there exists a $G$-graph $\Gamma \in \Gamma(I)$ that satisfies the following conditions.

1. Let $x^i y^j \in \Gamma$, $0 \leq i \leq \alpha$, and $0 \leq j \leq \beta$. If $x^i y^j$ belongs to some 3 dimensional representation then $x^i y^j$ is in $\Gamma$. Otherwise at least one of the three polynomials $x^i y^j + \omega^k y^j z^j + \omega^{2k} z^j x^j$ belongs to $\Gamma$.

2. Let $x^i y^j + \omega y^j z^j + \omega^{2j} z^j x^j \in \Gamma$, $0 \leq i \leq \mu$, $0 \leq j \leq \nu$ and $ij < \mu \nu$. If $x^i y^j$ belongs to some 3 dimensional representation then $x^i y^j \in \Gamma$. Otherwise the three polynomials $x^i y^j + \omega^k y^j z^j + \omega^{2k} z^j x^j$ can belong to $\Gamma$.

**Proof.** We show how to prove Part 1. The proof of Part 2 is similar. Define a set $\Lambda_{M,N}$ as follows.

$$\Lambda_{M,N} := \left\{ x^n y^m, y^n z^m, z^n x^m \mid m \geq M, n \geq N \right\}$$

Let $x^\alpha y^\beta$ be basic, $0 \leq i \leq \alpha$, $0 \leq j \leq \beta$ and $ij < \alpha \beta$. Suppose that $x^i y^j$ is not basic. Let $x^i y^j$ belong to some 3 dimensional representation. Then

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there exists a polynomial

\[ x^i y^j - a_1 f_1 - a_2 f_2 \]

in \( I \). If \( a_1 = a_2 = 0 \) then \( x^i y^j \in I \) and \( x^\alpha y^\beta \in I \). Therefore \( x^\alpha y^\beta \) cannot be basic. From now on assume that \( a_1 \neq 0 \). If \( f_1 \in \Lambda_{i,j} \) then replace \( x^i y^j \) with \( f_1 \) in the basis. Therefore without loss of generality we may assume \( a_1 \neq 0 \) and \( f_1 \notin \Lambda_{i,j} \). It follows that

\[ x^\alpha y^\beta - a_1 f_1 x^{\alpha-i} y^{\beta-j} - a_2 f_2 x^{\alpha-i} y^{\beta-j}. \]

If \( z \nmid f_1 \) then \( f_1 x^{\alpha-i} y^{\beta-j} \notin \Lambda_{\alpha,\beta} \) and we may replace \( x^\alpha y^\beta \) with \( f_1 x^{\alpha-i} y^{\beta-j} \) in our basis. If \( f_1 x^{\alpha-i} y^{\beta-j} \) is already basic then \( a_2 \) must also be nonzero and repeat this argument for \( f_2 \).

If \( z \mid f_1 \) then use the relation \( Q \) to replace \( f_1 x^{\alpha-i} y^{\beta-j} \) a monomial of the form \( x^k y^\eta z^\zeta \) where \( \xi \eta \zeta = 0 \). This is also not in \( \Lambda_{\alpha,\beta} \) and we may replace \( x^\alpha y^\beta \) with \( x^k y^\eta z^\zeta \) in the basis.

Suppose \( x^i y^j + \omega^k y^j z^i + \omega^2 z^i y^j \) belongs to one of the 1 dimensional representations. If it cannot be chosen to be basic for each \( 0 \leq k \leq 2 \). Then \( x^i y^j + \omega^k y^j z^i + \omega^2 z^i x^j \in I \) for all \( k \) and so \( x^i y^j \in I \). Therefore \( x^\alpha y^\beta \in I \) and so cannot be basic. \( \square \)

**Claim 3.15** For every ideal \( I \) which defines a G-cluster there exists a G-graph \( \Gamma \in \Gamma(I) \) that satisfies the following conditions.

1. If \( x^\lambda y^\mu + \omega y^\lambda z^\mu + \omega^2 z^\lambda x^\mu \in L_{0,1} \) belongs to \( \Gamma \) then \( x^\lambda y^\mu + \omega^2 y^\lambda z^\mu + \omega z^\lambda x^\mu \in L_{0,2} \) belongs to \( \Gamma \).

2. Similarly if \( x^\lambda y^\mu + \omega^i y^\lambda z^\mu + \omega^{2j} z^\lambda x^\mu \in L_{i,j} \) for \( i = 1, 2 \) and some \( j = 1, 2 \) or 3 then \( x^\lambda y^\mu + \omega^j y^\lambda z^\mu + \omega^{2j} z^\lambda x^\mu \) in \( \Gamma \) for all \( j \).
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Proof. Again we prove the first part. The proof of the second part is analogous. Note that $x^R + y^R + z^R \in L_{0,0}$ so does not belong to $\Gamma$ we therefore assume $\lambda \leq R$ and $\mu < R$. If $\lambda = R$ then $\mu = 0$ because otherwise $x^R + y^R + z^R \in \Gamma$ by Claim 3.14. Note that $\mathbb{C}[x, y, z]^A \subset \mathbb{C}[x^r, y^r, z^r, xyz]$ and so $r \mid \lambda$ and $r \mid \mu$.

Suppose that

$$x^\lambda y^\mu + \omega y^\lambda z^\mu + \omega^2 z^\lambda x^\mu \in L_{0,1}$$

belongs to $\Gamma$ but $x^\lambda y^\mu + \omega y^\lambda z^\mu + \omega z^\lambda x^\mu \in L_{0,2}$ does not belong to $\Gamma$. This means that $x^\lambda y^\mu + \omega^2 y^\lambda z^\mu + \omega z^\lambda x^\mu = 0$. Thus

$$x^\eta y^\xi + \omega y^\eta z^\xi + \omega z^\eta x^\xi \in L_{0,2}$$

belongs to $\Gamma$. Without loss of generality we assume that $\mu < \zeta < R$ and $\eta < \lambda \leq R$.

If $\mu = 0$ then $\lambda = R$ and assume also that $r = R$. Then $\mathbb{C}[x, y, z]^A = \mathbb{C}[x^r, y^r, z^r, xyz]$ and so $x^r y^r \mid x^\eta y^\xi$. Thus $(x^r, y^r, z^r)$ must be contained in the vector space spanned by $\Gamma$ by Claim 3.14. Therefore $x^r + y^r + z^r \in \Gamma$ and $\Gamma$ contains two polynomials belonging to the trivial representation and so $\mathbb{C}[x, y, z] / I \neq \mathbb{C}[G]$.

Now assume $\mu = 0$ and $R > r > 1$. Since $\mathbb{C}[x, y, z]^A \subset \mathbb{C}[x^r, y^r, z^r, xyz]$ it follows that $r \mid \eta$ and $r \mid \zeta$. The triples of monomials

$$(x^{R-r}, y^{R-r}, z^{R-r}), \quad (z^{R-kr}, x^{R-kr}, y^{R-kr}), \quad (y^{R-k^2r}, z^{R-k^2r}, x^{R-k^2r})$$

all belong to the same representation $V$. These three triples are distinct if $k \neq 1$ and are basic by Claim 3.14. If $k = 1$ the three polynomials

$$x^{R-r} + \omega y^{R-r} + \omega^2 z^{R-r}$$

belong to the representations $L_{2,3}$. 

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Now consider the triples
\[
(x^{\eta-r}y^\zeta, y^{\eta-r}z^{\zeta}, z^{\eta-r}x^\zeta), \quad (x^{\eta}x^{\zeta-r}, x^{\eta}y^{\zeta-r}, y^{\eta}z^{\zeta-r}).
\]
These also belong to the same representation \( V \) and are basic by Claim 3.14. At least one of these triples is distinct from the three given above unless \( r = \eta = \zeta \). If \( r = \eta = \zeta \) then \( x^r y^r \) is invariant under \( A \). In particular \( x^r y^r \) is invariant under \( \left( \frac{1}{R}(a, b, c) \right) \) and so \( \lambda(a + b) \equiv 0 \) for all \( 1 \leq \lambda \leq R \) which means \( c = 0 \). For \( A \), which is generated by \( \frac{1}{R}(1, -1, 0) \) and \( \frac{1}{R}(a, b, c) \), to be a normal subgroup it must contain the element \( \frac{1}{R}(c, a, b) \). This is clearly impossible since \( c = 0 \).

Therefore if \( k > 1 \) we have at least four triples of monomials in \( \Gamma \) belonging to the same 3 dimensional representation. If \( k = 1 \) the polynomials
\[
x^{\eta-r}y^\zeta + \omega^{i}y^{R-r}z^\zeta + \omega^{2i}z^{\eta-r}x^\zeta, z^{\eta}x^{\zeta-r} + \omega^{i}x^{\eta}y^{\zeta-r} + \omega^{2i}y^{\eta}z^{\zeta-r} \in L_{2,i}
\]
and at least one of these polynomials is distinct from \( x^{R-r} + \omega^{i}y^{R-r} + \omega^{2i}z^{R-r} \) so \( \Gamma \) contains two polynomials belonging to each of the 1 dimensional representations \( L_{2,i} \). In either case \( \mathbb{C}[x, y, z] / I \not\cong \mathbb{C}[G] \).

Now assume \( \mu = 0 \) and \( r = 1 \). If \( x^R + \omega y^R + \omega^2 z^R \in \Gamma \) then
\[
\{1, (x, y, z), \ldots, (x^{R-1}, y^{R-1}, z^{R-1}), x^R + \omega y^R + \omega^2 z^R \} \subset \Gamma.
\]
This set is missing only one polynomial belonging to \( L_{0,2} \). By Claim 3.14 we are only able to add a polynomial which is a linear combination of the monomials \( xy, yz, zx \) or \( x^R + \omega^2 y^R + \omega z^R \in \Gamma \). Since \( xy, yz, zx \not\in \mathbb{C}[x, y, z]^A \) no linear combination of these monomials can be in \( L_{0,2} \) so \( \eta = R \) and \( \zeta = 0 \).

Finally if \( \mu > 0 \) then by Claim 3.14 the four distinct triples of monomials
\[
(x^{\lambda-1}y^{\mu}, y^{\lambda-1}z^{\mu}, z^{\lambda-1}x^{\mu}), \quad (z^{\lambda}x^{\mu-1}, x^{\lambda}y^{\mu-1}, y^{\lambda}z^{\mu-1}),
\]
\[
(x^{\eta-1}y^{\zeta}, y^{\eta-1}z^{\zeta}, z^{\eta-1}x^{\zeta}), \quad (z^{\eta}x^{\zeta-1}, x^{\eta}y^{\zeta-1}, y^{\eta}z^{\zeta-1})
\]

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are all in $\Gamma$. Again these triples belong to the same representation. Therefore $\mathbb{C}[x, y, z]/I \not\cong \mathbb{C}[G]$. □

In the next definition we construct set $S$ and show how to construct from this set a $G$-graph. Let $T := \{x^i y^j \mid i > 0, j \geq 0\}$ and $S \subset T$.

Definition 3.16 Define a set $S$ as follows.

- For each irreducible 3 dimensional representation $V_{i,j}$ take three monomials in $S$ which belong to $T \cap V_{i,j}$.

- Take one monomial $f_0$ in $S$ belonging to $T$ which is invariant under $A$.

- If $|A| \equiv 0 \pmod{3}$ take one monomial $f_1$ in $S$ belonging to $T$ such that $g(f_1) = \omega f_1$ and one monomial $f_2$ in $S$ belonging to $T$ such that $g(f_2) = \omega^2 f_2$.

We say $S$ is a $G$-set if it satisfies the following conditions

1. If $x^i y^j \in S$ then $x^i, x^i y^j \in S$ for all $1 \leq i \leq \eta$ and $0 \leq j \leq \zeta$.

2. Let $f_0 = x^\lambda y^\mu \in S$ be invariant under $A$. If $\mu \neq 0$ then no element of the form $x^\alpha y^\beta f_0 \in S$ for $\alpha \beta \neq 0$. Either take neither of the monomials $x^\alpha f_0, y^\alpha f_0$ or both. If we take both of them then count them with multiplicity 1/2 in $S$.

3. If $f_0 = x^R \in S$ is invariant under $A$ then take none, two or three of the elements in $x^{\alpha+R}, x^R y^\alpha, x^\alpha y^R$. If two of them then count each with multiplicity a 1/2 in $S$. If three of them then count each with multiplicity 2/3 in $S$.

For $i, j > 0$ either take neither of the monomials $x^{i+R} y^{j+1}, x^{i+1} y^{R+j}$ or both. If both then count each with multiplicity a 1/2 in $S$. 

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3.2 $G$-graphs

We now show how $S$ generates a $G$-graph. Fix a $G$-set $S$. In the following algorithm we define a $G$-graph, $\Gamma(S)$, and a set $\mathcal{B}(S) \supset \Gamma(S)$ which is a $\mathbb{C}$-basis for the vector space $\mathbb{C}[x, y, z]$.

**Algorithm 3.17**

1. Take $1 \in \Gamma(S)$.

2. For $f_0 \in S$ take the polynomials

   \[ F_j := f_0 + \omega^j \tau(f_0) + \omega^{2j} \tau^2(f_0) \in \Gamma(S) \]

   for $j = 1, 2$ and

   \[ F_0 := f_0 + \tau(f_0) + \tau^2(f_0) \in \mathcal{B}(S). \]

   These belong to the representation $L_{0,j}$.

3. If $|A| \equiv 0 \pmod{3}$ take the polynomials

   \[ f_i + \omega^j \tau(f_i) + \omega^{2j} \tau^2(f_i) \in \Gamma(S) \]

   for $i = 1, 2$ and $j = 0, 1, 2$. These belong to $L_{i,j}$.

4. For $f \in S$ such that $f$ belongs to one of the 3 dimensional representations, $V_{i,j}$, and $f_0 \mid f$ and $y^k \mid f_0$ take

   \[ f, \tau(f), \tau^2(f) \in \Gamma(S). \]

   These monomials base a copy of $V_{i,j}$.

5. For $f \in S$ such that $f$ belongs to one of the 3 dimensional representations, $V_{i,j}$ and $f_0 = x^\lambda y^\mu \mid f$ where $\mu > 0$. Then $x^\alpha f_0$ and $y^\alpha f_0$ belong to $S$. In this case take

   \[ x^\alpha (f_0 - \tau^2(f_0)), y^\alpha (\tau(f_0) - f_0), z^\alpha (\tau^2(f_0) - \tau(f_0)) \in \Gamma(S) \]
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and

$$x^\alpha (f_0 + \tau^2(f_0)), y^\alpha (\tau(f_0) + f_0), z^\alpha (\tau^2(f_0) + \tau(f_0)) \in \mathcal{B}(S).$$

$$x^\alpha \tau(f_0), y^\alpha \tau^2(f_0), z^\alpha f_0$$

Each of these polynomials base a copy of $V_{i,j}$.

6. For $f \in S$ such that $f$ belongs to one of the 3 dimensional representations $f_0 = x^R \mid f$ or $y^R \mid f$ and $f \in \{x^{R+\alpha}, x^R y^\alpha, x^\alpha y^R\}$. If the three monomials in this set belong to $S$ then take

$$x^\alpha F_1, y^\alpha \omega^2 F_1, z^\alpha \omega F_1 \in \Gamma(S)$$

$$x^\alpha F_2, y^\alpha \omega F_2, z^\alpha \omega^2 F_2$$

and

$$x^\alpha F_0, y^\alpha F_0, z^\alpha F_0 \in \mathcal{B}(S).$$

If only two monomials in $\{x^{R+\alpha}, x^R y^\alpha, x^\alpha y^R\}$ belong to $S$ then take

$$x^\alpha F_1, y^\alpha \omega^2 F_1, z^\alpha \omega F_1 \in \Gamma(S)$$

and

$$x^\alpha F_0, y^\alpha F_0, z^\alpha F_0 \in \mathcal{B}(S).$$

$$x^\alpha F_2, y^\alpha \omega F_2, z^\alpha \omega^2 F_2$$

7. Finally for $f \in S$ such that $f$ belongs to one of the 3 dimensional representations, $f_0 = x^R$ and $f = x^\alpha y^\beta f_0$ where $\alpha \beta \neq 0$. Then $x^\alpha y^\beta y^R \in S$. Take

$$x^\alpha y^\beta (f_0 - \tau(f_0)), y^\alpha z^\beta (\tau(f_0) - \tau^2(f_0)), z^\alpha x^\beta (\tau^2(f_0) - f_0) \in \Gamma(S)$$

$$x^\alpha y^\beta (f_0 + \tau(f_0)), y^\alpha z^\beta (\tau(f_0) + \tau^2(f_0)), z^\alpha x^\beta (\tau^2(f_0) + f_0)$$

$$x^\alpha y^\beta \tau^2(f_0), y^\alpha z^\beta f_0, z^\alpha x^\beta \tau(f_0) \in \mathcal{B}(S)$$
3.2 $G$-graphs

8. Take every element in $\Gamma_T$ in $\mathcal{B}(S)$. If $m$ is any monomial in $\mathbb{C}[x, y, z]$ which does not belong to the vector space spanned by the elements already in $\mathcal{B}(S)$ then take $m \in \mathcal{B}(S)$.

Denote by $I(S)$ or sometimes $I(\Gamma(S))$ the ideal generated by the elements in $\mathcal{B}(S) \setminus \Gamma(S)$.

**Proposition 3.18** An ideal $I$ defines a $G$-cluster if and only if there exists a $G$-set $S$ such that $\Gamma(S) \in \Gamma(I)$.

**Proof.** Let $S$ be a $G$-set. The set $\Gamma(S)$ contains three triples of monomials in each 3 dimensional representation and one polynomial in each 1 dimensional representation so $\Gamma(S)$ is a $G$-graph. By construction the ideal $I(S)$, generated by the elements in $\mathcal{B}(S) \setminus \Gamma(S)$, contains no element of $\Gamma(S)$. Therefore $\Gamma(S)$ is a basis for $\mathbb{C}[x, y, z] / I(S)$ and so $I(S)$ defines a $G$-cluster.

Now let $I$ be an ideal defining a $G$-cluster and $\Gamma$ its $G$-graph.

- By Claim 3.10 we can always take 1 in $\Gamma$.

- By Claim 3.11 we never have to take a polynomial containing a monomial of the form $x^\lambda y^\mu z^\nu$ in $\Gamma$ for $\lambda \mu \nu \neq 0$.

- By Claim 3.12 if a monomial $x^\eta y^\xi \in \Gamma$ then we can choose $y^\rho z^\zeta$ and $z^\eta x^\zeta$ in $\Gamma$.

- By Claim 3.13 we may assume that all the polynomials belonging to the 3 dimensional representations are monomial.

- By Claim 3.14 we may assume that if $x^\lambda y^\mu \in \Gamma$ or $x^\lambda y^\mu + \omega^j y^\lambda x^\mu + \omega^{2j} z^\lambda x^\mu$ then $x^j y^k \in \Gamma$ for all $i \leq \lambda$, $j \leq \mu$ and $ij < \lambda \mu$.

- By Claim 3.15 the polynomials belonging to $L_{0,1}$ and $L_{0,2}$ can be chosen to be $x^\lambda y^\mu + \omega^j y^\lambda z^\mu + \omega^{2j} z^\lambda x^\mu$ in $\Gamma$. We may also assume that the
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polynomials belonging to $L_{1,j}$, can also be chosen to be of the form $x^\lambda y^\mu + \omega^j y^\lambda z^\mu + \omega^{2j} z^\lambda x^\mu$ for some $x^\lambda y^\mu$ such that $g(x^\lambda y^\mu) = \omega x^\lambda y^\mu$. Similarly for $L_{2,j}$.

Therefore we may assume that $\Gamma$ is of the form $\Gamma = \Gamma(S)$ for some $G$-set $S$. □

**Claim 3.19** Let $I$ be an ideal which defines a $G$-cluster then there exists $\Gamma \in \Gamma(I)$ such that the triples $(x^\lambda, y^\lambda, z^\lambda)$ are in $\Gamma$ for $0 < \lambda \leq (|A|)^{1/2}$.

**Proof.** Now $\Gamma$ must contain $3|A|$ elements. If one of the triples $(x^\lambda, y^\lambda, z^\lambda)$ for $0 < \lambda \leq (|A|)^{1/2}$ cannot be chosen to be in $\Gamma$ then $x^\lambda, y^\lambda, z^\lambda \in I$ and therefore $x^i, y^j, z^j \in I$ for all $j \geq \lambda$. Thus $\Gamma$ is contained in the set

$$\{ x^i y^j, y^i z^j, z^i x^j \mid 0 \leq i, j < \lambda \}.$$

This has only $1 + 3(\lambda - 1)\lambda < 3|A|$ elements. Therefore the triples $(x^\lambda, y^\lambda, z^\lambda)$ can always be chosen to be in $\Gamma$ for $0 < \lambda \leq (|A|)^{1/2}$. □

**Definition 3.20** If $S$ is a $G$-set then $f \in S$ is called a horn of $S$ if $xf, yf, zf$ do not belong to the corresponding $G$-graph $\Gamma(S)$.

If $f = x^i y^j$ for any $j > 0$ then automatically $zf \not\in \Gamma$ so it only needs to be checked that $xf, yf \not\in S$. For $f = x^i$ to be a horn we also need to check $zf \not\in \Gamma$. This is equivalent to $xyi$ not being in $S$. The horns of $S$ generate $\text{Soc}(I(\Gamma))$ in the same way that a monomial $f \in S$ generates a subset of $\Gamma(S)$ in Algorithm 3.17. This subset defines a submodule of $\mathbb{C}[x, y, z] / I(S)$ which belongs to $\text{Soc}(I(S))$. For example if $f \in V_{i,j}$ is a horn of $S$ and $f_0 \not\in f$ then the submodule generated by $f, \tau(f), \tau^2(f)$ belongs to $\text{Soc}(I(S))$. 64
3.3 Classification of $G$-graphs for the group $\langle \frac{1}{31}(1, 5, 25) \rtimes \mathbb{Z}_3 \rangle$

**Conjecture 3.21** Let $G$ be any trihedral group then $G$-Hilb $\mathbb{C}^3$ is a smooth rational 3 dimensional variety. Then there exists a $G$-set $S$ such that $\Gamma(S) \in \Gamma(I)$ and $S$ has at most 5 horns.

To prove this I try to compute an explicit cover of $G$-Hilb $\mathbb{C}^3$ as in Chapter 2. However the cover depends on the particular family to which the trihedral group belongs and this method does not allow us to construct the general case. It does compute $G$-Hilb $\mathbb{C}^3$ for interesting families of trihedral groups.

### 3.3 Classification of $G$-graphs for the group $\langle \frac{1}{31}(1, 5, 25) \rtimes \mathbb{Z}_3 \rangle$

In this section we classify $G$-graphs when the maximal normal Abelian subgroup $A$ of $G$ is generated by $g = \frac{1}{31}(1, 5, 25)$. We begin by computing the irreducible representations of $G$. For $i = 0, \ldots, 30$ define a 3 dimensional representation $V_i$ by the following formula.

$$V_i(g) = \begin{pmatrix} \varepsilon^i & 0 & 0 \\ 0 & \varepsilon^{5i} & 0 \\ 0 & 0 & \varepsilon^{25i} \end{pmatrix} \quad \text{and} \quad V_i(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The representation $V_0$ decomposes as the sum of three 1 dimensional representations $L_0$, the trivial representation, $L_1$ and $L_2$ which are defined as follows.

$$L_i(g) = 1 \quad \text{and} \quad L_i(\tau) = \omega^i.$$

The remaining representations are irreducible but not distinct. For example $V_1 \cong V_3$. Two representations are isomorphic if and only if they have the same character so the subset

$$\{V_i \mid \text{for } i \in \Lambda = \{0, 1, 2, 3, 4, 6, 8, 11, 12, 16, 17\} \}$$

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is a set of distinct $G$-representations and only the representation $V_0$ is reducible. Thus

$$\mathbb{C}[G] = \bigoplus_{i \in \Lambda \setminus \{0\}} V_i^{\oplus 3} \oplus \bigoplus_{i=0}^{2} L_i.$$
3.3 Classification of G-graphs for the group \( \langle \frac{1}{30}(1, 5, 25) \times \mathbb{Z}_3 \rangle \)

Figure 3.1 shows to which representation \( V_k \) the monomial \( x^iy^j \) belongs for \( i > 0 \) and \( j \geq 0 \). For example \( x^5y \in V_2 \) and \( x^y^5 \in V_6 \). Note that if \( xy^6 \in S_0 \) then \( xy^6 + yz^6 + zx^6 - a \in I \) and so \( x^2y^7 \notin S \) by Claim 3.14.

![Monomial Table]

Figure 3.1: Decomposition of the monomials in \( T \).

For \( k \in \Lambda \) let \( S_k \) be the subset of \( S \) defined by \( S_k := \{ f \mid f \in V_k \cap S \} \).

Note that \( x^y^5 \) cannot be the element in \( S_0 \) because if \( x^y^5 \in S \) then each of the monomials \( x^y, x^y^5, x^y^4, xy \in S \) by Claim 3.14 and therefore there would be 4 monomials in \( S_0 \). For this reason the only possible monomials in \( S_0 \) are \( x^{31} \) and \( xy^6 \). Similarly any monomials in the shaded area of Figure 3.1 cannot be in \( S \) as there would be too many monomials in some \( S_i \).
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### 3.3.1 Step 0

If $x^{31}$ is in $S_0$ then $S_A = \{ x, \ldots, x^{31} \}$ and so the graph which $S_A$ generates is

$$
\Gamma_A = \{ 1, x, \ldots, x^{30}, y, \ldots, y^{30}, z, \ldots, z^{30},
\quad x^{31} + \omega y^{31} + \omega^2 z^{31}, x^{31} + \omega^2 y^{31} + \omega z^{31} \}\). 
$$

The ideal $I(\Gamma_A)$, is as follows.

$$
I(\Gamma_A) = \langle x^{31} + y^{31} + z^{31}, xy, yz, zx \rangle \quad (3.5)
$$

Only $x^{31}$ is a horn of $S_A$ and since $x^{31}$ is invariant under $A$ it generates the two representations $L_1$ and $L_2$ in $\Gamma(S_A)$.

Let $I$ be any ideal with $\Gamma(S_A) \in \Gamma(I)$ as its $G$-graph. Then

$$
A := x^{31} + y^{31} + z^{31} - a
$$

is in $I$ for $a \in \mathbb{C}$. The triple $xy, yz, zx$ is a basis of $V_6$. Now in $\Gamma$ there are three copies of $V_6$ with bases $\{ x^6, y^6, z^6 \}, \{ y^{26}, z^{26}, x^{26} \}$ and $\{ z^{30}, x^{30}, y^{30} \}$. Therefore the following three polynomials

$$
B := xy - b_1 x^6 - c_1 y^{26} - d_1 z^{30}, \\
C := yz - b_1 y^6 - c_1 z^{26} - d_1 x^{30}, \\
D := zx - b_1 z^6 - c_1 x^{26} - d_1 y^{30}
$$

belong to $I$. In order to define an ideal we need polynomials which express every monomial $x^i y, y^i z, z^i x, xy^i, yz^i, zx^i$ for $i < 31$ in terms of the elements in $\Gamma(S_A)$. As in Chapter 2 we can show that these polynomials generate an ideal which defines a $G$-cluster. In other words these polynomials allow any element in $\mathcal{B}(S_A) \setminus \Gamma(S_A)$ to be written as a linear combination of the polynomials in $\Gamma(S_A)$. 

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3.3 Classification of $G$-graphs for the group $\langle \frac{1}{3}(1, 5, 25) \times \mathbb{Z}_3 \rangle$

3.3.2 Step 1

If $x^{31} \notin S$ then $xy^6 \in S$. By Claim 3.14 $xy, \ldots, x^5y \in S$ and so $x^{26}, \ldots, x^{30} \notin S$. If $x^{25} \in S$ then

$$
xy^6 \\
S_B = \\
x^{25}
$$

and $I(\Gamma_B) = \langle xy^6 + yz^6 + zx^6, x^{26}, y^{26}, z^{26}, x^2y, y^2z, z^2x, xyz \rangle$.

Let $I$ be any ideal with $\Gamma(S_B) \in \Gamma(I)$. Again $xy^6 + yz^6 + zx^6$ and $xyz$ belong to $L_0$ so $I$ contains polynomials of the form

$$xy^6 + yz^6 + zx^6 - a, \; xyz - h.$$

The triple $\{y^{26}, z^{26}, x^{26}\}$ is a basis of $V_6$. In $\Gamma$ the three copies of $V_6$ are based by $\{x^6, y^6, z^6\}$, $\{xy, yz, zx\}$ and $\{yz^5, zx^5, xy^5\}$. Therefore $I$ contains

$$
y^{26} - bx^6 - cxy - dy^5, \\
z^{26} - by^6 - cyz - dx^5, \\
x^{26} - bz^6 - cxx - dxy^5.
$$

The triple $\{y^2z, z^2x, x^2y\}$ are a basis for $V_4$. In $\Gamma$ the three copies of $V_4$ are based by $\{x^4, y^4, z^4\}$, $\{y^5, z^7, x^7\}$ and $\{z^{20}, x^{20}, y^{20}\}$. Therefore

$$
y^2z - e_1x^4 - f_1y^7 - g_1z^{20}, \\
z^2x - e_1y^4 - f_1z^7 - g_1x^{20}, \\
x^2y - e_1z^4 - f_1x^7 - g_1y^{20}
$$

belong to $I$. To generate an ideal we also need polynomials that express each monomial $x^iy, y^iz, z^ix$ for $i < 26$ in terms of monomials in $\Gamma(S_B)$.

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The horns of $S_B$ are $xy^6$ and $x^{25}$. Again $xy^6$ is invariant under $A$ so defines two subsets of $\Gamma(S_B)$ corresponding to 1 dimensional representations. The monomial $x^{25}$ defines one subset of $\Gamma(S_B)$ corresponding to a 3 dimensional representation. Thus the horns of $S_B$ correspond to

$$L_1 \oplus L_2 \oplus V_1.$$ 

Now assume that $x^{25} \notin S$. There must still be three elements in $S_1$. The monomials $x, x^5 \in S$ by Claim 3.19. Figure 3.1 shows that the remaining element is $x^5y^4$, $xy^7$ or $x^6y^2$. By Proposition 3.18 the last two monomials will only correspond to one 3 dimensional representation

$$(x \left(xy^6 - zx^6\right), y \left(yz^6 - xy^6\right), z \left(zx^6 - yz^6\right)) \in \Gamma.$$ 

If $x^5y^4 \in S_1$ then all of the elements contained in the shaded area of Figure 3.2 are in $S$.

```
2
1
0 1 2
6 11 16 12 6
12 17 17 11 1 6
16 17 16 2 4 12
11 12 3 8 3 16
6 4 8 8 2 11 12 3
1 2 3 4 1 6 4 8 8 2
```

Figure 3.2: The set obtained by choosing $x^5y^4$ in $S$

For $S$ to be a $G$-set there only needs to be one more element in each of the sets $S_{11}$, $S_{12}$ and $S_{16}$. Figure 3.2 shows that there are two ways of doing
3.3 Classification of $G$-graphs for the group $\langle \frac{1}{25}(1,5,25) \rtimes \mathbb{Z}_3 \rangle$

this. Firstly

\[ xy^6 \]
\[ xy^6 \quad x^2 y^5 \quad x^3 y^5 \quad x^4 y^5 \]
\[ xy^4 \quad x^2 y^4 \quad x^3 y^4 \quad x^4 y^4 \quad x^5 y^4 \]
\[ S_C = \quad x^3 y^3 \quad x^3 y^3 \quad x^4 y^3 \quad x^5 y^3 \]
\[ xy^2 \quad x^2 y^2 \quad x^3 y^2 \quad x^4 y^2 \quad x^5 y^2 \]
\[ xy \quad x^2 y \quad x^3 y \quad x^4 y \quad x^5 y \]
\[ x \quad x^2 \quad x^3 \quad x^4 \quad x^5 \quad x^6 \]

then

\[ I(\Gamma_C) = \langle xy^6 + yz^6 + zx^6, x^7, y^7, z^7, x^6 y, y^6 z, z^6 x, x^5 y^5, y^5 z^5, z^5 x^5, xyz \rangle. \]

Any ideal $I_C$ having $\Gamma(S_C) \in \Gamma(I_C)$ is generated by polynomials that express each of the generators of $I(\Gamma_C)$ as a linear combination of elements in $\Gamma(S_C)$. In other words the generators of $I_C$ are

\[ xy^6 + yz^6 + zx^6 - a, \]
\[ x^7 - bx^4 - cz^4 \quad x^6 y - cy^2 x - f z^5 y^2 - gx^4 z^4, \]
\[ y^7 - bx^4 - cy^2 z - dx^5 x^3 \quad y^6 z - ez^2 y - f x^5 z^2 - gy^4 x^4 \]
\[ z^7 - by^4 - cz^2 x - dx^5 y^3 \quad z^6 x - ex^2 z - f y^5 x^2 - gx^4 y^4 \]
\[ x^5 y^5 - hy^6 - iyz - jx^5 z, \]
\[ y^5 z^5 - hx^6 - izx - jy^5 x, \]
\[ z^5 x^5 - hx^6 - ixy - jz^5 y, \]
\[ xyz - k \]

Note that $x^6$ is not a horn of $\Gamma_C$ since $xy^6 \in \Gamma_C$ and so $x^6 z \neq 0$. The only horns are $xy^6 \in S_0$, $x^4 y^5 \in S_{12}$ and $x^5 y^4 \in S_1$. These correspond to the representations

\[ L_1 \oplus L_2 \oplus V_{12} \oplus V_1. \]
Explicit construction of $G$-Hilb $\mathbb{C}^3$ for trihedral groups

Alternatively if

$$
S_D = \begin{array}{c}
y^6 \\
y^5 \\
x^4y^2 \\
x^4 \end{array}
$$

then the ideal is

$$
I(\Gamma_D) = \langle xy^6 + yz^6 + zx^6, x^7, y^7, z^7, x^6y^4, y^6z^4, z^6x^4, x^2y^5, y^2z^5, z^2x^5, xyz \rangle.
$$

As with $\Gamma(S_C)$ any ideal $I_D$ having $\Gamma(S_D) \in \Gamma(I_D)$ is generated by polynomials that express each of the generators of $I(\Gamma_D)$ as a linear combination of elements in $\Gamma(S_D)$.

The four horns of $S_D$ correspond to the representations

$$L_1 \oplus L_2 \oplus V_1 \oplus V_6 \oplus V_{12}.$$

### 3.3.3 Step 2

All the remaining clusters must have $x^7y$ and $x^6y^2$ in $S_1$. Now consider the space $S_2$, the monomial $x^2 \in S_2$. If $x^{19} \in S$ then

$$
S_E = \begin{array}{c}
y^7 \\
y^6 \\
x^2y^6 \\
x^6 \\
x^2y \\
x^2 \\
\vdots \\
x^{19}.
\end{array}
$$
3.3 Classification of $G$-graphs for the group $\langle \frac{1}{5}(1,5,25) \rtimes \mathbb{Z}_3 \rangle$

Then

$$I(\Gamma_E) = \langle xy^6 + yz^6 + zx^6, x^{20}, y^{20}, z^{20}, x^3y, y^3z, z^3x, xyz \rangle.$$ 

As with $\Gamma(S_A)$ and $\Gamma(S_B)$ a general ideal will have the generators of $I(\Gamma_E)$ as well as the monomials $x^iy, y^iz, z^ix$ for $i < 20$ expressed as a linear combination of the elements of $\Gamma(S_E)$.

The two horns of $S_E$ correspond to the representations

$$V_1 \oplus V_2.$$ 

If $x^{19} \notin S_2$ then consider the possibilities for the remaining two elements in $S_2$. From Figure 3.3 it follows that $xy^8$, $x^6y^3$, $x^{10}$, $x^4y^3$ and $x^5y$ are the other possibilities for monomials in $S_2$. As before the first two monomials

![Figure 3.3: If $x^2y^6 \in S$ and $x^{19} \notin S$](image)

count as one element in $S$. The following pairs cannot be contained in $S$ as it would mean having too many elements in $S$: $\{x^5y, x^{10}\}$, $\{x^4y^3, x^{10}\}$ or $\{x^4y^3, xy^8, x^3y^6\}$.

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If the monomials $xy^8, x^6y^3, x^{10} \in S_2$ then

$$S_F = \begin{bmatrix}
xy^8 \\
xy^7 \\
x^6y^6 \\
x^2y^6 \\
x^3y^6 \\
x^2y \\
x^2y \\
x^3y \\
x^2 \\
x^3 \\
\vdots
\end{bmatrix}$$

$$I(\Gamma_F) = \langle xy^6 + yz^6 + zx^6, x^{14}, y^{14}, z^{14}, x^4y, y^4z, z^4x, xyz \rangle.$$  

Any ideal having $\Gamma(S_F)$ for its $G$-graph is generated by polynomials that express each of the generators of $I(\Gamma_F)$ as a linear combination of elements in $\Gamma(S_F)$. The two horns of $S_F$ correspond to the representations $V_2 \oplus V_3$.

Let $x^5y, x^4y^3 \in S_2$ then the shaded region in Figure 3.4 is in $S$.

![Figure 3.4: If $x^5y, x^4y^3 \in S$](image)

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3.3 Classification of $G$-graphs for the group $\langle \frac{1}{12}(1, 5, 25) \times \mathbb{Z}_3 \rangle$

Note that there are already three monomials in $S_8$ so $x^8 \notin S$. Similarly there are already three monomials in $S_4$ so $x^5y^3 \notin S$. To make this into a $G$-set there must be one more monomial in each of the spaces $S_3$, $S_{11}$, $S_{12}$, $S_{16}$ and $S_{17}$. The only choice for a monomial in $S_3$ is $x^3y^2$ and then there are four possible ways of adding the others.

The first possibility is that

$$S_G = \begin{align*}
x y^7 \\
x y^6 & \quad x^2 y^6 \\
x y^5 & \quad x^2 y^5 \\
x y^4 & \quad x^2 y^4 \\
x y^3 & \quad x^2 y^3 \\
x y^2 & \quad x^2 y^2 \\
x y & \quad x^2 y \\
x & \quad x^2 \end{align*}$$

so

$$I(G) = \langle x y^6 + y z^6 + z x^6, y^8, z^5, x^6 y, y^5 z, z^6 x, x^3 y^3, y^5 z^3, z^5 x^3, x y z \rangle$$

Any ideal having $\Gamma(S_G)$ for its $G$-graph is generated by polynomials that express each of the generators of $I(G)$ as a linear combination of elements in $\Gamma(S_G)$. The three horns of $S_G$ correspond to the representations

$$V_1 \oplus V_3 \oplus V_12.$$
Explicit construction of $G$-Hilb $\mathbb{C}^3$ for trihedral groups

If

$$S_H =\begin{cases} xy^7 \\ xy^6 \ x^2y^5 \\ xy^5 \ x^2y^6 \\ xy^3 \ x^2y^5 \ x^3y^6 \\ xy \ x^2y \ x^3 \ x^4 \ x^5 \ y \ x^7y \\ x \ x^2 \ x^3 \ x^4 \ x^5 \ x^6 \ x^7, \end{cases}$$

then

$$I(\Gamma_H) = \left\{ xy^6 + yz^6 + zx^6, y^2x^6, z^2y^6, x^2z^6, x^8, y^8, z^8, \right\}.$$  

Any ideal having $\Gamma(S_H)$ for its $G$-graph is generated by polynomials that express each of the generators of $I(\Gamma_H)$ as a linear combination of elements in $\Gamma(S_H)$. The five horns of $S_H$ correspond to the representations

$$V_1 \oplus V_2 \oplus V_3 \oplus V_{12} \oplus V_{16}.$$  

If

$$S_I =\begin{cases} xy^7 \\ xy^6 \ x^2y^6 \\ xy^5 \ x^2y^5 \\ xy^4 \ x^2y^4 \ x^3y^4 \\ xy^3 \ x^2y^3 \ x^3y^3 \ x^4y^3 \\ xy^2 \ x^2y^2 \ x^3y^2 \ x^4y^2 \ x^5y^2 \ x^6y^2 \\ xy \ x^2y \ x^3y \ x^4y \ x^5y \ x^6y \ x^7y \\ x \ x^2 \ x^3 \ x^4 \ x^5 \ x^6 \ x^7, \end{cases}$$

then

$$I(\Gamma_I) = \left\{ xy^6 + yz^6 + zx^6, x^8, y^8, z^8, y^2x^7, z^2y^7, x^2z^7, \right\}.$$  

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3.3 Classification of G-graphs for the group \( \langle \frac{1}{10}(1, 5, 25) \times \mathbb{Z}_3 \rangle \)

Any ideal having \( \Gamma(S_t) \) for its G-graph is generated by polynomials that express each of the generators of \( I(\Gamma_t) \) as a linear combination of elements in \( \Gamma(S_t) \). The five horns of \( S_t \) correspond to the representations

\[
V_1 \oplus V_2 \oplus V_{12} \oplus V_{16} \oplus V_{17}.
\]

Finally if

\[
S_J = \begin{align*}
xy^7 \\
xy^6 & \quad x^2 y^6 \\
xy^5 & \quad x^2 y^5 \\
x^4 y^4 & \\
x^3 y^3 & \quad x^3 y^3 & x^4 y^3 \\
x^2 y^2 & \quad x^3 y^2 & x^4 y^2 & x^5 y^2 & x^6 y^2 & x^7 y^2 \\
xy & \quad x^2 y & x^3 y & x^4 y & x^5 y & x^6 y & x^7 y \\
x & \quad x^2 & x^3 & x^4 & x^5 & x^6 & x^7.
\end{align*}
\]

Then

\[
I(\Gamma_J) = \langle xy^6 + yz^6 + zx^6, x^8, y^8, z^8, x^5 y^3, y^6 z^3, z^5 x^3, y^4 x^3, z^4 y^3, x^4 z^3, xy z \rangle.
\]

Any ideal having \( \Gamma(S_t) \) for its G-graph is generated by polynomials that express each of the generators of \( I(\Gamma_J) \) as a linear combination of elements in \( \Gamma(S_J) \). The three horns of \( S_J \) correspond to the representations

\[
V_1 \oplus V_2 \oplus V_{17}.
\]

Finally if \( xy^8, x^6 y^3, x^5 y \in S_2 \) and a monomial must be added to each of the spaces \( S_5, S_{11}, \) and \( S_{12} \). Figure 3.3 shows that there is only one way of
Explicit construction of $G$-Hilb $\mathbb{C}^3$ for trihedral groups

doing this.

\[
\begin{align*}
xy^8 \\
xy^7 \\
x^2y^6 & x^3y^6 \\
x^2y^5 & x^3y^5 \\
S_K & = \begin{array}{c}
xy^4 \\
x^2y^3 & x^3y^3 \\
x^2y^2 & x^3y^2 \\
xy & x^2y & x^3y & x^4y & x^5y & x^6y & x^7y & x^8y \\
x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 & x^8
\end{array}
\end{align*}
\]

Then

\[I(\Gamma_K) = \langle xy^6 + yz^6 + zx^6, x^9, y^9, z^9, x^4y^2, z^2y^4, x^2z^4, xyz \rangle.\]

Any ideal having $\Gamma(S_K)$ for its $G$-graph is generated by polynomials that express each of the generators of $I(\Gamma_K)$ as a linear combination of elements in $\Gamma(S_K)$. The two horns of $S_K$ correspond to the representations

\[V_2 \oplus V_3.\]

Thus by Proposition 3.18 we have classified the $G$-graphs. For any ideal $I$ defining a $G$-cluster there exists $\Gamma \in \Gamma(I) \cap \{\Gamma_A, \ldots, \Gamma_K\}$.

### 3.4 Syzygies

To compute the relations between the generators we have to use two different methods. The first method uses syzygies and is the same as the method used in the previous chapter. The second method uses $A$-Hilb $\mathbb{C}^3$. 
3.4 Syzygies

3.4.1 Using syzygies

We show how to do this in two examples. Let $I_G$ be any ideal which defines a $G$-cluster and has $\Gamma(S_G) \in \Gamma(I_G)$. Then $I_G$ is generated by

$$A = xy^6 + yz^6 + zx^6 - a,$$
$$B = x^6 y - bxy^2 - cz^4 x^4 - dz^5 y^2,$$
$$C = y^6 z - byz^2 - cx^4 y^4 - dx^5 z^2,$$
$$D = z^6 x - bzx^2 - cy^4 z^4 - dy^5 x^2,$$
$$E = x^8 - ex^3 y - fzx^4 - gy^4 z^2,$$
$$F = y^8 - ey^3 z - fyx^4 - gz^4 x^2,$$
$$G = z^8 - ez^3 x - fzy^4 - gx^4 y^2,$$
$$H = x^5 y^3 - hyx^4 - iz^7 - jz^2 x,$$
$$I = y^5 z^3 - hzx^4 - ix^7 - jx^2 y,$$
$$J = z^5 x^3 - hx^4 - iy^7 - jy^2 z,$$
$$K = x^3 yz - k.$$

To find the syzygies between the generator of this ideal first find syzygies for the generators of the ideal $I(\Gamma_G)$. For example the syzygy

$$0 = xE - y^2 B$$

of $I(\Gamma_G)$ becomes

$$0 = xE - y^2 B = -xhy^4 - xiz^7 - jz^2 x^2 + y^4 bx + y^2 cz^4 x^4 + y^4 dz^5$$

for the general $I_G$. This can be written as a linear combination of the monomials in $\Gamma_G$, that is of $xy^4, z^2 x^2$ and $z^5 y^4$ by using the above equations to remove the unwanted terms.

$$0 = xE - y^2 B + izD - (cx^3 yz^3 - c_k x^2 z^2 + idxy^4)K$$
$$= (-h + b - idk) xy^4 + (-j - bi + c_k^2) x^2 z^2 + (d - ci) y^4 z^5$$
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Now since $xy^4, z^2 x^2, z^5 y^4$ form part of a basis for $\mathbb{C}[x,y,z]/I_G$ the coefficients of each of these monomials must be zero so

$$-h + b - idk = 0$$
$$-j - bi + ek^2 = 0$$
$$d - ci = 0.$$  

Consider three other syzygies

$$0 = -x^4 y^2 K + zH + iG$$
$$= k x^4 y^2 - ig x^4 y^2 - h z y^4 - if z y^4 - j z^5 x - i e z^3 x,$$

$$0 = x^3 z^5 K + x^2 z E + y^3 H - x^4 A + i y^2 z A + (k + f)J +$$
$$+ (e x^4 - iy^4 + jy^2 z - ix^5 y z + g x y^3 z^2) K - (i k x^4 - g k y^2 z) K$$
$$= -f i y^7 - h y^7 + a x^4 - f h x^4 - k h x^4 - e k x^4 + i k^2 x^4 -$$
$$i a y^2 z - f j y^2 z - 2 k j y^2 z - g k y^2 z$$

and

$$0 = y E - x^2 B - c z^3 A + cy z G +$$
$$+ (-d x y z^4 + (f + ce - dk) z^3 + cz^2 y^5 + cg x^3 y^2) K$$
$$= -f k z^3 + a c z^3 + d k^2 z^3 - c e k z^3$$
$$+ b x^3 y^2 - e x^3 y^2 - c g k x^3 y^2 - g y^5 z^2 - c f y^5 z^2 - c k z^2 y^5.$$  

These imply the following relations between the generators.

$$a = fh + kh + ek - ik^2\quad b = h + idk\quad d = ci\quad e = b - cgk$$
$$h = -if\quad j = -ie\quad k = ig\quad 0 = -g - cf - ck$$

Thus this affine open set is the smooth three fold in $\mathbb{C}^4_{c,f,g,i}$ defined by the equation

$$g + c g i + c f.$$  

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Let \( I_C \) be any ideal which defines a \( G \)-cluster and has \( \Gamma(S_C) \in \Gamma(I_C) \). Then the generators for \( I_C \) are shown in Equation 3.6.

The syzygies

\[
x^4y^4Q - zK - hH = (-q + hg)x^4y^4 + (i + eh)yz^2 + (j + hf)x^5z^2
\]

\[
(-xy^5 - z^6)Q - z^2B + xzA - dL - cxzQ =
\]

\[
(q + dj)xy^5 + (q + b + dh)z^6 + (-a + di + gq)xz
\]

\[
-x^5N + zG + fyA - fxC + ((e - fc)y - fx^5)Q =
\]

\[
(q + fb + fq)x^5 + (-g + fd)x^4z^5 + (-af - eq + fcq)y
\]

\[
-yB + xG - dz^2H + f z^4yN - x^2y^2dg(xyz + q)Q =
\]

\[
(b + ed - f q)yz^4 + (c - e + dgq^2)x^2y^2 + (-g + df)x^5z^4
\]

\[
-y^4G + xK - yf L + i(1 - f)Q - g(x^3y^3z^3 + qx^2y^2z^2 + q^2xyz + q^3)Q - hf A =
\]

\[
(e - h + fj - hf)y^6x + (-j - hf)x^6z - iq + ifq + gq^4 + hf a
\]

give the relations

\[
q = hg \quad i = -he \quad j = -hf \quad b = -q - hd
\]

\[
a = di + cq \quad g = fd \quad c = e - dgq^2 \quad e = (h + hf + f^2h).
\]

Hence this affine piece is a copy of \( \mathbb{C}^3 \) with coordinates \( d, h, f \).

For the ideals \( I_C, I_H \) and \( I_I \) calculations of this type determine show that this affine piece is a copy \( \mathbb{C}^3 \). For the remaining ideals this method determines only on threefolds in \( \mathbb{C}^n \).
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3.5 Proof of Conjecture 3.21

We prove Conjecture 3.21 when $A = \langle \frac{1}{1+k+kr}(1, k, k^2) \rangle$. In Section 3.6 we develop an algorithm which constructs sets $S$, in Section 3.7 we prove that these sets are $G$-sets and that these are the only $G$-sets and in Section 3.9 we show that the $G$-sets $S$ have at most 5 horns.

The Hirzebruch continued fraction

$$\frac{1 + k + k^2}{k^2 + 1} = 2 - \frac{1}{2 - \frac{1}{\frac{1}{k+1}}} = [2, 2, \ldots, 2, k + 1]$$

can be used to compute the generators of the $A$-invariant monomials contained in $T$. The generators are therefore $x^{1+k+k^2}, x^{k^2+1}y, \ldots, x^{k+1}y, xy^{k+1}$. The monomials 1 and $x^{k+1}z$ also need to be considered.

For $i = 1, \ldots, k-1$ a monomial is in the $i$th representation space of $G$ if it is given by multiplying an invariant monomial by $x^i, y^i$ or $z^i$. Notice $x^m y^n z^i$ and $x^{m-i} y^{n-i}$ belong to the same representation space if $m \geq i$, $n \geq i$. An invariant multiplied by $x^{ik}$ and $x^{ik^2} = x^{k^2-(i-1)(k+1)}$ will also be in $V_i$ but this would mean taking too many elements in $S$ unless the invariant was 1.

If $x^n y^m \in S$ then $nm \leq 1 + k + k^2$. The only monomials which can be in $S_0$ are

$$a_0 := x y^{k+1} \quad e_0 := x^{1+k+k^2}.$$ 

For $i < k$ the only monomials which can be in $S_i$ are

$$a_i := x y^{k+i+1} (= y^i x y^{k+1}) \quad d_i := x^{i+1} y^{k+1} (= x^i x y^{k+1})$$

$$b_i := x^{k-i+1} y^{k-i} (= z^i x^{k+1} y^k) \quad c_i := x^k y^{i-1} (= y^i x^{k+1} z)$$

$$d_i := x^k \quad e_i := x^{k^2-(i-1)(k+1)}$$

$$f_i := x^{2k+1-i} y^{k-1-i} (= z^i x^{2k+1} y^{k-1}) \quad g_i := x^i.$$ 

Let $i = \lfloor (k-1)/2 \rfloor$. If $f_i \in S$ then $b_{i+1}, c_{i+1}, f_{i+1}, g_{i+1} \in S$. This is too many monomials in $S_{i+1}$ so we may assume that $f_j \notin S$ for all $j \leq (k-1)/2$. 

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3.6 Construction of possible sets $S$

Figure 3.5 shows the positioning of these monomials in $T$. The monomials $a_i$ and $a'_i$ count as one element of $S$ by Proposition 3.1.8. By Condition 1 of the same proposition this can only happen if $x^{k+i+1} \in S$.

```
     a_{k-1}
       .
       .
     a_1

     a_0  a'_1  \cdots  a'_{k-1}

       b_{ly}
       b_1

       c_{k-1}

     b_k-1  c_2  f_{k-2}

  g_1  g_2  \cdots  g_{k-1}  c_1  f_{k-1} = e_{k-1}
```

Figure 3.5: The monomials in $T$

Eigenspace will describe the representation space of $A$ and representation space the representation space of $G$ to which a monomial belongs. The monomial $x^iy^j$ belongs to the $(i + jk)$th eigenspace of $A$. The $i$th and $j$th eigenspaces of $A$ are the same representation space of $G$ if $i = j$ or, in the case when $3i \not\equiv 0 \pmod{1+k+k^2}$, also if $i \equiv jk$ or $jk^2 \pmod{1+k+k^2}$.

### 3.6 Construction of possible sets $S$

#### 3.6.1 Step Zero

First choose a monomial in $S_0$. If $e_0 \in S_0$ then the set

$$S(e_0) = x, \ldots, x^{1+k+k^2} = e_0$$

contains $1+k+k^2$ elements. If $k = 1$ then this is the only possibility. From now on assume that $k > 1$. If $e_0 \not\in S$ then $a_0 = xy^{k+1} \in S_0$ and also $x^{k+1} \in S$.
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by Claim 3.14.

3.6.2 Step One

Now consider the possible monomials in $S_1$. Since $x$, $x^k \in S_1$, we only choose one monomial and there are three possibilities $a_1, b_1$ and $e_1$. If $e_1 \in \Gamma$ then this determines a set

$$S(a_0, e_1) = \begin{cases} 
xy^{k+1} \\
\vdots \\
x y \\
x \\
x^2 \\
\cdots \\
x^k
\end{cases}$$

If $b_1 \in S_1$ then $a_1 \not\in S$ and every monomial below and to the left of $b_1$ must be in $S$, that is, the shaded area of Figure 3.6 must be contained in $S$. The monomial $x^{k+2} = f_{k-1}$, which is marked with one of the crosses, cannot be in $S_{k-1}$ because the monomials $b_{k-1}, c_{k-1}$ and $g_{k-1}$ are all in the shaded area. A $G$-set is given by choosing one of the two blocks marked in Figure 3.6. Write $S(b_1)$ for a set of this type.

![Figure 3.6: S(b_1)](image)

If $b_1 \not\in S$ then $a_1 \in S_1$ and so $f_{k-1} = x^{k+2} \in S$. If $k = 2$ then $a_1$ cannot be in $S_1$ and the first three choices determine all the possible $G$-sets. Also

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if $k = 3$ then choosing $a_1$ uniquely determines a $G$-set and so from now on assume that $k > 3$.

### 3.6.3 Step Two

The only monomial that must be in $S_2$ is $x^2$. Defining a $G$-set involves choosing two more monomials in $S_2$. If $e_2 \in S$ then $d_2 \in S$ and

$$S(e_2, a_1) = \begin{bmatrix}
  xy^{k+2} \\
  xy^{k+1} & x^2y^{k+1} \\
  \vdots & \vdots \\
  x & x^2 & \ldots & x^{k^2-k-1}.
\end{bmatrix}$$

If $b_2 \in S$ then $c_2 \in S$. This is because if $b_2 \in S$ then the monomials $x^{k-1}y^{k-3}, x^3y, x^{k-1}y \in S$ and so $x^{k+3} \not\in S$ since these monomials all belong to the same representation spaces. Therefore $a_2, d_2 \not\in S$.

If $x^{k+3} \not\in S$ then $f_{k-2}, a_{k-2} \not\in S$ and so $c_{k-2} \in S$. Thus $S$ contains the shaded region shown in Figure 3.7. The crosses show monomials that cannot be in $S$. Now $c_{k-1} \not\in S$ because $g_{k-1}, b_{k-1}, f_{k-1} \in S$.

![Figure 3.7: $S(b_i, c_i)$](image)

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These crosses imply that the only monomials that can be added to $S$ must be from the Blocks $A$ or $B$. In what follows count the columns of $A$ from the right and the rows of $A$ from the top but count the columns of $B$ from the left and the rows of $B$ from the bottom. This is because monomials can only be removed from the top row or right hand column of $A$ and they must be added to $B$ starting from the bottom row or left hand column.

As we will prove later monomials can only be taken from these blocks in the following way. For the first set take all of Block $A$. The next set is given by removing the first column of $A$ and replacing it with the first row of $B$. Next remove what remains of the first row of $A$ and replace it with the first column of $B$. Continue this way until all of $A$ has been replaced with $B$. In other words take sets that are made up of a rectangle in $A$ and an $L$ shape in the $B$ where the ends of the $L$ shape reach the edges of $B$ and the number of rows of the $L$ shape minus the number of columns is equal to 0 or 1. When the rectangle and the $L$ shape are put together they must form one of the larger rectangles. This is shown by the dotted lines in Figure 3.7. Write $S(b_2, c_2)$ for such a set containing $(b_2, c_2)$.

If $k = 4$ then choosing $a_2$ is impossible and there are no more possibilities for $G$-sets. From now on assume that $k > 4$.

If $b_2, e_2 \not\in S$ then $a_2$ must be one of the monomials in $S_2$. If $d_2$ is the other monomial in $S_2$ then cannot have $x^{k-1}y \in S$ since there are already three monomials in this representation space and so $c_{k-j}, f_{k-j}, a_{k-2} \not\in S$ for $j > 1$. Thus $e_{k-2} \in S$ and two monomials from $a_{k-j}, b_{k-j}, e_{k-j}, d_{k-j}$ must be chosen for $j = 3, \ldots, k - 3$. Set $i = 3$.

If $e_i \in S$ then $d_i \in S$ and call this set $S(e_i, a_{i-1})$. If $b_i \in S$ then $a_i \in S$. This is because there are already 3 elements in the $(k+i+1)$th representation space so $x^{k-1} \not\in S$ and $d_i \not\in S$. One more element in the $(k-i)$th representat-
3.6 Construction of possible sets \( S \)

A construction space is required. It cannot be \( d_{k-i} \) or \( e_{k-i} \) because \( d_i \notin S \) so \( a_{k-i} \in S \). This set is denoted by \( S(e_{k-i+1}, a_{k-i}) \). If \( i = k/2 \) and \( a_i \in S \) then \( b_i \in S \) and this is the final possibility. Else \( a_i \) and \( d_i \) are in \( S \) and so \( e_{k-i+1} \in S \). If \( i = (k - 1)/2 \) then \( e_{i+1} \in S \) and there are no more possibilities. This set is called \( S(e_{i+1}, a_i) \). Otherwise let \( i \mapsto i + 1 \) and repeat this paragraph. Thus

\[
xy^{k+j} \\
\vdots \\
S(e_j, a_{j-1}) = \begin{array}{cccc}
xy^{k+1} & \cdots & x^iy^{k+1} \\
\vdots \\
xy & \cdots & x^iy \\
x & \cdots & x^i & \cdots & x^{k^2-j(k+1)}
\end{array}
\]

If \( d_2 \notin S_2 \) then \( a_2, c_2 \in S_2 \) and set \( i = 2 \).

3.6.4 Step \( i + 1 \)

The monomial \( g_{i+1} \in S_{i+1} \) by Claim 3.19 so it remains to choose two monomials in \( S_{i+1} \). There are two basic types of sets depending on whether \( S \) is determined by the fact it contains \( a_j \) and \( b_j \) or \( b_j \) and \( c_j \) for some \( j \). The details are sketched below.

If \( 2i + 1 < k \) then let \( i \mapsto i + 1 \). In order for \( S \) to be a \( G \)-set we need to choose two monomials in \( S_i \). We have already shown that \( f_j \notin S \) for \( 2j \leq k - 1 \). Therefore \( f_j \notin S \). Now \( d_2 \notin S \) and since \( d_2 \mid d_i \) and \( d_2 \mid e_i \) it follows that \( d_i \) and \( e_i \) do not belong to \( S \) by Claim 3.14. Therefore we must choose two monomials from the set \( \{ a_i, b_i, c_i \} \).

If \( b_i, c_i \in S_i \) then this is the case shown in Figure 3.7 where \( 2 \) is replaced with \( j \). We construct \( G \)-sets \( S(b_i, c_i) \) in the same that the sets \( S(b_2, c_2) \) were determined.
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If $a_i, b_i \in S_i$ then the shaded area shown in Figure 3.8 is in $S$. This is because $c_i \notin S$ so $f_{k-i}, c_{k-i} \notin S$ and so $a_{k-i} \in S$. Also because $b_{i-1}, c_i \notin S$ so $a_{k-i+1}, c_{k-i+1} \notin S$ and $f_{k-i+1} \in S$. The crosses are due to the monomials $x^{2k-i+2}, x^{k+i+1}y^{-2}, x^{k-1}y^{i-1}$ and $x^{k-i+2}, y^{k-i}$ that cannot be in $S$ since the shaded area has enough monomials in these representation spaces.

![Figure 3.8: $S(a_i, b_i)$](image)

We construct $G$-sets in a similar manner to the case $S(b_2, c_2)$. First add Block $A$ to the shaded area in Figure 3.8. Then from this set remove the first row of $A$ and replace it with the first column of $B$. From this remove what remains of the first column of $A$ and replace it with the first row of $B$. Continue this process until all of $A$ has been replaced with $B$.

If $b_i$ does not belong to $S$ then it must be that $a_i, c_i$ belong to $S_i$. In this case return to the start of Step $i + 1$ and repeat this process.
3.7 Proof that each of these sets is a $G$-set

If $2i + 1 = k$ then there exists a unique set $S$(end) shown in Figure 3.9. This is because $x^{i+2}y^i, x^{k+i+2} \notin S$.

![Figure 3.9: S(end)](image)

Finally when $2i + 1 > k$ there are no more cases.

3.7 Proof that each of these sets is a $G$-set

I will now show that each of these sets is a $G$-set and that these are the only ones. There are two standard domains $S'$ and $S''$ in $T$ which contains all the monomials necessary for a $G$-set. These are

$$S' := \{x, \ldots, x^{1+k+k^2}\} \quad \text{and} \quad S'' := \{xy^{k+1}, x^iy^j \mid 0 \leq i \leq k, 0 < j \leq k\}. $$

Now $S''$ is not a $G$-set because it contains $xy^{k+1}$ but not $x^{k+1}$.

First show that a set $S(c_j, a_{j-1})$ is a $G$-set. It is obtained from $S'$ by replacing the blocks

$$A_0 := \{x^{k^2+1}, \ldots, x^{k^2+k+1}\}, \ldots, A_{j-1} := \{x^{k^2-j(k+1)+1}, \ldots, x^{k^2-(j-1)(k+1)}\}$$

with the blocks

$$B_0 := \{xy, \ldots, xy^{k+1}\}, \ldots, B_{j-1} := \{x^{j+1}y, \ldots, x^{j+1}y^{k+1}\}.$$
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We claim that the monomials in Block $A_i$ belong to the same representation spaces as those in Block $B_i$ and that they occur in the same order. The $n$th monomial in $A_i$ is $x^{k^2-(i+1)(k+1)+n}$ which is in the $(k^2-(i+1)(k+1)+n) \equiv (n+(i+2)k^2)$th eigenspace of $A$. The $n$th monomial in $B_i$ is $x^{i+2}y^n$ which is in the $(i+2+nk)$th eigenspace of $A$. These belong to the same representation space of $G$ since $(n+(i+2)k^2)k \equiv i + 2 + nk \pmod{1 + k + k^2}$.

The first part of this claim proves that the set given by replacing $A_i$ with $B_i$ still has all the characters. The fact that they occur in the same order means that part of some Block $A_i$ cannot be replaced with part of $B_i$.

Consider the two sets $S(b_1)$. Take the set containing Block $A$ then this is obtained from $S''$ by replacing $x^ky^k$ with $x^{k+1}$ which is in the same representation space. This set is a $G$-set. Block $A$ has monomials

$$xy^k, \ldots, x^{k-1}y^k$$

which belong to the eigenspaces

$$2 + k^2, 3 + k^2, \ldots, k - 1 + k^2.$$ 

Block $B$ has monomials

$$x^{k+1}y, \ldots, x^{k+1}y^{k-2}$$

which belong to the eigenspaces

$$k+1+k \equiv k(2+k^2), k+1+2k \equiv k(3+k^2), \ldots, k+1+k(k-2) \equiv k(k-1+k^2).$$

So replacing $A$ with $B$ is still a $G$-set. Since the representation spaces in Block $A$ occur in the same order as those in Block $B$ and are distinct, part of Block $A$ cannot be replaced with part of Block $B$. Hence these are the only two sets containing $b_1$ which are $G$-sets.
3.7 Proof that each of these sets is a $G$-set

Next consider sets of the form $S(b_i, c_i)$. First show that the set which contains all of Block $A$ is a $G$-set. The shaded area in Figure 3.10 shows the set $S(b_i, c_i)$ where all of Block $A$ has been chosen. If the Blocks $C$ and $E$ are replaced with $D$ in this set then the set $S''$ is obtained. This is a $G$-set if the monomials in $C$ and $E$ are in the same representation spaces as those in $D$. This is true since $x^{k-j}y^{k-j-1} \in D$ belongs to the same representation space as $x^{j+2}y^{j+1} \in C$ for $j = 0, \ldots, i - 1$. Also $x^{k-m-n}y^{k-m} \in D$ belongs to the same representation space as $x^{k+1+n}y^{n+m} \in E$ for $n = 0, \ldots, i - 1$, and $n = 0, \ldots, i - 1 - m$ and $x^{k+2+n+m}y^{n} \in D$ belongs to the same representation space as $x^{k-m}y^{k-2-n-m} \in E$ for $m = 0, \ldots, i - 1$ and $n = 0, \ldots, i - 1 - m$. Thus we have constructed a bijection $\theta: D \to C \cup E$ such that $f \in D$ belongs to the same representation as $\theta(f)$.

![Figure 3.10: $S(b_i, c_i)$](image)

Then show that any of the sets $S(b_i, c_i)$ are $G$-sets. Block $A$ contains the
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monomials

\[
x^{i+2}y^k \quad x^{i+3}y^k \quad x^{i+4}y^k \quad \ldots \quad x^{k-i+1}y^k \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
x^{i+2}y^{k-i+1} \quad x^{i+3}y^{k-i+1} \quad x^{i+4}y^{k-i+1} \quad \ldots \quad x^{k-i+1}y^{k-i+1}
\]

and Block $B$ contains the monomials

\[
x^{k+1}y^{k-i-1} \quad x^{k+2}y^{k-i-1} \quad \ldots \quad x^{i+k}y^{k-i-1} \\
\vdots \quad \vdots \quad \vdots \\
x^{k+1}y^{i-1} \quad x^{k+2}y^{i-1} \quad \ldots \quad x^{i+k}y^{i-1}.
\]

Once $n$ rows and $n$ columns have been removed from $A$ and added to $B$, the monomials that remain in the $(n+1)$th column of $A$ are

\[
x^{k-i-n+1}y^{k-n} \\
\vdots \\
x^{k-i-n+1}y^{k-i-1}.
\]

These monomials belong to the same representation spaces as those monomials

\[
x^{k+n+1}y^{i+n-1} \ldots x^{i+k}y^{i+n-1}
\]

that will be added to the $(n+1)$th row of $B$. This is checked as in the case of $S(b_1)$. The same method shows that the monomials remaining in the $(n+1)$th row of $A$ belong to the same representation space as those that will be added to the $(n+1)$th column of $B$.

Take monomials $x^\lambda y^\mu$ and $x^{\lambda+i}y^{\mu+j}$ in $A$. Now $\lambda + \mu k \neq \lambda + i + (\mu + j)k$, $\lambda k + \mu k^2 \neq \lambda + i + (\mu + j)k$ and $\lambda k^2 + \mu k^3 \neq \lambda + i + (\mu + j)k$ so $x^\lambda y^\mu$ and $x^{\lambda+i}y^{\mu+j}$ in Block $A$ belong to distinct representations. This is the only way to move monomials from $A$ to $B$ and these are the only sets containing $b_i$ and $c_i$ which are $G$-sets.
### 3.7 Proof that each of these sets is a $G$-set

Now consider the sets of the form $S(a_i, b_i)$. Again start by showing that the set which contains all of Block $A$ is a $G$-set. This set is the shaded area in Figure 3.11. Replacing Blocks $C$, $D$ and $E$ with Block $F$ gives $S''$. As in the previous example there is a one-to-one correspondence between the representation space to which monomials in $C \cup D \cup E$ belongs and the representation space to which monomials in $F$ belongs. So this set is a $G$-set.

![Figure 3.11: S(a_i, b_i)](image)

The proof that all other sets $S(a_i, b_i)$ are $G$-sets is the same as the proof given for all other sets $S(b_i, c_i)$. Again the monomials in $A$ belong to distinct representation spaces so the sets listed above are the only possible $G$-sets containing $a_i$ and $b_i$.

The same methods show that $S($end$)$ is a $G$-set.
3.8 The ideals

Let $\Gamma(e_j, a_{j-1})$ be the $G$-graph generated by $S(e_j, a_{j-1})$. If $j = 1$ then the generators of $I(\Gamma(e_j, a_{j-1}))$ consist of polynomials

\[
\begin{align*}
x^{1+k+k+2} + y^{1+k+k+2} + z^{1+k+k+2} - a, \\
x^iy + b_{i1}x^{i+k} + c_{i1}y^{i+k+1} + d_{i1}z^{i+k+2}, \\
y^iz + b_{i1}y^{i+k} + c_{i1}z^{i+k+1} + d_{i1}x^{i+k+2}, \\
z^ix + b_{i1}z^{i+k} + c_{i1}x^{i+k+1} + d_{i1}y^{i+k+2}, \\
x^iy^i + b_{i1}x^{i+k} + c_{i1}y^{i+k+1} + d_{i1}z^{i+k+2}, \\
y^iz^i + b_{i1}y^{i+k} + c_{i1}z^{i+k+1} + d_{i1}x^{i+k+2}, \\
z^ix^i + b_{i1}z^{i+k} + c_{i1}x^{i+k+1} + d_{i1}y^{i+k+2},
\end{align*}
\]

for $i < 1 + k + k^2$ where $b_{r,s}, c_{r,s}, d_{r,s} \in \mathbb{C}$.

If $j > 1$ then the corresponding ideal, $I(\Gamma(e_j, a_{j-1}))$ has generators

\[
\begin{align*}
xy^{k+1} + yz^{k+1} + zx^{k+1} - a, \\
x^{k^2-(j-1)(k+1)+1} - yz^{j+1} (xz^{j+1} - yz^{j+1}) - cz^{j+1} - dy^{(j+1)k}, \\
y^{k^2-(j-1)(k+1)+1} - bx^{j+1} (xy^{j+1} - xz^{j+1}) - cx^{j+1} - dx^{(j+1)k}, \\
z^{k^2-(j-1)(k+1)+1} - by^{j+1} (yz^{j+1} - xy^{j+1}) - cy^{j+1} - dx^{(j+1)k}, \\
x^iy + b_{i1}x^{i+k} + c_{i1}y^{i+k+1} + d_{i1}z^{i+k+2}, \\
y^iz + b_{i1}y^{i+k} + c_{i1}z^{i+k+1} + d_{i1}x^{i+k+2}, \\
z^ix + b_{i1}z^{i+k} + c_{i1}x^{i+k+1} + d_{i1}y^{i+k+2}, \\
xyz - q
\end{align*}
\]

for $i < k^2 - (j - 1)(k + 1) + 1$ where $b_i, c_i, d_i \in \mathbb{C}$.

Let $\Gamma(b_1)$ be the $G$-graph which is generated by $S(b_1)$ where Block A is chosen. The corresponding ideal $I(\Gamma(b_1))$ has generators

\[
xy^{k+1} + yz^{k+1} + zx^{k+1} - a,
\]
3.8 The ideals

\[
x^{k+2} - bx^2y - cy^kz^k - dz^k, \\
y^{k+2} - by^2z - cz^kx^k - dx^k, \\
z^{k+2} - bz^2x - cx^ky^k - dy^k, \\
x^{k+1}y - cxy^2 - f x^{k-1}z^{k-1} - gy^2 z^k, \\
y^{k+1}z - cyz^2 - f y^{k-1}x^{k-1} - gz^2 x^k, \\
z^{k+1}x - czx^2 - f z^{k-1}y^{k-1} - gx^2 y^k, \\
x^k y^k - lyz - mz^{k+1} - nx^k z, \\
y^k z^k - lzx - mx^{k+1} - ny^k x, \\
z^k x^k - lxy - my^{k+1} - nz^k y, \\
xyz = q.
\]

If Block B is chosen swap the triple \((x^{k-1}y, y^{k-1}z, z^{k-1}x)\) in the middle three generators with \((y^2 z^k, z^2 x^k, x^2 y^k)\) and replace the triple \((x^k y^k, y^k z^k, z^k x^k)\) with \((z^{k+1} x^{k-1}, x^{k+1} y^{k-1}, y^{k+1} z^{k-1})\) in the last triple of polynomials.

Let \(\Gamma(b_j, c_j)\) be the G-graph generated by \(S(b_j, c_j)\) where \(i_1\) columns and \(i_2\) rows of Block A are chosen and \(i_1 = i_2\) or \(i_1 = i_2 + 1\). Then the ideal \(I(\Gamma(b_j, c_j))\) has generators

\[
xy^{k+1} + yz^{k+1} + zx^{k+1} - a, \\
x^{k+j+1} - bx^{j+1} - cy^{k-j+1} z^{k-j-1} - dx^{j-1} z^{k-1}, \\
y^{k+j+1} - by^{j+1} z - cz^{k-j+1} x^{k-j-1} - dy^{j-1} x^{k-1}, \\
z^{k+j+1} - bz^{j+1} x - cx^{k-j+1} y^{k-j-1} - dz^{j-1} y^{k-1}, \\
x^{k-j+2} y^{k-j} - cz^{k-j} - f xz^{j} - gy^{k-1} z^{j-2}, \\
y^{k-j+2} z^{k-j} - cx^{k-j} - f yx^{j} - gz^{k-1} x^{j-2}, \\
z^{k-j+2} x^{k-j} - cy^{k-j} - f zy^{j} - gx^{k-1} y^{j-2}, \\
x^{k-j+2-i} y^{k-j+1} - ly^{k+i} z^{j-2+i} - mz^{j-1+i} = - m x^{k-1-i} z^{j}, \\
y^{k-j+2-i} z^{k-j+1} - lx^{k+i} y^{j-2+i} - mz^{i} x^{j-1+i} = - n y^{k-1} x^{j}, \\
z^{k-j+2-i} x^{k-j+1} - lx^{k+i} y^{j-2+i} - mz^{i} y^{j-1+i} = - n z^{k-1} y^{j},
\]
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\[
x^{j+1}y^{k+1-i_2} - ox^{j+i_2}z^{k+i_2} - py^{k-j+1-i_2}z^{k-j} - r_{j+i_2}z^{i_2}, \\
y^{j+1}z^{k+1-i_2} - oy^{j+i_2}x^{k+i_2} - px^{k-j+1-i_2}x^{k-j} - r_{j+i_2}z^{i_2}, \\
z^{j+1}x^{k+1-i_2} - oz^{j+i_2}y^{k+i_2} - px^{k-j+1-i_2}y^{k-j} - r_{j+i_2}z^{i_2}, \\
x^{k+1+i_2}y^{j-1+i_1} - sx^{1+i_2}y^{j+i_1} - tx^{k-j+1}z^{k-j+1-i_1} - u_{j+i_2}z^{i_2}, \\
y^{k+1+i_2}z^{j-1+i_1} - sy^{1+i_2}z^{j+i_1} - ty^{k-j+1}x^{k-j+1-i_1} - u_{j+i_2}x^{i_2}, \\
z^{k+1+i_2}x^{j-1+i_1} - sz^{1+i_2}x^{j+i_1} - tx^{k-j+1}y^{k-j+1-i_1} - u_{j+i_2}y^{i_2}. \\
y_{j} = q
\]

if $i_1 = i_2$. Otherwise replace the final triple of polynomials with

\[
x^{k+1+i_2}y^{j-1+i_1} - sx^{1+i_2}y^{j+i_1} - ty^{j+1}z^{k-i_2} - u_{j}z^{k-j-i_2}x^{k-j-1}, \\
y^{k+1+i_2}z^{j-1+i_1} - sy^{1+i_2}z^{j+i_1} - tx^{j+1}y^{k-i_2} - u_{j}x^{k-j-i_2}y^{k-j-1}, \\
z^{k+1+i_2}x^{j-1+i_1} - sz^{1+i_2}x^{j+i_1} - tx^{j+1}y^{k-i_2} - u_{j}y^{k-j-i_2}z^{k-j-1}. \\
\]

Let $\Gamma(a_j, b_j)$ be the $G$-graph generated by $S(a_j, b_j)$ where $i_1$ rows are taken from Block A and $i_2$ rows from Block B. Then ideals $I(\Gamma(a_j, b_j))$ have generators

\[
xy^{k+1} + yz^{k+1} +zx^{k+1} - a, \\
x^{2k-j+2} - bx^{k-j+1}y - cy^{j-2} - dy^{k+j}z^{j-3}, \\
y^{2k-j+2} - by^{k-j+1}z - cz^{j}x^{j-2} - dx^{k+j}y^{j-3}, \\
z^{2k-j+2} - bx^{k-j+1}x - cx^{j}y^{j-2} - dx^{k+j}y^{j-3}, \\
x^{k-j+2}y^{k-j+1} = ex^{1+i_2}z^{i_1+i_2} - f x^{1+i_2}z^{i_1+j} - g y^{k-j+1-i_2}z^{j-2}, \\
y^{k-j+2}z^{k-j} = f y^{1+i_2}x^{i_1+j} - g z^{k-j-1}x^{j-2}, \\
z^{k-j+2}x^{k-j} = f z^{1+i_2}y^{i_1+j} - g x^{k-j-1}y^{j-2}, \\
x^{k+1-i_2}y^{j-1} = l y^{j+1+i_2}z^{i_1} - m y^{j+1+i_2}z^{1+i_2} - n x^{k-j+i_2-1}z^{k-j+1}, \\
y^{k-1-i_2}j^{j-1} = l z^{k-j+1+i_2}x^{i_1} - m z^{j+1+i_2}x^{1+i_2} - n y^{k-j+i_2-1}x^{k-j+1}, \\
z^{k-1-i_2}z^{j-1} = l z^{k-j+1+i_2}y^{i_1} - m z^{j+1+i_2}y^{1+i_2} - n z^{k-j+i_2-1}y^{k-j+1}, \\
\]

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3.9 Conclusion of the proof

\[ x^{k+j+1+i}y^{k+1} - \alpha x^{i+1+1}y^{2+i} - p x^{j-2+i}z^{k-2-t_2} - r y^{k-j+2-i_1+i_2}z^{k-j-1-t_1}, \]
\[ y^{k+j+1+i}z^{k+1} - o y^{i+1+i}z^{2+i} - p y^{j-2+i_1-i_2}x^{k-2-t_2} - r z^{k-j+2-i_1+i_2}y^{k-j-1-t_1}, \]
\[ z^{k+j+1+i}x^{k+1} - o z^{i+1+i}x^{2+i} - p z^{j-2+i_1-i_2}y^{k-2-t_2} - r z^{k-j+2-i_1+i_2}y^{k-j-1-t_1}, \]
\[ x^{k+j+1}y^{i-2} - s z^{k-j+1}x - t z^{2k-j+1} - u x^{j+1}y^{i-1}, \]
\[ y^{k+j+1}z^{i-2} - s z^{k-j+1}y - t x^{2k-j+1} - u y^{j+1}z^{i-1}, \]
\[ z^{k+j+1}x^{i-2} - s y^{k-j+1}z - t y^{2k-j+1} - u z^{j+1}x^{i-1}, \]
\[ xyz - q. \]

The ideals of the form \( I(\Gamma(\text{end})) \) have generators

\[ xy^{k+1} + yz^{k+1} + zx^{k+1} - a, \]
\[ x^{(3k+3)/2} - bx^{(k+3)/2}y - cy^{(k+1)/2}z^{(k-3)/2} - dy^{(3k+1)/2}z^{(k-5)/2}, \]
\[ y^{(3k+3)/2} - by^{(k+3)/2}z - cz^{(k+1)/2}x^{(k-3)/2} - dx^{(3k+1)/2}x^{(k-5)/2}, \]
\[ z^{(3k+3)/2} - bz^{(k+3)/2}x - cx^{(k+1)/2}y^{(k-3)/2} - dx^{(3k+1)/2}y^{(k-5)/2}, \]
\[ x^{(k+3)/2}y^{(k-1)/2} - ez^{(k+1)/2} - f xz^{(k+1)/2} - gy^{k-1}z^{(k-3)/2}, \]
\[ y^{(k+3)/2}z^{(k-1)/2} - ez^{(k+1)/2} - f yz^{(k+1)/2} - gx^{k-1}y^{(k-3)/2}, \]
\[ z^{(k+3)/2}x^{(k-1)/2} - ey^{(k+1)/2} - f xz^{(k+1)/2} - gx^{k-1}y^{(k-3)/2}, \]
\[ xyz - q. \]

In each case syzygies between these generators can be computed using the methods outlined in Section 3.4.

### 3.9 Conclusion of the proof

The set \( S(\epsilon_0) \) has only one horn, this is invariant. A set \( S(\epsilon_i, a_{i-1}) \) has two horns one of which is invariant if \( i = 1 \) Both of the sets \( S(b_1) \) have three horns and one of these horns is invariant. Note that \( x^{k+1} \) is not a horn since \( xy^{k+1} \in S \). The set \( S(\text{end}) \) has only two horns and neither of these are invariant.
**Explicit construction of $G$-Hilb $\mathbb{C}^3$ for trihedral groups**

The set given by choosing Block $A$ to be in $S(b_i, c_i)$ has four horns. Removing the first column of $A$ and replacing it with the first row of $B$ adds one horn. All the other moves leave the number of horns unchanged, except the final move which reduces the number of horns by 2. Thus a set $S(b_i, c_i)$ has at most five horns.

The set $S(a_i, b_i)$ given by choosing Block $A$ has three horns. Again note that $x^{2k-i+1}$ is not a horn. Removing the first row from $A$ increases the number of horns by one so this set has four horns. Removing the first column increases the number of horns by one more. The other moves will not alter the number of horns so all the sets of this type will have five horns. Except the final move which will decrease the number of horns by two. Thus a set $S(a_i, b_i)$ has at most five horns.

Thus a $G$-set $S$ has at most five horns.

### 3.10 Conclusion

It is therefore possible to give an explicit description of an affine cover of $G$-Hilb $\mathbb{C}^3$. This shows that $G$-Hilb $\mathbb{C}^3$ is again a smooth rational variety. It has an affine cover but again most of the affines are not isomorphic to $\mathbb{C}^3$.

The same methods as those used in the proof of Conjecture 3.21 when

$$A = \langle \frac{1}{1+k+2k'}(1, k, k^2) \rangle$$

can be used to prove Conjecture 3.21 for other families of trihedral groups. For example when $A$ is a cyclic group generated by an element of the form $\frac{1}{3k^2+3k+1}(1, 3k+1, (3k+1)^2)$ or $A = \mathbb{Z}_r \times \mathbb{Z}_r$. We conclude this chapter by making some remarks about these cases.

Consider the group $A$ generated by $\frac{1}{1+k+2k'}(1, k, k^2)$. When $k \equiv 1 \pmod{3}$ the group $A$ has a subgroup $A'$ generated by $\frac{1}{3k^2+3k+1}(1, 3k'+1, 3k'^2 - 1)$ where $k = 3k' + 1$. The group $A' \times \mathbb{Z}_3$ is a trihedral group. When $G$ is a
triheiral group of this type there are three choices for the invariant monomial in $S$ these are $xy^{3k+2}$, $x^{2k+1}y^k$ and $x^{1+3k+3k^2}$. When $G$ is a triheiral group of type $\mathbb{Z}_r \times \mathbb{Z}_r \times \mathbb{Z}_3$ then $x^r$ must be invariant monomial in $S$ we can show that any $G$-set $S$ has 1, 2 or 3 horns.
Chapter 4

Equivariant and Orbifold

Riemann–Roch

4.1 Introduction

4.1.1 Notation

Fix the following notation throughout this chapter. Let $G$ be a finite group acting on a smooth projective variety $Z$. For a nontrivial element $g \in G$ of order $r$ say, let $H := \langle g \rangle$ be the subgroup generated by $g$. Fix $\zeta$ a primitive $r$th root of unity. Let $\pi: Z \to X = Z/H$ be the quotient morphism. Then $X$ is normal and $\pi$ is a finite morphism.

For $h \in H$ let $Z^{h} = \{ z \in Z \mid hz = z \}$ and define $Z^{H}$ by the following formula $Z^{H} = \bigcup_{h \in H \setminus 1} Z^{h}$. The action is coprime if $Z^{H} = Z^{h}$ for any $h \neq 1 \in H$.

If $D$ is a prime divisor and $D_{1}$ any $\mathbb{Q}$-Cartier divisor we write $\lambda D \in \text{Supp } D_{1}$ to mean $D \in \text{Supp } D_{1}$ but $D \notin \text{Supp } (D_{1} - \lambda D)$. We also write $P \in \text{Supp } D_{1}$ to mean $P \in D$ and $D \in \text{Supp } D_{1}$. 

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Let $\mathcal{F}$ be a coherent sheaf on a variety $X$ then denote by $\mathcal{F}^*$ the dual $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ of the sheaf $\mathcal{F}$.

4.1.2 Background to the equivariant Riemann–Roch theorem

Let $x$ be an irreducible component of $Z^g$ and $\mathcal{N}_x = \mathcal{N}_x^Z$ the normal sheaf of $x$ in $Z$ which is a locally free sheaf of rank $\text{codim}(x, Z)$ on the trivial $G$-variety $x$. By Proposition 1.18 this decomposes into eigensheaves. We denote this decomposition by

$$\mathcal{N}_x = \bigoplus_i \mathcal{N}_x^i \otimes \chi_i,$$

where $\chi_i$ is the character of $\langle g \rangle$ given by $\chi_i(g) = e^i$. This gives a decomposition of $\mathcal{N}^g$, the normal sheaf of $Z^g \in Z$, as follows, $\mathcal{N}^g = \bigoplus_i \mathcal{N}^g_i \otimes \chi_i$.

For a locally free $G$-sheaf $\mathcal{F}$ on a trivial $G$-space $X$ write $\lambda_{-1}(\mathcal{F})$ for the element $\bigoplus(-1)^i \Lambda^i(\mathcal{F})$ in $K^G(X)$ and write $\text{td}(X)$ for the Todd class of $X$.

Let $G$ be a finite group acting on a smooth projective variety $Z$ and $\mathcal{F}$ a $G$-sheaf. The vector space $\mathbb{H}^i(Z, \mathcal{F})$ is a representation of $G$ so the character of any element $g$ acting on the virtual representation $\bigoplus_{i \geq 0} (-1)^i \mathbb{H}^i(Z, \mathcal{F})$ can be defined. Equivariant Riemann–Roch gives an expression for this in terms of the irreducible components of the fixed locus.

**Theorem 4.1 (Equivariant Riemann–Roch [AS68])** Let $Z$ be a non-singular projective variety, $G$ a finite group acting on $Z$ and $\mathcal{F}$ a locally free $G$-sheaf. Then

$$\text{Tr} \left( g : \bigoplus_{p \geq 0} (-1)^p \mathbb{H}^p(Z, \mathcal{F}) \right) = \int_{Z^g} \frac{\text{ch}(\mathcal{F}|_{Z^g})(g) \cdot \text{td}(Z^g)}{\text{ch} \lambda_{-1}(\mathcal{N}^g)(g)} \cdot \text{ch} \lambda_{-1}(\mathcal{N}^g)(g).$$ (4.1)
Equivariant and Orbifold Riemann–Roch

In this chapter we give an explicit proof of this theorem when $Z$ is a curve or a surface and $\mathcal{F}$ is an invertible $G$-sheaf. We then use these results to generate a Riemann–Roch theorem for an orbifold of dimension $n$ having only isolated cyclic quotient singularities.

4.1.3 Technical preliminaries on higher direct image sheaves

We begin by recalling some properties of higher direct image sheaves. See [Har77, III.8] for details. The results in this section are not new but some proofs are included here.

**Definition 4.2** Let $f : Y \to X$ be a continuous map of topological spaces, $i \geq 0$ and $\mathcal{F}$ a sheaf of Abelian groups on $Y$. The $i$th higher direct image of $\mathcal{F}$ is the sheaf, $\mathcal{R}^i f_*(\mathcal{F})$, associated to the presheaf

$$V \mapsto H^i\left(f^{-1}V, \mathcal{F}(f^{-1}V)\right)$$

on $X$. Notice that $\mathcal{R}^0 f_*(\mathcal{F}) = f_* \mathcal{F}$.

**Lemma 4.3** If $f : Y \to X$ is an affine morphism and $\mathcal{F}$ a coherent $\mathcal{O}_Y$-module then $\mathcal{R}^i f_*(\mathcal{F}) = 0$ for all $i > 0$.

**Proof.** If $V \subset X$ is affine then $f^{-1}(V)$ is affine and so $H^i\left(f^{-1}V, \mathcal{F}|_{f^{-1}V}\right) = 0$. Thus $\mathcal{R}^i f_*(\mathcal{F}) = 0$ for all $i > 0$. \qed

The Leray spectral sequence relates the cohomology of the sheaf $\mathcal{F}$ with the cohomology of the sheaves $\mathcal{R}^i f_*(\mathcal{F})$.

**Theorem 4.4** (Leray spectral sequence [God73]) Let $f : Y \to X$ be a morphism between ringed spaces and $\mathcal{F}$ an $\mathcal{O}_Y$-module. Then there exists a
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spectral sequence with

\[ E_\infty \Rightarrow H^*(Y, \mathcal{F}) \text{ where } E_2^{p,q} = H^p(X, \mathcal{R}^q f_* \mathcal{F}). \]

Taking alternating sums in the Leray spectral sequence shows that

\[ \chi(Y, \mathcal{F}) = \sum_{q} (-1)^{q} \chi(X, \mathcal{R}^q f_* \mathcal{F}). \]  \hfill (4.2)

It follows immediately from Theorem 4.4 that if \( \mathcal{R}^i f_*(\mathcal{F}) = 0 \) for all \( i > 0 \) then \( H^i(Y, \mathcal{F}) \cong H^i(X, f_* \mathcal{F}) \) and so \( \chi(Y, \mathcal{F}) = \chi(X, f_* \mathcal{F}) \). In particular this is true if \( f \) is an finite morphism by Lemma 4.3.

It could be hoped that if \( f : Y \to X \) is a resolution of rational singularities on \( X \) and \( \mathcal{M} = f^* \mathcal{L}/(\text{torsion}) \) then \( \chi(Y, \mathcal{M}) = \chi(X, f_* \mathcal{M}) \). However this is only true when \( \dim Y \leq 2 \) as we will see in Lemma 4.5. In higher dimensional cases we will also have terms \( \chi(X, \mathcal{R}_i f_*(\mathcal{M})) \), where \( \mathcal{R}_i f_*(\mathcal{M}) \) is a sheaf supported only over the singular locus of \( X \). This will be proved in the following lemma.

**Lemma 4.5** Let \( X \) be a projective variety having at worst rational singularities and \( f : Y \to X \) be a resolution of \( X \). Let \( \mathcal{L} \) be a divisorial sheaf on \( X \). Then \( \mathcal{M} = (f^* \mathcal{L})/(\text{torsion}) \) is a rank one torsion free sheaf on \( Y \) and

\[ \chi(Y, \mathcal{M}) = \chi(X, f_* \mathcal{M}) + \sum_{i=1}^{n-2} (-1)^i \chi(X, \mathcal{R}^i f_* \mathcal{M}). \]

If \( i > 0 \) and \( X \) has only isolated singularities then

\[ \chi(X, \mathcal{R}^i f_* \mathcal{M}) = h^i(X, \mathcal{R}^i f_* \mathcal{M}). \]
Equivariant and Orbifold Riemann–Roch

**Proof.** For any coherent sheaf $\mathcal{L}$ on a projective variety there exists a surjective morphism

$$\mathcal{O}_X^n(-n) \to \mathcal{L} \to 0$$

for some $n, N \in \mathbb{Z}$. See [Har77, II.5 Cor 5.18]. Since $f^*\mathcal{O}_X = \mathcal{O}_Y$ this gives a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-n) \to \mathcal{M} \to 0$$

where $\mathcal{M} = (f^*\mathcal{L})/(\text{torsion})$ as in the definition and $\mathcal{K}$ is the defined as the kernel of the surjective morphism $\mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-n) \to \mathcal{M}$. Applying $f_*$ gives the following long exact sequence.

$$0 \to f_*(\mathcal{K}) \to f_*(\mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-n)) \to f_*\mathcal{M} \to \cdots$$

By the Projection Formula [Har77, III Ex 8.3] $\mathcal{R}^i f_*(\mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-n)) = \mathcal{R}^i f_*(\mathcal{O}_Y^N) \otimes \mathcal{O}_X(-n)$ and if $i > 0$ then this is zero by definition of rational singularities. Thus $\mathcal{R}^i f_*(\mathcal{M}) \cong \mathcal{R}^{i+1} f_*(\mathcal{K})$. Since $\mathcal{K}$ is supported on the fibres of $f$, the sheaves $\mathcal{R}^i f_*(\mathcal{M}) = 0$ for $i \geq n - 1$.

Now $f_*\mathcal{M} = \mathcal{L}$ since $\mathcal{M}$ is generated by the sections of $\mathcal{L}$. Thus by Equation 4.2

$$\chi(Y, \mathcal{M}) = \sum_{i=0}^{n-2} (-1)^i \chi(X, \mathcal{R}^i f_*(\mathcal{M})) = \chi(X, \mathcal{L}) + \sum_{i=1}^{n-2} (-1)^i \chi(X, \mathcal{R}^i f_*(\mathcal{M})).$$

For $i > 0$ the sheaves $\mathcal{R}^i f_*(\mathcal{M})$ are supported only over the singularities of $X$. If the singularities are all isolated then the sheaves $\mathcal{R}^i f_*(\mathcal{M})$ are supported only over points so all its higher cohomology groups vanish. □

**Lemma 4.6** In notation of Lemma 4.5 if $X$ is 2 dimensional then $\mathcal{M}$ is invertible and $\mathcal{R}^i f_*(\mathcal{M}) = 0$ for all $i > 0$ so

$$\chi(X, \mathcal{L}) = \chi(Y, \mathcal{M}).$$
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Proof. The fact that $\mathcal{R}^i f_*(\mathcal{M}) = 0$ follows immediately from Lemma 4.5. Since $\mathcal{M}$ is torsion free there exists a short exact sequence

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{M}^{**} \rightarrow \mathcal{Q} \rightarrow 0$$

where $\mathcal{Q}$ is defined to be the cokernel of the embedding $\mathcal{M} \rightarrow \mathcal{M}^{**}$. Applying $f_*$ gives the long exact sequence

$$0 \longrightarrow f_* \mathcal{M} \longrightarrow f_*(\mathcal{M}^{**}) \longrightarrow f_* \mathcal{Q} \longrightarrow \mathcal{R}^1 f_*(\mathcal{M}) = 0.$$

As in the proof of Lemma 4.5 $f_* \mathcal{M} = \mathcal{L}$. The sheaf $f_*(\mathcal{M}^{**})$ is torsion free, contains the reflexive sheaf $\mathcal{L}$ and $\text{Supp}(f_*(\mathcal{M}^{**})/\mathcal{L})$ is contained in the singular locus of $X$. Therefore by Proposition 1.9 $f_*(\mathcal{M}^{**}) = \mathcal{L}$. This implies that $f_* \mathcal{Q} = 0$ and so $\mathcal{Q} = 0$. Hence $\mathcal{M} \cong \mathcal{M}^{**}$ and so $\mathcal{M}$ is reflexive. A reflexive rank 1 sheaf on a smooth variety is invertible so $\mathcal{M}$ is an invertible sheaf. 

In particular the next proposition implies that if $\pi: Z \rightarrow X$ is the quotient morphism and $\mathcal{F}$ a locally free sheaf then $\pi_* \mathcal{F}$ is locally free away from the singularities.

Proposition 4.7 (II.5 Cor 2 [Mum85]) Let $f: Z \rightarrow X$ be a proper morphism of Noetherian schemes, $X$ integral and $\mathcal{F}$ a coherent sheaf on $Z$ which flat over $X$. If for some $i \geq 0$ the function $x \mapsto h^i(Z_x, \mathcal{F}_x)$ is constant then the sheaf $\mathcal{R}^i f_*(\mathcal{F})$ is locally free of rank $h^i(Z_x, \mathcal{F}_x)$. 

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4.1.4 Roots of unity

It will also be useful to recall the following properties of roots of unity. For any nontrivial \( r \)-th root of unity, \( \varepsilon \), we have

\[
1 + \varepsilon + \varepsilon^2 + \cdots + \varepsilon^{r-1} = 0
\]

(4.3)

\[
\frac{-\varepsilon - 2\varepsilon^2 - \cdots - (r-1)\varepsilon^{r-1}}{r} = \frac{1}{1 - \varepsilon}
\]

(4.4)

4.2 Results on \( G \)-sheaves

The proof of Theorem 4.1 also uses the following results.

Lemma 4.8 Let \( Z \) be a smooth projective variety, \( G \) a finite cyclic group acting on \( Z \) such that only the trivial element of \( G \) acts trivially on \( Z \) and \( \mathcal{F} \) an invertible \( G \)-sheaf on \( Z \). Let \( \pi : Z \to X = Z/G \) be the quotient morphism. Then

\[
\pi_* \mathcal{F} = \bigoplus_i \mathcal{F}_i
\]

and each \( \mathcal{F}_i \) is a divisorial sheaf which is invertible away from the singularities of \( X \).

Proof. Let \( \mathcal{F}_i \) be the torsion free sheaf \( \text{Hom}(V_i, \pi_* \mathcal{F})^G \). We show that this is invertible away from the singularities. Since \( X \) is normal and the singular locus has codimension \( \geq 2 \) this means that the \( \mathcal{F}_i \) are divisorial by Proposition 1.13 and therefore \( \mathcal{F}_i = \mathcal{O}(D_i) \) for some \( \mathbb{Q} \)-Cartier divisor \( D_i \).

Let \( X^0 \) be the nonsingular locus of \( X \) and \( Z^0 = \pi^{-1}(X^0) \). Then the restriction of the morphism \( \pi \) to \( Z^0 \), is proper and \( \mathcal{F}_{|Z^0} \) is flat over \( X^0 \). The function \( x \mapsto h^0(Z_x, \mathcal{F}_x) \) is equal to \( |G| \) for all \( x \in X^0 \). Therefore by Proposition 4.7 the sheaf \( \pi_* (\mathcal{F}_{|X^0}) \) is locally free of rank \( |G| \). Since \( X^0 \) is a trivial \( G \)-space \( \pi_* (\mathcal{F}_{|X^0}) \) decomposes into eigensheaves \( \mathcal{F}_i^0 \) which are locally free by Proposition 1.18 and \( \mathcal{F}_i^0 \cong \mathcal{F}_{i|X^0} \).

\[\square\]
4.2 Results on $G$-sheaves

**Lemma 4.9** Let $G$ be a finite group acting on a smooth projective variety $Z$ such that only the trivial element of $G$ acts trivially on $Z$ and $\mathcal{F}$ an invertible $G$-sheaf. Then $\mathcal{F}$ is isomorphic to a subsheaf of $k(Z)$ as $G$-sheaves.

**Proof.** Let $\mathcal{F}$ be an invertible $G$-sheaf. As in [Har77, II Prop 6.13] we know that $\mathcal{F} \otimes k(Z)$ is 1 dimensional vector space over $k(Z)$. The compatibility conditions of $\lambda_g$ imply that the map $\mathcal{F} \otimes k(Z) \rightarrow g^*\mathcal{F} \otimes k(Z)$ is quasilinear. Then by [Sha94, Appendix 3 Prop 1] we may assume that $\mathcal{F} \otimes k(Z)$ is based by an invariant element of $\mathcal{F} \otimes k(Z)$ over $k(Z)$. Thus $\mathcal{F} \subset \mathcal{F} \otimes k(Z) \cong k(Z)$ as $G$-sheaves and so $\mathcal{F}$ is isomorphic to a subsheaf of $k(Z)$ as $G$-sheaves. □

**Lemma 4.10** ("Hilbert’s Theorem 90") Let $G$ be a cyclic group, $\mathcal{F}$ an invertible $G$-subsheaf of $k(Z)$ and assume that the map $\lambda_g : \mathcal{F} \rightarrow g^*\mathcal{F}$ is induced by the action of $g$ on $k(Z)$. Then $\mathcal{F} \cong \mathcal{O}(D)$ where $D$ is $G$-invariant.

**Proof.** Compare with [Lan93, proof of VI Thm 6.1]. Let $G$ be a cyclic group of order $r$ and $g$ a generator of $G$. Let $\mathcal{F}$ be an invertible $G$-sheaf and $f$ a generator of $\mathcal{F}_P$ where $P$ is a fixed point. Now $\mathcal{F}_P = (g^*\mathcal{F})_P$ and since $\lambda_g$ is an isomorphism $\lambda_g(f)$ is also a generator of the stalk $\mathcal{F}_P$. Therefore $\beta = \lambda_g(f)/f \in \mathcal{O}_P^*$. We want to find $\gamma \in \mathcal{O}_P^*$ such that $\lambda_g(\gamma)/\gamma = \varepsilon^{-i}\beta$ and then $f/\gamma$ is a generator of $\mathcal{F}_P$ and $\lambda_g(f/\gamma) = \varepsilon^i f/\gamma$.

By Artin’s Theorem on characters [Lan93, VI Thm 4.1] the map $\Theta$ on $k(Z)$ given by

$$\Theta(\xi) = \text{id}^*(\xi) + \beta^{-1}g^*(\xi) + \cdots + \beta^{-1}g^*(\beta^{-1}) \cdots (g^{r-2})^*(\beta^{-1}) g^{r-1}*(\xi)$$

for $\xi \in k(Z)$ is not identically zero. So there exists $\eta \in k(Z)$ such that $\gamma = \Theta(\eta) \neq 0$ and then $\lambda_g(\gamma)/\gamma = \beta$.

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Now \(\Theta(\eta)\) may have a zero or pole at \(P\). Define a map \(\Theta_i\) by

\[
\Theta_i(\xi) = \text{id}^*(\xi) + \varepsilon_i^\beta^{-1} g^*(\xi) + \cdots + \varepsilon_i^\beta^{-1} g^* (\beta^{-1}) \cdots (g^{-2})^* (\beta^{-1}) (g^{-1})^* (\xi)
\]

for \(\xi \in k(Z)\). Then \(\lambda_g(\Theta_i(\eta))/\Theta_i(\eta) = \varepsilon^{-i} \beta\). If \(\Theta_i(\eta)\) has a zero at \(P\) for all \(i\) then \(\eta\) has a zero at \(P\). If \(\eta\) has a zero at \(P\) then \(\eta + n\) does not for any \(n \in Z\). At least one of \(\Theta_i(\eta + n)\) or \(\Theta_i(n)\) is nonzero and for some \(i\) this has no zero at \(P\). Set \(\gamma\) equal to \(\Theta_i(\eta + n)\) or \(\Theta_i(n)\). Then \(\gamma\) does not have a zero at \(P\) and \(\lambda_g(\gamma)/\gamma = \varepsilon^{-i} \beta\). Similarly we can find \(\gamma\) such that \(\gamma\) has no pole at \(P\).

Consider an open set \(U \subset Z\) on which \(G\) acts freely. By [Mum85, 2.7 Prop 2] \(\mathcal{F}^\mu \cong \pi^* \left( (\pi_* \mathcal{F})^G \right) \mid_U\). Thus for any \(P \in Z \setminus Z^g\) the stalks \(\mathcal{F}_P\) and \(\pi^* (\pi_* \mathcal{F})^G \mid_P\) are isomorphic as \(G\)-modules over \(\mathcal{O}_P\). Therefore we can choose a generator of \(\mathcal{F}_P\) which is \(G\)-invariant.

We have shown that for all \(P \in Z\) we can find a generator of \(\mathcal{F}_P\) such that \(\lambda_P(f) = \varepsilon f\). Thus \(F\) is a \(G\)-invariant divisor. \(\square\)

### 4.3 Proof of Theorem 4.1 on a smooth projective curve

Let \(Z = C\) be a smooth projective curve. Theorem 4.1 can be restated as follows.

**Theorem 4.11** Let \(G\) be a finite group, \(C\) a smooth curve on which \(G\) acts and \(\mathcal{F}\) an invertible \(G\)-sheaf on \(C\). Assume that only the trivial element of \(G\) acts trivially on \(C\). Then for all nontrivial \(g \in G\)

\[
\text{Tr} \left( g : \bigoplus_{i \geq 0} (-1)^i H^i(C, \mathcal{F}) \right) = \sum_{P \in C^g} \frac{\text{Tr} (g : \mathcal{F}_P)}{1 - dg_P} \tag{4.5}
\]

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4.3 Proof of Theorem 4.1 on a smooth projective curve

where \( dg_P: T_PC \rightarrow T_P^*C \) is the induced map on the cotangent space and \( \mathcal{F}_P \in K_G(P) = R(G) \).

The trace of the identity acting on a virtual representation is its dimension. Thus, by Hirzebruch Riemann–Roch for curves,

\[
\text{Tr} \left( 1: \bigoplus_{i \geq 0} (-1)^i H^i(C, \mathcal{F}) \right) = \chi(C, \mathcal{F}) = \deg \mathcal{F} + 1 - p_a(C). \tag{4.6}
\]

Recall the notation, \( g \in G \) is of order \( r \), \( H = \langle g \rangle \) and \( \pi: C \rightarrow X = C/H \) is the quotient morphism.

Let \( \pi_* \mathcal{F} = \bigoplus_{i=0}^{r-1} \mathcal{F}_i \) be the decomposition of \( \pi_* \mathcal{F} \) into eigensheaves. Now \( X \) is smooth and \( H \) is cyclic so each of the sheaves \( \mathcal{F}_i \) is invertible by Lemma 4.8. If \( \mathcal{F} \) is the trivial \( G \)-sheaf then denote the decomposition of \( \pi_* \mathcal{O}_C \) into eigensheaves by \( \pi_* \mathcal{O}_C = \bigoplus_i \mathcal{L}_i \) and write \( \mathcal{L}_i = \mathcal{O}_X(D_i) \). Since the sheaves \( \mathcal{L}_i \) are invertible \( \mathcal{L}_i \otimes \mathcal{L}_j \cong \mathcal{O}(D_i + D_j) \).

Using the previous results

\[
\text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(C, \mathcal{F}) \right) = \text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(X, \pi_* \mathcal{F}) \right) \quad \text{by Lemma 4.3}
\]

\[
= \sum_{k=0}^{r-1} \text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(X, \mathcal{F}_k) \right) \quad \text{by Lemma 4.8}
\]

\[
= \sum_{k=0}^{r-1} \varepsilon^k \left( \deg \mathcal{F}_k + 1 - p_a(X) \right) \quad \text{by Equation 4.6}
\]

\[
= \sum_{k=0}^{r-1} \varepsilon^k \deg \mathcal{F}_k \quad \text{by Equation 4.3.}
\]
So it remains to calculate the degrees of the invertible sheaves \( \mathcal{F}_k \).

### 4.3.1 Proof of Theorem 4.1 for the trivial \( G \)-sheaf when the action is coprime.

**Example 4.12** Let the action of \( G = \mathbb{Z}_r \) on \( C = \mathbb{P}^1 \) be generated by \( g = \frac{1}{r}(0, 1) \) and let \( \mathcal{F} \) be the trivial \( G \)-sheaf. The vector space \( H^0(C, \mathcal{O}_C) \cong \mathbb{C} \) and \( g \) acts trivially on it so
\[
\text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(C, \mathcal{O}_C) \right) = 1.
\]

The fixed locus of \( g \) acting on \( C \) is \( C^g = \{ P_0 = (1 : 0), P_1 = (0 : 1) \} \). At \( P_0 \) the local coordinate is \( x_1/x_0 \) and so the induced action on the cotangent space is \( d_{g_P}d(x_1/x_0) = \varepsilon d(x_1/x_0) \). At \( P_1 \) the coordinate is \( x_0/x_1 \) and so the induced action on the cotangent space is \( d_{g_P}d(x_0/x_1) = \varepsilon^{-1} d(x_0/x_1) \). Thus
\[
\sum_{P \in C^g} \frac{\text{Tr}(g: \mathcal{O}_P)}{1 - d_{g_P}} = \frac{1}{1 - \varepsilon} + \frac{1}{1 - \varepsilon^{-1}} = 1.
\]

The \( i \)th eigensheaf \( \mathcal{L}_i \) of \( \pi_* (\mathcal{O}_X) \) is not a subsheaf of \( k(X) \) but the sheaf \( \mathcal{L}_i^{\otimes r} \) is. Therefore its Cartier divisor can be recovered along with its embedding into \( k(X) \) [Har77, II Prop 6.13]. In what follows we show how to construct the divisor corresponding to \( rD_i \).

Let \( P \) be any point in \( C \), \( Q = \pi(P) \) and \( u_i \) a local generator of \( \mathcal{L}_{iQ} \). If \( P \) is not fixed by \( g \), then \( u_i^r \in \mathcal{O}_Q \) is a unit. Therefore \( \mathcal{L}_{iQ}^{\otimes r} \) is generated by \( 1 \) as an \( \mathcal{O}_Q \)-module and so \( Q \notin \text{Supp } rD_i \).

If \( P \) is fixed by \( g \) then the quotient morphism \( \pi: C \to X \) is ramified with order \( r \) at \( P \). If \( t \) is a local coordinate of \( C \) at \( P \) then \( s = t^r \) is a local coordinate of \( X \) at \( Q \). The element \( g \) induces an action on the cotangent space of \( C \) at \( P \) which is given by \( d_{g_P}(dt) = \varepsilon^a dt \) for some \( a \) coprime to \( r \).
4.3 Proof of Theorem 4.1 on a smooth projective curve

Let \( u_i \) be a local generator of \( \mathcal{L}_{iQ} \) then \( u_i^r \in \mathcal{O}_Q \) is no longer a unit so \( u_i^r = (\text{unit})s^\lambda \). Since \( dg dt = \varepsilon^a dt \) it follows that \( u_i = t^b \) where \( b_i \equiv i \pmod{r} \). Therefore \( u_i^r \) has a zero of order \( b_i \) at \( Q \) and \( -b_i Q \in \text{Supp } rD_i \).

Continuing this process over all \( P \in C \) we obtain

\[
\mathcal{L}^r = \mathcal{O}_X \left( -\sum_{Q} b_{i,Q}Q \right)
\]

where \( 0 \leq b_{i,Q} \leq r - 1 \) is defined as follows. If \( Q \) is not a branch point then \( b_{i,Q} = 0 \). If \( Q \) is a branch point and the induced action on the cotangent space to the point \( P = \pi^{-1}(Q) \) is \( d\gamma_P(dt) = \varepsilon^{aP} dt \) then \( b_{i,Q} \) is defined such that \( b_{i,Q} a_P \equiv i \pmod{r} \). Hence

\[
\deg(D_i) = -\sum_{Q} \frac{b_{i,Q}}{r}.
\]

The divisor \( D_i \) is Cartier so \( \deg D_i \) is an integer and the following corollary is immediate.

**Corollary 4.13** The order of \( g \) divides \( \sum_{Q} b_{i,Q} \).

This corollary restricts the way a group may act on a curve. For example if \( \mathbb{Z}_r \) acts on a curve then any \( g \in \mathbb{Z}_r \) cannot fix only one point. If \( g \) fixes exactly two points then \( g \) acts by \( \varepsilon^a \) at one point and \( \varepsilon^{-a} \) at the other.

**Example 4.14** Continuing with the above example. Let \( Q_0 = \pi(P_0) \). Now \( x_1/x_0 \) is a local coordinate of \( P_0 \), \( g(x_1/x_0) = \varepsilon x_1/x_0 \) and \( (x_1/x_0)^r \) is a local coordinate at \( Q_0 \). If \( u_i \) is the generator of \( \mathcal{L}_{iQ_0} \) then \( u_i^r \) is invariant so \( u_i^r = (x_1/x_0)^{r\lambda} \) and since \( u_i = (x_1/x_0)^{\lambda} \) it follows that \( \lambda = i \). Similarly if \( Q_1 = \pi(P_1) \) and \( u_i \) is the generator of \( \mathcal{L}_{iQ_1} \) then \( u_i = (x_1/x_0)^{r-i} \). Thus

\[
\deg \mathcal{L}_i = -\frac{i}{r} - \frac{r - i}{r} = -1
\]
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and

$$\sum_i \varepsilon^i \deg \mathcal{L}_i = -\varepsilon - \varepsilon^2 - \cdots - \varepsilon^{r-1} = 1.$$ 

So far Equation 4.5 has been shown to equal

$$\text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(C, \mathcal{O}_C) \right) = \sum_{k=0}^{r-1} \varepsilon^k \deg \mathcal{L}_k = -\sum_{k=0}^{r-1} \sum_{Q} \frac{b_{kQ} \varepsilon^k}{r} \quad (4.7)$$

where \( Q \) runs through all the branch points of \( \pi \). Now fix a branch point \( Q \), the contribution this point makes to \( \sum_i \varepsilon^i \deg \mathcal{L}_i \) is

$$- \sum_{k=0}^{r-1} \frac{b_{kQ} \varepsilon^k}{r} = - \sum_{i=1}^{r-1} \frac{i \varepsilon^{iap}}{r}$$

$$= -\varepsilon^{a_p} - 2\varepsilon^{2a_p} - \cdots - (r-1)\varepsilon^{(r-1)a_p} = \frac{1}{1 - \varepsilon^{a_p}}$$

by Equation 4.4. Substituting this into Equation 4.7 gives

$$\text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(C, \mathcal{O}_C) \right) = \sum_{Q} \frac{1}{1 - \varepsilon^{d_Q}} = \sum_{p \in C^g} \frac{1}{1 - \varepsilon^{g_p}}.$$ 

\[ \square \]

### 4.3.2 Proof of Theorem 4.1 for any invertible \( G \)-sheaf

**Example 4.15** Let \( C \) and \( G \) be as above. Let \( \mathcal{F} = \mathcal{O}_C(\mu P_0) \) and assume \( \mu > 0 \) so that \( \mathcal{F}_{P_1} \) is generated by 1 and \( \mathcal{F}_{P_0} \) is generated by \( (x_0/x_1)^\mu \). Let the action \( \mathcal{F} \to g^*\mathcal{F} \) be induced by the action of \( g \) on \( k(C) \). Then isomorphism \( \mathcal{F}_{P_1} \to \mathcal{F}_{P_1} \) is multiplication by 1 and the isomorphism \( \mathcal{F}_{P_0} \to \mathcal{F}_{P_0} \) is multiplication by \( \varepsilon^\mu \). The rational functions \( 1, x_0/x_1, \ldots, (x_0/x_1)^\mu \) generate the global sections of the sheaf \( \mathcal{F} \) and so

$$\text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(C, \mathcal{F}) \right) = 1 + \varepsilon^{-1} + \cdots + \varepsilon^{-\mu}.$$
4.3 Proof of Theorem 4.1 on a smooth projective curve

At the fixed points $\text{Tr} \left( g; \mathcal{F}_{|P_0} \right) = \varepsilon^{-\mu}$ and $\text{Tr} \left( g; \mathcal{F}_{|P_1} \right) = 1$ so

$$
\sum_{P \in C_g} \frac{\text{Tr} \left( g; \mathcal{F}_{|P} \right)}{1 - dg} = \varepsilon^{-\mu} + \frac{1}{1 - \varepsilon^{-1}} = 1 + \varepsilon^{-1} + \cdots + \varepsilon^{-\mu}.
$$

Now let $\mathcal{F} = \mathcal{O}(F)$ be an invertible $G$-sheaf and so $\mathcal{F}$ is also a $H$-sheaf. By Lemma 4.10 we may assume $F$ is $H$-invariant so $F = \sum_S \lambda_S S + \sum_R \mu_R R$ where the $S$ are isolated fixed points and the $R$ are free $H$-orbits. Let $\mathcal{F}_i$ be the $i$th eigensheaf of $\pi_*(\mathcal{F})$. As in the case when $\mathcal{F}$ was the trivial $G$-sheaf the theorem is proved by computing the degree of the sheaves $\mathcal{F}_i$. Let $f$ be the generator of $\mathcal{F}_p$.

If $P \notin \text{Supp } F$ then $f = 1$ and so if $f_i$ is a generator of $\mathcal{F}_{iQ}$ then $f_i = u_i$. If $\lambda P \in \text{Supp } F$ and $P \notin C_g$ then $f_i = u_i s^{-\lambda}$ where $s$ is the local coordinate at $Q = \pi(P)$. If $P \in C_g$ then $f_{ia} = u_{ia} s^{-\left\lfloor \frac{i-1}{r} \right\rfloor}$. If $\mathcal{L}_i = \mathcal{O}_X \left( - \sum_Q \frac{b_{ia} \alpha_i}{r} Q \right)$ then

$$
\mathcal{F}_i = \mathcal{O}_X \left( - \sum_Q \frac{b_{ia} \alpha_i}{r} Q + \sum_S \left[ \frac{i + \lambda_S}{r} \right] \pi(S) + \sum_R \mu_R \pi(R) \right).
$$

Now $\pi(R)$ makes the same contribution to each of the sheaves $\mathcal{F}_i$ so makes no contribution to $\sum \varepsilon^i \deg \mathcal{F}_i$ by Equation 4.3. If $P$ is fixed by $g$ but $P \notin \text{Supp } F$ then $P$ makes the same contribution $\sum \varepsilon^i \deg \mathcal{F}_i$ as it made to $\sum \varepsilon^i \deg \mathcal{L}_i$ and since $\text{Tr}(g; \mathcal{F}_{|P}) = \text{Tr}(g; \mathcal{O}_{|P})$ the theorem holds.

If $P \in C_g$ and $\lambda P \in \text{Supp } F$ then $P$ contributes

$$
\sum_{i=0}^{r-1} \left( \left[ \frac{i + \lambda}{r} \right] - \frac{i}{r} \right) \varepsilon^i a
$$

to $\sum \varepsilon^i \deg \mathcal{F}_i$. 

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Now there exists $0 \leq \alpha < r$ such that $\alpha \equiv -\lambda \pmod{r}$. Thus

$$
\sum_{i=0}^{r-1} \left( \frac{i + \lambda}{r} - \frac{i}{r} \right) \varepsilon^a
= \frac{\lambda + \alpha - \alpha}{r} \varepsilon^a + \cdots + \frac{\lambda + \alpha - \alpha + r - 1}{r} \varepsilon^{(\alpha + r - 1)a}
= \frac{\varepsilon^{a a}}{r} + \frac{\lambda - 1}{r} \varepsilon^{(a + 1)a} + \cdots + \frac{\lambda - (r - 1)}{r} \varepsilon^{(\alpha + r - 1)a}
= \frac{\varepsilon^{a a} + 2 \varepsilon^{2a} + \cdots + (r - 1) \varepsilon^{(r - 1)a}}{1 - d_{g^P}}
\square
$$

4.3.3 Proof of Theorem 4.1 when the action is not coprime

In this case we show that the points in $C^H \setminus C^g$ will contribute to $\deg \mathcal{F}_i$ but these terms will cancel in the summation $\sum_i \varepsilon^i \deg \mathcal{F}_i$. Assume for simplicity of notation that $\mathcal{F} = \mathcal{O}_C$ but the proof of the general case is the same.

Take a point $P \in C^H \setminus C^g$. Then $\text{Stab}_H P$ is generated by some element $g^a \in H$ and let $a$ be the least such $a$. Now $g^a t_P = \varepsilon^{\lambda a} t_P$ and assume for simplicity that $\lambda = 1$. Let $a' = r/a$ and consider the short exact sequence

$$
0 \to \mathbb{Z}_{a'} \to \mathbb{Z}_r \to \mathbb{Z}_a \to 1.
$$

Then consider the chain of quotients

$$
C \to C_1 = C/\mathbb{Z}_{a'} \to C_1/\mathbb{Z}_a = X.
$$

Let $\pi_1 : C \to C_1$ and $\pi_1(P) = Q_1$, then $\pi_1_* \mathcal{O}_C = \bigoplus_{i=0}^{a'-1} \mathcal{L}_i'$. The morphism $\pi_2 : C_1 \to X$ is unramified at $Q_1$ and $\pi_2_* \mathcal{L}_i' = \bigoplus_{j=0}^{a-1} \mathcal{L}_{i+j a'}$. Then as in the previous subsection, for fixed $i$, the sheaf $\mathcal{L}_{i+j a'}^{\otimes r}$ has the same degree $b_i$ at
4.4 Proof of Theorem 4.1 on a smooth projective surface.

\[ Q = \pi_2(Q_j) \text{ for all } j. \] Thus this point contributes

\[
\sum_{i=0}^{a'-1} \sum_{j=0}^{a-1} b_i \epsilon^{i+j \alpha'} = \sum_{i=0}^{a-1} b_i \left( \epsilon^i + \epsilon^{i+\alpha'} + \ldots + \epsilon^{i+a'(a-1)} \right) = 0
\]

to \( \text{Tr} \left( g; \bigoplus_{i \geq 0} (-1)^i H^i(C, \mathcal{F}) \right) \). The same argument holds for any invertible \( G \)-sheaf \( \mathcal{F} \) on \( C \).

Hence for any nontrivial \( g \in G \)

\[
\text{Tr} \left( g; \bigoplus_{i \geq 0} (-1)^i H^i(C, \mathcal{F}) \right) = \sum_{\nu \in C^g} \frac{\text{Tr} \left( g; \mathcal{F}_{[\nu]} \right)}{(1 - dg_{\nu})}. 
\]

\[ \square \]

4.4 Proof of Theorem 4.1 on a smooth projective surface.

Let \( Z = S \) a smooth projective surface then Theorem 4.1 can be rewritten in the following way.

**Theorem 4.16** Let \( G \) be a finite group acting on \( S \) and \( \mathcal{F} \) an invertible \( G \)-sheaf on \( S \). Assume that only the trivial element of \( G \) acts trivially on \( S \). Then for any nontrivial \( g \in G \)

\[
\text{Tr} \left( g; \bigoplus_{p \geq 0} (-1)^p H^p(S, \mathcal{F}) \right) = \sum_{\nu \in S^g} \frac{\text{Tr} \left( g; \mathcal{F}_{[\nu]} \right)}{\det(I - dg_{\nu})}
+ \sum_{C \subset S^g} \text{Tr} \left( g; \mathcal{F}_{[C]} \right) \left( \frac{1 - p_a(C) + c_1(\mathcal{F}_{[C]})}{\det(I - dg_{C})} - \frac{dg_C C^2}{\det(I - dg_C)^2} \right). \tag{4.8}
\]

The first sum runs through the isolated fixed points of \( g \) and the second sum runs though prime divisors \( C \) fixed by \( g \). The genus of the curve \( C \) is denoted by \( p_a(C) \) and \( dg_C: N^*_{C/S} \to N^*_{C/S} \) is the induced action of \( g \) on the conormal space to \( C \subset S \).
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As in Theorem 4.11 if $g = 1$ then by Hirzebruch Riemann–Roch

$$\text{Tr} \left( 1: \bigoplus_{p \geq 0} (-1)^p H^p(S, \mathcal{F}) \right) = \chi(S, \mathcal{F}) = \frac{c_1(\mathcal{F})(c_1(\mathcal{F}) - K_S)}{2} + 1 + p_a(S).$$

To prove this theorem first show that there exists a formula of the form

$$\text{Tr} \left( g: \bigoplus_{p \geq 0} (-1)^p H^p(S, \mathcal{O}_S) \right) = \sum_{F \subset S^g} \text{RR}(F)$$

and then compute the terms RR(F) for each irreducible $F \subset S^g$.

4.4.1 Notation

Let $S^g = \{C_1, \ldots, C_n, P_1, \ldots, P_m\}$, where $C_i$ are divisors and $P_i$ are isolated points. The $C_i$ are disjoint smooth curves and $P_j \notin C_i$ for all $i, j$. Let $X = S/H$ and $\pi: S \to X$ be the quotient morphism. Let $B_i = \pi(C_i)$ and $Q_i = \pi(P_i)$.

The sheaf $\pi_*\mathcal{F}$ on $X$ decomposes as a direct sum of eigensheaves $\bigoplus_{i=0}^{r-1} \mathcal{F}_i$. Each of the $\mathcal{F}_i$ are divisorial sheaves on $X$ which are invertible away from the singularities of $X$ by Lemma 4.8. Again the decomposition of $\pi_*\mathcal{O}_S$ into eigensheaves is denoted by $\bigoplus_{i=0}^{r-1} \mathcal{L}_i$ when $\mathcal{O}_S$ is the trivial $G$-sheaf. Denote the $\mathbb{Q}$-Cartier divisor corresponding to $\mathcal{L}_i$ by $D_i$.

Let $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ be reflexive sheaves on $X$ and let $X^0$ be the nonsingular locus of $X$. It may be that $\mathcal{O}(D_1) \otimes \mathcal{O}(D_2) \neq \mathcal{O}(D_1 + D_2)$ but $(\mathcal{O}(D_1) \otimes \mathcal{O}(D_2))^{**} = \mathcal{O}(D_1 + D_2)$. Since $\text{codim}(X \setminus X^0, X) = 2$ the sheaf $(\mathcal{O}(D_1) \otimes \mathcal{O}(D_2))^{**}$ is the unique reflexive sheaf on $X$ such that

$$(\mathcal{O}(D_1) \otimes \mathcal{O}(D_2))^{**}_{|X^0} = \mathcal{O}(D_1)_{|X^0} \otimes \mathcal{O}(D_2)_{|X^0}.$$

See [KM98] for details. We denote by $\mathcal{L}^{[r]}$ the reflexive sheaf $(\mathcal{L}^{**})^{**}$.

Let $f: Y \to X$ be a resolution of $X$ such that $f$ is a log resolution of each pair $(X, D_i)$. Let $\mathcal{M}_i = (f^*\mathcal{L}_i)/(\text{torsion})$. Then by Lemma 4.6 $\mathcal{M}_i$ is
4.4 Proof of Theorem 4.1 on a smooth projective surface.

an invertible sheaf. Write $M_i$ for the corresponding divisor. Let $A_i = f^*B_i$ and $E_i = \sum_j E_{i,j} = f^{-1}Q_i$ where each $E_{i,j}$ is an irreducible curve. Lemma 4.5 implies that

$$\chi(M_i) = \chi(L_i). \quad (4.9)$$

As in the case of curves

$$\text{Tr} \left( g: \bigoplus_{p \geq 0} (-1)^p H^p(S, \mathcal{F}) \right) = \sum_{i=0}^{r-1} \varepsilon^i \chi(L_i) = \sum_{i=0}^{r-1} \varepsilon^i \chi(M_i)$$

$$= \sum_{i=0}^{r-1} \varepsilon^i \left( \chi(O_Y) + \frac{1}{2} M_i(M_i - K_Y) \right)$$

$$= \sum_{i=0}^{r-1} \varepsilon^i \frac{1}{2} M_i(M_i - K_Y).$$

Theorem 4.16 is proved in three steps. First assume that the action is coprime and that the invertible sheaf is trivial, then prove the theorem for any invertible $G$-sheaf and finally prove it for any action.

4.4.2 Proof of Theorem 4.1 for the trivial $G$-sheaf when the action is coprime

If $S^g = S^g$ then the points $Q_i$ are the only singularities of $X$.

Example 4.17 Let $\mathbb{Z}_r$ act on $S = \mathbb{P}^2$ by $g = \frac{1}{r}(0,0,1)$. Let $L_2: (x_2 = 0)$ and $P_2 = (0 : 0 : 1)$ then $S^g = \{L_2, P_2\}$. Again let $\mathcal{F}$ be the trivial $G$-sheaf so

$$\text{Tr} \left( g: \bigoplus_{p \geq 0} (-1)^p H^p(S, \mathcal{O}_S) \right) = 1.$$

The coordinate on the conormal space to $L_2$ in $S$ is $d(x_2/x_0)$ and so $dg_{L_2} = \varepsilon$. The coordinates on the cotangent space to $P$ in $S$ are $d(x_0/x_2)$
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and \( d(x_1/x_2) \) so \( dg = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \). Therefore

\[
\sum_{C \subset S^g} \text{Tr} \left( g: \mathcal{O}_{L_2} \right) \left( \frac{1 - p_a(L_2) + c_1(\mathcal{O}_{L_2})}{\det(I - dg_{L_2})} - \frac{dg_{L_2}L_2}{\det(I - dg_{L_2})^2} \right) + \\
\sum_{P \in S^g} \frac{\text{Tr} \left( g: \mathcal{O}_{[P]} \right)}{\det(I - dg_P)} = \frac{1}{1 - \varepsilon} \frac{\varepsilon}{(1 - \varepsilon)^2} + \frac{1}{1 - \varepsilon^{-1}} = 1.
\]

The theorem will be proved by constructing the divisor \( M_i \) and computing \( M_i^2 \) and \( M_iK_Y \). Let \( P \) be any point on \( S \) which is not fixed by \( g \) so \( Q = \pi(P) \) is a nonsingular point of \( X \) and let \( u_iQ \) be the generator of the free \( \mathcal{O}_Q \)-module \( \mathcal{L}_{iQ} \). As in the case of curves \( \mathcal{L}_{iQ}^{[P]} \) is generated by 1 as an \( \mathcal{O}_Q \)-module. In a neighbourhood of \( \bar{Q} = f^{-1}(Q) \), the resolution \( Y \) is isomorphic to \( X \) so \( u_iQ \) is also a generator for \( \mathcal{M}_{i\bar{Q}} \).

Next let \( P \in C \), one of the fixed divisors, again \( Q = \pi(P) \) is a nonsingular point of \( X \) and \( \mathcal{L}_{iQ} \) is a free \( \mathcal{O}_Q \)-module generated by \( u_iQ \). Now \( \mathcal{L}_{iQ} \) is a submodule of \( \mathcal{O}_P \) and choose \( s, t \in m_P/m_P^2 \) such that \( s \) is the local equation for \( C \) and \( g(s, t) = (\varepsilon^as, \varepsilon^b) \). Therefore \( u_iQ = \text{(unit)} s^{b_i} \) where \( b_i \equiv i \pmod{r} \). Now \( u_i^rQ = s^{rb_i} \in \mathcal{O}_Q \) and \( s^r \) is locally the equation of \( B = \pi(C) \). Thus \(-b_iB \in \text{Supp } rD_i\).

\[ L_0 \quad E_1 \quad L_1 \]

Figure 4.1: Exceptional locus for the blowup of \( Q \)

Finally let \( P = P_j \) be an isolated fixed point. Then \( s, t \in m_P/m_P^2 \) can be chosen so that \( g(s, t) = (\varepsilon^as, \varepsilon^b) \) and, since we are assuming the action is

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coprime, a and b are coprime to r. Now \( Q = \pi(P) \) is a singular point of \( X \). Figure 4.1 shows the exceptional locus of \( Y \) in a neighbourhood of \( f^{-1}Q \).

Let \( \widetilde{Q} \) be any point in \( f^{-1}Q \). At \( \widetilde{Q} \) the local coordinates are \( s_k \) and \( s_{k+1} \) where \( s_k \) is the local coordinate along the line \( E_k \). Now \( s_k = s^\alpha / t^\beta \) and \( s_{k+1} = t^{\beta k+1} / s^{\alpha k+1} \) where \( \alpha_i \) and \( \beta_i \) depend only on the local analytic nature of the singularity \( Q \), that is only on how \( g \) acts in a local analytic neighbourhood of \( P \).

Now \( \mathcal{M}_{iQ} \) is a free \( \mathcal{O}_{\widetilde{Q}} \)-module generated by some \( u_i \in \mathcal{L}_{iQ} \). Since \( u_i^r \in \mathcal{O}_{\widetilde{Q}} \) it follows that \( u_i^r = (\text{unit})s_k^{c_i,k} s_{k+1}^{c_i,k+1} \) where \( c_i,k+1 \) and \( c_i,k \) are defined such that

\[
ac_i,k+1(\alpha_k - \alpha_{k+1}) + bc_i,k(\beta_{k+1} - \beta_k) \equiv i \pmod{r}.
\]

Thus they depend only on \( a, b, r, s_k \) and \( s_{k+1} \).

Let \( L_0 \) be the strict transform of the curve \( s^r = 0 \) shown in Figure 4.1 and \( \widetilde{Q} \) a point in a neighbourhood of \( L_0 \cap E_1 \). Now \( s_0 = t^r \) and \( s_1 = s / t^\mu \) are local coordinates for \( \widetilde{Q} \) for some \( \mu \). Since \( t^\lambda \in \mathcal{M}_{iQ} \) for some \( 1 \leq \lambda \leq r - 1 \) it follows that \( u_i^r = s_0^\lambda \) and \( L_0 \notin \text{Supp} \, rM_i \). Similarly \( L_1 \notin \text{Supp} \, rM_i \).

Thus the local generator of the sheaf \( \mathcal{M}_{iQ} \) depends only on how \( g \) acts on a local analytic neighbourhood of the fixed point.

By continuing this process over all \( \widetilde{Q} \in Y \) it follows that

\[
rM_j = \sum_i -b_{i,j}A_i + \sum_{i,k} -c_{i,k,j}E_{i,k} = f^*(rD_i) + \sum_{i,k} -c_{i,k,j}E_{i,k}.
\]

The terms \( c_{i,k,j} \) depend only on how \( g \) acts on a local analytic neighbourhood of the points \( P_i \).

Also \( rK_Y = f^*(rK_X) - \sum d_{i,k}E_{i,k} \) and \( d_{i,k} \) depends on only the local analytic nature of the singularities of \( X \). By the projection formula \( f^*A \cdot B =

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\( A \cdot f_* B \) and so if \( B \) is exceptional then \( f^* A \cdot B = 0 \). Therefore

\[
\chi(\mathcal{M}_j) = \chi(\mathcal{O}_Y) + \frac{1}{2}(M_j^2 - M_j K_Y)
\]

\[
= \chi(\mathcal{O}_Y) + \sum_i \left( b_{i,j}^2 A_i^2 - r b_{i,j} A_i f^* K_X \right)
\]

\[
+ \sum_i \left( \frac{\sum_k c_{i,k,j} E_{i,k}}{2r^2} \right)^2 - \left( \sum_k c_{i,k,j} E_{i,k} \right) \left( \sum_k d_{i,k} E_{i,k} \right)
\]

\[
= \chi(\mathcal{O}_Y) + \sum_{F \subset S^g} \text{RR}(F, j)
\]

where the sum runs through all irreducible components of the fixed locus. Again \( E_{i,j} E_{i,k} \) depends only on the local analytic nature of the singularities. Hence if \( F \) is an isolated fixed point then \( \text{RR}(F, j) \) depends only on how \( g \) acts on a local analytic neighbourhood of the singularity.

Define \( \text{RR}(F) := \sum_j \varepsilon^j \text{RR}(F, j) \). We complete the proof by computing these terms for each \( F \subset S^g \).

If \( F = C_i \subset S \) is a fixed curve then as shown above

\[
\text{RR}(C_i, j) = \frac{b_{i,j}^2}{2r^2} A_i^2 - \frac{b_{i,j}}{2r} A_i K_Y = \frac{b_{i,j}^2}{2r^2} B_i^2 - \frac{b_{i,j}}{2r} B_i K_X
\]

where \( b_{i,j} \) is defined such that \( b_{i,j} a_i \equiv j \) (mod \( r \)).

For ease of notation assume \( a_i = 1 \) and so \( b_{i,1} = 1 \). If instead \( b_{i,1} = 1 \) then replace \( \varepsilon \) with \( \varepsilon^\lambda \) in what follows. Thus

\[
\text{RR}(C_i) = \sum_{j=0}^{r-1} \frac{\varepsilon^j}{2r} \left( \frac{j^2}{r} B_i^2 + j B_i K_X \right).
\]

By the ramification formula \( K_S = \pi^* K_X + (r - 1) \sum C_i \). For any divisors \( D_1 \) and \( D_2 \) in \( X \) such that \( D_1 \) does not pass through the singularities of \( X \) then

\[
rD_1 D_2 = \pi^* (D_1) \pi^* (D_2).
\]

Thus

\[
\text{RR}(C_i) = \sum_{j=0}^{r-1} \frac{\varepsilon^j}{2r} \left( j^2 C_i^2 + j K_S C_i - j (r - 1) C_i^2 \right)
\]

\[
= -\frac{\varepsilon C_i^2}{(1 - \varepsilon)^2} + \frac{1 - \text{pa}(C_i)}{1 - \varepsilon}.
\]

(4.10)
4.4 Proof of Theorem 4.1 on a smooth projective surface.

The final equality holds since

$$\sum_{i=1}^{r-1} i^2 \varepsilon^i = \frac{-r^2}{1 - \varepsilon} + \frac{-2r \varepsilon}{(1 - \varepsilon)^2}.$$  

Let $F = P_i$ be an isolated fixed point in $S$. Choose local analytic coordinates at $P_i$ such that $g$ acts by $g(s, t) = (\varepsilon^a s, \varepsilon^b t)$. We have shown above that RR($P_i$) depends only on the action of $g$ on a local analytic neighbourhood of $P_i$. Thus RR($P_i$) can be computed for a point $P_i \in C \times C$ where $C$ is a smooth curve acted on by $\mathbb{Z}_r$.

Let $\mathbb{Z}_r$ act on $\mathbb{P}^2$ by $h = \frac{1}{r}(1, 1, 0)$. We showed in Example 4.17 that $\mathbb{P}^{2h} = \{L_2, P_2\}$. Let $C_r$ be a smooth $\mathbb{Z}_r$-invariant curve of degree $r$ in $\mathbb{P}^2$ which does not pass through the point $P_2 = (0 : 0 : 1)$. For example let $C_r: (x_0^r + x_1^r + x_2^r = 0)$. Then $h$ fixes the $r$ points where $C_r$ meets the line $L_2: (x_2 = 0)$. At each of these points the induced action on the cotangent space is $dhdt_P = \varepsilon dt_P$. Thus by equivariant Riemann–Roch for curves

$$\text{Tr} \left( h^a : \bigoplus_{j \geq 0} (-1)^j \mathbb{H}^j(C_r, O_{C_r}) \right) = \sum_{P \in C_r^h} \frac{1}{1 - dh^a_P}$$

$$= \frac{r}{1 - \varepsilon^a}.$$

Let $g' = (h^a, h^b) \in \mathbb{Z}_r \times \mathbb{Z}_r$ act on $C_r \times C_r$ so that the induced action on the cotangent space is given by $dg(ds, dt) = (\varepsilon^a ds, \varepsilon^b dt)$. This action fixes $r^2$ points in $C_r \times C_r$. The Künneth formula says that

$$\mathbb{H}^i(C_r \times C_r, O_{C_r \times C_r}) = \sum_{j+k=i} \mathbb{H}^j(C_r, O_{C_r}) \otimes \mathbb{H}^k(C_r, O_{C_r}).$$

Thus

$$\text{Tr} \left( g : \bigoplus_{i \geq 0} (-1)^i \mathbb{H}^i(C_r \times C_r, O_{C_r \times C_r}) \right)$$

$$= \text{Tr} \left( h^a : \bigoplus_{j \geq 0} (-1)^j \mathbb{H}^j(C_r, O_{C_r}) \right) \text{Tr} \left( h^b : \bigoplus_{k \geq 0} (-1)^k \mathbb{H}^k(C_r, O_{C_r}) \right)$$

$$= \frac{r}{1 - \varepsilon^a} \frac{r}{1 - \varepsilon^b} = \frac{r^2}{(1 - \varepsilon^a)(1 - \varepsilon^b)}.$$
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Since there are \( r^2 \) fixed points in \( C_r \times C_r \) each point contributes \( \frac{1}{(1-e^a)(1-e^b)} \) to \( \text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(C_r \times C_r, \mathcal{O}_{C_r \times C_r}) \right) \) and so

\[
\text{RR}(P_i) = \frac{1}{(1-e^a)(1-e^b)} \left( \frac{1}{\det(I-dg_{R})} \right).
\]

(4.11)

Together Equations 4.10 and 4.11 imply

\[
\text{Tr} \left( g: \bigoplus_{p \geq 0} (-1)^p H^p(S, \mathcal{O}_S) \right) = \sum_{i=0}^{r-1} \frac{e^i}{2} M_i(M_i - K_Y) = \sum_{F \subset S^g} \text{RR}(F)
\]

\[
= \sum_{p \in S^g} \frac{1}{\det(I-dg_{p})} + \sum_{C \subset S^g} \frac{1 - p_{a}(C)}{\det(I-dg_{C})} - \frac{d_{g_{C}} C^2}{\det(I-dg_{C})^2}.
\]

\( \square \)

### 4.4.3 Proof of Theorem 4.1 for any invertible \( G \)-sheaf

Let \( \mathcal{F} \) be an invertible \( G \)-sheaf on \( S \). By Lemma 4.9 we may assume that \( \mathcal{F} \subset k(S) \) and so \( \mathcal{F} = \mathcal{O}_S(F) \). Then \( \mathcal{F} \) is also an \( H \)-sheaf and so \( F \) is an \( H \)-invariant divisor by Lemma 4.10. If \( P \), an isolated fixed point, belongs to \( \text{Supp } F \) then we can replace \( F \) by a linearly equivalent \( H \)-invariant divisor \( F' \) where \( P \notin \text{Supp } F' \). Then \( \mathcal{F} \cong \mathcal{F}' \otimes V_i \) where \( \mathcal{F}' = \mathcal{O}(F') \) and \( V_i \) is an irreducible representation of \( H \). Since

\[
\text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(S, \mathcal{F}) \right) = V_i(g) \text{Tr} \left( g: \bigoplus_{i \geq 0} (-1)^i H^i(S, \mathcal{F}') \right)
\]

and for any fixed point \( P \)

\[
\text{Tr} (g: \mathcal{F}_P) = V_i(g) \text{Tr} (g: \mathcal{F}'_P)
\]

it is sufficient to prove the theorem when \( \mathcal{F} = \mathcal{O}_S(F) \) and no isolated fixed point belongs to the support of \( F \).

Let \( \pi_* \mathcal{F} = \bigoplus \mathcal{F}_i \) and \( \mathcal{E}_i = f^* \mathcal{F}_i/(\text{torsion}) \). As before let \( \pi_* \mathcal{O}_S = \bigoplus \mathcal{L}_i \) and \( \mathcal{M}_i = f_* \mathcal{L}_i/(\text{torsion}) \). We prove Theorem 4.1 when \( F \) is an irreducible
4.4 Proof of Theorem 4.1 on a smooth projective surface.

G-divisor this includes the case that \( F = \sum_{i=1}^{r} F_i \) where each \( F_i \) is irreducible and \( F_{i+1} = gF_i \).

Let \( f \) be the generator of \( \mathcal{F}_P \) as an \( \mathcal{O}_P \)-module and \( u_i \) a generator of \( \mathcal{M}_{i\bar{Q}} \) as an \( \mathcal{O}_Q \)-module for some \( \bar{Q} \in \pi^{-1}(P) \). If \( P \notin \text{Supp} \ F \) then \( \mathcal{E}_{i\bar{Q}} \) is also generated by \( u_i \). In particular this implies that \( \text{RR}_{\mathcal{F}}(P) = \text{RR}_{\mathcal{O}}(P) \) for any of the isolated fixed points.

Let \( P \in \text{Supp} \ F \). By Lemma 4.10 we can choose a generator of \( \mathcal{F}_P \) such that \( g^* f = \varepsilon^\lambda f \) so \( \pi_r \mathcal{F}_Q \) is generated by \( f u_1, \ldots, f u_r \) as an \( \mathcal{O}_Q \)-module. If \( \lambda \equiv 0 \) then \( f \in k(X) \) and so \( \mathcal{E}_i = \mathcal{M}_i(f^*\pi(F)) \). In this case

\[
\text{RR}_{\mathcal{F}}(C_j) = \sum_{k=0}^{r-1} \frac{\varepsilon^k}{2r} \left( \frac{k^2}{r} A_j^2 - k A_j f^*\pi(F) + r (f^*\pi(F))^2 \right.
\]

\[
+ k A_j K_Y - r f^*\pi(F) K_Y \biggr) = \text{RR}_{\mathcal{O}_S}(C_j) + \frac{C_j F}{1 - \varepsilon}.
\]

If \( \lambda \neq 0 \) then \( f \notin k(X) \) but \( f^r \in k(X) \) and so \( f u_{i-\lambda} \in \mathcal{F}_{i\bar{Q}} \). In this case \( \mathcal{E}_{i\bar{Q}}^\otimes r = \mathcal{O}_Y \left( f^*\pi(F) + r \sum_j c_{i,j} A_j \right) \) where \( c_{i,j} = b_{i,j} \) if \( A_j \) does not meet \( f^*\pi(F) \) and \( c_{i,j} = b_{i-\lambda,j} \) otherwise. Again if \( A_j \) does not meet \( f^*\pi(F) \) then \( \text{RR}_{\mathcal{F}}(C_j) = \text{RR}_{\mathcal{O}_S}(C_j) \). If \( f^*\pi(F) \) meets some \( A_j \) then

\[
\text{RR}_{\mathcal{F}}(C_j) = \sum_{k=0}^{r-1} \frac{\varepsilon^k}{2r} \left( \frac{(k - \lambda)^2}{r} A_j^2 - \frac{k - \lambda}{r} A_j f^*\pi(F) + (f^*\pi(F))^2 \right.
\]

\[
+ k A_j K_Y - f^*\pi(F) K_Y \biggr) = \varepsilon^\lambda \left( \text{RR}_{\mathcal{O}_S}(C_j) + \frac{C_j F}{1 - \varepsilon} \right) = \text{Tr}(g: \mathcal{F}_C) \left( \text{RR}_{\mathcal{O}_S}(C_j) + \frac{C_j F}{1 - \varepsilon} \right).
\]

Note this shows that if \( f^*\pi(F) \) does not meet any of the \( A_j \), or equivalently \( F \) does not meet any of the fixed curves then, \( F \) does not affect \( \text{Tr}(g: \oplus(-1)^p H^p(S, \mathcal{F})) \).

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The argument for any invertible $G$-sheaf is identical. □

4.4.4 Proof of Theorem 4.1 when the action is not coprime

In this case there will be curves and points which are fixed by some subgroup of $\langle g \rangle = H$. There are the following five possibilities.

1. $P$ is a point which is not fixed by any element of the group.

2. $P$ is a point on a curve $C$ which is fixed by some nontrivial subgroup of $H$ and $\text{Stab}_H C = \text{Stab}_H P$.

3. $P$ is a point in the intersection of two curves $C_1$ and $C_2$ which are fixed by nontrivial subgroups of $H$. Now $\text{Stab}_H C_1 \cap \text{Stab}_H C_2 = 1$. Suppose $\text{Stab}_H C_1 \times \text{Stab}_H C_2 = \text{Stab}_H P$.

4. As in Case 2 but $\text{Stab}_H C \neq \text{Stab}_H P$.

5. As in Case 3 but $\text{Stab}_H C_1 \times \text{Stab}_H C_2 \neq \text{Stab}_H P$.

In the first three cases the point $\pi(P) = Q$ is smooth.

We will only show the proof when $\mathcal{F} = \mathcal{O}_S$ but the case of a general invertible sheaf follows easily. First compute the generator of $\mathcal{M}_{\tilde{u}\tilde{Q}}$ for $\tilde{Q} \in f^{-1}Q$ in each of the above cases.

In Case 1 it follows as before that $u_i' = 1$ is a unit and so $P \notin \text{Supp} r D_i$.

Case 2 is analogous to the case of curves when $C^g \neq C^H$. We show that $\text{RR}(C) = 0$. Let $C$ be a curve fixed by $g^a$ say, where $a$ is the least such $a$, $\pi(C) = B$ and $a' = r/a$. Consider the short exact sequence

$$0 \to \mathbb{Z}_{a'} \to H \to \mathbb{Z}_a \to 1.$$
4.4 Proof of Theorem 4.1 on a smooth projective surface.

Decompose the morphism \( \pi : S \to X \) as follows:

\[
S \to S_1 := S/\mathbb{Z}_d \to X = S_1/\mathbb{Z}_a.
\]

Now \( \pi_1 : S \to S_1 \) is ramified along \( C \) and \( \pi_1^* \mathcal{O}_S = \bigoplus_{i=0}^{d-1} \mathcal{L}_i' \). The morphism \( \pi_2 : S_1 \to X \) is unramified along \( \pi_1 (C) \). Let \( u \) be a generator of \( \mathcal{L}_{i+j' a' Q} \) and \( v \) a generator of \( \mathcal{L}_{i+j' a' Q} \). Then \( u^r = v^r \) so if \( a'|i - i' \) then \( \text{RR}(C, i) = \text{RR}(C, i') \) and so

\[
\text{RR}(C) = \sum_i \varepsilon^i \text{RR}(C, i)
\]

\[
= \left( 1 + \varepsilon^a + \cdots + \varepsilon^{(a-1) a'} \right) \text{RR}(C, 0)
\]

\[
+ \left( \varepsilon + \varepsilon^{1 + a'} + \cdots + \varepsilon^{1 + (a-1) a'} \right) \text{RR}(C, 1)
\]

\[
+ \cdots
\]

\[
+ \left( \varepsilon^{a'-1} + \varepsilon^{a'-1 + a'} + \cdots + \varepsilon^{a'-1 + (a-1) a'} \right) \text{RR}(C, a - 1)
\]

\[
= 0.
\]

Simplify Case 3 by assuming that \( P \) is fixed by \( H \). Again this assumption is to simplify the notation. Let \( g^a \) be the generator of \( \text{Stab}_H C_1 \) and \( g^b \) the generator of \( \text{Stab}_H C_2 \). This case is similar to the above but \( \pi_2 : S_1 \to X \) is ramified over \( \pi_1 (C_2) \). Let \( a' = r/a \) and \( b' = r/b \). Choose local coordinates for the point so that \( g(s, t) = (\varepsilon^a s, \varepsilon^b t) \) and assume, for ease of notation, that \( \lambda = \mu = 1 \). Thus

\[
\text{RR}(P)
\]

\[
= \frac{1}{2} \sum_{j=0}^{a-1} \sum_{k=0}^{b-1} \frac{\varepsilon^{d_j j + b' k}}{r^2} \left( j^2 A_1^2 + 2jk A_1 A_2 + k^2 A_2^2 + j r A_1 K_Y + k r A_2 K_Y \right)
\]

\[
= \left( \sum_{j=0}^{a-1} \frac{\varepsilon^{d_j j}}{r} \right) \left( \sum_{k=0}^{b-1} \frac{k \varepsilon^{b' k}}{r} \right)
\]

\[
= \frac{1}{1 - \varepsilon^a} \frac{1}{1 - \varepsilon^b}.
\]

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Also simplify Case 4 by assuming that the point $P$ is fixed by $g$. Let $\text{Stab}_H C$ be generated by $g^a$ and simplify further by assuming that $g(s, t) = (\varepsilon_d s, \varepsilon t)$ where $s$ and $t$ are the local coordinates at the point $P$. Let $A = f^* \pi(C)$.

![Diagram](image)

Figure 4.2: Exceptional locus for the blowup of $Q$

Figure 4.2 shows the exceptional locus of $Y$ in a neighbourhood of $f^{-1}(Q)$. If $\bar{Q} \in E_1$ then as in the prime case $u_i = t^i$ and so $L_0 \not\in \text{Supp} M_i$. If $\bar{Q} \in E_n$, then $u_i = (t^i / s^m) \lambda_i (s^a)^\mu_i$ and again if $\alpha^i i - i'$ then $\lambda_i = \lambda_i'$.

Again $r K_Y = f^* (r K_X) - \sum d_{i,k} E_{i,k}$. Therefore

$$
\text{RR}(P) = \sum_i \frac{\varepsilon^i}{2r^2} \left( \left( \sum_{j=1}^n a_{i,j} E_j + \lambda_i A \right)^2 + r \sum_{j=1}^n a_{i,j} E_j K_Y + r \lambda_i A K_Y \right)
$$

$$
= \sum_i \frac{\varepsilon^i}{2r^2} \left( \left( \sum_{j=1}^n a_{i,j} E_j \right)^2 + 2 \lambda_i a_{n,i} - \sum a_{i,j} E_j \sum d_k E_k \right)
$$

where $a_{i,j}$, $\lambda_i$, $E_j E_k$ and $d_j$ depend only on the local analytic nature of the singularity $Q = \pi(P)$. Thus RR$(P)$ is determined only by how $g$ acts in a local analytic neighbourhood of the point $P$.

Let $C_r$ be the curve in $\mathbb{P}^2$ of degree $r$ defined previously. Let the action of $Z_r$ on $\mathbb{P}^2$ be generated by $h = \frac{1}{r}(0, 0, 1)$ and $g = (h, h^a)$ acts on $C_r \times C_r$. Then $g^a$ fixes the one of the curves $C_r$ and $g$ fixes $r^2$ points on $C_r \times C_r$. One of these fixed points $P'$ contributes

$$
\text{RR}(P') = \frac{1}{1 - \varepsilon^a} \frac{1}{1 - \varepsilon}.
$$

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to \( \text{Tr}(g : \bigoplus(-1)^i H^i(C_r \times C_r, \mathcal{O}_{C_r \times C_r})) \). Since \( \text{RR}(P) \) depends only on how \( g \) acts in a local analytic neighbourhood of the point \( P \) it follows that

\[
\text{RR}(P) = \text{RR}(P') = \frac{1}{1 - \varepsilon^a} \frac{1}{1 - \varepsilon^b}.
\]

Exactly as in the previous case, for the final possibility we can show that \( \text{RR}(P) \) depends only on how \( g \) acts on the local analytic neighbourhood of \( P \). So compute \( \text{RR}(P) \) for \( P \in C_r \times C_r \) and \( g = (h^a, h^b) \). Thus

\[
\text{RR}(P) = \frac{1}{1 - \varepsilon^a} \frac{1}{1 - \varepsilon^b}.
\]

\( \Box \)

4.5 Orbifold Riemann–Roch

Let \( X \) be a orbifold with at worst isolated cyclic quotient singularities. Let \( P \) be a singular point of \( X \) such that there exists a neighbourhood \( U \) of \( P \) such that \( U \cong V/\mathbb{Z}_r \) where \( \mathbb{Z}_r \) is generated by \( \frac{1}{r}(a_1, \ldots, a_n) \) with \( a_i \) coprime to \( r \). Let \( \mathcal{L} = \mathcal{O}(D) \) be a divisorial sheaf on \( X \) such that \( \mathcal{L}|_U \) is isomorphic to the \( i \)th eigensheaf of \( \pi_* \mathcal{O}_V \). In this case we say \( P \) with the sheaf \( \mathcal{L} \) is of type \( i \left( \frac{1}{r}(a_1, \ldots, a_n) \right) \).

Let \( f : Y \to X \) be a log resolution of the pair \((X, D)\). The sheaf \( \mathcal{M} = f^* \mathcal{L}/(\text{torsion}) \) is the torsion free rank one sheaf on \( Y \) such that \( f_* \mathcal{M} = \mathcal{L} \) by Lemma 4.5. The sheaf \( \mathcal{M}^{**} \) is reflexive and hence invertible. The proof of Lemma 4.6 implies that \( f_*(\mathcal{M}^{**}) = f_*(\mathcal{M}) \). Hence from now on we replace \( \mathcal{M} \) with \( \mathcal{M}^{**} \).

Now \( \mathcal{M} = \mathcal{O}(f^*D + \sum b_i E_i) \) and \( K_Y = f^*K_X + \sum a_i E_i \). Thus

\[
\chi(Y, \mathcal{M}) = \int_Y \text{ch}(f^*D + \sum b_i E_i) \text{td}(Y) = \int_X \text{ch}(D) \text{td}(X) + \sum \text{c}_P(D)
\]

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where \( c'_p(D) \) depends only on the local analytic nature of the singularity. The expression \( \int_X \text{ch}(D) \text{td}(X) \) is interpreted as follows. Take \( \text{td}(X) \) on the nonsingular locus of \( X \) and intersect with \( D \). This makes sense because some multiple of \( D \) is Cartier so can be moved away from the singularities.

By Lemma 4.5 the sheaves \( \mathcal{R}^i f_*(\mathcal{M}) \) are supported only on the singularities so its space of global sections and hence \( h^0(X, \mathcal{R}^i f_*(\mathcal{M})) \) depends only on the analytic nature of the singularities. Thus

\[
\chi(X, \mathcal{L}) = \int_X \text{ch}(D) \text{td}(X) + \sum_p c_i,p
\]

where \( c_i,p = c'_i(p) + \sum_{i=1}^{n-2} (-1)^{i+1} \chi(P, \mathcal{R}^i f_*(\mathcal{M})_p) \).

To compute the term \( c_i,p \) for a singularity of type \( i(\frac{1}{r}(a_1, \ldots, a_n)) \) with \( a_1, \ldots, a_n \) coprime to \( r \) calculate them for a global orbifold \( Z/G \) having only singularities of this type.

We may assume that \( \mathcal{F} \cong \mathcal{F}' \otimes V \) where \( \mathcal{F}' \) is not supported at \( P \) and \( V \) is an irreducible representation of \( \mathbb{Z}_r \). The contribution that \( P \) makes to \( \text{Tr} \left( g: \bigoplus_{p \geq 0} (-1)^p H^p(Z, \mathcal{F}) \right) \) depends only on how \( g \) acts on a local analytic neighbourhood of \( P \). Therefore the contribution that \( P \) makes to \( \text{Tr} \left( g: \bigoplus_{p \geq 0} (-1)^p H^p(Z, \mathcal{F}) \right) \) equals the contribution a point \( P' \in C_1 \times \cdots \times C_n \) makes to \( \text{Tr} \left( g': \bigoplus_{p \geq 0} (-1)^p H^p(C_1 \times \cdots \times C_n, V \otimes \mathcal{O}_{C_1 \times \cdots \times C_n}) \right) \) for suitable \( g' \).

As in the case of a surface each curve can be taken to be \( C_r \subset \mathbb{P}^2 \) a curve of degree \( r \) invariant under the action of \( h = \frac{1}{r}(0, 0, 1) \) not passing through the point \( (0 : 0 : 1) \). This curve has induced action \( \frac{1}{r}(1) \) at each of the fixed points.

The Künneth formula implies that

\[
H^p(C_r \times \cdots \times C_r, \mathcal{O}_{C_1 \times \cdots \times C_n}) = \bigoplus_{p_1 + \cdots + p_n = p} H^{p_1}(C_1, \mathcal{O}_{C_1}) \otimes \cdots \otimes H^{p_n}(C_n, \mathcal{O}_{C_n}).
\]

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Let $g = (h^{a_1}, \ldots, h^{a_n})$ act on $C_r \times \cdots \times C_r$. This fixes $r^n$ points in $C_r \times \cdots \times C_r$. Since each of the $a_i$ is coprime to $r$ it follows that $C_r^h = C_r^H$.

Then Theorem 4.11 gives
\[
\text{Tr} \left( g: \bigoplus_{p \geq 0} (-1)^p \mathbb{H}^p(C_1 \times \cdots \times C_n, V \otimes \mathcal{O}_{C_1 \times \cdots \times C_n}) \right)
\]
\[
= \text{Tr}(h^{a_1}: \bigoplus_{p \geq 0} (-1)^p \mathbb{H}^p(C_r, V \otimes \mathcal{O}_{C_r})) \prod_{i=2}^{n} \text{Tr}(h^{a_i}: \bigoplus_{p \geq 0} (-1)^p \mathbb{H}^p(C_r, \mathcal{O}_{C_r}))
\]
\[
= \left( \sum_{P \in C_r^h} \frac{\text{Tr}(h^{a_1}: V \otimes \mathcal{O}_{C_r}|_P)}{1 - dh^{a_1}_P} \right) \prod_{i=2}^{n} \left( \sum_{P \in C_r^h} \frac{\text{Tr}(h^{a_i}: \mathcal{O}_{C_r}|_P)}{1 - dh^{a_i}_P} \right)
\]
\[
= r^n \frac{V(g)}{1 - \varepsilon^{a_1}} \cdots \frac{1}{1 - \varepsilon^{a_n}}.
\]

Therefore $P \in Z$ contributes
\[
\frac{V(g)}{(1 - \varepsilon^{a_1}) \cdots (1 - \varepsilon^{a_n})} \text{ to } \text{Tr} \left( g: \bigoplus_{p \geq 0} (-1)^p \mathbb{H}^p(Z, \mathcal{F}) \right).
\]

Thus if $G$ is a cyclic group acting on a smooth variety $Z$ of any dimension and having only isolated fixed points then
\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
\varepsilon & \varepsilon^2 & \cdots & \varepsilon^{r-1} \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon^{r-1} & \varepsilon^{r-2} & \cdots & \varepsilon
\end{pmatrix}
\begin{pmatrix}
\chi(X, \mathcal{L}_0) \\
\chi(X, \mathcal{L}_1) \\
\vdots \\
\chi(X, \mathcal{L}_{r-1})
\end{pmatrix} =
\begin{pmatrix}
\chi(Z, \mathcal{F}) \\
\frac{\text{Tr}(g: \mathcal{F}_r)}{\det(I - dg_{r}^{-1})} \\
\vdots \\
\frac{\text{Tr}(g^{r-1}: \mathcal{F}_r)}{\det(I - dg_{r}^{-1})}
\end{pmatrix}.
\]

(4.12)

Inverting the Vandermonde matrix gives a Riemann–Roch expression for a sheaf $\mathcal{L}_i$ on a global orbifold $X$.

\[
\frac{1}{r}
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \varepsilon^{-1} & \varepsilon^{-2} & \cdots & \varepsilon \\
\vdots & \vdots & \ddots & \vdots \\
1 & \varepsilon & \varepsilon^2 & \cdots & \varepsilon^{r-1}
\end{pmatrix}
\begin{pmatrix}
\chi(Z, \mathcal{F}) \\
\frac{\text{Tr}(g: \mathcal{F}_r)}{\det(I - dg_{r}^{-1})} \\
\vdots \\
\frac{\text{Tr}(g^{r-1}: \mathcal{F}_r)}{\det(I - dg_{r}^{-1})}
\end{pmatrix} =
\begin{pmatrix}
\chi(X, \mathcal{L}_0) \\
\chi(X, \mathcal{L}_1) \\
\vdots \\
\chi(X, \mathcal{L}_{r-1})
\end{pmatrix}.
\]

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Equivariant and Orbifold Riemann–Roch

This proves the following theorem.

**Theorem 4.18 (Orbifold Riemann–Roch)** Let $X$ be a cyclic orbifold having at worst isolated quotient singularities. Let $\mathcal{L}$ be a divisorial sheaf such that $\mathcal{L}|_{U_P}$ is isomorphic to the $i$th eigensheaf of $\pi_*(\mathcal{O}_U)$. Then

$$
\chi(X, \mathcal{L}) = \int_X \text{ch}(D) \, \text{td}(X) + \sum_{P \in \text{Sing } X} c_{i,P}.
$$

If $P$ is a point of type $i \left( \frac{1}{r} \alpha_1, \alpha_2, \ldots, \alpha_n \right)$ for $i = 0, \ldots, r - 1$ then

$$
c_{i,P} = \frac{1}{r} \sum_{l=1}^{r-1} \frac{\varepsilon^{-il}}{(1 - \varepsilon^{l \alpha_1}) \cdots (1 - \varepsilon^{l \alpha_n})}.
$$

By Equation 4.12 when $X = Z/G$ then

$$
\int_X \text{ch}(D) \, \text{td}(X) = \frac{1}{r} \int_Z \text{ch}(D') \, \text{td}(Z)
$$

where $D' = \pi^* D$.

This expression can be simplified to show that

$$
c_{i,P} = (-1)^{n+1} \left( \frac{(r-1)^n}{2^n r} - \sum_{(\lambda_1, \ldots, \lambda_n) \in \Lambda_i, r} \frac{\lambda_1 \cdots \lambda_n}{r^n} \right),
$$

$$
\Lambda_{i,P} := \left\{ (\lambda_1, \ldots, \lambda_n) \mid \sum a_j \lambda_j \equiv i \pmod{r}, \ 0 \leq \lambda_k \leq r - 1 \right\}.
$$

**Remark 4.19** The following two observations follow immediately from this definition.

- If $(\lambda_1, \ldots, \lambda_n) \in \Lambda_{i,P}$ then $(r - \lambda_1, \ldots, r - \lambda_n) \in \Lambda_{-i,P}$.

- $\sum_i c_{i,P} = (-1)^{n+1} \left( \frac{r(r-1)^n}{2^n r} - (1 + \cdots + r - 1)^n / r^n \right) = 0$.

If $n = 1$ then there are no singular points but we can still compute $c_{i,P}$. In Section 4.3 we computed the degree of the sheaves $\mathcal{L}_j$ by calculating the degree of the divisors $D_j$ as $\mathbb{Q}$-divisors. If we let $\mathcal{L}'_j = \mathcal{O}([D_j])$ then the
difference between $\chi(X, \mathcal{L}_j)$ and $\chi(X, \mathcal{L}_j')$ is the sum of the terms $c_{i,P}$ for all $P$ appearing in Supp $D_j$. For $P$ a point of type $i \left( \frac{1}{r}(1) \right)$, Equation 4.13 implies

$$c_{i,P} = \frac{r - 1}{2r} - \frac{i}{r}.$$  

Let $P$ be a singularity of type $i \left( \frac{1}{r}(1, -1) \right)$. Then

$$\Lambda_0 = \{(1, 1), (2, 2), \ldots, (r - 1, r - 1)\}$$

and

$$c_{0,P} = \frac{1 + 4 + \cdots + (r - 1)^2}{r^2} - \frac{(r - 1)^2}{4r} = \frac{r(r - 1)(2r - 1)}{6r^2} - \frac{(r - 1)^2}{4r} = \frac{(r - 1)(r + 1)}{12r}. \quad (4.14)$$

The following result is taken from Reid’s “Young person’s guide to canonical singularities” [Rei87]

$$c_{i,P} \left( \frac{1}{r}(a_1, \ldots, a_n) \right) - c_{-a_n,P} \left( \frac{1}{r}(a_1, \ldots, a_n) \right) = c_{i,P} \left( \frac{1}{r}(a_1, \ldots, a_{n-1}) \right).$$

Together with Equation 4.14 this implies

$$c_{i,P} = \frac{r^2 - 1}{12r} - \frac{i(r - i)}{2r}.$$  

This agrees with [Rei87].

If $P$ is a singularity of type $i \left( \frac{1}{r}(1, 1) \right)$ then

$$\Lambda_{0,P} = \{(1, r - 1), (2, r - 2), \ldots, (r - 1, 1)\}$$

and in the same way it follows that

$$c_{i,P} = \frac{(r - 1)(5 - r)}{12r} + \frac{i(r - i - 2)}{2r}.$$  

Let $G$ be any finite group such that $Z^g$ is zero dimensional for all $g \in G$. Then we will obtain a similar matrix expression to Equation 4.12 where the square matrix will be the character table of $G$. In the same way inverting this matrix will imply an orbifold Riemann–Roch formula for any orbifold having isolated singularities.


Bibliography


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