Framed Stratified Sets in Morse Theory

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Contents

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Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

Abstract

Let f be a Morse function on a closed manifold M. In this thesis, we investigate the relations between the attaching maps in the CW complex determined by f, and the moduli spaces of gradient flow lines of f, with respect to some Riemannian metric on M.

These moduli spaces are naturally framed submanifolds of the unstable spheres of the gradient flow. The set of moduli spaces contained in a given unstable sphere determines a stratification of this sphere. The unstable spheres are thus stratified sets, with framed strata. It is possible to associate to such a structure a homotopy class of maps with values in some CW complex. Thus, the framed moduli spaces determine maps and it is shown in this thesis that one can use them to construct the classical Morse CW complex associated to f.

Our main tools are stratified sets and transverse CW complexes. We give smooth versions of known results in the PL category. In particular, we establish a transversality result and a Pontryagin-Thom correspondence for maps from manifolds with faces to CW complexes.

As an application, we determine the cup product of successive critical points in terms of linking of moduli spaces of flow lines.

1 Introduction and statement of the main results

Let $f : M \to \mathbf{R}$ be a Morse function on a closed Riemannian manifold M. Thus, f is a smooth function, with isolated, non-degenerate critical points. The *gradient vector field* $\mathrm{grad}(f)$ of f is the vector field on M corresponding to the 1-form df, under the duality given by the Riemannian metric. A *flow line* of $\mathrm{grad}(f)$ is by definition a curve γ in M which satisfies the global differential equation

$$\frac{d\gamma}{dt} = -\mathrm{grad}(f).$$

By the compactness of M, flow lines are actually maps from the entire real line $\mathbf{R}$ to the manifold M. Given $x \in M$, let $\gamma_x(t)$ be the unique flow line of $\mathrm{grad}(f)$ which satisfies $\gamma_x(0) = x$. It is not difficult to show that both $\lim_{t\to\infty} \gamma_x(t)$ and $\lim_{t\to-\infty} \gamma_x(t)$ exist and are critical points of f. We denote the set of critical points of f by $\mathrm{Crit}(f)$. Since f is a Morse function, it is a finite subset of M.

An important quantity attached to a critical point a of f is its *index*. It can be defined as the number of independent directions in the tangent space of a, such that f decreases along the corresponding curves. More precisely, it is the number of negative eigenvalues of the *Hessian matrix* of f at a

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_a$$

in any system of local coordinates. The index of the critical point a will be denoted by λ_a.

Let a be any element of $\mathrm{Crit}(f)$. Consider the following subsets of M:

$$W^u(a) = \{x \in M | \lim_{t\to-\infty} \gamma_x(t) = a\},$$

$$W^s(a) = \{x \in M | \lim_{t\to\infty} \gamma_x(t) = a\},$$

the so-called *unstable* and *stable manifolds* of grad(f). In words, $W^u(a)$ is the set of points of M which flow toward a in negative time under the action of the gradient flow, while $W^s(a)$ consists of those points which flow toward a in positive time. The following *stable manifold theorem* is classical and a proof can be found in [16].

Theorem 1 *Both $W^u(a)$ and $W^s(a)$ are submanifolds of M. They are diffeomorphic to Euclidean spaces. Their dimensions are related to the index λ_a of a by the equalities*

$$\dim[W^u(a)] = \lambda_a = \operatorname{codim}[W^s(a)]. \blacksquare$$

A Morse function f on a Riemannian manifold M is said to be *Morse-Smale* if it satisfies the following *transversality condition*: for any pair a, b of critical points of f, $W^u(a)$ is transverse to $W^s(b)$. If f is Morse-Smale, we will denote by $W(a,b)$ the transverse intersection of $W^u(a)$ with $W^s(b)$. It follows that $W(a,b)$ is a smooth manifold of dimension $\lambda_a - \lambda_b$. Let t be a regular value of f such that $t < f(a)$ and the interval $[t, f(a))$ contains no critical value. The stable manifold theorem and the Morse lemma imply that the intersection $W^u(a) \cap f^{-1}(t)$ is a sphere of dimension $\lambda_a - 1$. This is the *unstable sphere* of the critical point a. Thus, any flow which "starts" at the critical point a meets $S^u(a)$ in a unique point.

One very important consequence of this transversality condition is the *transitivity property*, proved by Smale [22]. It says that if there is a flow from a to b and a flow from b to c, then there is a flow from a to c. Hence, there is a partial order on Crit(f), the so-called *Smale order*, defined as follows: $a \prec b$ if and only if $W(a,b)$ is not empty. Alternative formulations of this

relation are (i) there is a flow from a to b, and (ii) there is a flow γ satisfying $\lim_{t\to-\infty}\gamma(t) = a$, $\lim_{t\to+\infty}\gamma(t) = b$.

The next theorem shows that Morse-Smale functions are "generic". A good reference for this result is the paper of Austin and Braam [3].

Theorem 2 *The set of Morse-Smale functions $f : M \to \mathbf{R}$ is a Baire subset of the set of all smooth functions $C^\infty(M, \mathbf{R})$, i.e. is an intersection of a countable family of dense open sets.* ∎

From now on, our functions will always be Morse-Smale.

There is a smooth, free action of the real numbers on the manifold $W(a,b)$ given by the gradient flow of f:

$$W(a,b) \times \mathbf{R} \to W(a,b)$$

$$(x,t) \mapsto \gamma_x(t)$$

where γ_x is the unique flow line of grad(f) satisfying $\gamma_x(0) = x$. Choose any regular value t of f, belonging to the interval $(f(b), f(a))$, and consider the level hypersurface $f^{-1}(t)$. It is clear that $W(a,b) \cap f^{-1}(t)$ is a submanifold of M. Moreover, there is a natural diffeomorphism

$$(W(a,b) \bigcap f^{-1}(t)) \times \mathbf{R} \to W(a,b)$$

$$(x,t) \mapsto \gamma_x(t).$$

Thus, the quotient space

$$W(a,b)/\mathbf{R} = M(a,b)$$

is a manifold of dimension $\lambda_a - \lambda_b - 1$, diffeomorphic to $W(a,b) \cap f^{-1}(t)$. The manifold $M(a,b)$ is called the *moduli space of flow lines from a to b*. As

we have seen, it can be embedded in suitable regular level hypersurfaces of f, and in particular, in the unstable sphere $S^u(a)$. For our purposes, this is the best way to think of it.

The moduli spaces of flow lines have been used by many authors, in several Morse theoretic contexts, notably in infinite dimensional Morse theory (see [11] and [13]). The main goal of this thesis is to exploit the properties of these spaces in classical Morse theory.

The main tool will be *framed stratified sets*. It is an adaptation of the concept of *framified set*, introduced by Buoncristiano, Rourke and Sanderson [7] in the PL category. Roughly speaking, a framed stratified set is an abstract stratified set, in the sense of Thom-Mather [20], in which the tubular neighbourhood of any stratum is equipped with a trivialisation. Moreover, these trivialisations must satisfy some compatibility conditions. More formally, a framed stratified set is a pair (A, F) where A is the underlying topological space and F is itself a pair $(\mathcal{S}, H)$ with $\mathcal{S}$ the set of strata and H the set of trivialisations of the tubular neighbourhoods of the strata.

Our definition allows the strata to be *manifolds with faces*. Verona [24] considers stratified sets with faces, but with no trivialisations of the tubular neighbourhoods while Buoncristiano and Dedo [6] work with framed stratified sets in our sense, but their strata are ordinary manifolds or manifolds with boundary.

A framed stratified set is a natural extension of the idea of a framed manifold. While a framed manifold can be constructed out of a function with values in a sphere, a framed stratified set can be build out of a nice map with values in a nice CW complex. In order to make precise what one means by "nice", we must introduce the concept of a transverse map from

a manifold to a CW complex. It is basically a natural generalization of the idea of a Pontryagin-Thom map. As any map from a sphere to a sphere is homotopic to a Pontryagin-Thom map, in the same manner, any map from a manifold with faces to a CW complex is homotopic to a transverse one. This is the content of the basic transversality result proved in section 2.2. A proof in the PL category is supplied in [7].

We will say that a CW complex X is *transverse*, if all the attaching maps of X are transverse to the relevant skeletons. The transversality result says that a map from a manifold with faces to a transverse CW complex is homotopic to a transverse map. We then show that any transverse map $\phi : M \to X$ determines a framed stratification of M by submanifolds with faces, in much the same way that a map from a manifold to a sphere determines a framed submanifold. This framed stratification will be denoted by $F(\phi, X) = (\mathcal{S}(\phi, X), H(\phi, X))$. Here, $\mathcal{S}(\phi, X)$ is the set of strata of the stratification and $H(\phi, X)$ is a set of framings for the strata. With the above notations, $(M, F(\phi, X))$ is a framed stratified set. It is *modelled* on X in the sense that the strata of $F(\phi, X)$ correspond to the cells of some subcomplex of X and that the attaching map of a cell in this subcomplex is "encoded" into the link of the corresponding stratum. The precise definitions are given in 2.4.

The outcome of this work is a generalized "Pontryagin-Thom" correspondence. Let $[M, X]$ denotes the set of homotopy classes of maps from the manifold M to the transverse CW complex X and $\Omega_X(M)$ the set of cobordism classes of framed stratifications of M modelled on X. The cobordism class of a framed stratification F of M will be denoted by $[F]$. The following theorem is a smooth version of a result in [7].

Theorem A *There is a bijection*

$$\mathcal{F} : [M, X] \to \Omega_X(M)$$

given by $[\phi] \mapsto [F(\phi, X)]$.

Let us now illustrate how one can apply these concepts to Morse theory. We consider first the simplest case. Let a and b be two critical points of f which are *successors* in the Smale order, i.e. such that $a \prec b$ and there is no critical point z satisfying $a \prec z \prec b$. We will denote this relation by $a \prec_s b$. By the Morse lemma, there is a coordinate neighbourhood U of b such that f restricted to U can be written as a quadratic polynomial:

$$f(x_1, \ldots, x_m) = f(b) - x_1^2 - \ldots - x_{\lambda_b}^2 + x_{\lambda_b+1}^2 + \ldots + x_m^2.$$

The *stable sphere* $S^s(b)$ of b is simply the unstable sphere of b for the Morse function $-f$. It is easy to see that, in the coordinate system introduced above, the stable sphere $S^s(b)$ can be identified with

$$S^s(b) = \{(0, \ldots, 0, x_{\lambda_b+1}, \ldots, x_m) \in U \mid x_{\lambda_b+1}^2 + \ldots + x_m^2 = \epsilon\},$$

for $\epsilon > 0$ sufficiently small.

It then becomes clear that the stable sphere $S^s(b)$ has a natural framing in the level hypersurface $f^{-1}(f(b) + \epsilon)$, and that this framing extends to a framing of the stable manifold $W^s(b)$ in M. Now, recall that the critical point a satisfies $a \prec_s b$. It follows that the moduli space $M(a, b)$ is a compact closed manifold. This is explained in 3.1. It can be seen as the intersection $S^u(a) \cap W^s(b)$. It is not difficult to show that the framing of $W^s(b)$, when

restricted to $M(a,b)$, determines a framing of $M(a,b)$ in $S^u(a)$. We denote this framing by $F(a,b)$.

This is the simplest example of a framed stratified subset of the unstable sphere $S^u(a)$, determined by the gradient flow. It is easy to show, see section 3, that any moduli space of the form $M(a,b)$, can be seen as a framed submanifold (generally non-compact) of $S^u(a)$ even if b is not a successor of a.

Let us denote by $\mathcal{S}(a)$ the set of moduli spaces $M(a,z)$ with $a \prec z$. Let $H(a)$ be the corresponding set of framings and let us denote by $F(a)$ the pair $(\mathcal{S}(a), H(a))$.

Theorem B *The pair $(S^u(a), F(a))$ is a framed stratified set.*

Franks shows in [14] how to construct a CW complex X_f, well-defined up to cell equivalence, out of a Morse-Smale function f, when the gradient field of f is linear about the critical points. Actually, Franks phrases his results in terms of *gradient-like* vector fields as we do in 3.4. For the sake of clarity, we will carry on our discussion in terms of Morse functions.

The complex X_f is homotopy equivalent to M and is called the *Morse complex* of f. The set of cells of X_f is in bijection with $\mathrm{Crit}(f)$. The construction of X_f is inductive and at each step cells corresponding to critical points are attached. Let us be more precise. Consider a critical point a of f. Let t_1 and t_2 be regular values of f satisfying $f(t_1) < f(a) < f(t_2)$ and $f([t_1,t_2])$ contains no other critical values than $f(a)$. The stable manifold theorem and the Morse lemma imply that

$$W^u(a) \bigcap f^{-1}([t_1,t_2])$$

is a λ_a-disk D^{λ_a}. Notice that the boundary ∂D^{λ_a} of D^{λ_a} can be identified with the unstable sphere $S^u(a)$ of a.

In the CW decomposition of M induced by f, one attaches D^{λ_a} to the $(\lambda_a - 1)$-skeleton $X_f^{\lambda_a - 1}$ of X_f by a map

$$\phi_a : \partial D^{\lambda_a} = S^u(a) \to X_f^{\lambda_a - 1},$$

the *attaching map* of a. Now, let $r < \lambda_a$ be the largest integer for which there is a critical point b of index r and a flow from a to b. It is not difficult to show that the attaching map ϕ_a can be factorized through the r-skeleton of X_f^r:

$$\phi_a : S^u(a) \to X_f^r \subset X_f^{\lambda_a - 1}.$$

If one collapses the $(r-1)$-skeleton of X_f^r, one gets a wedge of r-spheres, and each of these spheres correspond to a critical point of index r. Consider now the composition

$$\varphi_{a,b} : S^u(a) \xrightarrow{\phi_a} X_f^r \to X_f^r / X_f^{r-1} \simeq \bigvee S^r \to S_b^r$$

where the second and the third maps are the natural projections. Here, S_b^r is the image of the cell corresponding to the critical point b. Notice that b is a successor of a. The map $\varphi_{a,b}$ will be called the *primary attaching map* of the critical point a with respect to the critical point b. In [14], Franks interprets these maps in terms of the gradient flow of f.

Theorem 3 (Franks) *The closed framed submanifold $M(a,b)$ of $S^u(a)$ corresponds to the map $\phi_{a,b} : S^u(a) \to S_b^r$ under the (classical) Pontryagin-Thom correspondence.* ■

Thus, up to homotopy, the framed compact moduli spaces determine the *primary* attaching maps of X_f. In this thesis, we show that one can actually recover the *absolute* attaching maps of the complex X_f if one considers the framed stratified sets given by Theorem B.

Recall that $\Omega_X(M)$ denotes the set of cobordism classes of framed stratifications of M that are modelled on X.

Let $(S^u(a), F(a))$ be the framed stratified set constructed in Theorem B, and let $F(\varphi_a, X_f)$ be the framed stratification constructed out of the attaching map of the critical point a (it exists, by Theorem A).

Theorem C *The cobordism classes* $[F(a)]$ *and* $[F(\phi_a, X_f)]$ *coincide in* $\Omega_{X_f}(S^u(a))$.

In other words, the framed stratifications by moduli spaces of flow lines determine the attaching maps in the Morse complex X_f.

The CW decomposition of M introduced above determines cellular cohomology groups which are of course isomorphic to the singular cohomology groups of the manifold M. Hence, there are well-defined integral cochains ω_a, ω_b and ω_c associated to the critical points a, b, c (we are assuming $\lambda_a > \lambda_b \geq \lambda_c$). We will denote by $S^u(a, b)$ the following union of moduli spaces

$$S^u(a, b) = \bigcup_{a \prec z \preceq b} M(a, z).$$

Notice that $S^u(a, b)$ is a framed stratified set.

Let a, b and c be critical points of f such that ω_a, ω_b and ω_c represents non-zero cocycles and $\lambda_a = \lambda_b + \lambda_c$. What can we say about the cup product of ω_b with $\omega_c$in terms of the gradient flow ? In order to get some information,

we must unfortunately make the following assumptions.

- The manifold M is simply-connected.
- If z is a critical point of f such that $z \in f^{-1}([c,a])$, then $W(z,b) = W(z,c) = \emptyset$.
- The primary attaching map g of b with respect to c has finite order.

Assuming this, we can interpret geometrically the cup product $\omega_b \smile \omega_c$. Recall that $\dim[S^u(a)] = \lambda_a - 1$. It is clear that $S^u(a,b)$ and $S^u(a,c)$ determine chains in $S^u(a)$, of dimension $\lambda_a - \lambda_b - 1$ and $\lambda_a - \lambda_c - 1$ respectively. Since $(\lambda_a - \lambda_b - 1) + (\lambda_a - \lambda_c - 1) = 2\lambda_a - (\lambda_b + \lambda_c) - 2 = \lambda_a - 2$, one can take the linking number of $S^u(a,b)$ and $S^u(a,c)$ in $S^u(a)$. Consider the sum $\omega_b \smile \omega_c = \sum_{\lambda_x = \lambda_b + \lambda_c} \alpha_x \omega_x$.

Theorem D *If a, b and c satisfy the above hypotheses, then*

$$\alpha_a = \mathrm{lk}(S^u(a,b), S^u(a,c)).$$

This provides a partial answer to a question of Franks [14]. The proof of this last result relies more on classical topological techniques (triad homotopy groups, Whitehead product) than on the theory of stratified sets. Alternative approaches to this question can be found in [3] and [5].

To conclude this introduction, let us establish a connection between the framed stratified sets $S^u(a,b)$ and the compactified moduli spaces $\mathcal{C}(a,b)$. For the latter, see [10] and also section 3.1 of this thesis. We recall briefly the definition of $\mathcal{C}(a,b)$.

It is easy to show that a Morse function f is strictly decreasing along the flow lines of $\mathrm{grad}(f)$. We can use this fact to reparametrize the flow lines as follows. Consider two critical points a, b with $a \prec b$ and pick a flow

$\gamma \in M(a,b)$. The function f determines a diffeomorphism of the image of the flow γ with the open interval $(f(b), f(a))$. One can use this diffeomorphism to reparametrize γ as a smooth curve

$$\omega : (f(b), f(a)) \to M$$

which satisfies $f(\omega(t)) = t$. By setting $\omega(f(b)) = b$ and $\omega(f(a)) = a$, one gets a continuous curve, that we also call ω,

$$\omega : [f(b), f(a)] \to M.$$

A straightforward calculation shows that ω satisfy the global differential equation

$$\frac{d\omega}{dt} = \frac{\mathrm{grad}(f)}{||\mathrm{grad}(f)||^2} \tag{1}$$

with boundary conditions

$$\omega(f(b)) = b, \ \omega(f(a)) = a. \tag{2}$$

The space $\mathcal{C}(a,b)$ is defined to be the space of all continous curves in M which are smooth on $M - \mathrm{Crit}(f)$ and are solutions of (1) on $M - \mathrm{Crit}(f)$, with boundary conditions (2). It is topologized as a subspace of the space of continuous curves $\mathrm{Map}([f(b), f(a)], M)$, where the latter has the compact-open topology.

Thus, elements of $\mathcal{C}(a,b)$ are in a very natural sense *piecewise flow lines* from a to b. If one removes the critical points from the image of a piecewise flow line, then what is left is a finite union of images of genuine flow lines.

It has been established that $\mathcal{C}(a,b)$ has the structure of a manifold with corners (actually, of a manifold with faces) with its interior diffeomorphic to $M(a,b)$, see 3.1 for some details. Notice that it is certainly possible to embed

$\mathcal{C}(a,b)$ into the unstable sphere $S^u(a)$, by simply pushing the boundary along collar lines and noting that $M(a,b)$ is naturally a submanifold of $S^u(a)$. However, this way of embedding $\mathcal{C}(a,b)$ does not say much about the relative positions of the moduli spaces inside $S^u(a)$. Perhaps more interesting in this respect is the "quotient" or "evaluation map" (see 3.2)

$$\mathcal{C}(a,b) \to S^u(a,b),$$

which is basically given by taking the intersection of the image of a piecewise flow line with the unstable sphere of a.

This map could be interpreted as some kind of "resolution of singularities" in the smooth category (a manifold with corners being less "singular" than a smooth stratified set). This is perhaps an illustration of the following difficult theorem of Verona [24]: "Any abstract stratification of finite depth can be obtained from a manifold with faces by making certain identifications on the faces". But, this is mere speculation ...

This thesis is organized as follows. In 2.1 we state and justify some well-known facts about manifolds with corners, faces, etc. In 2.2 we prove the basic transversality result which is central to the whole work. Framed stratified sets are introduced in 2.3 and the proof of Theorem A is given in section 2.4. In 3.1, we introduce some basic constructions in Morse theory. In 3.2 and 3.3, we prove that the unstable spheres are framed stratified sets (Theorem B). Theorem C, on the attaching maps, is proved in 3.4. Section 4 is entirely devoted to the proof of Theorem D. Since the ideas involved in this proof have a somewhat different "flavor" from those used in the previous sections, we have included a separate introduction for this last section.

2 A Pontryagin-Thom correspondence

2.1 Manifolds with faces

A *k-sector* of $\mathbf{R}^n$ ($k \leq n$) is a subset of $\mathbf{R}^n$ (with the subspace topology) of the form $\mathbf{Q}_k^n = \mathbf{R}_+^k \times \mathbf{R}^{n-k}$, where $\mathbf{R}_+ = [0, \infty)$. Let U be an open subset of $\mathbf{Q}_k^n$ and U' an open subset of $\mathbf{Q}_l^m$. A map $f : U \to U'$ is *smooth* if it is the restriction of a smooth map from a neighbourhood of U in $\mathbf{R}^n$ to a neighbourhood of U' in $\mathbf{R}^m$. According to Whitney [25], f is smooth if f restricted to $\mathrm{Int}(U)$ (with respect to $\mathbf{R}^n$) has the following property: it is smooth and all its partial derivatives (of all orders) have continuous extensions to U. The definition of diffeomorphism is as one expects. If $f : U \to U'$ is a diffeomorphism (U and U' are as above), then $n = m$ and $k = l$.

A *manifold with corners* M of dimension m is a manifold M, locally modelled on sectors $\mathbf{Q}_k^m$ of $\mathbf{R}^m$, such that all the transition maps are diffeomorphisms in the above sense. For any $x \in M$, one can always choose a chart $\phi : U \to \mathbf{Q}_k^m$ such that $\phi(x) = 0$. Such a chart will be called a *centered chart* at x. By the preceding remarks, the number k is independent of the chart, and is called the *index* of x. This assignment of an integer to each point of M determines a map $\iota : M \to \mathbf{Z}_+$.

Let M be a manifold with corners. For the sake of simplicity, we will often take our charts with values in the full sector $\mathbf{Q}_m^m = \{x \in \mathbf{R}^m | x_1 \geq 0, \ldots, x_m \geq 0\}$. It is clear that $\iota(x)$ is then simply the number of zeroes in the coordinates of $\phi(x) \in \mathbf{Q}_m^m$. From now on, we will write $\mathbf{Q}_m$ for $\mathbf{Q}_m^m$.

A closed subset N of M is a *submanifold with corners* of dimension n if for any $y \in N$, one can find an open set U in M containing y and

a chart $\phi : U \to \mathbf{Q}_m$ with ϕ restricted to $U \cap N$ a diffeomorphism into $\mathbf{Q}_m \cap \{x \in \mathbf{R}^m | x_{n+1} = \ldots x_m = 1\}$. Clearly, N is then itself a manifold with corners. When the ambient manifold is a manifold with boundary, this concept of submanifold coincides with the standard definition of *neat submanifold*, as given, for example, in [17] or [19]. Notice that if $y \in N$, the index $\iota(y)$ of y as a point of N coincides with its index as a point of M.

As usual, one has the concepts of normal bundle and tubular neighbourhood, see [12] and [18]. We recall quickly the points which are relevant for us. The *normal bundle* of N in M is defined to be the quotient bundle $\nu = TM/TN \to N$. A *tubular neighbourhood* of N in M is a diffeomorphism h from an open neighbourhood U of the zero section N in ν, onto an open neighbourhood U' of N in M which satisfies the following properties.

1. The map h restricted to the zero section is the inclusion map.

2. The derivative of h induces the identity map $Id : \nu \to \nu$ of the normal bundle of N in U into the normal bundle of N in U'.

It has been shown by Cerf (see [9] or [12]) that any submanifold admits such a tubular neighbourhood.

Following Jänich [18], we introduce the category of (k)-manifolds, which is the natural one for our purpose. The idea is quite simple. If one removes a small tubular neighbourhood of a submanifold of some closed manifold, then the result is a manifold with boundary. Removing a neighbourhood of a neat submanifold (in the above sense) of this manifold with boundary yields a certain type of manifold with corners, etc. Roughly speaking, a (k)-manifold is a manifold with corners that can be obtained by this process in k steps. Thus, a (0)-manifold is simply an ordinary closed manifold, a

(1)-manifold is a manifold with boundary, and so on. To be more precise, one needs to introduce the important intermediate concept of manifold with faces.

A manifold with corners M is a *manifold with faces* if each $x \in M$ belongs to the closure of exactly $\iota(x)$ connected components of the set $\{x \in M | \iota(x) = 1\}$. Recall that $\iota(x)$ is by definition the number of zeroes in any expression of x in local coordinates. The closure of a connected component of this set is called a *connected face* of M and is a manifold with corners. A *face* of M is a union of pairwise disjoint connected faces of M. A face is itself a manifold with faces in a natural way.

A good example of a manifold with faces is the compactification $\mathcal{C}(a, b)$ of the moduli space of flow lines from a to b, mentioned in the preceding section (see also section 3). In this case, the index $\iota(\gamma)$ can be interpreted as the number of "breaks" in the piecewise flow line γ.

Let M be a manifold with faces. A *(k)-manifold structure* on M is a decomposition of the topological boundary ∂M of M into k pieces $\partial_i M$ $(i = 0, \ldots, k-1)$ which satisfy the following two conditions.

1. The pieces $\partial_i M$ are faces of M, and

2. $\partial_i M \cap \partial_j M$ is a face of both $\partial_i M$ and $\partial_j M$.

A *(k)-manifold* is a manifold with faces with some fixed (k)-structure. Notice that any face of a (k)-manifold M is a (k-1)-manifold in a natural way.

The decomposition of ∂M is in not unique and a face may be empty. For example, consider the 3-cube $[-1, 1]^3$. The decomposition given by the 6 connected faces determines a structure of (6)-manifold on $[-1, 1]^3$. But if one takes as a face the union of two antipodal faces, the 3-cube becomes a

(3)-manifold. It seems however that the n-simplex has an essentially unique structure of (n+1)-manifold if $n > 1$.

Each face $\partial_i M$ admits a *collar neighbourhood*

$$F_i : U_i \to \partial_i M \times \mathbf{R}_+$$

given by $F_i(x) = (p_i(x), r_i(x))$ where both p_i and r_i are smooth and F_i is a diffeomorphism from the open neighbourhood U_i of $\partial_i M$ in M to $F_i(U_i)$. These collars must satisfy the usual compatibility conditions. For our purposes, the most important property is the commutation relation

$$p_i \circ p_j = p_j \circ p_i,$$

whenever both sides are defined, and we will freely refer to it as the *compatibility condition* for the collar system $\{F_i\}$. For more details, we refer to [18] and [24].

If N is a submanifold of a (k)-manifold M (more precisely, a submanifold of the underlying manifold with corners), it is easy to see that N has a natural structure of (k)-manifold.

The structure of (k)-manifold is interesting for the following reason. If E is a tubular neighbourhood of N in M, the set $\mathrm{cl}[M - E]$ has a natural structure of a (k+1)-manifold, see [18]. Notice that if the decomposition of ∂M is given by

$$\partial M = \partial_0 M \cup \ldots \cup \partial_{k-1} M$$

then a decomposition of the topological boundary of $M_0 = \mathrm{cl}[M - E]$ will be given by

$$\partial M_0 = \partial_0 M_0 \cup \ldots \cup \partial_{k-1} M_0 \cup \partial_k M_0.$$

For us, the most important point here is that the k-th face $\partial_k M_0$ can be identified with the sphere bundle of the normal bundle of N in M:

$$\partial_k M_0 \xrightarrow{\simeq} \Sigma(\nu_M(N)).$$

This is an essential fact, and it will be used many times in what follows.

If $f : M \to Z$ is a map from a (k)-manifold to an arbitrary topological space Z, we will write $\partial_i f$ for the restriction of f to the face $\partial_i M$, and ∂f for its restriction to ∂M.

Consider any smooth map $f : \mathbf{Q}_m \to \mathbf{R}^l$. Recall that $\mathbf{Q}_m = \mathbf{R}_+ \times \ldots \times \mathbf{R}_+$ (m times). Suppose that $y \in \mathbf{R}^l$ is a regular value of f. Assume additionaly that y is a regular value of f restricted to the bounding hyperplanes of $\mathbf{Q}_m$ and of f restricted to all the possible multi-intersections of these hyperplanes. We will say in this case that y is a *nice regular value* of the map f. Clearly, the definition of nice regular value is independent of coordinates. Thus, the concept of nice regular value for maps between manifolds with corners is well-defined. The following fact is easy to prove.

Lemma 4 *Let f be a smooth map from a manifold with corners M to $\mathbf{R}^l$. Consider any nice regular value y of f. Then, $f^{-1}(y) = N$ is a submanifold with corners of M, of codimension l.*

Sketch of the proof. We work in local coordinates. Assume $y = 0 \in \mathbf{R}^l$. Choose any x in N and consider the index $\iota(x)$ of x. This integer has the following interpretation: x belongs to a stratum X of $\mathbf{R}_+ \times \ldots \times \mathbf{R}_+ = \mathbf{Q}_m$, which is a $\iota(x)$-fold intersection of bounding hyperplanes. A 1-fold intersection is an hyperplane, a 2-fold intersection has dimension $m-2$,... and clearly, $\dim[X] = m - \iota(x)$. Now, by hypothesis, x is a regular point of f restricted to X and consequently $\iota(x) + l \leq m$.

There are exactly $\imath(x)$ coordinates of x which are 0 and one can certainly assume (permuting) that they occupy the first $\imath(x)$ entries in $x = (x_1, \ldots, x_m)$, i.e. $x = (0, \ldots, 0, x_{k+1}, \ldots, x_m)$ where $k = \imath(x)$. Choose a neighbourhood U of x in $\mathbf{R}^m$, so small that any $z \in U \cap \mathbf{Q}_m$ satisfies $\imath(x) \geq \imath(z)$, and that $f(U \cap \mathbf{Q}_m)$ is in some sufficently small box of $\mathbf{R}^l$ centered at $f(x) = 0$. For example, take the box $[-\epsilon, \epsilon]^l$ with $0 < \epsilon < 1$. Consider now the following self-map of $\mathbf{R}^m$: $\phi(x) = (x_1, \ldots, x_k, f_1(x) + 1, \ldots, f_l(x) + 1, x_{k+l+1}, \ldots, x_m)$. Writing the Jacobian of ϕ at x, one sees that it is a linear isomorphism and thus ϕ is a local diffeomorphism. Shrinking if necessary, one can assume that ϕ restricted to U is a diffeomorphism onto its image. It is easy to see that $\phi(U \cap N) \subset \{x \in \mathbf{R}^n | x_1 \geq 0, \ldots, x_k \geq 0, x_{k+1} = \ldots = x_{k+l} = 1, x_{k+l+1} \geq 0, \ldots, x_m \geq 0\}$, and this shows that N is a submanifold. ∎

A most important fact about manifold with faces is that they admit *partitions of unity*, see [24]. It follows that many constructions and techniques from classical differential topology can still be performed. An important example is the following approximation lemma. Let K be a closed subset of a manifold with faces M and f be a continuous map from M to some manifold N. We will say that f is *smooth on the closed set* K if for every $x \in K$, there is an open neighbourhood U in M containing x, and a smooth function g from U to N such that f restricted to $U \cap K$ coincides with g.

Lemma 5 *Let $f : M \to N$ be a continuous map from a manifold with faces M to a manifold N which is smooth on a closed subset K of M. Let $\epsilon > 0$ be any positive real number. Then, there is a smooth map $g : M \to N$ that*

agrees with f on K. Moreover

$$|f(x) - g(x)| < \epsilon$$

for any $x \in M$. In particular, g can be chosen to be homotopic to f.

Proof. A proof, for M closed, can be found in [19]. ∎

It is also quite important to note that Sard's theorem holds in this context. Indeed, if a point y in N is a critical value of f (i.e. is not a nice regular value), then there is a stratum of M (by stratum we mean either $\mathrm{Int}[M]$ or the interior of an intersection of faces of M) such that $f|_X$ is not a submersion on the set $(f|_X)^{-1}(y)$. But since these strata are ordinary smooth manifolds, the set of such y is a finite union of sets of measure zero, and is thus itself of measure zero. It follows that the set of nice regular values is dense.

To end this section, let us say a few words about *global collars* for manifolds with corners i.e. smooth neighbourhood of the topological boundary of M. For our purpose, there is a nice way to construct them. The construction, given in [12], is simple but important, since it also gives a way to embed any manifold with corners M into some open smooth manifold, namely its own interior!

Let x be any point in the manifold with corners M. Consider a centered chart $\psi : U \to \mathbf{Q}_k^n$, where U is an open set of M containing x. The image by $(d\psi_x)^{-1}$ of $\mathbf{Q}_k^n$ is a convex region of the tangent space T_xM, denoted by P_xM. Let θ be a smooth vector field on M such that θ_x belongs to the *interior* of P_xM for all $x \in M$. A vector field satisfying this property is called an *inner vector field* on M. Using a partition of unity, one can show that there is always such a vector field on a manifold with corners. To each

x is associated an interval I_x (half-open, or open containing 0), and the integral curve of θ through x, a map from I_x to M. It is not difficult to modify θ to make sure that the upper bound of I_x is greater than 1, for all x. Let γ be the flow generated by θ. One can define a smooth map

$$c : \partial M \times [0, 1] \to M$$

in the following way: $c(x, s) = \gamma_x(s)$. Notice that $c(-, s)$ is a diffeomorphism. Clearly, $c(x, 0) = x$ and $c(\partial M \times [0, 1])$ is a neighbourhood of ∂M in M. We will refer to c as a *global collar* for ∂M in M to distinguish it from the collars of the faces of M.

2.2 Transversality and control

In the following two sections, we transfer some constructions and results of [7] from the PL to the smooth category (with corners or faces). The concept of a *transverse map* from a (k)-manifold to a CW complex is crucial.

Let $f : M \to Z$ be a continuous map from a (k)-manifold M to a topological space Z. Assume that a system $\{F_i\}$ of compatible collars of the faces of M has been chosen. We will say that f is *controlled* (or "compatible with the faces", in Verona's terminology [24]) if

$$\partial_i f \circ p_i = f,$$

whenever the left-hand side of the equation is defined. We recall that $\partial_i f$ is the restriction of f to the face $\partial_i M$ of M and that p_i is determined by a given collar F_i of $\partial_i M$, see 2.1. Thus, in the neighbourhood of a face, a controlled map is determined by its values on the boundary. One has the following result.

Lemma 6 *Let $f : M \to Z$ be a continuous map from a (k)-manifold M to a topological space Z. If $\partial_i f$ is controlled for all i, then f is homotopic rel ∂M to a controlled map. If Z is a smooth manifold and if $\partial_i f$ is smooth and controlled for all i, then f is homotopic rel ∂M to a smooth controlled map.*

Proof. First, one solves the problem in a neighbourhood of the boundary of M. Consider an open neighbourhood of ∂M given by a union of collars of faces of M. One can assume that this neighbourhood is of the form

$$U = \bigcup_i F_i^{-1}(\partial_i M \times [0, 1)),$$

and consider the closed subset

$$V = \bigcup_i F_i^{-1}(\partial_i M \times [0, 1/2]).$$

Define a map $g : V \to Z$ in the following way: if $y \in V$ can be written as $y = F_i^{-1}(x, s)$ for some i, then $g(y)$ is defined to be $\partial_i f(x)$. Let us check that g is well-defined. If $F_i^{-1}(x, s) = y = F_j^{-1}(z, t)$ for some $j \neq i$, then clearly $x = p_i(y)$ and $z = p_j(y)$. By the compatibility conditions, $p_j(x) = p_j p_i(y) = p_i p_j(y) = p_i(z)$. Since $\partial_i f$ and $\partial_j f$ are controlled (by hypothesis), one gets

$$\partial_i f(x) = \partial_j \partial_i(f) \circ p_j(x) = \partial_i \partial_j(f) \circ p_i(z) = \partial_j f(z)$$

and g is well-defined, controlled and smooth, provided that $\partial_i f$ is smooth for all i. Clearly, g is homotopic rel ∂M to $f|_V$. Since V is a retract of U, the homotopy extends to U. By the Homotopy Extension Property ([23], p. 57), the homotopy extends to M. By the approximation lemma 5, one is finished. ∎

We now introduce the notion of a transverse map from a (k)-manifold to a CW complex. It is a natural generalization of the idea of a Pontryagin-Thom map associated to a framed submanifold. In fact, when one considers maps from closed manifolds to spheres, our definition gives exactly those maps which are obtained by the classical Pontryagin-Thom construction applied to framed submanifolds.

Let X be a CW complex with a fixed cellular decomposition. Then X is a disjoint union of cells

$$\bigcup_{\lambda \in \Lambda} e_\lambda$$

and each cell has a characteristic map $h_\lambda : D_\lambda \to X$. Let f be a continuous map from a manifold M to X and set

$$T_\lambda = \mathrm{cl}[f^{-1}(e_\lambda)].$$

We define transversality inductively and give the definition for (0)-manifold first. Remember that a (0)-manifold is simply an ordinary closed manifold.

The map $f : M \to X$ from the closed manifold M to the CW complex X is *transverse* if the decomposition $M = \bigcup_{\lambda \in \Lambda} T_\lambda$ has the following properties.

(t1) For any $\lambda \in \Lambda$, T_λ is a manifold with faces.

(t2) Let $t_\lambda : T_\lambda \to D_\lambda$ be the unique map whose restriction to $f^{-1}(e_\lambda)$ is equal to $h_\lambda^{-1} \circ f$. Then

$$t_\lambda : T_\lambda \to D_\lambda$$

is a smooth fibre bundle, whose fibres are manifolds with faces.

(t3) The bundle map $t_\lambda : T_\lambda \to D_\lambda$ is controlled.

(t4) The set $M_\lambda = \mathrm{cl}[M - T_\lambda]$ is a manifold with faces and f restricted to M_λ is controlled.

We will say that X is a *transverse CW complex* if all the attaching maps in X are transverse. Assume now that we know what is a transverse map from a (k-1)-manifold to a CW complex X. A map $f : M \to X$ from a (k)-manifold to a CW complex X is *transverse* if the decomposition $M = \bigcup_{\lambda\in\Lambda} T_\lambda$ satisfies the conditions $t1, \ldots, t4$ and if moreover the restrictions $\partial_i f : \partial_i M \to X$ are transverse (recall that $\partial_i M$ can be given the structure of a (k-1)-manifold) for $i = 0, \ldots, k-1$.

It follows from this definition that

(t5) there is a factorization $f|_{T_\lambda} : T_\lambda \xrightarrow{t_\lambda} D_\lambda \xrightarrow{h_\lambda} X$ and f is a controlled map.

As a simple example, notice that a constant map is transverse if and only if it maps to a 0-cell (otherwise it would have to surject on some k-cell with $k > 0$ and this is clearly impossible).

Proposition 7 *Let $f : M \to X$ be a map from a compact (k)-manifold M to a transverse CW complex X. Suppose that $\partial_i f$ is transverse to X, for $i = 0, \ldots, k-1$. Then f is homotopic rel ∂M to a transverse map.*

Proof.

Step 1: proof in a neighbourhood of ∂M.

By hypothesis, the restrictions $\partial_i f$ of f are transverse, and thus controlled by t5. By the preceding lemma, one can assume that f is itself controlled, up to a homotopy which does not change f on the boundary. By definition

of controlled map, there is an open neighbourhood V of ∂M in M such that

$$\partial_i f \circ p_i(x) = f(x)$$

for all $x \in V \cap U_i$. We recall that U_i is a collar neighbourhood of the face $\partial_i M$.

We claim that this forces $f|_V$ to be transverse. To see this, pick any cell e in X and consider

$$T = \mathrm{cl}[(f|_V)^{-1}(e)].$$

For each face $\partial_i M$ of M, which intersects T, one gets a smooth controlled bundle map

$$t_i : \mathrm{cl}[(\partial_i f)^{-1}(e)] = T_i \to D$$

satisfying (t1), ..., (t4), since $\partial_i f$ is transverse. In order to prove that $f|_V$ is transverse, one must define a map

$$t : T \to D$$

which also satisfies (t1), ..., (t4). This is easily done by setting

$$t(x) = t_i \circ p_i(x)$$

when $x \in U_i \cap V$. Let us check that t is well-defined, i.e. that for any $x \in U_i \cap U_j \cap V$, $t_i \circ p_i(x) = t_j \circ p_j(x)$. First, since t_i is controlled with respect to p_j restricted to $\partial_i M$, it follows that

$$t_i \circ p_i(x) = t_i \circ (p_j \circ p_i(x)).$$

However,

$$t_i \circ (p_j \circ p_i(x)) = t_j \circ (p_j \circ p_i(x))$$

by (t5). By the compatibility conditions for collars,

$$t_j \circ (p_j \circ p_i(x)) = t_j \circ (p_i \circ p_j(x)).$$

Applying a last time the control condition, one gets

$$t_j \circ (p_i \circ p_j(x)) = t_j \circ p_j(x)$$

and this shows that t is well-defined.

It remains to check that t satisfies the four transversality conditions. Let us verify that T is a manifold with faces. We can assume that $V \subset \cup_i U_i$. Thus, any $x \in T$ belongs to some U_i. It is clear that one can take a coordinate neighbourhood $O \subset T_i$ such that x is in the open set $p_i^{-1}(O)$. Since F_i is a diffeomorphism, $F_i(p_i^{-1}(O))$ is open in $\partial_i M \times \mathbf{R}_+$. But clearly, $F_i(p_i^{-1}(O)) = O \times r_i(p_i^{-1}(O))$. Since r_i is an open map, this is an open set of $T_i \times \mathbf{R}_+$. Thus, one can use F_i to construct a coordinate neighbourhood around $x \in T$. It follows that T is a manifold with faces, i.e. (t1) holds.

Using the same idea, one gets that $V_0 = \mathrm{cl}[V - T]$ is also a manifold. If $x \in V_0$, then $p_i(x) \in \mathrm{cl}[\partial_i M - T_i]$ for some i. One knows that this space is a manifold. Thus the trick used above applies and V_0 is a manifold. This proves half of (t4).

The map $t : T \to D$ is clearly a surjective submersion and its fibres are all diffeomorphic. Thus condition (t2) holds. Condition (t3) follows from the equalities

$$\begin{aligned} t \circ p_i(x) &= t_j \circ p_j \circ p_i(x) \text{ (by definition)} \\ &= t_j \circ p_i \circ p_j(x) \text{ (compatibility)} \\ &= t_j \circ p_j(x) \qquad \text{(control)} \\ &= t(x) \qquad \text{(definition)} \end{aligned}$$

which show that t is a controlled map.

Finally, condition (t4) holds for the following reason. Pick any compatible collar p for the face $\mathrm{cl}[\partial T - \partial M]$ of $\mathrm{cl}[V - T]$, and consider the restriction of $f|_V$ to $\mathrm{cl}[V - T]$. Since $f|_V$ is controlled, and since $\partial_i f$ restricted to $\mathrm{cl}[\partial_i M - T_i]$ is controlled,

$$\begin{aligned} f \circ p(x) &= \partial_i f \circ p_i \circ p(x) \\ &= \partial_i f \circ p \circ p_i(x) \\ &= \partial_i f \circ p_i(x) \\ &= f(x) \end{aligned}$$

Thus $f|_V$ is controlled with respect to p. It is clear that f is controlled with respect to the other collars, since they are obtained by restricting the original collars of ∂M.

Step 2: proof for $X = S^p$, with cellular decomposition $* \cup e$.
Let us first approximate f by a smooth map, without perturbing it on ∂M (where it is already transverse). Let $h : D \to D/\partial D = S^p$ be a smooth characteristic map for the CW complex S^p. By Sard's theorem, one can assume that $h(0)$ is a nice regular value of f. It follows that $N = f^{-1}(h(0))$ is a smooth submanifold with corners of M, which inherits from M a structure of (k)-manifold.

Now, it is necessary to prove that $f : M \to S^p$ can be identified with a projection map on the inverse image of a sufficently small neighbourhood of any nice regular value. When M is closed, it is a standard fact, see for example [19]. Examining the proof, one sees that is easily generalizes to maps from manifolds with corners to ordinary manifolds.

Consider an open subdisc $D(\epsilon) \subset D$ of radius ϵ. Let $e(\epsilon)$ be the image of $D(\epsilon)$ by the characteristic map $h : D \to S^p$. It follows from the preceding remark that if $\epsilon > 0$ is small enough, then f restricted to $f^{-1}(e(\epsilon))$ is a bundle map. It can be factorized as

$$f^{-1}(e(\epsilon)) \xrightarrow{f_\epsilon} D(\epsilon) \xrightarrow{h|} S^p.$$

Consider a smooth homotopy $\phi_t : S^p \times [0,1] \to S^p$ of the identity of S^p such that ϕ_1 maps $h(D(\epsilon))$ diffeomorphically onto $e \subset S^p$ and shrinks the complement to the 0-cell $*$. Such a homotopy always exists. Notice that in the standard Pontryagin-Thom construction, one simply considers the homotopy $\phi_t \circ f$ from f to $\phi_1 \circ f$ and one is finished. But recall that in our situation, we do not want to change the values of f on the boundary, since this is our basis for induction. By step 1, we know that f restricted to a suitable collar V of ∂M is transverse. In order to exploit this fact, choose an inner vector field on M and let γ_x be the integral curve of this vector field through x. Choose also a $\delta > 0$ such that γ_x maps the interval $[0, \delta]$ into V. For simplicity, assume that $\delta = 1$. Finally, consider a smooth function

$$\epsilon : \mathbf{R} \to \mathbf{R}$$

satisfying $\epsilon(t) \geq 0, \epsilon'(t) \geq 0, \epsilon(t) = 0$ whenever $t \leq 0$ and $\epsilon(t) = 1$ whenever $t \geq 1$. Clearly, ϵ restricts to a map from $[0,1]$ to $[0,1]$.

Consider the homotopy

$$F(x,t) = \begin{cases} \phi_{\epsilon(s)t} \circ f(y) & \text{if } y = \gamma_x(s), x \in \partial M \text{ and } 0 \leq s \leq 1, \\ \phi_t \circ f(y) & \text{otherwise.} \end{cases}$$

The map $F(x,1)$ is transverse.

Step 3: Proof for X arbitrary.

We proceed by induction on the number of cells in X. The basis of the induction is step 2. Consider a transverse CW complex $X = X_0 \cup e$, where X_0 is the subcomplex $X - e$ and e is a top-dimensional cell. Let $h : D \to X$ be a characteristic map for e and let $q : X \to X/X_0$ be the standard collapsing map. The characteristic map of the cell e gives an identification of X/X_0 with $D/\partial D$ and we choose an identification of $D/\partial D$ with S^p such that the collapsing map q is smooth on the top cell. With these choices, the composite

$$D \to X \to X/X_0 \to D/\partial D \to S^p$$

is smooth since it is the above collapsing map.

By step 1, we can assume that the map $f : M \to X$, restricted to some suitable neighbourhood V of ∂M, is transverse to X. There is thus a factorization

$$f|_K : K \xrightarrow{t_V} D \xrightarrow{h} X$$

where $K = cl[(f|_V)^{-1}(e)]$. Recall that t_V is smooth. It follows that $q \circ f|_K = q \circ (h \circ t_V) = (q \circ h) \circ t_V$ is smooth. Moreover, on $U_i \cap V$

$$q \circ f|_V = q \circ \partial_i f \circ p_i.$$

So $q \circ f$ is smooth on $U_i \cap V$ for all i and therefore on V. Notice that the complement of K in V goes to one point.

Let $W \subset V$ be some closed neighbourhood of ∂M. Since $q \circ f|_W$ is smooth, by the approximation lemma, it extends to a smooth map $g : M \to S^p$ which is homotopic to $q \circ f$, and coincides with $q \circ f$ on the closed set W.

By Sard's theorem, one can suppose that $y = q(h(0)) \in S^p$ is a nice regular value of g.

As before, let $D(\epsilon)$ be an open subdisc of D, of radius ϵ and denote by $e(\epsilon)$ the subset of S^p which is the image of $D(\epsilon)$ by the smooth map $q \circ h$. By the same argument as in step 2, if ϵ is small enough, then g is a projection when restricted to $T_\epsilon = g^{-1}(e(\epsilon))$. Consider now the map $k : T_\epsilon \cup W \to X$ given by

$$k(x) = \begin{cases} q^{-1} \circ g(x) & \text{if } x \in T_\epsilon, \\ f(x) & \text{if } x \in W. \end{cases}$$

This map is well-defined since $g = q \circ f$ on W. It is not difficult to show that the map k is homotopic to f restricted to $T_\epsilon \cup W$, by a homotopy which does not perturb f on the boundary ∂M. Since $T_\epsilon \cup W$ is a retract of some open set of M, an application of the Homotopy Extension Property shows that k extends to a map $k : M \to X$, which is homotopic to f, coincides with f on ∂M and is smooth on $\mathrm{cl}[(q \circ k)^{-1} e(\epsilon)]$.

To summarize, one can suppose that f is smooth and trivial when restricted to the inverse image of a small subset of the top cell e. Consider such a small open set $e' \subset e \subset X$. There is an homotopy ϕ_t of the identity map of X which map e' diffeomorphically onto e, and which is the identity on X_0. Using the same function $\epsilon : [0,1] \to [0,1]$ and the same inner vector field as in step 2, one gets a homotopy

$$F(x,t) = \begin{cases} \phi_{\epsilon(s)t} \circ f(y) & \text{if } y = \gamma_x(s) \text{ and } 0 \leq s \leq 1, \\ \phi_t \circ f(y) & \text{otherwise.} \end{cases}$$

Clearly, F is a homotopy rel ∂M of $F(y,0) = f(y)$. The map $F(y,1) = F_1$ has the property that

$$F_1 : cl[F_1^{-1}(e)] \xrightarrow{t} D \xrightarrow{h} X$$

and is thus transverse to the top cell e. Now, $M_0 = M - F_1^{-1}(e)$ is a (k+1)-manifold and the map F_1, restricted to the boundary of M_0 is transverse to X_0. Applying the induction hypothesis completes the proof. ∎

2.3 Framed stratified sets

As such, there does not seem to be any discussion of framed stratifications with faces in the literature. However while formulating our definitions, we relied heavily on [6], [7] and [24].

A *stratified set with faces* $(A, \mathcal{S})$ ([24]) with underlying set A consists of the following data.

1. A Hausdorff, locally compact, paracompact topological space A, with a countable basis.

2. A locally finite family $\mathcal{S}$ of locally closed subsets of A, the *strata*, such that A is the disjoint union of the strata.

3. A *tubular neighbourhood* T_X for each stratum X. More precisely, each T_X is an open neighbourhood of X in A, and there is a continuous retraction $\Pi_X : T_X \to X$ (i.e. Π_X restricted to X is the identity), and a continous *tubular function* $\rho_X : T_X \to [0, \infty)$ with $\rho_X^{-1}(0) = X$.

4. A collection $\{A_i\}$ of closed subsets of A, called the *faces*.

These objects must satisfy the following axioms.

(ss1) (Frontier condition) If X, Y are two strata of A and if $X \cap \mathrm{cl}_A Y \neq \emptyset$ then $X = Y$ or $X \subset \mathrm{cl}_A Y$.

We write $X < Y$ if $X \subset \mathrm{cl}_A Y$ and $X \neq Y$. Note that the relation $\leq$ defines a partial order on the set of strata.

(ss2) Let A_i be a face of A. Then there is an open neighbourhood U_i of A_i in A, and a homeomorphism (a *collar*)

$$F_i : U_i \to A_i \times \mathbf{R}_+$$

onto an open subset of $A_i \times \mathbf{R}_+$, satisfying the standard conditions:

1. $F_i(a) = (a, 0)$ for all $a \in A_i$.
2. For any stratum X of A, $F_i(X \cap U_i) \subset (X \cap A_i) \times \mathbf{R}_+$.

It follows that one has two maps $p_i : U_i \to A_i$ and $r_i : U_i \to \mathbf{R}_+$, which are completely determined by the equation $F_i(a) = (p_i(a), r_i(a))$. One can choose p_i and p_j so that for all i, j, $p_i \circ p_j = p_j \circ p_i$.

(ss3) Each stratum is a smooth manifold with faces. Its faces are obtained by simply taking intersections with the faces of A.

The set $A_i \cap X$ is a face of X and will be denoted by X_i. The collars for the strata are obtained by restricting the suitable collars of A and they are collars in the sense of 2.1.

(ss4) For any strata X, Y with $Y \neq X$,

$$T_X \cap Y \neq \emptyset \quad \Rightarrow \quad X < Y.$$

Furthermore, there is a smooth function $\epsilon = \epsilon_X : X \to \mathbf{R}$ such that, if one sets $T_X^\epsilon = \{a \in T_X | \rho_X(a) < \epsilon(\Pi_X(a))\}$, then the map

$$(\Pi_X, \rho_X) : T_X^\epsilon \cap Y \to X \times (0, \epsilon)$$

is a smooth submersion. Here, $X \times (0, \epsilon) = \{(x, t) \in X \times \mathbf{R} | 0 < t < \epsilon(x)\}$. Notice that ϵ_X can be the constant map.

(ss5) If X and Y are strata and if $T_X \cap T_Y \neq \emptyset$ then X and Y are comparable in the partial order $\leq$.

(ss6) (Control conditions) For any pair of strata $X < Y$, the tubular neighbourhoods T_X, T_Y satisfy the following conditions : for any $a \in T_X^{\epsilon_X} \cap T_Y^{\epsilon_Y}$, the retractions

$$\Pi_X : T_X \to X, \ \Pi_Y : T_Y \to Y$$

satisfy

1. $\Pi_Y(a) \in T_X$.
2. $\Pi_X(\Pi_Y(a)) = \Pi_X(a)$.
3. $\rho_X(\Pi_Y(a)) = \rho_X(a)$.

where $\rho_X : T_X \to [0, \infty)$ is the tubular function of X.

(ss7) If X is a stratum of A, then

$$\Pi_X^{-1}(X \cap A_i) = A_i \cap T_X.$$

Furthermore,

$$F_i \circ \Pi_X = ((\Pi_X | T_X \cap A_i) \times 1_{R^+}) \circ F_i$$

on the the intersection of the collar neighbourhood U_i with T_X.

(ss8) For any stratum X, the tubular function ρ_X satisfies

$$\rho_X = \rho_X \circ p_i$$

on the intersection of the collar neighbourhood U_i with T_X.

The *depth* of a stratum X of A is the integer

$$\mathrm{depth}_A[X] = \sup\{n \mid \text{there is a chain } X = X_0 < X_1 < \ldots < X_{n-1} < X_n\}.$$

One can also define

$$\mathrm{depth}[A] = \sup\{\mathrm{depth}_A[X] \mid X \text{ is a stratum of } A\},$$

and

$$\dim[A] = \sup\{\dim[X] \mid X \text{ is a stratum of } A\}.$$

Clearly, if $\dim[A]$ is finite, then so is $\mathrm{depth}[A]$.

Consider now any map $f : A \to B$ between the stratified sets $(A, \mathcal{S})$ and $(B, \mathcal{S}')$ which is a homeomorphism of the underlying topological spaces. The map f is an *isomorphism of stratified sets* if there is a bijection

$$\mathcal{S} \to \mathcal{S}', \quad X \mapsto X'$$

such that

1. f restricts to a diffeomorphism between X and X', and between $T_X \cap Y$ and $T_{X'} \cap Y'$, for any stratum Y satisfying $X \leq Y$;

2. $f \circ \Pi_X = \Pi_{X'} \circ f$ (f is "controlled");

3. $\rho_{X'} \circ f = \rho_X$.

Let A be a compact stratified set. An *open cone* on A is by definition the set $A \times [0, \infty)/\sim$, where $(x, 0) \sim (x', 0)$, with the obvious stratification.

A *trivial stratified set* (see [6]) is a stratified set $(A, \mathcal{S})$ (with faces if required) such that for each stratum X, there is an open cone $D(X)$ and a homeomorphism $h_X : T_X \to X \times D(X)$, such that $\Pi_X = p_1 \circ h_X$. The maps h_X are called *trivialisations*.

Let $(A, \mathcal{S})$ be a trivial stratified set. Consider the pair $F = (\mathcal{S}, H)$ where H is a set of trivialisations for the tubular neighbourhoods of the strata of A. More precisely, for each $X \in \mathcal{S}$, one has chosen a homeomorphism $h_X \in H$:

$$h_X : T_X \to X \times D(X).$$

The pair $(A, F) = (A, (\mathcal{S}, H))$ is a *framed stratified set* if the following conditions hold. If $X \leq Y$, then

(fs1) The open cone $D(X)$ associated to the stratum X of A is a trivial stratified set with one stratum for each stratum of Y of A satisfying $X \leq Y$.

We will denote the stratum corresponding to Y by $D(X)_Y$. Thus the set of strata of $D(X)$ is $\{D(X)_Y | X \leq Y\}$.

(fs2) The homeomorphism h_X maps $T_X \cap Y$ into $X \times D(X)_Y$ and is furthermore a diffeomorphism between these two manifolds.

(fs3) The cone associated to Y is equal to the cone associated to $D(X)_Y$ i.e. $D(Y) = D(D(X)_Y)$.

(fs4) $h_X(T_X \cap T_Y) = X \times T_{D(X)_Y}$.

(fs5) On the intersection $T_X \cap T_Y$, the following equalities hold:

(A) $(1_X \times \Pi_{D(X)_Y}) \circ h_X = h_X \circ \Pi_Y$ i.e.

$$\begin{array}{ccc} T_X \cap Y & \xrightarrow{h_X} & X \times D(X)_Y \\ \Pi_Y \downarrow & & \downarrow Id \times \Pi_{D(X)_Y} \\ T_X \cap T_Y & \xrightarrow{h_X} & X \times T_{D(X)_Y} \end{array}$$

Notice that $\Pi_Y(T_X \cap T_Y) \subset T_X \cap Y$ makes sense by (ss6).

(B) $(1_X \times h_{D(X)_Y}) \circ h_X = (h_X \times 1_{D(Y)}) \circ h_Y$ i.e.

$$\begin{array}{ccc} T_X \cap T_Y & \xrightarrow{h_X} & X \times T_{D(X)_Y} \\ {\scriptstyle h_Y}\downarrow & & \downarrow{\scriptstyle Id \times h_{D(X)_Y}} \\ (T_X \cap Y) \times D(Y) & \xrightarrow{h_X \times Id} & X \times D(X)_Y \times D(Y) \end{array}$$

By (ss6) and the fact that $\Pi_Y = p_1 \circ h_Y$ (p_1 is the projection on the first factor), one has $h_Y(T_X \cap T_Y) \subset (T_X \cap Y) \times D(Y)$.

(C) $\rho_Y = \rho_{D(X)_Y} \circ p_2 \circ h_X$ i.e.

$$\begin{array}{ccc} T_X \cap T_Y & \xrightarrow{h_X} & X \times T_{D(X)_Y} \\ {\scriptstyle \rho_Y}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbf{R}_+ & \xrightarrow{\rho_{D(X)_Y}} & T_{D(X)_Y} \end{array}$$

(D) $\rho_{D(X)_Y} = \rho_0 \circ p_2 \circ h_{D(X)_Y}$ i.e.

$$\begin{array}{ccc} T_{D(X)_Y} & \xrightarrow{\rho_{D(X)_Y}} & \mathbf{R}_+ \\ {\scriptstyle h_{D(X)_Y}}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ D(X)_Y \times D(Y) & \xrightarrow{p_2} & D(Y) \end{array}$$

Here, p_2 is the projection on the second factor and ρ_0 is the tubular function of $D(Y)$. Notice that $D(X)$ is an open cone $C_0L(X)$ over $L(X)$. The stratified set $L(X)$ is called the *link* of X and is identified with $L(X) \times \{1\}$ in $C_0L(X)$.

These equations are from [6]. The presence of faces in our context does not modify the compatibility conditions.

An *isomorphism of framed stratified sets* is an isomorphism of stratified sets $f : (A, \mathcal{S}) \to (B, \mathcal{S}')$ which satisfies the following conditions. For any stratum X of A, corresponding under f to the stratum X' of B, there is a commutative diagram

$$\begin{array}{ccc} T_X & \xrightarrow{h_X} & X \times D(X) \\ f \downarrow & & \downarrow f_{X,X'} \\ T_{X'} & \xrightarrow{h_{X'}} & X' \times D(X') \end{array}$$

where $f_{X,X'}$ is an isomorphism of stratified sets which, when restricted to $X \times \{*\}$ is given by $(x, *) \mapsto (f(x), *)$. Moreover, f and $f_{X,X'}$ induce isomorphisms between the diagrams A,B,C and D and the corresponding diagrams for $(B, \mathcal{S}')$, i.e. one has commutative "*cubes*".

The *trivial framed stratification* on a manifold M is the one which has for unique stratum the manifold M itself i.e. $\mathcal{S} = \{M\}$. If F is the trivial framed stratification of M, one will make no distinction between M and (M, F).

Let M be a manifold and (A, F) be a framed stratified set. Then the product $M \times A$ has an obvious structure of framed stratified set, denoted by $(M \times A, M \times F)$. If X is a stratum of (A, F), with tubular neighbourhood T_X, the product $M \times X$ is a stratum of $M \times F$ and its tubular neighbourhood $M \times T_X$ has the obvious trivialisation

$$M \times T_X \xrightarrow{Id \times h_X} M \times X \times D(X),$$

where h_X is the trivialisation of T_X. Notice that the link of $M \times X$ is $L(X)$. Checking the axioms presents no difficulty.

Let A and A' be two stratified sets and suppose that $A' \subset A$. One says that A' is a *restriction* of A if

1. for any stratum X' of A', there is a stratum X of A such that X' is a proper framed submanifold of X, and

2. $T_{X'} = T_X \cap A'$, $\Pi_{X'} = \Pi_X|_{T_{X'}}$, and $\rho_{X'} = \rho_X|_{T_{X'}}$.

In the case where A and A' are both framed stratified sets, then A' is a restriction of A if moreover

3. $L(X, A) = L(X', A')$ and $h_{X'} = h_X|_{T_{X'}}$.

Let (A, F) be a framed stratified set of depth n. For any stratum Y of A, let $A_Y = \rho_Y^{-1}[0, 1]$ and $\dot{A}_Y = \rho_Y^{-1}(1)$. For the sake of simplicity, let us assume that A has a unique stratum X of depth n. The next lemma contains some useful facts.

Lemma 8 *With the above hypotheses and notations, one has the following.*

1. *The sets T_X, $L(X)$, A_X, $\mathrm{cl}[A - A_X]$ and $\dot{A}_X$ are all framed stratified sets. Moreover, both $\dot{A}_X$ and $\mathrm{cl}[A - A_X]$ have depth $< n$.*

2. *The homeomorphism h_X induces isomorphisms of framed stratified sets $T_X \to X \times D(X)$ and $\dot{A}_X \to X \times L(X)$.*

3. *If $f : (A, F) \to (B, G)$ is an isomorphism of framed stratified sets, then f determines an identification $L(X) \to L(f(X))$.*

The framed stratifications of $D(X)$ and of $L(X)$ induced by the framed stratification F of A will be also denoted by F. Thus $(D(X), F)$ and $(L(X), F)$ are both framed stratified sets.

Proof. The proof follows immediately from the definitions. ∎

2.4 Cobordism of framed stratified sets

In this section, we are going to consider framed stratified sets up to framed cobordism. This will allow us to prove a Pontryagin-Thom correspondence for maps from manifolds with faces to CW complexes.

Let us first make an important remark. We are going to work with framed stratified sets whose underlying topological spaces will be manifolds with faces. We are going to suppose that the closed strata are submanifolds in the sense of section 2.1. We make the same remark for the tubular neighbourhoods of the strata. It follows from this assumption that the links of the strata will always be spheres.

Proposition 9 *Let $\phi : M \to X$ be a transverse map from a compact (k)-manifold M to a transverse CW complex X. There is a structure of framed stratified set with faces on M determined up to isomorphism by the map ϕ.*

The framed stratification given by the proposition will be denoted by $F(\phi, X) = (S(\phi, X), H(\phi, X))$, and $(M, F(\phi, X))$ is a framed stratified set.

Proof. Notice that the uniqueness assertion means that the isomorphism class of framed stratifications constructed out of ϕ and X does not depend on any of the non-canonical choices made. The proof of the result is by induction on the number of cells in X.

Proof when $X = S^p$, with cellular structure $S^p = * \cup e$.

Due to transversality, ϕ restricted to $\mathrm{cl}[\phi^{-1}(e)]$ is a smooth bundle map (t2) and ϕ maps the complement of $\mathrm{cl}[\phi^{-1}(e)]$ to the base point $*$ of S^p. Thus, all the fibres of ϕ restricted to $\mathrm{cl}[\phi^{-1}(e)]$ are difeomorphic. It follows that the fibre $N = \phi^{-1}(y)$ is independant of the choice of y as long as y belongs to $S^p - *$. Moreover, the map ϕ induces a trivialisation of $\mathrm{cl}[\phi^{-1}(e)]$, or equivalently, a framing of N in M. We set $\mathcal{S}(\phi, X) = \{N, M - N\}$. The set $H(\phi, X)$ of trivialisations is the obvious one. Checking the axioms is easy.

Proof for X arbitrary.

First, one considers the case when X has a unique top dimensional cell e. Write X as $X_0 \cup e$ where X_0 is the subcomplex $X - e$. By transversality, one has a factorization:

$$\phi|_T : T \xrightarrow{t} D \xrightarrow{h} X$$

where $T = \mathrm{cl}[\phi^{-1}(e)]$ and h is the characteristic map of the cell e. Then, construct $F(\phi, X)$ starting with the closed stratum $N = t^{-1}(0)$. By transversality (t2), it is a compact submanifold with faces of M.

By (t4), $M_0 = \mathrm{cl}[M - T]$ is also manifold with faces. Since M has a (k)-manifold structure, M_0 can be seen as a (k+1)-manifold, and the decomposition of its topological boundary is given by

$$\partial M_0 = \partial_0 M_0 \cup \partial_1 M_0 \cup \ldots \cup \partial_{k-1} M_0 \cup \partial_k M_0,$$

where $\partial_i M_0 \subset \partial_i M$, for $i = 0, \ldots, k-1$, and $\partial_k M_0 = \mathrm{cl}[\partial T - \partial M]$.

Let $\phi_0 = \phi|_{M_0}$. Checking the axioms, one sees that ϕ_0 is a transverse map from M_0 to X_0 (this follows from the above factorization and the hypothesis that the attaching maps of the cells of X are transverse). By induction, M_0 has a framed stratification $F(\phi_0, X_0)$, well-defined up to isomorphism. Let

$\{N_\alpha\} = \mathcal{S}(\phi_0, X_0)$ be the set of strata of $F(\phi_0, X_0)$. Notice that a stratum N_α has a face of the form $N_\alpha \cap \partial_k M_0$ (which could be empty).

As mentioned in 2.1, the important point here is to note that one can identify the face $\partial_k M_0 = \mathrm{cl}[\partial T - \partial M]$ with the sphere bundle $\Sigma(\nu_N)$ of the normal bundle of N in M. Moreover, the map ϕ determines a trivialisation of $\Sigma(\nu_N)$:

$$\Phi : N \times \partial D \to \partial_k M_0 \simeq \Sigma(\nu_N)$$

such that

$$N \times \partial D \xrightarrow{\Phi} \partial_k M_0 \xrightarrow{\phi|} \partial D$$

is the projection p_2 on the second factor. By induction, the attaching map $h| : \partial D \to X$ of the cell e determines a framed stratification $F(h|, X_0)$ of ∂D, unique up to isomorphism.

On the other hand, ϕ_0 is transverse on $\partial_k M_0$, and it can be factorized as

$$\partial_k M_0 \xrightarrow{\phi|} \partial D \xrightarrow{h|} X_0.$$

Since $\phi_0 \circ \Phi = h| \circ p_2$, one has

$$F(\phi_0 \circ \Phi, X_0) = F(h| \circ p_2, X_0)$$

and it follows that Φ determines an isomorphism of framed stratified sets

$$(\partial_k M_0, F(\phi_0|, X_0)) \simeq (N \times \partial D, N \times F(h|, X_0)).$$

Let $(\partial D)_\alpha$ be any stratum of $\mathcal{S}(h|, X_0)$. In order to construct $\mathcal{S}(\phi, X)$, one simply extends smoothly the strata $\{N_\alpha\}$ of $\mathcal{S}(\phi_0, X)$ by taking products of the form $N \times C(\partial D)_\alpha - N \times \{cone\ point\}$, where $C(\partial D)_\alpha$ is the cone over $(\partial D)_\alpha$. The framings of these strata are defined in the obvious way.

The uniqueness is easy to see, since $t : cl[\phi^{-1}(e)] \to D$ is a bundle map. The proof when X has more than one top cell is no more difficult. ∎

Let X be any CW complex. There is a natural partial order on the set of cells of X defined as follows: given two cells e and e' in X, we write $e' \leq e$ if $e' \cap \overline{e} \neq \emptyset$. The transitive closure of $\leq$ will also be denoted by $\leq$. If e is a cell of X, the *base* of e is by definition the subcomplex $X(e)$ of X given by $X(e) = \cup_{e' \leq e} e'$.

Let $h : X \to X'$ be a homotopy equivalence between two CW complexes X and X'. We will say that h is a *cell equivalence* if there is a bijection

$$\{cells\ of\ X\} \xrightarrow{b} \{cells\ of\ X'\}$$

such that h restricts to a homotopy equivalence between $X(e)$ and $X'(b(e))$, for any cell e in X. Two CW complexes X and X' are *cell equivalent* if there is a cell equivalence $h : X \to X'$. Cell equivalence has two important properties:

1. it is an equivalence relation on finite CW complexes;

2. it preserves the partial order $\leq$.

A cell equivalence is by definition a homotopy equivalence, but the opposite is of course not true. A *cell homeomorphism* is a cell equivalence that is also a homeomorphism.

Let M be a (k)-manifold and X a transverse CW complex. Consider a framed stratified set (M, F). We will say that $F = (\mathcal{S}, H)$ is *modelled on* X if the following holds: there is a bijective map c

$$\mathcal{S} \xrightarrow{c} \{cells\ of\ X\}$$

satisfying the following conditions.

1. The bijection c is a poset anti-isomorphism (the partial order on the set of strata is given by the condition (ss1)).

2. Let $L(N)$ be the framed stratified link of the stratum N. There is an isomorphism of framed stratified sets

$$(L(N), F) \simeq (S^n, F(h, X^n))$$

where $h : S^n = \partial c(N) \to X^n$ is the attaching map of the cell $c(N)$.

The framed stratification (M, F) of M is said to be *weakly modelled* on X if it is modelled on some subcomplex X' of X. The set of isomorphism classes of framed stratifications of M weakly modelled on X will be denoted by $Fram(M/X)$. Notice that if (M, F) is weakly modelled on X, then $\dim(e) = m - \dim(c^{-1}(e)) \leq m$, for any cell e in X.

Proposition 10 *Let M be a compact (k)-manifold and X a transverse CW complex.*

1. *Let $\phi : M \to X$ be any transverse map. Then the framed stratification $F(\phi, X)$ belongs to $Fram(M/X)$, i.e. is weakly modelled on X.*

2. *Conversely, if $F \in Fram(M/X)$, then it is of the form $F = F(\phi, X)$ for some transverse map $\phi : M \to X$.*

3. *Let F be a framed stratification of M. Then, there is a transverse CW complex X, unique up to cell homeomorphism, and a transverse map $\phi : M \to X$ such that $F \simeq F(\phi, X)$.*

The uniqueness statement in 3 has the following meaning : if F is modelled on another complex Y (i.e. $F \simeq F(\psi, Y)$ in addition to $F \simeq F(\phi, X)$),

then there is an isomorphism of framed stratified sets $h : (M, F(\phi, X)) \to (M, F(\psi, Y))$ and a cell homeomorphism $H : X \to Y$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & X \\ h \downarrow & & \downarrow H \\ M & \xrightarrow{\psi} & Y \end{array}$$

commutes. Note that this implies that $H \circ \phi$ is a transverse map.

Proof. All along this proof, we will freely use the notations introduced in the proof of the preceding proposition.

Proof of 1. We will prove the statement by induction over the number of cells. For simplicity, we assume that X has a unique 0-cell. As usual, we begin with the sphere S^p, with cellular structure $* \cup e$. If $\phi : M \to S^p$ is a transverse map, then either $\phi(M) = *$ or ϕ is surjective. In the first case, $F(\phi, S^p)$ is clearly modelled on the subcomplex $* \subset S^p$. For the second case, the set of strata consists of a framed submanifold $N \subset M$ and of its complement $M - N$. The corresponding framed stratification $F(\phi, S^p)$ is clearly modelled on $S^p = * \cup e$.

Let us assume that the result holds for complexes with fewer than n cells and suppose that X_0 is such a transverse CW complex. Consider now a transverse map $\phi : M \to X = X_0 \cup e$, where X is transverse. Then, either ϕ surjects on $e \in X$ or ϕ maps into X_0. In the second case, we are finished since X_0 is a subcomplex of X and $F(\phi, X) = F(\phi, X_0) \in Fram(M/X)$, by induction. So, let us suppose that ϕ surjects on e. We recall the notations

$T = \mathrm{cl}[\phi^{-1}(e)]$ and $M_0 = \mathrm{cl}[M - T]$. Consider the restriction $\phi_0 : M_0 \to X_0$. It is a transverse map. By induction, there is an anti-isomorphism c_0 onto the cells of some subcomplex X_0' of X_0

$$\mathcal{S}(\phi_0, X_0) \xrightarrow{c_0} \{cells\ of\ X_0'\}.$$

It is clear that X_0' is also a subcomplex of X. We are going to show that $F(\phi, X)$ is modelled on $X_0' \cup e$.

First, notice that the attaching map $h| : \partial D \to X$ of the cell e actually maps into X_0'. To see this, notice that the face $\partial_k M_0$ of M_0 has a framed stratification, determined by the map

$$\phi|_{\partial_k M_0} : \partial_k M_0 \xrightarrow{t|} \partial D \xrightarrow{h|} X_0.$$

By the same argument as in the preceding proposition, there is an isomorphism of framed stratified sets

$$(\partial_k M_0, F(\phi_0|, X_0)) \simeq (\partial_k M_0, N \times F(h|, X_0)).$$

It follows that $N \times F(h|, X_0)$ is modelled on X_0', and so is $F(h|, X_0)$. This implies that the attaching map $h| : \partial D \to X$ maps into X_0'.

We now define an anti-isomorphism c

$$\mathcal{S}(\phi, X) \xrightarrow{c} \{cells\ of\ X\}$$

in the following way. Recall that, by construction of $F(\phi, X)$ (see the proof of the preceding proposition), each stratum $S_{\gamma'}$ of $F(\phi_0, X_0)$ which intersects $\partial_k M_0$ determines a unique stratum S_γ of $F(\phi, X)$. We set $c(S_\gamma) = c_0(S_{\gamma'})$ and $c(N) = e$ where N is the closed stratum corresponding to e, i.e. is the fibre of $t| : T \to D$. The new map c is clearly an anti-isomorphism (condition 1), and it obviously satisfies condition 2. Thus,

$F(\phi, X) \in Fram(M/X_0' \cup e) \subset Fram(M/X)$. The proof for more than one 0-cell presents no new difficulty.

Proof of 2. Let us assume that 3 holds. It follows that there is a transverse CW complex Y and a transverse map $\psi : M \to Y$ such that $F \simeq F(\psi, Y)$. By 1,

$$F \in Fram(M/X) \cap Fram(M/Y).$$

Let X_0 and Y_0 be subcomplexes of respectively X and Y such that F is modelled on both X_0 and Y_0. An easy induction shows that there exists a cell homeomorphism $h : Y_0 \to X_0$. If we set

$$\phi : M \xrightarrow{\psi} Y_0 \xrightarrow{h} X_0 \hookrightarrow X$$

then $F \simeq F(\phi, X)$.

Proof of 3. We will construct a transverse map $\phi : M \to X$ from M to some transverse CW complex X such that $F \simeq F(\phi, X)$. We proceed by induction on the depth of F.

Suppose that depth$[F] = 1$. Take $X = S^p = * \cup e$ where p is the codimension of the closed stratum N. Now, let $\phi : M \to S^p$ be the map given by applying the Pontryagin-Thom construction to the framed manifold N. The statement clearly holds.

Assume that 3 *holds for* depth$[F] < n$. Let F be a framed stratification of depth n. Let us first assume that there is a single stratum N of depth n.

As before, we use the notation $M_N = \rho_N^{-1}([0, 1])$ and $\dot{M}_N = \rho_N^{-1}(\{1\})$ and

we write M as $M_N \cup_{\dot{M}_N} E_N$ where E_N is the framed stratified set $\mathrm{cl}[M - M_N]$ (recall that ρ_N is the tubular function of the stratum N). Notice that $\dot{M}_N$ and E_N are framed stratified sets and that both have depth $< n$. Moreover, there is a isomorphism of framed stratified sets $\dot{E}_N = \dot{M}_N \to N \times L(N)$. Notice that the link $L(N)$ of N is a sphere, since the underlying set is a manifold (see the remark at the beginning of 2.4). We will denote it by $S(N)$ in what follows. By induction, there are maps

$$\lambda : S(N) \to Y', \ \psi : E_N \to Y$$

inducing the framed stratification F on $S(N)$ and on E_N, since these spaces have depth $< n$. By the uniqueness condition, there is a commutative diagram

$$\begin{array}{ccccc} \dot{M}_N & \to & S(N) & \xrightarrow{\lambda} & Y' \\ h \downarrow & & & & \downarrow H \\ \dot{M}_N & & \xrightarrow{\psi|} & & Y \end{array}$$

where H is a cell equivalence and h is an isomorphism of framed stratified sets. Let X be the CW complex

$$Y \bigcup_{H \circ \lambda} D(N).$$

Consider a map

$$\phi : M = E_N \cup M_N \to Y \cup_{H \circ \lambda} D(N) = X$$

defined as follows: on E_N, one uses $\psi \circ h$ (having prealably extended h to E) while on M_N one uses the composite

$$M_N \to N \times D(N) \to D(N) \xrightarrow{\lambda'} Y' \cup D(N) \xrightarrow{H'} Y \cup D(N)$$

where λ' and H' are the obvious extensions of H and λ. The map ϕ is transverse and one has that $F(\phi, X) \simeq F$. The proof for more than one closed strata presents no new difficulty. ■

Let $Tran(M, X)$ be the set of all transverse maps from M to X. As a corollary of the preceding result, we get the existence of a well-defined map

$$Tran(M, X) \to Fram(M/X)$$

given by $\phi \mapsto F(\phi, X)$. We can now prove Theorem A. First, let us give some additional definitions. Let F_0 and F_1 be two framed stratifications of the same manifold M. In other words, (M, F_0) and (M, F_1) are two framed stratified sets, with the same underlying (k)-manifold M. We will say that (M, F_0) is *cobordant* to (M, F_1) if there is a framed stratified set $(M \times [0, 1], F)$ which induces the framed stratifications F_0 on $M \times 0$ and F_1 on $M \times 1$. In other words, both $(M \times \{0\}, F_0)$ and $(M \times \{1\}, F_1)$ are restrictions of $(M \times [0, 1], F)$. We will say that (M, F_0) is *X-cobordant* to (M, F_1) if the framed stratification F is weakly modelled on X. Clearly, cobordism and X-cobordism are equivalence relations. The set of X-cobordism classes of elements of $Fram(M/X)$ will be denoted by $\Omega_X(M)$.

Theorem 11 (Theorem A) *Let M be a (k)-manifold and X a transverse CW complex. The map*

$$\mathcal{F} : [M, X] \to \Omega_X(M)$$

given by $[\phi] \mapsto [F(\phi, X)]$, is a bijection.

Proof. By proposition 10, the map $\mathcal{F}$ is surjective. It remains to show that $\mathcal{F}$ is injective. Suppose that $F(\phi, X)$ is X-cobordant to $F(\psi, X)$. We

want to show that ϕ is homotopic to ψ. Let F be a framed stratification of $M \times [0,1]$, weakly modelled on X, which gives the cobordism between $F(\phi, X)$ and $F(\psi, X)$. Thus F induces $F(\phi, X)$ on $M \times 1$ and $F(\psi, X)$ on $M \times 0$. By proposition 10, the framed stratification F is of the form $F(\Phi, X)$ for some map transverse map $\Phi : M \times [0,1] \to X$. One has that

$$F(\Phi|_{M\times 1}, X) = F(\phi, X)$$

and

$$F(\Phi|_{M\times 0}, X) = F(\psi, X).$$

It follows that ϕ is homotopic to ψ. ∎

3 The Morse complex

3.1 Preliminaries

Throughout this section, $f : M \to \mathbf{R}$ will be a Morse-Smale function on a closed Riemannian manifold M. We recall from the first section that a flow line of $\mathrm{grad}(f)$ is simply a curve in M

$$\gamma : \mathbf{R} \to M$$

satisfying the gradient flow equation

$$\frac{d\gamma}{dt} = -\mathrm{grad}(f).$$

It is easily checked, by simply computing the derivative of $f(\gamma(t))$, that f is strictly decreasing along any flow line of the gradient field which is not a critical point.

Consider two critical points a and b of f with $a \prec b$ and any flow γ from a to b. It follows from the above remark that f restricted to $\gamma(\mathbf{R})$ is a diffeomorphism onto the open interval $(f(b), f(a))$. Thus, one can consider its inverse ω, given by $\omega(t) = \gamma \circ (f \circ \gamma)^{-1}(t)$. It is a map

$$\omega : (f(b), f(a)) \to M$$

satisfying $f(\omega(t)) = t$. It certainly extends to a continuous map

$$\omega : [f(b), f(a)] \to M$$

by setting $\omega(f(b)) = b$ and $\omega(f(a)) = a$. It is quite important to notice that ω is *not* a solution of the gradient flow equation, but that it satisfies on $(f(b), f(a))$ the differential equation

$$\frac{d\omega}{dt} = \frac{\mathrm{grad}(f)}{||\mathrm{grad}(f)||^2}$$

with boundary conditions $\omega(f(b)) = b$ and $\omega(f(a)) = a$. We will refer to this equation as the *modified gradient flow equation of f*. A solution ω of the modified gradient flow equation satisfies $\omega(f(x)) = x$ and $f(\omega(t)) = t$.

Let us consider the space $\mathrm{Map}([f(b), f(a)], M)$ of all continuous curves from $[f(b), f(a)]$ to M, equipped with the compact-open topology. We now define the *compactified moduli space of flow lines from a to b*, denoted by $\mathcal{C}(a, b)$. It is the subspace of $\mathrm{Map}([f(b), f(a)], M)$ consisting of all continuous curves ω which are smooth on the complement of the set of critical values of f (or more precisely on the complement of $f(\mathrm{Crit}(f)) \cap [f(b), f(a)]$) and satisfy the modified gradient flow equation with boundary conditions $\omega(f(b)) = b$, $\omega(f(a)) = a$.

Notice that if one removes the critical points from the image of a curve $\omega \in \mathcal{C}(a, b)$, what is left is a finite number of genuine geometric flow lines of f. This is why ω is often called a *piecewise flow line*.

It is a remarkable fact that the moduli spaces $\mathcal{C}(a, b)$ admit another description, given in terms of *gluing* of flows lines. Let us explain this briefly. We first quote the following standard lemma. A proof can be found in [3]. Recall that the moduli space $M(a, b)$, defined in the first section as

$$M(a, b) = W(a, b)/\mathbf{R} = W^u(a) \cap W^s(b)/\mathbf{R},$$

can be also thought of as

$$\{\gamma : \mathbf{R} \to M \ C^\infty \mid \lim_{t \to -\infty} \gamma(t) = a, \lim_{t \to \infty} \gamma(t) = b, \frac{d\gamma}{dt} = -\mathrm{grad}(f)\}/\sim$$

where "$\sim$" is an equivalence relation given by additive reparametrization of γ: $\gamma \sim \gamma_\tau$ where $\gamma_\tau(t) = \gamma(t + \tau)$, for $\tau \in \mathbf{R}$. The correspondence between these two versions of $M(a, b)$ is given by the evaluation map.

Lemma 12 *Let $a \prec b$ be two critical points of f and (γ_i) be any sequence in $M(a,b)$. Then, either (γ_i) converges in $M(a,b)$, or there is*

1. *a subsequence (γ_j) of (γ_i),*

2. *an ordered set of critical points $a = a_1 \prec \ldots \prec a_k \prec a_{k+1} = b$,*

3. *a finite set of real numbers $r_1 > \ldots > r_k$,*

such that the points $x_{i,j} = \gamma_j(s)$ satisfying $f(x_{i,j}) = r_i$ converge to a regular point of M which belongs to $W(a_i, a_{i+1}) \cap f^{-1}(r_i) \simeq M(a_i, a_{i+1})$. ∎

It follows immediately from this result that $M(a,b)$ is a closed manifold if and only if $a \prec_s b$ i.e. if a is a successor of b. If a is not a successor of b, the next lemma gives a local parametrization of the ends of $M(a,b)$. Proofs of this result can be found in [3] and [21]. Set $\mathrm{Int}[\mathcal{C}(a,b)] = \mathcal{M}(a,b)$.

Lemma 13 *Let $a \prec b \prec b$. There exists an $\epsilon > 0$ and a map*

$$\mu : \mathcal{M}(a,b) \times \mathcal{M}(b,c) \times (0,\epsilon] \to \mathcal{M}(a,c)$$

which is a diffeomorphism onto an open set of $\mathcal{M}(a,c)$. ∎

These maps are called *gluing maps*. According to Austin and Braam [3], the last two lemmas imply that $\mathcal{C}(a,b)$ is a manifold with corners. It is important to note that μ extends to a map

$$\mu : \mathcal{C}(a,b) \times \mathcal{C}(b,c) \times [0,\epsilon] \to \mathcal{C}(a,c),$$

in the sense that

$$\lim_{t \to 0} \mu(\omega^1, \omega^2, t) = \mu(\omega^1, \omega^2, 0)$$

is the obvious piecewise flow from a to c. For the sake of simplicity, we will suppose that the parameter ϵ can be chosen to be 1.

The work of Cohen, Jones and Segal [10] refines this description of $\mathcal{C}(a,b)$ in the following way. First, we need some notation. Let $a \prec b \prec c \prec d$ be critical points and let us denote by μ_{abc} the gluing map

$$\mu : \mathcal{C}(a,b) \times \mathcal{C}(b,c) \times [0,1] \to \mathcal{C}(a,c).$$

The *clean intersection condition* [10] states that the various gluing maps satisfy

$$\mathrm{im}[\mu_{abd}] \bigcap \mathrm{im}[\mu_{acd}] = \mathrm{im}[\mu_{abcd}] = \mathrm{im}[\mu_{acbd}]$$

where μ_{abcd} and μ_{acbd} are both maps

$$\mathcal{C}(a,b) \times \mathcal{C}(b,c) \times \mathcal{C}(c,d) \times [0,1]^2 \to \mathcal{C}(a,d)$$

given respectively by

$$\mu_{abcd} : (x,y,z,(t,s)) \mapsto \mu_{acd}(\mu_{abc}(x,y,t),z,s),$$

and

$$\mu_{acbd} : (x,y,z,(t,s)) \mapsto \mu_{abd}(x,\mu_{bcd}(y,z,s),t).$$

The next result is from [10].

Theorem 14 *It is possible to choose the gluing maps so that they satisfy the clean intersection condition and the following associativity property:*

$$\mu(\mu(u,v,t),w,s) = \mu(u,\mu(v,w,s),t),$$

for any $(u,v,w) \in \mathcal{C}(a,b) \times \mathcal{C}(b,c) \times \mathcal{C}(c,d)$ *and* $s,t \in [0,1]$. ∎

Notice that the clean intersection condition says essentially two things:

1. If $a \prec b \prec c \prec d$ then $\mathrm{im}[\mu_{abcd}] = \mathrm{im}[\mu_{acbd}]$.

2. Whenever $\mathrm{im}[\mu_{abd}] \bigcap \mathrm{im}[\mu_{acd}] \neq \emptyset$, then it is equal to $\mathrm{im}[\mu_{abcd}]$ (or to $\mathrm{im}[\mu_{acbd}]$, by 1).

It also follows from [10] that if $\mathrm{im}[\mu_{abd}] \cap \mathrm{im}[\mu_{acd}] \neq \emptyset$, then b and c are comparable in the Smale order (i.e. $b \prec c$ or $c \prec b$). *In what follows, we will always suppose that the gluing maps satisfy the properties stated in this theorem.*

Another crucial property of the moduli spaces of flow lines of a Morse-Smale function is the fact that they are *framed manifolds*. This means that they can be embedded in spheres (or equivalently, in Euclidean spaces) with trivial normal bundles. As already mentioned, the moduli spaces can be embedded in any suitable regular level hypersurface of the function f. In particular, $M(a,b)$ can be embedded in $S^u(a)$ and one has the following standard result.

Lemma 15 *The normal bundle of $M(a,b)$ in $S^u(a)$ is trivial. In other words, $M(a,b)$ is a framed submanifold of $S^u(a)$.*

The particular framing of $M(a,b)$ in $S^u(a)$ constructed in the proof of this lemma will be called the *Morse framing*.

Proof. Recall the following fact from basic differential topology. If $f : M \to N$ is a smooth map which is transverse (in the classical sense) to the submanifold L of N, then the pull-back of the normal bundle of L in M by f is isomorphic to the normal bundle of the submanifold $f^{-1}(L)$ of M:

$$\nu_M(f^{-1}(L)) \simeq f^*\nu_N(L).$$

Notice that the Morse-Smale condition implies that the inclusion map

$$S^u(a) \xhookrightarrow{j} M$$

is transverse to the submanifold $W^s(b)$ of M. It follows that

$$\begin{aligned} j^*\nu_M(W^s(b)) &\simeq \nu_{S^u(a)}(j^{-1}(W^s(b)) \\ &= \nu_{S^u(a)}(M(a,b)) \end{aligned}$$

and this implies that the normal bundle of $M(a,b)$ in $S^u(a)$ is trivial, since $W^s(b)$ is contractible and thus has a framing, which is unique up to homotopy. ■

Notice that the framing of $M(a,b)$ constructed in the lemma is obtained by "restricting" the unique framing of $W^s(b)$ in M.

3.2 Construction of the tubular neighbourhoods

We recall from the last section that $\mathcal{C}(a,b)$ has the subspace topology induced from the compact-open topology on $\mathrm{Map}([f(b), f(a)], M)$. It follows that the evaluation map

$$E : \mathrm{Map}([f(b), f(a)], M) \times [f(b), f(a)] \to M$$

is continuous, and so is its restriction to the subspace $\mathcal{C}(a,b) \times [f(b), f(a)]$.

Let us now choose a $\delta > 0$ which has the following property: if $f^{-1}([f(a)-\delta, f(a)+\delta])$ contains a critical points b not equal to a, then $f(a) = f(b)$. It is clear that one can choose a δ which has this property for all the critical points of f, since they are only finitely many. For the sake of simplicity, we will assume that $\delta = 1$.

From now on, we will work with the *unstable disk*

$$D^u(a) = f^{-1}([f(a)-1, f(a)]) \cap W^u(a),$$

and we will identify the unstable sphere $S^u(a)$ with $\partial D^u(a)$. For each critical point a, there is a smooth function

$$t_a : D^u(a) \to [0,1]$$

given by

$$x \mapsto f(a) - f(x)$$

and satisfying $t_a^{-1}(0) = \{a\}$ and $t_a^{-1}(1) = S^u(a)$.

Let us now consider the topological sum $\coprod_{a \prec b} \mathcal{C}(a,b)$, taken over the set of critical points b satisfying $a \prec b$. The evaluation map induces a continuous map

$$E_a : \coprod_{a \prec b} \mathcal{C}(a,b) \to S^u(a)$$

given by $\omega \mapsto \omega(f(a)-1)$. Finally, consider the map

$$I_a : S^u(a) \times [0,1] \to D^u(a)$$

given by $(x,t) \mapsto \omega_x(f(a)-t)$, where ω_x is a solution of the modified gradient equation satisfying $\omega_x(f(x)) = x$. Notice that ω_x is simply a reparametrization of the unique flow line γ_x containing x. Clearly $I_a(x,0) = a$ and $I_a(x,1) = x$.

Before proving the next lemma, we want to make a simple but important remark. The evaluation map

$$E_a : \coprod_{a \prec b} \mathcal{C}(a,b) \to S^u(a)$$

is a surjective continuous map. It is also a proper map since $\coprod_{a \prec b} \mathcal{C}(a,b)$ is a compact space. Consequently, E_a is a closed map. This implies that $S^u(a)$ has the quotient topology.

Lemma 16 *The map I_a is continuous.*

Proof. Consider the following composition:

$$\coprod_{a \prec b} \mathcal{C}(a,b) \times [0,1] \xrightarrow{E_a \times Id} S^u(a) \times [0,1] \xrightarrow{I_a} D^u(a).$$

By the universal property of quotient, if the composite is continuous, then I_a is continuous. Let $(\omega, s) \in \coprod_{a \prec b} \mathcal{C}(a,b) \times [0,1]$. One has that

$$\begin{aligned} I_a \circ (E_a \times Id)(\omega, s) &= I_a(E_a(\omega), s) \\ &= I_a(\omega(f(a)-1), s) \\ &= \omega_x(f(a)-s) \end{aligned}$$

where $x = \omega(f(a)-1)$. But since $f(\omega(f(a)-s)) = f(a)-s = f(\omega_x(f(a)-s))$ and $x \in \mathrm{Im}[\omega] \cap \mathrm{Im}[\omega_x]$, it follows that $\omega_x = \omega|_{\mathrm{dom}[\omega_x]}$. But the map $(\omega, s) \mapsto \omega(f(a)-s)$ is clearly continuous. ∎

Consider now the map

$$\tau_b^a : \mathcal{C}(a,b) \times D^u(b) \to S^u(a)$$

given by

$$\tau_b^a(\omega, x) = \begin{cases} \mu(\omega, \omega_x, t_b(x))(f(a)-1) & \text{if } x \neq b \\ \omega(f(a)-1) & \text{if } x = b \end{cases}$$

Note that if $x \neq b$, $\omega_x \in \mathcal{C}(b,c)$ (and is thus a solution of the modified gradient flow equation) for some critical point c of f. Clearly, $f(\tau_b^a(\omega, x)) = f(a)-1$ and τ_b^a maps into $S^u(a)$.

Lemma 17 *The map τ_b^a is continuous.*

Proof. Consider the following maps:

$$Id \times E_b \times Id : \mathcal{C}(a,b) \times \coprod_{b \prec_s c} \mathcal{C}(b,c) \times [0,1] \to \mathcal{C}(a,b) \times S^u(b) \times [0,1],$$

and

$$Id \times I_b : \mathcal{C}(a,b) \times S^u(b) \times [0,1] \to \mathcal{C}(a,b) \times D^u(b).$$

By the universal property of quotient, in order to prove the lemma, it is sufficient to check that

$$\tau_b^a \circ (Id \times I_b) \circ (Id \times E_b \times Id)$$

is continuous. Let us calculate. First, set $z_1 = f(b) - 1$, $z_s = f(b) - s$, and $y_a = f(a) - 1$. Consider $(\omega^1, \omega^2, s) \in \mathcal{C}(a,b) \times \mathcal{C}(b,c) \times [0,1]$. We have that

$$\begin{aligned}
\tau_b^a \circ (Id \times I_b) \circ (Id \times E_b \times Id)(\omega^1, \omega^2, s) &= \tau_b^a \circ (Id \times I_b)(\omega^1, \omega^2(z_1), s) \\
&= \tau_b^a(\omega^1, \omega_{\omega^2(z_1)}(z_s)) \\
&= \tau_b^a(\omega^1, \omega^2(z_s)) \\
&= \mu(\omega^1, \omega_{\omega^2(z_s)}, t_b(\omega^2(z_s))(y_a),
\end{aligned}$$

where the last equality holds if $s \neq 0$. Since $t_b(\omega^2(f(b) - s)) = f(b) - f(\omega^2(f(b) - s)) = f(b) - (f(b) - s) = s$, and $\omega_{\omega^2(f(b)-s)} = \omega^2$, we finally get

$$\tau_b^a \circ (Id \times I_b) \circ (Id \times E_b \times Id)(\omega^1, \omega^2, s) = \mu(\omega^1, \omega^2, s)(f(a) - 1).$$

On the other hand we have $E_a \circ \mu(\omega^1, \omega^2, s) = \mu(\omega^1, \omega^2, s)(f(a) - 1)$, and $E_a \circ \mu$ is clearly continuous. The case $s = 0$ is also easily obtained. ∎

We consider now some properties of the map τ_b^a. First, let us look at its restriction to $\mathcal{C}(a,b) \times \{b\}$. As above, $\mathrm{Int}[\mathcal{C}(a,b)] = \mathcal{M}(a,b)$. Let

$\mathcal{C}(a, a_1, \ldots, a_l, b)$ be the part of $\mathcal{C}(a, b)$ consisting of piecewise flows which "stop" at the critical points $a_1 \prec \ldots \prec a_l$.

Lemma 18 *One has the equality*

$$\tau_b^a(\mathcal{C}(a, b) \times \{b\}) = \bigcup_{a \prec z \preceq b} M(a, z).$$

Furthermore, $\tau_b^a(\mathcal{M}(a, b) \times \{b\}) = M(a, b)$ *and more generally, one has that* $\tau_b^a(\mathcal{C}(a, a_1, \ldots, a_l, b) \times \{b\}) = M(a, a_1)$.

Proof. This is obvious since $\tau_b^a(\omega, b) = \omega(f(a) - 1)$. ■

The next lemma tells us that the various maps τ_b^a are compatible. It is a direct consequence of the associativity of the gluing maps.

Lemma 19 *If* $a \prec b \prec c$, *then one has the equality*

$$\tau_c^a(\mu(\omega, \omega', s), x) = \tau_b^a(\omega, I_b(\tau_c^b(\omega', x), s))$$

for any flow $(\omega, \omega', s) \in \mathcal{M}(a, b) \times \mathcal{M}(b, c) \times (0, 1)$ *and* $x \in D^u(c)$.

Proof One has that

$$\begin{aligned} \tau_c^a(\mu(\omega, \omega', s), x) &= \mu(\mu(\omega, \omega', s), \omega_x, t_c(x))(f(a) - 1) \\ &= \mu(\omega, \mu(\omega', \omega_x, t_c(x)), s)(f(a) - 1) \end{aligned}$$

while

$$\begin{aligned} \tau_b^a(\omega, I_b(\tau_c^b(\omega', x), s)) &= \tau_b^a(\omega, I_b(\mu(\omega', \omega_x, t_c(x))(f(b) - 1)), s) \\ &= \tau_b^a(\omega, \omega_z(f(b) - s)) \\ &= \mu(\omega, \omega_z, t_b(\omega_z(f(b) - s)))(f(a) - 1) \end{aligned}$$

where $z = \mu(\omega', \omega_x, t_c(x))(f(b)-1)$. The equality holds since $\omega_z = \mu(\omega', \omega_x, t_c(x))$, and $t_b(\omega_z(f(b)-s)) = s$. ∎

Remember that in the last section, we constructed a particular framing of the moduli space $M(a,b)$ in $S^u(a)$. We called that framing the Morse framing of $M(a,b)$.

Proposition 20 *The image of τ_b^a restricted to $\mathcal{M}(a,b) \times D^u(b)$ is a tubular neighbourhood of $M(a,b)$ in $S^u(a)$. Moreover, the induced framing is the Morse framing.*

We will denote the open tubular neighbourhood $\tau_b^a(\mathcal{M}(a,b) \times \mathrm{Int}[D^u(b)])$ by $T_{a,b}$.

Proof. For the first assertion, it is sufficient to show that

$$\tau_b^a|_{\mathcal{M}(a,b)\times D^u(b)} : \mathcal{M}(a,b) \times D^u(b) \to S^u(a)$$

is a local homeomorphism (see the definition of stratified set in 2.3). Let us check first that $\tau_b^a|$ is injective. Clearly, $\tau_b^a(\omega^1, x) = \tau_b^a(\omega^2, y)$ if and only if $\mu(\omega^1, \omega_x, t_b(x)) = \mu(\omega^2, \omega_y, t_b(y))$, and the latter belongs $\mathcal{M}(a,k)$, for some critical point k of f. It follows that ω_x and ω_y belongs to $\mathcal{M}(b,k)$. But

$$\mu : \mathcal{M}(a,b) \times \mathcal{M}(b,k) \times (0,1] \to \mathcal{M}(a,k)$$

is a diffeomorphism onto its image. Hence, $\omega^1 = \omega^2$ and $x = y$.

In order to conclude, let us check that $\tau_b^a|$ is a local homeomorphism (and thus an open map). We proceed by induction. First, it is clear that this statement holds if $\mathcal{M}(a,b)$ is compact, since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Assume

inductively that it holds for the maps $\tau^a_{a'}|$ and $\tau^{a'}_b|$, with $a \prec a' \prec b$. Then, the equation

$$\tau^a_b(\mu(\omega, \omega', s), x) = \tau^a_{a'}(\omega, I_{a'}(\tau^{a'}_b(\omega', x), s))$$

(from lemma 19) forces $\tau^a_b|$ to be a local homeomorphism.

Let us finally check that the framing induced by τ^a_b is the same as the Morse framing. One easily sees that the map τ^a_b extends to a map

$$W^s(b) \times D^u(b) \to M$$

by considering the inclusions

$$\mathcal{M}(a, b) \times D^u(b) \subset \coprod_{z \prec b} \mathcal{M}(z, b) \times D^u(b) \subset W^s(b) \times D^u(b).$$

Thus, it determines a framing of the stable manifold $W^s(b)$ in M. We are done since $W^s(b)$ is contractible and consequently, its is framing unique up to homotopy. ∎

3.3 Stratification of the unstable spheres

In the preceding section, we have constructed a tubular neighbourhood $T_{a,z}$ for each moduli space $M(a, z)$ in $S^u(a)$. Recall that

$$T_{a,z} = \tau^a_z(\mathcal{M}(a, z) \times \mathrm{Int}[D^u(z)])$$

and in particular,

$$M(a, z) = \tau^a_z(\mathcal{M}(a, z) \times \{z\}).$$

There is an obvious *retraction map*

$$\Pi_{a,z} : T_{a,z} \to M(a, z)$$

given by $\tau_z^a(\omega, y) \mapsto \omega(f(a) - 1)$ and a continuous *tubular function*

$$\rho_{a,z} : T_{a,z} \to [0, 1)$$

given by $\tau_z^a(\omega, y) \mapsto t_z(y) = f(z) - f(y)$. Moreover, it is clear that $\rho_{a,z}^{-1}(0) = M(a, z)$.

Lemma 21 *If $a \prec b \prec c$, then*

$$T_{a,b} \cap T_{a,c} = \tau_b^a(\mathcal{M}(a, b) \times I_b(T_{b,c} \times (0, 1))).$$

In particular, $T_{a,b} \cap M(a, c) = \tau_b^a(\mathcal{M}(a, b) \times I_b(M(b, c) \times (0, 1)))$.

Proof. Let us check the first assertion. It is clear, by lemma 19, that the inclusion

$$T_{a,b} \cap T_{a,c} \supseteq \tau_b^a(\mathcal{M}(a, b) \times I_b(T_{b,c} \times (0, 1)))$$

holds. To show the inclusion"$\subseteq$", pick a point in the intersection $T_{a,b} \cap T_{a,c}$. It is necessarily of the form $\omega(f(a) - 1)$ for some flow $\omega \in \mathcal{M}(a, d)$. Looking at the definitions of $T_{a,b}$ and $T_{a,c}$, one sees that

$$\omega \in \mathrm{im}[\mu_{abd}] \cap \mathrm{im}[\mu_{acd}].$$

By the clean intersection condition, one has that

$$\omega = \mu_{acd}(\mu_{abc}(\alpha, \beta, t), \gamma, s)$$

for some $\alpha \in \mathcal{M}(a, b)$, $\beta \in \mathcal{M}(b, c)$, $\gamma \in \mathcal{M}(c, d)$ and $s, t \in (0, 1)$. This shows that $\omega(f(a) - 1) = \tau_c^a(\mu(\alpha, \beta, t), z) = \tau_b^a(\alpha, I_b(\tau_c^b(\beta, z), t))$ for some $z \in D^u(c)$. The inclusion "$\subseteq$" follows. The second assertion is an easy consequence of the first. ∎

In the same manner, from lemma 21 and the clean intersection condition, one sees that $T_{a,b} \cap T_{a,c} \neq \emptyset$ if and only if $a \prec b \prec c$ or $a \prec c \prec b$. .

Let $\mathcal{S}(a)$ be the set of all moduli spaces of the form $M(a, z) \subset S^u(a)$.

Proposition 22 *The pair $(S^u(a), \mathcal{S}(a))$ is a stratified set.*

Proof. We must check that the five axioms (ss1), (ss3), (ss4), (ss5) and (ss6) hold (we are not concerned with the other axioms, since the strata of $S^u(a)$ have no faces). In what follows, $M(a,z)$ and $M(a,z')$ will denote two distinct strata of $S^u(a)$ (i.e. z and z' are two distinct critical points).

(ss1) If $M(a,z) \cap \overline{M(a,z')} \neq \emptyset$ then $M(a,z) \subset \overline{M(a,z')}$.

By lemma 18, one has that

$$M(a,z') \subset \overline{M(a,z')} \subset \bigcup_{a \prec k \preceq z'} M(a,k),$$

since $\cup_{a \prec k \preceq z'} M(a,k)$ is closed and contains $M(a,z')$. But the last inclusion is obviously an equality. To see this, consider any $x \in M(a,k)$ with $a \prec k \prec z'$. We want to show that x is the limit of some sequence of points $x_n \in M(a,z')$. For that purpose, consider any sequence y_n in $D^u(k) \cap W(k,z')$ such that $\lim_{n\to\infty} y_n = k$. Then, a trivial calculation shows that

$$\lim_{n\to\infty} \tau_k^a(\omega, y_n) = \omega(f(a)-1) = x,$$

where $\omega \in \mathcal{M}(a,k)$ is such that $\omega(f(a)-1) = x$. The required sequence is then given by $x_n = \tau_k^a(\omega, y_n)$. Axiom (ss1) follows immediately since the moduli spaces form a partition of the unstable sphere. Notice that (ss1) is equivalent to the following statement:

$$M(a,z) \cap \overline{M(a,z')} \neq \emptyset \;\Rightarrow\; z \prec z'.$$

(ss3) Each stratum is a smooth manifold.

This is obvious. Recall that $M(a,z) < M(a,z')$ means $M(a,z) \cap \overline{M(a,z')} \neq \emptyset$.

(ss4) If $T_{a,z} \cap M(a,z') \neq \emptyset$, then $M(a,z) < M(a,z')$. Furthermore, $(\Pi_{a,z}, \rho_{a,z}) : T_{a,z} \cap M(a,z') \to M(a,z) \times (0,1)$ is a smooth submersion.

The first assertion follows from the preceding lemma. The second one is easily obtained by noting that $(E_a^{-1} \times Id) \circ (\Pi_{a,z}, \rho_{a,z}) \circ \tau_z^a = Id \times t_z$.

(ss5) If $T_{a,z} \cap T_{a,z'} \neq \emptyset$, then $M(a,z)$ and $M(a,z')$ are comparable strata.

This is an immediate consequence of the preceding lemma (or of the clean intersection condition).

(ss6) If $M(a,z) < M(a,z')$ and if $x \in T_{a,z} \cap T_{a,z'}$ then

1. $\Pi_{a,z'}(x) \in T_{a,z}$
2. $\Pi_{a,z}(\Pi_{a,z'}(x)) = \Pi_{a,z}(x)$.
3. $\rho_{a,z}(\Pi_{a,z'}(x)) = \rho_{a,z}(x)$.

These equalities follow easily form the preceding lemma. ∎

We can now prove Theorem B, i.e. we can show that the stratification of the preceding proposition is actually a *framed stratification.*

First, one must associate with each stratum $M(a,z)$ in $S^u(a)$ an open cone $D(z)$. An obvious candidate for $D(z)$ is the open unstable disk $\mathrm{Int}[D^u(z)]$. It is then clear that $D(z)$ is homeomorphic to the open cone $C(S^u(z))$ (with z as cone point). By definition, $\tau_z^a(\mathcal{M}(a,z) \times D(z)) = T_{a,z}$. The required homeomorphism (trivialisation)

$$h_{a,z} : T_{a,z} \to M(a,z) \times D(z)$$

is defined to be the restriction of the inverse of τ_z^a, followed by the obvious identification $\mathcal{M}(a,z) \simeq M(a,z)$. It follows that $S^u(a)$ is a trivial stratified

set. Let us now check the axioms. We recall that $F(a) = (\mathcal{S}(a), H(a))$, where $H(a)$ is the set of all trivialisations $h_{a,z}$. Consider two comparable strata $M(a,z) < M(a,z')$.

(fs1) $D(z)$ is a trivial stratified set.

The open cone $D(z)$ associated to the stratum $M(a,z)$ of $F(a)$ is a trivial set with strata $\{D(z,c) | z \prec c\} \cup \{z\}$, where $D(z,c)$ stands for $\mathrm{Int}[D^u(z) \cap W(z,c)]$. The tubular neighbourhood $T_{D(z,c)}$ of the stratum $D(z,c)$ is given by

$$T_{D(z,c)} = I_z(T_{z,c} \times (0,1)).$$

(fs2) $h_{a,z}$ restricts to a diffeomorphism from $T_{a,z} \cap M(a,z')$ to $M(a,z) \times D(z,z')$.

This is obvious since $h_{a,z}(T_{a,z} \cap M(a,z)) = M(a,z) \times I_z(M(z,z') \times (0,1)) = M(a,z) \times D(z,z')$.

(fs3) $D(z') = D(D(z,z'))$.

One has that $D(z') = \mathrm{Int}[(D^u(z')]$. But the trivialisation of $T_{D(z,z')}$ is given by

$$T_{D(z,z')} \to T_{z,z'} \times (0,1) \to M(z,z') \times D(z') \times (0,1) \to D(z,z') \times D(z').$$

Axiom (fs3) follows easily.

(fs4) $h_{a,z}(T_{a,z} \cap T_{a,z'}) = M(a,z) \times T_{D(z,z')}$.

This is clear by the preceding lemma and by definition of $T_{D(z,z')}$. Finally, we must check the equalities from (fs5). Let $p \in T_{a,z} \cap T_{a,z'}$, i.e. p can be written as $p = \tau_z^a(x, I_z(\tau_{z'}^z(y,w), s)) = \tau_{z'}^a(\mu(x,y,s), w)$ (by lemma 19 and

21).

•Equation (A)

$$
\begin{aligned}
(1_{a,z} \times \Pi_{D(z,z')}) \circ h_{a,z}(p) &= (1_{a,z} \times \Pi_{D(z,z')}) \circ h_{a,z}(\tau_z^a(x, I_z(\tau_{z'}^z(y,w),s))) \\
&= (1_{a,z} \times \Pi_{D(z,z')})(x(f(a)-1), I_z(\tau_{z'}^z(y,w),s)) \\
&= (x(f(a)-1), I_z(y(f(z)-1),s)) \\
&= (x(f(a)-1), I_z(\tau_{z'}^z(y,z'),s)) \\
&= h_{a,z}(\tau_z^a(x, I_z(\tau_{z'}^z(y,z'),s))) \\
&= h_{a,z}(\tau_{z'}^a(\mu(x,y,s),z')) \\
&= h_{a,z}(\mu(x,y,s)(f(a)-1)) \\
&= h_{a,z} \circ \Pi_{a,z'}(\tau_{z'}^a(\mu(x,y,s),w)) \\
&= h_{a,z} \circ \Pi_{a,z'}(p).
\end{aligned}
$$

•Equation (B)

$$
\begin{aligned}
(1_{a,z} \times h_{D(z,z')}) \circ h_{a,z}(p) &= (1_{a,z} \times h_{D(z,z')})(x(f(a)-1), I_z(\tau_{z'}^z(y,w),s)) \\
&= (x(f(a)-1), I_z(y(f(z)-1),s), w) \\
&= (h_{a,z}(\tau_z^a(x, I_z(\tau_{z'}^z(y,z'),s))), w)) \\
&= (h_{a,z}(\tau_{z'}^a(\mu(x,y,s),z'), w) \\
&= (h_{a,z}(\mu(x,y,s)(f(a)-1), w) \\
&= (h_{a,z} \times 1_{D(z')})(\mu(x,y,s)(f(a)-1), w) \\
&= (h_{a,z} \times 1_{D(z')}) \circ h_{a,z'}(p).
\end{aligned}
$$

•Equation (C)

$$
\rho_{a,z'}(p) = \rho_{a,z'}(\tau_{z'}^a(\mu(x,y,s),w))
$$

$$
\begin{aligned}
&= t_{z'}(w) \\
&= \rho_{z,z'}(\tau^z_{z'}(y,w)) \\
&= \rho_{D(z,z')}(I_z(\tau^z_{z'}(y,w),s)) \\
&= \rho_{D(z,z')} \circ p_2(x(f(a)-1), I_z(\tau^z_{z'}(y,w),s)) \\
&= \rho_{D(z,z')} \circ p_2 \circ h_{a,z}(p).
\end{aligned}
$$

•Equation (D)

Let $p \in T_{D(z,z')}$ i.e. $p = I_z(\tau^z_{z'}(y,w),s)$.

$$
\begin{aligned}
\rho_0 \circ p_2 \circ h_{D(z,z')}(p) &= \rho_0 \circ p_2 \circ h_{D(z,z')}(I_z(\tau^z_{z'}(y,w),s)) \\
&= \rho_0 \circ p_2((y(f(a)-1),s),w) \\
&= \rho_0(w) \\
&= t_{z'}(w) \\
&= \rho_{D(z,z')}(p).
\end{aligned}
$$

Here, p_2 is the projection on the second factor and ρ_0 is the tubular function of $D(z')$. We have proved our second result.

Theorem 23 (Theorem B) *The pair $(S^u(a), F(a))$ is a framed stratified set.* ■

An easy modification of the preceding proof shows that the union

$$S^u(a,b) = \bigcup_{a \prec z \preceq b} M(a,z)$$

determines also a framed stratification $F(a,b) = (\mathcal{S}(a,b), H(a,b))$ of the unstable sphere $S^u(a)$.

3.4 Attaching maps

Let $f : M \to \mathbf{R}$ be a Morse function on a closed manifold M of dimension m. This means that all the critical points of f are nondegenerate. It follows that for any critical point a, there is a neighbourhood U of a and a coordinate map $h : U \to V \subset \mathbf{R}^m$ (a *Morse chart*) such that f restricted to U can be written as a quadratic polynomial:

$$f(h^{-1}(x_1, \ldots, x_m)) = f(a) - x_1^2 - \ldots - x_{\lambda_b}^2 + x_{\lambda_b+1}^2 + \ldots + x_m^2.$$

A vector field X on M is *gradient-like with respect to* f (see [14]) if

1. $X(f) > 0$ throughout the complement of the set of critical points of f.

2. For any critical point a of f, there is an open neighbourhood U of M containing a, a Morse chart $h : U \to V \subset \mathbf{R}^m$ as above, such that

$$Dh_p(X_p) = 2(-x_1(p), \ldots, -x_k(p), x_{k+1}(p), \ldots, x_m(p))$$

   for any $p \in U$.

To construct gradient-like vector fields on M, one can proceed as follows. Pick any Morse function f on M. Take a Riemannian metric on M which is induced by a Morse chart near the critical points of f. Then, the gradient of f, with respect to this metric, is a gradient-like vector field on M. Actually, a theorem of Smale [22] says that any gradient-like field is the gradient of some Morse function.

A gradient-like vector field X satisfies the *transversality condition* if for any pair of rest points a and b of X, $W^u(a)$ is transverse to $W^s(b)$. Here,

the stable and unstable manifolds of rest points of vector fields are defined in the obvious way.

Recall that a Morse function $f : M \to \mathbf{R}$ is *self-indexing* if $f(a) = \lambda_a$ for any $a \in \mathrm{Crit}(f)$.

Theorem 24 (Smale) *In the space of gradient-like flows with the C^r topology, those which satisfy the transversality condition form a dense open subset. Any gradient-like field X in this subset is of the form $X = \mathrm{grad}(f)$ for some Riemannian metric and some self-indexing Morse function $f : M \to \mathbf{R}$.* ∎

We can now quote a fundamental theorem of Morse theory which, in this form, is due to Franks [14].

Theorem 25 (Franks) *If X is a gradient-like vector field on M, there exists a CW complex Z, unique up to cell equivalence, and a homotopy equivalence $g : M \to Z$, such that for each rest point a of index k, $g(W^u(a))$ is contained in the base $Z(e)$ of a single k-cell e.*

The map g establishes a bijection between rest points of X of index k and k-cells of Z. Moreover, the Smale partial order on the set of rest points of X corresponds to the natural partial order on the cells of Z, i.e. g induces a poset anti-isomorphism. ∎

Let us describe briefly the construction of the CW complex Z. We follow closely the exposition of Franks [14], where more details can be found.

Take a self-indexing Morse-Smale function f whose gradient is X. Its existence is ensured by Smale's theorem quoted above. Set $M_k = f^{-1}([0, k+1/2])$ and suppose that one has constructed a homotopy equivalence

$$g_{k-1} : M_{k-1} \to Z^{k-1}$$

for some CW complex Z^{k-1}. Notice that if $k-1=0$, the definitions of g_0 and Z^0 are clear. Classical Morse theory tells us that there is a retraction

$$r_k : M_k \to M_{k-1} \cup \{D^u(a_i)\}$$

where, for each critical point a_i of index k,

$$D^u(a_i) = W^u(a_i) \cap \mathrm{cl}[M - M_{k-1}].$$

The construction of the retraction r_k will be indicated below. We identify the unstable sphere $S^u(a_i)$ of a_i with $\partial D^u(a_i)$. Clearly, g_{k-1} extends to a homotopy equivalence

$$(g_{k-1}) : M_{k-1} \cup \{D^u(a_i)\} \to Z^{k-1} \cup \{e_{a_i}\}$$

where the k-cells $\{e_{a_i}\}$ are attached to Z^{k-1} by the restrictions of g_{k-1} to the unstable spheres $\{S^u(a_i)\}$. Set

$$Z^k = Z^{k-1} \cup \{e_{a_i}\}.$$

Composing the retraction r_k with the homotopy equivalence (g_{k-1}) yields a homotopy equivalence $g_k = (g_{k-1}) \circ r_k : M_k \to Z^k$. The *attaching map* of a critical point a of index $k+1$ is the map

$$\phi_a : S^u(a) \to Z^k$$

obtained by restricting g_k to $S^u(a) \subset \partial M_k$. Since there are finitely many critical points, the procedure eventually terminates and one gets a CW complex Z, together with a homotopy equivalence $g : M \to Z$. Recall that Z has been denoted by X_f in the first section.

On the other hand, one knows from the preceding sections (proposition 10) that there is a transverse CW complex $Y(a)$ and a transverse map

$$\psi_a : S^u(a) \to Y(a)$$

such that $F(\psi_a, Y(a)) = F(a)$. Recall that the following three sets are in bijection:

- The set of critical points of f satisfying $a \prec z$.
- The set of strata of $F(a)$.
- The set of cells of $Y(a)$.

As we have seen, these sets are equipped with natural partial orders, and they are isomorphic (or anti-isomorphic) as posets.

What are the relations between the CW complexes Z^k and $Y(a)$? The answer lies in the following lemma.

Proposition 26 *There exists a transverse CW complex Y^k, of dimension at most k, which has the following properties.*

1. *The complex Y^k has a filtration*

$$Y^0 \subset Y^1 \subset \cdots \subset Y^l \subset \cdots \subset Y^k$$

   by transverse subcomplexes and each Y^l has dimension at most l.

2. *For any critical point z of index $\lambda_z = l + 1 \leq k + 1$, the transverse complex $Y(z)$ can be identified with a subcomplex of Y^l.*

3. *For any critical point z of index $\lambda_z = l + 1 \leq k + 1$, there is a cell equivalence $H_l : Z^l \to Y^l$ such that $H_l \circ \phi_z$ is homotopic to ψ_z.*

Before embarking on the proof, let us show how one can get Theorem C as a corollary of this proposition. Clearly, Y^k is simply a transverse version of Z^k. It is thus obvious that $[F(\phi_a, X_f)] = [F(\phi_a, Z)] = [F(H_k \circ \phi_a, Y^k)]$. But since ψ_a is homotopic to $H_k \circ \phi_a$, it follows from Theorem A that

$$[F(\phi_a, X_f)] = [F(H_k \circ \phi_a, Y^k)] = [F(\psi_a, Y(a))] = [F(a)].$$

This leads to

Theorem 27 (Theorem C) *The cobordism classes* $[F(\phi_a, X_f)]$ *and* $[F(a)]$ *coincide in* $\Omega_{X_f}(S^u(a))$. ∎

This proposition will be proved by induction. If $k = 0$, then the definition of Y^0 is obvious (one zero cell for each critical point of index 0), and if a is a critical point of index 1, it is plain that $\phi_a \simeq \psi_a$. Assume that the lemma holds for any critical point z with $\lambda_z < \lambda_a = k + 1$. Thus, if z is such a critical point of index $l + 1$, there is a homotopy equivalence $H_l : Z^l \to Y^l$ such that $H_l \circ \phi_z \simeq \psi_z$ and $F(\psi_z, Y(z)) = F(\psi_z, Y^l) = F(z)$.

Let b be a critical point of index $\lambda_b = k$. For the sake of clarity, let us assume for the moment that there is no other critical point of index k. Let

$$F : S^u(b) \times [0, 1] \to Y^{k-1}$$

be a homotopy from $H_{k-1} \circ \phi_b$ to ψ_b, such that $F_0 = H_{k-1} \circ \phi_b$ and $F_1 = \psi_b$. This homotopy between $H_{k-1} \circ \phi_b$ and ψ_b determine a homotopy equivalence $H : Z^{k-1} \cup_{\phi_b} e \to Y^{k-1} \cup_{\psi_b} e$. The map H is given by the following composite:

$$Z^{k-1} \cup_{\phi_b} e \xrightarrow{H'_{k-1}} Y^{k-1} \cup_{H_{k-1} \circ \phi_b} e \xrightarrow{k} Y^{k-1} \cup_{\psi_b} e$$

Here, H'_{k-1} is the obvious extension of H_{k-1}. The map k is given by $k|_{Y^{k-1}} = Id$, and

$$k(tu) = \begin{cases} 2tu & \text{if } u \in \partial e \text{ and } 0 \leq t \leq 1/2 \\ F_{2-2t}(u) & \text{if } u \in \partial e \text{ and } 1/2 \leq t \leq 1 \ . \end{cases}$$

Let $\{b_i\}$ be the set of critical points of index k. If we repeat the above construction for each b_i, and set $Z^k = Z^{k-1} \cup_{\phi_{b_i}} \{e_{b_i}\}$ and $Y^k = Y^{k-1} \cup_{\psi_{b_i}} \{e_{b_i}\}$, we get a homotopy equivalence

$$H_k : Z^k \to Y^k.$$

We must check that $H_k : Z^k \to Y^k$ satisfies the three conditions of the proposition.

The first condition trivially holds, since $Y^k = Y^{k-1} \cup_{\psi_{b_i}} \{e_{b_i}\}$ fits naturally at the end of the filtration of Y^{k-1}, which exists by induction. Now, if one refers to the construction of the framed stratification $F(a)$ of $S^u(a)$, one sees that the link of the stratum $M(a,z)$ is the unstable sphere $S^u(z)$, with the framed stratification $F(z)$. By induction $F(z) = F(\psi_z, Y^l)$ if $\lambda_z = l+1$. Thus, $F(a)$ is clearly weakly modelled on Y^k, and this precisely means that $Y(a)$ can be identified with a subcomplex of Y^k. Thus $\psi_a : S^u(a) \to Y^k$. To complete the proof, it remains to show that $H_k \circ \phi_a$ is homotopic to ψ_a. The rest of this subsection is devoted to this problem.

Since ϕ_a is defined as the composite of a retraction r_k with a homotopy equivalence, we must first describe the retraction $r = r_k : M_k \to M_{k-1} \cup \{D^u(b_i)\}$ with some details. This construction is classical and we follow closely [17]. Let us assume for clarity that there is a unique critical point b of index k. We can suppose that $f(b) = 0$ (replacing f by $f - k$). It is then

sufficient to give a retraction

$$r : f^{-1}((-\infty, \epsilon]) \to f^{-1}((-\infty, -\epsilon]) \cup D^k$$

for some well chosen $\epsilon > 0$. We pick once for all a chart

$$h : U \to V \subset \mathbf{R}^m = \mathbf{R}^k \times \mathbf{R}^{m-k}$$

such that U contains b and h satisfies condition 2 in the definition of gradient-like vector field. It then follows that $f \circ h^{-1}(x) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_m^2$. Let $0 < \delta < 1$ be such that the open set V contains the box

$$\Gamma = B^k(\delta) \times B^{m-k}(\delta).$$

Here, $B^i(\delta) \subset \mathbf{R}^i$ is the closed ball of radius δ centered at the origin. Set $g(x) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_m^2$. It is clear that our choice of coordinates implies that

$$Dh(\mathrm{grad}(f)) = \mathrm{grad}(f \circ h^{-1}) = \mathrm{grad}(g),$$

where the gradient of g is taken with respect to the Euclidean metric on $\mathbf{R}^m$. It follows that flow lines of $\mathrm{grad}(f) = X$ correspond to flow lines of $\mathrm{grad}(g)$ under the chart h.

Choose an $\epsilon > 0$ much smaller than δ and set

$$B^k = \{(x, 0) \in \mathbf{R}^k \times \mathbf{R}^{m-k} : |x|^2 \leq \epsilon\}$$

and

$$D^k = h^{-1}(B^k) \subset M.$$

We will describe a deformation of $f^{-1}([-\epsilon, \epsilon])$ to $f^{-1}(-\epsilon) \cup D^k$ which is the identity on $f^{-1}(-\epsilon) \cup D^k$, and consequently extends by the identity map to

a deformation of $f^{-1}((-\infty,\epsilon])$ to $f^{-1}((-\infty,-\epsilon])\cup D^k$, as required. We first describe a "local model" i.e. a retraction of $g^{-1}([-\epsilon,\epsilon])$ to $g^{-1}(-\epsilon)\cup B^k$.

Roughly speaking, it is obtained by following the flow lines of some vector field on V which coincides with the field $(0,-2y)$ near $\{0\}\times\mathbf{R}^m$, and which, sufficiently far from this coordinate plane, corresponds to the gradient field of $g(x)$. The deformation is then pulled back to M by the coordinate map $h:U\to V$.

More precisely, consider the concentric boxes

$$\Gamma\supset\Gamma_2\supset\Gamma_1$$

where $\Gamma_1 = B^k(\sqrt{\epsilon})\times B^{m-k}(\sqrt{\epsilon})$ and $\Gamma_2 = B^k(\sqrt{2\epsilon})\times B^{m-k}(\sqrt{2\epsilon})$. Let $\mu : V\to[0,1]$ be the restriction to V of some smooth function $\mu:\mathbf{R}^k\times\mathbf{R}^{m-k}\to[0,1]$ which vanishes on Γ_1 and equals 1 outside Γ_2. Consider the vector field on V given by

$$Y(x,y) = 2(\mu(x,y)x,-y).$$

The deformation of $g^{-1}([-\epsilon,\epsilon])$ into $g^{-1}(-\epsilon)\cup B^k$ is obtained by moving each point of $\Gamma\cap g^{-1}([-\epsilon,\epsilon])$ along the flow lines of the field Y until it reaches $g^{-1}(-\epsilon)\cup B^k$. Clearly, one can transport this motion to $f^{-1}([-\epsilon,\epsilon])$ by h. We obtain in this way a deformation of $f^{-1}([-\epsilon,\epsilon])$ into $f^{-1}(-\epsilon)\cup D^k$ which outside $h^{-1}(\Gamma_2)$ is given by following the flow lines of $-X = -\mathrm{grad}(f)$. More details can be found in [17].

Thus, for any critical point z, there is an interval $[\alpha_z,\beta_z]$, about $f(z)$ and a retraction

$$r_z : f^{-1}((-\infty,\beta_z])\to f^{-1}((-\infty,\alpha_z])\cup D^u(z)$$

constructed as above. Notice that our notation has changed slighty, since $D^u(z)$ is now

$$D^u(z) = W^u(z) \cap \mathrm{cl}[M - f^{-1}((-\infty, \alpha_z])]$$

rather than $W^u(z) \cap \mathrm{cl}[M - M_{\lambda_z - 1}]$, as in the beginning of this section. The construction of this retraction will be used in the proof of lemma 29.

We must now recall the following concepts from section 2.2, and also introduce some new terminology. Let $X = \cup_{\lambda \in \Lambda} e_\lambda$ be a transverse CW complex and let

$$h_\lambda : D_\lambda \to X$$

be a characteristic map for the cell $e_\lambda \subset X$. To a transverse map $f : M \to X$ from a (k)-manifold M to X, one has associated a family of bundles

$$t_\lambda : T_\lambda \to D_\lambda$$

such that $f|_{T_\lambda} = h_\lambda \circ t_\lambda$. We recall that T_λ is simply $\mathrm{cl}[f^{-1}(e_\lambda)]$. We will say that T_λ is a *block* of M and that $M = \cup_{\lambda \in \Lambda} T_\lambda$ is the *block decomposition* of M associated to $F(f, X)$.

The fibre $N_\lambda = t_\lambda^{-1}(0)$ of t_λ, which is a manifold with faces, will be called a *truncated stratum* of the framed stratified set $(M, F(f, X))$. An easy induction argument shows that the truncated stratum N_λ is given by $M_\lambda \cap T_\lambda$, where M_λ is the stratum corresponding to the cell e_λ. The various properties of these objects are described by (t1),...,(t5) of section 2.2.

If f and g are two transverse maps from M to X, we will say that the block decompositions associated to $F(f, X)$ and $F(g, X)$ are *comparable* if they induce the same set-theoretic decomposition $M = \cup_{\lambda \in \Lambda} T_\lambda$ of M and diffeomorphic truncated strata. If f and g induce comparable block

decompositions, it does not of course follow that f is homotopic to g, since they could give rise to different framings of the strata. But this is the only obstruction, as we can see from the following lemma.

Lemma 28 *Let $\phi, \psi : M \to X$ be two transverse maps from the (k)-manifold M to the transverse CW complex X. Let us assume that ϕ and ψ induce comparable block decompositions of M. If ϕ and ψ induce equivalent framings of the truncated strata, then ϕ is homotopic to ψ.*

Proof. The proof is by induction on the number of cells in X.

Proof for $X = S^p$, with cellular structure $S^p = * \cup e$.
This case is trivial, since there is a unique non-open stratum, which is framed equivalently by ϕ and ψ.

Proof for X arbitrary.
As usual, let $X = X_0 \cup e$ and $M_0 = \mathrm{cl}[M - T]$, where T is the tubular neighbourhood of the stratum N corresponding to the cell e. First, notice that the restrictions ϕ_0 and ψ_0 of ϕ and ψ to M_0 both map M_0 into X_0. By induction, there is a homotopy

$$H_0 : M_0 \times [0, 1] \to X_0$$

such that $H_0(x, 0) = \phi_0(x)$ and $H_0(x, 1) = \psi_0(x)$. Since $\phi|_T$ is homotopic to $\psi|_T$ through maps which are at each time t trivialisations of the neighbourhood T of the closed stratum $N \subset T$, H_0 extends to a homotopy H from ϕ to ψ. ∎

Let us now come back to the Morse-theoretic context. We have a block decomposition

$$S^u(a) = \bigcup_{a \prec z} S_{a,z}$$

induced by $\psi_a : S^u(a) \to Y(a)$, bundle maps $t_{a,z} : S_{a,z} \to D^u(z)$ such that $h_z \circ t_{a,z} = \psi_a|_{S_{a,z}}$, see section 3.3. We recall that h_z is a characteristic map for the cell associated to the critical point z. Moreover, the truncated strata are obviously given by the intersections $N(a,z) = M(a,z) \cap S_{a,z}$. We must study the behaviour of the Morse attaching map $\phi_a : S^u(a) \to Z^k$ with respect to the block decomposition induced by ψ_a.

Lemma 29 *One can construct the attaching map ϕ_a in such a way that for each truncated stratum $N(a,z)$, there is an open neighbourhood U of $N(a,z)$ in $S^u(a)$ with the following properties:*

1. *$H_k \circ \phi_a|_U : U \to e_z \subset Y^k$ is smooth.*
2. *$H_k \circ \phi_a(N(a,z)) = h_z(0) \in e_z$.*
3. *$H_k \circ \phi_a(S_{a,w}) \cap e_z = \emptyset$ if $\lambda_w < \lambda_z$.*

Proof. The lemma clearly holds for the closed strata. Notice that if $M(a,z)$ is a closed stratum (i.e. if $a \prec_s z$), then $M(a,z) = N(a,z)$.

Consider the set of critical points $A \subset \mathrm{Crit}(f)$ determined by the condition

$$z \in A \quad \Leftrightarrow \quad f(a) > f(z) > f(b).$$

Set $S(a,b) = \mathrm{cl}[S^u(a) - \cup_{z \in A} S_{a,z}]$. Let us now show that one can construct the attaching map $\phi_a : S^u(a) \to Z^k$ in such a way that ϕ_a, restricted to $S(a,b)$, can be factorized as

$$S(a,b) \xrightarrow{R} f^{-1}((-\infty, \beta_b]) \xrightarrow{(g_b)\circ r_b} Z^{\lambda_b - 1} \cup_{\phi_b} e_b \xrightarrow{i} Z^k$$

Here, i is the obvious inclusion and $R(x)$ is by definition the unique point on $f^{-1}(\beta_b)$ which is on the same orbit as x. Let us explain why there is such a factorization. If b is any successor of a, there is no problem. Let $b \prec_s c$ and assume that one has the factorization given above for ϕ_a restricted to $S(a,b)$. Let $S(a,c) = \mathrm{cl}[S^u(a) - \cup_{z\in A'} S_{a,z}]$, where $A' = \{z \in \mathrm{Crit}(f) : f(a) > f(z) > f(c)\}$. Notice that $S(a,c) \subset S(a,b)$. To show that one can get the same factorization, with c replacing b, we will apply the following principle, which follows directly from the method used to construct the retraction.

Claim: Given any $\delta > 0$, the retraction

$$r_b : f^{-1}((-\infty, \beta_b]) \to f^{-1}((-\infty, \alpha_b]) \cup D^u(b)$$

can be chosen to satisfy the following condition. If $x \in f^{-1}((-\infty, \beta_b]) - f^{-1}((-\infty, \alpha_b])$ and is not within δ of $W^s(b)$, then $r_b(x)$ is the unique point on $f^{-1}(\alpha_b)$ on the same orbit as x.

Let us apply $R : S(a,b) \to f^{-1}((-\infty, \beta_b])$ to $S(a,c) \subset S(a,b)$. Clearly,

$$R(S(a,c)) \cap f^{-1}(\beta_b) \cap W^s(b) = \emptyset.$$

Since $R(S(a,c))$ and $f^{-1}(\beta_b) \cap W^s(b) = S^s(b)$ are both closed in $f^{-1}(\beta_b)$, there is an open set U containing $S^s(b)$ and disjoint from $R(S(a,c))$. Now, one can choose $\delta > 0$ so small that a δ-neighbourhood U_b of $S^s(b)$ will be contained in the open set U, and in particular, $R(S(a,c)) \cap U_b = \emptyset$. By the claim, when restricted to $S(a,c)$, R maps into $f^{-1}(\beta_c)$. This implies that ϕ_a restricted to $S(a,c)$ can be factorized as

$$S(a,c) \xrightarrow{R} f^{-1}((-\infty, \beta_c]) \xrightarrow{(g_c)\circ r_c} Z^{\lambda_b - 1} \cup_{\phi_c} e_c \xrightarrow{i} Z^k,$$

as required.

We now show that the existence of such a factorization leads to the 3 statements of the lemma. Set $z = c$.

For the first statement, notice that by the above factorization, there is a neighbourhood V of $N(a,c)$ in $S(a,c)$ such that $H_k \circ \phi_a|_V$ is smooth. But since one can certainly modify the factorization, replacing $S(a,c)$ by some open set containing $S(a,c)$, 1 holds.

The second statement is clear since $H_k \circ \phi_a(N(a,c)) = H_k \circ (g_c)(c) = h_c(0)$.

Finally, if $\lambda_w < \lambda_c$, then $S_{a,w} \subset S(a,c)$. But we have a similar factorization of ϕ_a restricted to $S(a,w)$, and 3 follows immediately. ∎

We now "abstract" the situation in the next lemma, where we use the notation introduced above.

Lemma 30 *Let ϕ and ψ be two maps from the (k)-manifold M to the transverse CW complex $X = \bigcup_{\lambda \in \Lambda} e_\lambda$. Let us assume that ψ is transverse and that for any $\lambda \in \Lambda$, there is an open neighbourhood U_λ of the truncated stratum N_λ (of $F(\psi, X)$) in M such that*

1. *$\phi|_{U_\lambda} : U_\lambda \to e_\lambda \subset X$ is smooth.*

2. *$\phi(N_\lambda) = h_\lambda(0) \in e_\lambda$.*

3. *$\phi(T_\gamma) \cap e_\lambda = \emptyset$ if $\dim[e_\gamma] < \dim[e_\lambda]$.*

Then, ϕ is homotopic to a transverse map $\phi' : M \to X$ whose block decomposition is comparable with the block decomposition induced by ψ.

Notice that in our situation, $M = S^u(a)$, $H_k \circ \phi_a = \phi$, $\psi_a = \psi$ and $X = Y^k$.

Proof. As usual, we prove the lemma by induction on the number of cells in X.

Proof when $X = S^p$, with cellular structure $S^p = * \cup e$.

Let $t : T \to D$ be the block corresponding to the cell e of X, $h : D \to X$ a characteristic map for e and $N \subset T$ the stratum corresponding to e. Since $X = S^p$ is a manifold, one can apply lemma 5 and assume that ϕ is a smooth map satisfying $\phi(N) = h(0) \in e \subset X$. Applying step 2 in the proof of proposition 7, one can make ϕ transverse (and the homotopy can be taken rel N). Thus, one gets a tubular neighbourhood T' of N in M such that

$$\phi|_{T'} : T' \xrightarrow{t'} D \xrightarrow{h} S^p$$

and $(t')^{-1}(0) = N$. By [12], there is a self-diffeomorphism $\gamma : M \to M$ of M such that $\gamma(T) = T'$, $\gamma|_N$ is the identity of N and γ is isotopic to the identity. It is easy to check that the map $\phi \circ \gamma$ is transverse to S^p and induces the same block decomposition as ψ.

Proof for X arbitrary.

Let $X = X_0 \cup e$. We denote by ϕ_0 and ψ_0 the restrictions of ϕ and ψ to $M_0 = \mathrm{cl}[M - T]$. Here, T is the block of $F(\psi, X)$ corresponding to the top cell e. Clearly, ψ_0 and ϕ_0 map into X_0 (by 3). By induction, one can suppose that ϕ_0 is homotopic to a transverse map with the same block decomposition as ψ_0. But by the Homotopy Extension Property, this homotopy can be extended to M and it is easy to check that one gets a transverse map which has the same block decomposition as ψ. ∎

It follows that $H_k \circ \phi_a$ is homotopic to a map which has the same block decomposition as ψ_a. Moreover, the framing that this map induces on $N(a, c)$ is clearly the Morse framing, since it is given by the splitting of a Morse chart about the critical point c. Thus, it is the restriction of the framing of $W^s(c)$ in M to $N(a, c)$. By lemma 28, $H_k \circ \phi_a$ is homotopic to ψ_a.

This completes the proof of the proposition and ends this section.

4 On the cup product in Morse theory

4.1 Introduction

Consider the following isomorphism from standard homotopy theory:

$$\pi_{2p-1}(S^p \vee S^p) \simeq \pi_{2p-1}(S^p) \oplus \pi_{2p-1}(S^p) \oplus \mathbf{Z}, \tag{3}$$

see for example [23], p. 419. It is a well-known fact that the third summand on the right-hand side is generated by the Whitehead product

$$W = [i_p, i_p] : S^{2p-1} \to S^p \vee S^p.$$

We recall that W can be seen as the attaching map of the top cell in the CW decomposition $S^p \vee S^p \cup_W e^{2p}$ of $S^p \times S^p$. It is easy to see that the only non-trivial relation in the cellular cohomology ring of $S^p \vee S^p \cup_W e^{2p}$ is given by $\omega_p \smile \omega'_p = \omega_{2p}$, where ω_p and ω'_p are the generators corresponding to the two bottom cells of $S^p \vee S^p \cup_W e^{2p}$ and ω_{2p} is the generator corresponding to the top cell e^{2p}. More generally, there is a "cup product" homomorphism

$$\mathcal{C} : \pi_{2p-1}(S^p \vee S^p) \to \mathbf{Z}$$

defined as follows: given any map ϕ, first construct the associated mapping cone $S^p \vee S^p \cup_\phi e^{2p}$. The integer $\mathcal{C}(\phi)$ (which only depends on the homotopy class of ϕ) is then determined by the equality

$$\omega_p \smile \omega'_p = \mathcal{C}(\phi)\omega_{2p}$$

where ω_p, ω'_p and ω_{2p} are the obvious generators of the cellular cohomology ring of $S^p \vee S^p \cup_\phi e^{2p}$. The above remark about $H^*(S^p \vee S^p \cup_W e^{2p})$ implies that $\mathcal{C}(W) = 1$.

On the other hand, given any map $\phi : S^{2p-1} \to S^p \vee S^p$, alternatively collapsing both p-spheres yields two maps from S^{2p-1} to S^p and thus, two framed $(p-1)$-submanifolds M_ϕ and N_ϕ of S^{2p-1}. We can take their linking number $\mathrm{lk}(M_\phi, N_\phi)$, which depends only on the homotopy class of ϕ. In this way, one gets a second homomorphism

$$\mathcal{L} : \pi_{2p-1}(S^p \vee S^p) \to \mathbf{Z}$$

given by $[\phi] \mapsto \mathrm{lk}(M_\phi, N_\phi)$. An easy geometric argument shows that $\mathcal{L}(W) = 1$. Using the above isomorphism (3), it follows that $\mathcal{L} = \mathcal{C}$, i.e. both homomorphisms coincide. This gives a simple geometric interpretation of the cup product for 3-complexes of the form $S^p \vee S^p \cup_\phi e^{2p}$. Notice that the same argument gives the analogous result for $S^q \vee S^r \cup_\phi e^{q+r}$, $q > r > 2$. This is of course a mild generalization of the *Hopf invariant* homomorphism.

Motivated by Morse theory, we generalize this remark to the following situation. Let us assume that $p + 1 = q + r$ and $p > q > r$. Consider a representative g of a homotopy class $[g]$ in $\pi_{q-1}(S^r)$ such that $[g]$ is of *finite order* (Serre's theorem implies that this is always the case except if r is even and $q = 2r$, or if $q - 1 = r$). As above, we can construct the mapping cone $S^r \cup_g e^q$. Again, there is a cup product homomorphism $\mathcal{C} : \pi_p(S^r \cup_g e^q) \to \mathbf{Z}$. In the next section, we define a *linking number* for $[g]$ which coincides with the standard linking number when the order of g is 1 (i.e. when g is null-homotopic). In this way, one obtains a second homomorphism $\mathcal{L} : \pi_p(S^r \cup_g e^q) \to \mathbf{Z}$. The main result is the following.

Theorem 31 *The homomorphisms $\mathcal{C}$ and $\mathcal{L}$ coincide.*

This section is organized as follows. Subsection 4.2 is devoted to the definition and properties of $\mathcal{L}$. In 4.3, we prove some homotopy-theoretic

lemmas while the proof of the main result occupies 4.4. The last subsection is devoted to the proof of Theorem D.

4.2 Linking numbers

Let ϕ be an element of $\pi_p(S^r \cup_g e^q)$. The *linking number* of ϕ, $\mathcal{L}(\phi)$, is by definition the standard linking number $\mathrm{lk}(k_\phi, l_\phi)$ of two disjoint null homologous simplicial cycles k_ϕ, l_ϕ in S^p. For the definition and properties of the standard linking number, we refer to [1].

We first explain the construction of k_ϕ and l_ϕ. Let us assume that both g and ϕ are transverse maps, in the sense of section 2.2, and let N be the framed $(q-r-1)$-submanifold of $\partial D^q = S^{q-1}$ associated to the transverse map $g : S^{q-1} \to S^r$.

- Construction of the cycle l_ϕ.

Consider the composite $c \circ \phi$

$$S^p \xrightarrow{\phi} S^r \cup_g e^q \xrightarrow{c} S^q$$

where c collapses the bottom cell of $S^r \cup_g e^q$. Consider also the unique closed stratum L_ϕ of the framed stratification $F(c \circ \phi, S^q)$ of S^p. It is a framed closed $(p-q)$-submanifold of S^p, and it is thus naturally oriented ("the orientation plus the framing" yields the standard orientation of S^p). Using the orientation, one can orient the $(p-q)$-simplexes of any triangulation of L_ϕ, and this yields an oriented cycle l_ϕ which is a generator of the top (simplicial) homology of L_ϕ. It is clear that l_ϕ is a null homologous simplicial $(p-q)$-cycle in S^p.

- Construction of the cycle k_ϕ.

Consider the framed stratification $F(\varphi, S^r \cup_g e^q)$ of S^p. There are only two

non open strata S_1, S_2 and clearly $\mathrm{cl}[S_2] = S_1 \cup S_2$. The closed set $S_1 \cup S_2$ can be triangulated and since S_2 is framed, one can again orient the top dimensional simplexes. We thus get a generator of the top homology of $S_1 \cup S_2$ and this is precisely the cycle k_ϕ.

We can give a more precise description of the cycle k_ϕ in the following way. Pick any regular value $z \in S^q$ of $c \circ \phi$, and as above, consider the framed submanifold $c \circ \phi^{-1}(z) = M$ of S^p. There is a trivialisation

$$h_M : T_M \to M \times D(M)$$

of the tubular neighbourhood of M in S^p. Here, $D(M)$ is an open q-disk D^q. Let ϕ_0 be the restriction of ϕ to the manifold with boundary $\mathrm{cl}[S^p - T_M] = S^p_0$. Pick a regular value $y \in S^r$ of ϕ_0 and consider the $(p-r)$ framed submanifold with boundary

$$Q = \phi_0^{-1}(y)$$

with $\partial Q = Q \cap \partial S^p_0$. It is clear that h_M determines a diffeomorphism from $\partial \mathrm{cl}[T_M]$ to $M \times S^{q-1}$, and that $h_M(\partial Q) = \{(m,x) \in M \times S^{q-1} : g(x) = y\} = M \times g^{-1}(y) = M \times N$. Consider the cone CN over the closed manifold N (with cone point $0 \in D^q$) and the product $M \times CN$ in $M \times D^q$. Pull it back to $\mathrm{cl}[T_M]$ using the trivialisation h_M and consider the space K_ϕ given by

$$K_\phi = Q \cup_{M \times N} M \times CN.$$

It is a *manifold with a singularity of type* M, see [4]. It is easy to show that each connected component of this manifold has a fundamental class (by excision and the long exact sequence in homology), provided that Q is oriented, which is the case since Q is framed.

We will denote by k_ϕ the oriented simplicial cycle given by any triangulation of K_ϕ which is a generator of the top homology of K_ϕ. Clearly, k_ϕ is a null homologous simplicial $(p-r)$-cycle in S^p.

Recall that $p+1 = q+r$ and thus $\dim[L_\phi] + \dim[K_\phi] = \dim[S^p] - 1$. By a simple transversality argument, we can make L_ϕ and K_ϕ disjoint. The *linking number* of ϕ is defined to be

$$\mathcal{L}(\phi) = \mathrm{lk}(k_\phi, l_\phi),$$

where lk is the standard linking number of cycles in S^p. We will sometimes write $\mathrm{lk}(K_\phi, L_\phi)$ instead of $\mathrm{lk}(k_\phi, l_\phi)$ in what follows.

Lemma 32 *The linking number $\mathcal{L}(\phi)$ depends only on the homotopy class of ϕ and thus determines a map $\mathcal{L} : \pi_p(S^r \cup_g e^q) \to \mathbf{Z}$.*

Proof. Let Φ be a transverse homotopy

$$\Phi : S^p \times I \to S^r \cup_g e^q$$

and set $\Phi_0 = \phi$ and $\Phi_1 = \psi$. It is necessary to show that $\mathcal{L}(\phi) = \mathcal{L}(\psi)$, i.e. that $\mathrm{lk}(K_\phi, L_\phi) = \mathrm{lk}(K_\psi, L_\psi)$. Consider the composition

$$S^p \times I \xrightarrow{\Phi} S^r \cup_g e^q \xrightarrow{c} S^q$$

where c is th usual collapsing map. Choose two distinct regular values z and z' of $c \circ \Phi$. Let L and M be respectively the inverse images of z and z'. Both are smooth manifolds with boundary. Set $\partial L = \partial_0 L \cup \partial_1 L$ and $\partial M = \partial_0 M \cup \partial_1 M$, where $\partial_i M, \partial_i L \subset S^p \times \{i\}, i = 0, 1$. Consider the manifold with faces

$$(S^p \times I)_0 = \mathrm{cl}[(S^p \times I) - T_M]$$

where T_M is the tubular neighbourhood of M (the inverse image of e^q under Φ). Let Φ_0 be the restriction of Φ to $(S^p \times I)_0$. Choose a regular value y of Φ_0 in S^r and set $Q = \Phi_0^{-1}(y)$. Notice that

$$\begin{aligned} \partial Q &= (Q \cap S^p \times \{0\}) \cup (Q \cap S^p \times \{1\}) \cup (Q \cap \partial \mathrm{cl}[T_M]) \\ &\simeq \partial_0 Q \cup \partial_1 Q \cup M \times N \end{aligned}$$

with $\partial\partial_0 Q \simeq \partial_0 M \times N$ and $\partial\partial_1 Q \simeq \partial_1 M \times N$. Let $K = Q \cup M \times CN$. The manifold with singularity K has its topological boundary equal to

$$\partial_0 K \sqcup \partial_1 K = \partial_0 Q \cup \partial_0 M \times CN \ \sqcup \ \partial_1 Q \cup \partial_1 M \times CN.$$

Clearly $\mathcal{L}(\phi) = \mathrm{lk}(\partial_0 K, \partial_0 L)$ and $\mathcal{L}(\psi) = \mathrm{lk}(\partial_1 K, \partial_1 L)$. Moreover, it is clear by construction that $K \cap L = \emptyset$. We are thus finished modulo the following easy lemma.

Lemma 33 *Let v and w be two (relative) simplicial chains in $S^p \times I$ which satisfy*

1. $\dim v + \dim w = p + 1$.
2. $\mathrm{carr}\ v \cap \mathrm{carr}\ w = \emptyset$,
3. $\partial v = v_0 - v_1$, $\partial w = w_0 - w_1$

where v_i, w_i are chains in $S^p \times \{i\}$, $i = 0, 1$ and carr *denotes the carrier of a chain. Then* $\mathrm{lk}(v_0, w_0; S^p \times \{0\}) = \mathrm{lk}(v_1, w_1; S^p \times \{1\})$.

Proof. To show the lemma, first embed $S^p \times [0,1]$ in $\mathbf{R}^{p+1}$ such that $S^p \times \{1\}$ corresponds to the unit sphere and $S^p \times \{0\}$ to the set $\{x \in \mathbf{R}^{p+1} : ||x|| = 1/2\}$. Since w_0 is homologically trivial in $S^p \times \{0\}$, one can choose a chain a

with $\partial a = w_0$. Now, "cone off" v_0 and w_0 in the set $\{x \in \mathbf{R}^{p+1} : ||x|| \leq 1/2\}$. More precisely, take the cone Cv_0, with cone point $0 \in \mathbf{R}^{p+1}$ and define $Cyl\ w_0$ to be the chain

$$Cyl\ w_0 = w_0 \times [1/4, 1/2] \ \cup \ a \times \{1/4\}.$$

Clearly,

$$\mathrm{Int}(Cv_0, Cyl\ w_0) = \mathrm{lk}(v_0, w_0),$$

where Int denotes the *intersection number* for chains (this trick is from [2]). Since carr $v \cap$ carr $w = \emptyset$, we have that

$$\begin{aligned} \mathrm{lk}(v_1, w_1) &= \mathrm{Int}(v \cup_{v_0} Cv_0, w \cup_{w_0} Cyl\ w_0) \\ &= \mathrm{Int}(Cv_0, Cyl\ w_0) \\ &= \mathrm{lk}(v_0, w_0). \blacksquare \end{aligned}$$

Lemma 34 *The map $\mathcal{L}$ is a group homomorphism.*

Proof. If ϕ, ψ are elements of $\pi_p(S^r \cup_g e^q)$, their sum $\phi + \psi$ can be represented by the composite

$$S^p \xrightarrow{P} S^p \vee S^p \xrightarrow{\phi \vee \psi} (S^r \cup_g e^q) \vee (S^r \cup_g e^q) \xrightarrow{F} S^r \cup_g e^q,$$

where P is the standard "pinching map" and F is the "folding map". Assuming that all the maps involved are transverse, $\phi + \psi$ is also transverse. Clearly, $L_{\phi+\psi} = L_\phi \ \sqcup \ L_\psi \subset E^p_+ \ \sqcup \ E^p_-$ where E^p_+, E^p_- are respectively the upper and lower hemisphere of S^p. It is also easy to see that $K_{\phi+\psi} = K_\phi \ \sqcup \ K_\psi \subset E^p_+ \ \sqcup \ E^p_-$ and thus, using the "bilinearity" of lk, together with the fact that $\mathrm{lk}(K_\phi, L_\psi) = \mathrm{lk}(K_\psi, L_\phi) = 0$, one gets $\mathcal{L}(\phi + \psi) = \mathcal{L}(\phi) + \mathcal{L}(\psi)$. $\blacksquare$

4.3 Homotopy theory

Recall that $[g]$ is a finite order element in $\pi_{q-1}(S^r)$. From now on, we will write g for both the map and the homotopy class. Let n be the order of g . Consider the homotopy exact sequence of the pair $(S^r \cup_g e^q, S^r)$:

$$\cdots \to \pi_q(S^r \cup_g e^q) \xrightarrow{j_*} \pi_q(S^r \cup_g e^q, S^r) \xrightarrow{\partial} \pi_{q-1}(S^r) \to \cdots .$$

Lemma 35 *There exists $\theta_q \in \pi_q(S^r \cup_g e^q)$ such that $j_*\theta_q = n\overline{g}$. Here, $\overline{g}$ is the characteristic map of the q-cell in $S^r \cup_g e^q$. The element $\overline{g}$ generates $\pi_q(S^r \cup_g e^q, S^r)$*

Proof. By the homotopy excision property (see [23] p. 487), the collapsing map c induces an isomorphism

$$c_* : \pi_q(S^r \cup_g e^q, S^r) \to \pi_q(S^q) \simeq \mathbf{Z}$$

and it is clear that $c_*(\overline{g}) = i_q$, the standard generator of $\pi_q(S^q)$. This shows the second statement. Since $\partial\overline{g} = g$, we have

$$\partial(n\overline{g}) = n\partial\overline{g} = ng = 0.$$

By exactness, there exists $\theta_q \in \pi_q(S^r \cup_g e^q)$ such that $j_*(\theta_q) = n\overline{g}$. ■

Lemma 36 *There is an short exact sequence*

$$0 \to \mathbf{Z} \xrightarrow{d} \pi_p(S^r \cup_g e^q, S^r) \xrightarrow{c_*} \pi_p(S^q) \to 0$$

where c is the standard collapsing map $S^r \cup_g e^q \to S^r \cup_g e^q/S^r$.

Proof. Consider the triad

$$(X; A, B) = (S^r \cup_g e^q \cup_i e^{r+1}; S^r \cup_g e^q, e^{r+1})$$

where i is is the natural inclusion $i : S^r \to S^r \cup_g e^q$. Clearly $C = A \cap B \simeq S^r$. It is easy to show that this triad is excisive. Again, it is clear that

$$\begin{aligned}(X, A) &= (S^r \cup_g e^q \cup_i e^{r+1}, S^r \cup_g e^q),\\ (X, B) &= (S^r \cup_g e^q \cup_i e^{r+1}, e^{r+1})\end{aligned}$$

are respectively r-connectened and $(q-1)$-connected. By the Blakers-Massey theorem (see [8]), $(X; A, B)$ is $(q+r-1) = p$-connected (recall $p+1 = q+r$), and the Whitehead product

$$[\ ,\ ] : \pi_q(A, C) \otimes \pi_{r+1}(B, C) \to \pi_{p+1}(X; A, B)$$

is an isomorphism. Now, the collapsing map c induces an isomorphism

$$c_* : \pi_q(A, C) = \pi_q(S^r \cup_g e^q, S^r) \xrightarrow{\sim} \pi_q(S^q) \simeq \mathbf{Z}$$

by the homotopy excision theorem. By the homotopy exact sequence, the boundary operator also induces an isomorphism

$$\partial : \pi_{r+1}(B, C) = \pi_{r+1}(e^{r+1}, S^r) \xrightarrow{\sim} \pi_r(S^r) \simeq \mathbf{Z}.$$

Take $\overline{g} \in \pi_q(S^r \cup_g e^q, S^r)$ as in the preceding lemma and let $\iota_r \in \pi_{r+1}(e^{r+1}, S^r)$ be such that $\partial \iota_r = i_r$. One has that $\pi_{p+1}(X : A, B) \simeq \mathbf{Z}$ and is generated by $[\overline{g}, \iota_r]$. But the triad homotopy groups fit in an exact sequence

$$\cdots \to \pi_{p+1}(X, B) \to \pi_{p+1}(X; A, B) \xrightarrow{d} \pi_p(A, C) \to \pi_p(X, B) \to \cdots$$

and one has a sequence of isomorphisms

$$\begin{aligned}\pi_{p+1}(X, B) &= \pi_{p+1}(S^r \cup_g e^q \cup_i e^{r+1}, e^{r+1})\\ &\simeq \pi_{p+1}(S^r \cup_g e^q \cup_i e^{r+1})\\ &\simeq \pi_{p+1}(S^q).\end{aligned}$$

Moreover, $\pi_p(X; A, B) = 0$, since $(X; A, B)$ is p-connected. Thus the triad exact sequence becomes

$$\cdots \to \pi_{p+1}(S^q) \to \mathbf{Z} \xrightarrow{d} \pi_p(S^r \cup_g e^q, S^r) \to \pi_p(S^q) \to 0.$$

Since $q - r > 1$, $\pi_{p+1}(S^q)$ is torsion and one gets the result. ∎

This proof shows that the copy of $\mathbf{Z}$ in the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{d} \pi_p(S^r \cup_g e^q, S^r) \xrightarrow{c_*} \pi_p(S^q) \to 0$$

is generated by the generalized Whitehead product $[\overline{g}, \imath_r]$, where $\imath_r$ is such that $\partial \imath_r = i_r$ and i_r is the standard generator of S^r. Let us compare with the ordinary Whitehead product

$$[\ ,\ ] : \pi_q(S^r \cup_g e^q) \otimes \pi_r(S^r \cup_g e^q) \to \pi_p(S^r \cup_g e^q).$$

Lemma 37 *The following identities hold :*

1. $d[\overline{g}, \imath_r] = \pm[\overline{g}, i_r]$,
2. $j_*[\theta_q, i_*(i_r)] = \pm n[\overline{g}, i_r]$.

Proof. These are standard formulae and proofs can be found in [8]. ∎

By the preceding lemma, $j_*[\theta_q, i_*(i_r)] = \pm n d[\overline{g}, \imath_r]$ and this is sufficient to show that $[\theta_q, i_*(i_r)]$ is a torsion-free element in $\pi_p(S^r \cup_g e^q)$. Recall that $\mathcal{C}$ is the cup product homomorphism

$$\mathcal{C} : \pi_p(S^r \cup_g e^q) \to \mathbf{Z}.$$

Lemma 38 $\mathcal{C}([\theta_q, i_*(i_r)]) = n$

Proof. The proof is an easy exercise in the naturality properties of the cup product. ∎

4.4 Final calculations

We can now prove a fundamental lemma which will allow us to complete the proof of the main result. Let us recall that n is the order of $[g]$ in $\pi_{q-1}(S^r)$.

Lemma 39 $\mathcal{L}([\theta_q, i_*(i_r)]) = n$

Proof. Let us denote $[\theta_q, i_*(i_r)]$ by ω. We will also write θ for θ_q and i for $i_*(i_r)$. Then, the map ω can be represented by the composite

$$S^p \xrightarrow{W} S^q \vee S^r \xrightarrow{\theta \vee i} (S^r \cup_g e^q) \vee (S^r \cup_g e^q) \xrightarrow{F} S^r \cup_g e^q$$

where W is the ordinary Whitehead product and F is the "folding" map. By definition, $\mathcal{L}(\omega) = \mathrm{lk}(K_\omega, L_\omega)$ and we now construct K_ω and L_ω.

It is easy to see that if x is in $S^r \cup_g e^q$ then

$$(F \circ (\theta \vee i))^{-1}(x) = \theta^{-1}(x) \times \{*\} \cup \{*\} \times i^{-1}(x).$$

It follows that we must first examine $\theta^{-1}(x)$ and $i^{-1}(x)$.

Suppose that $z \in e^q$. It is then clear that $i^{-1}(z) = \emptyset$ and

$$\theta^{-1}(z) = \{p_1, \ldots, p_n\}$$

where $\{p_1, \ldots, p_n\}$ is a set of n distinct points of S^q. Now, since one can take θ transverse, we get $\theta^{-1}(e^q) = D_1 \cup \cdots \cup D_n$, where each D_i is an open q-disk containing p_i, with $D_i \cap D_j = \emptyset$ if $i \neq j$.

Let θ_0 be the restriction of θ to the manifold with boundary $S_0^q = S^q - D_1 \cup \ldots \cup D_n$. Clearly, θ_0 maps into S^r. Let us set $A = \theta_0^{-1}(y)$ for $y \in S^r$ a regular value. It is obvious that $\partial A = N_1 \cup \cdots \cup N_n$, where each N_i is diffeomorphic to the framed manifold N associated to $g : S^{q-1} \to S^r$. By definition

$$K_\theta = A \cup_{\partial A} C(\partial A)$$

where $C(\partial A) = CN_1 \cup \cdots \cup CN_n$ and the cone points are the p_i's.

Let us write $S^p = \partial(D^q \times D^r)$ as the union

$$S^p = D^q \times S^{r-1} \cup_{S^{q-1} \times S^{r-1}} S^{q-1} \times D^r.$$

We are now ready to construct K_ω and L_ω. We begin with L_ω. If $y \in e^q$, then it easily follows from what we have seen above that

$$L_\omega = W^{-1}(\theta^{-1}(y) \times \{*\}) \cup W^{-1}(\{*\} \times i^{-1}(y))$$

and since $i^{-1}(y) = \emptyset$ and $\theta^{-1}(y) = \{p_1, \ldots, p_n\}$, we get

$$L_\omega = p_1 \times S^{r-1} \cup \ldots \cup p_n \times S^{r-1},$$

using the above decomposition of S^p.

On the other hand, it is not difficult to see that

$$K_\omega = K_\theta \times S^{r-1} \cup S^{q-1} \times \{y\}.$$

To make K_ω and L_ω disjoint, it is sufficient to use another regular value $y' \in e^q$, $y' \neq y$, rather than y in the construction of K_ω. Now, having triangulated both spaces (see 4.2), we easily get

$$\begin{aligned} \mathrm{lk}(K_\omega, L_\omega) &= \mathrm{lk}(S^{q-1} \times \{y\} \sqcup K_\theta \times S^{r-1}, L_\omega) \\ &= \mathrm{lk}(S^{q-1} \times \{y\}, L_\omega) + \mathrm{lk}(K_\theta \times S^{r-1}, L_\omega) \\ &= n + \mathrm{lk}(K_\theta \times S^{r-1}, L_\omega). \end{aligned}$$

Claim: $\mathrm{lk}(K_\theta \times S^{r-1}, L_\omega) = 0$

Recall that $L_\omega = p_1 \times S^{r-1} \cup \cdots \cup p_n \times S^{r-1}$. Now K_θ and $\{p_1, \ldots, p_n\}$ can be separated in S^q, since the codimension of K_θ in S^q is equal to r,

which is greater than one. This shows that $\mathrm{lk}(K_\theta \times S^{r-1}, L_\omega) = 0$ and that $\mathcal{L}(\omega) = n$. Lemma 9 is proved. ■

Looking at the homotopy exact sequence of the pair $(S^r \cup_g e^q, S^r)$, we get

$$\cdots \to \pi_p(S^r) \xrightarrow{i_*} \pi_p(S^r \cup_g e^q) \xrightarrow{j_*} \pi_p(S^r \cup_g e^q, S^r) \xrightarrow{\partial} \pi_{p-1}(S^r) \to \cdots .$$

Notice that the hypotheses on p, q and r imply that $\pi_p(S^r)$ and $\pi_{p-1}(S^r)$ are torsion groups. Now

$$0 \to \pi_p(S^r \cup_g e^q)/im\ i_* \xrightarrow{\bar{j}_*} \pi_p(S^r \cup_g e^q, S^r) \xrightarrow{\partial} im\ \partial \to 0$$

is exact. By lemma 36, the rank of $\pi_p(S^r \cup_g e^q, S^r)$ is 1 and thus the rank of $\pi_p(S^r \cup_g e^q)/im\ i_*$ is also 1. Since rk $im\ i_* = 0$, one has that the rank of $\pi_p(S^r \cup_g e^q)$ is 1. But, by lemmas 29 and 30,

$$n = \mathcal{C}([\theta_q, i_*(i_r)]) = \mathcal{L}([\theta_q, i_*(i_r)]).$$

This shows that $\mathcal{C} = \mathcal{L}$ and the theorem is proved. ■

4.5 Proof of Theorem D

We are now ready to prove Theorem D. We first recall some well-known constructions in Morse theory for which we refer to [13] and [21].

Let $f : M \to \mathbf{R}$ be a Morse-Smale function on a closed Riemannian manifold M. Consider the free $\mathbf{Z}$-module C_λ on the set of critical points of f of index λ. There are homomorphisms (boundary operators)

$$\ldots \xrightarrow{\partial} C_{\lambda+1} \xrightarrow{\partial} C_\lambda \xrightarrow{\partial} C_{\lambda-1} \xrightarrow{\partial} \ldots$$

satisfying

(i) $\partial \circ \partial = 0$,

(ii) $H(C_\lambda, \partial) \simeq H_\lambda(M, \mathbf{Z})$.

As usual, if one sets $C^\lambda = \mathrm{Hom}(C_\lambda, \mathbf{Z})$, $\lambda_a = \lambda$, $\lambda_b = \lambda + 1$ and

$$< \delta a, b > = < a, \partial b >,$$

we do get a sequence of modules and operators

$$\ldots \xrightarrow{\delta} C^{\lambda-1} \xrightarrow{\delta} C^\lambda \xrightarrow{\delta} C^{\lambda+1} \xrightarrow{\delta} \ldots$$

such that

(i) $\delta \circ \delta = 0$,

(ii) $H(C^\lambda, \delta) \simeq H^\lambda(M, \mathbf{Z})$.

We will denote by ω_a the homology (or cohomology, depending on the context) class corresponding to the critical point a. The boundary operator ∂ in the above complex can be described as follows. If a and b are critical points of f with $\lambda_a = \lambda_b + 1$, then let $n(a, b)$ be the degree of the primary attaching map (see section 1) of a with respect to b. This is by definition the degree of a map

$$S^u(a) \simeq S^{\lambda_a - 1} \to W^u(b)^+ \simeq S^{\lambda_b},$$

where $W^u(b)^+$ is the one-point compactification of the unstable manifold $W^u(b)$. Then, by definition

$$\partial(a) = \sum_{\lambda_b = \lambda_a - 1} n(a, b) b.$$

Recall from 3.3 that $S^u(a,b) \subset S^u(a)$ has been defined as

$$S^u(a,b) = \bigcup_{a \prec z \preceq b} M(a,z).$$

Recall also that $\dim[S^u(a)] = \lambda_a - 1$. It is clear that $S^u(a,b)$ and $S^u(a,c)$ determine simplicial cycles in $S^u(a)$, of dimension $\lambda_a - \lambda_b - 1$ and $\lambda_a - \lambda_c - 1$ respectively. Suppose that $\lambda_a = \lambda_b + \lambda_c$. Since $(\lambda_a - \lambda_b - 1) + (\lambda_a - \lambda_c - 1) = 2\lambda_a - (\lambda_b + \lambda_c) - 2 = \lambda_a - 2$, one can take the linking number of $S^u(a,b)$ and $S^u(a,c)$ in $S^u(a)$. If ω_a, ω_b and ω_c are non zero cocycles, their cup product $\omega_b \smile \omega_c$ can be written as the sum

$$\omega_b \smile \omega_c = \sum_{\lambda_x = \lambda_b + \lambda_c} \alpha_x \omega_x.$$

We would like to understand geometrically the coefficients α_x, in terms of the moduli spaces of flow lines. We make the following conjecture.

Conjecture $\alpha_a = \mathrm{lk}(S^u(a,b), S^u(a,c))$

We are unfortunately unable to prove this conjecture in its full generality. In order to get some partial results, one must assume that the following three somewhat stringent conditions hold.

- The critical point a represents a cycle in $H(C_{\lambda_a}, \partial)$ and M is simply-connected.

We will immediately deduce a useful consequence of this hypothesis. Let us denote by X_f^n the n-skeleton of X_f. Since the critical point a determines a cycle ω_a, i.e. $\partial(a) = 0$, then all the primary attaching maps of a are of degree zero (i.e. are null-homotopic). Now since we have also assumed that M is simply-connected, the Blakers-Massey theorem implies that the

attaching map

$$\phi_a : S^u(a) \to X_f^{\lambda_a - 1}$$

can be factorized through $X_f^{\lambda_a - 2}$. The next condition is the following.

• If z is a critical point of f ($z \neq a, b, c$) such that $z \in f^{-1}([c, a])$, then $W(z, b) = W(z, c) = \emptyset$.

This has the following consequence. Consider the primary attaching map of e_b with respect to c. It is a map

$$g : S^u(b) \to W^u(c)^+ \simeq S^{\lambda_c}$$

where $W^u(c)^+$ is the one point compactification of the unstable sphere of c. There is a homotopy commutative diagram

$$\begin{array}{ccc} X^{\lambda_a - 2} & \xrightarrow{\pi} & X^{\lambda_a - 2}/X^{\lambda_c - 1} \\ \phi_a \uparrow & & \downarrow p \\ S^u(a) & \xrightarrow{\phi} & e_b \cup_g S^{\lambda_c} \end{array}$$

where π is the obvious projection and p is a collapsing map. To see this, notice that in general, if the attaching map ϕ_w of the cell e_w hits another cell e_z, then there must be a flow line from w to z. Now if z is a critical point of f such that $z \in f^{-1}([c, a])$, it follows from our hypotheses that there cannot be any flow from z to b or c. From this remark, one easily deduces the existence of the map p, and consequently, of the map ϕ. The latter could be called a *secondary attaching map*.

• The primary attaching map g of b with respect to c has finite order.

This last condition is essential in order to apply the main result of this section.

Theorem 40 (Theorem D) *If the above hypotheses hold then*

$$\alpha_a = \mathrm{lk}(S^u(a,b), S^u(a,c)).$$

Proof. Notice that our second hypothesis imply that either $S^u(a,c) = M(a,c)$ (if $a \prec_s b$ and $a \prec_s c$) or $S^u(a,c) = M(a,c) \cup M(a,b)$ (if $a \prec_s b \prec_s c$). We consider only the case where $S^u(a,c) = M(a,c) \cup M(a,b)$.

Let c be the standard collapsing map

$$c : e_b \cup_g S^{\lambda_c} \to S^{\lambda_b}.$$

By the work of the preceding sections, it is clear that the framed manifold $M(a,b)$ can be represented by the map $c \circ \phi$ and that the framed stratified set $S^u(a,c) = M(a,b) \cup M(a,c)$ can be represented by the map ϕ. One can make them disjoint and take their linking number. Set

$$\mathrm{lk}(S^u(a,b), S^u(a,c)) = k.$$

By the main result of this section, we know that k determines the cohomology ring of $e_a \cup e_b \cup S^{\lambda_c}$. More precisely, if θ_a, θ_b and θ_c are the obvious generators of the cohomology ring of $e_a \cup e_b \cup S^{\lambda_c}$, then

$$\theta_b \smile \theta_c = k\theta_a.$$

The transfer of this result to the cohomology ring of the manifold M is a straightforward exercise in algebraic topology. We give the main steps only.

From the exact sequence of a pair, one gets a monomorphism

$$H^{\lambda_a}(X/X^{\lambda_c-1}) \to H^{\lambda_a}(X^{\lambda_a}/X^{\lambda_c-1}),$$

and the exact sequence of a triple determines an epimorphism

$$H^{\lambda_a}(X^{\lambda_a}/X^{\lambda_c-1}) \to H^{\lambda_a}(e_a \cup_{\phi_a} X^{\lambda_a-2}/X^{\lambda_c-1}),$$

which has the property that the class corresponding to e_a is not in the kernel. Finally, there is an isomorphism

$$H^{\lambda_a}(e_a \cup e_b \cup S^{\lambda_c}) \to H^{\lambda_a}(e_a \cup X^{\lambda_a-2}/X^{\lambda_c-1}).$$

This comes from the exact sequence of the pair

$$(e_a \cup X^{\lambda_a-2}/X^{\lambda_c-1}, A/X^{\lambda_c-1})$$

where A is the subcomplex collapsed by the natural extension $p' : e_a \cup X^{\lambda_a-2}/X^{\lambda_c-1} \to e_a \cup e_b \cup S^{\lambda_c}$ of the map

$$p : X^{\lambda_a-2}/X^{\lambda_c-1} \to e_b \cup_g S^{\lambda_c}.$$

One concludes easily, using the naturality properties of the cup product. ∎

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