Optimal Investment and Consumption under Infinite Horizon Epstein–Zin Stochastic Differential Utility

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ABSTRACT

This thesis deals with the infinite-horizon optimal investment and consumption problem for a utility maximising agent with Epstein–Zin stochastic differential utility (EZ-SDU) preferences. In particular, it achieves four main goals.

The first achievement is to provide a detailed introduction to the problem of continuous-time optimal investment and consumption under constant relative risk aversion (CRRA) utility preferences—a restriction of EZ-SDU preferences—in a Black–Scholes–Merton financial market. This is significantly simpler than the optimal investment-consumption problem under EZ-SDU, but even in this case, features of the problem take it outside the standard settings of classical primal stochastic control. This means that existing primal verification proofs rely on parameter restrictions, restrictions on the *admissible* strategies, or intricate approximation arguments. We show in Chapter II that these complications can be overcome using a simple and elegant argument involving a stochastic perturbation of the utility function.

The second achievement is to provide a detailed introduction to infinite-horizon EZ-SDU, including a discussion of which parameter combinations lead to a well-posed investment-consumption problem. To do this, we introduce a slightly different formulation of EZ-SDU to that which is traditionally used in the literature. This highlights the necessity and appropriateness of certain restrictions on the parameters governing EZ-SDU. We provide a thorough comparison of our formulation of EZ-SDU to the classical formulation.

Thirdly, and most importantly, we tackle the existence and uniqueness of infinite horizon EZ-SDU—a result currently lacking from the literature. To do this, it is necessary to make case distinctions depending on the parameters governing the agent's temporal and risk preferences. Specifically, if R is the agent's risk aversion and S is the agent's elasticity of complementarity (which are both defined in Chapter I), we must distinguish between the case when $\vartheta = \frac{1-R}{1-S} \in (0,1)$ and the case when $\vartheta > 1$. We argue in Chapter III that $\vartheta < 0$ does not have a well-formulated solution—and the case $\vartheta = 1$ reduces EZ-SDU to CRRA utility.

Existence poses more challenges than uniqueness when $\vartheta \in (0,1)$, but we show that by generalising the solution to the EZ-SDU equation so that it becomes either the minimal supersolution or the maximal subsolution, we may assign a unique (but not necessarily finite) utility to *all consumption streams*. In the case $\vartheta > 1$, it is uniqueness that proves difficult, and the only consumption stream with a unique utility process is the zero consumption stream. To overcome this, we introduce the economically motivated notion of a *proper solution*, and argue that it corresponds to the "true" utility process. We show that under certain conditions, the proper solution exists and is unique.

The final achievement is a verification theorem for the optimal strategy. When $\vartheta > 1$, we optimise over the right-continuous attainable consumption streams with a unique proper solution. When $\vartheta \in (0,1)$, we prove verification under the necessary and sufficient conditions for the investment-consumption problem to be well-posed.

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Firstly, I would like to thank my supervisors David Hobson and Martin Herdegen. I feel fortunate to have had the privilege of working with two such brilliant mathematicians throughout my PhD, each of whom has a keen eye for the intricacies and aesthetics of a mathematical proof. They have shared with me tools for thinking about problems in both mathematics and in life.

Amongst the many things that they have taught me are: that deriving a result yourself from first principles is the best way to really understand an abstract idea; that a change of perspective can be invaluable; and that both specifics and generalities are of utmost importance: specific examples guide intuition, whilst proving a result in maximal generality confirms the underlying mathematical structure. They have always expected the best from me, and I believe those expectations have helped me perform to the highest of my ability. The supervisions have been a highlight of my week, and David and Martin have encouraged and supported me with humour and insight.

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DECLARATION

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. Work based on collaborative research has all been conducted with my supervisors David Hobson and Martin Herdegen and is declared as follows:

- Chapter two is based on the paper [HHJ21a] titled "An elementary approach to the Merton problem" to appear in the journal *Mathematical Finance*.
- Chapters three and four are based on the paper [HHJ21b] titled "The Infinite Horizon Investment-Consumption Problem for Epstein-Zin Stochastic Differential Utility" which has been submitted to the journal *Finance and Stochastics*.
- Chapter five is based upon a working paper titled "Proper solutions for Epstein–Zin Stochastic Differential Utility".

Place: University of Warwick
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LIST OF ABBREVIATIONS

BSDE: backwards stochastic differential equation

CRRA : constant relative risk aversion

EIC : elasticity of intertemporal complementarity

EIS : elasticity of intertemporal substitution

EZ-SDU : Epstein–Zin stochastic differential utility

HARA : hyperbolic absolute risk aversion

RRA : relative risk aversion

SDU : stochastic differential utility

CHAPTER

I

INTRODUCTION

The problem of how to invest optimally is of obvious interest to anyone trading in financial markets. When formulated mathematically, this problem—which has long been studied in the fields of mathematical finance and financial economics—is known as the *Merton investment-consumption problem* after Robert C. Merton, the first person to find a solution in a continuous-time financial market [Mer69].

More generally, the Merton problem has come to be a name used to describe a variety of investment and consumption optimisation problems that broadly ask the following common question:

> *How should an agent investing in a financial market—whilst consuming from their wealth—behave, so as to maximise their subjective utility over a given period of time?*

The most common variations of this problem deal with either: optimising the expected utility of wealth (modelled by a controlled stochastic process $X = (X_t)_{t \geq 0}$) at

some fixed time T in the future (in which case, consumption is omitted); or, additionally allowing the agent to consume from their wealth and maximising the expected utility gained from the consumption (modelled by a stochastic process $C = (C_t)_{t\geq 0}$) over some (possibly infinite) time horizon. Mathematically, if $U : \mathbb{R}_+ \to \mathbb{R}$ is a concave increasing (utility) function, the agent seeks to maximise either the objective function $J(X) = \mathbb{E}\left[U(X_T)\right]$ in the former case, or $J(C) = \mathbb{E}\left[\int_0^T e^{-\delta t} U(C_t)\,\mathrm{d}t\right]$ in the latter. In this thesis, we will focus on variants of the latter problem. We will further assume that $T = \infty$, which results in the so-called *lifetime problem.* The aim of this first chapter is to provide a brief overview of the problem with which this thesis deals—*the lifetime optimal investment and consumption problem for an agent with Epstein–Zin stochastic differential utility.* In this chapter, we omit formal definitions and mathematical rigour, deferring them to the later chapters, and instead try to help the reader develop an intuitive understanding of the concepts that will be used throughout the thesis—and an explanation of why they are useful.

1 The Merton problem with CRRA utility

The standard (primal—as opposed to dual) approach to solving the Merton problem consists of two steps: first, deriving a "candidate" optimal strategy and an associated "value-process" by solving the Hamilton–Jacobi–Bellman equation or otherwise; and then, verifying mathematically that this is in fact optimal. In practice, it is often quite easy to derive the candidate optimal strategy, but the second step can be quite mathematically complicated.

This thesis begins with an introduction to the original problem studied by Merton with $U : \mathbb{R}_+ \to \mathbb{R}$ given by the constant relative risk aversion utility function, $U(c) = \frac{c^{1-R}}{1-R}$. We first derive the solution to the investment-consumption problem. Then we explain the standard approach to verification taken in the literature, paying particular attention to any additional conditions imposed to make the proof go through. Finally, we show that perturbing the value function by a multiple of

the optimal wealth process results in an elegant verification proof that requires no constraints on parameters in addition to the strictly necessary.

1.1 Risk aversion and elasticity of intertemporal complementarity

The principal reason that one may choose to apply a utility function to consumption and maximise the integral of discounted expected *utility* of consumption—as opposed to just maximising the integral of discounted expected consumption—is to model the agent as exhibiting risk aversion. We illustrate this with a simple example.

Suppose that our expected utility maximising agent is approached by a wealthy benefactor who offers to give them £1000. Since the benefactor also enjoys gambling, he offers them another choice: a gamble in which they either receive £2000 or £0 depending on the outcome of a coin toss. Both options will give the agent an expected value of £1000, but which will they choose?

The answer can be explained by *risk aversion*, the agent's preference for certainty. Explicitly, to determine their preferences, we compare the value of $\mathbb{E}[U(X)]$ for each gamble (random variable) X. The first option can be represented by the deterministic $y = 1000$, and the second can be represented by the transformed Bernoulli random variable

$$X = \begin{cases} 0, & \text{with probability } \frac{1}{2} \\ 2000, & \text{with probability } \frac{1}{2} \end{cases}, \qquad \text{with} \quad \mathbb{E}\left[X\right] = y. \tag{1.1}$$

Now, because the utility function is strictly concave and X is random, Jensen's inequality implies that $\mathbb{E}\left[U(X)\right] < U(\mathbb{E}\left[X\right]) = U(y)$. Hence, the agent strictly prefers the risk-free option. This is illustrated graphically in Figure 1.1.

The certainty equivalent and the risk premium

For a utility function U and a gamble X, we may define the *certainty equivalent* $c_U(X)$ to be the deterministic amount that gives the same expected utility as the gamble X under U; $c_U(X) = U^{-1}(\mathbb{E}[U(X)])$. Clearly, since U^{-1} is increasing, there is a direct equivalence between the preference ordering induced by the agent's expected

utilities and the associated certainty equivalent.

We may further define the *risk premium* $\rho_U(X)$ of X to be the difference between the expected value of X and its certainty equivalent; $\rho_U(X) = \mathbb{E}[X] - c_U(X)$. Using Jensen's inequality gives $c_U(X) \leq U^{-1}(U(\mathbb{E}[X])) = \mathbb{E}[X]$ and $\rho_U(X) \geq 0$.

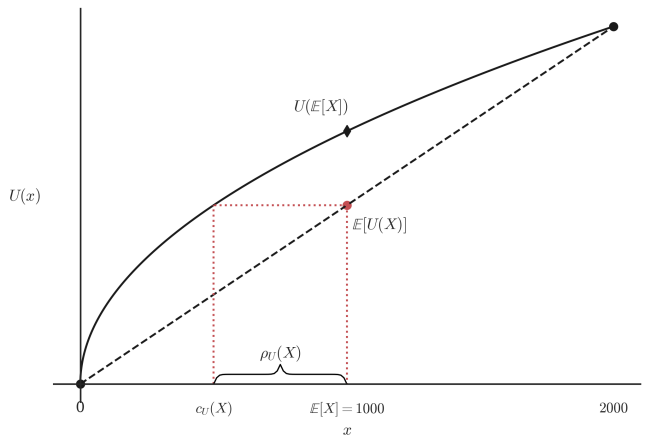


Figure 1.1: An illustration of the utility assigned to the simple gamble given in (1.1). Since $U : \mathbb{R}_+ \to \mathbb{R}$ is concave, $\mathbb{E}[U(X)] \leq U(\mathbb{E}[X])$. The certainty equivalent $c_U(X)$ and the risk premium $\rho_U(X)$ of the gamble are represented graphically.

Relative risk aversion and CRRA utility

There are a variety of ways of quantifying how risk averse an agent with expected utility preferences is, with one of the most commonly used being their relative risk aversion (RRA). Suppose that an agent has utility function $U : \mathbb{R}_+ \to \mathbb{R}$. Then, their relative risk aversion[1] (at a point x) is given by

$$R_U(x) = -\frac{xU''(x)}{U'(x)}.$$

Let $R \in \mathbb{R}_+ \backslash \{1\}$. As the name suggests, the constant relative risk aversion (CRRA) utility function[2] $U(x) = \frac{x^{1-R}}{1-R}$ has a constant RRA given by R at all points x,

[1] RRA contrasts with the agent's *absolute risk aversion* given by $A_U(x) = -\frac{U''(x)}{U'(x)}$.

[2] When $R = 1$, the utility function become $U(x) = \log(x)$.

$R_U(x) = R$ for $x > 0$.

Comparative risk aversion

It can be shown (see for example [FS16, Proposition 2.44]) that the risk premium is increasing in R. Consequently, the certainty equivalent $c_R(\cdot)$ associated to the CRRA utility function with risk aversion R is decreasing in R. This means that if we have two agents—agent A and agent B—with relative risk aversion parameters R_A and R_B respectively, and if $R_A \leq R_B$, then necessarily agent B will reject any gamble that agent A rejects.

To see this, fix an arbitrary random variable X and deterministic value y. Then, agent A rejects gamble X in favour of the deterministic y (denoted $X \prec_A y$) if and only if $c_{R_A}(X) < c_{R_A}(y) = y$. Since the certainty equivalent is decreasing in R, $c_{R_B}(X) \leq c_{R_A}(X) < y$ and agent B will also reject the gamble ($X \prec_B y$).

1.2 Assigning utility to stochastic consumption streams

Discounted expected CRRA utility

As we mentioned previously, we will first consider agents with discounted expected CRRA utility preferences over the infinite time-horizon. This means that the agent will aim to maximise

$$J(C) = \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(C_t)\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty e^{-\delta t} \frac{C_t^{1-R}}{1-R}\,\mathrm{d}t\right]. \tag{1.2}$$

To solve this problem, the value process $V = (V_t)_{t\geq 0}$ is introduced, where

$$V_t = \mathbb{E}\left[\int_t^\infty e^{-\delta t} U(C_t)\,\mathrm{d}t \,\middle|\, \mathcal{F}_t\right]. \tag{1.3}$$

This represents the utility that the agent gains from time t onwards from the consumption stream $C = (C_t)_{t\geq 0}$.

Provided the integral in (1.2) is well-defined, an application of Fubini's theorem

and then Jensen's inequality gives

$$J(C) = \int_0^\infty e^{-\delta t}\mathbb{E}\left[\frac{C_t^{1-R}}{1-R}\right] \mathrm{d}t \;\leq\; \int_0^\infty e^{-\delta t}\frac{(\mathbb{E}\left[C_t\right])^{1-R}}{1-R}\,\mathrm{d}t = J(\mathbb{E}\left[C\right]). \tag{1.4}$$

As the risk premium is increasing in R, so too the difference between the left and right hand sides of (1.4) is increasing in R (provided C is stochastic).

However, this is not the only effect that the parameter R has upon the agent's preferences. Suppose now that the agent is evaluating the discounted expected utility of a *deterministic* consumption stream $c = (c(t))_{t\geq 0}$, and suppose that $\delta > 0$. Then, we may introduce a measure $\mathbb{Q}_\delta$, defined on the nonnegative real line and given by $\mathbb{Q}_\delta(A) = \delta \int_A e^{-\delta t}\,\mathrm{d}t$. In this case, we may again use Jensen's inequality to find that

$$J(c) = \int_0^\infty e^{-\delta t}\frac{(c(t))^{1-R}}{1-R}\,\mathrm{d}t \;=\; \frac{1}{\delta}\mathbb{E}^{\mathbb{Q}_\delta}\left[U(c)\right] \leq \frac{1}{\delta}U(\mathbb{E}^{\mathbb{Q}_\delta}\left[c\right]) = J(\mathbb{E}^{\mathbb{Q}_\delta}\left[c\right]). \tag{1.5}$$

Furthermore, the larger the value of R, the larger the difference between the right and left hand sides of (1.5).

Since we are considering deterministic consumption streams here, we can no longer be capturing risk aversion. Is there a similar quantity to risk aversion that captures this observed preference for consumption streams that are constant throughout time as opposed to constant over states of the world?

The answer to this question is found in the economics literature on intertemporal choice. In particular, the quantity that captures the difference between the left hand and right hand sides of (1.5) is referred to as the *elasticity of intertemporal complementarity* (EIC), the inverse of which is the more commonly used *elasticity of intertemporal substitution* (EIS). A clear graphical explanation of how these quantities affect preferences can be found in [Bom05].

Thus, for discounted expected CRRA utility, the parameter R plays two roles, and is equal to both the agent's RRA and their EIC. As a consequence, the agent's risk preferences and intertemporal variance preferences are inflexibly interrelated.

This "entanglement" is undesirable from a modelling perspective, and there is no empirical evidence to suggest that the two concepts must be linked so strictly. It is for this reason that we will consider Epstein–Zin stochastic differential utility—a natural extension to CRRA utility that permits these two types of preferences to differ.

Epstein–Zin stochastic differential utility

Stochastic differential utility [DE92]—and its discrete-time analogue, recursive utility [EZ89, Wei89]—aim to resolve the entanglement issue outlined in the previous section. To do this, the discounted utility function $e^{-\delta t}U(c)$ is replaced by an aggregator $g(t, c, v)$ so that instead of evaluating (1.3) to calculate the agent's utility from time t onwards, one solves the fixed point equation

$$V_t = \mathbb{E}\left[\int_t^\infty g(s, C_s, V_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]. \tag{1.6}$$

The utility of a consumption stream $C = (C_t)_{t\geq 0}$ is then given by V_0. In particular, the aggregator may depend on V in a nonlinear way. It is precisely this nonlinearity that permits the disentanglement of risk preferences from intertemporal variance preferences.

Epstein–Zin stochastic differential utility (EZ-SDU) is a particular type of SDU, whose aggregator is given by $g_{EZ}(t, c, v) = be^{-\delta t}\frac{c^{1-S}}{1-S}((1-R)v)^{\frac{S-R}{1-S}}$. It will be shown in Chapter III that the parameter R controls the agent's risk aversion and the parameter S controls their aversion to temporal variance. It turns out that the parameter $\vartheta := \frac{1-R}{1-S}$ is critical. First, it is argued in Chapter III that $\vartheta > 0$ is necessary for the Epstein–Zin SDU equation to have a meaningful solution over the infinite horizon. Second, the mathematics of the problem is vastly different depending on whether $\vartheta \in (0, 1)$ or $\vartheta \in (1, \infty)$. When $\vartheta = 1$, $g_{EZ}(t, c, v) = be^{-\delta t}\frac{c^{1-R}}{1-R}$, which corresponds to a scaled version of discounted CRRA utility studied in the previous section. The different parameter combinations are illustrated in Figure 1.2.

The EZ-SDU process associated to a consumption stream C is therefore given by $V^C = (V_t^C)_{t\geq 0}$ which solves

$$V_t^C = \mathbb{E}\left[\int_t^\infty g_{EZ}(s, C_s, V_s^C)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \qquad \text{for all } t \geq 0, \tag{1.7}$$

provided a unique solution to (1.7) exists. One of the main goals of this thesis will be to investigate when this is the case and, when it is not the case, to explain what steps may be taken to assign an economically meaningful utility process to C.

The Merton problem for EZ-SDU is therefore to find the value of $\sup_{C\in\mathscr{C}} V_0^C$ (and ideally a maximising consumption stream $\hat{C}$), where $\mathscr{C}$ is a set of consumption streams that can be self-financed from a financial market and to which a unique meaningful utility process exists. Whilst $\mathscr{C}$ is left intentionally vague, it will become clearer what the this set should be as the thesis progresses.

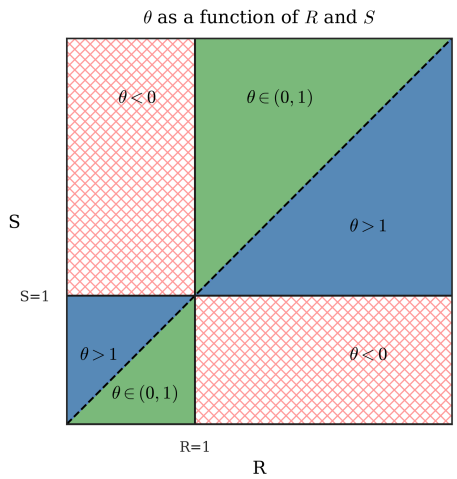


Figure 1.2: A plot of the different parameter combinations considered. In Chapter III it is argued that $\vartheta := \frac{1-R}{1-S} > 0$ is necessary for the Epstein–Zin SDU equation to have a meaningful solution over the infinite horizon. Chapter IV mainly deals with the case $\vartheta \in (0,1)$ and Chapter V focuses on the parameter combinations leading to $\vartheta > 1$. The CRRA case corresponding to $\vartheta = 1$ is shown by the dashed line.

2 Overview of the thesis and main contributions

II The Merton Investment-Consumption Problem under CRRA utility.

The second chapter will focus on the Merton problem [Mer69, Mer71] for an agent with preferences determined by discounted expected constant relative risk aversion utility. In this case, it is quite straightforward to write down the candidate value function. However, it is more difficult to give a complete primal verification argument: first; the application of Itô's formula to the candidate value function breaks down due to an infinite derivative at zero if the wealth process hits zero; second, the local martingale which arises from the application of Itô's formula may fail to be a martingale; third, and this holds even for constant proportional strategies, the value function may fail to satisfy an appropriate transversality condition.

For all these reasons, it is difficult to give a concise, rigorous verification proof via analysis of the value function, and many textbooks either sidestep the issues or restrict attention to a subclass of admissible strategies, and/or restrict attention to a subset of parameter combinations (especially $R < 1$, but even then there can be substantive points which are often overlooked). The need for such a verification argument has been obviated by the development of proofs using the dual method, which provides a powerful and intuitive alternative approach, see Biagini [Bia10] for a survey (and also [Kar89, KLSX91, KS98, Rog13]). Nonetheless, it would be satisfying to provide a short and complete proof based on the primal approach. Furthermore, the primal approach is well-suited to dealing with the Merton problem for an agent with EZ-SDU. In this case, dual approaches are more involved and do not cover all parameter combinations, so that the primal method is not redundant, and indeed may provide a more direct approach.

The goal of Chapter II is partly to introduce the Merton problem in its simplest form, and to give a simple, brief proof that the candidate value function is the value function via the primal approach. Moreover it aims to give a proof which is valid

for all parameter combinations for which the Merton problem is well-posed—a result lacking from the primal literature.

III Stochastic Differential Utility: An Introduction.

In this chapter, we introduce the optimal investment-consumption problem for agents whose preferences are given by EZ-SDU. We define SDU and EZ-SDU for infinite horizon consumption streams and provide a clear interpretation of all the parameters, with a focus on the feasible ranges for these parameters. We contrast EZ-SDU with CRRA utility described in Section 1.2 and show that it does manage to disentangle risk aversion from temporal variance aversion.

We take a different approach to that used in the rest of the infinite horizon SDU literature and use a different aggregator as well as considering the infinite horizon problem directly (we explain in Section III.5 in detail what the alternative approach is). The key property of the aggregator that we consider is that it takes only one sign. This means that issues regarding existence of the integral in the right hand side of (1.6) do not arise.

We compare and contrast our formulation with that taken in the rest of the literature in Section III.5 and explain why we believe that our approach has many advantages. In particular, we show that where there exist utility processes associated with both our aggregator and the classical aggregator, then the utility processes agree, but crucially any utility process associated to the traditional aggregator is also a utility process associated with our modified aggregator, whereas the converse is not true. Moreover, when specialised to the case of additive utility, our aggregator corresponds to the classical formulation of the Merton problem, whereas the traditional aggregator has a nonstandard specification in this context.

Our reformulation of the problem brings significant new insights concerning the set of feasible parameters for the problem with Epstein–Zin preferences. In particular we conclude that the RRA co-efficient (R) and the EIC co-efficient (S) must lie on

the same side of unity for the problem to make sense, at least for infinite horizon problems. (In the classical Merton problem for power law utility the RRA and EIC are necessarily equal.) This can be stated as $\vartheta := \frac{1-R}{1-S} > 0$. We further argue in Section III.5 that the putative solutions which have been found previously in the literature (in the case $\vartheta < 0$) correspond to a bubble-like behaviour, where the value associated with a consumption stream comes not from the utility of consumption in the short and medium term, but rather from a perceived and unrealisable value in the distant future.

IV Stochastic Differential Utility: Existence and Uniqueness Results.

This chapter gives a rigorous treatment of the Merton problem for Epstein–Zin stochastic differential utility, with the main focus being on parameter combinations such that $\vartheta \in (0, 1)$. The first main contribution is an existence result which shows that there exists a well-defined utility process for a large class of consumption streams. Since there are no existence results for infinite horizon EZ-SDU, this is a significant addition to the literature. Then, provided $\vartheta \in (0, 1)$, we show how to extend the existence result further to give a well-defined (though not necessarily finite) utility process for *any* consumption stream. We show that depending on the sign of the aggregator, it either corresponds to the minimal supersolution or the maximal subsolution. Again, key to our proofs is the fact that under our formulation the aggregator takes one sign.

We then turn to uniqueness. Under the same parameter restriction, we show that the EZ-SDU process associated to a consumption stream is unique.[3] The main idea is to use a comparison theorem for (sub- and super-) solutions to ensure a unique representation of the utility process.

Finally, we turn to the identification of the optimal investment and consumption strategy, and the optimal utility process. The candidate optimal strategy and candi-

[3]When this condition fails, and despite claims to the contrary in the literature, there are simple examples showing nonuniqueness. This is discussed in detail in Chapter V.

date optimal utility process are known (see [SS99, MMKS20, KSS17]), and the main techniques behind a verification argument are also well established in the literature. But, what distinguishes our results is the fact that we optimise over *all* admissible consumption streams, i.e., all consumption streams which can be financed from an initial wealth x. Typically in the extant literature optimisation only takes place over an abstract sub-family of consumption streams for which the utility process exists, is unique, and possesses certain regularity and integrability conditions. Further, since there are very few existence results in the literature, often the only strategies for which it can be verified that the utility process indeed satisfies the required regularity conditions are the constant proportional investment-consumption strategies. Since we optimise over *all* admissible consumption streams, this is a significant advance.

V Stochastic Differential Utility: Proper Solutions When $\vartheta > 1$.

Under the parameter combinations leading to $\vartheta = \frac{1-R}{1-S} \in (0,1)$, studied in Chapter IV, we showed that there exists a unique utility process for every consumption process (perhaps taking values in $[-\infty, \infty]$). However, when $\vartheta > 1$, uniqueness fails. Thus, to make progress we must first decide which utility process to associate to a given consumption process. Only then can we attempt to optimise over investment-consumption pairs.

We begin by studying constant proportional investment consumption strategies. For such strategies, an explicit (time-homogeneous) solution is known. However, we show that when $\vartheta > 1$, *there exists a family of solutions* to the EZ-SDU equation (1.7). These (time-inhomogeneous) solutions can be characterised by an absorption time—the first time that they hit zero and are absorbed. In particular, the utility process corresponding to absorption time T ignores all utility gained from consumption from time T onwards. To rule out these degenerate solutions, we introduce the notion of a *proper* solution and argue that it gives the 'true' utility process associated to a consumption stream.

We then proceed to show that: first, for a very wide class of consumption pro-

cesses C there exists a proper utility process V associated to C; second, for a wide class of consumption streams C the proper utility process V is unique; third, for a wide class of consumption streams, the proper solution corresponds to two other solution concepts with desirable mathematical properties. Finally, we solve the optimal investment-consumption problem in a constant parameter financial market, where we optimise over the right-continuous attainable consumption streams that have a unique proper solution associated to them.

3 Probabilistic setup and notation

We will work throughout on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions[4] of right-continuity and completeness and where $\mathcal{F}_0$ is $\mathbb{P}$-trivial. Let $\mathscr{P}$ be the set of progressively measurable processes, and let $\mathscr{P}_+$ and $\mathscr{P}_{++}$ be the restrictions of $\mathscr{P}$ to processes that take nonnegative and positive values, respectively. Moreover, denote by $\mathscr{S}$ the set of all semimartingales. We identify processes in $\mathscr{P}$ or $\mathscr{S}$ that agree up to indistinguishability.

We set $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_{++} = (0, \infty)$.

4 The Black–Scholes–Merton financial market

Throughout this thesis we will assume a Black–Scholes–Merton financial market consisting of a risk-free asset with interest rate $r \in \mathbb{R}$ whose price process $S^0 = (S^0_t)_{t\geq 0}$ is given by $S^0_t = \exp(rt)$ and a risky asset whose price process $S = (S_t)_{t\geq 0}$ follows a geometric Brownian motion with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$:

$$\frac{dS_t}{S_t} = \mu \, \mathrm{d}t + \sigma \, \mathrm{d}B_t, \quad S_0 = s > 0.$$

An agent operating with this investment opportunity set and initial wealth $x > 0$

[4] There are a couple of times that we relax this assumption to aid an example, but these points will be clearly highlighted.

chooses an *admissible investment-consumption strategy* $(\vartheta^0, \vartheta, C) = (\vartheta^0_t, \vartheta_t, C_t)_{t\geq 0}$, where $\vartheta^0_t \in \mathbb{R}$ denotes the number of *riskless* assets held at time t, $\vartheta_t \in \mathbb{R}$ denotes the number of shares of the risky asset held at time t, and $C_t \in \mathbb{R}_+$ represents the rate of consumption at time t. We require that $\vartheta^0, \vartheta, C$ are progressively measurable processes, ϑ^0 is integrable with respect to S^0, ϑ^1 is integrable with respect to S, C is integrable with respect to the identity process[5], the *wealth process* $X = (X_t)_{t\geq 0}$ defined by

$$X_t := \vartheta^0_t S^0_t + \vartheta_t S_t \tag{4.1}$$

is $\mathbb{P}$-a.s. nonnegative and the self-financing condition,

$$X_t = x + \int_0^t \vartheta^0_s \,\mathrm{d}S^0_s + \int_0^t \vartheta_s \,\mathrm{d}S_s - \int_0^t C_s \,\mathrm{d}s, \quad t \geq 0,$$

is satisfied. We then denote by $\Pi^0_t := \frac{\vartheta^0_t S^0_t}{X_t}$ and $\Pi_t := \frac{\vartheta_t S_t}{X_t}$ the fraction of wealth invested in the riskless and risky asset at time t, respectively.[6] Noting that $\Pi^0_t + \Pi_t = 1$ by (4.1), it follows that X satisfies the SDE

$$\begin{aligned} \mathrm{d}X_t &= \vartheta^0_t \,\mathrm{d}S^0_t + \vartheta_t \,\mathrm{d}S_t - C_t \,\mathrm{d}t \\ &= X_t \Pi^0_t r \,\mathrm{d}t + X_t \Pi_t (\mu \,\mathrm{d}t + \sigma \,\mathrm{d}B_t) - C_t \,\mathrm{d}t \\ &= X_t \Pi_t \sigma \,\mathrm{d}B_t + (X_t(r + \Pi_t(\mu - r)) - C_t) \,\mathrm{d}t, \end{aligned} \tag{4.2}$$

subject to $X_0 = x$. This means that we can describe an admissible investment-consumption strategy for initial wealth $x > 0$ more succinctly by a pair $(\Pi, C) = (\Pi_t, C_t)_{t\geq 0}$ of progressively measurable processes. In particular, we introduce the following two definitions.

Definition 4.1. Given $x > 0$ an *admissible investment-consumption strategy* is a pair $(\Pi, C) = (\Pi_t, C_t)_{t\geq 0}$ of progressively measurable processes, where Π is real-valued

[5] By saying that a process X is integrable with respect to the identity process we mean that $\int_0^t |X_s| ds < \infty$ $\mathbb{P}$-a.s. for each $t > 0$.

[6] Strictly speaking, Π^0_t and Π^1_t are not defined for $X_t = 0$, but this does not matter. We can for example set $\Pi^0_t := 0$ and $\Pi_t := 1$ for $X_t = 0$.

and C is nonnegative, such that the SDE (4.2) has a unique strong solution $X^{x,\Pi,C}$ that is $\mathbb{P}$-a.s. nonnegative. We denote the set of admissible investment-consumption strategies for $x > 0$ by $\mathscr{A}(x; r, \mu, \sigma)$.

The objective criteria by which the strategy is evaluated will depend only upon the consumption and not upon the investment portfolio in the financial assets. This motivates the following definition:

Definition 4.2. A consumption stream $C \in \mathscr{P}_+$ is called *attainable* for initial wealth $x > 0$ if there exists a progressively measurable process $\Pi = (\Pi_t)_{t\geq 0}$ such that (Π, C) is an admissible investment-consumption strategy. Denote the set of attainable consumption streams for $x > 0$ by $\mathscr{C}(x; r, \mu, \sigma)$.

When it is clear which financial market we are considering, we simplify the notation and write $\mathscr{A}(x) = \mathscr{A}(x; r, \mu, \sigma)$ and $\mathscr{C}(x) = \mathscr{C}(x; r, \mu, \sigma)$.

Remark 4.3. If $(\Pi, C) \in \mathscr{A}(x)$ is such that the corresponding wealth process $X^{x,\Pi,C}$ has $\mathbb{P}$-a.s. *positive* paths, we can define the process $\Gamma = (\Gamma_t)_{t\geq 0}$ by $\Gamma_t := C_t / X_t^{x,\Pi,C}$. Here, Γ_t denotes the fraction of wealth consumed per unit time at time t. Conversely, for any pair $(\Pi, \Gamma) = (\Pi_t, \Gamma_t)_{t\geq 0}$ of progressively measurable processes such that Π is real-valued and square-integrable with respect to the identity process and Γ is nonnegative and integrable with respect to the identity process, there exists a unique positive process $Z^{x,\Pi,\Gamma} = (Z_t^{x,\Pi,\Gamma})_{t\geq 0}$ such that $Z^{x,\Pi,\Gamma} = X^{x,\Pi,C}$ for $C := \Gamma Z^{x,\Pi,\Gamma}$. We conclude that we can parameterise consumption via the fraction of wealth consumed per unit time rather than an absolute value.

CHAPTER

II

THE MERTON INVESTMENT-CONSUMPTION PROBLEM UNDER CRRA UTILITY

In the Merton investment-consumption problem (Merton [Mer69, Mer71]) a risk-averse agent seeks to maximise the expected integrated discounted utility of consumption over the infinite horizon in a Black–Scholes–Merton financial market. In this chapter, we will give a rigorous account of the problem, paying particular attention to any assumptions made for the argument to go through.

The structure of this chapter is as follows. In the first section we introduce the problem, and in Section 2 we derive the candidate value function. In Section 3 we give a proof of the main result under a set of clearly-stated assumptions which are designed precisely to make the proof work. Often, this is the approach taken in the

stochastic control literature (see, for example, Davis and Norman [DN90], Fleming and Soner [FS06, Example 5.2] and Pham [Pha09]), where the authors artificially impose such restrictions on the set of admissible strategies or on the parameter values to ensure that these assumptions are satisfied by default.

In Section 4, we summarise two of the most general primal approaches in the literature—Karatzas et al [KLSS86] and Davis and Norman [DN90]—both of which use a perturbation argument to weaken some of the assumptions. In particular, [DN90] provided inspiration for our approach taken in Section 5.

In Section 5 we give our proof of the Verification Theorem, which works for all parameter combinations for which the problem is well-posed and allows for all admissible strategies. At its heart, our idea is a modification of the approach in [DN90]. We perturb the utility function, which leads to a perturbed value function. However, rather than perturbing by the addition of a deterministic constant, we perturb by adding a multiple of the optimal wealth process. The great benefit is that the optimal consumption and the optimal investment are unchanged under the perturbation, which means that mathematical calculations remain strikingly simple. Moreover, these arguments are valid whenever the Merton problem is well-posed.

Finally, the Appendix to Chapter II contains: an example which illustrates how one of the clearly-stated assumptions may easily fail; a section detailing a convenient numéraire change that allows us to weaken the parameter restrictions imposed by [KLSS86] and [DN90] and illustrate the role played by the discount factor more clearly; a discussion the case of logarithmic utility; and, for completeness, a brief discussion of duality methods for the Merton problem.

Our proof of the Verification Theorem is an improvement on the existing primal results in at least three important ways. First, it places no restrictions on the class of admissible strategies: for example, unlike much of the stochastic control literature, it does not require the fraction of wealth invested in the risky asset to be bounded. Second, the proof covers all parameter combinations for which the Merton problem is well-posed (and does not assume that interest rates and discounting are positive—

as we shall argue these quantities depend on the choice of accounting units, and therefore are not absolutes in themselves). Third, our proof is simple, elegant and concise and not counting the derivation of the candidate solution and candidate value function can be written up in just over one page (Theorem 5.1 and Corollary 5.4).

1 The Merton objective function

The objective of the agent is to maximise the expected discounted utility of consumption over an infinite time horizon for a given initial wealth $x > 0$. To any attainable consumption stream $C \in \mathscr{C}(x)$, they associate a value $J(C) \in [-\infty, \infty]$, where

$$J(C) := \mathbb{E}\left[\int_0^\infty e^{-\delta t} U\left(C_t\right) \mathrm{d}t\right].$$

Remark 1.1. Here, $\delta \in \mathbb{R}$ can loosely be interpreted as a discount or impatience parameter. Note that, unlike much of the literature, we do not assume that $\delta > 0$. One reason for this is that a deterministic change of accounting units leads to the same Merton investment-consumption optimisation problem, but under a different set of parameters, including a modified value of δ. We will see in Section II.B that the problem with parameters $R, r, \mu, \sigma, \delta$ is equivalent to the problem with parameters $R, r+\gamma, \mu+\gamma, \sigma, \delta-(R-1)\gamma$ where $\gamma \in \mathbb{R}$ is an arbitrary constant. In particular, it is possible to make a perfectly a reasonable change of units, and change the sign of δ. A second reason is that when $R > 1$ the interpretation of the restriction $\delta > 0$ is less clear. Note that $e^{-\delta t}\frac{C_t^{1-R}}{1-R} = \frac{(e^{-\delta t/(1-R)}C_t)^{1-R}}{1-R}$. Then, when $R > 1$, the discounted utility of consumption is equivalent to the utility of *upcounted* consumption. Equally plausible might be to take the utility of discounted consumption which would correspond to upcounting the utility of consumption. In the absence of a unequivocal rationale for a sign restriction on δ, we allow δ to be unrestricted.

We assume that the agent has constant relative risk aversion (CRRA) or equiva-

lently that $U : [0, \infty) \to [-\infty, \infty)$ takes the form $U(c) = \frac{c^{1-R}}{1-R}$, where $R \in (0, \infty) \backslash \{1\}$ is the coefficient of relative risk aversion;[1] $R = 1$ is the case of logarithmic utility $U(c) = \log(c)$ and is discussed in Section II.C. Note that since $R \neq 1$, the sign of $U(c)$ is uniquely determined. Thus, if $\int_0^\infty e^{-\delta t} U(C_t)\, \mathrm{d}t$ is not integrable, we can define $J(C) := +\infty$ when $R < 1$ and $J(C) := -\infty$ when $R > 1$.

In summary, the problem facing the agent is to determine

$$V(x) := \sup_{C \in \mathscr{C}(x)} J(C) = \sup_{C \in \mathscr{C}(x)} \mathbb{E}\left[\int_0^\infty e^{-\delta t} \frac{C_t^{1-R}}{1-R}\, \mathrm{d}t\right].$$

2 The candidate value function

From the homogeneous structure of the problem we expect (see for example, Rogers [Rog13, Proposition 1.2]) that $V(\kappa x) = \kappa^{1-R} V(x)$ and that if $(\hat{\Pi}, \hat{C})$ is an optimal strategy in $\mathscr{A}(x)$ then $(\hat{\Pi}_\kappa = \hat{\Pi}, \hat{C}_\kappa = \kappa \hat{C})$ is optimal in $\mathscr{A}(\kappa x)$ for $\kappa > 0$. For this reason, we may guess that it is optimal to invest a constant fraction of wealth in the risky asset, and to consume a constant fraction of wealth. (Of course, this will be verified later.) So, consider an investment-consumption strategy that at each time t, invests a constant proportion of wealth $\Pi_t = \pi$ into the risky asset and consumes a constant fraction $\xi > 0$ of wealth per unit time, i.e., $C_t = \xi X_t$.[2]

Then, the agent's wealth process $X = X^{x,\pi,\xi X}$ is given by

$$X_t = x \exp\left(\pi \sigma B_t + \left(r + \pi(\mu - r) - \xi - \frac{\pi^2 \sigma^2}{2}\right) t\right). \tag{2.1}$$

Denoting the market price of risk or *Sharpe ratio* by $\lambda := \frac{\mu - r}{\sigma}$, we obtain

$$\frac{C_t^{1-R}}{1-R} = \frac{x^{1-R} \xi^{1-R}}{1-R} \exp\left(\pi \sigma (1-R) B_t + (1-R)\left(r + \lambda \sigma \pi - \xi - \frac{\pi^2 \sigma^2}{2}\right) t\right).$$

[1] We follow the convention that $0^{1-R} := \infty$ for $R > 1$.
[2] This is well defined by Remark I.4.3.

Multiplying this by $e^{-\delta t}$ and taking expectations gives

$$\mathbb{E}\left[e^{-\delta t}\frac{C_t^{1-R}}{1-R}\right] = x^{1-R}\frac{\xi^{1-R}}{1-R}e^{-H(\pi,\xi)t}, \tag{2.2}$$

where

$$H(\pi,\xi) = H(\pi,\xi;R,\delta,\lambda,r,\sigma) := \delta - (1-R)\left(r + \lambda\sigma\pi - \frac{\pi^2\sigma^2}{2}R - \xi\right).$$

Provided that $H(\pi,\xi) > 0$, we find that

$$J(\xi X) = \mathbb{E}\left[\int_0^\infty e^{-\delta t}\frac{\xi^{1-R}X_t^{1-R}}{1-R}\,\mathrm{d}t\right] = \frac{x^{1-R}}{1-R}\frac{\xi^{1-R}}{H(\pi,\xi)}. \tag{2.3}$$

We want to maximise this expression considered as a function of π and ξ, where the maximisation is restricted to pairs (ξ,π) for which $H(\pi,\xi) > 0$.

In order to maximise this over π, we want to minimise $(1-R)H(\pi,\xi)$, which is equivalent to maximising $\lambda\sigma\pi - \frac{\pi^2\sigma^2}{2}R$. This is achieved at $\hat{\pi} = \frac{\lambda}{\sigma R}$. In this case $\lambda\sigma\hat{\pi} - \frac{\hat{\pi}^2\sigma^2}{2}R = \frac{\lambda^2}{2R}$ and the problem then becomes to maximise

$$\frac{x^{1-R}}{1-R}\frac{\xi^{1-R}}{H(\hat{\pi},\xi)} = \frac{x^{1-R}}{1-R}\frac{\xi^{1-R}}{\left(\delta - (1-R)(r + \frac{\lambda^2}{2R} - \xi)\right)} =: \frac{x^{1-R}}{1-R}f(\xi)$$

over ξ. This is equivalent to maximising $\log(f(\xi))$, since $\log(x)$ is increasing in x. Taking derivatives of $\log(f(\xi))$ shows that the maximum is attained at $\hat{\xi} = \eta$, where

$$\eta := \frac{1}{R}\left[\delta - (1-R)\left(r + \frac{\lambda^2}{2R}\right)\right], \tag{2.4}$$

provided that $\eta > 0$. (If $\eta \leq 0$, no maximum exists.)

Therefore, when $\eta > 0$, the agent's optimal behaviour (at least over constant proportional strategies) and corresponding value function are given by

$$\hat{\pi} = \frac{\mu - r}{\sigma^2 R}, \qquad \hat{\xi} = \eta, \qquad \hat{V}(x) := J(\hat{\xi}X) = \frac{\eta^{-R}x^{1-R}}{1-R}. \tag{2.5}$$

When $\eta \leq 0$, the problem is ill-posed. Indeed, if $R < 1$, then $H(\hat{\pi}, \xi) \downarrow 0$ as $\xi \downarrow -\frac{\eta R}{(1-R)}$ and hence $J(\xi X) \uparrow \infty$ by (2.3). If $R > 1$, then $H(\pi, \xi) \leq H(\hat{\pi}, \xi) = R\eta + (1-R)\xi \leq R\eta \leq 0$ for every $\pi \in \mathbb{R}$ and $\xi \geq 0$. Hence, at least for constant proportional strategies $J(\xi X) = -\infty$. We will see in Corollary 5.5 that $J(C) = -\infty$ for *every* admissible consumption stream $C \in \mathscr{C}(x)$.

3 The verification argument under fiat conditions

In this section, we prove that our candidate optimal strategy $(\hat{\pi}, \hat{\xi} X)$ from (2.5) is optimal in a subset of the class of all admissible strategies. Since the conditions defining that class are chosen precisely in such a way that the proof works, we call them *fiat* conditions.

Definition 3.1. Fix $x > 0$. An investment-consumption strategy $(\Pi, C) \in \mathscr{A}(x)$ is called *fiat admissible* if the following three conditions are satisfied:

(P) The wealth process $X^{x,\Pi,C}$ is $\mathbb{P}$-a.s. positive.

(M) The local martingale $\int_0^\cdot e^{-\delta t} \sigma \Pi_t (X_t^{x,\Pi,C})^{1-R} \, \mathrm{d}B_t$ is a supermartingale.

(T) The transversality condition $\liminf_{t\to\infty} \mathbb{E}[e^{-\delta t} \frac{(X_t^{x,\Pi,C})^{1-R}}{1-R}] \geq 0$ is satisfied.

We denote the set of all fiat admissible investment-consumption strategies for $x > 0$ by $\mathscr{A}^*(x)$. A consumption stream $C \in \mathscr{C}(x)$ is called *fiat attainable* for $x > 0$ if there is an investment process Π such that $(\Pi, C) \in \mathscr{A}^*(x)$. We denote the set of *fiat attainable* consumption streams by $\mathscr{C}^*(x)$.

Remark 3.2. As far as we are aware, the above notion of fiat admissible strategies has not been explicitly used in the literature before. However, the conditions (P), (M) and (T) or stronger versions thereof have been used explicitly or implicitly throughout the stochastic control literature on the Merton problem:

1. Condition (P) is (implicitly) assumed throughout most of the stochastic control literature dealing with the Merton problem; a notable exception is [KLSS86].

However, for $R > 1$, (P) can be assumed without loss of generality because any admissible strategy $(\Pi, C) \in \mathscr{A}(x)$ violating (P) has $J(C) = -\infty$.

2. Condition (M) is implied by the stronger condition

   (M1) The local martingale $\int_0^\cdot \sigma \Pi_t (X_t^{x,\Pi,C})^{1-R} e^{-\delta t} \, \mathrm{d}B_t$ is a martingale.

   It is not difficult to check that for $R < 1$, (M1) is implied by the even stronger condition

   (B) Π is uniformly bounded.

   A common approach in the stochastic control literature is to assume (B), see e.g. Davis and Norman [DN90, Equation (2.1)(B)], Fleming and Soner [FS06, Equation IV.5.2], or Pham [Pha09, Equation (3.2)], and then prove (M1) for $R < 1$.[3]

3. Condition (T) is implied by the stronger standard transversality condition[4]

   (T1) $\lim_{t\to\infty} \mathbb{E}[e^{-\delta t} \frac{(X_t^{x,\Pi,C})^{1-R}}{1-R}] = 0$.

   When $R < 1$, Davis and Norman [DN90, page 682] prove that (T1) is satisfied for any admissible strategy satisfying (B). Pham [Pha09, Equation (3.39)] and Fleming and Soner [FS06, Equation IV.5.11] require (T1), and prove that the candidate optimal strategy has this property.

It is clear that $\mathscr{C}^*(x) \subset \mathscr{C}(x)$. The following result shows that the candidate optimal strategy $(\hat{\pi}, \hat{\xi} X)$ from (2.5) is optimal in the class of fiat admissible strategies.

Theorem 3.3. *Suppose* $\eta := \frac{1}{R}[\delta - (1-R)(r + \frac{\lambda^2}{2R})] > 0$. *Let the function* $\hat{V} : (0,\infty) \to \mathbb{R}$ *be given by* $\hat{V}(x) = \frac{x^{1-R}}{1-R}\eta^{-R}$. *Then, for* $x > 0$,

$$V^*(x) := \sup_{C \in \mathscr{C}^*(x)} J(C) = J(\hat{C}) = \hat{V}(x),$$

[3] Davis and Norman [DN90, Proof of Theorem 2.1] argue that (B) implies (M1) also in the case $R > 1$ but this is not the case. See Example A.1.

[4] Note, however, that if $R > 1$, (T) and (T1) are equivalent.

where the corresponding optimal investment-consumption strategy is $(\Pi, C) = (\hat{\Pi}, \hat{C})$*;*

$$\hat{\Pi} = \frac{\lambda}{\sigma R}, \quad \hat{C} = \eta X^{x,\hat{\Pi},\hat{C}}. \tag{3.1}$$

Proof. First, we show that $V^*(x) \geq \hat{V}(x) = J(\hat{C})$. By the arguments in Section 2, it only remains to show that $\hat{C}$ is *fiat* attainable. It follows from the construction of $\hat{C}$, that the wealth process $X^{x,\hat{\Pi},\hat{C}}$ is $\mathbb{P}$-a.s. positive. Next, a similar calculation as in (2.2) shows that for each $T > 0$,

$$\mathbb{E}\left[\int_0^T e^{-2\delta t}\sigma^2\hat{\pi}^2\left(X_t^{x,\hat{\Pi},\hat{C}}\right)^{2-2R}\mathrm{d}t\right] = \sigma^2\hat{\pi}^2\int_0^T \exp\left(\left(\frac{\lambda^2(1-R)^2}{R^2} - 2\eta\right)t\right)\mathrm{d}t < \infty.$$

This implies that the local martingale $\int_0^\cdot \exp(-\delta t)\sigma\hat{\Pi}_t(X_t^{x,\hat{\Pi},C})^{1-R}\,\mathrm{d}B_t$ is a (square-integrable) martingale and hence a supermartingale. Finally, (2.2) together with the fact that $H(\hat{\pi}, \eta) = \eta > 0$, implies that $(\hat{\Pi}, \hat{C})$ satisfies the transversality condition (T1).

Next, we show that $V^*(x) \leq \hat{V}(x)$. Let $(\Pi, C) \in \mathscr{A}^*(x)$ be arbitrary. If $R > 1$, we may in addition assume without loss of generality that C^{1-R} is integrable with respect to the identity process; for otherwise $J(C) = -\infty$. It suffices to argue that $J(C) \leq \hat{V}(x)$.

Set $X := X^{x,\Pi,C}$ for brevity and define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \int_0^t e^{-\delta s}U(C_s)\,\mathrm{d}s + e^{-\delta t}\hat{V}(X_t).$$

We want to apply Itô's formula to M. This is indeed possible as $\hat{V}$ is in $C^2(0,\infty)$ and X is positive by fiat admissibility of (Π, C). Note that $\hat{V}_x(X_t)$ is positive and $\hat{V}_{xx}(X_t)$ is negative. Then, noting that the argument of $\hat{V}$ and its derivatives is X_t throughout, we obtain

$$\begin{aligned} dM_t &= \sigma\Pi_t X_t e^{-\delta t}\hat{V}_x\,\mathrm{d}B_t \\ &\quad + e^{-\delta t}\left[\frac{C_t^{1-R}}{1-R} - \delta\hat{V} + (X_t(r+\sigma\lambda\Pi_t) - C_t)\hat{V}_x + \frac{\sigma^2}{2}\Pi_t^2 X_t^2\hat{V}_{xx}\right]\mathrm{d}t \end{aligned}$$

$$= \quad \mathrm{d}N_t + e^{-\delta t} L(\Pi_t, C_t; X_t, \hat{V})\,\mathrm{d}t.$$

where $N_t = \int_0^t \sigma \Pi_s X_s e^{-\delta s} \hat{V}_x \,\mathrm{d}B_s = \int_0^t \eta^{-R} \sigma \Pi_s X_s^{1-R} e^{-\delta s}\,\mathrm{d}B_s$ is a local martingale and

$$L(\pi, c; x, v = v(x)) = \frac{c^{1-R}}{1-R} - \delta v + (x(r + \sigma\lambda\pi) - c)v_x + \frac{\sigma^2}{2}\pi^2 x^2 v_{xx}. \tag{3.2}$$

Maximising (3.2) over $\pi \in \mathbb{R}$ and $c \geq 0$, shows that the optimisers are attained at $\hat{\pi} = \frac{\lambda}{\sigma}\frac{-v_x}{x v_{xx}}$ and $\hat{c} = v_x^{-1/R}$. Plugging in $\hat{V}$ shows that $L(\hat{c}, \hat{\pi}; x, \hat{V}) = 0$, which implies that $\hat{V}$ solves the Hamilton-Jacobi-Bellman equation

$$\sup_{\pi \in \mathbb{R}, c \geq 0} L(\pi, c; x, v) = 0. \tag{3.3}$$

It follows that

$$M_t \leq \hat{V}(x) + N_t, \quad t \geq 0. \tag{3.4}$$

Taking expectations and using fiat admissibility of (Π, C) to ensure that N is a supermartingale, we find for each $t \geq 0$,

$$\mathbb{E}\left[M_t\right] \leq \mathbb{E}\left[\hat{V}(x) + N_t\right] \leq \hat{V}(x).$$

Taking the limit as t goes to infinity, and using the monotone convergence theorem as well as the transversality condition, we obtain

$$\begin{aligned} J(C) &= \lim_{t\to\infty} \mathbb{E}\left[\int_0^t e^{-\delta s}\frac{C_s^{1-R}}{1-R} ds\right] = \lim_{t\to\infty} \mathbb{E}\left[M_t - e^{-\delta t}\hat{V}(X_t^{x,\Pi,C})\right] \\ &\leq \limsup_{t\to\infty} \mathbb{E}\left[M_t\right] - \liminf_{t\to\infty} \mathbb{E}\left[e^{-\delta t}\hat{V}(X_t^{x,\Pi,C})\right] \leq \limsup_{t\to\infty} \mathbb{E}\left[M_t\right] \leq \hat{V}(x). \end{aligned} \tag{3.5}$$

This establishes the claim. □

Remark 3.4. A close inspection of the proof of Theorem 3.3 shows that for the optimal strategy $(\hat{\Pi}, \hat{C})$, the process $\hat{M} = (\hat{M}_t)_{t\geq 0}$ given by $\hat{M}_t := \int_0^t e^{-\delta s} U(\hat{C}_s)\,\mathrm{d}s +$

$e^{-\delta t}\hat{V}(X^{x,\hat{\Pi},\hat{C}})$ is a uniformly integrable martingale. Indeed, in this case $\hat{N}$ is a martingale and $\hat{M} = \hat{V}(x) + \hat{N}$. Hence, $\hat{M}$ is a martingale. It is uniformly integrable because, by the transversality condition (T1) and monotone convergence, equation (3.5) implies that $\hat{M}_t$ converges in L^1 to $\hat{M}_\infty := \int_0^\infty e^{-\delta s} U(\hat{C}_s)\,\mathrm{d}s$.

For $R < 1$, the above fiat verification theorem can be easily turned into to a general verification theorem that does not restrict to fiat attainable consumptions.

Corollary 3.5. *Suppose $R < 1$ and $\eta > 0$. Then, $V(x) = \hat{V}(x)$.*

Proof. It is sufficient to show that (P), (M) and (T) are satisfied for general strategies, or to find a way of bypassing the relevant part of the argument. First, (T) is automatically satisfied by the fact that $X^{1-R}/(1-R)$ is nonnegative. Next, M is nonnegative and hence N is bounded below by $-\hat{V}(x)$ by (3.4). Therefore, N is always a supermartingale and (M) is automatically satisfied.

Finally, to avoid imposing (P), one has to refine the argument in Theorem 3.3 by a stopping argument. To wit, fix an admissible strategy $(\Pi, C) \in \mathscr{A}(x)$. Then, for $n \in N$, set $\tau_n := \inf\{t \geq 0 : X^{x,\Pi,C} \leq \frac{1}{n}\}$ and let $\tau_\infty := \lim_{n\to\infty} \tau_n$. Then it is not difficult to check that $X_t = X_t^{x,\Pi,C} \geq 1/n > 0$ if $t \leq \tau_n$ and $X_t = 0 = C_t$ if $t \geq \tau_\infty$.[5] Moreover, for each n, we get

$$\mathbb{E}\left[M_t^{\tau_n}\right] \leq \mathbb{E}\left[\hat{V}(x) + N_t^{\tau_n}\right] \leq \hat{V}(x).$$

Now first taking the limit $t \to \infty$, we obtain

$$\mathbb{E}\left[\int_0^{\tau_n} e^{-\delta s} \frac{C_s^{1-R}}{1-R} ds\right] \leq \limsup_{t\to\infty} \mathbb{E}\left[M_t^{\tau_n}\right] \leq \hat{V}(x).$$

Next, taking the limit $n \to \infty$, the result follows from the monotone convergence theorem and the fact that $\int_{\tau_\infty}^\infty C_s\,\mathrm{d}s = 0$ $\mathbb{P}$-a.s. □

Remark 3.6. The above approach of avoiding (P) is taken in [KLSS86, Theorem 4.1]. Note, however, that there the stopping argument is slightly more involved as

[5] More precisely, we have $\int_{\tau_\infty}^\infty C_s\,\mathrm{d}s = 0$ $\mathbb{P}$-a.s.

it also requires stopping when the wealth process $X^{x,\Pi,C}$ or the quadratic variation of $\int_0^\cdot \sigma\Pi \,\mathrm{d}W$ gets too large. But this additional stopping rather obfuscates the argument.

Remark 3.7. If $R > 1$, extending Theorem 3.3 to general admissible strategies is far more involved. While condition (P) can be assumed without loss of generality (recall Part 1 of Remark 3.2), condition (M) is in general not satisfied as there are investment strategies Π and consumption strategies C such that N fails to be a supermartingale, see Section II.A. Note that these strategies are suboptimal because $L(\Pi_t, C_t; X_t, \hat{V})$ is (very) negative. Finally, we have no reason to expect that the transversality condition (T) is satisfied. Indeed, (T) even fails for constant proportional strategies: If $\xi > \frac{\eta R}{R-1}$, then $H(\hat{\pi}, \xi) < 0$, and it follows from (2.2) that $\lim_{t\to\infty} \mathbb{E}\left[\frac{e^{-\delta t}}{1-R} X_t^{x,\hat{\pi},\xi X}\right] = -\infty$.

4 Verification approaches for $R > 1$

As we have explained in Remark 3.7, a verification argument for general admissible strategies requires some additional ideas for the case $R > 1$. In this section, we discuss the two most general approaches in the extant stochastic control literature. Both approaches first consider a perturbation of the problem (or the candidate solution) and then let the perturbation disappear.

The first full verification of the solution to the Merton problem of which we are aware (under an assumption of strictly positive discounting and interest rates) is Karatzas et al [KLSS86], which built on the previous work of Lehoczky et al [LSS83]. There, the idea is to solve a perturbation of the original problem in which the agent may go bankrupt, at which point they receive a residual value P. (Part of their motivation was to better understand the results of Merton [Mer71] on HARA utilities, see also Sethi and Taskar [ST88].) The solution to the perturbed problem is very clever, and is developed in the case of a general utility function, but it is also very intricate and takes many pages of calculation. Moreover, when specialised to the case of CRRA utilities, it places some assumptions on the parameter values beyond

the necessary assumption of well-posedness of the Merton problem. The problem with bankruptcy is of independent interest, but more important for our purposes is the fact that, given the solution to the problem with bankruptcy for a CRRA utility, by letting $P \downarrow 0$ $(R < 1)$ or $P \downarrow -\infty$ $(R > 1)$ Karatzas et al [KLSS86] recover the solution to the original Merton problem.

In their seminal paper on transaction costs, Davis and Norman [DN90, Section 2] briefly consider the Merton problem without transaction costs. They assume that the proportion of wealth invested in the risky asset is bounded, and for $R < 1$ they go on to prove a verification theorem for strategies restricted to this class. Further, in the case $R > 1$ they propose a different perturbation, this time a deterministic perturbation of the candidate value function. The key point is that in the perturbed problem the candidate value function has a finite lower bound, and this allows Davis and Norman [DN90] to re-apply arguments from the $R < 1$ case, although the restriction to uniformly bounded investment strategies remains. The candidate value for the perturbed problem can be used to give an upper bound on the true value function, which converges to the candidate solution to the Merton problem as the perturbation disappears. Unlike the argument in Karatzas et al [KLSS86], the proof is quite short, but again it only works for certain parameter combinations, and more importantly it restricts attention to a subclass of admissible strategies.

4.1 Perturbation with finite bankruptcy

The first perturbation approach is by Karatzas et al [KLSS86] who study an optimal investment-consumption problem with bankruptcy for a general utility function which is of interest in its own right, building on earlier work [LSS83] by a subset of the authors. In the following, we only describe their contribution towards the solution of the Merton problem for CRRA utilities. We assume $R > 1$, and we use our notation.

Assume that $\delta > 0$ and $r > 0$. For an admissible strategy $(\Pi, C) \in \mathscr{A}(x)$, denote the *bankruptcy time* $\tau_0 = \tau_0^{x,\Pi,C} = \inf\{t : X_t^{x,\Pi,C} = 0\}$. Then choose a finite

bankruptcy value $P \in (-\infty, 0)$ and consider the problem with bankruptcy:

$$V^P(x) := \sup_{C \in \mathscr{C}(x)} J^P(C) = \sup_{C \in \mathscr{C}(x)} \mathbb{E}\left[\int_0^{\tau_0^{x,\Pi,C}} e^{-\delta t} U(C_t)\,\mathrm{d}t + e^{-\delta \tau_0^{x,\Pi,C}} P\right]. \quad (4.1)$$

Note that the classical Merton problem corresponds to the limiting case $P = -\infty$.

Karatzas et al [KLSS86] show the following:

(A) Suppose that a C^2-function $\hat{V}^P : (0,\infty) \to (P, 0)$ solves the HJB equation corresponding to the optimisation problem (4.1) given by

$$\delta \tilde{V}(x) = \sup_{c \geq 0, \pi} \left[\tilde{V}'(x)((\mu - r)\pi x + (rx - c)) + \frac{1}{2}\pi^2\sigma^2 x^2 \tilde{V}''(x) + U(c)\right], \quad (4.2)$$

for $x > 0$. Subject to $\lim_{x \downarrow 0} \tilde{V}(x) = P$.

Then, $\hat{V}^P(x) = V^P(x)$ for all $x \in [0, \infty)$.

(B) For each $P \in (-\infty, 0)$, there exists a C^2-function $\hat{V}^P : (0,\infty) \to (P, \infty)$ that solves the HJB equation (4.2) with $\lim_{x \downarrow 0} \hat{V}^P(x) = P$.

(C) $V(x) \leq \lim_{P \downarrow -\infty} \hat{V}^P(x) = \hat{V}(x)$, which together with $\hat{V}(x) \leq V(x)$ establishes the claim.

Here, the argument for (A) is relatively straightforward; see [KLSS86, Theorem 4.1] and not more difficult than the proof of our Theorem 4.3. Similarly, the argument for (C) is easy: the first inequality follows from the fact that $V(x) \leq V^P(x) \leq \hat{V}^P(x)$ for each $x > 0$ and $P \in \mathbb{R}_-$ by the definition of V^P and (A); the second inequality is straightforward using the explicit form for $\hat{V}^P$.

But the main difficulty—and great ingenuity—of the argument in [KLSS86] is (B). Indeed, a direct calculation for $r > 0$ case takes at least two pages and yields the answer:[6]

$$\hat{V}^P(x) = \frac{\nu}{\eta(R-\nu)}\left(\frac{\eta}{R}\frac{R-\nu}{1-\nu}(1-R)P\right)^{\frac{1-\nu}{1-R}} (\hat{C}^P(x))^{\nu - R} + \eta^{-1}\frac{(\hat{C}^P(x))^{1-R}}{1-R}, \quad (4.3)$$

[6] For $r = 0$, we get the simpler answer $\hat{V}(x, P) = \eta^{-1}(\eta x + (\eta(1-R)P)^{1/(1-R)})^{1-R}/(1-R)$.

where the function $\hat{C}^P(x)$ describing the optimal consumption is the inverse of the function

$$I^P(c) = -\eta^{-1}\left(\frac{\eta}{R}\frac{\nu - R}{\nu - 1}(1-R)P\right)^{\frac{1-\nu}{1-R}} c^\nu + \frac{c}{\eta},$$

and ν is the negative root of the equation $\frac{\lambda^2}{2}\zeta^2 + (r - \delta - \frac{\lambda^2}{2})R\zeta - rR^2 = 0$.

4.2 Perturbation of the value function

The second perturbation approach is by Davis and Norman [DN90] who study the Merton problem with transaction costs; the perturbation argument for $R > 1$ in the frictionless case is a fortunate by-product, and not the main contribution of the paper. Again we will use our notation to describe their approach.

Assume that $\delta > 0$ and $r > 0$. Denote by $\mathscr{A}^b(x)$ the set of all admissible strategies (Π, C) for which Π is uniformly bounded, write $\mathscr{C}^b(x)$ for the corresponding set of attainable consumption strategies and set $V^b(x) := \sup_{C\in\mathscr{C}^b(x)} J(C)$. For $\zeta > 0$, consider the perturbed value function $\hat{V}^\zeta(x) = \hat{V}(x+\zeta)$ and for $(\Pi, C) \in \mathscr{A}^b(X)$ (such that C^{1-R} is integrable with respect to the identity process), consider the process M^ζ defined by

$$M^\zeta = \int_0^t e^{-\delta t}U(C_s)\,\mathrm{d}s + e^{-\delta t}\hat{V}^\zeta(X_t).$$

Then the same argument as in the proof of Theorem 3.3 but with $\hat{V}$ replaced by $\hat{V}^\zeta$ yields

$$\mathrm{d}M_t^\zeta = \mathrm{d}N_t^\zeta + L(\Pi_t, C_t; X_t, \hat{V}^\zeta)\,\mathrm{d}t$$

Using that $\hat{V}^\zeta(x) = \hat{V}(x+\zeta)$, it is straightforward to check that

$$\sup_{\pi\in\mathbb{R}, c\geq 0} L(\pi, c; x, \hat{V}^\zeta) = -r\zeta\hat{V}_x^\zeta(X_t)e^{-\delta t} \leq 0.$$

It follows that, under the crucial assumption that $r \geq 0$, we have $L(\Pi_t, C_t; X_t, \hat{V}^\zeta) \leq 0$. Finally, using that Π and $\hat{V}_x^\zeta$ are bounded, it is not difficult to check that N^ζ is

a square integrable martingale. Now following the proof of Theorem 3.3, and using that $|\hat{V}^\zeta|$ is bounded and $\delta > 0$ it follows that

$$J(C) \leq \limsup \mathbb{E}\left[M_t^\zeta\right] - \liminf \mathbb{E}\left[e^{-\delta t}\hat{V}^\zeta(X_t^{x,\Pi,C})\right] \leq \limsup \mathbb{E}\left[M_t^\zeta\right] \leq \hat{V}^\zeta(x).$$

We may conclude that $V^b(x) \leq \hat{V}^\zeta(x)$ and taking the limit as $\zeta \downarrow 0$, it follows that $V^b(x) = \hat{V}(x)$.

5 The general verification argument

In this section, we present our general verification argument. It is inspired by the perturbation argument of Davis and Norman, see Section 4.2. The key idea is to use the *candidate optimal consumption strategy* as a *stochastic perturbation* of the utility function. This yields a very elegant and simple argument that has the trio of advantages that it is no more difficult than the fiat verification argument in Theorem 3.3, it does not need to distinguish between the case $R > 1$ and $R < 1$ and it does not involve any stopping argument.

The following theorem contains the solution to the stochastically perturbed Merton problem. The subsequent corollary then lets this perturbation disappear. Recall the notations of Theorem 3.3: $\eta = \frac{1}{R}[\delta - (1-R)(r + \frac{\lambda^2}{2R})]$, $\hat{\Pi} = \frac{\lambda}{\sigma R}$ and $\hat{V}(x) = \frac{x^{1-R}}{1-R}\eta^{-R}$.

Theorem 5.1. *Suppose $\eta > 0$. Denote by $Y = (Y_t)_{t\geq 0}$ the candidate optimal wealth process started from unit initial wealth 1, i.e., $Y_t := X_t^{1,\hat{\Pi},\eta X}$, and by $G = (G_t)_{t\geq 0}$, the corresponding optimal consumption stream, i.e., $G_t = \eta Y_t$. Fix $\varepsilon > 0$, define the function $U_\varepsilon : [0,\infty) \times (0,\infty) \to (-\infty,\infty)$ by $U_\varepsilon(c,g) = \frac{(c+\varepsilon g)^{1-R}}{1-R}$, and for an attainable consumption stream C consider*

$$J_\varepsilon(C) := \mathbb{E}\left[\int_0^\infty e^{-\delta t} U_\varepsilon(C_t, G_t)\,\mathrm{d}t\right] = J(C + \varepsilon G).$$

Then, for $x > 0$,

$$V_\varepsilon(x) := \sup_{C \in \mathscr{C}(x)} J_\varepsilon(C) = \hat{V}(x+\varepsilon).$$

Moreover, the supremum is attained when $\Pi = \hat{\Pi}$ and $C = \hat{C}$ where $\hat{C} = \eta X^{x,\hat{\Pi},\hat{C}}$.

Proof. First, from the SDE for the wealth process (I.4.2) we have that $X^{x,\hat{\Pi},\eta X} + \varepsilon Y = X^{x+\varepsilon,\hat{\Pi},\eta X}$. It follows that $\hat{C} + \varepsilon G = \eta X^{x+\varepsilon,\hat{\Pi},\eta X} \in \mathscr{C}(x+\varepsilon)$, which together with Theorem 3.3 implies that $J_\varepsilon(\hat{C}) = J(\hat{C} + \varepsilon G) = \hat{V}(x+\varepsilon)$.

It remains to show that $V_\varepsilon(x) \leq \hat{V}(x+\varepsilon)$. The argument is very similar to the one in the proof of Theorem 3.3. Let $(\Pi, C) \in \mathscr{A}(x)$ be arbitrary and set $X := X^{x,\Pi,C}$ for brevity. The dynamics of $X + \varepsilon Y$ are given by

$$\mathrm{d}(X_t + \varepsilon Y_t) = \left(\sigma \Pi_t X_t + \frac{\lambda}{R}\varepsilon Y_t\right)\mathrm{d}B_t + \left(X_t(r + \Pi_t \sigma\lambda) - C_t + \left(r + \frac{\lambda^2}{R} - \eta\right)\varepsilon Y_t\right)\mathrm{d}t.$$

Define the process $M^\varepsilon = (M_t^\varepsilon)_{t\geq 0}$ by

$$M_t^\varepsilon = \int_0^t e^{-\delta s} U_\varepsilon(C_s, G_s)\,\mathrm{d}s + e^{-\delta t}\hat{V}(X_t + \varepsilon Y_t).$$

We proceed to apply Itô's formula to M^ε. Noting that the argument of $\hat{V}$ and its derivatives is $(X_t + \varepsilon Y_t)$ throughout, we obtain

$$\begin{aligned}
\mathrm{d}M_t^\varepsilon &= e^{-\delta t} U(C_t + \varepsilon\eta Y_t) + e^{-\delta t}\left[-\delta \hat{V}\,\mathrm{d}t + \hat{V}_x\,\mathrm{d}(X_t + \varepsilon Y_t) + \frac{1}{2}\hat{V}_{xx}\,\mathrm{d}[X + \varepsilon Y]_t\right] \\
&= \mathrm{d}N_t^\varepsilon + e^{-\delta t} L^\varepsilon(\Pi_t, C_t; X_t, Y_t, \hat{V})\,\mathrm{d}t
\end{aligned}$$

where $N_t^\varepsilon = \int_0^t e^{-\delta s}\eta^{-R}(X_s + \varepsilon Y_s)^{-R}(\sigma\Pi_s X_s + \frac{\lambda\varepsilon}{R}Y_s)\,\mathrm{d}B_s$ and, with $z = x + \varepsilon y$,

$$\begin{aligned}
L^\varepsilon(\pi, c; x, y, v = v(z)) &= \frac{(c + \varepsilon\eta y)^{1-R}}{1-R} + \left[x(r + \pi\sigma\lambda) - c + (r + \frac{\lambda^2}{2} - \eta)\varepsilon y\right] v_z \\
&\quad + \frac{1}{2}\left(\sigma\pi x + \frac{\lambda\varepsilon y}{R}\right)^2 v_{zz} - \delta v \\
&= L\left(\frac{\pi x}{z} + \frac{\lambda\varepsilon y}{\sigma R z}, c + \varepsilon\eta y; z, v = v(z)\right).
\end{aligned}$$

Here L is the operator defined in (3.2). Then[7]

$$\sup_{\pi\in\mathbb{R},c\geq 0} L^{\varepsilon}(\pi,c;x,y,\hat{V}=\hat{V}(z)) = \sup_{\pi\in\mathbb{R},c\geq 0} L\left(\frac{\pi x}{z}+\frac{\lambda\varepsilon y}{\sigma R z}, c+\varepsilon\eta y; z, \hat{V}=\hat{V}(z)\right)$$
$$= \sup_{\tilde{\pi}\in\mathbb{R},\tilde{c}\geq\varepsilon\eta y} L\left(\tilde{\pi},\tilde{c},z,\hat{V}=\hat{V}(z)\right) \leq \sup_{\tilde{\pi}\in\mathbb{R},\tilde{c}\geq 0} L\left(\tilde{\pi},\tilde{c};z,\hat{V}=\hat{V}(z)\right) = 0$$

where the final equality follows from (3.3). This gives

$$M_t^{\varepsilon} \leq \hat{V}(x+\varepsilon) + N_t^{\varepsilon}, \quad t\geq 0. \tag{5.1}$$

Next, define the process $\Lambda^{\varepsilon} = (\Lambda_t^{\varepsilon})_{t\geq 0}$ by

$$\Lambda_t^{\varepsilon} := \int_0^t e^{-\delta s} U_{\varepsilon}(0,G_s)\,\mathrm{d}s + e^{-\delta t}\hat{V}(0+\varepsilon Y_t) = \int_0^t e^{-\delta s} U(\varepsilon G_s)ds + e^{-\delta t}\hat{V}(\varepsilon Y_t).$$

Then, $\Lambda^{\varepsilon} \leq M^{\varepsilon}$ by monotonicity of U and $\hat{V}$. Using that Λ^{ε} is a (UI) martingale by Remark 3.4, it follows that N^{ε} is bounded below by the (UI) martingale $\Lambda^{\varepsilon}-\hat{V}(x+\varepsilon)$ and is hence a supermartingale.

Taking expectation in (5.1), we find for each $t\geq 0$,

$$\mathbb{E}\left[M_t^{\varepsilon}\right] \leq \mathbb{E}\left[\hat{V}(x+\varepsilon)+N_t^{\varepsilon}\right] \leq \hat{V}(x+\varepsilon). \tag{5.2}$$

Next, note that $X+\varepsilon Y$ satisfies the transversality condition (T) since

$$\liminf_{t\to\infty} \mathbb{E}\left[e^{-\delta t}\frac{(X_t+\varepsilon Y_t)^{1-R}}{1-R}\right] \geq \varepsilon^{1-R}\liminf_{t\to\infty}\mathbb{E}\left[e^{-\delta t}\frac{Y_t^{1-R}}{1-R}\right] = 0. \tag{5.3}$$

Taking the limit in (5.2) as t goes to infinity and using (5.3), we may conclude that for any $C\in\mathscr{C}(x)$,

$$J_{\varepsilon}(C) = \lim_{t\to\infty}\mathbb{E}\left[\int_0^t e^{-\delta s}\frac{(C_s+\varepsilon G_s)^{1-R}}{1-R}ds\right] = \lim_{t\to\infty}\mathbb{E}\left[M_t^{\varepsilon} - e^{-\delta t}\hat{V}(X_t+\varepsilon Y_t)\right]$$

[7] The inequality is in fact an equality since the maximum over $\tilde{c}$ is attained at $\hat{V}^{-1/R}(z) = \eta z = \eta(x+\varepsilon y) \geq \varepsilon\eta y$.

$$\leq \limsup_{t\to\infty} \mathbb{E}\left[M_t^\varepsilon\right] - \liminf_{t\to\infty} \mathbb{E}\left[e^{-\delta t}\eta^{-R}\frac{(X_t+\varepsilon Y_t)^{1-R}}{1-R}\right]$$
$$\leq \limsup_{t\to\infty} \mathbb{E}\left[M_t^\varepsilon\right] \leq \hat{V}(x+\varepsilon). \qquad \square$$

Remark 5.2. The perturbation of the problem by the additional consumption of εG elegantly and simply transforms the problem to one in which the fiat conditions (P), (M) and (T) are satisfied. Since Y is positive $\mathbb{P}$-a.s., the same is trivially true for $X+\varepsilon Y$. Moreover, $J(\varepsilon G) = \varepsilon^{1-R}J(G) > -\infty$ and this allows us to easily find an integrable lower bound on N^ε and hence conclude it is a supermartingale. Again Y satisfies a transversality condition (T) and so the same is trivially true for $X+\varepsilon Y$.

Remark 5.3. One interpretation of the theorem is that a financially-savvy benefactor gives the agent an additional consumption stream based on an initial wealth ε which is invested optimally by the benefactor. Then, if the agent behaves optimally with their own wealth, the two consumption streams and investment strategies remain perfectly aligned to each other, and the derivation and valuation of the candidate optimal strategy is simple and immediate.

Corollary 5.4. *Suppose $\eta > 0$. Then, for $x > 0$,*

$$V(x) := \sup_{C\in\mathscr{C}(x)} J(C) = J(\hat{C}) = \hat{V}(x).$$

Proof. The equality $J(\hat{C}) = \hat{V}(x)$ follows from Theorem 3.3. It remains to establish that $V(x) \leq \hat{V}(x)$. Using the notation of Theorem 5.1, for any $C \in \mathscr{C}(x)$, we get $J(C) \leq J_\varepsilon(C) \leq V_\varepsilon(x) = \hat{V}(x+\varepsilon)$. Letting $\varepsilon \downarrow 0$, we conclude that $V(x) \leq \hat{V}(x)$. $\square$

We finish this section by showing that in the case $R > 1$ if $\eta \leq 0$, *every* $C \in \mathscr{C}(x)$ has $J(C) = -\infty$.

Corollary 5.5. *Suppose that $R > 1$ and $\eta \leq 0$. Then*

$$V(x) = \sup_{C\in\mathscr{C}(x)} J(C) = -\infty.$$

Proof. Fix $C \in \mathscr{C}(x)$. It suffices to show that $J(C) = -\infty$. We use an approximation argument. For $n \in \mathbb{N}$ set $\delta_n := \delta + R(\frac{1}{n} - \eta)$. Then, $\delta_n > \delta$ and $\eta_n := \frac{1}{R}[\delta_n - (1 - R)(r + \frac{\lambda^2}{2R})] = \frac{1}{n} > 0$. Then, as $U(c) < 0$ for $c \geq 0$, it follows from Corollary 5.4 that

$$J(C) = \mathbb{E}\left[\int_0^\infty e^{-\delta s} U(C_s)\,\mathrm{d}s\right] \leq \mathbb{E}\left[\int_0^\infty e^{-\delta_n s} U(C_s)\,\mathrm{d}s\right] \leq \frac{x^{1-R}}{1-R}(\eta_n)^{-R}, \quad n \in \mathbb{N}.$$

Taking the limit as n goes to ∞, it follows that $J(C) = -\infty$. $\square$

APPENDIX TO CHAPTER II

II.A An example where N fails to be a supermartingale

For $R > 1$, the process N in the proof Theorem 3.3 can fail to be supermartingale. We first give an abstract version of an example and then two concrete specifications.

Example A.1. Let $(\Pi, C) \in \mathscr{A}(x)$ be such that $X = X^{x,\Pi,C}$ has $\mathbb{P}$-a.s. positive paths. Define the stopping time

$$\tau := \inf\left\{t \geq 0 : \int_0^t \eta^{-R}\sigma\Pi_s X_s^{1-R} e^{-\delta s}\,\mathrm{d}B_s = 1\right\}.$$

If τ is bounded, then N fails to be a supermartingale because $\mathbb{E}\left[N_\tau\right] = 1 > 0 = \mathbb{E}\left[N_0\right]$.

The above abstract situation can be achieved either by "wild" investment or by "too fast" consumption, or a combination of the two.

For an example of a "wild" investment strategy Π, assume that $\mu \geq r > 0$ and

define the stopping time

$$\tilde{\tau} := \inf\left\{t \geq 0 : \int_0^t \frac{\eta^{-R}\sigma e^{-\delta s}}{1-s}\,\mathrm{d}B_s = 1\right\}.$$

Note that $\tilde{\tau} < 1$ $\mathbb{P}$-a.s. since $\int_0^1 \left(\frac{\eta^{-R}\sigma e^{-\delta s}}{1-s}\right)^2 \mathrm{d}s = \infty$. Then define $(\Pi, C) \in \mathscr{A}(x)$ by

$$\Pi_t = \frac{1}{1-t} X_t^{R-1} \mathbf{1}_{\{t \leq \tilde{\tau}\}}, \quad C_t := rX_t + \Pi_t X_t(\mu - r).$$

Then, the corresponding wealth process X is a stopped and time changed CEV process:

$$\mathrm{d}X_t = X_t^R \frac{\sigma}{1-t} \mathbf{1}_{\{t \leq \tilde{\tau}\}}\,\mathrm{d}B_t, \quad X_0 = x.$$

Since $R > 1$, X remains positive. Since $\tau = \tilde{\tau}$ $\mathbb{P}$-a.s. we have $\tau < 1$ $\mathbb{P}$-a.s. and N fails to be a supermartingale.

For an example of a "too fast" consumption strategy C (with bounded investment strategy Π), assume that $\mu \geq r > 0$ and define the stopping time

$$\bar{\tau} := \inf\left\{t \geq 0 : \int_0^t \frac{x^{1-R}\eta^{-R}\sigma e^{\sigma(1-R)B_s - (\delta + \frac{(1-R)}{2}\sigma^2)s}}{1-s}\,\mathrm{d}B_s = 1\right\}.$$

Note that $\bar{\tau} < 1$ $\mathbb{P}$-a.s. since $\int_0^1 (\frac{x^{1-R}\eta^{-R}\sigma e^{\sigma(1-R)B_s - (\delta + \frac{(1-R)}{2}\sigma^2)s}}{1-s})^2\,\mathrm{d}s = \infty$, $\mathbb{P}$-a.s. Then define $(\Pi, C) \in \mathscr{A}(x)$ by

$$\Pi_t = \mathbf{1}_{\{t \leq \bar{\tau}\}}, \quad C_t := \frac{1}{R-1}\frac{X_t}{1-t}\mathbf{1}_{\{t \leq \bar{\tau}\}} + rX_t + \Pi_t X_t(\mu - r).$$

Then, the corresponding wealth process satisfies the SDE

$$\mathrm{d}X_t = \sigma X_t \mathbf{1}_{\{t \leq \bar{\tau}\}}\,\mathrm{d}B_t - \frac{1}{R-1}\frac{X_t}{1-t}\mathbf{1}_{\{t \leq \bar{\tau}\}}\,\mathrm{d}t, \quad X_0 = x.$$

It is not difficult to check that this has the solution

$$X_t = x(1 - t \wedge \bar{\tau})^{\frac{1}{R-1}} e^{\sigma B_{t\wedge\bar{\tau}} - \frac{1}{2}\sigma^2(t\wedge\bar{\tau})}$$

which is well-defined and positive by the fact that $\bar{\tau} < 1$ $\mathbb{P}$-a.s. Since $\tau = \bar{\tau}$ $\mathbb{P}$-a.s., we have $\tau < 1$ $\mathbb{P}$-a.s. and N fails to be a supermartingale.

II.B Change of numéraire arguments and the role of the discount factor

It is interesting to study how the Merton problem behaves under a *change of numéraire*. As we have seen in Section 4, using the perturbation arguments of Karatzas et al [KLSS86] or Davis and Norman [DN90], we get verification arguments for the case $R > 1$ under the parameter restrictions $\delta > 0$ and $r > 0$. The goal of this section is to show using a change of numéraire that this parameter restriction can be weakened, although not to the extent that it covers all the parameter combinations for which $\eta > 0$. We then discuss how these arguments shed some light on the interpretation of the parameter δ.

A pair $(\tilde{S}^0, \tilde{S}) = (\tilde{S}^0_t, \tilde{S}_t)_{t\geq 0}$ of semimartingales is said to be *economically equivalent* to (S^0, S) if there exists a positive continuous semimartingale $D = (D_t)_{t\geq 0}$ such that $\tilde{S}^0 = DS^0$ and $\tilde{S} = DS$. Here, the interpretation of D is an exchange rate process and $(\tilde{S}^0, \tilde{S})$ describes the financial market in a different currency unit; see [Her17, Section 2.1] for more details. We will restrict attention to deterministic processes D in which case D is better described as a change in accounting units.

Next, recall that if $(\vartheta^0, \vartheta, C)$ is a admissible investment-consumption strategy for initial wealth $x > 0$, (where ϑ^0 and ϑ^1 denote the *number of shares* held in the riskless and risky asset, respectively), then the corresponding wealth process $X = \vartheta^0_t S^0_t + \vartheta_t S$ satisfies the SDE

$$\mathrm{d}X_t = \vartheta^0_t \,\mathrm{d}S^0_t + \vartheta_t \,\mathrm{d}S_t - C_t \,\mathrm{d}t.$$

Now if $(\tilde{S}^0, \tilde{S})$ is economically equivalent to (S^0, S) with corresponding exchange rate process D, it is not difficult to check that the corresponding wealth process $\tilde{X} := \vartheta^0 \tilde{S}^0 + \vartheta \tilde{S} = DX$ satisfies the SDE

$$\mathrm{d}\tilde{X}_t = \vartheta_t^0 \,\mathrm{d}\tilde{S}_t^0 + \vartheta_t \,\mathrm{d}\tilde{S}_t - \tilde{C}_t \,\mathrm{d}t,$$

where $\tilde{C} = DC$. This means that if C describes an attainable consumption strategy in units corresponding to (S^0, S), then $\tilde{C} = DC$ describes the *same* consumption strategy in units corresponding to $(\tilde{S}^0, \tilde{S})$ (which is also attainable for those units).

Consider now the case that $D_t = e^{\gamma t}$ for some $\gamma \in \mathbb{R}$. Then, $(\tilde{S}^0, \tilde{S})$ is again a Black–Scholes–Merton model with interest rate $\tilde{r} = r + \gamma$, drift $\tilde{\mu} = \mu + \gamma$ and volatility $\tilde{\sigma} = \sigma$. Let C be an attainable consumption strategy in units corresponding to (S^0, S) and $\tilde{C} = DC$ the corresponding attainable consumption strategy in units corresponding to $(\tilde{S}^0, \tilde{S})$. Then $\tilde{C}/\tilde{S}^0 = DC/DS^0 = C/S^0$ and

$$\begin{aligned}
J(C;\delta) &:= \mathbb{E}\left[\int_0^\infty \frac{e^{-\delta t}}{1-R} C_t^{1-R} \,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty \frac{e^{-(\delta + r(R-1))t}}{1-R} \left(\frac{C_t}{S_t^0}\right)^{1-R} \mathrm{d}t\right] \\
&= J(C/S^0; \delta + r(R-1)) = J(\tilde{C}/\tilde{S}^0; \delta + (\tilde{r} - \gamma)(R-1)) \\
&= \mathbb{E}\left[\int_0^\infty \frac{e^{-(\delta - (R-1)\gamma + \tilde{r}(R-1))t}}{1-R} \left(\frac{\tilde{C}_t}{\tilde{S}_t^0}\right)^{1-R} \mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^\infty \frac{e^{-(\delta - (R-1)\gamma)t}}{1-R} \tilde{C}_t^{1-R} \,\mathrm{d}t\right] = J(\tilde{C}; \delta - (R-1)\gamma). \qquad (2.1)
\end{aligned}$$

It follows from the above calculation that the Merton problem for $R, r, \mu, \sigma, \delta$ is equivalent to the Merton problem for $R, r+\gamma, \mu+\gamma, \sigma, \delta - (R-1)\gamma$ for each $\gamma \in \mathbb{R}$. Note in particular, that the well-posedness parameter η from (2.4) is independent of the choice of accounting units. This means that if we have a verification argument for the parameters $R, r+\gamma, \mu+\gamma, \sigma, \delta - (R-1)\gamma$, we also have verification argument for the parameters $R, r, \mu, \sigma, \delta$. Hence, if $\delta + r(R-1) > 0$ we can choose $\gamma = \frac{\delta - r(R-1)}{2(R-1)}$ so that $\tilde{\delta} = \tilde{r} = \frac{\delta + r(R-1)}{2} > 0$ and then we can extend the verification arguments

of Karatzas et al [KLSS86] or Davis and Norman [DN90] to this case. It follows that instead of needing to assume $\delta > 0$ and $r > 0$ as in [KLSS86] and [DN90] it is sufficient to assume only that $\delta + r(R-1) > 0$. We provide more details on using a numéraire change for the problem with bankruptcy studied by Karatzas et al. [KLSS86] in Section II.B.2.

II.B.1 What is the role of the discount factor?

The above ideas also shed some light on the interpretation of the parameter δ. To this end, consider an alternative formulation of the Merton problem and associate to an attainable consumption stream C the expected utility

$$K(C;\phi) = \mathbb{E}\left[\int_0^\infty \frac{e^{-\phi t}}{1-R}\left(\frac{C_t}{S_t^0}\right)^{1-R} \mathrm{d}t\right], \tag{2.2}$$

where $\phi := \delta + r(R-1)$ is the *impatience rate*. Then, $K(C;\phi) = J(C, \phi - r(R-1))$. In order to emphasise the dependence of the problem on the accounting units which are being used, we might expand the notation to write $J(C; S^0, S; \delta)$ and $K(C; S^0, S; \phi)$ and then (2.1) becomes

$$J(C; S^0, S; \delta) = J(\tilde{C}; \tilde{S}^0, \tilde{S}; \delta - (R-1)\gamma),$$

whilst, for $K(C,\phi) = K(C; S^0, S, \phi)$ we find

$$K(\tilde{C}; \tilde{S}^0, \tilde{S}, \phi) = \mathbb{E}\left[\int_0^\infty \frac{e^{-\phi t}}{1-R}\left(\frac{D_t C_t}{D_t S_t^0}\right)^{1-R} \mathrm{d}t\right] = K(C; S^0, S, \phi).$$

In particular, K defined via (2.2) has the advantage that (unlike J) it is numéraire-independent in the sense that a change of accounting unit leaves the problem value unchanged.[8] With this in mind it makes sense to focus on the impatience rate ϕ rather than the discount rate δ. Note that $\eta = \frac{1}{R}\phi + \frac{R-1}{R}\frac{\lambda^2}{2R}$ so that the optimal

[8] This applies not only to deterministic changes of accounting units, but also to stochastic changes of numéraire.

consumption rate is a linear (convex if $R > 1$) combination of the impatience rate and (half of) the squared Sharpe ratio per unit of risk aversion, with the weights depending on the risk aversion.

II.B.2 The Merton problem with bankruptcy revisited

Using the ideas of this section, we can revisit the argument of Karatzas et al [KLSS86] to give a much simpler proof for $V(x) = \hat{V}(x)$ in the case that $\delta + (1-R)r > 0$.

The idea is to consider the case $r = 0$, which is not studied in Karatzas et al [KLSS86]. For $r = 0$, it is not difficult to check that the HJB equation (4.2) has the solution

$$\hat{V}_\eta^{P,r=0} = \frac{(\eta x + (\eta(1-R)P)^{1/(1-R)})^{1-R}}{\eta(1-R)},$$

which is substantially simpler than the solution (4.3) for $r > 0$. Since the argument in (A) and (C) of Section 4.1 carry verbatim over to $r = 0$, we have a verification argument for the Merton Problem in the case that $\delta > 0$ and $r = 0$. Now if $\delta + r(R-1) > 0$, we choose $\gamma = -r$, so that $\tilde{\delta} = \delta + r(R-1) > 0$ and $\tilde{r} = 0$, and the above change of numéraire argument in Section II.B give a verification argument also in this case.

II.C The Merton problem with logarithmic utility

The case of logarithmic utility $U(c) = \log(c)$ corresponds to the case of unit coefficient of relative risk-aversion. The function log differs from the other CRRA utility functions in that it may take both signs but much of the analysis goes through in exactly the same way.

Under logarithmic utility, the problem facing the agent is to choose an admissible strategy $(\Pi, C) \in \mathscr{A}(x)$ so as to find

$$V(x) := \sup_{C \in \mathscr{C}(x)} J(C) := \sup_{C \in \mathscr{C}(x)} \mathbb{E}\left[\int_0^\infty e^{-\delta t} \log(C_t)\, \mathrm{d}t\right].$$

Since log takes both positive and negative values, we make the definition[9]

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}\log(C_t)\,\mathrm{d}t\right] := \mathbb{E}\left[\left(\int_0^\infty e^{-\delta t}\log(C_t)\,\mathrm{d}t\right)^+\right] - \mathbb{E}\left[\left(\int_0^\infty e^{-\delta t}\log(C_t)\,\mathrm{d}t\right)^-\right],$$

where for each ω,

$$\int_0^\infty e^{-\delta t}\log(C_t)\;\mathrm{d}t := \int_0^\infty e^{-\delta t}\log(C_t)^+\;\mathrm{d}t - \int_0^\infty e^{-\delta t}\log(C_t)^-\;\mathrm{d}t,$$

with the standard convention that $\infty - \infty := -\infty$.

As before, we postulate a constant proportion of wealth for both our optimal investment and our optimal consumption, $\Pi_t = \pi$ and $C_t = \xi X_t$. In this case, our wealth process is given by (2.1). Taking logarithms we find that

$$\log(C_t) = \log(\xi x) + \pi\sigma B_t + \left(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}\right)t. \tag{3.1}$$

This implies that

$$\mathbb{E}[e^{-\delta t}\log(\xi X_t)] = e^{-\delta t}\log(\xi x) + e^{-\delta t}\left(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}\right)t.$$

The well-posedness condition $\eta > 0$ for $R = 1$ is equivalent to $\delta > 0$. So, suppose first that $\delta > 0$. Then

$$\begin{aligned} J(\xi X) &= \mathbb{E}\left[\int_0^\infty e^{-\delta s}\left(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}\right)s + e^{-\delta s}\log(\xi x)ds\right] \\ &= \frac{1}{\delta^2}\left(\delta\log(\xi) + \delta\log(x) + \left(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}\right)\right). \end{aligned}$$

By taking derivatives with respect to π and ξ, we find that this is maximised at

[9] There are other ways $\mathbb{E}\left[\int_0^\infty e^{-\delta t}\log(C_t)\;\mathrm{d}t\right]$ might be defined. For example, one could make the subtly different definition $\mathbb{E}\left[\int_0^\infty e^{-\delta t}\log(C_t)\;\mathrm{d}t\right] := \mathbb{E}\left[\int_0^\infty e^{-\delta t}\log(C_t)^+\;\mathrm{d}t\right] - \mathbb{E}\left[\int_0^\infty e^{-\delta t}\log(C_t)^-\;\mathrm{d}t\right]$, with the convention $\infty - \infty := -\infty$. The advantage of our definition is that in the case $\delta \le 0$ it leads to a much cleaner final statement of results. In the case $\delta > 0$, the two definitions are equivalent because then $\mathbb{E}\left[\int_0^\infty e^{-\delta t}\log(C_t)^+\;\mathrm{d}t\right] < \infty$ for all admissible consumption streams C.

$\pi = \hat{\pi} := \frac{\lambda}{\sigma}$ and $\xi = \hat{\xi} := \delta$. Note that this corresponds to the candidate optimal strategy given in (3.1) for $R = 1$. Then, the candidate value function is given by

$$\hat{V}(x) := J\left(\hat{\xi}X\right) = \frac{1}{\delta^2}\left(\delta \log(\delta x) + r + \frac{\lambda^2}{2} - \delta\right).$$

To prove optimality, one considers the stochastically perturbed Merton problem corresponding to the aggregator $U_\varepsilon(c, g) = \log(c + \varepsilon g)$ and $G = \delta Y$ for Y the wealth process under the candidate optimal strategy. The corresponding version of Theorem 5.1 then goes through exactly as when $R \in (0, \infty) \setminus \{1\}$.

When $\delta \leq 0$ the problem becomes delicate. Set $\kappa := r + \frac{\lambda^2}{2}$.

If $\kappa > 0$ choose $\Pi_t := \hat{\pi} = \frac{\lambda}{\sigma}$ and $C_t := \frac{\kappa}{2} X_t$. Then, (3.1) gives

$$\log(C_t) = \log\left(\frac{\kappa}{2}x\right) + \lambda B_t + \frac{\kappa}{2}t.$$

It follows from the strong law of large numbers, that for $\mathbb{P}$-a.e. ω, $\log(C_t(\omega)) \geq 1$ for all t sufficiently large. Hence

$$\int_0^\infty e^{-\delta t} \log\left(C_t\right) \, \mathrm{d}t = +\infty \quad \mathbb{P}\text{-a.s.}$$

whence $J(C) = +\infty$.

If $\kappa \leq 0$, we show that for *every* admissible consumption stream

$$\int_0^\infty e^{-\delta t} \left(\log\left(C_t\right)\right)^- \, \mathrm{d}t = +\infty \quad \mathbb{P}\text{-a.s.},$$

and hence $J(C) = -\infty$. Here we only consider the case that $r < 0$; the case $r = 0$ (which implies that $\lambda = 0$) is similar but easier.

First, we show that for any $(\Pi, C) \in \mathscr{A}(x)$, there exists $(\tilde{\Pi}, 0) \in \mathscr{A}(x)$ such that

$$e^{rt} \int_0^t e^{-ru} \, C_u \, \mathrm{d}u \leq X_t^{x,\tilde{\Pi},0} \quad \mathbb{P}\text{-a.s.}, \quad t \geq 0. \tag{3.2}$$

Let $\vartheta_t := \frac{\Pi_t X_t^{x,\Pi,C}}{S_t}$ and define the process $G = (G_t)_{t \geq 0}$ by $G_t = x_0 + \int_0^t \vartheta_u S_u (\sigma \, \mathrm{d}B_u +$

$(\mu - r)\,\mathrm{d}u)$. Then, using the product formula and the dynamics of $X^{x,\Pi,C}$, we obtain

$$\begin{aligned} 0 \le e^{-rt} X_t^{x,\Pi,C} &= x_0 + \int_0^t \vartheta_u S_u \sigma(\,\mathrm{d}B_u + \lambda\,\mathrm{d}u) - \int_0^t e^{-ru} C_u\,\mathrm{d}u \\ &= G_t - \int_0^t e^{-ru} C_u\,\mathrm{d}u = e^{-rt} X_t^{x,\tilde{\Pi},0} - \int_0^t e^{-ru} C_u\,\mathrm{d}u, \end{aligned}$$

where $\tilde{\Pi}_t := \frac{\vartheta_t S_t}{G_t} \mathbf{1}_{\{G_t > 0\}}$. Rearranging gives (3.2).

Next, we show that

$$\liminf_{t\to\infty} X^{x,\tilde{\Pi},0} = 0 \ \ \mathbb{P}\text{-a.s.} \tag{3.3}$$

Itô's formula and the fact that $\kappa \le 0$ gives

$$\begin{aligned} &\log(X^{x,\tilde{\Pi},0}) \\ &= \log(x_0) + \int_0^t \tilde{\Pi}_u \sigma\,\mathrm{d}B_u + \int_0^t \left(\kappa - \frac{(\tilde{\Pi}_u\sigma - \lambda)^2}{2}\right)\mathrm{d}u \le \log(x_0) + \int_0^t \tilde{\Pi}_u \sigma\,\mathrm{d}B_u. \end{aligned}$$

There are two cases: On the event $\{\int_0^\infty \tilde{\Pi}_u^2\,\mathrm{d}u < \infty\}$, $\int_0^\infty (\kappa - \frac{(\tilde{\Pi}_u\sigma-\lambda)^2}{2})\,\mathrm{d}u = -\infty$ and $\lim_{t\to\infty} \int_0^t \tilde{\Pi}_u \sigma\,\mathrm{d}B_u$ exists in $\mathbb{R}$, whence $\lim_{t\to\infty} \log(X_t) = -\infty$. On the event $\{\int_0^\infty \tilde{\Pi}_u^2\,\mathrm{d}u = \infty\}$, $\liminf_{t\to\infty} \int_0^t \tilde{\Pi}_u \sigma\,\mathrm{d}B_u = -\infty$ by the law of iterated logarithm, whence $\liminf_{t\to\infty} \log(X_t) = -\infty$. So we have (3.3).

Finally, combining (3.3) with (3.2) yields

$$\liminf_{t\to\infty} e^{rt} \int_0^t e^{-ru} C_u\,\mathrm{d}u = 0 \ \ \mathbb{P}\text{-a.s.}$$

This implies that the random set $A := \{u \in [0,\infty) : C_u < 1/2\}$ has $\mathbb{P}$-a.s. infinite Lebesgue-measure. But this implies that

$$\int_0^\infty e^{-\delta t} \log\,(C_t)^- \ \mathrm{d}t \ge \int_0^\infty \log\,(C_t)^- \ \mathrm{d}t \ge \int_A \log(2)\,\mathrm{d}t = +\infty \ \ \mathbb{P}\text{-a.s.}$$

Putting the results together we have the following result for logarithmic utility:

Theorem C.1. *Suppose $\delta > 0$. Then, for $x > 0$,*

$$V^*(x) := \sup_{C \in \mathscr{C}^*(x)} J(C) = J(\hat{C}) = \frac{1}{\delta^2}\left(\delta \log(\delta x) + r + \frac{\lambda^2}{2} - \delta\right),$$

where the corresponding optimal investment-consumption strategy is given by $(\Pi, C) = (\hat{\Pi}, \hat{C})$, where $\hat{\Pi} = \frac{\lambda}{\sigma R}$ and $\hat{C} = \delta X^{x,\hat{\Pi},\hat{C}}$.

Suppose $\delta \leq 0$. Then, the problem is ill-posed. For $\kappa := r + \frac{\lambda^2}{2} > 0$ we have $V^*(x) = +\infty$, whereas for $\kappa \leq 0$ we have $V^*(x) = -\infty$.

II.D The dual approach

For completeness, we include a brief description of the dual approach to the Merton problem. This is a static argument in the sense that we replace the dynamic admissibility condition—that the wealth process is negative at all times—with a static *budget feasibility* condition. As before, we only deal with a Black–Scholes–Merton financial market, and in this case the argument is particularly simple since the market is complete and hence there is exactly one equivalent martingale measure.

Define the state-price density process $\zeta = (\zeta_t)_{t \geq 0}$ by

$$\zeta_t = e^{-rt}\mathcal{E}(-\lambda W)_t = \exp\left(-\lambda B_t - \left(r + \frac{\lambda^2}{2}\right)t\right). \tag{4.1}$$

The following proposition gives a neat equivalent criterion for a consumption process to be admissible in terms of the state price density.

Proposition D.1. *A nonnegative progressively measurable process C is in $\mathscr{C}(x)$ if and only if the budget feasibility condition holds:*

$$\mathbb{E}\left[\int_0^\infty \zeta_s C_s \, \mathrm{d}s\right] \leq x. \tag{4.2}$$

Proof. Suppose that $C \in \mathscr{C}(x)$. Let Π be the corresponding investment process such that $(\Pi, C) \in \mathscr{A}(x)$. Denote the corresponding wealth process by $X = X^{x,\Pi,C}$ and

define the nonnegative process $Y = (Y_t)_{t\geq 0}$ by $Y_t = \zeta_t X_t + \int_0^t \zeta_s C_s \,\mathrm{d}s$. The product rule, the definition of ζ in (4.1) and the dynamics of X from (I.4.2) give

$$\mathrm{d}Y_t = \zeta_t \,\mathrm{d}X_t + X_t \,\mathrm{d}\zeta_t + \,\mathrm{d}\langle \zeta, X\rangle_t + \zeta_t C_t \,\mathrm{d}t = \zeta_t X_t(\Pi_t \sigma - \lambda)\,\mathrm{d}B_t. \tag{4.3}$$

Hence, Y is a nonnegative local martingale and therefore a supermartingale. Using $\zeta_t X_t \geq 0$ and Fatou's Lemma we conclude that

$$x = Y_0 \geq \liminf_{t\to\infty} \mathbb{E}\left[Y_t\right] \geq \mathbb{E}\left[\int_0^\infty \zeta_s C_s \,\mathrm{d}s\right].$$

Therefore, Equation (4.2) holds.

Conversely, suppose that Equation (4.2) holds. Then, since $\xi = \int_0^\infty \zeta_t C_t \,\mathrm{d}t$ is integrable, we may define the uniformly integrable martingale $Z = (Z_t)_{t\geq 0}$ by $Z_t = \mathbb{E}[\xi|\mathcal{F}_t]$. By the Brownian martingale representation theorem, there exists a predictable process $H = (H_t)_{t\geq 0} \in L(W)$ such that

$$Z_t = \mathbb{E}\left[\xi\right] + \int_0^t H_s \,\mathrm{d}B_s.$$

Let $\Pi = (\Pi_t)_{t\geq 0}$ be given by

$$\Pi_t X_t = \frac{H_t + \lambda \zeta_t}{\zeta_t \sigma}.$$

Then, using (4.3) we find that the wealth process started from initial wealth $x' = \mathbb{E}\left[\xi\right] \leq x$ satisfies

$$\zeta_t X_t + \int_0^t \zeta_s C_s \,\mathrm{d}s = Y_t = Z_t = \mathbb{E}\left[\int_0^\infty \zeta_s C_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right].$$

Hence, $\zeta_t X_t = \mathbb{E}[\int_t^\infty \zeta_s C_s \,\mathrm{d}s|\mathcal{F}_t] \geq 0$ since both ζ and C are nonnegative. Consequently, the pair (Π, C) is admissible and C is an attainable consumption stream. $\square$

The verification argument then goes as follows. First, the candidate consumption

process $\hat{C}$ from (3.1) satisfies

$$e^{-\delta t}U'(\hat{C}_t) = \hat{V}_x(x)\zeta_t, \tag{4.4}$$

as well as

$$\mathbb{E}\left[\int_0^\infty \zeta_s \hat{C}_s \,\mathrm{d}s\right] = x. \tag{4.5}$$

Then, for an admissible $C \in \mathscr{C}(x)$, using the budget condition (4.2) for C as well as the budget condition (4.5) for $\hat{C}$, together with (4.4) and the simple fact that the concave function U is bounded above by its tangent, i.e., $U(b) \leq U(a) + (b-a)U'(a)$ for $a > 0, b \geq 0$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-\delta t}U(C_t)\,\mathrm{d}t\right] &\leq \mathbb{E}\left[\int_0^\infty \left(e^{-\delta t}U(\hat{C}_t) + e^{-\delta t}U'(\hat{C}_t)(C_t - \hat{C}_t)\right)\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-\delta t}U(\hat{C}_t)\,\mathrm{d}t\right] + \hat{V}_x(x)\mathbb{E}\left[\int_0^\infty \zeta_t(C_t - \hat{C}_t)\,\mathrm{d}t\right] \\
&\leq \mathbb{E}\left[\int_0^\infty e^{-\delta t}U(\hat{C}_t)\,\mathrm{d}t\right] = \hat{V}(x).
\end{aligned}$$

Here, the last inequality from the fact that since since C is admissible, it is budget feasible, and that the budget constraint in (4.2) is satisfied with equality for $\hat{C}$. This proves optimality of $\hat{C}$.

CHAPTER

III

STOCHASTIC DIFFERENTIAL UTILITY: AN INTRODUCTION

In this chapter, we introduce stochastic differential utility (SDU) and explain how it permits a wider range of risk aversion and temporal variance aversion preferences than the time-additive utility studied in Chapter II. In particular, we focus on infinite-horizon Epstein–Zin stochastic differential utility (EZ-SDU).

Within the economics literature, SDU (introduced by Duffie and Epstein [DE92] as the continuous-time analogue of recursive utility (Epstein and Zin, [EZ89], Weil [Wei89], Kraft and Seifreid [KS14]), and further developed by Duffie and Lions [DL92], Skiadas [Ski98] and Schroder and Skiadas [SS99]) is viewed as an extension to classical additive utilities, and recognised as having the potential to explain several of the inconsistencies between the predictions of the Merton model and agent behaviour

(for example, the equity premium puzzle [Mehra and Prescott, MP85] and the risk-free rate puzzle [Wei89]).[1] However, with several honourable exceptions (including [KSS13, KS14, SS16, KSS17, Xin17, MX18, MMKS20]), SDU has not been widely studied in the mathematical finance literature. Given the deep connections with many areas of modern probability theory (for example, backward stochastic differential equations (BSDEs)) this is in some ways surprising, but given the technical challenges involved it is also understandable.

The fact that we concentrate on the infinite horizon brings several issues into focus. Over the infinite horizon, it is not possible to work backwards from the terminal horizon and it is necessary to consider some form of transversality condition as an alternative. Moreover, integrability (and uniform integrability) become much more significant challenges. However, although the infinite-horizon problem brings potentially different (and greater) technical challenges when compared with the finite-horizon problem, it can lead to a time-homogeneous Merton problem and therefore to a dimension reduction and the greater prospect of closed-form solutions.

The structure of this chapter is the following: In Sections 1 and 2 we introduce SDU and EZ-SDU, explaining the roles of the parameters involved; in Section 3 we define the Merton problem for EZ-SDU and derive a candidate for the optimal strategy; in Section 4 we compare our EZ aggregator with the *difference* form of the EZ aggregator; and in Section 5 we compare our formulation of EZ-SDU with that used in the rest of the infinite-horizon SDU literature. In particular, in Section 5.1 we outline the conventional approach to infinite-horizon SDU (see for example Duffie and Epstein [DE92] and Melnyk, Muhle-Karbe and Seifried [MMKS20]) in which the infinite-horizon problem is replaced with a family of finite-horizon problems coupled with a transversality condition. We show that any utility process formulated in this manner is necessarily a utility process under our formulation (after a change of numéraire), whereas the converse is not true. In Section 5.2 we define the concept of

[1] Summaries of these "puzzles" in the asset pricing economics literature can be found in [Cam99, Cam00].

a utility bubble for CRRA utility—essentially, a scenario in which the utility process and the utility gained from consumption differ in sign—and show that choosing a mismatched transversality condition can lead to such bubbles arising. In section 5.3, we show that utility bubbles can also arise for stochastic differential utility, and that the utmost care must be taken if one wishes to use the conventional approach and deviate from the 'natural' transversality condition.

1 Stochastic differential utility

Stochastic diffential utility is a generalisation of time-additive discounted expected utility and is designed to allow a separation of risk preferences from time preferences. The goal in this section is to explain how this statement should be interpreted.

Under discounted expected utility, the value or utility of a consumption stream is given by $J_U(C) = \mathbb{E}\left[\int_0^\infty U(t, C_t)\,\mathrm{d}t\right]$ and the value or utility process is given by $V_t = \mathbb{E}[\int_t^\infty U(s, C_s)|\mathcal{F}_t]$. Under SDU, the function $U = U(s, C_s)$ is generalised to become an *aggregator* $g = g(s, C_s, V_s)$, and the stochastic differential utility process $V^C = (V_t^C)_{t\geq 0}$ associated to a consumption stream C solves

$$V_t^C = \mathbb{E}\left[\int_t^\infty g(s, C_s, V_s^C)ds \,\middle|\, \mathcal{F}_t\right]. \tag{1.1}$$

This creates a feedback effect in which the value at time t may depend in a nonlinear way on the value at future times. This feature leads to a separation of the two phenomena mentioned in the previous section: risk aversion and temporal variance aversion.

Note that if g takes positive and negative values, the conditional expectation on the right hand side of (1.1) may not be well-defined. With this in mind, we introduce the following definitions.

Definition 1.1. An *aggregator* is a function $g : [0, \infty) \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. For $C \in \mathscr{P}_+$, define $\mathbb{I}(g, C) := \left\{V \in \mathscr{P} : \mathbb{E}\int_0^\infty |g(s, C_s, V_s)|\,\mathrm{d}s < \infty\right\}$. Further, let $\mathbb{UI}(g, C)$ be

the set of elements of $\mathbb{I}(g, C)$ which are uniformly integrable. Then, $V \in \mathbb{I}(g, C)$ is a *utility process* associated to the pair (g, C) if it has càdlàg paths and satisfies (1.1) for all $t \in [0, \infty)$.

Remark 1.2. All utility processes are necessarily semimartingales and uniformly integrable. Indeed, let $M = (M_t)_{t \geq 0}$ be the (càdlàg) martingale given by $M_t = \mathbb{E}\left[\int_0^\infty g(s, C_s, V_s^C) ds \,\middle|\, \mathcal{F}_t\right]$ and $A = (A_t)_{t \geq 0}$ the continuous adapted process given by $A_t = \int_0^t g(s, C_s, V_s^C) ds$. Then, $V^C = M - A \in \mathscr{S}$. Moreover, let $\tilde{M} = (\tilde{M}_t)_{t \geq 0}$ be the uniformly integrable martingale given by $\tilde{M}_t = \mathbb{E}\left[\int_0^\infty |g(s, C_s, V_s^C)| ds \,\middle|\, \mathcal{F}_t\right]$. Then, $V^C \in \mathbb{UI}(g, C)$ since

$$|V_t^C| \leq \mathbb{E}\left[\int_t^\infty |g(s, C_s, V_s^C)| ds \,\middle|\, \mathcal{F}_t\right] \leq \mathbb{E}\left[\int_0^\infty |g(s, C_s, V_s^C)| ds \,\middle|\, \mathcal{F}_t\right] = \tilde{M}_t, \quad t \geq 0.$$

Definition 1.3. C is *g-evaluable* if there exists a utility process $V \in \mathbb{I}(g, C)$ associated to the pair (g, C). The set of *g-evaluable* consumption streams C is denoted by $\mathscr{E}(g)$.

Furthermore, if the utility process is unique (up to indistinguishability), then C is *g-uniquely evaluable*. The set of *g-uniquely evaluable* C is denoted by $\mathscr{E}_u(g)$.

Throughout this chapter (with a few exceptions where we explictly state otherwise), we will only consider uniquely evaluable consumption streams. Provided that C is uniquely evaluable, we may therefore define the *stochastic differential utility* of a consumption stream C and aggregator g by $J_g(C) := V_0^C$ where V^C satisfies (1.1).

The restriction to evaluable or uniquely evaluable consumption streams is a very real restriction. We will see examples, in this chapter and the next, of many reasonable consumption streams which are not evaluable when $\vartheta \in (0, 1)$. We will see in Chapter V that, when $\vartheta > 1$, the only uniquely evaluable consumption stream is the zero consumption stream.

2 Epstein–Zin stochastic differential utility

The goals of this section are: firstly, to introduce Epstein–Zin stochastic differential utility, which is a generalisation of the discounted CRRA utility that was the focus of Chapter II; secondly, to define the associated aggregator; thirdly, to examine some of the properties of EZ-SDU; and finally, to justify any restrictions on coefficients that must be imposed to make EZ-SDU well-founded. We will see in Section 2.1 that EZ-SDU allows a disentanglement of risk preferences from temporal variance preferences.

The Epstein–Zin aggregator corresponding to the vector of parameters (b, δ, R, S) is a function $g_{EZ} : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$, given by

$$g_{EZ}(t, c, v) := be^{-\delta t} \frac{c^{1-S}}{1-S} \left((1-R)v\right)^{\frac{S-R}{1-R}} . \tag{2.1}$$

Here $\mathbb{V} = (1-R)\mathbb{R}_+$ is the domain of the Epstein–Zin utility process and both R and S lie in $(0,1) \cup (1,\infty)$. It is convenient to introduce the parameters $\vartheta := \frac{1-R}{1-S}$ and $\rho = \frac{S-R}{1-R} = \frac{\vartheta-1}{\vartheta}$, so that (2.1) becomes

$$g_{EZ}(t, c, v) = be^{-\delta t} \frac{c^{1-S}}{1-S} \left((1-R)v\right)^{\rho} . \tag{2.2}$$

Note that when $S = R$ the aggregator reduces to the discounted CRRA utility function. This case corresponds to $\vartheta = 1$ and $\rho = 0$.

Remark 2.1. The expression in (2.2) is a reformulation of the classical Epstein–Zin stochastic differential utility. Other authors use the *difference* form aggregator g^{Δ}_{EZ} given by

$$g^{\Delta}_{EZ}(c, v) := b\frac{c^{1-S}}{1-S}((1-R)v)^{\rho} - \delta\vartheta v. \tag{2.3}$$

When we want to emphasise the difference between the two formulations we will call (2.2) the *discounted* form of the EZ-SDU aggregator. As might be expected there is a very close relationship between solutions of the two different forms, and we will

discuss this further in Section 4. Note immediately however, that the discounted form is easily recognised as the natural generalisation of CRRA utility as given in Chapters I and II. Indeed, when $R = S$, the aggregator becomes $U(t, c) = be^{-\delta t}\frac{c^{1-R}}{1-R}$.

Let g_{EZ} be the aggregator in (2.2). We begin by trying to give interpretations of the various parameters and to show that (despite appearances) R captures the agent's risk aversion whereas S captures agent's elasticity of intertemporal complementarity, or temporal variance aversion. In addition, δ represent the agent's subjective discount rate, and b is a scaling parameter which has no effect on the agent's preferences (as long as it is positive) - see Remark 2.2. We have included b to facilitate comparison with other forms of Epstein–Zin SDU used in the literature, but it may be set to 1 without loss of generality (alternatively, sometimes it is set equal to δ).

Standing Assumption 1 (Rational Parameter Assumption)**.** *We assume* $b > 0$*,* $\delta \in \mathbb{R}$ *and* $R \neq S \in \mathbb{R}_+ \setminus \{1\}$*.*

The case $S = R$ corresponds to CRRA utility. We exclude the case $R = S$ as it has been extensively studied and is well understood.

In addition to excluding $R = S$ we also exclude $R = 1$ and $S = 1$. Just as power law utility becomes logarithmic utility when $R = S = 1$, EZ-SDU also changes form. The parameter combination when $S = 1$ is considered by Chacko and Viceira [CV05]. (It is less clear how to extend EZ-SDU to the case $R = 1$.) Rather than study these limiting cases we focus on the case $R \neq 1 \neq S$, where the issues are already substantial.

Positivity of b corresponds to monotone preferences which are increasing in consumption. We will show in Section 2.1 via a pair of examples that the condition $R > 0$ corresponds to the agent being risk averse (rather than risk seeking) to variance of consumption over ω, and the condition $S > 0$ corresponds to the agent being averse to variance (rather than variance seeking) in consumption over time. The parameter δ is left unrestricted. Whilst it is natural based on its interpretation as a discount factor to expect δ to be positive, when EZ-SDU is associated with a finan-

cial market model a deterministic change of consumption units leads to a change in the value of δ and potentially to a change in sign, see Section 3.2 (also see Remark II.1.1 and Section II.B from Chapter II). Since typically the choice of accounting units is arbitrary there is no economic or mathematical reason to require or expect that $\delta \geq 0$.

If g_{EZ} is the Epstein–Zin aggregator given in (2.2) then the utility process $V^C = V = (V_t)_{t\geq 0}$ associated to consumption C and aggregator g_{EZ} solves

$$V_t = \mathbb{E}\left[\int_t^\infty be^{-\delta s}\frac{C_s^{1-S}}{1-S}\left((1-R)V_s\right)^\rho \,\mathrm{d}s\middle|\mathcal{F}_t\right]. \tag{2.4}$$

Remark 2.2. The parameter b has no effect on preferences, provided it is positive. To see this, suppose that V is a solution to (2.4) with $b = 1$. For arbitrary $d > 0$ it follows that $d^\vartheta V = (d^\vartheta V_t)_{t\geq 0}$ is a solution to (2.4) with $b = d$. Since preferences remain unchanged by a multiplicative scaling of the utility function, it does not matter which value of b we choose.

2.1 Risk aversion and temporal variance aversion

In Section I.1.2, we explained the dual roles of the parameter R in CRRA discounted expected utility and how it controls both the agent's risk aversion and their aversion to variance of temporal consumption. In this section, we shall see that under EZ-SDU, the parameter R governs risk preferences and the parameter S governs preferences over deterministic consumption streams that vary throughout time.

Consider a deterministic consumption stream $c = (c(t))_{t\geq 0}$. Then, $V^c = V = (V(t))_{t\geq 0}$ can be found by solving the ordinary differential equation

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = -be^{-\delta t}\frac{c(t)^{1-S}}{1-S}((1-R)V(t))^\rho,$$

subject to $\lim_{t\to\infty} V(t) = 0$. Making the change of variables to $W(t) = (1-R)V(t)$

and dividing through by $W(t)^\rho$, we find (recall $\vartheta = \frac{1-R}{1-S} = \frac{1}{1-\rho}$)

$$\frac{1}{W(t)^\rho}\frac{\mathrm{d}W(t)}{\mathrm{d}t} = -be^{-\delta t}\vartheta c(t)^{1-S}, \qquad \lim_{t\to\infty} W(t) = 0. \tag{2.5}$$

Assuming that $e^{-\delta s}c(s)^{1-R}$ is integrable at infinity, a solution to (2.5) is $W(t) = \left(\int_t^\infty be^{-\delta s}c(s)^{1-S}\,\mathrm{d}s\right)^\vartheta$. Therefore, a utility process $V = V^c$ associated to c is

$$V(t) = \frac{1}{1-R}\left(b\int_t^\infty e^{-\delta s}c(s)^{1-S}\,\mathrm{d}s\right)^\vartheta. \tag{2.6}$$

In particular, when $C^{a,\gamma} = (C_t^{a,\gamma})_{t\geq 0}$ is the deterministic, exponentially decaying consumption stream given by $C_t = C_t^{a,\gamma} = ae^{-\gamma t}$ and $\delta + \gamma(1-S) > 0$ we find

$$V(t) = V_t^{C^{a,\gamma}} = e^{-(\delta+\gamma(1-S))\vartheta t}\left(\frac{b}{\delta+\gamma(1-S)}\right)^\vartheta \frac{a^{1-R}}{1-R}$$

and $J_{g_{EZ}}(C^{a,\gamma}) := V_0^{C^{a,\gamma}} = \left(\frac{b}{\delta+\gamma(1-S)}\right)^\vartheta \frac{a^{1-R}}{1-R}$.

Now consider a 'purely random' consumption stream, whose paths have no variance over time, except for an exponential decay. Suppose that the nonnegative random variable Y is such that Y and Y^{1-R} are integrable. Let $\mathcal{F}_t = \sigma(Y)$ for all $t > 0$.[2] Consider the (progressively measurable) consumption stream $C_t^{Y,\gamma} \equiv Ye^{-\gamma t}$ for $t > 0$. All uncertainty is resolved instantaneously at $t = 0$. The value of such a consumption stream is given by

$$\begin{aligned} J_{g_{EZ}}(C^{Y,\gamma}) &= \left(\frac{b}{\delta+\gamma(1-S)}\right)^\vartheta \mathbb{E}\left[\frac{Y^{1-R}}{1-R}\right] \\ &\leq \left(\frac{b}{\delta+\gamma(1-S)}\right)^\vartheta \frac{(\mathbb{E}[Y])^{1-R}}{1-R} = J_{g_{EZ}}(C^{\mathbb{E}[Y],\gamma}), \end{aligned}$$

where the inequality follows directly from Jensen's inequality. The loss in utility from the uncertainty is captured by the risk-aversion R of the agent and the larger value of R, the stronger the agent's preference for certainty. Thus R may interpreted

[2]For the exposition, we temporarily drop the assumption that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous.

as the agent's aversion to risk. Looking at (2.4) or (2.6) one might expect that the risk aversion comes from the value of S but, contrary to naive intuition, this is not the case.

Now consider the agent's preferences over deterministic consumption streams that vary over time. Assume temporarily and for the purposes of exposition that $\delta > 0$ and $\vartheta > 0$ and define a new (probability) measure $\mathbb{Q} = \mathbb{Q}_\delta$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$ by

$$\mathbb{Q}_\delta(A) = \int_A \delta e^{-\delta t}\,\mathrm{d}t.$$

The choice of δ accounts for the agent's temporal preferences for consumption in the sense that the higher the value of δ, the greater the weighting on consumption which occurs earlier.

Now compare a (deterministic) consumption stream $c = (c(t))_{t\geq 0}$ with its $\mathbb{Q}_\delta$-average value $\mathbb{E}^{\mathbb{Q}_\delta}[c] = \int_0^\infty \delta e^{-\delta t} c(t)\,\mathrm{d}t$ which we suppose to be finite. From (2.6) we know that the value at time 0 is given is given by

$$V^c(0) = \frac{1}{1-R}\left(\frac{b}{\delta}\right)^\vartheta \left(\int_0^\infty \delta e^{-\delta t} c(t)^{1-S}\,\mathrm{d}t\right)^\vartheta = \vartheta\left(\frac{b}{\delta}\right)^\vartheta \frac{\left(\mathbb{E}^{\mathbb{Q}_\delta}\left[c^{1-S}\right]\right)^\vartheta}{1-S}.$$

Again, Jensen's inequality (and $\vartheta > 0$) gives $\frac{1}{1-S}(\mathbb{E}^{\mathbb{Q}_\delta}[c^{1-S}])^\vartheta \leq \frac{1}{1-S}[(\mathbb{E}^{\mathbb{Q}_\delta}[c])^{1-S}]^\vartheta$, which implies that $V_0^c \leq V_0^{\mathbb{E}^{\mathbb{Q}_\delta}[c]}$. Note that all of the variance aversion (after changing the Lebesgue measure to an equivalent probability measure) comes from S. This justifies considering S as the parameter governing aversion to variance over time. In the economics literature S is named the elasticity of intertemporal complementarity (EIC).

Note that if $(1-R)V_t < 0$ then the integrand on the right hand side of (2.4) is ill-defined for non-integer ρ. This justifies the choice $\mathbb{V} = (1-R)\mathbb{R}_+$. Further, the integrand is either positive ($S < 1$) or negative ($S > 1$). It is therefore necessary to impose a link between the co-efficient of RRA R and co-efficient of EIC S to ensure agreement in the sign of the left-hand-side of (2.4) and the right hand side. Recall that $\vartheta = \frac{1-R}{1-S}$.

Theorem 2.3. *For EZ-SDU over the infinite-horizon with aggregator given by* (2.2) *we must have* $\vartheta > 0$ *for there to exist solutions to* (2.4).

The condition $\vartheta > 0$, or equivalently $\rho \in (-\infty, 1)$ means that either both R and S are greater than unity, or both R and S are smaller than unity.

In the finite time horizon problem the parity issue can be overcome by adding a bequest function so that (2.4) is replaced by $V_t = \mathbb{E}[\int_t^T be^{-\delta s}\frac{C_s^{1-S}}{1-S}((1-R)V)^\rho + e^{-\delta T}\frac{B(X_T)}{1-R}|\mathcal{F}_t]$ where $B : \mathbb{R}_+ \mapsto \mathbb{R}_+$ assigns a value to terminal wealth. But, even over the finite horizon this leads to conceptual issues: for example, when $S < 1 < R$ the utility process is negative at time t, even though the term corresponding to consumption over (t, T) is everywhere positive, because this positive term is outweighed by the contribution from the bequest. Moreover if we let the terminal horizon tend to infinity the problem becomes even more stark—in order to outweigh the increasing (as terminal horizon T increases) contribution from consumption the contribution from the bequest must also grow, and must become more (not less) influential as the terminal horizon increases. In Section 4.2 we argue that in the limit $T \nearrow \infty$ we end up with bubble-like behaviour which cannot be justified economically, and which is not consistent with any notion of transversality. This further justifies the requirement $\vartheta > 0$.

3 Optimal investment and consumption in a Black–Scholes–Merton financial market

3.1 The financial market and attainable consumption streams

The financial market that we consider in this chapter (and throughout the thesis) is the Black–Scholes–Merton financial market given in Section I.4 on page 13. It is shown in Section I.4 that, given such a financial market, we can define the set of attainable consumption streams $\mathscr{C}(x)$ to be the consumption streams that the agent can self finance from initial wealth $x > 0$.

The goal of an agent with Epstein–Zin stochastic differential utility preferences is to maximise $J_{g_{EZ}}(C)$ over attainable consumption streams. However, $J_{g_{EZ}}(C)$ is currently only defined for $C \in \mathcal{E}_u(g_{EZ})$ and therefore, we can currently only optimise over uniquely evaluable consumption streams. Thus, we seek to find

$$V^*_{\mathscr{E}_u(g_{EZ})}(x) = \sup_{C \in \mathscr{C}(x) \cap \mathscr{E}_u(g_{EZ})} V_0^C = \sup_{C \in \mathscr{C}(x) \cap \mathscr{E}_u(g_{EZ})} J_{g_{EZ}}(C). \tag{3.1}$$

This is very restrictive. For $\vartheta > 1$, one can show that $\mathscr{E}_u(g_{EZ}) = \{0\}$ and so the problem (3.1) is meaningless. Further, even when $\vartheta \in (0,1)$, there are many attainable consumptions streams which are nonevaluable and therefore to which we currently cannot assign them a utility. For example, when $S > 1$, the zero consumption stream is not evaluable. Since it might reasonably be argued that the zero consumption stream is clearly suboptimal (and when $S > 1$ should give a utility process with negative infinite utility), we would like to eliminate this choice of consumption stream because it is suboptimal and not because we cannot evaluate it. The same applies to other nonevaluable consumption streams. Ideally, we would like *every* attainable consumption stream to be considered, and not just the "nice" ones for which we can define a unique utility process. For $\vartheta \in (0,1)$, this problem will be considered in Chapter IV.

3.2 Changes of numéraire

One apparent advantage of the difference form g^{Δ}_{EZ} of the EZ-SDU aggregator given in (2.3) over the discounted form g_{EZ} given in (2.2) is that g^{Δ}_{EZ}, unlike g_{EZ}, has no explicit time-dependence, i.e. $g^{\Delta}_{EZ} = g^{\Delta}_{EZ}(c,v)$ whereas $g_{EZ} = g_{EZ}(t,c,v)$. However, when we consider EZ-SDU in the constant parameter Black–Scholes–Merton model a simple change of accounting unit leads to a modification of the discount factor δ, but leaves the problem otherwise unchanged. It follows that by an appropriate choice of units we can switch to a coordinate system in which the aggregator becomes time-independent. The change of accounting units has an effect upon the financial

market model, but it remains a Black–Scholes–Merton financial market, albeit with modified interest rate and market drift.

Let C be a consumption stream with corresponding utility process V for g_{EZ}. Let $\chi \in \mathbb{R}$ and define the the *discounted consumption stream* $\tilde{C}$ by $\tilde{C}_t = e^{-\chi t} C_t$. Then, V satisfies

$$\begin{aligned} V_t &= \mathbb{E}\left[\int_t^\infty b e^{-\delta s} \frac{C_s^{1-S}}{1-S} \left((1-R)V_s\right)^\rho \, \mathrm{d}s \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^\infty b e^{-(\delta - \chi(1-S))t} \frac{\tilde{C}_s^{1-S}}{1-S} \left((1-R)V_s\right)^\rho \, \mathrm{d}s \middle| \mathcal{F}_t\right]. \end{aligned}$$

This implies V is the utility process for $\tilde{C}$ with the aggregator $g_{\chi,EZ}$ defined by

$$g_{\chi,EZ}(t,c,v) = b e^{-(\delta - \chi(1-S))t} \frac{c^{1-S}}{1-S} \left((1-R)v\right)^\rho.$$

Choosing $\chi = \frac{\delta}{1-S}$, we find that V is the utility process for the time independent aggregator

$$f_{EZ} = f_{EZ}(c,v) = g_{\chi,EZ}(t,c,v) = b \frac{c^{1-S}}{1-S} \left((1-R)v\right)^\rho.$$

Furthermore, $V \in \mathbb{I}(f_{EZ}, \tilde{C} = (C_t e^{-\frac{\delta}{1-S}t})_{t\geq 0})$ if and only if $V \in \mathbb{I}(g_{EZ}, C) = \mathbb{I}(g_{0,EZ}, C)$ and $\tilde{C} \in \mathscr{E}_u(f_{EZ})$ in and only if $X \in \mathscr{E}_u(g_{EZ})$.

If we consider the discounted wealth process $\tilde{X}_t^{\Pi,\tilde{C}} := e^{-\frac{\delta}{1-S}t} X_t^{\Pi,C}$ then, by applying Itô's lemma, we find that with $\tilde{r} = r - \frac{\delta}{1-S}$ and $\tilde{\mu} = \mu - \frac{\delta}{1-S}$,

$$d\tilde{X}_t^{\Pi,\tilde{C}} = \tilde{X}_t^{\Pi,\tilde{C}} \Pi_t \sigma \, \mathrm{d}B_t + \left(\tilde{X}_t^{\Pi,\tilde{C}} \left(\tilde{r} + \Pi_t(\tilde{\mu} - \tilde{r})\right) - \tilde{C}_t\right) \mathrm{d}t, \quad \tilde{X}_0^{\Pi,\tilde{C}} = x.$$

This means that our control problem (3.1) admits the equivalent formulation,

$$\begin{aligned} V^*_{\mathscr{E}_u(g_{EZ})}(x) &= \sup_{C \in \mathscr{C}(x;r,\mu,\sigma) \cap \mathscr{E}_u(g_{EZ})} V_0^{C,g_{EZ}} \\ &= \sup_{\tilde{C} \in \mathscr{C}(x;\tilde{r},\tilde{\mu},\sigma) \cap \mathscr{E}_u(f_{EZ})} V_0^{\tilde{C},f_{EZ}} \quad = \quad V^*_{\mathscr{E}_u(f_{EZ})}(x). \end{aligned}$$

In particular, by an appropriate change of accounting units the problem for EZ-SDU in discounted form reduces to an equivalent form with no discounting. This simplification result will be used extensively in Chapter IV on existence and uniqueness, but whilst we are comparing and contrasting the discounting and difference forms we will continue to allow δ to be any real number.

3.3 The candidate optimal strategy

Suppose now $\vartheta > 0$. We seek to heuristically find an admissible (and uniquely evaluable) consumption stream C that maximises the value of V_0^C, where

$$V_t^C = \mathbb{E}\left[\int_t^\infty be^{-\delta s}\frac{C_s^{1-S}}{1-S}\left((1-R)V_s^C\right)^\rho \,\mathrm{d}s\middle|\mathcal{F}_t\right]. \tag{3.2}$$

As in the Merton problem with CRRA utility, it is reasonable to expect that the optimal strategy is to invest a constant proportion of wealth in the risky asset and consume a constant proportion of wealth. Consider the investment-consumption strategy $\Pi \equiv \pi \in \mathbb{R}$ and $C \equiv \xi X$ for $\xi \in \mathbb{R}_{++}$. Then, solving (I.4.2), the wealth process $X^{x,\pi,\xi} = X = (X_t)_{t\geq 0}$ is given by $X_t = x\exp\left(\pi\sigma B_t + \left(r + \pi(\mu - r) - \xi - \frac{\pi^2\sigma^2}{2}\right)t\right)$, and then, for $s > t$

$$X_s^{1-R} = x^{1-R}\exp\left(\pi\sigma(1-R)B_t + (1-R)\left(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}\right)t\right). \tag{3.3}$$

As in the Merton problem, consider a value process of the form $V_t = V(t, X_t) = Ae^{-\beta t}\frac{X_t^{1-R}}{1-R}$ for some constant β to be determined. Substituting this expression into (3.2), and using $1 - S + \rho(1-R) = 1 - R$ yields

$$\begin{aligned} V_t &= \mathbb{E}\left[\int_t^\infty be^{-\delta s}\frac{(\xi X_s)^{1-S}}{1-S}\left(Ae^{-\beta s}X_s^{1-R}\right)^\rho \,\mathrm{d}s\middle|\mathcal{F}_t\right] \\ &= bA^\rho\frac{\xi^{1-S}}{1-S}\mathbb{E}\left[\int_t^\infty e^{-(\delta+\beta\rho)s}X_s^{1-R}\,\mathrm{d}s\middle|\mathcal{F}_t\right] \end{aligned} \tag{3.4}$$

Then, for $s > t$, $\mathbb{E}[e^{-(\delta+\beta\rho)s}X_s^{1-R}|\mathcal{F}_t] = e^{-(\delta+\beta\rho)t}X_t^{1-R}e^{-H_{\delta+\beta\rho}(\pi,\xi)(s-t)}$, where for $\nu \in \mathbb{R}$, $H_\nu : \mathbb{R} \times \mathbb{R}_{++} \mapsto \mathbb{R}$ is given by

$$H_\nu(\pi,\xi) \;=\; \nu + (R-1)\left(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}R\right). \tag{3.5}$$

Remark 3.1. If we consider the constant proportional investment-consumption (π,ξ), then the drift of $(e^{-\nu t}X_t^{1-R})_{t\geq 0}$ is given by $-H_\nu(\pi,\xi)$. This means that $H_\nu(\pi,\xi)$ is a critical quantity for both the well-definedness of the integral $\mathbb{E}[\int_0^\infty e^{-\nu t}X_t^{1-R}\,\mathrm{d}t]$ and the transversality condition $\lim_{t\to\infty}\mathbb{E}[e^{-\nu t}X_t^{1-R}] = 0$ which will feature heavily in Section 5.

Provided that $H_{\delta+\beta\rho}(\pi,\xi) > 0$, so that the integral in (3.4) is well-defined, it follows that

$$V_t = \frac{be^{-(\delta+\beta\rho)t}A^\rho\xi^{1-S}}{H_{\delta+\beta\rho}(\pi,\xi)}\frac{X_t^{1-R}}{1-S}.$$

Since V was postulated to be of the form $V_t = Ae^{-\beta t}\frac{X_t^{1-R}}{1-R}$, it must be the case that $\beta = \delta + \beta\rho$ (i.e. $\beta = \delta\vartheta$) and $A = A(\pi,\xi) = \left(\frac{b\vartheta\xi^{1-S}}{H_\beta(\pi,\xi)}\right)^\vartheta > 0$. Then, $\delta + \beta\rho = \delta\vartheta$ and $H := H_{\delta\vartheta}$ satisfies

$$H(\pi,\xi) = \delta\vartheta + (R-1)\left(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}R\right).$$

It follows that any proportional investment strategy ($\Pi = \pi,\ C = \xi X$) is evaluable provided that $H(\pi,\xi)$ is positive.

To find the optimal strategy amongst constant proportional strategies (and hence to find the candidate optimal strategy) it remains to maximise $\frac{A(\pi,\xi)}{1-R}$ over $(\pi,\xi) \in \mathbb{R}\times\mathbb{R}_{++}$ such that $H(\pi,\xi) > 0$. There is a turning point of $\frac{A(\pi,\xi)}{1-R} = \frac{1}{1-R}\left(\frac{b\vartheta\xi^{1-S}}{H(\pi,\xi)}\right)^\vartheta$ at $(\hat{\pi},\hat{\xi}) = (\frac{\lambda}{\sigma R},\eta)$ where

$$\eta = \frac{1}{S}\left(\delta + (S-1)r + (S-1)\frac{\lambda^2}{2R}\right) \tag{3.6}$$

and this point is such that $H(\hat{\pi},\hat{\xi}) = H(\frac{\lambda}{\sigma R},\eta) > 0$ provided $\eta > 0$. Under the

condition $\eta > 0$ it is easily checked that $(\pi = \frac{\lambda}{\sigma R}, \xi = \eta)$ is a maximum of $\frac{A(\pi,\xi)}{1-R}$ over $\{(\pi,\xi) : H(\pi,\xi) > 0\}$; it then follows that $\max_{\{\xi>0:H(\hat{\pi},\xi)>0\}} V_0 = b^{\vartheta}\eta^{-\vartheta S}\frac{x^{1-R}}{1-R}$. Considering this as a function of the initial wealth, for $\eta > 0$ the candidate value function is defined by

$$\hat{V}(x) = b^{\vartheta}\eta^{-\vartheta S}\frac{x^{1-R}}{1-R}. \tag{3.7}$$

The results of this section are summarised in the following proposition:

Proposition 3.2. *Define $D = \{(\pi,\xi) \in \mathbb{R} \times \mathbb{R}_+ : H(\pi,\xi) > 0\}$. Consider constant proportional strategies with parameters $(\pi,\xi) \in D$. Suppose $\vartheta > 0$ and $\eta > 0$, where η is given in* (3.6)*.*

(*i*) *For $(\pi,\xi) \in D$, one solution $V = (V_t)_{t\geq 0}$ to* (3.2) *is given by*

$$V_t = e^{-\delta\vartheta t}\left(\frac{b\vartheta\xi^{1-S}}{H(\pi,\xi)}\right)^{\vartheta}\frac{X_t^{1-R}}{1-R}. \tag{3.8}$$

(*ii*) *The global maximum of $h(\pi,\xi) = \frac{1}{1-R}\left(\frac{b\vartheta\xi^{1-S}}{H(\pi,\xi)}\right)^{\vartheta}$ over the set D is attained at $(\pi,\xi) = (\frac{\lambda}{\sigma R}, \eta)$ and the maximum is $\frac{b^{\vartheta}\eta^{-\vartheta S}}{1-R}$.*

(*iii*) *The optimal strategy for* (3.8) *is $(\hat{\pi},\hat{\xi}) = (\frac{\lambda}{\sigma R}, \eta)$ and satisfies $\hat{V}_0 = b^{\vartheta}\eta^{-\vartheta S}\frac{x^{1-R}}{1-R} = \hat{V}(x)$, where x denotes initial wealth.*

The candidate well-posedness condition for the investment-consumption problem is $\eta > 0$, where η is given in (3.6). We shall see in Corollary IV.4.3 that when $\vartheta \in (0,1)$ this is a necessary and sufficient condition for the well-posedness of the problem. The agent's (candidate) optimal investment in this case is a constant fraction $\hat{\pi} = \frac{\lambda}{\sigma R}$ of their wealth—a proportion which is independent of their EIC. The agent's investment preferences are controlled solely by the risk aversion coefficient R. The agent's (candidate) optimal consumption is a constant proportion η of their wealth.

To interpret η, it is insightful to perform a change of numèraire. As in Section

II.B, the problem may be rewritten in equivalent form as

$$V_t = \mathbb{E}\left[\int_t^\infty \frac{be^{-(\delta+r(S-1))s}}{1-S}\left(\frac{C_s}{S_s^0}\right)^{1-S}((1-R)V_s)^\rho \, \mathrm{d}s \middle| \mathcal{F}_t\right].$$

With this in mind, it makes sense to call $\phi := \delta + r(S-1)$ the *impatience rate*. Then, the optimal proportional consumption rate is given by

$$\eta = \frac{\phi}{S} + \frac{S-1}{S}\frac{\lambda^2}{2R}.$$

This is a linear (convex if $S > 1$) combination of the impatience rate and (half of) the squared Sharpe ratio per unit of risk aversion, with the weights depending on the elasticity of intertemporal complementarity S.

Remark 3.3. The well-posedness condition $\eta > 0$ is equivalent to $\delta > (1-S)\left(r + \frac{\lambda^2}{2R}\right)$ (or $\phi > (1-S)\frac{\lambda^2}{2R}$). This means that when $S > 1$ (or $r < 0$), the problem can be well-posed even for negative values of δ (or ϕ).

Remark 3.4. When $\vartheta > 1$, uniqueness of a utility process fails (for example $V_t = 0$ always solves (3.2)). In this case, the first issue is to decide which utility process to associate to a consumption stream; this in turn has implications for the optimal value function and optimal consumption stream, and ultimately for the well-posedness of the problem. Since this is a delicate issue and deserves a full discussion, we postpone it to a later chapter covering the case $\vartheta > 1$.

4 A comparison of the discounted and difference formulations

The goal of this section is to compare the discounted and difference formulations of the aggregator for EZ-SDU. Despite the ubiquity of the latter in the literature, we will argue that the discounted form has many advantages. As demonstrated in Section 3.2, its main disadvantage, the fact that it has an explicit dependence on

time, is easily overcome by a change in accounting unit.

4.1 The difference form of CRRA utility

Additive utilities such as CRRA may be thought of as special cases of SDU in which the aggregator has no dependence on v. In this sense CRRA utility may be identified with the aggregator

$$g_{CRRA}(t,c,v) = g_{CRRA}(t,c) = e^{-\delta t}\frac{c^{1-R}}{1-R}.$$

Note that, provided $\mathbb{E}\left[\int_0^\infty e^{-\delta s}\left|C_s^{1-R}\right| ds\right] < \infty$, it follows that

$$V_t^C = \mathbb{E}\left[\int_t^\infty e^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \tag{4.1}$$

is the unique utility process associated with consumption C for aggregator g_{CRRA} and then $J_{g_{CRRA}}(C) = V_0^C$. Further, if $\mathbb{E}[\int_0^\infty e^{-\delta s}|C_s^{1-R}|ds] = \infty$ we can set $J(C) = \infty$ if $R < 1$ and $J(C) = -\infty$ if $R > 1$.

In particular, two subtle but important questions which are crucial to the study of SDU are absent from the additive utility setting: first, what value to assign to nonevaluable strategies, and second which utility process to assign to consumptions which are not uniquely evaluable.

Suppose C is such that $\mathbb{E}\left[\int_0^\infty e^{-\delta s}\left|C_s^{1-R}\right| ds\right] < \infty$. Then, the martingale $M = (M_t)_{0\le t\le\infty}$ given by $M_t := \mathbb{E}\left[\int_0^\infty e^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$ is uniformly integrable and satisfies $M_t = \int_0^t e^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s + V_t$ where V is the utility process in (4.1). Using that $M_\infty = \int_0^\infty e^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s$ and rearranging, we find that $V_t = \int_t^\infty e^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s - \int_t^\infty \mathrm{d}M_t$. Then, applying Itô's formula to V^Δ given by $V_t^\Delta := e^{\delta t}V_t$ and integrating yields $V_t^\Delta = \int_t^\infty\left(\frac{C_s^{1-R}}{1-R} - \delta V_s^\Delta\right)\mathrm{d}s + \int_t^\infty e^{\delta s}\,\mathrm{d}M_s$, provided such a solution is well-defined. Taking expectations, and assuming that $M^\delta = (M_t^\delta)_{t\ge 0}$ given by $M_t^\delta = \int_0^t e^{\delta s}dM_s$ is a uniformly integrable martingale we get the *difference form*

of discounted expected utility,

$$V_t^\Delta = \mathbb{E}\left[\int_t^\infty \left(\frac{C_s^{1-R}}{1-R} - \delta V_s^\Delta\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t\right]. \tag{4.2}$$

Modulo the technical issues, under CRRA preferences, it is possible to define the value associated to a consumption stream C as the initial value V_0^Δ of the utility process $V^\Delta = (V_t^\Delta)_{t\geq 0}$ where V^Δ solves (4.2), rather than using (4.1). However, doing so brings several immediate disadvantages. It is no longer obvious if solutions to (4.2) are unique or even exist. This may result in a smaller class of evaluable strategies. Indeed there are simple deterministic counter-examples to existence of a solution to (4.2), see Example 4.1. The counterexamples arise because the integrand $\frac{C_s^{1-R}}{1-R} - \delta V_s^\Delta$ takes both signs and so the integral on the right hand side of (4.2) may not be well-defined. (In contrast, $\mathbb{E}[\int_0^\infty e^{-\delta s}\frac{C_s^{1-R}}{1-R}]$ is always well defined, at least in $[-\infty,\infty]$.) Further, whenever $\mathbb{E}[\int_0^\infty e^{-\delta s}|C_s^{1-R}|ds] < \infty$, we have that M is a uniformly integrable martingale. But M^δ may not be uniformly integrable, and the representation (4.2) may fail.

Example 4.1. Suppose $\delta > 0$ and let $A = \cup_{n\geq 0}[2n, 2n+1)$. Consider the deterministic consumption stream $c = (c(t))_{t\geq 0}$ which satisfies

$$U(c(t)) := \frac{c(t)^{1-R}}{1-R} = \frac{2\delta}{1-R} e^{\delta(\lceil t\rceil - t)} \mathbf{1}_{A^c}(t).$$

It is easily checked (consider the cases $t \in A$ and $t \in A^c$ separately) that V^Δ defined by $V^\Delta(t) = \frac{1}{1-R} e^{\delta(t-\lfloor t\rfloor)(\mathbf{1}_A(t) - \mathbf{1}_{A^c}(t))}$ satisfies $\mathrm{d}V^\Delta(t) = \left[\delta V^\Delta(t) - \frac{c(t)^{1-R}}{1-R}\right] \mathrm{d}t$ (at least for non-integer t).

Clearly, $\int_t^\infty \left(\frac{c(s)^{1-R}}{1-R} - \delta V^\Delta(s)\right) \mathrm{d}s$ is *not* well-defined since both the positive part and the negative part are infinite and hence it is *not* the case that V^Δ solves $V^\Delta = \int_t^\infty \left(\frac{c(s)^{1-R}}{1-R} - \delta V^\Delta(s)\right) \mathrm{d}s$. On the other hand, $V(t) = e^{-\delta t} V^\Delta(t)$ is a solution to the discounted formulation $V(t) = \int_t^\infty e^{-\delta s}\frac{c(s)^{1-R}}{1-R} \mathrm{d}s$. (Note that since $U(c(s))$ is bounded and $\delta > 0$, $V(0)$ is finite.)

Thus, if we set $g_{CRRA}^\Delta(t,c,v) = \frac{c^{1-R}}{1-R} - \delta v$ and $g_{CRRA} = e^{-\delta t}\frac{c^{1-R}}{1-R}$, then $\mathscr{E}(g_{CRRA}^\Delta) \subsetneq$

$\mathscr{E}(g_{CRRA})$. In particular, there are consumption streams which can be evaluated under the formulation (4.1) but which cannot be evaluated using (4.2).

4.2 The difference form of Epstein–Zin stochastic differential utility

In the previous section we argued that for additive CRRA preferences, the discounted form was better than the difference form for three reasons: first, existence and uniqueness of the utility process are guaranteed; second, there is a wider class of consumption streams to which it is possible to assign a (finite) value; and third, it is possible to assign a value (possibly infinite) to any consumption stream even when $\int_0^\infty g_{CRRA}(s, C_s)ds$ is not integrable. The goal in this section is to show that, although the first property in this list no longer applies, when we move to EZ-SDU preferences the second and third advantages of the discounted form remain. Indeed, much of the discussion is as in the additive case.

Suppose that $C \in \mathscr{E}_u(g_{EZ})$ and set $M_t := \mathbb{E}[\int_0^\infty be^{-\delta s}\frac{C_s^{1-S}}{1-S}\left((1-R)V_s\right)^\rho \,\mathrm{d}s|\mathcal{F}_t]$. After a re-arrangement, (3.2) becomes

$$\begin{aligned} V_t &= M_t - \int_0^t be^{-\delta s}\frac{C_s^{1-S}}{1-S}\left((1-R)V_s\right)^\rho \,\mathrm{d}s \\ &= \int_t^\infty be^{-\delta s}\frac{C_s^{1-S}}{1-S}\left((1-R)V_s\right)^\rho \,\mathrm{d}s \; - \int_t^\infty \mathrm{d}M_s. \end{aligned}$$

Furthermore, applying Itô's lemma to the upcounted utility process $V^\Delta = (V_t^\Delta)_{t\geq 0}$ defined by $V_t^\Delta := e^{\delta\vartheta t}V_t$, we find that V^Δ satisfies[3]

$$V_t^\Delta = \int_t^\infty \left(b\frac{C_s^{1-S}}{1-S}\left((1-R)V_s^\Delta\right)^\rho - \delta\vartheta V_s^\Delta\right)\mathrm{d}s \; - \int_t^\infty e^{\delta\vartheta s}\,\mathrm{d}M_s,$$

and we may reasonably hope to be able to define the (upcounted) utility process as

[3]Here, we have assumed that all integrals are well-defined and finite. In practice this is not always true.

the solution to

$$V_t^{\Delta} = \mathbb{E}\left[\int_t^{\infty}\left(b\frac{C_s^{1-S}}{1-S}\left((1-R)V_s^{\Delta}\right)^{\rho} - \delta\vartheta V_s^{\Delta}\right)\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]. \tag{4.3}$$

This is the utility process associated to the difference form of the Epstein–Zin aggregator, g_{EZ}^{Δ}.

As in Section 4.1, for some consumption streams (4.3) is not well defined because the integrand takes both positive and negative values. If the utility process is defined via the difference aggregator g_{EZ}^{Δ} then it is necessary to restrict the class of consumption streams, when compared with those which can be evaluated under g_{EZ}.

Example 4.2. This example is similar to Example 4.1. Recall the definition of A, and consider the deterministic consumption stream $c = (c(t))_{t\geq 0}$ such that $\frac{c(t)^{1-S}}{1-S} := 2\frac{\delta}{b(1-S)}e^{\delta(\lceil t\rceil - t)}\mathbf{1}_{A^c}(t)$. Let $V^{\Delta} = (V^{\Delta}(t))_{t\geq 0}$ be given by $V^{\Delta}(t) = \frac{1}{1-R}\exp(\delta\vartheta(t - \lfloor t\rfloor)(\mathbf{1}_A(t) - \mathbf{1}_{A^c}(t)))$. Then,

$$\mathrm{d}V^{\Delta}(t) = \left[\delta\vartheta V^{\Delta}(t) - b\frac{c(t)^{1-S}}{1-S}((1-R)V^{\Delta}(t))^{\rho}\right]\mathrm{d}t.$$

For this consumption stream, both the positive and negative part of the integral

$$\int_t^{\infty}\left(b\frac{c(t)^{1-S}}{1-S}((1-R)V^{\Delta}(t))^{\rho} - \delta\vartheta V^{\Delta}(s)\right)\mathrm{d}s = \int_t^{\infty}\delta\vartheta V^{\Delta}(s)\left[\mathbf{1}_A(s) - \mathbf{1}_{A^c}(s)\right]\mathrm{d}s$$

are infinite for all $t \geq 0$. Hence, it cannot be the case that V^{Δ} solves (4.3). On the other hand, if $V(t) = e^{-\delta\vartheta t}V^{\Delta}(t)$, then

$$\int_0^{\infty} be^{-\delta t}\frac{c(t)^{1-S}}{1-S}((1-R)V(t))^{\rho}\,\mathrm{d}t = \int_0^{\infty} 2e^{-\delta t}\frac{\delta}{1-S}e^{\delta\vartheta(\lceil t\rceil - t)}\mathbf{1}_{A^c}(t)\,\mathrm{d}t < \infty$$

and $V = (V(t))_{t\geq 0} \in \mathbb{I}(g_{EZ}, c)$. Furthermore, it can be shown that V solves (3.2). Thus, $\mathcal{E}(g_{EZ}^{\Delta}) \subsetneq \mathcal{E}(g_{EZ})$.

5 Alternative formulations of SDU

5.1 A family of finite-horizon problems

Our approach to investment-consumption problems for EZ-SDU over the infinite horizon differs from the conventional approach in two important ways. First, we use the discounted aggregator given by (2.2) whereas the standard approach is to use the difference form. Second, we define the value function over the infinite horizon directly (with the natural transversality condition that the value process tends to zero in expectation following as a consequence), whereas the standard approach (formulated by Duffie, Epstein and Skiadas in the appendix to [DE92], and developed further by Melnyk et al. [MMKS20]) is to look for utility processes which solve a family of finite-horizon problems (where now the form of the transversality condition is not so clear, and may be part of the definition of a utility process). We have already compared the aggregators, so the goal in this section is to explain why we believe that it is better to define utility processes over the infinite horizon directly, and why, as a corollary, parameter combinations corresponding to $\vartheta < 0$ cannot make economic sense.

For the sake of exposition, we introduce some additional pieces of notation. Fix an aggegrator g and $C \in \mathscr{P}_+$. Then, for $T > 0$, let

$$\mathbb{I}_T(g, C) = \left\{ W \in \mathscr{P} : \int_0^T |g(s, C_s, W_s)| ds < \infty \right\}$$

and $\mathbb{J}_T = \mathbb{J}_T(g, C)$ be a subset of $\mathbb{I}_T(g, C)$ such that elements of $\mathbb{J}_T$ have additional regularity and/or integrability properties. Let $\mathbb{J} := \bigcap_{T>0} \mathbb{J}_T$. Examples of suitable sets $\mathbb{J}_T$ will be given below.

As an alternative to defining utility processes directly over the infinite horizon, [DE92] and [MMKS20] define utility processes as solutions to a family of finite-horizon problems.

Definition 5.1. V is the $(\nu, \mathbb{J})$-utility process associated to the consumption stream

C and aggregator g if it has càdlàg paths, lies in $\mathbb{J}$, satisfies the transversality condition $\lim_{t\to\infty} e^{-\nu t}\mathbb{E}[|V_t|] = 0$, and for all $0 \le t \le T < \infty$,

$$V_t = \mathbb{E}\left[\int_t^T g(s, C_s, V_s)\,\mathrm{d}s + V_T \middle| \mathcal{F}_t\right]. \tag{5.1}$$

Remark 5.2. It follows as in Remark 1.2 that a $(\nu, \mathbb{J})$-utility process is automatically a semimartingale.

Let $\mathscr{E}^{\nu,\mathbb{J}}(g)$ be the set of consumption streams C such that there exists a $(\nu, \mathbb{J})$-utility process associated to C for aggregator g, and let $\mathscr{E}_u^{\nu,\mathbb{J}}(g)$ be the subset of $\mathscr{E}^{\nu,\mathbb{J}}(g)$, where there exists a exists a unique $(\nu, \mathbb{J})$-utility process. Moreover, let $\mathscr{C}_0(x)$ be some subset of $\mathscr{C}(x)$, the set of attainable consumption streams given initial wealth x. Additional regularity conditions on the consumption streams may be encoded in $\mathscr{C}_0(x)$.

In order to avoid the technical challenges of dealing with the infinite-horizon problem directly, the idea in [DE92, MMKS20] is to replace the problem of finding $V(x)$ with the problem of finding $V_{\mathscr{C}_0,\mathscr{E}_u^{\nu,\mathbb{J}}(g)}(x) = \sup_{C\in\mathscr{C}_0(x)\cap\mathscr{E}_u^{\nu,\mathbb{J}}(g)} V_0^C$, for an appropriate transversality parameter ν and appropriate sets $\mathscr{C}_0(x)$ and $\mathbb{J}$. But this immediately raises several issues. What exactly are the spaces $\mathscr{C}_0(x)$, $\mathscr{E}^{\nu,\mathbb{J}}(g)$ and $\mathscr{E}_u^{\nu,\mathbb{J}}(g)$? How do we (easily) check whether $C \in \mathscr{C}_0(x)$ and/or $C \in \mathscr{E}_u^{\nu,\mathbb{J}}(g)$?

Regarding the choice of transversality condition, the issue crystalises as: first, how do we know that $\mathscr{E}^{\nu,\mathbb{J}}(g)$ is nonempty?; second, how do we know that a utility process V associated with a consumption C makes economic sense? As regards the first issue, if $\nu < \nu'$, any $(\nu, \mathbb{J})$-utility process is also a $(\nu', \mathbb{J})$-utility process. Hence, $\mathscr{E}^{\nu,\mathbb{J}}(g) \subseteq \mathscr{E}^{\nu',\mathbb{J}}(g)$ and if ν is chosen too small, then it may easily follow that $\mathscr{E}^{\nu,\mathbb{J}}(g)$ does not include the candidate optimal solution. As regards the second issue, in Section 5.2 below we introduce the concept of a *bubble solution* and argue that bubble solutions do not make economic sense.

Duffie et al. [DE92] impose Lipschitz-style conditions which exclude EZ-SDU. Melnyk et al. [MMKS20] do study EZ-SDU but the main focus of [MMKS20] is to

understand the impact of market frictions on the investment-consumption problem for SDU-preferences. Nonetheless, in the frictionless case which is the subject of this thesis, Melnyk et al. prove some of the most complete results for Epstein–Zin preferences currently available in the literature. Melnyk et al. [MMKS20] only consider $R > 1$ but this is mainly to limit the number of cases rather than because their methods do not extend to the general case.

Definition 5.3 (Melnyk et al. [MMKS20, Definition 3.1])**.** Suppose $R > 1$ and $\delta > 0$. For $T > 0$, let

$$\begin{aligned}
\mathbb{S}^1_T &= \{V : V \in \mathscr{S} \text{ with } \mathbb{E}\left[\sup_{0\le t\le T} |V_t|\right] < \infty\} \\
\mathbb{J}^1_T &= \mathbb{S}^1_T \cap \mathbb{I}_T(g^{\Delta}_{EZ}, C). \\
\mathbb{J}^2_T &= \left\{V : V \in \mathbb{J}^1_T : V_t \le -\tfrac{C_t^{1-R}}{R-1} \le 0 \ \text{ for all } 0 \le t \le T\right\}.
\end{aligned}$$

For $k \in \{1, 2\}$ set $\mathbb{J}^k := \bigcap_{T>0} \mathbb{J}^k_T$ and let $\mathscr{C}_0(x)$ be the set of $C \in \mathscr{C}(x)$ for which there exists Π such that $\Pi(X^{x,\Pi,C})^{1-R} \in \mathbb{S}^1_T$ for all $T > 0$ and $\frac{1}{1-R}(X^{x,\Pi,C})^{1-R} \in \mathbb{J}^1$. Moreover, if $0 < \vartheta < 1$, set $\mathbb{J}^{MMS} := \mathbb{J}^1$ and $\mathscr{E}^{MMS} = \mathscr{E}^{MMS}(g^{\Delta}_{EZ}) := \mathscr{E}^{\delta\vartheta,\mathbb{J}^{MMS}}(g^{\Delta}_{EZ})$; if $\vartheta > 1$ or $\vartheta \in (-\infty, 0)$, set $\mathbb{J}^{MMS} := \mathbb{J}^2$ and $\mathscr{E}^{MMS} = \mathscr{E}^{MMS}(g^{\Delta}_{EZ}) := \mathscr{E}^{\delta,\mathbb{J}^{MMS}}(g^{\Delta}_{EZ})$.

Note that as we move from $\vartheta \in (0, 1)$ to $\vartheta \notin (0, 1)$ the transversality parameter ν changes from $\delta\vartheta$ to δ. Moreover, an additional restriction that $V \le -\frac{C^{1-R}}{R-1}$ is imposed.

Melnyk et al. [MMKS20] take $b = \delta$. Then, from (3.7) we have that for $\eta > 0$ the candidate value function is given by $\hat{V}(x) = \eta^{-\vartheta S}\delta^{\vartheta}\frac{x^{1-R}}{1-R}$.

Theorem 5.4 (Melnyk et al. [MMKS20, Corollary 2.3, Theorem 3.4])**.** *Suppose $R > 1$ and $\delta > 0$. Then, $\mathscr{E}^{MMS} = \mathscr{E}^{MMS}_u$. Moreover, suppose $\frac{\mu-r}{R\sigma^2} \notin \{0, 1\}$ and $\eta > 0$.*

(i) If $\vartheta \in (0, 1)$ (i.e. $1 < R < S$), then $V_{\mathscr{C}_0,\mathscr{E}^{MMS}_u}(x) = \hat{V}(x)$.

(ii) If $\vartheta \in (1, \infty)$ (i.e. $1 < S < R$) and $\frac{R-S}{R-1}\delta < \eta < \delta$, then $V_{\mathscr{C}_0,\mathscr{E}^{MMS}_u}(x) = \hat{V}(x)$.

(iii) If $\vartheta \in (-\infty, 0)$ *(i.e.* $S < 1 < R$*) and* $\delta < \eta < \delta \frac{R-S}{R-1}$*, then* $V_{\mathscr{C}_0, \mathscr{E}_u^{MMS}}(x) = \hat{V}(x)$.

The results of Melnyk et al. [MMKS20] on the frictionless problem are amongst the few rigorous results on the investment-consumption problem over the infinite horizon. Nonetheless, they are incomplete in several respects. For all values of ϑ, there is no existence result; although it is possible (at least under the conditions of the theorem) to verify that the candidate optimal consumption stream is a member of $\mathscr{C}_0(x) \cap \mathscr{E}_u^{MMS}$, in general little is said about which consumption streams are evaluable by Definition 5.3, and it is unclear if the space of evaluable strategies goes beyond the set of constant proportional strategies. The fact that the wealth process must satisfy transversality and integrability conditions means that many plausible consumption streams are excluded by assumption, rather than because they are sub-optimal.

When $\vartheta \notin (0, 1)$ there are additional issues. In that case, the transversality condition in Definition 5.3 is that $\nu = \delta$. This condition leads to simple mathematics, but does not necessarily make economic sense—in Section 5.3 we will argue that the economically-correct transversality condition is $\nu = \delta\vartheta$. Moreover, the restriction to consumption streams for which there exists a utility processes with $V \leq \frac{1}{1-R} C^{1-R}$ seems both hard to verify in general and hard to interpret. Finally, the analysis in [MMKS20] leaves several parameter combinations uncovered, including the case $\{\vartheta > 1, \eta \in (0, \delta\rho] \cup [\delta, \infty)\}$.

Although the space $\mathscr{E}^{MMS}$ is difficult to describe, the following result, whose proof is given in Appendix IV.D, says that if C has an associated utility process in the sense of Melnyk et al., then automatically it has an associated utility process in the sense of a solution to (1.1). The converse is not true.

Proposition 5.5. *Suppose* $\vartheta \in (0, 1)$ *or* $\vartheta \in (1, \infty)$ *and suppose* $\delta > 0$*. Suppose* $C \in \mathscr{E}^{MMS}$ *and let* V^Δ *be a* $(\delta\vartheta, \mathbb{J}^{MMS})$*-utility process associated to consumption stream* C *and aggregator* g_{EZ}^Δ*. Then,* V *given by* $V_t = e^{\delta\vartheta t} V_t^\Delta$ *is a utility process associated to consumption stream* C *and aggregator* g_{EZ} *in the sense of Definition 1.1. In particular,* $\mathscr{E}^{MMS}(g_{EZ}^\Delta) \subset \mathscr{E}(g_{EZ})$.

Although Melnyk et al. [MMKS20] also define utility processes in the case $\vartheta < 0$ we will argue that the solutions in this case do not make sense.

5.2 The transversality condition and utility bubbles in the additive case

Our goal is to show that, when coupled with the switch from the infinite-horizon problem to the family of finite-horizon problems approach, a mismatched transversality condition can lead to peculiar behaviour. We conclude that the modeller is not free to choose the transversality condition, at least in the framework of Definition 5.1, and electing to use the wrong condition can either rule out perfectly reasonable admissible strategies (and possibly rule out all strategies, including the candidate optimal strategy) or it can allow utility processes to be defined which have the characteristics of a bubble.

In this section we consider the simpler case of time-additive CRRA utility. We will assume throughout this section that: the well-posedness condition $\eta_a := \frac{\delta}{R} - \frac{1-R}{R}(r + \frac{\lambda^2}{2R}) > 0$ holds (note that this is a necessary and sufficient condition for the well-posedness of the Merton problem with additive utility by Corollary II.5.4 and Corollary II.5.5 on page 34); also, that $R > 1$. The latter condition is only imposed to avoid case distinctions and similar behaviour is observed when $R < 1$.

In this case it is clear that for g_{CRRA}-evaluable consumption stream, the infinite-horizon formulation

$$V_t = \mathbb{E}\left[\int_t^\infty e^{-\delta s}\frac{C_s^{1-R}}{1-R}ds \,\middle|\, \mathcal{F}_t\right], \qquad 0 \le t < \infty,$$

is equivalent to the finite-horizon formulation:

$$V_t = \mathbb{E}\left[\int_t^T e^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s + V_T \,\middle|\, \mathcal{F}_t\right], \qquad 0 \le t \le T < \infty, \tag{5.2}$$

if and only if the transversality condition $\lim_{T\to\infty}\mathbb{E}[V_T] = 0$ is met. Define $V_t^\Delta =$

$e^{\delta t}V_t$. By arguing as in the proof of Proposition 5.5 (specialised to the case $\vartheta = 1$), V satisfies (5.2) if and only if V^Δ satisfies

$$V_t^\Delta = \mathbb{E}\left[\int_t^T \left(\frac{C_s^{1-R}}{1-R} - \delta V_s^\Delta\right) \mathrm{d}s + V_T^\Delta \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t \le T < \infty, \tag{5.3}$$

where the transversality condition is $e^{-\delta t}\mathbb{E}[V_t^\Delta] \to 0$.

The above observation suggests that the 'correct' transversality condition for the problem with the difference aggregator is $e^{-\delta t}\mathbb{E}[V_t^\Delta] \to 0$. But, what happens if the transversality condition is modified to become $e^{-\nu t}\mathbb{E}[V_t^\Delta] \to 0$ for some $\nu \neq \delta$?

For $\hat{\pi} = \frac{\lambda}{\sigma R}$ and $\xi > 0$ with $H_\delta(\hat{\pi}, \xi) = \delta + (R-1)(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}R) > 0$, it follows from (3.3) that the constant proportional strategy with $\Pi \equiv \hat{\pi}$ and $C = \xi X$ satisfies $\mathbb{E}[C_t^{1-R}] = \xi^{1-R}\mathbb{E}[X_t^{1-R}] = \xi^{1-R}x^{1-R}e^{(1-R)(r+\frac{\lambda^2}{2R}-\xi)t}$ and the solution to (5.2) is

$$V_t = V_t^\xi = \frac{K(\xi)}{1-R}e^{-\delta t}X_t^{1-R},$$

where $K(\xi) := \frac{\xi^{1-R}}{H_\delta(\hat{\pi},\xi)} = \frac{\xi^{1-R}}{R\eta_a + (1-R)\xi}$. This implies that a solution to (5.3) is given by

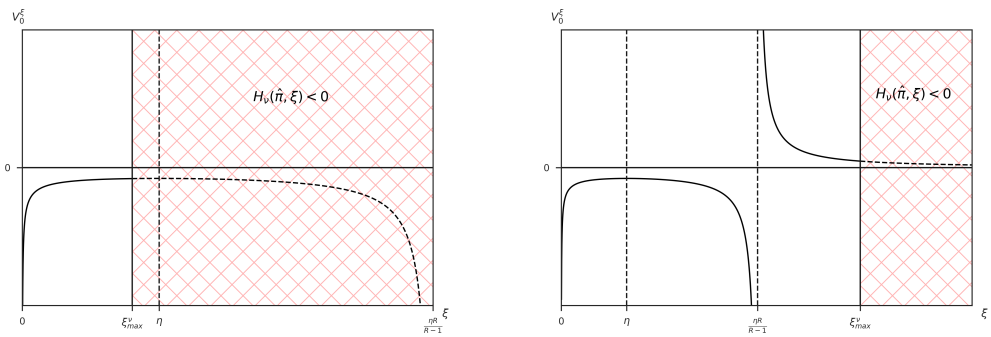
$$V_t^\Delta = V_t^{\Delta,\xi} = e^{\delta t}V_t = \frac{K(\xi)}{1-R}X_t^{1-R}. \tag{5.4}$$

On the other hand, $e^{-\nu t}\mathbb{E}[V_t^\Delta] \to 0$ is equivalent to $e^{(\delta-\nu)t}\mathbb{E}[V_t] \to 0$, which in turn is equivalent to $H_\nu(\hat{\pi}, \xi) > 0$. We can therefore define the *maximum* value of ξ such that the transversality condition $e^{-\nu t}\mathbb{E}[V_t^\Delta] \to 0$ is satisfied. This is given by

$$\xi_{\max}^\nu := \sup\{\xi > 0 : \text{there is } \pi \in \mathbb{R} \text{ with } H_\nu(\pi, \xi) > 0\} = \left(r + \tfrac{\lambda^2}{2} + \tfrac{\nu}{R-1}\right)_+ < \infty.$$

First, consider a stronger transversality condition, $e^{-\nu t}\mathbb{E}[V_t^\Delta] \to 0$ for $\nu < \delta$. This means that $H_\delta(\hat{\pi}, \xi) > H_\nu(\hat{\pi}, \xi)$. In this case, if $H_\delta(\hat{\pi}, \xi) > 0 \ge H_\nu(\hat{\pi}, \xi)$, or equivalently if ξ is such that $R\eta_a > (R-1)\xi \ge \nu + (R-1)\big(r + \frac{\lambda^2}{2R}\big)$, then V^Δ defined in (5.4) satisfies (5.3) but it does not satisfy the transversality condition $e^{-\nu t}\mathbb{E}[V_t^\Delta] \to 0$. In particular, if $\eta_a > \xi_{\max}^\nu$ then the candidate optimal strategy leads to a utility process which does not satisfy the transversality condition and

hence does not lie in the set of consumption streams over which the optimisation takes place. This is illustrated in Figure 5.1a for the case $R > 1$ (but can also occur when $R < 1$).



(a) When the transversality condition is too small ($\nu < \delta$) the candidate optimal strategy may *not be evaluable*.

(b) When the transversality condition is too large ($\nu > \delta$), the candidate optimal strategy is not optimal and some consumption streams lead to bubble-like utility processes.

Figure 5.1: Plots of the solution to (5.4) associated to the constant proportional investment-consumption strategy $(\hat{\pi}, \xi)$ along with blocked out region where the transversality condition is not met ($H_\nu(\hat{\pi}, \xi) \leq 0$).

Second, consider solving (5.3) under a weaker transversality condition $e^{-\nu t}\mathbb{E}[V_t^\Delta] \to 0$ for $\nu > \delta$. In this case, $H_\nu(\hat{\pi}, \xi) > H_\delta(\hat{\pi}, \xi)$. Let $\xi \neq \frac{R\eta_a}{R-1}$ be such that $H_\nu(\hat{\pi}, \xi) > 0 > H_\delta(\hat{\pi}, \xi)$ (for example $\xi = \xi_\varepsilon := \frac{\delta+\varepsilon}{R-1} + \left(r + \frac{\lambda^2}{2R}\right) = \frac{\varepsilon+R\eta_a}{R-1} > 0$ for $\varepsilon \in (0, \nu - \delta)$). Again, it follows that $V^{\Delta,\xi_\varepsilon}$ as defined (5.4) solves (5.3) for the constant proportional investment-consumption strategy $(\pi, \xi) = (\hat{\pi}, \xi_\varepsilon)$. As $H_\nu(\hat{\pi}, \xi_\varepsilon) > 0$, the transversality condition $e^{-\nu t}\mathbb{E}[V_t^{\Delta,\xi_\varepsilon}] \to 0$ is met.

Further $V^{\Delta,\xi_\varepsilon} = -\frac{K(\xi_\varepsilon)}{R-1}X^{1-R}$ where $K(\xi_\varepsilon) = -\frac{\xi_\varepsilon^{1-R}}{\varepsilon}$. In particular, $V_0^{\xi_\varepsilon} = \frac{\xi^{1-R}}{\varepsilon}\frac{x^{1-R}}{R-1} > 0$. By comparison, $V_0^\eta = b\eta^{-\vartheta S}\frac{x^{1-R}}{1-R} < 0$. Hence, the candidate optimal strategy no longer maximises the initial value of the utility process over constant proportional strategies, in contradiction to the well-established theory for this case.

In the case $R > 1$ where we would expect to assign a negative utility, we may actually obtain an arbitrarily large *positive* utility (see Figure 5.1b). This can be done by letting $\varepsilon \searrow 0$ in the above. What is happening is that—whilst the integrand

in (5.2) is always negative—the discounted expected future utility $\mathbb{E}\left[V_T^{\Delta} \,\middle|\, \mathcal{F}_t\right]$ is diverging to positive infinity as $T \nearrow \infty$. The agent is always receiving a negative utility from consumption, but this is offset by an ever-increasing positive contribution from expectations of future utility. The endless optimism that things will always be better in the future creates bubble-like behaviour.

Although there are special features in the additive case, the study of CRRA utility does show that some delicacy is needed when defining infinite-horizon utility to be the solution to the finite-horizon utilities paired with a transversality condition. If we wish to define stochastic differential utility in this manner, we must be very careful that we use the appropriate transversality condition.

In preparation for the move beyond the additive case we record the following definition and proposition summarising the results of this section.

Definition 5.6. V is a *CRRA-bubble* for a consumption stream C if V solves (5.2) for each $0 \leq t \leq T < \infty$ but V and $U = U(t, C)$ are of opposite sign.

Proposition 5.7. (i) *For constant proportional strategies, there are no CRRA-bubbles which satisfy the transversality condition* $e^{-\delta t}\mathbb{E}[V_t^{\Delta}] \to 0$.

(ii) *If* $\nu < \delta$ *then there is a financial market such that the candidate optimal investment-consumption strategy does not satisfy the transversality condition.*

(iii) *If* $\nu > \delta$*, there is a financial market such that there is a consumption strean for which the associated utility process satisfies the transversality condition but is a CRRA-bubble. When* $R > 1$*, the candidate optimal consumption stream does not maximise* V_0^C *over attainable strategies.*

5.3 Transversality, the case $\vartheta < 0$, and the family of finite-horizon problems.

For the EZ-SDU aggregator in discounted form over the infinite horizon it is not possible to define a utility process in the case $\vartheta < 0$. However, several authors have

attempted to define a utility process for $\vartheta < 0$ using the difference form with the family of finite-horizon problems approach or otherwise. Motivated by the analysis of the additive case, in this section we explain why the mathematical results they find may not have a sensible economic interpretation.

The only strategies for which we can hope to find a nontrivial utility process in explicit form are constant proportional investment-consumption strategies. Moreover, the candidate optimal strategy is of this form. In consequence, and for this section only, we make the following assumption so we can see explicitly the issues which arise when $\vartheta < 0$.

Temporary Standing Assumption (for Section 5.3 only). *Consumption plans under consideration in this section are generated by constant proportional investment-consumption strategies (π, ξ). If an associated utility process exists, then it is of the form $V_t^\Delta = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$ where $B = B(\pi, \xi) \in \mathbb{R}_{++}$ is a positive constant. If there is no solution of the form $V_t^\Delta = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$ for $B \in (0, \infty)$, then the consumption stream is not evaluable.*

Remark 5.8. Note that if $\vartheta \in (0, 1)$, Corollary IV.2.9 below shows that if a utility process exists for a consumption stream C, then it is unique. If $\vartheta \notin [0, 1]$, then this need not be the case. In that case we must decide which utility process to assign to a given consumption stream. Typically the literature makes additional assumptions to ensure that the time-homogeneous solution $V_t^\Delta = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$ is the utility process associated with C, if such a solution exists. Without discussing what these assumptions might be, the impact of the temporary standing assumption is to assign the utility process V^Δ given by $V_t^\Delta = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$ to the constant proportional strategy.

Finding an explicit solution for the finite-horizon formulation

Consider g_{EZ}^{Δ} and a constant proportional investment-consumption strategy (π, ξ). Suppose $V^{\Delta} = (V_t^{\Delta})_{t \geq 0}$ is a solution to

$$V_t^{\Delta} = \mathbb{E}\left[\int_t^T \left[b\frac{\xi^{1-S}X_s^{1-S}}{1-S}\left((1-R)V_s^{\Delta}\right)^{\rho} - \delta\vartheta V_s^{\Delta}\right] \mathrm{d}s + V_T^{\Delta}\middle| \mathcal{F}_t\right] \tag{5.5}$$

for all $0 \leq t \leq T < \infty$. We look for a solution of the form $V_t^{\Delta} = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$ where $B = B(\pi, \xi)$ is a positive constant which we seek to identify—we need $B \geq 0$ since we require V^{Δ} is $\mathbb{V}$-valued. For a constant proportional strategy (π, ξ), we have that $\mathbb{E}[X_s^{1-R}|\mathcal{F}_t] = X_t^{1-R}e^{-H_0(s-t)}$ where $H_0 = H_0(\pi, \xi)$ is as in (3.5) with $\nu = 0$. Then, substituting the candidate form for V^{Δ} into (5.5) and dividing by $\xi^{1-R}X_t^{1-R}$ yields

$$\frac{B}{1-R} = \int_t^T \left[\frac{b}{1-S}B^{\rho} - \frac{\delta\vartheta B}{1-R}\right] e^{-H_0(s-t)}\,\mathrm{d}s + \frac{B}{1-R}e^{-H_0(T-t)},$$

and, provided $H_0(\pi, \xi) \neq 0$,

$$B = (b\vartheta B^{\rho} - \delta\vartheta B)\frac{1-e^{-H_0(T-t)}}{H_0(\pi, \xi)} + Be^{-H_0(T-t)}. \tag{5.6}$$

It follows that there is a solution of the given form if there is a solution to

$$BH_{\delta\vartheta}(\pi, \xi) = B(\delta\vartheta + H_0(\pi, \xi)) = b\vartheta B^{\rho}, \tag{5.7}$$

where $H_{\delta\vartheta}(\pi, \xi)$ is as in (3.5) with $\nu = \delta\vartheta$. (If $H_0(\pi, \xi) = 0$, instead of (5.6), we get $B = (T-t)(b\vartheta B^{\rho} - \delta\vartheta B) + B$ which means that again B solves (5.7).) Since $b > 0$, there can only be a positive solution to (5.7) if $\vartheta H_{\delta\vartheta}(\pi, \xi) > 0$.

Note that already this is different to the additive case ($\rho = 0$ and $\vartheta = 1$) in the way that it was presented in Section 5.2. In the additive case we (effectively) looked for solutions to $B(\delta + H_0(\pi, \xi)) = b$ but did not require that $B > 0$; indeed we sometimes found (genuine) solutions with $B > 0$ and sometimes bubble solutions with $B < 0$. Solutions in the additive case with $B < 0$ do not satisfy $V \in \mathbb{V}$ and are

automatically excluded when we consider utility processes in the EZ-SDU framework. We now argue that similar ideas mean that the case $\vartheta < 0$ does not make sense if bubble solutions are excluded.

Suppose $\vartheta \neq 1$ (equivalently $\rho \neq 0$ or $R \neq S$) and consider nonnegative solutions to (5.7). If $\vartheta \in (0,1)$ (equivalently $\rho < 0$), then this equation has a solution if and only if $H_{\delta\vartheta}(\pi,\xi) > 0$ and then the solution is unique and given by $B = \left(\frac{b\vartheta}{H_{\delta\vartheta}(\pi,\xi)}\right)^{\vartheta}$. If $\vartheta > 1$, then $B = 0$ is always a solution to (5.7) (and so is $B = \infty$ if $H_{\delta\vartheta}(\pi,\xi) > 0$) and there exists a strictly positive, finite solution if and only if $H_{\delta\vartheta}(\pi,\xi) > 0$, whence again $B = \left(\frac{b\vartheta}{H_{\delta\vartheta}(\pi,\xi)}\right)^{\vartheta}$. If $\vartheta < 0$, then $B = 0$ is always a solution to (5.7), $B = \infty$ is a solution if $H_{\delta\vartheta}(\pi,\xi) < 0$ and there exists a further solution if and only if $H_{\delta\vartheta}(\pi,\xi) < 0$ whence $B = \left(\frac{b|\vartheta|}{|H_{\delta\vartheta}(\pi,\xi)|}\right)^{\vartheta}$. By the Temporary Standing Assumption, we exclude zero and infinity as solutions.

A change of accounting units

For a constant proportional strategy $(\hat{\pi} = \frac{\lambda}{\sigma R}, \xi)$, a change of accounting units will have the effect of changing the discount parameter. Fix δ and g^{Δ}_{EZ} but introduce also $g^{\gamma} = g^{\gamma}_{EZ}$ and V^{γ} where $g^{\gamma} := b\frac{c^{1-S}}{1-S}((1-R)v)^{\rho} - \gamma\vartheta v$ and $V^{\gamma} = (V^{\gamma}_t)_{t\geq 0}$ is a solution to

$$V^{\gamma}_t = \mathbb{E}\left[\int_t^T \left[be^{(\gamma-\delta)s}\frac{\xi^{1-S}X^{1-S}_s}{1-S}((1-R)V^{\gamma}_s)^{\rho} - \gamma\vartheta V^{\gamma}_s\right] \mathrm{d}s + V^{\gamma}_T \middle| \mathcal{F}_t\right] \tag{5.8}$$

for all $0 \leq t \leq T < \infty$. (Then also $(g^{\delta}, V^{\delta}) \equiv (g^{\Delta}_{EZ}, V^{\Delta})$.) As before, we look for a solution of the form $V^{\gamma}_t = B_{\gamma}\xi^{1-R}\frac{X^{1-R}_t}{1-R}$ where $B_{\gamma} = B_{\gamma}(\pi,\xi) \in (0,\infty)$.

Lemma 5.9. *Let $(X^{\gamma}_t)_{t\geq 0}$ be given by $X^{\gamma}_t = X_t e^{-\frac{(\gamma-\delta)}{1-S}t}$ so that X^{γ} is the wealth process which arises from a change of accounting unit.*

(i) V^{Δ} solves (5.5) if and only if V^{γ} defined by $V^{\gamma}_t = e^{(\gamma-\delta)\vartheta t}V^{\Delta}_t$ solves (5.8).

(ii) V^{γ} solves (5.8) if and only if it also solves

$$V^{\gamma}_t = \mathbb{E}\left[\int_t^T \left[b\frac{\xi^{1-S}(X^{\gamma}_s)^{1-S}}{1-S}((1-R)V^{\gamma}_s)^{\rho} - \gamma\vartheta V^{\gamma}_s\right] \mathrm{d}s + V^{\gamma}_T \middle| \mathcal{F}_t\right]$$

Proof. The proof of (i) follows by a similar argument to the one used in the proof of Proposition 5.5. Statement (ii) is a simple renaming of variables. □

In particular, taking $\gamma = 0$, V_t^0 solves

$$V_t^0 = \mathbb{E}\left[\int_t^T b\xi^{1-S}\frac{(X_s^0)^{1-S}}{1-S}\left((1-R)V_s^0\right)^\rho \,\mathrm{d}s + V_T^0 \,\middle|\, \mathcal{F}_t\right]. \tag{5.9}$$

Considering solutions of (5.9) it is clear that the aggregator g^0 takes only one sign (except possibly on the boundary where it may not be defined) in the sense that either $g^0 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{V} \mapsto \mathbb{R}_+$ or $g^0 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{V} \mapsto \mathbb{R}_-$.

Utility bubbles

As in Section 5.2, we define bubble solutions to be solutions which differ in sign from the aggregator.

Definition 5.10. V is a *bubble solution* for a consumption stream C and aggregator g if V solves

$$V_t = \mathbb{E}\left[\int_t^T g(s, C_s, V_s)\,\mathrm{d}s + V_T \,\middle|\, \mathcal{F}_t\right]$$

for each $0 \le t \le T < \infty$ and either $V \ge 0$ and $g \le 0$ or $V \le 0$ and $g \ge 0$, so that V and $g = (g(s, C_s, V_s))_{s\ge 0}$ are of opposite sign.

Hypothesis 1. *There are no bubble solutions under any choice of accounting units.*

Theorem 5.11. *Under Hypothesis 1 we must have $\vartheta > 0$.*

Proof. Consider the constant proportional strategy (π, ξ).

Suppose there exists a utility process V^Δ which solves (5.5). Then, by Lemma 5.9, we can switch accounting units so that V^0 solves (5.9). There g has one sign. Since there are no bubble solutions under any accounting units, V^0 is not a bubble and therefore has the same sign as g^0. Hence, $(1-S)V_t^0 \ge 0$. Further, since the integral in (5.9) is monotonic in T and $\mathbb{E}[V_T^0]$ always has exponential growth (or decay) for proportional investment-consumption strategies, we must have $\mathbb{E}[V_T^0] \to 0$ for V^0 (and V^Δ) to take finite values.

But, $\mathbb{E}[V_T^0] \to 0$ if and only if $e^{-\delta\vartheta t}\mathbb{E}[V_t^\Delta] \to 0$ which is equivalent to $H_{\delta\vartheta}(\pi, \xi) > 0$. Since there exists a solution to (5.7) if and only if $\vartheta H_{\delta\vartheta}(\pi, \xi) > 0$ it must be the case that $\vartheta > 0$. □

Now we want to consider which transversality condition we should associate with (5.5). Suppose the transversality condition is

$$e^{-\nu t}\mathbb{E}[V_t^\Delta] \to 0. \tag{5.10}$$

It is easy to see that $\mathbb{E}[e^{-\nu t}V_t^\Delta] \to 0$ if and only if $e^{-(\nu-\delta\vartheta)t}\mathbb{E}[e^{-\gamma\vartheta t}V_t^\gamma] \to 0$, and the transversality condition (5.10) becomes $e^{-(\nu-\delta\vartheta)t}\mathbb{E}[V_t^0] \to 0$.

Hypothesis 2. (*i*) *The transversality condition associated with the aggregator g should depend on the aggregator, but not on the financial market.*

(*ii*) *Whenever the problem is well-posed, the utility process associated with the candidate optimal consumption stream satisfies the transversality condition* (5.10).

Proposition 5.12. *Under Hypothesis 2 we must have that $\nu \geq \delta\vartheta$.*

Proof. Suppose $\nu < \delta\vartheta$ and define $\varepsilon = \delta\vartheta - \nu > 0$. Then, the candidate optimal strategy $(\hat{\pi}, \eta)$ satisfies the transversality condition $e^{-\nu t}\mathbb{E}[V_t^\Delta] \to 0$ if and only if it satisfies $e^{\varepsilon t}\mathbb{E}[e^{-\delta\vartheta t}V_t^\Delta] \to 0$, which in turn is equivalent to $H_{\delta\vartheta}(\hat{\pi}, \eta) > \varepsilon$. Suppose the market parameters are such that $\eta \in (0, \frac{\varepsilon}{\vartheta})$. Then, $H_{\delta\vartheta}(\hat{\pi}, \eta) = \vartheta\eta < \varepsilon$ and the candidate optimal utility process fails to satisfy the transversality condition. □

In general the larger the value of ν, the weaker the admissibility condition and the more processes which will satisfy the transversality condition. However, for the Epstein–Zin aggregator, there is a point where increasing ν further makes no difference to the set of evaluable consumption streams.

Lemma 5.13. *Fix C and suppose that Hypothesis 1 holds. If there exists a solution V^Δ to* (5.5)*, then V^Δ satisfies* (5.10) *for $\nu = \delta\vartheta$.*

Proof. Let V^Δ be a solution to (5.5). Then, by Lemma 5.9, V^0 solves (5.9). Since $V^\Delta \in \mathbb{V}$ and there are no bubble solutions, $\vartheta > 0$ and $\mathbb{E}[V_t^0] \to 0$. Hence, $e^{-\delta\vartheta t}\mathbb{E}[V_t^\Delta] \to 0$. □

The final hypothesis says that we choose the smallest possible value for ν which allows us to evaluate all the strategies that we want.

Hypothesis 3. *The transversality parameter should be the smallest parameter ν such that every solution to* (5.5) *satisfies* (5.10).

Proposition 5.14. *Under Hypotheses 1, 2 and 3, the parameter ν in the transversality condition* (5.10) *must take the value $\nu = \delta\vartheta$.*

Remark 5.15. By construction there cannot be any bubble solutions in the infinite-horizon discounted version. If $\mathbb{E}[|\int_t^\infty g(s, C_s, V_s)\,\mathrm{d}s|] < \infty$ then $\mathbb{E}[V_T] \to 0$. Then, since g has one sign, V and g must have the same sign.

Remark 5.16. For $\vartheta > 1$, Melnyk et al. [MMKS20] take the transversality condition to be (5.10) with $\nu = \delta < \delta\vartheta$. For some parameter values, the candidate optimal strategy may not be admissible because it fails the transversality condition. However, these parameter combinations are ruled out by the extra parameter restrictions imposed in [MMKS20]. In particular, the authors restrict attention to financial models for which $\eta > \delta\rho$. This is precisely enough to ensure that $e^{-\delta t}\mathbb{E}[X_t^{1-R}] \to 0$ for the candidate optimal strategy. For $0 < \eta \leq \delta\rho$, the utility process for the candidate optimal strategy would fail the transversality condition. Further, both in the case $\eta > \delta\rho \geq 0$ and in the case $0 < \eta \leq \delta\rho$, many reasonable strategies are unnecessarily excluded because they fail the transversality condition, and not because they are suboptimal.

For $\vartheta < 0$ (and $R > 1$), Melnyk et al. [MMKS20] define candidate solutions V^Δ as solutions to (5.5). It follows that $V = (V_t)_{t\geq 0}$ given by $V_t = e^{-\delta\vartheta t}V_t^\Delta$ solves the family of finite-horizon problems given in (5.9). However, relative to the aggregator g^0, the solution V is a bubble and would be ruled out by Hypothesis 2.

The same bubble feature can be observed without the switch in accounting units. For $\vartheta < 0$, Melnyk et al. [MMKS20] define candidate solutions V^Δ of the form $V_t^\Delta = -B\frac{1}{R-1}X_t^{1-R}$ where $B = B(\pi, \xi)$ solves (5.6). Since $H_{\delta\vartheta}(\hat{\pi}, \hat{\xi} = \eta) = \eta\vartheta$, the condition $\eta > 0$ implies that $\vartheta H_{\delta\vartheta}(\hat{\pi}, \hat{\xi}) = \eta\vartheta^2 > 0$. Furthermore, the condition $\eta < \delta\rho$ ensures that $H_\delta(\hat{\pi}, \hat{\xi}) = \eta\vartheta + \delta(1-\vartheta) = \vartheta(\eta - \delta\rho) > 0$. Then, for $C_s = \eta X_s$, the proposed solution does indeed solve

$$V_t^\Delta = \mathbb{E}\left[\int_t^T g_{EZ}^\Delta(C_s, V_s^\Delta)\,\mathrm{d}s + V_T^\Delta \middle| \mathcal{F}_t\right] \tag{5.11}$$

for all $0 \le t \le T < \infty$ together with the transversality condition $e^{-\delta t}\mathbb{E}[X_t^{1-R}] \to 0$. However, [MMKS20] impose the additional admissibility condition $V_s^\Delta \le -\frac{C_s^{1-R}}{R-1} \le 0$ (which for the optimal strategy amounts to the condition $\eta > \delta$). This is precisely the condition under which $g_{EZ}^\Delta(C_s, V_s^\Delta) = \frac{\delta C_s^{1-S}}{1-S}((1-R)V_s^\Delta)^\rho - \delta\vartheta V_s^\Delta \ge 0$ (recall that [MMKS20] take $\delta = b$). Therefore, if (C_s, V_s) is the candidate optimal strategy, it follows that g_{EZ}^Δ and V^Δ have the opposite sign, and so corresponds to a bubble, *even in the original units.*

Due to the results in this section, we make the following standing assumption for the remainder of the thesis.

Standing Assumption 2. (*Positive ϑ Assumption*) *The parameters R and S are such that $\vartheta = \frac{1-R}{1-S} > 0$.*

5.4 The dual approach

Dual methods have proved spectacularly successful for the Merton problem with additive utility. They work for general utility functions, and in principle they make it possible to move beyond the setting of constant parameter financial markets to non-Markovian settings and incomplete markets. However, it is not immediately clear how to extend dual methods to the SDU setting. One promising idea is based on stochastic variational utility as formulated by Dumas, Uppal and Wang [DUW00].

Building on work of Geoffard [Geo96] for deterministic consumption streams,

Dumas et al.[DUW00] define the *felicity function* G to be the Fenchel–Legendre transform of the aggregator $g(c, v)$ in v, so that for $c > 0$ and $(1 - \vartheta)\nu > 0$, $G(c, \nu) = \inf_{(1-R)u>0}(g(c, u) + \nu u)$. ([DUW00] assume that g is convex in its second argument, but a similar argument works if g is concave.) Then, the *stochastic variational utility* (SVU) is given by

$$U_t^C := \sup_{(1-\vartheta)\nu>0} \mathbb{E}\left[\int_t^T e^{-\int_t^s \nu_u \,\mathrm{d}u} G(C_s, \nu_s)\,\mathrm{d}s + U_T(X_T) \,\middle|\, \mathcal{F}_t\right], \tag{5.12}$$

where $U_T(\cdot)$ is a bequest function. [DUW00] consider consumption streams C that satisfy $\mathbb{E}[\int_0^T C_t^2\,\mathrm{d}t] < \infty$ and aggregators $g(c, v)$ that have linear growth in c and are Lipschitz in v. Then, under these conditions, they show that U is the stochastic variational utility associated to the pair (g, c), if and only it is the finite-horizon stochastic differential utility associated to the pair (g, c).

Matoussi and Xing [MX18] take the approach of [DUW00] and extend it to the case of Epstein–Zin SDU in the finite-horizon case. They show that if $\vartheta < 1$ and the consumption stream is such that a utility process exists and is uniformly integrable, then the solution to (5.11) is equal to the solution to (5.12) for G the Fenchel–Legendre transform of g_{EZ}^{Δ} and $V_T = U_T(X_T)$.

Exploiting the equivalence of [DUW00] between SDU and SVU, [MX18] show that if the bequest function is of an appropriate power law form, the maximisation problem of finding $\sup_{C\in\mathscr{C}(x)\cap\mathscr{E}_u(g)} V_0^C$ where V^C solves (5.11) becomes that of finding $\sup_{C\in\mathscr{C}(x)\cap\mathscr{E}_u(g)} U_0^C$, where U^C solves (5.12). Exchanging the order of suprema, the problem becomes to find

$$\sup_{(1-\vartheta)\nu>0} \sup_{C\in\mathscr{C}(x)\cap\mathscr{E}_u(g)} \mathbb{E}\left[\int_t^T e^{-\int_t^s \nu_u \,\mathrm{d}u} G(C_s, \nu_s)\,\mathrm{d}s + U_T(X_T) \,\middle|\, \mathcal{F}_t\right]. \tag{5.13}$$

For EZ-SDU both $G(\cdot, \nu)$ and U_T are power law functions, and hence standard duality techniques can be applied to the inner problem in (5.13) with fixed ν. Finally, by taking the dual with respect to the second argument again, the dual stochastic variational problem can be transformed back into what Matoussi and Xing call the

stochastic differential dual. They then prove that

$$\sup_{C \in \mathscr{C}(x) \cap \mathscr{E}_u(g)} V_0^C \leq \inf_{k>0} \left(\inf_{D \in \mathscr{D}_a} Y_0^{kD} + xk \right). \tag{5.14}$$

where $\mathscr{D}_a$ is the class of state-price densities and Y^{kD} is the stochastic differential dual associated to a state-price density D and a positive real number k. Matoussi and Xing show that under certain restrictions on the financial market (for example, bounded market price of risk) there is no duality gap and that (5.14) is satisfied with equality. Finally, they show that the optimal strategy is defined in terms of a BSDE and in particular it exists.

The papers of Dumas et al. [DUW00] and especially Matoussi and Xing [MX18] provide great insights and a potential route-map describing how dual methods might be extended to the investment-consumption problem for SDU. However, there are several obstacles which make it difficult to apply these ideas to the infinite-horizon problem. First, at present, the dual method has little to say about existence of solutions, and typically for existence it relies on results from the primal approach—in turn these have traditionally involved imposing restrictive assumptions on the consumption stream which are not satisfied in the infinite-horizon problem. Second, the equivalence between the SDU and SVU formulations may be challenging to prove in the infinite-horizon setting, without imposing substantive technical assumptions. Third, we shall see that there are major issues of non-uniqueness when $\vartheta > 1$; these issues do not disappear simply by a change of viewpoint.

5.5 Summary

The conclusions from Chapter III are twofold.

First, for Epstein–Zin stochastic differential utility over the infinite horizon, combined with a constant parameter Black–Scholes–Merton frictionless financial model, certain restrictions on the parameters are necessary to have a well-founded problem. In particular, in addition to $b > 0$, for the problem to make sense it must the

case that the coefficient of relative risk aversion and the coefficient of elasticity of intertemporal complimentarity both lie on the same side of unity, i.e. $\vartheta > 0$. (However, the condition that the discount parameter δ must be positive can sometimes be weakened. Indeed, since this parameter depends on the accounting units it is sometimes natural to consider a case where it takes a negative value.)

Second, for the infinite-horizon problem, it is preferable to consider a discounted aggregator rather than a difference aggregator. The one-sign property of the discounted form of the EZ-SDU aggregator means that the integral $\int_0^\infty g(s, C_s, V_s)\,\mathrm{d}s$ and its expectation are always well defined in $[-\infty, \infty]$ whereas this is not always the case for the difference aggregator. Then, in addition to the fact that the discounted aggregator is the natural generalisation of the standard form of the Merton problem for additive utility, for the discounted aggregator there are no issues over bubble solutions. In Chapter IV we shall strengthen this result further by showing that, at least when $\vartheta \in (0, 1)$, for the aggregator of discounted form it is possible to define a (generalised) utility process for *every* consumption stream. This means that we can prove the optimality of the candidate optimal strategy within the class of *all* admissible investment-consumption strategies, and not just a subclass satisfying certain integrability properties.

APPENDIX TO CHAPTER III

III.A Proof of Proposition 5.5

Proof of Proposition 5.5. Let V^Δ be a $(\delta\vartheta, \mathbb{J}^{MMS})$-utility process associated to consumption stream C and aggregator g^Δ_{EZ}. Then, $V^\Delta \in \mathbb{S}^1_T \cap \mathbb{I}_T(g^\Delta_{EZ}, C)$, $\lim_{t\to\infty} e^{-\delta\vartheta t}\mathbb{E}[V^\Delta_t] = 0$, and V^Δ solves (5.1) with aggregator g^Δ_{EZ} for all $0 \le t \le T < \infty$.

Define the process $V = (V_t)_{t\ge 0}$ by $V_t := e^{-\delta\vartheta t}V^\Delta_t$. Then, $V \in \mathbb{S}^1_T$ and $V_t \xrightarrow{L^1} 0$ by the transversality condition of V^Δ. We proceed to show that $V \in \mathbb{I}_T(g_{EZ}, C)$ and V satisfies

$$V_t = \mathbb{E}\left[\int_t^T be^{-\delta u}\frac{C_u^{1-S}}{1-S}((1-R)V_u)^\rho \,\mathrm{d}u + V_T \,\middle|\, \mathcal{F}_t\right] \tag{1.1}$$

for all $T > 0$. So, fix $T > 0$. Using that $V^\Delta \in \mathbb{S}^1_T \cap \mathbb{I}_T(g^\Delta_{EZ}, C)$ and $e^{-\delta t}|V_t|^\rho \le e^{\delta\vartheta T}|V^\Delta_t|^\rho$ for $t \in [0,T]$, we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T \left| be^{-\delta s}\frac{C_s^{1-S}}{1-S}((1-R)V_s)^\rho \right| \mathrm{d}s \\
&\quad\le\ e^{\delta\vartheta T}\mathbb{E}\left[\int_0^T \left| b\frac{C_s^{1-S}}{1-S}((1-R)V^\Delta_s)^\rho - \delta\vartheta V^\Delta_s \right| \mathrm{d}s\right] + e^{\delta\vartheta T}T\,|\delta\vartheta|\,\mathbb{E}\left[\sup_{s\in[0,T]}|V^\Delta_s|\right],
\end{aligned}$$

where the right hand side is finite. Thus, $V \in \mathbb{I}_T(g_{EZ}, C)$. Next, define the martingale $M = (M_t)_{t\in[0,T]}$ by

$$M_t = \mathbb{E}\left[\int_0^T \left[b\frac{C_s^{1-S}}{1-S}((1-R)V_s)^\rho - \delta\vartheta V_s\right] \mathrm{d}s + V_T \,\middle|\, \mathcal{F}_t\right]$$

As V^Δ satisfies (5.1), it satisfies the BSDE

$$V_t^\Delta = V_T^\Delta + \int_t^T \left(b\frac{C_u^{1-S}}{1-S}((1-R)V_u^\Delta)^\rho - \delta\vartheta V_u^\Delta\right) \mathrm{d}u - \int_t^T \mathrm{d}M_u.$$

Applying the product rule to $V_t = e^{-\delta\vartheta t}V_t^\Delta$ we find that

$$V_t = V_T + \int_t^T be^{-\delta u}\frac{C_u^{1-S}}{1-S}((1-R)V_u)^\rho \,\mathrm{d}u - \int_t^T e^{-\delta\vartheta u}\,\mathrm{d}M_u.$$

Since $\mathbb{E}[(1-e^{-\delta\vartheta T})|M_T|] < \infty$, $N_t = \int_0^t e^{-\delta\vartheta s}\,\mathrm{d}M_s$ is a martingale by [HMK19, Lemma A.1.]. Now taking expectations gives (1.1).

Next, using that V and the integrand in (1.1) have the same sign, it follows from the monotone convergence theorem and $\lim_{T\to\infty}\mathbb{E}[V_T] = 0$ that V satisfies (3.2). Since V_0 is finite, this also gives $V \in \mathbb{I}(g_{EZ}, C)$.

Finally, if $\vartheta > 1$ then $\delta\vartheta > \delta$ and any $(\delta, \mathbb{J}^{MMS})$-utility process is automatically a $(\delta\vartheta, \mathbb{J}^{MMS})$- utility process. Hence $\mathscr{E}^{MMS}(g_{EZ}^\Delta) \subseteq \mathscr{E}(g_{EZ})$. □

CHAPTER

IV

STOCHASTIC DIFFERENTIAL UTILITY: EXISTENCE AND UNIQUENESS RESULTS

This chapter is concerned with rigorously proving well-posedness of the investment-consumption problem under Epstein–Zin stochastic differential utility and verifying that the candidate optimal investment-consumption strategy which we derived in Section III.3.3 is optimal. To do this, there are three main issues which we must address: first, the existence of a utility process associated to a general consumption stream; second, the uniqueness of such a utility process; and third, optimality of the candidate optimal investment-consumption strategy over all attainable consumption streams.

We mainly consider parameter combinations leading to $\vartheta := \frac{1-R}{1-S} \in (0,1)$. (The combinations of R and S satisfying this condition are illustrated in Figure I.1.2 on page 8.) Importantly, we will show in this chapter that when $\vartheta \in (0,1)$, if the utility process exists then it is unique.

Our results and approach are as follows. From the arguments in Section III.3 (page 58), we have existence of a utility process for proportional investment and consumption strategies $\Pi = \pi \in \mathbb{R}$ and $C = \xi X$ for $\xi \in \mathbb{R}_{++}$ (provided that $H_{\delta\vartheta}(\pi, \xi) > 0$) in a Black–Scholes–Merton financial market. The first major contribution is an extension of this existence result to all strictly positive consumption streams $C = (C_t)_{t \geq 0}$ which satisfy $kC_t^{1-R} \leq \mathbb{E}\left[\int_t^\infty e^{-\delta\vartheta(s-t)} C_s^{1-R} ds \,\middle|\, \mathcal{F}_t\right] \leq KC_t^{1-R}$, for some constants $0 < k \leq K < \infty$. In particular, we may evaluate strategies that are, in a very precise sense, within a multiplicative constant of a constant proportional investment-consumption strategy. Moreover, for each such C there is a unique utility process $V = (V_t^C)_{t \geq 0}$ such that $k_V e^{-\delta\vartheta t} C_t^{1-R} \leq (1-R)V_t \leq K_V e^{-\delta\vartheta t} C_t^{1-R}$ for a different pair of constants (k_V, K_V). (*Note that this does not preclude the existence of other utility processes which do not satisfy such bounds.*) The proof relies on the construction of a contraction mapping and a fixed point argument.

To make further progress, we assume that $\vartheta \in (0,1)$. In this case, we can show that any utility process is unique (in fact we show uniqueness for a wide class of aggregators, the main restriction being that they are decreasing in v). The key idea is to use concepts from the theory of BSDEs to extend the concept of a solution to (III.2.4) to include subsolutions and supersolutions, depending (roughly speaking) on whether the equality in (III.2.4) is replaced by $\leq$ or $\geq$. Then, again under the assumption that the aggregator is decreasing in v, we prove a comparison theorem which tells us that any subsolution always lies below any supersolution. Uniqueness of solutions follows—any solution is simultaneously both a sub-solution and a supersolution so if V^1 and V^2 are solutions then $V^1 \leq V^2$ and $V^2 \leq V^1$ and hence $V^1 = V^2$.

For EZ-SDU, when $\vartheta > 1$ the comparison argument fails and the uniqueness

argument does not hold. Note that it is not merely that we need to look for a different strategy of proof—instead, it is simple to give examples for which there are multiple solutions to (III.2.4). In this case, a different comparison theorem and a modification of the definition of the utility process is required. For these reasons, we defer discussion of this case to Chapter V.

Returning to the case of $\vartheta \in (0,1)$, in order to remove the constraints $k > 0$ and $K < \infty$ we again exploit the comparison theorem to obtain a monotonicity property for solutions. Provided we allow utility processes to take values in the extended real line, we can exploit the fact that the aggregator takes one sign to show that it is possible to define a unique, possibly infinite, utility process for *any* admissible consumption stream. Here we make use of the notion of generalised supermartingales.

Finally, still under the assumption that $\vartheta \in (0,1)$, we turn to the verification argument. By the arguments of the previous paragraphs, for any attainable consumption stream $C = (C_t)_{t\geq 0}$, we can define a utility process $V^C = (V_t^C)_{t\geq 0}$ and time-zero value $J(C) = V_0^C$. Our goal is to find $\sup_{C\in\mathscr{C}(x)} J(C)$. Note that here the supremum is taken over *all* admissible consumption streams; not just over consumption streams for which there exists a finite value function, or consumption/utility process pairs lying in some special set as is common in much of the literature. (In many cases, the only strategy known to lie in this special set is the candidate optimal strategy.)

From the results of Section III.3.3, we have candidates for the optimal strategy and value function, but several issues remain. The key is proving that $\hat{V}(X^{\Pi,C}) = (\hat{V}(X_t^{\Pi,C}))_{t\geq 0}$ is a supersolution for any admissible C, where $X^{\Pi,C}$ is the wealth process arising from the investment-consumption strategy (Π, C). Then, by the comparison theorem, $V_t^C \leq \hat{V}(X_t^{\Pi,C})$ and $J(C) = V_0^C \leq \hat{V}(x)$. (Further, for $(\hat{\Pi}, \hat{C})$ the candidate optimal investment-consumption strategy, $V_0^{\hat{C}} = \hat{V}(x)$ and so $\sup_{C\in\mathscr{C}(x)} J(C) = \hat{V}(x)$.) However, as in the case of rigorous primal verification arguments for the Merton problem, there are several challenges to overcome. First $X^{\Pi,C} \in \mathscr{P}_+$ but is not necessarily a member of $\mathscr{P}_{++}$ and so we cannot naively

apply Itô's formula to $\hat{V}(X^{\Pi,C})$. Second, for general (Π, C), $\hat{V}(X^{\Pi,C})$ does not (always) satisfy a transversality condition (and we do not want to artificially restrict the class of admissible C by requiring that it does). Third, the local martingale term arising from applying Itô's formula to $\hat{V}(X_t^{\Pi,C})$ is in general not a true martingale and cannot be assumed to have constant expectation. Nonetheless, as we show, these challenges can all be overcome by using a similar perturbation to that used in Section II.5. Where proofs are not given in the main text, they are given in the appendix.

The structure of this chapter is as follows. In Section 1, we prove existence of EZ-SDU for a wide class of consumption streams, including all constant proportional consumption streams for which the problem is well-posed, and any strategies which are 'close' to constant proportional streams, in a sense to be made precise. Still, this is not all consumption streams, so in Sections 2 and 3, we show how the utility process for an arbitrary consumption stream can be obtained by approximation and taking limits. Finally, in Section 4, we prove optimality of the candidate optimal strategy (Theorem 4.1) first derived in Section III.3.3, where the optimisation is taken over all attainable consumption streams and not just those satisfying regularity and integrability conditions. Key results along the way include a comparison result (Theorem 2.8), existence and uniqueness results (Theorem 1.6, Theorem B.2) and a result that generalises the utility process by approximation (Theorem 3.4).

1 Existence of Epstein–Zin SDU

For the Epstein–Zin aggregator g_{EZ}, we showed in Section III.3.3 that the candidate optimal strategy—along with many other proportional consumption streams—is evaluable. The goal of this section is to prove existence for a much larger class of consumption streams. The author is not aware of any results on the existence of infinite-horizon Epstein–Zin stochastic differential utility, so this is an essential result that is currently missing from the literature.

A transformation of the coordinate system leads to a simplified problem. Define

the $[0,\infty]$-valued processes $W = (W_t)_{t\geq 0}$ and $U = (U_t)_{t\geq 0}$ by[1]

$$W_t = (1-R)V_t \quad \text{and} \quad U_t = u(t,C) = b\vartheta e^{-\delta t} C_t^{1-S}. \tag{1.1}$$

Let $h_{EZ}(u,w) : [0,\infty) \times (0,\infty) \to [0,\infty)$ be defined by $h_{EZ}(u,w) = uw^\rho$ and extend the definition of h_{EZ} to the domain $[0,\infty]^2$ and co-domain $[0,\infty]$ as follows:

$$h_{EZ}(u,w) := \begin{cases} uw^\rho, & (u,w) \in (0,\infty) \times (0,\infty), \\ w^\rho, & (u,w) \in (0,\infty) \times \{0,\infty\}, \\ u, & (u,w) \in \{0,\infty\} \times [0,\infty], \end{cases}$$

with the standard convention $0^\rho := \infty$ and $\infty^\rho = 0$ for $\rho < 0$. The motivation behind the definition on the boundary is to ensure continuity in w for fixed u.

Note that $V \in \mathbb{I}(g_{EZ}, C)$ if and only if $W \in \mathbb{I}(h_{EZ}, U)$. Consequently, V^C is a utility process associated to consumption stream C with aggregator g_{EZ} if and only if W^U is a utility process associated to consumption stream U with aggregator h_{EZ}.

We next define an operator F_U from an appropriate subset of $\mathscr{P}_{++}$ to itself satisfying[2]

$$F_U(W)_t := \mathbb{E}\left[\int_t^\infty h_{EZ}(U_s, W_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t \right]. \tag{1.2}$$

Note that V is a solution to (III.1.1) with aggregator g_{EZ} and consumption C if and only W is a fixed point of the operator F_U for the transformed consumption U. In particular, every fixed point of the operator F_U has càdlàg paths.

Definition 1.1. Suppose that $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are nonnegative progressive processes. We say that X has the same order as Y (and write $X \overset{\mathbb{O}}{=} Y$, noting that $\overset{\mathbb{O}}{=}$ is an equivalence relation) if there exist constants $k, K \in (0,\infty)$ such that

$$0 \leq kY \leq X \leq KY. \tag{1.3}$$

[1] Here, we agree that $U_t := \infty$ if $C_t = 0$ and $S > 1$.
[2] Here, we always choose a càdlàg version for the right-hand side of (1.2).

Denote the set of progressive processes with the same order as Y by $\mathbb{O}(Y) \subseteq \mathscr{P}_+$. Further, denote by $\mathbb{O}(Y; k, K)$ the set of processes such that (1.3) holds for a pre-specified k and K.

Definition 1.2. Let $\mathbb{SO}$ be the class of positive progressively measurable processes of *self-order*, i.e. such that X is of the same order as $J^X = (J_t^X)_{t\geq 0}$, where $J_t^X = \mathbb{E}\left[\int_t^\infty X_s ds \,\middle|\, \mathcal{F}_t\right]$. More precisely, define

$$\mathbb{SO} := \left\{ X \in \mathscr{P}_{++} : \ \mathbb{E}\left[\int_0^\infty X_t \, \mathrm{d}t\right] < \infty \text{ and } X \overset{\mathbb{O}}{=} J^X \right\}.$$

For arbitrary $\alpha \in \mathbb{R}$, let $\mathbb{SO}^\alpha = \{X \in \mathscr{P}_{++} : X^\alpha \in \mathbb{SO}\}$.

Remark 1.3. If $Y \in \mathbb{SO}^\alpha$ then $KY \in \mathbb{SO}^\alpha$ for all $K > 0$.

Example 1.4. Geometric Brownian motion raised to a power remains a geometric Brownian motion. Let $Z = (Z_t)_{t\geq 0}$ be a geometric Brownian motion such that $\mathbb{E}[Z_t^\vartheta] = Z_0 e^{-\gamma t}$, where $\gamma > 0$. Then, $Z^\vartheta = \frac{1}{\gamma} J^{Z^\vartheta}$. Hence, $Z \in \mathbb{SO}^\vartheta$.

If $\eta > 0$ and if $\hat{C}$ is the candidate optimal strategy, then $\hat{U} = u(t, \hat{C})$ is a geometric Brownian motion, and $\hat{U}^\vartheta$ has drift $-\eta\vartheta < 0$. Hence, $\hat{U} \in \mathbb{SO}^\vartheta$. Similarly, all the constant proportional investment-consumption strategies (π, ξ) with $H_{\delta\vartheta}(\pi, \xi) > 0$ lie in $\mathbb{SO}^\vartheta$ (after a suitable transformation). Roughly speaking, the same holds true for any strategy which is close to a constant proportional strategy (for which $H_{\delta\vartheta}(\pi, \xi) > 0$).

Lemma 1.5. *Let $U \in \mathbb{SO}^\vartheta$. Then, $F_U(\cdot)$ maps from $\mathbb{O}(U^\vartheta)$ to itself.*

Proof. This follows from the more general Lemma B.1 in Appendix IV.B. □

We may now state a first existence result. Whilst it is not the strongest existence result we prove in this chapter (Theorem 1.6 is a special case of Theorem B.2), it forms the backbone of further existence arguments. The idea of the proof is to transform the problem to an alternative space where the transformed form of F_U is a contraction mapping. The existence of a fixed point then follows from the Banach Fixed Point Theorem.

Theorem 1.6. *Let $U \in \mathbb{SO}^{\vartheta}$. Then, F_U defined by* (1.2) *has a unique fixed point $W \overset{\mathbb{O}}{=} U^{\vartheta}$, which has càdlàg paths.*

Proof. This is a specific version of the more general Theorem B.2. For a stand-alone proof, one just needs to set $\varepsilon = 0$ and $\Lambda = U$ in the proof of Theorem B.2. □

The following theorem is a direct corollary to Theorem 1.6 and the definitions of W and U in terms of V and C given in (1.1).

Theorem 1.7. *Suppose $C \in \mathscr{P}_{++}$ satisfies $\mathbb{E}[\int_0^\infty e^{-\delta\vartheta s} C_s^{1-R} ds] < \infty$, and for some $0 < k < K < \infty$,*

$$k\mathbb{E}\left[\int_t^\infty e^{-\delta\vartheta s} C_s^{1-R} ds \,\middle|\, \mathcal{F}_t\right] \leq e^{-\delta\vartheta t} C_t^{1-R} \leq K\mathbb{E}\left[\int_t^\infty e^{-\delta\vartheta s} C_s^{1-R} ds \,\middle|\, \mathcal{F}_t\right]$$

for all $t \geq 0$. Then, there exists a utility process $V = (V_t^C)_{t\geq 0}$ associated with g_{EZ} and C. Moreover this utility process is unique in the class of processes with the property that $V_t / \mathbb{E}\left[\int_t^\infty e^{-\delta\vartheta s} \frac{C_s^{1-R}}{1-R} ds \,\middle|\, \mathcal{F}_t\right]$ is bounded above and below by strictly positive constants.

Proof. Take $U_t = \Lambda_t = e^{-\delta t} C_t^{1-S}$. Then, U satisfies the conditions of Theorem 1.6 and so there exists a utility process W associated to (h_{EZ}, U) which is unique in $\mathbb{O}(\Lambda^{\vartheta})$. Therefore, $V = \frac{W}{1-R}$ is a utility process associated to (g_{EZ}, C); uniqueness in the appropriate class is also inherited. □

Relative to the extant literature, Theorem 1.7 massively expands the set of consumption streams which are known to be evaluable (the only consumption stream known in the literature to have a utility process associated to it is the candidate optimal consumption stream $\hat{C}$). However, it still does not allow us to assign a utility to every consumption stream. For example, the zero consumption stream is excluded. Note also that Theorem 1.7 does not exclude the possibility of other utility processes which do not satisfy the condition that $V_t / \mathbb{E}\left[\int_t^\infty e^{-\delta\vartheta s} \frac{C_s^{1-R}}{1-R} ds \,\middle|\, \mathcal{F}_t\right]$ is bounded.

Finally, note that no restriction on ϑ has been imposed apart from $\vartheta > 0$ due to Standing Assumption 2 on page 83.

2 Subsolutions and supersolutions

The aim of this section is to introduce the notions of subsolutions and supersolutions and then to prove a comparison theorem for aggregators that take only one sign and are nonincreasing in v. As a consequence, all evaluable consumption streams for such aggregators are *uniquely* evaluable.

Let $\mathbb{V} \subseteq [-\infty, \infty]$ denote the set in which V may take values. For EZ-SDU we have that either $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$ or $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$. This one-sign property ensures that integrals are always well defined. For the rest of this thesis, we make this a standing assumption.

Standing Assumption 3 (One-sign property of the aggregator)**.** *Either* $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$ *or* $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$.

The following definition extends the notion of an aggregator, allowing it also to depend on the state of the world $\omega \in \Omega$.

Definition 2.1. An *aggregator random-field* $g : [0, \infty) \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ is a product measurable mapping such that $g(\cdot, \omega, \cdot, \cdot)$ is an aggregator for fixed $\omega \in \Omega$, and for progressively-measurable processes $C = (C_t)_{t \geq 0}$ and $V = (V_t)_{t \geq 0}$, the process $(g(t, \omega, C_t(\omega), V_t(\omega)))_{t \geq 0}$ is progressively-measurable.

Example 2.2. Let $G : \mathbb{R}_+ \times \mathbb{V} \times \mathbb{R} \to \mathbb{V}$ be continuous and $Y : [0, \infty) \times \Omega \to \mathbb{R}$ a progressively measurable process. Then, $g(t, \omega, c, v) := G(c, v, Y_t(\omega))$ is an aggregator random field.

Let g be an aggregator random field. The definitions of $\mathbb{I}(g, C)$, $\mathbb{UI}(g, C)$, the utility process associated to the pair (g, C), and the sets of evaluable and uniquely evaluable consumption streams $\mathscr{E}(g)$ and $\mathscr{E}_u(g)$ follow verbatim from Definitions III.1.1 and III.1.3 on page 51.

We now introduce the notion of subsolutions and supersolutions.

Definition 2.3. Let $C \in \mathscr{P}_+$ and g be an aggregator random field. A $\mathbb{V}$-valued, làd, optional process V is called

- a *subsolution* for the pair (g, C) if $\limsup_{t\to\infty} \mathbb{E}\left[V_{t+}\right] \leq 0$, and for all bounded stopping times $\tau \leq \sigma$,

$$V_\tau \;\leq\; \mathbb{E}\left[V_{\sigma+} + \int_\tau^\sigma g(s, \omega, C_s, V_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right]. \tag{2.1}$$

- a *supersolution* for the pair (g, C) if $\liminf_{t\to\infty} \mathbb{E}\left[V_{t+}\right] \geq 0$, and for all bounded stopping times $\tau \leq \sigma$,

$$V_\tau \;\geq\; \mathbb{E}\left[V_{\sigma+} + \int_\tau^\sigma g(s, \omega, C_s, V_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right]. \tag{2.2}$$

- a *solution* for the pair (g, C) if it is both a subsolution and a supersolution and $V \in \mathbb{I}(g, C)$.

Remark 2.4. (a) V is a supersolution associated to the pair (g, C) if and only if $\widetilde{V} := -V$ (which is valued in $\widetilde{\mathbb{V}} := -\mathbb{V}$) is a subsolution for the pair $(\widetilde{g}, C)$, where $\widetilde{g}(t, \omega, c, \widetilde{v}) = -g(t, \omega, c, -\widetilde{v})$.
(b) While we do not to require sub- or supersolutions to be in $\mathbb{I}(g, C)$, we require this integrability for solutions.
(c) It might be expected that the definition would require subsolutions and supersolutions to be càdlàg. However, we will construct the utility process for a general consumption stream by taking limits and the monotone limit of càdlàg processes is not necessarily càdlàg. In contrast, optionality is preserved in the limit.

If V is a solution for the pair (g, C), then $V \in \mathbb{I}(g, C)$ by definition. By Remark III.1.2 it then follows that V is uniformly integrable. Similar results hold for sub- and supersolutions by the following lemma.

Lemma 2.5. *Suppose that $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$ and V is a subsolution or $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$ and V is a supersolution for the pair (g, C). If $V \in \mathbb{I}(g, C)$ then $V \in \mathbb{UI}(g, C)$.*

Proof. We only consider the case that $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$ and $V \in \mathbb{I}(g, C)$ is a subsolution. The other case is symmetric. Define the UI martingale $M = (M_t)_{t\geq 0}$ by $M_t :=$

$\mathbb{E}\left[\int_0^\infty g(s,\omega,C_s,V_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$. Since $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$, setting $\tau := t$ and $\sigma := u$ in (2.1) and taking the limsup as $u \to \infty$ gives $0 \leq V_t \leq \mathbb{E}\left[\int_t^\infty g(s,\omega,C_s,V_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \leq M_t$. Hence, V is uniformly integrable. □

It is useful to introduce two monotonicity conditions on an aggregator random field.

Definition 2.6. Let $g : [0,\infty) \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ be an aggregator random field. Then, g is said to satisfy

- (c↑) if it is nondecreasing in c, its third argument, $\mathbb{P} \otimes \mathrm{d}t$-a.e.
- (v↓) if it is nonincreasing in v, its fourth argument, $\mathbb{P} \otimes \mathrm{d}t$-a.e.

Remark 2.7. For EZ-SDU, (v↓) is satisfied if and only if $\vartheta \in (0,1]$; if $\vartheta > 1$ then the aggregator is increasing in its fourth argument.

The following result shows that under condition (v↓), a comparison result holds for sub- and supersolutions.

Theorem 2.8 (Comparison Theorem for Subsolutions and Supersolutions)**.** *Let $C \in \mathscr{P}_+$ and let g be an aggregator random field satisfying* (v↓)*. If V^1 is a subsolution and V^2 is a supersolution to the pair (g,C), and V^1 or V^2 is in $\mathbb{UI}(g,C)$, then $V^1_\tau \leq V^2_\tau$ $\mathbb{P}$-a.s. for all finite stopping times τ.*

We deduce two simple but important corollaries. The first one shows that under condition (v↓), all g-evaluable strategies are *g-uniquely* evaluable. The second one shows that for aggregators g satisfying (c↑) and (v↓), the utility associated to (g,C) is increasing in g and C.

Corollary 2.9. *Let g be an aggregator random field satisfying* (v↓)*. Then, $\mathscr{E}(g) = \mathscr{E}_u(g)$.*

Proof. Clearly, $\mathscr{E}(g) \supseteq \mathscr{E}_u(g)$. For the converse inclusion, fix $C \in \mathscr{E}(g)$. Suppose there are two utility processes V^1 and V^2 for the pair (g,C). Since V^1 and V^2 are both solutions, they are in $\mathbb{UI}(g,C)$ by Lemma 2.5. Since they are both sub- and supersolutions, we may apply Theorem 2.8 twice to show $V^1_\tau \geq V^2_\tau$ $\mathbb{P}$-a.s. and

$V^2_\tau \geq V^1_\tau$ $\mathbb{P}$-a.s. for all finite stopping times $\tau \geq 0$. Thus, $V^1_\tau = V^2_\tau$ $\mathbb{P}$-a.s. for all finite stopping times τ. Since V^1 and V^2 are both optional, this implies that they are indistinguishable (see e.g. [Nik06, Theorem 3.2]). □

Corollary 2.10. *Let $C^1, C^2 \in \mathscr{P}_+$ and $g^1, g^2 : [0,\infty) \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ be aggregator random fields satisfying* (c↑) *and* (v↓). *Suppose that $C^2 \geq C^1$ $\mathbb{P} \otimes \mathrm{d}t$-a.e. and $g^2(\cdot,\cdot,c,v) \geq g^1(\cdot,\cdot,c,v)$ $\mathbb{P} \otimes \mathrm{d}t$-a.e. for $(c,v) \in \mathbb{R}_+ \times \mathbb{V}$. Moreover suppose there exists a utility process $V^i \in \mathbb{I}(g^i, C^i)$ for the pair (g^i, C^i), $i \in \{1,2\}$. Then, $V^1_\tau \leq V^2_\tau$ for all finite stopping times τ.*

Remark 2.11. If g_1, g_2 are both nonincreasing rather than nondecreasing in c but otherwise the hypotheses of the corollary are unchanged, then $V^1_\tau \geq V^2_\tau$.

3 Removing the bounds on evaluable strategies when $\vartheta \in (0,1)$

The goal of this section will be to show that if $\vartheta \in (0,1)$ we may: first, remove the lower bound restriction from Theorem 1.6; and second, generalise the notion of a utility process, allowing us to evaluate the Epstein–Zin stochastic differential utility of *any* consumption stream. The following assumption holds for the remainder of this chapter.

Standing Assumption 4. *Henceforth we assume that $\rho < 0$, or equivalently $\vartheta \in (0,1)$.*

Theorem 3.1. *Suppose that $U \in \mathscr{P}_+$ is such that $U \leq \Lambda \in \mathbb{SO}^\vartheta$. Then, F_U defined by* (1.2) *has a unique fixed point $W \in \mathbb{I}(h_{EZ}, U)$.*

Recall that $\hat{X} = X^{\hat{C},\hat{\Pi}}$ is the candidate optimal wealth process—the solution to (I.4.2) under the candidate optimal strategy $\hat{\Pi} \equiv \frac{\mu - r}{\sigma R}$ and $\hat{C} = \eta \hat{X}$—and that $\hat{C} = \eta \hat{X}$ is the associated candidate optimal consumption.

Corollary 3.2. *Suppose that $C \in \mathscr{P}_+$ is such that there exists $K \in \mathbb{R}_+$ with $C^{1-S} \leq K(\hat{C})^{1-S}$. Then, $C \in \mathscr{E}_u(g_{EZ})$.*

Proof. Since, $C^{1-S} \leq K(\hat{C})^{1-S}$, it follows that $U = u(t, C_t) \leq Ku(t, \hat{C}_t) = K\hat{U}_t$ where $\hat{U}_t := u(t, \hat{C}_t)$. Furthermore, $K\hat{U} \in \mathbb{SO}^{\vartheta}$ by Example 1.4 and Remark 1.3. Finally, using Theorem 3.1 we may deduce that $U \in \mathscr{E}_u(h_{EZ})$ and consequently that $C \in \mathscr{E}_u(g_{EZ})$. □

Corollary 3.2 gives us a large class of evaluable consumption streams. The rest of this section is dedicated to generalising the notion of a utility process. In particular, for any aggregator g satisfying (c↑) and (v↓), the results of this section make it possible to assign a utility to any process $C \in \mathscr{P}_+$ that we can express as the monotone limit of processes $C^n \in \mathscr{E}_u(g)$. For the Epstein–Zin aggregator this includes *all* consumption streams.

Definition 3.3. For a general aggregator $g : [0, \infty) \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$, let $\overline{\mathscr{E}}(g)$ denote the set of consumption streams $C \in \mathscr{P}_+$ that are monotone limits of a sequence $(C^n)_{n\in\mathbb{N}}$ of processes in $\mathscr{E}(g)$ and either

1) $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$ and $(C^n)_{n\in\mathbb{N}}$ is nondecreasing, or
2) $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$ and $(C^n)_{n\in\mathbb{N}}$ is nonincreasing.

We now state the central result of this section—that we may extend the notion of a utility process and evaluate processes in $\overline{\mathscr{E}}(g)$.

Theorem 3.4. *Let g be an aggregator random field satisfying* (c↑) *and* (v↓)*, and let $C \in \overline{\mathscr{E}}(g)$. Let $(C^n)_{n\in\mathbb{N}}$ be a monotone approximating sequence. Let V^n be the utility process associated to C^n for each $n \in \mathbb{N}$. Then, there exists a càdlàg adapted process $V^\dagger = \lim_{n\to\infty} V^n$ that is independent of the approximating sequence. Moreover, if $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$, then $V^\dagger$ is the minimal supersolution and if $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$, then $V^\dagger$ is the maximal subsolution.*

Definition 3.5. We call the unique process $V^\dagger = (V^\dagger_t)_{t\geq 0}$ constructed in Theorem 3.4 the *generalised solution* or the *generalised utility process* associated to (g, C).

The following theorem tells us that the notion of a generalised solution *extends* the notion of a solution, in the sense that if a solution exists, then it is equal to the generalised solution.

Theorem 3.6. *Let g satisfy* (c↑) *and* (v↓). *If there exists a solution V associated to the pair (g, C) then it agrees with the generalised solution $V^\dagger$.*

Proof. We only prove the result in the case $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$. The case $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$ follows by a symmetric argument. By Theorem 3.4, $V^\dagger$ is the minimal supersolution. Let τ be an arbitrary finite stopping time. Since $V \in \mathbb{UI}(g, C)$ is a subsolution and $V^\dagger$ is a supersolution, $V_\tau \leq V_\tau^\dagger$ by Theorem 2.8. Since V is a supersolution and $V^\dagger$ is minimal in the class of supersolutions, $V_\tau^\dagger \leq V_\tau$. Hence, $V_\tau^\dagger = V_\tau$. Since $V^\dagger$ and V are both optional ($V^\dagger$ by Theorem 3.4, and V by definition) and they agree for all bounded stopping times, $V^\dagger$ is equivalent to V up to indistinguishability (see, for example, [Nik06, Theorem 3.2]). □

We therefore drop the superscript † and denote the *generalised utility process* by V. The next proposition shows that the generalised solution is increasing in C.

Proposition 3.7. *Let g be an aggregator random field satisfying* (c↑) *and* (v↓) *and let $C^1, C^2 \in \overline{\mathscr{E}}(g)$. Suppose further that C^2 dominates C^1 $\mathbb{P} \otimes \mathrm{d}t$-a.e. For $i = 1, 2$, let V^i be the generalised solution associated to the pair (g, C^i). Then, $V_\tau^2 \geq V_\tau^1$ for all bounded stopping times τ.*

If we consider Epstein–Zin aggregator g_{EZ}, we may assign a generalised utility process to *any* consumption stream.

Theorem 3.8. *Let $C \in \mathscr{P}_+$. There exists a unique generalised utility process associated to the pair (g_{EZ}, C).*

Proof. Suppose that $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$. We therefore want to find a nondecreasing sequence of consumption streams $(C^n)_{n\in\mathbb{N}}$ such that $C^n \in \mathscr{E}_u(g_{EZ})$ for all $n \in \mathbb{N}$ and $C^n \nearrow C$. Let $\hat{C} = \eta\hat{X}$ be the candidate optimal strategy. Let $C^n = C \wedge n\hat{C}$. Then, $(C^n)_{n\in\mathbb{N}} \in \mathscr{E}_u(g_{EZ})$ by Corollary 3.2 and $C^n \nearrow C$. Therefore, by Theorem 3.4, there exists a unique generalised utility process associated to C.

The proof in the case $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$ goes through in exactly the same manner if we consider the sequence of processes $C^n = C \vee \frac{1}{n}\hat{C}$. □

We can now extend the definition of Epstein–Zin utility to any consumption stream.

Definition 3.9. Let $C \in \mathscr{P}_+$. Define the Epstein–Zin utility process associated to C to be the generalised utility process $V^{C,g_{EZ}}$ associated to the pair (g_{EZ}, C). Define the Epstein–Zin utility of the consumption stream to be $J_{g_{EZ}}(C) := V_0^{C,g_{EZ}}$.

This allows us to consider the infinite-horizon investment-consumption problem for EZ-SDU over all admissible strategies:

$$\sup_{C \in \mathscr{C}(x)} J_{g_{EZ}}(C) \;=\; \sup_{C \in \mathscr{C}(x)} V_0^{C,g_{EZ}}.$$

This definition of the stochastic control problem is different to that considered by Schroder and Skiadas [SS99], Xing [Xin17], Matoussi and Xing [MX18], Melnyk et al. [MMKS20] and the rest of the literature on the Merton problem for Epstein–Zin SDU in the fact that it optimises over all consumption streams and *does not impose any regularity or integrability conditions* beyond attainability.

4 The verification argument for the candidate optimal strategy

The goal of this final section is to verify that the candidate optimal strategy is indeed optimal. The general structure of a primal verification argument for recursive optimal investment problems is as follows: first, apply Itô's lemma to $\hat{V}(X^{\Pi,C})$ for a general strategy (Π, C); next, use the HJB equation to show that $\hat{V}(X^{\Pi,C})$ is a supersolution associated to the pair (g_{EZ}, C); finally, the Comparison Theorem (Theorem 2.8) for sub- and supersolutions implies $\hat{V}(x) \geq V_0^C$ for any admissible strategy $C \in \mathscr{C}(x)$. Optimality follows since we showed in Section III.3.3 that $V_0^{\hat{C}} = \hat{V}(x)$.

Unfortunately, there are at least three difficulties with this approach. The first difficulty is that the candidate value function $\hat{V}(x)$ defined in (III.3.7) does not have a well-defined derivative at zero, meaning that we cannot apply Itô's lemma to $\hat{V}(X_t^{\Pi,C})$ for a general admissible wealth process $X_t^{\Pi,C}$. The second difficulty is

that for a general strategy (Π, C), the standard proof that $\hat{V}(X_t^{\Pi,C})$ corresponds to a supersolution involves showing that the local martingale part of $\hat{V}(X^{\Pi,C})$ is a supermartingale, and in the case $R > 1$ this is not true in general. The third difficulty is that a utility process V^C might fail to exist.

The first two issues arise also in the case of CRRA utility and we showed in Chapter II that they may be overcome using a stochastic perturbation of the value function. We now extend the ideas in Chapter II to the setting of EZ-SDU. The third issue has been dealt with in Section 3.

Theorem 4.1 (Verification Theorem)**.** *Suppose that $\eta > 0$ and $\vartheta \in (0,1)$. If V^C is the (generalised) utility process associated to the pair (g_{EZ}, C) and $\hat{V}(x)$ is the candidate optimal utility given in* (III.3.7) *then $\sup_{C\in\mathscr{C}(x)} V_0^C = V_0^{\hat{C}} = \hat{V}(x)$, and the optimal investment-consumption strategy is given by $(\hat{\Pi}, \hat{C})$.*

To prove this theorem, we will use the following lemma. In its assumptions, we assume without loss of generality that $\delta = 0$.

Lemma 4.2. *Let $\varepsilon > 0$ and let $\hat{X} = X^{\hat{\Pi},\hat{C}}$ be the wealth process under our candidate optimal strategy. If $X = X^{\Pi,C}$ is the wealth process associated to an admissible strategy (Π, C), then $\hat{V}(X + \varepsilon\hat{X})$ is a supersolution for the pair $(f_{EZ}, C + \eta\varepsilon\hat{X})$.*

We may now prove Theorem 4.1

Proof of Theorem 4.1. We showed in Section III.3.2 that $\sup_{C\in\mathscr{C}(x;r,\mu,\sigma)} V_0^{g_{EZ},C} = \sup_{C\in\mathscr{C}(x;\tilde{r},\tilde{\mu},\sigma)} V_0^{f_{EZ},C}$ for $\tilde{r} = r - \frac{\delta}{1-S}$ and $\tilde{\mu} = \mu - \frac{\delta}{1-S}$. Hence, without loss of generality we may assume $\delta = 0$. It follows from Section III.3.3 that $V_0^{f_{EZ},\hat{C}} = \hat{V}(x)$, so it only remains to prove that $\hat{V}(x) \geq \sup_{c\in\mathscr{C}(x)} V_0^{f_{EZ},C}$.

Suppose $R < 1$. Then, by Lemma 4.2 and since $C + \eta\varepsilon\hat{X} > C$ and f_{EZ} is increasing in its first argument, $\hat{V}(X + \varepsilon\hat{X})$ is a supersolution for the pair (f_{EZ}, C). Furthermore, the (generalised) utility process $V^{f_{EZ},C}$ associated to (f_{EZ}, C) is the minimal supersolution by Theorem 3.4. Consequently, $\hat{V}(X + \varepsilon\hat{X}) \geq V^{f_{EZ},C}$.

Suppose $R > 1$, and hence also $S > 1$ by Standing Assumption 2. Then, since $(C + \eta\varepsilon\hat{X})^{1-S} \leq (\eta\varepsilon)^{1-S}\hat{X}^{1-S}$, $C + \eta\varepsilon\hat{X} \in \mathscr{E}_u(f_{EZ})$ by Corollary 3.2. Hence, there

exists a utility process $V^{f_{EZ},C+\eta\varepsilon\hat{X}} \in \mathbb{UI}(f_{EZ}, C+\eta\varepsilon\hat{X})$ associated to $C+\eta\varepsilon\hat{X}$. Since also $\hat{V}(X+\varepsilon\hat{X})$ is a supersolution associated to $(f_{EZ}, C+\eta\varepsilon Y)$, applying Theorem 2.8 and then Proposition 3.7 gives $\hat{V}(X+\varepsilon\hat{X}) \geq V^{f_{EZ},C+\eta\varepsilon\hat{X}} \geq V^{f_{EZ},C}$.

In both cases, taking the supremum over attainable consumption streams at time zero gives $\hat{V}(x(1+\varepsilon)) \geq \sup_{c\in\mathscr{C}(x)} V_0^{f_{EZ},C}$. Letting $\varepsilon \searrow 0$ gives the result. □

We conclude this section by showing that the correct well-posedness condition of the investment-consumption problem is $\eta > 0$.

Corollary 4.3. *Suppose that $\vartheta \in (0,1)$. Then, the infinite-horizon investment consumption problem for EZ-SDU is well-posed if and only if $\eta > 0$.*

In particular, suppose that $\eta \leq 0$ and let V^C be the (generalised) utility process associated to the pair (g_{EZ}, C). If $R < 1$, then $\sup_{C\in\mathscr{C}(x)} V_0^C = \infty$. If $R > 1$, then, $\sup_{C\in\mathscr{C}(x)} V_0^C = -\infty$.

Proof. When $\eta > 0$ the problem is well-posed by Theorem 4.1.

Now suppose $\eta \leq 0$. Since $\vartheta \in (0,1)$, the utility process is unique, and if $H(\pi,\xi) > 0$ then V given by (III.3.8) is the utility process for a constant proportional strategy.

Suppose $R < 1$ and then also $S < 1$. Let $h(\pi,\xi) = \frac{\xi^{1-R}}{1-R}\left(\frac{b\vartheta}{H_{\delta\vartheta}(\pi,\xi)}\right)^{\vartheta}$ and $D = \{(\pi,\xi) \in \mathbb{R}\times(0,\infty) : H_{\delta\vartheta}(\pi,\xi) > 0\}$. Note that $\vartheta(H_{\delta\vartheta}(\hat{\pi},\xi))^{-1} = (\eta S + (1-S)\xi)^{-1}$. Letting $\xi \searrow -\eta\frac{S}{1-S}$ yields $\vartheta(H_{\delta\vartheta}(\hat{\pi} = \frac{\mu-r}{\sigma R},\xi))^{-1} \nearrow \infty$. It follows that $h(\pi,\xi) \nearrow \infty$ and the supremum of V_0^C over constant proportional strategies is $+\infty$. Hence, $\sup_{C\in\mathscr{C}(x)} V_0^C = \infty$.

Now suppose $R > 1$ and fix an arbitrary $C \in \mathscr{C}(x; r, \mu, \sigma)$ with associated wealth process X. Denote by V the generalised utility process associated to the pair (g_{EZ}, C). It suffices to show that $V_0 = -\infty$. For $n \in \mathbb{N}$, let $\alpha_n := \frac{S}{S-1}(\frac{1}{n} - \eta) > 0$, $r_n := r + \alpha_n$ and $\mu_n := \mu + \alpha_n$. Consider the modified consumption stream C^n, given by $C_t^n := e^{\alpha_n t}C_t$. Then, by calculating the dynamics of $X_t^n := e^{\alpha_n t}X_t$ as in Section III.3.2, it can be shown that $C^n \in \mathscr{C}(x; r_n, \mu_n, \sigma)$. Furthermore, $\eta_n = \frac{1}{S}[\delta - (1-S)(r_n + \frac{\lambda^2}{2R})] = \frac{1}{n} > 0$. Then, considering the Black–Scholes–Merton

financial market with parameters (r_n, μ_n, σ) and applying Theorem 4.1 gives $V_0^n \leq \hat{V}^n(x) = \eta_n^{-\vartheta S} b^{\vartheta} \frac{x^{1-R}}{1-R}$. It follows from Proposition 3.7 that if V^n is the (generalised) solution associated for the pair (g_{EZ}, C^n), then $C \leq C^n$ implies $V \leq V^n$. Combining the inequalities and taking limits yields $V_0 \leq \lim_{n\to\infty} n^{\vartheta S} b^{\vartheta} \frac{x^{1-R}}{1-R} = -\infty$. $\square$

APPENDIX TO CHAPTER IV

IV.A Proof of the Comparison Theorem

Lemma A.1. *Let $-\infty < a < b < \infty$. Every uncountable set $U \subseteq [a, b)$ contains at least one of its right accumulation points.*

Proof. Seeking a contradiction, suppose U contains none of its right accumulation points. Then, for each $x \in U$, we may find $\varepsilon_x > 0$ such that $[x, x + \varepsilon_x) \cap U = \{x\}$. Let $U_n := \{x \in U : \ \varepsilon_x > \frac{1}{n}\}$. Then, each U_n is finite since the pairwise disjoint union $\bigcup_{x \in U_n} [x, x + \frac{1}{n})$ is contained in the interval $[a, b + \frac{1}{n})$. Hence, $U = \bigcup_{n \in \mathbb{N}} U_n$ is countable, and we arrive at a contradiction. $\square$

Proof of Theorem 2.8. We prove the result when $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$. The case $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$ is symmetric.

Suppose for contradiction that there exists a finite stopping time τ and a set $A \in \mathcal{F}_\tau$ of positive measure with $V^1_\tau(\omega) > V^2_\tau(\omega)$ for $\omega \in A$. Then, $\mathbb{E}\left[\mathbf{1}_A \left(V^1_\tau - V^2_\tau\right)\right] > 0$. Since V^1 and V^2 are làd, the processes $(V^1_{t+})_{t \geq 0}$ and $(V^2_{t+})_{t \geq 0}$ exist and are right-

continuous. Moreover, $\sigma := \inf\{s \geq \tau : V^1_{s+} - V^2_{s+} \leq 0\}$ is a stopping time. The right continuity of $(V^1_{t+})_{t\geq 0}$ and $(V^2_{t+})_{t\geq 0}$ gives $(V^1_{\sigma+} - V^2_{\sigma+})\mathbf{1}_{\{\sigma<\infty\}} \leq 0$ $\mathbb{P}$-a.s.

For each $\omega \in A$, we have $V^1_s(\omega) \geq V^2_s(\omega)$ for almost all $s \in [\tau(\omega), \sigma(\omega))$. Indeed, seeking a contradiction suppose there is $\omega \in A$ and a set of positive Lebesgue measure U such that $V^1_s(\omega) < V^2_s(\omega)$ for $s \in U \subseteq [\tau(\omega), \sigma(\omega))$. Since U is uncountable, it has a right accumulation point $q \in U$ by Lemma A.1. Then, $q < \sigma(\omega)$ and $V^1_{q+}(\omega) \leq V^2_{q+}(\omega)$, and we arrive at a contradiction.

Next, fix $n \in \mathbb{N}$. By subtracting (2.2) from (2.1) for the bounded stopping times $\tau := \tau \wedge n$ and $\sigma := \sigma \wedge n$, noting that the expectations are well defined since either V^1 or V^2 is in $\mathbb{UI}(g, C)$, and using the fact that g is a.s. decreasing in v and $V^1_s(\omega) \geq V^2_s(\omega)$ for almost all $s \in [\tau(\omega), \sigma(\omega))$ for $\omega \in A$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_A \mathbf{1}_{\{\tau\leq n\}}\left(V^1_\tau - V^2_\tau\right)\right] \\
&\quad\leq \mathbb{E}\left[\mathbf{1}_A \mathbf{1}_{\{\tau\leq n\}}\left(V^1_{(\sigma\wedge n)+} - V^2_{(\sigma\wedge n)+} + \int_{\tau\wedge n}^{\sigma\wedge n} g(s,\omega,C_s,V^1_s) - g(s,\omega,C_s,V^2_s)\,\mathrm{d}s\right)\right] \\
&\quad\leq \mathbb{E}\left[\mathbf{1}_A \mathbf{1}_{\{\tau\leq n\}}\left(V^1_{(\sigma\wedge n)+} - V^2_{(\sigma\wedge n)+}\right)\right].
\end{aligned}$$

Finally, taking the limsup as $n \to \infty$, monotone convergence, the fact that $(V^1_{t+})_{t\geq 0}$ and $(V^2_{t+})_{t\geq 0}$ are $\overline{\mathbb{R}}_+$-valued, the transversality condition for subsolutions and $(V^1_{\sigma+} - V^2_{\sigma+})\mathbf{1}_{\{\sigma<\infty\}} \leq 0$ $\mathbb{P}$-a.s. give

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_A (V^1_\tau - V^2_\tau)\right] \\
&\quad\leq \limsup_{n\to\infty} \mathbb{E}\left[\mathbf{1}_A \mathbf{1}_{\{\tau\leq n<\sigma\}}(V^1_{n+} - V^2_{n+})\right] + \limsup_{n\to\infty} \mathbb{E}\left[\mathbf{1}_A \mathbf{1}_{\{\sigma\leq n\}}(V^1_{\sigma+} - V^2_{\sigma+})\right] \\
&\quad\leq \limsup_{n\to\infty} \mathbb{E}\left[V^1_{n+}\right] + \mathbb{E}\left[\mathbf{1}_A \mathbf{1}_{\{\sigma<\infty\}}(V^1_{\sigma+} - V^2_{\sigma+})\right] \leq 0.
\end{aligned}$$

We arrive at a contradiction. □

Proof of Corollary 2.10. Suppose that $\mathbb{V} = \overline{\mathbb{R}}_+$; the proof for $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$ is symmetric. As $g_2(\cdot,\cdot,c,v) \geq g_1(\cdot,\cdot,c,v)$ $\mathbb{P}\otimes \mathrm{d}t$-a.e. and g_1 and g_2 are increasing in c, we have $g_2(s,\omega,C^2_s,V^2_s) \geq g_1(s,\omega,C^2_s,V^2_s) \geq g_1(s,\omega,C^1_s,V^2_s) \geq 0$ for $\mathrm{d}t \otimes \mathbb{P}$-a.e. (s,ω).

It then follows that, for all bounded stopping times $\tau \leq \sigma$,

$$V_\tau^2 = \mathbb{E}\left[V_{\sigma+}^2 + \int_\tau^\sigma g_2(s,\omega,C_s^2,V_s^2)\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right] \geq \mathbb{E}\left[V_{\sigma+}^2 + \int_\tau^\sigma g_1(s,\omega,C_s^1,V_s^2)\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right].$$

Since $\lim_{t\to\infty} V_{t+}^2 = 0$ $\mathbb{P}$-a.s., V^2 satisfies the definition of a supersolution associated to the pair (g_1, C^1). As $V^2 \in \mathbb{UI}(g_2, C^2) \subseteq \mathbb{UI}(g_1, C^1)$ and V^1 is a (sub)solution associated to (g_1, C^1), it follows that $V_\tau^1 \leq V_\tau^2$ for all finite stopping times τ by Theorem 2.8. □

IV.B Proving Existence and Uniqueness of a Utility Process

For $\Lambda \in \mathbb{SO}^\vartheta$, define the ε-perturbed operator $F_{U,\Lambda}^\varepsilon : \mathbb{I}(h_{EZ}, U) \to \mathscr{P}_+$ by[3]

$$F_{U,\Lambda}^\varepsilon(W)_t = \mathbb{E}\left[\int_t^\infty (U_s W_s^\rho + \varepsilon \Lambda_s^\vartheta)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]. \tag{2.1}$$

A key property of $F_{U,\Lambda}^\varepsilon$ is, when $\varepsilon > 0$ and $\Lambda \in \mathbb{SO}^\vartheta$, $F_{0,\Lambda}^\varepsilon$ is bounded away from zero. Another property is the following.

Lemma B.1. *Let $\varepsilon \geq 0$, $\Lambda \in \mathbb{SO}^\vartheta$ and $U \overset{\mathbb{O}}{=} \Lambda$. Then, $F_{U,\Lambda}^\varepsilon(\cdot)$ maps from $\mathbb{O}(\Lambda^\vartheta)$ to itself.*

Proof. Fix arbitrary $W \overset{\mathbb{O}}{=} \Lambda^\vartheta$. It follows that there exist $k_W, K_W \in (0,\infty)$ such that $k_W \Lambda^\vartheta \leq W \leq K_W \Lambda^\vartheta$. Similarly, since $U \overset{\mathbb{O}}{=} \Lambda$ and $\Lambda^\vartheta \overset{\mathbb{O}}{=} J^{\Lambda^\vartheta}$, there exist $k_U, K_U, k_\Lambda, K_\Lambda \in (0,\infty)$ such that $k_U \Lambda \leq U \leq K_U \Lambda$ and $k_\Lambda J^{\Lambda^\vartheta} \leq \Lambda^\vartheta \leq K_\Lambda J^{\Lambda^\vartheta}$. We only prove that $F_{U,\Lambda}^\varepsilon(W) \geq \kappa \Lambda^\vartheta$ for $\rho < 0$; the argument for $\rho > 0$ involves $W^\rho \geq (k_W \Lambda)^{\vartheta\rho}$ and the argument for the upper bound is symmetric. By the definition of $F_{U,\Lambda}^\varepsilon(\cdot)$ in (2.1) and since $U \geq k_U \Lambda$, $W \leq K_W \Lambda^\vartheta$ and $\Lambda^\vartheta \leq K_\Lambda J^{\Lambda^\vartheta}$, and $1 + \vartheta\rho = \vartheta$,

[3] Here, we always choose a càdlàg version for the right-hand side of (2.1).

we see that

$$\begin{aligned} F^{\varepsilon}_{U,\Lambda}(W)_t &\geq \mathbb{E}\left[\int_t^\infty k_U \Lambda_s (K_W \Lambda^\vartheta)^\rho + \varepsilon \Lambda^\vartheta_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\ &= (k_U K_W^\rho + \varepsilon)\mathbb{E}\left[\int_t^\infty \Lambda^\vartheta_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \geq \left(\frac{k_U K_W^\rho + \varepsilon}{K_\Lambda}\right)\Lambda^\vartheta. \qquad \square \end{aligned}$$

The subsequent theorem is the preliminary existence result and includes Theorem 1.6 as a special case.

Theorem B.2. *Let $\varepsilon \geq 0$, $\Lambda \in \mathbb{S}\mathbb{O}^\vartheta$ and $U \overset{\mathbb{O}}{=} \Lambda$. Then, $F^{\varepsilon}_{U,\Lambda}$ defined by* (2.1) *has a unique fixed point $W \overset{\mathbb{O}}{=} \Lambda^\vartheta \subseteq \mathbb{I}(h_{EZ}, U)$, which has càdlàg paths.*

Let Prog denote the progressive σ-algebra on $\Omega \times \mathbb{R}_+$ and set $\mathscr{B} := L^\infty(\Omega \times \mathbb{R}_+, \mathrm{Prog}, \mathbb{P} \otimes \mathrm{d}t)$. For the proof of Theorem B.2, we use the following sufficient condition for an operator $T : \mathscr{B} \to \mathscr{B}$ to be a contraction.

Lemma B.3 (Blackwell's sufficient conditions for a contraction)**.** *Let $\mathscr{B}$ be a Banach space of bounded functions and $T : \mathscr{B} \to \mathscr{B}$ an operator that is nonincreasing. Suppose there exists $\beta \in (0,1)$ with*

$$T(X + a) \geq T(X) - \beta a \qquad \text{for all } X \in \mathscr{B},\ a > 0. \tag{2.2}$$

Then, T is a contraction with Lipchitz constant β. Similarly, T is a contraction if it is monotonely nondecreasing and there exists $\beta \in (0,1)$ such that $T(X+a) \leq TX + \beta a$ for all $X \in \mathscr{B}$, $a > 0$.

Proof. We only prove the case when T is monotonely decreasing and (2.2) holds. The other case follows by a symmetric argument.

For any $X, Y \in \mathscr{B}$, $X \leq Y + ||Y - X||_\infty$. Since T is monotonely decreasing, $TX \geq T(Y + ||Y - X||_\infty)$. Furthermore, using (2.2) yields

$$T(Y + ||Y - X||_\infty) \geq TY - \beta ||Y - X||_\infty.$$

so that $TY - TX \leq \beta ||Y - X||_\infty$. By reversing the roles of Y and X one can show

that $TX - TY \leq \beta ||Y - X||_\infty$ and consequently that $||TX - TY||_\infty \leq \beta ||Y - X||_\infty$. Thus, T is a contraction mapping with Lipschitz constant β. □

Proof of Theorem B.2. Consider the change of variables

$$P_t = \log(U_t) - \log(\Lambda_t), \qquad Q_t = \log(W_t) - \vartheta \log(\Lambda_t).$$

Then, $U \overset{\mathbb{O}}{=} \Lambda$ if and only if $P \in \mathscr{B}$ and $W \overset{\mathbb{O}}{=} \Lambda^\vartheta$ if and only if $Q \in \mathscr{B}$.

The fixed point condition $W = F^\varepsilon_{U,\Lambda}(W)$ is equivalent to the fixed point condition $Q = G^\varepsilon_{P,\Lambda}(Q)$ where

$$G^\varepsilon_{P,\Lambda}(Q)_t := \log\left(\mathbb{E}\left[\int_t^\infty \Lambda_s^\vartheta \exp(P_s + \rho Q_s) + \varepsilon \Lambda_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]\right) - \vartheta \log(\Lambda_t). \tag{2.3}$$

Note that since the first term on the right-hand side of (2.3) has càdlàg paths, every fixed point Q to (2.3) corresponds to a W with càdlàg paths.

Since $G^\varepsilon_{P,\Lambda}(Q)$ is the difference of two continuous functions of progressive processes, it is progressive. Furthermore, as a consequence of Lemma B.1, $G^\varepsilon_{P,\Lambda}$ maps $\mathscr{B}$ to itself.

Suppose $\rho \in (-1, 0)$ and let $a > 0$. Then, $G^\varepsilon_{P,\Lambda}(Q)$ is decreasing. Furthermore,

$$\begin{aligned}
&G^\varepsilon_{P,\Lambda}(Q + a)_t \\
&\quad = \log\left(\exp(\rho a)\mathbb{E}\left[\int_t^\infty \Lambda_s^\vartheta \exp(P_s + \rho Q_s) + \varepsilon \frac{\Lambda_s^\vartheta}{\exp(\rho a)} \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]\right) - \vartheta \log(\Lambda_t) \\
&\quad \geq \log\left(\mathbb{E}\left[\int_t^\infty \Lambda_s^\vartheta \exp(P_s + \rho Q_s) + \varepsilon \Lambda_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]\right) - \vartheta \log(\Lambda_t) + \rho a \\
&\quad = G^\varepsilon_{P,\Lambda}(Q)_t + \rho a.
\end{aligned}$$

By Lemma B.3, this implies that $G^\varepsilon_{P,\Lambda}$ is a contraction with constant ρ. Hence, by the Contraction Mapping Theorem, there exists a unique $Q \in \mathscr{B}$ satisfying (2.3).

If $\rho \in (0, 1)$, then $G^\varepsilon_{P,\Lambda}(Q)$ is increasing and one can show that $G^\varepsilon_{P,\Lambda}(Q + a)_t \leq G^\varepsilon_{P,\Lambda}(Q)_t + \rho a$. Again, existence of a fixed point follows from Lemma B.3 and the Contraction Mapping Theorem.

Finally, to extend the result to $\rho \in (-\infty, -1]$, we borrow an idea from Schroder and Skiadas [SS99] and show by induction that the following holds for each $k \in \mathbb{N}$:

$$\text{For } 0 > \rho > -k \text{ and } P \in \mathscr{B}, \text{ there exists a unique fixed point } Q \in \mathscr{B} \text{ of } G^{\varepsilon}_{P,\Lambda}(Q). \tag{2.4}$$

The induction hypothesis ($k = 1$) holds by the above. For the induction step, suppose that (2.4) holds true for some $k \geq 1$. In order to show that (2.4) holds true for $k+1$, it suffices to consider $\rho \in (-(k+1), k]$. So fix $\rho \in (-(k+1), k]$ and choose $\chi \in (0,1)$ small enough that $-k < \rho + \chi < 0$. Now define the map $\widetilde{G}^{\varepsilon}_{P,\Lambda} : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ by[4]

$$\widetilde{G}^{\varepsilon}_{P,\Lambda}(Q,Z)_t = \log\left(\mathbb{E}\left[\int_t^{\infty} \Lambda_s^{\vartheta} \exp(P_s - \chi Q_s + (\rho+\chi)Z_s) + \varepsilon \Lambda_s^{\vartheta} \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]\right) - \vartheta \log(\Lambda_t)\,. \tag{2.5}$$

It suffices to show that there exists a unique $Q \in \mathscr{B}$ such that $Q = \widetilde{G}^{\varepsilon}_{P,\Lambda}(Q,Q)$. Note that since the first term on the right-hand side of (2.5) has càdlàg paths, every $Q \in \mathscr{B}$ satisfying $Q = \widetilde{G}^{\varepsilon}_{P,\Lambda}(Q,Q)$ corresponds to a W with càdlàg paths. By the induction hypothesis, for each fixed $Q \in \mathscr{B}$, and since $P - \chi Q \in \mathscr{B}$, there exists a unique $Z \in \mathscr{B}$ such that $Z = \widetilde{G}^{\varepsilon}_{P,\Lambda}(Q,Z)$. So, we can define the operator $Z^{\varepsilon}_{P,\Lambda} : \mathscr{B} \to \mathscr{B}$ implicitly by

$$Z^{\varepsilon}_{P,\Lambda}(Q) = \widetilde{G}^{\varepsilon}_{P,\Lambda}(Q, Z^{\varepsilon}_{P,\Lambda}(Q)). \tag{2.6}$$

If we can show that $Z^{\varepsilon}_{P,\Lambda}$ has a unique fixed point, we are done. To this end, arguing as above, it suffices to show that $Z^{\varepsilon}_{P,\Lambda}$ is nonincreasing and satisfies (2.2) for $\beta := \chi$.

To argue that $Z^{\varepsilon}_{P,\Lambda}$ is nonincreasing, let $Q^1, Q^2 \in \mathscr{B}$ with $Q^1 \leq Q^2$ $\mathbb{P} \otimes \mathrm{d}t$-a.e. For $i \in \{1,2\}$, set $\widetilde{C}^i := \Lambda^{\vartheta} \exp(Q^i)$ and $\widetilde{V}^i := \Lambda^{\vartheta} \exp(Z^{\varepsilon}_{P,\Lambda}(Q^i))$. Then, (2.6) implies that

$$\begin{aligned}
\widetilde{V}^i_t &= \mathbb{E}\left[\int_t^{\infty} \left(\Lambda_s^{\vartheta}\left(\frac{U_s}{\Lambda_s}\right)\left(\frac{\widetilde{C}^i_s}{\Lambda_s^{\vartheta}}\right)^{-\chi}\left(\frac{\widetilde{V}^i_s}{\Lambda_s^{\vartheta}}\right)^{\rho+\chi} + \varepsilon\Lambda_s^{\vartheta}\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^{\infty} \left(U_s\left(\widetilde{C}^i_s\right)^{-\chi}\left(\widetilde{V}^i_s\right)^{\rho+\chi} + \varepsilon\Lambda_s^{\vartheta}\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t\right].
\end{aligned}$$

[4]Here, we always choose a càdlàg version for the conditional expectation in (2.5).

Since $\widetilde{h}(t,\omega,c,v) = U_t(\omega)c^{-\chi}v^{\rho+\chi}+\varepsilon(\Lambda_t(\omega))^{\vartheta}$ satisfies (c$\downarrow$) and (v$\downarrow$), by Remark 2.11 it follows that $\widetilde{V}^1 \geq \widetilde{V}^2$, and consequently $Z^1 \geq Z^2$.

Finally, to show that $Z^{\varepsilon}_{P,\Lambda}$ satisfies (2.2) for $\beta := \chi$, let $a > 0$ and set $\Psi = (Z^{\varepsilon}_{P,\Lambda}(Q+a) - Z^{\varepsilon}_{P,\Lambda}(Q))/a \leq 0$. It suffices to show that $\Psi \geq -\chi$. Let $L := \Lambda^{\vartheta}\exp(Z^{\varepsilon}_{P,\Lambda}(Q))$. Then

$$\begin{aligned}
&L_t\exp(\Psi_t a)\\
&\quad= \Lambda_t^{\vartheta}\exp(Z^{\varepsilon}_{P,\Lambda}(Q)_t)\exp(\Psi_t a) = \Lambda_t^{\vartheta}\exp(Z^{\varepsilon}_{P,\Lambda}(Q+a)_t)\\
&\quad= \mathbb{E}\left[\int_t^{\infty}\left(\Lambda_s^{\vartheta}\exp(P_s-\chi(Q_s+a)+(\rho+\chi)Z^{\varepsilon}_{P,\Lambda}(Q+a)_s)+\varepsilon\Lambda_s^{\vartheta}\right)\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]\\
&\quad= \mathbb{E}_t\left[\int_t^{\infty}\left(\Lambda_s^{\vartheta}e^{-\chi a+(\rho+\chi)a\Psi_s}e^{P_s-\chi Q_s+(\rho+\chi)Z^{\varepsilon}_{P,\Lambda}(Q)_s}+\varepsilon\Lambda_s^{\vartheta}\right)\mathrm{d}s\right]\\
&\quad\geq L_t\exp(-\chi a),
\end{aligned}$$

where in the last line we have used that $(\rho+\chi)\Psi \geq 0$. Hence, $\exp(\Psi a) \geq \exp(-\chi a)$ and consequently $\Psi \geq -\chi$. □

We may now prove Theorem 3.1.

Proof of Theorem 3.1. The proof is formed of two parts. The first part removes the lower bound on U for $\varepsilon > 0$; the second part shows that we may remove the restriction $\varepsilon > 0$.

Let $U^n = \max\{U, \frac{1}{n}\Lambda\}$. Then, $U^n \overset{\mathbb{O}}{=} \Lambda$ for every $n \in \mathbb{N}$. Hence, by Theorem B.2, for each $n \in \mathbb{N}$, there exists W^n that satisfies $W^n_t = \mathbb{E}\left[\int_t^{\infty} U^n_s(W^n_s)^{\rho} + \varepsilon\Lambda_s^{\vartheta}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$. Since $\Lambda \in \mathbb{SO}^{\vartheta}$, there exists κ such that $\Lambda^{\vartheta} \leq \kappa J^{\Lambda^{\vartheta}}$. Hence, $W^n \geq \varepsilon J^{\Lambda^{\vartheta}} \geq \frac{\varepsilon}{\kappa}\Lambda^{\vartheta}$ and

$$U^n(W^n)^{\rho} \leq \Lambda(\varepsilon^{\rho}\kappa^{-\rho}\Lambda^{\vartheta-1}) = \kappa^{-\rho}\varepsilon^{\rho}\Lambda^{\vartheta}. \tag{2.7}$$

Since $\rho < 0$, g satisfies (v$\downarrow$). Hence, by Corollary 2.10, the sequence $(W^n)_{n\in\mathbb{N}}$ is decreasing (and positive) so it converges almost surely. Therefore, applying the Dominated Convergence Theorem with the bound in (2.7) and the condition $\Lambda \in \mathbb{SO}^{\vartheta}$, we

find that $W^* := \lim_{n\to\infty} W^n$ satisfies

$$W_t^* = \lim_{n\to\infty} \mathbb{E}\left[\int_t^\infty U_s^n (W_s^n)^\rho + \varepsilon\Lambda^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty U_s (W_s^*)^\rho + \varepsilon\Lambda_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right],$$

so that W^* is a fixed point of $F^\varepsilon_{U,\Lambda}(\cdot)$. Uniqueness follows from Corollary 2.9 since $h^{\varepsilon,\Lambda}_{EZ}(t,\omega,u,v) = uv^\rho + \varepsilon(\Lambda(t,\omega))^\vartheta$ satisfies (v$\downarrow$). This concludes the first part of the proof.

Let $U \in \mathscr{P}_+$ be such that $U \le \Lambda \in \mathbb{SO}^\vartheta$ and define the aggregator random field $h^{\varepsilon,\Lambda}_{EZ}$ by $h^{\varepsilon,\Lambda}_{EZ}(t,\omega,u,v) = uv^\rho + \varepsilon(\Lambda(t,\omega))^\vartheta$. By the preceding argument, for each $\varepsilon > 0$ there exists a utility process associated to the pair $(h^{\varepsilon,\Lambda}_{EZ}, U)$.

It follows from Corollary 2.10 that the fixed point W^ε to the operator $F^\varepsilon(\cdot)$ given in (2.1) is decreasing as $\varepsilon \searrow 0$. Define $W_t = \lim_{\varepsilon\to 0} W_t^\varepsilon$. Then,

$$\begin{aligned} W_t &= \lim_{\varepsilon\to 0} \mathbb{E}\left[\int_t^\infty U_s (W_s^\varepsilon)^\rho + \varepsilon\Lambda_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\ &= \lim_{\varepsilon\to 0} \mathbb{E}\left[\int_t^\infty h_{EZ}(U_s, W_s^\varepsilon)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] + \lim_{\varepsilon\to 0} \mathbb{E}\left[\int_t^\infty \varepsilon\Lambda_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^\infty h_{EZ}(U_s, W_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \end{aligned}$$

where the last line follows from the Monotone Convergence Theorem and the fact that h_{EZ} was chosen so that $\lim_{w\to w_0} h_{EZ}(u,w) = h_{EZ}(w,w_0)$ even for $(u,w_0) = (0,0)$ and $(u,w_0) = (\infty,\infty)$. Furthermore, $W \in \mathbb{I}(h_{EZ}, U)$ since $\mathbb{E}\left[\int_0^\infty U_s W_s^\rho \,\mathrm{d}s\right] = W_0 \le W_0^\varepsilon < \infty$. Uniqueness follows from Corollary 2.9 since h_{EZ} satisfies (v$\downarrow$). $\square$

IV.C Existence and Uniqueness of a Generalised Utility Process

To prove Theorem 3.4 we must first introduce generalisations of some well-known concepts. We focus on the supermartingale case, but the submartingale case is symmetric.

Definition C.1 (Generalised supermartingale, Doob [Doo53], Snell [Sne52])**.** A stochastic process $M = (M_t)_{t\geq 0}$ taking values in $(-\infty, \infty]$ is called a *generalised supermartingale* if, $M_t^- \in L^1$ for all $t \geq 0$, M is adapted and $M_s \geq \mathbb{E}[M_t \,|\, \mathcal{F}_s]$ for all $t \geq s \geq 0$.

Remark C.2. Since $M_t^- \in L^1$ (M_t is quasi-integrable), the conditional expectation $\mathbb{E}[M_t \,|\, \mathcal{F}_s]$ exists and is unique, even if $M_t \notin L^1$.

Compared to an (ordinary) supermartingale, a generalised supermartingale does not require $M_t \in L^1$ for all $t \geq 0$. So it is possible to have $M_s = +\infty \geq \mathbb{E}[M_t \,|\, \mathcal{F}_s]$. We now need to generalise this notion even further.[5]

Definition C.3 (Generalised Optional Strong Supermartingale, Mertens [Mer72])**.** A generalised supermartingale is called a *generalised optional strong supermartingale* if it is optional and for all bounded pairs of stopping times $\tau \leq \sigma$, $M_\sigma^- \in L^1$ and $\mathbb{E}[M_\sigma \,|\, \mathcal{F}_\tau] \leq M_\tau$.

Remark C.4. Note that every càdlàg supermartingale is a (generalised) optional strong supermartingale by the Optional Sampling Theorem.

Proposition C.5. *A generalised optional strong supermartingale M that is either bounded above or below is almost surely làdlàg and for a.e. ω, the path $t \mapsto M_t(\omega)$ is right-continuous outside a countable set.*

Proof. Suppose first that M is bounded below by a constant K and define the continuous bijection $f : [K, \infty] \to [1 - e^{-K}, 1]$ by $f(x) := 1 - e^{-x}$ with the convention that $e^{-\infty} = 0$. It follows from Jensen's inequality (note that f^{-1} is convex) that $M_\tau \geq \mathbb{E}[M_\sigma \,|\, \mathcal{F}_\tau] = \mathbb{E}\left[(f^{-1} \circ f)(M_\sigma) \,\middle|\, \mathcal{F}_\tau\right] \geq f^{-1}(\mathbb{E}[f(M_\sigma) \,|\, \mathcal{F}_\tau])$. Consequently, if $\widetilde{M} = f(M)$, then for all bounded pairs of stopping times $\tau \leq \sigma$ we have $\widetilde{M}_\tau = f(M_\tau) \geq \mathbb{E}[f(M_\sigma) \,|\, \mathcal{F}_\tau] = \mathbb{E}\left[\widetilde{M}_\sigma \,\middle|\, \mathcal{F}_\tau\right]$ and $\widetilde{M}$ is a bounded optional strong supermartingale. Hence, it is làdlàg (see for example [DM82, Theoreom

[5] In [Mer72], Mertens referred to the processes in the definition that follows simply as *supermartingales*.

A1.4]). Moreover, it has a Mertens decomposition (see, for example [DM82, Theorem A1.20]) given by $\widetilde{M} = \widetilde{N} - \widetilde{A}$, where $\widetilde{N} = (\widetilde{N}_t)_{t\geq 0}$ is a càdlàg local martingale and $\widetilde{A} = (\widetilde{A}_t)_{t\geq 0}$ is a nondecreasing adapted làdlàg process. Since a noncreasing làdlàg function is (right-)continuous up to a countable set, it follows that for for a.e. ω, the path $t \mapsto \widetilde{M}_t(\omega)$ is right-continuous outside a countable set. Then, using that f^{-1} is continuous, it follows that M is làdlàg and for a.e. ω, the path $t \mapsto M_t(\omega)$ is right-continuous outside a countable set.

For the case when M is bounded above, we use the concave function $g(x) = 1-e^x$. $\square$

The following results are generalised versions of the Backwards Martingale Convergence Theorem (BMCT) and Hunt's Lemma.

Proposition C.6 (Generalised Backwards Martingale Convergence Theorem)**.** *Suppose that X is a $[0,\infty]$-valued random variable and let $\mathcal{F} \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_{-1} \supseteq \mathcal{F}_{-2} \supseteq \cdots$ be a decreasing sequence of sub-σ-algebras and $\mathcal{F}_{-\infty} := \bigcap_{k=1}^{\infty} \mathcal{F}_{-k}$. Then,*

$$\lim_{n\to\infty} \mathbb{E}\left[X \,|\, \mathcal{F}_{-n}\right] = \mathbb{E}\left[X \,|\, \mathcal{F}_{-\infty}\right], \quad \mathbb{P}\text{-}a.s.$$

Proof. For $n \in \mathbb{N}$, set $Z_{-n} := \mathbb{E}\left[X \,|\, \mathcal{F}_{-n}\right]$, and let $Z_{-\infty} := \mathbb{E}\left[X \,|\, \mathcal{F}_{-\infty}\right]$. Since $\mathcal{F}_{-\infty} \subset \mathcal{F}_{-n}$ for all $n \in \mathbb{N}$, it suffices to show that $\lim_{n\to\infty} Z_{-n} = Z_{-\infty}$ $\mathbb{P}$-a.s. on $\{Z_\infty \leq k\}$ for all $k \in \mathbb{N}$ and $\lim_{n\to\infty} Z_{-n} = Z_{-\infty}$ $\mathbb{P}$-a.s. on $\{Z_\infty = \infty\}$. The case of finite k follows from the standard BMCT via

$$\begin{aligned}\lim_{n\to\infty} Z_{-n}\mathbf{1}_{\{Z_\infty\leq k\}} &= \lim_{n\to\infty} \mathbb{E}\left[X\mathbf{1}_{\{Z_\infty\leq k\}} \,\middle|\, \mathcal{F}_{-n}\right] \\ &= \mathbb{E}\left[X\mathbf{1}_{\{Z_\infty\leq k\}} \,\middle|\, \mathcal{F}_{-\infty}\right] = Z_{-\infty}\mathbf{1}_{\{Z_\infty\leq k\}}, \qquad \mathbb{P}\text{-a.s.}\end{aligned}$$

For the other case, by the standard BMCT for fixed $k \in \mathbb{N}$,

$$\liminf_{n\to\infty} Z_{-n} \geq \lim_{n\to\infty} \mathbb{E}\left[X \wedge k \,|\, \mathcal{F}_{-n}\right] = \mathbb{E}\left[X \wedge k \,|\, \mathcal{F}_{-\infty}\right], \qquad \mathbb{P}\text{-a.s.}$$

Now taking the monotone limit as $k \to \infty$ gives $\liminf_{n\to\infty} Z_{-n} \geq \mathbb{E}\left[X \,|\, \mathcal{F}_{-\infty}\right]$ $\mathbb{P}$-a.s.

Finally, on $\{Z_{-\infty} = \infty\}$ the liminf trivially coincides with the limsup. $\square$

Lemma C.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ a nondecreasing sequence of $[0,\infty]$-valued random variables with $\lim_{n\to\infty} X_n = X$ $\mathbb{P}$-a.s. Let $\mathcal{F} \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_{-1} \supseteq \mathcal{F}_{-2} \supseteq \cdots$ be a decreasing sequence of sub-σ-algebras and $\mathcal{F}_{-\infty} := \bigcap_{k=1}^{\infty} \mathcal{F}_{-k}$. Then, $\lim_{n\to\infty} \mathbb{E}\left[X_n \,|\, \mathcal{F}_{-n}\right] = \mathbb{E}\left[X \,|\, \mathcal{F}_{-\infty}\right]$ $\mathbb{P}$-a.s.*

Proof. For $n \in \mathbb{N}$, let $Y_n = \mathbb{E}\left[X_n \,|\, \mathcal{F}_{-n}\right]$. Then, $\mathbb{E}\left[X_m \,|\, \mathcal{F}_{-n}\right] \leq Y_n \leq \mathbb{E}\left[X \,|\, \mathcal{F}_{-n}\right]$ for $m \leq n$. Now taking taking the limit as $n \to \infty$ and applying Proposition C.6 gives

$$\mathbb{E}\left[X_m \,|\, \mathcal{F}_{-\infty}\right] \leq \liminf_{n\to\infty} Y_n \leq \limsup_{n\to\infty} Y_n \leq \mathbb{E}\left[X \,|\, \mathcal{F}_{-\infty}\right] \quad \mathbb{P}\text{-a.s.}$$

Taking the limit as $m \to \infty$, the result follows from the Monotone Convergence Theorem. $\square$

We may now prove Theorem 3.4, the central result of Section 3.

Proof of Theorem 3.4. We only prove the case that $(C^n)_{n\in\mathbb{N}}$ is an increasing sequence and $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$. For the case when $(C^n)_{n\in\mathbb{N}}$ is a decreasing sequence and $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$, the proof goes through by a symmetric argument. Since $(C^n)_{n\in\mathbb{N}}$ is increasing, so is $(V^n)_{n\in\mathbb{N}}$ by Corollary 2.10. Then, $V^\dagger = \lim_{n\to\infty} V^n$ exists and $V^n \leq V^\dagger$ for each $n \in \mathbb{N}$. Further, for any bounded stopping times τ and σ with $\tau \leq \sigma$ $\mathbb{P}$-a.s.,

$$\begin{aligned} V_\tau^\dagger &= \lim_{n\to\infty} \mathbb{E}\left[\int_\tau^\sigma g(s,\omega,C_s^n,V_s^n)\,\mathrm{d}s + V_\sigma^n \,\middle|\, \mathcal{F}_\tau\right] \\ &\geq \lim_{n\to\infty} \mathbb{E}\left[\int_\tau^\sigma g(s,\omega,C_s^n,V_s^\dagger)\,\mathrm{d}s + V_\sigma^n \,\middle|\, \mathcal{F}_\tau\right] \\ &= \mathbb{E}\left[\int_\tau^\sigma g(s,\omega,C_s,V_s^\dagger)\,\mathrm{d}s + V_\sigma^\dagger \,\middle|\, \mathcal{F}_\tau\right] \end{aligned} \tag{3.1}$$

It follows that $V_\tau^\dagger \geq \mathbb{E}[V_\sigma^\dagger|\mathcal{F}_\tau]$ so that $V^\dagger$ is a nonnegative generalised optional strong supermartingale. Hence, by Proposition C.5, it is làdlàg. Since $\mathbb{E}[V_\sigma^\dagger|\mathcal{F}_\tau] \geq$

$\mathbb{E}[V^\dagger_{\sigma+}|\mathcal{F}_\tau]$, (3.1) becomes $V^\dagger_\tau \geq \mathbb{E}\left[\int_\tau^\sigma g(s,\omega,C_s,V^\dagger_s)\,\mathrm{d}s + V^\dagger_{\sigma+}\,\middle|\,\mathcal{F}_\tau\right]$. Furthermore, since $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$, $\liminf_{t\to\infty} V^\dagger_{t+} \geq 0$ a.s. and $V^\dagger$ is a supersolution.

Now, take any other arbitrary monotone sequence $(\widetilde{C}^n)_{n\in\mathbb{N}}$ whose limit is equal to C. Let $\widetilde{V}^n$ be the utility process associated to $\widetilde{C}^n$ and $\widetilde{V}^\dagger := \lim_{n\to\infty}\widetilde{V}^n$. Then, since $\widetilde{V}^n \in \mathbb{UI}(g,C)$ is a subsolution associated to (g,C) because g satisfies (c$\uparrow$), we may apply Theorem 2.8 and deduce that $V^\dagger_\tau \geq \widetilde{V}^n_\tau$ for all finite stopping times τ. Taking limits gives that $V^\dagger_\tau \geq \widetilde{V}^\dagger_\tau$. Repeating the argument with the roles of $V^\dagger$ and $\widetilde{V}^\dagger$ reversed, we find that $\widetilde{V}^\dagger_\tau \geq V^\dagger_\tau$ for all finite stopping times τ. Therefore, since $V^\dagger$ and $\widetilde{V}^\dagger$ are optional processes that agree for all finite stopping times, they agree up to indistinguishability (see, for example, [Nik06, Theorem 3.2]).

Next, we show that $V^\dagger$ is the minimal supersolution for C. Let $\overline{V}$ be any supersolution. Then, since $V^n \in \mathbb{UI}(g,C)$ is a subsolution associated to (g,C), $\overline{V}_t \geq V^n_t$ for all $t \geq 0$ by Theorem 2.8. Taking limits gives $\overline{V}_t \geq V^\dagger_t$.

Finally, we show that $V^\dagger$ is càdlàg. To this end, it suffices to show that the right-continuous process $(V^\dagger_{t+})_{t\geq 0}$ is also a supersolution. Then, by the supermartingale property of $V^\dagger$ it follows that $V^\dagger_{\tau+} = \mathbb{E}[V^\dagger_{\tau+}|\mathcal{F}_\tau] \leq \mathbb{E}[V^\dagger_\tau|\mathcal{F}_\tau] = V^\dagger_\tau$ for each bounded stopping time, and thus by the minimality of $V^\dagger$, $(V^\dagger_t)_{t\geq 0} = (V^\dagger_{t+})_{t\geq 0}$ up to indistinguishability.

To show that $(V^\dagger_{t+})_{t\geq 0}$ is indeed a supersolution, fix bounded stopping times τ and σ with $\tau \leq \sigma$. We first assume that there is $\delta > 0$ such that $\tau + \delta \leq \sigma$. Then, for each $\varepsilon < \delta$, by the fact that $V^\dagger$ is a supersolution,

$$V^\dagger_{\tau+\varepsilon} \geq \mathbb{E}\left[\int_{\tau+\varepsilon}^\sigma g(s,\omega,C_s,V^\dagger_s)\,\mathrm{d}s + V^\dagger_{\sigma+}\,\middle|\,\mathcal{F}_{\tau+\varepsilon}\right].$$

Taking the limit as $\varepsilon \to 0$, and using the fact that for a.e. ω, the path $t \mapsto V^\dagger$ is right-continuous outside a countable set by Proposition C.5, we get by Hunt's lemma in the form of Lemma C.7,

$$V^\dagger_{\tau+} \geq \mathbb{E}\left[\int_\tau^\sigma g(s,\omega,C_s,V^\dagger_{s+})\,\mathrm{d}s + V^\dagger_{\sigma+}\,\middle|\,\mathcal{F}_\tau\right]. \tag{3.2}$$

Now if σ is general, for $\delta > 0$ set $\sigma^\delta := \sigma \vee (\tau + \delta)$. Then, applying (3.2) for σ^δ gives

$$\begin{aligned} V^\dagger_{\tau+}\mathbf{1}_{\{\sigma\geq\tau+\delta\}} &\geq \mathbb{E}\left[\int_\tau^{\sigma^\delta} g(s,\omega,C_s,V^\dagger_{s+})\,\mathrm{d}s + V^\dagger_{\sigma^\delta+}\,\middle|\,\mathcal{F}_\tau\right]\mathbf{1}_{\{\sigma\geq\tau+\delta\}} \\ &= \mathbb{E}\left[\int_\tau^{\sigma} g(s,\omega,C_s,V^\dagger_{s+})\,\mathrm{d}s + V^\dagger_{\sigma+}\,\middle|\,\mathcal{F}_\tau\right]\mathbf{1}_{\{\sigma\geq\tau+\delta\}} \end{aligned}$$

Taking the limit as $\delta \to 0$ gives by monotone convergence,

$$V^\dagger_{\tau+}\mathbf{1}_{\{\sigma>\tau\}} \geq \mathbb{E}\left[\int_\tau^{\sigma} g(s,\omega,C_s,V^\dagger_{s+})\,\mathrm{d}s + V^\dagger_{\sigma+}\,\middle|\,\mathcal{F}_\tau\right]\mathbf{1}_{\{\sigma>\tau\}}.$$

Since trivially, $V^\dagger_{\tau+}\mathbf{1}_{\{\sigma=\tau\}} = \mathbb{E}\left[\int_\tau^\sigma g(s,\omega,C_s,V^\dagger_{s+})\,\mathrm{d}s + V^\dagger_{\sigma+}\,\middle|\,\mathcal{F}_\tau\right]\mathbf{1}_{\{\sigma=\tau\}}$, we conclude

$$V^\dagger_{\tau+} \geq \mathbb{E}\left[\int_\tau^{\sigma} g(s,\omega,C_s,V^\dagger_{s+})\,\mathrm{d}s + V^\dagger_{\sigma+}\,\middle|\,\mathcal{F}_\tau\right]. \qquad \square$$

IV.D Additional proofs omitted from the main text

Proof of Proposition 3.7. Suppose $V \subseteq \overline{\mathbb{R}}_+$; the case of $V \subseteq \overline{\mathbb{R}}_-$ follows by a symmetric argument.

Let $C^{2,n}$ be a nondecreasing sequence of processes in $\mathscr{E}(g)$ with limit C^2 and let $C^{1,n} := C^{2,n} \wedge C^1$. Then, $C^{1,n}$ is a monotone sequence which approximates C^1. Furthermore, let $V^{1,n} \in \mathbb{UI}(g, C^{1,n}) \subseteq \mathbb{UI}(g, C^{2,n})$ and $V^{2,n} \in \mathbb{UI}(g, C^{2,n})$ be the utility processes associated to $C^{1,n}$ and $C^{2,n}$ respectively. Then, if $V^{1,\dagger}$ and $V^{2,\dagger}$ are the generalised solutions associated to C^1 and C^2, it follows from Theorem 3.4 that $V^{1,\dagger} = \lim_{n\to\infty} V^{1,n}$ and $V^{2,\dagger} = \lim_{n\to\infty} V^{2,n}$.

Since $C^{2,n} \geq C^{1,n}$ and g satisfies $(c\uparrow)$, $g(t,\omega,C^{2,n}_t,V^{2,n}_t) \geq g(t,\omega,C^{1,n}_t,V^{2,n}_t)$ for almost all (t,ω). Hence, for all bounded stopping times $\tau \leq \sigma$,

$$V^{2,n}_\tau = \mathbb{E}\left[V^{2,n}_{\sigma+} + \int_\tau^\sigma g(s,\omega,C^{2,n}_s,V^{2,n}_s)\,\mathrm{d}s\,\middle|\,\mathcal{F}_\tau\right]$$

$$\geq \mathbb{E}\left[V^{2,n}_{\sigma+} + \int_\tau^\sigma g(s,\omega,C^{1,n}_s,V^{2,n}_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right].$$

Since also $\lim_{t\to\infty}\mathbb{E}\big[V^{2,n}_{t+}\big] = 0 \quad a.s.$, $V^{2,n}$ satisfies the definition of a supersolution associated to the pair $(g, C^{1,n})$. Hence, by Theorem 2.8 it follows that $V^{2,n}_\tau \geq V^{1,n}_\tau$ for all finite stopping times τ. Taking the limit as $n \to \infty$ gives the result. □

Proof of Lemma 4.2. The dynamics of $\hat{X}$ are given by

$$\frac{d\hat{X}_t}{\hat{X}_t} = \frac{\lambda}{R}\,\mathrm{d}B_t + \left(r + \frac{\lambda^2}{R} - \eta\right)\mathrm{d}t, \qquad \hat{X}_0 = 1.$$

Fix $\varepsilon > 0$, and let $f^\varepsilon_{EZ}(c,y,v) = f_{EZ}(c+\varepsilon y, v) = b\frac{(c+\eta\varepsilon y)^{1-S}}{1-S}((1-R)v)^\rho$. Fix an arbitrary admissible strategy $(\Pi, C) \in \mathscr{C}(x)$. The dynamics of $X + \varepsilon\hat{X} = X^{\Pi,C} + \varepsilon\hat{X}$ are given by

$$d(X_t + \varepsilon\hat{X}_t) = \Big(\sigma\Pi_t X_t + \tfrac{\lambda\varepsilon}{R}\hat{X}_t\Big)\mathrm{d}B_t + \Big(X_t\big(r + \Pi_t(\mu - r)\big) - C_t + (r + \tfrac{\lambda^2}{R} - \eta)\varepsilon\hat{X}_t\Big)\mathrm{d}t.$$

Let $\mathscr{L}^{\pi,c}$ denote the infinitesimal generator of the diffusion $X + \varepsilon\hat{X}$ when the instantaneous rates of investment and consumption are, respectively, π and c: for $h = h(x,y)$,

$$\mathscr{L}^{\pi,c}h := \left[x\left(r + \pi\sigma\lambda\right) - c + \left(r + \frac{\lambda^2}{R} - \eta\right)\varepsilon y\right]h' + \frac{1}{2}\left(\sigma\pi x + \frac{\lambda}{R}\varepsilon y\right)^2 h''.$$

The first aim is to show that $\hat{V}$ satisfies a perturbed HJB equation

$$\sup_{c\in\mathbb{R}_+,\pi\in\mathbb{R}}\left[\mathscr{L}^{\pi,c}\hat{V}(x+\varepsilon y) + f^\varepsilon_{EZ}(c,y,\hat{V}(x+\varepsilon y))\right] = 0. \tag{4.1}$$

This follows from the fact that for general $c \in \mathbb{R}_+$ and $\pi \in \mathbb{R}$

$$\mathscr{L}^{\pi,c}\hat{V}(x+\varepsilon y) + f^\varepsilon_{EZ}(c,y,\hat{V}(x+\varepsilon y)) = A^1(c,x,y) + A^2(\pi,x,y) + A^3(x,y),$$

where (noting that the argument of $\hat{V}$ and its derivatives is $(x+\varepsilon y)$ throughout),

$$\begin{aligned}
A^1(c,x,y) &= b\frac{(c+\eta\varepsilon y)^{1-S}}{1-S}((1-R)\hat{V})^\rho - \hat{V}_x\left(c+\eta\varepsilon y+\eta\frac{S}{1-S}(x+\varepsilon y)\right),\\
A^2(\pi,x,y) &= \hat{V}_x\left(x\pi\sigma\lambda+\frac{\lambda^2}{R}\varepsilon y\right)+\frac{1}{2}\hat{V}_{xx}\left(\pi\sigma x+\frac{\lambda}{R}\varepsilon y\right)^2+\frac{\lambda^2}{2}\frac{(\hat{V}_x)^2}{\hat{V}_{xx}},\\
A^3(x,y) &= (x+\varepsilon y)r\hat{V}_x-\frac{\lambda^2}{2}\frac{(\hat{V}_x)^2}{\hat{V}_{xx}}+\eta\frac{S}{1-S}(x+\varepsilon y)\hat{V}_x,
\end{aligned}$$

and the trio of inequalities $A^1 \le 0$, $A^2 \le 0$, $A^3 = 0$. Taking the derivative with respect to c we find that the maximum of $A^1(c,x,y)$ is attained when $c = \left(\frac{b((1-R)\hat{V}(x+\varepsilon y))^\rho}{\hat{V}_x(x+\varepsilon y)}\right)^{\frac{1}{S}} - \eta\varepsilon y$ and then using the explicit form of $\hat{V}$ we find that the maximising value of c is $c = \eta x$ and that $A^1(\eta x,x,y) = 0$. Similarly, by taking the derivative with respect to π, the maximum of $A^2(\pi,x,y)$ is attained when $\pi = \frac{\lambda}{\sigma x}\left(\frac{\varepsilon y}{R} - \frac{\hat{V}_x(x+\varepsilon y)}{\hat{V}_{xx}(x+\varepsilon y)}\right) = \frac{\lambda}{\sigma R}$ and then $A^2(\frac{\lambda}{\sigma R},x,y) = 0$. Finally, by using the definition of $\hat{V}$ and η we find that $A^3(x,y) = 0$. Consequently, (4.1) is satisfied and the supremum is attained. Note that, since $\varepsilon\hat{X}$ is just a scaling of the wealth process under the optimal strategy, it follows that $(\hat{V}(\varepsilon\hat{X}_t))_{t\ge 0} \in \mathbb{UI}(f_{EZ},\eta\varepsilon\hat{X})$ is the utility process associated to the consumption stream $\eta\varepsilon\hat{X}$. Consequently, $\lim_{t\to\infty}\mathbb{E}[\hat{V}(\varepsilon\hat{X}_{t+})] = 0$.

Fix arbitrary bounded stopping times $\tau \le \sigma$, define $N = (N_t)_{t\ge 0}$ by

$$N_t = \int_0^t \hat{V}_x(X_u+\varepsilon\hat{X}_u)\left(\sigma\Pi_u X_u+\frac{\lambda}{R}\varepsilon\hat{X}_u\right)dW_u$$

and for $n \in \mathbb{N}$, set $\zeta_n := \inf\{s \ge \tau : \langle N\rangle_s - \langle N\rangle_\tau \ge n\}$. It follows by Itô's lemma, (4.1) and the definition of f^ε_{EZ} that

$$\begin{aligned}
&\hat{V}(X_\tau+\varepsilon\hat{X}_\tau)\\
&= \hat{V}(X_{\sigma\wedge\zeta_n}+\varepsilon\hat{X}_{\sigma\wedge\zeta_n}) - \int_\tau^{\sigma\wedge\zeta_n}\mathscr{L}^{C_s,\Pi_s}\hat{V}(X_s+\varepsilon\hat{X}_s)\,\mathrm{d}s + N_\tau - N_{\sigma\wedge\zeta_n}\\
&\ge \hat{V}(X_{\sigma\wedge\zeta_n}+\varepsilon\hat{X}_{\sigma\wedge\zeta_n}) + \int_\tau^{\sigma\wedge\zeta_n} f^\varepsilon_{EZ}(C_s,\hat{X}_s,\hat{V}(X_s+\varepsilon\hat{X}_s))\,\mathrm{d}s + N_\tau - N_{\sigma\wedge\zeta_n}
\end{aligned}$$

$$= \hat{V}(X_{\sigma\wedge\zeta_n} + \varepsilon\hat{X}_{\sigma\wedge\zeta_n}) + \int_\tau^{\sigma\wedge\zeta_n} f_{EZ}(C_s + \eta\varepsilon\hat{X}_s, \hat{V}(X_s + \varepsilon\hat{X}_s))\,\mathrm{d}s + N_\tau - N_{\sigma\wedge\zeta_n}.$$

Taking conditional expectations and using that $(N_{t\wedge\zeta_n} - N_{t\wedge\tau})_{t\geq 0}$ is an L^2-bounded martingale, the Optional Sampling Theorem gives

$$\begin{aligned}
&\hat{V}(X_\tau + \varepsilon\hat{X}_\tau) \\
&\quad \geq \mathbb{E}\left[\hat{V}(X_{\sigma\wedge\zeta_n} + \varepsilon\hat{X}_{\sigma\wedge\zeta_n}) + \int_\tau^{\sigma\wedge\zeta_n} f_{EZ}(C_s + \eta\varepsilon\hat{X}_s, \hat{V}(X_s + \varepsilon\hat{X}_s))\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right].
\end{aligned}$$

Since $\hat{V}$ is increasing and wealth is nonnegative, $\hat{V}(X_{\sigma\wedge\zeta_n} + \varepsilon\hat{X}_{\sigma\wedge\zeta_n}) \geq \hat{V}(\varepsilon\hat{X}_{\sigma\wedge\zeta_n})$. Using that $(\hat{V}(\varepsilon\hat{X}_t))_{t\geq 0}$ is uniformly integrable, taking the liminf as $n \to \infty$, the generalised conditional version of Fatou's Lemma and the conditional Monotone Convergence Theorem yield

$$\hat{V}(X_\tau + \varepsilon\hat{X}_\tau) \geq \mathbb{E}\left[\hat{V}(X_\sigma + \varepsilon\hat{X}_\sigma) + \int_\tau^{\sigma} f_{EZ}(C_s + \eta\varepsilon\hat{X}_s, \hat{V}(X_s + \varepsilon\hat{X}_s))\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right].$$

Furthermore, $\liminf_{t\to\infty} \mathbb{E}[\hat{V}(X_{t+} + \varepsilon\hat{X}_{t+})] \geq \lim_{t\to\infty} \mathbb{E}[\hat{V}(\varepsilon\hat{X}_{t+})] = 0$. Consequently, $\hat{V}(X + \varepsilon\hat{X})$ is a supersolution associated to the pair $(f_{EZ}, C + \eta\varepsilon\hat{X})$. $\square$

CHAPTER

V

STOCHASTIC DIFFERENTIAL UTILITY: PROPER SOLUTIONS WHEN $\vartheta > 1$

Epstein–Zin stochastic differential utility is parametrised by two coefficients, R and S, taking values in $(0,1) \cup (1,\infty)$ and corresponding to the agent's risk aversion and their elasticity of intertemporal complementarity. A detailed explanation of the roles of R and S is provided in Chapter III. The parameter $\vartheta := \frac{1-R}{1-S}$ is critical.

First, it is argued in Chapter III that $\vartheta > 0$ is necessary for the EZ-SDU equation to have a meaningful solution over the infinite horizon. Second, the mathematics of the problem is vastly different depending on whether $\vartheta \in (0,1)$ or $\vartheta \in (1,\infty)$. The boundary case of $\vartheta = 1$ (corresponding to CRRA utility) was studied in Chapter II and the case $\vartheta \in (0,1)$ was studied in Chapters III and IV. This chapter will be

dedicated to parameter combinations leading to $\vartheta > 1$, a case that has been neglected in the literature.

One of the desirable properties of EZ-SDU when $\vartheta \in (0, 1]$ is that all evaluable consumption streams are uniquely evaluable—if there exists a solution to the EZ-SDU equation (I.1.7), then it is necessarily unique. This follows from a comparison theorem (Theorem IV.2.8) which shows that a *subsolution* to the EZ-SDU equation always lies below a *supersolution*; since a solution is both a subsolution and a supersolution, applying the comparison theorem twice to two such solutions yields uniqueness (Corollary IV.2.9). In the case $\vartheta > 1$, however, the requirements of the comparison theorem are not met since the EZ aggregator f_{EZ} is increasing in v.

One might hope that this is a technical issue and that, by being smarter, it is possible to adapt the comparison theorem to the case $\vartheta > 1$, thus resolving issues of uniqueness. However, this is not the case—the problem with $\vartheta > 1$ is fundamentally different to the problem with $\vartheta < 1$, and when $\vartheta > 1$ it is not just that the comparison theorem fails but rather that nonuniqueness is endemic to the problem. Note that the same issue of nonuniqueness arises in finite-horizon EZ-SDU unless a nonzero bequest function is added at the terminal time.

The main goals of this chapter are to illuminate *how* nonuniqueness may occur and to provide an economically motivated criterion for selecting the "true" solution before finally solving the investment-consumption problem.

We begin by studying the utility process associated to constant proportional investment-consumption strategies. In this case, an explicit, time-homogeneous utility process is provided in Proposition III.3.2 (on page 63) which is valid in the case $\vartheta > 1$. However, we show in Section 1 that this solution is *not the only solution*, and there exists an infinite family of (equally explicit but time-inhomogeneous) utility processes.

It is clear that to formulate Merton's optimal investment-consumption problem for EZ-SDU, there must be a rule that assigns a particular utility process to each consumption stream over which we maximise. Various candidates for this assignation

rule are plausible. Perhaps the most obvious choice is the maximal utility process. The rationale behind this would be that the agent gets to choose which utility process they associate to a given consumption stream, naturally choosing the best one. However, when $R > 1$, the maximal utility process associated to any consumption stream is the zero process, rendering the problem degenerate. An alternative choice might be the "game theoretic" or minimax version of the Merton problem, where the agent maximises the worst utility process associated to each consumption stream. However, when $R < 1$, the minimal utility process associated to any consumption stream is the zero process, again rendering the problem degenerate. Instead, one of the main contributions of the chapter is to introduce the notion of a *proper* solution.

It will follow from the discussion of constant proportional investment-consumption strategies in Section 1 below that if C is the consumption stream which arises from a constant proportional strategy (and if $\mathbb{E}[\int_0^\infty C_s^{1-R} ds] < \infty$), then for each $T \in [0, \infty]$ there exists a utility process associated to C which is strictly positive for $t < T$ and zero for $t \geq T$. (In particular, zero is a solution and corresponds to $T = 0$). Economically, this may be interpreted as saying that the amount consumed after time T has no effect on the agent's utility. It is hard to justify that this represents a meaningful version of the utility process, and motivates the definition of a proper solution: a solution $V = (V_t)_{t \geq 0}$ (of the EZ-SDU equation, given below in (1.2)) is proper if $\mathbb{E}\big[\int_t^\infty C_s^{1-R} \, \mathrm{d}s \;\big|\; \mathcal{F}_t\big] > 0$ implies that $(1-R)V_t > 0$.

We also introduce other notions of a solution of a more mathematical nature, namely the concepts of an *extremal solution* (which corresponds either to the maximal solution or the minimal solution, depending on the sign of $\mathbb{V} = (1-R)\mathbb{R}_+$) and the *CRRA-order solution* (a solution such that $(1-R)V \overset{\mathbb{O}}{=} J^{C^{1-R}}$, where the order relation $\overset{\mathbb{O}}{=}$ is given in Definition IV.1.1).

The main results of the chapter are the following—and more precise statements follow as Theorems 3.3, 3.5, 3.6 and 3.8 respectively.

Main Result 1. *For a very wide class of consumption processes C, there exists a proper utility process V associated to C.*

Main Result 2. *For a wide class of consumption processes, the proper utility process is unique.*

Main Result 3. *For a wide class of consumption processes, the three solution concepts agree.*

Main Result 4. *In the constant parameter financial model, if we maximise over attainable consumptions streams which have an associated unique proper utility process, then the investment-consumption problem is solved by a constant proportional investment-consumption strategy* (*whose parameters may be identified in terms of the parameters of the EZ-SDU and the financial market*).

The remainder of the chapter is structured as follows. In Section 1, we introduce EZ-SDU and Merton's investment consumption problem under EZ-SDU. We show that we may associate to each constant proportional investment-consumption strategy a family of EZ-SDU processes indexed by $T \in [0, \infty]$, each of which corresponds to ignoring consumption from time T onwards. In Section 2, we define the notions of a *proper solution*, an *extremal solution* and a *CRRA-order solution*. The main results are restated precisely in Section 3.

Section 4 shows how a change of coordinates can simplify the problem—and until we consider the verification theorem we work in this new coordinate system. Section 5 is dedicated to proving the existence of a *CRRA-order solution*. It relies heavily on the existence results of Chapter IV.

In Section 6, we recall the definition of subsolutions and supersolutions to the EZ-SDU equation from Chapter IV. We then prove a comparison theorem that provides a sufficient criterion under which subsolutions are dominated from above by supersolutions.

In Section 7, we prove existence and uniqueness of the extremal solution for a class of consumption streams.

Section 8 considers existence and uniqueness of proper solutions. Furthermore, we show that all three solution concepts agree for consumption streams that are

bounded above and below by constant multiples of constant proportional strategies.

Finally, in Section 9 we verify that the candidate solution we propose in Section 1 is optimal over the class of right-continuous attainable consumption streams to which we may assign a unique proper solution.

Throughout this chapter we make the following additional assumption.

Standing Assumption 5. *The filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ *is continuous.*[1]

1 Infinite-horizon Epstein–Zin stochastic differential utility and the Merton investment-consumption problem

The Merton investment-consumption problem that we consider in this chapter will be similar to that outlined in Chapter III on page 58. In particular, the goal of the Merton investment-consumption problem is (for some as yet undefined set $\mathscr{U}$) to find

$$V^*(x) = \sup_{C\in\mathscr{C}(x)\cap\mathscr{U}} V_0^C, \tag{1.1}$$

where $V^C = V = (V_t)_{t\geq 0}$ solves

$$V_t = \mathbb{E}\left[\int_t^\infty \frac{C_s^{1-S}}{1-S}\left((1-R)V_s\right)^\rho \,\mathrm{d}s \middle| \mathcal{F}_t\right], \quad \text{for } t \geq 0. \tag{1.2}$$

Remark 1.1. It is worth noting here that we have set $b = 1$ and $\delta = 0$ in the Epstein–Zin aggregator g_{EZ} (given in (III.2.2) on page 53) without loss of generality. We can set $b = 1$ because of the scaling properties of EZ-SDU (see Remark III.2.2 on page 55) and we can set $\delta = 0$ by absorbing the discount factor into the financial market as in Section III.3.2.

When $\vartheta > 1$, (1.2) may not possess a unique solution, so the problem in (1.1)

[1] This is slightly stronger than the *right-continuity* assumed in the *usual conditions*. However, it is a necessary assumption for the existence arguments in Appendix V.A to go through. Whilst we do not believe that left-continuity of the filtration is a necessary assumption for existence of a proper solution to hold in general, it is a pragmatic and convenient assumption, and it does cover the main application of a Brownian filtration and a constant parameter Black-Scholes-Merton financial market.

is not well formulated until we have decided *which* utility process to assign to each consumption and which consumption streams we want to optimise over. Section 1.2 illustrates what can go wrong by providing a family of utility processes associated to the constant proportional investment-consumption strategies. We will then explain in Section 2 how to select the economically meaningful utility process from the many available and how this impacts the control problem (1.1). We reformulate the problem in Section 9.

1.1 A time-homogeneous utility process for constant proportional strategies

We recall from Section III.3.3 on page 61 that we may find explicit (time-homogeneous) solutions associated to constant proportional strategies. This will provide useful in illustrating what can go wrong when $\vartheta > 1$.

Consider the proportional investment-consumption strategy $\Pi \equiv \pi \in \mathbb{R}$ and $C = \xi X$ for $\xi \in \mathbb{R}_+$. Fixing (π, ξ), and using Itô's lemma and the dynamics of the wealth process X given in (I.4.2), we find $X_t^{1-R} = x^{1-R} e^{\pi\sigma(1-R)B_t + (1-R)\left(r + (\mu - r)\pi - \xi - \frac{\pi^2\sigma^2}{2}\right)t}$. In particular, for $s \geq t$, $\mathbb{E}\left[X_s^{1-R} \,\middle|\, \mathcal{F}_t\right] = X_t^{1-R} e^{-H(\pi,\xi)(s-t)}$, where

$$H(\pi, \xi) = (R-1)\left(r + \lambda(\mu - r)\pi - \xi - \frac{\pi^2\sigma^2}{2}R\right).$$

Furthermore, by Proposition III.3.2 (on page 63), one EZ-SDU process $V = (V_t)_{t \geq 0}$ associated to the strategy $(\pi, \xi X)$ such that $H(\pi, \xi) > 0$ is given by

$$V_t = h(\pi, \xi) X_t^{1-R}, \qquad \text{where} \qquad h(\pi, \xi) = \frac{\xi^{1-R}}{1-R}\left(\frac{\vartheta}{H(\pi, \xi)}\right)^{\vartheta}. \tag{1.3}$$

However, the next section shows that it is far from being the unique utility process.

Remark 1.2. For future reference note that, provided $H(\pi, \xi) > 0$, if C is the constant proportional consumption stream associated with parameters (π, ξ), then C^{1-R} is a geometric Brownian motion and $\mathbb{E}\left[\int_t^\infty C_s^{1-R} ds \,\middle|\, \mathcal{F}_t\right] = \frac{1}{H(\pi,\xi)} C_t^{1-R}$.

1.2 A family of utility processes indexed by absorption time for constant proportional strategies

In this section, we show that for each proportional consumption stream $(\pi, \xi X)$, where $\pi \in \mathbb{R}$, $\xi \in \mathbb{R}_{++}$ satisfy $H(\pi, \xi) > 0$, the utility process given in (1.3) is not the only associated utility process. In particular, there exists a family of utility processes—which we can write in explicit form—parametrised by the first time they hit zero (and are absorbed).

We postulate a time-dependent form of the utility process $V = (V_t)_{t \geq 0}$ given by $V_t = \frac{A(t)\xi^{1-R}}{1-R} X_t^{1-R}$. Finding a solution associated to the constant proportional investment-consumption strategy with parameters (π, ξ) then becomes that of solving

$$\frac{A(t)\xi^{1-R}}{1-R} X_t^{1-R} = V_t = \mathbb{E}\left[\int_t^\infty \frac{\xi^{1-S}}{1-S} X_s^{1-S} \left(A(s)\xi^{1-R} X_s^{1-R}\right)^\rho \, \mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$$

Using the expression for $\mathbb{E}\left[X_s^{1-R} \,\middle|\, \mathcal{F}_t\right]$ given above we find that A solves $A(t)e^{-H(\pi,\xi)t} = \vartheta \int_t^\infty e^{-H(\pi,\xi)s} A(s)^\rho \, \mathrm{d}s$ and taking derivatives with respect to t yields the ODE for $A(t)$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = H(\pi, \xi)A(t) - \vartheta A(t)^\rho. \tag{1.4}$$

Note that one solution is the constant solution $A(t) = \left(\frac{\vartheta}{H(\pi,\xi)}\right)^\vartheta$ (which leads to (1.3)). More generally, the ODE (1.4) is separable and can be solved to give

$$A(t) = \left(\frac{\vartheta - \left(\vartheta - H(\pi, \xi)A(0)^{1/\vartheta}\right) e^{\frac{H(\pi,\xi)}{\vartheta} t}}{H(\pi, \xi)}\right)^\vartheta$$

If we assume that $A(0) \in \left(0, \left(\frac{\vartheta}{H(\pi,\xi)}\right)^\vartheta\right)$, then A hits zero at $t = T$ where $T = \frac{\vartheta}{H(\pi,\xi)} \ln\left(\frac{\vartheta}{\vartheta - H(\pi,\xi)A(0)^{1/\vartheta}}\right)$. Since $(A(t))_{t \geq T} \equiv 0$ is a solution on $[T, \infty)$ we can define a family of solutions to (1.4) and hence to (1.2), indexed by $A(0)$ such that

$$A(t) = \begin{cases} \left(\frac{\vartheta - \left(\vartheta - H(\pi,\xi)A(0)^{1/\vartheta}\right) e^{\frac{H(\pi,\xi)}{\vartheta} t}}{H(\pi,\xi)}\right)^\vartheta & t < T = \frac{\vartheta}{H(\pi,\xi)} \ln\left(\frac{\vartheta}{\vartheta - H(\pi,\xi)A(0)^{1/\vartheta}}\right) \\ 0 & t \geq T \end{cases}$$

(Note that if $A(0) > \left(\frac{\vartheta}{H(\pi,\xi)}\right)^{\vartheta}$ then A diverges to plus infinity, but this is not consistent with the fact that $\mathbb{E}[V_t] \to 0$.) Alternatively, the family of solutions can be thought of as indexed by T where $T = \inf\{t \geq 0 : V_t = 0\}$. Effectively, for the solution indexed by T, consumption after T does not yield any utility, and the utility process is zero thereafter.

It is hard to argue that, when considering the infinite time horizon, this represents an economically meaningful reduction of the problem. Hence, intuitively the "correct" utility process should correspond to $T = \infty$ and be given by (1.3).

Remark 1.3. This issue also arises when trying to evaluate finite-horizon EZ-SDU. A variant of the Merton problem for finite-horizon EZ-SDU and $\vartheta > 1$ is considered by [MX18, SS99, SS16, Xin17], among others. In [MX18, SS16, Xin17] the issue of uniqueness is finessed by incorporating a strictly positive bequest function $U_\varepsilon(C_T) = \varepsilon \frac{C_T^{1-R}}{1-R}$ with $\varepsilon > 0$ at the finite time horizon T (and either restricting to consumption streams that are strictly positive or only considering the case $R > 1$), so that the EZ-SDU equation in this case becomes

$$V_t = \mathbb{E}\left[\int_t^T \frac{C_s^{1-S}}{1-S}\left((1-R)V_s\right)^{\rho}\,\mathrm{d}s + U_\varepsilon(C_T)\middle|\mathcal{F}_t\right], \quad \text{for all } 0 \leq t \leq T. \tag{1.5}$$

This is not a viable approach in the infinite-horizon case, as a bequest "at infinity" has no meaning. In [SS99], the authors claim that EZ-SDU processes are unique by finding a solution to (1.5), letting $\varepsilon \searrow 0$, and then claiming that the limiting process—which is an EZ-SDU process for (1.5) with zero bequest—is the unique EZ-SDU process. As we have seen in this section, this is not the case.

The approach taken in this chapter is to embrace the multiple utility processes and distinguish the *proper* utility process (which we introduce in the next section) from other utility processes. The aim of the proper utility process is to rule out solutions which ignore the utility gained from consumption from some finite time onwards.

2 Three solution concepts and three spaces of consumption streams

2.1 Solution concepts

The following definition of a *proper* solution to the EZ-SDU equation is motivated by the arguments of the previous section.

Definition 2.1. Let $C \in \mathscr{P}_+$ and suppose that $V = (V_t)_{t\geq 0}$ is a solution to (III.2.4). V is a *proper solution* if, for all $t \geq 0$, $(1-R)V_t > 0$ when $\mathbb{E}\left[\int_t^\infty C_s^{1-R}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] > 0$ up to null sets.

The notion of a proper solution immediately excludes the time-inhomogeneous utility processes found in Section 1.2.

Remark 2.2. Note that the reverse implication automatically holds, i.e. $(1-R)V_t > 0$ implies that $\mathbb{E}\left[\int_t^\infty C_s^{1-R}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] > 0$ up to null sets. Suppose instead that there were $D \in \mathcal{F}_t$ such that $\mathbf{1}_D(1-R)V_t > 0$ but $\mathbf{1}_D\mathbb{E}\left[\int_t^\infty C_s^{1-R}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = 0$. Then, $\mathbb{E}\left[\mathbf{1}_D \int_t^\infty C_s^{1-S}\,\mathrm{d}s\right] = 0$ and $(C_s(\omega))^{1-S} = 0$ for $\mathrm{d}t\times\mathrm{d}\mathbb{P}$ almost all (s,ω) in $[t,\infty)\times D$. Consequently, $\mathbb{E}[\mathbf{1}_D V_t] = \mathbb{E}\left[\mathbf{1}_D \int_t^\infty f(C_s, V_s)\,\mathrm{d}s\right] = 0$. Since $(1-R)V_t > 0$ on D, it must follow that the set D has zero measure.

We also define the *extremal* solution. This corresponds to either the maximal solution or the minimal solution depending on the sign of $\mathbb{V} = (1-R)\mathbb{R}_+$. Such solutions frequently appear in BSDE theory (see [DHK13, DKRGT16, Pen99]).

Definition 2.3. Let $C \in \mathscr{P}_+$ and suppose that $V = (V_t)_{t\geq 0}$ is a solution to (1.2). V is an *extremal* solution if $(1-R)V \geq (1-R)Y$ for any other solution $Y = (Y_t)_{t\geq 0}$.

Remark 2.4. If there exists a proper solution, then extremal solutions are proper. This follows from Definitions 2.1 and 2.3; if Y is a proper solution and V is an extremal solution, then up to null sets

$$\left\{\omega : \mathbb{E}\left[\int_t^\infty C_s^{1-R}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] > 0\right\} \subseteq \{\omega : (1-R)Y_t > 0\} \subseteq \{\omega : (1-R)V_t > 0\}.$$

The third solution concept that will be useful is a CRRA-order solution. It relies on the order equivalence relation given in Definition IV.1.1 on page 93. We also recall the notation $J^X = (J^X)_{t\geq 0}$ for $X \in \mathscr{P}_+$, where $J_t^X = \mathbb{E}\left[\int_t^\infty X_s ds \,\middle|\, \mathcal{F}_t\right]$.

Definition 2.5. Let $C \in \mathscr{P}_+$ and suppose that $V = (V_t)_{t\geq 0}$ is a solution to (1.2). We say that V is a CRRA-order solution if $(1-R)V \overset{\mathbb{O}}{=} J^{C^{1-R}}$.

Remark 2.6. If V is a CRRA-order solution, then V is a proper solution.

2.2 Spaces of consumption streams

Let $\mathbb{BC}^{\pi,\xi}$ be the class of consumption streams that are the same order as a constant proportional strategy with parameters (π, ξ), $\mathbb{BC}^{\pi,\xi} := \left\{C \in \mathscr{P}_{++} :\ C \overset{\mathbb{O}}{=} \xi X^{\pi,\xi X}\right\}$, and set $\mathbb{BC} = \cup_{\pi,\xi: H(\pi,\xi)>0} \mathbb{BC}^{\pi,\xi}$. Then, $\mathbb{BC}$ is the set of consumption streams which are the same order as a constant proportional strategy with a finite utility process.

Another useful class of consumption streams will be those of *self-order*. We recall from Definition IV.1.2 that the space of self-order processes is given by

$$\mathbb{SO} := \left\{X \in \mathscr{P}_{++} :\ \mathbb{E}\left[\int_0^\infty X_t \,\mathrm{d}t\right] < \infty \text{ and } X \overset{\mathbb{O}}{=} J^X\right\}.$$

Furthermore, define $\mathbb{SO}(k, K) = \left\{X \in \mathbb{SO} :\ kJ^X \leq X \leq KJ^X\right\}$. On some occasions we need a slightly stronger condition. Define

$$\mathbb{SO}_\nu := \left\{X \in \mathbb{SO} :\ \left(e^{\nu t} X_t\right)_{t\geq 0} \in \mathbb{SO}\right\} \quad \text{for } \nu \geq 0, \qquad \text{and} \qquad \mathbb{SO}_+ = \cup_{\nu>0} \mathbb{SO}_\nu.$$

For arbitrary $\alpha \in \mathbb{R}$, let $\mathbb{SO}^\alpha$, $\mathbb{SO}_\nu^\alpha$ and $\mathbb{SO}_+^\alpha$ be the sets of processes X such that X^α is in $\mathbb{SO}$, $\mathbb{SO}_\nu$ and $\mathbb{SO}_+$ respectively.

Remark 2.7. If $Y \in \mathbb{SO}$, then $KY \in \mathbb{SO}$ for any $K > 0$.

The following two lemmas provide some set inclusions.

Lemma 2.8. *For* $0 \leq \mu \leq \nu$, $\mathbb{SO}_\mu \supseteq \mathbb{SO}_\nu$.

Proof. Suppose that $X \in \mathbb{SO}_\nu$ and $\mu \leq \nu$. Then, there exists $k > 0$ such that

$$X_t \geq k\mathbb{E}\left[\int_t^\infty e^{\nu(s-t)} X_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \geq k\mathbb{E}\left[\int_t^\infty e^{\mu(s-t)} X_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]. \tag{2.1}$$

Furthermore, since $X \in \mathbb{SO}$,

$$X_t \leq K\mathbb{E}\left[\int_t^\infty X_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \leq K\mathbb{E}\left[\int_t^\infty e^{\mu(s-t)} X_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]. \qquad \square$$

Remark 2.9. For $\nu \geq 0$, let $(k(\nu), K(\nu))$ be the tightest interval such that $(e^{\nu t} X_t)_{t \geq 0} \in \mathbb{SO}(k(\nu), K(\nu))$. Then, $k(\nu)$ is decreasing in ν by (2.1). Furthermore, since $X \in \mathbb{SO}$, $k(\nu) \leq k(0) < \infty$. By a symmetric argument, one can show that $K(\nu)$ is decreasing in ν as well.

Lemma 2.10. $\mathbb{BC} \subseteq \mathbb{SO}_+^{1-R} \subseteq \mathbb{SO}^{1-R}$.

Proof. Take $C \in \mathbb{BC}$. Then, there exists $\pi \in \mathbb{R}$ and $\xi \in \mathbb{R}_{++}$ and $C^\dagger = \xi X^{\pi,\xi X}$ such that $C \overset{\mathbb{O}}{=} C^\dagger$. Note that $Z^{(\nu)}$ defined by $Z_t^{(\nu)} = e^{\nu t}(C_t^\dagger)^{1-R}$ is a geometric Brownian motion and $\mathbb{E}\big[Z_t^{(\nu)} \big| \mathcal{F}_s\big] = Z_s^{(\nu)} e^{(\nu - H(\pi,\xi))(s-t)}$ for $s \geq t$. Let $\nu < H(\pi, \xi)$ and $Z = Z^{(\nu)}$. Then, $Z \overset{\mathbb{O}}{=} J^Z$ by Remark 1.2.

Define Y by $Y_t = e^{\nu t} C^{1-R}$. Then, $Y \overset{\mathbb{O}}{=} Z$ since $C \overset{\mathbb{O}}{=} C^\dagger$. By integrating from time t onwards and taking conditional expectations $J^Y \overset{\mathbb{O}}{=} J^Z$. Since $\overset{\mathbb{O}}{=}$ is an equivalence relation, combining these gives $Y \overset{\mathbb{O}}{=} Z \overset{\mathbb{O}}{=} J^Z \overset{\mathbb{O}}{=} J^Y$ and $Y \in \mathbb{SO}$. Thus, $C \in \mathbb{SO}_\nu^{1-R} \subset \mathbb{SO}_+^{1-R}$. $\square$

3 Main theorems

The four theorems in this section correspond to the four main results presented in the introduction. Proving the results in this section will constitute the majority of the rest of this chapter.

The first proposition states that all consumption streams such that C^{1-R} is a self-order process have a CRRA-order solution associated to them.

Proposition 3.1. *Suppose that $C^{1-R} \in \mathbb{SO}$. Then, there exists a unique CRRA-order solution to* (1.2).

The next result says that we may relax the lower bound on the consumption streams in Proposition 3.1 and still find a solution V. In particular, we may evaluate consumption streams such that C^{1-R} is bounded above by a process in $\mathbb{SO}$. We find an extremal solution in these cases.

Proposition 3.2. *For each $C \in \mathscr{P}_+$ such that $C^{1-R} \leq Y \in \mathbb{SO}_+$, there exists a unique extremal solution V^C to* (1.2). *Furthermore, V^C is increasing in C.*

We now turn to *proper solutions*. The following result shows that we may find a proper solution associated to a large class of consumption streams.

Theorem 3.3. *Suppose that $C \in \mathscr{P}_+$ is a right-continuous consumption stream such that $C^{1-R} \leq Y \in \mathbb{SO}_+$. Then, there exists a proper solution to* (1.2).

Proper solutions are an economically meaningful concept that allow us to choose from the many solutions to the EZ-SDU equation and Theorem 3.3 provides a large class of consumption streams which have proper solutions. However, we have not yet discussed their uniqueness. If the property of being proper does not provide a criteria for selecting a unique solution, then it does not help to overcome the issues of nonuniqueness intrinsic to EZ-SDU when $\vartheta > 1$. The following definition is therefore of great importance.

Definition 3.4. We say that $C \in \mathscr{P}_+$ is *uniquely proper* if there exists a unique proper solution to (1.2). Let $\mathbb{UP}$ denote the set of uniquely proper consumption streams.

Theorem 3.5. $\mathbb{SO}_+^{1-R} \subseteq \mathbb{UP}$.

Theorem 3.6. *Suppose that $C \in \mathbb{SO}_+^{1-R}$. Then, the following three solutions to* (1.2) *all coincide:*

1. *the CRRA-order solution associated to C given in Proposition 3.1;*

2. *the unique extremal solution;*

3. *the unique proper solution.*

Finally, in Section 9, we prove a verification theorem that shows that the candidate optimal proportional investment-consumption strategy given in Proposition III.3.2 (on page 63) is optimal over all attainable and uniquely proper right-continuous consumption streams.

Definition 3.7. Let $\mathbb{UP}^*$ be the restriction of $\mathbb{UP}$ to the right-continuous processes.

Theorem 3.8 (Verification Theorem)**.** *Let initial wealth be $x > 0$ and suppose that $\vartheta > 1$ and $\eta > 0$ where η is defined in* (III.3.6)*. If V^C is the unique proper utility process associated to C and $\hat{V}(x) = \eta^{-\vartheta S}\frac{x^{1-R}}{1-R}$ is the candidate optimal utility, then*

$$\sup_{C \in \mathscr{C}(x) \cap \mathbb{UP}^*} V_0^C = V_0^{\hat{C}} = \hat{V}(x)$$

and the optimal investment-consumption strategy is given by $(\hat{\Pi}, \hat{C})$.

4 A change of coordinates

In this section we make a similar change of coordinates to that used in Section IV.1. Define the nonnegative processes $W = (1-R)V$ and $U = U(C) = \vartheta C^{1-S}$ and the aggregator

$$h_{EZ}(u, w) = uw^{\rho}. \tag{4.1}$$

Further define $J = J^{U^\vartheta}$ by

$$J_t = \mathbb{E}\left[\int_t^\infty U_s^\vartheta \, \mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \quad \text{for all } t \geq 0. \tag{4.2}$$

Remark 4.1. Note that $V \in \mathbb{I}(f_{EZ}, C)$ if and only if $W \in \mathbb{I}(h_{EZ}, U(C))$. Hence, V^C is a utility process associated to consumption stream C with aggregator f_{EZ} if and

only if $W = W^{U(C)}$ is a utility process associated to consumption stream $U(C)$ with aggregator h_{EZ}.

$V = (V_t)_{t\geq 0}$ is a proper solution associated to (f_{EZ}, C) if and only if $W = (W_t)_{t\geq 0}$ is a solution associated to (h_{EZ}, U) such that $\{\omega : \mathbb{E}\left[\int_t^\infty U_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right](\omega) > 0\} \setminus \{\omega : W_t(\omega) > 0\}$ is a null set for all $t \geq 0$. In a slight abuse of the definition, we then also refer to W as being proper.

Furthermore, V is a CRRA-order solution if and only if W is solution such that $W \overset{\mathbb{O}}{=} J$ and V is an extremal solution if and only if W is a maximal solution. We will work in this coordinate system until Section 9 and prove existence and uniqueness results in these coordinates; translation to the original coordinate system is immediate.

5 The CRRA-order solution

This section will be dedicated to proving the existence and local uniqueness of CRRA-order solutions, as well as giving some explicit bounds on the associated utility process. The main result has already been proved and is given by Theorem IV.B.2. Since it is crucial to this section and Section 7, we restate it. The original theorem states that $W \overset{\mathbb{O}}{=} \Lambda^\vartheta$, but since $\Lambda^\vartheta \overset{\mathbb{O}}{=} J^{\Lambda^\vartheta} \overset{\mathbb{O}}{=} J^{U^\vartheta} =: J$, the result holds with $W \overset{\mathbb{O}}{=} J$.

Theorem 5.1. *Let $\varepsilon \geq 0$ and let $U \overset{\mathbb{O}}{=} \Lambda \in \mathbb{SO}^\vartheta$. Let $J = (J_t)_{t\geq 0}$ be defined by* (4.2). *Then, $F^\varepsilon_{U,\Lambda} : \mathbb{I}(h_{EZ}, U) \to \mathscr{P}_+$, defined by*

$$F^\varepsilon_{U,\Lambda}(W)_t = \mathbb{E}\left[\int_t^\infty U_s W_s^\rho + \varepsilon \Lambda_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right],$$

has a fixed point $W \in \mathbb{I}(h_{EZ}, U)$ which has càdlàg paths. It is the unique fixed point such that $W \overset{\mathbb{O}}{=} J$.

Remark 5.2. Often this theorem will be applied with $\Lambda = U$ and $\varepsilon = 0$. In that case if $U^\vartheta \in \mathbb{SO}$ then there exists a fixed point W of the operator F_U and $W \overset{\mathbb{O}}{=} J$. In particular, Proposition 3.1 holds.

The following Corollary gives explicit bounds $\hat{k}, \hat{K} > 0$ such that the fixed point W found in Theorem 5.1 satisfies $\hat{k}J \leq W \leq \hat{K}J$.

Corollary 5.3. *Let $\varepsilon \geq 0$ and suppose that $U^\vartheta \in \mathbb{SO}\big(k, K\big)$. For $\varepsilon > 0$, suppose that A and B solve*

$$A = K^{-1}(A^\rho + \varepsilon) \qquad B = k^{-1}(B^\rho + \varepsilon) \tag{5.1}$$

and, if $\varepsilon = 0$, set $A = K^{-\vartheta}$ and $B = k^{-\vartheta}$ (the positive solution to (5.1)). Then, the fixed point W of $F^\varepsilon_{U,U}$ found in Theorem 5.1 is in $\mathbb{O}(J; kA, KB)$.

Proof. We first show that $F^\varepsilon_{U,U}$ maps from $\mathbb{O}(U^\vartheta; A, B)$ to itself. We only prove the upper bound as the lower bound is symmetric. Suppose that $W \leq BU^\vartheta$. Then,

$$\begin{aligned}
F^\varepsilon_{U,U}(W)_t &= \mathbb{E}\left[\int_t^\infty U_s W_s^\rho + \varepsilon U_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[\int_t^\infty U_s B^\rho U_s^{\vartheta\rho} + \varepsilon U_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= (B^\rho + \varepsilon)\, \mathbb{E}\left[\int_t^\infty U_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= (B^\rho + \varepsilon)\, J_t \leq \frac{1}{k}\,(B^\rho + \varepsilon)\, U_t^\vartheta = BU_t^\vartheta.
\end{aligned}$$

The proof of Theorem 5.1/Theorem IV.B.2 given on page 111 first shows that $F^\varepsilon_{U,U} : \mathbb{O}(U^\vartheta) \to \mathbb{O}(U^\vartheta)$ is a contraction mapping and then uses Banach's fixed point theorem. Hence, if we choose an initial process $W^0 \in \mathbb{O}(U^\vartheta; A, B)$, then repeated application of $F^\varepsilon_{U,U}$ yields a fixed point $W^* \in \mathbb{O}(U^\vartheta; A, B)$. Since the fixed point W found in Theorem 5.1 is unique in the class $\mathbb{O}(U^\vartheta)$, $W = W^* \in \mathbb{O}(U^\vartheta; A, B)$. Finally, since $U^\vartheta \in \mathbb{SO}(k, K)$, $\mathbb{O}(U^\vartheta; A, B) \subseteq \mathbb{O}(J; kA, KB)$. □

6 Subsolutions and supersolutions

In this section we prove comparison results for subsolutions and supersolutions which we defined in Chapter IV. We recall the definition for the reader's benefit.

Definition 6.1. Let $C \in \mathscr{P}_+$ and g be an aggregator random field. A $\mathbb{V}$-valued,

làd, optional process V is called

- a *subsolution* for the pair (g, C) if $\limsup_{t\to\infty} \mathbb{E}\left[V_{t+}\right] \leq 0$, and for all bounded stopping times $\tau \leq \sigma$,

$$V_\tau \;\leq\; \mathbb{E}\left[V_{\sigma+} + \int_\tau^\sigma g(s, \omega, C_s, V_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right].$$

- a *supersolution* for the pair (g, C) if $\liminf_{t\to\infty} \mathbb{E}\left[V_{t+}\right] \geq 0$, and for all bounded stopping times $\tau \leq \sigma$,

$$V_\tau \;\geq\; \mathbb{E}\left[V_{\sigma+} + \int_\tau^\sigma g(s, \omega, C_s, V_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right].$$

- a *solution* for the pair (g, C) if it is both a subsolution and a supersolution and $V \in \mathbb{I}(g, C)$.

Remark 6.2. By taking the limit as $\sigma \to \infty$ and using the transversality condition, it is clear that a solution V for the pair (g, C) satisfies (III.1.1). We then choose a càdlàg version of V so that V is a utility process associated to (g, C). Similarly, since utility processes are càdlàg, $V_\tau = \mathbb{E}\left[\int_\tau^\infty g(s, \omega, C_s, V_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_\tau\right] = \mathbb{E}\left[\int_\tau^\sigma g(s, \omega, C_s, V_s)\,\mathrm{d}s + V_{\sigma+} \,\middle|\, \mathcal{F}_\tau\right]$ and the converse is also true.

6.1 Comparison of subsolutions and supersolutions

It was shown in Theorem IV.2.8 on page 98 that when g is decreasing in its last argument, if V^1 is a subsolution and V^2 is a supersolution (both associated to g and some $C \in \mathscr{P}_+$) and either V^1 or V^2 is in $\mathbb{UI}(g, C)$, then $V^1_\sigma \leq V^2_\sigma$ for all finite stopping times σ. However, when $\vartheta > 1$, the Epstein–Zin aggregator f_{EZ} is increasing in its last argument so this theorem does not hold.

The next proposition shows that we may weaken the condition that the aggregator has a negative derivative with respect to its last argument, and instead assume that it has a derivative which is bounded above by some positive decreasing exponential.

The two corollaries that follow then amount to providing conditions under which the assumptions of the theorem are met.

We introduce the following condition on a pair (V^1, V^2) of stochastic processes which will be a requirement for our comparison theorems.

Condition A. Let $g : [0,\infty) \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ be an aggregator random field and $C \in \mathscr{P}_+$. The pair (V^1, V^2) satisfies Condition A for the pair (g, C) if either V^1 or V^2 is in $\mathbb{UI}(g, C)$ and $(V^1 - V^2)^+$ is L^1 bounded.

Remark 6.3. Note that for a subsolution V^1 and a supersolution V^2 associated to the pair (g, C), a sufficient condition for the pair (V^1, V^2) to satisfy Condition A for the pair (g, C) is that $V^1, V^2 \in \mathbb{UI}(g, C)$. This is because $\mathbb{E}\left[(V^1 - V^2)^+\right] \leq |V_0^1| + |V_0^2| + \mathbb{E}\left[\int_0^\infty |g(s, \omega, C_s, V_s^1)| \, \mathrm{d}s\right] + \mathbb{E}\left[\int_0^\infty |g(s, \omega, C_s, V_s^2)| \, \mathrm{d}s\right] < \infty$. However, Condition A is more general.

Proposition 6.4 (Comparison Theorem)**.** *Let $g : [0,\infty) \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ be an aggregator random field that is concave and nondecreasing in its last argument. Let $C \in \mathscr{P}_+$. Suppose that V^1 is a subsolution and V^2 is a supersolution for the pair (g, C) and that the pair (V^1, V^2) satisfies Condition A for the pair (g, C). Suppose further that $(g_v(t, \omega, C_t(\omega), V_t^2(\omega)))_{t \geq 0} \leq (\kappa e^{-\nu t})_{t \geq 0}$ for some $\kappa, \nu > 0$. Then, $V_\tau^1 \leq V_\tau^2$ $\mathbb{P}$-a.s. for all finite stopping times $\tau \geq 0$.*

Proof. Seeking a contradiction, suppose there is a finite stopping time τ such that $\mathbb{P}[V_\tau^1 > V_\tau^2] > 0$. By replacing τ with $\tau \wedge T$ for T sufficiently large, we may assume without loss of generality that τ is bounded. Set $A := \{V_\tau^1 > V_\tau^2\}$. Since V^1 and V^2 are làd, we may define the right-continuous processes $(V_{t+}^1)_{t \geq 0}$ and $(V_{t+}^2)_{t \geq 0}$. Further, define the stopping time $\sigma := \inf\{t \geq \tau : V_{t+}^1 \leq V_{t+}^2\}$. Then $(V_{\sigma+}^1 - V_{\sigma+}^2)\mathbf{1}_{\{\sigma < \infty\}} \leq 0$ by the right-continuity of $(V_{t+}^1)_{t \geq 0}$ and $(V_{t+}^2)_{t \geq 0}$.

First, we show that $\mathbb{P}[A \cap \{\tau < \sigma\}] > 0$. Indeed, otherwise if $\mathbf{1}_{\{\tau = \sigma\} \cap A} = \mathbf{1}_A$ $\mathbb{P}$-a.s., the definition of sub- and supersolutions yields

$$\mathbf{1}_A \left(V_\tau^1 - V_\tau^2\right) \leq \mathbb{E}\left[\mathbf{1}_A \left(V_{\tau+}^1 - V_{\tau+}^2\right) \middle| \mathcal{F}_\tau\right] = \mathbb{E}\left[\mathbf{1}_A \left(V_{\sigma+}^1 - V_{\sigma+}^2\right) \mathbf{1}_{\{\sigma < \infty\}} \middle| \mathcal{F}_\tau\right] \leq 0,$$

and we arrive at a contradiction.

Next, by the definition of sub- and supersolutions and by Jensen's inequality for the convex function $f(x) = x^+ = \max\{x, 0\}$, for $t \leq T$, letting $B_t = A \cap \{\tau \leq t < \sigma\}$

$$\begin{aligned}
&\mathbf{1}_{B_t} \left(V_t^1 - V_t^2\right)^+ \\
&\leq \mathbb{E}\left[\mathbf{1}_{B_t}\left(V^1_{(T\wedge\sigma)+} - V^2_{(T\wedge\sigma)+} + \int_t^{T\wedge\sigma} g(s,\omega,C_s,V_s^1) - g(s,\omega,C_s,V_s^2)\,\mathrm{d}s\right)^+ \middle| \mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[\mathbf{1}_{B_t}\big(V^1_{(T\wedge\sigma)+} - V^2_{(T\wedge\sigma)+}\big)^+ + \mathbf{1}_{B_t}\int_t^{T\wedge\sigma}\big(g(s,\omega,C_s,V_s^1) - g(s,\omega,C_s,V_s^2)\big)^+\mathrm{d}s \middle| \mathcal{F}_t\right]
\end{aligned}$$

where the right hand side is well-defined since either V^1 or V^2 is in $\mathbb{UI}(g, C)$. Taking expectations yields

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{B_t}\left(V_t^1 - V_t^2\right)^+\right] \\
&\quad\leq \mathbb{E}\left[\mathbf{1}_{B_t}\big(V^1_{(T\wedge\sigma)+} - V^2_{(T\wedge\sigma)+}\big)^+ + \mathbf{1}_{B_t}\int_t^{T\wedge\sigma}\big(g(s,\omega,C_s,V_s^1) - g(s,\omega,C_s,V_s^2)\big)^+\mathrm{d}s\right]
\end{aligned} \tag{6.2}$$

Taking the lim sup as $T \to \infty$ and using the fact that $\mathbf{1}_{B_t}\mathbf{1}_{\sigma\leq T}\left(V^1_{\sigma+} - V^2_{\sigma+}\right)^+ = 0$ $\mathbb{P}$-a.s. for all $T \geq 0$ and the transversality condition of sub- and supersolutions gives

$$\begin{aligned}
&\limsup_{T\to\infty}\mathbb{E}\left[\mathbf{1}_{B_t}\left(V^1_{(T\wedge\sigma)+} - V^2_{(T\wedge\sigma)+}\right)^+\right] \\
&= \limsup_{T\to\infty}\mathbb{E}\left[\mathbf{1}_{B_t}\mathbf{1}_{T<\sigma}\left(V^1_{T+} - V^2_{T+}\right)^+\right] + \limsup_{T\to\infty}\mathbb{E}\left[\mathbf{1}_{B_t}\mathbf{1}_{\sigma\leq T}\left(V^1_{\sigma+} - V^2_{\sigma+}\right)^+\right] \\
&\leq \limsup_{T\to\infty}\mathbb{E}\left[\left(V^1_{T+} - V^2_{T+}\right)^+\right] \;\leq\; \limsup_{T\to\infty}\mathbb{E}\left[(V^1_{T+})^+ + (V^2_{T+})^-\right] \;\leq\; 0,
\end{aligned}$$

where the last inequality follows since either $\mathbb{V} = \mathbb{R}_+$ (and $(V^2_{T+})^- = 0$) along with the transversality condition for supersolutions, or $\mathbb{V} = \mathbb{R}_-$ (and $(V^1_{T+})^+ = 0$) along with the transversality condition for supersolutions. Hence, by taking the lim sup as

$T \to \infty$ and using the positivity of the integrand, (6.2) becomes

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{B_t}\left(V_t^1 - V_t^2\right)^+\right] \\
&\leq \quad \mathbb{E}\left[\mathbf{1}_{B_t}\int_t^\sigma \left(g(s,\omega,C_s,V_s^1) - g(s,\omega,C_s,V_s^2)\right)^+ \,\mathrm{d}s\right] \\
&\leq \quad \mathbb{E}\left[\int_t^\infty \mathbf{1}_{B_s} g_v(s,\omega,C_s,V_s^2)\left(V_s^1 - V_s^2\right)^+ \,\mathrm{d}s\right] \\
&\leq \quad \mathbb{E}\left[\int_t^\infty \kappa e^{-\nu s}\mathbf{1}_{B_s}\left(V_s^1 - V_s^2\right)^+ \,\mathrm{d}s\right],
\end{aligned}$$

where in the middle line we have used that g is concave and increasing in its last argument. If $\Gamma(t) := \mathbb{E}\left[\mathbf{1}_{B_t}\left(V_t^1 - V_t^2\right)^+\right]$, then $\Gamma = (\Gamma(t))_{t\geq 0}$ is a nonnegative process such that $\Gamma(t) \leq \int_t^\infty \kappa e^{-\nu s}\Gamma(s)\,\mathrm{d}s$. Note that $\Gamma(t) \leq \mathbb{E}\left[(V_t^1 - V_t^2)^+\right] \leq \gamma$ for some $\gamma > 0$, by the L^1-boundedness of $(V^1 - V^2)^+$. Therefore, since $\int_0^\infty \kappa e^{-\nu t}\,\mathrm{d}t = \frac{\kappa}{\nu}$ and $\int_0^\infty \kappa e^{-\nu s}\Gamma(s)\,\mathrm{d}s \leq \frac{\gamma\kappa}{\nu}$, we can apply Grönwall's inequality for Borel functions ([HH17, Theorem 2.5] with $y(t) = \Gamma(-t)$ and $\mu(A) = \int_{A\cap\mathbb{R}_-} \kappa e^{\nu t}\,\mathrm{d}t$) to conclude that $\Gamma(t) = 0$ for all $t > 0$.

Note that $\mathbf{1}_{B_t}\left(V_t^1 - V_t^2\right)^+ \geq 0$ for each $t \geq 0$. Hence, by Fatou's Lemma,

$$0 \leq \mathbb{E}\left[\mathbf{1}_{B_t}\left(V_{t+}^1 - V_{t+}^2\right)^+\right] \leq \liminf_{s\downarrow\downarrow t} \mathbb{E}\left[\mathbf{1}_{B_s}\left(V_s^1 - V_s^2\right)^+\right] = 0. \tag{6.6}$$

Furthermore, since $\mathbf{1}_{B_t}\left(V_{t+}^1 - V_{t+}^2\right) = \mathbf{1}_{B_t}\left(V_{t+}^1 - V_{t+}^2\right)^+ \geq 0$ for each $t \geq 0$ by the definition of σ, it follows from (6.6) that $P_t = \mathbf{1}_{B_t}\left(V_{t+}^1 - V_{t+}^2\right) = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. Since $(V_{t+}^1)_{t\geq 0}$, $(V_{t+}^2)_{t\geq 0}$ and $\mathbf{1}_{B_t}$ are right-continuous, $P = (P_t)_{t\geq 0}$ is right continuous and is therefore indistinguishable from zero. In particular, $\mathbf{1}_A\mathbf{1}_{\tau<\sigma}\left(V_{\tau+}^1 - V_{\tau+}^2\right) = P_\tau = 0$ $\mathbb{P}$-a.s. But then the definition of sub-and supersolutions implies that

$$\mathbf{1}_A\mathbf{1}_{\tau<\sigma}\left(V_\tau^1 - V_\tau^2\right) \leq \mathbb{E}\left[\mathbf{1}_A\mathbf{1}_{\tau<\sigma}\left(V_{\tau+}^1 - V_{\tau+}^2\right) \,\middle|\, \mathcal{F}_\tau\right] = 0,$$

and we arrive at a contradiction. □

The first corollary will be used in the Verification Theorem later. Hence, it is

stated in the original coordinate system.

Corollary 6.5. *Let f_{EZ} be the Epstein–Zin aggregator and suppose that $R < 1$. Let $C \in \mathscr{P}_+$. Suppose that V^1 is a subsolution and V^2 is a supersolution for the pair (f_{EZ}, C) and that the pair (V^1, V^2) satisfies Condition A for the pair (f_{EZ}, C). Suppose further that $C_t^{1-R} \leq Ke^{-\gamma t}(1-R)V_t^2$ for some $K, \gamma > 0$ and all $t \geq 0$. Then, $V_\tau^1 \leq V_\tau^2$ for all finite stopping times $\tau \geq 0$.*

Proof. Taking derivatives of f_{EZ} with respect to its second argument gives

$$\begin{aligned} \frac{\partial f_{EZ}}{\partial v}(c,v) &= (\vartheta - 1)c^{1-S}((1-R)v)^{-\frac{1}{\vartheta}} \\ \frac{\partial^2 f_{EZ}}{\partial v^2}(c,v) &= -\rho(1-R)c^{1-S}((1-R)v)^{-(1+\frac{1}{\vartheta})} < 0. \end{aligned} \tag{6.8}$$

Hence, using (6.8), $\frac{\partial f_{EZ}}{\partial v}(C_s, V_s^2) \leq (\vartheta - 1)K^{\frac{1}{\vartheta}}e^{-\frac{\gamma}{\vartheta}t}$ and the conditions of Proposition 6.4 are met with $\kappa = (\vartheta - 1)K^{\frac{1}{\vartheta}}$ and $\nu = \frac{\gamma}{\vartheta}$. □

Remark 6.6. Note that the utility process associated to constant proportional strategies is given in (1.3) by $h(\pi, \xi)X_t^{1-R}$. Since $C_t = \xi X$, the conditions of Corollary 6.5 are not met, and we cannot use it to give a uniqueness result (even over constant proportional strategies). We instead use Corollary 6.5 in the proof of the Verification Theorem (Theorem 3.8), in which we perturb the candidate value function beforehand.

Corollary 6.7. *Fix $\nu > 0$. Suppose $\Lambda \in \mathbb{SO}_\nu^\vartheta$ and define the perturbed aggregator*

$$h_{EZ}^{\varepsilon,\nu,\Lambda}(t,\omega,u,w) = uw^\rho + \varepsilon e^{\nu t}\Lambda_t^\vartheta(\omega), \quad \text{for } \varepsilon > 0. \tag{6.9}$$

Fix $\varepsilon_2 > 0$ and $0 \leq \varepsilon_1 \leq \varepsilon_2$. Let $U^1, U^2 \in \mathscr{P}_+$ satisfy $U^1 \leq U^2 \leq \Lambda$. Suppose that W^1 is a subsolution for the pair $(h_{EZ}^{\varepsilon_1,\nu,\Lambda}, U^1)$ and W^2 is a supersolution for the pair $(h_{EZ}^{\varepsilon_2,\nu,\Lambda}, U^2)$ such that $W^1, W^2 \in [0,\infty)$ and such that (W^1, W^2) satisfies Condition A for the pair $(h_{EZ}^{\varepsilon_1,\nu,\Lambda}, U^1)$. Then, $W_\tau^1 \leq W_\tau^2$ for all finite stopping times $\tau \geq 0$.

Proof. First note that W^2 is a supersolution for $(h_{EZ}^{\varepsilon_1,\nu,\Lambda}, U^1)$, since

$$W_\tau^2 \geq \mathbb{E}\left[\int_\tau^\sigma h_{EZ}^{\varepsilon_2,\nu,\Lambda}(U_s^2, W_s^2)\,\mathrm{d}s + W_{\sigma+}^2 \,\middle|\, \mathcal{F}_t\right] \geq \mathbb{E}\left[\int_\tau^\sigma h_{EZ}^{\varepsilon_1,\nu,\Lambda}(U_s^1, W_s^2)\,\mathrm{d}s + W_{\sigma+}^2 \,\middle|\, \mathcal{F}_t\right].$$

As $\Lambda^\vartheta \in \mathbb{SO}_\nu$, there exists K_Λ such that $W_t^2 \geq \mathbb{E}\left[\int_t^\infty \varepsilon_2 \mathrm{e}^{\nu s}\Lambda_s^\vartheta\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \geq \frac{\varepsilon_2}{K_\Lambda} e^{\nu t}\Lambda_t^\vartheta$. Therefore, since $U^1 \leq \Lambda$,

$$\frac{\partial h_{EZ}^{\varepsilon_1,\nu,\Lambda}}{\partial w}\left(t,\omega,U_t^1,W_t^2\right) = \rho U_t^1 (W_t^2)^{-\frac{1}{\vartheta}} \leq \rho\left(\frac{K_\Lambda}{\varepsilon_2}\right)^{\frac{1}{\vartheta}} e^{-\frac{\nu}{\vartheta}t}.$$

Furthermore, $\frac{\partial^2 h_{EZ}^{\varepsilon_1,\nu,\Lambda}}{\partial w^2}(u,w) = -\frac{\vartheta-1}{\vartheta^2} u w^{-(1+\frac{1}{\vartheta})} < 0$, so that $h_{EZ}^{\varepsilon_1,\nu,\Lambda}$ is concave. Since (W^1, W^2) satisfies Condition A for the pair $(h_{EZ}^{\varepsilon_1,\nu,\Lambda}, U^1)$, the assumptions of Proposition 6.4 are met, and the conclusion follows. □

Remark 6.8. The utility process associated to $(h_{EZ}^{\varepsilon,\nu,\Lambda}, U)$ for $U \leq \Lambda \in \mathbb{SO}_\nu^\vartheta$ is unique when $\varepsilon, \nu > 0$. This follows from Corollary 6.7 and the fact that a utility process is both a subsolution and a supersolution which lies in $\mathbb{UI}(h_{EZ}^{\varepsilon,\nu,\Lambda}, U)$.

7 The extremal solution

In this section, we investigate the *extremal* solution, from Definition 2.3; in the changed coordinate system it corresponds to the maximal solution. We first note that it is unique.

Proposition 7.1. *If the maximal solution exists, it is unique.*

Proof. Suppose for contradiction that there are two maximal solutions W^1 and W^2. Then, for all $t \geq 0$, $W_t^1 \geq W_t^2$ since W^1 is maximal in the class of solutions and $W_t^1 \geq W_t^2$ since W^1 is maximal also. Thus, $W_t^1 = W_t^2$ for all $t \geq 0$. Since both W^1 and W^2 are càdlàg, they are indistinguishable. □

We now turn to existence of a maximal solution associated to (h_{EZ}, U).

Proposition 7.2. *Suppose that $U \in \mathscr{P}_+$ satisfies $U \leq \Lambda$ for $\Lambda \in \mathbb{SO}_+^\vartheta$. Then, there exists a unique maximal solution associated to (h_{EZ}, U). It is also maximal in the class of L^1-bounded subsolutions.*

Proof. Since $\Lambda \in \mathbb{SO}_+^\vartheta$, there exists $\nu > 0$ such that $\Lambda \in \mathbb{SO}_{\nu\vartheta}^\vartheta$. For such ν, let $\Lambda_t^{(\nu)} = e^{\nu t}\Lambda_t$. Then, $\Lambda^{(\nu)} = (\Lambda_t^{(\nu)})_{t\geq 0} \in \mathbb{SO}^\vartheta$. For each $n \in \mathbb{N}$, let $U^n := \max\left\{U, \frac{1}{n}\Lambda^{(\nu)}\right\}$. Then, $U^n \overset{\mathrm{O}}{=} \Lambda^{(\nu)}$ as $U^n \leq \Lambda \leq \Lambda^{(\nu)}$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a positive-valued sequence such that $\varepsilon_n \searrow 0$.

By Proposition 5.1, for each $n \in \mathbb{N}$, there exists a solution W^n associated to $(h_{EZ}^{\varepsilon_n,\nu\vartheta,\Lambda}, U^n)$, where $h_{EZ}^{\varepsilon_n,\nu\vartheta,\Lambda}$ is defined in (6.9). Furthermore, W^n is decreasing in n by Corollary 6.7 and $U^n(W^n)^\rho$ is dominated by $U^1(W^1)^\rho$. Hence, by the Dominated Convergence Theorem, we find that $W := \lim_{n\to\infty} W^n$ satisfies

$$W_t = \lim_{n\to\infty} \mathbb{E}\left[\int_t^\infty U_s^n (W_s^n)^\rho + \varepsilon_n e^{\nu\vartheta s}\Lambda_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty U_s W_s^\rho \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right],$$

so that $W \in \mathbb{I}(h_{EZ}, U)$ is a solution associated to (h_{EZ}, U).

Suppose that $W' \in \mathbb{I}(h_{EZ}, U)$ is a solution (or an L^1-bounded subsolution) associated to (h_{EZ}, U). Then, the pair (W', W^n) satisfies Condition A for the pair (h_{EZ}, U), since $W^n \in \mathbb{UI}(h_{EZ}^{\varepsilon_n,\nu\vartheta,\Lambda}, U^n) \subset \mathbb{UI}(h_{EZ}, U)$ and $(W' - W^n)^+ \leq W'$, where W' is L^1-bounded. Hence, $W^n \geq W'$ for each $n \in \mathbb{N}$ by Corollary 6.7 and $W \geq W'$ is a maximal (L^1-bounded sub-) solution. Uniqueness in the class of maximal solutions follows from Proposition 7.1. □

Proposition 3.2 is a direct result of the Proposition 7.2 and the following comparison result for maximal solutions.

Proposition 7.3. *Let h_{EZ} be the aggregator defined in* (4.1) *and suppose that $U^1, U^2 \in \mathscr{P}_+$ satisfy $U^1 \leq U^2 \leq \Lambda \in \mathbb{SO}_+^\vartheta$. If W^1 and W^2 are the maximal solutions associated to h_{EZ} and consumptions U^1 and U^2 respectively, then $W_t^1 \leq W_t^2$ for all $t \geq 0$.*

Proof. Let ν be such that $\Lambda \in \mathbb{SO}_{\nu\vartheta}^\vartheta$. Define $\Lambda^{(\nu)} = \left(\Lambda_t^{(\nu)}\right)_{t\geq 0}$ by $\Lambda_t^{(\nu)} = e^{\nu t}\Lambda_t$ and,

for $n \in \mathbb{N}$ and $i \in \{1,2\}$, define $U^{i,n} = \max\{U^i, \frac{1}{n}\Lambda^{(\nu)}\}$ and $\varepsilon_n = \frac{1}{n}$. Then, by Proposition 5.1, there exists a solution $W^{i,n}$ associated to $(h_{EZ}^{\varepsilon_n,\nu\vartheta,\Lambda}, U^{i,n})$. Furthermore, for all $n \in \mathbb{N}$ and $t \geq 0$, $W^{1,n} \leq W^{2,n}$ by Corollary 6.7. It is unique by Remark 6.8.

As in Proposition 7.2, the unique maximal solution associated to U^i is given by $W^i := \lim_{n\to\infty} W^{i,n}$ for $i \in \{1,2\}$. Thus, $W_t^1 = \lim_{n\to\infty} W_t^{1,n} \leq \lim_{n\to\infty} W_t^{2,n} = W_t^2$ for all $t \geq 0$. □

We may also deduce the following Corollary to Proposition 7.2.

Corollary 7.4. *Let $C \in \mathscr{P}_+$ be such that $C^{1-R} \leq Y \in \mathbb{SO}_+$. Then, the extremal solution associated to (f_{EZ}, C) is the maximal L^1-bounded subsolution when $R < 1$ and the minimal L^1-bounded supersolution when $R < 1$.*

The final result of this section shows that the solution found by fixed point argument in Proposition 5.1 is the unique maximal solution.

Proposition 7.5. *Let h_{EZ} be the aggregator defined in* (4.1)*. Suppose that $U \in \mathbb{SO}_+^\vartheta$. Then the solution associated to (h_{EZ}, U) found in Proposition 5.1 is the maximal solution.*

Proof. Fix $\vartheta > 1$. Since $U \in \mathbb{SO}_+^\vartheta$, there exists $\hat{\nu} > 0$ such that $U \in \mathbb{SO}_{\hat{\nu}\vartheta}^\vartheta$. By Lemma 2.8, it follows that $U \in \mathbb{SO}_{\nu\vartheta}^\vartheta$ for $\nu \leq \hat{\nu}$. For each $\nu \leq \hat{\nu}$, define $U_t^{(\nu)} = e^{-\nu t} U_t$ and $J^{(\nu)} = (J_t^{(\nu)})_{t\geq 0}$ by $J_t^{(\nu)} = \mathbb{E}\left[\int_t^\infty e^{\nu s} U_s^\vartheta \, \mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$. We can then find $k(\nu), K(\nu)$ such that $\left(U^{(\nu)}\right)^\vartheta \in \mathbb{SO}(k(\nu), K(\nu))$. By Remark 2.9, we may choose $k(\nu), K(\nu)$ such that $0 < k(\hat{\nu}) \leq \lim_{\nu\to 0} k(\nu) = k(0) =: k < \infty$ and $0 < K(\hat{\nu}) \leq \lim_{\nu\to 0} K(\nu) = K(0) =: K < \infty$, where both limits are decreasing in ν.

For each $\varepsilon > 0$ and $0 < \nu \leq \hat{\nu}$, there exists a solution $W^{\varepsilon,\nu}$ associated to $(h_{EZ}^{\varepsilon,\nu\vartheta,\Lambda}, U^{(\nu)})$ by Proposition 5.1. Furthermore, by Corollary 5.3, $W_t^{\varepsilon,\nu} \leq K(\nu) B^{\varepsilon,\nu} J_t^{(\nu)}$ where $B = B^{\varepsilon,\nu}$ solves $B = k(\nu)^{-1}(B^\rho + \varepsilon)$. By Proposition 7.2, the unique maximal solution associated to U is given by $W := \lim_{\varepsilon\to 0} W^{\varepsilon,\nu}$. Therefore, since $\lim_{\varepsilon\to 0} B^{\varepsilon,\nu} = B^{0,\nu} = k(\nu)^{-\vartheta}$, $W_t \leq K(\nu) k(\nu)^{-\vartheta} J^{(\nu)}$ for all $\nu \leq \hat{\nu}$. Taking the limit as $\nu \searrow 0$, gives $W_t \leq K k^{-\vartheta} J^{(0)} =: K k^{-\vartheta} J$.

Similarly, by maximality of W and the lower bound found in Corollary 5.3, $W \geq kK^{-\vartheta}J$. Hence, the maximal solution is in $\mathbb{O}(J)$. Since the solution found in Proposition 5.1 is unique in $\mathbb{O}(J)$, it is equal to the maximal solution. □

8 The proper solution

We first focus on existence of proper solutions and then turn to uniqueness.

8.1 Existence of proper solutions

The goal of this section is to prove Theorem 3.3. To do this we first prove that there exists a proper solution W associated to the aggregator h_{EZ} and consumption stream U given by a discounted indicator function of a stochastic interval, i.e. $U_t = e^{-\gamma t}\mathbf{1}_{\tau \leq t < \sigma}$ for τ and σ stopping times such that $\tau < \sigma$. Since any right-continuous consumption stream can be bounded from below by (a scaled version of) these processes, we then show by taking limits that there exists a proper solution associated to right-continuous processes.

Proposition 8.1. *Let $\gamma > 0$ and τ and σ be stopping times such that $\tau \leq \sigma$. Let $U = (U_t)_{t \geq 0}$ be given by $U_t := e^{-\gamma t}\mathbf{1}_{\tau \leq t < \sigma}$. Then, there exists a proper solution $W = (W_t)_{t \geq 0}$ associated to U, for which $W_t \geq \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma(t\vee\tau)} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}_t\right]\right)^{\vartheta}$.*

The proof of Proposition 8.1 is long and technical and therefore relegated to the appendix. To prove Theorem 3.3, we introduce two lemmas.

Lemma 8.2. *Suppose that W is a utility process associated to (h_{EZ}, U) and suppose that there exists $t_0 \geq 0$ such that $U_t = 0$ for $t < t_0$. If $A \in \mathcal{F}_{t_0}$, and $\widetilde{U} = (\widetilde{U}_t)_{t \geq 0}$ is given by $\widetilde{U}_t = \mathbb{E}\left[\mathbf{1}_A \,|\, \mathcal{F}_t\right] U_t$ then $\widetilde{W}_t = \mathbb{E}\left[\mathbf{1}_A W_{t\vee t_0} \,|\, \mathcal{F}_t\right]$ is a utility process associated to $(h_{EZ}, (\widetilde{U}_t)_{t \geq 0})$.*

Proof. Suppose that $t \geq t_0$. Then, since W is a utility process associated to U,

$$\widetilde{W}_t = \mathbf{1}_A W_t = \mathbf{1}_A^{1+\rho} W_t = \mathbb{E}\left[\int_t^\infty \mathbf{1}_A U_s (\mathbf{1}_A W_s)^\rho \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty \widetilde{U}_s \widetilde{W}_s^\rho \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right].$$

Conversely, suppose that $t < t_0$. Then, since $\widetilde{U}_s = 0$ for $s < t_0$ and both $\widetilde{W}_{t_0} = \mathbf{1}_A W_{t_0}$ and $\widetilde{W}_{t_0} = \mathbb{E}\left[\int_{t_0}^{\infty} \widetilde{U}_s \widetilde{W}_s^{\rho} \,\mathrm{d}s \,\middle|\, \mathcal{F}_{t_0}\right]$,

$$\widetilde{W}_t = \mathbb{E}\left[\mathbf{1}_A W_{t_0} \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_{t_0}^{\infty} \widetilde{U}_s \widetilde{W}_s^{\rho} \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_{t}^{\infty} \widetilde{U}_s \widetilde{W}_s^{\rho} \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]. \qquad \square$$

Lemma 8.3. *Suppose $U \in \mathscr{P}_+$ is right-continuous. Fix $t \geq 0$ and let $A_t = \{J_t^{U^\vartheta} > 0\}$ and $B_t = \bigcup_{\substack{T \geq t \\ T \in \mathbb{Q}}} \bigcup_{\substack{\varepsilon > 0 \\ \varepsilon \in \mathbb{Q}}} \left\{\mathbb{E}\left[\mathbf{1}_{\{U_T \geq \varepsilon\}} \,\middle|\, \mathcal{F}_t\right] > 0\right\}$. Then, $\mathbb{P}(A_t \setminus B_t) = 0$.*

Proof. Seeking a contradiction, suppose $C_t := A_t \setminus B_t$ has positive measure. Then $\mathbb{E}[\mathbf{1}_{C_t} J_t^U] > 0$ by the definition of A_t. Moreover, for each *rational* $T \geq t$, the definition of B_t gives $\mathbb{E}\left[\mathbf{1}_{C_t} U_T \,|\, \mathcal{F}_t\right] \leq \mathbb{E}\left[\mathbf{1}_{B_t^c} U_T \,\middle|\, \mathcal{F}_t\right] = 0$ which yields $\mathbf{1}_{C_t} U_T = 0$ $\mathbb{P}$-a.s. Since U is right-continuous, $\mathbf{1}_{C_t} U_T = 0$ for all $T \geq t$ $\mathbb{P}$-a.s. Taking expectations yields $\mathbb{E}[\mathbf{1}_{C_t} J_t^U] = \mathbb{E}\left[\int_t^{\infty} \mathbf{1}_{C_s} U_s \,\mathrm{d}s\right] = 0$ and we arrive at a contradiction. $\square$

We may now prove Theorem 3.3. To show that W is proper, we must show that if $A_t = \{J_t^{U^\vartheta} > 0\}$ and $C_t = \{W_t > 0\}$, then $\mathbb{P}(A_t \setminus C_t) = 0$. By Lemma 8.3, since $A_t \subseteq B_t$ up to null sets, we may instead prove that $\mathbb{P}(B_t \setminus C_t) = 0$.

Proof of Theorem 3.3. Since $C^{1-R} \leq Y$ for $Y \in \mathbb{SO}_+$, $U := \vartheta C^{1-S} \leq \vartheta Y^{\frac{1}{\vartheta}}$. As $Y \in \mathbb{SO}_+$ and by Remark IV.1.3, $\vartheta Y^{\frac{1}{\vartheta}} \in \mathbb{SO}_+^{\vartheta}$. Therefore, by Proposition 7.2 there exists a maximal solution W associated to (h_{EZ}, U). We now show that W is proper.

Fix $t^* \geq 0$. Set $A_{t^*} := \{J_{t^*}^{U^\vartheta} > 0\}$ and $C_{t^*} = \{W_{t^*} > 0\}$. By Lemma 8.3, it suffices to show that $\mathbb{P}(B_T^\varepsilon \setminus C_{t^*}) = 0$ for all rational $T \geq t^*, \varepsilon > 0$, where $B_T^\varepsilon = \{\mathbb{E}\left[\mathbf{1}_{\{U_T \geq \varepsilon\}} \,\middle|\, \mathcal{F}_{t^*}\right] > 0\}$. So, fix rational $T \geq t^*$ and $\varepsilon > 0$. Define the stopping time $\sigma_\varepsilon = \inf\{t \geq T : U_t \leq \frac{\varepsilon}{2}\}$ and note that $\{\sigma_\varepsilon > T\}$ on $\{U_T \geq \varepsilon\}$ by right-continuity of U. Define the process $\widetilde{U} = (\widetilde{U}_t)_{t \geq 0}$ by $\widetilde{U}_t := \frac{\varepsilon}{2} e^{-\gamma t} \mathbb{E}\left[\mathbf{1}_{\{U_T \geq \varepsilon\}} \,\middle|\, \mathcal{F}_t\right] \mathbf{1}_{\{t \in [T, \sigma_\varepsilon)\}}$. Then $\widetilde{U}$ is dominated by U. Moreover, Proposition 8.1 for $\hat{U}_t = e^{-\gamma t} \mathbf{1}_{\{T \leq t < \sigma_\varepsilon\}}$ with corresponding solution $\hat{W}$, Lemma 8.2 for $t_0 = T$ and $A = \{\hat{U}_T \geq \varepsilon\}$ and Jensen's inequality show that there exists a solution $\widetilde{W}$ associated to $\widetilde{U} = (\widetilde{U}_t)_{t \geq 0}$ such that

for all $t \geq 0$

$$\begin{aligned}\widetilde{W}_t &\geq \mathbb{E}\left[\mathbf{1}_{\{U_T \geq \varepsilon\}}\left(\frac{\varepsilon}{2\gamma\vartheta}\mathbb{E}\left[\left(e^{-\gamma(t\vee T)} - e^{-\gamma(t\vee\sigma_\varepsilon)}\right)\middle|\mathcal{F}_t\right]\right)^{\vartheta}\middle|\mathcal{F}_t\right] \\ &\geq \left(\frac{\varepsilon}{2\gamma\vartheta}\mathbb{E}\left[\mathbf{1}_{\{U_T \geq \varepsilon\}}\,\middle|\,\mathcal{F}_t\right]\mathbb{E}\left[\left(e^{-\gamma(t\vee T)} - e^{-\gamma(t\vee\sigma_\varepsilon)}\right)\middle|\mathcal{F}_t\right]\right)^{\vartheta}.\end{aligned}$$

Since $\{\sigma_\varepsilon > T\}$ on $\{U_T \geq \varepsilon\}$, it follows that $\tilde{W}_{t^*} > 0$ on $\{U_T \geq \varepsilon\}$. Now the claim follows from the fact that $W_{t^*} \geq \widetilde{W}_{t^*}$ by Proposition 7.3. □

8.2 Uniqueness of proper solutions

We now turn to uniqueness of proper solutions. The aim of this section will be to prove Theorem 3.5 and then, as a corollary, Theorem 3.6. The following two lemmas will be useful.

Lemma 8.4. *Fix $X \in \mathbb{SO}$, and let $J^X = (J_t^X)_{t\geq 0}$ be defined by $J_t^X = \mathbb{E}\left[\int_t^\infty X_s ds \,\middle|\, \mathcal{F}_t\right]$. Then, there exists a martingale $M = (M_t)_{t\geq 0}$ such that $J_t^X = M_t e^{-\int_0^t (X_s/J_s^X)ds}$.*

Proof. Let $N = (N_t)_{t\geq 0}$ be the uniformly integrable martingale given by $N_t = \mathbb{E}\left[\int_0^\infty X_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$. Then $J_t^X = N_t - \int_0^t X_s \,\mathrm{d}s$. Define the increasing process A by $A_t = \int_0^t (X_s/J_s^X)\,\mathrm{d}s$. Since $X \in \mathbb{SO}$ we have $0 < X_t \leq K J_t^X$ for some K and hence $0 < A_t \leq Kt$.

Define M via $M_t = e^{A_t} J_t^X$. Then $dM_t = e^{A_t} dJ_t^X + X_t e^{A_t} dt = e^{A_t} dN_t$ and all that remains to show is that the local martingale M is a martingale. Since A is increasing and $A_t \leq Kt$, we have $\mathbb{E}[\|e^A\|_T |N_T|] \leq \mathbb{E}[(e^{KT} - 1)|N_T|] < \infty$ for $T \geq 0$, where $\|e^A\|_T$ is the total variation of $e^A = (e^{A_t})_{t\geq 0}$ at time T. Hence, M is a martingale by [HMK19, Lemma A.1]. □

Lemma 8.5. *Let $\alpha > 0$, $\beta \in (0,1)$ and let $G = (G_t)_{t\geq 0}$ be a càdlàg submartingale. Suppose that $X = (X_t)_{t\geq 0}$ is is a right-continuous process such that $X_0 = \beta$ and $X_t \leq 1$ for all $t \geq 0$. Define $\sigma = \inf\{t \geq 0 : \ X_t = 1\}$ and suppose that*

$$\mathrm{d}X_t \geq \alpha X_t \,\mathrm{d}t + \mathrm{d}G_t, \quad \textit{for all } t < \sigma. \tag{8.1}$$

Then, there exists $\nu \in (0,1)$ and $T \in (0,\infty)$ such that $\mathbb{P}(\sigma < T) > \nu$.

Proof. First note that X is a (local) submartingale bounded above by 1 and so converges almost surely to an $\mathcal{F}_\infty$-measurable random variable $X_\infty \leq 1$ by the Martingale Convergence Theorem.

Fix $\xi \in (0,\beta)$. Let $\tau = \inf\{t \geq 0 : \ X_t \notin (\xi, 1)\} \leq \sigma$. From the dynamics of X given in (8.1),

$$X_{t\wedge\tau} \geq X_0 + \int_0^{t\wedge\tau} \alpha X_s \, \mathrm{d}s + G_{t\wedge\tau} - G_0 \geq \beta + \alpha\xi(t\wedge\tau) + G_{t\wedge\tau} - G_0.$$

Then, using that G is a càdlàg submartingale and the Optional Sampling Theorem,

$$\mathbb{E}\left[X_{t\wedge\tau}\right] \geq \beta + \alpha\xi\mathbb{E}\left[t\wedge\tau\right]. \tag{8.2}$$

Since $X \leq 1$, taking the lim sup and using the Reverse Fatou's Lemma on the left hand side of (8.2) and the Monotone Convergence Theorem on the right hand side gives

$$1 \geq \mathbb{E}\left[X_\tau\right] = \mathbb{E}\left[\mathbf{1}_{\tau<\infty}X_\tau\right] + \mathbb{E}\left[\mathbf{1}_{\tau=\infty}X_\infty\right] \geq \beta + \alpha\xi\mathbb{E}\left[\tau\right] \geq \beta.$$

Therefore, $\mathbb{E}[\tau] \leq \frac{1-\beta}{\alpha\xi}$ and $\mathbb{P}(\tau = \infty) = 0$. Consequently, since X is right-continuous, $X_\tau \in (-\infty, \xi] \cup \{1\}$ $\mathbb{P}$-a.s. and

$$1 - (1-\xi)\mathbb{P}(X_\tau \leq \xi) = \mathbb{P}(X_\tau = 1) + \xi\mathbb{P}(X_\tau \leq \xi) \geq \mathbb{E}\left[X_\tau\right] \geq \beta.$$

In particular, $\mathbb{P}(X_\tau \leq \xi) \leq \frac{1-\beta}{1-\xi}$ and $\mathbb{P}(\tau = \sigma) = \mathbb{P}(X_\tau = 1) \geq 1 - \frac{1-\beta}{1-\xi} = \frac{\beta-\xi}{1-\xi}$. Furthermore,

$$\mathbb{P}(\tau \geq T; \tau = \sigma) \leq \mathbb{E}\left[\frac{\tau}{T}\mathbf{1}_{\tau\geq T}\mathbf{1}_{\tau=\sigma}\right] \leq \frac{1}{T}\mathbb{E}\left[\tau\right] \leq \frac{1-\beta}{\alpha\xi T}, \quad \text{for all } T \geq 0,$$

and

$$\mathbb{P}(\sigma < T) \geq \mathbb{P}(\tau < T; \tau = \sigma) = \mathbb{P}(\tau = \sigma) - \mathbb{P}(\tau \geq T; \tau = \sigma) \geq \frac{\beta-\xi}{1-\xi} - \frac{1-\beta}{\alpha\xi T}.$$

Choose $\nu = \frac{1}{2}\left(\frac{\beta-\xi}{1-\xi}\right)$ and $T = \frac{1-\beta}{\alpha\xi\nu}$. Then, $\mathbb{P}(\sigma < T) \geq \nu$. $\square$

We may now prove Theorem 3.5 in the (U, W) coordinates.

Proof of Theorem 3.5. Fix $U \in \mathbb{SO}_+^\vartheta$ and recall $J = (J_t)_{t\geq 0}$ is given by $J_t = \mathbb{E}\left[\int_t^\infty U_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$. Suppose that $U^\vartheta \in \mathbb{SO}(k, K)$ and let $W \in \mathbb{O}(J)$ be the solution associated to (h_{EZ}, U) found in Theorem 5.1 (after setting $\varepsilon = 0$ and $U = \Lambda$). By Proposition 7.5, W is the unique maximal solution, and as we saw in the proof of Theorem 3.3 it is also proper. We now prove uniqueness.

For contradiction, assume that there exists a proper solution $Y = (Y_t)_{t\geq 0}$ such that $Y \neq W$. Since W is maximal, $Y \leq W$. Then, since W is unique in the class $\mathbb{O}(J)$, it follows that $Y \overset{\mathbb{O}}{\neq} J$. Hence, there exists $t \geq 0$, $B \in \mathcal{F}_t$ and $\varepsilon > 0$ such that $\mathbb{P}(B) > \varepsilon$ and $Y_t < kK^{-\vartheta}J_t$ on B. For ease of exposition, assume that $t = 0$, $B = \Omega$ and $Y_0 = y = (1-\varepsilon)kK^{-\vartheta}J_0$. The general case is similar.

Define $Z = (Z_t)_{t\geq 0}$ by $Z_t = Y_t(kK^{-\vartheta}J_t)^{-1}$ and note that Z is càdlàg by Remark 6.2 and $Z_0 = (1-\varepsilon)$. Since $kK^{-\vartheta}JZ \equiv Y$ is a utility process and since $U^\vartheta \geq kJ$, for all $t \leq T$,

$$\begin{aligned} kK^{-\vartheta}J_tZ_t = Y_t &= \mathbb{E}\left[\int_t^T U_sY_s^\rho \,\mathrm{d}s + Y_T \,\middle|\, \mathcal{F}_t\right] \\ &\geq \mathbb{E}\left[\int_t^T (kJ_s)^{\frac{1}{\vartheta}}(kK^{-\vartheta}J_sZ_s)^\rho \,\mathrm{d}s + kK^{-\vartheta}J_TZ_T \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

By Lemma 8.4, we find that $J_t = M_te^{-A_t}$ for $A_t = \int_0^t (U_s^\vartheta/J_s)ds$. Hence, dividing by $kK^{-\vartheta}M_t$ and collecting terms gives

$$e^{-A_t}Z_t \geq \frac{1}{M_t}\mathbb{E}\left[\int_t^T KM_se^{-A_s}Z_s^\rho \,\mathrm{d}s + M_Te^{-A_T}Z_T \,\middle|\, \mathcal{F}_t\right].$$

Define $\widetilde{Z}_t = e^{-A_t}Z_t$ and consider an equivalent measure $\widetilde{\mathbb{P}}$ defined by $\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t} = M_t$, so that

$$\widetilde{Z}_t \geq \widetilde{\mathbb{E}}\left[\int_t^T Ke^{-(1-\rho)A_s}\widetilde{Z}_s^\rho \,\mathrm{d}s + \widetilde{Z}_T \,\middle|\, \mathcal{F}_t\right], \quad \text{for all } t \leq T.$$

If we define O by $O_t = \widetilde{Z}_t + \int_0^t Ke^{-(1-\rho)A_s}\widetilde{Z}_s^\rho \,\mathrm{d}s$, then O is a càdlàg supermartingale.

By the Doob-Meyer decomposition, O can therefore be decomposed as $O = N + P$, where N is a local martingale and P is a decreasing process such that $P_0 = 0$, both of which are càdlàg. In particular, rearranging gives

$$\mathrm{d}\widetilde{Z}_t = -Ke^{-(1-\rho)A_t}\widetilde{Z}_t^{\rho}\,\mathrm{d}t + \mathrm{d}N_t - \mathrm{d}P_t.$$

Let $\tau := \{t \geq 0 : Z_t \notin (0,1)\}$. Let $\hat{Z}_t = Z_{t\wedge\tau}$, $\hat{N}_t = \int_0^{t\wedge\tau} e^{A_s}\,\mathrm{d}N_s$ and $\hat{P}_t = \int_0^{t\wedge\tau} e^{A_s}\,\mathrm{d}P_s$. Then, applying the product rule to $\hat{Z}_t = e^{A_t}\widetilde{Z}_t$ up to $t \leq \tau$, and noting that $\frac{\mathrm{d}A_t}{\mathrm{d}t} = \frac{U_t^{\vartheta}}{J_t} \leq K$,

$$\mathrm{d}\hat{Z}_t = \hat{Z}_t\,\mathrm{d}A_t - K\hat{Z}_t^{\rho}\,\mathrm{d}t + \mathrm{d}\hat{N}_t - \mathrm{d}\hat{P}_t \leq K(\hat{Z}_t - \hat{Z}_t^{\rho})\,\mathrm{d}t + \mathrm{d}\hat{N}_t \leq \mathrm{d}\hat{N}_t.$$

Since $\hat{N}_t \geq \hat{Z}_t - \hat{Z}_0 \geq -\hat{Z}_0$, $\hat{N}$ is a supermartingale.

Let $X_t = 1 - \hat{Z}_t^{1-\rho} \leq 1$. Then, $X = (X_t)_{t\geq 0}$ is càdlàg and, for $t < \tau$,

$$\begin{aligned}
\mathrm{d}X_t &= -(1-\rho)\hat{Z}_t^{-\rho}\,\mathrm{d}\hat{Z}_t + \frac{\rho(1-\rho)}{2}\hat{Z}_t^{-(\rho+1)}\,\mathrm{d}\langle Z\rangle_t \\
&= -(1-\rho)(\hat{Z}_t^{1-\rho}\,\mathrm{d}A_t - K\,\mathrm{d}t) + \mathrm{d}L_t + \mathrm{d}Q_t \\
&\geq K(1-\rho)X_t\,\mathrm{d}t + \mathrm{d}L_t + \mathrm{d}Q_t \geq \mathrm{d}L_t
\end{aligned}$$

where

$$L_t := -(1-\rho)\int_0^t \hat{Z}_s^{-\rho}\,\mathrm{d}\hat{N}_s \quad \text{and} \quad Q_t := (1-\rho)\int_0^t \hat{Z}_s^{-\rho}\,\mathrm{d}\hat{P}_s + \int_0^t \frac{\rho(1-\rho)}{2}\hat{Z}^{-(\rho+1)}\,\mathrm{d}\langle Z\rangle_s.$$

Since $L_t \leq X_t - X_0 \leq 1$, L is a (continuous) submartingale. Hence, $G := L + Q$ is a continuous submartingale. The result that X explodes to 1 in finite time with positive probability follows from Lemma 8.5. This implies that Z hits zero in finite time and, consequently, that Y is not proper. □

Proof of Theorem 3.6. Let W be the unique solution such that $W \overset{\mathbb{O}}{=} J$ given by Proposition 3.1 ($V = \frac{W}{1-R}$ is the CRRA-order solution). Since $J > 0$, W is proper

and uniqueness in the class of proper solutions follows from Theorem 3.5. Finally, W is the maximal solution by Proposition 7.5. □

9 Verification of the candidate optimal strategy

This section aims to prove that the candidate optimal strategy given in Proposition III.3.2 is indeed optimal. We will roughly follow the approach detailed in Chapter IV which goes as follows: first, show that if $\hat{X} = X^{\hat{\Pi},\hat{C}}$ is the wealth process under the candidate optimal strategy, then $\hat{V}(X + \varepsilon\hat{X})$ is a supersolution for (f_{EZ}, C); next, use a version of the Comparison Theorem (Corollary 6.5) for sub- and supersolutions to conclude that $\hat{V}(x(1+\varepsilon)) \geq V_0^C$; finally, let $\varepsilon \searrow 0$ to give $\hat{V}(x) \geq V_0^C$. Optimality follows since $V_0^{\hat{C}} = \hat{V}(x)$ by Proposition III.3.2.

However, the approach is not quite this simple for two main reasons. The most pressing reason is that when $\vartheta > 1$ the utility process V^C fails to be unique. The optimisation therefore takes place over the attainable and right-continuous consumption streams for which there exists a unique proper solution to the EZ-SDU equation. As we have argued in Section 8, the proper solution is the economically meaningful solution, and we may only consider the consumption streams that have a unique proper solution associated to them. The assumption of right-continuity, which is necessary for the proof, is not overly restrictive.

The next issue is that the hypotheses of the relevant comparison theorem (Corollary 6.5) are not satisfied, even for right-continuous consumption streams with a unique proper solution. To overcome this issue, one must approximate an arbitrary consumption stream in $\mathbb{UP}^*$ by a series of consumption streams satisfying the conditions and then take limits. The requirement of right-continuity ensures that we may choose right-continuous approximating consumption streams, which then have an associated proper solution V^n by Theorem 3.3. Since the limiting process $V = \lim_{n\to\infty} V^n$ is a proper solution associated to C, it must agree with the unique proper solution V^C.

To prove Theorem 3.8, we will use Lemma IV.4.2, which we restate for the readers benefit.

Lemma 9.1. *Let $\varepsilon > 0$ and let $\hat{X} = X^{\hat{\Pi},\hat{C}}$ be the wealth process under our candidate optimal strategy. If $X = X^{\Pi,C}$ is the wealth process associated to an admissible strategy (Π, C), then $\hat{V}(X + \varepsilon\hat{X})$ is a supersolution for the pair $(f_{EZ}, C + \eta\varepsilon\hat{X})$.*

We may then prove Theorem 3.8. Note that $\hat{V}(\hat{X})$ is a solution for the pair $(f_{EZ}, \eta\hat{X})$ and, by scaling, $\hat{V}(\varepsilon\hat{X})$ is a solution for the pair $(f_{EZ}, \eta\varepsilon\hat{X})$. We expect that $\hat{V}(X^{\Pi,C})$ is a supersolution for (f_{EZ}, C) but, when $R > 1$, the transversality condition might not hold. Furthermore, the conditions required for Proposition 6.4 to hold may be impossible to verify. However, as we show in the proof below, by considering the perturbed problem, the transversality condition is guaranteed and the comparison theorem can be applied.

Proof of Theorem 3.8. It follows from Proposition III.3.2 that $\hat{V}(\hat{X})$ is a utility process associated to candidate optimal strategy $(\hat{\Pi}, \hat{C})$. Since $\hat{V}(\hat{X})$ is a CRRA-order solution to (III.2.4), it is the unique proper solution to (III.2.4) by Theorem 3.6. Hence, $V_0^{\hat{C}} = \hat{V}(x)$. It therefore only remains to show that $V_0^C \leq \hat{V}(x)$ for all $C \in \mathscr{C}(x) \cap \mathbb{UP}^*$. Fix an arbitrary $C \in \mathscr{C}(x) \cap \mathbb{UP}^*$ and let $\Pi = (\Pi_t)_{t\geq 0}$ be an associated investment process.

We first prove the result when $R > S > 1$. In this case V^C is the minimal solution (since $C \in \mathbb{UP}$ and the minimal solution is proper). By Lemma IV.4.2, for each $\varepsilon > 0$, $\hat{V}(X^{C,\Pi} + \varepsilon\hat{X})$ is a supersolution associated to $C^\varepsilon = C + \eta\varepsilon\hat{X}$. Since $(C^\varepsilon)^{1-S} \leq (\eta\varepsilon)^{1-S}\hat{X}^{1-S}$, there exists an extremal solution V^{C^ε} associated to C^ε by Proposition 3.2 which is increasing in ε by Proposition 7.3. It is the minimal supersolution by Corollary 7.4. Hence, by minimality, $V_t^{C^\varepsilon} \leq \hat{V}(X_t^{C,\Pi} + \varepsilon\hat{X}_t) < 0$ for all $t \geq 0$ and V^{C^ε} is proper.

Let $V^* = \lim_{\varepsilon\to 0} V^{C^\varepsilon}$. Then, $V_0^* \leq \hat{V}(x)$. Consequently, since f_{EZ} is increasing in both arguments, and C^ε and V^ε are increasing in ε, $f_{EZ}(C^\varepsilon, V^\varepsilon)$ is increasing in ε, and applying the Monotone Convergence Theorem for conditional expectations

yields

$$V_t^* = \lim_{\varepsilon\to 0} V_t^{C^\varepsilon} = \lim_{\varepsilon\to 0} \mathbb{E}\left[\int_t^\infty f_{EZ}(C_s^\varepsilon, V_s^{C^\varepsilon})\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty f_{EZ}(C_s, V_s^*)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right].$$

Therefore, V^* is a solution associated to (f_{EZ}, C). Since $V_t^* = \lim_{\varepsilon\to 0} V_t^{C^\varepsilon} < 0$ for all $t \geq 0$, V^* is proper. It therefore agrees with the unique proper solution V^C so that $V_0^C \leq \hat{V}(x)$.

We now prove the result when $R < S < 1$. Fix an arbitrary $C \in \mathscr{C}(x) \cap \mathbb{UP}^*$ with associated investment process $\Pi = (\Pi_t)_{t\geq 0}$. Let $0 < \zeta < \eta\frac{S}{1-S}$ and define $\widetilde{X}_t = e^{\zeta t} X_t^{C,\Pi}$, $Y_t = e^{\frac{\zeta}{S}t}\hat{X}_t$ and $\widetilde{C}_t = e^{\zeta t} C_t$ for $t \geq 0$. Note that if $r_\zeta = r + \zeta$ and $\mu_\zeta = \mu + \zeta$, then

$$\mathrm{d}\widetilde{X}_t \;=\; \widetilde{X}_t \Pi_t \tau\,\mathrm{d}B_t + \Big(\widetilde{X}_t(r_\zeta + \Pi_t(\mu_\zeta - r_\zeta)) - \widetilde{C}_t\Big)\,\mathrm{d}t$$

We may think of $\widetilde{X} = (\widetilde{X}_t)_{t\geq 0}$ as being the wealth process associated to the strategy $(\Pi, \widetilde{C} = (\widetilde{C}_t)_{t\geq 0})$ in a more favourable financial market with risk-free rate r_ζ, drift of the risky asset μ_ζ, and well-posedness parameter $\eta_\zeta = -\frac{1-S}{S}(r_\zeta + \frac{\lambda^2}{2R}) = \eta - \frac{1-S}{S}\zeta \in (0, \eta)$. The volatility is unchanged. Furthermore, since

$$\frac{dY_t}{Y_t} = \frac{\lambda}{R}\,\mathrm{d}B_t + \left(\left(r + \frac{\lambda^2}{R} - \eta\right) + \frac{\zeta}{S}\right)\mathrm{d}t = \frac{\lambda}{R}\,\mathrm{d}B_t + \left(r_\zeta + \frac{\lambda^2}{R} - \eta_\zeta\right)\mathrm{d}t,$$

$Y = (Y_t)_{t\geq 0}$ is the wealth process under the optimal strategy in the new financial market with parameters r_ζ and μ_ζ. Define $\hat{V}^\zeta(x) = \eta_\zeta^{-\vartheta S}\frac{x^{1-R}}{1-R}$. Then, $\hat{V}^\zeta(\widetilde{X} + \varepsilon Y)$ is a supersolution for $(f_{EZ}, \widetilde{C} + \eta\varepsilon Y)$ by Lemma IV.4.2 and then also for (f_{EZ}, C) since $\widetilde{C} + \eta\varepsilon Y \geq C$.

Let $C^n := C \wedge n\hat{X}$. Then, there exists an extremal solution V^{C^n} associated to C^n by Proposition 3.2 which is monotonely increasing in n by Proposition 7.3. Also, $\hat{V}^\zeta = \hat{V}^\zeta(\widetilde{X} + \varepsilon Y)$ is a supersolution for (f_{EZ}, C^n) since $C \geq C^n$. Furthermore, since C is right-continuous, C^n is right-continuous. The extremal solution V^{C^n}

associated to C^n is therefore proper by Theorem 3.3 and Remark 2.4. In particular, $\{C_t > 0\} = \{C_t^n > 0\} \subseteq \{V_t^{C^n} > 0\}$ for all $t \geq 0$ up to null sets.

Note that $(C^n)^{1-R} \leq n^{1-R}\hat{X}^{1-R}$. Then,

$$(1-R)\hat{V}^\zeta \geq \eta_\zeta^{-\vartheta S}(\varepsilon Y)^{1-R} = \eta_\zeta^{-\vartheta S}\varepsilon^{1-R}e^{\frac{\zeta(1-R)}{S}t}\hat{X}^{1-R} \geq \frac{\eta_\zeta^{-\vartheta S}\varepsilon^{1-R}}{n^{1-R}}e^{\frac{\zeta(1-R)}{S}t}(C^n)^{1-R}.$$

Furthermore, $V^{C^n} \in \mathbb{UI}(f_{EZ}, C^n)$ by Remark III.1.2 and $\mathbb{E}[(V_t^{C^n} - \hat{V}_t^\zeta)^+] \leq \mathbb{E}[V_t^{C^n}] = \mathbb{E}[\int_t^\infty f_{EZ}(C_s^n, V_s^{C^n})\,\mathrm{d}s] \leq \mathbb{E}[\int_0^\infty f_{EZ}(C_s^n, V_s^{C^n})\,\mathrm{d}s] < \infty$. Hence, the conditions of Corollary 6.5 are met and $\hat{V}_t^\zeta \geq V_t^{C^n}$ for all $t \geq 0$. In particular, $\hat{V}_0^\zeta \geq V_0^{C^n}$. If $V^* = \lim_{n\to\infty} V^{C^n}$ is the monotone limit, then

$$V_t^* = \lim_{n\to\infty} V_t^{C^n} = \lim_{n\to\infty} \mathbb{E}\left[\int_t^\infty f_{EZ}(C_s^n, V_s^{C^n})\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty f_{EZ}(C_s, V_s^*)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right].$$

Hence, V^* is a solution. It is a proper solution since $\{C_t > 0\} \subseteq \{V_t^{C^n} > 0\} \subseteq \{V_t^* > 0\}$ up to null sets. Therefore, it must agree with the unique proper solution V^C associated to C. In particular, $\hat{V}_0^\zeta = \hat{V}^\zeta(x(1+\varepsilon)) \geq \lim_{n\to\infty} V_0^{C^n} = V_0^* = V_0^C$. Finally, taking $\zeta, \varepsilon \searrow 0$ gives $\hat{V}(x) \geq V_0^C$. □

10 Future directions

In Chapter IV, we showed that when $\vartheta \in (0,1)$, we were able to assign a unique (generalised if nonfinite) utility process associated to *any* consumption stream. This meant that we could solve the optimal investment and consumption problem under *all* parameter combinations for which it is well-posed, and optimise over *all* attainable consumption streams. However, in this chapter, we explained that extra difficulties arise under parameter combinations leading to $\vartheta > 1$, since uniqueness of solutions fails. To deal with this, we introduced the economically motivated notion of a *proper solution*, which rules out solutions that ignore the agent's consumption and assign zero utility from a finite time onwards. This lead us to consider the class $\mathbb{UP}$ of

consumption streams with a *unique* proper solution associated to them. We showed that a large class of consumption streams lie in $\mathbb{UP}$. In Theorem 3.3 we showed that the class of proper solutions is even larger than the class which we proved is contained in $\mathbb{UP}$.

We only treated parameter combinations leading to $\eta > 0$, where η is defined by (III.3.6), but this was only due to time limitations. We expect that cases leading to $\eta \leq 0$ lead to an ill-posed investment-consumption problem. In the case $R < 1$, by considering a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $H(\hat{\pi}, \xi_n) \to 0$, one should be able to show that the problem is ill posed when $\eta \leq 0$; when $R > 1$, it will be harder.

The first direction for extending the results of the chapter would be to enlarge the class $\mathbb{UP}$. By Theorem 3.3, all right-continuous consumption streams $C \in \mathscr{P}_+$ such that $C^{1-R} \leq Y \in \mathbb{SO}_+$ have a proper solution associated to them. However, we only showed (in Theorem 3.5) that this proper solution is unique when $C^{1-R} \in \mathbb{SO}_+$. It seems reasonable to believe that, when a proper solution exists, it is unique, but we have not yet been able to prove this result.

The next direction in which the results could be extended would be to investigate what we can say about consumption streams for which there does not exist a finite proper solution. When $\vartheta \in (0, 1)$ it was shown in Chapter IV that the utility process may be generalised to consider utility processes that lie in $\overline{\mathbb{V}}$ (which is either $\overline{\mathbb{R}}_+$ or $\overline{\mathbb{R}}_-$ depending on the sign of $1 - R$), permitting the evaluation of *all* consumption streams. This generalision corresponds to the minimal supersolution when $R < 1$ and the maximal subsolution when $R > 1$.

Such a generalisation is a natural extension to the CRRA case, where it is perfectly reasonable to talk about consumption streams yielding positive or negative infinite utilities. To reduce the number of case distinctions we will assume that $R > 1$ from this point onwards, so that $\mathbb{V} = \overline{\mathbb{R}}_-$; the other case is symmetric. Three obvious candidates that generalise the utility process come to mind when $\vartheta > 1$: the natural extension of the minimal solution which is allowed to takes values in $\overline{\mathbb{R}}_-$; a proper solution which imposes a further restriction at $-\infty$; or, the maximal proper

solution.

Since the unique proper solution corresponds to the extremal solution when it is finite-valued, it seems reasonable to believe that—by extending the notion of the extremal solution to allow it to take infinite values—the extremal solution will help us evaluate consumption streams for which there is no finite proper solution. However, an issue arises here, that shares common characteristics with the non-proper solutions examined in Section 2.5: we may always assign the utility process $V \equiv -\infty$ to any consumption stream. Since this would be the minimal solution (extended to take values in $\overline{\mathbb{R}}_-$) associated to every consumption stream, it would render EZ-SDU degenerate as it does not distinguish between consumption streams. To get around this, one could further impose a transversality condition, but since there are perfectly reasonable consumption streams for the CRRA case and the case $\vartheta \in (0,1)$ which do yield negative infinite utilities, this doesn't seem like a desirable approach.

The next candidate for the utility process associated to consumption streams $C \in \mathscr{P}_+$ for which no finite proper solution exists is a modification of the definition of the proper solution. For example, one could define the proper solutions (which may take values in $\overline{\mathbb{R}}_-$) to be only those solutions such that

$$\left\{\omega : \mathbb{E}\left[\int_t^\infty C_s^{1-R}\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] < \infty\right\} \setminus \{\omega : (1-R)V_t(\omega) < \infty\} \quad \text{is a null set.} \tag{10.1}$$

However, one can show that this rules out a large class of deterministic consumption streams. It is possible to show (by taking derivatives) that an explicit solution $v = (v(t))_{t\geq 0}$ associated to a deterministic consumption stream $c = (c(t))_{t\geq 0}$ is given by

$$v(t) = \frac{1}{1-R}\left(\int_t^\infty c(s)^{1-S}\,\mathrm{d}s\right)^\vartheta. \tag{10.2}$$

It is also possible to show that the family of improper solutions, indexed by $T \geq 0$,

$$v(t,T) = \frac{\mathbf{1}_{\{t<T\}}}{1-R}\left(\int_t^T c(s)^{1-S}\,\mathrm{d}s\right)^\vartheta \tag{10.3}$$

are also all utility processes associated to (f_{EZ}, c). Intuitively, we want to rule out the utility processes in (10.3) and only consider the utility process in (10.2). However, the utility process in (10.2) does not satisfy the (extended) definition of a proper utility process at infinity given in (10.1). Explicitly, let $c(t) = t^{\frac{1}{S-1}}$. Then, for each $t \geq 0$, $\int_t^\infty c(t)^{1-R}\,\mathrm{d}s = \int_t^\infty t^{-\vartheta}\,\mathrm{d}s = \frac{t^{1-\vartheta}}{1-\vartheta}$ for $t > 0$. However, $(1-R)v(t) = (\frac{1}{\vartheta}\int_t^\infty t^{-1}\,\mathrm{d}s)^\vartheta = \infty$. This means that v does not satisfy (10.1). If one wants to consider such consumption streams, the definition of a proper solution must be different. It is less clear what the restriction should be.

The final candidate for the solution associated to consumption streams when no finite proper utility process exists is the *maximal proper solution.* This definition is similar to that used in Chapter IV, and is promising for a number of reasons. Firstly, the maximal proper solution agrees with the unique proper solution for all consumption streams $C \in \mathbb{UP}$. Secondly, if we define $\overline{\mathbb{UP}}$ to be the consumption streams C such that there exists a sequence of consumption streams $(C^n)_{n\in\mathbb{N}}$ with $C^n \nearrow C$ and $C^n \in \mathbb{UP}$, then if V^n is the unique proper utility process associated to C^n for each $n \in \mathbb{N}$, it is simple to see via monotonicity arguments that $V^* = \lim_{n\to\infty} V^n$ satisfies $V_t^* = \mathbb{E}\left[\int_t^\infty f_{EZ}(C_s, V_s^*)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$ for all $t \geq 0$. Also, since V^* is approximated from above by a monotonely decreasing sequence of supersolutions associated to (f_{EZ}, C), it corresponds to what we expect is the maximal proper solution—although we currently have no comparison theorem to confirm this. Finally, having the solution defined in this way could be extremely beneficial mathematically, since we can show that V^* is increasing in C and prove a verification theorem by using the comparison theorem on the approximating processes and then taking limits.

APPENDIX TO CHAPTER V

V.A Existence of proper solutions associated to discounted indicators of stochastic intervals

This section will be dedicated to proving Proposition 8.1. In this section, we will emphasise the role that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ plays in determining the utility process $V = (V_t)_{t\geq 0}$ associated to a pair (g, C). To this end, if $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration used in Definition III.1.1, then we will refer to V as the utility process associated to *the triple* $(g, C, \mathbb{F})$.

Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{T} = \{t_0, t_1, t_2, \cdots, t_n\}$ be an ordered set. We assume without loss of generality that $t_0 = 0$ and $t_n = \infty$.

Condition B. The pair $(\mathbb{F}, \mathbb{T})$ satisfies Condition B if $\mathcal{F}_t = \mathcal{F}_{t_{i(t;\mathbb{T})}}$ for all $t \geq 0$.

Definition A.1. An $(\mathbb{F}, \mathbb{T})$-stopping time is an $\mathbb{F}$-stopping time σ that can be written as $\sigma = \sum_{i=0}^{n} t_i \mathbf{1}_{A_i}$ for some family $(A_i)_{i\in\{0,\cdots,n\}}$ of disjoint sets such that $\mathbb{P}(\cup_{i=0}^{n} A_i) = 1$ and $A_i \in \mathcal{F}_{t_i}$.

Throughout this section, we define $B_i^{\mathbb{T},\sigma} := \{\sigma > t_i\} = \cup_{j=i+1}^n A_j$ and $i(t;\mathbb{T}) := \max\{i : t_i \leq t\}$. When it is clear which $\mathbb{T}$ and σ we are referring to, we drop the subscript and write $i(t) = i(t;\mathbb{T})$ and $B_i = B_i^{\mathbb{T},\sigma}$. Note that $\{\sigma > t\} = \{\sigma > t_{i(t;\mathbb{T})}\}$ for all $t \geq 0$.

We first prove the existence of a proper solution associated to $(h_{EZ}, U, \mathbb{F})$ where $U = (U_t)_{t\geq 0}$ is given by $U_t = e^{-\gamma t}\mathbf{1}_{t<\sigma}$. Here $\gamma > 0$, and σ is an $(\mathbb{F},\mathbb{T})$-stopping time where $(\mathbb{F},\mathbb{T})$ satisfies Condition B.

Proposition A.2. *Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{T}$ be an ordered set such that $(\mathbb{F},\mathbb{T})$ satisfies Condition B. Let σ be a $(\mathbb{F},\mathbb{T})$-stopping time and define $U = (U_t)_{t\geq 0}$ by $U_t = e^{-\gamma t}\mathbf{1}_{t<\sigma}$. Then, there exists a proper solution $W = (W_t)_{t\geq 0}$ associated to $(h_{EZ}, U, \mathbb{F})$ such that*

$$W_t \geq \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}_t\right]\right)^{\vartheta}, \quad \textit{for all } t \geq 0. \tag{1.1}$$

The proof of Proposition A.2 relies on the following lemma.

Lemma A.3. *Suppose that $\sigma = \sum_{i=0}^n t_i \mathbf{1}_{A_i}$ is a $(\mathbb{F},\mathbb{T})$-stopping time. Then,*

$$\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}_t\right] = \mathbf{1}_{B_{i(t)}}(e^{-\gamma t} - e^{-\gamma t_{i(t)+1}}) + \sum_{j=i(t)+1}^{n-1} \mathbb{E}\left[\mathbf{1}_{B_j} \,\middle|\, \mathcal{F}_t\right]\left(e^{-\gamma t_j} - e^{-\gamma t_{j+1}}\right)$$

Proof. Using the definition of σ and B_j, the fact that $B_{i(t)} \in \mathcal{F}_t$, and rearranging the telescoping sum gives

$$\begin{aligned}
&\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}_t\right] \\
&\quad = \sum_{j=i(t)+1}^{n} (e^{-\gamma t} - e^{-\gamma t_j})\mathbb{P}(A_j|\mathcal{F}_t) \\
&\quad = e^{-\gamma t}\mathbb{P}(B_{i(t)}|\mathcal{F}_t) - \sum_{j=i(t)+1}^{n} e^{-\gamma t_j}(\mathbb{P}(B_{j-1}|\mathcal{F}_t) - \mathbb{P}(B_j|\mathcal{F}_t)) \\
&\quad = \mathbf{1}_{B_{i(t)}}(e^{-\gamma t} - e^{-\gamma t_{i(t)+1}}) + \sum_{j=i(t)+1}^{n-1} \mathbb{E}\left[\mathbf{1}_{B_j} \,\middle|\, \mathcal{F}_t\right]\left(e^{-\gamma t_j} - e^{-\gamma t_{j+1}}\right). \qquad \square
\end{aligned}$$

Proof of Proposition A.2. Define $W = (W_t)_{t\geq 0}$ recursively backwards by $W_\infty = 0$ $\mathbb{P}$-a.s. and, for $t < \infty$, $W_t = \mathbf{1}_{B_{i(t)}} w\big(t, \xi_{t_{i(t)+1}}\big)$, where

$$w(t,y) = \left(y^{\frac{1}{\vartheta}} + \frac{1}{\gamma\vartheta}\left(e^{-\gamma t} - e^{-\gamma t_{i(t)+1}}\right)\right)^{\vartheta} \quad \text{and} \quad \xi_{t_{i(t)+1}} = \mathbb{E}\left[W_{t_{i(t)+1}} \,\middle|\, \mathcal{F}_{t_{i(t)}}\right]. \tag{1.2}$$

We will first show that W is a solution associated to $(h_{EZ}, U, \mathbb{F})$. We then show that (1.1) holds and W is proper.

First, note that $F_U(W)_\infty := \lim_{t\to\infty} F_U(W)_t \leq \lim_{t\to\infty} \int_t^\infty e^{-\gamma s}(\frac{e^{\gamma\vartheta s}}{\gamma\vartheta})^\rho \,\mathrm{d}s = 0 = W_\infty$. We will now show that $W_t = F_U(W)_t = \mathbb{E}\left[\int_t^\infty U_s W_s^\rho \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$ for all $t \geq 0$ by backwards induction. For the inductive step, fix $k \in \{0, \cdots, n-1\}$ and assume that W satisfies $W_{t_{k+1}} = F_U(W)_{t_{k+1}}$ and that $t_k \leq t < t_{k+1}$. By the definition of $W_{t_{k+1}}$, since $\mathbf{1}_{B_k}\mathbf{1}_{B_{k+1}} = \mathbf{1}_{B_{k+1}}$, and by Condition B, $\mathbb{E}\left[W_{t_{k+1}} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\mathbf{1}_{B_k} W_{t_{k+1}} \,\middle|\, \mathcal{F}_t\right] = \mathbf{1}_{B_k}\mathbb{E}\left[W_{t_{k+1}} \,\middle|\, \mathcal{F}_{t_k}\right] = \mathbf{1}_{B_k}\xi_{t_{k+1}}$. Hence, combining this with the inductive hypothesis yields

$$\begin{aligned} F_U(W)_t &= \mathbb{E}\left[\int_t^{t_{k+1}} U_s W_s^\rho \,\mathrm{d}s + W_{t_{k+1}} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbf{1}_{B_k}\left(\int_t^{t_{k+1}} e^{-\gamma s}(w(s, \xi_{t_{k+1}}))^\rho \,\mathrm{d}s + \xi_{t_{k+1}}\right). \end{aligned}$$

If $\omega \in B_k^c$, then clearly $W_t(\omega) = 0 = F_U(W)_t(\omega)$. Assume instead that $\omega \in B_k$. Then, $\xi := \xi_{t_{k+1}}(\omega)$ is known. Since $\lim_{t\nearrow t_{k+1}} w(t,\xi) = \xi$ and $\frac{\partial w}{\partial t}(t,\xi) = -e^{-\gamma t}(w(t,\xi))^\rho$ for $t_k \leq t < t_{k+1}$, integrating yields $w(t,\xi) = \int_t^{t_{k+1}} e^{-\gamma s}(w(s,\xi))^\rho \,\mathrm{d}s + \xi$. In particular,

$$W_t(\omega) = w(t,\xi) = \int_t^{t_{k+1}} e^{-\gamma s}(w(s,\xi))^\rho \,\mathrm{d}s + \xi = F_U(W)_t(\omega).$$

Consequently, $W_t = F_U(W)_t$ for $t_k \leq t \leq t_{k+1}$, and hence for all $t \geq 0$ by induction.

We now show that $W = (W_t)_{t\geq 0}$ defined in (1.2) is proper by proving the following

statement recursively backwards for $k \in \{0, \cdots, n-1\}$: If $t_k \leq t < t_{k+1}$, then

$$W_t^{\frac{1}{\vartheta}} \geq \frac{\mathbf{1}_{B_k}}{\gamma\vartheta} \left(\sum_{j=k+1}^{n-1} \mathbb{E}\left[\mathbf{1}_{B_j} \,\middle|\, \mathcal{F}_t\right] \left(e^{-\gamma t_j} - e^{-\gamma t_{j+1}}\right) + e^{-\gamma t} - e^{-\gamma t_{k+1}} \right). \tag{1.3}$$

Here, we define $\sum_n^{n-1} a = 0$ for arbitrary $a \in \mathbb{R}$. Hence, the statement holds true for $k = n-1$ by the definition of W in (1.2). For the induction step, assume that it holds true for $k+1$ and let $t_k \leq t < t_{k+1}$ so that $i(t) = k$. Using the definition of W given in (1.2), Jensen's inequality, the inductive hypothesis and the fact that $\mathbf{1}_{B_i}\mathbb{E}\left[\mathbf{1}_{B_j} \,\middle|\, \mathcal{F}_{t_i}\right] = \mathbb{E}\left[\mathbf{1}_{B_i}\mathbf{1}_{B_j} \,\middle|\, \mathcal{F}_{t_i}\right] = \mathbb{E}\left[\mathbf{1}_{B_j} \,\middle|\, \mathcal{F}_{t_i}\right]$ for $i \leq j$,

$$\begin{aligned}
W_t^{\frac{1}{\vartheta}} &= \mathbf{1}_{B_k} \left(\left(\mathbb{E}\left[W_{t_{k+1}} \,\middle|\, \mathcal{F}_t\right]\right)^{\frac{1}{\vartheta}} + \frac{e^{-\gamma t} - e^{-\gamma t_{k+1}}}{\gamma\vartheta} \right) \\
&\geq \frac{\mathbb{E}\left[\mathbf{1}_{B_{k+1}} \,\middle|\, \mathcal{F}_t\right]}{\gamma\vartheta} \left(\sum_{j=k+2}^{n-1} \mathbb{E}\left[\mathbf{1}_{B_j} \,\middle|\, \mathcal{F}_t\right] \left(e^{-\gamma t_j} - e^{-\gamma t_{j+1}}\right) + e^{-\gamma t_{k+1}} - e^{-\gamma t_{k+2}} \right) \\
&\quad + \mathbf{1}_{B_k} \frac{e^{-\gamma t} - e^{-\gamma t_{k+1}}}{\gamma\vartheta} \\
&= \frac{1}{\gamma\vartheta} \sum_{j=k+1}^{n-1} \mathbb{E}\left[\mathbf{1}_{B_j} \,\middle|\, \mathcal{F}_t\right] \left(e^{-\gamma t_j} - e^{-\gamma t_{j+1}}\right) + \frac{\mathbf{1}_{B_k}}{\gamma\vartheta} \left(e^{-\gamma t} - e^{-\gamma t_{k+1}}\right).
\end{aligned}$$

Hence, (1.3) holds for all t for which $t_k \leq t < t_{k+1}$, and hence by induction for all $t < t_n = \infty$.

Since the right hand side of (1.3) is equal to $\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t \vee \sigma)} \,\middle|\, \mathcal{F}_t\right]$ by Lemma A.3, Equation (1.1) holds and W is proper. □

We now show that for any continuous filtration $\mathbb{F}$ and $\mathbb{F}$-stopping time σ, we can find a proper utility process associated to $U_t = e^{-\gamma t}\mathbf{1}_{t<\sigma}$ by approximating $(\mathbb{F}, \mathbb{T})$ by a monotone sequence of pairs $(\mathbb{F}^n, \mathbb{T}^n)$ satisfying Condition B.

Lemma A.4. *Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a continuous filtration and let σ be a $\mathbb{F}$-stopping time. Let $\mathbb{T}^n = \{k2^{-n} : k = 0, 1, \cdots, n2^n\} \cup \{\infty\}$ and define $\mathbb{F}^n = (\mathcal{F}_t^n)_{t\geq 0}$ by $\mathcal{F}_t^n = \mathcal{F}_{2^{-n}\lfloor 2^n t\rfloor \wedge n}$, for $t \geq 0$. Then, $\sigma_n = \mathbf{1}_{\sigma\leq n}2^{-n}\lceil 2^n\sigma\rceil + \infty\mathbf{1}_{\sigma>n}$ is a $(\mathbb{F}^n, \mathbb{T}^n)$-stopping time. Furthermore, $(\mathbb{F}^n, \mathbb{T}^n)$ satisfies Condition B for each $n \in \mathbb{N}$, $\sigma_n \searrow \sigma$*

and $\mathcal{F}_t^n \nearrow \mathcal{F}_t$ *for all* $t \geq 0$.

Proof. Note that σ_n takes values in $\mathbb{T}^n$ and $\{\sigma_n \leq t\} = \{\sigma \leq n\} \cap \{2^{-n}\lceil 2^n \sigma \rceil \leq t\} = \{\sigma \leq 2^{-n}\lfloor 2^n t \rfloor \wedge n\} \in \mathcal{F}_t^n$, so that σ_n is a $(\mathbb{F}^n, \mathbb{T}^n)$-stopping time. In addition, for each $n \in \mathbb{N}$ and $t \geq 0$, $t_{i(t;\mathbb{T}_n)} = 2^{-n}\lfloor 2^n t \rfloor \wedge n$. Hence, $\mathcal{F}_t^n = \mathcal{F}_{t_{i(t;\mathbb{T}_n)}} = \mathcal{F}^n_{t_{i(t;\mathbb{T}_n)}}$ so that $(\mathbb{F}^n, \mathbb{T}^n)$ satisfies Condition B. It is easily checked that $\sigma_n \searrow \sigma$ and $\mathcal{F}_t^n \nearrow \mathcal{F}_t$ for all $t \geq 0$. □

To prove Proposition 8.1, we will need the following lemma, which is a variant of Hunt's Lemma and the Reverse Fatou Lemma.

Lemma A.5. *Let* $(\Omega, \mathcal{G}, \mathbb{P})$ *be a probability space and* $(X_n)_{n\in\mathbb{N}}$ *a sequence of random variables bounded in absolute value by an integrable random variable* Y. *Let* $(\mathcal{G}_n)_{n\in\mathbb{N}}$ *be an increasing family of* τ*-algebras and* $\mathcal{G}_\infty := \bigcup_{n=1}^\infty \mathcal{G}_n$. *Then,*

$$\limsup_{n\to\infty} \mathbb{E}\left[X_n \,|\, \mathcal{G}_n\right] \leq \mathbb{E}\left[\limsup_{n\to\infty} X_n \,\middle|\, \mathcal{G}_\infty\right], \ \mathbb{P}\text{-}a.s.$$

Proof. Let $Z_m = \sup_{n\geq m} X_n \in L^1$. Then, $\mathbb{E}\left[X_n \,|\, \mathcal{G}_n\right] \leq \mathbb{E}\left[Z_m \,|\, \mathcal{G}_n\right]$ for $n \geq m$. Taking the lim sup and using the L^1-Martingale Convergence Theorem yields

$$\limsup_{n\to\infty} \mathbb{E}\left[X_n \,|\, \mathcal{G}_n\right] \leq \lim_{n\to\infty} \mathbb{E}\left[Z_m \,|\, \mathcal{G}_n\right] = \mathbb{E}\left[Z_m \,|\, \mathcal{G}_\infty\right]. \tag{1.4}$$

Furthermore, by the conditional version of the Reverse Fatou Lemma (and since $Z_m \leq Y$),

$$\limsup_{m\to\infty} \mathbb{E}\left[Z_m \,|\, \mathcal{G}_\infty\right] \leq \mathbb{E}\left[\lim_{m\to\infty} Z_m \,\middle|\, \mathcal{F}_\infty\right] = \mathbb{E}\left[\limsup_{n\to\infty} X_n \,\middle|\, \mathcal{G}_\infty\right]. \tag{1.5}$$

Combining (1.4) and (1.5) yields the result. □

Proposition A.6. *Let* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ *be a continuous filtration,* σ *be a* $\mathbb{F}$*-stopping time and* $\gamma > 0$. *Let* $U = (U_t)_{t\geq 0}$ *be given by* $U_t = e^{-\gamma t}\mathbf{1}_{t<\sigma}$. *Then, there exists a proper solution* $W = (W_t)_{t\geq 0}$ *associated to* $(h_{EZ}, U, \mathbb{F})$ *such that* $W_t \geq \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}_t\right]\right)^\vartheta$ *for all* $t \geq 0$.

Proof. By Lemma A.4, we may choose a sequence $(\mathbb{F}^n = (\mathcal{F}_t^n)_{t\geq 0}, \mathbb{T}^n)_{n\in\mathbb{N}}$ such that $(\mathbb{F}^n, \mathbb{T}^n)$ satisfies Condition B for each $n \in \mathbb{N}$ and a $(\mathbb{F}^n, \mathbb{T}^n)$-stopping time σ_n such that $\sigma_n \searrow \sigma$, and $\mathcal{F}_t^n \nearrow \mathcal{F}_t$ for $t \geq 0$. Since, for each $n \in \mathbb{N}$, the pair $(\mathbb{F}^n, \mathbb{T}^n)$ satisfies the conditions of Proposition A.2, for $U^n = (U_t^n)_{t\geq 0}$ defined by $U_t^n = e^{-\gamma t}\mathbf{1}_{t<\sigma_n}$, there exists a proper solution $W^n = (W_t^n)_{t\geq 0}$ associated to the triple $(h_{EZ}, U^n, \mathbb{F}^n)$ such that

$$W_t^n \geq \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma_n)} \,\middle|\, \mathcal{F}_t^n\right]\right)^{\vartheta}. \tag{1.6}$$

Since W^n is a solution (and is càdlàg by Remark 6.2), for all bounded stopping times $\tau \leq \sigma$,

$$W_\tau^n = \mathbb{E}\left[\int_\tau^\sigma U_s^n (W_s^n)^\rho \,\mathrm{d}s + W_\sigma^n \,\middle|\, \mathcal{F}_\tau^n\right]. \tag{1.7}$$

Consider $\overline{u} = (\overline{u}(t))_{t\geq 0}$ defined by $\overline{u}(t) = e^{-\gamma t}$ for $t \geq 0$. Then, by taking derivatives, one finds that $W^{\overline{u}} = (W_t^{\overline{u}})_{t\geq 0}$ defined by $W_t^{\overline{u}} = \frac{e^{-\gamma\vartheta t}}{\gamma^\vartheta\vartheta^\vartheta}$ is a solution associated to $(h_{EZ}, \overline{u})$ (and any filtration). Furthermore, as $J_t^{\overline{u}^\vartheta} = \int_t^\infty (\overline{u}(s))^\vartheta \,\mathrm{d}s = \frac{e^{-\gamma\vartheta t}}{\gamma\vartheta}$, $W^{\overline{u}} \overset{\mathrm{O}}{=} J^{\overline{u}^\vartheta} = (J_t^{\overline{u}^\vartheta})_{t\geq 0}$ and $W^{\overline{u}}$ is the maximal solution associated to $\overline{u}$ by Proposition 7.5. Therefore, since $U_t^n \leq \overline{u}(t)$ for $t \geq 0$, it follows from Proposition 7.3 that $W_t^n \leq W_t^{\overline{u}} \leq W_0^{\overline{u}} = \frac{1}{\gamma^\vartheta\vartheta^\vartheta} < \infty$ for all $t \geq 0$ and W^n is bounded. Similarly, $U_t(W_t^n)^\rho \leq \overline{u}(t)(W_t^{\overline{u}})^\rho$ and $\mathbb{E}[\int_0^\infty \overline{u}(t)(W_t^{\overline{u}})^\rho \,\mathrm{d}t] = W_0^{\overline{u}} < \infty$.

Define $W_t^* = \limsup_{n\to\infty} W_t^n$ for each $t \geq 0$. We now show that $W^* = (W_t^*)_{t\geq 0}$ is a subsolution associated to $(h_{EZ}, U, \mathbb{F})$. Taking the lim sup in (1.7) and using Lemma A.5 gives

$$W_\tau^* = \limsup_{n\to\infty} \mathbb{E}\left[\int_\tau^\sigma U_s^n (W_s^n)^\rho \,\mathrm{d}s + W_\sigma^n \,\middle|\, \mathcal{F}_\tau^n\right] \leq \mathbb{E}\left[\int_\tau^\sigma U_s (W_s^*)^\rho \,\mathrm{d}s + W_\sigma^* \,\middle|\, \mathcal{F}_\tau\right]. \tag{1.8}$$

Let $Y_t = W_t^* + \int_0^t U_s (W_s^*)^\rho \,\mathrm{d}s$. Then, by (1.8), $Y = (Y_t)_{t\geq 0}$ is an optional strong submartingale. It is therefore làdlàg (see [DM82, Theorem A1.4]) and, by the strong submartingale property, $Y_\sigma \leq \mathbb{E}\left[Y_{\sigma+} \,|\, \mathcal{F}_\sigma\right]$ for all stopping times σ. Consequently,

W^* is làdlàg, and

$$W^*_\sigma = Y_\sigma - \int_0^\sigma U_s(W^*_s)^\rho \,\mathrm{d}s \le \mathbb{E}\left[Y_{\sigma+} \,|\, \mathcal{F}_\sigma\right] - \int_0^\sigma U_s(W^*_s)^\rho \,\mathrm{d}s = \mathbb{E}\left[W^*_{\sigma+} \,\middle|\, \mathcal{F}_\sigma\right].$$

Thus, $W^*_\tau \le \mathbb{E}\left[\int_\tau^\sigma U_s(W^*_s)^\rho \,\mathrm{d}s + W^*_{\sigma+} \,\middle|\, \mathcal{F}_\tau\right]$. In addition, the transversality condition for subsolutions holds since $\limsup_{t\to\infty} \mathbb{E}\left[W^*_{t+}\right] \le \lim_{t\to\infty} W^{\overline{u}}_t = 0$, so that W^* is a subsolution for $(h_{EZ}, U, \mathbb{F})$.

Since W^* is nonnegative and $\mathbb{P}$-a.s. bounded above by $W^{\overline{u}}_0 < \infty$, it is ($L^1$-) bounded. Furthermore, by (1.6), the fact that $\sigma_n \ge \sigma$ for all $n \in \mathbb{N}$ and the L^1-Martingale Convergence Theorem,

$$\begin{aligned} W^*_t &\ge \lim_{n\to\infty} \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma_n)} \,\middle|\, \mathcal{F}^n_t\right]\right)^\vartheta \\ &\ge \left(\frac{1}{\gamma\vartheta}\lim_{n\to\infty}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}^n_t\right]\right)^\vartheta \\ &= \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}_t\right]\right)^\vartheta. \end{aligned} \tag{1.9}$$

Let $W^U = (W^U_t)_{t\ge 0}$ be the *maximal* solution associated to $(h_{EZ}, U, \mathbb{F})$. Then, by Proposition 7.2, W^U is the maximal L^1-bounded subsolution. Combining this with (1.9) gives that W^U is a proper solution and

$$W^U_t \ge W^*_t \ge \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}_t\right]\right)^\vartheta, \quad \text{for all } t \ge 0. \qquad \square$$

Finally, we remove the assumption that $\tau = 0$ and prove Proposition 8.1.

Proof of Proposition 8.1. Let $U_t = e^{-\gamma t}\mathbf{1}_{\tau\le t<\sigma}$ and let $\widehat{U}_t = e^{-\gamma t}\mathbf{1}_{t<\sigma}$. Then, by Proposition A.6, there exists a proper solution $\widehat{W}$ associated to the pair $(h_{EZ}, \widehat{U})$ such that $\widehat{W}_t \ge \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}_t\right]\right)^\vartheta$. Suppose first that $t \ge \tau$. Then, since $U_s = \widehat{U}_s$ for all $s \ge t$, the (unique) maximal solutions W and $\widehat{W}$ associated to U and $\widehat{U}$ coincide at t. Hence, $W_t = \widehat{W}_t \ge \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma t} - e^{-\gamma(t\wedge\sigma)} \,\middle|\, \mathcal{F}_t\right]\right)^\vartheta$.

Suppose now that $t < \tau$ and note that $W_\tau \ge \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[(e^{-\gamma\tau} - e^{-\gamma\sigma}) \,|\, \mathcal{F}_\tau\right]\right)^\vartheta$ as

above. Then,

$$W_t = \mathbb{E}\left[W_\tau \,|\, \mathcal{F}_t\right] \geq \mathbb{E}\left[\left(\tfrac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma\tau} - e^{-\gamma\sigma} \,\middle|\, \mathcal{F}_\tau\right]\right)^\vartheta \,\middle|\, \mathcal{F}_t\right] \geq \left(\tfrac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma\tau} - e^{-\gamma\sigma} \,\middle|\, \mathcal{F}_t\right]\right)^\vartheta.$$

The final inequality follows from Jensen's inequality and the tower property of the conditional expectation. Combining the inequalities above yields

$$W_t \geq \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma(t\vee\tau)} - e^{-\gamma(t\vee\sigma)} \,\middle|\, \mathcal{F}_t\right]\right)^\vartheta. \qquad \square$$

V.B Additional results

Lemma B.1. *Let $\rho \in (0,1)$ and $\phi > 0$, $\varepsilon \geq 0$. Then, there is a solution $x = x(\phi, \varepsilon)$ to*

$$x = \phi(x^\rho + \varepsilon), \tag{2.1}$$

that is continuous and increasing in ε and ϕ and is such that $x(\phi, 0) = \phi^\vartheta$.

Proof. Let $g_\phi(x) = \phi^{-1}x - x^\rho$. Then, x solves (2.1) if and only if $g_\phi(x) = \varepsilon$. To find x, we must therefore show that $g_\phi(x)$ has an inverse defined on the nonnegative real line.

To begin with, note that g_ϕ is twice continuously differentiable, with $g'_\phi(x) = \phi^{-1} - \rho x^{\rho-1}$ and $g''_\phi(x) = (1-\rho)\rho x^{\rho-2} > 0$ for $x > 0$. Since $g''_\phi(x) > 0$ for $x > 0$, g_ϕ is strictly convex on $[0, \infty)$. There are two zeros of g_ϕ; one at $x = 0$ and one at $x = x_0 := \phi^{\frac{1}{1-\rho}} = \phi^\vartheta$. These are the only two zeros, since a strictly convex function can have at most two zeros. Furthermore, for each $x \geq x_0$, $g'_\phi(x) = \phi^{-1} - \rho x^{\rho-1} \geq \phi^{-1}(1-\rho) > 0$ so that the restriction $g_\phi|_{[x_0,\infty)}$ is a strictly increasing function with domain $[0, \infty)$. It is therefore a bijection from $[\phi^\vartheta, \infty)$ to $[0, \infty)$. This implies that there exists an inverse function $g_\phi^{-1} : [0, \infty) \to [\phi^\vartheta, \infty)$. Furthermore, by the Inverse Function Theorem for differentiable mappings, for all $y > 0$, $(g_\phi^{-1})'(y) = \frac{1}{g'_\phi(g_\phi^{-1}(y))} > 0$.

Therefore, for all $\varepsilon \in [0, \infty)$, there exists a unique solution $x = x(\phi, \varepsilon) \in [\phi^\vartheta, \infty)$ that solves (2.1). In addition $x(\phi, \varepsilon)$ is increasing in ε. We can explicitly calculate

$x(\phi, 0) = \phi^{\vartheta}$.

Finally, by taking derivatives of (2.1) with respect to ϕ, we find that

$$\frac{\partial x}{\partial \phi} = x^{\rho} + \varepsilon + \rho\phi x^{\rho-1}\frac{\partial x}{\partial \phi}.$$

Rearranging yields

$$\frac{\partial x}{\partial \phi} = \frac{1}{1 - \rho\phi x^{\rho-1}}(x^{\rho} + \varepsilon) \geq x^{\rho} + \varepsilon > 0,$$

where $1 - \rho\phi x^{\rho-1} \geq 1 - \rho > 0$ for $x \geq \phi^{\vartheta}$. Hence, $x(\phi, \varepsilon)$ is increasing in ϕ. □

Proposition B.2. *Let C be a $\mathbb{R}_+$-valued progressively-measurable process and let $g : [0, \infty) \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ be an aggregator random field. Let W be a càdlàg adapted processes such that $W_0 < \infty$. Then, the following are equivalent:*

1. *W is a supersolution associated to the pair (g, C).*

2. *There exists a càdlàg uniformly integrable martingale M and a càdlàg integrable decreasing predictable process A such that, for all bounded stopping times $\sigma \geq t$,*

$$W_t = W_\sigma + \int_t^\sigma g(s, \omega, C_s, W_s)\,\mathrm{d}s + M_t - M_\sigma + A_t - A_\sigma.$$

3. *There exists a càdlàg uniformly integrable martingale M such that, for all bounded stopping times $\sigma \geq t$,*

$$W_t \geq W_\sigma + \int_t^\sigma g(s, \omega, C_s, W_s)\,\mathrm{d}s + M_t - M_\sigma \tag{2.2}$$

There is a corresponding result for subsolutions $W \in \mathbb{I}(g, C)$, provided we replace A with increasing predictable process and change the direction of the inequality in (2.2).

Proof. We only prove the result for supersolutions. The result for subsolutions is very similar.

[1 $\implies$ 2] If we define Y by $Y_t = W_t + \int_0^t g(s, \omega, C_s, W_s)\,\mathrm{d}s$, then Y is a class (D) supermartingale. By the Doob-Meyer decomposition, Y can therefore be decomposed as $Y = M + A$, where M is a uniformly integrable martingale and A is an integrable decreasing process both of which are càdlàg. Therefore, using the definition of Y, we find that for $\sigma \geq t$ a stopping time,

$$\begin{aligned} W_t - W_\sigma &= Y_t - Y_\sigma + \int_t^\sigma g(s, \omega, C_s, W_s)\,\mathrm{d}s \\ &= \int_t^\sigma g(s, \omega, C_s, W_s)\,\mathrm{d}s + M_t - M_\sigma + A_t - A_\sigma, \end{aligned}$$

and the result holds.

[2 $\implies$ 3] This follows directly, since A is a decreasing process.

[3 $\implies$ 1] By taking conditional expectations at time t, we see that W is a supersolution. $\square$

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