Manuscript version: Published Version
The version presented in WRAP is the published version (Version of Record).

Persistent WRAP URL:
http://wrap.warwick.ac.uk/166168

How to cite:
The repository item page linked to above, will contain details on accessing citation guidance from the publisher.

Copyright and reuse:
The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions.

Copyright © and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable the material made available in WRAP has been checked for eligibility before being made available.

Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

Publisher’s statement:
Please refer to the repository item page, publisher’s statement section, for further information.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk
QUANTUM MEAN-FIELD GAMES

BY VASSILI N. KOLOKOLTSOVA

Department of Statistics, University of Warwick and Higher School of Economics RF, v.kolokoltsov@warwick.ac.uk

In this paper we are merging the two new branches of game theory: quantum games and mean-field games (MFG). Building a quantum analog of MFGs requires the full reconstruction of its foundations and methodology, because in $N$-particle quantum evolution particles are not separated in individual dynamics and the key concept of the classical MFG theory, the empirical measure defined as the sum of Dirac masses of the positions of the players, is not applicable in a quantum setting.

As a preliminary result we derive the new nonlinear stochastic Schrödinger equation, as the limit of the quantum filtering equation describing continuously observed and controlled system of a large number of interacting particles, the result that may have an independent value. We then show that to a control quantum system of interacting particles there corresponds a special system of classical interacting particles with the identical limiting MFG system, defined on an appropriate Riemannian manifold. Solutions of this system are shown to specify approximate Nash equilibria for $N$-agent quantum games.

1. Introduction. Quantum games represent the really 21st century branch of game theory, tightly linked to the modern development of quantum computing and quantum technologies. Initiated by Meyer [53], Eisert, Wilkens and Lewenstein [24], and Marinatto and Weber [50], the theory now boasts of many beautiful results obtained by various authors in numerous publications, see, for example, surveys [30, 34], and a mathematically oriented survey in textbook [43]. However, the main accent in these developments was made on stationary or repeated games. In [39] the author developed the truly dynamic quantum game theory with strategies chosen by players in real time. Since direct continuous observations are known to destroy quantum evolutions (so-called quantum Zeno paradox) the necessary new ingredient for quantum dynamic games represented the theory of nondirect observations and the corresponding quantum filtering. This theory was essentially developed by Belavkin in the 1980s, in [7, 9, 10]; see [16] for a readable modern account. There is an important work under way on the technical side of organising feedback quantum control in real time, see, for example, [1, 19] and [56].

Another recently emerged branch of game theory represent the so-called mean-field games (MFG). They were initially introduced by M. Huang, R. Malhamé, P. Caines in [32] and by J.-M. Lasry, P.-L. Lions in [47] and have an impressive and ever growing development, see, for example, recent monographs [13, 20, 28, 44] and references therein.

In this paper we are merging these two exciting new branches of game theory. Building a quantum analog of MFGs requires the full reconstruction of its foundations and methodology, because in $N$-particle quantum evolution particles are not separated in individual dynamics and the key concept of the classical MFG theory, the empirical measure defined as the sum of Dirac masses of the positions of the players, is not applicable in a quantum setting.
As a preliminary result we derive (first heuristically and then rigorously) the new nonlinear stochastic Schrödinger equation, as the limit of continuously observed and controlled system of a large number of interacting quantum particles, the result that may have an independent value. There is a huge literature on the derivation of deterministic nonlinear equations like Hartree, Gross–Pitaevskii equations, see some review in [27]. There is a parallel development on the mathematical properties of various nonlinear Schrödinger equations including stochastic and controlled ones, see, for example, [2, 3, 17, 29] and references therein. Our equation is different from the equations considered in these papers, as the linearity depends on the expectations of the correlations calculated with respect to the solution. It arises naturally from continuously observed systems and is reminiscent to the McKean–Vlasov nonlinear diffusion. It can be in fact considered as some quantum analog of the latter.

Motivated by this result, we build a correspondence between quantum \( N \)-agent dynamic games and classical \( N \)-player dynamic games on appropriate Riemannian manifolds in such a way that the corresponding games have the same limiting MFG system that describes these games in the limit of infinite number of agents. Of course the precise link between the limiting game and the pre-limit \( N \)-agent games is quite different for quantum and classical games. Our main result shows that, similar to the classical setting, solutions to the limiting MFG system specify approximate \( \epsilon \)-Nash equilibria for \( N \)-agent quantum games, though with \( \epsilon \) of order \( N^{-1/4} \), which is quite different from classically available convergence rates of type \( \epsilon \sim N^{-1/2} \) (see [32]) or \( \epsilon \sim N^{-1} \) (see [45]).

The content of the paper is as follows. In the next section we recall the basic theory of quantum continuous measurement and filtering. In Section 3 our new nonlinear equations are introduced in a heuristic manner and the rigorous convergence results are formulated. The result concerning controlled dynamics is seemingly new even in the deterministic case.

In Section 4 the limiting forward-backward MFG system is introduced and the main result of the paper is formulated stating that the solutions to the limiting forward-backward systems (if they exist) determine the \( \epsilon \)-Nash equilibria for the corresponding \( N \)-agent quantum game. Sections 5, 6 and 7 are devoted to the proof of some of the main results.

Section 8 can be looked at as an introduction to classical MFGs on compact Riemannian manifolds on the example of complex projective spaces, which play the role of the state spaces for finite-dimensional quantum mechanics. Based on the discovery from [39], that allows one to organise special homodyne detection schemes of continuous observation on finite-dimensional quantum systems in such a way that the resulting diffusion operator turns to the standard Laplace–Beltrami operator on a complex projective space, we prove under these arrangements the global existence and local well-posedness of the limiting forward-backward MFG system on complex projective spaces thus supplying the missing existence result from our main theorem at least for finite-dimensional quantum games.

Finally, Section 9 presents some questions arising from our analysis. The Appendix contains the proofs of two auxiliary results, which represent extensions of the deterministic results from [35] to the present stochastic case.

2. Prerequisites: Nondemolition observation and quantum filtering. The general theory of quantum nondemolition observation, filtering and resulting feedback control was built essentially in papers [7, 9, 10]. A very good readable introduction is given in [16]. We shall describe briefly the main result of this theory.

The nondemolition measurement of quantum systems can be organised in two versions: photon counting and homodyne detection. One of the first mathematical results on the control with photon counting measurement was given in [36], and the game-theoretic framework was developed in [41]. Here we fully concentrate on the homodyne (mathematically speaking, diffusive type) detection. Under this type of measurement the output process \( Y_t \) is a
usual Brownian motion (under appropriate probability distribution). There are several (by now standard) ways of writing down the quantum filtering equation for states resulting from the outcome of such process. The one which is the most convenient to our purposes is the following linear Belavkin filtering equation (which is a particular version of the stochastic Schrödinger equation) describing the a posteriori (pure but not normalized) state

\[ d\chi = -\left[iH\chi + \frac{1}{2}L^*L\chi\right]dt + L\chi dY_t, \]

where \( i^2 = -1 \), the unknown vector \( \chi \) is from the Hilbert space of the observed quantum system, which we shall sometimes referred to as the atom, the self-adjoint operator \( H \) is the Hamiltonian of the corresponding initial (nonobserved) quantum evolution and the operator \( L \) is the coupling operator of the atom to the optical measurement device specifying the chosen version of the homodyne detection. The Hilbert space we will deal with in this paper will be the space \( L^2(X) \) of square integrable functions defined on some metric space \( X \) with a fixed Borel measure that we denote by \( dx \). The particular case with the Hilbert space \( C^2 \) (corresponding to a two-point set \( X \)) is referred to as a qubit, and the case with the Hilbert space \( C^3 \) (respectively the general finite-dimensional case \( C^{d+1} \)) is often referred to as a qutrit (respectively, qudit).

REMARK 1. For the case of bounded operators \( H, L \), which we mostly adhere to in this paper for simplicity, the well-posedness of the linear equation (1) is a standard fact of the theory of Hilbert-space valued SDEs. For the unbounded case the story is more complicated, see, for example, [37] and [26] for reviews and some results.

The initial derivation of the quantum filtering equation was carried out via the method of quantum stochastic calculus. Later on more elementary derivations appeared. It can be obtained from an appropriate limit of sequential discrete observation scheme, see, for example, [11] or [54]. A derivation from the theory of instruments was given in [5] and [31].

An important role in the theory is played by the so-called innovation process

\[ dB_t = dY_t - \langle L + L^*\rangle\chi dt, \]

where for an operator \( A \) and a vector \( v \) in the phase space we use the (more or less standard) notation for the average value of \( A \) in the state \( v \):

\[ \langle A \rangle_v = (v, Av)/(v, v). \]

It is shown in the theory of quantum filtering and quantum probability (see, e.g., [16] and [31]) that the innovation process \( B_t \) is the standard Brownian motion (or the Wiener process) with respect to the fixed (initial vacuum) state of the homodyne detector. Let us recall here that a family of commuting self-adjoint operators on a Hilbert space and a pure state (a unit vector in this space) define a family of classical random variables via the spectral decomposition (this fact is one of the starting points of quantum probability). On the other hand, the output process \( Y_t \) is a Brownian motion with respect to the states transformed by the (quite complicated) interaction of the quantum system and optical device, which can also be obtained by the Girsanov transformation from the innovation process \( B_t \).

Due to (2), \( dY_t \) satisfies the usual Itô rule: \( dY_t dY_t = dt \), which is the basic tool in all calculations.

A very particular case represent the equations with anti-Hermitian operators \( L: L^* = -L \). As seen from (2), in this case the innovation process coincides with the output process, which thus becomes the standard Brownian motion on its own. This means that the noise does not
properly interact with the atom, and therefore this case is the less interesting one for continuous measurement, see the discussion in [6]. Nevertheless the filtering theory still applies to this case and control can be analysed via the averaging with respect to the noise. This case is referred to in the theory as conservative, because in this case (as seen from direct application of Itô’s formula) solutions to (1) preserve the norm almost surely, that is, the resolving operators for the Cauchy problem of this equation are unitary almost surely. In the present paper the rigorous analysis will be carried out only for the conservative case. Extension to arbitrary coupling operators is carried out in [42].

The theory extends naturally to the case of several, say \( N \), coupling operators \( \{L_j\} \), where the quantum filtering is described by the following direct extension of equation (1):

\[
d\chi = -\left[ iH\chi + \frac{1}{2} \sum_j L_j^* L_j \chi \right] dt + \sum_j L_j \chi dY^j_t,
\]

with the \( N \)-dimensional output process \( Y_t = \{Y^j_t\} \), different \( Y_j \) and the corresponding \( L_j \) standing for various “channels” of observation. The corresponding innovation process is the standard \( N \)-dimensional Brownian motion with the coordinate differentials

\[
dB^j_t = dY^j_t - \langle L_j + L_j^* \rangle \chi dt.
\]

Recall that the density matrix or density operator \( \gamma \) (physicists often use the term “matrix” even in infinite-dimensional case) corresponding to a unit vector \( \chi \in L^2(X) \) is defined as the orthogonal projection operator on \( C\chi \). This operator is usually expressed either as the tensor product \( \gamma = \chi \otimes \bar{\chi} \) or in the most common for physics bra-ket Dirac’s notation as \( \gamma = |\chi\rangle \langle \chi| \). Of course in the tensor notation \( \gamma \) is formally an element of the tensor product \( L^2(X^2) \). However, considered as an integral kernel, it is identified with the corresponding integral operator.

As one checks by direct application of Itô’s formula, in terms of the density matrix \( \gamma \), equation (1) rewrites as

\[
d\gamma = -i[H, \gamma] dt + \left( L\gamma L^* - \frac{1}{2} L^* L \gamma - \frac{1}{2} \gamma L^* L \right) dt + (\gamma L^* + L\gamma) dY,
\]

where \([\cdot, \cdot]\) denotes the commutator of the operators. In particular, the expectation \( E\gamma \) satisfies the following master equation (sometimes referred to as the Lindblad equation):

\[
dE\gamma = -i[H, E\gamma] dt + \left( LE\gamma L^* - \frac{1}{2} L^* LE\gamma - \frac{1}{2} E\gamma L^* L \right) dt.
\]

The theory of quantum filtering reduces the analysis of quantum dynamic control and games to the controlled version of evolutions (4), where operators \( H \) and \( L_j \) may become dependent on some control parameters of one or several players. The simplest situation concerns the case when the homodyne devices are fixed, that is, the operators \( L_j \) are fixed, and the players can control the Hamiltonian \( H \), say, by applying appropriate electric or magnetic fields to the atom. Thus equation (4) is modified by allowing \( H \) to depend on one or several control parameters. One can prove a rigorous mathematical result, the so-called separation principle (see [15]), that shows that the effective control of an observed quantum system (that can be based in principle on the whole history of the interaction of the atom and optical devices) can be reduced to the Markovian feedback control of the quantum filtering equation, with the feedback at each moment depending only on the current (filtered) state of the atom.
3. A new nonlinear stochastic Schrödinger equation. The well developed theory of the so-called nonlinear stochastic Schrödinger equations deals with the stochastic equations of the type

$$d\psi(t) = -iH\psi dt - if(t, \psi(t)) dt - g(t, \psi(t)) dW(t),$$

with a Hamiltonian operator $H$, some nonlinear function $f$, $g$ and various noises $dW$ (including infinite-dimensional), see, for example, [2, 3, 17, 29] and references therein.

For our theory a different type of nonlinear equation is needed, with nonlinearity depending additionally on the distribution specified by the wave function solving the equation. It bears some analogy to the classical McKean–Vlasov diffusions, though the role of the law of the diffusion is now played by its quantum analog.

To write it properly, it is handy to introduce the following standard general notation for the lifting of one-particle operators to multiparticle states. Namely, let $X$ be a Borel space with a fixed Borel measure that we denote by $dx$. For any linear operator $B$ acting on the space of square integrable functions $L^2(X)$, let us denote by $B_j$ the linear operator on the space $L^2(X^N)$ that acts as $B$ on functions $f(x_1, \ldots, x_N)$ considered as the functions of the variable $x_j$.

Let $H$ be a self-adjoint operator on $L^2(X)$ (a single particle Hamiltonian) and $L$ an operator on $L^2(X)$ (serving as a coupling with a homodyne measurement device) and $V(x, y)$ a symmetric function of two variables $x, y \in X$ (interaction potential). Let us now consider the quantum evolution of $N$ particles driven by the standard interaction Hamiltonian $H(N)$ on $L^2(X^N)$ acting according to the formula

$$H(N)f(x_1, \ldots, x_N) = \sum_{j=1}^{N} H_j f(x_1, \ldots, x_N) + \frac{1}{N} \sum_{i<j \leq N} V(x_i, x_j) f(x_1, \ldots, x_N),$$

and observed via symmetric coupling with one-particle operators $L$. That is, we consider the filtering equation (4) of the type

$$d\Psi_{1N,t} = -\frac{i}{2} \sum_{j=1}^{N} L_j^* L_j \Psi_{1N,t} dt + \sum_{j=1}^{N} L_j \Psi_{1N,t} dY_j.$$
be written approximately as the product of individual functions $\psi(x_j)$ with the same $\psi$ for all $j$. Each $j$th particle is influenced by the average interaction potential $V(x_j, x_m)$ over the position of all particles $m \neq j$. But the distribution of the position of the $m$th particle is given by the density $|\psi(x_m)|^2$. Thus the total potential acting on the $j$th particle is

$$\frac{1}{N} \sum_{m \neq j} \int V(x_j, x_m)|\psi(x_m)|^2 dx_m = \frac{N-1}{N} \int V(x_j, y)|\psi(y)|^2 dy.$$  

For large $N$ this equals $\int V(x_j, y)|\psi(y)|^2 dy$, approximately. With such interaction potential the Schrödinger equation for each particle gets the form

$$\dot{\psi}_t(x) = -i H \psi_t(x) - i (V|\psi_t|^2)(x) \psi_t(x),$$

where $V|\psi|^2$ denotes the function $\int V(x, y)|\psi(y)|^2 dy$, which is the standard nonlinear Schrödinger or the Hartree equation.

**Remark 2.** The idea of the limiting decoupling of interacting particles in the limit of a large number of these particles and the related notion of the propagation of chaos go back to [49] and [33], see also [8, 12]. Mathematically the propagation of chaos and related mean-field limit for classical particles (leading probabilistically to the so-called nonlinear Markov process) was analysed in numerous papers, see, for example, monograph [40] and references therein. The property of the propagation of chaos is related to exchangeable probability distributions, see, for example, [21]. The quantum analog of classical exchangeable distributions represent the bosons, identical particles with symmetric wave functions, which we are dealing with here. For fermions the story is a bit different, see, for example, [25].

Very often $V$ is assumed to depend on the difference of the arguments, that is, to be of the form $V(x - y)$ with an even function $V$. In this case the Hartree equation takes its most familiar form

$$\dot{\psi}_t(x) = -i H \psi_t(x) - i (V*|\psi_t|^2)(x) \psi_t(x),$$

where

$$(V*|\psi|^2)(x) = \int V(x - y)|\psi(y)|^2 dy$$

denotes the convolution.

In the case of evolution (4) there can be no question of having identical wave functions in the product, because they are controlled by different noises. However, assuming the initial condition is the product of identical functions, and that noises $Y_j$ are independent, we may assume that the terms in the product $\psi_j(x_j)$ are independent and identically distributed and hence, by the strong law of large numbers, there may exist an almost sure deterministic limit

$$\xi_t(x) = \lim \frac{1}{N} \sum_{j=1}^{N} |\psi_{j,t}(x)|^2 = E|\psi_{m,t}(x)|^2.$$  

Relating the observations given by $L_j$ to the evolution of the $j$th particle, we may suggest the following limiting equations for the individual particles:

$$d\psi_{j,t}(x) = -\left[i H \psi_{j,t}(x) + i (V^{\xi_t})(x) \psi_{j,t}(x) + \frac{1}{2} L^* L \psi_{j,t}(x) + L \psi_{j,t} dY_{i}^j \right] dt + L \psi_{j,t} dY_{i}^j,$$

$$\xi_t(x) = E|\psi_{j,t}(x)|^2.$$  


Here, as in (10), \( V^\xi_t \) denotes the function \( \int V(x,y)\xi_t(y)\,dy \).

The corresponding equation for the density operator (5) has the form

\[
\begin{align*}
d\gamma_{j,t} &= -i[H,\gamma_{j,t}]\,dt - i[V^\xi_t,\gamma_{j,t}]\,dt \\
&+ \left(L\gamma_{j,t}L^* - \frac{1}{2}L^*L\gamma_{j,t} - \frac{1}{2}\gamma_{j,t}L^*L\right)\,dt \\
&+ (\gamma_{j,t}L^* + L\gamma_{j,t})\,dY^j_t, \\
\xi_t(x) &= E_{\gamma_{j,t}}(x,x),
\end{align*}
\]

(12)

where, with usual abuse of notation, \( V^\xi_t \) is considered as the operator of multiplication by the function \( V^\xi_t \).

Below we shall give a rigorous derivation of (11) from (8) under the strong simplifying assumption of conservativity, namely under the condition \( L^* = -L \), in which case the output processes \( Y_j \) are standard independent Brownian motions (see Section 2). For general \( L \) the derivation is performed in [42].

Apart from the pairwise interaction expressed by a multiplication operator, another standard class of binary interactions (specifically often used in finite-dimensional quantum mechanics) is expressed by integral operators \( A \) with kernels \( A(x,y;x',y') \) that act on the functions of two variables as

\[
A\psi(x,y) = \int_{X^2} A(x,y;x',y')\psi(x',y')\,dx'dy'.
\]

It is usually assumed (and we shall do it) that \( A \) are symmetric in the sense that they take symmetric functions \( \psi(x,y) \) (symmetric with respect to permutation of \( x \) and \( y \)) to symmetric functions. For the kernels this means that they are symmetric with respect to the simultaneous exchange of the first pair of variables and the second one:

\[
A(x,y;x',y') = A(y,x;y',x').
\]

**Remark 3.** For example, the standard physics choice of the interaction of two qubits is the operator arising from possible exchange of photons, \( A = a_1^*a_2 + a_2^*a_1 \), with the annihilation operators \( a_1 \) and \( a_2 \) of the two atoms, see, for example, Chapter 7 of [48] or [23]. This interaction is given by the kernel \( A(j,k;m,n) \) such that \( A(1,0;0,1) = A(0,1;1,0) = 1 \) with other elements vanishing. Also in quantum calculations the main role is played by the 2-gate operators, which are of this type, see, for example, [52].

With such interaction the \( N \) particle Hamiltonian becomes

\[
\begin{align*}
H(N) f(x_1, \ldots, x_N) &= \sum_{j=1}^N H_j f(x_1, \ldots, x_N) + \frac{1}{N} \sum_{i<j} A_{ij} f(x_1, \ldots, x_N),
\end{align*}
\]

(13)

with \( A_{ij} \) denoting the operator \( A \) acting on the variables \( x_i, x_j \) of \( f \), and the equation for the density matrix \( \Gamma_{N,t} = \Psi_{N,t} \otimes \overline{\Psi}_{N,t} \) is given by (9) with \( A_{ij} \) instead \( V_{ij} \).

By symmetry, one can rewrite (13) equivalently as

\[
\begin{align*}
H(N) f(x_1, \ldots, x_N) &= \sum_{j=1}^N H_j f(x_1, \ldots, x_N) + \frac{1}{2N} \sum_{i \neq j} A_{ij} f(x_1, \ldots, x_N).
\end{align*}
\]

(14)

The corresponding analog of nonlinear equation (11) can be written in the form

\[
\begin{align*}
d\psi_{j,t}(x) &= -\left[iH\psi_{j,t}(x) + i\overline{\eta}_t \psi_{j,t}(x) + \frac{1}{2}L^*L\psi_{j,t}(x)\right]dt + L\psi_{j,t}(x)\,dY^j_t,
\end{align*}
\]

(15)
where $A\bar{\eta}_t$ is the integral operator in $L^2(X)$ with the integral kernel

$$A\bar{\eta}_t(x; x') = \int_{X^2} A(x, y; x', y') \eta_t(y, y') dy dy'$$

and

$$\eta_t(y, z) = E(\psi_{j,t}(y)\bar{\psi}_{j,t}(y'))$$

(16)For the expectation $\eta_t(y, z) = E\gamma_{j,t}(y, z)$ we get the following nonlinear version of the Lindblad equations:

$$d\gamma_{j,t} = -i[H, \gamma_{j,t}] dt - i[A\bar{\eta}_t, \gamma_{j,t}] dt + \left( L\gamma_{j,t}L^* - \frac{1}{2}L^*L\gamma_{j,t} - \frac{1}{2}\gamma_{j,t}L^*L \right) dt$$

The density matrix $\eta_t$ can be considered as the quantum analog of the empirical measure of classical particles, and equation (17) as the quantum analog of the Kolmogorov–Fokker–Planck equation of nonlinear diffusion.

Formally the case of the multiplication operators by $V(x - y)$ can be considered as a particular case of the integral operator with the singular kernel $V(x - y)\delta(x - x')\delta(y - y')$.

In a finite-dimensional setting this just means that the matrix of the corresponding integral operator is diagonal.

The story extends naturally to the case when the measurement related to each particle is multidimensional, that is, the operator $L$ is vector-valued, $L = (L^1, \ldots, L^k)$, in which case each noise $dY^j_t$ is also $k$-dimensional, so that the term $L^j_\psi_{j,t} dY^j_t$ should be understood as the inner product,

$$L^j_\psi_{j,t} dY^j_t = \sum_{l=1}^k L^j_{1_l} \psi_{j,t} dY^j_{1,l},$$

with all other terms containing $L$ understood in the same way.

For the rigorous derivation of the Hartree equation (10) from the corresponding $N$-particle evolution several ingenues methods were developed recently, see a review in [27]. Our analysis of the stochastic situation will be carried out via the method suggested by Pickl, see [55] and [35], appropriately adapted and modified to address the stochastic setting. In Pickl’s approach the main measures of the deviation of the solutions $\Psi_{N,t}$ to $N$-particle systems from the product of the solutions to the Hartree equations are the following positive numbers from the interval $[0, 1]$:

$$E^{(k)}_{N,t} = 1 - (\psi_t^{\otimes k}, \Gamma^{1_1, \ldots, k}_{N,t} \psi_t^{\otimes k})$$

and in particular,

$$\alpha_N(t) = E^{(1)}_{N,t},$$

where, for a subset $I$ of $\{1, \ldots, N\}$, $\Gamma^{I}_{N,t}$ denotes the partial trace of $\Gamma_{N,t}$ with respect to all variables outside the set $I$. 
Remark 4. For convenience of the readers let us recall the key notion of the partial trace. For an integral operator $\gamma_N$ with the integral kernel $\gamma_N(x,x')$, $x = (x_1, \ldots, x_N)$, $x' = (x'_1, \ldots, x'_N)$, in $L^2(X^N)$, the partial trace over the variables $x_{N-k} = (x_{k+1} \cdots, x_N)$ is defined as the operator $\gamma^{1,\ldots,k}_N = \text{tr}_{k+1,\ldots,N} \gamma_N$ in $L^2(X^k)$ with the kernel

$$
\gamma^{1,\ldots,k}_N(x_k; x'_k) = \int_{X^{N-k}} \gamma_N(x_k, x_{N-k}; x'_k, x_{N-k}) \, dx_{N-k}
$$

The operators $\gamma^{1,\ldots,k}_N$ are also referred to as the reduced states (on the subsystems of $k$ particles). The key property of the reduced state is the following, more or less obvious, identity:

$$
\text{tr}\left[ (A^{1,\ldots,k} \otimes \mathbf{1}^{(N-k)}) \gamma_N \right] = \text{tr}[A^{1,\ldots,k} \gamma^{(k)}_N]
$$

for any operator $A^{1,\ldots,k}$ that acts only on the space $L^2(X^k)$ generated by the first $k$ variables.

Similarly one defines the partial trace $\gamma^I_N = \text{tr}_{[1,\ldots,N]\setminus I} \gamma_N$ with respect to all variables outside the set $I \subset \{1, \ldots, N\}$.

Clearly, if $\Gamma_{N,t}$ were the tensor product of $\psi_t$, then one would have $E^{(k)}_{N,t} = 0$. Hence the convergence $E^{(k)}_{N,t} \to 0$, as $N \to \infty$, expresses some kind of convergence of $\Gamma_{N,t}$ to the product state. As was shown in [35],

$$
E^{(k)}_{N,t} \leq kE^{(1)}_{N,t},
$$

so that for the convergence of all $E^{(k)}_{N,t}$ to zero it is sufficient to show the convergence $\alpha_N(t) \to 0$.

In the present stochastic case, the quantities $E^{(k)}_{N,t}$ depend not just on the number $k$ of particles in the product, but on the concrete choice of these particles. For instance, the proper stochastic analog of the quantity $\alpha_N(t)$ is the collection of random variables

$$
\alpha_{N,j}(t) = 1 - (\psi_{j,t}, \Gamma^{(j)}_{N,t} \psi_{j,t}) = 1 - \text{tr}(\gamma_{j,t} \Gamma^{(j)}_{N,t}),
$$

where $\Gamma^{(j)}_{N,t}$ is the partial trace $\Gamma^I_{N,t}$ with the set $I$ consisting of just one index $j$. By (18), identifying $\gamma_{j,t}$ with the operator in $L^2(X^N)$ acting on the $j$th variable, one can also write

$$
\alpha_{N,j}(t) = 1 - \text{tr}(\gamma_{j,t} \Gamma_{N,t}).
$$

The stochastic analogs of $E^{(k)}_{N,t}$ are the collections of random variables

$$
E^I_{N,t} = 1 - (\psi^I_{t}, \Gamma^I_{N,t} \psi^I_t),
$$

where $\psi^I_t = \otimes_{j \in I} \psi_{j,t}$, for all subsets $I \subset \{1, \ldots, N\}$.

Due to the i.i.d. property of the solutions to (11) or (15), the expectations $\mathbb{E}_t E^I_{N,t}$ depend only on the size $|I|$ of $I$ and do not depend on a particular choice of $|I|$ particles. In particular $\mathbb{E}_t \alpha_N(t) = \mathbb{E}_N \alpha_{N,j}(t)$ are well defined (they do not depend on $j$).

In the Appendix we prove Proposition A.1 generalising Lemma 2.1 from [35], which implies that, in place of inequality (19), we have in our stochastic setting the inequality

$$
\mathbb{E}_t E^I_{N,t} \leq |I| \mathbb{E}_N \alpha_{N,t},
$$

so that, analogously to the deterministic case, for the convergence of all $\mathbb{E}_t E^I_{N,t}$ to zero it is sufficient to show the convergence $\mathbb{E}_N \alpha_{N}(t) \to 0$. 
Expressions $\alpha_{N,j}$ can be linked with the traces by the following inequalities, due to Knowles and Pickl:

$$\alpha_{N,j}(t) \leq \text{tr}|\Gamma_{N,t}^{(j)} - \gamma_{j,t}| \leq 2\sqrt{2\alpha_{N,j}(t)},$$

see Lemma 2.3 from [35]. Here, for a bounded operator $A$, we use the standard notation $|A| = \sqrt{A^*A}$, so that $\text{tr}|B|$ is the standard trace norm of an operator $B$.

The following result shows that heuristical arguments given above can be corroborated by the rigorous analysis.

**Theorem 3.1.** Let the operators $H, L$ be bounded and the interaction be given either (i) by the multiplication by a bounded symmetric function $V(x, y) \in L^\infty(X^2)$ or (ii) by a symmetric self-adjoint integral operator $A$ with a Hilbert–Schmidt kernel, that is, a kernel $A(x, y; x', y')$ such that

$$\|A\|_{HS}^2 = \int_{X^4} |A(x, y; x', y')|^2 dx dy dx' dy' < \infty,$$

(23)

$$A(x, y; x', y') = A(y, x; y', x'), \quad A(x, y; x', y') = \overline{A(x', y'; x, y)}.$$

(24)

Let $L$ be anti-Hermitian, $L^* = -L$, and the noises $Y_j$ be independent standard Brownian motions.

Let $\Psi_{N,t}$ be a solution to the $N$-particle equation (8) with $H(N)$ of type either (7) or (13), with some initial condition $\Psi_{N,0}, \|\Psi_{N,0}\|_2 = 1$. Let $\psi_{j,t}$ be solutions to equations (11) with the identical initial conditions $\psi_{j,0} = \psi \in L^2(X), \|\psi_{j,0}\| = 1$.

Then, in case (i)

$$E\alpha_{N}(t) \leq e^{12t\|V\|_\infty} \left(\alpha_{N}(0) + \frac{1}{N}\right),$$

(25)

$$+ \frac{3\|V\|_\infty}{\sqrt{N}} \int_0^t e^{12(t-s)}\|V\|_\infty \int_X \sqrt{E(|\psi_{j,s}(z)|^4)} dz ds,$$

and in case (ii)

$$E\alpha_{N}(t) \leq e^{12t\|A\|_{HS}} \alpha_{N}(0) + (e^{12t\|A\|_{HS}} - 1) \frac{1}{\sqrt{N}}.$$

(26)

The proof will be given in Section 5. Let us make some comments. The assumption that $H$ and $L$ are bounded is not essential, and was made only to simplify the presentation. In fact, as seen from the proof, neither $H$ nor $L$ enter any essential calculations or bounds, so that to include unbounded $H$ and $L$ one just has to carefully describe all domains.

The assumption of boundedness of $V$ can be possibly relaxed, because in the deterministic case developed in [35] the assumption $V \in L^2(R^d) + L^\infty(R^d)$ is sufficient.

Unlike the simple estimate (26), application of (25) requires some additional estimates of the r.h.s., which we are not dealing with here, paying the main attention to the integral type interactions.

Of course, everything remains unchanged for a vector-valued $L$.

**Remark 5.** Our key restrictive assumption is the conservativity $L^* = -L$. As was mentioned above, extension to arbitrary $L$ is carried out in [42].

For the application to the control theory some further extension is needed. Let us formulate this result (proof will be given in Section 6) for the case of the integral interaction only. Let us assume that the individual Hamiltonian $H$ has a control component, that is, it can be
written as $H + u \hat{H}$ with two self-adjoint operators $H$ and $\hat{H}$ and $u$ a real control parameter. Suppose that, for the limiting evolution, $u$ is chosen as a certain function of an observed density matrix $\gamma_{j,t}$, that is, $u = u(t, \gamma_{j,t})$, while in the approximating $N$ particle evolution one chooses $u$ based on the approximation $\Gamma_{N,t}^{(j)}$ of $\gamma_{j,t}$ (recall (22)), that is, as $u = u(t, \Gamma_{N,t}^{(j)})$, where $\Gamma_{N,t}^{(j)}$ denotes the partial trace of $\Gamma_{N,t}$ over all variables except for the $j$th. Thus the $N$ particle evolution (9) generalizes to the following nonlinear evolution (where we again omit the index $t$):

\[
\frac{d \Gamma_N}{dt} = -i \sum_j \left[ H_j + u(t, \Gamma_N^{(j)}) \hat{H}_j, \Gamma_N \right] dt - \frac{i}{N} \sum_{l < j \leq N} [A_{lj}, \Gamma_N] dt \\
+ \sum_j \left( L_j \Gamma_N L_j^* - \frac{1}{2} L_j^* L_j \Gamma_N - \frac{1}{2} \Gamma_N L_j^* L_j \right) dt \\
+ \sum_j (\Gamma_N L_j^* + L_j \Gamma_N) dY^j,
\]

and the equation (15) generalizes to the following equation:

\[
\frac{d \psi_{j,t}(x)}{dt} = -\left[ i(H + u(t, \gamma_{j,t}) \hat{H}) \psi_{j,t}(x) + \frac{1}{2} L_j^* L \psi_{j,t}(x) \right] dt + L \psi_{j,t}(x) dY^j.
\]

**Remark 6.** (i) The choice of linear control $u \hat{H}$ is of course the simplest possible. It also makes sense physically (see, e.g., [15] and [14]). In fact, operators that contributed to the Hamiltonian stand for various magnetic or electric fields that can be applied, and choosing the linear dependence on the control parameters means the ability of the controlling agents to choose the strength of the fields inside given bounds. (ii) The feedback to the control based on partial traces is natural, because partial traces in complex quantum systems are the analogs of the individual states for classical systems of interacting particles. They are widely used to characterise properties of individual particles inside a complex system, see, for example, [4] or [23]. (iii) Equation (28) is the main object we need for the study of mean field games below.

**Theorem 3.2.** Under all the assumptions of Theorem 3.1, but assuming evolutions (27) and (28) instead of (9) and (15) respectively, and assuming that the function $u(\gamma)$ is Lipschitz in the sense that

\[
|u(t, \gamma) - u(t, \tilde{\gamma})| \leq \varkappa \text{tr} |\gamma - \tilde{\gamma}|,
\]

with a positive constant $\varkappa$, it follows that the estimate (26) generalizes to the estimate

\[
\mathbb{E} \alpha_N(t) \leq \exp \{12(\|A\|_{HS} + \varkappa \|\hat{H}\|)t\} \alpha_N(0) + (\exp \{12(\|A\|_{HS} + \varkappa \|\hat{H}\|)t\} - 1) \frac{1}{\sqrt{N}}.
\]

The comments related to the previous theorem remain valid. However the boundedness of the control part $\hat{H}$ of the control Hamiltonian is essential (unlike the boundedness of $H$ and $L$): its norm $\|\hat{H}\|$ explicitly enters the final estimate.

**4. Quantum MFG: Main result.** Let us consider the quantum dynamic game of $N$ players, where the dynamics of the density matrix $\Gamma_{N,t}$ is given by the controlled dynamics of type (27), though the control of each player can be chosen independently (we write here
$u_j(t, \Gamma_{N,t}^{(j)})$ rather than $u(t, \Gamma_{N,t}^{(j)})$ in (27):
\[
d\Gamma_{N,t} = -i \sum_j [H_j + u_j(t, \Gamma_{N,t}^{(j)})\hat{H}_j, \Gamma_{N,t}] dt - \frac{i}{N} \sum_{l<j\leq N} [A_{lj}, \Gamma_{N,t}] dt \\
+ \sum_j \left( L_j\Gamma_{N,t}L_j^* - \frac{1}{2}L_j^*L_j\Gamma_{N,t} - \frac{1}{2}\Gamma_{N,t}L_j^*L_j \right) dt \\
+ \sum_j (\Gamma_{N,t}L_j^* + L_j\Gamma_{N,t}) dY_j^t.
\]

**Remark 7.** Here we are working with density operators only. In fact, all proofs of convergence are based on the density operators representation. The wave functions are playing only an auxiliary role. However, the proofs of well posedness (which, for the infinite-dimensional case, is given in [42]) are easier to perform for the wave functions, because wave functions live in a Hilbert space and density operators in a Banach space (of trace class operators). For the latter the theory of SDEs is much more involved, see [18] and references therein.

Assume further that control $u$ can be chosen from some bounded closed interval $U$ of the real line, that the initial matrix is the product of identical states
\[
\Gamma_{N,0}(x_1, \ldots, x_n; y_1, \ldots, y_N) = \prod_{j=1}^N \psi(x_j)\bar{\psi}(y_j)
\]
and that the payoff of each player on the interval $[t, T]$ is given by the expression
\[
P_j(t, Y; u(\cdot)) = \int_t^T \left( \text{tr}(J_j\Gamma_{N,s}) - \frac{c}{2}u_j^2(s) \right) ds + \text{tr}(F_j\Gamma_{N,T}),
\]
where $J$ and $F$ are some operators in $L^2(X)$ expressing the current and the terminal costs of the agent, $J_j$ and $F_j$ denote their actions on the $j$th variable, $c$ measures the cost of applying the control.

**Remark 8.** The choice of payoff (32) bears analogy with the linear-quadratic control of classical optimal control theory. Physically the quadratic term in $u_j$ expresses the cost of applying the field corresponding to the operator $\hat{H}$, which is quite common in physics, see, for example, [57] and [14]. The structure of current and terminal payoffs in (32) is natural, because all observables in quantum physics are represented by some linear operators $B$ and their values in states $\gamma$ are given by the traces $\text{tr}(\gamma B)$.

Notice for clarity that by the property of the partial trace, the payoff (32) rewrites as
\[
P_j(t, Y; u(\cdot)) = \int_t^T \left( \text{tr}(J_j\Gamma_{N,s}^{(j)}) - \frac{c}{2}u_j^2(s) \right) ds + \text{tr}(F_j\Gamma_{N,T}^{(j)}),
\]
so that it really depends explicitly only on the individual partial traces $\Gamma_{N,t}^{(j)}$, which can be considered as quantum analogs of the states of classical particles.

The limiting evolution of each player can be expected to be described by the equations
\[
d\gamma_{j,t} = -i[H + u_j(t, \gamma_{j,t})\hat{H}, \gamma_{j,t}] dt - i[A_{\gamma_j,t}, \gamma_{j,t}] dt \\
+ \left( L\gamma_{j,t}L^* - \frac{1}{2}L^*L\gamma_{j,t} - \frac{1}{2}\gamma_{j,t}L^*L \right) dt
\]
Let us stress that we are not stating that evolution (34) is in fact the limiting one for the $N$-agent quantum evolution in the general case (when all players use different strategies), because we really do not need it. We need it only in the case when almost all players (actually except for one only) are playing the same strategy $u_t^{\text{com}}$, in which case $\eta_t = E\gamma_{j,t}$ for all $j$ adhering to the common strategy. In which sense this is the limiting evolution will be made explicit in the proof of our main Theorem 4.1.

Let us say that the pair of functions $u_t^{\text{MFG}}(\gamma) = u^{\text{MFG}}(t, \gamma)$ with $t \in [0, T]$ (satisfying the Lipschitz assumptions of Theorem 3.2) and $\gamma$ from the set of density operators on $L^2(X)$, $u \in U$, and $\eta_t^{\text{MFG}}(x, y)$ with $x, y \in X, t \in [0, T]$, solve the limiting MFG problem if (i) $u_t(\gamma)$ is an optimal feedback strategy for the stochastic control problem (34), (35) under the fixed function $\eta_t = \eta_t^{\text{MFG}}$ and (ii) $\eta_t^{\text{MFG}}$ arises from the solution of (34) under fixed $u_t = u_t^{\text{MFG}}$.

We can formulate it also in another equivalent way. For a function $u_t^{\text{com}}(\gamma)$ (index “com” from “common”) suppose we can solve the Cauchy problem for SDE (34) with $u_t = u_t^{\text{com}}$ defining the correlations $\eta_t(x, y) = E\gamma_{j,t}(x, y)$. Given these correlations we may be able to find an optimal feedback control for the individual control problem (34), (35) under the fixed function $\eta_t$ defining the individually optimal feedback control $u_t^{\text{ind}}(w)$ (index “ind” from “individual”) from equation (65). The main MFG consistency equation is then expressed by the equation $u_t^{\text{com}} = u_t^{\text{ind}}$. If it is fulfilled, the pair $u_t^{\text{com}}$ and $\eta_t$ solves the limiting MFG in the sense defined above.

Formulated in this way, the MFG problem is fully standard, though the state space is the sphere in the Hilbert space (or the space of density matrices). In Section 8 we shall write down explicitly the corresponding classical forward backward system for the case of finite-dimensional quantum mechanics (i.e., for finite set $X$). A bit new moment (but not very essential one) is that the initial conditions are identical to all players meaning that the initial measure in the forward equation is a Dirac atom and thus has no density, unlike what is usually assumed in MFG analysis. Therefore, the theorem on the existence of the solutions (given below in Section 8 for finite $X$) can be considered as an existence result for classical MFGs on manifolds. What makes this story truly quantum is the completely different link with the $N$-agent quantum game arising essentially from Theorem 3.2. Expressed otherwise, we established the correspondence that to each quantum $N$-agent game assigns a standard $N$-agent game on some Riemannian manifold (possibly infinite-dimensional), so that the limiting MFG forward-backward system is identical for both games.

**THEOREM 4.1.** Let the conditions on $H, L, A$ from Theorem 3.1 hold and let $\hat{H}$ be a bounded self-adjoint operator in $L^2(X)$. Assume that the pair $u_t^{\text{MFG}}(\gamma)$ and $\eta_t^{\text{MFG}}(x, y)$ solves the limiting MFG problem and moreover $u_t^{\text{MFG}}$ is Lipschitz in the sense of inequality (29). Then the strategies

$$u_j(t, \Gamma_{N,t}) = u_t^{\text{MFG}}(\Gamma^{(j)}_{N,t}),$$
where $\Gamma_{N,t}^{(j)}$ is the partial trace of $\Gamma_{N,t}$ with respect to all variables except of the $j$th, form a symmetric $\epsilon$-Nash equilibrium for the $N$-agent quantum game described by (31) and (32), with $\epsilon$ being of order $N^{-1/4}$.

**Remark 10.** We assume the readers are familiar with the main concepts of game theory like the Nash equilibrium, see any textbook on game theory, for instance [43].

A proof will be given in Section 7.

Let us briefly describe an important extension concerning the information space of the players. In the game above the players were allowed to have access to their individual partial traces $\Gamma_{N,t}^{(j)}$. In the spirit of classical MFGs one could imagine them to have access to “empirical measures”, which in our case represent the average operators

\begin{equation}
\frac{1}{N} \sum_{j=1}^{N} \Gamma_{N,t}^{(j)}
\end{equation}

considered as operators in $L^2(X)$. This is of course not exactly the same, as in the classical case, where empirical measures are averages of the Dirac $\delta$-functions placed at the positions of the particles.

**Remark 11.** Notice that this notion is very close, but different from the “empirical measures” introduced in [27], where it is defined as the average in $L^2(XN)$ of secondly quantized operators evolved according to Heisenberg equations.

Since the averages (36) approach in expectation the expected correlations $\eta_t$, allowing $u_j$ in (31) to depend on this average amounts to the dependence on $\eta_t$ in the limit, which is already taken into account in the construction of the MFG consistency problem. Consequently this additional information possibility will not change the result of Theorem 4.1.

Let us also comment that the standard $\epsilon$-Nash equilibria concerns the result of one player deviating from the common strategy. Here, one can show quite similarly that even if a finite (but bounded) number of players deviate from the common MFG strategy, they cannot improve their payoff more than by an $\epsilon$.

5. **Proof of Theorem 3.1.** We shall use the notation from Section 3 without further reminder. For definiteness, we shall give the argument for the case of the integral operator $A$ of interaction, noting occasionally some specific features of another case.

**Step 1.** In case $L^* = -L$ equation (9) for the density matrix of $N$ particle evolution takes the form

\begin{equation}
d\Gamma_{N,t} = -i \sum_j [H_j, \Gamma_{N,t}] dt - i \sum_{l<j \leq N} \frac{A_{lj}}{N} \gamma_{N,t} dt
\end{equation}

\begin{equation}
+ \sum_j \left( \frac{1}{2} L_j L_j \Gamma_{N,t} + \frac{1}{2} \Gamma_{N,t} L_j L_j - L_j \Gamma_{N,t} L_j \right) dt
\end{equation}

\begin{equation}
+ \sum_j [L_j, \Gamma_{N,t}] dY_j,
\end{equation}

or with $V_{lj}$ instead of $A_{lj}$ in case of an integral operator of interaction.

Our first objective is to calculate the differential $d\alpha_{N,j}$ for $\alpha_{N,j}$ defined by (20).

One of the key property of the conservative case is the preservation of the trace, so that $\text{tr} \Gamma_{N,t} = 1$ and $\text{tr} \gamma_{j,t} = \|\psi_{j,t}\|^2 = 1$ for all $t$ almost surely, because it was assumed that
tr$\Gamma_{N,0} = 1$ and $tr\gamma_{j,0} = \|\psi_{j,0}\|^2 = 1$. Hence the operators $q_{j,t} = I - \gamma_{j,t}$ are orthogonal projectors in $L^2(X)$, which are also identified with the orthogonal projectors in $L^2(X^N)$ by making them act on the $j$th variable.

An important role in calculations belongs to the averaging operator

$$\hat{m}(t) = \hat{m}_N(t) = \frac{1}{N} \sum_{j=1}^N q_{j,t},$$

on $L^2(X^N)$. In terms of these operators one can rewrite (20) in the following equivalent form:

$$\alpha_{N,j}(t) = 1 - tr(\gamma_{j,t} / \Gamma_{N,t}) = tr(q_{j,t} / \Gamma_{N,t}) = (\psi_{1N,t}, q_{j,t} \psi_{1N,t})$$

so that (by the i.i.d. property of $q_{j,t}$)

$$E\alpha_{N,j}(t) = Etr(q_{j,t} / \Gamma_{N,t}) = Etr(m_N(t) / \Gamma_{N,t}).$$

Using definition (20) and Itô’s product rule we derive that

$$d\alpha_{N,j}(t) = -tr(d / \Gamma_{N,t} \gamma_{j,t}) - tr(\gamma_{j,t} / \Gamma_{N,t} d) - tr(d / \Gamma_{N,t} d \gamma_{j,t}).$$

The first nice observation is that the stochastic part (the terms with differentials $dY^j_t$) vanishes in this expression. In fact, it equals

$$-tr\left(\sum_{k=1}^N [L_k, \Gamma_{N,t}] \gamma_{j,t} dY^k_t + \Gamma_{N,t} [L_j, \gamma_{j,t}] dY^j_t\right).$$

All terms with $k \neq j$ vanish, because of the commutativity of the trace and because $L_k$ and $\gamma_{j,t}$ commute (since they act on different variables). Thus this stochastic part reduces to

$$-tr(-\Gamma_{N,t} L_j \gamma_{j,t} + L_j \Gamma_{N,t} \gamma_{j,t} - \Gamma_{N,t} \gamma_{j,t} L_j + \Gamma_{N,t} L_j \gamma_{j,t}) dY^j_t = 0.$$

Therefore we can write further $\dot{\alpha}_{N,j}(t)$ instead of $d\alpha_{N,j}(t)$. The next remarkable fact is that the operators $L_j$ cancel completely from the expression for $\dot{\alpha}_{N,j}(t)$. In fact, as follows from (41) and Itô’s rule, their contribution to $\dot{\alpha}_{N,j}(t)$ equals

$$-tr\left(\sum_k \left(\frac{1}{2} L_k^2 \Gamma_{N,t} + \frac{1}{2} \Gamma_{N,t} L_k^2 - L_k \Gamma_{N,t} L_k\right) \gamma_{j,t}\right)$$

$$- tr\left(\Gamma_{N,t} \left(\frac{1}{2} L_j^2 \gamma_{j,t} + \frac{1}{2} \gamma_{j,t} L_j^2 - L_j \gamma_{j,t} L_j\right)\right) - tr([L_j, \Gamma_{N,t}] [L_j, \gamma_{j,t}]).$$

Again terms with $k \neq j$ vanish and thus this expression reduces to

$$-tr\left(\left(\frac{1}{2} L_j^2 \Gamma_{N,t} + \frac{1}{2} \Gamma_{N,t} L_j^2 - L_j \Gamma_{N,t} L_j\right) \gamma_{j,t} + \Gamma_{N,t} \left(\frac{1}{2} L_j^2 \gamma_{j,t} + \frac{1}{2} \gamma_{j,t} L_j^2 - L_j \gamma_{j,t} L_j\right) + [L_j, \Gamma_{N,t}] [L_j, \gamma_{j,t}]\right)$$

$$= -tr(\Gamma_{N,t} (L_j^2 \gamma_{j,t} + \gamma_{j,t} L_j^2 - 2 L_j \gamma_{j,t} L_j + [L_j, \gamma_{j,t}] L_j - L_j [L_j, \gamma_{j,t}])) = 0.$$
Thus, denoting $A_j^{\tilde{n}_j}$ the operator $A_j^{\tilde{n}}$ acting on the $j$th variable, we obtain

$$\dot{\alpha}_{N,j}(t) = i\text{tr}(\left[ H_j + A_j^{\tilde{n}_j}, \gamma_j,t \right] \Gamma_{N,t}) + i\text{tr}(\left[ \gamma_j,t, H(N) \right] \Gamma_{N,t})$$

$$= -i\text{tr}(\left[ H_j + A_j^{\tilde{n}_j}, q_j,t \right] \Gamma_{N,t}) + i\text{tr}(\left[ H(N), q_j,t \right] \Gamma_{N,t})$$

$$= i\text{tr}(\left[ \frac{1}{N} \sum_{m \neq j} A_{mj} - A_j^{\tilde{n}_j}, q_j,t \right] \Gamma_{N,t}).$$

(42)

Note that all $H_k$ vanish from the final expression, because $[H_k, q_j,t] = 0$ for all $k \neq j$ (since $H_k$ and $q_j,t$ act on different variables).

Since $|\text{tr}(ABC)| = |\text{tr}(BCA)|$ for any self-adjoint operators $A, B, C$, it follows

$$|\dot{\alpha}_{N,j}(t)| \leq 2\left| \text{tr}\left( \left( \frac{1}{N} \sum_{m \neq j} A_{mj} - A_j^{\tilde{n}_j} \right) q_j,t \Gamma_{N,t} \right) \right| \leq 2\left| \text{tr}\left( \sum_{m \neq j} A_{mj} - (N-1) A_j^{\tilde{n}_j} q_j,t \Gamma_{N,t} \right) \right| + \frac{2}{N} |\text{tr}(A_j^{\tilde{n}_j} q_j,t \Gamma_{N,t})|.$$

(43)

Step 2. The main observation allowing to achieve some cancellation in the first term of (43) is the following equation:

$$\gamma_{m,t} A_{jm} \gamma_{m,t} = \gamma_{m,t} A_j^{\tilde{m},t}. \quad (44)$$

This is proved by inspection. In fact, the operator $\gamma_{m,t} A_{jm} \gamma_{m,t}$ acts as

$$\gamma_{m,t} A_{jm} \gamma_{m,t} f(x_j, x_m)$$

$$= \int_{X^2} \psi_{m,t}(x_m) \bar{\psi}_{m,t}(z_m) A(x_j, z_m; x'_j, w_m) \psi_{m,t}(w_m) \bar{\psi}_{m,t}(x'_m)$$

$$\times f(x'_j, x'_m) dx'_j dx'_m dz_m dw_m,$$

and hence it has the kernel

$$\int_{X^2} \psi_{m,t}(x_m) \bar{\psi}_{m,t}(z) A(x_j, z; x'_j, w) \psi_{m,t}(w) \bar{\psi}_{m,t}(x'_m) dz dw.$$

The operator $\gamma_{m,t} A_j^{\tilde{m},t}$ acts on $f(x_j, x_m)$ as

$$\left( \gamma_{m,t} A_j^{\tilde{m},t} f \right)(x_j, x_m)$$

$$= \int_{X^2} \psi_{m,t}(x_m) \bar{\psi}_{m,t}(x'_m) A(x_j, z_m; x'_j, w_m)$$

$$\times \bar{\psi}_{m,t}(z_m) \psi_{m,t}(w_m) f(x'_j, x'_m) dz_m dw_m dx'_j dx'_m.$$

and (44) follows.

In the case of the multiplication operator of interaction, formulas (43) and (44) remain valid with the operators $V_{ml}$ and $V_j^{\tilde{n}_j}$ instead of the operators $A_{mj}$ and $A_j^{\tilde{n}_j}$. In fact for the case of the multiplication operator this formula was introduced and exploited in [35].

Step 3. Let us introduce the random function

$$\delta_{N,t}(z, w) = \frac{1}{N-1} \sum_{m \neq j} \bar{\psi}_{j,t}(z) \psi_{j,t}(w) - \eta(z, w) = \frac{1}{N-1} \sum_{m \neq j} \gamma_{j,t}(z, w) - \eta(z, w).$$
By the law of large numbers $\delta^{j}_{N,t}$ tends to 0, as $N \to \infty$ for any $j$. More precisely, $E\delta^{j}_{N,t} = 0$ and

\[ E|\delta^{j}_{N,t}(z, w)|^2 = \text{Var}(\delta^{j}_{N,t}(z, w)) \leq \frac{1}{N-1} \text{Var}(\gamma_{j,t}(z, w)). \]

We can write

\[ A\tilde{\eta}_j = \frac{1}{N-1} \sum_{m \neq j} A\tilde{\gamma}_{m,t} - \delta^{j}_{N,t}, \]

and therefore

\[
\dot{\alpha}_{N,j}(t) = \frac{i}{N} \text{tr}(\left[ \sum_{m \neq j} (A_{mj} - A\tilde{\gamma}_{j,t}), q_{j,t} \right] \Gamma_{N,t}) + \frac{i(N-1)}{N} \text{tr}(\left[ A\delta^{j}_{N,t}, q_{j,t} \right] \Gamma_{N,t}) - \frac{i}{N} \text{tr}(\left[ A\tilde{\eta}_j, q_{j,t} \right] \Gamma_{N,t}).
\]

Consequently,\n
\[ E|\dot{\alpha}_{N}(t)| \leq 2E(I + II + III), \]

with

\[
I = \frac{1}{N} \left| \text{tr}(\sum_{m \neq j} (A_{mj} - A\tilde{\gamma}_{j,t}) q_{j,t} \Gamma_{N,t}) \right|,
\]

\[ II = |\text{tr}(A\delta^{N,t} q_{j,t} \Gamma_{N,t})|, \quad III = \frac{1}{N} \left| \text{tr}(A\tilde{\eta}_j q_{j,t} \Gamma_{N,t}) \right|. \]

In the case of the multiplication operator of interaction we define

\[ \delta^{j}_{N,t}(z) = \frac{1}{N-1} \sum_{m \neq j} (|\psi_{j,t}(z)|^2 - \xi_t(z)), \]

so that $E\delta^{j}_{N,t}(z) = 0$ and

\[ \text{Var}(\delta^{j}_{N,t}(z)) = E(|\delta^{j}_{N,t}(z)|^2) = E(|\psi_{j,t}(z)|^4)/(N-1). \]

**Step 4.** Since $|\text{tr}(BC)| \leq \text{tr}|B||C|$ for any operators $B, C$, we have

\[ III \leq \frac{1}{N} \text{tr}|q_{j,t} \Gamma_{N,t}| \|A\tilde{\eta}_j\| \leq \frac{1}{N} \|A\tilde{\eta}_j\|. \]

By Lemma 5.1 (proved at the end of this Section),

\[ \|A\tilde{\eta}_j\| \leq \|\eta\|_{L^2(X^2)} \|A\|_{HS}. \]

But

\[ \|\eta\|_{L^2(X^2)} = \text{tr}(\eta^2) \leq 1 \]

(here $\eta^2$ means the square of $\eta$ as an operator in $L^2(X)$). Consequently

\[ III \leq \frac{1}{N} \|A\|_{HS}. \]

Similarly,

\[ II \leq \|A\|_{HS} \|\delta^{j}_{N,t}\|_{L^2(X^2)}. \]
Consequently, by (45),

\[ E^{II} \leq \sqrt{\mathbb{E}(II^2)} \leq \|A\|_{HS} \|\delta^{II}_{N,t}\|_{L^2(X^2)}^{1/2} \]

\[ = \|A\|_{HS} \left( \int_{X^2} \mathbb{E}\left|\delta^{II}_{N,t}(z,w)\right|^2 \, dz \, dw \right)^{1/2} \]

\[ \leq \frac{1}{\sqrt{N-1}} \|A\|_{HS} \left( \int_{X^2} \text{Var}(\gamma_{j,t}(z,w)) \, dz \, dw \right)^{1/2}. \]

Since

\[ \int_{X^2} \text{Var}(\gamma_{j,t}(z,w)) \, dz \, dw \leq \int_{X^2} \mathbb{E}\left|\gamma_{j,t}(z,w)\right|^2 \, dz \, dw = \mathbb{E} \int_{X^2} \left|\gamma_{j,t}(z,w)\right|^2 \, dz \, dw = \text{Etr} \gamma_{j,t}^2 = 1, \]

where in the last term \( \gamma_{j,t}^2 \) is the square of \( \gamma_{j,t} \) as an operator in \( L^2(X) \), it follows that

\[ E^{II} \leq \frac{1}{\sqrt{N-1}} \|A\|_{HS}. \]

In case of the multiplication operator of interaction we obtain

\[ III \leq \frac{1}{N} \|V\|_{\infty}, \quad II \leq \|V\|_{\infty} \int_X |\delta^{II}_{N,t}(z)| \, dz, \]

and

\[ E^{II} \leq \|V\|_{\infty} \int_X \mathbb{E}\left|\delta^{II}_{N,t}(z)\right| \, dz \]

\[ \leq \|V\|_{\infty} \int_X \left( \mathbb{E}\left|\delta^{II}_{N,t}(z)\right|^2 \right)^{1/2} \, dz \]

\[ \leq \frac{1}{\sqrt{N-1}} \|V\|_{\infty} \int_X \left[ \mathbb{E}\left(|\psi_{j,t}(z)|^4\right) \right]^{1/2} \, dz. \]

Thus the estimate in this case becomes more involved.

**Step 5.** Dealing with \( I \) we plan to use the cancellation formula (44). To this end, we write

\[ I = \frac{1}{N} \left| \left( \Psi_{N,t}, \sum_{m \neq j} (A_{mj} - A_{jm}^\dagger) q_{j,t} \Psi_{N,t} \right) \right| \]

\[ \leq \frac{1}{N} \sum_{m \neq j} \left| (\Psi_{N,t}, (q_{m,t} + \gamma_{m,t})(A_{mj} - A_{jm}^\dagger)(q_{m,t} + \gamma_{m,t})q_{j,t} \Psi_{N,t}) \right|. \]

By (44), the term containing two multipliers \( \gamma_{m,t} \) vanishes, so that \( I \leq I_1 + I_2 \) with

\[ I_1 = \frac{1}{N} \sum_{m \neq j} \left| (\Psi_{N,t}, q_{m,t}(A_{mj} - A_{jm}^\dagger)q_{j,t} \Psi_{N,t}) \right|, \]

\[ I_2 = \frac{1}{N} \sum_{m \neq j} \left| (\Psi_{N,t}, \gamma_{m,t}(A_{mj} - A_{jm}^\dagger)q_{m,t}q_{j,t} \Psi_{N,t}) \right|. \]
For the first term we get the estimate
\[ I_1 \leq \frac{1}{N} \sum_{m \neq j} \| q_{m,t} \Psi_{N,t} \| \| A_{mj} - A_{j}^{\Psi_{N,t}} \| \]
\[ \leq \frac{2}{N} \sum_{m \neq j} (\Psi_{N,t}, q_{m,t} \Psi_{N,t}) \| A \|_{HS} \]
\[ \leq \frac{2}{N} \| A \|_{HS} \sum_{m \neq j} \alpha_{N,m}(t). \]

Consequently
\[ E I_1 \leq 2 \| A \|_{HS} E \alpha_N(t). \]

With \( I_2 \) we repeat the transformation above writing
\[ I_2 \leq \frac{1}{N} \sum_{m \neq j} |(\Psi_{N,t}, q_{j,t} \Psi_{N,t}) (A_{mj} - A_{j}^{\Psi_{N,t}}) q_{m,t} q_{j,t} \Psi_{N,t})| \]
\[ + \frac{1}{N} \sum_{m \neq j} |(\Psi_{N,t}, \gamma_{j,t} \Psi_{N,t}) (A_{mj} - A_{j}^{\Psi_{N,t}}) q_{m,t} q_{j,t} \Psi_{N,t})|. \]

The first term is estimated as \( I_1 \) above and in the second term the operator \( A_{j}^{\Psi_{N,t}} \) cancels, since it commutes with \( q_{m,t} \). Thus we obtain that
\[ E I_2 \leq 2 \| A \|_{HS} E \alpha_N(t) + \frac{1}{N} E \sum_{m \neq j} |(\Psi_{N,t}, \gamma_{j,t} \Psi_{N,t}) (A_{mj}q_{m,t}q_{j,t} \Psi_{N,t})|. \]

The second term here is similar the one that appears in both [55] and [35], where nice specific tricks were invented to deal with it. For the convenience of the readers, we present in the Appendix Proposition A.2 the arguments of [35], slightly modified to take into account the present stochastic situation. This gives the following estimate:
\[ E I_2 \leq 2 \| A \|_{HS} (2E \alpha_N(t) + \frac{1}{N}). \]

In the case of the multiplication operator of interaction we obtain all estimates in Step 5 with \( \| V \|_{\infty} \) instead of \( \| A \|_{HS} \).

Step 6. Putting all estimates above together we get (for \( N > 1 \)) that
\[ E |\dot{\alpha}_N(t)| \leq 12 \| A \|_{HS} \left( E \alpha_N(t) + \frac{1}{\sqrt{N}} \right), \]
and
\[ E |\dot{\alpha}_N(t)| \leq 12 \| V \|_{\infty} \left( E \alpha_N(t) + \frac{1}{N} \right) + \frac{3}{\sqrt{N}} \| V \|_{\infty} \int_X \sqrt{E(|\psi_{j,t}(z)|^4)} \, dz, \]
for the case of the integral operator and the multiplication operator of interaction respectively.

Applying Gronwall’s lemma yields (26) and (25).

**Lemma 5.1.** Let \( A, \eta \) be measurable real or complex-valued functions on \( X^4 \) and \( X^2 \), for some Borel space \( X \) with a Borel measure \( dx \), and let
\[ a(x, x') = \int A(x, y; x', y') \eta(y, y') \, dy \, dy'. \]

Then
\[ \| a \|_{L^2(X^2)}^2 \leq \| \eta \|_{L^2(X^2)}^2 \| A \|_{L^2(X^4)}^2. \]
PROOF. We have
\[ |a(x, x')|^2 \leq \int |A(x, y; x', y')\eta(y, y')| dy dy' \int |A(x, z; x', z')\eta(z, z')| dz dz'. \]
Applying the Cauchy–Schwarz inequality for both integrals yields
\[ |a(x, x')|^2 \leq \int |A(x, y; x', y')|^2 dy dy' \int |\eta(z, z')|^2 dz dz' \]
\[ = \|\eta\|^2_{L^2(X^2)} \int |A(x, y; x', y')|^2 dy dy'. \]
Integrating with respect to \( dx dx' \) yields (51). \( \square \)

6. Proof of Theorem 3.2. Following the lines of the proof of Theorem 3.1, we arrive at the equation
\[ \dot{\alpha}_{N,j}(t) = i \text{tr} \left( \left[ \frac{1}{N} \sum_{m \neq j} A_{mj} - \tilde{A}_j^\eta + (u(t, \Gamma_{N,t}^{(j)}) - u(t, \gamma_{j,t})) \hat{H}, q_{j,t} \right] \Gamma_{N,t} \right), \]
generalizing equation (52). Thus we need to get the estimate for the term
\[ |\text{tr}((u(t, \Gamma_{N,t}^{(j)}) - u(t, \gamma_{j,t}))\hat{H}, q_{j,t}, \Gamma_{N,t})| \]
\[ \leq 2|\text{tr}((u(t, \Gamma_{N,t}^{(j)}) - u(t, \gamma_{j,t}))\hat{H} q_{j,t}, \Gamma_{N,t})| \]
\[ \leq 2|u(t, \Gamma_{N,t}^{(j)}) - u(t, \gamma_{j,t})||\hat{H}||\sqrt{\alpha_{N,j}(t)}. \]
By (29),
\[ |u(t, \Gamma_{N,t}^{(j)}) - u(t, \gamma_{j,t})| \leq \kappa |\gamma_{N,t} - \gamma_{j,t}|. \]
By (22),
\[ \text{tr}|\Gamma_{N,t}^{(j)} - \gamma_{j,t}| \leq 2\sqrt{2\alpha_{N,j}(t)}. \]
Therefore,
\[ |\text{tr}((u(t, \Gamma_{N,t}^{(j)}) - u(t, \gamma_{j,t}))\hat{H} q_{j,t}, \Gamma_{N,t})| \leq 4\sqrt{2}\kappa \|\hat{H}\|\alpha_{N,j}(t), \]
and thus
\[ E|\text{tr}((u(t, \Gamma_{N,t}^{(j)}) - u(t, \gamma_{j,t}))\hat{H} q_{j,t}, \Gamma_{N,t})| \leq 4\sqrt{2}\kappa \|\hat{H}\|E\alpha_N(t). \]
Adding this term to the r.h.s. of (50) and using again Gronwall’s lemma yields (30).

7. Proof of Theorem 4.1. Assume that all players, except for one of them, say the first one, are playing according to the MFG strategy \( u_{MFG}(t, \Gamma_{N,t}^{(j)}), j > 1 \), and the first player is following some other strategy \( \tilde{u}(t, \Gamma_{N,t}^{(1)}). \)
For all \( j > 1 \) the functions \( \eta_j \) are equal and are given by the formula \( \eta_j = E\gamma_{j,t}. \) Moreover, \( E\alpha_{N,j}(t) = E\alpha_N(t) \) are the same for all \( j > 1. \)

REMARK 12. To conclude that \( \eta_j \) are equal, we use of course the uniqueness in law for the solutions of equations (34). Recall that all required well-posedness results are proved in Section 8 for the finite-dimensional case. For the general case they are given in [42].
Following the proof of Theorem 3.2 we obtain

\[ \dot{\alpha}_{N,j}(t) = i \text{tr} \left( \left[ \frac{1}{N} \sum_{m \neq j} A_{mj} - \check{A}_{j} + (u_{\text{MFG}}(t, \Gamma_{N,t}^{(j)}) - u_{\text{MFG}}(t, \gamma_{j,t})) \hat{H}, q_{j,t} \right] \Gamma_{N,t} \right), \]

for all \( j > 1 \). Up to an additive correction of magnitude not exceeding \( 4/N \) the r.h.s. can be substituted by the expression

\[ \dot{\alpha}_{N,j}(t) = i \text{tr} \left( \left[ \frac{1}{N} \sum_{m \neq j, 1} A_{mj} - \check{A}_{j} + (u_{\text{MFG}}(t, \Gamma_{N,t}^{(j)}) - u_{\text{MFG}}(t, \gamma_{j,t})) \hat{H}, q_{j,t} \right] \Gamma_{N,t} \right), \]

which is then dealt with exactly as in the proof of Theorem 3.2 yielding the same estimate (30) (with a corrected multiplier) for \( E\alpha_{N}(t) = E\alpha_{N,j}(t), j > 1 \), that is,

\[ E\alpha_{N}(t) \leq \exp \left\{ 12(\|A\|_{\text{HS}} + \varkappa \|\hat{H}\|) t \right\} \alpha_{N}(0) + \left( \exp \left\{ 12(\|A\|_{\text{HS}} + \varkappa \|\hat{H}\|) t \right\} - 1 \right\} \frac{5}{\sqrt{N}}. \]

Similarly the same estimate is obtained for \( E\alpha_{N,1}(t) \). Since our initial conditions are supposed to be the product of identical functions, the initial \( \alpha_{N,j}(0) \) vanish yielding

\[ E\alpha_{N,j}(t) \leq C(T)N^{-1/2} \]

for all \( j \) and a constant \( C(T) \) depending on \( \|A\|_{\text{HS}}, \varkappa, \|\hat{H}\| \).

We can now compare the expected payoffs (33) received by the players in the \( N \)-player quantum game with the expected payoff (35) received in the limiting game. For each \( j \)th player the difference is bounded by

\[ E \int_{t}^{T} \left| \text{tr} \left( J (\Gamma_{N,s}^{(j)} - \gamma_{j,s}) \right) \right| ds + E \left| \text{tr} \left( F (\Gamma_{N,T}^{(j)} - \gamma_{j,T}) \right) \right|. \]

Since,

\[ \left| \text{tr} \left( J (\Gamma_{N,s}^{(j)} - \gamma_{j,s}) \right) \right| \leq \|J\| \left| \text{tr} \left( \Gamma_{N,s}^{(j)} - \gamma_{j,s} \right) \right|, \]

and by (22),

\[ \text{tr} \left| \Gamma_{N,s}^{(j)} - \gamma_{j,s} \right| \leq 2 \sqrt{2\alpha_{N,j}(s)}, \]

it follows that the expectation of the difference of the payoffs is bounded by

\[ 2\sqrt{2}(\|J\|T + \|F\|) \sup_{t} E \sqrt{\alpha_{N,j}(t)} \leq 2\sqrt{2}(\|J\|T + \|F\|) \sup_{t} \sqrt{E\alpha_{N,j}(t)} \leq \left( \|J\|T + \|F\| \right) C(T)N^{-1/4}, \]

with a constant \( C(T) \) depending on \( \|A\|_{\text{HS}}, \varkappa, \|\hat{H}\| \).

But by the assumption of the Theorem, \( u_{t}^{\text{MFG}} \) is the optimal choice for the limiting optimization problem. Hence the claim of the theorem follows.

8. Limiting MFG problem in finite-dimensional case. Here we derive some existence results for the limiting MFG problem to finite-dimensional quantum systems, referred to as atoms. These results yield some conditions under which Theorem 4.1 applies. This section can be also considered as an introduction to classical MFGs with a drift control on compact Riemannian manifolds, as exemplified by the complex projective space.
The state space of each atom is a finite-dimensional Hilbert space $\mathbf{C}^{n+1}$. The interaction will be given by the tensor $A(j, k; j', k')$, $j, k, j', k' \in \{0, \ldots, n\}$ such that

$$A(j, k; j', k') = A(k, j; k', j').$$

The case of a multiplication operator of interaction is now fully included, as it corresponds to the diagonal tensor $A$.

As in Section 4, each atom is controlled by an agent that can control a part of the individual Hamiltonians. Therefore the $N$-particle Hamiltonian (7) will have the form

$$H(N)f(i_1, \ldots, i_N) = \sum_{j=1}^{N} (H_j + u_j \hat{H}_j) f(i_1, \ldots, i_N) + \frac{1}{N} \sum_{l<j \leq N} A_{lj} f(i_1, \ldots, i_N),$$

where $H$ and $\hat{H}$ are self-adjoint matrices in $\mathbf{C}^{n+1}$ and $H_j, \hat{H}_j$ denote their actions on the variables $i_j, u_j$ is a control parameter of the $j$th agent defining the strength of the field (say electric or magnetic) described by the Hamiltonian $\hat{H}_j$. For simplicity we assume that each $u_j$ can be chosen from some fixed interval $U = [-U_0, U_0]$.

Assuming that observations of each atom are performed by the coupling with the same anti-Hermitian vector-valued operator $L = (L^1, \ldots, L^K)$ and assuming that the initial conditions and controls of the agents are identical, it follows from Theorem 3.1 that the limiting evolution, as $N \to \infty$, will be described by the nonlinear equation (15), that is,

$$d\psi_{j,k,t} = -\left[i(H + u_j(t) \hat{H})\psi_{j,k,t} + iA\eta_{j,k,t} - \frac{1}{2}L^2\psi_{j,k,t}\right]dt + L\psi_{j,k,t}dY^j_t,$$

where

$$L^2 = \sum_{p=1}^{P} (L^p)^2, \quad LdY^j_t = \sum_{p=1}^{P} L^p dY^{j,p}_t,$$

with $Y^{j,p}_t, j = 1, \ldots, N, p = 1, \ldots, P$, being independent standard Brownian motions.

Since quantum states are defined up to a complex multiplier, so that the state space is effectively the complex projective space $\mathbf{CP}^n$, rather than the linear space $\mathbf{C}^{n+1}$, it is convenient to rewrite equation (28) in projective coordinates $w = (w_1, \ldots, w_n)$, with $w_k = w_{k,t} = \psi_{k,1}/\psi_{0,0,t}$ (where we omit index $j$ for brevity). To shorten formulas it is also handy to use the $n + 1$-dimensional vector $W = (1, w_1, \ldots, w_n)$ with the additional coordinate $w_0 = 1$. This rewriting is done by direct application of Itô’s formula (details of simple calculations given in paper [39]) yielding the following equation:

$$dw_k = i\left[w_k((H + u \hat{H} + A\tilde{\eta})W)_0 - ((H + u \hat{H} + A\tilde{\eta})W)_k\right]dt$$

$$+ \frac{1}{2} \sum_p \left[\left((L^p)^2W\right)_k - w_k((L^p)^2W)_0\right]dt$$

$$+ \sum_p \left[w_k(L^pW)_0^2 - (L^pW)_0(L^pW)_k\right]dt$$

$$+ \sum_p \left[(L^pW)_k - w(k)(L^pW)_0\right]dY^{j,p}_t.$$

Recall that the coordinates $w = (w_1, \ldots, w_n)$ cover the open dense subset $V_0$ of $\mathbf{CP}^n$ arising from the vectors $\psi = (\psi_0, \ldots, \psi_n) \in \mathbf{C}^{n+1}$ with $\psi_0 \neq 0$. The whole $\mathbf{CP}^n$ is covered by $n + 1$ such as charts $V_j$, each describing the vectors with $\psi_j \neq 0$.

As was discovered in [39], if one chooses as the coupling operators $L^p$ the $(n^2 + 2n)$ generalized Gell–Mann matrices (in case of a qubit these are 3 Pauli matrices), the third and
fourth terms in (58) will vanish and the second order diffusion operator arising from the last term in (58) will coincide (up to a multiplier 2) with the major (second order) part of the Laplace–Beltrami operator on the corresponding complex projective space $\mathbb{C}P^n$. In our case $L^p$ are assumed to be anti-Hermitian, rather than Hermitian operators in [39]. However, as seen directly, multiplying all operators by the imaginary unit $i$ (turning Hermitian matrices to anti-Hermitian) does not affect these properties.

**REMARK 13.** Here we exploit only the mathematical similarity of the diffusions arising from Hermitian and anti-Hermitian coupling operators, which is due to a particular setting of homodyne detection. Of course, multiplying the coupling operators by $i$ changes the physics. As was already noted, the case of Hermitian coupling operators is much more natural physically, than the simplest case of anti-Hermitian operators that we are treating in this paper. A systematic treatment of Hermitian coupling operators will be performed in a subsequent paper (in preparation).

Hence, choosing $L^p$ as the generalized Gell–Mann matrices multiplied by $i$, equation (58) simplifies to

$$
\begin{align*}
  dw_k &= i \left[ w_k \left( (H + u \hat{H} + A \hat{\eta}) W \right)_0 - \left( (H + u \hat{H} + A \hat{\eta}) W \right)_k \right] dt \\
  &\quad + \sum_p \left[ (L^p W)_k - w_k (L^p W)_0 \right] dY^{j,p}_k, \quad k = 1, \ldots, n,
\end{align*}
$$

and the second order diffusion operator arising from the last term of this equation equals $2\Delta_{\text{pro}}$, where $\Delta_{\text{pro}}$ is the major (second order) part of the Laplace–Beltrami operator on the corresponding complex projective space $\mathbb{C}P^n$. This operator is both invariant (it looks the same in the projective coordinates of all $n + 1$ charts $V_j$ covering $\mathbb{C}P^n$) and nondegenerate, which makes it the most handy for the analysis of optimal control.

In particular, for the most important case of a qubit, that is, for $n = 1$, the following formula holds in real coordinates $x, y$ (with $w = x + iy$):

$$
2\Delta_{\text{pro}} S(x, y) = \frac{1}{2} \left( 1 + x^2 + y^2 \right)^2 \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right),
$$

that is, the l.h.s. coincides exactly (up to a multiplier 2) with the Laplace–Beltrami operator on the two-dimensional sphere $S^2$. In case of a qudit, that is, for $n = 2$, the formula for $\Delta_{\text{pro}}$ on $\mathbb{C}P^2$ is as follows:

$$
\Delta_{\text{pro}} S(w_1, w_2) = \left( 1 + \sum_j |w_j|^2 \right) \left[ (1 + |w_1|^2) \frac{\partial^2 S}{\partial w_1 \partial \bar{w}_1} + (1 + |w_2|^2) \frac{\partial^2 S}{\partial w_2 \partial \bar{w}_2} \right] \\
+ w_1 \bar{w}_2 \frac{\partial^2 S}{\partial w_1 \partial \bar{w}_2} + \bar{w}_1 w_2 \frac{\partial^2 S}{\partial w_1 \partial \bar{w}_2}.
$$

**REMARK 14.** More generally, the same effect occurs (the diffusion operator of evolution (58) coincides with the second order term of the Laplace–Beltrami operator), if one takes as $L^p$ an orthonormal basis of the Lie algebra of the group of unitary matrices $U(n)$.

To derive the MFG equation let us assume that some deterministic “empirical measure” $\eta_t$ is given and $u_j$ has to be chosen by the $j$th agent to maximize the payoff

$$
P(t, W; u(\cdot)) = \mathbb{E} \int_t^T \left( \langle J \rangle_{W(s)} - \frac{c}{2} u^2(s) \right) ds + \langle F \rangle_{W(T)}.
$$
where \( J = (J_{pm}) \) and \( F = (F_{pm}) \) are some \((n + 1) \times (n + 1)\) matrices expressing the current and the terminal costs of the agent, \( c \) measures the cost of applying the control and \( E \) denotes the expectation with respect to the random trajectories \( W(s) \) arising from dynamic (59) under the strategic choice of the control \( u \). Recall that

\[
\langle J \rangle_W = \frac{(W, J W)}{(W, W)} = \frac{\sum_{p,m} W_p J_{pm} W_m}{\sum_m |W_m|^2} = \frac{\sum_{p,m} W_p J_{pm} W_m}{1 + \sum_{m>0} |w_m|^2},
\]

with similar formula for \( \langle F \rangle_W \).

The problem of dynamic maximization of (62) is a standard problem of controlled diffusion, and we can write down the standard Hamilton–Jacobi–Bellman (HJB) equation describing the optimal payoff \( S \), taking into account that the (uncontrolled) diffusion part is given by

\[
0 = \frac{\partial S_t}{\partial t} + 2\Delta_{\text{pro}} S_t + \langle J \rangle_W + \sup_u \left\{ u \Pi(\nabla S_t) - \frac{c}{2} u^2 \right\}
\]

(63)

\[
+ \sum_k \text{Re}[i w_k(H + A \hat{\eta} W)_0 - i(H + A \hat{\eta} W)_k] \frac{\partial S_t}{\partial x_k}
\]

\[
+ \sum_k \text{Im}[i w_k(H + A \hat{\eta} W)_0 - i(H + A \hat{\eta} W)_k] \frac{\partial S_t}{\partial y_k}
\]

with

\[
(64) \quad \Pi(\nabla S) = \sum_k \left[ \text{Re}[i w_k(\hat{H} W)_0 - i(\hat{H} W)_k] \frac{\partial S}{\partial x_k} + \text{Im}[i w_k(\hat{H} W)_0 - i(\hat{H} W)_k] \frac{\partial S}{\partial y_k} \right].
\]

Here \( \sup_u = \sup_{u \in [-U_0, U_0]} \) can be calculated explicitly yielding

\[
(65) \quad u = u_t(w, \nabla S_t(w)) = \min\left[ \max(U_0, \Pi(\nabla S_t)/c), -U_0 \right].
\]

The optimal cost function \( S_t \) is expected to satisfy the backward Cauchy problem for this equation, that is, it is specified by the terminal condition

\[
S_T(W) = \langle F \rangle_{W(T)}.
\]

Next, for a given feedback control \( u = u_t(w) \), the process (59) is a nondegenerate diffusion in \( \mathbb{C} P^n \) and consequently its distribution has density (for \( t > 0 \)) with respect to the Lebesgue measure. In the coordinates \( w = \{w_j = x_j + iy_j\}, j = 1, \ldots, n, \) of the chart \( V_0 \) this density \( \mu_t(w) \) satisfies the forward Kolmogorov equation

\[
\frac{\partial \mu_t}{\partial t} = 2\Delta_{\text{pro}} \mu_t - \sum_k \frac{\partial}{\partial x_k} \left[ \text{Re}(i w_k(H + u(w) \hat{H} + A \hat{\eta} W)_0
\]

\[
- i(H + u(w) \hat{H} + A \hat{\eta} W)_k) \mu_t \right]
\]

(66)

\[
- \sum_k \frac{\partial}{\partial y_k} \left[ \text{Im}(i w_k((H + u(w) \hat{H} + A \hat{\eta} W)_0
\]

\[
- i((H + u(w) \hat{H} + A \hat{\eta} W)_k) \mu_t \right].
\]

Since our process starts with some fixed initial state \( \psi_0 = (\psi_{00}, \ldots, \psi_{n0}) \), with the corresponding initial vector \( w(0) \) such that \( w(0)_k = \psi_{j,k,0}/\psi_{j,0,0}, k = 1, \ldots, n, \) the density \( \mu_t \) satisfies the Dirac initial condition \( \mu_0(dw) = \delta(w - w(0)). \)
As the control $u_t(\mu)$ is usually not a smooth function, we cannot expect to have strong solutions of either the HJB equation (63) or the Kolmogorov equation (66). Hence it is convenient to rewrite them in the so called mild (or integral) forms, that already include the corresponding initial and terminal conditions, that is, as the equations

$$S_t(w) = e^{2(T-t)\Delta_{\text{pro}}} S_T + \int_t^T e^{2(s-t)\Delta_{\text{pro}}} \left( \langle J \rangle W + \sup_u \left\{ u \Pi(\nabla S_s) - c u^2 \right\} \right) ds$$

(67)

$$+ \int_t^T e^{2(s-t)\Delta_{\text{pro}}} \sum_k \text{Re} \left[ i w_k (H + A_{\tilde{n}} W)_0 - i (H + A_{\tilde{n}} W)_k \right] \frac{\partial S_s}{\partial x_k} ds$$

$$+ \int_t^T e^{2(s-t)\Delta_{\text{pro}}} \sum_k \text{Im} \left[ i w_k (H + A_{\tilde{n}} W)_0 - i (H + A_{\tilde{n}} W)_k \right] \frac{\partial S_s}{\partial y_k} ds,$$

and

$$\mu_t(w) = e^{2t\Delta_{\text{pro}}} \mu_0(w)$$

$$- \int_0^t e^{2(t-s)\Delta_{\text{pro}}} \sum_k \frac{\partial}{\partial x_k} \left[ \text{Re} \left( i w_k (H + u(w) \hat{H} + A_{\tilde{n}} W)_0 \right) \right] ds$$

(68)

$$- i \left( (H + u(w) \hat{H} + A_{\tilde{n}} W)_k \right) \mu_s] ds$$

$$- \int_0^t e^{2(t-s)\Delta_{\text{pro}}} \sum_k \frac{\partial}{\partial y_k} \left[ \text{Im} \left( i w_k ((H + u(w) \hat{H} + A_{\tilde{n}} W)_0 \right) \right] ds$$

$$- i \left( (H + u(w) \hat{H} + A_{\tilde{n}} W)_k \right) \mu_s] ds,$$

respectively.

**REMARK 15.** An alternative approach could be to use the viscosity solutions to equations (63) and (66).

The forward-backward system of MFG that express the consistency of the individual optimal control and the dynamics is therefore the pair of equations (63) and (66), or more generally (67) and (68), coupled via control (65). Namely, this system consists of the backward HJB equation (63) with

$$\eta_t(k, l) = E \gamma_t(k, l) = E(\psi_{k,t} \overline{\psi_{l,t}}), \quad k, l = 0, \ldots, n,$$

arising from equation (66), and the forward equation (66) with $u_t(w)$ arising from (63) via formula (65). Notice also that $\psi$ is obtained by normalization from the vector $W$, so that

$$\psi_{k,t} = \frac{W_{k,t}}{\sqrt{1 + |w_t|^2}}.$$  

(70)

We can formulate it also in another equivalent way, as the consistency condition $u_t^{\text{com}}(w) = u_t^{\text{ind}}(w)$, as in Section 4.

Let us finally represent the MFG problem as a single anticipating equation. First of all let $K(t, w, v)$ be the heat kernel related to the operator $2\Delta_{\text{pro}}$ on $\mathbb{C}P^n$, that is, $K(t, v, w)$ is the solution of the corresponding heat equation $(\partial K/\partial t) = 2\Delta_{\text{pro}} K$ as a function of $(t > 0, v \in \mathbb{C}P^n)$ and has the Dirac initial condition $K(0, v, w) = \delta_w(v)$. It is well known that $K(t, v, w)$ is a (infinitely) smooth function of $v, w$ for $t > 0$ and that the Cauchy problem for this heat equation is well-posed in $M$. Its resolving operators

$$e^{2t\Delta_{\text{pro}}} f(v) = \int_{\mathbb{C}P^n} K(t, v, w) f(w) d_n w = \int_{\mathbb{C}P^n} K(t, v, w) f(w) d_n w,$$

(71)
form a strongly continuous semigroup of contractions in the space $C(CP^n)$ of bounded continuous functions on $CP^n$, equipped with the sup-norm. Here $d_n w = \sqrt{\det g(w)} \, dw$ is the Riemannian volume on $CP^n$, where $dw = \prod dw_j = \prod (dx_j \, dy_j)$ and $g(w)$ denotes the Riemannian metric on $CP^n$ in coordinate $w$). Notice that the chart $V_0$ (of vectors $\psi$ with $\psi_0 \neq 0$) is isomorphic to $C^n$ and covers $CP^n$ up to a set of zero measure, so that integrating over $C^n$ and $CP^n$ in (71) is equivalent. Having in mind that the spaces $L_1(CP^n)$ of integrable (with respect to $dv$) functions are inserted into the space $M(CP^n)$ of Borel measures on $CP^n$, it follows that the semigroup $S_t$ extends to the semigroup $e^{2t\Delta_{\text{pro}}}$ on $M(CP^n)$ that maps $M(CP^n) \to L_1(CP^n)$ for any $t > 0$ and acts according to the formula

$$e^{2t\Delta_{\text{pro}}} \mu(v) = \int_{CP^n} K(t, v, w) \mu(dw).$$

Let us note for clarity that when we consider this transformation on measures, we identify the function $e^{2t\Delta_{\text{pro}}} \mu(v)$ on the l.h.s. with the measure $e^{2t\Delta_{\text{pro}}} \mu(v) \, d_n v$ on $CP^n$.

**Remark 16.** As mentioned above, $\Delta_{\text{pro}}$ coincides with the Laplace–Beltrami operator in case $n = 1$. In case $n > 1$ $\Delta_{\text{pro}}$ is in fact only the major second order part of the Laplace–Beltrami operator defining the standard Brownian motion on $CP^n$. However, since they differ only by the bounded smooth first order parts, all well known asymptotic and smoothness properties of the Laplace–Beltrami operator remain valid for $\Delta_{\text{pro}}$.

Hence equation (68) rewrites in the following form:

$$\mu_t(v) = \int_{C^n} K(t, v, w) \mu_0(dw) - \int_0^t \int_{C^n} K(t - s, v, w) \, ds \, d_n w$$

$$\times \left( \sum_k \frac{\partial}{\partial x_k} \left[ \text{Re}(iw_k (H + u_s(w) \hat{H} + A^{\tilde{y}_s} W))_0 - i (H + u_s(w) \hat{H} + A^{\tilde{y}_s} W)_k \mu_s(w) \right] \right. \right.$$

$$\left. \left. + \sum_k \frac{\partial}{\partial y_k} \left[ \text{Im}(iw_k ((H + u_s(w) \hat{H} + A^{\tilde{y}_s} W)_k) \mu_s(w) \right] \right).$$

Let us reiterate that with some abuse of notation we identify measures with their densities (with respect to Riemannian volume) so that $\mu(dw) = \mu(w) \, d_n w$. Thus in (73) only $\mu_0(dw)$ denotes the initial Dirac measure (that has no density), and all other $\mu_t(w)$ denote the densities.

Using integration by parts this rewrites as

$$\mu_t(v) = \int_{C^n} K(t, v, w) \mu_0(dw)$$

$$+ \int_0^t \int_{C^n} \sum_k \frac{\partial}{\partial x_k} \left[ K(t - s, v, w) \sqrt{\det g(w)} \right] \, dw \, ds$$

$$\times \left[ \text{Re}(iw_k (H + u_s(w) \hat{H} + A^{\tilde{y}_s} W)_0 - i (H + u_s(w) \hat{H} + A^{\tilde{y}_s} W)_k \mu_s(w) \right]$$

$$\left. + \int_0^t \int_{C^n} \sum_k \frac{\partial}{\partial y_k} \left[ K(t - s, v, w) \sqrt{\det g(w)} \right] \, dw \, ds$$

$$\times \left[ \text{Im}(iw_k (H + u_s(w) \hat{H} + A^{\tilde{y}_s} W)_0 - i (H + u_s(w) \hat{H} + A^{\tilde{y}_s} W)_k \mu_s(w) \right].$$

The advantage of this equation as compared to (73) and (66) is clear: no smoothness of the function $u(w)$ is required for this equation to make sense.
REMARK 17. Everything is well defined in (74) due to the well known differentiability of the heat kernel $K$ and the bounds for its derivatives, see [22].

Next, let $u_t(w, \nabla S_t(w; \eta_{\geq t})$ be given by (65) (existence of $\nabla S_t$ follows from Theorem 8.1 below) and be considered as a functional of $w$ and the curve $\eta_{\geq t}$, by which we denote the piece of curve $\eta_s$ for $s \in [t, T]$. Of course $S_t$ denotes here the solution of (67). Plugging this into the forward equation (74) we get the following single nonlinear equation with the anticipating (depending on the future) r.h.s.:

\begin{equation}
\mu_t(v) = \int_{C^n} K(t, v, w) \mu_0(dw) \\
+ \int_0^t \int_{C^n} \sum_{k} \frac{\partial}{\partial x_k} [K(t - s, v, w)\sqrt{\det g(w)}] \, dw \, ds \\
\times [\text{Re}(iw_k(H + u_s(w, \nabla S_s(w; \eta_{\geq s}))\hat{H} + A_{\eta_s} W)_0] \\
- i(H + u_s(w, \nabla S_s(w; \eta_{\geq s}))\hat{H} + A_{\eta_s} W)_k] \mu_s(w) \\
+ \int_0^t \int_{C^n} \sum_{k} \frac{\partial}{\partial y_k} [K(t - s, v, w)\sqrt{\det g(w)}] \, dw \, ds \\
\times [\text{Im}(iw_k((H + u_s(w, \nabla S_t(w; \eta_{\geq s}))\hat{H} + A_{\eta_s} W)_0] \\
- ((H + u_s(w, \nabla S_s(w; \eta_{\geq s}))\hat{H} + A_{\eta_s} W)_k] \mu_s(w),
\end{equation}

where

\begin{equation}
\eta_t(k, m) = E_{\mu_t} (\psi_{m,t} \bar{\psi}_{m,t}) = \int_{C^n} (\psi_{m,t} \bar{\psi}_{m,t}) \mu_t(w) \, dw, \quad t > 0,
\end{equation}

and

\begin{equation}
\eta_0(k, m) = \psi_{m,0} \bar{\psi}_{m,0} = \int_{C^n} (\psi_{m,0} \bar{\psi}_{m,0}) \mu_0(dw).
\end{equation}

By (70), we also have

\begin{equation}
\eta_{km}^t = E_{\mu_t} \frac{W_{m,t} \overline{W_{m,t}}}{1 + |w_t|^2}, \quad k, m = 0, \ldots, n,
\end{equation}

with $|w_t|^2 = \sum_{k=1}^n |w_{k,t}|^2$.

Now we formulate the global existence and local well-posedness for equation (75), which is equivalent to the coupled system of equations (67) and (68).

**THEOREM 8.1.** (i) For sufficiently small $T$ equation (75) is well-posed, that is, for any $R > 0$ there exist $T > 0$ such that for all $S_T$ with $\|S_T\|_{C^1(C^n)} \leq R$, equation (75) has a unique solution. (ii) For any $T > 0$ and any continuously differentiable function $S_T$ on $C^n$ there exists a solution to equation (75).

**PROOF.** As was mentioned this result can be considered as belonging to the theory of classical MFGs on manifolds. Though being seemingly new (the author is unaware of any papers dealing with MFGs on manifolds), the proof can be performed by the (by now) standard approach, of course enhanced by some specific geometric analysis. We will follow closely the method from [46].
(i) We reformulate equation (75) as a fixed point problem in the following way. Let $C_0([0, T], \mathcal{M}(CP^n))$ be the space of weakly continuous functions from $[0, T]$ to $\mathcal{M}(CP^n)$, equipped with distance

$$\|\mu_1 - \mu_2\|_{\mathcal{M}, T} = \sup_{t \in [0, T]} \sup_{\|f\| \leq 1} |(f, \mu_1 - \mu_2)|$$

such that the initial point is fixed as the Dirac measure $\mu_0(dw) = \delta(w - w(0))$.

**Remark 18.** It is straightforward to see that this space is a complete metric space. It is a bit nonstandard, because continuity in $t$ is defined as the weak one, while the distance is defined via the strong Banach topology. The necessity to use such a hybrid arises because curves starting with a Dirac measure and having densities otherwise (that eventually would solve our problem) cannot be strongly continuous.

Equation (75) is the fixed point problem for the mapping $\Phi(\mu)$ in $C_0([0, T], \mathcal{M}(CP^n))$ expressed by the r.h.s. of (75).

To any curve $\mu_t$ in $C_0([0, T], \mathcal{M}(CP^n))$ there corresponds the curve $\eta_t$ given by (76). It follows that for two curves $\mu_1^t, \mu_2^t$ we have for the corresponding matrix-valued curves $\eta_1^t, \eta_2^t$ the estimate

$$\|\eta_1^t - \eta_2^t\| \leq (\mu_1^t - \mu_2^t)\|_{\mathcal{M}, T}.$$ (77)

Next we solve the backward HJB equation (67) finding the function $S(t, w)$ that depends on $\mu$. Actually $S(t, w)$ depends on the future $\mu_{\geq t}$ only, but this is not very essential for the argument.

The well-posedness of this HJB equation was proved in [39]. Namely, it was shown that for any function $S_T \in C^1(CP^n)$ (the space of continuously differentiable functions on $CP^n$), there exists a unique curve $S_t \in C([0, T], C^1(CP^n))$ (the space of continuous curves with values in $C^1(CP^n)$) that solves (67).

In the fact, this well-posedness is a consequence of a general well-posedness result from [38] and the following statement expressing the key smoothing property of the semigroup $e^{2t\Delta_{\text{pro}}}$:

$$\|e^{2t\Delta_{\text{pro}}} f\|_{C^1(M)} \leq C t^{-1/2} \| f \|_{C(M)},$$ (78)

$$\|e^{2t\Delta_{\text{pro}}} f\|_{C^1(M)} \leq C \| f \|_{C^1(M)},$$ (79)

with a constant $C$, uniformly for any compact interval of time. Moreover, by the same arguments as in the proof of Theorem 6.1.2 of [38] (devoted to the case of HJB in $\mathbb{R}^d$) it follows that the solution depends Lipschitz continuously on a parameter, if the corresponding Hamiltonian function depends Lipschitz continuously on this parameter. In our case the role of the parameter is played by the matrix-valued curves $\eta_t$, and it follows that for the solutions $S_1^t$ and $S_2^t$ corresponding to the curves $\eta_1^t$ and $\eta_2^t$, the following estimate holds:

$$\max_{t \in [0, T]} \|S_1^t - S_2^t\|_{C^1(CP^n)} \leq K \max_{t \in [0, t]} \|\eta_1^t - \eta_2^t\|,$$ (80)

with a constant $K$ depending continuously on the tensor $A$, the time interval $T$ and the norm $\|S_T\|_{C^1(CP^n)}$.

Next, by (77) and (80), the square bracket on the r.h.s. of (75) depend Lipschitz continuously on $\mu_\cdot$, and by (78), the derivatives $(\partial / \partial x_k)[\cdots]$ and $(\partial / \partial y_k)[\cdots]$ are of order $(t - s)^{-1/2}$. Hence, for two curves $\mu_1^t$ and $\mu_2^t$ we get

$$\|\Phi(\mu_1^t) - \Phi(\mu_2^t)\|_{\mathcal{M}, T} \leq \sqrt{T} K_T \|\mu_1^t - \mu_2^t\|_{\mathcal{M}, T}.
with $K_T$ depending continuously on $H, A, \hat{H}, \propto$ and the norm $\|S_T\|_{C^1(CP^n)}$. For sufficiently small $T$, we get $\sqrt{T} K_T < 1$, and for this $T$ by the Banach contraction principle we derive the existence and uniqueness of the fixed point thus proving part (i) of the theorem.

(ii) Let us now consider $C_0([0, T], \mathcal{M}(CP^n))$ with a different topology. Namely, the distance in $\mathcal{M}(CP^n)$ will be defined from the space $(C^1(CP^n))^*$, dual to the space of smooth functions on $\mathcal{M}(CP^n)$. This distance defines the weak topology, and therefore $C_0([0, T], \mathcal{M}(CP^n))$ becomes a closed convex subset of the set of continuous functions from $[0, T]$ to the compact space of probability measures on the compact space $CP^n$. The mapping $\Phi_1(\mu) \cdot$ from part (i) clearly maps $C_0([0, T], \mathcal{M}(CP^n))$ into itself. To deduce the existence of a fixed point from the Schauder fixed point principle we have to show that the image of $\Phi_1$ is compact in $C_0([0, T], \mathcal{M}(CP^n))$. But as seen directly from the definition of $\Phi$ this image consists of uniformly 1/2-Hölder curves, and such curves form a compact set due to the Arzela theorem. □

9. Conclusion. We have developed a new framework for studying the control problems of a continuously observed system of large number of interacting quantum particles, by developing quantum analog for the theory of mean-field games. By-passing we proved a rigorous convergence result deriving a limiting nonlinear stochastic equation for individual particles from a large stochastic system of interacting particles.

Let us point out to some questions and open problems arising from this developments.

We developed the theory on the assumption of the existence of solutions to the limiting MFG problems. The existence was proved only for finite-dimensional state spaces of a single particle (and only for special homodyne arrangements). The method was based on the forward-backward system on manifolds. For the standard infinite-dimensional cases such systems would become systems of equations in variational derivatives. Therefore possible other methods (like stochastic Pontryagin maximum principle) can be used here to establish existence of solutions to the limiting MFG problem.

Finally, the development of numerical schemes for solving forward-backward MFG systems on manifolds would be of interest. They can be based, for instance, on some extensions of the technique from [51].

APPENDIX

PROPOSITION A.1. Let $\phi_1, \ldots, \phi_N$ be arbitrary $N$ unit vectors (not necessary different) in a separable Hilbert space $\mathcal{H}$ and $\gamma$ a positive operator in $\mathcal{H} \otimes \mathcal{N}$ of trace one. For any subset $I \subset \{1, \ldots, N\}$ let $\phi_I = \bigotimes_{i \in I} \phi_i$ and $\gamma^I$ be the partial trace of $\gamma$ with respect to all variables outside $I$. We shall write simply $\gamma^j$ for $\gamma^{\{j\}}$.

Let $E^I = 1 - (\phi_I, \gamma^I \phi_I)$ and we shall write simply $E^j$ for $E^{\{j\}}$.

Then, for any $i \in I$,

$$E^I \leq E^i + E^{I \setminus i}. \quad (81)$$

PROOF. For any $I \subset \{1, \ldots, N\}$ let us fix some orthonormal basis $\Phi^I_j, j \geq 1$, in the Hilbert space $\mathcal{H} \otimes |I|$ such that $\Phi^I_1 = \phi_I$.

We have

$$(\phi_I, \gamma^I \phi_I) = \sum_{j \geq 1} (\phi_i \otimes \Phi^I_{j \setminus i}, \gamma^I \phi_i \otimes \Phi^I_{j \setminus i}) - \sum_{j \geq 2} (\phi_i \otimes \Phi^I_{j \setminus i}, \gamma^I \phi_i \otimes \Phi^I_{j \setminus i}).$$

Notice that

$$\sum_{j \geq 1} (\phi_i \otimes \Phi^I_{j \setminus i}, \gamma^I \phi_i \otimes \Phi^I_{j \setminus i})$$
\[
E^I - E^i = (\phi_i, \gamma^i \phi_i) - (\phi_I, \gamma^I \phi_I)
\]
\[
= \sum_{j \geq 2} (\phi_i \otimes \Phi^j_{\setminus i}, \gamma^i \phi_i \otimes \Phi^j_{\setminus i})
\]
\[
\leq \sum_{k \geq 1} \sum_{j \geq 2} (\Phi_k \otimes \Phi^j_{\setminus i}, \gamma^i \Phi_k \otimes \Phi^j_{\setminus i})
\]
\[
= \sum_{k \geq 1} \sum_{j \geq 1} (\Phi_k \otimes \Phi^j_{\setminus i}, \gamma^i \Phi_k \otimes \Phi^j_{\setminus i})
- \sum_{k \geq 1} (\Phi_k \otimes \Phi^j_{\setminus i}, \gamma^i \Phi_k \otimes \Phi^j_{\setminus i})
\]
\[
= 1 - (\phi_{I \setminus i}, \gamma^i \phi_{I \setminus i}) = E^I_{\setminus i},
\]

where \(1^I_{\setminus i}\) is of course the unit operator in the Hilbert space \(H^\otimes(|I| - 1)\).

Therefore
\[
E^I - E^i = (\phi_i, \gamma^i \phi_i) - (\phi_I, \gamma^I \phi_I)
\]
\[
\leq \sum_{k \geq 1} \sum_{j \geq 2} (\Phi_k \otimes \Phi^j_{\setminus i}, \gamma^i \Phi_k \otimes \Phi^j_{\setminus i})
\]
\[
= \sum_{k \geq 1} \sum_{j \geq 1} (\Phi_k \otimes \Phi^j_{\setminus i}, \gamma^i \Phi_k \otimes \Phi^j_{\setminus i})
- \sum_{k \geq 1} (\Phi_k \otimes \Phi^j_{\setminus i}, \gamma^i \Phi_k \otimes \Phi^j_{\setminus i})
\]
\[
= 1 - (\phi_{I \setminus i}, \gamma^i \phi_{I \setminus i}) = E^I_{\setminus i},
\]

implying (81). □

Though we do not need it, let us mention that a straightforward extension of the argument above shows the following “convexity property” of the function \(E^I\):

\[(82)\]
\[
E^I \leq E^J + E^I_{\setminus J}
\]
for any \(J \subset I \subset \{1, \ldots, N\}\).

The remaining part of this appendix is devoted to the following result, which represents a slight modification of the arguments from [35].

**Proposition A.2.** Under the setting of Theorem 3.1, the second term on the r.h.s. of (49) enjoys the estimate

\[(83)\]
\[
\frac{1}{N} \mathbb{E} \sum_{m \neq j} |(\Psi_{N,t}, \gamma_{j,t} \gamma_{m,t} A_{mjq_{j,t}q_{j,t}} \Psi_{N,t})| \leq 2 \|A\|_{HS} \left( \mathbb{E} a_{N,t} + \frac{1}{N} \right).
\]

**Proof.** Let us systematically omit the index \(t\) in all calculations below. By symmetry it is sufficient to show that

\[(84)\]
\[
\mathbb{E} |(\Psi_N, \gamma_1 \gamma_2 A_{12q_1q_2} \Psi_N)| \leq 2 \|A\|_{HS} \sqrt{N/(N-1)} (\mathbb{E} a_N + 1/N).
\]

Recall the definition of the average operator (38):

\[
\hat{m} = \hat{m}_N = \frac{1}{N} \sum_{j=1}^N q_j.
\]

The key objects in the method are the projectors

\[
P_k = \sum_{I \subset \{1, \ldots, N\}: |I| = k} \prod_{i \in I} q_i \prod_{i \notin I} \gamma_i,
\]
for $k \in \{0, \ldots, N\}$ and $P_k = 0$ for other integer $k$.

Clearly $P_k$ are orthogonal projectors such that

$$P_k P_l = \delta_k^l P_k, \quad \sum_k P_k = 1,$$

and

$$\hat{m} = \frac{1}{N} \sum_{j=1}^{N} q_j = \frac{1}{N} \sum_{j=1}^{N} \sum_{k=0}^{N} q_j P_k = \frac{1}{N} \sum_{k=0}^{N} k P_k,$$

because the composition of $q_j$ with each term of the sum entering the expression for $P_k$ either coincides with this term or equals zero.

For functions $f : \mathbb{Z} \to \mathbb{C}$ with support contained in $\{0, \ldots, N\}$ one defines

$$\hat{f} = \sum_k f(k) P_k.$$  \hfill (85)

The mapping $f \to \hat{f}$ is an algebraic homomorphism in the sense that $\hat{f} g = \hat{f} \hat{g}$. Thus $\hat{m}$ arises from the function $m(k) = k/N$.

Of importance are the powers of this operator:

$$\hat{m}^j = \sum_{k=0}^{N} \left( \frac{k}{N} \right)^j P_k$$  \hfill (86)

for any $j \geq 0$. It follows from the properties of the projectors that

$$\hat{m}^j \hat{m}^l = \hat{m}^{j+l}$$

for all $l, j \geq 0$. Since $\hat{m}$ is not invertible, the inverse power is not defined. However, since

$$\hat{m} P_k = \frac{k}{N} P_k,$$

for any $k$, $\hat{m}^{-1}$ can be defined on the image of $1 - P_0$. Let us thus denote (with some abuse of notation)

$$\hat{m}^{-j} = \sum_{k=1}^{N} \left( \frac{k}{N} \right)^{-j} P_k,$$  \hfill (87)

for any $j > 0$. It is seen directly that

$$\hat{m}^j \hat{m}^{-j} = \hat{m}^{-j} \hat{m}^j = 1 - P_0$$

and thus

$$\hat{m}^j \hat{m}^{-j} q_i = q_i$$  \hfill (88)

for all $i \in \{1, \ldots, N\}$ and $j \neq 0$.

Let us introduce the operators

$$P_{k}^{r} = \sum_{I \subseteq \{r+1, \ldots, N\} : |I| = k} \prod_{i \in I} q_i \prod_{i \notin I} \gamma_i$$

setting $P_k^r = 0$ if $k \notin \{0, \ldots, N - r\}$. These operators appear by composing $\gamma_i$ and $q_i$ with $P_k$. For instance,

$$\gamma_1 \cdots \gamma_r P_k = P_k^r, \quad k \leq N - r,$$

and $\gamma_1 \cdots \gamma_r P_k = 0$ otherwise.
More generally, let \( Q_r = \omega_1 \cdots \omega_r \), where each \( \omega_l \) is either \( q_l \) or \( \gamma_l \). By direct inspection one checks that

\[
(91) \quad Q_r P_k = P_k Q_r = Q_r P_k^{r-n},
\]

where \( n \) is the number of \( q_s \) in the product \( Q_r \). In particular, both sides vanish if \( k \notin \{n, \ldots, N+n-r\} \).

The key property of \( Q_r \) is given by the following auxiliary result.

**Lemma A.1.** Let two products \( Q_r^1 \) and \( Q_r^2 \) be given with the numbers \( n_1 \) and \( n_2 \) of \( q_s \) respectively, and \( A_r \) an operator acting on the first \( r \) variables in \( L^2(\mathbb{X}^N) \). Then

\[
(92) \quad Q_r^1 A_r \hat{f} Q_r^2 = Q_r^1 \tau_n A_r Q_r^2,
\]

where \( \tau_n f(k) = f(k+n) \) and \( n = n_2 - n_1 \).

**Proof.** For any \( Q_r \),

\[
Q_r \hat{f} = \hat{f} Q_r = \sum_k f(k) Q_r P_k = \sum_k f(k) Q_r P_k^{r-n} = \sum_k f(k) P_k^{r-n} Q_r.
\]

Since \( A_r \) and \( P_k^{r-n} \) commute and using \( Q_r^1 P_{k+n_1} = Q_r^1 P_k^{r-n} \), it follows that

\[
Q_r^1 A_r \hat{f} Q_r^2 = Q_r^1 A_r \sum_k f(k) P_k^{r-n_2} Q_r^2 = Q_r^1 A_r \sum_k f(k + n_2) P_k^{r-n_2} Q_r^2 = Q_r^1 \sum_k f(k + n_2) P_k A_r Q_r^2 = Q_r \sum_k f(k + n_2 - n_1) P_k A_r Q_r^2
\]

as required. \( \square \)

All tools are now ready to get (83). The key idea is to write, using (88) and Lemma A.1, that

\[
(\Psi_N, \gamma_1 \gamma_2 A_{12} q_1 q_2 \Psi_N) = (\Psi_N, \gamma_1 \gamma_2 A_{12} \hat{m}^{1/2} \hat{m}^{-1/2} q_1 q_2 \Psi_N) \\
\times (\Psi_N, \gamma_1 \gamma_2 \tau_2 m^{1/2} A_{12} \hat{m}^{-1/2} q_1 q_2 \Psi_N)
\]

yielding the estimate

\[
(90) \quad |(\Psi_N, \gamma_1 \gamma_2 A_{12} q_1 q_2 \Psi_N)| \leq \|A_{12} \tau_2 m^{1/2} p_1 p_2 \Psi\| \|\hat{m}^{-1/2} q_1 q_2 \Psi_N\|,
\]

and therefore

\[
(91) \quad E|(\Psi_N, \gamma_1 \gamma_2 A_{12} q_1 q_2 \Psi_N)| \leq (E\|A_{12} \tau_2 m^{1/2} p_1 p_2 \Psi\|^2)^{1/2} (E\|\hat{m}^{-1/2} q_1 q_2 \Psi_N\|^2)^{1/2}.
\]

To estimate the second term in (91) we write

\[
E\|\hat{m}^{-1/2} q_1 q_2 \Psi_N\|^2 = E(\Psi_N, \hat{m}^{-1/2} q_1 q_2 \Psi_N) = \frac{1}{N(N-1)} E(\Psi_N, \sum_{l \neq r} \hat{m}^{-1} q_l q_r \Psi_N) \\
\leq \frac{1}{N(N-1)} E(\Psi_N, \sum_{l,r} \hat{m}^{-1} q_l q_r \Psi_N) = \frac{N}{N-1} E(\Psi_N, \hat{m}^{-1} \hat{m}^2 \Psi_N) \\
= \frac{N}{N-1} E(\Psi_N, \hat{m} \Psi_N) \leq \frac{N}{N-1} E\alpha_N.
\]
For the first term in (90) we can write
\[ E\|A_12\tau_2m^{1/2}p_1p_2\psi\|^2 = E(\Psi_N, \tau_2m^{1/2}p_1p_2A_12p_1p_2\tau_2m^{1/2}\Psi_N) \]
\[ \leq \|A\|^2_{HS}E\|\tau_2m^{1/2}\Psi_N\|^2 = \|A\|^2_{HS}(\Psi_N, \tau_2m\Psi_N) \]
\[ \leq \|A\|^2_{HS}\left[E(\Psi_N, \hat{m}\Psi_N) + \frac{2}{N}\right] = \|A\|^2_{HS}\left[E\alpha_N + \frac{2}{N}\right], \]
where (39) was used and because
\[ \tau_2\hat{m} = \sum_{k=0}^{N} \frac{k+2}{N} P_k = \hat{m} + \frac{2}{N} \sum_{k=0}^{N} P_k = \hat{m} + \frac{2}{N}. \]

Putting these estimates together we get that
\[ E|\langle\Psi_N, \gamma_1\gamma_2 A_12q_1q_2\psi_N\rangle| \leq \|A\|_{HS}\sqrt{N/(N-1)}\sqrt{E\alpha_N(E\alpha_N + 2/N)} \]
\[ \leq 2\|A\|_{HS}\sqrt{N/(N-1)}(E\alpha_N + 1/N), \]
as required. \(\square\)

Acknowledgments. The author is also affiliated with Higher School of Economics RF, and this article was prepared within the framework of the HSE University Basic Research Program.

The author is grateful to the anonymous referees for carefully reading the first draft of the manuscript and making lots of useful comments and suggestions.

REFERENCES


