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A Metric Rendition of Subadditive Probability

David Miller
Department of Philosophy
University of Warwick
COVENTRY CV4 7AL UK
e-mail: dwmiller57@yahoo.com

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ABSTRACT

It is known that the usual axioms of probability on a Boolean algebra are interderivable with a set of axioms for a pseudometric operation on that algebra. A set of axioms for subadditive probability is here demonstrated to be interderivable with a suitably weakened set of pseudometric axioms.

0 Introduction

Throughout this paper, a, b, c, \dots are elements of the domain \mathbf{B} of an arbitrary Boolean lattice $\mathfrak{B} = \langle \mathbf{B}, \leq \rangle$ with zero \perp , unit \top , meet \cdot , join $+$, and complement $'$. The operation Δ of *symmetric difference* (exclusive disjunction) may be defined on the elements of \mathbf{B} by either of the identities $a \Delta c = a \cdot c' + c \cdot a'$ and $a \Delta c = (a + c) \cdot (c' + a')$. Its simpler properties, and the other Boolean identities and inequalities called on, may be checked by truth tables.

An (unnormalized) *measure* or *valuation* on \mathfrak{B} is a real-valued function \mathfrak{p} obeying the axioms

- (0) $0 = \mathfrak{p}(\perp) \leq \mathfrak{p}(b)$
- (1) $\mathfrak{p}(a + c) + \mathfrak{p}(a \cdot c) = \mathfrak{p}(a) + \mathfrak{p}(c)$.

It is well known that the axiom system for the elementary theory of probability introduced by Kolmogorov (1933), Chapter I, § 1, is obtained by adding to these axioms the further requirement that $\mathfrak{p}(\top) = 1$, but this normalization will not be relevant to anything that follows here.

It is also reasonably well known, and was established in detail in Miller (1984), that the extended set \mathbb{A} that follows, which contains also a definition (A0) of the operator \mathfrak{d} ,

- (A0) $\mathfrak{d}(a, c) =_{\text{Df}} \mathfrak{p}(a \Delta c)$
- (A1) $0 = \mathfrak{p}(\perp) \leq \mathfrak{p}(b)$
- (A2) $\mathfrak{p}(a + c) + \mathfrak{p}(a \cdot c) = \mathfrak{p}(a) + \mathfrak{p}(c)$,

Most of the results of this paper date from around 1979–1981.

is logically equivalent to the following set \mathbb{C} of axioms for a pseudometric operation \mathfrak{d} :

$$\begin{aligned} (\mathbb{C}0) \quad & \mathfrak{p}(b) =_{\text{Df}} \mathfrak{d}(b, \perp) \\ (\mathbb{C}1) \quad & \mathfrak{d}(a, c) \leq \mathfrak{d}(b, a) + \mathfrak{d}(b, c) \\ (\mathbb{C}2) \quad & \mathfrak{d}(b + a, b + c) + \mathfrak{d}(b \cdot a, b \cdot c) \leq \mathfrak{d}(a, c), \end{aligned}$$

which contains a corresponding definition (C0) of the operator \mathfrak{p} in terms of \mathfrak{d} .

Three easily derived consequences, of the system \mathbb{A} state the *monotony* of the measure \mathfrak{p} (with respect to the ordering \leq on \mathfrak{B}), a special type of *subadditivity*, and its *complementarity*:

$$\begin{aligned} (\mathbb{A}3) \quad & \mathfrak{p}(a) \leq \mathfrak{p}(a + c) \\ (\mathbb{A}4) \quad & \mathfrak{p}(a + c) \leq \mathfrak{p}(a) + \mathfrak{p}(c) \\ (\mathbb{A}5) \quad & \mathfrak{p}(a) + \mathfrak{p}(a') = \mathfrak{p}(\top). \end{aligned}$$

PROOF The required proofs are familiar and undemanding. Since $a \cdot (c \cdot a') = \perp$ and $a + (c \cdot a') = a + c$ for all $a, c \in \mathfrak{B}$, it follows from (A2) and the first part of (A1) that $\mathfrak{p}(a + c) = \mathfrak{p}(a) + \mathfrak{p}(c \cdot a')$, from which (A3) follows by use of the second part of (A1), and (A5) follows by putting $c = \top$. What is more, $c \cdot a' \leq c \cdot a' + c \cdot a = c$, and so by (A3) and what has been proved, $\mathfrak{p}(a + c) = \mathfrak{p}(a) + \mathfrak{p}(c \cdot a') \leq \mathfrak{p}(a) + \mathfrak{p}(c)$, which yields (A4). ■

The set $\mathbb{A}^* = \{(\mathbb{A}0), (\mathbb{A}3), (\mathbb{A}4), (\mathbb{A}5)\}$ is strictly weaker than the set \mathbb{A} , for it admits interpretations in which $\mathfrak{p}(a + c) = \mathfrak{p}(a) + \mathfrak{p}(c)$ for all $a, c \in \mathfrak{B}$. It should be noted that, like \mathbb{A} , it admits interpretations in which \mathfrak{p} is identically 0.

From the system \mathbb{C} we can also derive with ease the following two inequalities concerning \mathfrak{d} :

$$\begin{aligned} (\mathbb{C}3) \quad & \mathfrak{d}(b + a, b + c) \leq \mathfrak{d}(a, c) \\ (\mathbb{C}4) \quad & \mathfrak{d}(a, c) + \mathfrak{d}(a, c') \leq \mathfrak{d}(c, c'). \end{aligned}$$

PROOF It follows from suitable substitutions in (C1) and (C2) that $\mathfrak{d}(b, b) \leq 2\mathfrak{d}(b, b) \leq \mathfrak{d}(b, b)$, and hence that $\mathfrak{d}(b, b) = 0$; and then, from two further applications of (C1), that $0 \leq \mathfrak{d}(c, a) = \mathfrak{d}(a, c)$. (C3) now follows from (C2). By (C1), (C2), and (C1) again,

$$\begin{aligned} \mathfrak{d}(\top, \perp) &\leq \mathfrak{d}(c, \top) + \mathfrak{d}(c, \perp) = \mathfrak{d}(c + c, c + c') + \mathfrak{d}(c \cdot c, c \cdot c') \\ &\leq \mathfrak{d}(c, c') \leq \mathfrak{d}(a, c) + \mathfrak{d}(a, c') \leq \mathfrak{d}(\top, \perp), \end{aligned}$$

by writing \top for a and \perp for c . The identity of the two sides of (C4) now follows. ■

To obtain the set \mathbb{C}^* we add to (C3) and (C4) the definition (C0) of \mathfrak{p} in terms of \mathfrak{d} and the triangle inequality (C1).

In the next two sections we shall show that the sets \mathbb{A}^* and \mathbb{C}^* are logically equivalent.

1 Proof of the pseudometric axioms \mathbb{C}^* from the measure axioms \mathbb{A}^*

In the present section the four axioms (A0), (A3), (A4), (A5), of \mathbb{A}^* are assumed.

LEMMA 1.0 $\mathfrak{p}(b) = \mathfrak{d}(b, \perp)$.

PROOF This follows immediately from (A0), given that $b \triangle \perp = b$. ■

LEMMA 1.1 $\mathfrak{d}(a, c) \leq \mathfrak{d}(b, a) + \mathfrak{d}(b, c)$.

PROOF Since $(b \triangle a) + (b \triangle c) = (a \triangle c) + ((b \triangle a) + (b \triangle c))$, we have from (A3) and (A4) that $\mathfrak{p}(a \triangle c) \leq \mathfrak{p}((b \triangle a) + (b \triangle c)) \leq \mathfrak{p}(b \triangle a) + \mathfrak{p}(b \triangle c)$. By (A0), $\mathfrak{d}(a, c) \leq \mathfrak{d}(b, a) + \mathfrak{d}(b, c)$. ■

LEMMA 1.2 $\mathfrak{d}(b + a, b + c) \leq \mathfrak{d}(a, c)$.

PROOF It follows from (A3) that $\mathfrak{p}(a \cdot c) \leq \mathfrak{p}(a)$. Since $(b + a) \triangle (b + c) = b' \cdot (a \triangle c)$, we may conclude by (A0) that $\mathfrak{d}(b + a, b + c) \leq \mathfrak{d}(a, c)$. ■

LEMMA 1.3 $\mathfrak{d}(a, c) + \mathfrak{d}(a, c') \leq \mathfrak{d}(c, c')$.

PROOF By (A5), $\mathfrak{p}(a \triangle c) + \mathfrak{p}(a \triangle c') = \mathfrak{p}(a \triangle c) + \mathfrak{p}((a \triangle c)') = \mathfrak{p}(\top) = \mathfrak{p}(c \triangle c')$. The result follows by (A0). ■

It may be noted that (A5) is needed only for the proof of (C4) in Lemma 1.3. Matters are quite different in the converse direction, where (C4) plays a central role throughout.

2 Proof of the measure axioms \mathbb{A}^* from the pseudometric axioms \mathbb{C}^*

In the present section the four axioms (C0), (C1), (C3), (C4), of \mathbb{C}^* are assumed.

LEMMA 2.0 $0 = \mathfrak{d}(b, b) \leq \mathfrak{d}(a, c) = \mathfrak{d}(c, a) = \mathfrak{d}(a', c') \leq \mathfrak{d}(b, b')$.

PROOF It follows immediately from (C1) that $\mathfrak{d}(b, b) \leq 2\mathfrak{d}(b, b)$, and hence that $0 \leq \mathfrak{d}(b, b)$ for any $b \in \mathbb{B}$. Writing b for both a and c in (C4), we see that $\mathfrak{d}(b, b) \leq 0$. This proves that $\mathfrak{d}(b, b) = 0$. It follows from (C1) that $\mathfrak{d}(a, c) \leq \mathfrak{d}(c, a)$, and therefore that $\mathfrak{d}(a, c) = \mathfrak{d}(c, a)$ for every $a, c \in \mathbb{B}$. It follows also from (C1) that $\mathfrak{d}(c, c) \leq 2\mathfrak{d}(a, c)$, and hence that $\mathfrak{d}(b, b) \leq \mathfrak{d}(a, c)$.

To conclude we note first that (C1) allows (C4) to be strengthened to an identity, and so

$$\begin{aligned} 2\mathfrak{d}(c, c') &= \mathfrak{d}(a, c) + \mathfrak{d}(a, c') + \mathfrak{d}(a', c) + \mathfrak{d}(a', c') = \\ &\mathfrak{d}(a, c) + \mathfrak{d}(a', c) + \mathfrak{d}(a, c') + \mathfrak{d}(a', c') = 2\mathfrak{d}(a, a'), \end{aligned}$$

by the commutative property of \mathfrak{d} already proved. We obtain $\mathfrak{d}(c, c') = \mathfrak{d}(a, a')$ for every $a, c \in \mathbb{B}$. A further application of (C4) completes the proof that $\mathfrak{d}(a, c) \leq \mathfrak{d}(b, b')$. It now follows from (C4) that $\mathfrak{d}(a, c) + \mathfrak{d}(a, c') = \mathfrak{d}(a, c') + \mathfrak{d}(a', c')$, and accordingly that $\mathfrak{d}(a, c) = \mathfrak{d}(a', c')$. ■

COROLLARY 2.0.0 $\mathfrak{d}(a, c) + \mathfrak{d}(a, c') = \mathfrak{d}(b, b') = \mathfrak{d}(a, d) + \mathfrak{d}(a', d)$.

PROOF Immediate. ■

LEMMA 2.1 $\mathfrak{d}(a, c) = \mathfrak{p}(a \triangle c)$.

PROOF By (C3), $\mathfrak{d}(b' + a', b' + c') \leq \mathfrak{d}(a', c')$, and hence by Lemma 2.0,

$$(C5) \quad \mathfrak{d}(b \cdot a, b \cdot c) \leq \mathfrak{d}(a, c).$$

Since $(a \cdot c) + (a \triangle c) = a + c$, it follows from (C3) again that $\mathfrak{d}(a + c, a \cdot c) = \mathfrak{d}(a \cdot c + (a \triangle c), a \cdot c + \perp) \leq \mathfrak{d}(a \triangle c, \perp)$. Since $(a \triangle c) \cdot (a + c) = a \triangle c$, and $(a \triangle c) \cdot (a \cdot c) = \perp$, it follows from (C5) also that $\mathfrak{d}(a \triangle c, \perp) = \mathfrak{d}((a \triangle c) \cdot (a + c), (a \triangle c) \cdot (a \cdot c)) \leq \mathfrak{d}(a + c, a \cdot c)$. We may conclude that $\mathfrak{d}(a \triangle c, \perp) = \mathfrak{d}(a + c, a \cdot c)$.

By Corollary 2.0.0, $\mathfrak{d}(a \triangle c, \perp) + \mathfrak{d}(a' \triangle c, \perp) = \mathfrak{d}(a + c, a \cdot c) + \mathfrak{d}(a' \cdot c', a \cdot c)$, from which it follows that $\mathfrak{d}(a' \triangle c, \perp) = \mathfrak{d}(a' \cdot c', a \cdot c)$. By writing a' for a in this equation, we may conclude that $\mathfrak{d}(a \triangle c, \perp) = \mathfrak{d}(a \cdot c', a' \cdot c)$. It now follows from (C3), (C5), and some Boolean identities that $\mathfrak{d}(a, c) = \mathfrak{d}((a \cdot c) + (a \cdot c'), (a \cdot c) + (a' \cdot c)) \leq \mathfrak{d}(a \cdot c', a' \cdot c) = \mathfrak{d}((a \triangle c) \cdot a, (a \triangle c) \cdot c) \leq \mathfrak{d}(a, c)$. This completes the proof that $\mathfrak{d}(a, c) = \mathfrak{d}(a \triangle c, \perp)$, and hence, by (C0), that (A0) holds. ■

COROLLARY 2.1.0 $\mathfrak{d}(a, c) = \mathfrak{d}(a + c, a \cdot c)$.

PROOF Immediate. ■

LEMMA 2.2 $\mathfrak{p}(a) \leq \mathfrak{p}(a + c)$.

PROOF Suppose that $c \leq b \leq a$. It follows at once from (C5) that $\mathfrak{d}(b, c) \leq \mathfrak{d}(a, c)$. Since $\perp \leq a \leq a + c$, we may conclude that $\mathfrak{d}(a, \perp) \leq \mathfrak{d}(a + c, \perp)$. The result follows by (C0). ■

LEMMA 2.3 $\mathfrak{p}(a + c) \leq \mathfrak{p}(a) + \mathfrak{p}(c)$.

PROOF By (C1) and (C3), $\mathfrak{d}(a + c, \perp) \leq \mathfrak{d}(a, a + c) + \mathfrak{d}(a, \perp) = \mathfrak{d}(a + \perp, a + c) + \mathfrak{d}(a, \perp) \leq \mathfrak{d}(\perp, c) + \mathfrak{d}(a, \perp)$. The result follows from the commutativity of \mathfrak{d} (Lemma 2.0) and (C0). ■

LEMMA 2.4 $\mathfrak{p}(a) + \mathfrak{p}(a') = \mathfrak{p}(\top)$.

PROOF By Corollary 2.0.0, $\mathfrak{d}(a, \perp) + \mathfrak{d}(a', \perp) = \mathfrak{d}(\top, \perp)$. The result follows by (C0). ■

3 Conclusion

Axiom (A4) is only one way of formulating the subadditivity of a measure \mathfrak{p} , and by no means obviously the best. Other formulations are

- (2) $\mathfrak{p}(a + c) + \mathfrak{p}(a \cdot c) \leq \mathfrak{p}(a) + \mathfrak{p}(c)$
- (3) $a \cdot c = \perp \rightarrow \mathfrak{p}(a + c) \leq \mathfrak{p}(a) + \mathfrak{p}(c)$.

It is clear that (2) is at least as strong as (A4), which itself is at least as strong as (3). It is not known whether there are attractive sets of pseudometric axioms that are equivalent to $\{(\text{A0}), (\text{A3}), (2), (\text{A5})\}$ or to $\{(\text{A0}), (\text{A3}), (3), (\text{A5})\}$.

References

- Kolmogorov, A. N. (1933). ‘Grundbegriffe de Wahrscheinlichkeitsrechnung’. *Ergebnisse der Mathematik und ihrer Grenzgebiete* 3. Berlin: Springer. English translation by Nathan Morri-son, 1950. *Foundations of the Theory of Probability*. New York: Chelsea Publishing Company. 2nd edition 1956. <http://www.mathematik.com/Kolmogorov/index.html/>.
- Miller, D. W. (1984). ‘A Geometry of Logic’. In H. J. Skala, S. Termini, & E. Trillas, editors (1984), pp.91–104. *Aspects of Vagueness*. Dordrecht: D. Reidel Publishing Company