Effects of Tobin Taxes in Minority Game Markets

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We show that the introduction of Tobin taxes in agent-based models of currency markets can lead to a reduction of speculative trading and reduce the magnitude of exchange rate fluctuations at intermediate tax rates. In this regime revenues for the market maker obtained from speculators are maximal. We here focus on Minority Game models of markets, which are accessible by exact techniques from statistical mechanics. Results are supported by computer simulations. Our findings suggest that at finite systems sizes the effect is most pronounced in a critical region around the phase transition of the infinite system, but much weaker if the market is operating far from criticality and does not exhibit anomalous fluctuations.

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I. INTRODUCTION

In 1972 James Tobin proposed to “throw some sand in the wheels of our excessively efficient international money markets” [2] by imposing a tax of 0.05 to 0.5% on all international currency transactions. The Bretton Woods agreement – a system of fixed foreign exchange rates tied to the price of gold – at that time was gradually being dismantled, with the USA stepping out in 1971. This system had been introduced in the wake of World War II in order to rebuild global capitalism. Tobin feared the effects of countries exposed to freely fluctuating exchange rates and suggested, as a second best solution, the introduction of what is now called a ‘Tobin tax’ in order to suppress speculative behaviour and thus containing exchange rates volatility within tenable levels.

Since then, under floating exchange rates, the trading volume on international currency markets has grown sharply, especially after the introduction of electronic trading, reaching a level of 1.5 trillion US-Dollar per day in 2000 [1]. Most of these transactions are effected on time-scales of less than seven days, more than 40% involve round-trips within two days or less, and 90% are of speculative nature, only one tenth are carried out within the production sector [2].

While the Tobin tax has never been implemented in reality the discussion of this issue is still lively and opinions are widely divided between proponents who claim that a Tobin tax would improve the situation of countries damaged by international currency speculation [3], and opponents who reject the proposal on various grounds. They claim that its implementation would hardly be feasible and extremely expensive, that it would mostly damage developing countries and that through a reduction of market liquidity, Tobin taxes might indeed result in more, not less volatility [2].

The question of whether a tax on financial transactions reduces volatility or not is indeed a subtle one. On the one hand excess volatility is related to trading volume, hence reducing the currency trading activity through the introduction of a tax on currency transactions can be expected to reduce volatility. On the other hand speculators provide liquidity and eliminate market inefficiencies, hence volatility might increase when a tax is levied and speculators drop out of the market due to the increasing trading costs. The latter consideration is particularly relevant given that the margin on which speculators live is extremely small and even a 0.5% tax could turn their marginal gains negative.

In order to make these considerations quantitative some authors have investigated the effects of Tobin taxes in agent-based models of financial markets [4, 6, 7]. These model are capable to reproduce typical features of price fluctuations in real markets – so-called stylized facts [3] – and are therefore well suited to shed light on the effects of introducing a Tobin tax. In the context of zero-intelligence percolation models [8], Ehrenstein and co-workers found that generally the introduction of Tobin tax brings about a reduction in volatility, as long as the tax rate is not too high to cause liquidity problems [4, 6]. Westerhoff [5] comes to similar conclusions building on the Chiarella-Hommes approach of modeling financial markets with systems of heterogeneous agents [10]. At variance with previous models, this study introduces considerations of agents’ rationality through heuristic expectation models of future returns of a chartist or fundamentalist nature.

The present paper addresses similar general questions. In contrast with previous works, however, we take the view of a financial market as an ecology of different types of agents interacting along an ‘information food chain’. In our picture speculators ‘predate’ on market inefficiencies (arbitrage opportunities) created by other investors (the so-called producers). This approach has been recently formalized in the Minority Game (MG) [11, 12, 13]. The MG captures the interplay between

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volatility and market efficiency in a vivid though admit-
tedly simplified and stylized way. Indeed the analysis
of the MG has revealed that within this model frame-
work excess volatility and market efficiencies are iden-
tified as two sides of the same coin, both resulting as
consequences of speculative trading. The MG exhibits
two different regimes, one in which the market is fully
efficient and another in which arbitrage opportunities are
not entirely eliminated by the dynamics of the agents.
These regimes are separated sharply in the parameter
space of the model, and it turns out that the boundary
at which the market becomes efficient coincides precisely
with the locus of a phase transition in the language of
statistical physics. At the same time critical fluctuations
very similar to the stylized facts observed in real mar-
ket data emerge in the vicinity of this transition but
not further away [14]. Hence, at odds with previous mod-
els, the MG might offer a perspective of understanding
how the introduction of transactions taxes affects the in-
formation ecology and market efficiency. It will here be
important to distinguish between regimes close to and
far away from the above transition between efficient and
inefficient phases of the market. A further advantage of
the MG over other more elaborate agent-based models
lies in its analytic tractability. Despite its stylized setup
the phenomenology of the MG is remarkably rich, but
at the same time the MG model can be understood
fully analytically with tools of statistical physics of disor-
dered systems [11, 12]. This analytical solution provides
an understanding of the model, which goes much deeper
than approaches to other models purely based on nu-
umerical simulations. In particular it is possible to derive
closed analytical expressions for key observables such as
the market volatility, the trading activity of the agents
and the revenue for the market maker.

In brief, our main result is that within the picture of
the MG model a small tax decreases volatility whenever
a finite market exhibits anomalous fluctuations. Indeed
the fundamental effect of the tax is to draw the market
away from its critical point. At the same time, the tax in-
roduces an information inefficiency and thus too high a
tax might not be advisable. The total revenue from the
tax exhibits a maximum for intermediate rates similar
to what was found in earlier works on different models
[4]. Furthermore, the effects of imposing a tax mate-
rialize in the market behaviour only after times which
scale inversely with the tax rate. Extremely small tax
rates may thus need a long time to stabilize turbulent
markets. This can be quite relevant if this time scale
becomes comparable to that over which market’s com-
position changes. In particular, if speculators leave and
enter the population of traders at a too fast rate a small
Tobin tax may fail to stabilize the market. Large taxes in
such a scenario reduce the time the population needs to
co-ordinate below the rate with which new agents enter
the market. Under such circumstances a transaction tax
may thus prevent high volatility states and reduce the
volatility significantly even in an infinite system.

In the following we shall first introduce the grand-
canonical MG (GCMG) and re-iterate its known main
features. In the main sections we then comment on how
a tax on transactions can be introduced and discuss the
effects on the market within the present model. We then
turn to a brief discussion on how to relate these results to
real markets, summarize our results and give some final
concluding remarks.

II. THE GRAND-CANONICAL MINORITY
GAME

A. The model

The so-called grand-canonical MG (GCMG) describes
a simple market of $N$ agents $i = 1, \ldots, N$ who at each
round of the game make a binary trading decision (to
buy or to sell) or who each may decide to refrain from
trading. They thus each submit bids $b_i(t) \in \{-1, 0, 1\}$
in every trading period $t = 0, 1, 2, \ldots$ resulting in a total
excess demand of $A(t) = \sum_{i=1}^{N} b_i(t)$.

These trading decisions are taken to be based on a
stream of information available to the agents. This com-
mon information on the state of the market (or on other
questions relevant to the market) is encoded in an integer
variable $\mu(t)$ at time $t$ taking values in $\{1, 2, \ldots, P\}$ [13].
Here we assume that $\mu(t)$ models an exogenous news ar-
ival process, and that the $\{\mu(t)\}$ are drawn at random
from the set $\{1, \ldots, P\}$ independently and with equal
probabilities at each time $t$. The objective of each
agent is to be in the minority at each time-step, i.e. to
place a bid $b_i(t)$ which has the opposite sign of the total
bid $A(t)$. This minority setup corresponds to contrarian
behaviour and can be derived from a market mechanism
taking into account the expectations of the traders on the
future behaviour of the market [17]. In order to do so,
each agent has a ‘trading strategy’ at his disposal. Trader
$i$’s strategy is labelled by $a_i = (a_i^\mu)_{\mu=1, \ldots, P} \in \{-1, 1\}^P$ and provides a map from all values of the information $\mu$
on the binary set $\{-1, 1\}$ of actions (buy/sell). Upon
receiving information $\mu$ the trading strategy of agent $i$
is thus prescribes to take the trading action $a_i^\mu \in \{-1, 1\}$.
These strategies are assigned at random and with no cor-
relations at the beginning of the game, and then remain
fixed [13]. Agents in the GCMG are adaptive and may
decide not to trade if they do not consider their strategy
adequate. More precisely, each agent keeps a score $u_i(t)$
measuring the performance of his strategy vector. He
then trades at a given time-step $t$ only if his strategy
has a positive score $u_i(t) > 0$ at that time. Therefore,
the bids of agents take the form $b_i(t) = n_i(t)a_i^\mu(t)$ with
$n_i(t) = 1$ if $u_i(t) > 0$ and $n_i(t) = 0$ otherwise. Accord-
ingly the excess demand is given by

$$A(t) = \sum_{i=1}^{N} n_i(t)a_i^\mu(t). \quad (1)$$
Agent $i$ keeps a record of the past performance of his strategy $a_i^\mu$ by updating the score $u_i(t)$ as follows

$$u_i(t+1) = u_i(t) - a_i^\mu A(t) - \varepsilon_i,$$

at each step, with constant $\varepsilon_i$. The first term $-a_i^\mu A(t)$ is the Minority Game payoff, it is positive whenever the trading action $a_i^\mu$ proposed by $i$’s strategy vector and the aggregate bid $A(t)$ are of opposite signs, and negative whenever $i$ joins the majority decision. The idea of Eq. (2) is that whenever the payoff $-a_i^\mu A(t)$ is larger than $\varepsilon_i$ the score of player $i$’s strategy is increased, otherwise it is decreased. The constant $\varepsilon_i$ in (2) thus captures the inclination of agent $i$ to trade in the market. This inclination will in general be heterogeneous across the population of agents, with agents with high values of $\varepsilon_i$ being more cautious to trade than agents with low $\varepsilon_i$. In our simplified model we only consider two types of agents. First we assume that there are $N_s \leq N$ speculators who trade only if their perceived market profit obtained by using their strategy exceeds a given threshold, and hence we set $\varepsilon_i = \epsilon \geq 0$ for such agents. Here $\epsilon$ can be considered as the speculative margin of gain in a single transaction.

The remaining $N_p = N - N_s$ agents – the so-called producers or institutional investors – are assumed to trade no matter what. Mathematically this is implemented by setting $\varepsilon_i = -\infty$ for this second group of agents. They have $n_i(t) = 1$ for all times $t$. For convenience we will order the agents such that speculators carry the indices $i = 1, \ldots, N_s$ and producers the labels $i = N_s+1, \ldots, N$.

**B. Price process, volatility and predictability**

Within the MG setup the market volatility is then given by

$$\sigma^2 = \frac{\langle A(t)^2 \rangle}{N},$$

where $\langle \ldots \rangle$ will stand for a time-average in the stationary state of the model from now on. The normalization to the number of agents $N$ is here introduced to guarantee a finite value of $\sigma^2$ in the infinite-system limit, with which the statistical mechanics analysis of the model is concerned.

The information variable $\mu(t)$ allows one to quantify information-efficiency of the model market by computing the predictability

$$H = \frac{1}{PN} \sum_{\mu=1}^P \langle A|\mu \rangle^2$$

where $\langle \ldots |\mu \rangle$ denotes an average conditional on the occurrence of information pattern $\mu(t) = \mu$. A value $H \neq 0$ indicates that for some $\mu$ the minority payoff is statistically predictable $\langle A(t)|\mu \rangle \neq 0$, whereas the market is unpredictable and fully efficient when $H = 0$.

The simplest way to relate this picture to a financial market is to postulate a simple linear impact of $A(t)$ on the (logarithmic) price (or the exchange rate), i.e to assume that

$$p(t+1) = p(t) + \frac{A(t)}{\lambda},$$

where $\{p(t)\}$ denotes a price (exchange rate) process and where $\lambda$ is the liquidity. In a derivation of Eq. (5) from a market clearing condition $\lambda$ turns out to be inversely proportional to the number of active traders $\sum_i n_i(t)$ [17]. In the following we set the liquidity to $\lambda = \sqrt{N}$, so that $\sigma^2$ becomes the volatility of the price process, $\sigma^2 = \langle (p(t+1) - p(t))^2 \rangle$. It is found simulations that the effects of introducing a Tobin tax on the behaviour of the model are qualitatively similar for either choice of $\lambda$, so that we will stick with the technically more convenient first definition.

**C. The behaviour of the GCMG**

The GCMG has been studied in great detail in [14, 19] with methods well established in statistical mechanics. The analysis is here generally concerned with the stationary states of the system, i.e. the behaviour which is reached after running the learning dynamics of the agents for some sufficiently long transient equilibration time.

The statistical mechanics approach provides exact results for the model in the limit of infinite market sizes, where one takes the number of agents $N = N_s + N_p$ and the number $P$ of possible different information states to infinity, while at the same time keeping the ratios
\( n_s = N_s/P \) and \( n_p = N_p/P \) fixed and finite. \( n_s \) and \( n_p \) along with \( \varepsilon \) are thus control parameters of the model. This approach makes it possible to derive exact expressions for several quantities, including the predictability \( H \) and upon neglecting time-dependent correlations accurate approximations for the volatility \( \sigma^2 \) can be found. We will here not enter the detailed mathematics of the calculations, the resulting equations for the key quantities in the stationary states as well as a sketch of their derivation are found in the appendix. Further details regarding the statistical mechanics analysis of MGs and GCMGs can be found e.g. in references therein.

The overall picture which emerges is the following: at fixed \( n_p \), the statistical behavior of the model is characterized by a critical line at \( \varepsilon = 0 \) which extends from some critical value \( n_s^*(n_p) \) to larger values \( n_s > n_s^*(n_p) \) than this threshold. This is illustrated in Fig. 1. As this line (segment) is approached in parameter space the market becomes more and more efficient, i.e. \( H \to 0 \) as \( \varepsilon \to 0 \) for \( n_s > n_s^* \). On the critical line \( (\varepsilon = 0, n_s > n_s^*) \) itself the market is fully efficient and one finds \( H = 0 \) exactly in the limit of infinite system size. In addition, numerical simulations of the model at finite sizes close to the critical line reveal fluctuation properties which are similar to those observed in real markets. In particular \( A(t) \) has a fat tailed distribution and one observes volatility clustering. These effects become weaker as the system size is increased at constant values of \( n_s, n_p \) and \( \varepsilon \), and similarly they disappear gradually when one moves away from the critical line at fixed system size.

The case \( \varepsilon = 0 \) and \( n_s > n_s^*(n_p) \) is peculiar because it turns out that the stationary state is here not unique, but rather that it depends on the initial conditions from which simulations are started. In the following we will not consider the case of a strictly vanishing \( \varepsilon \), but will assume instead that speculators have a positive profit margin \( \varepsilon > 0 \), even if the latter may be small. All simulations on which we report are started from zero initial conditions \( u_i(t = 0) = 0 \) for \( i = 1, \ldots, N_s \).

We finally remark that, a detailed analysis of the transient dynamics demonstrates that for small \( \varepsilon \) the stationary state is reached after a characteristic equilibration time which scales as \( 1/\varepsilon \). This long equilibration time is responsible for some relevant effects if the market composition changes in time, as discussed in Section IV.

III. TOBIN TAX IN THE GCMG

Within the model setup the introduction of a tax \( \tau \) on each transaction can be accounted for by a change \( \varepsilon_i \to \varepsilon_i + \tau \) for all \( i = 1, \ldots, N \). Indeed by raising \( \varepsilon \) by an amount \( \tau \), an additional cost \( \tau \) incurs for any given agent every time they trade and no costs for agents who refrain from trading. Note that the trading volume of any fixed (active) agent is one unit in our simple setup so that \( \tau \) indeed corresponds to a transaction tax per unit traded. Hence we will assume

\[
\varepsilon_i = \begin{cases} \varepsilon + \tau & i \leq N_s, \\ -\infty & i > N_s \end{cases}
\]

in the following. While this will discourage speculators from trading (via the reduction of their strategy score) such a tax will have no effect on the participation of producers. They will trade at every time step as before \((n_s(t) = 1 \forall i > N_s \text{ at all times} t)\).

The total revenue from this tax \( \tau \) is then given by

\[
R = \frac{\tau}{N} \sum_i (a_i(t)) = \frac{\tau}{N} \sum_{i=1}^{N_s} (a_i(t) + \tau \frac{n_p}{n_s + n_p}) \equiv R_s + R_p,
\]

where the first term corresponds to the revenue \( R_s \) from speculators and the second \( (R_p) \) to that obtained from the producers.

An evaluation of the effects of levying a tax \( \tau \) on the GCMG then amounts to studying the behavior of the model as a function of \( \varepsilon \) at fixed model parameters \( n_s \) and \( n_p \). It turns out that here one has to distinguish between two different regimes, namely close and far away from the phase transition.

Fig. 2 reports the effects of introducing a tax \( \tau \) on markets whose parameters are far from the critical line. We here consider \( n_p = 1 \) and \( n_s = 1 < n_s^*(n_p = 1) \approx 4.15 \ldots \) so that one operates sufficiently far to the left of the critical region depicted in Fig. 1. For such parameter values the results of numerical simulations follow the curves predicted by the analytical theory perfectly, and no anomalous fluctuations are present in the corresponding price time-series. The tax has very mild effect both on volatility and on the information efficiency, as long as \( \varepsilon + \tau \ll 1 \).
such values of the parameters one is within the critical region (which is valid only in the limit of infinite systems). As illustrated in the lower panel of Fig. 3, imposing a sufficiently large tax may in such markets have a pronounced effect on the volatility, whereas smaller transaction fees may influence the time-series of the market only marginally. Fig. 4 presents a systematic account of these effects and shows the dependence of the volatility, the predictability and the revenue from the tax on the system size and the tax rate $\tau$ at $n_s \gg \bar{n}_s(n_p)$. In particular, a significant reduction of the market volatility can be obtained, while still keeping the market relatively information efficient. Furthermore, the contribution to the tax revenue of speculators largely outweighs that of producers and it is peaked at a value close to that where the volatility is minimal. The effect of a tax, as shown in Fig. 4 also depends on the size of the market. The volatility at low $\varepsilon + \tau$ indeed decreases with the size of the market and approaches the theoretical line, making a tax more effective in small than in large markets.

Fig. 4 also shows that the contribution of speculators to the tax revenue has a peak at intermediate tax rates, but that at the same time the revenue $R_s$ obtained from speculators is smaller than that from institutional investors.

Fig. 5 relates these two extremes and discusses the dependence of the volatility on $n_s$ at intermediate number of speculators for fixed $n_p = 1$. We here fix the (effective) system size by keeping $L = PN_s$ constant. For small values of $n_s$ one is then well outside the critical region and the numerical results follow the analytical predictions (solid lines in Fig. 4). As discussed in Ref. [14] the simulations then deviate systematically from the theory at large $n_s$ when the system has entered the zone near the phase transition line. More precisely, one finds a threshold value $\bar{n}_s(L) > n_s^\star$ so that numerical simulations agree with the theoretical lines for $n_s < \bar{n}_s(L)$ but deviations and anomalous fluctuations are observed for $n_s > \bar{n}_s(L)$. As $L$ is increased $\bar{n}_s(L)$ is found to grow as well in simulations (not shown here), and in particular one has $\lim_{L \to \infty} \bar{n}_s(L) = \infty$ ($\varepsilon + \tau \neq 0$) so that the critical region vanishes in the limit of infinite systems.
IV. MARKETS WITH EVOLVING COMPOSITION OF AGENTS

In the previous sections we have assumed that all traders stay in the market for an infinite amount of time and that their trading strategies remain fixed forever. Individual agents have the option to abstain from trading at intermediate times and to join the market again at a later stage, but no agent in the setup considered so far can actually modify his strategy vector \( \{ a_i^n \} \). Thus the composition of the population of traders does not change over time. In real-market situations however it would appear more sensible to expect some fluctuation in the population of traders and to assume that the market composition will evolve and/or that strategies get replaced after some time. In the latter case, one would expect poorly performing strategies to be removed from the market and replaced by new ones.

In this section we consider the simplest case of an evolving composition of the market, namely a situation in which agents (or equivalently their strategies) are replaced randomly, irrespective of their performance. More precisely, at each time step each speculator is removed with a probability \( 1/(\theta N_s P) \) and replaced by a new one with randomly drawn strategy and zero initial score. Here \( \theta \) is a constant, independent of the system size. This choice \( \theta = \mathcal{O}(L^0) \) guarantees that the expected survival time of any individual agent scales as \( N_s P \) so that one exit/entry event occurs in the entire population on average over a period of \( \theta P \) transaction time steps. Relaxation times in Minority Games are known to be of the order of \( P \) so that the above scaling of \( \theta \) results in the composition of the market changing slowly on times comparable with those on which the system relaxes. Indeed, extensive numerical simulations show that the behaviour of the volatility on \( \theta \) as well as that of other quantities characterizing the collective behaviour of the system is independent of the system size (see Fig. 4). This is in sharp contrast with the strong finite size effects observed for \( n_s \gg n_s^* \) at a fixed composition of the population of agents (Fig. 1).

The main feature of the MG market with an evolving population of agents is a pronounced minimum of the volatility as a function of \( \varepsilon + \tau \) in the crowded regime \( n_s > n_s^* \). In particular the volatility increases as \( \varepsilon + \tau \) is decreased, even in the limit of large system sizes (in which a corresponding system with fixed agent population would equilibrate to the flat theoretical line as shown in Fig. 1). This behaviour of the system with changing agent structure can be related to the fact that relaxation time of the GCMG scales as \( 1/(\varepsilon + \tau) \) \( \mathcal{O} \). Thus, when \( \varepsilon + \tau \) is very small, the time it would take a fixed population of agents to equilibrate collectively can be much larger than the time scale \( \theta P \) over which the market composition changes. In this case, the market remains in a high volatility state indefinitely because the agents do not have sufficient time to ‘coordinate’ and to adjust their respective behaviour as the strategy pool represented in the evolving population of agents changes too quickly. Fig. 6 demonstrates that introducing a tax in such markets with dynamically evolving trader structure can reduce the volatility considerably.

V. CONCLUSIONS

We have shown how the theoretical picture derived for GCMG \( \mathcal{G}_{\text{CMG}} \) can be used to fully characterize the impact of a Tobin tax on this toy model of a currency market. The main results of our study (see Fig. 4) is that within the GCMG the introduction of a Tobin Tax reduces the market volatility whenever the market is operating close to the critical line at \( \varepsilon = 0, n_s > n_s^* \) which marks perfect efficiency. In this region close to criticality the efficiency scales as \( H \sim \varepsilon^\alpha \) (with a theoretically predicted exponent of \( \alpha = 2 \)) and the market is close to full efficiency \( H \approx 0 \). The model is known to exhibit stylized facts such as broad non-Gaussian return distributions and volatility clustering for such values of the model parameters. Thus the reduction of the volatility by an additional tax is most pronounced when the market is operating close to efficiency and exhibits anomalous fluctuations. Within the GCMG the effect of a trading tax is due to a significant amount of excess volatility close to a critical line. The tax \( \tau \) draws the market away from criticality in parameter space and thus reduces volatility.

The picture is different when a tax is introduced to a market which operates far from criticality. For a market with few speculators \( n_s < n_s^* \) the predictability \( H \) attains a finite limit when \( \varepsilon + \tau \to 0 \). In this case the tax has a small effect on volatility.

In both regimes we find that the revenue for the market
maker from the tax attains a maximal value at intermediate tax rates. In the case of a market near criticality this occurs approximately at a tax rate at which the reduction in volatility is largest. In both shown examples the revenue for the market maker appears to weigh more on institutional investors than on speculators. Measuring the revenues from producers and speculators respectively at other values of \( n_s \) and \( n_p \) confirms this observation and we find \( R_p > R_s \) (not shown here). Only when the number of producers is extremely small or when \( \varepsilon < 0 \) can one observe instances in which the revenue from speculators is higher than that from producers. Both an extremely small number of producers and/or a negative value of the profit margin \( \varepsilon \) seem somewhat unrealistic in the present context so that we do not report further details here.

Thirdly our findings demonstrate that imposing a tax can also reduce the market volatility in cases where the composition of the population of traders changes slowly in time. In this case, the tax allows the agents to reach coordinated faster so that the market can reach a stationary state of relatively low volatility.

Given the stylized nature of the MG, it is hard to make a connection between the model parameter \( \tau \) and an actual tax rate in a real-world market. At any rate it seems reasonable to assume that a realistic tax rate should be of the order or smaller than the margin of profit of speculators, which is gauged by \( \varepsilon \) in the GCMG. The optimal tax rate \( \tau \) might be unrealistically large compared to \( \varepsilon \). E.g. in Fig. 4 if \( \varepsilon = 0.01 \) volatility can be substantially reduced only for tax rates \( \tau \) which are more than ten times larger.

The GCMG can at best be seen a minimalistic simplified version of a real market. Hence our conclusions on the behaviour of the MG are at most suggestive with respect to what might happen in the real world. Still, the Minority Game is able to capture the interplay between stochastic fluctuations and information efficiency in a system of adaptive agents in a simplified way. At the same time it is analytically solvable with the tools of statistical mechanics so that quantitative predictions for example of the volatility can be made based on an exact theory. In this sense, the analysis of MGs goes far beyond the results of zero-intelligence models. Most importantly in the present context our study provides a coherent theoretical picture of the effects of ‘throwing some sand in the wheels’ of markets operating close to information efficiency. The picture developed here can be extended in a number of directions toward more realistic market modes, but without giving up analytical tractability. One of the most interesting future directions might be to endow agents with individual wealth variables which evolve according to their relative success when trading in the model market.

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[15] This is an assumption on the cognitive limitation of agents because the game itself will generate a much richer information than just the sequence \( \mu(t) \).
[16] This contrasts with the definition of \( \mu(t) \) in the original MG, where the information was endogenously generated by the market, with \( \mu(t) \) encoding the sign of the past \( M = \log_2 P \) price changes. Most collective properties of the model have been shown to be affected only weakly by the origin of information. We focus here on the simpler case of exogenous information which is analytically more convenient. Note however that analytical approaches to MGs with endogenous information are also feasible, but involve much more intricate mathematics [14].
[18] We here assume that each agent holds only one trading strategy. Generalizations to multiple strategies per player are straightforward, and can be seen not to alter the qualitative behaviour of the model. Hence we here restrict to the simplest case.
[22] D. Challet private communication.
Appendix

We here sketch the theoretical analysis of the model with $\tau = 0$ and general values of $\varepsilon$. The introduction of a tax $\tau$ can be accounted for by replacing $\varepsilon \to \varepsilon + \tau$ in all equations below. The starting point of the statistical mechanics approach is the function

$$ H_z[\{\phi_i\}] = \frac{1}{P} \sum_{\mu=1}^{P} \left[ \sum_{i=1}^{N} a^n_i \phi_i \right]^2 + \frac{2e}{P} \sum_{i=1}^{N} \phi_i $$

(8)

of the mean activities $\{\phi_i = \langle n_i(t) \rangle\}$ of the speculators $i = 1, \ldots, N_s$. The $\phi_i$, $i = 1, \ldots, N_s$ are continuous variables within the interval $[0, 1]$, recall that producers are always active and have $\phi_i = 1$, $i = N_s + 1, \ldots, N = N_s + N_p$. Note that this function depends explicitly on the strategy assignments $\{a^n_i\}$ so that $H_z$ is a stochastic quantity. The strategy vectors which are fixed at the beginning of the game correspond to what is known as 'quenched disorder' in statistical mechanics. It turns out that the learning dynamics minimizes the function $H_z$ in terms of the $\{\phi_i\}$ for any fixed choice of the strategy vectors. Computing the stationary states of the model thus reduces to identifying the minima of $H_z$. It is here possible to characterize these minima using the so-called replica method of statistical physics. A different statistical mechanics approach is based on so-called generating functionals and deals directly with the update dynamics, see [2]. Both methods ultimately lead to the same equations describing the stationary states of the model, so that we here restrict the discussion to the former approach.

In the following we give a brief sketch of the so-called replica analysis of the model, which allows to compute the minima of the random function $H_z$. To this end one first introduces the partition function

$$ Z_z(\beta) = \int_0^1 d\phi_1 \cdots \int_1^1 d\phi_N \exp (\beta H_z[\phi_1, \ldots, \phi_N]) $$

(9)

at an ‘annealing temperature’ $T = 1/\beta$. In the limit $\beta \to \infty$ these integrals are dominated by configurations $\{\phi_1, \ldots, \phi_N\}$ which minimize $H_z$, so that the evaluation of $\lim_{\beta \to \infty} Z_z(\beta)$ allows one to characterize the minima of $H_z$.

This procedure is in general not feasible for individual realizations of the random strategy assignments as the dependence of $H_z$ on the $\{a^n_i\}$ is quite intricate. Instead we will compute ‘typical’ quantities in the limit of infinite systems, i.e. averages over the space of all strategy assignments.

The key quantity to compute here is the free energy density

$$ f_z(\beta) = -\frac{1}{\beta N} \ln Z_z(\beta). $$

(10)

The limit $N \to \infty$ is here taken at fixed ratios $n_s = N_s/P$ and $n_p = N_p/P$ (so that $N_s, N_p, P$ are taken to infinity as well). All relevant properties of the typical minima of $H_z$ can be read off from the disorder-average of $\lim_{\beta \to \infty} f_z(\beta)$. Using the identity $\ln Z = \lim_{n \to \infty} \frac{2^n - 1}{n}$ this problem can be reduced to computing averages of $Z^*_n$, corresponding to an $n$-fold replicated systems with no interactions between the individual copies. This is referred to as the replica method in statistical physics and is a standard tool for the analysis of problems involving quenched disorder [11, 12, 23]. The averaging procedure leads to an effective interaction between the replicas, and requires an assumption regarding the symmetry of the solution with respect to permutations of the replicas. In principle this symmetry may be broken, as different replica copies may end up in different minima of $H_z$. In the non-efficient phase of this model, instead, the so-called ‘replica symmetric’ ansatz is exact, simplifying the analysis considerably. We will here not report the detailed intermediate steps of the calculation, but will only quote the final outcome, namely a set of closed equations for the variables characterizing the minima of $H_z$ (and hence the stationary states of the GCMG). Further details of the replica analysis are found in [11, 12, 14].

The minima of $H_z$ turn out to be described by two independent variables $K$ and $\zeta$, uniquely determined from the following two relations:

$$ \zeta = \frac{1}{\sqrt{n_s(Q(\zeta, K) + n_p/n_s)}} $$

$$ K = \varepsilon \left[ 1 - \frac{n_s}{2} \left( \text{erf}((1 + K)\zeta/\sqrt{2}) - \text{erf}(K\zeta/\sqrt{2}) \right) \right]^{-1} $$

(11)

with

$$ Q(\zeta, K) = \frac{1}{2} \text{erfc}(1 + K)\zeta/\sqrt{2} $$

$$ + \frac{1}{\zeta^2} \left[ (K - 1)e^{(1 + K)\zeta^2/2} - Ke^{-K\zeta^2/2} \right] $$

$$ + \frac{1}{2} \left( K^2 + \frac{1}{\zeta^2} \right) \left( \text{erf}((1 + K)\zeta/\sqrt{2}) - \text{erf}(K\zeta/\sqrt{2}) \right) $$

These equations are easily solved numerically and one obtains $K$ and $\zeta$ as functions of the model parameters $\{n_s, n_p, \varepsilon\}$. The disorder-average of quantities such as the predictability $H$ or the mean activity of the speculators $\phi = \lim_{N \to \infty} N_s^{-1} \sum_{i=1}^{N_s} \phi_i$ can then be expressed in terms of $K$ and $\zeta$. One finds:

$$ H = \varepsilon \frac{n_p + n_s Q(\zeta, K)}{(n_s + n_p)K^2} $$

$$ \phi = \frac{1}{2} \text{erfc}(1 + K)^\zeta/\sqrt{2} $$

$$ + \frac{1}{\zeta^2} \left( e^{-K^2/2} - e^{-\zeta^2/2} \right) $$

$$ + \frac{K}{2} \left( \text{erf}(K\zeta/\sqrt{2}) - \text{erf}((1 + K)\zeta/\sqrt{2}) \right) $$

(12)

These results are fully exact in the thermodynamic limit, with no approximations (except for the replica-
symmetric ansatz) made at any stage. Finally, neglecting certain dynamical correlations between agents the volatility can be approximated as

$$\sigma^2 = \varepsilon^2 n_p + n_s Q(\zeta, K) \frac{1}{(n_s + n_p)K^2} + n_s \phi - Q(\zeta, K) \frac{\phi - Q(\zeta, K)}{n_s + n_p}. \quad (13)$$

As shown in the main text of the paper this approximation is in excellent agreement with numerical simulations (up to finite-size and equilibration effects).
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