Topics in

DYNAMICAL SYSTEMS AND GAME THEORY

by

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SUMMARY

This thesis consists of two parts, which deal with different topics in dynamical systems.

Part I (DYNAMICS FROM GAMES) is the main scope of the work. There we study a family of flows which are often applied in studies of some game dynamics in animal competition and evolutionary biochemistry. These flows are the solutions, on simplexes, of cubic differential equations determined by "pay-off" matrices. The main result in this part is a proof for a classification of stable flows in this family, in dimension 2, first conjectured by Zeeman in 1979 (stability under small perturbations in the pay-off matrix). We add necessary and sufficient conditions for stability, which decide the exact class for each stable flow in the family. We also give as preliminary properties some simple expressions to calculate eigenvalues at fixed points and prove that hyperbolicity of these is necessary for stability, in all dimensions.

In order to complete Zeeman's classification we had to adapt, in dimension 2, some techniques of structural stability for flows not satisfying the usually required transversality condition.

We discuss some aspects and difficulties present when one attempts to study cases in dimension ≥ 3.

One important three-dimensional example, involving a Hopf bifurcation, is discussed in detail.

In the final chapter, we present some three-dimensional cases to which a discussion of stability is feasible.

Part II (LIAPUNOV FUNCTIONS FOR DIFFEOMORPHISMS) has as its purpose the construction of Liapunov functions for diffeomorphisms. A local construction is presented in neighbourhood of compact isolated invariant sets. A globalization is obtained for Axiom A diffeos with no cycles.

NOTE

PART II of this thesis was first presented as M.Sc. dissertation at Warwick University. It is not related to PART I and it is here included in its original form. It has its own separate bibliography and pagination. For easy handling, PART II is presented after a coloured partition page.

ACKNOWLEDGEMENTS

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PART I

DYNAMICS FROM GAMES

To

Ewaldo, Daniel, David

for their love

CONTENTS - PART I

INTRODUCTION

The aim of this part is to state and prove some results in dynamical systems, mainly refering to a family of differential equations which are often studied for applications in game theory, chiefly games of animal conflicts and biochemistry. More specifically, we want to study a family of cubic differential equations which give a flow on an n-simplex Δ, each element of this family being determined by an $(n+1) \times (n+1)$ matrix A which (as will be seen in 1.5.1) can be assumed to have zero diagonal.

Explicitly, we take

$$\Delta = \{x = (x_0,\dots,x_n) \in \mathbb{R}^{n+1} \ ; \ x_i \geq 0 \ , \ \sum_0^n x_i = 1\}$$

and equations

$$(*) \quad \dot{x}_i = x_i \, ((Ax)_i - xAx)$$

(where x represents ambiguously both row and column matrices of elements $x_0, x_1, \dots, x_n$).

In game theory language, x_i is interpreted as the proportion of the population playing strategy i, A is the pay-off matrix and x represents the distribution of strategies in the population, which can evolve with time (see e.g. [39], [41] or 1.2 in this work).

The link between game theory and animal conflicts was proposed by Maynard Smith and Price [20], but as a game static in time. They proposed to seek a distribution of strategies that is stable under natural selection called an "evolutionary stable strategy" or EES. Taylor and Jonker [39] introduced the idea of dynamics to this type of game by assuming the hypothesis that the growth rate of population playing each strategy is proportional to the advantage to that strategy. In mathematical words they introduced system (*) above. Then they proved that an EESis a stable equilibrium (or attractor) to the dynamic.Studies and applications of (*) were presented by Schuster, Sigmund, Wolff and Hofbauer in [32, 33, 17], and Zeeman [42]. Later, both Zeeman [41] and Hofbauer, Schuster, Sigmund [15] argued that the concept of EES was too restrictive, because although an EES is an attracting point of the game dynamics, not all of these are qualified to be an EES. Consequently, Zeeman [41] proposed the idea of equivalence between two dynamic games of type (*) more in the line of the mathematical theory of structural stability of dynamical systems. This led to the notion of "stable games" whose dynamics (or flow) are topological preserved by small perturbations in the pay-off matrix A (see definition 1.3.3). The stable equilibrium points for such stable games must then be,generically, the distribution of strategies in real life applications. This is the line of work that we will pursue in this thesis. So we will try to describe topologically the flow associated to (*), looking for conditions for stability, not for EES points.

Later, Hofbauer [13] proved that system (*) is equivalent to equations in $\mathbb{R}^n_+$, called the Lotka-Volterra equations (see also 2.5 in this work).

This equivalence opens a whole range of applications and also unifies two different approaches since the Lotka-Volterra equations are, too, often applied to the study of competition of species and biochemistry.

For other applications of this system, either in form (*) or in the Lotka-Volterra form, we refer to works by Schuster et. al. [34, 35, 36], Eigen [9], Rescigno [30], May, Leonard [19], Maynard-Smith [21, 22], Hofbauer et al. [16, 17], Zeeman [42], Akin [1,2], Fujii [11], Coste et al [8].

After proposing the new notion of stability, as above mentioned, Zeeman [41] proved a number of general and basic properties for system (*) (some of these will be stated in chapter 2), and also conjectured a classification for the two-dimensional case $(n=2)$ with the existence of 19 stable classes, up to flow reversal. One of our main results here is to finish the proof of his classification (Theorem I in 1.4.1. Proof is in Chapters 3 and 4). We also give a test on the elements of A which is easy to carry out and decides if A is stable or not and which stable class it belongs to. (This is in Theorem II, stated in 1.5.2, proved in Chapter 3.) In [41], Zeeman has also conjectured the non-existence of limit cycles for $n = 2$ for systems (*). This was later proved by Hofbauer [13] as a consequence of the equivalence to Lotka-Volterra equations, which do not admit limit cycles in dimension two. (See discussion in 3.2.) Also in [41], stability was proved in detail for some cases but for most of them it was suggested that stability would follow from the standard structural stability techniques of Palis-Smale or Peixoto (as in [24],

[25] or [27]). In fact, this does not work since those standard techniques always rely on the hypothesis of transversality of all the saddle connections (hence no such connections should occur for dimension 2). But this hypothesis does not hold in most of the cases to be studied. However, we noted that the non-transversal connections occur on the boundary of the simplex and are "robust" in the sense that they are preserved for all systems (*) with sufficiently near pay-off matrices. So, in Chapter 4, we have digressed from our main line of work, in order to adapt some techniques of structural stability in dynamical systems, given by Fleitas [10], in order to apply them to our equations. (So, conclusion of proof for the classification is given as an application at the end of Chapter 4.) The only class to which this modified technique would not apply is the one that presents a cycle of saddle connections in its phase-diagram, but this was exactly the class that Zeeman studied in detail in [41], proving stability. Therefore, the classification for $n = 2$ is now complete.

For dimension 3 or more, a classification seems still a long way off. There are many questions that should be answered before any attempt to a classification is made. Some of the questions that we think are important for a better understanding of (*) for $n \geq 3$ we have listed in Chapter 7.

In Chapter 6, one example (in fact a one-parameter family of examples) is studied in detail. This is an important example which has already appeared in [41] or [17] and which has practical applications, being a generalization of the hypercycle of [9, 14, 32]. Theoretically it is also important because this family presents a Hopf bifurcation giving rise to a limit cycle. Our intention was to prove that such limit cycles can occur for stable classes for $n = 3$. (They do not occur for $n = 2$.) In fact, we could only prove that, near the bifurcation point, the flow is globally stable in the interior $\mathring{\Delta}$ of the simplex and stable in the boundary $\partial\Delta$, but we could not prove stability on the closed simplex, though we kept this as a conjecture. Even so, we tried to give the most complete picture we could for the flows in this one-parameter family.

In Chapter 7, besides a discussion about the difficulties that are present for $n = 3$, we choose some cases having special properties that allow answers to be given to the proposed general questions and, for these, we give description of the possible flow diagrams, and draw the flows on Δ .

Some of the questions we put in Chapter 7 have to do with the understanding of possible periodic orbits occurring for (*) in the interior of Δ . So we ask: for stable cases, are periodic orbits hyperbolic? unique? These questions are suggested by the fact that similar ones for fixed points have affirmative answers.

Another main difficulty for $n \geq 3$ is the possible existence of "strange attractors". This was claimed (supported by computer drawings, but not proved mathematically) by Arneodo, Coullet, Tresser [4,5] for Lotka-Volterra equations. A better understanding of such a possibility should be attempted before one aims at a classification for $n \geq 3$.

It is our intention to answer, in the future, some of these questions.

Remarks (1) All along this work we will use established concepts and usual notations of dynamical systems, as e.g. flows, topological equivalence ($\sim$), hyperbolic fixed points or orbits, stable (= inset) and unstable (= outset) manifolds of x (denoted by $W^s x$, $W^u x$ resp.), attractor (= sink), repellor (= source), saddles, α- and ω-limits, non-wandering points, asymptotical stability, phase diagrams, etc. For these we refer to general texts on dynamical systems like, for instance [26], [38] and many others.

(2) In Chapter 5 we have also digressed from the main line of work in order to present a statement and some general remarks about the classical Hopf bifurcation theorem. We do this because, surprisingly, we could not find, in the vast literature about this theorem, references about the basin of attraction of the periodic orbit. Some information about it was needed for our study of example in Chapter 6. In fact, the property we wanted was contained in some known proofs for that theorem, but not explicitly stated. The version of Hopf theorem that we finally state is more general than we really need for the application in mind, but it contains all the information we need for it.

(3) In Chapter 1 we give some basic definitions and statements for all our main results to be proved and discussed in the subsequent chapters.

CHAPTER 1

DEFINITIONS AND STATEMENT OF MAIN RESULTS

1.1 Presentation of the problem

As we observed in the introduction, our purpose here is to study a family of differential equations in $\mathbb{R}^{n+1}$ which give rise to a family of flows on the n-dimensional simplex

$$\Delta = \{x = (x_0,\dots,x_n) \in \mathbb{R}^{n+1} \ ; \ x_i \geq 0 \ , \ \sum_0^n x_i = 1\} \ .$$

Let M_{n+1} be the space of real $(n+1) \times (n+1)$ matrices. For each $A \in M_{n+1}$ we define a vector field $X_A = (X_A^0,\dots,X_A^n)$ on $\mathbb{R}^{n+1}$, given by

$$X_A^i(x) = x_i((Ax)_i - xAx) \ \ \forall \ x \in \mathbb{R}^{n+1}$$

then we consider the system of differential equations

$$(*) \qquad \dot{x}_i = X_A^i(x) \qquad i = 0,\dots,n \ .$$

1.1.1 Remarks (1) In order not to make notation too heavy, x above represents either the point $x = (x_0,\dots,x_n)$ of $\mathbb{R}^{n+1}$ or the column or row matrices with elements $x_0,\dots,x_n$. The meaning of x is usually clear on the text or expression.

(2) Equation (*) is sometimes referred to as the

replicator equation (e.g. in [13]). We will use this name, when necessary for clarity in the text.

Equation (*) leaves Δ and all its faces (see 1.1.2 below) invariant. Since Δ is compact, equation (*) determines a global flow ϕ_A on Δ for vectorfield X_A, i.e. $\phi_A : \mathbb{R} \times \Delta \to \Delta$ such that $t \to \phi_A(t,x)$ is the solution for (*) with initial point x. ϕ_A leaves all faces invariant.

1.1.2 Notations (1) Generally, a face of n-dimensional simplex Δ means any i-dimensional sub-simplex of Δ $(i = 0,1,\dots,n)$ determined when $(n-i)$ x-coordinates are fixed to be zero, while the others vary, keeping x in Δ. For instance, if $x_i = 0$ for all $i \neq k$ (hence $x_k = 1$) we get a 0-dimensional simplex, which is a point of Δ called vertex X_k. If $x_i = 0$ for all $i \notin \{r,s\}$ (hence $x_r + x_s = 1$) we get a 1-dimensional simplex, which is the straight line segment with X_r and X_s as end points, called the edge $X_r X_s$ of Δ. Δ is considered as its own n-dimensional face. We denote by $\mathring{\Delta}$ the interior of Δ, and by $\partial\Delta$ its boundary, which is the union of all $(n-1)$-dimensional faces. If F is an i-dimensional face, $\mathring{F}$ will represent F minus its $(i-1)$-dimensional faces.

(2) We think of Δ with $n = 3$ as a tetrahedron immersed in $\mathbb{R}^3$ and, when necessary, we will draw it as such. See figure 1

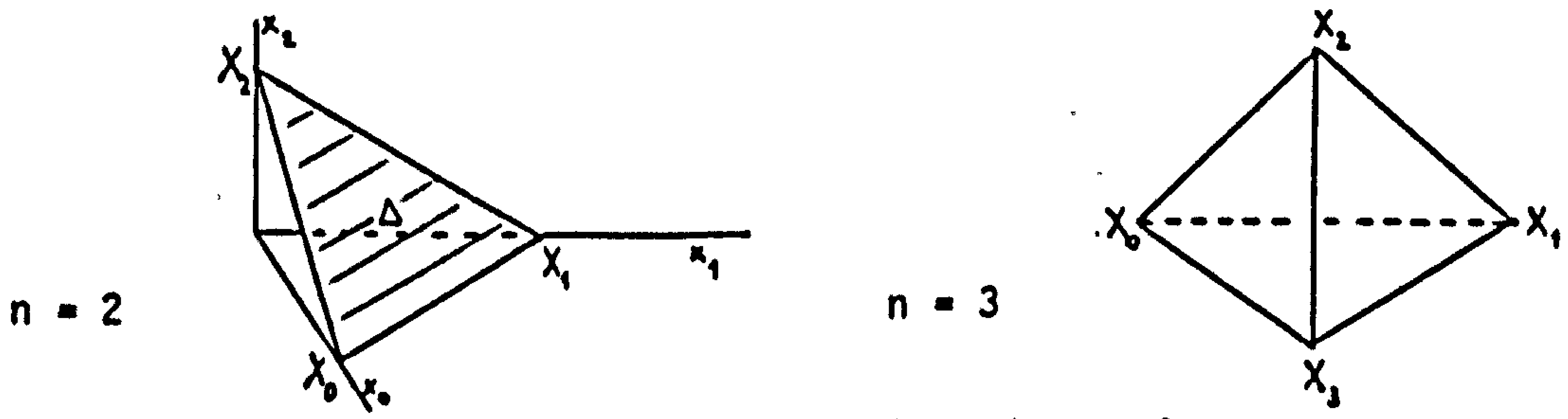


figure 1: representation of Δ for $n = 2$ and $n = 3$.

(3) When necessary we will denote by A_n the family of flows as above, i.e. $A_n = \{\phi_A \,;\, A \in M_{n+1}\}$.

1.1.3 Remark For general properties of this family of flows we refer mainly to Zeeman [41], Hofbauer [13, 14, 17], Akin [1], Taylor, Jonker [39]. Some of these properties we present here either in the rest of this chapter or in the next chapter. The literature on studies of particular cases of these equations, or their applications, is vast. So we refer only to a few, e.g. Schuster, Sigmund, Wolff [32, 33, 34, 35, 36], Eigen [9], Maynard Smith [20, 21, 22] and Zeeman [42].

1.2 Applications to game theory

The equations (*) presented in 1.1 have been widely applied, as we referred in the introduction, in game theory, in cases where the game is of a evolutionary dynamic type. The dynamic depends usually not only on the individual strategy, but also on the strategy of the population of players as a whole, and the future distribution of strategies in the population is determined, at all times, by the initial one.

Given such hypotheses for a game, of course this is not a play or parlour game. But these are approximately the conditions found by studies of animal evolution and conflicts between species or between different behaviour in a single specie. These conflicts or evolutions can be interpreted as games in animal society (e.g. in [8, 20, 21, 22, 34, 39, 42]). In these studies, it is supposed that:

1. pay-off to any strategy depends, at all times, on the distribution of strategies in the population;

2. behaviour is influenced genetically. This influence varies from almost complete genetic determination like in insect societies to partial determination like in mammals;

3. growth-rate for any strategy is proportional to the net advantage to that strategy in the population.

We show here how equations (*) represent the dynamic of a game satisfying 1,2,3.

Suppose that individuals in a population can play $n+1$ strategies, labelled $i = 0,1,\ldots,n$ and let x_i be the proportion of the total population playing strategy i. Then $\sum_0^n x_i = 1$ and $x_i \geq 0$ and $x = (x_0,\ldots,x_n)$ represents the distribution of strategies in the population. (Then $x \in \Delta$.)

Let $A = (a_{ij})$ be the matrix where a_{ij} measures the "expected gain" or "pay-off" of strategy i against strategy j.

Then we have:

pay-off to i, against $j = a_{ij}$

pay-off to i, against $x = \sum_j a_{ij}x_j = (Ax)_i$

pay-off to x, against $x = \sum_i x_i(Ax)_i = xAx =$

$=$ average pay-off to x.

Then the advantage to strategy i is $(Ax)_i - xAx$.

Then, hypothesis (3) is that growth rate of $x_i \sim$ advantage to i

i.e. $\frac{\dot{x}_i}{x_i} = (Ax)_i - xAx$ (by incorporating the proportionality constant into the time unit). Hence we have $x_i \geq 0$ $\sum\limits_0^n x_i = 1$ and

$\dot{x}_i = x_i\ ((Ax)_i - xAx)$.

1.2.1 Remark The language used above refers to application in animal competitions mainly. When applied in studies of biochemistry, or molecular biology, where self-replication and competition occur, e.g. as in [9, 16, 32, 33, 35] element of matrix A will measure the catalytic effect of one chemical upon the production of another.

The aim of the rest of our work is to study equations (*), with associated vectorfield X_A and flow ϕ_A , without much more reference to applications to game theory.

1.3 Some definitions and notations

In this paragraph we will establish the language necessary to state the main results we have about the replicator equations (*) of 1.1. Further definitions, notations and properties will be presented when necessary.

As noted in the introduction, we will follow here the line of investigation started by Zeeman [41], where he proposed to study how the flows ϕ_A change when A is perturbed in M_{n+1} , and, finally, when possible, classify all possible flows in this family A_n using the following concepts of <u>equivalence</u> and <u>stability</u>.

1.3.1 <u>Definition</u> $A, B \in M_{n+1}$ are <u>equivalent</u> (write $A \sim B$) if there exists a face-preserving homeomorphism of Δ onto itself taking ϕ_A-orbits onto ϕ_B-orbits, preserving orientation of orbits.

1.3.2 <u>Remarks</u> (1) The face preserving homeomorphism of 1.3.1 may permute the faces, but each k-dimensional face is taken onto itself or onto another k-dimensional face by the homeomorphism.

(2) Definition 1.3.1 could be written simply by saying $A \sim B \iff \phi_A \sim \phi_B$ (i.e. ϕ_A and ϕ_B are topologically equivalent in the usual sense of dynamical systems as in e.g. [26, 38].)

(3) $A \sim B$ is an equivalence relation in M_{n+1}.

1.3.3 <u>Definition</u> $A \in M_{n+1}$ is <u>stable</u> if it has a neighbourhood N in M_{n+1}, such that $B \in N$ implies $B \sim A$.

1.3.4 <u>More remarks</u> (4) A stable means, in terms of flow ϕ_A, that ϕ_A is structurally stable <u>inside family A_n</u>, i.e. in the space F of all flows on Δ, there is a neighbourhood $\tilde{N}$ of ϕ_A s.t. $\tilde{\phi} = \phi_B \in \tilde{N} \cap A_n \Rightarrow \phi_B \sim \phi_A$.

(5) When A is stable, we will say, following remark above, that ϕ_A is stable in A_n.

1.4 Classification for $n = 2$

In [41], Zeeman gave a full description of stable classes in M_2 $(n=1)$ with only 3 possible classes (or 2, up to flow reversals). He also conjectured a classification of stable matrices in M_3 $(n=2)$ (stability by definition 1.3.3) with 35 stable classes (or 19, up to flow reversals).

One of our main tasks here is to complete Zeeman's proof of this conjecture, i.e. we announce:

1.4.1 Theorem I For $n = 2$, stable matrices are dense in M_3 and there are 19 stable classes, up to flow reversals. Each of these is represented below by a matrix in the class and a drawing of its phase portrait in Δ (in figure 2 below).

1.4.2 Remarks (1) Proof of Theorem I will be done mainly in Chapter 3, but only completed in 4.8 using the technique of Chapter 4.

(2) In figure 2 (taken from [41]) attractors are marked by a solid dot, repellors by an open dot and saddles by their stable (inset) and unstable (outset) manifolds. All other orbits flow from a repellor to an attractor, except in class (1), where α-limit can be the cycle on $\partial\Delta$.

(3) The numbers denoting each stable class (as $(1),(2),\ldots,(5_1),(5_2),\ldots,(10_2)$) will be explained later. Roughly speaking they mean, for instance, that (5_1) and (5_2) are two classes whose flows are, on $\partial\Delta$, topologically equivalent. Classes denoted by a single number, like $(1),(2),(3),(8)$, mean that there is no other with the same topological type on $\partial\Delta$.

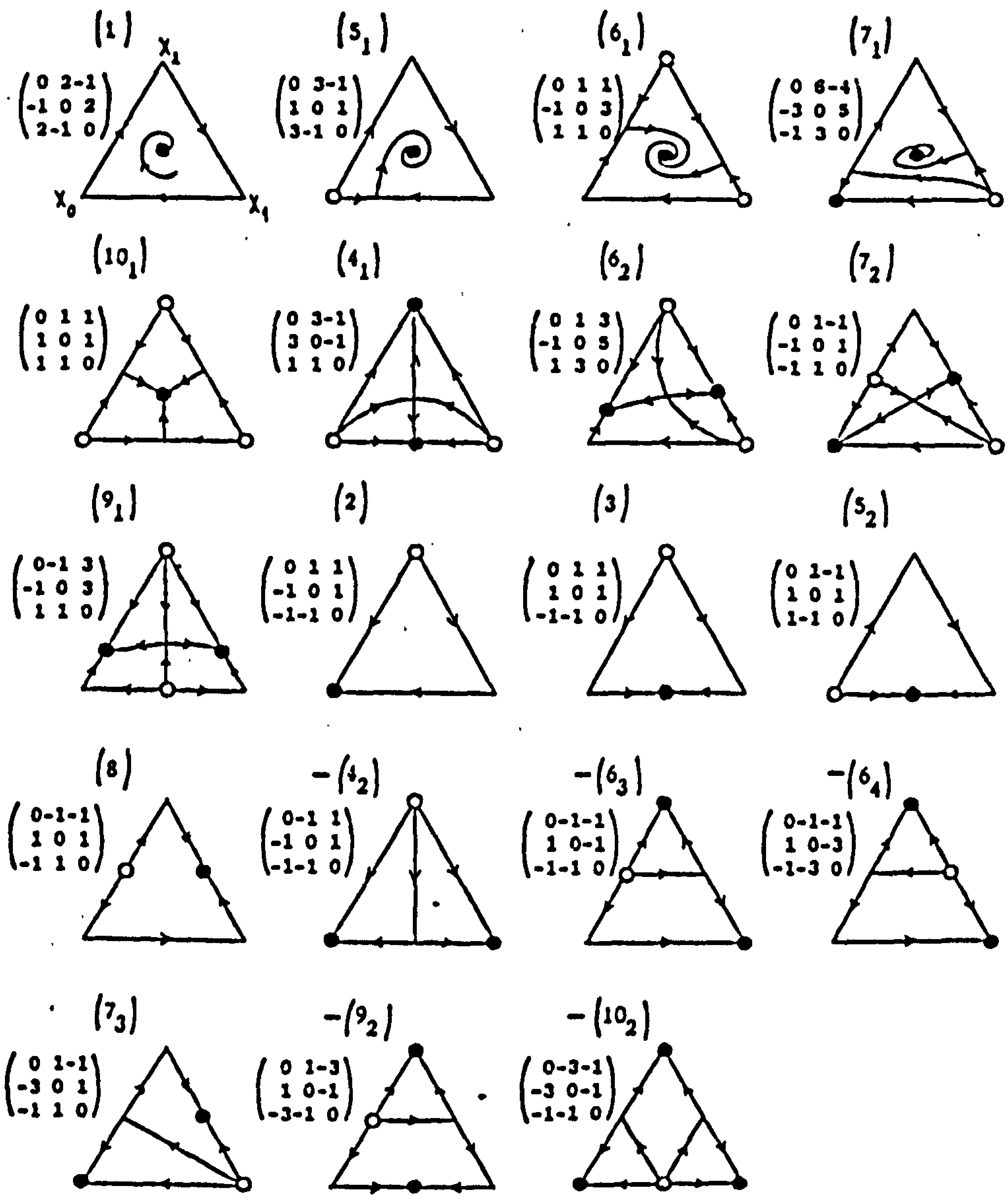


figure 2: phase portraits for stable classes in M_3 (n=2)
(up to flow reversal)

(4) The reversal of flow ϕ_A in A_n (denote reversal of ϕ_A by $-\phi_A$) is also in A_n with $-\phi_A = \phi_{-A}$ $(-X_A = X_{-A})$. To give all stable classes of M_{n+1} up to flow reversal means to give some stable classes in M_{n+1} so that all stable classes will be one of these or a reversal of one of these. This is equivalent to giving classes in M_{n+1} by a weaker equivalent relation given by

$$A \sim' B \iff \phi_A \sim \phi_B \quad \text{or} \quad \phi_{-A} \sim \phi_B \ .$$

The advantage to work "up to flow reversals" is that we have a much smaller number of classes and since all classes of M_{n+1} can be easily retrieved from this, there is no loss in the process.

(5) Theorem I shows that no stable matrix A in M_3 will present a periodic orbit for its associated flow ϕ_A. In fact this property is needed for the proof of the theorem. This was proved by Hofbauer [13]. Here we discuss this property in Chapter 3 (3.2).

1.5 Conditions for Stability

We will announce here some necessary conditions for A to be stable. In [41] we find many of such conditions.

We now want to state (Theorem III below) precise conditions on elements of A which allow us to decide stability (or not) of A and exactly to which class of Theorem I, stable matrix A belongs to. But before announcing this theorem we need some more definitions and notations from [41].

Let $Z_n = \{A \in M_n$; with zero diagonal$\}$

$Z_n^+ = \{A \in Z_n$; with no zeros off the diagonal$\}$

$K_n = \{A \in M_n$; all rows are identical$\}$.

So, all columns of K_n are multiples of the transpose of $u = (1,1,\dots,1)$.

Then, clearly $M_n = Z_n \oplus K_n$ and we have:

1.5.1 <u>Properties</u> (from [41])

(1) $A,B \in M_{n+1} : \phi_A = \phi_B \iff X_A = X_B \iff A-B \in K_{n+1}$

(2) $\forall\ A \in M_{n+1}$, there exists $\tilde{A} \in Z_{n+1}$ s.t. $\phi_A = \phi_{\tilde{A}}$
($\tilde{A}$ is obtained by subtracting a_{jj} from all elements a_{ij} of column j of A , i.e. $\tilde{A} = (\tilde{a}_{ij}),\ \tilde{a}_{ij} = a_{ij}-a_{jj}$)

(3) (equivalence class in M_{n+1}) = $K_{n+1} \oplus$ (equivalence class in Z_{n+1}).

Consequently, in order to give all classes in M_{n+1} , or to obtain conditions for stability, we can always assume $A \in Z_{n+1}$ if convenient.

(4) $A \in M_{n+1}$ stable $\Rightarrow$ ϕ_A has at most one fixed point in the interior of each face (including $\overset{\circ}{\Delta}$).

(5) $A \in Z_{n+1}$ stable $\Rightarrow A \in Z_{n+1}^+$.

(6) $A \in Z_{n+1} \Rightarrow$ the eigenvalues for ϕ_A at vertex X_j $(j = 0,\dots,n)$ are exactly the elements a_{ij} of column j of A off the diagonal.

Consequently, $A \in Z^+_{n+1}$ => vertices are hyperbolic fixed points and, by (5), this is true for all $A \in Z_{n+1}$ stable.

We will prove in Chapter 2, by combining Propositions 2.2.15 and 2.4.4 (see remark 2.4.9) the following:

1.5.2 <u>Theorem II</u> $A \in M_{n+1}$ stable => all fixed points of ϕ_A are hyperbolic.

1.5.3 <u>Remark</u> We remember that hyperbolicity of fixed points is a necessary condition for structural stability of a flow (among all vector-fields). In Λ_n we can only make perturbations on the matrix A, so there are no local perturbations. Therefore Theorem II really has to be proved and is not a consequence of the similar property for structurally stable flows.

1.5.4 <u>Definition</u> $A,B \in Z^+_{n+1}$ are <u>sign equivalent</u> if $a_{ij}\, b_{ij} > 0 \;\; \forall\, i \neq j$.

$A,B \in Z^+_{n+1}$ are <u>combinatorially equivalent</u> if there exists permutation σ of $\{0,1,\dots,n\}$ such that σA and B are sign equivalent.

1.5.5 <u>Remark</u> σA represents the matrix obtained from A by permuting both rows and columns by permutation σ.

1.5.6 <u>More Properties</u> (from [41])

(7) $A,B \in Z^+_{n+1}$: $A \sim B$ => A,B are combinatorially equivalent.

Hence, stable classes of Z^+_{n+1} refine combinatorial classes. So, to find stable classes we first must find combinatorial classes. Each of these is represented only by the signs of elements off the diagonal, up to permutations.

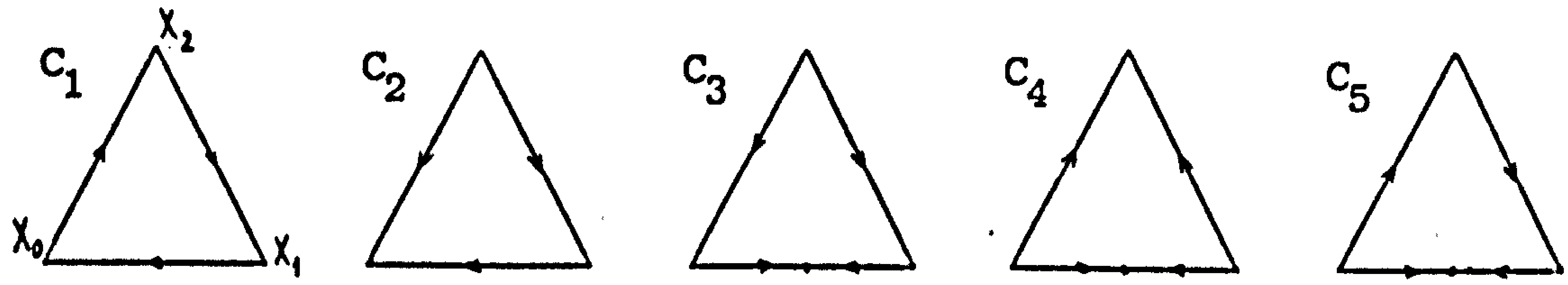
(8) The number of combinatorial classes is always finite, and for $n \leq 3$ we can give:

n	1	2	3
number of combinatorial classes	3	16	218
(up to sign reversal)	2	10	114

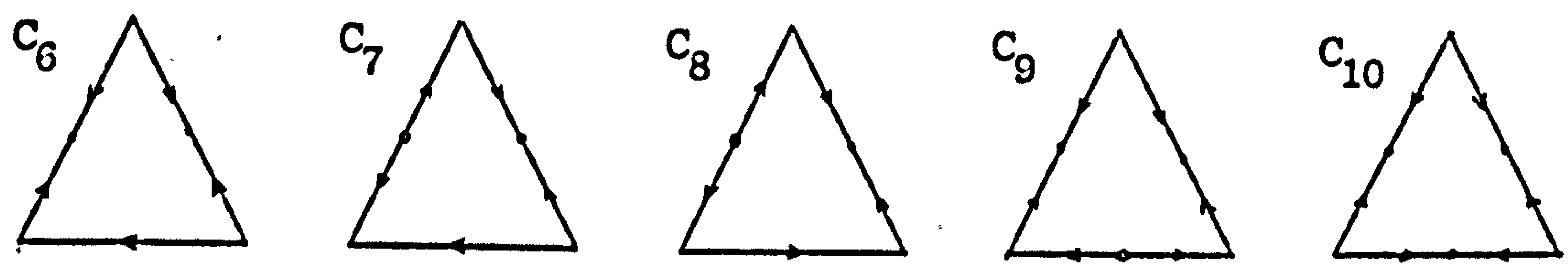
(9) A,B are combinatorially equivalent $\Leftrightarrow$ ϕ_A , ϕ_B , restricted to the <u>edges</u> of Δ , are topologically equivalent, by an "edge-preserving" homeomorphism.

1.5.7 Combinatorial classes of Z_3^+ (up to sign reversal) ([41])

For each $A \in Z_3^+$, either A or $-A$ belongs to one of 10 combinatorial classes $C_1,\dots,C_{10}$. Here we represent each one by a representing matrix of signs A_r, plus a drawing of phase portrait on $\partial\Delta$.



$$A_1 = \begin{pmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & + & + \\ - & 0 & + \\ - & - & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & + & + \\ + & 0 & + \\ - & - & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & + & - \\ + & 0 & - \\ + & + & 0 \end{pmatrix} \quad A_5 = \begin{pmatrix} 0 & + & - \\ + & 0 & + \\ + & - & 0 \end{pmatrix}$$



$$A_6 = \begin{pmatrix} 0 & + & + \\ - & 0 & + \\ + & + & 0 \end{pmatrix} \quad A_7 = \begin{pmatrix} 0 & + & - \\ - & 0 & + \\ - & + & 0 \end{pmatrix} \quad A_8 = \begin{pmatrix} 0 & - & - \\ + & 0 & + \\ - & + & 0 \end{pmatrix} \quad A_9 = \begin{pmatrix} 0 & - & + \\ - & 0 & + \\ + & + & 0 \end{pmatrix} \quad A_{10} = \begin{pmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{pmatrix}$$

figure 3: the combinatorial classes of Z_3^+ (up to sign reversal)

Now we are able to state necessary and sufficient conditions for $A \in M_3$ to be stable. From such conditions, density in Theorem I will be clear. In the following theorem we take each of the combinatorial classes C_r, $r = 1,\dots,10$, above and give conditions on elements a_{ij} for A to be stable, specifying which class of Theorem I A belongs to. When C_r contains only one stable class (up to reversal) we denote this class by (r). If C_r contains more than one stable class (up to reversal), these are denoted by $(r_1),(r_2),\ldots$. This justifies notation in figure 2 as we said in remark 1.4.2 (3).

To simplify statement in Theorem III below we let S_r be the <u>sign class</u> where matrices have signs exactly like A_r above. Let $S = \bigcup_{1}^{10} S_r$.

Now, for each $A \in Z_3^+$, by permutation of vertices, and reversing signs if necessary, we find that A has an equivalent matrix, up to reversal, in S . Consequently, we may give our necessary and sufficient conditions only for A in S .

And we have:

1.5.8 <u>Theorem III</u> Let $A \in S = S_1 \cup \ldots \cup S_{10}$ and

$$k_0 = \frac{a_{01}}{a_{21}} + \frac{a_{02}}{a_{12}} - 1 \quad , \quad k_1 = \frac{a_{10}}{a_{20}} + \frac{a_{12}}{a_{02}} - 1 \quad , \quad k_2 = \frac{a_{20}}{a_{10}} + \frac{a_{21}}{a_{01}} - 1 \quad .$$

Then

1) $\forall\ A \in S_1$, A is stable $\Leftrightarrow \det A \neq 0$. Here $\begin{cases} A \in (1) \Leftrightarrow \det A > 0 \\ A \in -(1) \Leftrightarrow \det A < 0 \end{cases}$

2) $\forall\ A \in S_2$, A is stable and $A \in (2)$

3) $\forall\ A \in S_3$, A is stable and $A \in (3)$

4) $\forall\ A \in S_4$, A is stable $\Leftrightarrow k_2 \neq 0$. Here $\begin{cases} A \in (4_1) \Leftrightarrow k_2 < 0 \\ A \in (4_2) \Leftrightarrow k_2 > 0 \end{cases}$

5) $\forall\ A \in S_5$, A is stable $\Leftrightarrow k_2 \neq 0$. Here $\begin{cases} A \in (5_1) \Leftrightarrow k_2 > 0 \\ A \in (5_2) \Leftrightarrow k_2 < 0 \end{cases}$

6) $\forall\ A \in S_6$, A is stable $\Leftrightarrow k_0, k_1 \neq 0$. Here $\begin{cases} A \in (6_1) \Leftrightarrow k_0, k_1 > 0 \\ A \in (6_2) \Leftrightarrow k_0, k_1 < 0 \\ A \in (6_3) \Leftrightarrow k_0 > 0,\ k_1 < 0 \\ A \in (6_4) \Leftrightarrow k_0 < 0,\ k_1 > 0 \end{cases}$

7) $\forall\ A \in S_7$, A is stable $\Leftrightarrow k_0, k_1 \neq 0$, and $\det A \neq 0$ in case $k_0, k_1 > 0$.

Here $\begin{cases} A \in (7_1) \Leftrightarrow k_0, k_1 > 0 , \ \det A > 0 \\ A \in -(7_1) \Leftrightarrow k_0, k_1 > 0 , \ \det A < 0 \\ A \in (7_2) \Leftrightarrow k_0, k_1 < 0 \\ A \in (7_3) \Leftrightarrow k_0 < 0 , \ k_1 > 0 \\ A \in -(7_3) \Leftrightarrow k_0 > 0 , \ k_1 < 0 . \end{cases}$

8) $\forall\ A \in S_8$, A is stable and $A \in (8)$.

9) $\forall\ A \in S_9$, A is stable $\Leftrightarrow k_0, k_1 \neq 0$. Here $\begin{cases} A \in (9_1) \Leftrightarrow k_0 k_1 > 0 \\ A \in (9_2) \Leftrightarrow k_0 k_1 < 0 . \end{cases}$

10) $\forall\ A \in S_{10}$, A is stable $\Leftrightarrow k_0, k_1, k_2 \neq 0$. Here $\begin{cases} A \in (10_1) \Leftrightarrow k_0, k_1, k_2 > 0 \\ A \in (10_2) \Leftrightarrow k_0 \text{ or } k_1 \text{ or } k_2 < 0 . \end{cases}$

1.5.9 Remark It will be clear, when we prove above Theorem III (in Chapter 3) that $k_0 \neq 0$ is necessary and sufficient condition for fixed point (if it exists) in the interior of edge X_1X_2 to be hyperbolic. Similarly $k_1 \neq 0$ and $k_2 \neq 0$ for fixed points in X_0X_2 and X_0X_1 .

1.6 Quasi-gradient flows in dimension two

As we have mentioned in the general introduction, in order to prove Theorem I, we have developed a technique for dealing with flows with some non-transversal saddle connections. We want to give a simple instrument with which to decide when two flows are topologically equivalent. Such an instrument was given by Fleitas [10] for gradient-like flows by means of some circular distributions. We will adapt his technique.

We recall ([10], [23], [37]).

1.6.1 Definition A flow ϕ on a compact manifold M is said to be gradient-like if:

a) ϕ has only a finite number of fixed points, which are all hyperbolic;

b) $\Omega = \mathrm{Fix}\ \phi$ (where $\mathrm{Fix}\ \phi$ is the set of fixed points);

c) all intersections of stable and unstable manifolds for fixed points are transversal.

Condition (c) in dimension 2 excludes the possibility of saddle connections.

We want to allow some saddle connections. So we modify definition above and give

1.6.2 Definition ϕ is quasi-gradient if it satisfies (a), (b) and

(c*) ϕ has no cycles, i.e., there is no sequence $p_0, p_1, \ldots, p_n$ of fixed points (saddles) with $p_0 = p_n$ and $W^u p_{i-1} \cap W^s p_i \neq \emptyset$.

Now we restrict ourselves to compact, connected M , with $\dim M = 2$.

To each quasi-gradient ϕ on M we will associate a collection of circles with some points and arrows attached which we will call circular distribution $\mathcal{D}(\phi)$ associated to ϕ . We do not give precise definition here since this would take too long. This will be presented in Chapter 4 (4.3.1). There we will also define isomorphism ($\tilde{\sim}$) between two distributions and will prove that $\mathcal{D}(\phi)$ determines the topological type of ϕ , i.e. we announce

1.6.3 Theorem IV If ϕ and ψ are quasi-gradient flows,

$$\phi \sim \psi \iff \mathcal{D}(\phi) \tilde{\sim} \mathcal{D}(\psi) .$$

1.6.4 Remark Gradient-like flows are particular cases of our quasi-gradient flows. When ϕ and ψ are gradient-like, $\mathcal{D}(\phi) \tilde{\sim} \mathcal{D}(\psi)$ is equivalent to isomorphism between Fleitas' circular distributions [10] for these flows. Hence Theorem IV is a generalization of Theorem 1b of [10].

1.7 Example with Limit Cycle for $n = 3$

In Chapter 6 we will discuss a family of flows in A_3 for which a Hopf bifurcation occurs. This is given by a one parameter family of matrices A_ε in Z_4 . Specifically, we let

$$A_\varepsilon = 4\begin{pmatrix} 0 & \gamma & -\varepsilon & -\delta \\ -\delta & 0 & \gamma & -\varepsilon \\ -\varepsilon & -\delta & 0 & \gamma \\ \gamma & -\varepsilon & -\delta & 0 \end{pmatrix} \quad \text{with } \gamma > \delta > 0 .$$

The same family has been presented before by Zeeman [41] (for $\gamma = 1, \delta = 0$) and by Hofbauer et al. [17] (where this is called a "circulant" matrix). In both these papers, the existence of an attracting periodic orbit in $\mathring{\Delta}$, for small $\varepsilon > 0$, is pointed out as the result of a Hopf bifurcation at $\varepsilon = 0$. However, neither of these discuss the flow ϕ_{A_ε} globally on Δ for such small values of ε, nor do they attempt to discuss stability of A_ε for small $\varepsilon \neq 0$. Our main contribution here was to prove that the periodic orbit (call it L_ε, $\varepsilon > 0$) is unique in $\mathring{\Delta}$ and has as its basin of attraction $\mathring{\Delta}$ minus a straight line which is the inset (i.e. stable manifold) of the saddle at the barycentre $e = \frac{1}{4}(1,\ldots,1)$ of. Δ.

We also describe ϕ_{A_ε} on $\partial\Delta$, and prove that ϕ_{A_ε} is globally stable (inside family A_3) in the interior $\mathring{\Delta}$ and on $\partial\Delta$.

We collect some of these properties (to be proved in Chapter 6) in the following:

1.7.1 <u>Theorem V</u> For A_ε as above, we have

(i) for $-(\gamma-\delta) < \varepsilon < 0$, the barycentre e is hyperbolic attracting with basin of attraction $W^s e = \mathring{\Delta}$;

(ii) for $\varepsilon = 0$, e is asymptotically attracting (not hyperbolically) with basin of attraction $= \mathring{\Delta}$;

(iii) for $\varepsilon > 0$ sufficiently small, e is a hyperbolic 1-saddle and ϕ_{A_ε} has a hyperbolic attracting periodic orbit L_ε in $\mathring{\Delta}$, with $W^s L_\varepsilon = \mathring{\Delta} - W^s e$;

(iv) for sufficiently small $\varepsilon \neq 0$, A_ε has a neighbourhood N_ε in M_4 such that $\forall\, B \in N_\varepsilon$ there exist homeomorphisms $h:\mathring{\Delta} \to \mathring{\Delta}$, $h_i : F_i \to F_i$ $i = 0,\dots,3$ (where F_i is face $x_i = 0$) taking ϕ_B-orbits onto ϕ_{A_ε}-orbits;

(v) drawings for ϕ_{A_ε} in $\mathring{\Delta}$ and $\partial\Delta$ are given in figure 4 below.

1.7.2 <u>Remark</u> Property (iv) above does not say that A_ε is stable for small $\varepsilon \neq 0$, according to Definition 1.3.3. By (iv) we say that ϕ_{A_ε} is stable (inside family A_3) if restricted to $\mathring{\Delta}$ or to $\partial\Delta$. The difficulty in constructing homeo $h:\Delta \to \Delta$ taking ϕ_B-orbits to ϕ_{A_ε}-orbits is mainly due to the cycle of saddle connections $X_3 \to X_2 \to X_1 \to X_0 \to X_3$ on $\partial\Delta$ which occur for ϕ_{A_ε} and for ϕ_B , $\forall\, B$ near A_ε .

1.8 Study of some combinatorial classes for $n = 3$

For $n = 3$ the classification problem is still unsolved. In Chapter 7 we discuss some of the many difficulties that arise in any attempted study of stable matrices of M_4 . The problem is even more bedevilled by the conjectured occurrence of strange attractors (see [4,5]).

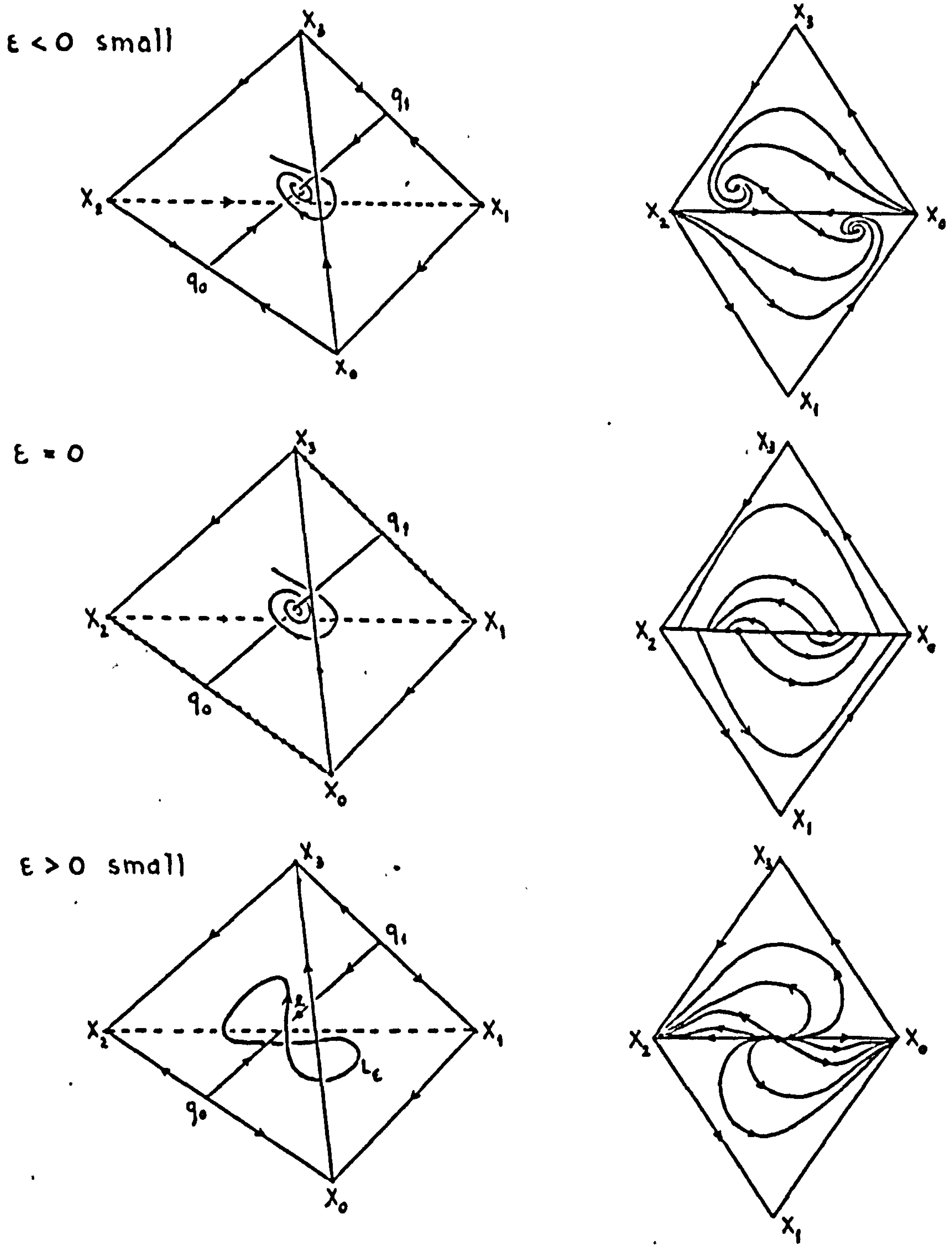


figure 4: phase portraits for ϕ_{A_ε} (small ε) in $\mathring{\Delta}$ and $\partial\Delta$.
($\partial\Delta$ is represented by two faces. Other faces are analogous.)

We will, however, take all combinatorial classes of Z_4^+ satisfying the condition that for all of their matrices A , ϕ_A has no fixed point in $\overset{o}{\Delta}$. We have exactly 18 such combinatorial classes (up to sign reversal) among a total of 114 classes. These are characterized by the fact that all their matrices have one negative row and one positive row (off the diagonal). We call $D_1, D_2, \ldots, D_{18}$ these classes.

Then we prove

1.8.1 <u>Theorem VI</u> For each $k = 1,\ldots,18$, D_k has an open dense subset $\tilde{D}_k$, and open subsets D_k^r $(r = 1,\ldots,\bar{r}(k))$ such that

1) $A \in D_k - \tilde{D}_k \Rightarrow A$ is not stable

2) $\tilde{D}_k = D_k^1 \cup \ldots \cup D_k^{\bar{r}(k)}$

3) each stable class in D_k is contained in D_k^r for some r $(1 \leq r \leq \bar{r}(k))$

4) $A,B \in D_k^r \Rightarrow \phi_A$ and ϕ_B have isomorphic phase diagram and are topologically equivalent on the two-dimensional faces of Δ .

Furthermore, we will prove (in 7.4.8) that some of these subsets D_k^r are in fact stable classes of Z_4^+ , i.e., in these cases, $\forall$ $A,B \in D_k^r$, A and B are stable with $A \sim B$.

Then we conjecture that in fact all the sets D_k^r of Theorem VI are stable classes. The difficulty in proving this conjecture is the presence in most D_k^r's of non-transversal intersections of invariant manifolds for saddles. We will discuss this problem further in 7.4.

CHAPTER 2

PRELIMINARY PROPERTIES

2.1 Introduction

During this chapter we intend to collect some properties of equations (*) of 1.1, which are valid in any dimension n. These properties will be needed for the rest of this work. Many of these are taken from [41], [13], [14] and [1]. We will only state some, but will present the proofs for others, mainly when the proof itself (or some detail in it) gives us any information or technique needed later.

Other properties, which we proved, are mainly technical, but useful in the tackling of later proofs. We also believe that some of these properties may prove useful for future studies of our family of flows. In this category are, for instance, the properties which help us to calculate eigenvalues at fixed points of ϕ_A (2.2.16 to 2.2.19 for fixed point in $\mathring{\Delta}$ when $n = 2,3$ or 4 ; 2.4.2 and 2.4.3 for fixed points on $\partial\Delta$) or to decide their topological types (as 2.2.20) without solving characteristic equation.

Among the known results, it will be important to mention those giving conditions on A for existence, or not, of fixed points of ϕ_A in $\mathring{\Delta}$ (in 2.2), and also the non-existence of non-wandering points in $\mathring{\Delta}$ in case where ϕ_A has no fixed points in $\mathring{\Delta}$ (in 2.3).

We will also include in 2.5 a proof by Hofbauer [13] of equivalence between the replicator equations (*) of 1.1, in Δ minus one $(n-1)$-dim. face, and the equations in $\mathbb{R}^n_+$ known as Lotka-Volterra equations. This property will be useful in Chapter 4 (4.2) when we discuss existence of periodic orbits of ϕ_A for $n = 2$.

2.2 Characterization of fixed points in $\overset{\circ}{\Delta}$

We want to study here properties of fixed points of ϕ_A in $\overset{\circ}{\Delta}$, when existing. Zeeman [41] proved that for A stable, any fixed point of ϕ_A in $\overset{\circ}{\Delta}$ is unique. We will prove (in 2.2.15) that it must be hyperbolic. For $n = 2,3$ or 4 we give expressions for its characteristic equation.

Many of the following properties are in [41] and are here included in a series of lemmas. In all the statements below we will be considering matrix $A \in M_{n+1}$ (or $A \in Z_{n+1}$ when convenient) and associated flow ϕ_A and vectorfield X_A. When we say that p is fixed point we mean that p is fixed for ϕ_A.

First we note that $p \in \overset{\circ}{\Delta}$ is fixed if, and only if, Ap is a multiple of $u = (1,1,\dots,1)$. This is clear from expression for X_A.

Then we have:

2.2.1 Lemma [41]

(i) If there are two fixed points in $\overset{\circ}{\Delta}$, the line joining them is pointwise fixed.

(ii) If p is isolated fixed point in $\overset{\circ}{\Delta}$, then it is unique.

(iii) Existence of isolated fixed point in $\mathring{\Delta}$ is a robust property.

(i.e., if ϕ_A has isolated fixed point p in $\mathring{\Delta}$, there is neighbourhood N of A in M_{n+1} such that $B \in N \Rightarrow \phi_B$ also has isolated fixed point in $\mathring{\Delta}$, and this is near p).

(iv) If p is fixed point in $\mathring{\Delta}$, then:

$$p \text{ is isolated} \iff \begin{cases} \det A \neq 0 \text{ , or} \\ Ap = 0 \text{ , } \operatorname{rank} A = n \text{ , } (\operatorname{adj} A)u \neq 0 \text{ .} \end{cases}$$

In either case $p = \mathring{\Delta} \cap [(\operatorname{adj} A)u]$

where $[(\operatorname{adj} A)u]$ is the subspace of $\mathbb{R}^{n+1}$ generated by $(\operatorname{adj} A)u$.

(v) ϕ_A has isolated fixed point in $\mathring{\Delta} \iff (\operatorname{adj} A)u$ has all components positive (or all negative).

Hence, if $(\operatorname{adj} A)u$ has both positive and negative components, there are no fixed points in $\mathring{\Delta}$ and this property is robust.

(vi) If there are no fixed points in $\mathring{\Delta}$, there are also no periodic orbits in $\mathring{\Delta}$.

2.2.2 <u>Corollary</u> If all fixed points on $\partial\Delta$ are isolated (in Δ), then a fixed point in $\mathring{\Delta}$, if it exists, is unique.

<u>Proof</u> In the proof of 2.2.1(i) (in [41]), it was shown that if there are $p_1, p_2 \in \mathring{\Delta}$ both fixed, then $p_t = tp_1 + (1-t)p_2$ is also fixed for all t with $p_t \in \mathring{\Delta}$. But there are t', t'' such that $p_t \in \mathring{\Delta}$ $\forall$ $t \in (t', t'')$ and $p_{t'}, p_{t''} \in \partial\Delta$. Then $p_{t'}, p_{t''}$ would be non-isolated fixed points

in $\partial\Delta$. Therefore conclusion must hold. □

Corollary above, though simple, is helpful when we impose condition that fixed points in $\partial\Delta$ are hyperbolic (as in 3.3).

2.2.3 Remark In 2.3 we will discuss the property that if ϕ_{A_0} has no fixed points in $\mathring{\Delta}$, it also has no non-wandering points in $\mathring{\Delta}$. This generalizes property 2.2.1 (vi) above.

2.2.4 Lemma [41] A stable => ϕ_A has at most one fixed point in the interior of each face of Δ (including $\mathring{\Delta}$) .

2.2.5 Remark By lemma above, A stable, with fixed point in $\mathring{\Delta}$ implies that this fixed point is isolated and by 2.2.1 (iv) rank A ≥ n . We want here to note that A may be stable with rank A < n as long as ϕ_A has no fixed point in $\mathring{\Delta}$. This can be seen by taking

$$A = \begin{pmatrix} 0 & -1 & -1 & -2 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 2 & -1 & 1 & 0 \end{pmatrix} \quad \text{for which} \quad \operatorname{adj} A = 0 , \quad \operatorname{rank} A = 2 .$$

A is stable. (This will be seen in 7.4, where A ∈ combinatorial class D_1 of 7.4.1, so stability of A is given by Proposition 7.4.8.)

2.2.6 Lemma [41] If $p_0, p_1, \ldots, p_n > 0$, let $P = \begin{pmatrix} p_0 & & 0 \\ & p_1 & \\ 0 & & \ddots\, p_n \end{pmatrix}$.

Then, $\forall\ A \in M_{n+1}$, $AP \sim A$.

2.2.7 Remark In the proof of this lemma, in [41], the homeomorphism of Δ giving the topological equivalence between ϕ_{AP} and ϕ_A is denoted by p and given by $(px)_i = \frac{1}{\pi(x)} p_i x_i$, $i = 0,\ldots,n$, where $\pi(x) = \sum_0^n p_k x_k$.

So, p is in fact a diffeomorphism of Δ onto Δ , (its inverse being of the same form relative to numbers $1/p_0,\ldots,1/p_n$) preserving faces.

It was then shown that

$$(Dp)_x\, X_{AP}(x) = \pi(x)\, X_A(px) \quad .$$

From this we can take the following properties:

2.2.8 Corollary Let $\bar{x}$ be a fixed point of ϕ_{AP} . Then

(i) eigenvalues at $\bar{x}$ (for ϕ_{AP}) are the eigenvalues at $p\bar{x}$ (for ϕ_A) multiplied by $\pi(\bar{x})$;

(ii) $\bar{x}$ is hyperbolic for ϕ_{AP} $\Leftrightarrow$ $p\bar{x}$ is hyperbolic for ϕ_A .

In this case, $\bar{x}$ and $p\bar{x}$ have the same index.

Proof Differentiating $(Dp)_x\, X_{AP}(x) = \pi(x)\, X_A(px)$ at point $\bar{x}$ and using that $X_{AP}(\bar{x}) = X_A(p\bar{x}) = 0$ we get $(Dp)_{\bar{x}}(DX_{AP})_{\bar{x}} = \pi(\bar{x})(DX_A)_{p\bar{x}}(Dp)_{\bar{x}}$ from where (i) follows.

(ii) is consequence of (i). $\square$

Unlike for topological equivalence by a homeomorphism, by substituting AP for its equivalent A , we lose no information about local behaviour of the flow.

2.2.9 <u>Corollary</u> If $p = (p_0, p_1, \dots, p_n) \in \mathring{\Delta}$ is fixed for ϕ_A then $\bar{A} = (n+1)AP$ is equivalent to A and $\phi_{\bar{A}}$ has the barycentre $e = \frac{1}{n+1} u$ $(u = (1,\dots,1))$ as fixed and the eigenvalues at e, for $\phi_{\bar{A}}$, are equal to the eigenvalues at p, for ϕ_A.

<u>Proof</u> $\bar{A} \sim A$ by 2.2.7, and $p(e) = p$.

Also $\Pi(e) = \sum_{k=0}^{n} p_k \cdot \frac{1}{n+1} = \frac{1}{n+1}$.

Conclusion about eigenvalues follows, then, by 2.2.8.

□

2.2.10 <u>Definition</u> [41] A is <u>central</u> if the barycentre of Δ is an isolated fixed point for ϕ_A .

Corollary 2.2.9 says, then, that if ϕ_A has isolated fixed point p in $\mathring{\Delta}$, A is equivalent to a central matrix $\bar{A} = (n+1)AP$, called the <u>centralization</u> of A, and the eigenvalues at e for $\bar{A}$ are exactly the eigenvalues at p for A .

Because of this property, whenever we want to obtain eigenvalues at an isolated fixed point in $\mathring{\Delta}$ for A we suppose A is central (or take its centralization).

2.2.11 <u>Lemma</u> Barycentre e is fixed for ϕ_A , $A = (a_{ij}) \iff \exists\, S \in \mathbb{R}$ s.t. $\sum_{j=0}^{n} a_{ij} = S$ for all $i = 0,\dots,n$ (i.e. all rows of A have the same sum).

Proof e is fixed $\Leftrightarrow (Ae)_i = eAe \quad i = 0,\dots,n$.

Take $S = (n+1)eAe$. Then

$$e \text{ is fixed} \Leftrightarrow (Ae)_i = \sum_{j=0}^{n} a_{ij} \frac{1}{n+1} = eAe = \frac{1}{n+1} S \quad , \quad i = 0,\dots,n$$

$$\Leftrightarrow \sum_{j=0}^{n} a_{ij} = S \quad i = 0,\dots,n \ .$$

□

2.2.12 Proposition If barycentre e is fixed for ϕ_A , taking $y_i = x_i - x_0 \quad i=1,\dots,n$ as coordinates, the linear part of X_A , at e , is $\frac{1}{n+1} By$ where B is the $n\times n$ matrix given by $b_{ij} = a_{ij}-a_{0j}$ $i,j = 1,\dots,n$.

Proof Since $\sum_{i=0}^{n} x_i = 1$,

$$y = (y_1,\dots,y_n) = 0 \Leftrightarrow x = (x_0,x_1,\dots,x_n) = e$$

$$\dot{y}_i = \dot{x}_i - \dot{x}_0 = x_i((Ax)_i - xAx) - x_0((Ax)_0 - xAx) =$$

$$= (x_i - x_0)((Ax)_i - xAx) + x_0((Ax)_i - (Ax)_0) \ .$$

But, for $i = 1,\dots,n$

$$\begin{aligned}(Ax)_i - (Ax)_0 &= \sum_{j=0}^{n} a_{ij}x_j - \sum_{j=0}^{n} a_{0j}x_j \\ &= a_{i0}x_0 + \sum_{j=1}^{n} a_{ij}(y_j + x_0) - a_{00}x_0 - \sum_{j=1}^{n} a_{0j}(y_j + x_0) \\ &= \sum_{j=1}^{n} (a_{ij} - a_{0j})y_j + x_0\left(\sum_{j=0}^{n} a_{ij} - \sum_{j=0}^{n} a_{0j}\right) \ .\end{aligned}$$

Using $\sum_{j=0}^{n} a_{ij} = \sum_{j=0}^{n} a_{0j} = S$ (by 2.2.11) we get

$$(Ax)_i - (Ax)_0 = (By)_i \qquad i = 1,\dots,n .$$

Also,

$$\begin{aligned}(Ax)_i - xAx &= \sum_{k=0}^{n} x_k((Ax)_i-(Ax)_k) = \\ &= x_0((Ax)_i-(Ax)_0) + \sum_{k=1}^{n} x_k((Ax)_i-(Ax)_0-(Ax)_k + (Ax)_0) = \\ &= x_0(By)_i + \sum_{k=1}^{n} x_k((By)_i - (By)_k) \\ &= (By)_i - \sum_{k=1}^{n} (y_k+x_0)(By)_k \\ &= (By)_i - yBy - x_0 \sum_{k=1}^{n} (By)_k .\end{aligned}$$

From $\sum_{i=0}^{n} x_i = 1$, we get $x_0 = \frac{1}{n+1}(1 - \sum_{k=1}^{n} y_k)$. Then

$$\begin{aligned}\dot{y}_i &= y_i((By)_i - yBy - \frac{1}{n+1}(1 - \sum_{k=1}^{n} y_k) \sum_{k=1}^{n} (By)_k) \\ &\quad + \frac{1}{n+1}(1 - \sum_{k=1}^{n} y_k)(By)_i \\ &= \frac{1}{n+1}(By)_i + y_i(By)_i - \frac{1}{n+1}(y_i \sum_{k=1}^{n}(By)_k + (By)_i \sum_{k=1}^{n} y_k) \\ &\quad + y_i(\frac{1}{n+1} \sum_{k,r=1}^{n} y_r(By)_k - yBy) .\end{aligned}$$

Linear part is $\dot{y}_i = \frac{1}{n+1}(By)_i$

i.e. $\dot{y} = (\frac{1}{n+1}B)y$.

$\square$

2.2.13 Remark Proposition above will be useful to calculate eigenvalues at e and to prove that A stable $\Rightarrow$ fixed point in $\overset{\circ}{\Delta}$, if it exists, is hyperbolic. (See remark 1.5.3.)

We also note property in 2.2.13 could be derived after showing equivalence to Lotka-Volterra equations (see remark 2.5.2 (2)).

2.2.14 Lemma If barycentre e is fixed for ϕ_A , with eigenvalues $\{\lambda_i;\ i = 1,\ldots,n\}$, then e is also fixed for $\phi_{A+\varepsilon I}$, $\forall\varepsilon$, with eigenvalues $\{\lambda_i + \frac{\varepsilon}{n+1}\}$.

Proof Let $A = (a_{ij})$ and $B = (b_{ij})$ as in Proposition 2.2.12.
i.e. $b_{ij} = a_{ij} - a_{0j}$, $i,j = 1,\ldots,n$. Let $S = \sum_{j=0}^{n} a_{ij}$.
Denote $A_\varepsilon = A + \varepsilon I = (a_{ij} + \varepsilon\delta_{ij})$ $(\delta_{ii} = 1$, $\delta_{ij} = 0$ $i \neq j)$.
Then $\sum_{j=0}^{n}(a_{ij} + \varepsilon\delta_{ij}) = S+\varepsilon$ $\forall\ i = 0,\ldots,n$, so, by 2.2.11, e is fixed for ϕ_{A_ε} .

By Proposition 2.2.12, eigenvalues at e for ϕ_A are zeros of $\det(\lambda I - \frac{1}{n+1}B_\varepsilon) = 0$ where

$$B_\varepsilon = (b^\varepsilon_{ij}),\ b^\varepsilon_{ij} = (a_{ij} + \varepsilon\delta_{ij}) - (a_{0j} + \varepsilon\delta_{0j}) = b_{ij} + \varepsilon\delta_{ij}$$

i.e. $B_\varepsilon = B + \varepsilon I$.

So $\det(\lambda I - \frac{1}{n+1} B) = 0 \iff \det((\lambda + \frac{\varepsilon}{n+1})I - \frac{1}{n+1} B_\varepsilon) = 0$

and

λ is eigenvalue at e for $\phi_A \iff \lambda + \frac{\varepsilon}{n+1}$ is eigenvalue, at e, for $\phi_{A+\varepsilon I}$.

□

2.2.15 <u>Proposition</u> Suppose ϕ_A has fixed point $p \in \overset{\circ}{\Delta}$.
Then: A stable $\Rightarrow$ p is hyperbolic.

<u>Proof</u> A stable with $p \in \overset{\circ}{\Delta}$ fixed $\Rightarrow$ (by 2.2.4) p is isolated $\Rightarrow$ (2.2.10) centralization $\bar{A}$ is stable with e fixed and by (2.2.9)

{eigenvalues at p for ϕ_A} = {eigenvalues at e for $\phi_{\bar{A}}$}.

So it is now enough to show that e is hyperbolic fixed point of $\phi_{\bar{A}}$.

Suppose it is not, then $\phi_{\bar{A}}$ has eigenvalue λ at e with real part $\mathrm{Re}(\lambda) = 0$.

By Lemma 2.2.14, $\phi_{\bar{A}+ I}$ has eigenvalue $\lambda + \frac{\varepsilon}{n+1}$, $\forall\varepsilon$. So, for all sufficiently small $\varepsilon > 0$, $\bar{A} + \varepsilon I$ and $\bar{A} - \varepsilon I$ are not equivalent, since $\phi_{\bar{A}+\varepsilon I}$ and $\phi_{\bar{A}-\varepsilon I}$ are not topologically equivalent, locally at e (because e has index for $\phi_{\bar{A}+\varepsilon I}$ < index for $\phi_{\bar{A}-\varepsilon I}$, if $\varepsilon > 0$). Therefore e must be hyperbolic fixed point for $\phi_{\bar{A}}$, and p must be hyperbolic fixed point for ϕ_A, if A is stable. □

2.2.16 <u>Remark</u> In [41], Zeeman gave a simple expression for the characteristic equation at e for a central matrix, when $n=2$, as follows:

if A is central in Z_3, A can be written as

$$A = \begin{pmatrix} 0 & \theta+\alpha_0 & \theta-\alpha_0 \\ \theta-\alpha_1 & 0 & \theta+\alpha_1 \\ \theta+\alpha_2 & \theta-\alpha_2 & 0 \end{pmatrix}$$

and, then, the eigenvalues at $e = \frac{1}{3}u$ for ϕ_A are the roots of $\lambda^2 + \frac{2\theta}{3}\lambda + \frac{\theta^2+\rho}{9} = 0$ where $\rho = \alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2$.

Letting S be as in 2.2.11, then $S = 2\theta$.

We take $P = \sum_{0\le i\le j\le 2} a_{ij}a_{ji} = a_{01}a_{10} + a_{02}a_{20} + a_{12}a_{21}$.

So $P = (\theta+\alpha_0)(\theta-\alpha_1) + (\theta-\alpha_0)(\theta+\alpha_2) + (\theta+\alpha_1)(\theta-\alpha_2)$

$= 3\theta^2 - \rho$.

Then the characteristic equation is written as

$$\lambda^2 + \frac{1}{3}S\ \lambda + \frac{1}{9}(S^2-P) = 0 \quad .$$

In the next Proposition, we give a similar expression for the characteristic equation for central matrix in Z_4 (i.e. $n = 3$), at fixed point e .

2.2.17 Notation Below, and in many other places in this work, we will use the following notation:

F_i = the (n-1)-dimensional face of Δ , given by $x_i = 0$.

If $A \in M_{n+1}$, A_i is the matrix in M_n obtained by eliminating in A row i and column i . Then $X_A|_{F_i} = X_{A_i}$.

2.2.18 <u>Proposition</u> If $A \in Z_4$ is central, then the characteristic equation at e for ϕ_A is

$$\lambda^3 + \tfrac{1}{4} S\lambda^2 + \frac{1}{16}(S^2-P)\lambda + \frac{1}{64}(S(S^2-P)-D) = 0$$

where

$$S = \sum_{j=0}^{n} a_{ij} \ ; \quad P = \sum_{0\le i<j\le 3} a_{ij}a_{ji} \ , \quad D = \sum_{i=0}^{3} D_i$$

with $D_i = \det A_i$.

<u>Proof</u> Using Proposition 2.2.12 we have that characteristic equation is $\det(\lambda I - \tfrac{1}{4} B) = 0$ where $B = (b_{ij}) \in M_3$ where $b_{ij} = a_{ij} - a_{0j}$. Making $\alpha = 4\lambda$ we have $\det(\alpha I - B) = 0$, i.e.

$$\det\begin{pmatrix} \alpha + a_{01} & -a_{12} + a_{02} & -a_{13} + a_{03} \\ -a_{21} + a_{01} & \alpha + a_{02} & -a_{23} + a_{03} \\ -a_{31} + a_{01} & -a_{32} + a_{02} & \alpha + a_{03} \end{pmatrix} = 0 .$$

Then, by calculating this determinant, and simplifying, using $\sum_{j=0}^{n} a_{ij} = S$, and $a_{ii} = 0 \ \forall\, i = 0,\dots,3$ we arrive at

$$\alpha^3 + S\,\alpha^2 + (S^2-P)\alpha + S(S^2-P)-D = 0 .$$

This proves our assertion. □

2.2.19 <u>Remark</u> A_i is the matrix determining flow ϕ_A restricted to the 2-dim face F_i (opposite to vertex X_i) . So D = sum of determinants on 2-dim faces. An edge X_iX_j is a 1-dim face, and on it ϕ_A is determined by matrix $\begin{pmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{pmatrix}$ whose determinant is $-a_{ij}a_{ji}$. So, $-P$ = sum of determinants on 1-dim faces.

Following analogous procedure for a central matrix A in Z_5 (i.e. $n = 4$) , we can show that characteristic equation at e for ϕ_A is

$$\lambda^4 + \frac{S}{5}\lambda^3 + \frac{1}{25}(S^2-P)\lambda^2 + \frac{1}{125}(S(S-P)-D)\lambda + \frac{1}{625}(S(S(S^2-P)-D)-\pi) = 0$$

where $S = \sum_{j=0}^{n} a_{ij}$

$-P$ = sum of determinants on 1-dim faces

D = sum of determinants on 2-dim faces

$-\pi$ = sum of determinants on 3-dim faces .

To prove this expression involves long calculations which we do not show here, since it is not relevant for the rest of the work. However, these expressions (for $n = 2,3,4$) give a hint on how such characteristic equations probably look like for greater values of n .

On the next proposition we give conditions (depending only on the values of S, P and D) for e to be hyperbolic attractor, repellor, 1- or 2-saddle for ϕ_A for central $A \in Z_4$. This is useful since (unlike for $A \in Z_3$) eigenvalues are not always easily obtained by solving characteristic equation.

2.2.20 <u>Proposition</u> Let $A \in Z_4$ and S, P and D be as in 2.2.18. Then barycentre e is hyperbolic fixed point $\Leftrightarrow$ $D \neq S(S^2-P)$ and $S^2 < P$ if $D = 0$. In this case

i) e is an attractor $\Leftrightarrow$ $S > 0$, $S^2 > P$, $S(S^2-P) > D > 0$.

ii) e is repellor $\Leftrightarrow$ $S < 0$, $S^2 > P$, $S(S^2-P) < D < 0$.

iii) e is saddle $\Leftrightarrow$ conditions on (i) and (ii) fail. Then

e is 1-saddle (i.e. $\dim W^s e = 1$) $\Leftrightarrow$ $S(S^2-P) > D$

e is 2-saddle (i.e. $\dim W^s e = 2$) $\Leftrightarrow$ $S(S^2-P) < D$.

<u>Proof</u> The conditions above are simple, based on the fact that a cubic polynomial equation

$$x^3 + Ax^2 + Bx + C = 0$$

has zero root $\Leftrightarrow$ $C = 0$

complex conjugate pair of imaginary roots $\Leftrightarrow$ $AB = C$, $B > 0$.

When $C \neq 0$ and $B < 0$ if $AB = C$, denoting α_i the real parts of roots λ_i, $i = 1,2,3$, we have

i) all $\alpha_i < 0$ $\Leftrightarrow$ $A, B, C > 0$ and $AB > C$

ii) all $\alpha_i > 0$ $\Leftrightarrow$ $A, C < 0$, $B > 0$ and $AB < C$

iii) if α_i do not all have the same sign, then one is < 0 , two are > 0 $\Leftrightarrow$ $C > 0$

one is > 0 , two are < 0 $\Leftrightarrow$ $C < 0$.

Applying this to

$$\lambda^3 + \frac{S}{4}\lambda^2 + \frac{1}{16}(S^2-P)\lambda + \frac{1}{64}(S(S^2-P)-D) = 0$$

we obtain conditions as asserted.

□

2.3 Flows ϕ_A without fixed points in $\overset{\circ}{\Delta}$

The aim of this paragraph is to present the important property that flows ϕ_A having no fixed points in $\overset{\circ}{\Delta}$, have all their non-wandering points on $\partial\Delta$. This property is in Theorem 2.3.1 below which was first proved by Hofbauer [14], and later by Akin [1]. We will include below (in 2.3.3) Hofbauer's proof, which is short and elegant, and also (in 2.3.6) a sketch of Akin's proof that, though longer, gives us an explicit expression for function V of the theorem, allowing more precise conclusions to be taken.

2.3.1 Theorem ([14], [1]) If ϕ_A has no fixed points in $\overset{\circ}{\Delta}$, then there exists a C^∞ function $V : \overset{\circ}{\Delta} \to \mathbb{R}$ which is strictly decreasing on orbits of ϕ_A in $\overset{\circ}{\Delta}$.

2.3.2 Remark A function decreasing on orbits is usually called a Liapunov function for the flow.

Consequently, ϕ_A has no non-wandering points in $\overset{\circ}{\Delta}$.

2.3.3 1st Proof of 2.3.1 ([14])

The convex set $C = \{Ax \; ; \; x \in \overset{\circ}{\Delta}\} \subset \mathbb{R}^{n+1}$ is disjoint from subspace M generated by vector $u = (1,1,\dots,1)$ because $p \in \mathring{\Delta}$ is fixed $\Leftrightarrow$ $Ap \in M$.

Hence there exists linear functional $f: \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x) = \sum_{i=0}^{n} c_i x_i$ such that $f|_M = 0$ and $f(x) < 0 \quad \forall\, x \in C$.

But $f|_M = 0 \Rightarrow \sum_{i=0}^{n} c_i = 0$. So, function $V : \mathring{\Delta} \to \mathbb{R}$, given by $V(x) = \prod_{i=0}^{n} x_i^{c_i}$, is well-defined, positive and has no equilibrium point in $\mathring{\Delta}$.

Also,

$$\dot{V}(x) = \frac{d}{dt} V(\phi_A(t,x))\Big|_{t=0} = V(x) \sum_{i=0}^{n} c_i \frac{\dot{x}_i}{x_i} =$$

$$= V(x)\left(\sum_{i=0}^{n} c_i (Ax)_i - \left(\sum_{i=0}^{n} c_i\right) xAx\right) =$$

$$= V(x) f(Ax) < 0 \qquad \forall\, x \in \mathring{\Delta}.$$

□

The second proof is longer, so we present here only a sketch. But first we need some new definitions, which we also take from [1].

2.3.4 <u>Definition</u> Let $q_1, q_2 \in \Delta$, $A \in M_{n+1}$. We say that q_1 <u>dominates</u> q_2 if $(q_1 A)_j \geq (q_2 A)_j \quad \forall\, j = 0,1,\dots,n$ with strict inequality for at least one j. (Note $(qA)_j = \sum_{i=0}^{n} a_{ij} q_i$.)

Also, q_1 <u>strictly dominates</u> q_2 if $(q_1 A)_j > (q_2 A)_j \quad \forall\, j = 0,\dots,n$.

2.3.5 Definition For $x \in \Delta$, we call support of x ,denoted by $\mathrm{supp}(x)$, the set $\{i \in \{0,\dots,n\}; x_i > 0\}$.

2.3.6 2nd Proof of 2.3.1 ([1])

This consists of a series of steps. We state what is proved in each step, without the actual proofs.

Step 1: For $p \in \Delta$, $Ap \in [u] \iff \tilde{A}p = 0$
where $\tilde{A}_{ij} = a_{ij} - \frac{1}{n+1} \sum_{k=0}^{n} a_{kj}$.

(Remember: $p \in \mathring{\Delta}$ is fixed for $\phi_A \iff Ap \in [u]$.)

Step 2: $\tilde{A}x = 0$ has no solution in $\mathring{\Delta}$ (Δ)
$\iff$ there exists q^+ , $q^- \in \Delta$, with disjoint support such that q^+ dominates q^- (or q^+ strictly dominates q^- , respectively).

(This is the crucial step in the proof.)

Step 3: If q^+ dominates q^- , let $I^+ = \mathrm{supp}(q^+)$, $I^- = \mathrm{supp}(q^-)$. If $I^+ \cap I^- = \emptyset$, we define

$$V(x) = \prod_{i \in I^-} (x_i)^{q_i^-} \Big/ \prod_{i \in I^+} (x_i)^{q_i^+} \qquad \text{for } x \in \mathring{\Delta} .$$

Then V is positive, with no equilibrium points in $\mathring{\Delta}$, and $\dot{V}(x) < 0 \quad \forall\, x \in \mathring{\Delta}$.

Steps 1,2 and 3 complete proof of 2.3.1. □

In [1], the proof goes on with two further steps, which we here present as propositions 2.3.6 and 2.3.7.

Let D (SD) be the subset of M_{n+1} consisting of the matrices A for which there are $q_1, q_2 \in \Delta$ with q_1 dominating q_2 (resp. q_1 strictly dominating q_2) . Let $D' = D - SD$.

2.3.6' <u>Proposition</u> [1] SD is open in M_{n+1} , $D' \subset \partial(SD)$ and $\forall A \in D'$, there exists a continuous one-parameter family $A^{\varepsilon} \in M_{n+1}$ with $A^0 = A$, $A^{\varepsilon} \in SD$ for $\varepsilon > 0$ and $A^{\varepsilon} \notin D$ for small $\varepsilon < 0$ (hence for $\varepsilon < 0$, ϕ_{A_ε} has fixed point $p_\varepsilon \in \overset{\circ}{\Delta}$) . Furthermore $p_\varepsilon \to p \in \partial\Delta$ as $\varepsilon \to 0^-$ where p is fixed for ϕ_A .

2.3.7 <u>Proposition</u> [1] Let $q^+, q^- \in \Delta$ with disjoint support. If q^+ strictly dominates q^- (for A) then:

$$\lim_{t\to+\infty} \prod_{i\in I^-} (x_i(t))^{q_i^-} = 0$$

and

$$\lim_{t\to-\infty} \prod_{i\in I^+} (x_i(t))^{q_i^+} = 0$$

(where $x(t) = \phi_A(t,x)$ for $x \in \overset{\circ}{\Delta}$) .

Hence, $\forall x \in \overset{\circ}{\Delta}$ we have

$$\omega(x) \subset \bigcup_{i\in I^-} F_i \quad , \quad \alpha(x) \subset \bigcup_{i\in I^+} F_i$$

(where α- and ω-limits are taken for flow ϕ_A).

So we can also take:

2.3.8 Corollary

(i) $A \in D' \Rightarrow A$ is not stable

(ii) A stable with no fixed points in $\overset{\circ}{\Delta} \Rightarrow A \in SD$

(iii) $A \in D' \Rightarrow \phi_A$ has fixed point $p \in \partial\Delta$ which is not hyperbolic.

Proof (i) and (ii) follow clearly from 2.3.6'. (iii) is also a consequence of 2.3.6, by taking $p = \lim_{\varepsilon \to 0^-} p_\varepsilon \in \partial\Delta$. If p was hyperbolic, there would exist neighbourhoods V of p in Δ and N of A in M_{n+1} such that $\forall B \in N$, ϕ_B has only one fixed point in V , which must be on $\partial\Delta$. This would contradict $A_\varepsilon \to A$ with p_ε fixed for ϕ_{A_ε} and $p_\varepsilon \to p$. So p is not hyperbolic. □

2.4 Fixed points of ϕ_A on $\partial\Delta$

Our main purpose in this paragraph is to prove that A stable implies that all fixed points of ϕ_A on $\partial\Delta$ are hyperbolic (in 2.4.4) and also to give explicit expressions for calculating eigenvalues at these points (in 2.4.2 or 2.4.3).

We already know (see 1.5.1 (6)) that A stable $\Rightarrow$ vertices are hyperbolic, and eigenvalues at X_i for ϕ_A are $\{a_{ij}-a_{ii} ; j \neq i\}$.

Now we will deal with fixed points q for ϕ_A $q \in \partial\Delta$-vertices.

So $I_q = \mathrm{supp}(q) \subsetneq \{0,1,\dots,n\}$ and I_q contains at least two indices i. Let $k = \# I_q - 1$. Then $1 \le k \le n-1$. We will denote by F the k-dimensional face of Δ, given by $F = \{x \in \Delta ;\ x_i = 0 \ \forall\, i \notin I_q\}$. Then $q \in$ interior of face F (denoted by $\mathring{F}$).

Let A_F be the $(k+1) \times (k+1)$ matrix obtained from A by eliminating all rows and columns with indices not in I_q.

Then flow ϕ_A restricted to face F is the flow ϕ_{A_F} on k-simplex F.

We know that A stable $\Rightarrow A_F$ stable in M_k. Hence, using Proposition 2.2.15 we get

2.4.1 <u>Lemma</u> A stable $\Rightarrow q$ is hyperbolic for restriction of ϕ_A to face F.

Now we calculate eigenvalues at q, i.e.

2.4.2 <u>Proposition</u> {eigenvalues of ϕ_A at q} =

= {eigenvalues of ϕ_{A_F} at q} $\cup$ $\{\lambda_i = (Aq)_i - qAq ; i \notin I_q\}$.

<u>Proof</u> To simplify notation, we can suppose that a permutation of vertices was made so that $I_q = \{0,1,\dots,k\}$. So $x \in F \Leftrightarrow x_i = 0 \ \forall\, i = k+1,\dots,n$ and $q = (q_0, q_1, \dots, q_k, 0, \dots, 0)$, $\sum_{i=0}^{k} q_i = 1$ and $q_i > 0$ for $0 \le i \le k$.

Let us take coordinates $y_1,\dots,y_n$ where

$$y = (y_0,y_1,\dots,y_n) = x - q\ , \quad \text{i.e.} \quad \begin{cases} y_i = x_i - q_i & 0 \le i \le k \\ y_i = x_i & k < i \le n\ . \end{cases}$$

Since q is fixed (and $q \in \overset{\circ}{F}$) $\Rightarrow$ $(Aq)_i = qAq$ for $0 \le i \le k$.

Also $y = 0 \iff x = q$, $\sum_{i=0}^{n} y_i = 0\ \forall y$,

and $\dot y_i = \dot x_i = x_i\ ((Ax)_i - xAx)$ $i = 1,\dots,n$. So,

$$1 \le i \le k \Rightarrow \dot y_i = (y_i+q_i)((Ay)_i-\cancel{(Aq)_i}-yAy-qAy-yAq-\cancel{qAq})$$

$$k < i \le n \Rightarrow \dot y_i = y_i((Ay)_i-(Aq)_i-yAy-qAy-yAq-qAq)\ .$$

Since $y_0 = -\sum_{i=1}^{n} y_i$, $(Ay)_i$, qAy and yAq are linear in $(y_1,\dots,y_n)$ and yAy is of second order. Therefore *linearization* at q is

$$\begin{cases} \dot y_i = q_i\ ((Ay)_i - qAy - yAq) & 1 \le i \le k \\ \dot y_i = y_i\ ((Aq)_i - qAq) = \lambda_i y_i & k < i \le n\ . \end{cases}$$

Hence $\lambda_i = (Aq)_i - qAq$ for $i \notin \{0,\dots,k\} = I_q$ are eigenvalues of ϕ_A at q with eigenvectors in the y_i-directions (respectively) which are transversal to face F .

So our assertion is proved.

$\square$

2.4.3 <u>Corollary</u> For any $i \notin I_q$, λ_i (as above) can also be calculated by $\lambda_i = (Aq)_i - (Aq)_j$ where j is any index in I_q.

<u>Proof</u> This follows from $\lambda_i = (Aq)_i - qAq$ plus the fact that $q \in \overset{\circ}{F} \Rightarrow (Aq)_j = qAq \ \forall j \in I_q$. □

2.4.4 <u>Proposition</u> A stable $\Rightarrow$ fixed point q for ϕ_A, in the interior of any k-dimensional face F of Δ, is hyperbolic.

<u>Proof</u> Take notations as before. By lemma 2.4.1 A stable $\Rightarrow$ q is hyperbolic for restriction to F. So it is sufficient now to prove that $\lambda_i \neq 0 \ \forall i \notin I_q$.

Suppose $\lambda_r = 0$ for some $r \notin I_q$ $r \in \{0,1,\dots,n\}$. We construct, below, a continuous one-parameter family $A^\varepsilon \in M_{n+1}$ such that $A^0 = A$ and q is fixed point for ϕ_{A^ε} with $\lambda_i+\varepsilon$ $(i \notin I_q)$ as eigenvalues in directions transversal to F. Then $\lambda_r = 0$ (for some $r \notin I_q$) would imply that $A^{-\varepsilon}$ and A^ε are not equivalent for all sufficiently small $\varepsilon > 0$, and, so, A would be not stable, contradicting the hypothesis.

Therefore $\lambda_i \neq 0 \ \forall i \in I_q$, if A is stable, and q is hyperbolic fixed point.

For the construction of family A^ε we suppose (as in 2.4.2) that $I_q = \{0,1,\dots,k\}$ and we take

$$C = (c_{ij}) = \begin{pmatrix} 1/q_0 & & & & & & 0 \\ & 1/q_1 & & & & & \\ & & \ddots & & & & \\ & & & 1/q_k & & & \\ & & & & 1 & & \\ 0 & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

and let $A^\varepsilon = A - \varepsilon C$

$q \in \overset{o}{F}$ fixed for $\phi_A \Rightarrow (Aq)_i = qAq$ for $i \in I_q = \{0,\dots,k\}$

$$\Rightarrow (A^\varepsilon q)_i = \sum_{j=0}^{n} (a_{ij} - \varepsilon c_{ij}) q_j = (Aq)_i - \varepsilon = qAq - \varepsilon =$$

$$= qA^\varepsilon q \quad \text{for } 0 \le i \le k$$

$\Rightarrow q$ is fixed for ϕ_{A^ε} .

Also for $k+1 \le i \le n$

$$(A^\varepsilon q)_i - (A^\varepsilon q)_0 = \sum_{j=0}^{n} (a_{ij} - \varepsilon c_{ij}) q_j - \sum_{j=0}^{n} (a_{0j} - \varepsilon c_{0j}) q_j =$$

$$= (Aq)_i - \varepsilon c_{ii} \cdot \underset{\substack{\| \\ 0}}{q_i} - (Aq)_0 + \varepsilon \underset{\substack{\| \\ 1/q_0}}{c_{00}} \cdot q_0$$

$$= \lambda_i + \varepsilon \quad .$$

Then, by 2.4.3, $\lambda_i + \varepsilon$ is eigenvalues at q for ϕ_{A^ε} , and this concludes the proof. □

We know by 2.2.1 (v) and 2.2.4 that A stable with fixed point p in $\overset{o}{\Delta} \Rightarrow \beta = (\text{adj } A)u$ has $\beta_i > 0 \ \forall\, 0 \le i \le n$ (or $\beta_i < 0 \ \forall\, 0 \le i \le n$).

But in remark 2.2.5 we showed that A may be stable (with no fixed point in $\overset{\circ}{\Delta}$) with $\beta_i = 0 \ \forall\ 0 \le i \le n$. Corollary 2.4.5 below is related to these properties.

2.4.5 Corollary Let $\beta = (\text{adj } A)u$.

If $\beta_i \ge 0$ (or $\beta_i \le 0$) for all $i = 0,\dots,n$ with $\beta_i = 0$ for some i, but not all, then A is not stable.

Proof By permutation of vertices, we may suppose $\beta_i > 0$ (or $\beta_i < 0$) for $0 \le i \le k$ and $\beta_i = 0$ for $k < i \le n$ (where $0 \le k < n$) .

Let $c = \sum_{i=0}^{k} \beta_i$, $q = \frac{1}{c}\beta$ and $F = \{x \in \Delta \ ; \ x_i = 0 \ \text{ for } k < i \le n\}$.

Then $q \in \overset{\circ}{F}$ and q is fixed for ϕ_A with

$$(Aq)_i = \frac{1}{c}\sum_{j=0}^{n} a_{ij}\beta_j = \frac{1}{c}\sum_{j=0}^{n} a_{ij}\sum_{k=0}^{n}(\text{adj } A)_{jk}$$

$$= \frac{1}{c}\sum_{k=0}^{n}(A.\text{adj } A)_{ik} = \frac{1}{c}\det A \quad \text{for } 0 \le i \le n \ .$$

So, by 2.4.3, $\lambda_i = 0 \ \forall\ i \notin I_q$, i.e., all eigenvalues for ϕ_A at q , in directions transversal to F , are zero. Therefore q is not hyperbolic and, by 2.4.4, A is not stable. □

In the next proposition, we treat the special case of a fixed point q in the interior of a $(n-1)$-dimensional face F of Δ . So, there exists (unique) vertex X_k opposite to F , i.e. $F = \{x \in \Delta ; x_k = 0\}$, i.e. $F = F_k$ of 2.2.17.

2.4.6 <u>Proposition</u> If q is fixed point in $\overset{o}{F}_k$, then:

q is hyperbolic $\iff$ q is hyperbolic for restriction to F_k and $\beta_k = (\text{adj } A\, u)_k \neq 0$.

<u>Proof</u> By Proposition 2.4.2, it is enough to prove that $\lambda_k = (Aq)_k - qAq$ is equal to β_k multiplied by a non-zero constant.

For simplification of notation, we suppose a permutation of vertices was made so that $k = 0$. We write $F = F_0$.

Then $q = (0, q_1, \ldots, q_n)$ with $q_i > 0$ for $1 \le i \le n$ and $\sum_{i=1}^{n} q_i = 1$. We denote $\bar{q} = (q_1, \ldots, q_n)$.

If q is hyperbolic fixed point for ϕ_{A_F} , then q is isolated fixed point in $\overset{o}{F}$, and, by 2.2.1(v), $(\text{adj } A_F\, u)_i > 0$ (or < 0) for all $i = 1, \ldots, n$. We let $c = \sum_{i=1}^{n} (\text{adj } A_F\, u)_i$. Then $c \neq 0$, and

$$\bar{q} = [\text{adj } A_F\, u] \cap \overset{o}{F} \quad \text{(see 2.1.1(iv))} \quad \Rightarrow$$

$$q_i = \frac{1}{c} (\text{adj } A_F\, u)_i \quad \forall\, i = 1, \ldots, n \ .$$

Also $(Aq)_i = (A_F \bar{q})_i = qAq \quad i = 1, \ldots, n$ implies $\det A_F = c(qAq)$.

By Proposition 2.4.2, the eigenvalue for ϕ_A at q in direction transversal to $F = F_0$ is $\lambda_0 = (Aq)_0 - qAq$. Now we will prove that $(\text{adj } A\, u)_0 = -c\, \lambda_0$ and this implies the assertion.

We denote by A^{rs} the matrix obtained from A by eliminating row r and column s . (Then $A_F = A^{00}$)

$$(\mathrm{adj}\ A\ u)_0 = \sum_{r=0}^{n} (\mathrm{adj}\ A)_{0r} = \sum_{r=0}^{n} (-1)^r \det A^{r0} =$$

$$= \det A_F + \sum_{r=1}^{n} (-1)^r \sum_{s=1}^{n} a_{0s}(-1)^{s+1} \det(A^{r0})^{0s} =$$

$$= \det A_F - \sum_{s=1}^{n} a_{0s} \sum_{r=1}^{n} (-1)^{r+s} \det(A_F)^{rs} =$$

$$= \det A_F - \sum_{s=1}^{n} a_{0s} \sum_{r=1}^{n} (\mathrm{adj}\ A_F)_{sr} =$$

$$= \det A_F - \sum_{s=1}^{n} a_{0s}(\mathrm{adj}\ A_F\ u)_s =$$

$$= c(qAq) - c \sum_{s=1}^{n} a_{0s}\, q_s =$$

$$= -c((Aq)_0 - qAq) =$$

$$= -c\ \lambda_0 .$$

So $(\mathrm{adj}\ A\ u)_0 = -c\lambda_0$ and

$$\lambda_0 \neq 0 \iff (\mathrm{adj}\ A\ u)_0 \neq 0 .$$

This completes the proof. □

2.4.7 Remarks

(1) Proposition above will be useful in next Chapter 3 (in 3.3), where we use corollary 2.4.8 below.

(2) When $\det A_F \neq 0$, also $qAq \neq 0$ and, then, $c = \det A_F/qAq$. But we may have $\det A_F = qAq = 0$ even for stable A .

2.4.8 Corollary If $A \in Z_3^+$ and $a_{ij}a_{ji} > 0$, then edge X_iX_j has fixed point q in its interior with eigenvalues $-a_{ij}a_{ji} / (a_{ij}+a_{ji})$ and $\beta_k/(a_{ij}+a_{ji})$ (where $\beta = (\text{adj } A)u$ and $k \neq i,j$) and

$\beta_k = a_{ki}a_{ij} + a_{kj}a_{ji} - a_{ij}a_{ji}$.

Proof Suppose $i = 1, j = 2, k = 0$. Then $a_{ij}a_{ji} > 0$
$\Rightarrow q = (0,a_{12},a_{21}) / (a_{12}+a_{21})$ is fixed in $\overset{\circ}{X_1X_2}$, with eigenvalue $-a_{12}a_{21} / (a_{12}+a_{21})$ for restriction to edge X_1X_2, with $A_0 = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$.

Also $\lambda_0 = (Aq)_0 - qAq = -\frac{1}{c}\beta_0$ (for $\beta_0 = (\text{adj}Au)_0$)

where $c = \frac{1}{qAq} \det A_0 = -(a_{12}+a_{21})$

hence $\lambda_0 = \beta_0/(a_{12}+a_{21})$.

Here adj A is easily obtained so we get expression for β_0 as stated. □

2.4.9 Remark We know that A stable $\Rightarrow$

- vertices are hyperbolic fixed points, by 1.5.1 (3), (6)
- fixed point in $\overset{\circ}{\Delta}$, if it exists, is unique and hyperbolic, by 1.5.1(4) and 2.2.15
- fixed points in interior of faces on $\partial\Delta$ are hyperbolic, by 2.4.4.

This completes proof of Theorem II, stated in 1.5.2.

2.5 Equivalence to Lotka-Volterra equations

The following property was proved by Hofbauer [13] in order to prove Zeeman's conjecture of non-existence of periodic orbits for ϕ_A, when A is stable, for $n=2$.

As noted in the introduction, the equivalence between the "replicator equations" (*) of 1.1 and the equations known as "Lotka-Volterra" is interesting since both are often applied to the same kind of studies, independently.

The equivalence to be shown is valid for any dimension n. So, we include it here in our preliminary properties, but we postpone the property of non-existence of periodic orbits in stable cases (for $n=2$) to next Chapter 3 where we deal exclusively with the two-dimensional case.

Given any $n\times(n+1)$ matrix (a_{ij}) $i=1,\dots,n$ $j=0,1,\dots,n$, the quadratic differential equation

$$(**)\quad \dot{y}_i = y_i(a_{i0} + \sum_{j=1}^{n} a_{ij}y_j) \qquad i=1,\dots,n$$

defined on $\mathbb{R}_+^n = \{y=(y_1,\dots,y_n) \in \mathbb{R}^n \;;\; y_i \geq 0\}$ is known as a Lotka-Volterra equation, or simply a Volterra equation.

2.5.1 Theorem ([13]) Letting $a_{0j}=0$ for all $j=0,1,\dots,n$ equation (**) above is equivalent to the replicator equation

$$(*)\quad \dot{x}_i = x_i(\sum_{j=0}^{n} a_{ij}x_j - \sum_{k,j=0}^{n} a_{kj}x_kx_j) = x_i((Ax)_i - xAx) = X_A(x)$$

where $A=(a_{ij})$, $\sum_{i=0}^{n} x_i = 1$, $x_i \geq 0$ $i=1,\dots,n$, $x_0 > 0$.

Proof Let, then, $a_{0j} = 0$ for $0 \leq j \leq n$, and make $y_0 = 1$.
Then (**) can be written as $\dot{y}_i = y_i(Ay)_i$

where $y = (y_0, y_1, \ldots, y_n) \in P = \{y \in \mathbb{R}^{n+1};\ y_0 = 1\ ,\ y_i \geq 0\ \ i = 1,\ldots,n\}$.

(Note that $\dot{y}_0 = 0$.)

Define transformation $h: P \to \Delta$ by

$$h(y) = x \text{ with } x_i = y_i \Big/ \sum_{j=0}^{n} y_j \ .$$

So $h(P) = \Delta - F_0$, F_0 = face $x_0 = 0$. Inverse of h (defined on $\Delta - F_0$) is given by $y_i = x_i/x_0$. Then

$$\dot{x}_i = \frac{\dot{y}_i}{\sum_j y_j} - \frac{y_i}{(\sum_j y_j)^2} \sum_j \dot{y}_j = \frac{y_i}{\sum_j y_j} \sum_k a_{ik} y_k - \frac{y_i}{(\sum_j y_j)^2} \sum_{j,k} a_{jk} y_j y_k =$$

$$= \frac{1}{x_0} x_i \Big(\sum_k a_{ik} x_k - \sum_{j,k} a_{jk} x_j x_k\Big)$$

which is replicator equation (*) above but for a change in time.

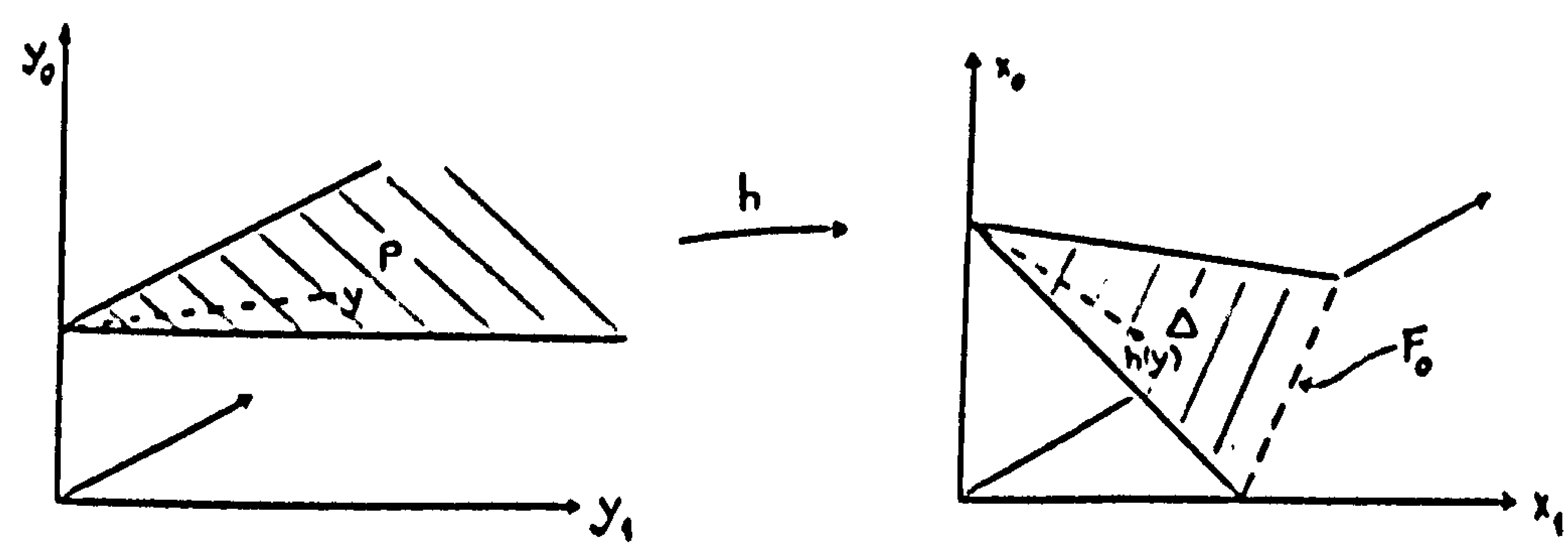


figure 5: the transformation $h: P \to \Delta - F_0$

2.5.2 Remarks

(1) Given $n\times(n+1)$ matrix $\begin{pmatrix} a_{10} & a_{11} & a_{12} \cdots a_{1n} \\ \cdots \\ a_{n0} & a_{n1} & a_{n2} \cdots a_{nn} \end{pmatrix}$

for (**), to obtain matrix $A \in M_{n+1}$ for the equivalent replicator equation (*) we added a first row of zeros.

So

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} \cdots & a_{1n} \\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n0} & a_{n1} & a_{n2} \cdots & a_{nn} \end{pmatrix}.$$

For the reversal process, having any $A \in M_{n+1}$, $A = (a_{ij})$ $i,j = 0,\ldots,n$ for (*), if we want the equivalent Lotka-Volterra equation (**), we subtract the first row from all rows, obtaining

$$\tilde{A} = \begin{pmatrix} 0 & 0 & & 0 \\ a_{10}-a_{00} & a_{11}-a_{01} & \cdots & a_{1n}-a_{0n} \\ \cdots\cdot \\ a_{n0}-a_{00} & a_{n1}-a_{01} & & a_{nn}-a_{0n} \end{pmatrix}$$

and by 1.5.1 (1) we have $X_{\tilde{A}} = X_A$. Eliminating, then, the first (zero) row of $\tilde{A}$ we get the $n\times(n+1)$ matrix for the equivalent (**).

(2) When, for $A \in M_{n+1}$, barycentre e of Δ is fixed for ϕ_A, we saw in 2.2.12 that $B = \frac{1}{n+1}(a_{ij} - a_{0j})$ $(i,j = 1,\dots,n)$ is the matrix for linear part of $X_A(x)$ at e (for coordinates $y_i = x_i - x_0$ $i = 1,\dots,n)$.

This is consistent with $(a_{ij}-a_{0j})_{i,j=1,\dots,n}$ being matrix for the linear part of the equivalent Lotka-Volterra equation at fixed point $(1,1,\dots,1)$. The factor $\frac{1}{n+1}$ agrees with the factor $\frac{1}{x_0}$ in the equivalence, at the point $e = \frac{1}{n+1}(1,1,\dots,1) = h(1,1,\dots,1)$.

(3) Equivalence obtained in 2.5.1 is useful for studying ϕ_A in $\overset{\circ}{\Delta}$, but not if we want to study stability of A as defined in 1.3.3 since, even if flow ψ_A associated to (**) in $\mathbb{R}^n_+$ is stable (inside its class of Lotka-Volterra equations), we may have A not stable. This may occur because to obtain the equivalence we had to remove one of the $(n-1)$-dimensional faces of Δ, and flow ϕ_A may be unstable on this face.

CHAPTER 3

CLASSIFICATION FOR DIMENSION 2

3.1 Introduction

The aim in this chapter is to give most of the proofs of theorems I and III, stated in 1.4.1 and 1.5.8, respectively. But part of the proofs will be here only stated (in 3.4.1), and only proved in 4.8 as application of techniques presented in chapter 4.

During the whole of this chapter we will consider only cases with $\underline{n=2}$.

We will take the following steps:

Step 1: $A \in M_3$ stable $\Rightarrow \phi_A$ has no periodic orbits in Δ . We prove this fact by finding conditions on A for ϕ_A to have periodic orbits and noting that these are not met for stable A's . This is done in 3.2.

Step 2: Description of phase portraits of ϕ_A depending on elements of $A \in Z_3^+$. By this we mean a description of all fixed points and of α- and ω-limits of all orbits of ϕ_A in Δ .

So we establish a finite collection of open subsets of Z_3^+ , with union dense in Z_3^+ , s.t. A,B in one of these, then ϕ_A and ϕ_B have "similar" phase portraits (meaning of this will be clear when we carry Step 2 out). This is done in 3.3.

Step 3: We will state, in 3.4.1, that $\forall$ A,B in one of the subsets of Step 2, we have that ϕ_A and ϕ_B are topologically equivalent in Δ , hence $A \sim B$. This is the Step whose proof is in 4.8. We then conclude that each of these subsets is in fact a stable class in Z_3^+ . Then Theorem I gives a geometrical description of these classes and Theorem III is the statement of conditions found in Step 2.

3.2 Non-existence of periodic orbits for stable classes (n=2)

This property was conjectured by Zeeman in [41]. When Hofbauer [13] proved equivalence to the Lotka-Volterra equations (see also 2.5 above), he noted that the non-existence, generically, of periodic orbits of ϕ_A in $\mathring{\Delta}$ follows from Dulac's test in $\mathbb{R}^2$ applied to the Lotka-Volterra equations.

Here we repeat this process in order to find conditions on elements of A for existence, or not, of periodic orbits.

3.2.1 Dulac's test ([3] pg.205, [7])

Let (I) $\frac{dx}{dt} = P(x,y)$ $\frac{dy}{dt} = Q(x,y)$ be

a C^1 dynamical system in a simply connected region G of $\mathbb{R}^2$. If there exists a C^1 function $B(x,y)$ in G such that

$\frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ)$ does not change sign in G , then system (I) has no periodic orbit in G . (Furthermore, if (I) is analytic, there is no simple closed curve in G which is a (finite) union of orbits of (I) .)

3.2.2 Application to Lotka-Volterra equations ([3] pg.213, or [7])

Consider $G = \{(x,y) \in \mathbb{R}^2 \; ; \; x,y > 0\}$ and system

$$(**) \qquad \begin{cases} \dot{x} = x(a_1 x + b_1 y + c_1) \\ \dot{y} = y(a_2 x + b_2 y + c_2) \end{cases}$$

where $\delta = a_1 b_2 - a_2 b_1 \neq 0$.

We take $B(x,y) = x^{k-1} \, y^{h-1}$ in G where $k = b_2(a_2 - a_1)/\delta$, $h = a_1(b_1 - b_2)/\delta$. Then, $\frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ) = (c_1 k + c_2 h) B(x,y)$. Letting $\sigma = (c_1 k + c_2 h)\delta = c_1 b_2 (a_2 - a_1) + c_2 a_1 (b_1 - b_2)$, we see that, generically, the system (**) above has $\delta \neq 0$ and $\sigma \neq 0$, and, in this case $\frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ)$ does not change sign in G .
So, generically, system (**) has no periodic orbit in the positive quadrant of $\mathbb{R}^2$.

(Note: In fact, the same argument can be applied analogously to any quadrant. Also, since x- and y- axis are invariant, we have that (**) has no periodic orbit in $\mathbb{R}^2$.)

Now, if we use this property above, plus the equivalence discussed in 2.5, we have that: arbitrarily near any $A \in M_3$ there is $B \in M_3$ for which ϕ_B has no periodic orbit. So, we have:

3.2.3 Proposition A stable in $M_3 \Rightarrow \phi_A$ has no periodic orbit in $\overset{\circ}{\Delta}$.

Moreover, 3.2.2 also shows that periodic orbits of (**) in $\mathbb{R}^2$ may only occur in cases where $\delta = 0$ or $\sigma = 0$. We want now to find out what these conditions mean in terms of the elements of A, in the equivalent system

(*) $\dot{x} = X_A(x)$, $A \in M_3$, $x \in \Delta$.

Our intention is, finally, to prove that non-existence of periodic orbits for ϕ_A is implied by condition of hyperbolicity of fixed point in $\overset{\circ}{\Delta}$.

Let $A = (a_{ij}) \in M_3$. By 2.5, the system

(*) $\dot{x}_i = x_i((Ax)_i - xAx)$ for $x \in \overset{\circ}{\Delta}$

is equivalent (in $\Delta - F_0$) (up to a change in time) to

$$\begin{cases} \dot{y}_1 = y_1((a_{10}-a_{00}) + (a_{11}-a_{01})y_1 + (a_{12}-a_{02})y_2) \\ \dot{y}_2 = y_2((a_{20}-a_{00}) + (a_{21}-a_{01})y_1 + (a_{22}-a_{02})y_2) \end{cases}$$

for $y_1, y_2 > 0$.

To simplify notation, without loss in generality, we may suppose that $A \in Z_3$ i.e. $a_{ii} = 0$. Then

$$\delta = (-a_{01})(-a_{02}) - (a_{12}-a_{02})(a_{21}-a_{01}) = a_{01}a_{12} + a_{02}a_{21} - a_{12}a_{21} = (\operatorname{adj} Au)_0$$

$$\sigma = a_{10}(-a_{02})a_{21} + a_{20}(-a_{01})a_{12} = -\det A .$$

We know, then, that $\delta = (\operatorname{adj} Au)_0 \neq 0$, $\sigma = -\det A \neq 0$ imply that ϕ_A has no periodic orbit in $\overset{\circ}{\Delta}$. We want to discuss cases $\delta = 0$ and $\sigma = 0$, $\delta \neq 0$.

3.2.4 Discussion for $\delta = 0$ or $\delta \neq 0$, $\sigma = 0$

$\delta = 0$, or $\sigma = 0$, do not imply in general that A is not stable. For instance, when ϕ_A has no fixed point in $\overset{\circ}{\Delta}$ then it certainly cannot have periodic orbits in $\overset{\circ}{\Delta}$ (see 2.2.1 (vi) or 2.3). In this case δ, or σ, or both can be zero with A stable.

Example: $$A = \begin{pmatrix} 0 & 1 & 2+\varepsilon \\ -1 & 0 & 1 \\ -(2+\varepsilon) & -1 & 0 \end{pmatrix} \quad \text{has } \delta = -\varepsilon\,,\ \sigma = 0$$

A is stable for all $\varepsilon > -2$ ($A \in$ stable class (2) of Theorem I). However for $\varepsilon = 0$, we have $\delta = \sigma = 0$, and for $\varepsilon \neq 0$, $\delta \neq 0$; $\sigma = 0$.

But, if A has fixed point $p \in \overset{\circ}{\Delta}$, taking centralization

$$\bar{A} = \begin{pmatrix} 0 & \theta+\alpha_0 & \theta-\alpha_0 \\ \theta-\alpha_1 & 0 & \theta+\alpha_1 \\ \theta+\alpha_2 & \theta-\alpha_2 & 0 \end{pmatrix} \qquad \text{the characteristic equation}$$

at e for $\phi_{\bar{A}}$ is $\lambda^3 + \frac{2\theta}{3}\lambda + \frac{\theta^2+\rho}{9} = 0$ (see 2.2.16).

Here $$\delta = (\theta+\alpha_0)(\theta+\alpha_1) + (\theta-\alpha_0)(\theta-\alpha_2) - (\theta+\alpha_1)(\theta-\alpha_2) =$$
$$= \theta^2 + \rho$$
$$\sigma = -\det A = -2\theta(\theta^2+\rho)\,.$$

Then $\delta = 0 \Rightarrow e$ is not hyperbolic for $\phi_{\bar{A}}$ (hence also p for ϕ_A). By Theorem II, A is not stable.

If $\delta = \theta^2+\rho \neq 0$, but $\sigma = 0$. Then $\theta = 0$, $\rho = \delta \neq 0$. So, characteristic equation at e for $\phi_{\bar{A}}$ is $\lambda^2 + \frac{1}{9}\rho = 0$.

By considering $\bar{A}$ or $-\bar{A}$ there are only two cases where this can happen.

Case I $\alpha_0, \alpha_1, \alpha_2 > 0$. Here A is in combinatorial class C_1 (see 1.5.7) where ϕ_A has a cycle of saddle connections on $\partial\Delta$. But Zeeman in [41] proved that $\det A = 0$ ($\sigma = 0$) implies A not stable. ($\mathring{\Delta}$-p is filled by periodic orbits of ϕ_A). Here $\rho > 0$ and p is not hyperbolic.

Case II $\alpha_0, \alpha_1 > 0$, $\alpha_2 < 0$. Here A is in combinatorial class C_7 (see 1.5.7), and we have two subcases.

(i) $\underline{\rho > 0}$ => A is not stable, since p is not hyperbolic. In [41], it was shown that a region of $\mathring{\Delta}$ is filled by periodic orbits of ϕ_A.

(ii) $\underline{\rho < 0}$ => p is hyperbolic saddle and ϕ_A has no periodic orbit in $\mathring{\Delta}$. In fact, $A \in$ stable class (7_2) of Theorem I (conditions may be checked byTheorem III).

3.2.5 Remark One of the purposes of the discussion above is to point out a mistake in [3] pg.213 (also in [7]) where it is asserted that $\delta \neq 0$, $\sigma = 0$ always imply a region around a center filled by periodic orbits. This does not happen either when there is no fixed point in $\mathring{\Delta}$ or, when there is, if $\delta < 0$ (Case II(ii) above).

Moreover this discussion allows us to state the following property:

3.2.6 Proposition For $A \in M_3$, suppose $p \in \mathring{\Delta}$ is fixed point for ϕ_A. Then:

$$p \text{ hyperbolic} \Rightarrow \phi_A \text{ has no periodic orbit in } \mathring{\Delta}.$$

3.2.7 Remark We can say, therefore, that non-existence of periodic orbits in $\mathring{\Delta}$ is implied either by non-existence of fixed point in $\mathring{\Delta}$ (2.1.1 (vi)) or by such a fixed point being hyperbolic (3.2.6 above). This is a stronger property than 3.2.3, and will be useful in the next paragraph (3.3), where we describe flow ϕ_A imposing condition of hyperbolicity of fixed points (not stability of A).

3.2.8 Remark In [3]pg.213 it also asserted that, since system (**) is analytic, Dulac's test (as in 3.2.2) also implies non-existence of "closed contours" (i.e. union of orbits forming a simple closed curve) in $\mathbb{R}^2$. In fact the test fails when $k < 1$ or $h < 1$ (see example 3.2.9 below). However, we will show that such "closed contours" can only occur with a non-hyperbolic fixed point in its interior, hence never in stable cases.

We will prefer, though, to prove this property using our systems ϕ_A in Δ $(A \in M_3)$ instead of their equivalent Lotka-Volterra systems. So, we postpone this proof till 3.3.8 and 3.3.10.

Now, we present an example that justifies remark 3.2.8.

3.2.9 Example Take system (**) with

$$a_1 = -2 \ , \ b_1 = -4 \ , \ a_2 = 4 \ , \ b_2 = 1 \ , \ c_1 = 6 \ , \ c_2 = -5$$

i.e.
$$\begin{cases} \dot{x} = x(-2x - 4y + 6) \\ \dot{y} = y(4x + y - 5) \end{cases}$$

which has fixed points $0 = (0,0)$, $q_0 = (3,0)$, $q_1 = (0,5)$, $p = (1,1)$ whose eigenvalues are, respectively, 6 and -5, -6 and 7, -12 and 5, $-\frac{1}{2}(1 \pm i\sqrt{55})$.

Hence 0, q_0 and q_1 are hyperbolic saddles, p is hyperbolic attractor (focus) (see figure 6(a) below). Also

$$\delta = a_1b_2 - a_2b_1 = 14 \neq 0$$

$$\sigma = a_1c_2(b_1-b_2) + b_2c_1(a_2-a_1) = -14 \neq 0$$

$$k = \frac{b_2(a_2-a_1)}{\delta} = \frac{6}{14} < 1$$

$$h = \frac{a_1(b_1-b_2)}{\delta} = \frac{10}{14} < 1 \quad .$$

Hence, Dulac's test, as in 3.2.2, does not exclude the possibility of an orbit γ going from q_0 to q_1 , forming, together with segments $0q_1$ and $0q_2$, a closed contour, like in figure 6(b) below.

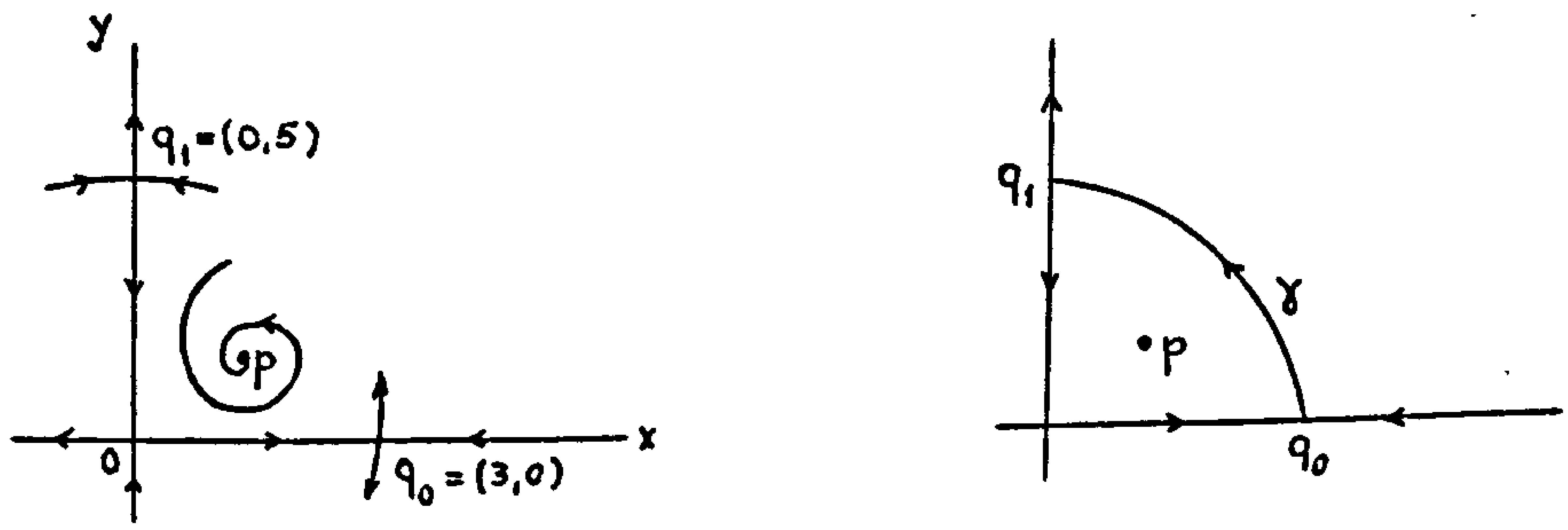


figure 6 (a) (b)

(a) the fixed points in example 3.2.9

(b) a cycle of saddles forming a closed contour for (**).

In order to help the proof of 3.3.8, we here establish the following lemmas.

3.2.10 Lemma If system (**) has a closed contour in $\bar{G}$, then the associated flow has a fixed point $p \in G$ and p is either a centre or a focus (i.e. eigenvalues at p are distinct complex conjugates).

Proof If there is a closed contour in $\bar{G}$, and $\delta \neq 0$, there is exactly one fixed point p in G and, exchanging x and y axis if necessary, we must have a situation as described in figure 6(b) above. Since (**) is quadratic, we must have that p is centre or focus. This can be proved exactly like Theorem 6 of [7]. (The proof there is for a fixed point in the interior of a periodic orbit of a quadratic system in $\mathbb{R}^2$, but, by careful inspection, we see that the property holds in our situation.)

$\square$

3.2.11 Lemma For $A \in M_3$, ϕ_A has an isolated fixed point p in $\overset{\circ}{\Delta}$ <=> (**) has an isolated fixed point $\bar{p}$ in G. Also λ is eigenvalue at $\bar{p}$ <=> $\frac{\lambda}{3}$ is eigenvalue at p (for respective systems).

Proof For simplicity, suppose $A \in Z_3$ and is central (taking centralization does not alter eigenvalues, by 2.2.9). Write $A = \begin{pmatrix} 0 & \theta+\alpha_0 & \theta-\alpha_0 \\ \theta-\alpha_1 & 0 & \theta+\alpha_1 \\ \theta+\alpha_2 & \theta-\alpha_2 & 0 \end{pmatrix}$

so eigenvalues at p are given by

$$\lambda^2 + \frac{2\theta}{3}\lambda + \frac{\theta^2+\rho}{9} = 0, \quad \rho = \alpha_0\alpha_1+\alpha_0\alpha_2+\alpha_1\alpha_2 \quad \text{(see 2.2.16).}$$

The corresponding Lotka-Volterra system (**)is (see 2.5.2):

$$\begin{cases} \dot{x} = x(\theta-\alpha_1 + (-\theta-\alpha_0)x + (\alpha_0+\alpha_1)y) & x \geq 0 \\ \dot{y} = y(\theta+\alpha_2 + (-\alpha_0-\alpha_2)x + (-\theta+\alpha_0)y) & y \geq 0 \end{cases}$$

where $\bar{p} = (1,1)$ has eigenvalues given by

$$\lambda^2 + 2\theta\lambda + (\theta^2+\rho) = 0 .$$

This finishes the proof. □

We note that a similar property can be proved for fixed points of ϕ_A and its equivalent Lotka-Volterra system, because the equivalence shown in 2.5 was given by a diffeomorphism between P and Δ minus one face. The factor $\frac{1}{3}$ multiplying eigenvalues in 3.2.11 results from the factor $\frac{1}{x_0}$ in 2.5 interpreted as a change in the time scale. But we do not need here this more general property.

From 3.2.10 and 3.2.11 we take

3.2.12 <u>Corollary</u> For $A \in M_3$, if ϕ_A has a cycle of (hyperbolic) saddles not intersecting one of the edges of Δ, then ϕ_A has a fixed point p in $\mathring{\Delta}$ with distinct complex conjugate eigenvalues.

3.3 <u>Phase portraits of</u> ϕ_A, <u>when fixed points are hyperbolic</u>

We know already that if $A \in Z_3$ is stable, then

1) $A \in Z_3^+$ (hence vertices are hyperbolic)

2) ϕ_A has at most one fixed point in the interior of each face

3) all fixed points are hyperbolic

4) ϕ_A has no periodic orbits.

In 1.5.7, we described all the combinatorial classes (up to sign reversal) $C_1, C_2, \ldots, C_{10}$ of Z_3^+ which contain (up to flow reversal) the stable classes of Z_3 .

Now we will establish, in terms of elements of A , for each C_r , conditions for fixed points to be hyperbolic. And we will prove that each C_r has an open, dense subset which is the union of open subsets $C_r^1, \ldots, C_r^{\bar{M}(r)}$, so that $A,B \in C_r^m \Rightarrow A \sim B$. So, subsets C_r^m will be the stable classes of Z_3 (up to flow reversal). But before proving $A,B \in C_r^m \Rightarrow A \sim B$ we will prove that A and B are "equivalent" in a weaker sense than that of 1.3.1 (see 3.3.4 below).

First we have:

3.3.1 <u>Proposition</u> For each $r = 1,\ldots,10$ there exists $\bar{M}(r) \geq 1$ and disjoint open subsets $C_r^1, \ldots, C_r^{\bar{M}(r)}$ of C_r such that

(a) $\tilde{C}_r = \bigcup_{m=1}^{\bar{M}(r)} C_r^m$ is open, dense in C_r .

(b) $A \in \tilde{C}_r \Leftrightarrow$ all fixed points of ϕ_A are hyperbolic.

(c) $A \in C_r - \tilde{C}_r \Rightarrow A$ is not stable.

(d) Any stable class in C_r is contained in one of the subsets C_r^m , $r = 1,\ldots,10$ $m = 1,\ldots,\bar{M}(r)$.

(e) Number $\bar{M}(r)$ of subsets in C_r is given by

r	1	2	3	4	5	6	7	8	9	10
$\bar{M}(r)$	2	1	1	2	2	4	5	1	2	2

(f) Taking $M(1) = 1$, $M(7) = 3$, $M(r) = \bar{M}(r)$ if $r \neq 1,7$ then any stable class of Z_3 is contained, <u>up to flow reversal</u>, in C_r^m for some $r = 1,\ldots,10$, some $m = 1,\ldots,M(r)$.

3.3.2 <u>Remarks</u>

(1) Subsets C_r^m will be described along the proof.

(2) We already know (1.5.6 (7)) that any stable class of Z_3 must be contained, up to flow reversal, in one of the combinatorial classes C_r . Along the proof of 3.3.1 presented below it will be clear that if $A \in C_r^{m_1}$, $B \in C_r^{m_2}$, $m_1 \neq m_2$, then $A \not\sim B$ because ϕ_A and ϕ_B will have different type of fixed points. Hence (d) will be valid by the construction (of C_r^m's) itself.

After proving 3.3.1 we will improve the result by describing the phase portrait of ϕ_A for A in each C_r^m $r = 1,\ldots,10$ $m = 1,\ldots,M(k)$. i.e. we will prove:

3.3.3 <u>Proposition</u>

(a) If $A,B \in C_r^m$ then there exists a face-preserving homeomorphism $h:\Delta \to \Delta$ such that

(i) h takes fixed point of ϕ_A to fixed point of ϕ_B , of the same topological type.

(ii) If $p \in \Delta$ is fixed point for ϕ_A, then $h(W^s p) = W^s h(p)$.

(b) The phase portraits of ϕ_A, up to homeomorphism, can be described by figure 7 below.

3.3.4 Remark We note that homeomorphism h of 3.3.3 (a) is not required to take ϕ_A-orbits to ϕ_B-orbits, although this will be true for some orbits. So, h is not a topological equivalence between ϕ_A and ϕ_B, and 3.3.3 (a) does not imply $A \sim B$ as in definition 1.3.1. This is the weaker form of equivalence we mentioned before the statement of Proposition 3.3.1. In 3.4.1 we will state that actually $A,B \in C_r^m \Rightarrow A \sim B$ in order to complete proofs of Theorems I and III. However the proof of this fact depends on the construction of a topological equivalence between ϕ_A and ϕ_B, and this property will be taken as an application of the methods developed in the next Chapter 4.

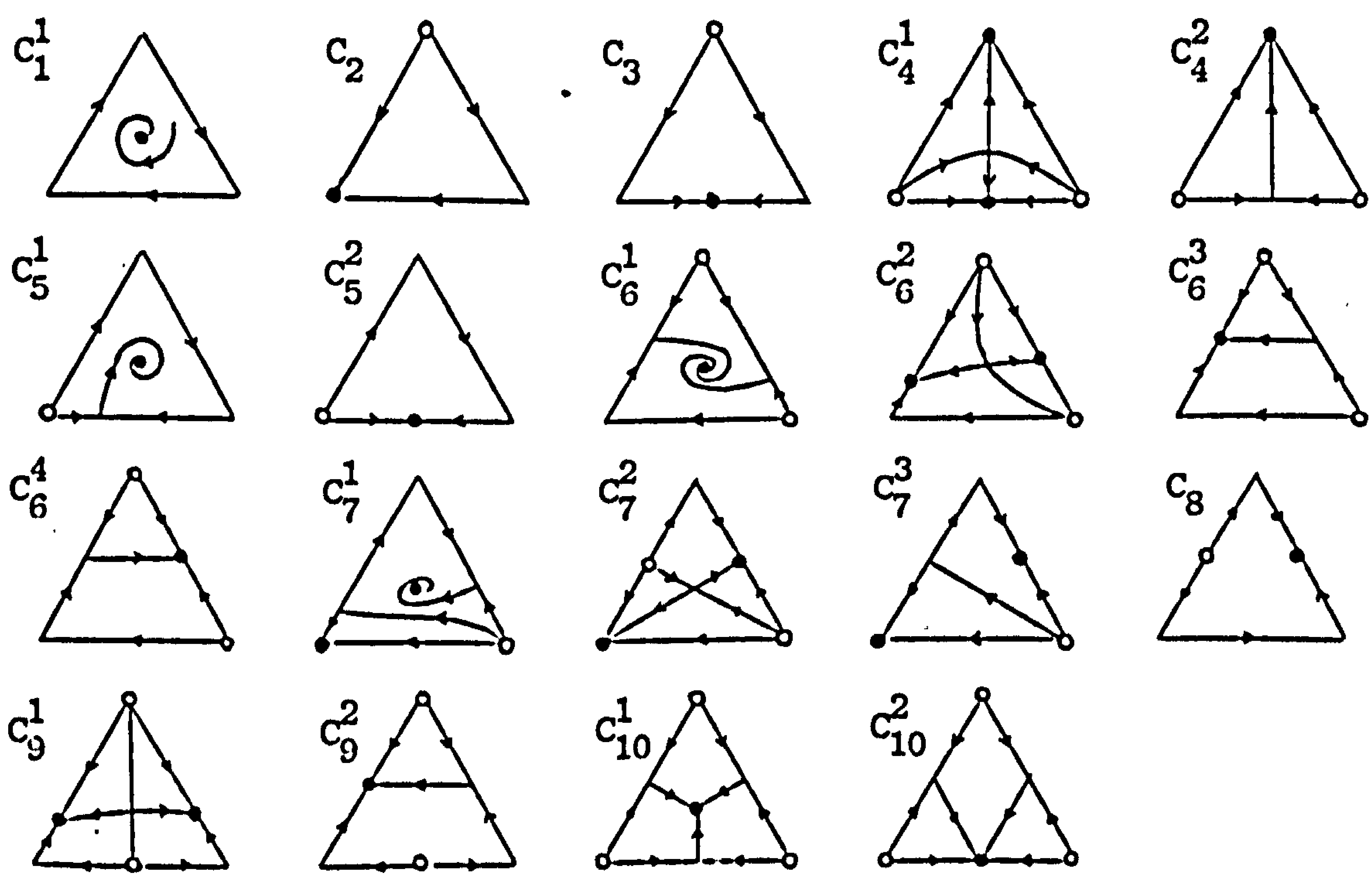


figure 7: Phase portraits for ϕ_A, for $A \in C_r^m$, where fixed

points are all hyperbolic, attractors are marked by solid dots, repellor by open dots and saddles by their invariant manifolds (insets and outsets). All other orbits flow from a repellor to an attractor, except in C_1^1 where the boundary $\partial\Delta$ is the α-limit for all regular orbits in $\overset{\circ}{\Delta}$.

3.3.5 Some useful facts

Under this heading we collect some known results but we arrange them in a form directly applicable during our proof of 3.3.1 below.

(1) First we remember that an edge X_iX_j of Δ $(i \neq j)$ has an isolated fixed point q for ϕ_A, in its interior, if and only if $a_{ij}a_{ji} > 0$ and, in this case, the eigenvalues at q are:

$$-\frac{a_{ij}a_{ji}}{a_{ij}+a_{ji}} \quad (= \text{eigenvalue at } q \text{ for } \phi_A|X_iX_j) \neq 0$$

and $\dfrac{\beta_k}{a_{ij}+a_{ji}}$ where $\beta = (\beta_0, \beta_1, \beta_2) = (\text{adj } A)u$ and $k \neq i,j$ (by Corollary 2.4.8).

So q is hyperbolic $\Leftrightarrow$ $\beta_k \neq 0$. And q is hyp. attractor (repellor or saddle) $\Leftrightarrow$ $a_{ij}, a_{ji} > 0$, $\beta_k < 0$ $(a_{ij}, a_{ji} < 0, \beta_k < 0$ or $\beta_k > 0$, respectively).

(2) ϕ_A has isolated fixed point p in $\overset{\circ}{\Delta}$ $\Leftrightarrow$ $\beta_0, \beta_1, \beta_2$ all have the same sign (by 2.2.1 (v)). In this case, $A \sim$ centralization $\bar{A}$ (by 2.2.9) and

$$\bar{A} = \begin{pmatrix} 0 & \theta+\alpha_0 & \theta-\alpha_0 \\ \theta-\alpha_1 & 0 & \theta+\alpha_1 \\ \theta+\alpha_2 & \theta-\alpha_2 & 0 \end{pmatrix} . \qquad \text{(by 2.2.16)}$$

Eigenvalues at p for ϕ_A are the roots of equation

$\lambda^2 + \frac{2\theta}{3}\lambda + \frac{\theta^2+\rho}{9} = 0$ where $\rho = \alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2$ (by 2.2.9 and 2.2.16)

Hence p is <u>not</u> hyperbolic $\Leftrightarrow \begin{cases} \theta^2 + \rho = 0 & \text{or} \\ \theta = 0 \,,\ \rho > 0 & . \end{cases}$

(3) $\bar{A} = 3AP$ where $P = \begin{pmatrix} p_0 & & 0 \\ & p_1 & \\ 0 & & p_2 \end{pmatrix}$, $p = (p_0, p_1, p_2)$.

So $\theta(\theta^2+\rho) = \det \bar{A} = 3(p_0p_1p_2)\det A$; and $\det A \neq 0 \Rightarrow$ p is hyperbolic.

(4) $\bar{\beta} = (\operatorname{adj} \bar{A})u \Rightarrow \bar{\beta}_0 = \bar{\beta}_1 = \bar{\beta}_2 = \theta^2 + \rho$.

(5) If X_iX_j has fixed point q , for ϕ_A , then (by 2.2.8) X_iX_j also contains fixed point $\bar{q}$ for $\phi_{\bar{A}}$, being q and $\bar{q}$ of same topological type. Then for $k \neq i,j$ β_k and $\bar{\beta}_k$ have the same sign.

(6) By direct calculation we see that:

$$\begin{cases} \beta_0 = a_{01}a_{12} + a_{02}a_{21} - a_{12}a_{21} \\ \beta_1 = a_{10}a_{02} + a_{12}a_{20} - a_{02}a_{20} \\ \beta_2 = a_{20}a_{01} + a_{21}a_{10} - a_{01}a_{10} . \end{cases}$$

During proof below we use these facts and will examine signs of β_i's depending on the signs of elements a_{ij} of A .

Proof of 3.3.1

First we take (b) as definition of $\tilde{C}_r$. Then $\tilde{C}_r$ is open and dense in C_r . (c) is implied by Theorem II, since $A \in C_r - \tilde{C}_r \iff \phi_A$ has non-hyperbolic fixed point $\Rightarrow A$ is not stable.

When necessary we will take permutation of vertices so as to throw A into a particular sign class (which we indicate by a matrix of signs) in C_r . This is done for simplification when we want to indicate which edges have fixed points and their types.

From now on the proof is a study, case by case, of combinatorial classes C_r .

Case by case discussion

Case 1: $A \in C_1$

There are no fixed points on edges since $a_{ij}a_{ji} < 0$ for all $i \neq j$.

We have $\beta_0, \beta_1, \beta_2 > 0$ always so ϕ_A has isolated fixed point p in $\overset{\circ}{\Delta}$. Taking centralization A , we must have $\alpha_0, \alpha_1, \alpha_2 > 0 \Rightarrow \rho > 0 \Rightarrow \theta^2 + \rho > 0$. (This can also be seen by $\bar{\beta}_i = \theta^2 + \rho > 0$ since $\beta_i\bar{\beta}_i > 0$.)

Now $\theta = 0 \iff \det A = 0$ and, in this case, p is not hyperbolic (since $\rho > 0$) .

So $A \in \tilde{C}_1 \iff \det A \neq 0$.

Take $C_1^1 = \{A \in C_1 \ ; \det A > 0\}$

$C_1^2 = \{A \in C_2 \ ; \det A < 0\}$.

So $A \in C_1^1 \Rightarrow p$ is an attractor

$A \in C_1^2 \Rightarrow p$ is a repellor.

We also note that $A \in C_1^2 \Leftrightarrow -A \in C_1^1$.

<u>Case 2</u>: $A \in C_2$

There are no fixed points on edges, and, also, no fixed point in $\overset{\circ}{\Delta}$ since A has positive and negative rows. (So, one vertex strictly dominates another vertex; see 2.3.4.) Hence, the only fixed points are the vertices (which are hyperbolic).

We take $C_2^1 = \tilde{C}_2 = C_2$.

<u>Case 3</u>: $A \in C_3$

There is no fixed point in $\overset{\circ}{\Delta}$ (as in case 2).

By permutation of vertices we may suppose that A is in sign class

$$S\begin{pmatrix} 0 & + & + \\ + & 0 & + \\ - & - & 0 \end{pmatrix}$$

$\Rightarrow \phi_A$ has fixed point q in X_0X_1 with $\beta_2 < 0$

$\Rightarrow q$ is hyperbolic attractor (always).

We take $C_3^1 = \tilde{C}_3 = C_3$.

<u>Case 4</u>: $A \in C_4$

By permutation, suppose $A \in S\begin{pmatrix} 0 & + & - \\ + & 0 & - \\ + & + & 0 \end{pmatrix}$

$\Rightarrow \phi_A$ has fixed point q in X_0X_1 and $\det A < 0$

$\Rightarrow$ fixed point in $\overset{\circ}{\Delta}$ (when it exists) is hyperbolic.

Then $A \in \tilde{C}_4 \iff \beta_2 \neq 0$.

We take $C_4^1 = \{A \in C_4 \ ; \ \beta_2 < 0\}$

$C_4^2 = \{A \in C_4 \ ; \ \beta_2 > 0\}$.

If $\underline{A \in C_4^1}$ $(\beta_2 < 0)$ $\Rightarrow$ q is hyp. attractor and

$0 < a_{20}a_{01} + a_{21}a_{10} < a_{01}a_{10}$

$$\Rightarrow \begin{cases} a_{21} < a_{01} \Rightarrow \beta_0 = a_{12}(a_{01}-a_{21}) + a_{02}a_{21} < 0 \\ a_{20} < a_{10} \Rightarrow \beta_1 = a_{02}(a_{10}-a_{20}) + a_{12}a_{20} < 0 \end{cases}$$

$\Rightarrow \phi_A$ has isolated fixed point p in $\overset{\circ}{\Delta}$ with

$$\begin{cases} \theta > 0 \text{ because } A \text{ (hence } \bar{A} \text{ too) has positive row} \\ \theta^2+\rho = \bar{\beta}_1 < 0 \text{ because } \beta_1 < 0 \end{cases}$$

$\Rightarrow$ p is hyp. saddle.

If $\underline{A \in C_4^2}$ $(\beta_2 > 0)$ $\Rightarrow$ q is hyp. saddle, we can see that

$\beta_1 \geq 0 \Rightarrow a_{10}a_{02} + a_{12}a_{20} \geq a_{02}a_{20} \Rightarrow a_{12} > a_{02} \Rightarrow \beta_0 = a_{01}a_{12}+a_{21}(a_{02}-a_{12}) < 0$.

Similarly $\beta_0 \geq 0 \Rightarrow \beta_1 < 0$.

Hence, always either β_0 or β_1 is < 0. Being $\beta_2 > 0$, ϕ_A has no fixed point in $\overset{\circ}{\Delta}$ (see 2.2.1 (v)).

<u>Case 5</u>: $A \in C_5$

By permutation we suppose $A \in S\begin{pmatrix} 0 & + & - \\ + & 0 & + \\ + & - & 0 \end{pmatrix}$

$\Rightarrow \phi_A$ has fixed point q in X_0X_1 which is hyperbolic if and only if $\beta_2 \neq 0$.

We always have $\beta_0 > 0$ and $\det A > 0$. So fixed point in $\overset{\circ}{\Delta}$ (if it exists) is hyperbolic.

Then $A \in \tilde{C}_5 \Leftrightarrow \beta_2 \neq 0$.

We take $C_5^1 = \{A \in C_5 \ ; \ \beta_2 > 0\}$

$C_5^2 = \{A \in C_5 \ ; \ \beta_2 < 0\}$.

If $\underline{A \in C_5^1}$ $(\beta_2 > 0) \Rightarrow q$ is hyp. saddle

and $a_{20}a_{01} > (-a_{21})a_{10} + a_{01}a_{10} \Rightarrow a_{20} > a_{10} \Rightarrow$

$\Rightarrow \beta_1 = (a_{10}-a_{20})a_{02} + a_{12}a_{20} > 0$.

So $\beta_0, \beta_1, \beta_2 > 0$ and ϕ_A has hyp. fixed point p in $\overset{\circ}{\Delta}$

with $\begin{cases} \theta > 0 & \text{(since } A \text{ has positive row)} \\ \theta^2+\rho = \bar{\beta}_1 > 0 & \text{(since } \beta_1 > 0) \end{cases}$

$\Rightarrow p$ is hyp. attractor.

If $\underline{A \in C_5^2}$ $(\beta_2 < 0) \Rightarrow q$ is hyp. attractor and ϕ_A has no fixed point in $\overset{\circ}{\Delta}$, because $\beta_0 > 0$, $\beta_2 < 0$ (by 2.2.1 (v)).

Case 6: $A \in C_6$

By permutation, we suppose $A \in S\begin{pmatrix} 0 & + & + \\ - & 0 & + \\ + & + & 0 \end{pmatrix}$

$\Rightarrow \phi_A$ has fixed points q_0 in X_1X_2 and $q_1 \in X_0X_2$ and q_0 (q_1) is hyperbolic $\Leftrightarrow$ $\beta_0 \neq 0$ $(\beta_1 \neq 0)$. It will be seen by process below that $\beta_0, \beta_1 \neq 0 \Rightarrow A \in \tilde{C}_6$.

We take define $C_6^1, \ldots, C_6^4$ by

$$A \in C_6^1 \Leftrightarrow \beta_0, \beta_1 > 0 \quad ; \quad A \in C_6^2 \Leftrightarrow \beta_0, \beta_1 < 0$$
$$A \in C_6^3 \Leftrightarrow \beta_0 > 0\ ,\ \beta_1 < 0 \quad ; \quad A \in C_6^4 \Leftrightarrow \beta_0 < 0\ ,\ \beta_1 > 0\ .$$

If $\underline{A \in C_6^1}$ $(\beta_0, \beta_1 > 0)$ $\Rightarrow$ q_0 and q_1 are hyp. saddle.

To show that $\beta_2 > 0$, we write $\beta_0 = a_{12}a_{21}\left(\frac{a_{01}}{a_{21}} + \frac{a_{02}}{a_{12}} - 1\right)$

$\beta_1 = a_{02}a_{20}\left(\frac{a_{10}}{a_{20}} + \frac{a_{12}}{a_{02}} - 1\right)$ and $\beta_2 = a_{01}a_{10}\left(\frac{a_{20}}{a_{10}} + \frac{a_{21}}{a_{01}} - 1\right)$

and, to simplify notation, we take

$$\alpha = \frac{a_{01}}{a_{21}} > 0\ ,\quad \beta = -\frac{a_{10}}{a_{20}} > 0\ ,\quad \gamma = \frac{a_{02}}{a_{12}} > 0$$

then $\beta_0 > 0 \Leftrightarrow \alpha + \gamma > 1 \Leftrightarrow \alpha > 1-\gamma$

$$\beta_1 > 0 \Leftrightarrow -\beta + \frac{1}{\gamma} > 1 \Rightarrow \frac{1}{\gamma} > 1 \Rightarrow 1-\gamma > 0\ .$$

So $\frac{1}{\alpha} < \frac{1}{1-\gamma}$ and $\frac{1}{\beta} > \frac{\gamma}{1-\gamma}$ $\Rightarrow$ $\frac{1}{\alpha} - \frac{1}{\beta} < 1$

$$\Rightarrow \beta_2 = a_{01}a_{10}\left(\frac{1}{\alpha} - \frac{1}{\beta} - 1\right) > 0\ .$$

Hence $\beta_0, \beta_1, \beta_2 > 0$ and ϕ_A has isolated fixed point p in $\overset{\circ}{\Delta}$ with $\begin{cases} \theta > 0 & \text{(because of positive row)} \\ \theta^2+\rho = \bar{\beta}_1 > 0 \end{cases}$

$\Rightarrow$ p is hyp. attractor.

If $\underline{A \in C_6^2}$ $(\beta_0, \beta_1 < 0)$ $\Rightarrow$ q_0 and q_1 are both hyp. attractors. Similarly to what was done in C_6^1 we can show that $\beta_2 < 0$, hence ϕ_A has isolated fixed point p in $\overset{\circ}{\Delta}$ with $\begin{cases} \theta > 0 \\ \theta^2+\rho = \bar{\beta}_1 < 0 \end{cases}$

$\Rightarrow$ p is hyp. saddle.

If $\underline{A \in C_6^3}$ $(\beta_0 > 0, \beta_1 < 0)$, q_0 is hyp. saddle, q_1 is hyp. attractor, and there is no fixed point in $\overset{\circ}{\Delta}$.

If $\underline{A \in C_6^4}$ $(\beta_0 < 0, \beta_1 > 0)$, q_0 is hyp. attractor, q_1 is hyp. saddle, and there is no fixed point in $\overset{\circ}{\Delta}$.

<u>Case 7</u>: $A \in C_7$

By permutation we suppose $A \in S \begin{pmatrix} 0 & + & - \\ - & 0 & + \\ - & + & 0 \end{pmatrix}$

$\Rightarrow$ ϕ_A has fixed points q_0 in X_1X_2 and q_1 in X_0X_2 which are hyperbolic $\Leftrightarrow$ $\beta_0, \beta_1 \neq 0$.

If $\underline{\beta_0, \beta_1 > 0}$, q_0 and q_1 are hyperbolic saddles and similarly to case 6 we can show that we must have $\beta_2 > 0$. Hence ϕ_A has an isolated fixed point p in $\overset{\circ}{\Delta}$ with $\theta^2+\rho = \bar{\beta}_1 > 0$.

If det $A > 0 \Rightarrow \theta > 0 \Rightarrow p$ is hyp. attractor.

If det $A < 0 \Rightarrow \theta < 0 \Rightarrow p$ is hyp. repellor.

If det $A = 0 \Rightarrow \theta = 0$, $\rho > 0 \Rightarrow p$ is not hyperbolic $\Rightarrow$

$\Rightarrow$ A is not stable. Zeeman [41] has proved that, in this case, p is a centre and there is a region of $\mathring{\Delta}$ filled by periodic orbits of ϕ_A. This corresponds to case II(i) in our discussion in 3.2.4.

If $\underline{\beta_0, \beta_1 < 0}$, then q_0 is hyp. attractor, q_1 is hyp. repellor, and in similarly to case 6 we can show that $\beta_2 < 0$. Hence ϕ_A has isolated fixed point p in $\mathring{\Delta}$ with $\theta^2 + \rho < 0$
$\Rightarrow$ p is hyp. saddle (even when det $A = 0$, which corresponds to case II(ii) in 3.2.4).

If $\underline{\beta_0 < 0, \beta_1 > 0}$, q_0 is hyp. attractor, q_1 is hyp. saddle, and there is no fixed point in $\mathring{\Delta}$.

If $\underline{\beta_0 > 0, \beta_1 < 0}$, q_0 is hyp. saddle, q_1 is hyp. repellor and there is no fixed point in $\mathring{\Delta}$.

We define C_7^r $r = 1, \dots, 5$ by

$A \in C_7^1 \Leftrightarrow \beta_0, \beta_1 > 0$, det $A > 0$; $A \in C_7^4 \Leftrightarrow \beta_0 > 0, \beta_1 < 0$

$A \in C_7^2 \Leftrightarrow \beta_0, \beta_1 > 0$; $A \in C_7^5 \Leftrightarrow \beta_0, \beta_1 > 0$, det $A < 0$

$A \quad C_7^3 \Leftrightarrow \beta_0 < 0, \beta_1 > 0$.

Then $\tilde{C}_7 = \bigcup_{r=1}^{5} C_7^r$.

But now we note that if A is in sign class S as indicated then, taking σ as the permutation $\binom{012}{102}$ then $\tilde{A} = \sigma(-A) \in S$ with $\tilde{\beta}_0 = \beta_1$ $\tilde{\beta}_1 = \beta_0$, $\det \tilde{A} = -\det A$.

Therefore $\begin{cases} A \in C_7^5 \iff -A \in C_7^1 \\ A \in C_7^4 \iff -A \in C_7^3 \end{cases}$

$(A \in C_7^2 \iff -A \in C_7^2)$.

Case 8: $A \in C_8$

By permutation, we suppose that $A \in S \begin{pmatrix} 0 & - & - \\ + & 0 & + \\ - & + & 0 \end{pmatrix}$.

$\Rightarrow \phi_A$ has fixed point q_0 in X_1X_2, q_1 in X_0X_2.

We always have $\beta_0 < 0$, $\beta_1 < 0$, $\beta_2 > 0$. So q_0 is hyp. attractor, q_1 is hyp. repellor and ϕ_A has no fixed point in $\mathring{\Delta}$. Then $\tilde{C}_8 = C_8$ and we make $C_8^1 = C_8$, too.

Case 9: $A \in C_9$

By permutation, we suppose that $A \in S \begin{pmatrix} 0 & - & + \\ - & 0 & + \\ + & + & 0 \end{pmatrix}$

$\Rightarrow \phi_A$ has fixed points q_0 in X_1X_2, q_1 in X_0X_2, q_2 in X_0X_1 which are hyperbolic $\iff \beta_0, \beta_1, \beta_2 \neq 0$.

But $\beta_2 < 0$ and $\det A < 0$ always. So, q_2 is hyp. repellor, and fixed point p in $\mathring{\Delta}$ (when it exists) is hyperbolic.

Then $A \in \tilde{C}_9 \iff \beta_0, \beta_1 \neq 0$.

If $\underline{\beta_0, \beta_1 < 0}$, then q_0 is hyp. attractor, q_1 is hyp. attractor and ϕ_A has isolated fixed point p in $\overset{\circ}{\Delta}$

with $\begin{cases} \theta > 0 & \text{(because of positive row)} \\ \theta^2+\rho = \bar{\beta}_1 < 0 \end{cases}$

$\Rightarrow$ p is hyp. saddle.

If $\underline{\beta_0 > 0}$ $\Rightarrow a_{02} > a_{12} \Rightarrow \beta_1 = a_{10}a_{02} + a_{20}(a_{12}-a_{02}) < 0$.

So, q_0 is hyp. saddle, q_1 is hyp. attractor and ϕ_A has no fixed point in $\overset{\circ}{\Delta}$.

If $\underline{\beta_1 > 0}$ $\Rightarrow a_{12} > a_{02} \Rightarrow \beta_0 = a_{01}a_{12} + a_{21}(a_{02}-a_{12}) < 0$.

So, q_0 is hyp. attractor, q_1 is hyp. saddle and ϕ_A has no fixed point in $\overset{\circ}{\Delta}$.

In this last situation we note that we can take permutation $\sigma = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$ (exchanging vertices X_0 and X_1, and points q_0 and q_1) obtaining $\tilde{A} = \sigma A$ still in the same sign class S with $\tilde{\beta}_0 = \beta_1$, $\tilde{\beta}_1 = \beta_0$ $\Rightarrow$ $\tilde{\beta}_0 > 0$, $\tilde{\beta}_1 < 0$.

Hence $\beta_0 > 0$, $\beta_1 < 0$ and $\beta_0 < 0$, $\beta_1 > 0$ are really equivalent possibilities.

We also note that we have shown that we cannot have β_0 and β_1 both positive.

We let $C_9^1 = \{A \in C_9 \ ; \ \beta_0, \beta_1 < 0\}$

$C_9^2 = \{A \in C_9 \ ; \ \beta_0 \, \beta_1 < 0\}$.

Case 10: $A \in C_{10}$

ϕ_A has fixed points q_0 in X_1X_2, q_1 in X_0X_2, q_2 in X_0X_1, which are hyperbolic $\iff \beta_0,\beta_1,\beta_2 \neq 0$. Since $\det A > 0$, fixed point p in $\overset{\circ}{\Delta}$ (when it exists) is hyperbolic.

Then $A \in \tilde{C}_{10} \iff \beta_0,\beta_1,\beta_2 \neq 0$.

If $\underline{\beta_0,\beta_1,\beta_2 > 0}$, q_0,q_1 and q_2 are all hyp. saddles and ϕ_A has isolated fixed point p in $\overset{\circ}{\Delta}$ with $\begin{cases} \theta > 0 & \text{and} \\ \theta^2+\rho = \bar{\beta}_1 > 0 \end{cases}$

$\Rightarrow$ p is hyp. attractor.

Now we see that

$$\beta_2 < 0 \Rightarrow \begin{cases} a_{01} > a_{21} & \Rightarrow \quad \beta_0 = a_{12}(a_{01}-a_{21}) + a_{02}a_{21} > 0 \\ a_{10} > a_{20} & \Rightarrow \quad \beta_1 = a_{02}(a_{10}-a_{20}) + a_{12}a_{20} > 0 \end{cases}$$

and, similarly, $\beta_0 < 0 \Rightarrow \beta_1,\beta_2 > 0$

and $\beta_1 < 0 \Rightarrow \beta_0,\beta_2 > 0$.

In case $\beta_0 < 0$ (or $\beta_1 < 0$) we can permute vertices X_0 and X_2 (or X_1 and X_2, respectively) so that we obtain $\tilde{A}$ (in same sign class as A) with $\tilde{\beta}_2 < 0$, $\tilde{\beta}_0,\tilde{\beta}_1 > 0$.

So these three cases are equivalent.

We put $C_{10}^1 = \{A \in C_{10} \ ; \ \beta_0,\beta_1,\beta_2 > 0\}$

$C_{10}^2 = \{A \in C_{10} \ ; \ \beta_0 < 0 \text{ or } \beta_1 < 0 \text{ or } \beta_2 < 0\}$.

For $A \in C_{10}^2$, ϕ_A has no fixed point in $\overset{\circ}{\Delta}$ and of the fixed points q_0,q_1,q_2, one is hyp. attractor, two are hyp. saddles. (q_i is the attractor if $\beta_i < 0$.)

We know that any stable class of Z_3 is contained, up to reversal, in C_r for some $r = 1,\dots,10$ (see 1.5.6(7)). Considering that $C_1^2 = -C_1^1$, $C_7^4 = -C_7^3$, $C_7^5 = -C_7^1$ we can then disregard C_1^2, C_7^4, C_7^5, i.e., taking $M(1) = 1$, $M(7) = 3$, $M(r) = \tilde{M}(r)$ for $r \neq 1,7$, then (d) implies that any stable class of Z_3 is contained up to flow reversal in C_r^m for some $r = 1,\dots,10$; $m = 1,\dots,M(r)$. i.e. (f) is valid.

$\square$

3.3.6 <u>Lemma</u> Let $A = \begin{pmatrix} 0 & \theta+\alpha_0 & \theta-\alpha_0 \\ \theta-\alpha_1 & 0 & \theta+\alpha_1 \\ \theta+\alpha_2 & \theta-\alpha_2 & 0 \end{pmatrix}$

where $\alpha_0, \alpha_1 > |\theta|$, $\alpha_2 < -|\theta|$

and $\rho = \alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2$.

Taking points $Y = \left(\dfrac{\alpha_0}{\alpha_0-\alpha_2}, 0, \dfrac{-\alpha_2}{\alpha_0-\alpha_2}\right) \in X_0X_2$

and $Z = \left(0, \dfrac{\alpha_1}{\alpha_1-\alpha_2}, \dfrac{-\alpha_2}{\alpha_1-\alpha_2}\right) \in X_1X_2$

then flow ϕ_A is transversal to segment YZ if and only if $\theta \neq 0$.

Denoting by Δ_1 and Δ_2 the components of Δ-YZ, containing X_0X_1 and X_2 respectively (see figure 8 below) then, for $\rho > 0$ we have:

if $\theta > 0$, flow ϕ_A crosses YZ from Δ_1 to Δ_2

if $\theta < 0$, flow ${}_A$ crosses YZ from Δ_2 to Δ_1

if $\theta = 0$, YZ is invariant, with Y and Z being fixed for ϕ_A and an orbit along YZ goes from Z to Y (so there is a cycle of saddles $Z \to Y \to X_2 \to Z$).

Also, barycentre e of Δ is fixed for ϕ_A, hyperbolic $\Leftrightarrow \theta = 0$; and $\rho > 0 \Rightarrow e \in \Delta_2$.

<u>Proof</u> For $x = (x_0, x_1, x_2) \in \Delta$, let

$$f(x) = \frac{\alpha_2}{\alpha_0} x_0 + \frac{\alpha_2}{\alpha_1} x_1 + x_2 .$$

So $x \in YZ \Leftrightarrow f(x) = 0$.

Take $\Delta_1 = \{x \in \Delta; f(x) < 0\}$; $\Delta_2 = \{x \in \Delta; f(x) > 0\}$.

We have

$$\dot{f}(x)\Big|_{x\in YZ} = \frac{\alpha_2}{\alpha_0}\dot{x}_0 + \frac{\alpha_2}{\alpha_1}\dot{x}_1 + \dot{x}_2 =$$

$$= \frac{\alpha_2}{\alpha_0} x_0(Ax)_0 + \frac{\alpha_2}{\alpha_1} x_1(Ax)_1 + x_2(Ax)_2 - \underset{\overset{\shortparallel}{0}}{f(x)}(xAx) =$$

$$= \frac{\alpha_2}{\alpha_0} x_0((\theta+\alpha_0)x_1 + (\theta-\alpha_0)x_2) + \frac{\alpha_2}{\alpha_1} x_1((\theta-\alpha_1)x_0 + (\theta+\alpha_1)x_2) +$$

$$+ x_2((\theta+\alpha_2)x_0 + (\theta-\alpha_2)x_1) +$$

$$= \theta\left(\frac{\alpha_2}{\alpha_0}x_0x_1 + \frac{\alpha_2}{\alpha_0}x_0x_2 + \frac{\alpha_2}{\alpha_1}x_0x_1 + \frac{\alpha_2}{\alpha_1}x_1x_2 + x_0x_2 + x_1x_2\right) +$$

$$+ \alpha_2(\cancel{x_0x_1} - \cancel{x_0x_2} - \cancel{x_0x_1} + \cancel{x_1x_2} + \cancel{x_0x_2} - \cancel{x_1x_2}) =$$

$$= \theta\left(- \frac{\alpha_2}{\alpha_0} x_0^2 - \frac{\alpha_2}{\alpha_1} x_1^2 - x_2^2\right) =$$

$$= -\theta\left(\frac{\alpha_2}{\alpha_0}\left(1 + \frac{\alpha_2}{\alpha_0}\right)x_0^2 + \frac{\alpha_2}{\alpha_1}\left(1 + \frac{\alpha_2}{\alpha_1}\right)x_1^2 + 2\frac{\alpha_2^2}{\alpha_0\alpha_1} x_0x_1\right)$$

$$= k\,\theta\, g(x)$$

where $\quad K = \dfrac{-\alpha_2}{\alpha_0^2\alpha_1^2} > 0$

and $\quad g(x) = \alpha_1^2(\alpha_0+\alpha_2)x_0^2 + \alpha_0^2(\alpha_1+\alpha_2)x_1^2 + 2\alpha_0\alpha_1\alpha_2x_0x_1)$.

But $\quad \rho = \alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2 > 0$

$$\Rightarrow \begin{cases} \alpha_1(\alpha_0+\alpha_2) > -\alpha_0\alpha_2 > 0 & \Rightarrow \quad \alpha_0+\alpha_2 > 0 \\ \alpha_0(\alpha_1+\alpha_2) > -\alpha_1\alpha_2 > 0 & \Rightarrow \quad \alpha_1+\alpha_2 > 0 \end{cases}$$

and $\quad (\alpha_0\alpha_1\alpha_2)^2 - \alpha_1^2(\alpha_0+\alpha_2)\alpha_0^2(\alpha_1+\alpha_2) = -\rho\,\alpha_0^2\alpha_1^2 < 0$

$\Rightarrow \quad g(x) > 0 \;\; \forall\, x$.

So, $\theta > 0 \Rightarrow \dot{f}(x) > 0 \;\; \forall\, x \in YZ \Rightarrow$ orbits of ϕ_A are transversal to YZ , going from Δ_1 to Δ_2 .

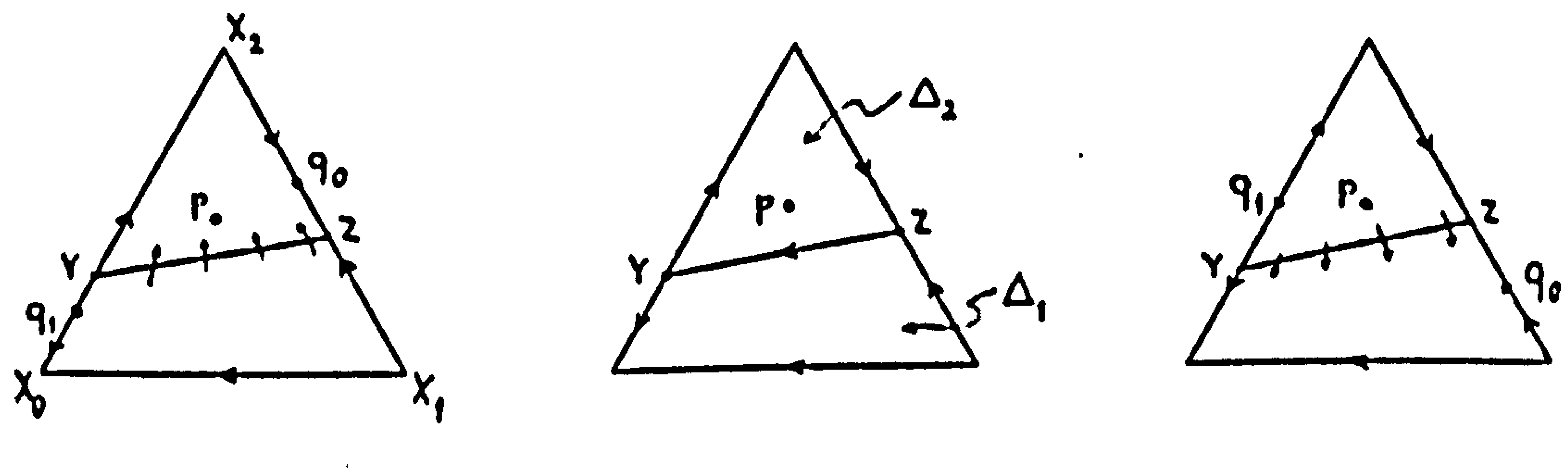
Analogously, $\theta < 0 \Rightarrow \dot{f}(x) < 0 \;\; \forall\, x \in YZ \Rightarrow \phi_A$ crosses YZ from Δ_2 to Δ_1 .

If $\theta = 0$, YZ is invariant, and clearly Y and Z are fixed, both saddles of ϕ_A .

Also $\quad f(e) = \frac{1}{3}\left(\frac{\alpha_2}{\alpha_0} + \frac{\alpha_2}{\alpha_1} + 1\right) = \frac{\rho}{3\alpha_0\alpha_1}$.

$\Rightarrow e \in \Delta_2$ for $\rho > 0$.

Then, for $\theta = 0$, there is an orbit of ϕ_A (on YZ) going from Z to Y. □



(a) $\theta > 0$, $\rho > 0$ (b) $\theta = 0$, $\rho > 0$ (c) $\theta < 0$, $\rho > 0$

figure 8: illustration of Lemma 3.3.6.

3.3.7 Remark The proof of 3.3.6 above, although quite different in style, was inspired by the proof of Theorem 7 of [41].

Let C_7^1 and C_7^5 be the subsets of Z_3^+ given in the proof of proposition 3.3.1. Then if $A \in C_7^1$ or C_7^5, ϕ_A has an isolated fixed point p in $\mathring{\Delta}$, and by taking centralization and permuting vertices if necessary we can suppose

$$A = \begin{pmatrix} 0 & \theta+\alpha_0 & \theta-\alpha_0 \\ \theta-\alpha_1 & 0 & \theta+\alpha_1 \\ \theta+\alpha_2 & \theta-\alpha_2 & 0 \end{pmatrix}$$

(as in 3.3.6) with $\alpha_0, \alpha_1 > |\theta|$, $\alpha_2 < -|\theta|$ and $\beta_i = \text{adj } Au \Rightarrow \beta_i = \theta^2+\rho > 0$

Then, X_0, X_1 and X_2 are, respectively, attractor, repellor and saddle. Also ϕ_A has fixed points $q_0 \in X_1X_2$, $q_1 \in X_0X_2$, with

$$W^s q_0 \subset X_1X_2 \ , \ W^u q_0 - q_0 \subset \mathring{\Delta} \ ,$$
$$W^u q_1 \subset X_0X_2 \ , \ W^s q_1 - q_1 \subset \mathring{\Delta} \ .$$

Also, ϕ_A has fixed point p in $\mathring{\Delta}$, and

$A \in C_7^1 \iff \det A > 0 \ (\iff \theta > 0) \iff p$ is attractor

$A \in C_7^5 \iff \det A < 0 \ (\iff \theta < 0) \iff p$ is repellor.

Now we prove:

3.3.8 Proposition

$$A \in C_7^1 \Rightarrow W^u q_0 - q_0 \subset W^s p \ , \ W^s q_1 - q_1 \subset W^u X_1$$
$$A \in C_7^5 \Rightarrow W^u q_0 - q_0 \subset W^s X_0 \ , \ W^s q_1 - q_1 \subset W^u p \ .$$

<u>Proof</u> Take $A \in C_7^1$. Fixed point p for ϕ_A in $\mathring{\Delta}$ is a hyperbolic attractor. Without loss in generality we can suppose A is central and can be written as above indicated.

So $\beta_i = \theta^2+\rho > 0$ and $\det A = 2\theta(\theta^2+\rho) > 0$ (i.e. $\theta > 0$) and eigenvalues at p are given by

$$\lambda^2 + \frac{2\theta}{3} + \frac{\theta^2+\rho}{9} = 0 .$$

Let $\gamma_0 = W^u q_0 - q_0$ and $\gamma_1 = W^s q_1 - q_1$. γ_0 and γ_1 are orbits of ϕ_A in $\mathring{\Delta}$. We claim that $\gamma_0 \cap \gamma_1 = \emptyset$.

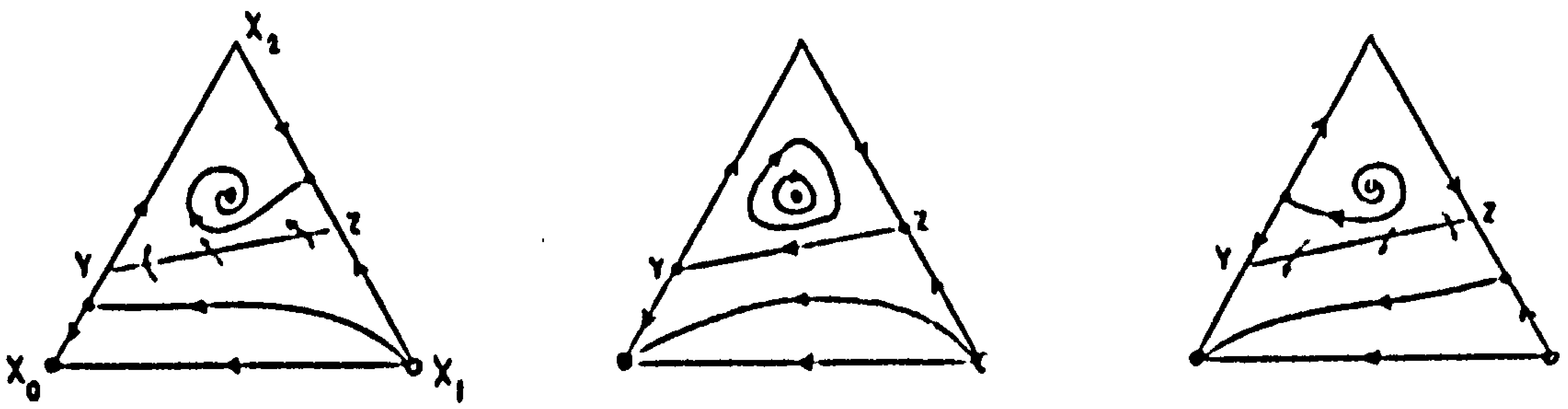
In fact, if this was not so, then $\gamma_0 = \gamma_1$. Then $\gamma_0 \cup q_1X_2 \cup q_0X_2$ would be a closed contour for ϕ_A (i.e. we would have a cycle of saddles $q_0 \to q_1 \to X_2 \to q_0$) not intersecting edge X_0X_1. By Corollary 3.2.12, p would have distinct complex conjugate eigenvalues and, so,

$$(2\theta)^2 - 4(\theta^2+\rho) = -4\rho < 0 \qquad \text{i.e. } \rho > 0 .$$

Then, taking $Y \in X_0X_2$, $Z \in \dot{X}_1X_2$ as in Lemma 3.3.6 we have that flow crosses YZ transversally from Δ_1 to Δ_2. Since, clearly, $q_0 \in ZX_2$ and $q_1 \in YX_0$ then orbit $\gamma_0 = \gamma_1$ cannot flow from q_0 to q_1 because it would cross YZ from Δ_2 to Δ_1 (see figure 8(a)).

Then $\gamma_0 = \gamma_1$ is absurd. Consequently $\gamma_0 \cap \gamma_1 = \emptyset$ and $\omega(\gamma_0) = p$, $\alpha(\gamma_1) = X_2$ (see figure 9(a)).

Case $A \in C_7^5$ can be treated analogously. (With help of Lemma 3.3.6 and figure 8(c).) □



(a) $A \in C_7^1$ (b) $A \in C_7$, $\sigma = 0$, $\rho > 0$ (c) $A \in C_7^5$

figure 9: illustration for 3.3.8 and 3.3.9.

3.3.9 Remark For case as above but with $\sigma = 0$ (hence $\det A = 0$) then $\beta_1 = \rho > 0$.

So, we have a cycle of saddles

$$q_0 = Z \to q_1 = Y \to X_2 \to q_0 \quad \text{and region } \overset{\circ}{\Delta}_2$$

is filled by periodic orbits of ϕ_A and all orbits in $\overset{\circ}{\Delta}_1$ flow from X_1 to X_0 (see figure 9(b)). This corresponds to Case II(i) of discussion in 3.2.4 and was also discussed in [41].

3.3.10 Corollary Taking (Lotka-Volterra) system

$$(**) \quad \begin{cases} \dot{x} = x(a_1x + b_1y + c_1) \\ \dot{y} = y(a_2x + b_2y + c_2) \end{cases}$$

let δ and σ be as in 3.2.2.

If a closed contour for ϕ_A (as in 3.2.8 and figure 6(b)) exists, then fixed point p inside it is not hyperbolic.

This corollary is related to remark 3.2.8.

Proof of 3.3.3 Again, this is mainly a case by case discussion. However, practically all the information needed to draw phase portraits of figure 7 are contained in the case by case description during the proof of 3.3.1. The only case where further discussion was necessary was C_7^1 and this was here discussed in proposition 3.3.8 above (see figure 9(a)). With this case discussed, all others are done by a simple inspection of where insets (and outsets) of saddles must come from (go to, respectively), and all phase portraits as in figure 7 are obtained with the information available.

To obtain homeomorphisms h as required in (a), we simply define h on all fixed points, extend it first to insets and outsets of saddles then to Δ . This should be done case by case. Just to illustrate, we choose one case (C_7^1) .

Let $\underline{A, B \in C_7^1}$. By permutation of vertices, if necessary, we can suppose A and B are in the same sign class $S\begin{pmatrix} 0 & + & - \\ - & 0 & + \\ - & + & 0 \end{pmatrix}$ as in proof of 3.3.1.

Then X_0 is attractor, X_1 is repellor and X_2 is saddle, for both ϕ_A and ϕ_B .

ϕ_A has saddles $q_0 \in X_1X_2$, $q_1 \in X_0X_2$ and attractor p in $\overset{\circ}{\Delta}$.

ϕ_B has saddles $\bar{q}_0 \in X_1X_2$, $\bar{q}_1 \in X_0X_2$ and attractor $\bar{p}$ in $\overset{\circ}{\Delta}$.

Let $\gamma_0 = W^u q_0 - q_0 \qquad \gamma_1 = W^s q_1 - q_1$

$\bar{\gamma}_0 = W^u \bar{q}_0 - \bar{q}_0 \qquad \bar{\gamma}_1 = W^s \bar{q}_1 - \bar{q}_1$

then $\omega_A(\gamma_0) = p \qquad \alpha_A(\gamma_1) = X_1$

$\omega_B(\bar{\gamma}_0) = \bar{p} \qquad \alpha_B(\bar{\gamma}_1) = X_1$

by Proposition 3.3.8 (see figure 9(a)).

Define $h(X_i) = X_i$ for $i = 0,1,2$

$$h(q_0) = \bar{q}_0 \quad , \quad h(q_1) = \bar{q}_1 \quad , \quad h(p) = \bar{p} \quad .$$

Now extend h to $C = \partial\Delta \cup \gamma_0 \cup \gamma_1 \cup p$ taking γ_0 onto $\bar{\gamma}_0$, γ_1 onto $\bar{\gamma}_1$, X_0q_1 onto $X_0\bar{q}_1$, etc. This can easily be done by choosing points $y \in \gamma_0$, $\bar{y} \in \bar{\gamma}_0$, making $h(y) = \bar{y}$ and extending timewise. (Similarly for γ_1 and $\bar{\gamma}_1$, X_0q_1 and $X_0\bar{q}_1$, etc.)

Now $\Delta - C = R_1 \cup R_2$ where R_1, R_2 are open, disjoint and $\omega_A(R_1) = p$ $\omega_A(R_2) = X_0$. Similarly $\Delta\text{-}h(C) = \bar{R}_1 \cup \bar{R}_2$, $\omega(\bar{R}_1) = \bar{p}$, $\omega_B(\bar{R}_2) = X_0$. We extend h to homeomorphism of Δ taking R_1 onto $\bar{R}_1$ and R_2 onto $\bar{R}_2$. This completes construction of h as in 3.3.3(a) for $A,B \in C_7^1$.

All other cases are done similarly. □

3.4 Proof of Theorems I and III

First we state

3.4.1 Proposition If $A,B \in C_r^m$ $r = 1,\ldots,10$, $m = 1,\ldots,M(k)$ (as in 3.3.1) then $A \sim B$.

This property is much stronger than that in 3.3.3(a). We leave proof of 3.4.1 to be done as application of techniques developed in next chapter 4 (in 4.8).

However, assuming Proposition 3.4.1 as true we can easily finish proofs for Theorems I (stated in 1.4.1) and III (stated in 1.5.8).

3.4.2 Proof of Theorem I

By 3.3.1 and 3.4.1 we have that each C_r^m is open and $A,B \in C_r^m \Rightarrow A \sim B$. This implies that $A \in C_r^m \Rightarrow A$ is stable and $C_r^m \subset$ stable class of A in Z_3 . By 3.3.1(d), we have that each stable class in Z_3 is contained (up to flow reversal) in C_r^m for some r,m . Hence C_r^m is the stable class, in Z_3 , of all its matrices. Then, by 1.5.1(3), all stable classes of M_3 are, up to flow reversal, exactly $C_r^m \oplus K_3$ for some r,m, and this we denote in 1.4.1 (figure 2) by (r_m) (or just (r) when $M(r) = 1$).

Since Z_3^+ is dense in Z_3 , $Z_3^+ = \bigcup_{r=1}^{10} (C_r \cup -C_r)$ (by 1.5.7), and $\bigcup_{m=1}^{M(r)} (C_r^m \cup -C_r^m)$ is dense in $C_r \cup -C_r$ (by 3.3.1) $\Rightarrow$ stable matrices are dense in M_3 .

This concludes Theorem I. □

3.4.3 <u>Proof of Theorem III</u>

Conditions for $A \in S = S_1 \dots S_{10}$ to belong to each C_r^m were established during the proof of Proposition 3.3.1.

We have, for $\beta = (\mathrm{adj}\ A)u$

$$\beta_0 = a_{01}a_{12} + a_{02}a_{21} - a_{12}a_{21} = a_{12}a_{21}\left(\frac{a_{01}}{a_{21}} + \frac{a_{02}}{a_{12}} - 1\right) = a_{12}a_{21}k_0$$

$$\beta_1 = a_{02}a_{20}k_1$$

$$\beta_2 = a_{01}a_{10}k_2 \ .$$

So, edge X_1X_2 has fixed point $\Leftrightarrow$ $a_{12}a_{21} > 0$ and, in this case, β_0 and k_0 have the same sign. Similarly for β_1 and k_1 , β_2 and k_2 when edges X_0X_2 or X_0X_1 , respectively, have fixed points. Now conditions in (1) to (10) of Theorem III are exactly the conditions established in 3.3.1. This finishes the proof. □

3.4.4 Corollary Let $A \in M_3$. Then:

A is stable <=> all fixed points of ϕ_A are hyperbolic.

Proof This is a consequence of the construction for subsets C_r^m.

3.4.5 Remark (about Proposition 3.4.1)

Since we have only 19 cases to consider, we could try to find homeomorphism giving topological equivalence between ϕ_A and ϕ_B, where $A,B \in C_r^m$, case by case.

Instead, we prefer to split the 19 cases into three groups:

(1) C_1^1 was treated by Zeeman in [41];

(2) for A in C_2, C_3, C_4^1, C_4^2, C_6^2, C_6^3, C_7^2, C_9^1, C_{10}^1 or C_{10}^2, ϕ_A has no saddle connections and is, therefore, a gradient-like flow in Δ. Gradient-like flows were classified by Peixoto [28] by means of "distinguished graphs" and by Fleitas [10] by "circular distributions of points around attractors". Either classification can be used to give conclusion in 3.4.1 for these cases;

(3) for A in C_5^1, C_5^2. C_6^1, C_6^4, C_7^1, C_8 or C_9^2, ϕ_A presents saddle connections, hence neither of the classifications mentioned above can be applied. If we tried to adapt Peixoto's technique we would have to work with lots of different types of "distinguished sets" and could not make it in a general way.

Fleitas' technique uses the idea of compatible tubular families of Palis-Smale [24,25] and so relies on transversality of saddle connections, which is not valid in our cases.

It is, hence, mainly to deal with group (3) that we develop the technique in next Chapter 4. There we give a classification of a family of flows (on two-dimensional manifolds) which can be roughly described as "gradient-like with saddle connections". We adapt Fleitas' technique of [10]. We think, however, that the classification may be useful in other situations besides the flows in our group (3) above.

CHAPTER 4

A CLASSIFICATION FOR QUASI-GRADIENT FLOWS IN DIMENSION TWO

4.1 Introduction

Usually (as in [10], [23], [28], [37]) a gradient-like flow ϕ on a compact manifold M is described as a "Morse-Smale flow without periodic orbits" meaning that ϕ satisfies:

(a) ϕ has a finite number of fixed points, all hyperbolic.

(b) $\Omega = \text{Fix}\,\phi$.

(c) $W^s p_1$ and $W^u p_2$ are transversal $\forall\, p_1, p_2 \in \Omega$ (where Ω is the non-wandering set for ϕ and $\text{Fix}\,\phi$ is the set of fixed points).

Condition (b) excludes the possibility of periodic orbits. Also, for gradient-like flows there is no cycle of saddles, where we remember that a "cycle" is a sequence of saddle points $p_0, p_1, \ldots, p_n$ with $p_0 = p_n$ and $W^s p_{i-1} \cap W^u p_i \neq \emptyset \;\; \forall \;\; i = 1, \ldots, n$ (see [26]).

If $\dim M = 2$, condition (c) excludes the possibility of any saddle-connection, i.e., if p_1 and p_2 are saddles, then $W^s p_1 \cap W^u p_2 = \emptyset$. Hence, for $\dim M = 2$, (c) is equivalent to saying that "there are no saddle-connections".

4.1.1 Remark Such flows are called gradient-like because it is possible to construct C^∞-functions $f: M \to \mathbb{R}$ such that the critical points of f are the fixed points of f and the regular orbits of ϕ are transversal to the level hypersurfaces of f (f decreasing on the regular orbits). See [23] and [37]. Such functions can be called Liapunov functions for ϕ, and they make ϕ to look like a gradient flow ψ (i.e. ψ has associated vectorfield X in the form of a gradient; $X = \text{grad } f$). In fact, Smale [37] proved that any gradient vectorfield is C^1 approximated by a vectorfield whose flow is gradient-like.

We want now to consider the following condition (c*) which is weaker than (c):

(c*) ϕ has no cycle.

4.1.2 Definition Flow ϕ is quasi-gradient if it satisfies (a), (b) and (c*).

Our aim in this chapter is to give a process of deciding when two quasi-gradient flows are, or not, topologically equivalent, for $\dim M = 2$. This process will be defined in 4.3 by means of what we call "circular distributions". These are adapted from similar distributions defined by Fleitas [10] for gradient-like flows. The main theorem is stated in 4.3.6 and proved in 4.5. In 4.7 and 4.8 we give applications aiming to complete the proof of Theorems I and III. First, in 4.2, we will recall the concept of "phase diagrams" to show that these are not sufficient to decide (topological) equivalence class of quasi-gradient flows.

We note that quasi-gradient flows may have non-transversal saddle connections. Hence, for $\dim M = 2$, saddle-connections are not excluded, only cycles of saddles are.

4.1.3 Remark In the construction of a "Liapunov" function for a gradient-like flow (see 4.11 above) in [37] or [23], the transversality of condition (c) is used to build a partial order on the fixed points of ϕ like in [38] (putting $p_1 < p_2$ if $W^s p_1 \cap W^u p_2 \neq \emptyset$) in order to construct a filtration for ϕ (Lemma 2.1 of [37]. Also in [38] or [26]). However, if we replace condition (c) by (c*) we can still give a partial ordering for the fixed points (by putting $p_1 < p_2$ if there are fixed points $q_0, \ldots, q_k$ of ϕ with $W^s q_{i-1} \cap W^u q_i \neq \emptyset$ $i = 1, \ldots, k$ and $q_0 = p_1$ $q_k = p_2$. See also Step 1 in 4.5). This ordering still allows the construction of a filtration for ϕ (with a process similar to the construction of $M_0 \subset M_1 \subset \ldots \subset M_\mu \subset M$ in Step 3 of 4.5). This filtration, together with a construction of local Liapunov functions near the fixed points (as in Meyer [23] or Wesley Wilson [40]), gives a global Liapunov function for flows satisfying (a), (b) and (c*). So we say that these flows are also similar (like in 4.1.1) to gradient flows, justifying the name "quasi-gradient" of Definition 4.1.2. We do not intend to present here any more details for the construction above indicated, but we claim that it can be successfully carried out.

Also, if we look carefully at the constructions in [37] we can see that any gradient flow can be C^1-approximated by a quasi-gradient flow. Note that a gradient flow <u>may</u> present non-transversal saddle connections, but it has no-cycles. Hence, our quasi-gradient flows of 4.1.2 are, in fact, much more "like" gradient-flows than the usual gradient-like flows, and these are particular cases of quasi-gradient flows. Condition (c), instead of (c*), is usually imposed to aim for structural stability of the flow.

4.1.4 <u>Convention</u> From now on, M will indicate a compact connected manifold with $\dim M = 2$, and ϕ will be quasi-gradient flow on M .

4.2 Phase diagrams

For each fixed point x of ϕ, we denote by $W^s x$ and $W^u x$ respectively the stable and unstable manifolds (inset and outset).

When either x or y is a saddle, and $W^u x \cap W^s y \neq \emptyset$ this intersection, since $\dim M = 2$, consists of:

- just one orbit (then we write $x \searrow y$), or
- two orbits (then we write $x \overset{2}{\searrow} y$).

When x is repellor and y attractor, $W^u x \cap W^s y$ is either empty or an open subset of M, and if there is saddle p with $x \searrow p \searrow y$, then the latter must happen.

4.2.1 Definition To each quasi-gradient flow ϕ, we associate a graph G (or $G(\phi)$) called the phase diagram of ϕ, as follows:

- take fixed points of ϕ as vertices
- for fixed points x, y, where either x or y is saddle, we put one edge (or two edges) from x to y if $x \searrow y$ (or $x \overset{2}{\searrow} y$, respectively)
- for repellor x, attractor y with $W^u x \cap W^s y \neq \emptyset$ we put one edge from x to y (and write $x \searrow y$) if there is no saddle p with $x \searrow p \searrow y$. (If p exists, this edge is not necessary.)

4.2.2 Remark In Palis [26], Peixoto [28], Smale [38], phase diagrams are considered but always with condition (c) imposed.

Let G and $\tilde{G}$ be graphs obtained as phase diagrams for flows ϕ and ψ.

4.2.3 Definition We say that G and $\tilde{G}$ are isomorphic (write $G \cong \tilde{G}$)

if there is a bijection π taking vertices (edges) of G to vertices (edges) of $\tilde{G}$ and such that: γ is edge from x to $y \Leftrightarrow \pi(\gamma)$ is edge from $\pi(x)$ to $\pi(y)$.

Hence $x \searrow y \Leftrightarrow \pi(x) \searrow \pi(y)$, $x \overset{2}{\searrow} y \Leftrightarrow \pi(x) \overset{2}{\searrow} \pi(y)$.

4.2.4 Lemma If $\phi \sim \psi \Rightarrow G(\phi) \simeq G(\psi)$.

Proof This is standard, π being induced by the homeomorphism h giving topological equivalence between the flows.

4.2.5 Remark For gradient-like flows, Peixoto [28] gave example showing that $G(\phi) \simeq G(\psi)$ does not imply $\phi \sim \psi$. Hence the same is valid for quasi-gradient flows. But, allowing saddle connections to exist, we can give, in 4.3.8, a simpler example than that in [28].

4.3 Circular distributions

To each quasi-gradient flow ϕ in two-dimensional M , we will associate a "circular distribution" of points (Definition in 4.3.2) which, roughly speaking, gives us information on how outsets of saddles flow to attractors or to other saddles of ϕ . In 4.3.6 we will state theorem saying that such distributions characterize classes of top. equivalence of quasi-gradient flows.

The idea of such circular distributions was taken from Fleitas [10], where he worked with gradient-like flows. In [10], the circular distributions are constructed considering, around each attractor A_i of a gradient-like flow ϕ , a small circle S_i transversal to ϕ , taking $S = \cup S_i$, and considering on S pairs of points P_n^1, P_n^2 which are the intersection of $W^u P_n$ with S for each saddle P_n of ϕ . To each such pair, arrows must be attached indicating how a neighbourhood of the arc $[P_n^1, P_n^2]$ on $W^u P_n$ rejoins S . Then the family of circles S_i , pairs P_n^1, P_n^2 and their arrows are the "circular distribution of points associated to gradient-like ϕ ". Two such distributions in [10] are said to be isomorphic if there exists homeomorphism between the unions of the circles taking pairs to pairs and compatible with arrows. Then theorem 1c of [10] says that two gradient-like flows are top. equivalent <=> their circular distributions are isomorphic.

One of the problems when we try a similar construction for quasi-gradient flows is that not all saddles will determine pairs of points on S . Some saddles determine just one point, some will have no associated points on S (when one, or two, of their unstable separatrices flow, not to attractors, but to other saddles). Even if we try to join the information retained by this construction with the information given by phase diagrams (as in 4.2), we are still not able to characterise equivalence class, as can be seen in example 4.3.8 below.

Therefore, we will extend the idea of circular distributions on circles around attractors to a circular distribution of points <u>around attractors and saddles</u>.

4.3.1 <u>The distribution</u>

Let ϕ be a quasi-gradient flow, with attractors $A_1, A_2, \ldots, A_\alpha$, repellors $B_1, B_2, \ldots, B_\beta$ and saddles $P_1, P_2, \ldots, P_\mu$.

Around each attractor A_i we take a small circle S_i , transversal to ϕ , limiting a ball D_i where A_i is the only fixed point.

(So $D_i \subset W^s A_i$.) See figure 10.

Around each saddle P_n we take a small circle C_n , limiting a ball E_n where P_n is the only fixed point. Also we can always take C_n small enough for each separatrix of $W^u P_n$ (and $W^s P_n$) to cut C_n just once and transversally, and, also, all points in $E_n - W^s P_n \cup W^u P_n$ must flow from a repellor to an attractor (i.e. separatrices of other saddles P_m will only intersect C_n either if $P_m \searrow P_n$ or $P_n \searrow P_m$). See figure 10.

Now we take $S = \bigcup_{i=1}^{\alpha} S_i$, $C = \bigcup_{n=1}^{\mu} C_n$.

For each saddle P_n we will consider the points of $W^u P_n \cap (S \cup C)$. There are always <u>four</u> points on this intersection, being <u>two</u> necessarily on C_n (points where $W^u P_n$ crosses C_n from inside to outside), and the other two <u>not</u> on C_n . These last two we will denote by P_n^1 , P_n^2 . The points of $W^u P_n \cap C_n$ we denote by $-P_n^1$, $-P_n^2$ where we take $-P_n^k \in \mathcal{O}(P_n^k)$ k = 1,2,. See figure 10.

To each of the points P_n^1 , P_n^2 , $-P_n^1$, $-P_n^2$ we will attach an <u>arrow</u>, whose orientation must indicate the way these points rejoin in a neighbourhood of arc $[P_n^1, P_n^2]$ on $W^u P_n$. Hence we choose arbitrarily the orientation of the arrow at any one of these points, then the other three arrows are determined by this choice (arrows at $-P_n^1$, $-P_n^2$ always having opposite orientations on C_n) in the following way: taking small transversal Σ to one of the stable separatices of P_n , for all sufficiently large $t > 0$, $\phi^t \Sigma$ must intersect $S \cup C$ at four points near P_n^1 , $-P_n^1$, $-P_n^2$, P_n^2 all at the tip side of the arrows, or all at the bottom side of the arrows. See figure 10.

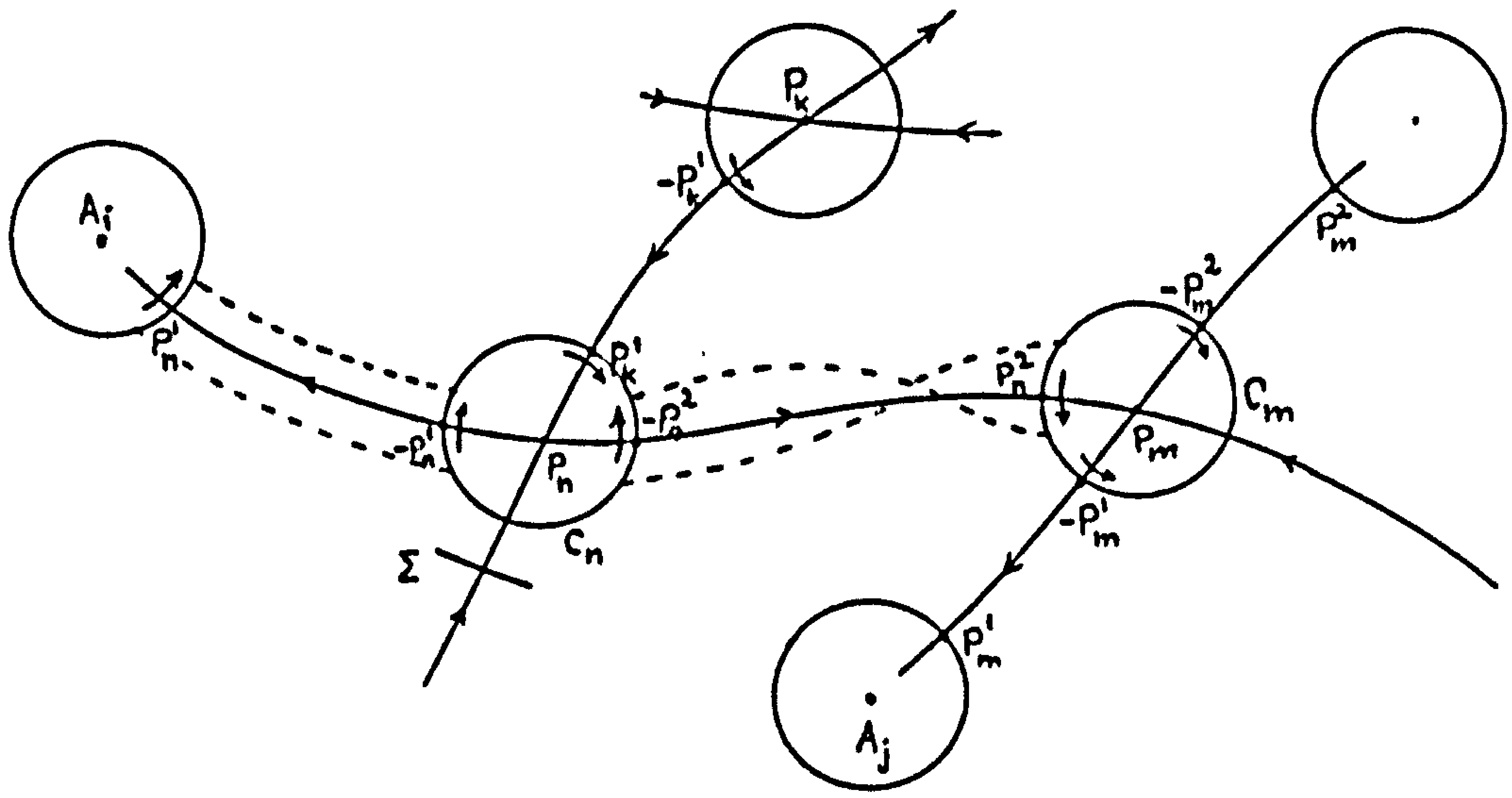


figure 10: circles $\{S_i\},\{C_n\}$, points $\{P_n^1,-P_n^1,-P_n^2,P_n^2\}$ and arrows

4.3.2 Definition The family of circles $S_1,\ldots,S_\alpha$, $C_1,\ldots,C_\mu$ with their distinguished points $\{P_n^1,P_n^2,-P_n^1,-P_n^2 \;;\; 1 \le n \le \mu\}$ plus the arrows attached to each of these points is a circular distribution of points associated to flow ϕ. We denote it by $\mathcal{D}(\phi)$.

4.3.3 Remark The choice of orientation for the arrows at the 4 points associated to P_n is of no particular importance, and we can reverse it, if convenient, as long as we reverse the orientation at the 4 points to keep them compatible in the way explained above.

4.3.4 Definition Let ϕ , ψ be quasi-gradient flows and $\mathcal{D} = \mathcal{D}(\phi)$, $\tilde{\mathcal{D}} = \mathcal{D}(\psi)$ their circular distribution of points, with circles S_i , C_n and $\tilde{S}_i$, $\tilde{C}_n$ respectively, and $S = \cup S_i$, $C = \cup C_n$, $\tilde{S} = \cup \tilde{S}_i$, $\tilde{C} = \cup \tilde{C}_n$. We say that $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are isomorphic (write $\mathcal{D} \approx \tilde{\mathcal{D}}$) if there is a homeomorphism $h: S \cup C \to \tilde{S} \cup \tilde{C}$ such that:

(i) h takes S onto $\tilde{S}$, C onto $\tilde{C}$

(ii) h is compatible with the points of $\mathcal{D}$ and $\tilde{\mathcal{D}}$

i.e. h takes the 4 points in $\mathcal{D}$, associated to a saddle P_n of ϕ , onto the 4 points in $\tilde{\mathcal{D}}$, associated to the corresponding (by (i)) saddle of ψ (which we denote by $h(P_n)$) , and

$$h(-P_n^k) = -h(P_n^k) \qquad k = 1,2 .$$

(iii) h is compatible with the arrows.

i.e. h is orientation preserving (or reversing) at P_n^1 in relation to local orientation given by arrows at P_n^1 and $h(P_n^1)$ if, and only if, the same is valid at $-P_n^1$, P_n^2 , $-P_n^2$.

4.3.5 Remarks

(1) Considering remark 4.3.3 we note that if $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are isomorphic, we can always suppose, without loss in generality, that the isomorphism h of 4.3.4 preserves orientation (reversing, whenever necessary, all arrows at points associated to $h(P_n)$) .

(2) The circular distribution of points of ϕ does not depend on the actual choice of circles S_i and C_n (as long as conditions in the construction are satisfied) because for any two choices, the flow itself induces an isomorphism between the distributions.

(3) $\mathcal{D} = \mathcal{D}(\phi) \cong \mathcal{D}(\psi) = \tilde{\mathcal{D}}$ induces a bijection between attractors (saddles) of ϕ and attractors (saddles) of ψ , by condition (i). We call h also this bijection, and, without loss in generality (permuting indices if necessary) we can write:

$$h(S_i) = \tilde{S}_i \ , \ \ h(C_n) = \tilde{C}_n \ , \ \ h(A_i) = \tilde{A}_i \ , \ \ h(P_n) = \tilde{P}_n \ .$$

4.3.6 Theorem (this is theorem IV announced in 1.6.3)

If ϕ and ψ are quasi-gradient flows on two-dimensional manifolds, then $\qquad \phi \sim \psi \iff \mathcal{D}(\phi) \cong \mathcal{D}(\psi)$

The proof of this theorem will take the whole of paragraphs 4.4 and 4.5. In 4.4 we prove some lemmas, in 4.5 we give the construction of the topological equivalence between ϕ and ψ . But the part => of 4.3.6 is easier so we prove it here, in the next Lemma.

4.3.7 Lemma $\phi \sim \psi \Rightarrow \mathcal{D}(\phi) = \mathcal{D}(\psi)$

Proof Let H be a homeomorphism giving top. equivalence between ϕ and ψ . Certainly ϕ and ψ must have the same number of attractors and saddles. (Denote these by $A_1,\dots,A_\alpha$, $P_1,\dots,P_\mu$ for ϕ , $\tilde{A}_1,\dots,\tilde{A}_\alpha$, $\tilde{P}_1,\dots,\tilde{P}_\mu$ for ψ , indexed in such a way that $H(Ai) = \tilde{A}_i$, $H(P_n) = \tilde{P}_n$.)

Let S_i , C_n be small C^1-circles around A_i , P_n satisfying conditions for a distribution $\mathcal{D}(\phi)$. H may, or may not, be differentiable on these circles.

For attractor $\tilde{A}_i$ we take any C^1-circle $\tilde{S}_i$ inside $H(S_i)$, transversal to ψ . The flow ψ induces a homeomorphism $\rho_i:H(S_i) \to \tilde{S}_i$ by putting $\rho_i(y) = \mathcal{O}^+(y) \cap \tilde{S}_i$ for $y \in H(S_i)$, where $\mathcal{O}^+(y) = \{\psi(t,y), t \geq 0\}$, since $\mathcal{O}^+(y)$ is inside $H(S_i)$ and must cross $\tilde{S}_i$ exactly once. See figure 11 (a).

For saddle $\tilde{P}_n$ we take any C^1-circle $\tilde{C}_n$, inside $H(C_n)$, and satisfying conditions for a distribution for ψ . Following orbits of ψ we get local homeomorphisms at the points where $W^u\tilde{P}_n \cup W^s\tilde{P}_n$ intersect $H(C_n)$ and $\tilde{C}_n$. See figure 11 (b). Let $\xi_n:H(C_n) \to \tilde{C}_n$ be a homeomorphism, extending these local ones.

(a)

(b)

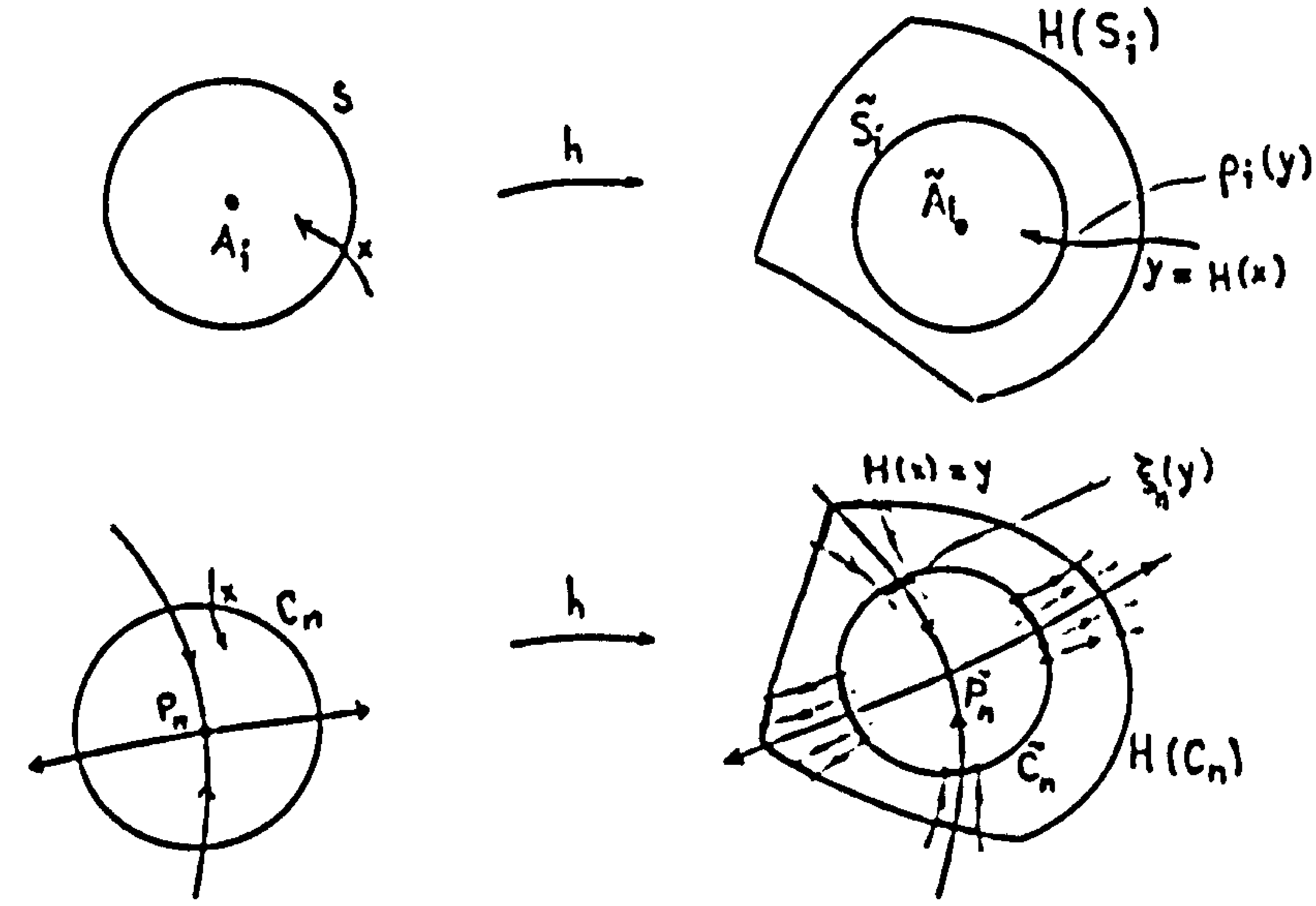


figure 11: construction of isomorphism h in 4.3.7.

Let S , C , $\tilde{S}$, $\tilde{C}$ be the unions of these circles. Define $h: S \cup C \to \tilde{S} \cup \tilde{C}$ by

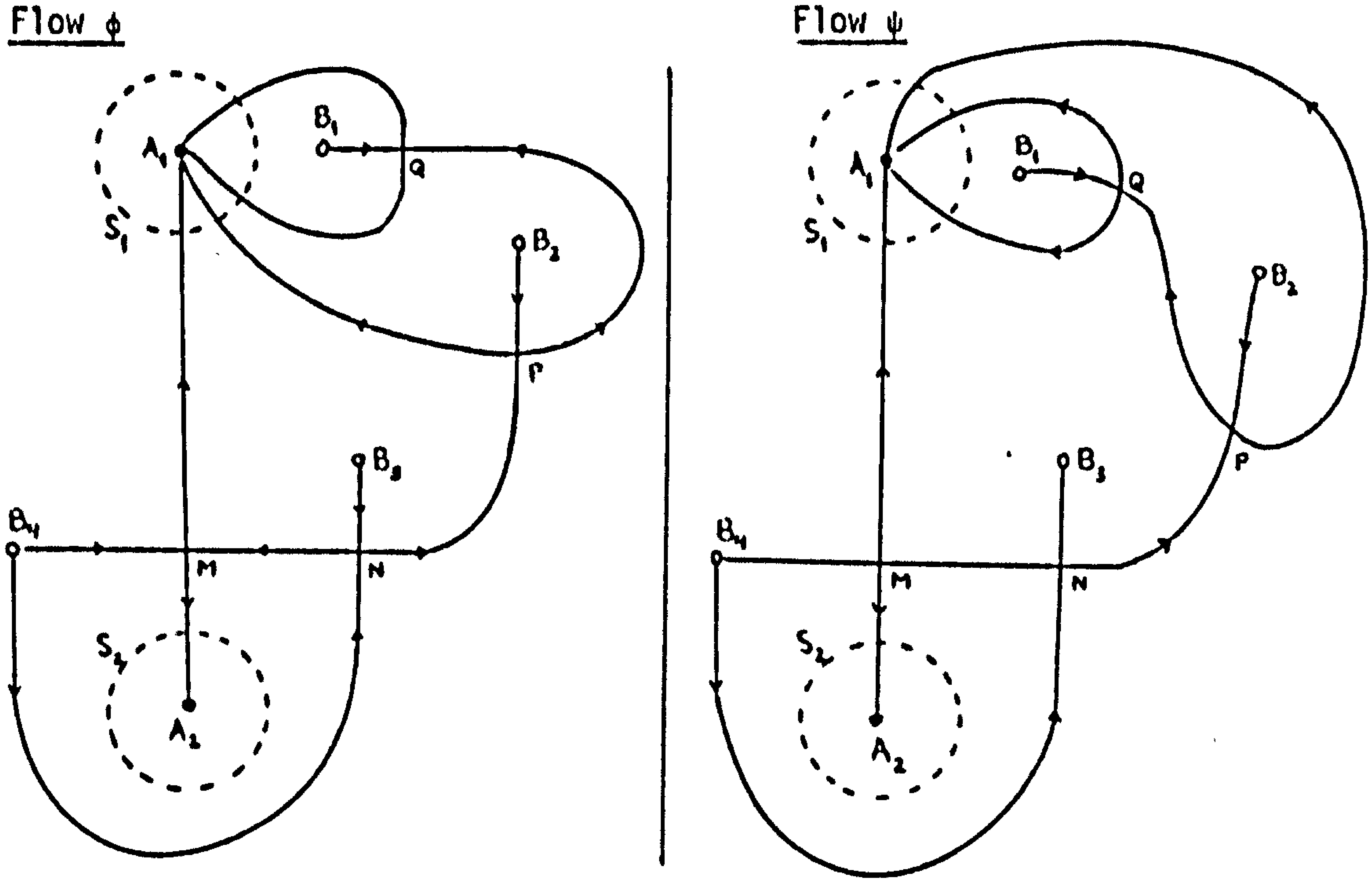
$$h(x) = \begin{cases} \rho_i \circ H(x) & \text{if } x \in S_i \\ \xi_n \circ H(x) & \text{if } x \in C_n . \end{cases}$$

Since any points of a distribution $\mathcal{D}(\psi)$ on $\tilde{C}_n$ are on $\tilde{C}_n \cap (W^u\tilde{P}_n \cup W^s\tilde{P}_n)$, we have that, h takes points of $\mathcal{D}(\phi)$ to points of $\mathcal{D}(\psi)$. It is easily checked that h is compatible with arrows. Hence h is an isomorphism between $\mathcal{D}(\phi)$ and $\mathcal{D}(\psi)$.

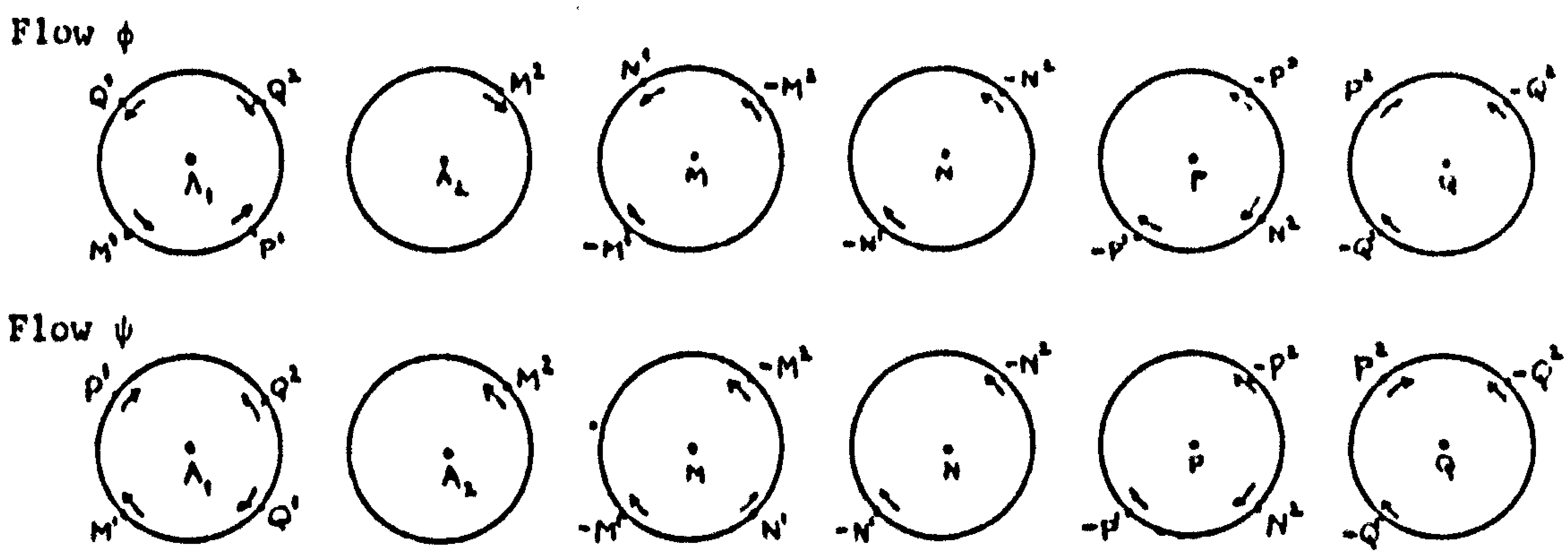
□

4.3.8 Example We give here two quasi-gradient flows ϕ and ψ which have isomorphic phase diagrams, but whose circular distributions $\mathcal{D}(\phi)$ and $\mathcal{D}(\psi)$ are not isomorphic.

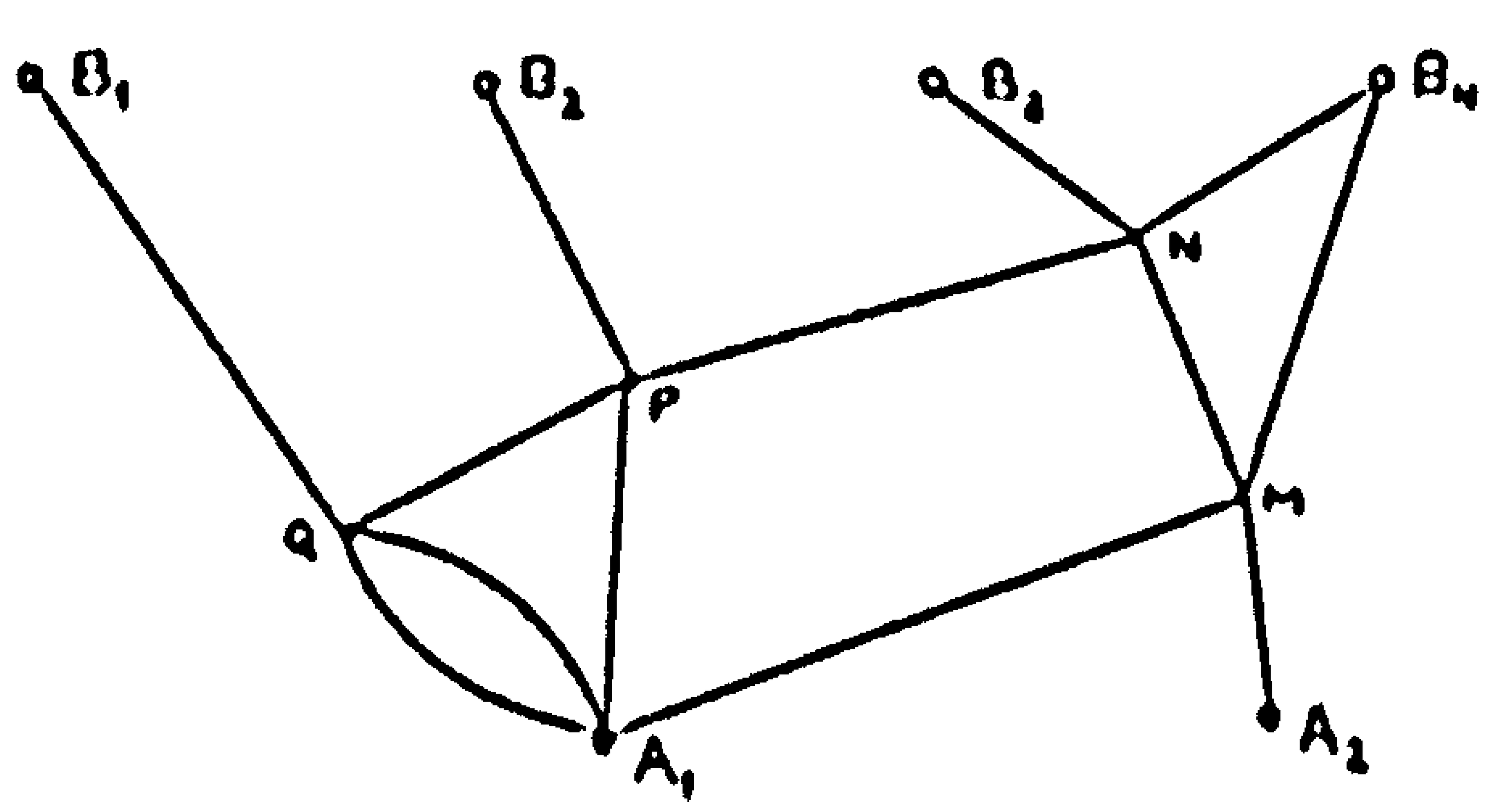
However, we note that if we were to consider only the distributions around attractors, these would be isomorphic.



(a) phase portraits of ϕ and ψ



(b) circular distributions



(c) phase diagrams for ϕ and ψ

figure 12

Here ϕ and ψ are supposed to be constructed on S^2, with attractors A_1, A_2, repellors B_1, B_2, B_3, B_4 and saddles M,N,P,Q. These flows are given by their phase portraits in figure 12 (a). Their circular distributions and phase diagrams are in figure 12 (b) and (c) respectively.

$\phi \not\sim \psi$: This can be seen in 2 ways:

(1) Noting that $\mathcal{D}(\phi) \neq \mathcal{D}(\psi)$ in figure 12(b). In 4.3.5 (1) we noted that if $\mathcal{D} \simeq \tilde{\mathcal{D}}$ we can suppose that isomorphism h preserves orientation. Fixing arrows at M^1 so that P^1 is at positive side of M^1 for both $\mathcal{D}(\phi)$ and $\mathcal{D}(\psi)$, then, for circles around M, N^1 is at positive side of $-M^1$, $-M^2$ for ϕ and at negative side for ψ. Hence, point N^1 of $\mathcal{D}(\phi)$ cannot be taken to N^1 of $\mathcal{D}(\psi)$ for any homeomorphism preserving orientations at $-M^1$ and $-M^2$.

(2) Directly, by noting that points $x \in W^uB_3 \cap W^sA_1$ will have orbit $\mathcal{O}^+(x)$ intersecting S_1 (around A_1) between $M^1 \in W^uM$ and $P^1 \in W^uP$ for ϕ, and between $M^1 \in W^uM$ and $Q^1 \in W^uQ$ for ψ, and P and Q are not interchangeable in the phase diagram.

4.4 Some handle lemmas

During the proof, in 4.5 next, of theorem 4.3.6, we will need the process of attaching handles to a given positively invariant (relative to flow ϕ) submanifold M_{n-1} of M, where each handle is a neighbourhood of a saddle P_n containing the arc of W^uP_n outside M_{n-1}. Also, the union of M_{n-1} with this handle must form a new positive-invariant submanifold. Then a topological equivalence between ϕ and ψ (with $\mathcal{D}(\phi) \simeq \mathcal{D}(\psi)$) will be constructed taking handles to handles. This process will be made clear in 4.5 but, in order to make the proof there clearer, we will establish here some technical lemmas, giving standard ways of constructing such handles and homeomorphism between them taking ϕ-orbits to ψ-orbits.

In the first lemma (4.4.1) we give a standard homeomorphism between standard handles for a standard flow. In the second (4.4.3) we show how handles are obtained for any flow ϕ (or ψ). In the third (4.4.4) we combine 4.4.1 and 4.4.3 to give topological equivalence between ϕ and ψ restricted to the handles of 4.4.3.

Also, we note that in 4.4.3 the handle for saddle P will be constructed so that its boundary intersects W^sP in 2 previously (arbitrarily) given points. This is not necessary for the construction itself, but it will be useful in Step 3 of 4.5.

For the first lemma, let us consider the (standard) flow θ on $\mathbb{R}^2$ induced by the linear vectorfield Y given by matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let R be the unit square $\{(x,y) \in \mathbb{R}^2 ; |x|,|y| \leq 1\}$. The boundary ∂R consists of 4 segments I^1, I^2, J^1, J^2 given, respectively, by intersection of R with lines $x = -1$, $x = 1$, $y = -1$, $y = 1$. We will denote by $\mathring{I}^i$, $\mathring{J}^i$ the interiors of I^i, J^i $(i = 1,2)$ as subsets of ∂R. Let $I = I^1 \cup I^2$ $J = J^1 \cup J^2$. See figure 13 (a).

The flow θ (given by $\theta(t,(x,y)) = (xe^t, ye^{-t})$) has origin 0 as a hyperbolic saddle with W^s0 = y-axis, W^u0 = x-axis. Let $p_1 = (-1,0) \in I^1$ $p_2 = (1,0) \in I^2$, $q_1 = (0,-1) \in J^1$, $q_2 = (0,1) \in J^2$. We consider segments I^1, I^2 (J^1, J^2) naturally oriented by increasing y (x).

4.4.1 <u>Lemma</u> (Standard handle)

Any homeomorphism $h_0 : I \to I$ sending I^i onto I^i, p_i to p_i

$i = 1,2$, preserving orientations, can be extended to homeomorphism $h:R \to R$ sending J^i onto J^i (q_i to q_i), preserving orientations, and such that h is topological equivalence (in R) of θ to itself.

Proof For any $\alpha \in [-1,1]$ consider the subset

$$D_\alpha = \{(x,y) \in R \ ; \ (1+\alpha)x^2 - (1-\alpha)y^2 = 2\alpha\} \ .$$

So $D_{-1} = J$, $D_1 = I$, D_0 = diagonals of R ($|x| = |y|$) and $C = I \cap J$ = corners of $R \subset D_\alpha$ $\forall \alpha \in [-1,1]$. See figure 13 (a).

D is a "continuous" family of subsets of R. To extend h to R we will use the following facts:

(1) $\forall$ $p = (x,y) \in R-C$, $\exists!$ $\alpha = \alpha(p)$ s.t. $p \in D_\alpha$, and $\alpha(p)$ depends continuously on p.

In fact $p = (x,y) \in D_\alpha \iff \alpha = \alpha(p) = \dfrac{x^2-y^2}{2-x^2-y^2}$.

(2) $\forall$ $p \in R-0$, the flow is transversal to family D_α .

In fact, putting $f_\alpha(x,y) = (1+\alpha)x^2 - (1-\alpha)y^2 - 2\alpha$, we have $D_\alpha = f_\alpha^{-1}(0)$ and $\forall$ $(x,y) \neq (0,0)$, $\alpha \in [-1,1]$

$$\frac{\partial f_\alpha}{\partial x}\dot{x} + \frac{\partial f_\alpha}{\partial y}\dot{y} = (1+\alpha)2x^2 + (1-\alpha)2y^2 > 0 \ .$$

(3) $\forall$ $p \in R-W^s0$, $\exists!$ $t^s(p) \geq 0$ such that $f(p) = \theta(t^s(p),p) \in I$. Also $t^s(p)$ and $f(p)$ are continuous in $R-W^s0$.

In fact, if $p = (x,y)$, $x \neq 0$, $|x|, |y| \leq 1$ then $t^s(p) = -\log|x|$, $f(p) = (\frac{x}{|x|}, y|x|) \in I$.

Now we define $h:R \to R$ by

$$h(p) = \begin{cases} p & , \text{ if } p \in C \text{ or } p \in W^sO \\ D_{\alpha(p)} \cap \mathcal{O}(h_0 \circ f(p)) , & \text{ if } p \in R - C \cup W^sO \ . \end{cases}$$

See figure 13 (b).

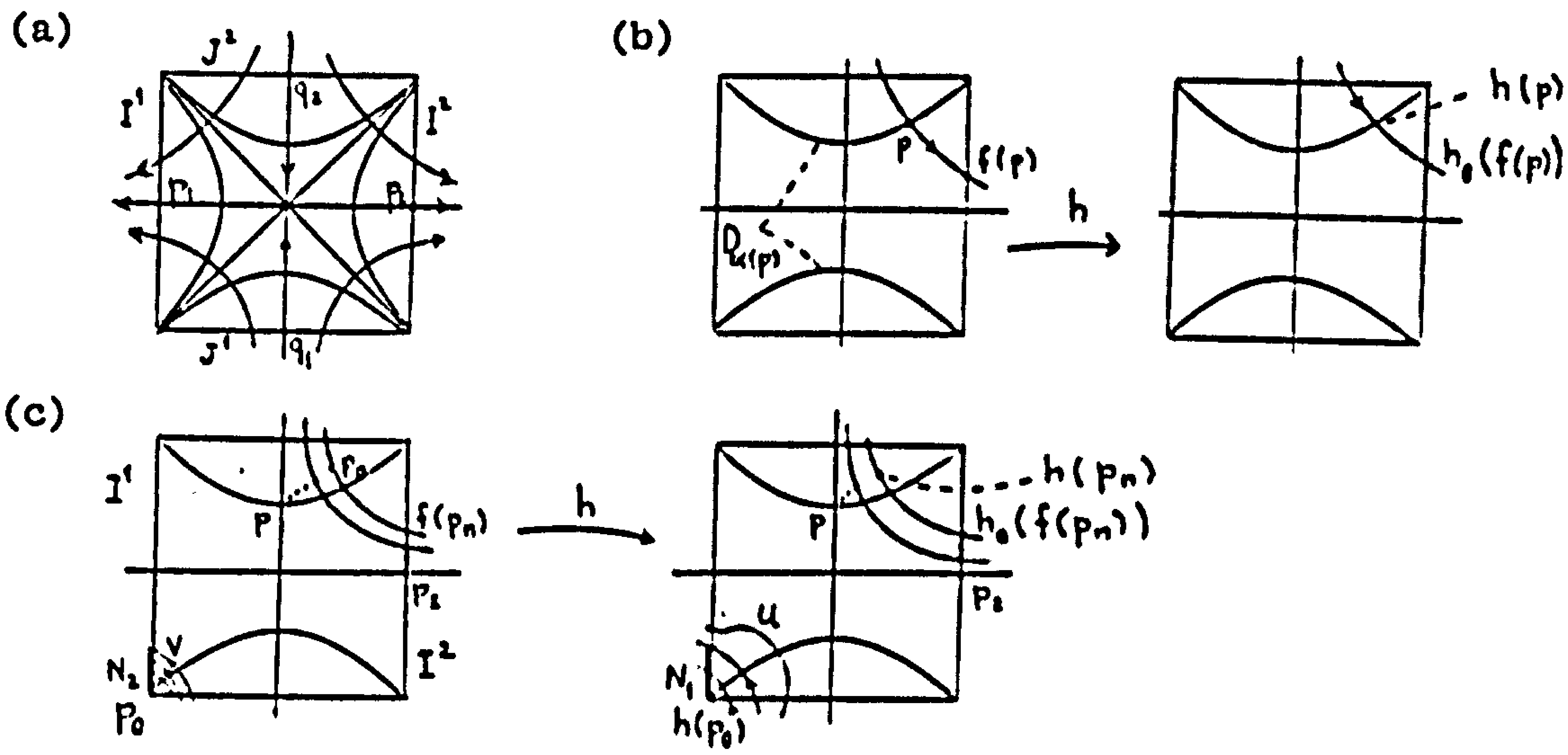


figure 13: (a) family D_α transversal to flow θ

(b) construction of h

(c) continuity of h on C and W^sO

We have to show that h is homeo, extends h_0 and takes orbits to orbits:

I) $h|_I = h_0$ because, if $p \in C$, $h(p) = p = h_0(p)$, and if $p \in \mathring{I}$, then $\alpha(p) = 1$, $t^s(p) = 0$, $f(p) = p$ hence $h(p) = D_1 \cap \mathcal{O}(h_0(p)) = h_0(p)$.

II) h takes orbits to orbits, by construction, since if p , $p' \in R\text{-}C \cup W^sO$ with $p' \in \mathcal{O}(p)$, then $f(p') = f(p)$ so $h(p') \in \mathcal{O}(h(p))$. The same is clear for $p, p' \in W^sO$.

III) Continuity of h on points of $R-C \cup W^s0$ is clear by the construction.

If $p \in W^s0$, let $p_n \to p$ with $p_n \in R-W^s0$. Then $\alpha(p_n) \to \alpha(p) \leq 0$. $p_n = (x_n,y_n)$ $x_n \neq 0$, $p = (0,y)$. If $x_n > 0$, $x_n \to 0^+$, then $f(p_n) \to p_2$, $t^s(p_n) \to +\infty$ and $h_0(f(p_n)) \to p_2$. So $h(p_n) =$ $= D_{\alpha(p_n)} \cap \mathcal{O}(h_0(f(p_n))) \Rightarrow h(p_n) \to D_{\alpha(p)} \cap W^s0 = \{p,-p\}$. Since h_0 preserves orientation, $h(p_n) \to p = h(p)$.

Analogously for $x_n < 0$. So h is continuous on W^s0. See figure 13 (c).

If $p_0 \in C = I \cap J$, $\forall$ neighbourhood U of $h(p_0) = p_0$ $\exists$ interval N_1 on I s.t. $p_0 \in N_1$, and $\forall$ $p \in N_1$, $\mathcal{O}(p) \cap R \subset U$. By continuity of h_0, $\exists$ interval N_2, $p_0 \in N_2 \subset I$ s.t. $p \in N_2 \Rightarrow h_0(p) \in N_1$. Taking $V = \{p \in R; f(p) \in N_2\}$ we have that V is a nbd. of p_0 in R and $\forall$ $p \in V$, $p \neq p_0 \Rightarrow h(p) = D_{\alpha(p)} \cap \mathcal{O}(h_0(f(p))) \in U$. So h is continuous on C. See figure 13 (c).

Hence h is continuous on R.

IV) h is homeo, because $h(\dot{R}) = R$ and h is injective since $h(p) = h(p') \Leftrightarrow \alpha(p) = \alpha(p')$ and $f(p) = f(p') \Leftrightarrow p = p'$. The inverse of h can be constructed by the same process starting with $h_0^{-1}: I \to I$.

4.4.2 <u>Remark</u> When $h_0: I \to I$ reverses orientation, an analogous lemma holds, with h taking J_1 to J_2, if we put $h(p) = -p$ if $p \in W^s(0)$, $h(p) = h_0(p)$ if $p \in C$.

Now we consider a 2-dim manifold M, and flow ϕ on M with hyperbolic saddle P. Let Σ^1, Σ^2 be C^1-arcs on M transversal to ϕ, intersecting W^uP at points P^1,P^2 respectively, each on one of the two separatrices of W^uP.

We denote by $[P_1,P_2]$ the arc of W^uP with P_1 and P_2 as endpoints. In the following lemma, R and θ are as in 4.4.1.

4.4.3 Lemma (Existence of handles)

Let Q^1 and Q^2 be any two points one on each separatrix of W^sP. Then, there exists a connected, compact neighbourhood N of P and homeomorphisms $T : N \to R$ such that:

(a) $[P_1,P_2] \subset N$

(b) $\partial N = \Sigma^1 \cup \Sigma^2 \cup L^1 \cup L^2$ where L^1 and L^2 are C^1-arcs transversal to ϕ, intersecting W^sP at Q^1 and Q^2, respectively.

(c) $\forall\ p \in N\text{-}W^sP$, orbit $\mathcal{O}(p)$ leaves N at a point of $\Sigma^1 \cup \Sigma^2$
$\forall\ p \in N\text{-}W^uP$, orbit $\mathcal{O}(p)$ enters N at a point of $L^1 \cup L^2$.

(d) T is top. equivalence between restrictions of ϕ to N, and θ to R.

(e) T takes $P^1,P^2,Q^1,Q^2,\Sigma^1,\Sigma^2,L^1,L^2$ respectively onto p_1,p_2,q_1,q_2, I^1,I^2,J^1,J^2.

Proof This construction uses standard techniques of dynamical systems. Since Σ^1, Σ^2 are transversals to ϕ intersecting W^uP on P^1,P^2, $W = W^sP \cup (\bigcup_{t\le 0} \phi^t(\Sigma^1 \cup \Sigma^2))$ is a neighbourhood of W^sP, fibrated over $[P^1,P^2]$ by W^sP and the iterates $\phi^t(\Sigma^i)$ as in [24], [25] or [26].

At Q^r $(r = 1,2)$ we take small C^1-arc S^r transversal to ϕ, $Q^r \in \mathring{S}^r$, $S^r \subset W$ and such that $\forall\ x \in S^r\text{-}Q^r$, $\exists\ t^+(x) > 0$ with $\phi(t^+(x),x) \in \mathring{\Sigma}^1 \cup \mathring{\Sigma}^2$, $\phi(t,x) \in W$ for $0 \le t \le t^+(x)$. See figure 14. ($\mathring{\Sigma}^i$ denotes Σ^i without endpoints.)

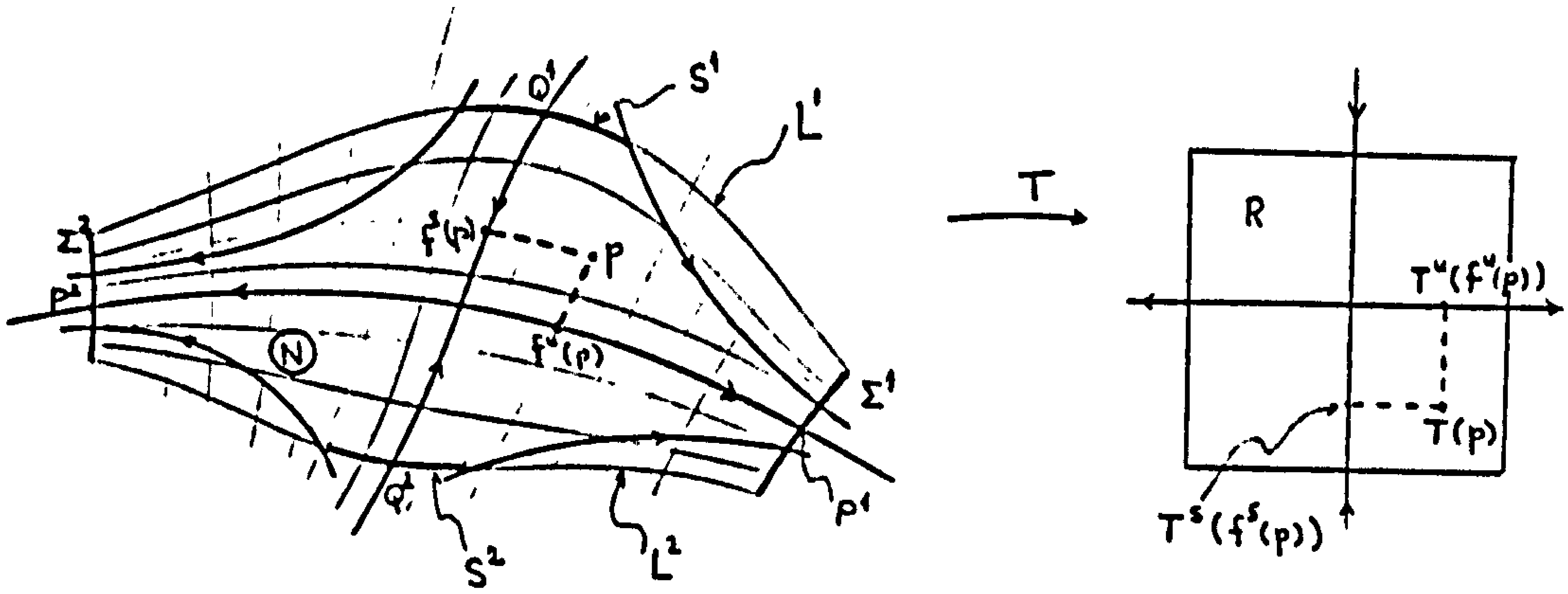


figure 14: construction of N and $T:N \to R$

We now extend S^r to a C^1-arc L^r whose end points are also end points, one of Σ^1, other of Σ^2 (take smaller S^r if necessary) with $\mathring{L}^r$ transversal to ϕ, not intersecting $[P^1,P^2]$ and $L^r \subset W$. See figure 14.

Now arcs $\Sigma^1,\Sigma^2,L^1,L^2$ bound a region N connected, compact containing P in its interior and satisfying (a), (b) and (c).

Now we define $T : N \to R$.

N is fibrated over $[P^1,P^2]$ with fibers F_x^u, $x \in [P^1,P^2]$,

where $F_x^u = \begin{cases} W^sP \cap N = [Q^1,Q^2], & \text{if } x = P \\ \phi(-t^+(x),\Sigma^r) \cap N, & \text{if } \phi(t^+(x),x) = P^r. \end{cases}$

Similarly, N is fibrated over $[Q^1,Q^2]$, with fibers F_y^s, $y \in [Q^1,Q^2]$,

where $F_y^s = \begin{cases} W^uP \cap N = [P^1,P^2] & \text{if } x = P \\ \phi(t^-(y),L^r) \cap N, & \text{if } \phi(-t^-(y),y) = Q^r. \end{cases}$

(This is the process of tubular families of Palis-Smale [24,25].)

Let f^u (f^s) be the projection of N onto $[P^1,P^2]$ ($[Q^1,Q^2]$) sending a F^u-fiber (F^s-fiber) to its intersection with $[P^1,P^2]$ ($[Q^1,Q^2]$).

Similarly, R can be fibrated by W^s0 plus iterates of I (fibers are vertical) and, also, by W^u0 plus iterates of J (fibers are horizontal). Then T must be constructed taking F^u-fibers to vertical fibers of R, and F^s-fibers to horizontal fibers of R. This can be achieved by taking homeomorphisms $T^u : [P^1,P^2] \to [p_1,p_2]$ and $T^s : [Q^1,Q^2] \to [q_1,q_2]$ given by:

$$\begin{cases} T^u(P) = 0 , \quad T^u(P^r) = p_r \\ T^u(x) = \theta(-t^+(x),p_r) \quad \text{if } \phi(t^+(x),x) = P^r \end{cases}$$

$$\begin{cases} T^s(P) = 0 , \quad T^s(Q^r) = q_r \\ T^s(y) = \theta(t^-(x),q_r) \quad \text{if } \phi(-t^-(y),y) = Q^r . \end{cases}$$

Now define $T:N \to R$ by

$$T(p) = (T^u(f^u(p)) , T^s(f^s(p))) .$$

See figure 14.

T is a homeomorphism and takes ϕ-orbits to θ-orbits, since $p \in N-[P^1,P^2] \cup [Q^1,Q^2]$ with $q = \phi(t_0,p)$ implies $f^u(q) = \phi(t_0,f^u(p))$, $f^s(q) = \phi(t_0,f^s(p))$, $t^+(q) = t^+(p)-t_0$ and $t^-(q) = t^-(p) + t_0$,

so $T^u(f^u(q)) = \theta(t_0,T^u(f^u(p)))$

$T^s(f^s(q)) = \theta(t_0,T^s(f^s(p)))$

$\Rightarrow \quad T(q) = \theta(t_0,T(p))$.

Hence, T is top. equivalence (in fact, a conjugacy i.e. time is preserved) between flows ϕ and θ restricted to N and R, i.e. (d) holds. (e) follows by the construction.

$\square$

Now we will combine lemmas 4.4.1 and 4.4.3 to obtain a topological equivalence between two flows ϕ and ψ restricted to neighbourhoods N, $\tilde{N}$ of saddles P, $\tilde{P}$ like in 4.4.3, the homeomorphism being previously fixed on arcs Σ^1, Σ^2 (taking them onto $\tilde{\Sigma}^1$, $\tilde{\Sigma}^2$).

So, take N as in 4.4.3 for ϕ and P with $\partial N = \Sigma^1 \cup \Sigma^2 \cup L^1 \cup L^2$ and suppose that compatible orientations of Σ^1, Σ^2 are given. By compatible we mean that arc L^r $(r=1,2)$ intersects both Σ^1 and Σ^2 at positive side (or both at negative side).

Similarly, take $\tilde{N}$ as in 4.4.3 for ψ and $\tilde{P}$ with $\partial\tilde{N} = \tilde{\Sigma}^1 \cup \tilde{\Sigma}^2 \cup \tilde{L}^1 \cup \tilde{L}^2$ and $\tilde{\Sigma}^1$, $\tilde{\Sigma}^2$ have compatible orientation.

4.4.4 <u>Lemma</u> (Equivalence on handles)

Given neighbourhoods N, $\tilde{N}$ of P, $\tilde{P}$, as above, and any homeomorphism $g_0:\Sigma^1 \cup \Sigma^2 \to \tilde{\Sigma}^1 \cup \tilde{\Sigma}^2$ taking Σ^r onto Σ^r $(r = 1,2)$, preserving orientation with $g_0(P^r) = \tilde{P}^r$ (where $P^r = \Sigma^r \cap W^uP$, $\tilde{P}^r = \Sigma^r \cap W^u\tilde{P}$), then g_0 can be extended to homeomorphism $g:N \to N$ which is a top. equivalence between flows ϕ on N and ψ on N, with $g(L^1 \cup L^2) = \tilde{L}^1 \cup \tilde{L}^2$.

4.4.5 <u>Remark</u> Re-indexing $\tilde{L}^r$ if necessary we can make $g(L^r) = \tilde{L}^r$.

<u>Proof</u> Let $T:N \to R$ and $\tilde{T}:\tilde{N} \to R$ be given by 4.4.3, for ϕ and ψ, respectively.

Exchanging Q^1 with Q^2, if necessary, we can suppose, in 4.4.3, that $x \in S^2-Q^2 \Rightarrow \phi(t^+(x),x) \in$ positive side of $\Sigma^1 \cup \Sigma^2$ (relative to given orientations). This makes T to take Σ^r onto I^r preserving orientation (orientation of I^r being the usual).

Analogously, we suppose $\tilde{T}$ takes $\tilde{\Sigma}^r$ onto I^r preserving orientation.

Let $h_0 = \tilde{T}\, g_0\, T^{-1}\big|_I : I^1 \cup I^2 \to I^1 \cup I^2$. h_0 satisfies hypothesis of 4.4.1, so it can be extended to homeomorphism $h:R \to R$ taking J^r to J^r .

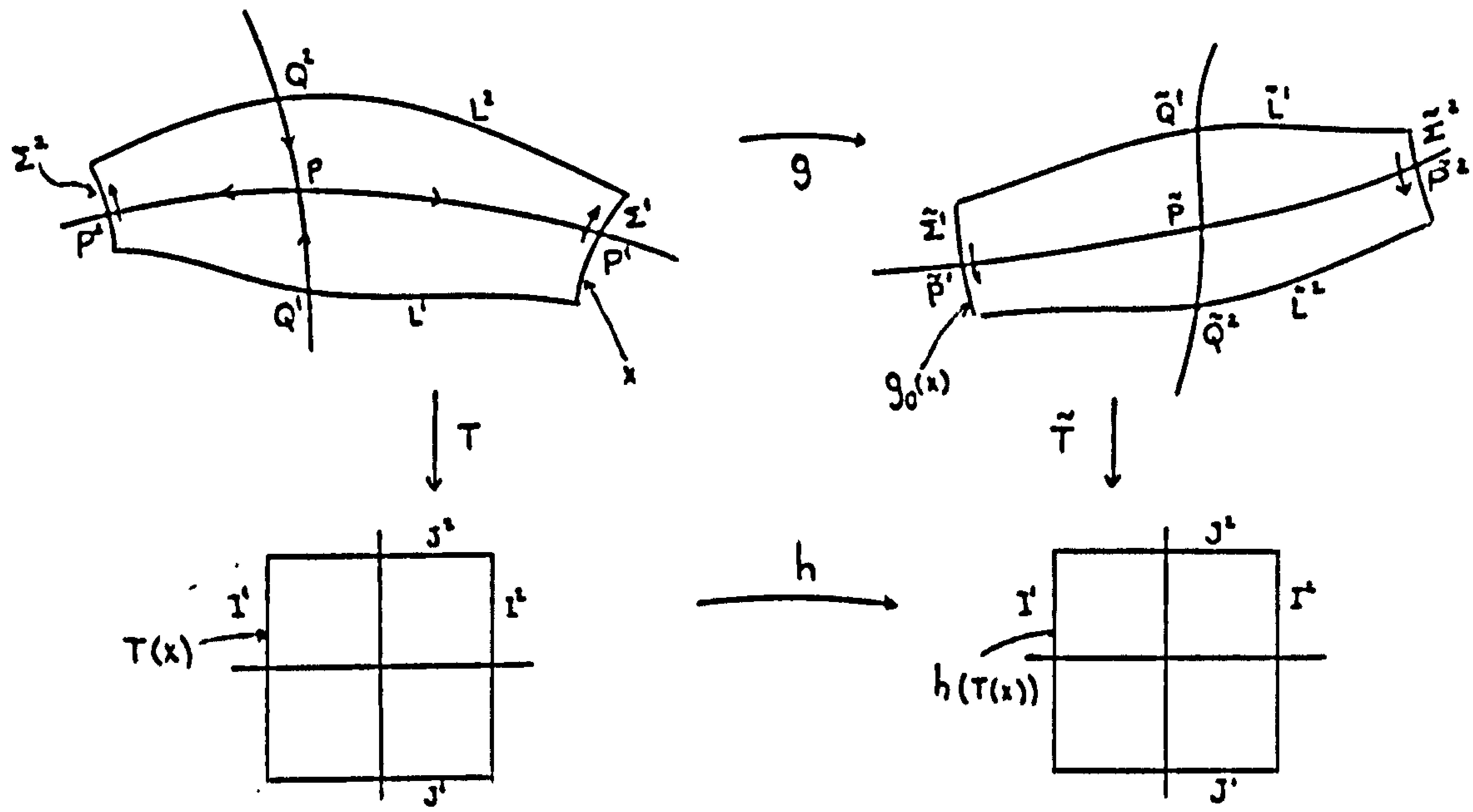


figure 15: construction of $g:N \to \tilde{N}$ in 4.4.4.

Take $g = \tilde{T}^{-1}\, h\, T : N \to \tilde{N}$. See figure 15. g is homeomorphism taking ϕ-orbits to ψ-orbits because T takes ϕ-orbits to θ-orbits, which are taken to θ-orbits by h , and these are taken to ψ-orbits by $\tilde{T}^{-1}$. Also $T(L^r) = J^r$, $\tilde{T}(\tilde{L}^r) = J^r$ so $g(L^r) = \tilde{L}^r$.

Hence g satisfies the lemma. □

4.5 <u>Proof of Theorem IV</u> (stated in 1.6.3 and 4.3.6)

Let ϕ and ψ be quasi-gradient in 2-dim manifolds M and $\tilde{M}$. Let $\mathcal{D}$, $\tilde{\mathcal{D}}$ be their circular distributions as in 4.3.

$\phi \sim \psi \Rightarrow \mathcal{D} \approx \tilde{\mathcal{D}}$ was proved in 4.3.7.

Now, let us suppose $\mathcal{D} \approx \tilde{\mathcal{D}}$, with isomorphism $h: S \cup C \to \tilde{S} \cup \tilde{C}$ where $\{S_i \ 1 \le i \le \alpha \ , \ C_n \ 1 \le n \le \mu\}$ and $\{\tilde{S}_i \ 1 \le i \le \alpha \ , \ \tilde{C}_n \ 1 \le n \le \mu\}$ are the circles of $\mathcal{D}$ and $\tilde{\mathcal{D}}$ and S , C , $\tilde{S}$, $\tilde{C}$ are their unions. (We recall definition 4.3.4 and use notation as in 4.3.5 (3) so that $h(S_i) = \tilde{S}_i$, $h(C_n) = \tilde{C}_n$, and denote by A_i , $P_n(\tilde{A}_i, \tilde{P}_n)$ the fixed points of ϕ (ψ) inside S_i , $C_n(\tilde{S}_i, \tilde{C}_n$; respectively). Write $h(A_i) = \tilde{A}_i$, $h(P_n) = \tilde{P}_n$.

As noted in 4.3.5(1) we can suppose that h <u>preserves orientation of arrows</u> at all points of the distributions.

Before going into the details we give here an idea of the construction of homeomorphism $H: M \to \tilde{M}$ which will give equivalence of ϕ and ψ . This will be done in 4 steps:

<u>Step 1</u>: Re-indexing of saddles $P_1, \ldots, P_\mu$ of ϕ so that each separatrix of $W^u P_n$ will always flow either to an attractor or to a saddle of smaller index. The same will be valid, then, for correspondent (relative to h) saddles $\tilde{P}_1, \ldots, \tilde{P}_\mu$ of ψ . This process is equivalent to a partial ordering of the saddles as referred in Remark 4.1.3.

<u>Step 2</u>: Call D_i , $\tilde{D}_i$ the discs in M , $\tilde{M}$ containing $A_i, \tilde{A}_i$, bounded by S_i , $\tilde{S}_i$ respectively. Extend h to $H_0: M_0 = \cup D_i \to \tilde{M}_0 = \cup \tilde{D}_i$, taking ϕ-orbits to ψ-orbits.

<u>Step 3</u>: Inductively construct $(M_1, \tilde{M}_1), \ldots (M_\mu, \tilde{M}_\mu)$ as subsets of $(M, \tilde{M})$ with $M_0 \subset M_1 \subset \ldots \subset M_\mu \subset M$, $\tilde{M}_0 \subset \tilde{M}_1 \subset \ldots \subset \tilde{M}_\mu \subset \tilde{M}$ and homeomorphisms $H_n: M_n \to \tilde{M}_n$, H_n extending H_{n-1} , and H_n taking ϕ-orbits to ψ-orbits. Also, ϕ-orbits (ψ-orbits) will cross $\partial M_n (\partial \tilde{M}_n)$ going into the interior of M_n $(\tilde{M}_n)$.

Step 4: Extend $H_\mu : M_\mu \to \tilde{M}_\mu$ to $H : M \to \tilde{M}$ where H is the required top. equivalence of ϕ and ψ.

Step 1: Ordering saddles

First we note that there must be at least one saddle (which we re-index as P_1) such that $W^uP_1 - P_1 \subset W_0 = \bigcup_{i=1}^{\alpha} W^sA_i$ (i.e. both separatrices of W^uP_1 flow to attractors). In fact, if there was no such saddle, for every P_n there would be another P_m with $P_n \searrow P_m$. Since the number of saddles is finite, there would be a sequence of saddles forming a cycle, which is not allowed by (c*) of 4.1.2.

Let $W_1 = W_0 \cup W^sP_1$. By similar argument, there exists saddle ($\neq P_1$) which we re-index as P_2 such that $W^uP_2 - P_2 \subset W_1$.

Inductively, take P_n s.t. $W^uP_n - P_n \subset W_{n-1}$ and put $W_n = W_{n-1} \cup W^sP_n$.

Therefore, saddles are put in order $P_1, P_2, \dots, P_\mu$ so that $P_n \searrow P_m \Rightarrow n > m$.

Now, re-index $\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_\mu$ by $\tilde{P}_n = h(P_n)$.

Without loss in generality we can suppose that $h(P_n^k) = \tilde{P}_n^k$ $k = 1,2$ (so, $h(-P_n^k) = -\tilde{P}_n^k$, too) by exchanging $\tilde{P}_n^1$, $-\tilde{P}_n^1$ with $\tilde{P}_n^2$, $-\tilde{P}_n^2$ if necessary. (This convention simplifies notation.)

By compatibility with points (4.3.4 (ii)) we have $\tilde{P}_n^k = h(P_n^k) \in \tilde{S}_i$ or $\tilde{C}_m \iff P_n^k \in S_i$ or C_m, respectively. Hence $\tilde{P}_n \searrow \tilde{P}_m \Rightarrow n > m$ (i.e. separatrices of $W^u\tilde{P}_n - \tilde{P}_n$ flow either to an attractor or to a saddle of smaller index).

Step 2: Construction of H_0

Let $M_0 = \bigcup_{i=1}^{\alpha} D_i$, $\tilde{M}_0 = \bigcup_{i=1}^{\alpha} \tilde{D}_i$ where D_i, $\tilde{D}_i$ are bounded by

S_i , $\tilde{S}_i$ containing A_i, $\tilde{A}_i$, respectively. Flows ϕ and ψ are transversal to ∂M_0 and $\partial \tilde{M}_0$, respectively. For $x \in D_i - A_i$, $\exists\, t(x) \geq 0$ s.t. $f(x) = \phi(-t(x), x) \in S_i$. $t(x)$ and $f(x)$ depend continuously on x , and $t(x) \to +\infty$ as $x \to A_i$. Define $H_0 : M_0 \to \tilde{M}_0$ by

$$H_0(x) = \begin{cases} \tilde{A}_i & \text{if } x = A_i \\ \psi(t(x), h(f(x))) & \text{if } x \in D_i - A_i . \end{cases}$$

See figure 16.

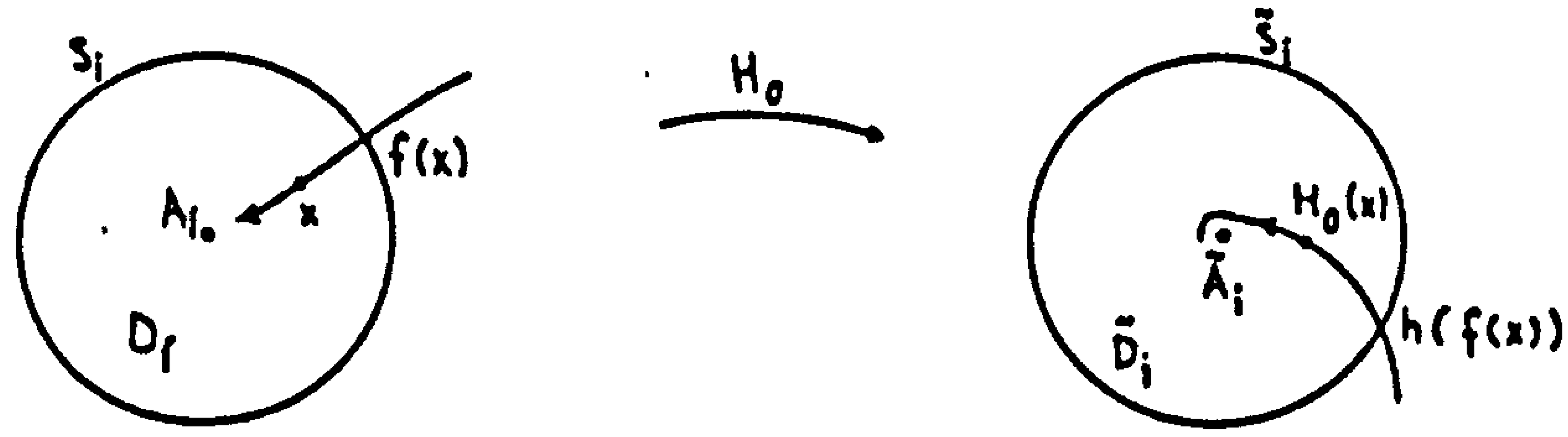


figure 16: construction of H_0

Then, $\lim_{x \to A_i} H_0(x) = \lim_{t \to +\infty} \tilde{\psi}(t, h(f(x)) = \tilde{A}_i$

since $\lim_{t \to +\infty} \psi(t, \tilde{S}_i) = \tilde{A}_i$.

H_0 is homeomorphism and takes ϕ-orbits to ψ-orbits by construction. $H_0|_S = h|_S$.

<u>Step 3</u>: <u>Construction of $H_n : M_n \to \tilde{M}_n$</u>

Consider the points P_n^k ($k = 1,2,\ n = 1,\ldots,\mu$) of $\mathcal{D}$. If $P_n^k \in S$, we take a small arc Σ_n^k on S containing P_n^k in its interior.

These arcs can be taken sufficiently small to be all disjoint. The arrow of $\mathcal{D}$ at P_n^k induces an orientation for Σ_n^k.

Now take $\tilde{\Sigma}_n^k = h(\Sigma_n^k) = H_0(\Sigma_n^k)$. So, $\tilde{\Sigma}_n^k$ is an arc on $\tilde{S}$ containing $\tilde{P}_n^k = h(P_n^k)$ in its interior, and the arcs $\tilde{\Sigma}_n^k$ (for $\tilde{P}_n^k \in \tilde{S}$) are disjoint.

For saddle P_n $(n = 1,\dots,\mu)$ of ϕ , $W^uP_n - P_n$ consists of two orbits (separatrices) which we denote by γ_n^1 , γ_n^2 , where $P_n^1, -P_n^1 \in \gamma_n^1$, $P_n^2, -P_n^2 \in \gamma_n^2$.

Now we will construct, inductively on $n = 1,\dots,\mu$, $M_n, \tilde{M}_n$ and $H_n : M_n \to \tilde{M}_n$, satisfying:

I) M_n is a compact, positively ϕ-invariant neighbourhood of $(\bigcup_{i=1}^{\alpha} A_i) \cup (\bigcup_{m=1}^{n} W^uP_m)$ with $P_m \notin M_n$ for $m > n$.

Analogously for $\tilde{M}_n$, relative to ψ .

II) ∂M_n is piecewise-C^1 one-dimensional submanifold, transversal to ϕ . By transversal here we mean that orbits of ϕ cut ∂M_n going from $M - M_n$ into int M_n (i.e., $\forall\, x \in \partial M_n$, $\phi(t,x) \notin M_n$ for $t < 0$ and $\phi(t,x) \in$ int M for $t > 0$) and the intersection of orbit with ∂M_n is C^1-transversal at point where ∂M_n is C^1 .

Analogously for $\partial\tilde{M}_n$ relative to ψ .

III) H_n is homeomorphism, $H_n(\partial M_n) = \partial\tilde{M}_n$ and H_n takes positive ϕ-orbits onto positive ψ-orbits.

IV) For $n < m \le \mu$, if W^uP_m has separatrix γ_m^k (k = 1 or 2) flowing into M_n , then

$$P_m^k \in \partial M_n \quad \text{and} \quad H_n(P_m^k) = h(P_m^k) = \tilde{P}_m^k \in \partial\tilde{M}_n \ .$$

V) For all $P_m^k \in \partial M_n$ (as in IV), $\tilde{P}_m^k \in \partial\tilde{M}_n$, there are (small) C^1-arcs Σ_m^k, $\tilde{\Sigma}_m^k$ contained in ∂M_n, $\partial\tilde{M}_n$ respectively, and containing P_m^k, $\tilde{P}_m^k$, respectively, in their interiors, with $H_n(\Sigma_m^k) = \tilde{\Sigma}_m^k$ preserving orientations given by arrows of $\mathcal{D}$ and $\tilde{\mathcal{D}}$ at P_m^k and $\tilde{P}_m^k$. Also, all such intervals on ∂M_n are disjoint.

We see that conditions (I)-(V) are valid for M_0, M_0 and H_0, where (I)-(III) follow from Step 2, (IV) follows from Step 1, and (V) from choice of arcs at beginning of this step, and compatibility of h with arrows.

By induction, let us suppose (I)-(V) are valid for M_{n-1}, $\tilde{M}_{n-1}$, H_{n-1}, for some $n \geq 1$.

By order established in Step 1, and condition IV of induction we must have $P_n^k \in \partial M_{n-1}$, $\tilde{P}_n^k = h(P_n^k) = H_{n-1}(P_n^k) \in \partial\tilde{M}_{n-1}$ for $k = 1,2$ and, by (V), $H_{n-1}(\Sigma_n^k) = \tilde{\Sigma}_n^k$ preserving orientation of arrows at P_n^k and $\tilde{P}_n^k$.

We denote by Q^1, Q^2 the points of $C_n \cap W^sP_n$ with Q^2 at positive side of $-P_n^1$, $-P_n^2$, relative to arrows at these points. Then $\tilde{Q}^1 = h(Q^1)$, $\tilde{Q}^2 = h(Q^2)$ are the points of $\tilde{C}_n \cap W^sP_n$ and, since h preserves orientations of arrows, $\tilde{Q}^2$ is at positive side of $-\tilde{P}_n^1 = h(-P_n^1)$, $-\tilde{P}_n^2 = h(-P_n^2)$. See figure 17.

Now, we use lemma 4.4.3 to construct N_n and $\tilde{N}_n$ containing, respectively, $[P_n^1,P_n^2]$, $[\tilde{P}_n^1,\tilde{P}_n^2]$ with $\partial N_n = \Sigma_n^1 \cup \Sigma_n^2 \cup L^1 \cup L^2$, L^1,L^2 as C^1-arcs transversal to ϕ, $\{Q^1\} = L^1 \cap W^sP_n$, $\{Q^2\} = L^2 \cap W^sP_n$ and, analogously, $\partial\tilde{N}_n = \tilde{\Sigma}^1 \cup \tilde{\Sigma}^2 \cup \tilde{L}^1 \cup \tilde{L}^2$, $\tilde{L}^1,\tilde{L}^2$ as C^1-arcs transversal to ψ, $\{\tilde{Q}^1\} = \tilde{L}^1 \cap W^s\tilde{P}_n$, $\{\tilde{Q}^2\} = \tilde{L}^2 \cap W^s\tilde{P}_n$.

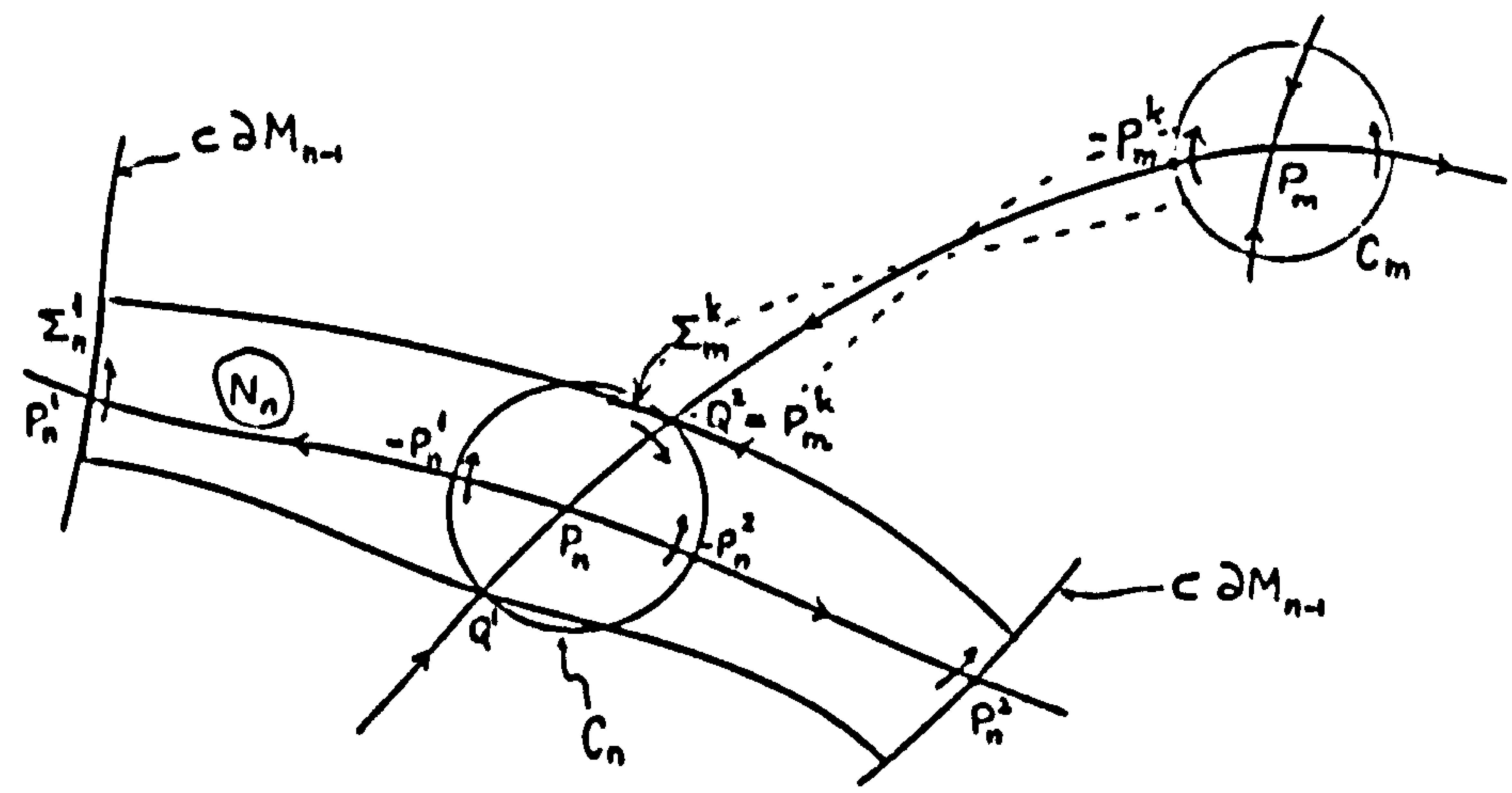


figure 17: construction of N_n and arc Σ_m^k

By lemma 4.4.4, $H_{n-1}\Big|_{\Sigma_n^1 \cup \Sigma_n^2}$ can be extended to $\bar{H}_n : N_n \to \tilde{N}_n$ taking ϕ-orbits to ψ-orbits and $\bar{H}_n(Q^r) = \tilde{Q}^r$, $r = 1,2$.

We take $M_n = M_{n-1} \cup N_n$ $\tilde{M}_n = \tilde{M}_{n-1} \cup \tilde{N}_n$ and $H_n : M_n \to \tilde{M}_n$ given by $H_n(x) = \begin{cases} H_{n-1}(x) & \text{if } x \in M_{n-1} \\ \bar{H}_n(x) & \text{if } x \in N_n \end{cases}$.

We have now to prove that (I)-(V) hold for these.

I) $\forall\ x \in N_n$, either $x \in W^s P_n$ (hence $\mathcal{O}^+(x) \subset N_n$) or

$\exists\ t_+(x) \geq 0$ s.t. $\begin{cases} \phi(t,x) \in N_n & \text{for } 0 \leq t \leq t_+(x) \\ \phi(t_+(x),x) \in \Sigma^1 \cup \Sigma^2 \subset \partial M_{n-1} \end{cases}$.

Since M_{n-1} is positively ϕ-invariant, it follows that M_n is positively ϕ-invariant, too.

Also, M_{n-1} is neighbourhood of $(\bigcup_{i=1}^{\alpha} A_i) \cup (\bigcup_{m=1}^{n-1} W^u P_m)$ containing positive orbits of P_n^1 and P_n^2, and N_n is neighbourhood of $[P_n^1, P_n^2]$. It follows that M_n is neighbourhood of $(\bigcup_{i=1}^{\alpha} A_i) \cup (\bigcup_{m=1}^{n} W^u P_m)$.

Analogously for $\tilde{M}_n$.

II) $\partial M_n = (\partial M_{n-1} - \Sigma_n^1 \cup \Sigma_n^2) \cup L^1 \cup L^2$. So, ∂M_n is formed by a finite number of C^1-arcs, all C^1-transversal to flow ϕ. ∂M_n only fails to be C^1 at the points joining two of these arcs, but by construction ϕ-orbits cross these points from $M - M_n$ into $\operatorname{int} M_n$.

Analogously for $\partial\tilde{M}_n$.

III) By induction, H_{n-1} is homeomorphism of M_{n-1} onto $\tilde{M}_{n-1}$ taking ϕ-orbits onto ψ-orbits, and, by construction H_n is homeo of N_n onto $\tilde{N}_n$ also taking ϕ-orbits to ψ-orbits. Since H_{n-1} and $\tilde{H}_n$ coincide on $\Sigma_n^1 \cup \Sigma_n^2$ $(= M_{n-1} \cap N_n)$, we get that H_n is homeomorphism and takes (positive) ϕ-orbits to (positive) ψ-orbits.

IV) Suppose P_m, with $m > n$, has separatrix γ_m^k ($k=1$ or 2) flowing into M_n. Then, either γ_m^k flows into M_{n-1} or $\gamma_m^k \subset W^s P_n$. In the first case, by induction, $P_m^k \in \partial M_{n-1}$ and interval Σ_m^k is disjoint of $\Sigma_n^1 \cup \Sigma_n^2$, then $P_m^k \in \Sigma_m^k \subset \partial M_n$. Hence $\tilde{P}_m^k = h(P_m^k) = H_{n-1}(P_m^k) = H_n(P_m^k) \in \partial\tilde{M}_n$, and, by induction $H_n(\Sigma_m^k) = \tilde{\Sigma}_m^k$ preserving orientations. In case $\gamma_m^k \subset W^s P_n$, γ_m^k must intersect ∂M_n at Q^1 or Q^2. Hence $P_m^k = Q^r \in L^r \subset \partial M_n$ for $r = 1$ or 2. See figure 17.

So, $H_n(P_m^k) = H_n(Q^r) = \tilde{Q}^r = h(Q^r) = \tilde{P}_m^k$ by construction of H_n on N_n. Then (IV) holds.

V) Now we take a small C^1-arc Σ_m^k containing P_m^k (as in IV above) in its interior, with $\Sigma_m^k \subset L^r \subset \partial M_n$. Hence Σ_m^k is transversal to ϕ . Take $\tilde{\Sigma}_m^k = H_n(\Sigma_m^k) \subset \tilde{L}^r \subset \partial\tilde{M}_n$ then $\tilde{P}_m^k \in \tilde{\Sigma}_m^k$.

At P_m^k and $\tilde{P}_m^k$ we have arrows given by the circular distributions. Because $h:S\cup C \to \tilde{S}\cup\tilde{C}$ preserves orientation of arrows, we have that H_n (by construction in 4.4.4) takes positive side of Σ_m^k (relative to arrow at P_m^k) to positive side of $\tilde{\Sigma}_m^k$ (relative to arrow at $\tilde{P}_m^k$) . Hence H_n takes Σ_m^k onto $\tilde{\Sigma}_m^k$ preserving orientations given by $\mathcal{D}$ and $\tilde{\mathcal{D}}$. Hence (V) holds.

Therefore Step 3 is complete.

<u>Step 4:</u> <u>Extending $H_\mu : M_\mu \to \tilde{M}_\mu$ to $H:M \to \tilde{M}$</u>

First we note that Step 3 implies that $M_\mu(\tilde{M}_\mu)$ is a compact positively ϕ- (ψ-) -invariant neighbourhood of $\overline{\bigcup_{n=1}^{\mu} W^u P_n}$ $(\overline{\bigcup_{n=1}^{\mu} W^u \tilde{P}_n})$.

Let $V = M - M_\mu$, $\tilde{V} = \tilde{M} - \tilde{M}_\mu$. Then V is negatively ϕ-invariant, containing all the repellors $B_1,\dots,B_\beta$ and no other fixed point of ϕ . Hence $V = \bigcup_{i=1}^{\beta} V_i$ where V_i is a negatively ϕ-invariant neighbourhood of B_i . So $V_i \subset W^u B_i$, and the V_i's are disjoint and ∂V_i is homeomorphic to a circle and ϕ-orbits cross ∂V_i from int V_i to $M - V_i$. Also $\partial V = \partial M_\mu = \bigcup_{i=1}^{\beta} \partial V_i$ and the "circles" ∂V_i are disjoint.

Analogously $\tilde{V}$ is negatively ψ-invariant containing all repellors $\tilde{B}_1,\dots,\tilde{B}_{\tilde{\beta}}$ and no other fixed point of ψ . Similarly to above,

$$\tilde{V} = \bigcup_{j=1}^{\tilde{\beta}} \tilde{V}_j \quad , \quad \tilde{B}_j \in \tilde{V}_j \quad , \quad \partial\tilde{V} = \partial\tilde{M}_\mu = \bigcup_{j=1}^{\tilde{\beta}} \partial\tilde{V}_j \ .$$

Since $H_\mu(\partial V) = \partial\tilde{V}$ we must have that ∂V and $\partial\tilde{V}$ have the same number of connected components, i.e. $\beta = \tilde{\beta}$. Hence ϕ and ψ have the same number of repellors. Re-indexing, if necessary we can suppose that $H_\mu(\partial V_j) = \partial\tilde{V}_j \quad j = 1,\dots,\beta$. Let $g_j = H_\mu|_{\partial V_j} : V_j \to \partial\tilde{V}_j$ $j = 1,\dots,\beta$. (g_j is homeomorphism)

g_j can be extended to homeomorphism

$G_j : V_j \to \tilde{V}_j$ by making

$G_j(B_j) = \tilde{B}_j$, and

$G_j(x) = \psi(-t_+(x), g_j(\phi(t_+(x),x))$ if $x \in V_j - B_j$ where $t_+(x) > 0$ is such that $\phi(t_+(x),x) \in \partial V_j$. (This process is similar to Step 1.)

Now, we define $H : M \to \tilde{M}$ by

$$H(x) = \begin{cases} H_\mu(x) & \text{if } x \in M_\mu \\ G_j(x) & \text{if } x \in V_j \quad j = 1,\dots,\beta \end{cases} .$$

H is continuous and bijective. Since M , $\tilde{M}$ are compact, H is homeomorphism.

H takes ϕ-orbits to ψ-orbits by construction. Therefore H is a topological equivalence between ϕ and ψ .

So $\mathcal{D}(\phi) = \mathcal{D}(\psi) \Rightarrow \phi \sim \psi$.

□

4.6 Application to Gradient-Like Flows

As noted in 4.1, gradient-like flows are a particular case of quasi-gradient flows.

Since gradient-like ϕ has no saddle connections for $\dim M = 2$, $\mathcal{D}(\phi)$ has, on every circle C_n (around saddle P_n), only the points $-P_n^1$, $-P_n^2$ and no P_m^k for $m \neq n$. The arrows at $-P_n^1$ and $-P_n^2$ always have opposite orientations on C_n. So, the information of how "handles" N_n of 4.5 Step 3 are attached is given only by arrows at P_n^1 and P_n^2, which are in $S = \cup S_i$. This means that all information given $\mathcal{D}(\phi)$ is carried by the circles $S_1,\ldots,S_\alpha$ with their distinguished points and arrows (and we can disregard all circles $C_1,\ldots,C_\mu$ around saddles). This coincides with the circular distributions for gradient-like flows as defined by Fleitas [10].

However, the equivalence H constructed in 4.5 does not agree in general with the equivalence constructed in [10] because there Fleitas has given a conjugacy (i.e. time preserving equivalence). But, in 4.5, we could not hope for a conjugacy because we were allowing saddle connections.

4.7 Application for 2-dim manifold with boundary

We are interested in applying Theorem 4.3.6 to a quasi-gradient flow ϕ on a compact smooth 2-dim. manifold M with piecewise smooth boundary ∂M, where ∂M is ϕ-invariant. We will consider cases where M can be "extended beyond ∂M" i.e. we want M to be a submanifold (with boundary) of another 2-dim manifold $\bar{M}$, with $\partial M \subset \operatorname{int} \bar{M}$ and also ϕ must be a restriction to M of a flow $\bar{\phi}$ of $\bar{M}$, with $\bar{\phi}$ invariant on ∂M. We could lessen such conditions, but then we would have to worry about what hyperbolicity of fixed points of ϕ on ∂M means. Having such conditions we say that fixed point $p \in \partial M$ is hyperbolic for ϕ if it is so for $\bar{\phi}$.

For such p, the invariant manifolds $W^s p$, $W^u p$, relative to flow ϕ, are the restrictions to M of $W^s p$, $W^u p$, relative to $\bar{\phi}$, respectively.

Suppose now that ϕ is quasi-gradient flow on M (i.e. ϕ satisfies (a), (b), (c*) of 4.1 where everything considered is restricted to M), invariant on ∂M. A circular distribution of points, $\mathcal{D}(\phi)$ is defined for ϕ, exactly as in 4.3, where the "circles" are restrictions to M of circles of $\bar{M}$. We note here that if saddle P_n of ϕ is on ∂M, then any sufficiently small neighbourhood of P_n will intersect ∂M on subsets of the invariant manifolds of P_n. So, near P_n, M is diffeomorphic to one of the possibilities (a) to (d) of figure 18.

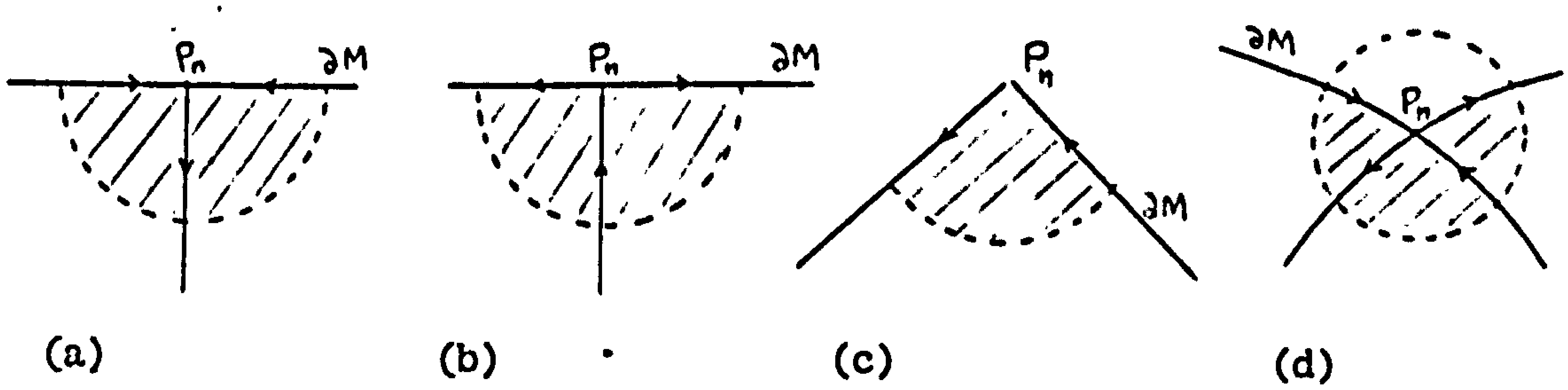


figure 18: Neighbourhood of saddle P_n on ∂M

Then every saddle P_n has at least one associated pair $(P_n^1, -P_n^1)$ on $\mathcal{D}(\phi)$ but it may or may not have the second pair $(P_n^2, -P_n^2)$.

Defining isomorphism ($\approx$) of distributions exactly as in 4.3.4 (remembering that some "circles" will be closed arcs of circle), we can go along proof in 4.5, step by step and noting that some "handles N_n" will be "half-handles" which can be extended to handles on $\bar{M}$, then we have, without giving more details:

4.7.1 <u>Theorem</u> For quasi-gradient flows ϕ and ψ on manifolds as above

$$\phi \sim \psi \iff \mathcal{D}(\phi) = \mathcal{D}(\psi)$$

4.8 <u>Application to Game Theory</u>

We consider here, with notation as in 4.7, $\bar{M}$ as the plane $\sum_{i=0}^{2} x_i = 1$ of $\mathbb{R}^3$ and flow $\bar{\phi}_A$ associated to vectorfield X_A given by $X_A^i(x) = x_i((Ax)_i - xAx)$ as in Chapters 1, 2 and 3 and let $M = \Delta = \{x \in \bar{M}, x_i \geq 0\}$ with ϕ_A as restriction of $\bar{\phi}_A$ to Δ.

We want to prove Proposition 3.4.1. First we note that for $A \in C_k^r$ $k = 2,\dots,10$ $r = 1,\dots,r(k)$ (as in 3.3), flow ϕ_A is quasi-gradient in Δ, invariant on $\partial\Delta$. (Note: $A \in C_1 \Rightarrow \phi_A$ is <u>not</u> quasi-gradient.)

<u>Proof of Proposition 3.4.1</u>

If $A, B \in C_1^1$, Zeeman [41], proved that $A \sim B$.

Now let $A,B \in C_k^r$ $k = 2,\dots,10$ $r = 1,\dots,r(k)$. In 3.3.3 we have described the phase portraits of ϕ_A, $A \in C_k^r$ (see also figure 7). From there we can extract sufficient information to determine circular distribution $\mathcal{D}_A = \mathcal{D}(\phi_A)$ which we give in table of figure 19. We then have:

$$A,B \in C_k^r \ (k = 2,\dots,10) \Rightarrow \mathcal{D}_A = \mathcal{D}_B \Rightarrow \phi_A \sim \phi_B \Rightarrow A \sim B$$

as we wanted.

$\square$

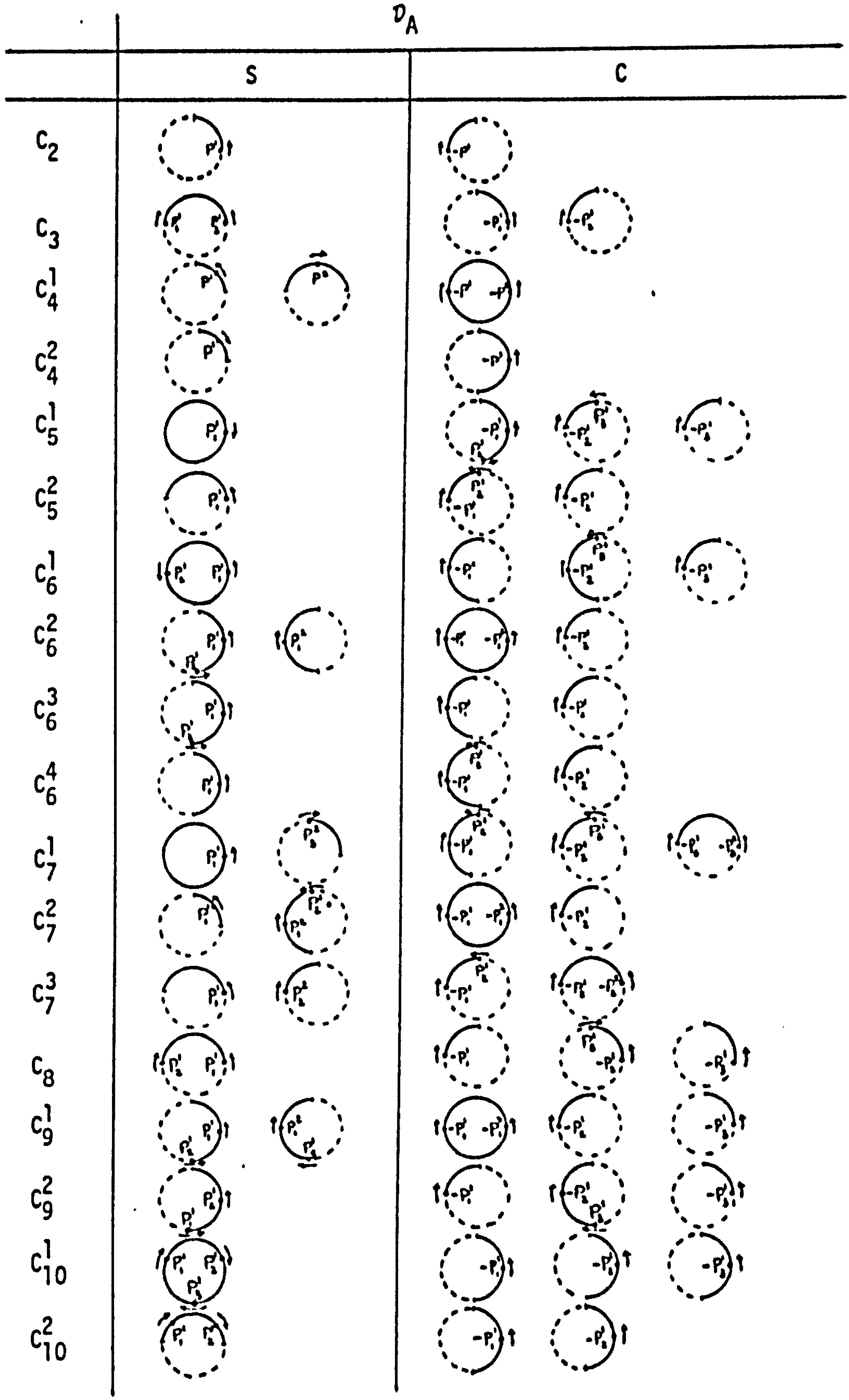


figure 19: circular distribution for ϕ_A with $A \in C_k^r$, $k \geq 2$.

CHAPTER 5

NOTES ON THE HOPF BIFURCATION THEOREM

5.1 Introduction

Our intention here is to present a statement of, and some remarks on, the important theorem known as the Hopf bifurcation theorem. Historically, the kind of bifurcation involved was already used by Poincaré, but the theorem was precisely stated and proved by Hopf in 1942 for a 1-parameter analytic family of differential equations. We have used, for reference, the translation of Hopf's original paper in Section 5 of [18]. The more modern proof, using center manifolds, is due to Ruelle and Takens [31] but more detailed versions can be found in books like Marsden, McCracken [18] or Hassard, Kazarinoff, Wan [12].

However, when we wanted to apply the theorem, using statements as in [12] or [18], to the particular family we had in mind (in next Chapter 6) we noted that some information we needed, about the local behaviour of the flow near the bifurcation point, was not included in the usual statements. Still we thought that the questions we asked were naturally linked to the theorem. Mainly, we asked this: if a 1-parameter family of vectorfields X_μ with fixed point 0 suffers a "Hopf bifurcation", resulting in the existence of attracting periodic orbits Λ_μ near 0, what can we say about the basin of attraction of Λ_μ ? Does Λ_μ attract all points in a neighbourhood of 0, except for those in W^s0 ? We would like to have this information included in the statement of the theorem, and, hence, did so in our statement presented in 5.2.1 below.

Also, the uniqueness of the periodic orbits as stated in [18] p.65 does not imply uniqueness of periodic orbit for each X_μ. We think that this is a relevant information and we included it in 5.2.1. (This is also done in [12] p.17.)

Mainly, in what follows, we will use notation of [18] and refer to the proof found in there.

We note that, in fact, the statement in 5.2.1, is much stronger than we really need for our application in Chapter 6 (where our family is analytic), but we give the statement in the way we think is best for applications in general.

Although most proofs we give are only sketched, the discussions and remarks we make have the intention of giving a full comprehension of Hopf's theorem. This chapter can be, therefore, considered as a small expository essay on the Hopf theorem and its proof, as in [18], with some additional properties.

5.2 Statements

5.2.1 Hopf's Theorem (C^k-version) : Let $n \geq 2$, $k \geq 2$.
Let X_μ be a C^{k+3}-vectorfield on $\mathbb{R}^n$ such that $X_\mu(0) = 0 \ \forall \mu$ and $X(x,\mu) = (X_\mu(x),0)$ is also C^{k+3}. Let $dX_\mu(0)$ have two distinct complex conjugate simple eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ where $\operatorname{Re}\lambda(0) = 0$, $\operatorname{Im}\lambda(0) \neq 0$, $\frac{d}{d\mu}\operatorname{Re}\lambda(\mu)\big|_{\mu=0} > 0$ and all other eigenvalues have negative real parts.

Then:

(A) There are a $c > 0$, a C^k-function $\mu:[0,c) \to \mathbb{R}$ with $\mu(0) = \mu'(0) = 0$

and a continuous family $L = \{\Lambda^r ; r \in (0,c)\}$ of closed simple curves in $\mathbb{R}^n$, such that, for each $r \in (0,c)$, Λ^r is a periodic orbit of $X_{\mu(r)}$, with period $\approx 2\pi/|\lambda(0)|$ and radius growing like r.

(See remark 5.2.2(I))

(B) There are a neighbourhood U of 0 in $\mathbb{R}^n$ and $\mu_0 > 0$ such that for any $\bar{\mu} \in (-\mu_0,\mu_0)$ all periodic orbits of $X_{\bar{\mu}}$ in U must belong to the family L of (A), i.e., if Λ is a periodic orbit of $X_{\bar{\mu}}$ in U, then $\bar{r} \in (0,c)$ s.t. $\bar{\mu} = \mu(\bar{r})$ and $\Lambda = \Lambda^{\bar{r}}$.

(See remark 5.2.2 (II) and (III) and figures 20-21 for comments and diagrams.)

(C) If there is $p \leq k$ s.t. $\mu(0) = \mu'(0) = \ldots = \mu^{(p-1)}(0) = 0$ and $b = \mu^{(p)}(0) \neq 0$, then p is even.

Moreover, if $b > 0$, then $\mu(r) > 0$ $\forall r \in (0,c)$ (taking smaller c, if necessary) and, also, there exists $\mu_0 > 0$ s.t.

(i) $\mu \in (-\mu_0,0] \Rightarrow X_\mu$ has no periodic orbit in U;

(ii) 0 is Liapunov (i.e. asymptotically) attracting for X_0;

(iii) $\mu \in (0,\mu_0) \Rightarrow X_\mu$ has one unique periodic orbit in U. (i.e. $\exists$ unique $r \in (0,c)$ with $\mu(r) = \mu$.) In this case we will denote Λ^r by Λ_μ;

(iv) orbit Λ_μ is hyperbolic attracting;

(v) radius of Λ_μ grows like $(\mu/b)^{1/p}$.

(D) With same conditions of (C), there exists a positively invariant neighbourhood U of 0 ($\forall \mu \in (-\mu_0,\mu_0)$) in $\mathbb{R}^n$ such that

$$\mu \in (-\mu_0, 0] \Rightarrow \begin{cases} \omega_\mu(U) = 0 \\ \alpha_\mu(U-0) \subset \mathbb{R}^n - U \end{cases}$$

$$\mu \in (0, \mu_0) \Rightarrow \begin{cases} \omega_\mu(x) = \Lambda_\mu & \text{for } x \in U - W^s 0 \\ \alpha_\mu(x) \subset \mathbb{R}^n - U & \text{for } x \in U - \Lambda_\mu \cup W^u 0 \end{cases}$$

(i.e. $U \subset$ basin of attraction of 0 for $\mu \leq 0$, and $U - W^s 0 \subset$ basin of attraction of Λ_μ for $\mu > 0$).

5.2.2 Remarks

(I) "Radius of Λ^r growing like r" means that

$$\lim_{r \to 0^+} \frac{\alpha(r)}{r} = \lim_{r \to 0^+} \frac{\beta(r)}{r} = 1 \quad \text{where}$$

$$\alpha(r) = \max\{|x| \; ; \; x \in \Lambda^r\} \quad ; \quad \beta(r) = \min\{|x| \; ; \; x \in \Lambda^r\} \; .$$

(II) After (A) and (B), without further assumptions we don't have much information about function $\mu(r)$ or family L. We could have the periodic orbits in L either (a) all for X_μ with $\mu > 0$; or (b) all for $\mu < 0$; or (c) all for $\mu = 0$; or (d) for some $\mu > 0$, some $\mu < 0$ even for arbitrarily small μ's. These possibilities correspond to $\mu(r)$ as in figure 20 below. In (C) and (D), conditions on $\mu(r)$ are imposed. These conditions are further discussed in 5.2.3 and 5.2.4.

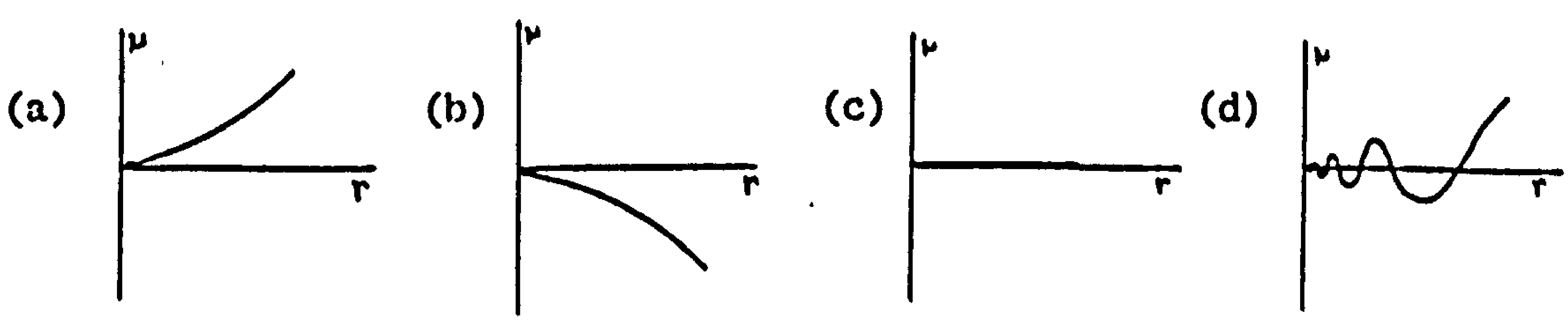


figure 20: possibilities for the graph of $\mu(r)$ near 0 .

(III) For $n=2$, we can draw the diagram in figure 21, where the periodic orbits Λ^r appear on a two-dimensional surface S in $\mathbb{R}^3 = \{(x_1,x_2,\mu)\}$. The intersection of S with horizontal plane of height μ gives the periodic orbits of X_μ. The parameter r is taken on the positive x_1-axis (justificative in 6.3.2).

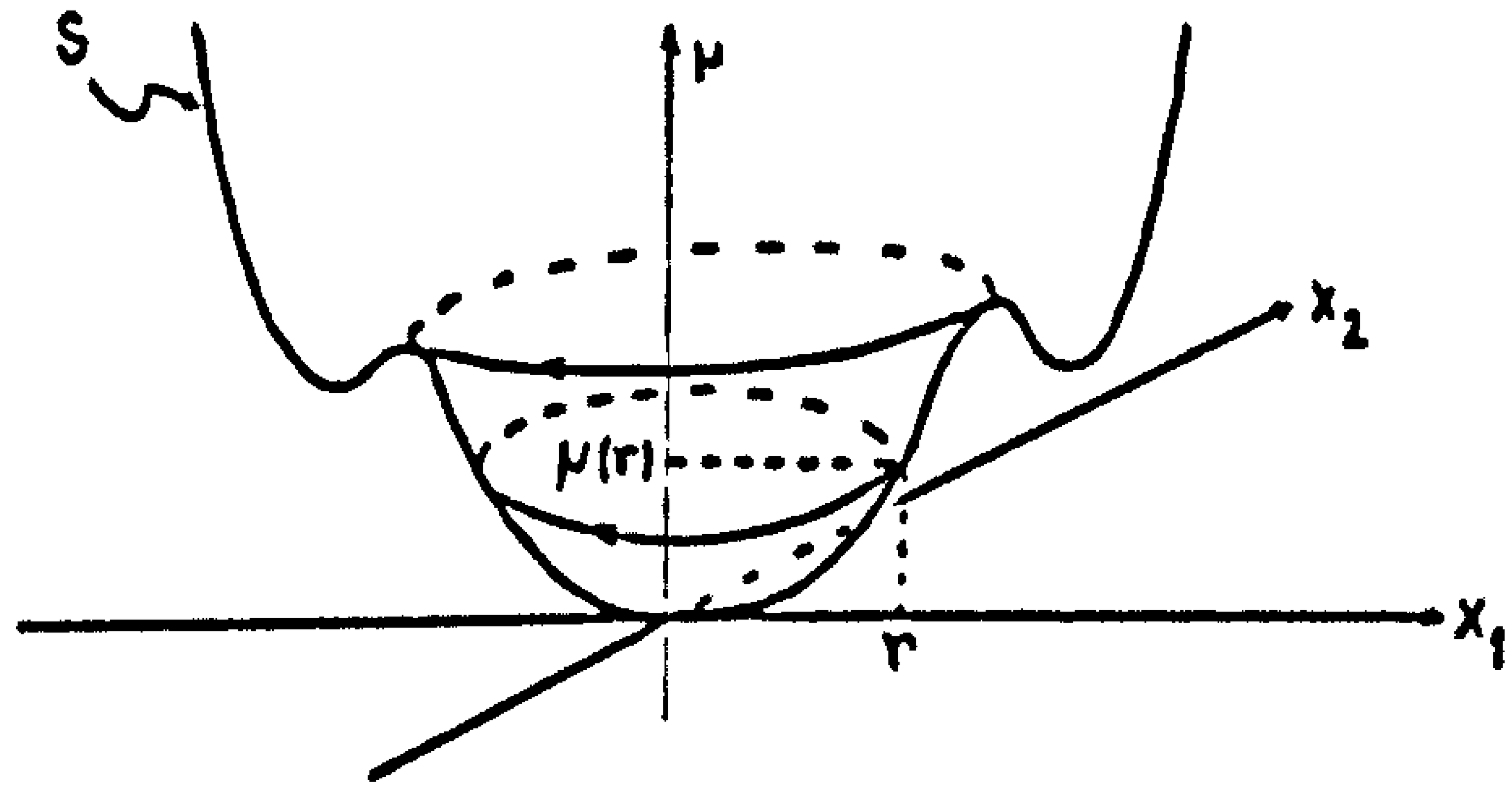


figure 21: family L of periodic orbits $(n=2)$

If $\mu(r)$ is not injective, some X_μ will have more than one periodic orbit in U.

For $n=3$, a diagram like this can be interpreted as being inside the center manifold for $X = (X_\mu, 0)$ (which is 3-dimensional).

Next propositions give us some situations where conditions of (C) above are met, in order for the conclusions there to hold.

For the first of these we recall that in Marsden, McCracken's book [18], for conclusions in (C), to be taken, a condition of 0 to be a <u>vague attractor</u> for X_0 is imposed, and later those authors prove that they can relax that condition (using higher order derivatives) and still conclusions will hold. This relaxed condition we will call here <u>weak vague</u>

attractor condition. The problem with this condition is that it refers to derivatives of a function $V(x_1,\mu)$ which is constructed during the proof of the theorem in [18]. In our statement, we have replaced this condition by the one about derivatives of $\mu(r)$ which we think it is easier to express at that stage. However, in the next proposition we say that in fact weak attractor condition for X_0 is equivalent to the condition about $\mu(r)$ in (C) and (D).

Proofs, or comments on proofs, of these will be given after we comment on the proof of 5.2.1.

5.2.3 Proposition For function $\mu:[0,c) \to \mathbb{R}$ as in 5.2.1 (A) we have that

$$\mu(0) = \mu'(0) = \dots = \mu^{(p-1)}(0) = 0 \quad \text{and} \quad b = \mu^{(p)}(0) > 0$$

if and only if

0 is a weak vague attractor for X_0 (in the sense of [18] p.78 or 92).

5.2.4 Proposition Suppose X in 5.2.1 is analytic and all other conditions on the eigenvalues are met. Then $\mu(r)$ (as in 5.2.1 (A)) is analytic and in a sufficiently small interval $(0,c)$ we have:

either $\mu(r) = 0 \ \forall r$; or $\mu(r) > 0 \ \forall r$; or $\mu(r) < 0 \ \forall r$.

If 0 is asymptotically attracting for X_0 , the first alternative does not happen and, so, $\exists$ p as in (C).

Therefore, if X is analytic and 0 is attractor for $\mu \le 0$, then conclusions of (C) and (D) hold.

5.3 Comments

5.3.1 Remark In [18] p.93, it is said that vague attractor condition (hence our conditions on existence of p and $b > 0$ in (C), by 5.2.3) can be replaced simply by the condition of 0 being Liapunov attractor for X_0, and still conclusions (C) (iii) and (iv) will hold. This is not true, unless X is analytic. (And for X analytic, p as in (C) must exist, because $\mu \not\equiv 0$.)

Below, in 5.5, we will give an example which shows that even in C^∞ cases we may have that X_μ (for arbitrarially small $\mu > 0$) has more than one periodic orbit in U and these are not attracting. Hence the conditions of item (C) are really necessary for the conclusions.

5.3.2 Remark (See [18]) (i) if $n = 2$, the variable r is taken as x_1 for any local coordinate (x_1, x_2), and Λ^{x_1} contains point $(x_1, 0)$. (ii) if $n \geq 3$, choosing coordinates (x_1, x_2, x_3) with $x_1, x_2 \in \mathbb{R}$, $x_3 \in \mathbb{R}^{n-2}$, where $x_3 = 0$ corresponds to the eigenspace relative to $\lambda(0)$, $\overline{\lambda(0)}$, then a center manifold M of X at 0 is represented locally by $(x_1, x_2, f(x_1, x_2, \mu), \mu)$. Then variable r is taken as x_1, and Λ^{x_1} must be contained in the section M of M with $\mu = \mu(x_1)$; and must contain point $(x_1, 0, f(x_1, 0, \mu(x_1)))$.

This is clear when one follows the proof of 5.2.1.

5.3.3 Remark In 5.2.1 (C), if we have $b < 0$, then $\mu(r) < 0$ for $r \in (0, c)$ (taking smaller c if necessary). Then all periodic orbits in L will be for X_μ with $\mu < 0$. Also, for all $\mu < 0$ small enough X_μ has unique periodic orbit Λ_μ in U with (C)(v) still valid.

However, Λ_μ will be repelling on M_μ (M_μ = intersection of center-manifold M of $X = (X_\mu, 0)$ with horizontal plane of height μ ; $\dim M_\mu = 2$) and Λ_μ is attracting on a $(n-1)$-submanifold transversal to M_μ . That is, Λ_μ is a periodic orbit of saddle type.

5.4 Proofs

Here, for most items in the propositions to prove, we will just make some comments, referring the reader to existing proofs. Our main reference is Marsden, McCracken's book [18] whose notation and proofs we will follow whenever possible, with the necessary adaptations.

Proof of 5.2.1

(A) and (B) We have nothing to add to the proof in [18] of existence and uniqueness of family L of periodic orbits. We will just note how function μ is obtained.

First, a suitable coordinate system is chosen (as noted in remark 5.3.2). Then X_μ is "reduced to two dimensions"

i.e. take
$$\hat{X}_\mu(x_1,x_2) = \begin{cases} X_\mu(x_1,x_2) & \text{if } n = 2 \\ (X_\mu^1, X_\mu^2)(x_1,x_2,f(x_1,x_2,\mu)) & \text{if } n \geq 3 \end{cases}$$

then "transform to polar coordinates", i.e. take

$$\tilde{X}_\mu(r,\theta) = (\psi_*)^{-1}\, \hat{X}_\mu\, \psi \quad \text{where} \quad \psi(r,\theta) = (r\cos\theta, r\sin\theta)$$

allowing r to take negative values, too. Interpret $\tilde{X} = (\tilde{X}_\mu, 0)$ as a flow in a thick cylinder (identifying planes $\theta = 0$ and $\theta = -2\pi$). $\tilde{X}$ has a periodic orbit γ along θ-axis with period $2\pi/|\lambda(0)|$.

Taking plane $\theta = 0$ as cross section, the Poincaré map $\tilde{P}$ for γ is

$$\tilde{P}(r,0,\mu) = (P(r,\mu),-2\pi,\mu) .$$

Then one takes the "displacement function" $V(r,\mu) = P(r,\mu) - r$.

The diagram in figure 22 (taken from [18]) helps us to visualise function P .

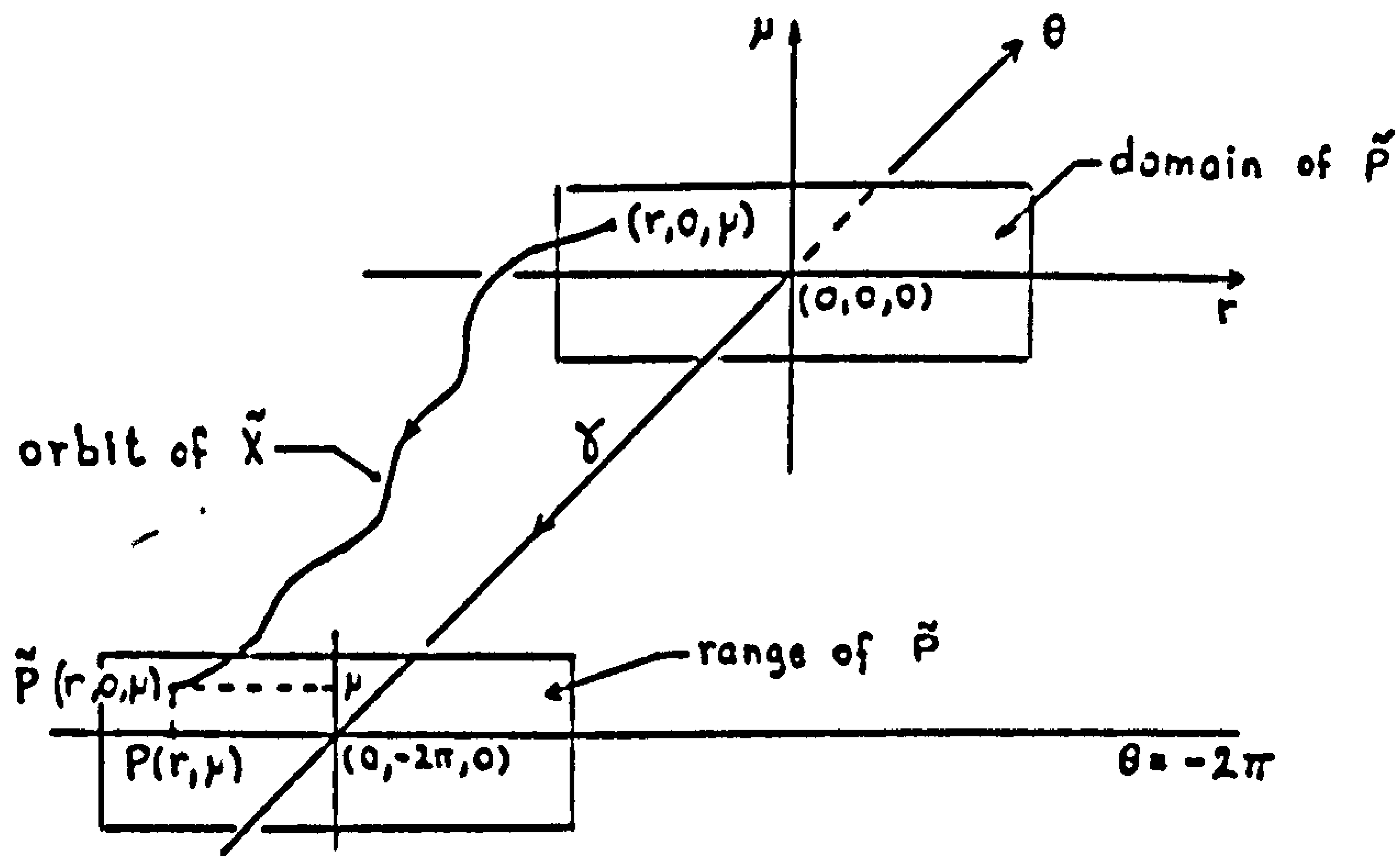


figure 22: construction of $P(r,\mu)$.

Then, $\mu(r)$ is given by the implicit function theorem as satisfying $V(r,\mu(r)) = 0$. Hence $P(\mu,\mu(r)) = r$ and orbit of $\tilde{X}$ through $(r,0,\mu(r))$ is periodic (period $\approx 2\pi/|\lambda(0)|$) . Since the r-axis for $\theta = 0$, or $\theta = -2\pi$, can be identified with x_1-axis, the point $(x_1,0,\mu(x_1))$ is on a periodic orbit for $X_{\mu(x_1)}$. This justifies remark 5.3.2.

In what refers to the rate of growth of Λ^r we think that the authors of [18] have not stated it precisely. Without further conditions (such as vague attractness or that in (C)) we can prove that radius of Λ^r

grows like r (as we stated) not like $\sqrt{\mu}$ as stated in [18]. Growth like $\sqrt{\mu}$ (better said like $k\sqrt{\mu}$) only is valid with (strong) vague attractness i.e. if $\mu''(0) \neq 0$.

Proof that $\lim\limits_{r\to 0^+} \frac{\alpha(r)}{r} = \lim\limits_{r\to 0^+} \frac{\beta(r)}{r} = 1$ is technical and involves writing the orbit of $\tilde{X}$ passing through $(r_0, 0, \mu_0)$ as

$$\tilde{\phi}(t) = (r_t(r_0,\mu_0) , \theta_t(r_0,\mu_0) , \mu_0)$$

and then noting that if $\phi(t) = (x_1(t), x_2(t), f(x_1(t), x_2(t), \mu_0))$ is the corresponding orbit in $\mathbb{R}^n$ for X_{μ_0} , then $|(x_1(t), x_2(t))| = |r_t|$. After this, it must be shown that $\lim\limits_{r_0\to 0} \frac{r_t(r_0,\mu_0)}{r_0} = 1$ for t taken in any compact interval. If $n = 2$, this completes the proof. If $n \geq 3$, we must take into account that $M_\mu = \{(x_1, x_2, f(x_1, x_2, \mu))\}$ is tangent to (x_1, x_2)-plane at $(0,0,0)$. We will not give more details here, since this is not our main purpose.

Proof of (C): The fact that the least i , for which $\mu^{(i)}(0) \neq 0$, must be even is proved in [18] Section 3B, resulting from the fact that all periodic orbits Λ^{x_1} must cross the x_1-axis twice, once on its positive side, once on its negative side. Hence $\mu(x_1)$ cannot be > 0 on one side of 0 and < 0 on the other side. (Note that function μ is defined on a neighbourhood of 0 .)

When $b > 0$, $\mu(r)$ has a local minimum at $r = 0$, hence $\mu(r) > 0$ for all small r's , and is injective on some interval $[0, c_0]$. Take $\mu_0 = \mu(c_0)$ and (iii) follows.

Proof of (iv) is in [18] p.79 for $n = 2$ and p.109 for $n \geq 3$. This proof depends on $\mu(r)$ having a minimum point at $r = 0$, and, hence, $\frac{\partial V}{\partial r}(r,\mu(r))$ having a local maximum at $r = 0$. So, for small $r > 0$, $\frac{\partial V}{\partial r}(r,\mu(r)) < 0$ and $|\frac{\partial P}{\partial r}(r,\mu(r))| < 1$.

But $x_1 \to P(x_1,\mu(r))$ is a Poincare return map for Λ^r, so its eigenvalues are inside the unit circle.

For (v) we write $\mu(r) = br^p + 0^{p+1}(r)$ (where $0^{p+1}(r)$ is a function of order $p+1$). Since $\lim_{r\to 0} \frac{\alpha(r)}{r} = \lim_{r\to 0} \frac{\beta(r)}{r} = 1$, writing $\bar{\alpha}(\mu) = \alpha(r)$ and $\bar{\beta}(\mu) = \beta(r)$ for $\mu(r) = \mu$, (i.e. $\bar{\alpha}(\mu) = \max\{|x|; x\in\Lambda_\mu\}$ and similarly for $\bar{\beta}(\mu)$), we get:

$$\lim_{\mu\to 0^+} \frac{(\bar{\alpha}(\mu))^p}{\mu} = \lim_{r\to 0^+} \frac{(\alpha(r))^p}{br^p+0^{p+1}(r)} = \frac{1}{b}\left(\lim_{r\to 0^+} \frac{\alpha(r)}{r}\right)^p = \frac{1}{b}$$

$$\Rightarrow \lim_{\mu\to 0^+} \frac{\bar{\alpha}(\mu)}{\mu^{1/p}} = \frac{1}{b^{1/p}} .$$

Similarly for $\bar{\beta}(\mu)$.

(i) follows immediately from $\mu(r) > 0$ for $r > 0$ plus (B).

For (ii) we have to show that for every neighbourhood U of 0 there is neighbourhood W of 0 with $\mathcal{O}_0^+(W) \subset U$ and $\omega_0(W) = 0$ (where $\mathcal{O}_0^+$ and ω_0 denote, respectively, positive orbit and ω-limit for X_0).

This property is not usually stated, so we here sketch a proof.

First, $\mu(0) = \mu'(0) = \ldots = \mu^{(p-1)}(0) = 0$ and $b = \mu^{(p)}(0) > 0$ implies $V(0,0) = \frac{\partial V}{\partial x_1}(0,0) = \ldots = \frac{\partial^p V}{\partial x_1^p}(0,0) = 0$ and $\frac{\partial^{p+1} V}{\partial x_1^{p+1}}(0,0) < 0$

(this is obtained by differentiating $V(x_1,\mu(x_1)) = 0$ $(p+1)$ times and using $\frac{\partial^2 V}{\partial x_1 \partial_\mu}(0,0) = e^{2\pi/|\lambda(0)|} - 1 > 0$ as in [18] p.92).

Then, for $\mu = 0$, the displacement function $V(x_1,0)$ for X_0 is negative for $x_1 > 0$ and positive for $x_1 < 0$.

Also, $\hat{X}_0$ (= restriction to two-dimensional cross-section M_0 of center manifold M) has eigenvalues $\lambda(0)$, $\overline{\lambda(0)}$, hence all orbits in any sufficiently small neighbourhood $\mathcal{U}$ of 0, starting at $(x_1,0)$ must go round 0 (transversally to all radius) crossing x_1-axis again, on the same side of 0 as x_1, at point $P(x_1,0)$. Taking a $\bar{x}_1$ sufficiently small, positive orbit of $(\bar{x}_1,0)$ up to point $P(\bar{x}_1,0)$ must be inside $\mathcal{U}$. Taking W as the region bounded by this arc of orbit plus segment $[P(\bar{x}_1,0),\bar{x}_1]$ on x_1-axis, W is positive invariant, hence $\mathcal{O}_0^+(W) \subset \mathcal{U}$. Also $\omega_0(W) = 0$ since in W there are no other critical elements of X_0. See figure 23 (a). So, (ii) holds for $n = 2$.

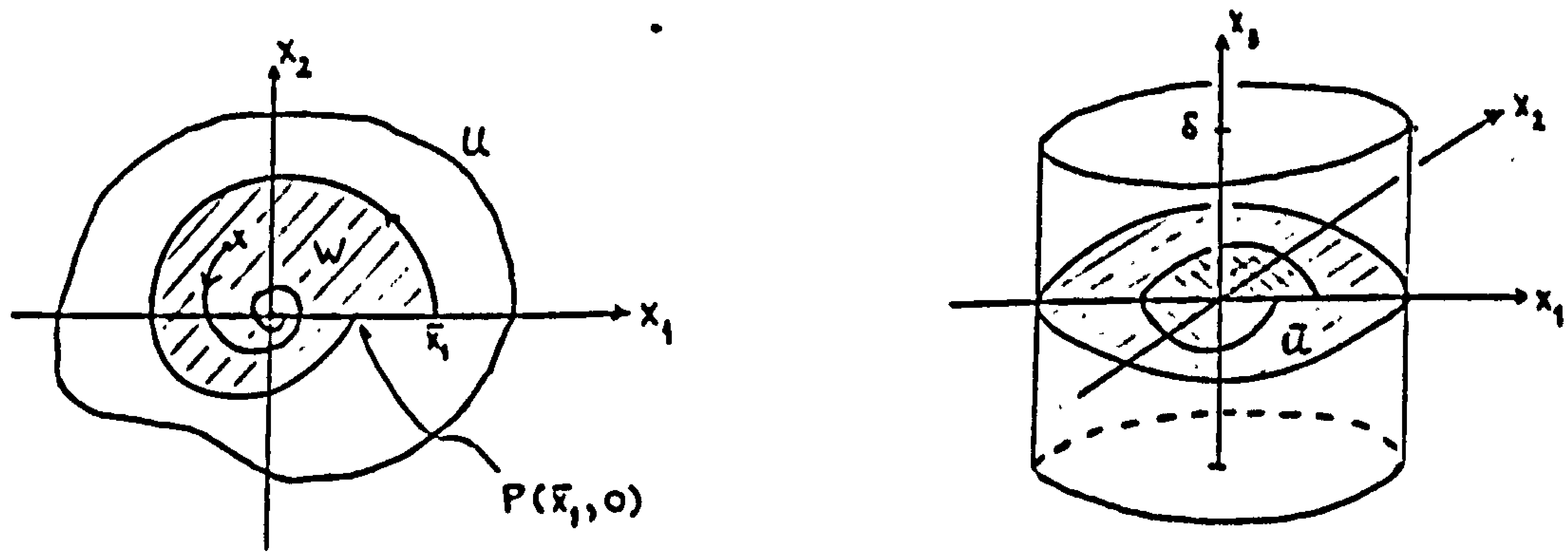


figure 23: (a) $n = 2$ (b) $n \geq 3$

If $n \geq 3$, on the x_3-direction all eigenvalues of X_0 have negative real parts. Since all neighbourhoods $\mathcal{U}$ of 0 contain a product of a neighbourhood $\bar{U}$ of 0 in M_0 and a disc $|x_3| < \delta$, we can get $\bar{W} \subset \bar{U}$ as in the case $n = 2$ and $\delta^1 \in (0,\delta)$ such that $W = \bar{W} \times \{|x_3| < \delta^1\}$ has $\mathcal{O}_0^+(W) \subset U$ and $\omega_0(W) = 0$. See figure 23 (b).

Proof of (D): There are two ways to take this property.

First, this is a consequence of a theorem by Chafee [6] (also mentioned in [18] Section 3A). This theorem gives us, for small $\mu > 0$, two periodic orbits $\gamma_1(\mu)$, $\gamma_2(\mu)$ for X_μ, bounding a 2-dimensional annular region R_μ on M_μ which contains all ω-limits of a neighbourhood of 0, except for orbits tending to 0. See figure 24 below.

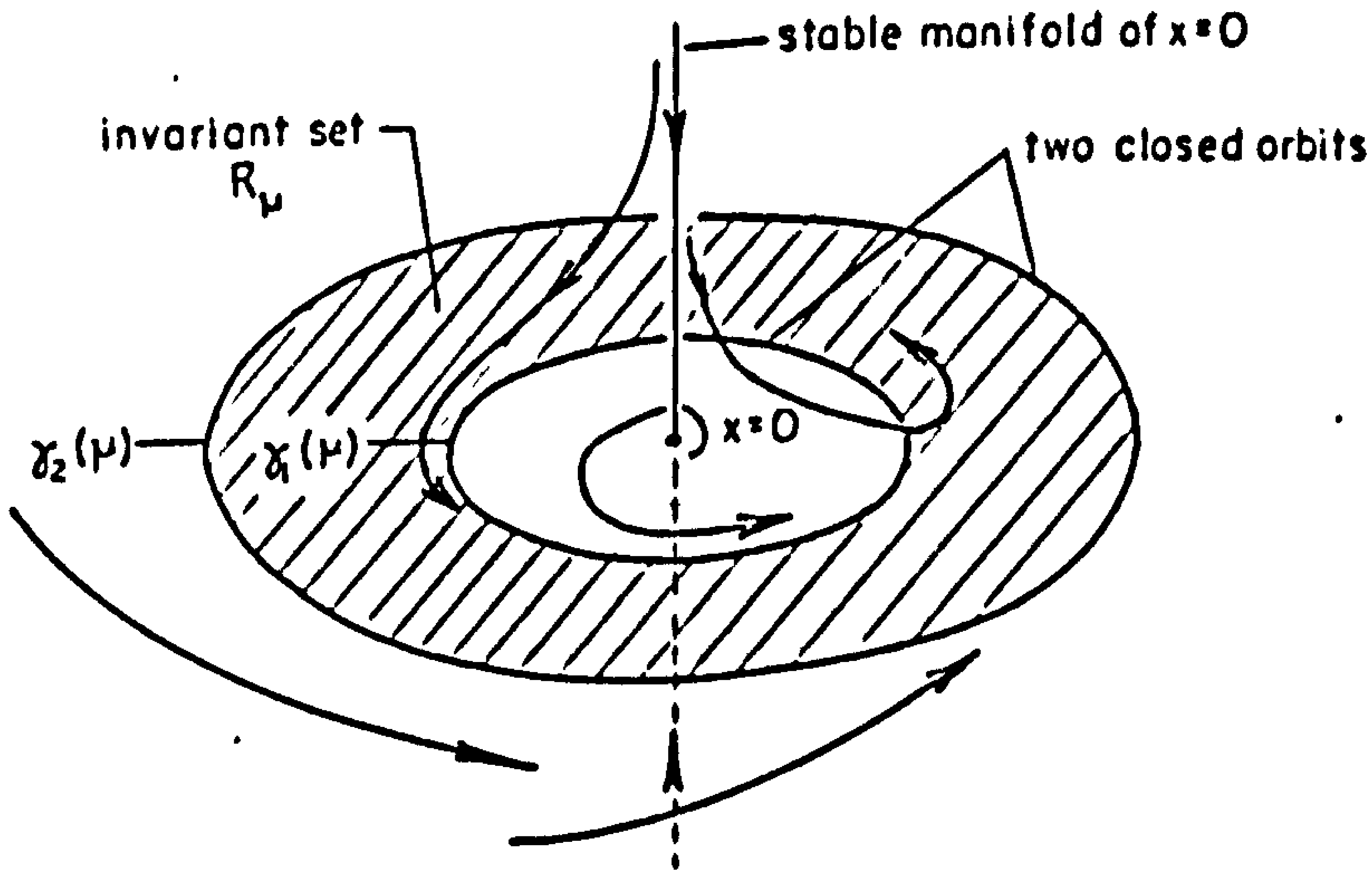


figure 24: Region R_μ in Chafee's theorem.

But (C) (iii) gives us that $\gamma_1(\mu) = \gamma_2(\mu) = \Lambda_\mu$. Taking U and μ_0 sufficiently small with U is positively invariant for all $\mu \in (-\mu_0,\mu_0)$, the assertion follows.

On the other hand, we can conclude the same property without using Chafee's theorem, by noting that (C) (ii) implies the existence of a C^1-Liapunov function $f: \mathcal{U} \to \mathbb{R}^+$ for flow ϕ_0 of X_0 , i.e. f is C^1 on $\mathcal{U}$, $f^{-1}(0) = 0$ and $\frac{d}{dt} f(\phi_0(t,x))\big|_{t=0} < 0 \quad \forall\, x \in -0$. (As in [40] or [23]). Taking $a > 0$ small, $W = f^{-1}[0,a] \subset \mathcal{U}$ and W is connected and positively invariant for ϕ_0 with ∂W transversal to X_0 . Then for sufficiently small μ_0 , ∂W will be transversal to X_μ , $\forall\, \mu \in (-\mu_0, \mu_0)$, and, so, $\omega_\mu(W) \subset M_\mu \cap W$. But the only critical elements of X_μ on M_μ are 0 and Λ_μ (when $\mu > 0$) . So,

$$\mu \le 0 \Rightarrow \omega_\mu(W) = 0 \quad \text{and} \quad \alpha_\mu(W-0) \subset \mathbb{R}^n - W$$

$$\mu > 0 \Rightarrow \omega_\mu(W-W^s 0) = \Lambda_\mu \quad \text{and} \quad \alpha_\mu(W-W^u 0 \cup \Lambda_\mu) \subset \mathbb{R}^n - W \ .$$

This finished the proof. □

<u>Proof of 5.2.3</u>: Although not stated as an equivalence this property is really proved in [18] Section 3B by successive derivations of $V(x_1, \mu(x_1)) = 0$. □

<u>Proof of 5.2.4</u>: The fact that $\mu(r)$ is analytic when X is analytic is part of the original proof by Hopf (as in [18] Section 5). Then, either $\mu \equiv 0$ on an interval round 0 , or there is a p such that $\mu(0) = \mu'(0) = \ldots = \mu^{(p-1)}(0) = 0$ and $b = \mu^{(p)}(0) \neq 0$.

If $b > 0$ (<0) $\Rightarrow \mu > 0$ (<0) on interval $(0,c)$ with small c . So the rest of the proposition is clear. □

5.5 <u>Example</u>

Here we want to show how to construct an example which contradicts a property stated in [18]. (See also remark 5.3.1 above.) It is said there (p.93) that under conditions for Hopf's theorem (as in 5.2.1) if 0 is Liapunov attracting for X_0 then orbits Λ_μ (for $\mu > 0$) are attracting. We show that this is not true in general, without more conditions.

5.5.1 <u>Lemma</u> For any C^k-function $f:[0,+\infty) \to \mathbb{R}$ $(k \geq 4)$ with $f(0) = 0$, there is a family X_μ of vectorfields in $\mathbb{R}^2$, satisfying:

1) $X = (X_\mu, 0)$ is C^k in $\mathbb{R}^3$ and $X_\mu(0) = 0 \ \forall \mu$.

2) $dX_\mu(0)$ has eigenvalues $\lambda(\mu)$, $\overline{\lambda(\mu)}$ with $\mathrm{Re}\,\lambda(0) = 0$, $\mathrm{Im}\,\lambda(0) \neq 0$, $\frac{d}{d\mu} \mathrm{Re}\,\lambda(\mu)\big|_{\mu=0} > 0$.

3) Function $\mu(r)$ of Hopf's theorem (as in 5.2.1(A)) is given by $\mu(r) = f(r^2)$.

4) If $f(r) > 0$ for $r \in (0, r_0)$, the origin is Liapunov attracting for X_0.

<u>Proof</u> Take f as in the hypothesis. Define $g:\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ by $g(r,\mu) = \mu - f(r)$. Then, let X_μ be given by:

$$X_\mu(x,y) = (X_\mu^1, X_\mu^2)(x,y) = (xg(x^2+y^2,\mu) + y \,,\, yg(x^2+y^2,\mu) - x) \quad .$$

So, $dX_\mu(0) = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$, hence $dX_\mu(0)$ has eigenvalues $\lambda(\mu) = \mu + i$ and $\overline{\lambda(\mu)} = \mu - i$. Then X_μ satisfies (1) and (2), and Hopf's theorem (5.2.1) gives function $\mu(r)$, such that $X_{\mu(r)}$ for $r > 0$ has a periodic orbit Λ^r crossing positive x-axis at $(r,0)$.

Transforming system $(\dot{x},\dot{y}) = X_\mu(x,y)$ to polar coordinates (r,θ) we get
$$\begin{cases} \dot{r} = rg(r^2,\mu) \\ \dot{\theta} = -1 \,. \end{cases}$$

For $r > 0$, let C^r be the circle $x^2+y^2 = r^2$. Then C^r is periodic orbit for X_μ whenever $\mu = f(r^2)$ because, on C^r, system $(\dot{x},\dot{y}) = X_\mu(x,y)$ has $\dot{r} = r(\mu-f(r^2)) = 0$, $\dot{\theta} = -1$. Hence $\mu(r) = f(r^2)$ and (3) holds.

To prove (4), suppose $f(r) > 0$ for $r \in (0,r_0)$. If $\mu = 0$, $(\dot{x},\dot{y}) = X_\mu(x,y)$ is represented by $\dot{r} = -rf(r^2)$, $\dot{\theta} = -1$. So, $\dot{r} < 0$ on all points with $|(x,y)|^2 < r_0$. Hence, (4) holds. □

For the example we want, it is now sufficient to choose any function $f(r)$, as in 5.5.1, so that f is not injective in any interval $(0,r_0)$ (no matter how small r_0 is), with $f(r) > 0$ for $r \in (0,r_0)$. Of course, $f(r)$ (and $\mu(r)$) must have $f^{(i)}(0) = 0$ for $0 \le i \le k$ and, so, the same construction cannot be made analytically.

We can even choose f to be C^∞ and having arbitrarily near 0, small intervals where f is constant. In this case, the family X_μ has, for arbitrarily small values of μ, bands of periodic orbits, corresponding to region R_μ of Chafee's theorem. See figure 25 below.

5.5.2 <u>Conclusion</u> The condition really necessary for uniqueness as in 5.2.1 (C) (iii) is that $\mu(r)$ be injective in some interval $(0,c)$.

The existence of p with $\mu^{(p)}(0) \neq 0$ is sufficient (but not always necessary). Even in example above, f can be constructed to be C^∞, injective on $(0,x_0)$ and yet $f^{(i)}(0) = 0 \;\forall i$. Only when X is analytic, condition $\mu^{(p)}(0) \neq 0$ is really necessary for (iii).

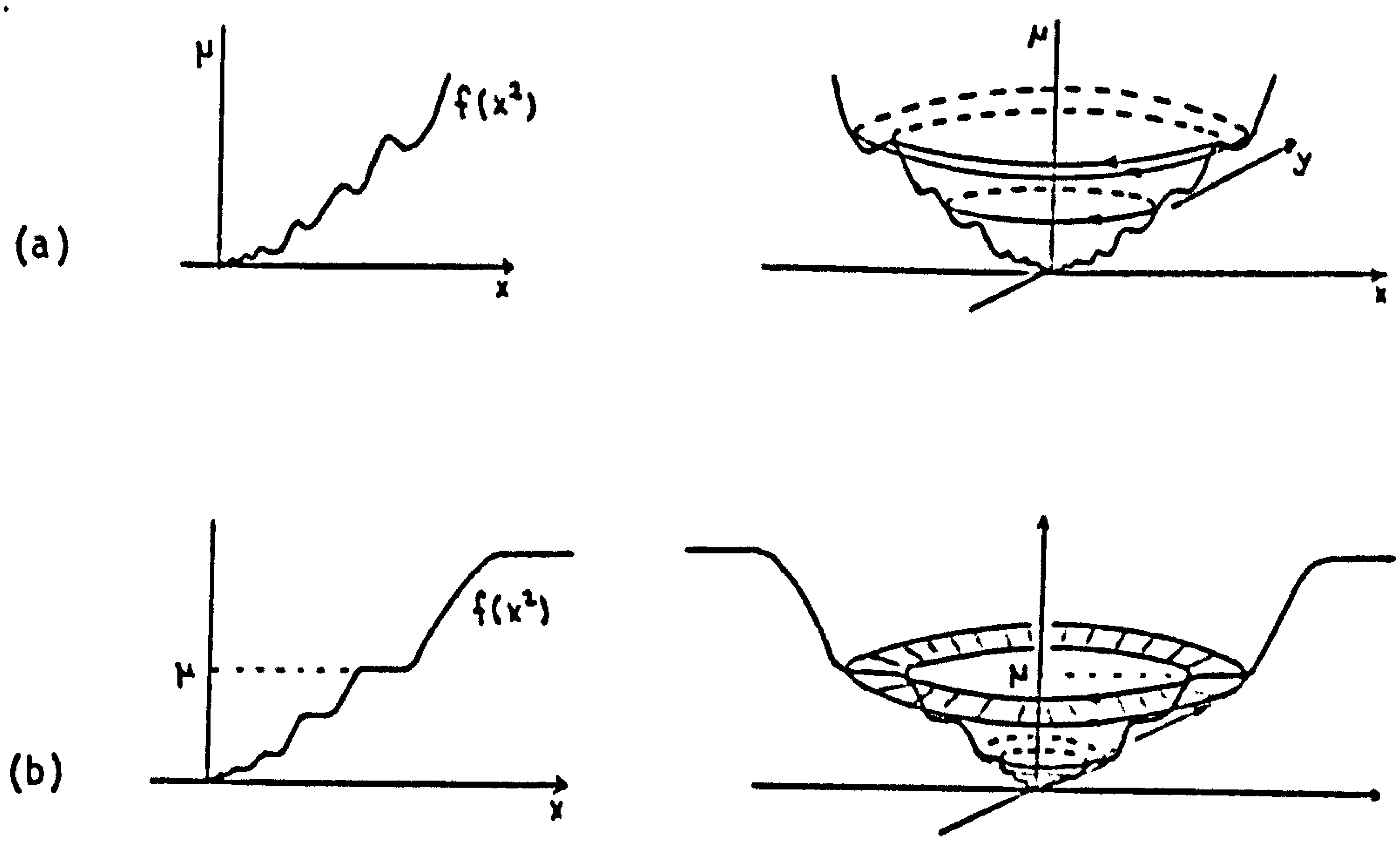


figure 25: graph for function $f(x^2)$ and its associated family of periodic orbits for X_μ .

CHAPTER 6

EXAMPLE WITH LIMIT CYCLE FOR $n = 3$

6.1 Introduction

The aim in this chapter is to study a family of matrices A in Z_4 for which ϕ_A (as in Chapters 1.2.3) present a Hopf bifurcation, resulting in the existence of an attracting periodic orbit in $\overset{\circ}{\Delta}$. The intention is to give a global description of the flow ϕ_A for values of the parameter near this bifurcation point, in order to show that this periodic orbit attracts "almost all $\overset{\circ}{\Delta}$" , i.e., we will show that it attracts $\overset{\circ}{\Delta}$ minus a line of points attracted to the fixed point at the barycentre e of Δ .

The matrices A in this family depend on three parameters which we denote by γ, δ and ε . We will take γ and δ as fixed and consider A_ε as a one-parameter family on ε , given by

$$A_\varepsilon = 4 \begin{pmatrix} 0 & \gamma & -\varepsilon & -\delta \\ -\delta & 0 & \gamma & -\varepsilon \\ -\varepsilon & -\delta & 0 & \gamma \\ -\delta & -\varepsilon & -\delta & 0 \end{pmatrix}$$

For simplification we write ϕ_{A_ε} also as ϕ_ε .

This family was presented by Hofbauer, Schuster, Sigmund, Wolff [17] as a case presenting <u>cyclic symmetry</u>, and also by Zeeman [41], (for $\gamma = 1$, $\delta = 0$) . This is also related to the studies of hypercycles as in [9, 16, 32, 33].

In [17] and [41] it was pointed out that for $\varepsilon = 0$, a Hopf bifurcation occurs at the barycentre $e = \frac{1}{4}(1,1,1,1)$, so that, for small positive values of ε , ϕ_ε has a periodic orbit L_ε in $\mathring{\Delta}$.

6.1.1 <u>Remark</u> In chapter 5, we have presented a discussion about Hopf bifurcations, to which we will refer when necessary.

In [17] and [41], the fact that the periodic orbit L_ε is attracting is taken from the fact that ϕ_ε has e as Liapunov (i.e. asymptotically) attracting for small $\varepsilon \le 0$. As we have noted in chapter 5, this is valid because the system is analytic (see 5.2.4). Below, in 6.3, we also show that L_ε is hyperbolic attracting by means of the "vague attractor" condition of [18] p.65,78. (See also 5.2.3.) In 6.4, we will prove that L_ε has its basin of attraction as $\mathring{\Delta}-W^s e$, where $W^s e$ (= stable manifold of e) is a segment with endpoints on $\partial\Delta$. We emphasise that this is a <u>global</u> statement, as opposed to the mere <u>local</u> statement of the Hopf theorem in the last chapter.

Also, for all values of ε , we will describe, in 6.5, the flow ϕ_ε on $\partial\Delta$. This is not done in [17] or [41].

In [17] we find a description of this flow in the interior $\mathring{\Delta}$ for various values of $\gamma,\delta,\varepsilon$, but for the most interesting of all cases ($\varepsilon > 0$, $\gamma-\delta > 0$) where the periodic orbit L occurs, the description is not complete. Also, we think that a complete, global description of ϕ_ε , at least for small values of ε , is important because many applications (e.g. [9, 16, 32, 33]) use the case $\varepsilon = 0$ $\delta = 0$, as a case presenting a hypercycle. Since this case is not stable we should look at all possible cases in a neighbourhood.

Therefore we declare that our main purpose during this chapter will always be to give a global description of flow ϕ_ε for small ε's , so proving Theorem V stated in 1.7.1 and gathering information to draw phase portraits of figure 4 in Chapter 1 (on p.26).

In what follows we will always keep figure 4 in mind.

6.1.2 Remark Before tackling the problem we note that in statement of Theorem V we supposed $\gamma > \delta > 0$. We want to justify this hypothesis. When $\gamma, \delta > 0$ we have a cycle of saddles on $\partial\Delta$ as:
$X_3 \searrow X_2 \searrow X_1 \searrow X_0 \searrow X_3$. If $\gamma, \delta < 0$, the same cycle occurs going on the opposite direction, and we can consider the reverse flow which is given by $-A_\varepsilon$, so we have a case equivalent to $\gamma, \delta > 0$. But if γ and δ have opposite signs, the cycle no longer exists, so we are not interested in this case. If $\delta > \gamma > 0$ we can permute vertices X_0 with X_2 , and X_1 with X_3 obtaining an equivalent matrix σA_ε , and by reversing signs we have

$$-\sigma A_\varepsilon = 4\begin{pmatrix} 0 & \delta & -\varepsilon & -\gamma \\ -\gamma & 0 & \delta & -\varepsilon \\ -\varepsilon & -\gamma & 0 & \delta \\ \delta & -\varepsilon & -\gamma & 0 \end{pmatrix} = 4\begin{pmatrix} 0 & \gamma' & \varepsilon' & -\delta' \\ -\delta' & 0 & \gamma' & \varepsilon' \\ \varepsilon' & -\delta' & 0 & \gamma' \\ \gamma' & \varepsilon' & -\delta' & 0 \end{pmatrix}$$

where $\gamma' = \delta > \gamma = \delta' > 0$.

Therefore, in everything that follows, we will always take $\underline{\gamma > \delta > 0}$.

6.1.3 Remark The vertices, the barycentre e , and points $q_0 = \frac{1}{2}(1,0,1,0) \in X_0X_2$, $q_1 = \frac{1}{2}(0,1,0,1) \in X_1X_3$ are fixed points of ϕ_ε , for all values of ε .

The vertices are hyperbolic saddles if $\varepsilon \neq 0$ with eigenvalues $\gamma, -\delta, -\varepsilon$.

In 6.2 we will discuss the local behaviour of ϕ_ε near e . In 6.3 we find the basin of attraction for the periodic orbit L_ε of ϕ_ε that occurs for small $\varepsilon > 0$. In 6.4 we study ϕ_ε restricted to the boundary $\partial\Delta$. In 6.5 we discuss stability.

6.2 Local behaviour at the barycentre

$e = \frac{1}{4}(1,1,1,1)$ is a fixed point for ϕ_ε , $\forall\ \varepsilon$, since all rows of A_ε have sum $S = 4(\gamma-\delta-\varepsilon)$. (See 2.2.11.)

$$\det A_\varepsilon = (\varepsilon^2-(\gamma-\delta)^2)(\varepsilon^2+(\gamma+\delta)^2) = 0 \iff \varepsilon = \pm(\gamma-\delta)$$

and for these values of ε , rank $A_\varepsilon = 3$, and

$$A_\varepsilon\, e = 4(\gamma-\delta-\varepsilon)e = (\gamma-\delta-\varepsilon)u = 0 \iff \varepsilon = \gamma-\delta$$

$$(\mathrm{adj}A_\varepsilon)u = -(\varepsilon+\gamma-\delta)(\varepsilon^2+(\gamma+\delta)^2) = 0 \iff \varepsilon = -(\gamma-\delta) .$$

Then, using 2.2.1(iv), we get:

6.2.1 <u>Lemma</u> e is isolated fixed point $\iff \varepsilon \neq -(\gamma-\delta)$.

6.2.2 <u>Lemma</u> e is hyperbolic fixed point if $\varepsilon \neq 0$ and $\varepsilon \neq -(\gamma-\delta)$. Also, e is

(i) attractor if $-(\gamma-\delta) < \varepsilon < 0$

(ii) 1-saddle if $\varepsilon > 0$

(iii) 2-saddle if $\varepsilon < -(\gamma-\delta)$.

<u>Proof</u> For $\varepsilon \neq -(\gamma-\delta)$, A is central. So, Proposition 2.2.18 says that the eigenvalues at e, for ϕ_ε, are given by:

$$\lambda^3 + \tfrac{1}{4}S\,\lambda^2 + \frac{1}{16}(S^2-P)\lambda + \frac{1}{64}(S(S^2-P)-D) = 0 \quad \text{where}$$

$$S = \sum_{j=0}^{3} a_{jj} = 4(\gamma-\delta-\varepsilon)\ , \quad P = \sum_{0\le i<j\le 3} a_{ij}a_{ji} = 16(2\varepsilon^2-4\gamma\delta)$$

and $D = \sum_{i=0}^{3} D_i$ where $D_i = \det(A_\varepsilon)_i = -64\,\varepsilon(\gamma^2+\delta^2)$ and, we have

$$\lambda^3+(\gamma-\delta-\varepsilon)\lambda^2 + ((\gamma+\delta)^2-\varepsilon^2-2\varepsilon(\gamma-\delta))\lambda + (\gamma-\delta+\varepsilon)(\varepsilon^2+(\gamma+\delta)^2) = 0$$

i.e. $(\lambda+(\gamma-\delta+\varepsilon))((\lambda-\varepsilon)^2 + (\gamma+\delta)^2) = 0$

giving eigenvalues

$$\lambda_\varepsilon = -(\gamma-\delta+\varepsilon) \quad \text{and} \quad \alpha_\varepsilon + \beta_\varepsilon i = \varepsilon \pm(\gamma+\delta)i\ .$$

Hence e is hyperbolic if $\varepsilon \neq 0\ ,\ -(\gamma-\delta)$.

$\varepsilon > 0 \Rightarrow \lambda_\varepsilon < 0\ ,\ \alpha_\varepsilon > 0 \Rightarrow e$ is 1-saddle

$0 > \varepsilon > -(\gamma-\delta) \Rightarrow \lambda_\varepsilon < 0\ ,\ \alpha_\varepsilon < 0 \Rightarrow e$ is attractor

$\varepsilon < -(\gamma-\delta) \Rightarrow \lambda_\varepsilon > 0\ ,\ \alpha_\varepsilon < 0 \Rightarrow e$ is 2-saddle.

□

It is clear from lemma above that conditions for the Hopf bifurcation theorem (as in 5.2.1) are met since $\lambda_\varepsilon < 0$ for small ε's, $\alpha_0 = 0$, $\left.\frac{d}{d\varepsilon}\alpha_\varepsilon\right|_{\varepsilon=0} = 1$. So by 5.2.1(A) we have:

6.2.3 <u>Corollary</u> There is a continuous family of simple closed curves in Δ , near e , each of these being a periodic orbit for ϕ_ε for some small value of ε .

This Hopf bifurcation occurring at $\varepsilon = 0$ will be studied in more detail in 6.2.7.

6.2.4 <u>Coordinates</u> We want to take, at e , a suitable system of coordinates, as in [41], given by $(z,y) = ((z_1,z_2),y)$ where $z_1 = x_0-x_2$, $z_2 = x_1-x_3$, $y = x_0+x_2-x_1-x_3$ (so $4x_0 = 1+y+2z_1$, $4x_1 = 1-y+2z_2$, $4x_2 = 1+y-2z_1$, $4x_3 = 1-y-2z_2$).

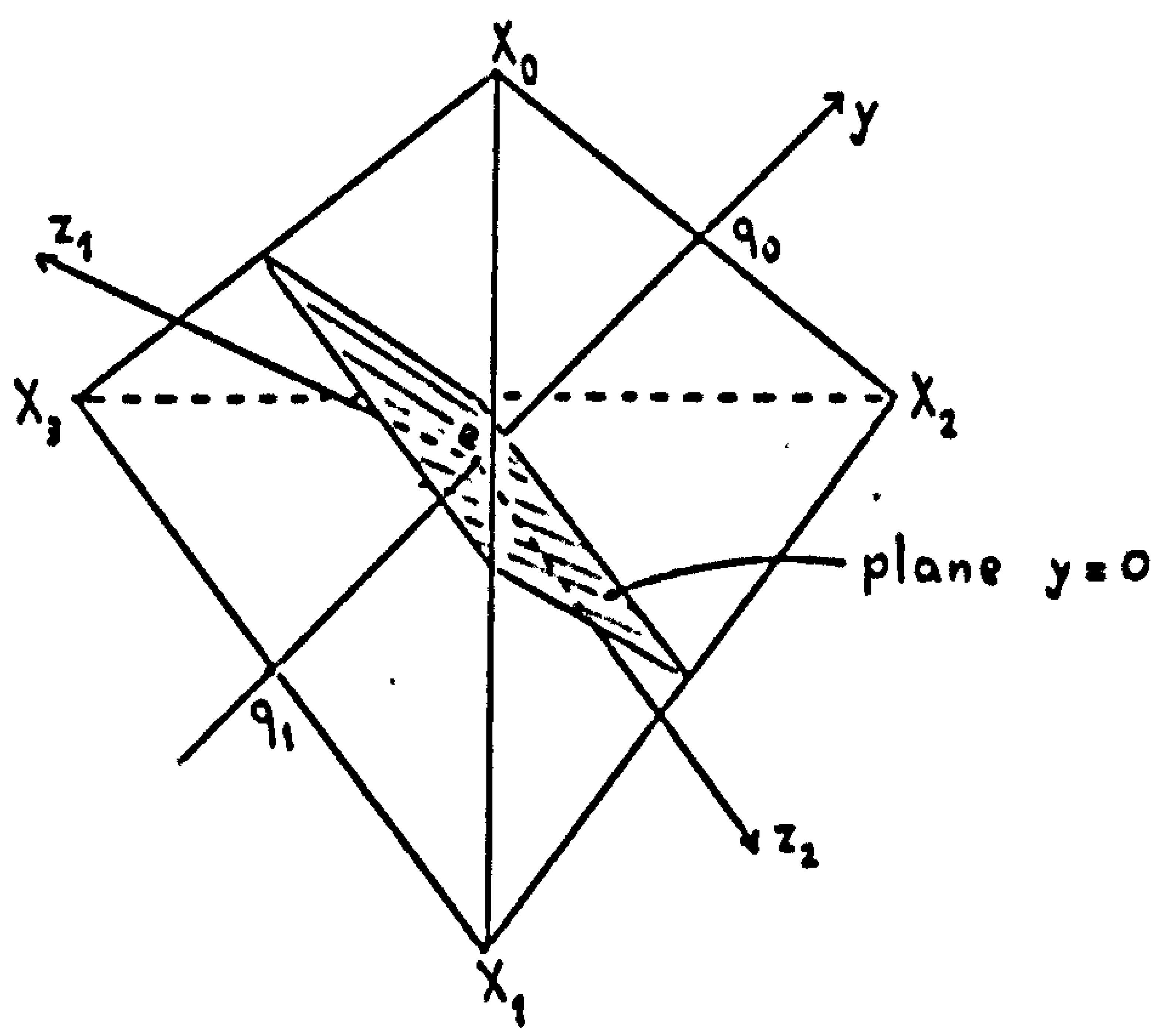


figure 26: coordinates (z_1,z_2,y) as in 6.2.4.

We have that:

$(z_1,z_2,y) = 0 \iff x = e$ and system

$\dot{x} = X_{A_\varepsilon}(x)$ is, then, represented by equations

$$
6.2(*)\quad\begin{cases}
\dot z_1 = (\varepsilon(1+y^2)-(\gamma-\delta)y(1-y)-2\varepsilon(z_1^2+z_2^2))z_1+(\gamma+\delta)(1+y)z_2 \\
\dot z_2 = -(\gamma+\delta)(1-y)z_1+(\varepsilon(1+y^2)+(\gamma-\delta)y(1+y)-2\varepsilon(z_1^2+z_2^2))z_2 \\
\dot y = -(\gamma-\delta+\varepsilon)y(1-y^2)+4(\gamma+\delta)z_1z_2+2\varepsilon(z_1^2-z_2^2)-2\varepsilon(z_1^2+z_2^2)y
\end{cases}
$$

which will be used in what follows.

6.2.5 Lemma For $-(\gamma-\delta) < \varepsilon \le 0$, e is Liapunov attracting with $\mathring{\Delta}$ as its basin of attraction.

Proof As in [41] or [17] we make use of function $V:\Delta \to \mathbb{R}$ given by $V(x) = x_0x_1x_2x_3$, which is zero on $\partial\Delta$, positive in $\mathring{\Delta}$ and has e as its only stationary (maximum) point in $\mathring{\Delta}$. Then, $\forall\, x \in \mathring{\Delta}$,

$$
\frac{\dot V(x)}{V(x)} = \sum_{i=0}^{3} \frac{\dot x_i}{x_i} = \sum_{i=0}^{3} (A_\varepsilon x)_i - 4xA_\varepsilon x = 4(\gamma-\delta-\varepsilon) - 4xA_\varepsilon x
$$

and $xA_\varepsilon x = 4(\gamma-\delta)(x_0+x_2)(x_1+x_3)-8\varepsilon(x_0x_2 + x_1x_3)$.

Using coordinates of 6.2.4, we get

$$
xA_\varepsilon x = (\gamma-\delta-\varepsilon) - (\gamma-\delta+\varepsilon)y^2 + 2\varepsilon(z_1^2+z_2^2)
$$

and, so, $\frac{1}{4}\frac{\dot V}{V} = (\gamma-\delta+\varepsilon)y^2 - 2\varepsilon|z|^2$.

For $-(\gamma-\delta) < \varepsilon < 0 \Rightarrow \dot V(x) > 0 \quad \forall\, x \in \mathring{\Delta} - e$

$\Rightarrow$ V increases (strictly) along ϕ_ε-orbits in $\mathring{\Delta} - e$,

$\Rightarrow$ all orbits in $\mathring{\Delta} - e$ have e as ω-limit,

and α-limit contained in $\partial\Delta$ (V is a Liapunov function for ϕ_ε) .

For $\varepsilon = 0$, $\frac{1}{4}\frac{\dot V}{V}(x) = (\gamma-\delta)y^2 > 0 \quad \forall\, x \in \mathring{\Delta} - e$

and $\dot{V}(x) = 0 \iff y = 0 \iff x_0+x_2 = x_1+x_3$.

So V increases (strictly) along ϕ_0-orbits out of plane $y = 0$.

If $y = 0$, equations 6.2(*) gives $\dot{y} = 4(\gamma+\delta)z_1z_2$.

If $y = 0$, $z_1 = 0$, $z_2 \neq 0 \Rightarrow \dot{z}_1 = (\gamma+\delta)z_2 \neq 0$.

If $y = 0$, $z_1 \neq 0$, $z_2 = 0 \Rightarrow \dot{z}_2 = -(\gamma+\delta)z_1 \neq 0$.

This means that, at points of $\mathring{\Delta}$-e , on plane $y = 0$, their ϕ_0-orbits cross this plane (though not transversally if $z_1z_2 = 0$). Hence we can say that, even in this case, V increases (strictly) along orbits.

Then for $-(\gamma-\delta) < \varepsilon \leq 0$, $\mathring{\Delta}$ = basin of attraction of e .

□

6.2.6 <u>Lemma</u> $Y = q_0q_1$ is invariant $\forall\, \varepsilon$ and

$Y = W^s e$ for $\varepsilon > 0$, $Y = W^u e$ for $\varepsilon < -(\gamma-\delta)$,

$Y \subset W^s e = \mathring{\Delta}$ for $-(\gamma-\delta) < \varepsilon \leq 0$

Y is pointwise fixed for $\varepsilon = -(\gamma-\delta)$.

<u>Proof</u> Points of segment $Y = q_0q_1$ corresponds, in (z_1,z_2,y) coordinates to y-axis ($z_1 = z_2 = 0$) with $|y| \leq 1$. But 6.2(*) gives: $z_1 = z_2 = 0 \Rightarrow \dot{z}_1 = \dot{z}_2 = 0 \Rightarrow Y$ is invariant.

On this axis, we have, $\dot{y} = -(\gamma-\delta+\varepsilon)y(1-y^2)$.

Hence $y \neq 0 \Rightarrow \dfrac{\dot{y}}{y} > 0$ if $\varepsilon < -(\gamma-\delta)$, $\dfrac{\dot{y}}{y} < 0$ if $\varepsilon > -(\gamma-\delta)$.

Together with 6.2.2 and 6.2.5, this implies the lemma.

□

Now we apply Hopf's theorem as in 5.2.1 to get:

6.2.7 <u>Proposition</u> There exist a neighbourhood U of e in $\mathring{\Delta}$, $\varepsilon_0 > 0$

and a function $\varepsilon:[0,c)\to\mathbb{R}$ with $\varepsilon(0)=\varepsilon'(0)=0$, $\varepsilon(r)>0$ for $r>0$, such that

A) $\forall\, r\in(0,c)$, $\phi_{\varepsilon(r)}$ has a periodic orbit L^r in U of period $\approx 2\pi/(\gamma+\delta)$;

B) $\forall\, \varepsilon\in(0,\varepsilon_0)$, ϕ_ε has exactly one periodic orbit L_ε in U , and $\exists!\ r\in(0,c)$ s.t. $\varepsilon=\varepsilon(r)$ and $L_\varepsilon=L^r$;

C) L_ε is hyperbolic attracting, with radius growing like $k\sqrt{\varepsilon}$ where $k^2=(4(\gamma+\delta)^2+(\gamma-\delta)^2)/2\ (\gamma-\delta)(\gamma+\delta)^2$;

D) $\forall\, \varepsilon\in(0,\varepsilon_0)$, $W^se=Y$, U is ϕ_ε-positively invariant $U-Y\subset$ basin of attraction of L_ε and $x\in U-L_\varepsilon\cup W^ue \Rightarrow \alpha_\varepsilon(x)\subset\Delta-U$.
(See figure 27 below.)

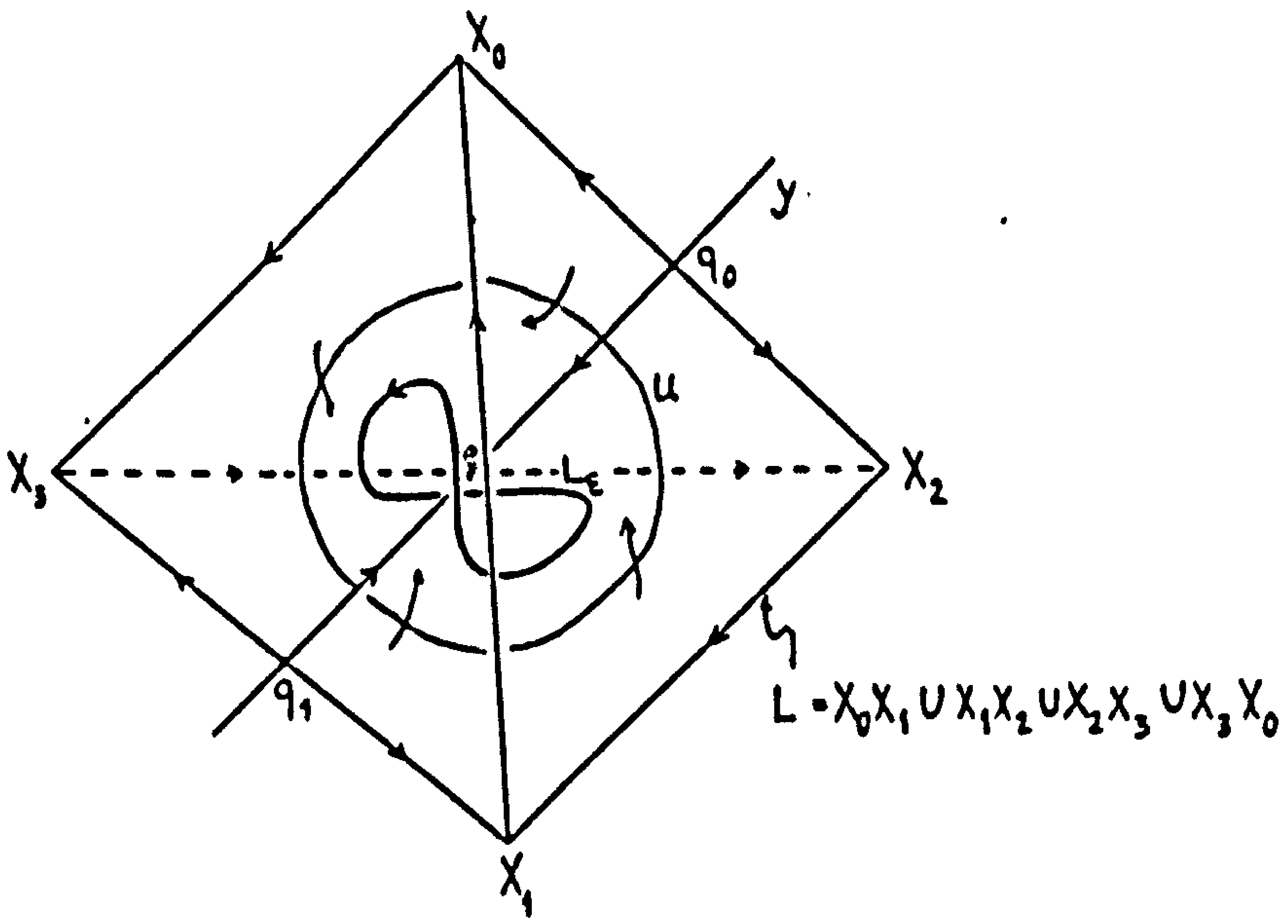


figure 27: flow ϕ_ε , small $\varepsilon>0$, near e .

Proof Existence of U and $\varepsilon:[0,c)\to\mathbb{R}$ with $\varepsilon(0)=\varepsilon'(0)=0$

satisfying (A) is a direct application of 5.2.1. Since, by 6.2.5, ϕ_ε for small $\varepsilon \le 0$ has no periodic orbit in $\overset{\circ}{\Delta}$, we must have $\varepsilon(r) > 0$ $\forall\, r > 0$. We will check, below, the (strong) vague attracting condition for $\varepsilon = 0$, as in [18] p.78. By Proposition 5.2.3 this will imply that $b = \varepsilon''(0) > 0$ and then (by 5.2.1 (C)(iii)-(v)), there exists $\varepsilon_0 > 0$ s.t. $\varepsilon \in (0,\varepsilon_0) \Rightarrow \phi_\varepsilon$ has exactly one periodic orbit L_ε in U, L_ε is hyperbolic attracting with radius growing like $k\sqrt{\varepsilon}$ where $k^2 = 1/b$. Then (C) will hold, and (D) follows from 5.2.1(D) plus lemma 6.2.6.

To check condition of vague attractor for ϕ_0 we follow procedure of Sections 4 and 4A of [18]. The calculations involved are lengthy and we just indicate some few steps.

At $\varepsilon = 0$, equations 6.2(*) give

$$6.2(^{**}) \begin{cases} \dot z_1 = -(\gamma-\delta)y(1-y)z_1 + (\gamma+\delta)(1+y)z_2 = Z^1(z_1,z_2,y) \\ \dot z_2 = -(\gamma+\delta)(1-y)z_1 + (\gamma-\delta)y(1+y)z_2 = Z^2(z_1,z_2,y) \\ \dot y = -(\gamma-\delta)y(1-y^2)+4(\gamma+\delta)z_1 z_2 = Z^3(z_1,z_2,y) . \end{cases}$$

We have that $b = \varepsilon''(0) = -V'''(0) \,/\, 3.\frac{\partial^2 V}{\partial\varepsilon\partial z_1}(0,0)$ where V is the displacement function of [18] (see also 5.4).

Taking $\alpha_\varepsilon + \beta_\varepsilon i$ as in 6.2.2, from [18]we get:

$$\frac{\partial^2 V}{\partial\varepsilon\partial z_1}(0,0) = \frac{2\pi}{|\beta_0|}\frac{d}{d\varepsilon}\alpha_\varepsilon\Big|_{\varepsilon=0} = \frac{2\pi}{\gamma+\delta}$$, and in [18] p.133 we find an expression for $V'''(0)$ in terms of the derivatives of Z^1, Z^2 and Z^3

up to third order always calculated at $(z_1, z_2, y) = (0,0,0)$. Using that

$$\frac{\partial Z^3}{\partial y} = -(\gamma-\delta) \,, \quad \frac{\partial^2 Z^3}{\partial z_1^2} = \frac{\partial^2 Z^3}{\partial z_2^2} = 0 \,, \quad \frac{\partial^2 Z^3}{\partial z_1 \, z_2} = 4(\gamma+\delta)$$

$$\frac{\partial^2 Z^1}{\partial z_1^2} = \frac{\partial^2 Z^1}{\partial z_1 \partial z_2} = \frac{\partial^2 Z^1}{\partial z_2^2} = 0 \quad , \quad \frac{\partial^2 Z^2}{\partial z_1^2} = \frac{\partial^2 Z^2}{\partial z_1 \partial z_2} = \frac{\partial^2 Z^2}{\partial z_2^2} = 0$$

$$\frac{\partial^2 Z^1}{\partial y \partial z_1} = - \frac{\partial^2 Z^2}{\partial y \partial z_2} = -(\gamma-\delta) \,, \quad \frac{\partial^2 Z^1}{\partial y \partial z_2} = \frac{\partial^2 Z^2}{\partial y \partial z_1} = \gamma+\delta$$

$$\frac{\partial^3 Z^i}{\partial z_1^3} = \frac{\partial^3 Z^i}{\partial z_2^3} = \frac{\partial^3 Z^i}{\partial z_1^2 \partial z_2} = \frac{\partial^3 Z^i}{\partial z_1 \partial z_2^2} = 0 \qquad i = 1,2 \,.$$

after a long calculation we get

$$V'''(0) = - \frac{12\pi(\gamma-\delta)(\gamma+\delta)}{4(\gamma+\delta)^2 + (\gamma-\delta)^2} < 0 \,.$$

Since $V'''(0) < 0$, vague attracting condition is satisfied at $\varepsilon = 0 \Rightarrow L_\varepsilon$ is hyperbolic attracting.

Also we get $\quad b = \dfrac{2\,(\gamma-\delta)(\gamma+\delta)^2}{4(\gamma+\delta)^2+(\gamma-\delta)^2}$

and the proof is complete.

$\square$

6.2.8 <u>Remark</u> The expression $\dot{V} = 4V((\gamma-\delta+\varepsilon)y^2 - 2\varepsilon|z|^2)$, obtained in 6.2.5, is in fact valid for all values of ε and in 6.3 it will be helpful in studying flow ϕ_ε in $\mathring{\Delta}$ for $\varepsilon > 0$.

For $\varepsilon > 0$, $x \in \mathring{\Delta}-e$, $\dot{V}(x) = 0 \iff (\gamma-\delta+\varepsilon)y^2 = 2\varepsilon(z_1^2+z_2^2)$. These points form a cone in $\mathring{\Delta}$. Inside the cone, V increases on orbits, outside it, V decreases on orbits. Hence the periodic orbit L_ε (for small $\varepsilon > 0$ as in 6.2.7) must cross the cone at least twice. In fact, since the system is invariant under any cyclic permutation of vertices of Δ , we can say that L_ε must, then, cross the cone at least four times (once in each quadrant). This justifies the way we have drawn L_ε on figure 4 (p.26) and figure 27.

6.3 <u>The basin of attraction for L_ε</u>

In 6.2.7 we showed that U-$Y \subset$ basin of attraction of $L_\varepsilon = W^sL_\varepsilon$. We want now to determine that, in fact, $W^sL_\varepsilon = \mathring{\Delta}$-$Y$ for small values of $\varepsilon > 0$. Let ε_0 be as in 6.2.7.

6.3.1 <u>Proposition</u> There exists $\bar{\varepsilon} \in (0,\varepsilon_0)$ s.t.

$$\forall \varepsilon \in (0,\bar{\varepsilon}) \quad \Rightarrow \begin{cases} \omega_\varepsilon(x) = L_\varepsilon & \text{if } x \in \mathring{\Delta}\text{-}Y \\ \omega_\varepsilon(x) = e & \text{if } x \in Y \end{cases}$$

(where $\omega_\varepsilon(x) = \omega$-limit of x by flow ϕ_ε) .

6.3.2 <u>Remark</u> In the following proof of 6.3.1 we will make use of some facts about the behaviour of ϕ_ε on $\partial\Delta$ (in lemma 6.3.5). These facts will be really proved in the next paragraph 6.4 but we do not want to postpone the property in 6.3.1 to after 6.4, since this is really more relevant than the study of ϕ_ε restricted to $\partial\Delta$.

After proving 6.3.1 we will show

6.3.3 Proposition For $\varepsilon \in (0,\bar{\varepsilon})$ ($\bar{\varepsilon}$ as in 6.3.1), and $x \in \overset{\circ}{\Delta} - L_\varepsilon$ we have either $\alpha_\varepsilon(x) = e$ (i.e. $x \in W^u e$) or $\alpha_\varepsilon(x) = q_0$ or q_1 (i.e. $x \in W^u q_0$ or $x \in W^u q_1$) or $\alpha_\varepsilon(x) = L = X_0X_1 \cup X_1X_2 \cup X_2X_3 \cup X_3X_0$.

Note that L is the union of the outsets for the vertices, forming a cycle of saddles. See figure 27. Points q_0 and q_1 are hyperbolic repellors, for $\varepsilon > 0$.

6.3.4 Corollary $\Omega_\varepsilon = e \cup q_0 \cup q_1 \cup L \cup L_\varepsilon$.

Proof of 6.3.1 Taking ε_0 and U as in 6.2.7 we have that, for $\varepsilon \in (0,\varepsilon_0)$, $x \in U$-Y then $\mathcal{O}^+_\varepsilon(x) \subset U$ and $\omega_\varepsilon(x) = L_\varepsilon$. We want now to show that all orbits of points in $\overset{\circ}{\Delta}$-U will, eventually in time, enter U .

We take, similarly to 6.2.5, function $V:\Delta \to \mathbb{R}$ given by $V(x) = 256(x_0x_1x_2x_3)$, for which $V(\partial\Delta) = 0$, $V(e) = 1$, $V(\overset{\circ}{\Delta}\text{-}e) = (0,1)$. (The factor 256 was only introduced in V to make $V(\Delta) = [0,1]$.) Now, for any $s \in [0,1]$ we let $N^s = \{x \in \Delta \,;\, V(x) \le s\}$ (hence $N^0 = \partial\Delta$, $N^1 = \Delta$) . N^s is a compact neighbourhood of $\partial\Delta$ and $e \notin N^s$ if $s < 1$.

At this point, we interrupt the proof in order to establish three lemmas (in 6.3.5, 6.3.6 and 6.3.7), after which we return to the main proof. In these lemmas, we will take coordinate system (z_1,z_2,y) as in 6.2.4.

6.3.5 Lemma Fix $\beta \in (0,1)$ and let $N_\beta = \{$points of $\partial\Delta$ with $|y| \le \beta\}$. Then, for all $\varepsilon \ge 0$, ϕ_ε satisfies the following properties on N_β :

(i) ϕ_ε has no fixed points;

(ii) every orbit crosses N_β in finite time, and finite length;

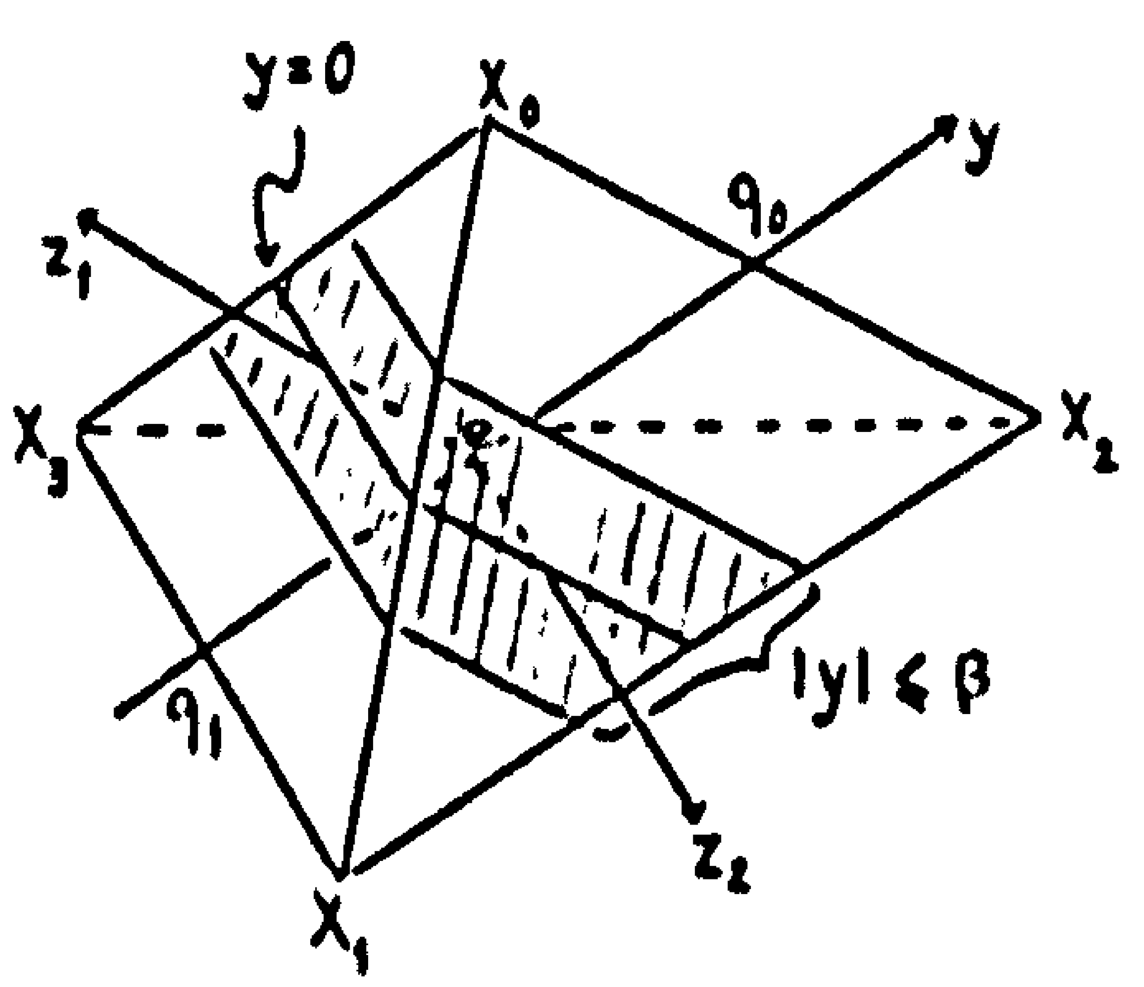
(iii) every orbit has at most one point where $\dot{y} = 0$, and at such a point $\ddot{y} \neq 0$.

(iii) means that orbits on $N_\beta \subset \partial\Delta$ intersect lines y = constant either transversally or parabolically, and parabolic contact can occur with at most one such line.

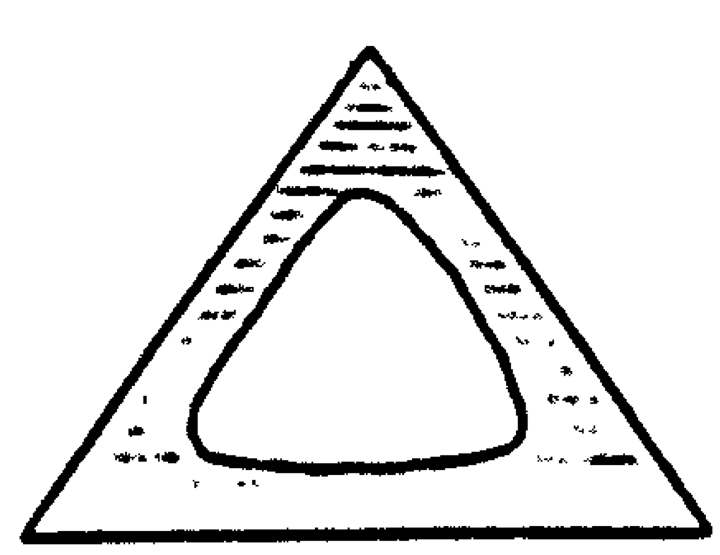
Proof First we note that N_β is the union of 4 strips, one on each face of $\partial\Delta$, which are parallel to X_0X_2 on faces F_1 and F_3 , and to X_1X_3 on F_0 and F_2 (F_i is given by $x_i = 0$ as in 2.2.17). See figure 28 (a). Since ϕ_ε restricted to one of the faces gives the flow on any other face by a cyclic permutation of the vertices of Δ , it is sufficient to show (i), (ii), iii) for just one face (e.g. $X_0X_1X_2 = F_3$) . This will be done in 6.4 where we detail the study of ϕ_ε restricted to this face. Particularly the properties we want here are in 6.4.8.

□

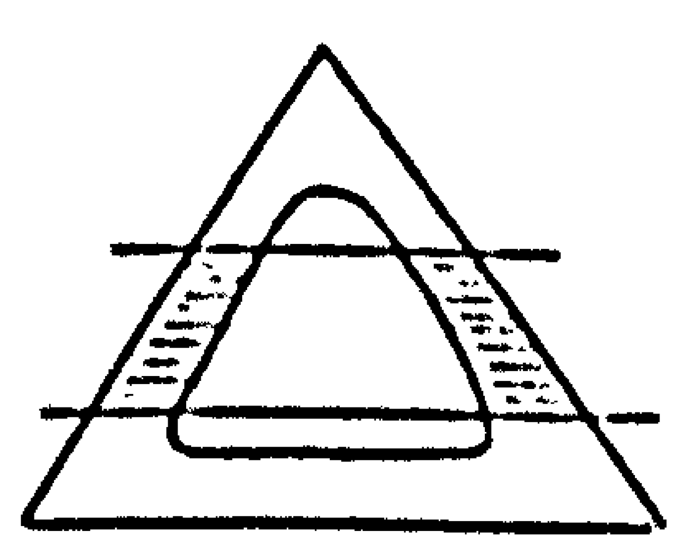
6.3.6 Lemma There exists $s \in (0,1)$ and $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0,\varepsilon_1)$ and $x \in N^S - \partial\Delta$, there is $t = t_\varepsilon(x) > 0$, for which $V(\phi_\varepsilon(t,x)) > V(x)$ (i.e. for small ε , and x near $\partial\Delta$, $\exists\, x' \in \sigma_\varepsilon^+(x)$ with $V(x') > V(x))$.



(a)



(b)



(c)

figure 28: (a) N_β (b) section of N^S (c) section of N^S_β .

<u>Proof</u> Take $\beta \in (0,1)$ and fix it. Let $N^s_\beta = N^s \cap \{$points of Δ with $|y| \leq \beta\}$. (Note that $N^0_\beta = N_\beta$ of previous lemma.) See figure 28.

So, properties (i), (ii), (iii) of Lemma 6.3.5 hold for N^0_β . By compactness of N^0_β we can choose $s \in (0,1)$ sufficiently small for these properties to hold in N^s_β , for all ϕ_ε with $0 \leq \varepsilon \leq \varepsilon_0$. Let v_M and v_m be, respectively, the maximum and minimum speeds of ϕ_ε-orbits in N^s_β , for all $\varepsilon \in [0,\varepsilon_0]$. (By speed we mean the norm of the vector $(\dot{z}_1,\dot{z}_2,\dot{y})$.) By property (i) of 6.3.5, plus compactness of N^s_β and $[0,\varepsilon_0]$, $0 < v_m \leq v_M < \infty$. For every $\alpha \in (0,\beta)$ we denote by ℓ_ε the length of the longest orbit in N^s_α for all $\varepsilon \in [0,\varepsilon_0]$. Again, ℓ_α is finite due to property (ii) plus compactness. Observe that by an "orbit in N^s_α" we mean a connected piece of a ϕ_ε-orbit (for some $\varepsilon \in [0,\varepsilon_0]$), this piece being contained in N^s_α . Clearly, ℓ_α decreases if α decreases. Property (iii) above implies that no piece of orbit can stay on any plane $y = \text{const}$. Then, all orbits intersecting plane $y = 0$, must do so either transversally or at a point for which $y(t)$ has (strict) local maximum or minimum. See figure 29. In any of these possibilities, making α tend to zero forces the length of any orbit in N^s_α to become arbitrarily small. So $\ell_\alpha \to 0$ as $\alpha \to 0^+$.

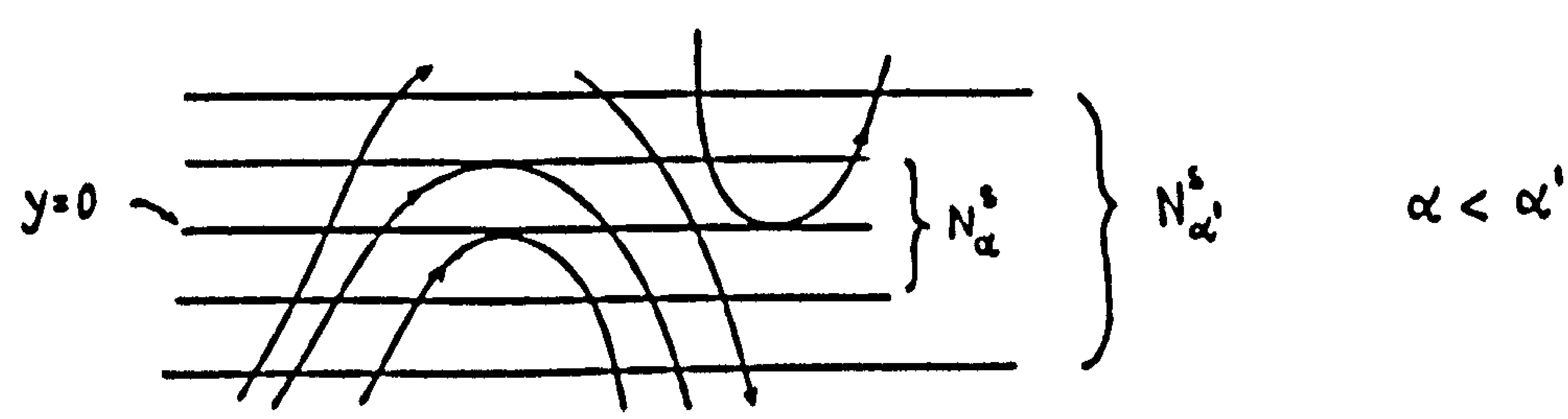


figure 29: orbits in N^s_α

Hence, we can choose some $\alpha > 0$ (with $\alpha < \beta/2$) for which $\ell_\alpha < \frac{1}{2}\beta\, v_m/v_M$ and then take $\varepsilon_1 < \min\{\varepsilon_0, \frac{1}{2}(\gamma-\delta)\alpha^2\}$, $\varepsilon_1 > 0$.

Let $\varepsilon \in [0,\varepsilon_1]$. Now we show that function V increases (strictly) along ϕ_ε-orbits in $M_\alpha^s = N^s - N_\alpha^s \cup \partial\Delta$. Take $x \in M_\alpha^s$. In (z,y) coordinates we must have $|y| > \alpha$. As in 6.2.5 we have
$\frac{1}{2}\frac{\dot V}{V}(x) = (\gamma-\delta+\varepsilon)y^2 - 2\varepsilon|z|^2$.

Using $\gamma-\delta+\varepsilon \geq \gamma-\delta$, $y^2 \geq \alpha^2$, $|z| \leq 1$ and $\varepsilon \leq \varepsilon_1 < \frac{1}{4}(\gamma-\delta)\alpha^2$, we get

$$\text{6.3(*)} \qquad \frac{1}{2}\frac{\dot V}{V}(x) > (\gamma-\delta)(\alpha^2-2\tfrac{1}{4}\alpha^2) = \tfrac{1}{2}(\gamma-\delta)\alpha^2 > 0 \quad \forall\, x \in M_\alpha^s$$

and since $V(x) > 0 \Rightarrow \dot V(x) > 0$.

Hence for $x \in M_\alpha^s$, and sufficiently small $t > 0$, we have $V(\phi_\varepsilon(t,x)) > V(x)$. So conclusion holds for $x \in M_\alpha^s$.

We have now to show that it also holds for $x \in N_\alpha^s - \partial\Delta$. For such points we have

$$\text{6.3(**)} \qquad \frac{1}{2}\frac{\dot V}{V}(x) = (\gamma-\delta+\varepsilon)y^2 - 2\varepsilon|z|^2 \geq -2\varepsilon|z|^2 > -\tfrac{1}{2}(\gamma-\delta)\alpha^2$$

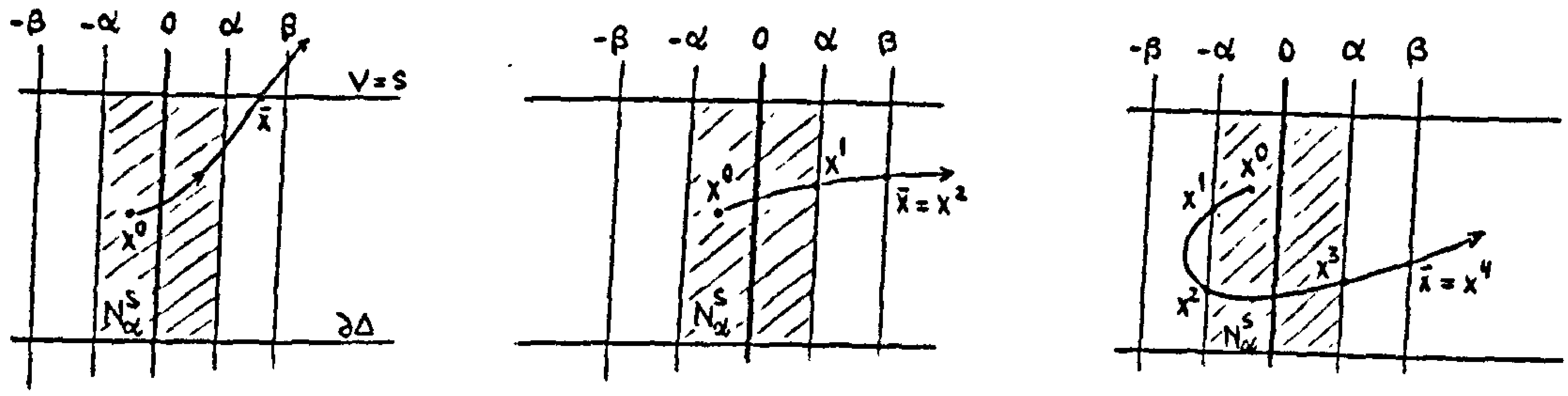
since $|z| < 1$ and $\varepsilon < \frac{1}{4}(\gamma-\delta)\alpha^2$.

By the choice of s , no ϕ_ε-orbit (for $\varepsilon \in [0,\varepsilon_0]$) stays in N_β^s for all positive times. Remember $N_\alpha^s \subset N_{\beta/2}^s \subset N_\beta^s$. Taking $x = x^0 \in N_\alpha^s - \partial\Delta$, we must then have some $\bar t > 0$ such that

$$\begin{cases} \phi_\varepsilon(t,x) \in N_\beta^s & \text{for } 0 \leq t \leq \bar t \\ \phi_\varepsilon(\bar t + t,x) \notin N_\beta^s & \text{for small } t > 0 . \end{cases}$$

i.e. $\bar x = \phi_\varepsilon(\bar t,x)$ is the first point of $\mathcal{O}_\varepsilon^+(x)$ at which this orbit leaves N_β^s .

Denoting by y_0 and $\bar{y}$ the y-coordinate at $x = x^0$ and $\bar{x}$, respectively, we must have $|y_0| \le \alpha$ and at $\bar{x}$, either $\bar{y} = \beta$ or $\bar{y} = -\beta$ or $V(\bar{x}) = s$. We will consider three cases of how $\mathcal{O}_\varepsilon^+(x)$ leaves N_β^s. See figure 30 below.



Case I Case II Case III

figure 30: How the orbit of $x^0 \in N_\alpha^s$ leaves N_β^s.

Case I: $\bar{y} \neq \pm\beta$ (hence $V(\bar{x}) = s$).

Case II: $\bar{y} = \pm\beta$, $\mathcal{O}_\varepsilon^+(x)$ leaves N_α^s at x^1, does not re-enter N_α^s before leaving N_β^s at $x^2 = \bar{x}$.

Case III: $\bar{y} = \pm\beta$, $\mathcal{O}_\varepsilon^+(x)$ leaves N_α^s at x^1, re-enters it at x^2 before leaving N_β^s, leaves N_α^s again at x^3, then gets to $x^4 = \bar{x}$.

We note that 6.3.5(iii), plus choice of s, implies that $\mathcal{O}_\varepsilon^+(x^0)$ cannot re-enter N_α^s again after x^3.

Now we study each of these cases:

<u>Case I</u> This is simple. $|\bar{y}| < \beta \Rightarrow \phi_\varepsilon(\bar{t}+t,x) \notin N_\beta^s$ for small $t > 0$ and this point has y-coordinate $< \beta$, too, if t is small enough. Then

$$V(\phi_\varepsilon(\bar{t}+t,x)) > s \geq V(x) \quad .$$

So, conclusion holds for x in this case.

Case II Let $V_i = V(x^i)$ $i = 0,1,2$

$$\begin{cases} \text{speed along arc } \widehat{x^0x^1} \text{ is } \geq v_m > 0 \\ \text{length of arc } \widehat{x^0x^1} \text{ is } \leq \ell_\alpha \quad . \end{cases}$$

So, if $x^1 = \phi_\varepsilon(t_1,x)$ we have

$$t_1 \leq \frac{\ell_\alpha}{v_m} < \tfrac{1}{4}\,\beta/v_M \quad \text{by the choice of } \alpha \quad .$$

Since arc $\widehat{x^0x^1} \subset N_\alpha^s$, inequality 6.3(**) holds for all its points, then

$$\int_0^{t_1} \frac{\dot V}{V}\,(\phi_\varepsilon(t,x))dt > -2(\gamma-\delta)\alpha^2\, t_1 > -\tfrac{1}{2}(\gamma-\delta)\alpha^2\beta/v_M \quad .$$

But $\displaystyle\int_0^{t_1} \frac{\dot V}{V}\,(\phi_\varepsilon(t,x))dt = \int_{V_0}^{V_1} \frac{dV}{V} = \log\frac{V_1}{V_0}$.

So $\displaystyle\frac{V_1}{V_0} > e^{-\frac{1}{2}(\gamma-\delta)\alpha^2\beta/v_M}$.

By the other hand

$$\begin{cases} \text{speed of arc } \widehat{x^1x^2} \leq v_M \\ \text{length of arc } \widehat{x^1x^2} \geq \text{(distance between planes } y = \alpha\text{ , } y = \beta) = \\ = \beta-\alpha > \tfrac{1}{2}\beta \quad . \quad (\text{Since } \alpha < \beta/2\text{ .}) \end{cases}$$

So, if $x^2 = \phi_\varepsilon(t_2,x^1)$, we have $t_2 \geq \tfrac{1}{2}\,\beta/v_M$.

Since $\widehat{x^1x^2}$ is outside N_α^s, 6.3(*) holds for all its points, and

$$\log \frac{V_2}{V_1} = \int_0^{t_2} \frac{\dot V}{V}\,(\phi_\varepsilon(t,x^1))dt > (\gamma-\delta)\alpha^2\beta/v_M$$

$$\Rightarrow \frac{V_2}{V_1} > e^{(\gamma-\delta)\alpha^2\beta/v_M}$$

and $$\frac{V_2}{V_0} = \frac{V_2}{V_1}\,\frac{V_1}{V_0} > e^{(\gamma-\delta)\alpha^2\beta/v_M}\quad e^{-\frac{1}{2}(\gamma-\delta)\alpha^2\beta/v_M} > 1 \ .$$

Hence $V(x^2) > V(x^0) = V(x)$ i.e. conclusion holds in this case.

<u>Case III</u> Let $V_i = V(x^i)$ $i = 0,\dots,4$.

Arcs $\widehat{x^0x^1}$ and $\widehat{x^2x^3}$ are in N_α^s. So, as in Case II

$$\frac{V_1}{V_0}\ ,\ \frac{V_3}{V_2} > e^{-\frac{1}{2}(\gamma-\delta)\alpha^2\beta/v_M} \ .$$

Arc $\widehat{x^3x^4}$ is similar to $\widehat{x^1x^2}$ of Case II. So,

$$\frac{V_4}{V_3} > e^{(\gamma-\delta)\alpha^2\beta/v_M} \ .$$

Arc $\widehat{x^1x^2}$ is outside N_α^s, so $\dot V(x) < 0$ for all its points. Then $\frac{V_2}{V_1} > 1$. Hence

$$\frac{V_4}{V_0} = \frac{V_4}{V_3}\,\frac{V_3}{V_2}\,\frac{V_2}{V_1}\,\frac{V_1}{V_0} > e^{(\gamma-\delta)\alpha^2\beta/v_M}e^{-\frac{1}{2}(\gamma-\delta)\alpha^2\beta/v_M}e^{-\frac{1}{2}(\gamma-\delta)\alpha^2\beta/v_M} = 1$$

i.e. $V(\bar x) = V_4 > V_0 = V(x)$ and conclusion holds in this case. This concludes the proof. □

6.3.7 <u>Lemma</u> Let s and ε_1 be as in lemma 6.3.6 and take $B^s = \{x \in \Delta;\ V(x) \geq s\} = \operatorname{clos}(\Delta - N^s)$. Then, for $\varepsilon \in [0,\varepsilon_1]$, any positive ϕ_ε-orbit in $\overset{\circ}{\Delta}$ meets B^s .

<u>Proof</u> Property could only fail for orbit of $x \in \overset{\circ}{\Delta}$ if $\mathcal{O}^+_\varepsilon(x) \subset \operatorname{int} N^s$, hence $V(\mathcal{O}^+_\varepsilon(x)) \subset (0,s)$. In this case, $\omega_\varepsilon(x)$ would be a compact invariant subset of N^s . Let $\bar{x} \in \omega_\varepsilon(x) \subset N^s$ be a point where $V|_{\omega_\varepsilon(x)}$ is maximum. $V(\bar{x}) \leq s$.

Using lemma 6.3.6 we can construct a sequence (t_n) strictly increasing with $0 < t_n \to +\infty$ such that, letting $x^n = \phi_\varepsilon(t_n,x)$, we have

$$V(x) < V(x^1) < V(x^2) < \dots < V(x^n) < \dots .$$

This implies that $V|_{\omega_\varepsilon(x)}$ cannot be identically zero. Hence $\bar{x} \notin \partial\Delta$.

Applying lemma 6.3.6 to $\bar{x}$, $\exists\ \bar{t} > 0$ s.t. $V(\phi_\varepsilon(\bar{t},\bar{x})) > V(\bar{x})$. But $\phi_\varepsilon(\bar{t},\bar{x}) \in \omega_\varepsilon(x)$ because $\omega_\varepsilon(x)$ is invariant, and this gives a contradiction to $V(\bar{x})$ being maximum of $V|_{\omega_\varepsilon(x)}$.

Hence $\omega_\varepsilon(x) \not\subset N^s$, and, so, $\mathcal{O}^+_\varepsilon(x)$ must meet B^s , proving the lemma.

□

<u>Proof of 6.3.1</u> (continuation)

Let $\varepsilon_0 > 0$ and $\mathcal{U}$ be as in the beginning of the proof. Take $s \in (0,1)$ and $\varepsilon_1 \in (0,\varepsilon_0)$ as in lemma 6.3.6 and B^s as in 6.3.7.

By lemma 6.2.5, e is Liapunov attractor for ϕ_0 with $\overset{\circ}{\Delta}$ as basin of attraction. Since B^s is compact and $B^s \subset \overset{\circ}{\Delta}$, there exists $T > 0$

such that $\phi_0(T,B^s) \subset U$, i.e. $\forall\, x \in B^s$ $\phi_0(T,x) \in U$ and since U is positively invariant $\phi_0(t,x) \in U$ $\forall\, t \geq T$.

Because B^s is compact and U is open there must exist $\bar{\varepsilon} \in (0,\varepsilon_1]$ such that $\phi_\varepsilon(T,B^s) \subset U$ for all $\varepsilon \in [0,\bar{\varepsilon}]$.

Then, $\forall$ $x \in \mathring{\Delta}$, $\mathcal{O}_\varepsilon^+(x)$ meets B^s (by 6.3.7) and so, it meets U, having $\omega_\varepsilon(x) \subset U$. Using Proposition 6.2.7, we have: either $x \in Y = W^s e$ or $\omega_\varepsilon(x) = L_\varepsilon$, concluding the proof. □

<u>Proof of 6.3.2</u> For $\varepsilon > 0$, e is hyperbolic 1-saddle and q_0, q_1 are hyp. repellors, hence $W^u q_0$ and $W^u q_1$ are open, connected subsets of Δ and $W^u e$ is 2-dimensional. Let $x \in \mathring{\Delta} - L_\varepsilon \cup W^u e$. The orbit of x always meets ∂U, crossing it, going inside U in the positive direction. So $\alpha_\varepsilon(x) \subset \Delta - U$. But $\mathring{\Delta} - U$ has no non-wandering points because $\forall\, x^1 \in \mathring{\Delta} - U$, $\mathcal{O}_\varepsilon^+(x^1)$ enters U (see proof of 6.3.1). Hence $\alpha_\varepsilon(x) \subset \partial\Delta$. Take face $F = X_0X_1X_2$. In the next paragraph we will see that $F - X_0X_1 \cup X_0X_3 \subset W^u q_0$ for $\varepsilon > 0$. See 6.4.1 and figure 31 (c). Similarly to other faces.
So all points in $\partial\Delta - L$ are either in $W^u q_0$ or $W^u q_1$ so they are wandering points and $L \subset \overline{W^u q_0} \cap \overline{W^u q_1}$. There must then be points $x^1 \in \mathring{\Delta} - W^u q_0 \cup W^u q_1$ with $\alpha_\varepsilon(x^1) \subset L$. In fact, then, for these points $\alpha_\varepsilon(x^1) = L$ because no proper subset of L can be α-limit of points of $\mathring{\Delta}$, since the invariant sets for the vertices (which are all 2-saddles) are contained in $\partial\Delta$.

This finishes the proof. □

6.4 <u>Flow ϕ_ε restricted to</u> $\partial\Delta$

We want to describe ϕ_ε on $\partial\Delta$. Since matrix A_ε has cyclic symmetry, matrices A_i associated to $\phi_\varepsilon|_{F_i}$ (F_i = face $x_i = 0$) are equal up to a permutation. So, we will describe ϕ_ε restricted to face F_3 and all others are the same by vertex permutation. $\psi_\varepsilon = \phi_\varepsilon|_{F_3}$ is given by

$$\dot{x}_i = x_i\,((B_\varepsilon x)_i - xB_\varepsilon x)\ , \quad i = 0,1,2\ , \quad x = (x_0,x_1,x_2)$$

$$\sum_{i=0}^{2} x_i = 1\ , \quad x_i \geq 0 \qquad \text{where} \qquad B_\varepsilon = 4\begin{pmatrix} 0 & \gamma & -\varepsilon \\ -\delta & 0 & \gamma \\ -\varepsilon & -\delta & 0 \end{pmatrix} \ .$$

In the next proposition we give topological description of ψ_ε .

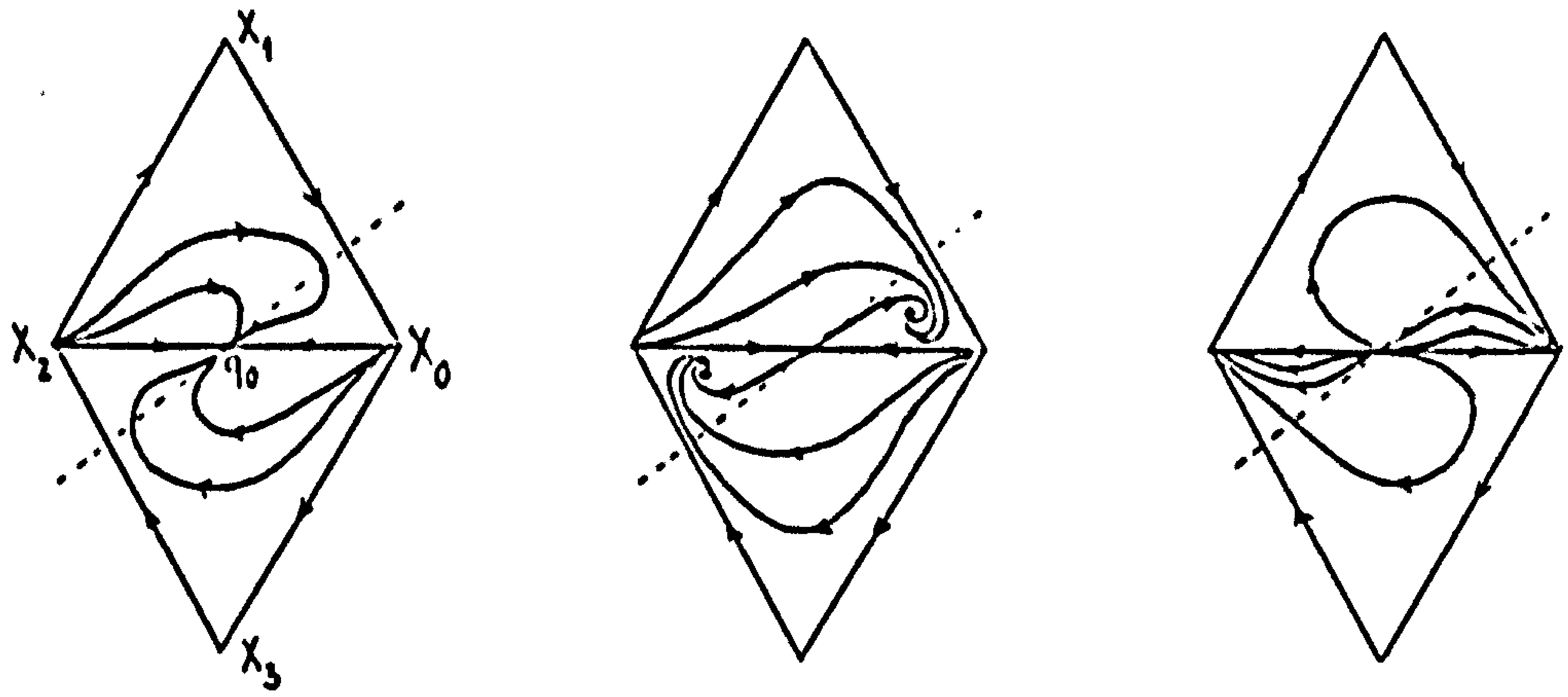
In figure 31 we represent the phase portraits of $\phi_\varepsilon|_{\partial\Delta}$, for all values of ε , drawing the flow on two faces. The other two faces are obtained just by cyclic permutation of vertices. Drawings will be justified during the paragraph.

6.4.1 <u>Proposition</u> If F_i is any 2-dimensional face of Δ , then $B_\varepsilon = (A_\varepsilon)_i$ is stable in $M_3 \Leftrightarrow \varepsilon \neq 0\ , -(\gamma-\delta)$. Moreover $B_\varepsilon \in -(5_2)$ for $\varepsilon > 0$, $B_\varepsilon \in (5_1)$ for $-(\gamma-\delta) < \varepsilon < 0$ and $B_\varepsilon \in (5_2)$ for $\varepsilon < -(\gamma-\delta)$. (Classes (5_1) and (5_2) as in Theorem I stated in 1.4.1.)

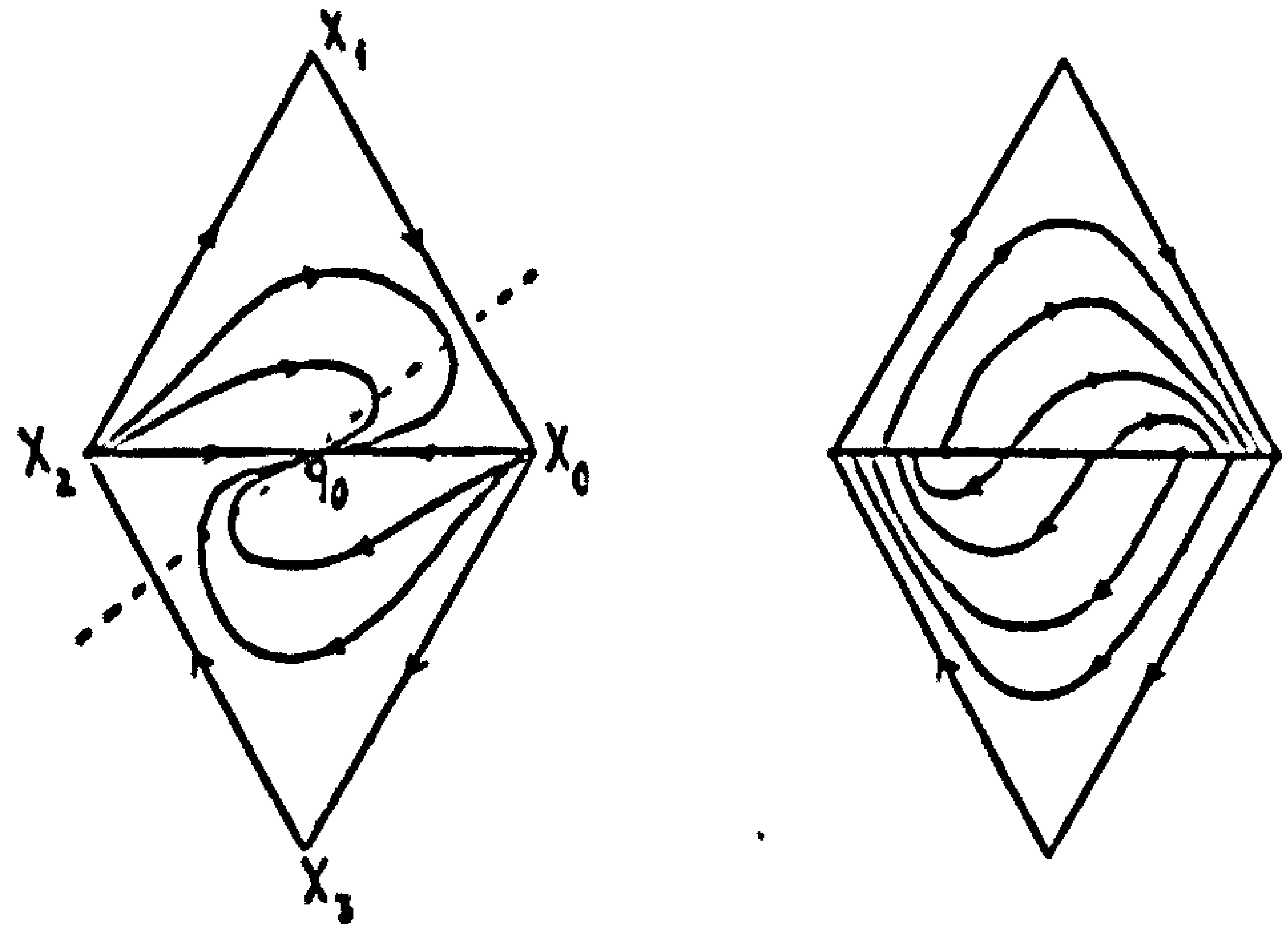
<u>Proof</u> We note that B_ε (as above) $\in Z_3^+$ belonging to combinatorial class C_5 if $\varepsilon < 0$ or $-C_5$ if $\varepsilon > 0$ (see 1.5.7). To determine stable class of B_ε we will use Theorem III (1.5.8). Taking σ as a suitable permutation of vertices, we put $\sigma B_\varepsilon \in S_5$ or $-S_5$ as in 1.5.8.

(i) for $\varepsilon < 0$,

$$D = \sigma B = \begin{pmatrix} 0 & -\varepsilon & -\delta \\ -\varepsilon & 0 & \gamma \\ \gamma & -\delta & 0 \end{pmatrix} \in S_5$$



<u>Stables</u> (a) $\varepsilon < -(\gamma-\delta)$ (b) $-(\gamma-\delta)<\varepsilon<0$ (c) $\varepsilon > 0$



<u>Unstables</u> (d) $\varepsilon = -(\gamma-\delta)$ (e) $\varepsilon = 0$

figure 31: Phase portraits for $\phi_\varepsilon|_{\partial\Delta}$

and, for D, $k_2 = \frac{d_{20}}{d_{10}} + \frac{d_{21}}{d_{01}} - 1 = -\frac{\gamma-\delta}{\varepsilon} - 1$.

Hence $\varepsilon < -(\gamma-\delta) \Rightarrow k_2 < 0 \Rightarrow B_\varepsilon \in (5_2)$

$-(\gamma-\delta) < \varepsilon < 0 \Rightarrow k_2 > 0 \Rightarrow B_\varepsilon \in (5_1)$

by Theorem III (1.5.8).

For $\varepsilon = -(\gamma-\delta)$, $k_2 = 0 \Rightarrow B_\varepsilon$ is not stable (q_0 not hyperbolic).

iii) for $\varepsilon > 0$,

$$D = -\sigma B_\varepsilon = \begin{pmatrix} 0 & \varepsilon & -\gamma \\ \varepsilon & 0 & \delta \\ \delta & -\gamma & 0 \end{pmatrix} \in S_5$$

and $k_2 = -\frac{\gamma-\delta}{\varepsilon} - 1 < 0 \quad \Rightarrow \quad B_\varepsilon \in -(5_2)$

For $\varepsilon = 0$, B_ε is not stable (with X_0X_2 pointwise fixed). □

The next lemmas will give other properties of ψ_ε , which, though not so important, help us to draw figure 31.

Taking, in $F = F_3$, coordinates (z_1,z_2) with $z_1 = x_0-x_2$, $z_2 = x_1$ (corresponding to $x_3 = 0$, and, so, $y = 1-2z_2$ in coordinates (z_1,z_2,y) of 6.2.4) we get

$$6.4(*) \begin{cases} \frac{1}{2}\dot{z}_1 = (\varepsilon(1-2z_2+z_2^2-z_1^2) - (\gamma-\delta)(1-2z_2)z_2)z_1 + (\gamma+\delta)(1-z_2)z_2 \\ \frac{1}{2}\dot{z}_2 = (\varepsilon(1-2z_2+z_2^2-z_1^2) + (\gamma-\delta)(1-2z_2)(1-z_2) - (\gamma+\delta)z_1)z_2 \end{cases}$$

where $0 \le z_2 \le 1$, $|z_1| \le 1 - z_2$.

6.4.2 <u>Lemma</u> For $\varepsilon \neq 0$, $q_0 = \frac{1}{2}(1,0,1)$ is isolated fixed point, being hyperbolic if $\varepsilon \neq -(\gamma-\delta)$. The eigenvalues at q_0 are $\lambda_1 = 2\varepsilon$ (with eigenspace on X_0X_2) and $\lambda_2 = 2(\gamma-\delta+\varepsilon)$ with eigenspace E given by $x_1 = -\frac{\gamma-\delta}{\gamma+\delta}(x_0-x_2)$.

<u>Proof</u> Taking (z_1,z_2) as above, $(z_1,z_2) = (0,0)$ corresponds to point q_0 . Linearization is

$$\begin{cases} \frac{1}{2}\dot{z}_1 = \varepsilon z_1 + (\gamma+\delta)z_2 \\ \frac{1}{2}\dot{z}_2 = (\gamma-\delta+\varepsilon)z_2 \end{cases}$$

which has eigenvalues and eigenspaces as claimed. Note that $\varepsilon = 0 \Rightarrow (\dot{z}_1, \dot{z}_2) = 0$ on $X_0X_2(z_2 = 0)$ and $\varepsilon = -(\gamma-\delta) \Rightarrow \lambda_2 = 0$. □

(The eigenvalues λ_1, λ_2 could also be determined using Proposition 2.4.2, but this would not give eigenspace E.)

Noting that $\forall\, \varepsilon, \lambda_1 = 2\varepsilon < 2(\gamma-\delta+\varepsilon) = \lambda_2$ and using known facts about "modes of approach" as, for instance, in [3], p.162-181, we have:

6.4.3 Corollary

(i) for $\varepsilon < -(\gamma-\delta)$, $\forall\, x \in \mathring{F}$, $\phi_\varepsilon(t,x) \to q_0$ as $t \to +\infty$ in the direction of E;

(ii) for $\varepsilon > 0$, $\forall\, x \in \mathring{F}$, $\phi_\varepsilon(-t,x) \to q_0$ as $t \to +\infty$ and, exactly one orbit leaves q_0 in the direction of E, all others leave q_0 tangentially to X_0X_2.

6.4.4 Lemma For $-(\gamma-\delta) < \varepsilon < 0$, ψ_ε has a fixed (isolated) point in $\mathring{F}$ (call it p_ε) such that:

(i) p_ε is hyperbolic attractor on $\dot{F}$ with $W^s p_\varepsilon = \mathring{F}$;

(ii) $p_\varepsilon \to \frac{1}{\gamma+\delta}(\gamma,0,\delta)$ as $\varepsilon \to 0$; $p_\varepsilon \to q_0$ as $\varepsilon \to -(\gamma-\delta)$;

(iii) for flow ϕ_ε in 3-dim Δ, p_ε is hyperbolic 2-saddle.

Proof p_ε is determined as $\mathring{F} \cap [(\mathrm{adj}B_\varepsilon)u]$ (by 2.2.1(iv)) and this gives $p_\varepsilon = v/k_\varepsilon$ where

$$v = (v_0,v_1,v_2) = (\mathrm{adj}B_\varepsilon)u = (\gamma(\gamma+\delta)+\varepsilon\delta, -\varepsilon(\gamma-\delta+\varepsilon), \delta(\gamma+\delta)-\varepsilon\gamma)$$

and $k_\varepsilon = v_0 + v_1 + v_2 = (\gamma+\delta)^2 - 2\varepsilon(\gamma-\delta)-\varepsilon^2$.

Then (i) holds since $B_\varepsilon \in (5_1)$ by 6.4.1, (ii) follows by calculating limits of above expression for p_ε .

By proposition 2.4.2 and corollary 2.4.3, the eigenvalue at p_ε in direction transversal to F is given by $\lambda = (A_\varepsilon p_\varepsilon)_3 - (A_\varepsilon p_\varepsilon)_i$ $i = 0,1$ or 2.

$$A_\varepsilon p_\varepsilon = \frac{4}{k_\varepsilon}\begin{pmatrix} 0 & \gamma & -\varepsilon & -\delta \\ -\delta & 0 & \gamma & -\varepsilon \\ -\varepsilon & -\delta & 0 & \gamma \\ \gamma & -\varepsilon & -\delta & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ 0 \end{pmatrix}$$

Then $\lambda = \frac{4}{k_\varepsilon}(\gamma v_0 - \varepsilon v_1 - \delta v_2 + \delta v_0 - \gamma v_2)$

$$= \frac{4}{k_\varepsilon}(\gamma-\delta+\varepsilon)((\gamma+\delta)^2 + \varepsilon^2) > 0 \quad \text{since} \quad -(\gamma-\delta) < \varepsilon < 0 .$$

So, (iii) holds and the proof is finished.

□

For $\varepsilon = -(\gamma-\delta)$ or $\varepsilon = 0$, B_ε is not stable (6.4.1) but in order to justify figure 31 (d) and (e), we study these cases in the two following lemmas.

6.4.5 <u>Lemma</u> For $\varepsilon = -(\gamma-\delta)$, ψ_ε has not fixed point in $\mathring{F}$, q_0 is an attractor (not hyperbolic) and $\forall\, x \in \mathring{F}$, $\alpha(x) = X_2$, $\omega(x) = q_0$.

<u>Proof</u> Let $B = B_{-(\gamma-\delta)}$. $\det B = (\gamma-\delta)(\gamma^2+\delta^2) \neq 0$. Hence, by 2.2.1(iv), if there was fixed point in $\mathring{F}$ for $\psi = \psi_{-(\gamma-\delta)}$, this would be isolated and, by 2.2.1(iii), this property is robust. Since for $\varepsilon < -(\gamma-\delta)$, ψ_ε has no fixed point in $\mathring{F}$, we cannot have fixed points in $\mathring{F}$ for ψ .

Consequently, by 2.3.1, ψ has no non-wandering points in $\mathring{F}$. In fact, proof of 2.3.1 [1], says that there exist points $q^+, q^- \in \partial F$ with disjoint support so that q^+ dominates q^- (not strictly, by 2.3.6, plus existence of fixed point in $\mathring{F}$ for $\varepsilon \in (-(\gamma-\delta),0))$. The only possible choice for q^+ and q^- for B is $q^+ = X_0 = (1,0,0)$ and $q^- = (0,1-\frac{\delta}{\gamma},\frac{\delta}{\gamma}) \in X_1X_2$ with $q^+B = (0,\gamma,\gamma-\delta)$, $q^-B = (0,-\frac{\delta^2}{\gamma},\gamma-\delta)$ giving that $V(x) = \frac{x_1}{x_0}(\frac{x_2}{x_1})^{\delta/\gamma}$ decreases (strictly) along orbits in $\mathring{F}$. (See 2.3.6 Step 3.)

Since $\lim_{x \to q_0} V(x) = 0$, no orbit of $\mathring{F}$ has q_0 as α-limit.

Noting that X_0 and X_1 are hyperbolic saddle with invariant sets (stable and unstable) contained in ∂F, and, also, that X_2 is hyperbolic repellor, we use Poincaré-Bendixson (as in [26]) to conclude that $\forall\, x \in \mathring{F}$, $\omega(x) = q_0$, $\alpha(x) = X_2$.

This concludes the lemma, justifying figure 31 (d). □

6.4.6 <u>Lemma</u> For $\varepsilon = 0$, let $B = B_0$, $\psi = \psi_0$.

(i) ψ has no non-wandering points in $\mathring{F}$, and $\forall\, x \in \mathring{F}$ $\dot{x}_0 > \dot{x}_2$ (hence orbits are transversal to straight lines passing through X_1);

(ii) $\forall\, x \in \mathring{F}$, $\omega(x)$ and $\alpha(x)$ are single points of X_0X_2;

(iii) letting $p = (a,0,b) \in X_0X_2$, we have:

if $\frac{\gamma}{\gamma+\delta} < a \le 1 \Rightarrow p$ is ω-limit of exactly one orbit in $\mathring{F}$ and no orbit starts at p;

if $0 \le a < \frac{\gamma}{\gamma+\delta} \Rightarrow p$ is α-limit of exactly one orbit in $\mathring{F}$ and no orbit finishes at p.

$p_0 = \frac{1}{\gamma+\delta}(\gamma,0,\delta)$ is not in the limit set of any orbit in $\overset{\circ}{F}$.

<u>Proof</u> Taking $q^+ = X_0 = (1,0,0)$ and $q^- = X_2 = (0,0,1)$ we have $q^+B = (0,\gamma,0)$, $q^-B = (0,-\delta,0)$. So, q^+ dominates q^- , not strictly (see 2.3.4). By 2.3.6 $V(x) = \frac{x_2}{x_0}$ has $\dot{V}(x) < 0$ (so $\dot{x}_2 < \dot{x}_0$) . Therefore, (i) holds.

Taking equations 6.4(*) at $\varepsilon = 0$ we get

$$6.4(**) \begin{cases} \dot{z}_1 = 2z_2((\gamma+\delta)(1-z_2) - (\gamma-\delta)(1-2z_2)z_1) \\ \dot{z}_2 = 2z_2((\gamma-\delta)(1-z_2)(1-2z_2) - (\gamma+\delta)z_1) \end{cases} .$$

In $\overset{\circ}{F}$ we have $0 < z_2 < 1$, $|z_1| < 1-z_2$.

$$\dot{z}_2 \gtrless 0 \iff z_1 \lessgtr \frac{\gamma-\delta}{\gamma+\delta}(1-z_2)(1-2z_2)$$

$z_1 = \frac{\gamma-\delta}{\gamma+\delta}(1-z_2)(1-2z_2)$ is a simple curve (we denote it by C) in F intersecting ∂F at X_1 ($z_1 = 0$, $z_2 = 1$) and $p_0 = \frac{1}{\gamma+\delta}(\gamma,0,\delta)$ ($z_1 = \frac{\gamma-\delta}{\gamma+\delta}$, $z_2 = 0$) . C is represented in figure 32.

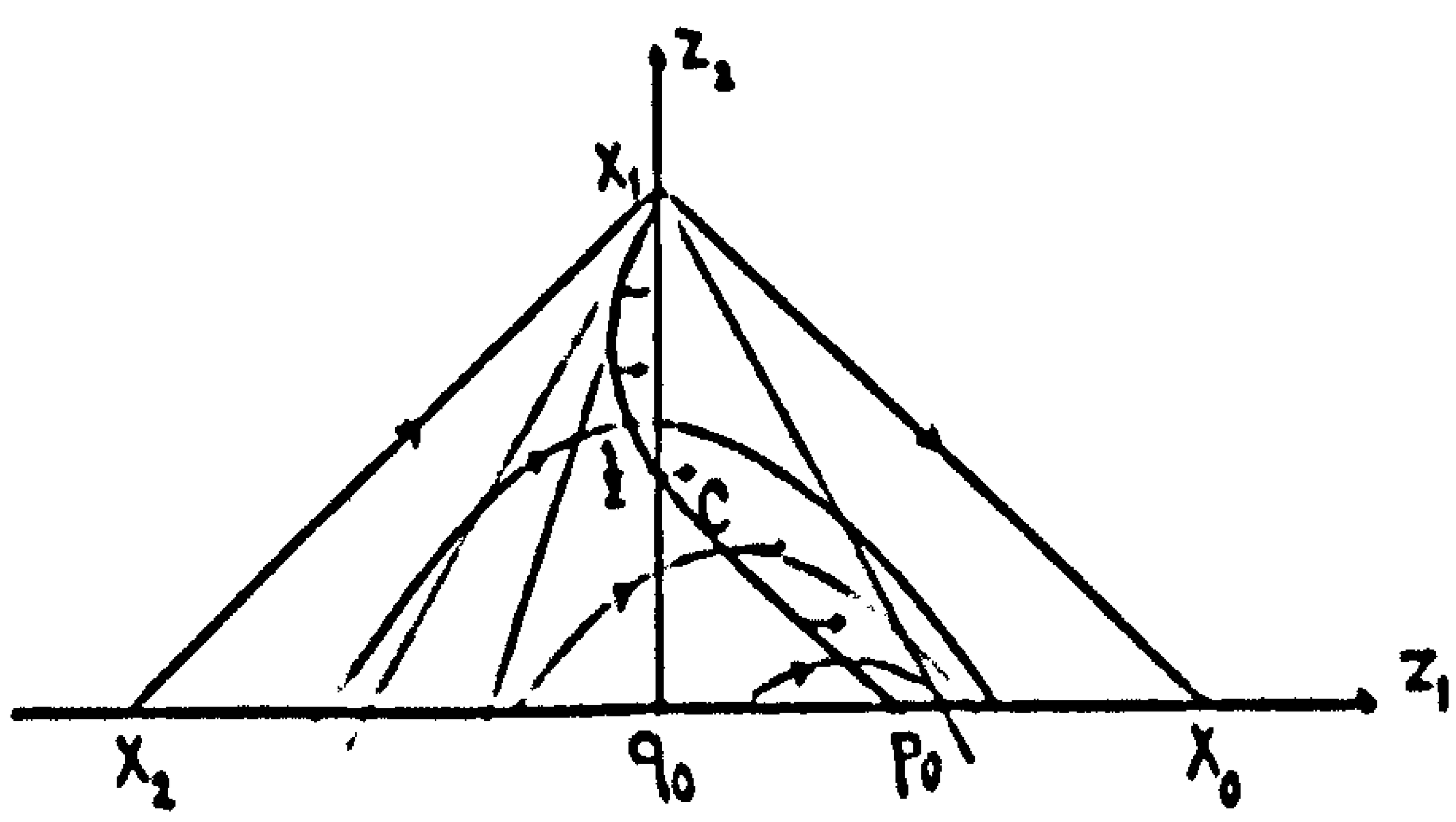


figure 32: flow ψ_0

For all $x \in \mathring{\Delta}$, $\dot{z}_1 = \dot{x}_0 - \dot{x}_2 > 0$ (see above). So, all ϕ_0-orbits in $\mathring{\Delta}$ have z_1 strictly increasing. This means, in figure 32, that these orbits go from left to right always and are transversal to curve C .

We can, then, conclude that orbits in $\mathring{F}$ must have α-limit in $X_2 p_0$, going upwards (i.e. $\dot{z}_2 > 0$) at the left of C (figure 32) till it crosses C transversally, then it goes downwards ($\dot{z}_2 < 0$) at the right of C , and has ω-limit in $p_0 X_0$. Also, ψ is transversal to lines $x_2 = cx_0$ (by (i)), so each $\alpha(x)$ or $\omega(x)$ consists of just one point of $X_0 X_2$. So (ii) holds.

To prove (iii) we will show that every orbit approaching (or leaving) a point $p = (a,0,b) \in X_0 X_2 - p_0$ must do so being contained locally on the graph of a function $z_1 = f(z_2)$ and this function depends solely on the point p . This implies that exactly one orbit in $\mathring{F}$ has p as its ω- or α-limit.

Suppose $\psi(t,x)$ has limit p as $t \to +\infty$ (or $-\infty$) . We write $\psi(t,x)$ as $(z_1(t), z_2(t))$ and p as $(r,0)$ where $r = a-b \in [-1,1]$. Using 6.4(**) and making $z_1(t) \to r$, $z_2(t) \to 0$ we have

$$\frac{\dot{z}_2}{\dot{z}_1}(t) = \frac{(\gamma-\delta)(1-2z_2(t))(1-z_2(t)) - (\gamma+\delta)z_1(t)}{(\gamma+\delta)(1-z_2(t)) - (\gamma-\delta)(1-2z_2(t))z_1(t)}$$

$$\frac{\dot{z}_2}{\dot{z}_1}(t) \to \alpha_r = \frac{(\gamma-\delta) - (\gamma+\delta)r}{(\gamma+\delta) - (\gamma-\delta)r} \neq 0 \quad \text{if } r \neq \frac{\gamma-\delta}{\gamma+\delta}$$

(The denominator is not zero for $r \in [-1,1]$.)

Also $\frac{d}{dr}\alpha_r < 0$, hence α_r decreases (strictly) with r, $\alpha(-1) = 1$, $\alpha(1) = -1$.

So, locally at $p \neq p_0$, $(z_1(t),z_2(t))$ must be contained in the graph of a function $z_1 = f(z_2)$ with $f(0) = r$ and $f'(0) = 1/\alpha_r$.

Graph of $z_1 = f(z_2)$ is invariant $\Leftrightarrow$ $(f'(z_2)\dot{z}_2-\dot{z}_1)_{z_1=f(z_2)} = 0$ from where we get

$$f'(z_2) = \frac{(\gamma+\delta)(1-z_2) - (\gamma-\delta)(1-2z_2)f(z_2)}{(\gamma-\delta)(1-2z_2)(1-z_2) - (\gamma+\delta)f(z_2)}$$

(which holds even for $z_2 = 0$ since $f'(0) = 1/\alpha_r$).

Defining $F(u,y) = \dfrac{(\gamma+\delta)(1-u) - (\gamma-\delta)(1-2u)y}{(\gamma-\delta)(1-2u)(1-u) - (\gamma+\delta)y}$

(F is well-defined, and differentiable, in a neighbourhood of points $(0,r)$ with $r \neq -(\gamma-\delta)/(\gamma+\delta)$.)

Then equation above, for $f'(z_2)$, can be written as $f'(u) = F(u,f(u))$ and this differential equation has a unique (local) solution $f(u)$ with $f(0) = r$ (and consequently $f'(0) = 1/\alpha_r$). Hence (iii) is proved for points in $X_0X_2-p_0$.

p_0 is the intersection of C with X_0X_2. Hence, if we had $\psi(t,x) \to p_0$ as $t \to +\infty$ (for $x \in \overset{o}{F}$) $\psi(t,x)$ should approach p_0 from the left of C (in figure 32), tangentially to edge X_0X_2, because for $r = (\gamma-\delta)/(\gamma+\delta)$, $\alpha_r = 0$. But this is impossible because to the left of C, $\dot{z}_2(t) > 0$.

Similarly $\psi(t,x) \to p_0$ as $t \to -\infty$ would give a contradiction. Hence (iii) is proved, completing the lemma.

6.4.7 Remark The point $p_0 \in X_0X_2$ as in 6.4.6 is the limit of points p_ε of 6.4.4 as $\varepsilon \to 0^-$.

For v_0, v_1, v_2 as in 6.4.4 we have that $\frac{v_1}{v_0 - v_2} = \frac{-\varepsilon}{\gamma+\delta} < \frac{\gamma-\delta}{\gamma+\delta}$.

This justifies p_ε in figure 31 (b) being drawn at the right of line E .

So properties in 6.4.1-6.4.6 justify drawings in figure 31.

In the next proposition, we collect some more properties of flow ϕ_ε for $\varepsilon \geq 0$ which were needed in 6.3 (announced in 6.3.5) in order to describe flow in $\overset{\circ}{\Delta}$.

6.4.8 Proposition Let $\beta \in (0,1)$ and N_β = {points of $F = F_3$ with $|y| \leq \beta$} . For all $\varepsilon \geq 0$ we have

(i) ψ_ε has no fixed points in N_β ;

(ii) the intersection of any orbit of ψ_ε with N_β has finite time and finite length;

(iii) every orbit has at most one point with $\dot{y} = 0$ and at this point $\ddot{y} \neq 0$.

Proof We observed (in 6.3.5) that N_β is a strip on F parallel to edge X_0X_2 , not containing X_1 or X_0X_2 . Hence (i) is immediate. (See figure 31 (c), (e).) For $\varepsilon \geq 0$ all orbits in $\overset{\circ}{F}$ start and end at points of X_0X_2 so they must cross N_β in finite time, and also finite length. The same is valid for orbits on X_0X_1 or X_1X_2 which go from vertex to vertex. So (ii) is true.

Proof of (iii) is technical. Using (z_1, z_2, y) -coordinates, $F(x_3 = 0)$ corresponds to $y = 1 - 2z_2$, hence $\dot{y} = -2\dot{z}_2$, $\ddot{y} = -2\ddot{z}_2$. N_β $(|y| \leq \beta)$ corresponds to $0 < \frac{1}{2}(1-\beta) \leq z_2 \leq \frac{1}{2}(1+\beta) < 1$. We recall equations 6.4(*) (in p.170). Suppose that, at a point (z_1, z_2) in $\mathring{\Delta}$ (so $z_2 \neq 0$) we have $\dot{y} = -2\dot{z}_2 = 0$. Then

$$\frac{1}{2}\ddot{z}_2 = \frac{1}{2}\frac{(\dot{z}_2)^2}{z_2} + z_2(-2\varepsilon(1-z_2)\dot{z}_2 - 2\varepsilon z_1\dot{z}_1 + (\gamma-\delta)(-3+2z_2)\dot{z}_2 - (\gamma+\delta)\dot{z}_1) .$$

So at a point where $\dot{y} = 0$ (i.e. $\dot{z}_2 = 0$) we have

$$\ddot{y} = -2\ddot{z}_2 = 4z_2\dot{z}_1(2\varepsilon z_1 + (\gamma+\delta)) .$$

Since in N_β there are no fixed points, $\dot{z}_2 = 0 \Rightarrow \dot{z}_1 \neq 0$. Then, at points where $\dot{y} = 0$, we get: $\ddot{y} = 0 \Leftrightarrow 2\varepsilon z_1 + (\gamma+\delta) = 0$.

For $\varepsilon = 0$ this is absurd, then $\dot{y} = 0 \Rightarrow \ddot{y} \neq 0$.

For $\varepsilon > 0$, note that $\dot{z}_2 = 0$ (with $z_2 \neq 0$) $\Leftrightarrow f(z_1) = g(z_2)$

where
$$\begin{cases} f(z_1) = \varepsilon z_1^2 + (\gamma+\delta)z_1 \\ g(z_2) = (1-z_2)(\varepsilon(1-z_2) + (\gamma-\delta)(1-2z_2)) . \end{cases}$$

But $g(z_2)$ is quadratic in z_2 with a minimum at

$$\tilde{z}_2 = \frac{3(\gamma-\delta) + 2\varepsilon}{4(\gamma-\delta) + 2\varepsilon} \in (\tfrac{1}{2}, 1) .$$

So $\forall z_2$, $g(z_2) \geq g(\tilde{z}_2) = -\dfrac{(\gamma-\delta)^2}{8(\gamma-\delta) + 4\varepsilon}$.

On the other hand

$$f(-\frac{\gamma+\delta}{2\varepsilon}) = \varepsilon\frac{(\gamma+\delta)^2}{4\varepsilon^2} - \frac{(\gamma+\delta)^2}{2\varepsilon} = -\frac{(\gamma+\delta)^2}{4\varepsilon} < g(\tilde{z}_2)$$

implying that $2\varepsilon z_1 + (\gamma+\delta) = 0 \Rightarrow f(z_1) < g(z_2) \ \forall z_2$.

Hence, in N_β , $\dot{y} = 0 \Rightarrow \dot{z}_2 = 0 \Rightarrow f(z_1) = g(z_2) \Rightarrow z_1 \neq -(\gamma+\delta)/2\varepsilon$ $\Rightarrow \ddot{y} \neq 0$.

Now we want to see that every orbit in N_β has at most one point with $\dot{y} = 0$. Let E_c be the line $z_2 = c \in (0,1)$. On E_c , $\frac{1}{2}\dot{z}_2 = c(g(c)-f(z_1)) = 0 \Leftrightarrow f(z_1) = g(c)$. Since $f(z_1)$ is quadratic, on any line E_c , there can be at most two points where $\dot{z}_2 = 0$. But, in fact, there is only <u>one</u> of these in $E_c \cap \Delta$ because on the point of intersection with X_2X_1 we have $\dot{z}_2 > 0$, and on the intersection with X_0X_1 , $\dot{z}_2 < 0$. This, plus the property that $\dot{z}_2 = 0 \Rightarrow \ddot{z}_2 \neq 0$, imply that on $E_c \cap \Delta$ there is an odd number of points with $\dot{z}_2 = 0$, hence exactly one. See figure 33 (a).

Therefore, there is a (continuous) curve C_ε in Δ , so that $\dot{z}_2 = 0$ on points of C_ε , $\dot{z}_2 > 0$ on the left of C_ε , $\dot{z}_2 < 0$ on the right of C_ε . See figure 33 (b). So fixed point q_0 must be at the left of C_ε .

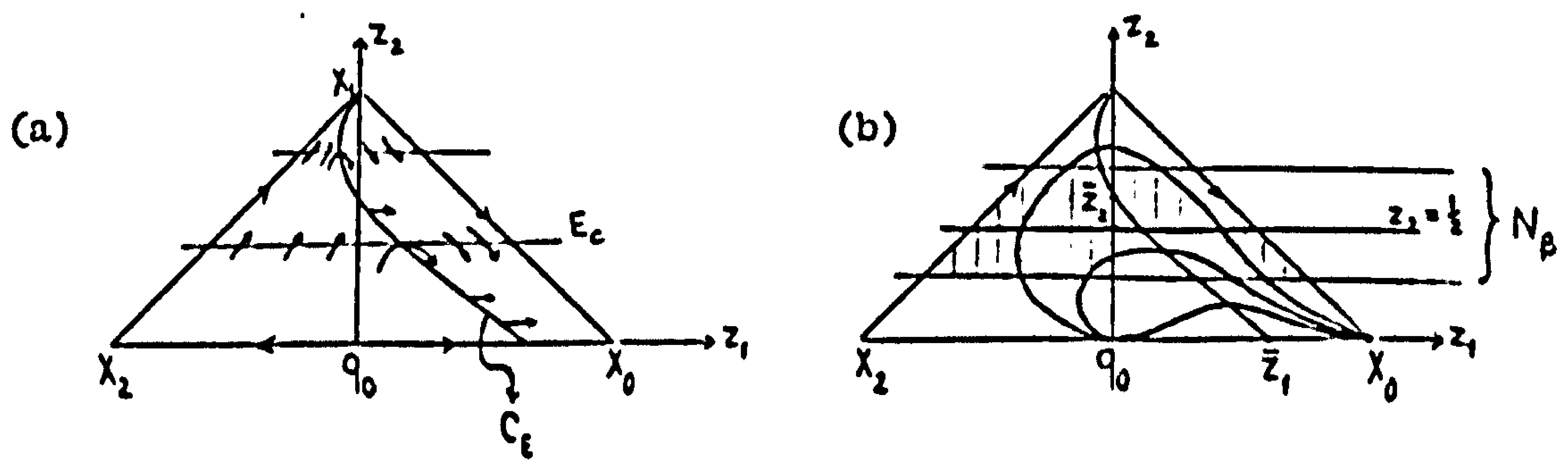


figure 33: (a) the curve C_ε of points with $\dot{z}_2 = 0$, $\varepsilon \geq 0$

(b) the flow ψ_ε crossing C_ε , $\varepsilon \geq 0$.

At points on C_ε , $\dot{z}_1 \neq 0$ (hence $\dot{z}_1 > 0$) and flow crosses C_ε transversally from left to right.

Now we can conclude part (iii):

Every orbit in $\mathring{\Delta}$ has q_0 as α-limit, X_0 as ω-limit (see figure 31 (c), (e)). So, every orbit in $\mathring{\Delta}$ must cross C_ε exactly once, and no orbit can have more than one point with $\dot{y} = 0$ ($\dot{z}_2 = 0$) . At this point, $\ddot{y} \neq 0$.

This finishes (iii) and the lemma. □

6.4.9 <u>Remark</u> C_ε , as above, intersects exactly once each line $E_c(z_2 = c \in (0,1))$. Hence, C_ε is the graph of some function $z_1 = F_\varepsilon(z_2)$ where $z_1 = F_\varepsilon(c)$ is the only root of $f(z_1) = g(c)$ with $|z_1| \leq 1-c$. It is not difficult to show that

$$F_\varepsilon(1) = 0 \quad , \text{ and}$$

$$F_\varepsilon(0) = \bar{z}_1 = \frac{1}{2\varepsilon}\left(\sqrt{(\gamma+\delta)^2+4\varepsilon(\gamma-\delta+\varepsilon)} - (\gamma+\delta)\right) \in (0,1) \quad \text{for } \varepsilon > 0$$

$$F_0(0) = \frac{\gamma-\delta}{\gamma+\delta} \quad \text{for } \varepsilon = 0 \qquad \text{(See figure 33 (b).)}$$

In fact for $\varepsilon = 0$, C_0 is exactly the curve C of lemma 6.4.6 and figure 32.

Also, C_ε cuts the z_1-axis at X_1 , plus the point with $(0,\bar{z}_2)$ with $\bar{z}_2 = \dfrac{\gamma-\delta+\varepsilon}{2(\gamma-\delta)+\varepsilon} \in [\frac{1}{2}, 1)$.

6.5 <u>About stability for ϕ_ε for small $\varepsilon \neq 0$</u>

In the last paragraphs we presented a global description of the flow ϕ_ε associated to Λ_ε at least for small values of ε . Our initial intention was to prove that A_ε is stable (by definition 1.3.3) for small $\varepsilon \neq 0$. We wanted to show that limit cycles (like L_ε) do occur for stable A in M_4. (No limit cycles occur for ϕ_A, when $A \in M_3$ is stable, by 3.2.3.)

However, any attempt we made to prove stability for A failed because the cycle L $(X_3 \to X_2 \to X_1 \to X_0 \to X_3)$ on $\partial\Delta$ hindered the construction of a global homeomorphism $h:\Delta \to \Delta$ giving equivalence between A_ε and B near it.

Nevertheless, we still conjecture that A_ε is stable for small $\varepsilon \neq 0$, as in definition 1.3.3, though we will not attempt here to prove this.

But we will show that A_ε (for small $\varepsilon \neq 0$) is "stable" in a lesser sense, meaning that there exists a neighbourhood N of A_ε s.t. $\forall B \in N$, there are homeomorphisms $h:\mathring{\Delta} \to \mathring{\Delta}$ and $h_i:F_i \to F_i$ $i = 0,\dots,3$ (where F_i is the face $x_i = 0$) all taking ϕ_B-orbits to ϕ_ε-orbits.

The existence of $h_i:F_i \to F_i$ $(i = 0,\dots,3)$ follows by Proposition 6.4.1 ($(A_\varepsilon)_i$ is stable in M_3).

The existence of $h:\mathring{\Delta} \to \mathring{\Delta}$, as required, is the purpose of proposition 6.5.3 for small $\varepsilon > 0$ and of proposition 6.5.8 for small $\varepsilon < 0$.

Our conjecture above means that we believe that h could be constructed in a way to be extended continuously to $\partial\Delta$.

Let us first consider the case $\underline{\varepsilon > 0}$. Take $\bar{\varepsilon}$ and U as in proposition 6.2.7. We remember that U is taken as ϕ_ε-positively invariant for all $\varepsilon \in (-\bar{\varepsilon},\bar{\varepsilon})$ (see (D) of Hopf's theorem in 5.2.1) and U (in 5.2.1) was constructed so that ∂U is a level surface of a C^1-Liapunov function f in a neighbourhood of 0 in $\mathbb{R}^3$ for ϕ_0 (as in [40]). So we can suppose, when necessary, that ∂U is a 2-dimensional compact surface transversal to ϕ_ε for all $\varepsilon \in (-\bar{\varepsilon},\bar{\varepsilon})$. First we prove:

6.5.1 <u>Lemma</u> Take $\bar{\varepsilon}$ and U as above, and $\varepsilon \in (0,\bar{\varepsilon})$. Then A_ε has a neighbourhood $N_1(\varepsilon)$ in M_4 s.t. $\forall\ B \in N_1$, $\exists$ top. equivalence $h_1:\bar{U} \to \bar{U}$ from ϕ_B to ϕ_ε .

<u>Proof</u> We can suppose, as explained above, that ϕ_ε is transversal to ∂U . This implies that ∂U is also transversal to all flows sufficiently C^1-near ϕ_ε .

We note that, inside U , ϕ_ε can be thought of as a Morse-Smale system [24, 25, 26] in S^3 for which ∂U plays the part of a fundamental neighbourhood associated to the outset of a repellor [24].

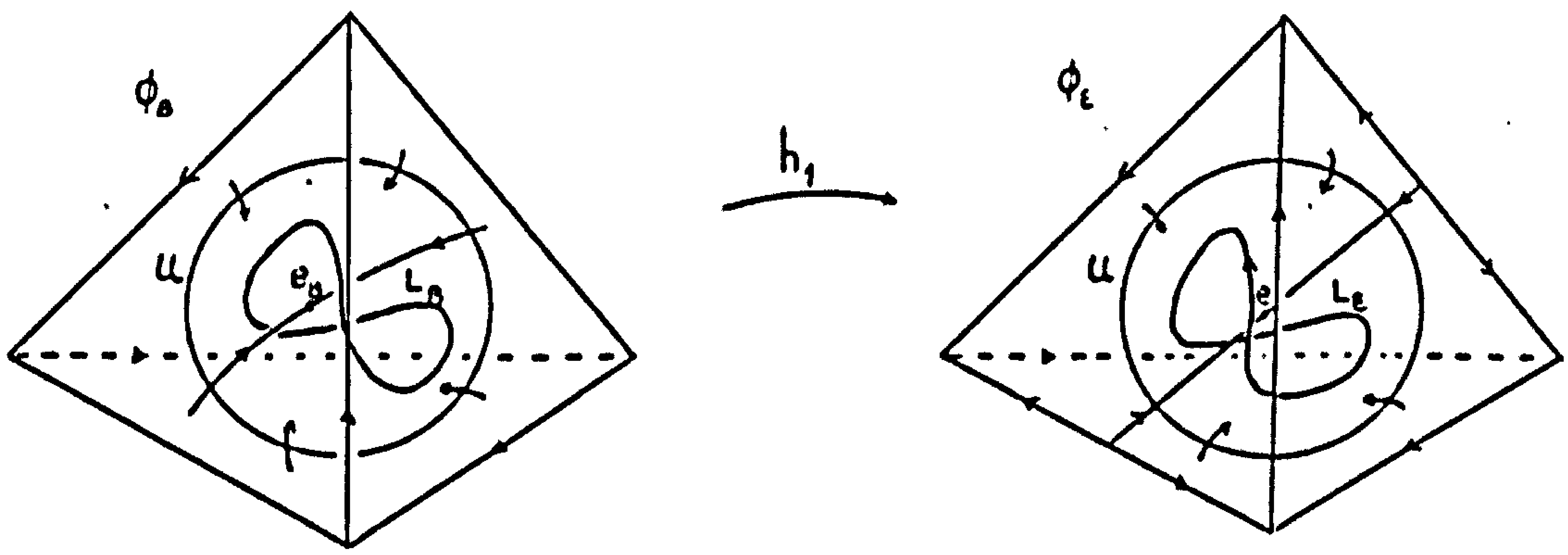


figure 34: equivalence of ϕ_B and ϕ_ε in U .

By standard methods of structural stability for Morse-Smale systems [24], there exists a neighbourhood N_1 of A_ε s.t. $\forall B \in N_1$, ϕ_B is transversal to ∂U and ϕ_B and ϕ_ε are top. equivalent in $\bar{U}$, i.e. $\exists\ h_1: \bar{U} \to \bar{U}$ as required. (Note that, in U, ϕ_B has fixed point e_B (1-saddle) near e and periodic orbit L_B (attracting) near L_ε. See figure 34.) This concludes the proof. □

In our next proposition we want to assert the property that, in fact, all ϕ_B-orbits (B in a neighbourhood of A_ε) of points in $\mathring{\Delta}-U$ will eventually in time intersect ∂U, hence having either e_B or L_3 as ω-limit. This is not a trivial assertion since near the cycle L of $\partial\Delta$ the behaviour of ϕ_B is not clear.

6.5.2 <u>Proposition</u> Take $\bar{\varepsilon}$ and U as above. Fix $\varepsilon \in (0,\bar{\varepsilon})$. Then A_ε has a neighbourhood $N_2(\varepsilon)$ in M_4 such that $\forall\ B \in N_2$, $\forall\ x \in \mathring{\Delta}-U$, the positive ϕ_B-orbit of x intersects ∂U.

The proof of 6.5.2 will follow the idea for proof of 6.3.1, so we will refer to that proof very often, indicating the necessary adaptations.

But before we do this, we show, as a consequence, "stability" of ϕ_ε in $\mathring{\Delta}$, i.e.

6.5.3 <u>Proposition</u> For all $\varepsilon \in (0,\bar{\varepsilon})$ ($\bar{\varepsilon}$ as in 6.2.7) A_ε has a neighbourhood $N(\varepsilon)$ in M_4 such that $\forall B \in N$, there exists homeomorphism $h: \mathring{\Delta} \to \mathring{\Delta}$ (h depending on ε and B) such that h takes ϕ_B-orbits onto ϕ_ε-orbits.

<u>Proof</u> Take N_1 and N_2 as in 6.5.1 and 6.5.2 respectively. Let $N = N_1 \cap N_2$. $\forall B \in N \subset N_1$, by 6.5.1, $\exists\ h_1: \bar{U} \to \bar{U}$ taking ϕ_B-orbits to

ϕ_ε-orbits (restricted to $\bar{U}$). We now extend h_1 to $h:\mathring{\Delta} \to \mathring{\Delta}$.

By 6.5.2, since $B \in N \subset N_2$, $\forall\, x \in \mathring{\Delta}-U$ there is $t_x \geq 0$ s.t. $\phi_B(t_x,x) \in \partial U$. t_x is unique and depends continuously on x because ∂U is transversal to ϕ_B, by 6.5.1.

We define $h:\mathring{\Delta} \to \mathring{\Delta}$ by

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in \bar{U} \\ \phi_\varepsilon(-t_x, h_1\phi_B(t_x,x)) & \text{if } x \in \mathring{\Delta}-U \end{cases}.$$

See figure 35.

h is the required homeomorphism. □

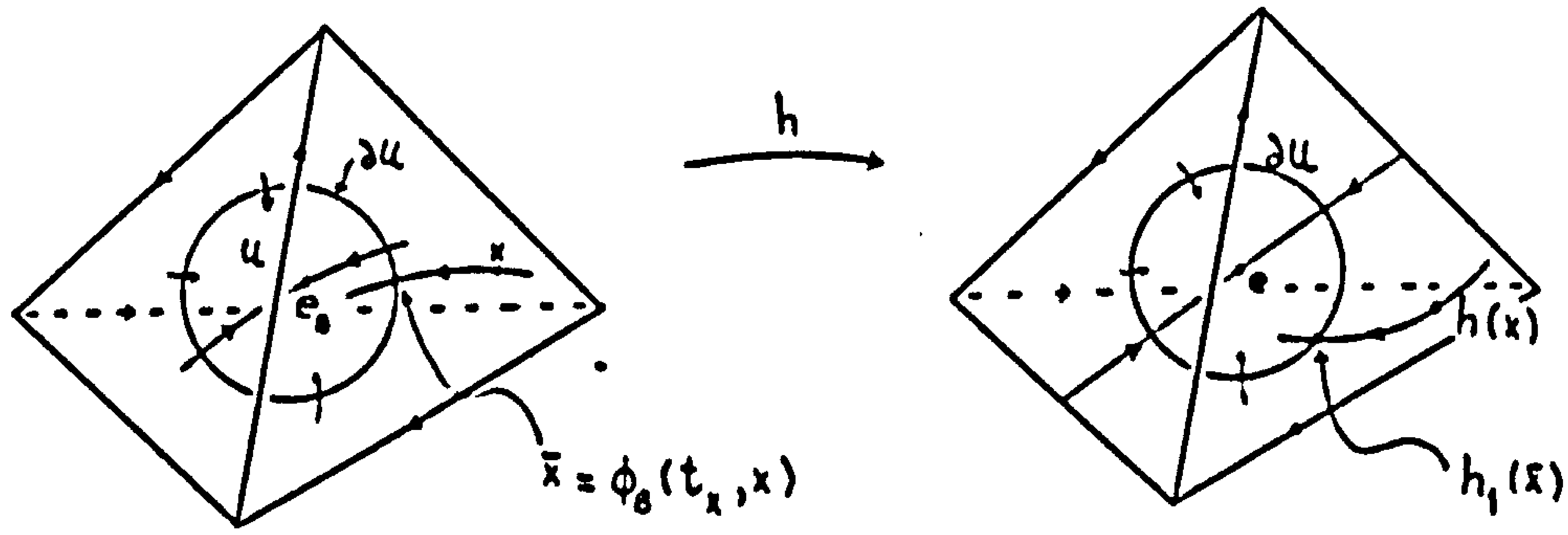


figure 35: construction of h in 6.5.3.

6.5.4 <u>Corollary</u> For any $\varepsilon',\varepsilon'' \in (0,\bar{\varepsilon})$, the restrictions to $\mathring{\Delta}$ of $\phi_{\varepsilon'}$ and $\phi_{\varepsilon''}$ are topologically equivalent.

<u>Proof</u> We can take $\varepsilon' = \varepsilon_0 < \varepsilon_1 <\ldots< \varepsilon_r = \varepsilon''$ such that $\forall\, j = 1,\ldots,r$ $\{A_\varepsilon;\varepsilon \in [\varepsilon_{j-1},\varepsilon_j]\} \subset N_j^1$ where N_j^1 is a neighbourhood of A_{ε_j} as in 6.5.3. Hence, in $\mathring{\Delta}$, $\phi_{\varepsilon'} = \phi_{\varepsilon_0} \sim \phi_{\varepsilon_1} \sim\ldots\sim \phi_{\varepsilon_r} = \phi_{\varepsilon''}$. □

In order to prove 6.5.2, we first prove some lemmas analogous to lemmas in 6.3.

Take $\beta \in (0,1)$ and N_β as in lemma 6.3.5, then take $s \in (0,1)$, $\varepsilon_1 > 0$ and N_β^s as in 6.3.6. We recall that properties (i)-(iii) of 6.3.5 hold in N_β^s for ϕ_ε if $0 \le \varepsilon \le \varepsilon_0$ (by choice of s in 6.3.6). Now we fix $\varepsilon \in (0,\bar{\varepsilon})$ (remember $\bar{\varepsilon} < \varepsilon_1 < \varepsilon_0$) .

6.5.5 <u>Lemma</u> A_ε has a compact neighbourhood $N_3(\varepsilon)$ in M_4 such that $\forall\, B \in N_3$, ϕ_B satisfies properties (i)-(iii) of 6.3.5 in N_β^s .

<u>Proof</u> ϕ_ε satisfies (i)-(iii) in N_β^s . By continuity, plus compactness of N_β^s , we will conclude that (i)-(iii) hold for all B sufficiently near A_ε as follows:

(i) if there were $B_n (B_n \to A_\varepsilon)$ with fixed point x_n in N_β^s any limit point x for (x_n) would be fixed for ϕ_ε ;

(ii) if there were $B_n (B_n \to A_\varepsilon)$ with $\mathcal{O}^+_{B_n}(x_n) \subset N_\beta^s$ (or $\mathcal{O}^-_{B_n}(x_n) \subset N_\beta^s$) , taking limit $x \in N_\beta^s$ for (x_n) , $\exists\, t_+, t_- > 0$ s.t. $\phi_\varepsilon(t_+,x)$, $\phi_\varepsilon(-t_-,x) \not\in N_\beta^s$. So, for all large n , $\phi_{B_n}(t_+,x_n), \phi_{B_n}(-t_-,x_n) \not\in N_\beta^s$ contradicting choice of B_n , x_n .

(iii) To take this property for all sufficiently near B we have to notice that B near A_ε (in metric of M_4) implies that associated vectorfields X_B and X_ε are C^k-near, for any $k \ge 0$. (We must use that X_B, X_ε are C^2-near at least on the finite pieces of orbits in N_β^s .) Suppose (iii) failed for B_n with $B_n \to A_\varepsilon$.

First, if, for ϕ_{B_n}, $\dot{y}(x_n) = \ddot{y}(x_n) = 0$, and $x_n \to x \in N^S$ then $\dot{y}(x) = \ddot{y}(x) = 0$ for ϕ_ε giving contradiction. Second, if, for ϕ_{B_n}, $\dot{y}(x_n) = \dot{y}(\bar{x}_n) = 0$ with $\bar{x}_n \in \sigma_{B_n}(x_n)$, for $x_n \to x$, $\bar{x}_n \to \bar{x}$, we would have, for ϕ_ε, either $\dot{y}(x) = \dot{y}(\bar{x}) = 0$ with $\bar{x} \in \sigma_\varepsilon(x)$ $\bar{x} \neq x$

or $\dot{y}(x) = \ddot{y}(x) = 0$ if $\bar{x} = x$

resulting in contradiction.

Therefore a neighbourhood N_3, as stated, must exist. □

6.5.6 Lemma For ε, s as above, $V(x) = 256\, x_0x_1x_2x_3$. There exists neighbourhood N_4 of A_ε s.t. $\forall\, B \in N_4$, $\forall\, x \in N^S - \partial\Delta$, $\exists\, x' \in \sigma_B^+(x)$ with $V(x') > V(x)$.

Proof As a first step we claim that for $A, B \in M_4$

$||A-B|| \leq \xi \Rightarrow |\, \dot{V}_A(x) - \dot{V}_B(x)\, | \leq 8\, \xi\, V(x)$ $\forall\, x \in \mathring{\Delta}$

(where $\dot{V}_A(x) = \frac{d}{dt} V(\phi_A(t,x))\Big|_{t=0} = \sum_{i=0}^{3} \frac{\partial V}{\partial x_i}(x) X_A^i(x)$ and similarly for $\dot{V}_B(x)$).

In fact, $X_A^i(x) - X_B^i(x) = x_i(((A-B)x)_i - x(A-B)x)$,

so $||A-B|| \leq \xi \Rightarrow |X_A^i(x) - X_B^i(x)| \leq 2\, \xi\, x_i$ $\forall\, x \in \Delta$.

Also $\frac{\partial V}{\partial x_i}(x) = \frac{V(x)}{x_i}$ $\forall\, x \in \mathring{\Delta}$, then

$$|\dot{V}_A(x) - \dot{V}_B(x)| = \Big|\sum_{i=0}^{3} \frac{V(x)}{x_i} (X_A^i(x) - X_B^i(x))\Big|$$

$$\leq V(x) \sum_{i=0}^{3} \frac{1}{x_i} 2\, \xi\, x_i = 8\, \xi\, V(x) .$$

Then $||A-B|| \leq \xi \Rightarrow \left|\frac{\dot{V}_A}{V}(x) - \frac{\dot{V}_B}{V}(x)\right| \leq 8\, \xi$ $\forall\, x \in \mathring{\Delta}$.

(Note that this property is not true in general.)

Now, taking $\alpha \in (0,\frac{1}{2}\beta)$ and $\varepsilon_1 < \min\{\varepsilon_0, \frac{1}{4}(\gamma-\delta)\alpha^2\}$ exactly as in 6.3.6, we have (notation as in 6.3.6)

$$\frac{1}{4}\frac{\dot{V}_A}{V}(x) = (\gamma-\delta+\varepsilon)y^2 - 2\varepsilon|z|^2 \qquad (\text{where } A = A_\varepsilon) \ .$$

Hence, for $x \in M^s_\alpha = N^s - N^s_\alpha \cup \partial\Delta \Rightarrow y^2 \geq \alpha^2$, $|z| \leq 1$

$\Rightarrow \frac{1}{4}\frac{\dot{V}_A}{V}(x) \geq \frac{1}{2}(\gamma-\delta)\alpha^2 + \varepsilon\alpha^2$ and, for $x \in N^s_\alpha - \partial\Delta$,

$$\frac{1}{4}\frac{\dot{V}_A}{V}(x) \geq -2\varepsilon = -2\varepsilon_1 + 2(\varepsilon_1-\varepsilon) > -\frac{1}{2}(\gamma-\delta)\alpha^2 + 2(\varepsilon_1-\varepsilon) \ .$$

If we take $0 < \xi < \min\{(\varepsilon_1-\varepsilon), \frac{1}{2}\varepsilon\alpha^2\}$, $\forall B$ with $||B-A|| \leq \xi$ we have

1) $\quad x \in M^s \Rightarrow \frac{1}{4}\frac{\dot{V}_B}{V}(x) \geq \frac{1}{4}\frac{\dot{V}_A}{V}(x) - 2\xi \geq \frac{1}{2}(\gamma-\delta)\alpha^2 + \varepsilon\alpha^2 - 2\xi$

$$> \frac{1}{2}(\gamma-\delta)\alpha^2 \ .$$

2) $\quad x \in N^s - \partial\Delta \Rightarrow \frac{1}{4}\frac{\dot{V}_B}{V}(x) \geq \frac{1}{4}\frac{\dot{V}_A}{V}(x) - 2\xi > -\frac{1}{2}(\gamma-\delta)\alpha^2 + 2(\varepsilon_1-\varepsilon-\xi)$

$$> -\frac{1}{2}(\gamma-\delta)\alpha^2 \ .$$

For any compact neighbourhood N_4 of $A = A_\varepsilon$, with $N_4 \subset N_3$ of 6.5.5, we let $\tilde{\ell}_\alpha$, $\tilde{v}_M$, $\tilde{v}_m$ denote, respectively, the length of the longest ϕ_B-orbit in N^s_α $\forall B \in N_4$, and the maximum and minimum speeds of ϕ_B-orbits in N^s_α $\forall B \in N_4$. If $B \to A$, $\ell_\alpha(B), v_M(B), v_m(B)$ will tend to ℓ_α, v_M, v_m (of 6.3.6) respectively. Since, by construction in 6.3.6 $\ell_\alpha < \frac{1}{2}\beta\, v_m/v_M$, we can take compact neighbourhood $N_4 \subset N_3$ such that $B \in N_4 \Rightarrow ||B-A|| \leq \xi$, so (1) and (2) hold $\forall B \in N_4$ and (3) $\tilde{\ell}_\alpha < \frac{1}{2}\beta\, \tilde{v}_m/\tilde{v}_M$ in N_4.

From this point, the proof continues exactly as the proof of 6.3.6 examining $x \in M_\alpha^S$ and three possible cases for $x \in N^S - \partial\Delta$. Hence we refer to that proof and say that the lemma is valid.

□

6.5.7 <u>Lemma</u> Take ε, s, N_4 as in 6.5.6 and $B^S = \mathrm{clos}(\Delta - N^S)$. Then $\forall\, B \in N_4$, any positive ϕ_B-orbit in $\overset{\circ}{\Delta}$ meets B^S.

<u>Proof</u> This is exactly like proof of 6.3.7, supposing, that for some $x \in \overset{\circ}{\Delta}$, $\mathcal{O}_B^+(x) \subset \mathrm{int}\, N^S$ and then using lemma 6.5.6 for $\bar{x} \in \omega_B(x)$ where $V\big|_{\omega_B(x)}$ is maximum.

<u>Proof of 6.5.2</u> Take N_4 and B^S as above. Take $T > 0$ as in proof of 6.3.1 i.e. $\phi_\varepsilon(T, B^S) \subset U$. Now we take neighbourhood N_2 of A_ε, $N_2 \subset N_4$ such that $\phi_B(T, B^S) \subset U \;\; \forall\, B \in N_2$.

But by 6.5.7, $\forall\, x \in \overset{\circ}{\Delta}$, $\exists\, t_x > 0$ with $\phi_B(t_x, x) \in B^S$, so $\phi_B(t_x + T, x) \in U$. Then, for $x \in \overset{\circ}{\Delta} - U$, $\mathcal{O}_B^+(x)$ must intersect ∂U. □

Now we look at case <u>$\varepsilon < 0$</u>. This is simpler.

6.5.8 <u>Proposition</u> For any $\varepsilon \in (-(\gamma-\delta), 0)$, A_ε has a neighbourhood N in M_4 such that $B \in N \Rightarrow \phi_B$ and ϕ_ε are topologically equivalent in $\overset{\circ}{\Delta}$.

<u>Proof</u> Fix ε $(-(\gamma-\delta) < \varepsilon < 0)$. Let $A = A_\varepsilon$. Take $V(x) = 256\, x_0 x_1 x_2 x_3$. Then

$$\frac{1}{4}\,\frac{\dot{V}_A}{V}(x) = (\gamma-\delta+\varepsilon)y^2 - 2\varepsilon|z|^2 > 0 \quad \forall\, x \in \overset{\circ}{\Delta} - e \;.$$

Take any $c \in (0,1)$ and $U = V^{-1}[0,c]$. U is neighbourhood of e and ϕ_ε is transversal to ∂U. We can take neighbourhood N_1 of A such that $B \in N_1$ implies

1) ϕ_B is transversal to ∂U .

2) ϕ_B has, inside U , one unique hyperbolic attracting point e_B and all ϕ_B-orbits in U-e_B have e_B as ω-limit and cross ∂U (transversally) in negative time. So, it is standard procedure to construct homeomorphism $h_1:\bar{U} \to \bar{U}$ taking ϕ_B-orbits to ϕ_ε-orbits.

To extend h_1 to $h:\dot{\Delta} \to \dot{\Delta}$, we prove that $x \in \overset{\circ}{\Delta}\text{-}U \Rightarrow \theta_B^+(x)$ meets ∂U, $\forall$ $B \in N \subset N_1$.

Let $\rho > 0$ be such that $\{x \in \overset{\circ}{\Delta}; y^2 + |z|^2 < \rho^2\} \subset U$. Take $K_\varepsilon = \min\{-2\varepsilon, (\gamma-\delta+\varepsilon)\} > 0$. Then, $\forall$ $x \in \overset{\circ}{\Delta}\text{-}U$

$$\tfrac{1}{2}\,\frac{\dot{V}_A}{V}(x) = (\gamma-\delta+\varepsilon)y^2 - 2\varepsilon|z|^2 \geq K_\varepsilon\rho^2 > 0 \ .$$

Taking $0 < \xi < \tfrac{1}{2}K_\varepsilon\rho^2$, we have $\forall$ $x \in \overset{\circ}{\Delta}\text{-}U$

$$||B-A|| \leq \xi \Rightarrow \tfrac{1}{2}\,\frac{\dot{V}_B}{V}(x) \geq \tfrac{1}{2}\,\frac{\dot{V}_A}{V}(x) - 2\xi \geq K_\varepsilon\rho^2 - 2\xi > 0 \ .$$

Letting $N \subset N_1$ be such that $B \in N \Rightarrow ||B-A|| \leq \xi$ we get $\dot{V}_B(x) > 0$ $\forall$ $x \in \overset{\circ}{\Delta}\text{-}U$, $\forall$ $B \in N$. i.e. function V is strictly increasing along ϕ_B-orbits in $\overset{\circ}{\Delta}\text{-}U$, and this implies that these will, in positive time, enter ∂U, crossing ∂U in finite time.

Now $h:\overset{\circ}{\Delta} \to \overset{\circ}{\Delta}$ can be defined as in 6.5.3, completing the proof. □

6.5.9 <u>Corollary</u> If $\varepsilon', \varepsilon'' \in (-(\gamma-\delta), 0)$, the restrictions to $\overset{\circ}{\Delta}$ of $\phi_{\varepsilon'}$ and $\phi_{\varepsilon''}$ are topologically equivalent.

<u>Proof</u> Analogous to 6.5.4.

6.5.10 <u>Conclusion</u> Propositions 6.5.3 and 6.5.8, plus stability on the 2-dimensional faces of Δ , make us believe that, apart from the difficulty in effectively constructing a global equivalence $h:\Delta \to \Delta$, A_ε must be stable for small $\varepsilon \neq 0$, so justifying our conjecture in the beginning of this section.

CHAPTER 7

FURTHER RESULTS AND COMMENTS ON FUTURE WORK

7.1 Introduction

As we noted in the general introduction, we have, in this work, completed, for $n = 2$, the classification of the replicator equations $\dot{x}_i = x_i((Ax)_i - xAx)$ $i = 0,\ldots,n$ (as in 1.1), by means of Theorems I, 1.4.1, and III, 1.5.8.

The next step would be to try a classification for $n = 3$. This is not yet possible, since many questions, related to the problem, remain unanswered. Some of these questions are presented in 7.2 next. In 7.4, we discuss some cases where such questions can be answered easily, but we do not intend to present any major results. The cases studied in 7.4 are chosen by picking, among the 114 existing combinatorial classes (up to flow reversal) in Z_4^+, those classes where one vertex strictly dominates another vertex (see [1], or 2.3.4). This domination implies that all orbits in $\mathring{\Delta}$ flow from one face to another (see 2.3.6). In order to specify these chosen classes in 7.3, we discuss a method of finding, for any n, all combinatorial classes of Z_{n+1}^+.

7.2 Questions

We know that if $A \in M_{n+1}$ is stable (by definition 1.3.3) we have that all fixed points for ϕ_A are hyperbolic (Theorem II, 1.5.2). Also, each fixed point must be unique in the interior of its face (see 1.5.1(4)). For $n = 2$, ϕ_A has no periodic orbits. Now we ask:

7.2.1 Question For $A \in M_{n+1}$ $n \geq 3$, stable, are periodic orbits of ϕ_A : (i) isolated? (ii) unique in the interior of its face? (like fixed points) (iii) hyperbolic?

With respect to (iii) in question above, we note that hyperbolicity of periodic orbits is a necessary condition for structural stability of a flow among all flows ([25], [26]). However, the condition might no longer be necessary if, instead of all flows, we consider only a particular family, as we had done in this work, by considering the family of flows ϕ_A as in 1.1.

But, even in this very restricted family, we saw (Theorem II) that hyperbolicity of fixed points keeps being a necessary condition for stability. Does it hold for periodic orbits? We conjecture that it does. If this is so, (i) is always answered. But uniqueness as in (ii) is not clear and we may ask:

7.2.2 Question For $A \in M_{n+1}$ $(n \geq 3)$ stable, if ϕ_A has periodic orbit in $\overset{\circ}{\Delta}$, how many can it have?

Now, we note that among the stable classes of M_3 (as in Theorem I, 1.4.1), some present saddle connections (which in dimension 2 cannot be transversal) i.e. $\exists$ fixed $p,q \in \Delta$ with $W^s p$ and $W^u q$ intersecting not transversally. But we also note that all these non-transversal saddle connections occur on $\partial\Delta$, and, still, when restricted to interior of faces, they are transversal. We would like to know if this property is still true when $n \geq 3$. Since A stable implies that its restriction to each face is also stable, it is enough to ask:

7.2.3 Question For $n \geq 3$. $A \in M_{n+1}$ stable. Suppose $p,q \in \Delta$ are fixed points for ϕ_A and $x \in W^s p \cap W^u q \cap \overset{o}{\Delta}$. Do $W^s p$ and $W^u q$ meet transversally along $\mathcal{O}(x)$?

It is also a well-known (e.g. [25], [26]) necessary condition for structural stability of flows that all saddle connections are transversal. The usual way to prove this is to make small local perturbations on the associated vectorfield near a point of non-transversal intersection. However, inside our family A_n of flows ϕ_A , $A \in M_{n+1}$, all perturbations of ϕ_A must be made perturbing matrix A . This implies that these perturbations are not local. In fact, any perturbation (even at just one element of A) perturbs vectorfield X_A at all points in $\overset{o}{\Delta}$, and also at all points in some of the faces of $\partial\Delta$. This makes question 7.2.3 more difficult to answer. Still, we conjecture that 7.2.3 has affirmative answer.

7.2.4 Questions Let $n \geq 3$ and $A \in M_{n+1}$.

(i) Are there strange attractors for ϕ_A ?

(ii) Can these occur for stable A ?

Arneodo, Coullet and Tresser gave in [4] a 1-parameter family of Lotka-Volterra equations with $n = 3$ for which computer drawings of the associated flow are presented for several values of the parameter. The family goes through a bifurcation of Hopf type, then, so it seems, through a series of doubling-period bifurcations after which the drawings seem to indicate the existence of a "strange attractor" in $\mathbb{R}^3_+$. This example was later generalised for $n \geq 3$ in [5]. We saw, in 2.5, that Lotka-Volterra equations in $\mathbb{R}^n_+$ are equivalent to our replicator equations in Δ minus a (n-1)-face. So, it seems that the answer to 7.2.4(i) is yes. However, this example in [4] would need a better mathematical understanding if we want to prove there is really a strange attractor or if we want to answer (ii).

These are some of the questions that need to be answered before a classification of stable classes in A_n, $n \geq 3$, is attempted. For the future, we plan to look at these questions in order to try a classification for $n \geq 3$.

Before this can be done, the most we can do is the study of some chosen particular cases, or, at most, families of cases. This is exactly what we present next. In 7.3 we discuss how to obtain, for each n, a full list of all combinatorial classes of Z^+_{n+1} (like in 1.5.7 for $n = 2$). Then, in 7.4 we choose some of these classes to study.

7.3 Combinatorial classes

In [41], Zeeman indicated the existence of 114 combinatorial classes in Z^+_4 up to sign reversal. His method of obtaining these classes (personally explained) is geometric. It consists of finding all possible combinations of configurations on the edges of Δ, then checking which are equivalent by permutation of vertices. (Each edge having either an arrow, a solid dot, or an open dot, representing edge, respectively, without fixed point, with an attractor, or with a repellor).

This geometric method would not be very suitable for higher dimensions. We found the same combinatorial classes by the more algebraic method of looking at A as a matrix with zeros on the diagonal and $+$ and $-$ signs off the diagonal, and then checking equivalence by permutation.

For $n = 3$, i.e. $A \in Z_4^+$, this consists of finding the number (denoted by N_s) of combinatorial classes when A has s minus signs. ($s = 0,1,\dots,12$.) By reversing signs $N_s = N_{12-s}$, so it is sufficient to look at $s = 0,1,\dots,6$. We get:

s	0	1	2	3	4	5	6	7	8	9	10	11	12
N_s	1	1	5	13	27	38	48	38	27	13	5	1	1

Among the 48 comb. classes for $s = 6$ it is easy to check (having the explicit list) that 10 are equivalent to their own reversals, 38 are equivalent to the reversal of another. This gives, for $s = 6$, 29 comb. classes up to sign reversal.

Therefore, up to sign reversal, we have $114 = \sum_{s=0}^{5} N_s + 29$ combinatorial classes in Z_4^+.

7.4 Study of some combinatorial classes of Z_4^+

As we said in 7.1 we will now choose among the combinatorial classes of Z_4^+ those where one vertex strictly dominates another vertex (by definition in [1] or 2.3.4) for all matrices in that class. If this must happen independently of the actual values for the elements in the matrices we must have $i_0, i_1 \in \{0,\dots,3\}$, $i_0 \neq i_1$, such that $a_{i_0 j} \le 0 \le a_{i_1 j}$ $j = 0,\dots,3$ for all $A = (a_{ij})$ in the class. i.e. A must have one positive row and one negative row (off the diagonal). By permutation we can suppose that $i_0 = 0$, $i_1 = 1$.

Among the 114 combinatorial classes (up to sign reversal) of Z_4^+ we find exactly 18 classes satisfying this condition. We will call these D_k , $k = 1,\dots,18$.

For $A \in \cup D_k$, ϕ_A has no fixed point (hence no-nonwandering point) in $\overset{\circ}{\Delta}$, by Theorem 2.3.1 plus 2.3.6 Step 3. We note that in all other combinatorial classes of Z_4^+ , there are some matrices A for which ϕ_A has fixed point in $\overset{\circ}{\Delta}$.

What we now intend to do with $D_1,\dots,D_{18}$ is similar to what we have done with the combinatorial classes $C_1,\dots,C_{10}$ of Z_3^+ in 3.3, i.e., we will take in each D_k a dense subset $\tilde{D}_k$ imposing condition of hyperbolicity of fixed points. Then $\tilde{D}_k$ will be the union of subsets D_k^r , so that each stable class in Z_4^+ must be contained in one of the subsets D_k^r .

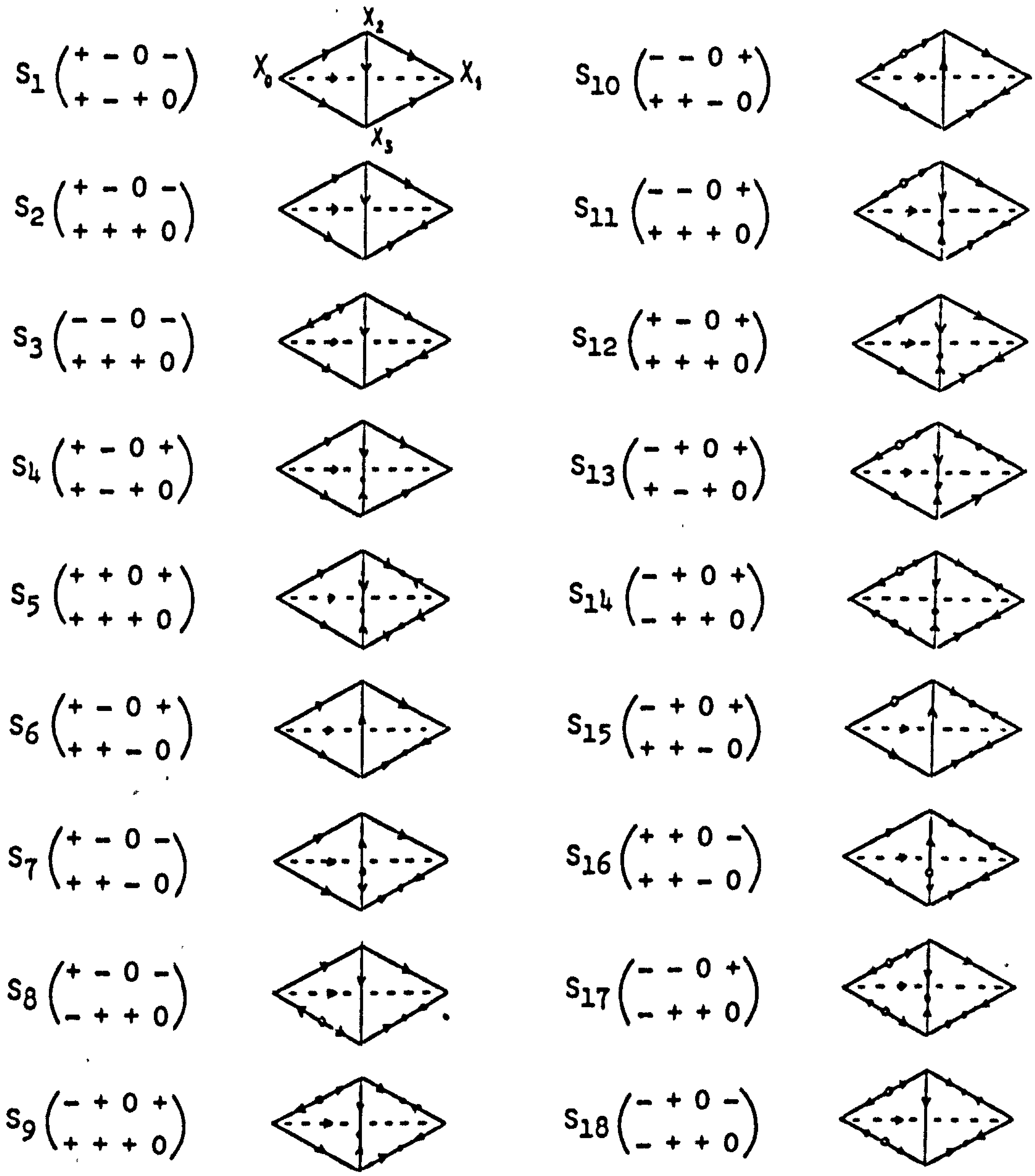
This property was announced as Theorem VI in 1.8.1, which we prove here in 7.4.5.

For $k = 1,\dots,5$ we will prove (in 7.4.8) that, in fact, the subsets D_k^r are the stable classes in D_k . For $6 \le k \le 18$ we believe that the subsets D_k^r are also stable classes, but we will leave this as a conjecture, for future work. The difficulty in proving this property is that the phase diagrams for ϕ_A , $A \in D_k^r$, $k = 6,\dots,18$, may present non-transversal intersections of invariant manifolds for saddles. For most cases, however, these intersections will occur only on $\partial\Delta$, will be robust, and of transversal type when restricted to the interior of the p-face where they occur.

So, we think that, in the future, we can produce a technique of classification of quasi-gradient flows (definition 4.1.2) for dimension 3, similar to the technique in Chapter 4, for dimension 2. We note that Fleitas [10] has also given a classification for gradient-like flows in dimension 3, by means of "Heegard diagrams" on small spheres around the attractors. If we limit the types of non-transversal saddle connections that may occur, we will probably be able to adapt Fleitas' technique for quasi-gradient flows, so that it can be applied to flows ϕ_A for $A \in D_k^r$. That we plan to try in the future. If we succeed, this technique can be applied to a much wider range of cases.

In what follows, we give, in 7.4.1, the combinatorial classes D_k, then we prove, in 7.4.2 to 7.4.4, some properties of ϕ_A for $A \in uD_k$, describe subsets D_k^r in 7.4.5, give phase diagrams in 7.4.7 (figure 38), and finally in 7.4.8 we show that some of these are stable classes of Z_4^+.

7.4.1 <u>The classes D_k</u> Here we give, in figure 36, each D_k by a representing matrix S_k of signs, plus a drawing of the flow on the edges of Δ. We suppose S_k has first row negative, second row positive, hence we indicate only the signs of third and fourth row of S_k.



$S_1\begin{pmatrix} + & - & 0 & - \\ + & - & + & 0 \end{pmatrix}$ $\quad$ $S_{10}\begin{pmatrix} - & - & 0 & + \\ + & + & - & 0 \end{pmatrix}$

$S_2\begin{pmatrix} + & - & 0 & - \\ + & + & + & 0 \end{pmatrix}$ $\quad$ $S_{11}\begin{pmatrix} - & - & 0 & + \\ + & + & + & 0 \end{pmatrix}$

$S_3\begin{pmatrix} - & - & 0 & - \\ + & + & + & 0 \end{pmatrix}$ $\quad$ $S_{12}\begin{pmatrix} + & - & 0 & + \\ + & + & + & 0 \end{pmatrix}$

$S_4\begin{pmatrix} + & - & 0 & + \\ + & - & + & 0 \end{pmatrix}$ $\quad$ $S_{13}\begin{pmatrix} - & + & 0 & + \\ + & - & + & 0 \end{pmatrix}$

$S_5\begin{pmatrix} + & + & 0 & + \\ + & + & + & 0 \end{pmatrix}$ $\quad$ $S_{14}\begin{pmatrix} - & + & 0 & + \\ - & + & + & 0 \end{pmatrix}$

$S_6\begin{pmatrix} + & - & 0 & + \\ + & + & - & 0 \end{pmatrix}$ $\quad$ $S_{15}\begin{pmatrix} - & + & 0 & + \\ + & + & - & 0 \end{pmatrix}$

$S_7\begin{pmatrix} + & - & 0 & - \\ + & + & - & 0 \end{pmatrix}$ $\quad$ $S_{16}\begin{pmatrix} + & + & 0 & - \\ + & + & - & 0 \end{pmatrix}$

$S_8\begin{pmatrix} + & - & 0 & - \\ - & + & + & 0 \end{pmatrix}$ $\quad$ $S_{17}\begin{pmatrix} - & - & 0 & + \\ - & + & + & 0 \end{pmatrix}$

$S_9\begin{pmatrix} - & + & 0 & + \\ + & + & + & 0 \end{pmatrix}$ $\quad$ $S_{18}\begin{pmatrix} - & + & 0 & - \\ - & + & + & 0 \end{pmatrix}$

figure 36: The combinatorial classes D_k , $1 \le k \le 18$, represented by matrices of signs S_k , plus a drawing of the flow on edges.

For the next lemmas we recall that we denote by F_i the face $x_i = 0$ and by A_i the matrix obtained from A by eliminating row i and column i . So $X_{A_i} = X_A\big|_{F_i}$.

7.4.2 <u>Lemma</u> Let $A = (a_{ij}) \in Z_4^+$ have $a_{0j} < 0$ $(j \neq 0)$ and $a_{1j} > 0$ $(j \neq 1)$.

(i) Taking $V: \Delta - F_0 \cup F_1 \to \mathbb{R}$; $V(x) = x_0/x_1$, then $\dot{V}(x) < 0$ (i.e. V is strictly decreasing on orbits).

(ii) $\forall\, x \in \Delta - F_0 \cup F_1 \Rightarrow \omega(x) \subset F_0$, $\alpha(x) \subset F_1$.

(iii) $\Omega_A \subset F_0 \cup F_1$.

(iv) If X_A has fixed point $p \in F_0 - X_2X_3$ (resp. $p \in F_1 - X_2X_3$) then the eigenvalue λ of p in direction transversal to F_0 (resp. F_1) is negative (resp. positive).

(v) If $p \in \overset{\circ}{X_2X_3}$ is fixed, p is either hyperbolic attractor on F_1 (if $a_{23}, a_{32} > 0$) or hyp. repellor on F_0 (if $a_{23}, a_{32} < 0$) .

(vi) A_2 and A_3 are stable in Z_3^+ (in classes (2), (3) or (8) of Theorem I).

(vii) A_0 and A_1 are not in combinatorial classes C_1 or C_7 , and, therefore, they are stable $\Leftrightarrow$ fixed points on edges of Δ are hyperbolic.

<u>Proof</u> (i) can be taken either directly by differentiating $V(x)$ and using hypothesis on signs of A , or from Step 3 of 2.3.6 with $q^+ = X_1$, $q^- = X_0$. (ii) and (iii) are consequences of (i). (iv) $p \in F_0 - X_2X_3 \Rightarrow p = (0, x_1, x_2, x_3)$ with $x_1 > 0$, $x_2, x_3 \geq 0$ and $x_1 + x_2 + x_3 = 1$. By corollary 2.4.3, the eigenvalue transversal to F_0 , at p , is $\lambda = (Ap)_0 - (Ap)_1$. But $(Ap)_0 < 0$ and $(Ap)_1 > 0 \Rightarrow \lambda < 0$.

For $p \in F_1 - X_2X_3$, proof is analogous.

(v) $p \in \overset{\circ}{X_2X_3} \Rightarrow p = (0,0,x_2,x_3)$ with $x_2, x_3 > 0$, $x_2 + x_3 = 1$.

Denoting by λ_0 (λ_1) the eigenvalue at p in direction transversal to F_0 (F_1 , resp.), we get (by 2.4.3) $\begin{cases} \lambda_0 = (Ap)_0 - (Ap)_2 = a_{02}x_2 + a_{03}x_3 - a_{23}x_2 \\ \lambda_1 = (Ap)_1 - (Ap)_2 = a_{12}x_2 + a_{13}x_3 - a_{23}x_2 \end{cases}$.

So, $a_{23} > 0 \Rightarrow \lambda_0 < 0$, and $a_{23} < 0 \Rightarrow \lambda_1 > 0$.

(vi) follows because both A_2 and A_3 have one negative row and one positive row (off the diagonal) so they must belong to C_2 , C_3 or C_8 of 1.5.7 and by 3.3.1, plus 4.8 we have $C_2 = (2)$, $C_3 = (3)$, $C_8 = (8)$. (See also Theorem III.)

(vii) follows because A_0 and A_1 have at least one row either positive or negative, so $A_0, A_1 \notin C_1$ or C_7 .

Other part results from study in 3.3.

□

7.4.3 <u>Remark</u> Properties above mean that all orbits in $\Delta - F_0 \cup F_1$ must start on F_1 , end on F_0 crossing transversally all planes $x_0/x_1 = c$. (These planes contain edge X_2X_3 and the point $q_c = (c,1,0,0)/(c+1)$ of edge X_0X_1 .

See figure 37.

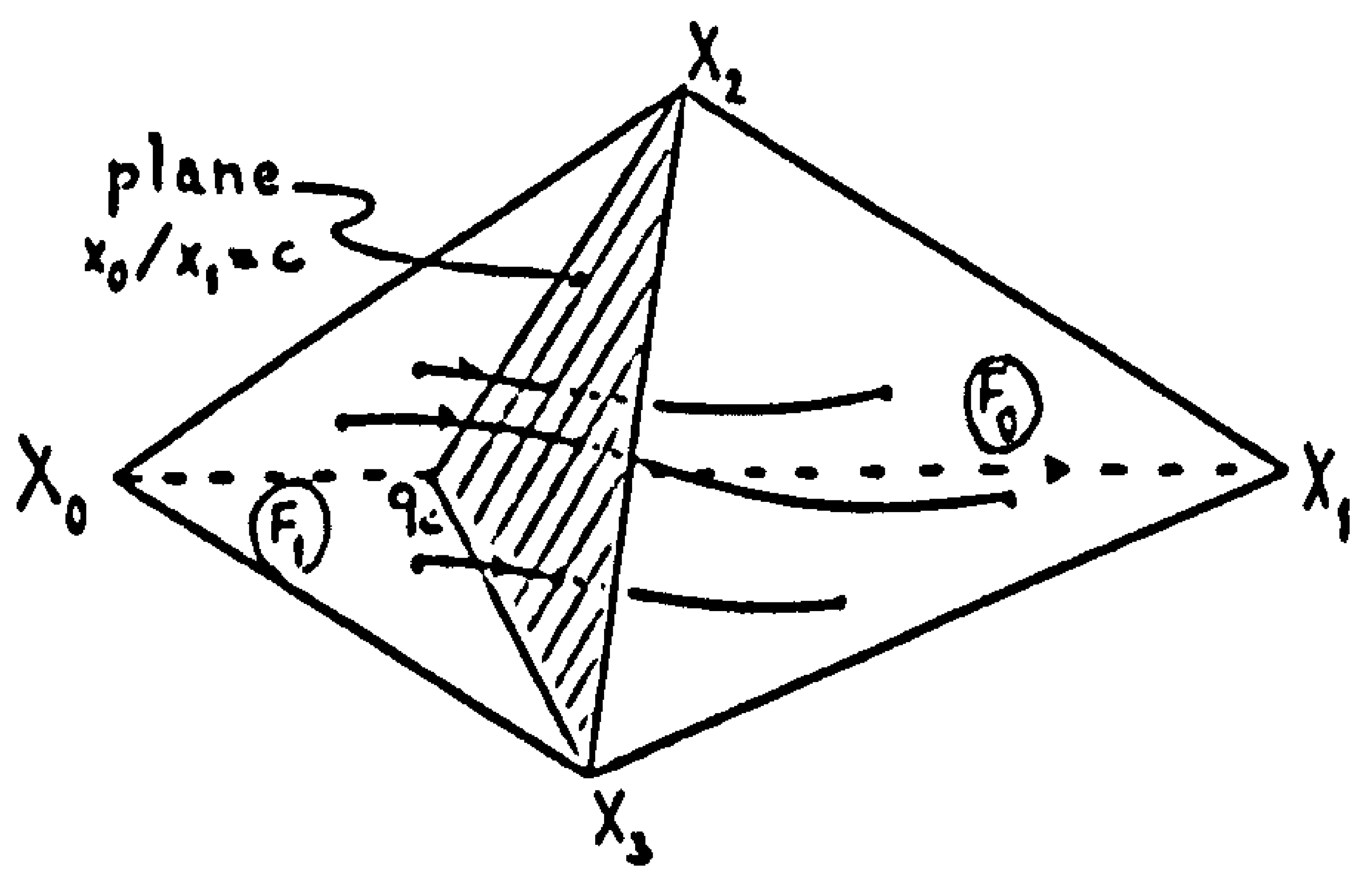


figure 37: orbits of ϕ_A , $A \in \cup D_k$ going from F_1 to F_0 .

Now, for each $k = 1,2,\dots,18$ we take $\tilde{D}_k = \{A \in D_k$; all fixed points of ϕ_A, on edges of Δ, are hyperbolic$\}$.

Also, let $\tilde{D} = \bigcup_{k=1}^{18} \tilde{D}_k$.

7.4.4 <u>Lemma</u> Let $A \in \tilde{D}$ with signs as in 7.4.2. Then:

(i) Fixed points in $\overset{o}{F}_0$ and $\overset{o}{F}_1$ are hyperbolic, if they exist, and $p \in F_0 - X_2X_3$ (resp. $p \in F_1 - X_2X_3$) fixed is attractor, saddle or repellor for ϕ_{A_0} (resp. ϕ_{A_1}) $\Leftrightarrow$ p is, for X_A, attractor, 2-saddle or 1-saddle (resp. 2-saddle, 1-saddle or repellor).

(ii) $\Omega_A = \mathrm{Fix}\ \phi_A$.

<u>Proof</u> (i) follows from 7.4.2(iv) and (vii), noting that $A_0(A_1) \in Z_3^+ - C_1 \cup C_7$, and, so, fixed points on edges being hyperbolic $\Rightarrow$ fixed point in F_0 (F_1) must also be hyperbolic (see Theorem III or Proof of 3.3.1).

(ii) we have, by 7.4.2 (iii) that $\Omega_A \subset F_0 \cup F_1$. Suppose $x \in F_0$ is not fixed. Examining all possibilities for stable classes for A_0 $(\notin C_1 \cup C_1)$ in Z_3^+ (in Theorem I) we have two cases.

Case 1: $\omega(x) = p$ where p is attractor for ϕ_{A_0} (hence, also for ϕ_A, by (i) above) hence $x \notin \Omega_A$.

Case 2: $\omega(x) = s$ where s is saddle for ϕ_{A_0} (hence s is 2-saddle for ϕ_A) and x has a neighbourhood $\mathcal{U}_0$ on F_0 s.t. $x' \in \mathcal{U}_0 - \mathcal{O}(x) \Rightarrow \omega(x')$ is an attractor for ϕ_{A_0} (hence also for ϕ_A) .

We can then conclude that x has a neighbourhood U in Δ s.t. $x' \in U \Rightarrow$ either $\omega(x') = s$ or $\omega(x')$ is an attractor for ϕ_A. Since ϕ_A has no cycle of saddles, x must be a wandering point, i.e. $x \notin \Omega_A$.

Analogously, $x \in F_1$ not fixed $\Rightarrow x \notin \Omega_A$.

Therefore $\Omega_A = \mathrm{Fix}\ \phi_A$. □

Now in our next proposition, we divide each $\tilde{D}_k$ in open subsets D_k^r.

7.4.5 Proposition $\tilde{D}_k$ is open and dense in D_k and $A \in D_k - \tilde{D}_k \Rightarrow A$ is not stable. Also $\tilde{D}_k = D_k^1 \cup \ldots \cup D_k^{\bar{r}(k)}$ where:

(1) each D_k^r is open;

(2) $A, B \in D_k^r \Rightarrow \phi_A$ and ϕ_B have isomorphic phase diagrams and are topologically equivalent on the 2-faces of Δ;

(3) each stable class in D_k is contained in D_k^r for some $r = 1, \ldots, \bar{r}(k)$;

(4) number $\bar{r}(k)$ of subsets, as above, for each k is:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\bar{r}(k)$	1	1	1	2	2	2	2	2	3	4	4	4	4	7	8	4	12	16

but up to flow reversal $\bar{r}(10)$ and $\bar{r}(18)$ can be reduced, respectively, to 3 and 10.

Proof $\tilde{D}_k$ is clearly open by its definition. Also, since ϕ_A, $A \in D_k$, has a finite number of fixed points on the edges (in fact, at most 5), $\tilde{D}_k$ is dense in D_k.

In what follows, whenever we take $A \in \tilde{D}_k$ we will assume that A is in the sign class represented by S_k, of figure 36, in order to specify stable classes for each A_i, $i = 0,\dots,3$.

By 7.4.4(i), $A \in \tilde{D} \Rightarrow$ all fixed points of ϕ_A are hyperbolic. Then $A \in D_k - \tilde{D}_k \Rightarrow \phi_A$ has non-hyperbolic fixed point $\Rightarrow$ A is not stable (by Theorem II, 1.5.2).

To describe subsets D_k^r, in a way that (1)-(4) are satisfied, we have to study sets D_k case by case, much as we did with $C_1,\dots,C_{10}$ in the proof of 3.3.1. Since we have here too many cases we will present below some, but not all, of them. For the ones we present, we characterize subsets D_k^r by giving the stable classes of A_0,A_1,A_2,A_3 in Z_3^+. In 7.4.7, figure 38, we also give the phase diagrams for these cases. All other cases can be treated similarly.

<u>Case 1</u> $A \in \tilde{D}_1 = D_1$. There are no fixed points on the edges, or on the faces. Also $A_0,A_1,A_2,A_3 \in (2)$. We take $D_1^1 = D_1$.

<u>Case 2</u> $A \in \tilde{D}_2 = D_2$. $A_0,A_2 \in (3)$, $A_1,A_3 \in (2)$. We take $D_2^1 = D_2$.

<u>Case 3</u> $A \in D_3 = \tilde{D}_3$. $A_0,A_2 \in (3)$, $A_1,A_3 \in -(3)$. We take $D_3^1 = D_3$.

<u>Case 4</u> $A \in D_4$. $A_1 \in (3)$, $A_2,A_3 \in (2)$, $A_0 \in C_4$. Take:

$A \in D_4^1 \Leftrightarrow A_0 \in (4_1)$, $A \in D_4^2 \Leftrightarrow A_0 \in (4_2)$.

<u>Case 5</u> $A \in D_5$. $A_1,A_2,A_3 \in (3)$, $A_0 \in C_{10}$. Take:

$A \in D_5^1 \Leftrightarrow A_0 \in (10_1)$; $A \in D_5^2 \Leftrightarrow A_0 \in (10_2)$.

<u>Case 6</u> $A \in D_6$. $A_1,A_3 \in (2)$, $A_2(3)$, $A_0 \in C_5$. Take:

$A \in D_6^1 \Leftrightarrow A_0 \in (5_1)$; $A \in D_6^2 \Leftrightarrow A_0 \in (5_2)$.

<u>Case 7</u> $A \in D_7$. $A_0 \in (8)$, $A_2 \in (3)$, $A_3 \in (2)$, $A_1 \in -C_4$. Take:

$A \in D_7^1 \iff A_1 \in -(4_1)$; $A \in D_7^2 \iff A_1 \in -(4_2)$

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<u>Case 9</u> $A \in D_9$. $A_1, A_3 \in (8)$, $A_2 \in (3)$, $A_0 \in C_{10}$. Take

$A \in D_9^1 \iff A_0 \in (10_1)$

$A \in D_9^2 \iff A_0 \in (10_2)$ with attractor on X_1X_3

$A \in D_9^3 \iff A_0 \in (10_2)$ with attractor on X_1X_2 or X_2X_3

<u>Case 10</u> $A \in D_{10}$. $A_2 \in (3)$, $A_3 \in -(3)$, A_0 , $-A_2 \in C_5$. Take:

$A \in D_{10}^1 \iff A_0, -A_1 \in (5_1)$; $A \in D_{10}^2 \iff A_0, -A_1 \in (5_2)$

$A \in D_{10}^3 \iff A_0 \in (5_1)$, $A_1 \in -(5_2)$; $A \in D_{10}^4 \iff A_0 \in (5_2)$, $A_1 \in -(5_1)$

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<u>Case 16</u> $A \in D_{16}$. $A_2, A_3 \in (3)$, $A_0 \in C_9$, $A_1 \in -C_4$. Take:

$A \in D_{16}^1 \iff A_0 \in (9_1)$, $A_1 \in -(4_1)$; $A \in D_{16}^2 \iff A_0 \in (9_2)$, $A_1 \in -(4_1)$

$A \in D_{16}^3 \iff A_0 \in (9_1)$, $A_1 \in -(4_2)$; $A \in D_{16}^4 \iff A_0 \in (9_2)$, $A_1 \in -(4_2)$

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All other cases are similar.

Phase diagrams are presented in figure 38. Then (1) and (2) hold. That each stable class in D_k is contained in one D_k^r follows from the fact that $A \in C_k^{r_1}$, $B \in C_k^{r_2}$, $r_1 \neq r_2 \Rightarrow \phi_A$ and ϕ_B have phase diagrams not isomorphic. Numbers $\bar{r}(k)$ in (4) are given during definition of subsets D_k^r . We note that $A \in D_{10}^4 \iff -A \in D_{10}^3$ hence, up to flow reversal, we can reduce $\bar{r}(10)$ to 3 (Case D_{18} is similar).

This concludes the proposition. □

7.4.6 <u>Remark</u> Among the cases, we have included above Cases 1 to 5 because for these we will prove in 7.4.8 that subsets D_1, D_2, D_3, D_4^1, D_4^2, D_5^1, D_5^2 are stable classes of Z_4^+. Cases 6 and 7 are the first ones in the list where non-transversal saddle connections occur. Case 9 is one where subsets D_9^r are not determined just by given stable classes on the faces. Case 10 has D_{10}^3 and D_{10}^4 equivalent by flow reversal. Case 16 is the first in the list where there exist a possibility of saddles occurring in the interior of both faces F_0 and F_1. So, for $A \in D_{16}^1$, we have a connected between such saddles, and this connection is transversal in $\mathring{\Delta}$. Similar saddle connections in $\mathring{\Delta}$ occur for $k = 17$ and 18. Transversality of these seems natural, but, mainly in D_{18}, it is not geometrically clear. This problem is related to question 7.2.3.

7.4.7 <u>Phase diagrams</u>

In the diagrams presented in figure 38 below we assume that $A \in$ sign class S_k as in 7.4.1. We indicate that fixed points have the same topological type (repellor, 1-saddle, 2-saddle or attractor) by putting them on a same horizontal level, with repellors at the top. We indicate, in the diagrams, the inset for 1-saddles, the outset for 2-saddles, plus all the saddle connections. These are indicated by arrows $\longrightarrow$ or $\multimap$ if they are transversal, or not, respectively.

If we were going to deal with top. equivalence between two cases with non-transversal saddle connections, it would also be necessary to indicate the type of non-transversality occurring. But we will not deal with this problem, though we have asserted that we plan to work on this problem, in the future.

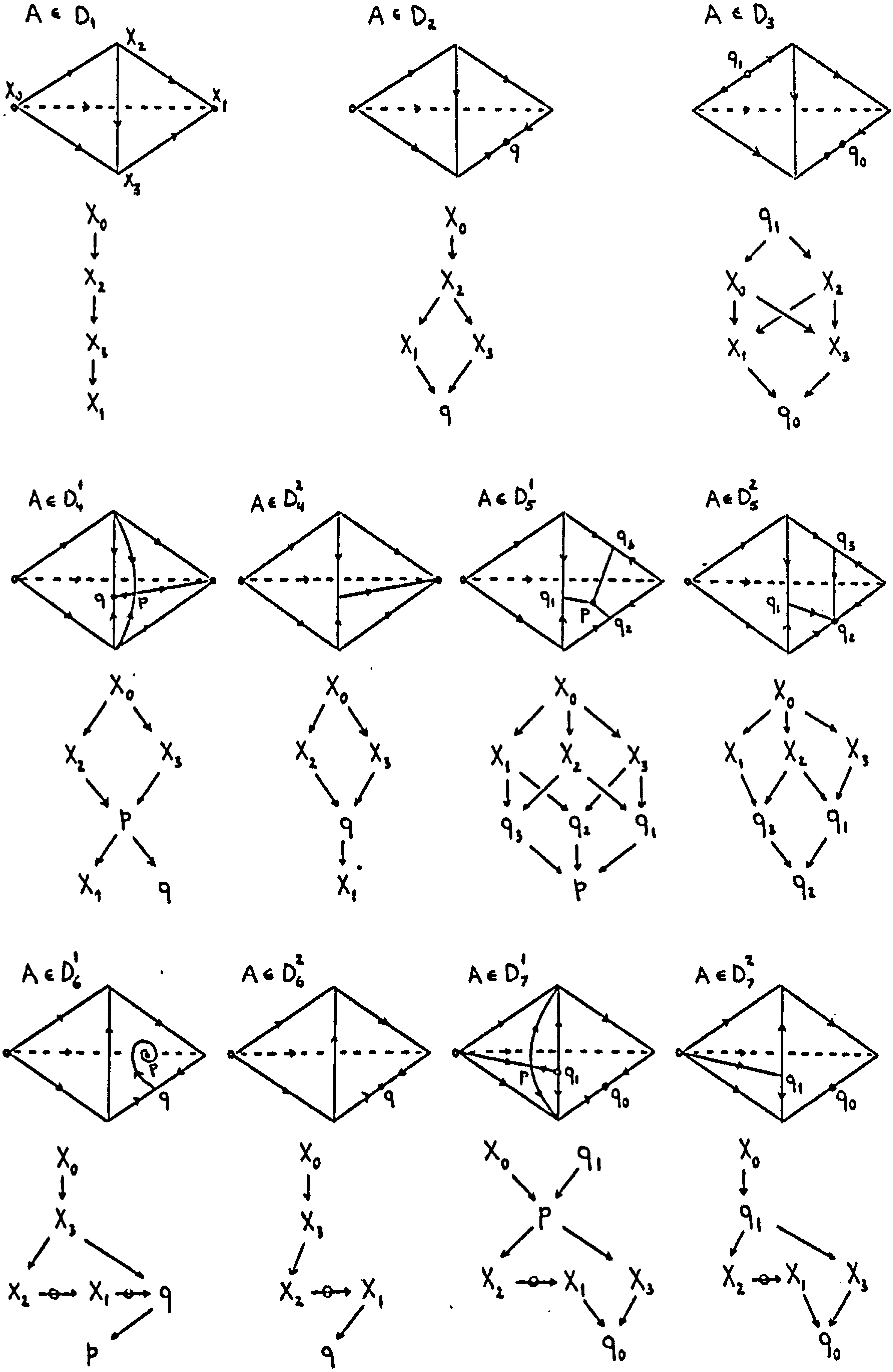

figure 38: Phase diagrams

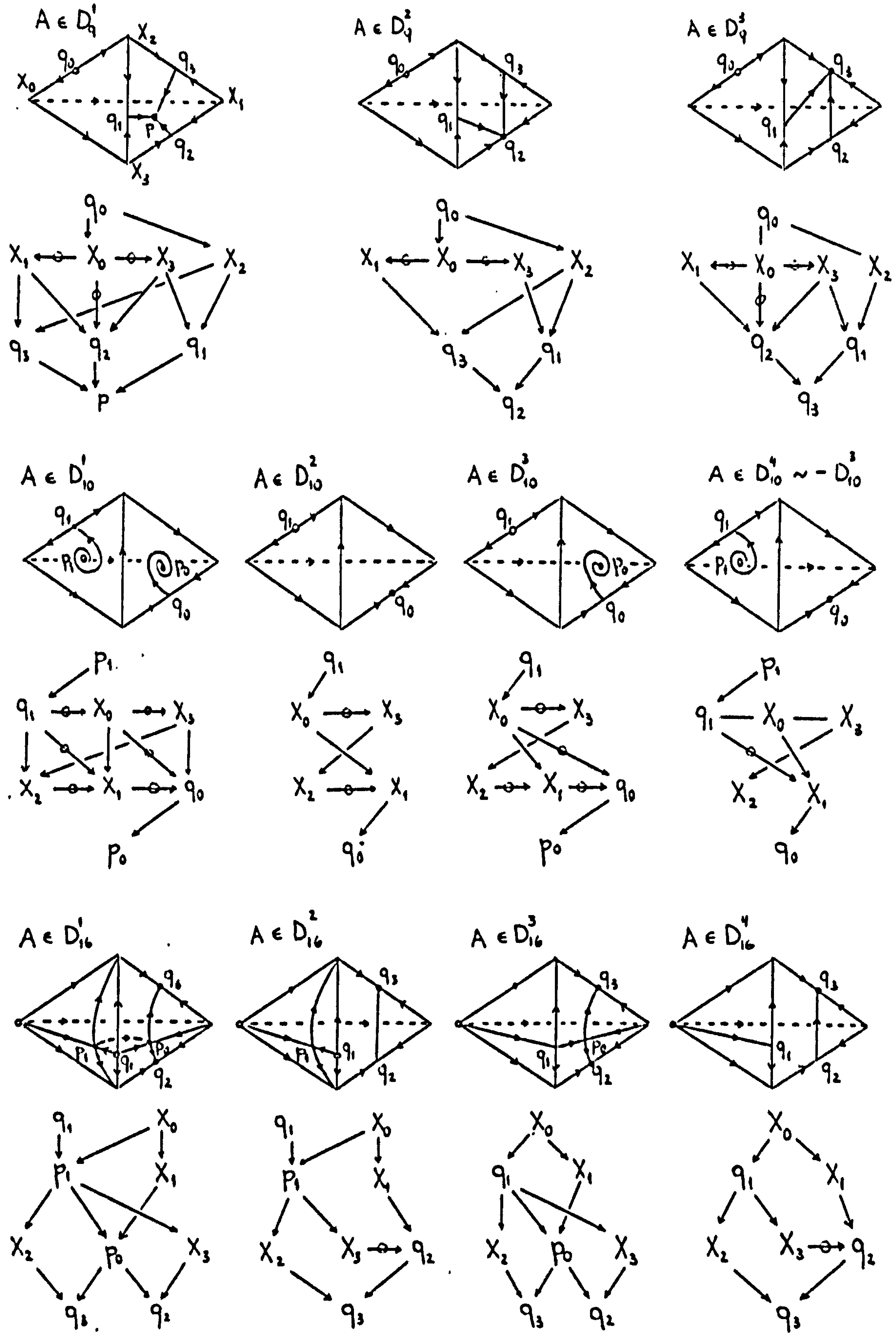

figure 38: (continuation)

Diagram for D_5^2 could have, equivalently, the attractor as q_1 or q_2 (by permuting vertices X_1, X_2, X_3) . Similarly, in D_9^2 , we can permute X_1, X_3 , making q_1 the attractor.

7.4.8 Proposition Subsets $D_1, D_2, D_3, D_4^1, D_4^2, D_5^1, D_5^2$ are stable classes of Z_4^+ .

Proof These sets are open. So we have only to prove that any two matrices A,B in one of these are equivalent i.e. $\phi_A \sim \phi_B$. But these flows are gradient-like, since all saddle connections are transversal. Topological equivalence between ϕ_A and ϕ_B is then obtained by standard techniques, for instance, by the "Heegard diagrams" of Fleitas [10] .

$\square$

7.4.9 Remark In other cases $D_6, \ldots, D_{18}$ we cannot as yet prove that all subsets D_k^r are stable classes,though we leave this as a conjecture. However by the same argument in 7.4.8 above we see that some of the subsets D_k^r $k \geq 6$ are stable classes. Among the ones we have presented above, this is clear for D_{16}^1 and $D_{16}^{3\cdot}$. But for most subsets D_k^r there are non-transversal saddle connections and argument in 7.4.8 does not apply.

Reader, I beg thy pardon, if I have kept thee long in reading this Discourse; but I hope thou wilt not be angry: for when I put Pen to Paper, I intended to be brief.

Andrew Yarranton, 1677

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PART II

LIAPUNOV FUNCTIONS FOR DIFFEOMORPHISMS

To my parents

PREFACE

The study of Liapunov functions for differential equations started with Liapunov [4] in 1907, and has been much developed since then. Initially, Liapunov tried to characterize stability of a singularity for a differential equation, given a positive definite continuous real function defined on a neighbourhood of the point. The reverse problem, that is the construction of continuous Liapunov function, given a stable singularity, only appeared later. In [3], Lefschetz adapted the ideas for a flow, with a stable compact invariant set. In [5], Massera gives some results on existence of differentiable Liapunov functions for differential equations. In [10], Wilson and Yorke took the problem of existence of C^∞ Liapunov function for a flow with a compact invariant set (which need not be stable) and gave a solution which is C^∞ outside the set, but (possibly) not on it.

The intention of this dissertation is to take the problem of existence of C^∞ Liapunov functions for diffeomorphisms with a compact invariant set, and we solve it in section 1, imposing conditions for the set being semi-isolated (notion which is there defined). We remark that the concept of isolated which we have introduced is different from the one used in [10]. We remark, too, that our definition of Liapunov function has, in some way, a more 'global' property.

The initial idea for the definition of the continuous Liapunov function in section 1 was taken from similar definition in [10], which we have here modified. However, our smoothing process is quite different from the one in that paper.

Section 2 treats two particular cases, applying results from section 1, to prove existence of local C^∞ Liapunov functions for attractor sets and basic sets.

In section 3, we deal with the problem of globalization of local Liapunov functions, that is, with certain hypothesis, given Liapunov functions defined on neighbourhoods of the basic sets of a diffeomorphism, we can construct a global Liapunov function which is an extension of all the local ones. Here we note that there are previous results in this direction, in the case of flows, e.g. [6] for Morse-Smale systems and [9] for gradient flows, but our method is quite different from both.

In section 4, we treat a particular case, the horseshoe, and construct a local Liapunov function for it.

Throughout this dissertation, we have used many concepts and results, some of which we recall here in the preliminaries of section 1, and which can be found in [7] and [8].

Since we believe we have done some original work, we tried to present it in form suitable for publication.

Most of all, I want to thank my supervisor, Professor E. C. Zeeman , for the suggestion of this problem for my dissertation, and for his help and encouragement in so many wonderful discussions.

CONTENTS - PART II

INTRODUCTION

We consider a C^∞ manifold M, and a diffeomorphism $f : M \longrightarrow M$.

Let $K \subset M$ be a closed invariant subset of M, relative to f (i.e. K is closed and $f(K) = K$ or, equivalently, $f^k(K) \subset K$ for all integers k).

Our objective is to construct a real function V, defined on a neighbourhood U of K, satisfying properties :

(1) V is constant on K

(2) $V(f^k x) \leq V(x)$ for $k \geq 1$, $x, f^k x \in U$

Such function is called a <u>Liapunov function</u> for f, relative to set K (or local Liapunov function) .

Moreover, we will look for a Liapunov function which is differentiable (in fact, C^∞) and which is also strictly decreasing on orbits, whenever that is possible, i.e. we want

(3) $V(f^k x) < V(x)$ for $k \geq 1$, $x, f^k x \in U$ and
$\omega(x) \cap K = \emptyset$ or $\alpha(x) \cap K = \emptyset$.

Remark : In the case of a flow φ , property (2) is usually stated as $V(\varphi(x,t)) \leq V(x)$ for $t \geq 0$ and $\varphi(x,s) \in U$, $s \in [0,t]$ or as $\frac{d}{dt} V(\varphi(x,t)) \geq 0$. ([1],[3],[4],[5],[10])
If we wanted simply adapt this condition for the diffeomorphism case, we would write property (2) with k=1 only. But then we note that such condition would permit us to study only local, rather than global, behaviour of f. Further, if we want to be able to extend V to a global Liapunov function

for f (i.e., V defined on M), we must allow k to be any positive integer. And, in fact, we will, in this work, extend V to a global Liapunov function at least for the important case where M is compact and f satisfies Axiom A and no-cycle condition.

Our exposition consists of four sections, which are:

1. Local Liapunov Functions,

   where we recall some preliminary definitions and results, define the new concepts of isolated and semi-isolated invariant sets and construct a C^∞ Liapunov function relative to a semi-isolated compact invariant set for f.

2. Liapunov Functions for Attractor Sets and Basic Sets,

   where we apply results from section 1 to these cases. More specifically, we prove that any attractor set for f is isolated and, then, there is C^∞ positive definite Liapunov function relative to it. Also, we prove that, if M is compact and f satisfies Axiom A and no-cycle condition, each basic set is isolated and, then, has a local C^∞ Liapunov function.

3. Globalization,

   where we prove our main theorem, which is:
   Theorem 8: <u>If M is compact and f satisfies Axiom A and no-cycle condition, there exist a global C^∞ Liapunov function for f, which is constant on each basic set and strictly decreasing on orbits outside the non-wandering set.</u>

4. Example: The Horseshoe,

   where we construct a local C^∞ Liapunov function for the horseshoe on a two-dimensional manifold.

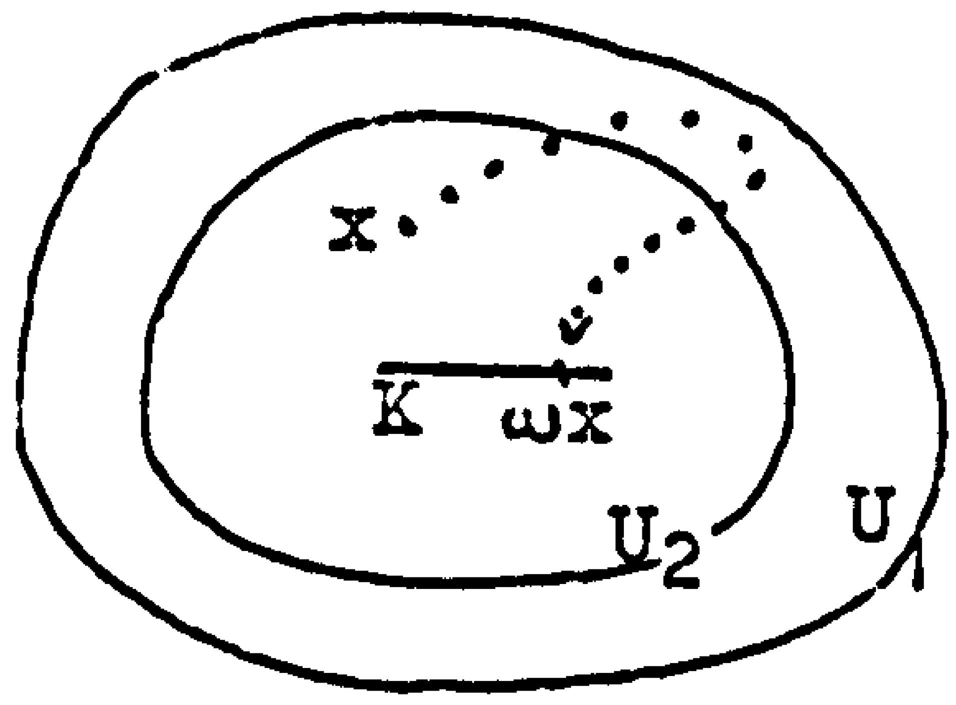
1. LOCAL LIAPUNOV FUNCTIONS

As a motivation, we first state and prove a theorem similar to the case usually studied for flows ([1],[3],[4],[5]) that is, when K is an attractor set.

Let K be a closed invariant set for f. We say that K is an <u>attractor set</u> if, for every neighbourhood U_1 of K, there is a second neighbourhood U_2 of K, with $U_2 \subset U_1$ and such that, for any point x in U_2, the forward orbit of x is contained in U_1 and the ω-limit set of x is contained in K; i.e. $K \subset U_2 \subset U_1$, $O^+(x) \subset U_1$ and $\omega(x) \subset K$ for any $x \in U_2$;

or equivalently,

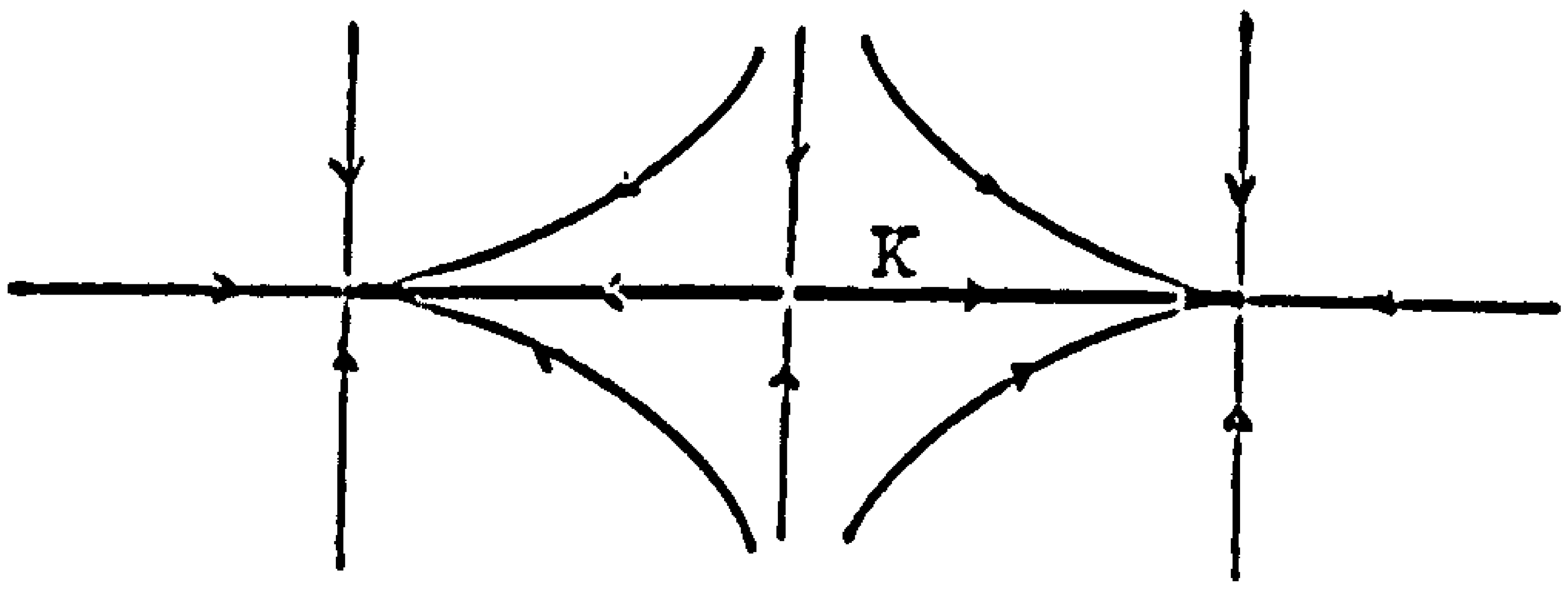
K is attractor set if the inset of K is neighbourhood of K, and the semi--outset of K equals K.



(Definitions of ω-limit, inset and semi-outset are given a after next theorem and equivalence for definition of attractor set is proved in Lemma 12, section 2.)

Note: The notion of <u>attractor</u> (rather than attractor set) is usually slightly more restrictive, namely it is required $\omega K = K$. In particular, an attractor must be contained in the non-wandering set, while an attractor set need not.

Example: In the following diagram, K is attractor set but not attractor.



If we substitute $O^+(x)$ for $O^-(x)$ and $\omega(x)$ for $\alpha(x)$ (or, inset for outset and viceversa) in above definition, we say that K is a <u>repellor set</u> for f. (Similarly for <u>repellor</u>.)

THEOREM 1: Let K be compact invariant set for f. If there exist a neighbourhood U of K and a continuous real function $V : U \longrightarrow \mathbb{R}$ such that

(1) $V \geqslant 0$; $V^{-1}(0) = K$

(2) $V(f^k x) < V(x)$ for $k \geqslant 1$, $x \in U \cap f^{-k}U - K$,

then K is an attractor set for f.

Proof:

Let U_1 be any neighbourhood of K. Without loss of generality, we suppose $\overline{U_1}$ is compact and $\overline{U_1} \subset U$.

By continuity of V, there is $\varepsilon > 0$ such that

$U_2 = (V|U_1)^{-1}(-\infty, \varepsilon) \subset U_1 \cap f^{-1}U_1$.

U_2 is open, $K \subset U_2$ and we claim that $O^+(U_2) \subset U_1$ and $\omega(U_2) \subset K$, thus proving that K is an attractor set.

In fact, let $x \in U_2$.

If $x \in K$, $O^+(x) \subset K \subset U_1$ and $\omega(x) \subset K$, because K is closed and invariant.

If $x \in U_2 - K$, we have $fx \in U_1 - K$. By (2), $V(fx) < V(x) < \varepsilon$, then $fx \in U_2$. Repeating the process, we obtain $O^+(x) \subset U_2$ for all $x \in U_2$; i.e. $O^+(U_2) \subset U_2 \subset U_1$.

As a consequence $\omega(U_2) \subset \overline{U_2} \subset U$. To see that $\omega(U_2) \subset K$, we first note that for all $x \in U_2$, $V|\omega(x)$ must be constant, because if $y, z \in \omega(x)$ with $V(y) < V(z)$, let $r_n \longrightarrow +\infty$, $s_n \longrightarrow +\infty$, with $f^{r_n}(x), f^{s_n}(x) \in U$ and $y = \lim f^{r_n}(x)$, $z = \lim f^{s_n}(x)$. We can always suppose that $s_n - r_n > 0$ and, so, by (2),

$V(f^{s_n}(x)) = V(f^{s_n - r_n}(f^{r_n}(x))) < V(f^{r_n}(x))$ which implies

$V(z) = \lim V(f^{s_n}(x)) \leqslant \lim V(f^{r_n}(x)) = V(y)$, contradicting $V(y) < V(z)$.

Now, supposing $y \in \omega(x) - K$ for some $x \in U_2$, we have $fy \in \omega(x) - K$, and $V(fy) = V(y)$. But, by (2), $V(fy) < V(y)$. Hence $\omega(U_2) \subset K$.

Remarks: (I) If condition (1) is substituted by

(1)' $V \leq 0$; $V^{-1}(0) = K$

and condition (2) is valid as stated, then K is repellor set.

(II) Using only half of the above proof, we can write the following property:

If $V : U \longrightarrow \mathbb{R}$ satisfies (1) as in Theorem 1, and

(2)' $V(f^k x) < V(x)$ for $k \geq 1$, $x \in U \cap f^{-k}U$

then for any neighbourhood U_1 of K, there is a second neighbourhood U_2 of K, with $U_2 \subset U_1$ and $O^+(U_2) \subset U_1$.

In the case of flows, Remark II and Theorem 1 are known as Liapunov theorems for stable and asymptotically stable invariant sets, respectively.

What we propose to do is a kind of reverse process of Theorem 1 and Remark I, i.e., given a compact invariant set K for f, we intend to construct, on a neighbourhood U of K, a C^∞ Liapunov function V relative to K, satisfying (2)', and in such a way that V will satisfy hypothesis of Theorem 1 (or Remark I) when K is attractor set (or repellor set) .

We need, first, to state some preliminary definitions, and recall some results which we will use throughout this work.

Let $x \in M$

<u>ω-limit of x</u> $= \omega(x) = \{y \in M;\ y = \lim f^{r_n}(x)$ for some $r_n \longrightarrow +\infty\}$

<u>α-limit of x</u> $= \alpha(x) = \{y \in M;\ y = \lim f^{r_n}(x)$ for some $r_n \longrightarrow -\infty\}$

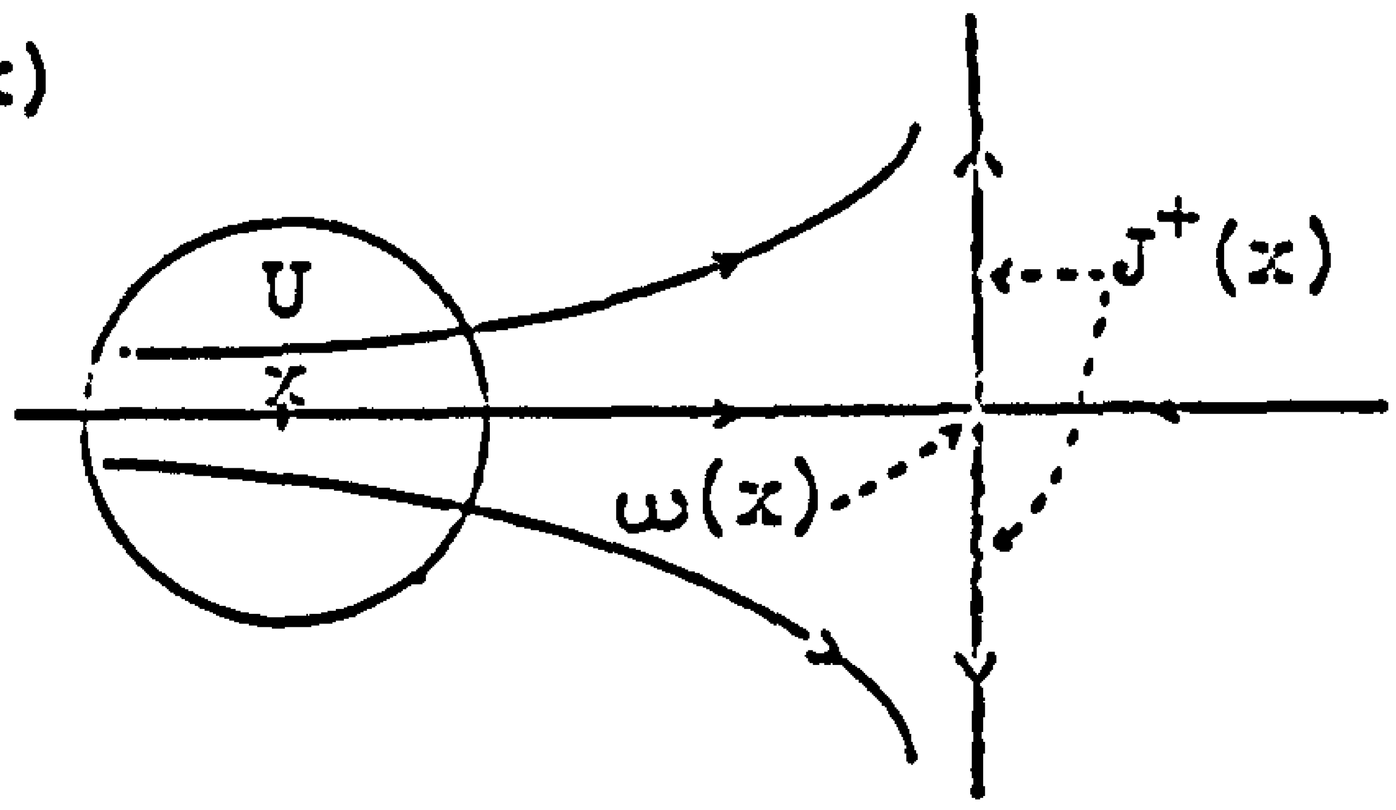
<u>Prolongational positive limit set of x</u> $= J^+(x) =$

$= \{y \in M;\ y = \lim f^{r_n}(x_n)$ for some $x_n \longrightarrow x$, $r_n \longrightarrow +\infty\}$

<u>Prolongational negative limit set of x</u> $= J^-(x) =$

$= \{y \in M;\ y = \lim f^{r_n}(x_n)$ for some $x_n \longrightarrow x$, $r_n \longrightarrow -\infty\}$

The definition of $J^+(x)$ and $J^-(x)$ is an adaptation for diffeomorphism of similar definitions in the case of flows and which can be found in [1] .



For any $U \subset M$, we write $\omega(U) = \bigcup_{x\in U} \omega(x)$ and similarly for $\alpha(U)$, $J^+(U)$, $J^-(U)$.

Let K be closed invariant set of M, relative to f.

Inset of K $= \omega^{-1}K = \{ x\in M; \quad \omega(x) \subset K \}$

Outset of K $= \alpha^{-1}K = \{ x\in M; \quad \alpha(x) \subset K \}$

Now we introduce some new concepts which we need for our exposition.

Semi-inset of K $= K^+ = \{ x\in M; \quad \omega(x)$ meets $K \}$

Semi-outset of K $= K^- = \{ x\in M; \quad \alpha(x)$ meets $K \}$

For any neighbourhood U of K, we write $U^+ = U \cap K^+$ and $U^- = U \cap K^-$.

Say that K is semi-isolated if there is a neighbourhood U of K such that $\begin{cases} J^+(U - U^+) \subset M - \overline{U} \\ J^-(U - U^-) \subset M - \overline{U} \end{cases}$

In this case, we say that U is an isolating neighbourhood of K.

Say that K is isolated if K is semi-isolated and, further, $U^+ \cap U^- = K$, for some isolating neighbourhood U.

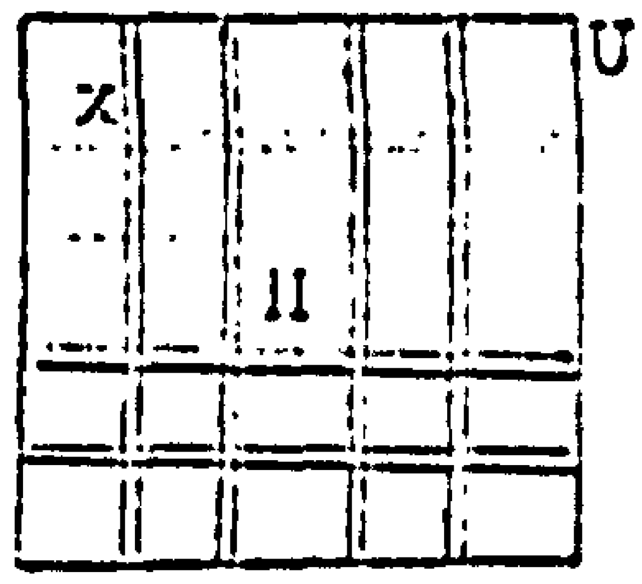
Examples : 1) If K is an attractor set for f, then K is isolated. (Proof in Lemma 13, section 2)

2) If M is compact and f satisfies Axiom A and no-cycle condition and K is a basic set for f (see following definitions), then K is isolated. (proof in Lemma 14, section 2)

3) If f is a horseshoe on M (with M two-dimensional) with horseshoe set H (see definition in section 4) and x is

the fixed point of f in H, we have that:

$$\begin{cases} H \text{ is isolated} \\ x \text{ is semi-isolated (but not isolated) saddle point} \end{cases}$$



$x^+ \cap U = H^+ \cap U = U^+$

We recall that for $x \in M$, x is <u>non-wandering</u> if for any beighbourhood U_x of x, there is $n \neq 0$ such that $f^n U_x \cap U_x \neq \emptyset$. We denote by Ω the set of non-wandering points for f in M. Ω is closed and invariant.

Say that $K \subset M$ is <u>hyperbolic</u> if the tangent bundle over K has an invariant (continuous) splitting under Tf, $T_K M = E^s \oplus E^u$, such that $Tf|E^s$ is contracting and $Tf|E^u$ is expanding.

Say that K is <u>transitive</u> if K has a dense orbit.

Say that a subset $K \subset \Omega$ is a <u>basic set</u> of f if K is invariant, transitive and it is open and closed in Ω.

Say that f satisfies <u>Axiom A</u> if:

(a) Ω is hyperbolic ; (b) the periodic points are dense in Ω

The <u>Spectral Decomposition Theorem</u> [8] says that, for compact M, if f satisfies Axiom A, then Ω can be decomposed (uniquely) in a finite union of basic sets ($\Omega = \Omega_1 \cup \Omega_2 \ldots \cup \Omega_r$) In this case, to be in conformity with the usual nomenclature and notation, we observe that

$$\begin{cases} \text{inset of } \Omega_i = \omega^{-1}\Omega_i = W^s\Omega_i \text{ (stable manifold of } \Omega_i) \\ \text{outset of } \Omega_i = \alpha^{-1}\Omega_i = W^u\Omega_i \text{ (unstable manifold of } \Omega_i) \end{cases}$$

As a corollary, M is disjoint union of the insets of the basic sets, i.e., $M = \bigcup_{i=1}^{r} W^s\Omega_i = \bigcup_{i=1}^{r} \omega^{-1}\Omega_i$.

Consequently, the notions of inset and semi-inset coincide for a basic set Ω_i, i.e., $W^s\Omega_i = \omega^{-1}\Omega_i = \Omega_i^+$.

Similarly, $M = \bigcup_{i=1}^{r} W^u\Omega_i = \bigcup_{i=1}^{r} \alpha^{-1}\Omega_i$ as disjoint union, and then $W^u\Omega_i = \alpha^{-1}\Omega_i = \Omega_i^-$.

Furthermore, we have [8] that $W^s\Omega_i \cap W^u\Omega_i = \Omega_i$.

If f has a spectral decomposition for Ω, as above, we say that a sequence $\Omega_0, \Omega_1, \dots, \Omega_{n+1}$ of basic sets is an <u>n-cycle</u> if $\Omega_0 = \Omega_{n+1}$, $\Omega_i \neq \Omega_j$ otherwise for $i \neq j$, and $W^s\Omega_i$ meets $W^u\Omega_{i+1}$ for $i = 0, 1, \dots, n$.

We say that f satisfies <u>no-cycle condition</u> if f has no n-cycles for $n \geqslant 1$.

If f satisfies no-cycle condition, it is possible to define a partial order on the basic sets by putting

$$\Omega_i < \Omega_j \Longleftrightarrow W^s\Omega_i \text{ meets } W^u\Omega_j \ .$$

After having stated these definitions and results, we turn back to our prime intention, that is, the construction of Liapunov functions for certain diffeomorphisms f with compact invariant subset K. And now we note that if V is any Liapunov function for set K, defined on U, V must be constant on $U^+ \cap U^-$ and this leads to restrict ourselves to look for Liapunov functions strictly decreasing on orbits outside $U^+ \cap U^-$, rather than outside K only, i.e., condition (2)' can be written as

(3) $V(f^k x) < V(x)$ for $k \geqslant 1$, $x \in U \cap f^{-k}U - U^+ \cap U^-$.

Our plan is to construct, for semi-isolated set K, two C^∞ functions $V_+, V_- : U \longrightarrow \mathbb{R}$ such that V_+ and V_- are non negative, which are zero only on U^+ and U^- respectively and V_+ is strictly increasing on orbits outside U^+ and V_- is strictly decreasing on orbits outside U^- .

First, we will prove the continuous case, i.e.

THEOREM 2 : If K is compact semi-isolated, there exist continuous Lipschitz functions

$\varepsilon_+, \varepsilon_- : U \longrightarrow \mathbb{R}$ on neighbourhood U of K, satisfying

(1) $\varepsilon_+ \geqslant 0 \quad ; \quad \varepsilon_+^{-1}(0) = U^+$

$\varepsilon_- \geqslant 0 \quad ; \quad \varepsilon_-^{-1}(0) = U^-$

(2) $\varepsilon_+(f^k x) > \varepsilon_+(x)$ for $k \geqslant 1$, $x \in U \cap f^{-k}U - U^+$

$\varepsilon_-(f^k x) < \varepsilon_-(x)$ for $k \geqslant 1$, $x \in U \cap f^{-k}U - U^-$

Applying some smoothing process, we will get

THEOREM 3 : If K is compact semi-isolated, there exist C^∞ functions $V_+, V_- : U \longrightarrow \mathbb{R}$ on neighbourhood U of K, satisfying

(1) $V_+ \geqslant 0 \quad ; \quad V_+^{-1}(0) = U^+$

$V_- \geqslant 0 \quad ; \quad V_-^{-1}(0) = U^-$

(2) $V_+(f^k x) > V_+(x)$ for $k \geqslant 1$, $x \in U \cap f^{-k}U - U^+$

$V_-(f^k x) < V_-(x)$ for $k \geqslant 1$, $x \in U \cap f^{-k}U - U^-$

Then, taking $V : U \longrightarrow \mathbb{R}$ defined by $V = V_- - V_+$

V is a C^∞ Liapunov function for f, strictly decreasing on orbits outside $U^+ \cap U^-$, i.e., we have

THEOREM 4 : If K is compact semi-isolated, there exist C^∞ function $V : U \longrightarrow \mathbb{R}$ on neighbourhood U of K, such that

(1) $V(K) = 0$

(2) $V(f^k x) < V(x)$ for $k \geqslant 1$, $x \in U \cap f^{-k}U - U^+ \cap U^-$

(3) $V|(U^+ - U^+ \cap U^-) > 0$

$V|(U^- - U^+ \cap U^-) < 0$

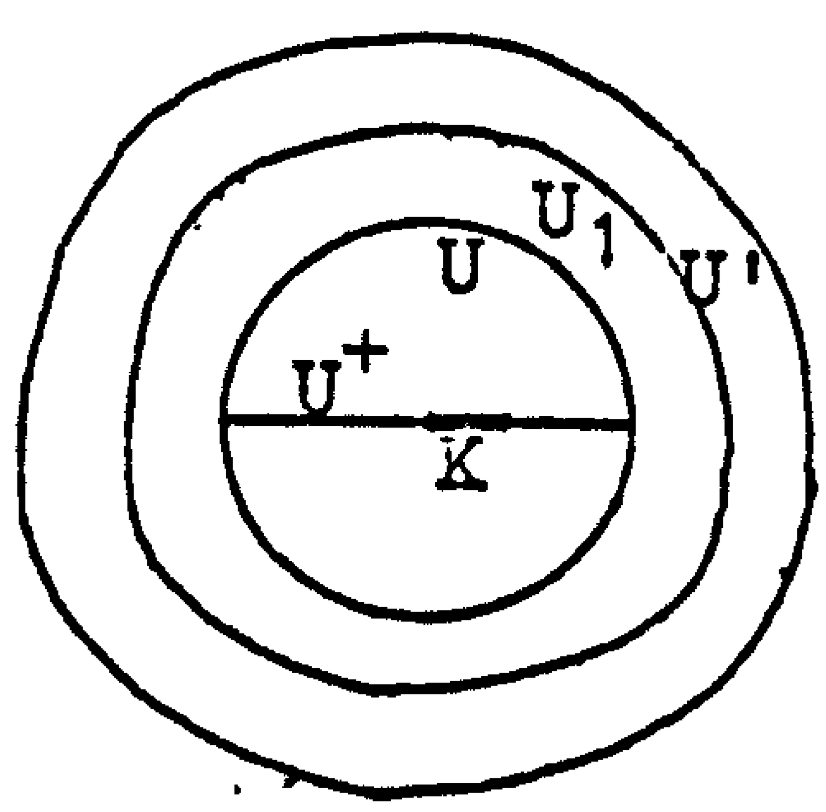
Now we start to carry out the construction of our Liapunov function. First of all we define functions g_+ and g_- and, in a sequence of lemmas, we prove they satisfy all the properties required in Theorem 2. So, let K be a compact invariant semi-isolated subset of M. Take neighbourhoods U and U' of K, such that $\overline{U'}$ is compact, $K \subset U \subset \overline{U} \subset U'$ and U' is an isolating neighbourhood (hence, so is U) of K.

$\overline{U^+}$ and $\overline{U^-}$ are closed subsets of U', thus we can choose continuous Lipschitz functions $F_+, F_- : M \longrightarrow \mathbb{R}$ such that

$$\begin{cases} F_+^{-1}(0) = \overline{U^+} \\ F_+ \geqslant 0 \\ F_+^{-1}([0,1)) \supset U \\ F_+^{-1}([0,2)) \subset U' \end{cases} \quad \text{and} \quad \begin{cases} F_-^{-1}(0) = \overline{U^-} \\ F_- \geqslant 0 \\ F_-^{-1}([0,1)) \supset U \\ F_-^{-1}([0,2)) \subset U' \end{cases}$$

(To prove F_+ exists, we take open U_1 with $\overline{U} \subset U_1 \subset \overline{U_1} \subset U'$ and continuous Lipschitz functions $h_1, h_2 : M \longrightarrow [0,1]$ such that $h_1^{-1}(0) = \overline{U^+}$; $h_1^{-1}(1) = M - U_1$

$h_2^{-1}(0) = \overline{U_1}$; $h_2^{-1}(1) = M - U'$

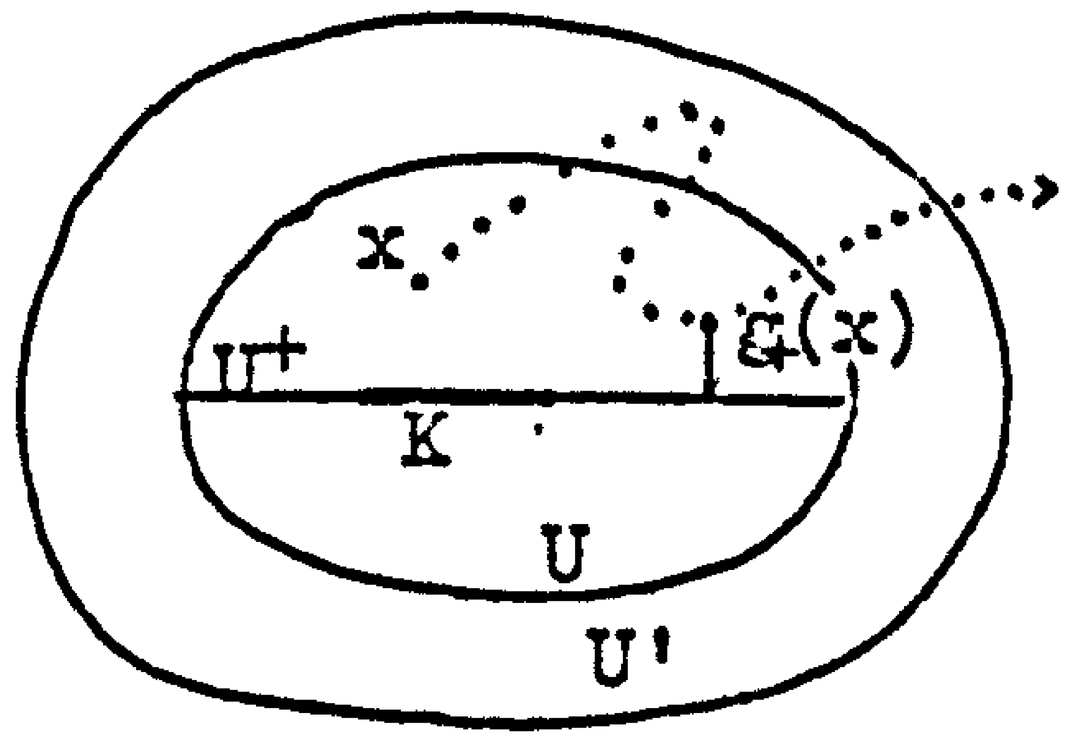


Then $F_+ = h_1 + h_2$ satisfies above properties. Existence of F_- is proved similarly.)

Take, also, some real sequence $(a_n)_{n \geqslant 0}$, satisfying $a_0 = 1$; $a_{n+1} < a_n$; $a_n > \frac{1}{2}$.

Define $g_+ , g_- : U \longrightarrow \mathbb{R}$ by

$$g_+(x) = \inf \{ a_n F_+ f^n(x) \quad ; n \geqslant 0 \}$$
$$g_-(x) = \inf \{ a_n F_- f^{-n}(x) \quad ; n \geqslant 0 \}$$



Before starting to prove our lemmas, let us make some remarks about our definitions and methods.

The original idea for definitions of g_+ and g_- came from similar definitions in [10], which here were modified to permit proving property (2) in Theorems 2,3 and 4 for

for $k \geq 1$, $x, f^k x \in U$ rather than

for $k \geq 1$, $f^i x \in U$ $(0 \leq i \leq k)$ which would be equivalent to take $k = 1$ only.

After proving existence of continuous Liapunov function (here, consequence of Theorem 2) the authors of [10] smoothed it by means of a theorem in [11], while we took a different smoothing process, given by our Lemmas 8,9,and 10.

LEMMA 1 : For isolating neighbourhood U of K,
$U^+ = U \cap \overline{K^+}$ and $U^- = U \cap \overline{K^-}$.
Consequently, U^+ and U^- are closed in U .

Proof:

We prove the assertions only for U^+, since proof for U^- is similar.

By definition, $U^+ = U \cap K^+ \subset U \cap \overline{K^+}$.

Suppose $x \in U \cap \overline{K^+}$. Then $x = \lim x_n$ where $\omega(x_n)$ meets K. We need to prove that $x \in U^+$. There is integer sequence (r_{nk}) such that $\lim_k r_{nk} = +\infty$ and $\lim_k f^{r_{nk}}(x_n) = y_n \in K$.

Taking subsequences, if necessary, we can suppose

$\lim y_n = y \in K$; $d(y_n,y) < \frac{1}{n}$; $d(f^{r_{nk}}(x_n),y_n) < \frac{1}{n}$; $r_{nk} \geq n$

Then, for $s_n = r_{nn}$, we have $\lim s_n = +\infty$ and

$d(f^{s_n}(x_n),y) \leq d(f^{s_n}(x_n),y_n) + d(y_n,y) < \frac{1}{n} + \frac{1}{n}$

i.e. $y = \lim f^{s_n}(x_n) \in J^+(x) \cap K \subset J^+(x) \cap \overline{U}$.

Since $x \in U$ and U is isolating neighbourhood, we must have that $x \in U^+$, as we wanted.

LEMMA 2 : $g_+^{-1}(0) = U^+$ and $g_-^{-1}(0) = U^-$

Proof:

$g_+(U^+) = 0$ by definition of g_+ .

If $g_+(x) = 0$ $(x \in U)$ one of two cases holds, (a) or (b)

(a) There is $k \geq 0$ such that $a_k F_+ f^k(x) = 0$. Therefore $f^k(x) \in \overline{U^+} \subset \overline{U} \cap \overline{K^+}$. This means there is sequence (y_n) in M with $\omega(y_n) \cap K \neq \emptyset$ and $\lim y_n = f^k x$. Let $x_n = f^{-k} y_n$, then $\omega(x_n) \cap K \neq \emptyset$. But $f^k x_n = y_n \longrightarrow f^k x$ implies $x_n \longrightarrow x$. Thus $x = \lim x_n$ with $\omega(x_n) \cap K \neq \emptyset$ i.e. $x \in U \cap \overline{K^+} = U^+$ by Lemma 1.

(b) There is integer sequence (r_n), $r_n \geq 0$, such that $r_n \longrightarrow +\infty$ and $a_{r_n} F_+ f^{r_n}(x) \longrightarrow 0$. Therefore $f^{r_n}(x) \longrightarrow y \in \overline{U^-}$ (at least in subsequence) and, so, $y \in \omega(x) \cap \overline{U^-} \subset J^+(x) \cap \overline{U}$. Since U is isolating neighbourhood of K, we have $x \in U^+$.

$g_-^{-1}(0) = U^-$ is proved similarly.

LEMMA 3 : Let $x \in U$

I) (a) $J^+(x) \subset M-\overline{U'}$ $\Longrightarrow$ (b) There are neighbourhood V_x^+ of x $(V_x^+ \subset U)$ and integer N_x^+ such that $f^n y \in M-\overline{U'}$ for $y \in V_x^+$, $n > N_x^+$ $\Longrightarrow$ (c) $J^+(x) \subset M-U'$

II)(a) $J^-(x) \subset M-\overline{U'}$ $\Longrightarrow$ (b) There are neighbourhood V_x^- of x $(V_x^- \subset U)$ and integer N_x^- such that $f^{-n} y \in M-\overline{U'}$ for $y \in V_x^-$, $n > N_x^-$ $\Longrightarrow$ (c) $J^-(x) \subset M-U'$

Proof:

I)($a \Rightarrow b$) Suppose (a) is valid and there are no V_x^+, N_x^+ satisfying (b). Then we can take $x_n \longrightarrow x$ and $r_n \longrightarrow +\infty$, such that $f^{r_n}(x_n) \in \overline{U'}$. Taking subsequence, if necessary, we suppose $f^{r_n}(x_n) \longrightarrow y \in \overline{U'}$. Then $y \in J^+(x) \cap \overline{U'}$, contrary to hypothesis.

($b \Rightarrow c$) Suppose (b) is valid. If $y \in J^+(x)$, there are sequences $x_n \longrightarrow x$, $r_n \longrightarrow +\infty$ with $f^{r_n}(x_n) \longrightarrow y$. For all sufficiently large n, we can suppose $x_n \in V_x^+$ and $r_n > N_x^+$ and then (b) implies $f^{r_n}(x_n) \in M-\overline{U'}$. Hence the limit y must be in $M-U'$. Then $J^+(x) \subset M-U'$.

II) is proved similarly.

LEMMA 4 : I) If $x \in U - U^+$, take V_x^+, N_x^+ as in Lemma 3(b) Then, for all $y \in V_x^+$, $g_+(y) = \min \{ a_n F_+f^n(y) \ ; 0 \leq n \leq N_x^+ \}$

II) If $x \in U - U^-$, take V_x^-, N_x^- as in Lemma 3(b) Then, for all $y \in V_x^-$, $g_-(y) = \min \{ a_n F_-f^{-n}(y) \ ; 0 \leq n \leq N_x^- \}$

Proof:

I) Let $x \in U - U^+$ and $y \in V_x^+$: For all $n > N_x^+$ we have $f^n y \in M - \overline{U'}$. By conditions on choice of F_+ and (a_n), $F_+f^n(y) \geq 2$ and $a_n F_+f^n(y) > 1$ for all $n > N_x^+$.

But $a_0 F_+f^0(y) = F_+(y) < 1$ since $y \in U$.

Then, $g_+(y) = \min \{ a_n F_+f^n(y) \ ; 0 \leq n \leq N_x^+ \}$.

II) is proved similarly.

LEMMA 5 : I) $g_+(f^k x) > g_+(x)$ for $k \geq 1$, $x \in U \cap f^{-k}U - U^+$

II) $g_-(f^k x) < g_-(x)$ for $k \geq 1$, $x \in U \cap f^{-k}U - U^-$

Proof:

I) For $k \geq 1$, $x \in U \cap f^{-k}U - U^+$,

$$\begin{aligned} g_+(f^k x) &= \inf \{ a_n F_+f^n(f^k x) \ ; n \geq 0 \} = \inf \{ a_{n-k} F_+f^n x \ ; n \geq k \} \\ &= a_{j-k} F_+f^j x \quad \text{for some finite } j \geq k \\ &> a_j F_+f^j x \geq \inf \{ a_n F_+f^n x \ ; n \geq 0 \} = g_+(x) . \end{aligned}$$

II) For $k \geq 1$, $x \in U \cap f^{-k}U - U^-$,

$$\begin{aligned} g_-(f^k x) &= \inf \{ a_n F_-f^{-n}(f^k x) ; n \geq 0 \} = \inf \{ a_{n+k} F_-f^{-n} x \ ; n \geq -k \} \\ g_-(x) &= a_j F_-f^{-j}(x) \quad \text{for some finite } j \geq 0 \\ &> a_{j+k} F_-f^{-j}(x) \geq \inf \{ a_{n+k} F_-f^{-n}(x) ; n \geq -k \} = g_-(f^k x) \end{aligned}$$

LEMMA 6 : g_+ and g_- are continuous

Proof:

If $x \in U^+$, $g_+(x) = 0$. Since $F_+(U^+) = 0$, and F_+ is continuous, given any $\varepsilon > 0$, there is neighbourhood V of x such that $y \in V \rightarrow F_+(y) < \varepsilon$.

Then, $y \in V \Longrightarrow g_+(y) = \inf \{ a_n F_+ f^n(y) \ ; \ n \geqslant 0 \} \leqslant F_+(y) < \varepsilon$

Hence g_+ is continuous at $x \in U^+$.

If $x \in U - U^+$, we have $J^+(x) \subset M-\overline{U'}$. Take N_x^+ and V_x^+ as in Lemma 3(b). By continuity of F_+ and f, for each n $(0 \leqslant n \leqslant N_x^+)$ the function $a_n \, F_+ f^n : V_x^+ \longrightarrow \mathbb{R}$ is continuous.

Therefore, $g_+ : V_x^+ \longrightarrow \mathbb{R}$, being the minimum of a finite set of continuous functions (by Lemma 4), is also continuous.

Hence g_+ is continuous.

Continuity of g_- is proved similarly.

LEMMA 7 : g_+ and g_- are Lipschitz

Proof:

We want to prove that g_+ is Lipschitz at each $x \in U$.

If $x \in U^+$, $g_+(x) = 0$ and $F_+(x) = 0$. Since we have required F_+ to be Lipschitz, there are constant $k_x > 0$ and a neighbourhood V_x of x such that

$|F_+(y) - F_+(x)| \leqslant k_x \, d(y,x)$ for all $y \in V_x$. Then

$$|g_+(y) - g_+(x)| = g_+(y) \leqslant F_+(y) = |F_+(y) - F_+(x)| \leqslant$$
$$\leqslant k_x \, d(y,x) \quad \text{for all } y \in V_x .$$

i.e. g_+ is Lipschitz at $x \in U^+$.

If $x \in U - U^+$, we have $J^+(x) \subset M-\overline{U'}$. Take N_x^+ and V_x^+ as in Lemma 3(b). Since F_+ and f are Lipschitz, for each n $(0 \leqslant n \leqslant N_x^+)$ $a_n F_+ f^n : V_x^+ \longrightarrow \mathbb{R}$ is Lipschitz at x.

Therefore, $g_+ : V_x^+ \longrightarrow \mathbb{R}$, being the minimum of a finite set of Lipschitz functions at x (by Lemma 4), is also Lipschitz at x. Hence g_+ is Lipschitz on U.

Proof that g_- is Lipschitz can be done similarly.

Proof of Theorem 2:

Take neighbourhood U of K and functions g_+ and g_- as previously defined. Then g_+ and g_- are continuous, by Lemma 6, and Lipschitz, by Lemma 7. Property (1) follows from Lemma 2 and definition; (2) is given by Lemma 5.

Now we want to approximate g_+ and g_- by C^∞ functions V_+ and V_-. For our approximation we will make use of a known lemma in analysis, which we here state and prove:

LEMMA 8 : If h_1 and h_2 are continuous real functions on M, with $h_1 < h_2$, then there exist a C^∞ function h such that $h_1 < h < h_2$.

Proof:

For each $x \in M$, there is open neighbourhood V(x) and a constant a(x) such that $y \in V(x) \Longrightarrow h_1(y) < a(x) < h_2(y)$ (e.g. take $a(x) = \frac{1}{2}(h_1(x)+h_2(x))$).

$\{V(x)\}_{x\in M}$ is open cover for M. Let $\{V_i\}_{i\in I}$ be a locally finite subcover (with $V_i = V(x_i)$). Let $a_i = a(x_i)$ and $\varphi_i : M \longrightarrow \mathbb{R}$ such that $\{\varphi_i\}_{i\in I}$ is C^∞ partition of unity, strictly subordinated to the cover $\{V_i\}$, i.e.

$$\sum_{i\in I}\varphi_i = 1 \quad \text{and} \quad \operatorname{clos}\{ y\in M \ ; \ \varphi_i(y) > 0 \} \subset V_i .$$

Define $h : M \longrightarrow \mathbb{R}$ by $h = \sum_{i\in I} a_i\varphi_i$. Then h is C^∞ and

$$h(y) = \sum a_i\,\varphi_i(y) = \sum_{y\in V_i} a_i\,\varphi_i(y) <$$

$$< \sum_{y\in V_i} h_2(y)\,\varphi_i(y) = h_2(y) \sum_{y\in V_i}\varphi_i(y) = h_2(y)$$

since $\varphi_i(y) > 0$ for some i with $y\in V_i$ and $a_i < h_2(y)$ for $y\in V_i$

Then, $h < h_2$. Similarly, $h > h_1$ and, hence, $h_1 < h < h_2$.

Remark : In fact, the proof of Lemma 8 is still valid if we suppose only that h_1 is upper semi-continuous and h_2 is lower semi-continuous. I first saw this lemma in lecture notes by E.L.Lima.

LEMMA 9 : <u>There exist neighbourhood U_1 of K and continuous Lipschitz functions $W_+, W_- : U_1 \longrightarrow \mathbb{R}$ such that</u>

(1) $W_+ \geq 0 \quad ; \quad W_+^{-1}(0) = U_1^+$

$W_- \geq 0 \quad ; \quad W_-^{-1}(0) = U_1^-$

(2) $W_+(f^k x) > W_+(x)$ for $k \geq 1$, $x \in U_1 \cap f^{-k}U_1 - U_1^+$

$W_-(f^k x) < W_-(x)$ for $k \geq 1$, $x \in U_1 \cap f^{-k}U_1 - U_1^-$

(3) W_+ is C^∞ on $U_1 - U_1^+$; W_- is C^∞ on $U_1 - U_1^-$

Proof:

Let $g_+, g_- : U \longrightarrow R$ be given by Theorem 2.

Take $U_1 = U \cap fU \cap f^{-1}U$. We have that $U_1^+ = U^+ \cap U_1$, $U_1^- = U^- \cap U_1$

Then $x \in U_1 \Longrightarrow fx, f^{-1}x \in U$, and by Theorem 2 we have

$x \in U_1 - U_1^+ \Longrightarrow g_+(fx) > g_+(f^{-1}x)$

$x \in U_1 - U_1^- \Longrightarrow g_-(fx) < g_-(f^{-1}x)$

So, $x \in U_1 - U_1^+ \Longrightarrow \frac{1}{2}(g_+(x)+g_+(f^{-1}x)) < \frac{1}{2}(g_+(x)+g_+(fx))$

$x \in U_1 - U_1^- \Longrightarrow \frac{1}{2}(g_-(x)+g_-(fx)) < \frac{1}{2}(g_-(x)+g_-(f^{-1}x))$

Applying Lemma 8, there is C^∞ function

$W_+ : U_1 - U_1^+ \longrightarrow \mathbb{R}$ satisfying

$\frac{1}{2}(g_+(x)+g_+(f^{-1}x)) < W_+(x) < \frac{1}{2}(g_+(x)+g_+(fx))$

We extend the definition of W_+ to $W_+ : U_1 \longrightarrow \mathbb{R}$ by putting $W_+(U_1^+) = 0$. Now we prove that W_+ satisfies the required properties.

(1) is valid because for $x \in U_1^+$, $W_+(x) = 0$ by definition, and for $x \in U_1 - U_1^+$, $W_+(x) > \frac{1}{2}(g_+(x)+g_+(f^{-1}x)) > 0$

(2) Let $k \geq 1$, $x \in U_1 \cap f^{-k}U_1 - U_1^+$. As consequence

$x,\ fx,\ f^kx,\ f^{k-1}x \in U - U^+$. Then

$$W_+(x) - W_+(f^kx) < \tfrac{1}{2}(g_+(x)+g_+(fx)) - \tfrac{1}{2}(g_+(f^kx)+g_+(f^{k-1}x)) =$$
$$= \tfrac{1}{2}(g_+(x)-g_+(f^{k-1}x)) + \tfrac{1}{2}(g_+(fx)-g_+(f^kx)) < 0 \quad \text{by}$$

Theorem 2 (2) . Hence $W_+(f^kx) > W_+(x)$.

(3) is valid by construction.

To prove continuity of W_+, we have only to prove it at $x \in U_1^+$. Let x_n be sequence in U_1 with $\lim x_n = x$. When $x_n \in U_1^+$, $W_+(x_n) = 0$. Suppose $x_n \in U_1 - U_1^+$. But $x_n \in U_1$ implies $f(x_n) \in U$ and $f^{-1}(x_n) \in U$. Therefore

$x_n \longrightarrow x$ implies $f(x_n) \longrightarrow f(x) \in U^+$ and $f^{-1}(x_n) \longrightarrow f^{-1}(x) \in U^+$

Then, $g_+(x_n) \longrightarrow g_+(x)=0$; $g_+(fx_n) \longrightarrow g_+(fx)=0$;

$g_+(f^{-1}x_n) \longrightarrow g_+(f^{-1}x) = 0$.

Since $\tfrac{1}{2}(g_+(x_n)+g_+(f^{-1}x_n)) < W_+(x_n) < \tfrac{1}{2}(g_+(x_n)+g_+(fx_n))$

we have $W_+(x_n) \longrightarrow 0 = W_+(x)$.

Hence W_+ is continuous.

To prove W_+ is Lipschitz, we have to prove it only at $x \in U_1^+$, since $W_+|(U_1-U_1^+)$ is C^∞ . We have that g_+ and f are Lipschitz at x. So, there are neighbourhood V_x of x and constants $k_1,\ k_2 > 0$ such that

$$g_+(y) = |g_+(y) - g_+(x)| \leq k_1\, d(y,x)$$
$$g_+(fy) = |g_+(fy) - g_+(fx)| \leq k_2\, d(y,x)$$

Taking $k = \tfrac{1}{2}(k_1+k_2)$ we have that, for all $y \in V_x \cap (U_1-U_1^+)$,

$$|W_+(y) - W_+(x)| = W_+(y) < \tfrac{1}{2}(g_+(y)+g_+(fy)) \leq \tfrac{1}{2}(k_1+k_2)\, d(y,x)$$

and for $y \in U_1^+$, $W_+(y) = 0$

Then, $y \in V_x \cap U_1 \Longrightarrow |W_+(y) - W_+(x)| \leq k\, d(y,x)$

Hence W_+ is Lipschitz.

For the construction of W_- we again apply Lemma 8 to obtain C^∞ function $W_- : U_1 - U_1^- \longrightarrow R$ such that

$$\tfrac{1}{2}(g_-(x)+g_-(fx)) < W_-(x) < \tfrac{1}{2}(g_-(x)+g_-(f^{-1}x))$$

and extend W_- continuously to U_1 by $W_-(U_1^-) = 0$.
Proof that W_- satisfies required properties can be done similarly to the proof for W_+ .

Proof of Theorem 3 :

Let $W_+, W_- : U \longrightarrow \mathbb{R}$ be given by Lemma 9. Define

$$V_+ : U \longrightarrow \mathbb{R} \text{ by } V_+(x) = \begin{cases} W_+(x) \exp(- W_+(x)^{-2}) & \text{if } x \in U - U^+ \\ 0 & \text{if } x \in U^+ \end{cases}$$

$$V_- : U \longrightarrow \mathbb{R} \text{ by } V_-(x) = \begin{cases} W_-(x) \exp(- W_-(x)^{-2}) & \text{if } x \in U - U^- \\ 0 & \text{if } x \in U^- \end{cases}$$

V_+ is C^∞ because W_+ is C^∞ on $U - U^+$ and Lipschitz on U^+.
Similarly, V_- is C^∞ .

(1) follows from definition of V_+, V_- and Lemma 9 (1) .

(2) If $k \geqslant 1$, $x \in U \cap f^{-k}U - U^+$, by Lemma 9(2), $W_+(f^k x) > W_+(x)$. Then

$$V_+(f^k x) = W_+(f^k x) \exp(- W_+(f^k x)^{-2}) >$$
$$> W_+(x) \exp(- W_+(x)^{-2}) = V_+(x)$$

Similarly, $V_-(f^k x) < V_-(x)$ for $k \geqslant 1$, $x \in U \cap f^{-k}U - U^-$.

The idea for the definitions of V_+ and V_- in the above proof was taken from [11] .

Theorem 4 is an easy consequence of Theorem 3, taking $V = V_- - V_+$.

2. LIAPUNOV FUNCTIONS FOR ATTRACTOR SETS AND BASIC SETS

In this section we want to apply Theorem 4 to some particular cases, to get the following theorems.

THEOREM 5 : If K is a compact attractor set for f, then there is a C^∞ function $V : U \longrightarrow \mathbb{R}$ on some neighbourhood U of K such that

(1) $V \geq 0$ and $V^{-1}(0) = K$

(2) $V(f^k x) < V(x)$ for $k \geq 1$, $x \in U \cap f^{-k}U - K$

Theorem 5 is a converse of Liapunov theorem for attractor sets (i.e. Theorem 1). Similarly we can get a converse for Remark I, i.e. for a compact repellor set of f there is C^∞ Liapunov function as in Theorem 5, changing $V \geq 0$ for $V \leq 0$.

THEOREM 6 : Suppose M is compact and f satisfies Axiom A and no-cycle condition. Let Ω_i be a basic set. Then, there is a C^∞ function $V_i : U_i \longrightarrow \mathbb{R}$ for some neighbourhood U_i of Ω_i, satisfying

(1) $V_i(\Omega_i) = 0$

(2) $V_i(f^k x) < V_i(x)$ for $k \geq 1$, $x \in U \cap f^{-k}U - \Omega_i$

(3) $V_i | (U_i^+ - \Omega_i) > 0$ (where $U_i^+ = U_i \cap W^s \Omega_i$)

$V_i | (U_i^- - \Omega_i) < 0$ (where $U_i^- = U_i \cap W^u \Omega_i$)

(4) $\operatorname{grad} V_i(x) = 0$ for $x \in \Omega_i$.

All we need is to apply Theorem 4 for each case. In order to be able to do that, we have to prove first that the sets considered are semi-isolated. In fact, we will prove that they are isolated. The proofs are in Lemmas 13 and 14 below. But we prove before some properties for J^+, J^- and attractor sets.

LEMMA 10 : For any $x \in M$, $J^+(x)$ and $J^-(x)$ are closed invariant sets for f .

Proof:

We prove the assertions only for $J^+(x)$, since proof for $J^-(x)$ is similar.

If $y \in J^+(x)$, there are sequences (x_n) and (r_n) with
$\lim x_n = x$; $\lim r_n = +\infty$; $\lim f^{r_n}(x_n) = y$
For any integer k, $\lim (k+r_n) = +\infty$, then
$f^k y = f^k(\lim f^{r_n}(x_n)) = \lim f^{k+r_n}(x_n) \in J^+(x)$.
Thus $f^k(J^+(x)) \subset J^+(x)$ for all integer k, i.e., $J^+(x)$ is invariant.

Suppose $y_n \in J^+(x)$ and $\lim y_n = y$. Then, there are x_{nk} in M and integers r_{nk} with $\lim_k x_{nk} = x$, $\lim_k r_{nk} = +\infty$ and $\lim_k f^{r_{nk}}(x_{nk}) = y_n$.
We can suppose that $d(y_n,y) < \frac{1}{n}$ and, for all k,
$d(f^{r_{nk}}(x_{nk}),y_n) < \frac{1}{n}$; $d(x_{nk},x) < \frac{1}{n}$; $r_{nk} \geqslant n$.
Consider sequences $z_n = x_{nn}$ and $t_n = r_{nn}$. So,
$\lim z_n = x$, $\lim t_n = +\infty$ and
$d(f^{t_n}(z_n),y) < d(f^{r_{nn}}(x_{nn}),y_n) + d(y_n,y) < \frac{1}{n} + \frac{1}{n}$.
Then $y = \lim f^{t_n}(z_n) \in J^+(x)$. Hence $J^+(x)$ is closed.

LEMMA 11 : Let K be compact invariant set for f. If K is attractor set, then $K^+ = \omega^{-1}K$ and $K^- = \alpha^{-1}K$.
(i.e. semi- inset of K = inset of K
and semi-outset of K = outset of K)

Proof:

We always have $\omega^{-1}K \subset K^+$ and $\alpha^{-1}K \subset K^-$.

Let $x \in K^+$, i.e. $\omega(x) \cap K \neq \emptyset$. There is sequence (r_n) such that $\lim r_n = +\infty$ and $\lim f^{r_n}(x) \in K$.
Suppose $y \in \omega(x) - K$. There is sequence (s_n) such that $\lim s_n = +\infty$ and $\lim f^{s_n}(x) = y$. We can suppose $s_n > r_n$.
Take open U_1 with $K \subset U_1$ and $y \notin \overline{U_1}$. Since K is attractor set, there is open U_2 such that $K \subset U_2 \subset U_1$ and $O^+(U_2) \subset U_1$.
For all sufficiently large n ($n \geq N$) we have $f^{r_n}(x) \in U_2$.
Then $f^{s_n}(x) = f^{s_n - r_n}(f^{r_n}(x)) \in O^+(U_2) \subset U_1$. Thus
$y = \lim f^{s_n}(x) \in \overline{U_1}$, contrary to choice of U_1.
Then we must have $\omega(x) - K = \emptyset$ i.e. $x \in \omega^{-1}K$.
Hence $K^+ \subset \omega^{-1}K$, implying $K^+ = \omega^{-1}K$.

Proof that $K^- \subset \alpha^{-1}K$ (hence $K^- = \alpha^{-1}K$) is similar.

LEMMA 12 : Let K be compact invariant for f. Then

$$\text{K is attractor set} \iff \begin{cases} \text{inset of K is neighbourhood of K} \\ \text{semi-outset of K} = K \end{cases}$$

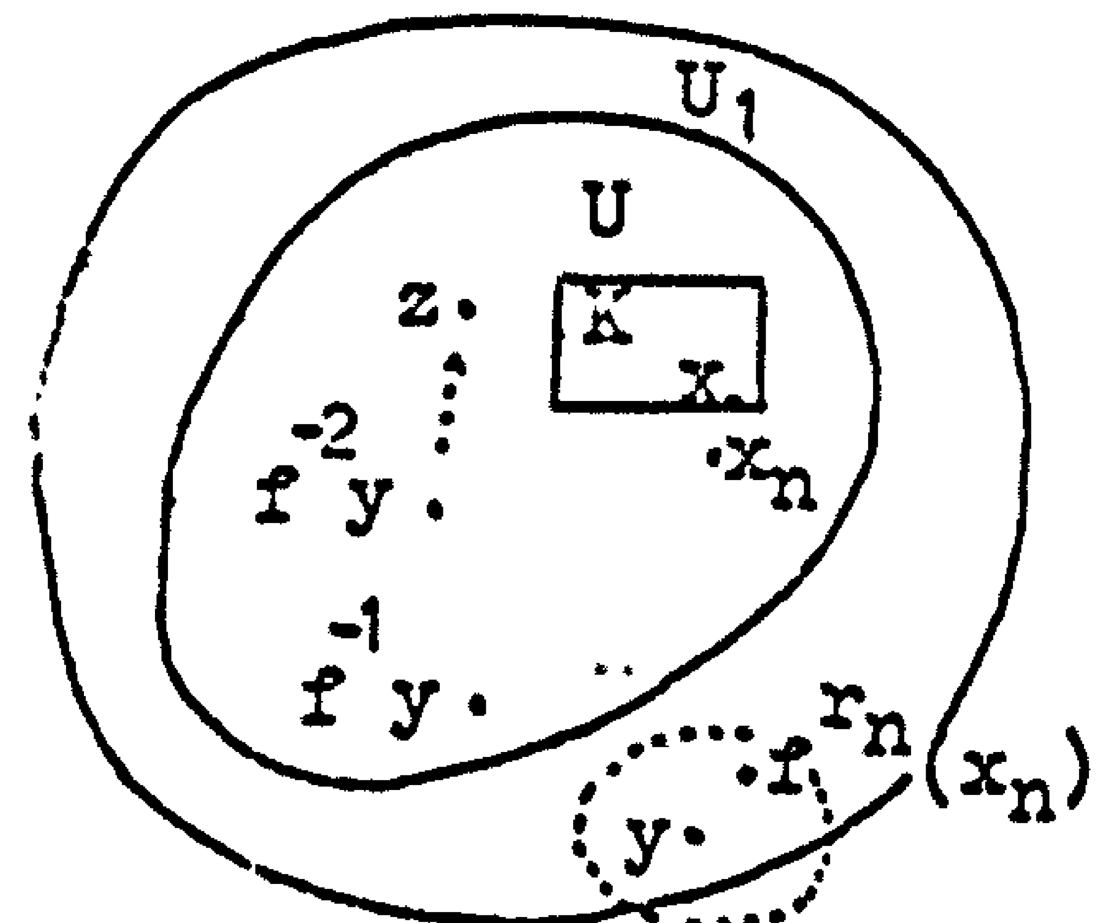
Proof:

($\Longrightarrow$) Let K be attractor set. There is open U with $K \subset U$ and $\omega(U) \subset K$. Then $U \subset \omega^{-1}K$, i.e. $\omega^{-1}K$ is neighbourhood of K.
Suppose there is $x \in K^- - K$. Then, there is sequence (r_n) with $r_n \geq 0$; $\lim r_n = +\infty$ and $\lim f^{-r_n}(x) \in K$. Take open U_1 with $K \subset U_1$ and $x \notin U_1$. There exist open U_2, with $K \subset U_2 \subset U_1$ and $O^+(U_2) \subset U_1$. Since $\lim f^{-r_n}(x) \in K$, we can suppose $f^{-r_n}(x) \in U_2$ for all n. Then
$x = f^{r_n}(f^{-r_n}(x)) \in O^+(U_2) \subset U_1$, contrary to choice of U_1.
Therefore $K^- - K = \emptyset$ and, so, $K^- = K$.
($\Longleftarrow$) We must prove that given open U_1 ($K \subset U_1$) there is open

U_2 with $K \subset U_2$, $O^+(U_2) \subset U_1$ and $\omega(U_2) \subset K$. Without loss of generality, we assume $\overline{U_1}$ is compact and $\overline{U_1} \subset \omega^{-1}K$, otherwise replace U_1 by a smaller neighbourhood of K satisfying these properties. Take open U with $K \subset U \subset U_1$ and $f(U) \subset U_1$. Suppose that for all open U_2 ($K \subset U_2$) there is $x \in U_2 - K$ and $n \geq 0$, with $f^n x \in M - U$. Then we could take (x_n) and (r_n) such that $r_n \geq 0$, $\lim x_n = x \in K$ and $f^{r_n}(x_n) \in M - U$ and $f^i(x_n) \in U$ for $0 \leq i < r_n$.

If (r_n) is bounded, there is constant subsequence, i.e. taking subsequence we can suppose $r_n = k$ for all n. Then $f^k x_n \in M - U$ implies $f^k x = \lim\limits_n f^k x_n \in M - U$. But $x \in K$ and K invariant give $f^k x \in K \subset U$, and we have a contradiction.

If (r_n) is not bounded, at least in subsequence we can assume $\lim r_n = +\infty$. But $f^{r_n - 1}(x_n) \in U$ and $f^{r_n}(x_n) \in M - U$ imply $f^{r_n}(x_n) \in U_1 - U$ (since $fU \subset U_1$) and $\overline{U_1}$ being compact, $f^{r_n}(x_n)$ converges, at least in subsequence, to $y \in \overline{U_1}$. For any fixed integer $k > 0$ for all n sufficiently large ($n \geq N(k)$) we have $r_n \geq k$ and, so, $0 \leq r_n - k < r_n$ and $f^{r_n - k}(x_n) \in U$, then $f^{-k}y = \lim\limits_n f^{r_n - k}(x_n) \in \overline{U}$. For some subsequence (k_n), $f^{-k_n}(y)$ converges to $z \in \overline{U} \cap \alpha(y)$. But in this case, $\omega(z) \subset K \cap \alpha(y)$ i.e. $y \in K^- - K$, contrary to hypothesis.



Hence, we conclude that there is some open U_2 such that $K \subset U_2 \subset U_1$ and $O^+(U_2) \subset U \subset U_1$. Also $\omega(U_2) \subset K$ since $U_2 \subset U_1 \subset \omega^{-1}K$. Therefore, K is attractor set.

As consequence of Lemmas 11 and 12, we have

$$K \text{ is attractor set} \Longrightarrow \begin{cases} \text{inset of } K \text{ is neighbourhood of } K \\ \text{outset of } K = K \end{cases}$$

We conjecture that this could also be an equivalence.

LEMMA 13 : Let K be attractor set for f. Then, there is neighbourhood U of K such that $U^+ = U$, $U^- = K$ and $J^-(U - K) \subset M - \overline{U}$.

In particular, K is isolated.

Proof :

K being attractor, by Lemma 12, $K^- = K$ and there is open U_1 with $K \subset U_1 \subset \overline{U_1} \subset \omega^{-1}K$. Also we can take open U and U_2 with $K \subset U \subset \overline{U} \subset U_2 \subset U_1$ and $O^+(U_2) \subset U_1$. Consequently, $\omega(U) \subset \omega(\overline{U_1}) \subset K$ and $U^+ = U$, $U^- = K$.

Now we want to prove that $J^-(U-K) \subset M-\overline{U}$. First we prove that $\alpha(U-K) \subset M-\overline{U_1}$. Suppose $y \in \alpha(x) \cap \overline{U_1}$ for $x \in U-K$, then $\omega(y) \subset \alpha(x) \cap K$ i.e. $x \in K^- - K = \emptyset$, which is absurd. Then $\alpha(U-K) \subset M-\overline{U_1}$. Now, if $y \in J^-(x) \cap \overline{U}$ for $x \in U-K$, there are (x_n) and (r_n) with $\lim x_n = x$, $\lim r_n = +\infty$ and $\lim f^{-r_n}(x_n) = y \in \overline{U}$. We can suppose $f^{-r_n}(x_n) \in U_2$ for all n. But $\alpha(x) \subset M-\overline{U_1}$ implies there is some $k > 0$ with $f^{-k}x \in M-\overline{U_1}$ and, so, for all sufficiently large n, $f^{-k}x_n \in M-\overline{U_1}$. Since $\lim r_n = +\infty$, we can take $r_n > k$. Then
$f^{-k}x_n = f^{r_n-k}(f^{-r_n}(x_n)) \in O^+(U_2) \subset U_1$, giving a contradiction.

Hence $J^-(x) \cap \overline{U} = \emptyset$ for all $x \in U-K$, i.e. $J^-(U-K) \subset M-\overline{U}$.

LEMMA 14 : Suppose M is compact and f satisfies Axiom A and no-cycle condition. Let Ω_i be a basic set for f . Then Ω_i is isolated .

Proof (*) :

We have noticed before that $W^s\Omega_i = \omega^{-1}\Omega_i = \Omega_i^+$, $W^u\Omega_i = \alpha^{-1}\Omega_i = \Omega_i^-$ and $W^s\Omega_i \cap W^u\Omega_i = \Omega_i$. Then, for any neighbourhood U of Ω_i we have $U^+ = W^s\Omega_i \cap U$, $U^- = W^u\Omega_i \cap U$ and $U^+ \cap U^- = \Omega_i$.

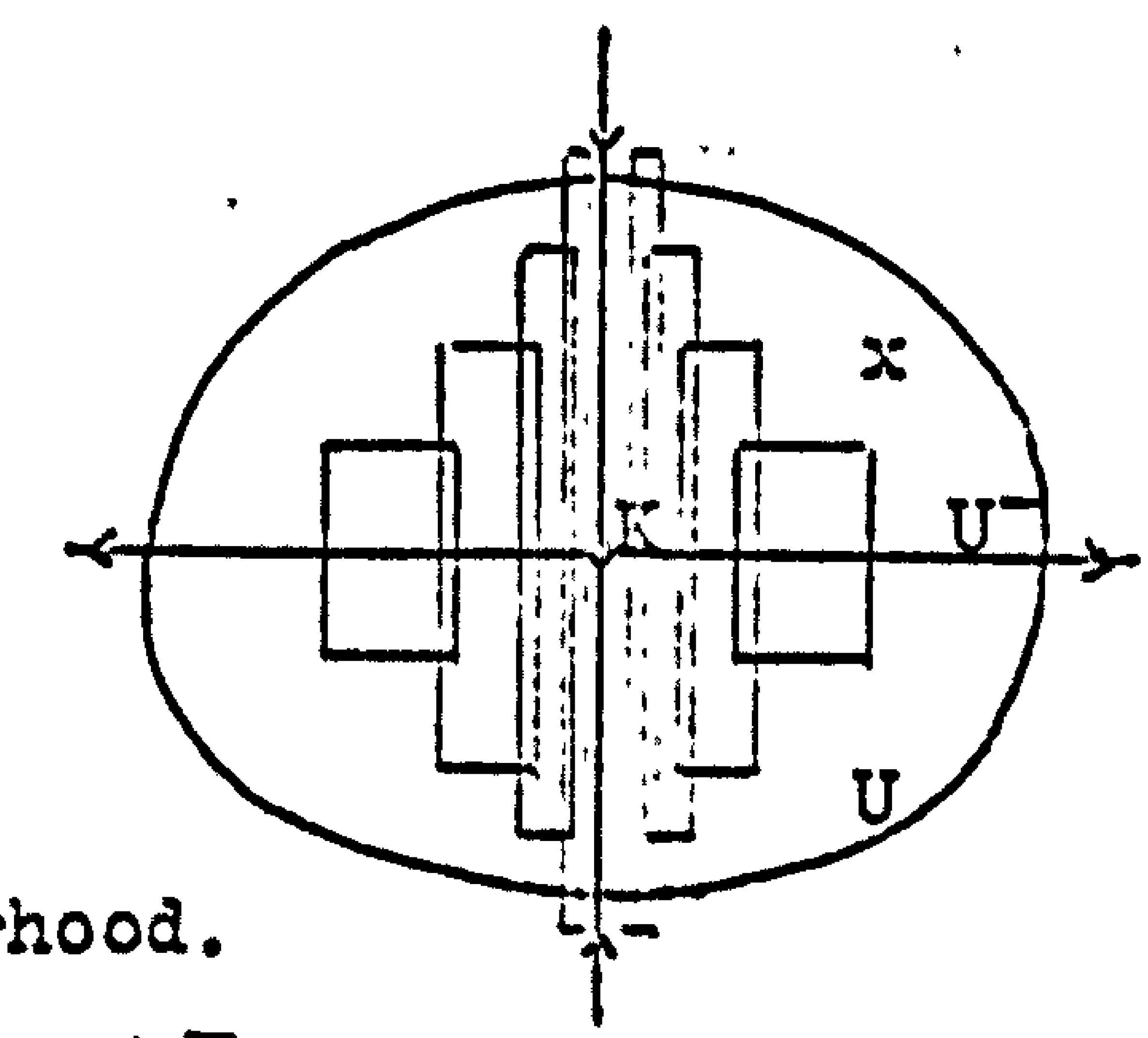
We must prove now that there is open U ($\Omega_i \subset U$) for which (a) $J^+(U-U^+) \subset M-\overline{U}$; (b) $J^-(U-U^-) \subset M-\overline{U}$. We will prove here (a) only, since (b) can be done similarly.

Let $\Omega_1, \Omega_2, \ldots, \Omega_i, \ldots, \Omega_r$ be all the basic sets of f, indexed in such a way that : $\Omega_j < \Omega_k \Longrightarrow j < k$

Denote $W_j = \bigcup\limits_{k=1}^{j} W^u\Omega_k$ and, then $W_1 \subsetneq W_2 \subsetneq \ldots \subsetneq W_r = M$

We can take open U ($\Omega_i \subset U$) such that $W_{i-1} \cap \overline{U} = \emptyset$ and $f^nU \longrightarrow W_i$, i.e. given $\varepsilon > 0$, there is integer $N(\varepsilon)$ such that $d(f^nx, W_i) < \varepsilon$ for all $x \in U$, $n > N(\varepsilon)$. Consequently, if $x_n \longrightarrow x \in U$ and $r_n \longrightarrow +\infty$, we have $d(f^{r_n}(x_n), W_i) < \varepsilon$ for all sufficiently large n. Hence $J^+(U) \subset W_i = W_{i-1} \cap W^u\Omega_i$. We will prove that, in fact, $J^+(U-U^+) \subset W_{i-1}$.

Suppose $y \in J^+(x) \cap \Omega_i$ for some $x \in U-U^+$. By [2], there exist open $D_o \subset U-U^+$ with $D_o \cap U^- \neq \emptyset$ such that $D = \bigcup\limits_{n \geq o} f^{-n}D_o \cup W^s\Omega_i$ is neighbourhood of $W^s\Omega_i$. In [2] D_o is called a fundamental neighbourhood.



D_o can be chosen in such a way that $x \notin \overline{D}$. In particular, D is neighbourhood of Ω_i (hence of y) and D is negative invariant. Since $y \in J^+(x)$, there are $x_n \longrightarrow x$, $r_n \longrightarrow +\infty$ ($r_n > 0$) with $f^{r_n}(x_n) \longrightarrow y$. For all large n, $f^{r_n}(x_n) \in D$ and,

(*) The idea of this proof was suggested to me by E.C.Zeeman

D being negative invariant, we have $x_n \in D$ and, so, $x = \lim x_n$ implies $x \in \overline{D}$, and we have a contradiction.
Hence $J^+(U-U^+) \cap \Omega_i = \emptyset$.

Now suppose $y \in J^+(x) \cap (W^u\Omega_i - \Omega_i)$ for $x \in U-U^+$. Since, by Lemma 10, $J^+(x)$ is closed and invariant, $\alpha(y) \subset J^+(x)$ But $y \in W^u\Omega_i \Longrightarrow \alpha(y) \subset J^+(x) \cap \Omega_i \subset J^+(U-U^+) \cap \Omega_i = \emptyset$, which is absurd. Then $J^+(U-U^+) \cap W^u\Omega_i = \emptyset$.

Therefore $J^+(U-U^+) \subset W_{i-1} \subset M-\overline{U}$.

Now we apply Theorem 4 and Lemmas 13 and 14 to prove Theorems 5 and 6 .

<u>Proof of Theorem 5</u>

Let K be an attractor set for f. By Lemma 13, K is isolated. Applying Theorem 4, there exist C^∞ function $V : U \longrightarrow \mathbb{R}$,for neighbourhood U of K, such that $V(K) = 0$;
$V(f^k x) < V(x)$ for $k \geq 1$, $x \in U \cap f^{-k}U - U^+ \cap U^-$;
$V|(U^+ - U^+ \cap U^-) > 0$; $V|(U^- - U^+ \cap U^-) < 0$.
Also by Lemma 13 we can take U, smaller if necessary, such that $U^+ = U$, $U^- = K$. Then $U^+ \cap U^- = K$ and
(1) $V \geq 0$; $V^{-1}(0) = K$
(2) $V(f^k x) < V(x)$ for $k \geq 1$, $x \in U \cap f^{-k}U - K$

<u>Proof of Theorem 6</u>

Suppose M, f and Ω_i as in the hypothesis. By Lemma 14, Ω_i is isolated. Applying Theorem 4, there exist C^∞ function $V_i : U_i \longrightarrow \mathbb{R}$ on neighbourhood U_i of Ω_i satisfying properties (1) to (3). Property (4) is consequence of (1),(3) and Ω_i being hyperbolic, since there is a splitting $T_{\Omega_i}M = E^s \oplus E^u$ such that $E^s = T_{\Omega_i}W^s\Omega_i$; $E^u = T_{\Omega_i}W^u\Omega_i$.

3. GLOBALIZATION

The main objective of this section is to prove the two following theorems, which give existence of global C^∞ Liapunov function for some particular cases.

THEOREM 7 : Suppose M is compact and the non-wandering set for f is hyperbolic, having a decomposition $\Omega = \Omega_1 \cup \ldots \cup \Omega_r$ in a finite union of basic sets, which satisfy no-cycle condition. Then, there exist a global C^∞ Liapunov function V for f, which is strictly decreasing on orbits outside Ω.

i.e. $V : M \longrightarrow R$ such that

(1) V is constant on each Ω_i

(2) $V(f^k x) < V(x)$ for $k \geqslant 1$, $x \notin \Omega$

As corollary of Theorem 7 and the Spectral Decomposition Theorem [8], we have :

THEOREM 8 : If M is compact and f satisfies Axiom A and no--cycle condition, there exist a global C^∞ Liapunov function for f, which is constant on each basic set Ω_i and strictly decreasing on orbits outside Ω.

Proof of Theorem 7 (*) :

Each Ω_i (i=1,...,r) is open and closed in Ω, then we can take open $\tilde{U}_i$, containing Ω_i and $\tilde{U}_i$'s disjoint.
The no-cycle condition allows us to partially order the basic sets by $\Omega_i < \Omega_j \iff W^s\Omega_i \cap W^u \Omega_j \neq \emptyset$.

(*) The idea of this proof was suggested to me by E.C.Zeeman, after a theorem by P.Purcell.

Then we can choose real numbers $a_1, a_2, \ldots, a_r$ such that $a_i < a_j$ if $\Omega_i < \Omega_j$. Moreover we can choose $\tilde{U}_i$ so small that $i \neq j$, $k \geq 1$, $x \in \tilde{U}_j$, $f^k x \in \tilde{U}_i \Longrightarrow \Omega_i < \Omega_j$.

As consequence of Theorem 4, for each $i=1,\ldots,r$, there exist (local) C^∞ Liapunov function $\tilde{V}_i$, defined on neighbourhood of Ω_i (which we can suppose to be as above $\tilde{U}_i$) and equal to a_i on Ω_i, i.e. $\tilde{V}_i : \tilde{U}_i \longrightarrow \mathbb{R}$

(a) $\tilde{V}_i(\Omega_i) = a_i$

(b) $\tilde{V}_i(f^k x) < \tilde{V}(x)$ for $k \geq 1$, $x \in \tilde{U}_i \cap f^{-k}\tilde{U}_i - \Omega_i$

Making U_i even smaller, if necessary, we can make

(c) $\sup \tilde{V}_i < \inf \tilde{V}_j$, whenever $a_i < a_j$

As a consequence, we can define

$\tilde{V} : \tilde{U} \longrightarrow \mathbb{R}$, for $\tilde{U} = \bigcup_{i=1}^{r} \tilde{U}_i$ by $\tilde{V}|\tilde{U}_i = \tilde{V}_i$.

Then $\tilde{V}$ is C^∞ on $\tilde{U}$ and satisfy

(i) $\tilde{V}(\Omega_i) = a_i$

(ii) $\tilde{V}(f^k x) < \tilde{V}(x)$ for $k \geq 1$, $x \in \tilde{U} \cap f^{-k}\tilde{U} - \Omega$

(ii) is a consequence of (b) and (c).

Now we take open sets U_i and U_i' such that

$\Omega_i \subset U_i \subset \overline{U}_i \subset U_i' \subset \overline{U_i'} \subset \tilde{U}_i$

Let $U = \bigcup_{i=1}^{r} U_i$; $U' = \bigcup_{i=1}^{r} U_i'$

We intend to construct a C^∞ function $W : M-U \longrightarrow \mathbb{R}$, satisfying

(iii) $W(x) = \tilde{V}(x)$ for $x \in U' - U$

(iv) $W(f^k x) < W(x)$ for $k \geq 1$, $x \in M-U$, $f^k x \in M-U$

(v) $W(f^k x) < \tilde{V}(x)$ for $k \geq 1$, $x \in U$, $f^k x \in M-U$

(vi) $\tilde{V}(f^k x) < W(x)$ for $k \geq 1$, $x \in M-U$, $f^k x \in U$

After constructing such W, we can define $V : M \longrightarrow \mathbb{R}$ by

$$\begin{cases} V(x) = \tilde{V}(x) & \text{for } x \in U' \\ V(x) = W(x) & \text{for } x \in M-U \end{cases}$$

Since $\tilde{V}$ and W are C^∞, we have, as consequence of condition(iii) that V is well defined and C^∞ on M.

Condition (i) implies

(1) $V(\Omega_i) = a_i$

Conditions (ii), (iv),(v),(vi) imply

(2) $V(f^k x) < V(x)$ for $k \geq 1$, $x \not\subset \Omega$

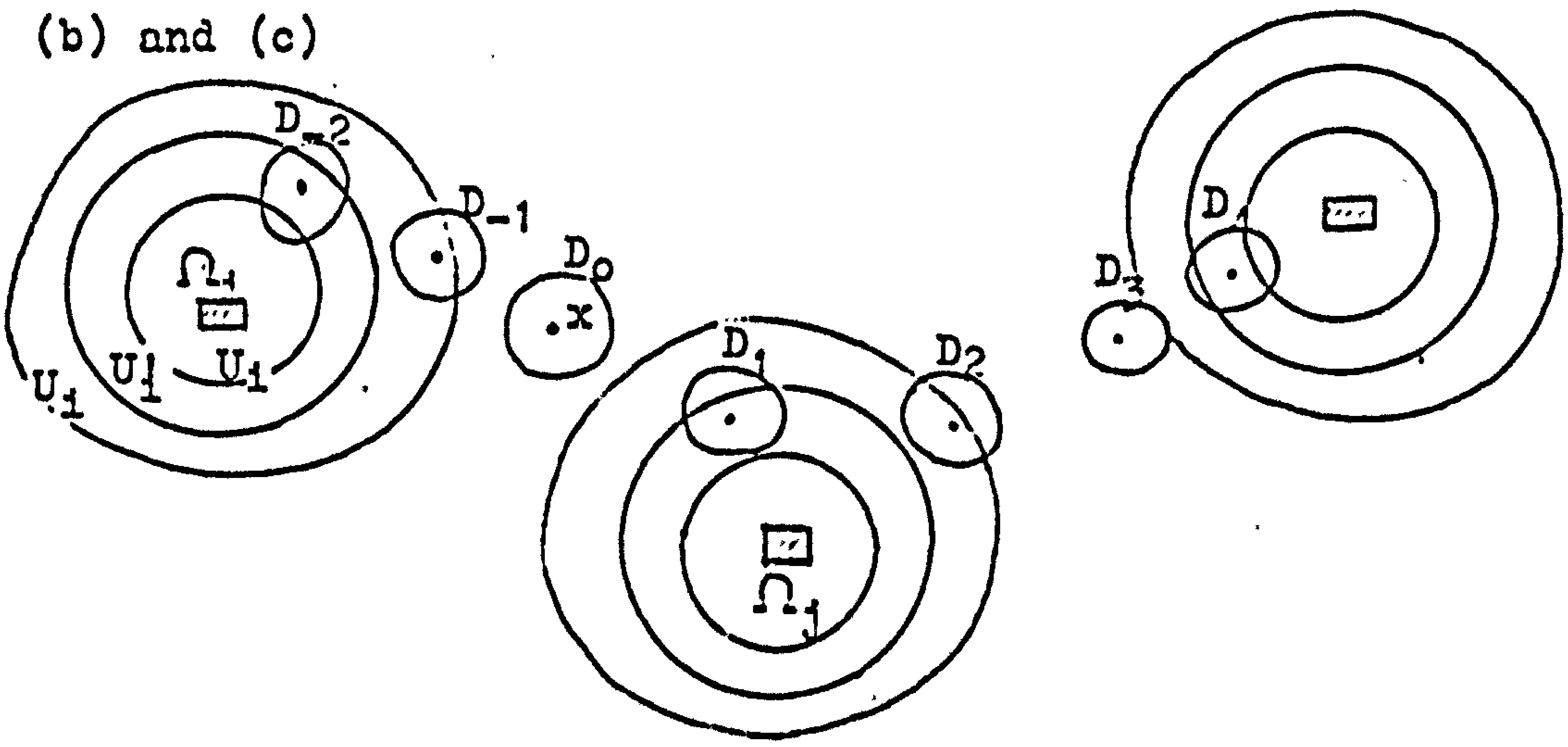
Now we have to carry out the construction of W.

For every $x \in M-U$, x is a wandering point and $\omega(x) \subset \Omega_{i_1}$, $\alpha(x) \subset \Omega_{i_2}$ for some $\Omega_{i_1} < \Omega_{i_2}$

Then we can take (and fix) integers $k_-(x) < 0 < k_+(x)$ such that
$$\begin{cases} f^k x \in U_{i_1} & \text{for all } k \geq k_+(x) \\ f^k x \in U_{i_2} & \text{for all } k \leq k_-(x) \end{cases}$$

and we can take an open neighbourhood $D_o(x)$ of x, in such a way that, writing $D_k(x) = f^k(D_o(x))$ for $k_-(x) \leq k \leq k_+(x)$, we have
$$\begin{cases} D_k \subset \tilde{U}_i & \text{if } f^k x \in U_i' \\ \overline{D_k} \cap \overline{D_m} = \emptyset & \text{if } k_- \leq k,m \leq k_+ \ , \ k \neq m \\ \inf(\tilde{V}_i | D_k) > \sup(\tilde{V}_j | D_m) & \text{if } k < m \text{ and } f^k x \in U_i' \\ & \qquad f^m x \in U_j' \end{cases}$$

Such choice of $D_o(x)$ is always possible, as consequence of (b) and (c)



Now, take open $E_o(x)$ such that $x \in E_o \subset \overline{E_o} \subset D_o$. Writing $E_k(x) = f^k(E_o(x))$ for $k_- \leq k \leq k_+$, it follows

$f^k x \in E_k \subset \overline{E_k} \subset D_k$. Let $\mathcal{D}_x = \bigcup_{k_-(x)}^{k_+(x)} D_k(x)$, $\mathcal{E}_x = \bigcup_{k_-(x)}^{k_+(x)} E_k(x)$

$\{\mathcal{E}_x\}_{x\in M-U}$ is open cover of $M-U$. By compactness, there is finite subcover, i.e., we can take $x_1, x_2, \ldots, x_N \in M-U$ such that $M-U \subset \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_N$ for $\mathcal{E}_n = \mathcal{E}_{x_n}$

Let $\mathcal{D}_n = \mathcal{D}_{x_n}$ and define

$W_n : \mathcal{D}_n \longrightarrow \mathbb{R}$ by defining W_n on $D_k(x_n)$ as follows :

- if $f^k x_n \in U_i'$ for some i $(1 \leq i \leq r)$ put $W_n|D_k = \tilde{V}_i|D_k$
- if $f^k x_n \notin U'$, there are (unique) $j_1, j_2 (k_- \leq j_1 < k < j_2 < k_+)$

  such that $\begin{cases} f^{j_1}(x_n) \in U' \ ; \ f^{j_2}(x_n) \in U' \text{ and} \\ f^j(x_n) \notin U' \text{ for all } j_1 < j < j_2 \end{cases}$

  Take $c_2 = \sup (\tilde{V}|D_{j_2})$; $c_1 = \inf (\tilde{V}|D_{j_1})$. Then $c_2 < c_1$.

  Define W_n as constant on D_k by putting

  $W_n(y) = c_1 - \dfrac{c_1 - c_2}{j_1 - j_2}(k - j_1)$ for all $y \in D_k(x_n)$.

Hence W_n is well-defined and C^∞ on $\mathcal{D}_n$, and satisfies

$(iv)^n$ $W_n(f^k y) < W_n(y)$ for $k \geq 1$, $y \in (M-U)\cap\mathcal{D}_n$, $f^k y \in M-U$

$(v)^n$ $\tilde{V}(f^k y) < W_n(y)$ for $k \geq 1$, $y \in (m-U)\cap\mathcal{D}_n$, $f^k y \in U$

$(vi)^n$ $W_n(f^k y) < \tilde{V}(y)$ for $k \geq 1$, $y \in U$, $f^k y \in (M-U)\cap\mathcal{D}_n$

Let $\beta_n : M \longrightarrow [0,1]$ be any C^∞ function such that

$$\begin{cases} \beta_n(E_o(x_n)) = 1 \\ \beta_n(M-D_o(x_n)) = 0 \end{cases}$$

Let $\gamma_n : M \longrightarrow \mathbb{R}$ be defined by

$$\begin{cases} \gamma_n(M-\mathcal{D}_n) = 0 \\ \gamma_n|D_k(x_n) = \beta_n f^{-k}|D_k(x_n) \quad \text{for } k_-(x_n) \leq k \leq k_+(x_n) \end{cases}$$

As consequences, we have

$$\begin{cases} \gamma_n \text{ is } C^\infty \text{ on } M \\ \gamma_n(\mathcal{E}_n) = 1 \\ \gamma_n(f^k y) = \gamma_n(y) \quad \text{if } y, f^k y \in M-U \\ \sum_{n=1}^{N} \gamma_n(y) > 0 \quad \text{for } y \in M-U \end{cases}$$

Define $\lambda_n : M \longrightarrow \mathbb{R}$ by $\begin{cases} \lambda_n(y) = 0 & \text{for } y \in M-\mathcal{D}_n \\ \lambda_n(y) = \dfrac{\delta_n(y)}{\sum_m \delta_m(y)} & \text{for } y \in M-U \end{cases}$

λ_n is well defined, non-negative and C^∞ on M, and satisfies

$$\sum_{n=1}^{N} \lambda_n(y) = 1 \quad \text{for all } y \in M-U ,$$

and $\lambda_n(f^k y) = \lambda_n(y)$ if $y, f^k y \in M-U$

Now we can define our wanted $W : M-U \longrightarrow \mathbb{R}$ by

$$W(y) = \sum_{n=1}^{N} \lambda_n(y)\, W_n(y)$$

W is well defined and C^∞ on $M-U$. We claim that W satisfies properties (iii) to (vi) and then our proof is complete.

(iii) Let $y \in U' - U$

If $y \in \mathcal{D}_n$, $W_n(y) = \tilde{V}(y)$, by definition of W_n

If $y \notin \mathcal{D}_n$, $\lambda_n(y) = 0$. Then

$$W(y) = \Sigma\, \lambda_n(y)\, W_n(y) = \sum_{y \in \mathcal{D}_n} \lambda_n(y)\, W_n(y) =$$

$$= \sum_{y \in \mathcal{D}_n} \lambda_n(y)\, \tilde{V}(y) = \Big(\sum_{y \in \mathcal{D}_n} \lambda_n(y)\Big)\, \tilde{V}(y) = \tilde{V}(y)$$

(iv) Let $k \geqslant 1$, $y \in M-U$, $f^k y \in M-U$

$$W(f^k y) = \Sigma\, \lambda_n(f^k y)\, W_n(f^k y) = \Sigma\, \lambda_n(y)\, W_n(f^k y) <$$

$$< \Sigma\, \lambda_n(y)\, W_n(y) = W(y)$$

since $W_n(f^k y) < W_n(y)$ and for some n, $\lambda_n(y) > 0$

(v) Let $k \geqslant 1$, $y \in U$, $f^k y \in M-U$

$$W(f^k y) = \Sigma\, \lambda_n(f^k y)\, W_n(f^k y) = \sum_{f^k y \in \mathcal{D}_n} \lambda_n(f^k y)\, W_n(f^k y) <$$

$$< \sum_{f^k y \in \mathcal{D}_n} \lambda_n(f^k y)\, \tilde{V}(y) = \sum_{f^k y \in \mathcal{D}_n} \lambda_n(f^k y)\, \tilde{V}(y) = \tilde{V}(y)$$

since $W_n(f^k y) < \tilde{V}(y)$ and for some n, $\lambda_n(f^k y) > 0$.

(vi) Let $k \geqslant 1$, $y \in M-U$, $f^k y \in U$

$$W(y) = \Sigma \lambda_n(y) W_n(y) = \sum_{y \in \mathcal{D}_n} \lambda_n(y) W_n(y) >$$

$$> \sum_{y \in \mathcal{D}_n} \lambda_n(y) \tilde{V}(f^k y) = \sum_{y \in \mathcal{D}_n} \lambda_n(y) \quad \tilde{V}(f^k y) = \tilde{V}(f^k y)$$

since $\tilde{V}(f^k y) < W_n(y)$ if $y \in \mathcal{D}_n$ and, for some n, $\lambda_n(y) > 0$.

If our goal is to construct C^∞ function $V: M \longrightarrow \mathbb{R}$ such that $V(f^k x) < V(x)$ whenever this is possible, we afirm that our results in Theorem 7 and 8 are the best possible in the respective cases. This means that V could never decrease on orbits which are inside the non-wandering set Ω, since any orbit in Ω must be entirely contained in one of the basic sets and V must be constant on each basic set (this is given by the Corollary of next Lemma 15) .

As a consequence of this observation and Theorem 7, if M is compact and f has a spectral decomposition in basic sets for Ω, we have :

$\Omega = \{x \in M$; $V(fx) = V(x)$ for all Liapunov function V for f on M $\}$

LEMMA 15 : Let V be Liapunov function for f, defined on neighbourhood of Ω . Then V is constant on $\overline{O(x)}$ for any $x \in \Omega$.

Proof:

V is constant on $\overline{O(x)} \Longleftrightarrow$ V is constant on $O(x) \Longleftrightarrow$ $V(f^{k+1}x) = V(f^k x)$ for all integer k .

Suppose that, for some k, $V(f^{k+1}x) < V(f^k x)$.

Call $y = f^k x$; then $V(fy) < V(y)$. Take $a = \frac{1}{2}(V(y)+V(fy))$ and open U ($y \in U$) such that $V(fU) < a < V(U)$.

For $n \geqslant 1$ and $z \in f^n U$, we have $z = f^n y'$ for some $y' \in U$. Then $V(z) = V(f^{n-1}(fy')) \leqslant V(fy') < a \Longrightarrow z \notin U$. Hence $U \cap f^n U = \emptyset$ for all $n \geqslant 1$.

For $n \geqslant 1$ and $z \in f^{-n} U$, we have $z = f^{-n} y'$ for some $y' \in U$. Then $y' = f^n z$, and, since $f^n U \cap U = \emptyset$, it follows that $z \notin U$ (because $z \in U \Longrightarrow y' \in U \cap f^n U$). Hence $U \cap f^{-n} U = \emptyset$.

Then $U \cap f^n U = \emptyset$ for all integer $n \neq 0$ i.e. y is a wandering point for f, contradictiong hypothesis, since $x \in \Omega$ implies $y = f^k x \in \Omega$.

Therefore $V(f^{k+1}x) = V(f^k x)$ for all k, and, so, V is constant on $\overline{O(x)}$.

COROLLARY : Let Ω_i be basic set for f and V be Liapunov function, defined on neighbourhood of Ω_i.
Then V is constant on Ω_i.

Proof:

We remember that, by definition, Ω_i is transitive, i.e. there is $x \in \Omega_i$ such that $\Omega_i = \overline{O(x)}$. By Lemma 15, V must be constant on Ω_i.

Here we remark that previous results on construction of global C^∞ Liapunov function (or similar) for dynamical systems, in the case of flows, can be found in [6] for Morse--Smale systems and in [9] for gradient flows. However, our method of construction is different from both.

4. EXAMPLE : THE HORSESHOE

We start with a short review on the horseshoe.

Consider $I = [0,1]$ and $R = I \times I \subset \mathbb{R}^2$.

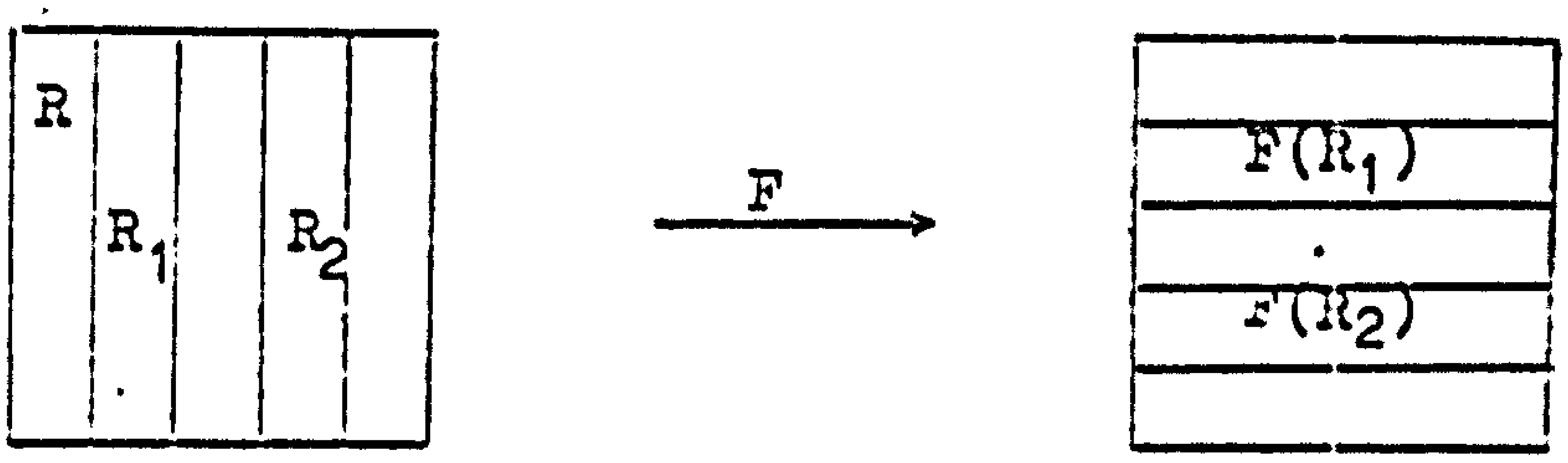
$$\pi_1 : R \longrightarrow I \qquad\qquad \pi_2 : R \longrightarrow I$$
$$(x,y) \longmapsto x \qquad\qquad (x,y) \longmapsto y$$

and $R_1 = \pi_1^{-1}[\frac{1}{5}, \frac{2}{5}]$ $\quad R_2 = \pi_1^{-1}[\frac{3}{5}, \frac{4}{5}]$

Let $F : R_1 \cup R_2 \longrightarrow R$ be such that

$F|R_1$ takes R_1 linearly onto $\pi_2^{-1}[\frac{1}{5}, \frac{2}{5}]$

$F|R_2$ takes R_2 linearly onto $\pi_2^{-1}[\frac{3}{5}, \frac{4}{5}]$



Let M be two-dimensional manifold and $\varphi : R \longrightarrow M$ a embedding.

Let $\tilde{R} = \varphi(R)$; $\tilde{R}_1 = \varphi(R_1)$; $\tilde{R}_2 = \varphi(R_2)$

Consider a diffeomorphism $f : M \longrightarrow M$ such that

(a)
$$\begin{tikzcd} R_1 \cup R_2 \arrow[r, "\varphi"] \arrow[d, "F"'] & M \arrow[d, "f"] \\ R \arrow[r, "\varphi"] & M \end{tikzcd}$$
is commutative diagram

(b) $f\,\varphi(R - R_1 \cup R_2) \subset M - \tilde{R}$

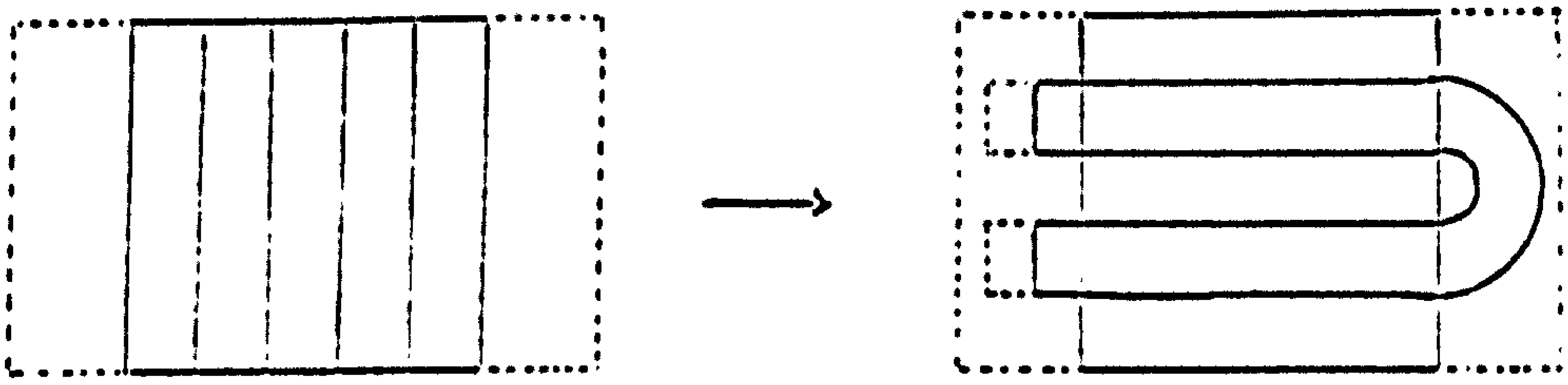
(c) If $p \in \tilde{R}$ and $fp \notin \tilde{R}$ then $f^i p \notin \tilde{R}$ for all $i \geq 1$

Then we say that the germ of f at $\Omega(f) \cap \tilde{R}$ is a <u>horseshoe</u> .

Call $H = \Omega(f) \cap \tilde{R}$ the <u>horseshoe set</u> for f .

As a consequence of (c) we have

(d) If $p \in \tilde{R}$ and $f^{-1}p \notin \tilde{R}$ then $f^{-i}p \notin \tilde{R}$ for all $i \geq 1$

Example : 

Take $\tilde{B}^{o}_{-1} = \tilde{R}$

$$\tilde{B}^{o}_{j} = f(\tilde{B}^{o}_{j-1}) \cap \tilde{R} \quad ; \quad \tilde{B}^{k}_{-1} = f^{-1}(\tilde{B}^{k-1}_{-1}) \cap \tilde{R}$$

$$\tilde{B}^{o}_{\infty} = \bigcap_{j\geq -1} \tilde{B}^{o}_{j} \quad ; \quad \tilde{B}^{\infty}_{-1} = \bigcap_{k\geq 0} \tilde{B}^{k}_{-1}$$

$$\tilde{B}^{k}_{j} = \tilde{B}^{o}_{j} \cap \tilde{B}^{k}_{-1} \quad ; \quad \tilde{B}^{\infty}_{\infty} = \tilde{B}^{o}_{\infty} \cap \tilde{B}^{\infty}_{-1}$$

and $B^{k}_{j} = \varphi^{-1}(\tilde{B}^{k}_{j})$ $\quad -1 \leq j \leq \infty \quad 0 \leq k \leq \infty$

Then the intersection of the non-wandering set of f with $\tilde{R}$ ($= H$) is given by $\tilde{B}^{\infty}_{\infty}$.

We known that H is a Cantor set, i.e., H is perfect and totally disconnected.

See definitions and proofs in [7] and [8] .

We want to construct a Liapunov function $\tilde{V} : \tilde{U} \longrightarrow \mathbb{R}$, relative to the set H i.e. $H \subset \tilde{U}$ and

(1) $\tilde{V}(H) = 0$

(2) $\tilde{V}(f^{k}p) < \tilde{V}(p)$ for $k \geq 1$, $p \in \tilde{U} \cap f^{-k}\tilde{U} - H$

We will construct first $V : U \longrightarrow \mathbb{R}$ such that

(1)' $V(B^{\infty}_{\infty}) = 0$

(2)' $V(Fx) < V(x)$ for $x \in \operatorname{int}(R_1 \cup R_2) - B^{\infty}_{\infty}$

where $U = \operatorname{int} R$.

Then we can define $\tilde{V} : \tilde{U} \longrightarrow \mathbb{R}$ for $\tilde{U} = \operatorname{int} \tilde{R}$ by $\tilde{V}(p) = V(\varphi^{-1}(p))$ for $p \in \tilde{U}$.

We claim that this $\tilde{V}$ is a Liapunov function for f , relative to H.

In fact,

(1) $p \in H \Rightarrow \varphi^{-1}(p) \in B_{\infty}^{\infty} \Rightarrow \tilde{V}(p) = V(\varphi^{-1}(p)) = 0$

(2) Suppose $p \in \tilde{U} \cap f^{-1}\tilde{U}$. Then, we must have

$p \in \operatorname{int}(\tilde{R}_1 \cup \tilde{R}_2) = \tilde{B}_0^0$ and $\varphi^{-1}(p) \in \operatorname{int}(R_1 \cup R_2)$

hence $\tilde{V}(fp) = V(\varphi^{-1}(fp)) = V(F(\varphi^{-1}(p)) <$ (by (2)')

$< V(\varphi^{-1}(p)) = \tilde{V}(p)$

If $k \geqslant 1$ and $p \in \tilde{U} \cap f^{-k}\tilde{U}$, by condition (c), we have that $f^i p \in \tilde{U}$ for all $0 \leqslant i \leqslant k$.

Then $\tilde{V}(f^k p) < \tilde{V}(f^{k-1}p) < \ldots < \tilde{V}(fp) < \tilde{V}(p)$

Therefore, $\tilde{V}$ is Liapunov function for f, relative to H.

Now we have to construct $V : U \longrightarrow \mathbb{R}$ satisfying (1)' and (2)' .

We call $C_k = \Pi_1(B_{-1}^k) = \Pi_1(B_j^k)$ for any $j \geqslant -1$ and $C = \bigcap_{k \geqslant 0} C_k$. Note that C is a Cantor set contained in I, and which can be characterized by the following lemma.

LEMMA 16 : $C = \{ x \in \mathbb{R} ;\ x = \lim \frac{a_n}{5^n}$ where $(a_n)_{n \geqslant 0}$ is sequence with $a_0 = 0$, $a_n = 5\, a_{n-1} + \delta_n$ and $(\delta_n)_{n \geqslant 1}$ such that $\delta_n = 1$ or $3 \}$

Proof:

$C = \Pi_1(B_{\infty}^{\infty}) = \bigcap_{n \geqslant 0} \Pi_1(B_{-1}^n)$ and B_{-1}^n is the union of 2^n disjoint intervals $(I_{n,j}\ ;\ 1 \leqslant j \leqslant 2^n)$ each of length $1/5^n$.

Say $I_{n,j} = [c_{n,j}\, ,\, c_{n,j} + \frac{1}{5^n}]$ $\qquad I_{0,1} = I = [0,1]$

$c_{n,2j-1} = c_{n-1,j} + \frac{1}{5^n}$; $c_{n,2j} = c_{n-1,j} + \frac{3}{5^n}$ for $0 \leqslant j \leqslant 2^n$

Now we note that $x \in C$ if and only if x is the limit of a sequence (c_{n,j_n}) with $2j_{n-1} - 1 \leqslant j_n \leqslant 2j_{n-1}$

$x = \lim c_{n,j_n}$. Take $\delta_n = \begin{cases} 1, & \text{if } j_n \text{ is odd} \\ 3, & \text{if } j_n \text{ is even} \end{cases}$

and $a_n = 5^n c_{n,j_n}$. Then $a_0 = 5^0 c_{0,j_0} = 0$

$$a_n = 5^n c_{n,j_n} = 5^n (c_{n-1,j_{n-1}} + \frac{\delta_n}{5^n}) = 5^n (\frac{a_{n-1}}{5^{n-1}} + \frac{\delta_n}{5^n}) =$$
$$= 5 a_{n-1} + \delta_n$$

Hence $x = \lim \frac{a_n}{5^n}$, as we wanted.

As a consequence of Lemma 16, we have that $\min C = \frac{1}{4}$ and $\max C = \frac{3}{4}$. Then, as C is closed, for every $x \in [\frac{1}{4}, \frac{3}{4}] - C$, there are unique $a(x)$, $b(x) \in C$ such that

$$x \in (a(x) , b(x)) \text{ and } (a(x),b(x)) \cap C = \emptyset .$$

To determine $a(x)$, $b(x)$ we note that, in this case, as $x \notin C$, then $x \notin C_k$ for some C_k . Take $n = \min\{ k;\ x \notin C_k \}$. Let a',b' such that $a' \in C_n$, $b' \in C_n$, $(a',b') \cap C = \emptyset$ and $x \in (a',b')$. Then

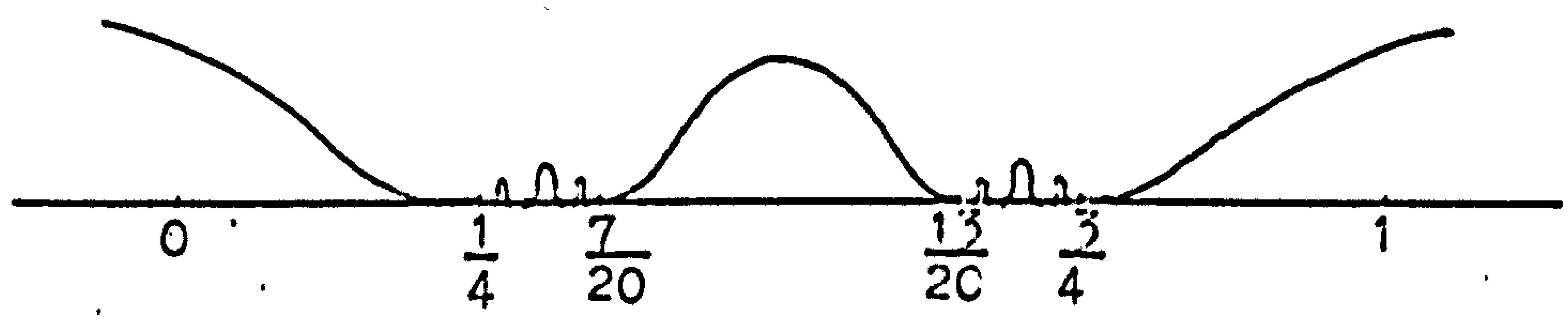
$$a(x) = a' - \frac{1}{4.5^n}$$
$$b(x) = b' + \frac{1}{4.5^n}$$

We note that there is positive integer k for which $b' - a' = \frac{k}{5^n}$. Therefore $b(x) - a(x) = \frac{2k+1}{2.5^n}$.

Now we go back to the actual construction of V, and for that we define an auxiliary function $W : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$W(x) = \begin{cases} \exp(-(\frac{1}{4} - x)^{-1}) & \text{if } x < \frac{1}{4} \\ \exp(-(x - \frac{3}{4})^{-1}) & \text{if } x > \frac{3}{4} \\ \exp(-((x-a(x))(b(x)-x))^{-1}) & \text{if } x \in (\frac{1}{4},\frac{3}{4}) - C \\ 0 & \text{if } x \in C \end{cases}$$

W is C^∞, non-negative, and $W^{-1}(0) = C$.



For $U = \text{int } R$, we define

$g_+ : U \longrightarrow \mathbb{R}$, $g_- : U \longrightarrow \mathbb{R}$, $V : U \longrightarrow \mathbb{R}$ by

$g_+(x,y) = W(x)$

$g_-(x,y) = (W(y))^4$

$V = g_- - g_+$

Remark: We take exponent 4 in definition of g_- to be able to prove (in Lemma 18) that B^∞_∞ is characterized as the only points where V and grad V are zero simultaneously. If we had taken $g_-(x,y) = W(y)$ instead all the other properties would still be true but not this one.

We have:

$g_+(x,y) = 0 \Longleftrightarrow W(x) = 0 \Longleftrightarrow x \in C \Longleftrightarrow (x,y) \in B^\infty_{-1}$ i.e.

$g_+^{-1}(0) = B^\infty_{-1} = W^s(H) \cap U = U^+$

Similarly, $g_-^{-1}(0) = B^0_\infty := W^u(H) \cap U = U^-$

LEMMA 17 : I) $\underline{g_+(F(x,y)) > g_+(x,y) \quad \text{for } (x,y) \in \text{int}(R_1 \cup R_2) - B^\infty_\infty}$

II) $\underline{g_-(F(x,y)) < g_-(x,y) \quad \text{for } (x,y) \in \text{int}(R_1 \cup R_2) - B^\infty_\infty}$

Proof:

$$\text{int}(R_1 \cup R_2) = \{(x,y) \; ; \; x \in (\tfrac{1}{5},\tfrac{2}{5}) \cup (\tfrac{3}{5},\tfrac{4}{5}) \; , \; y \in (0,1)\}$$

Write $F(x,y) = (x',y')$. We can consider several cases

Case 1 : $x \in (\frac{1}{4},\frac{7}{20}) \cup (\frac{13}{20},\frac{3}{4})$. Then we have that $x' \in (\frac{1}{4},\frac{3}{4})$

and $(x'-a(x')).(b(x')-x') = 25\ (x-a(x)).(b(x)-x)$. Thus

$$g_+(F(x,y)) = \exp\left(-\left((x'-a(x')).(b(x')-x')\right)^{-1}\right) =$$

$$= \exp\left(-\left(25\ (x-a(x)).(b(x)-x)\right)^{-1}\right) >$$

$$> \exp\left(-\left(\cdot(x-a(x)).(b(x)-x)\right)^{-1}\right) = g_+(x,y)$$

Case 2: $x \in (\frac{7}{20},\frac{2}{5}) \cup (\frac{3}{5},\frac{13}{20})$. Then we have $a(x) = \frac{7}{20}$, $b(x) = \frac{13}{20}$ and $x' \in (0,\frac{1}{4}) \cup (\frac{3}{4},1)$

For $x' \in (0,\frac{1}{4})$,

$$g_+(F(x,y)) = \exp\,(-\,(\tfrac{1}{4}-x)^{-1}) >$$
$$> \exp\,(-\,((x-a(x)).(b(x)-x))^{-1}) = g_+(x,y)$$

For $x' \in (\frac{3}{4},1)$,

$$g_+(F(x,y)) = \exp\,(-\,(x-\tfrac{3}{4})^{-1}) >$$
$$> \exp\,(-\,((x-a(x)),(b(x)-x))^{-1}) = g_+(x,y)$$

Case 3 ($x \in (\frac{1}{5},\frac{1}{4})$) and Case 4 ($x \in (\frac{3}{5},\frac{4}{5})$) can be done similarly.

II) is also proved similarly.

As consequence of Lemma 17 and previous remarks, we have that function $V = g_- - g_+$ satisfies conditions (1)',(2)' Therefore, V is a C^∞ Liapunov function for the diffeomorphism f, relative to the horseshoe set H .

Now we want to prove an additional property of the function $\tilde{V}$, which is :

$$p \in H \iff \begin{cases} \tilde{V}(p) = 0 & \text{and} \\ \operatorname{grad} \tilde{V}(p) = 0 \end{cases}$$

This will be a corollary of similar property for V, and which we prove in next lemma 18

LEMMA 18 : $\underline{(x,y) \in B^\infty_\infty} \iff \begin{cases} \underline{V(x,y) = 0 \quad \text{and}} \\ \underline{\operatorname{grad} V(x,y) = 0} \end{cases}$

Proof:

($\Longrightarrow$) If $(x,y) \in B^\infty_\infty$, then $x \in C$, $y \subset C$. So, $V(x,y) = 0$ and $\frac{\partial V}{\partial x}(x,y) = -\frac{dg_+^\omega}{dx}(x) = 0$; $\frac{\partial V}{\partial y}(x,y) = \frac{dg_-}{dy}(y) = 0$

($\Longleftarrow$) Suppose $(x,y) \in U - B^{\infty}_{\infty}$

If $x < \frac{1}{4}$ or $x > \frac{3}{4}$, we have $\frac{dg_+}{dx}(x) \neq 0$. Then $\frac{\partial V}{\partial x}(x,y) \neq 0$

If $y < \frac{1}{4}$ or $y > \frac{3}{4}$, we have $\frac{dg_-}{dy}(y) \neq 0$. Then $\frac{\partial V}{\partial x}(x,y) \neq 0$

If $\frac{1}{4} < x < \frac{3}{4}$,

$$\frac{\partial V}{\partial x}(x,y) = -\frac{dg_+}{dx}(x) = -\frac{d}{dx}\exp\left(-((x-a(x)).(b(x)-x))^{-1}\right) =$$

$$= -\frac{-2x+a(x)+b(x)}{(x-a(x))^2(b(x)-x)^2}\exp\left(-((x-a(x)).(b(x)-x))^{-1}\right) =$$

$$= 0 \quad \Longleftrightarrow \quad x = \tfrac{1}{2}(a(x)+b(x))$$

If $\frac{1}{4} < y < \frac{3}{4}$, similarly

$$\frac{\partial V}{\partial y}(x,y) = \frac{d}{dy}\exp\left(-((y-a(y)),(b(y)-y))^{-1}\right) = 0$$

$$\Longleftrightarrow y = \tfrac{1}{2}(a(y)+b(y))$$

But, $V(\frac{1}{2}(a(x)+b(x)), \frac{1}{2}(a(y)+b(y))) =$

$$= \exp(-16\,(b(y)-a(y))^{-2}) - \exp(-4\,(b(x)-a(x))^{-2}) = 0$$

$$\tfrac{1}{4}(b(y)-a(y))^2 = (b(x)-a(x))^2$$

$$b(y)-a(y) = 2\,(b(x)-a(x))$$

There are integers j,k,m,n such that $b(x)-a(x) = \frac{2j+1}{2.5^m}$, $b(y)-a(y) = \frac{2k+1}{2.5^n}$, as we have noted before. Then $(2k+1)\,5^m = 2\,(2j+1)\,5^n$, which is absurd, since left-hand side is odd and right-hand side is even.

Hence, $\left.\begin{array}{l} V(x,y) = 0 \\ \text{grad } V(x,y) = 0 \end{array}\right\} \Longleftrightarrow (x,y) \in B^{\infty}_{\infty}$.

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