On the Classification and Geometry of Finite Map-germs.

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Summary.

Summary of Chapter I:

1. We use Samuel's theory of multiplicity to describe the structure of $\theta(f)/T\mathcal{A}f$ in terms of the multiplicity $e(\theta(f)/T\mathcal{A}f)$ and the dimension of the instability locus $\Sigma$ (this dimension is also the Krull dimension of $\theta(f)/T\mathcal{A}f$).

2. We extend the theory of trivial unfoldings of map-germs to the theory of $k$-trivial unfoldings of $k$-jets.

3. We develop a method for obtaining normal forms for the $(k+1)$-jets having a given $k$-jet $f$ by inspection of certain submodules of $T\mathcal{A}f$. We give a test for sufficiency of a normal form and a method for constructing a $(k+1)$-trivial unfolding of the normal form.

4. We show that if $f$ is weighted homogeneous and the Krull dimension and multiplicity of $\theta(f)/T\mathcal{A}f$ are both 1 then $f$ is a weak stem and we can find an integer $k$ and a complete list of normal forms for finitely determined map-germs whose $k$-jet is $f$. The list has many similarities with the series found by Arnold and Mond.

Summary of Chapter II:

We extend Mond's classification of map-germs $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ under $\mathcal{A}$-equivalence. This chapter also demonstrates the use of the classification theory developed in I.2 and provides a large number of examples for use in the rest of the thesis.

Summary of Chapter III:

1. We prove that $f$ is a geometric stem if and only if $\Sigma$ is irreducible and the localised module $(\theta(f)/T\mathcal{A}f)$ has length $\ell = 1$ (localise with respect to the prime ideal defining $\Sigma$).

2. We also prove that if $\Sigma$ has transversal type $A_{2n}$ or $A_{2n+1}$ for some $n \geq 1$ then $\ell = n$. This makes it easier to determine $n$ if $\Sigma$ has transversal type of $A_{2n}$ or $A_{2n+1}$ since we shall calculate $\ell$ anyway.

3. If $f$ is a stem we show that $\Sigma$ is irreducible and give a list of transversal types that $\Sigma$ may have (although having one of these transversal types does not necessarily indicate that $f$ is a stem).

4. We look at how the numbers $C, T$ and $\mu(D_2(f)/\mathbb{Z}_2)$ behave for the families of map-germs associated with some weak stems. We observe that for a given family $\{f + p_s\}$ the integer $C(f + p_s) + T(f + p_s) + \mu(D_2(f + p_s)/\mathbb{Z}_2) - \text{cod}(\mathcal{A}, f + p_s)$ appears to be a constant.

Appendices A and B contain supplementary calculations.
Appendix C is a description of the computer programs written to calculate the modules used in classifying the map-germs of Chapter II.
## Contents.

**Introduction.**

### Chapter I: Submodules of $T_v f$.

<table>
<thead>
<tr>
<th>Subsections</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>I-1. Preliminaries</td>
<td>5</td>
</tr>
<tr>
<td>I-2. Classification Theory</td>
<td>23</td>
</tr>
<tr>
<td>I-3. Map-germs with 1-dimensional instability locus</td>
<td>44</td>
</tr>
</tbody>
</table>

### Chapter II: Map-germs $(C^2, 0) \to (C^3, 0)$.

<table>
<thead>
<tr>
<th>Subsections</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>II-1. Introduction and Summary of Results</td>
<td>53</td>
</tr>
<tr>
<td>II-2. $j^2 f = (x, y^2, 0)$</td>
<td>63</td>
</tr>
<tr>
<td>II-3. $j^2 f = (x, xy, 0)$</td>
<td>80</td>
</tr>
<tr>
<td>II-4. $j^2 f = (x, 0, 0)$</td>
<td>112</td>
</tr>
<tr>
<td>II-5. Map-germs with corank 2</td>
<td>125</td>
</tr>
</tbody>
</table>

### Chapter III: Geometry, Multiplicity and Stems.

<table>
<thead>
<tr>
<th>Subsections</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>143</td>
</tr>
<tr>
<td>III-1. Geometry and Multiplicity</td>
<td>145</td>
</tr>
<tr>
<td>III-2. Geometry of Stems</td>
<td>155</td>
</tr>
<tr>
<td>III-3. $\mathcal{A}$-invariants and weak stems</td>
<td>173</td>
</tr>
</tbody>
</table>

### Appendix A: Calculations for Chapter II.

<table>
<thead>
<tr>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
</tr>
</tbody>
</table>

### Appendix B: Calculations for section III-3.

<table>
<thead>
<tr>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
</tr>
</tbody>
</table>

### Appendix B: The Computer Programs.

<table>
<thead>
<tr>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
</tr>
</tbody>
</table>

### References:

<table>
<thead>
<tr>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
</tr>
</tbody>
</table>

### Figures 1–4.
List of Tables, Figures and Diagrams.

Tables.

Table 1. Some weak stems and their families ............... 49
Tables 2-5 contain data for map-germs: ................. 55+
  Table 2. \(j^2f = (x, y^2, 0)\)
  Table 3. Map-germs of the form \((x, xy + y^3, f_3)\)
  Table 4. Map-germs of the form \((x, f_2, f_3): f_2 \neq xy + y^3\)
  Table 5. Map-germs of corank 2
Table 6. Some transversal types ....................... 59.
Table 7. \(\mathcal{A}\)-invariants for some weak stems ........ 176.

Figures.

Figure 1. An \(\mathcal{A}\)-invariant stratification of \(\mathcal{A}^2(x, y^2, 0)\)
Figure 2. An \(\mathcal{A}\)-invariant stratification of \(\mathcal{A}^2(x, xy, 0)\)
Figure 3. An \(\mathcal{A}\)-invariant stratification of \(\mathcal{A}^2(x, 0, 0)\)
Figure 4. An \(\mathcal{A}\)-invariant stratification of \(\mathcal{A}^2(0, 0, 0)\)

These figures are located inside the back cover: an explanation is given in section II.1.1 (Results).

Diagrams.

Diagrams 1-8. The images of some non-finitely determined map-germs 56-58.
Diagram 9. Inequivalent map-germs with the same lower estimate for \(l\) 151.
Diagram 10. A map-germ which is not a stem ............... 155.
Diagram 11. A deformation of the map-germ in diagram 10. .... 156.
Introduction.

The most extensive lists of singularities known so far were produced by Arnold for functions up to $\mathcal{R}$-equivalence ([A.G.V]: section 15). A striking feature of these lists is the occurrence of series.

For example,

$$A_k: f(x, y) = x^{k+1} + y^2,$$
$$D_k: f(x, y) = x^2y + y^{k-1},$$
$$T_k, t, m: f(x, y, z) = axyz + x^k + y^t + z^m.$$  

We associate these series with the functions $f(x, y) = y^2$, $f(x, y) = x^2y$ and $f(x, y, z) = xyz$ respectively, all of which fail to be finitely determined for $\mathcal{R}$-equivalence.

We quote

'Although the series undoubtedly exist, it is not at all clear what a series of singularities is.' ([A.G.V], p. 243)

and

'However, a general definition of a series of singularities is not known. It is only clear that the series are associated with singularities of infinite multiplicities.' ([A.G.V], p. 244)

Series also occur in the classification of map-germs $f: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ up to $\mathcal{R}$-equivalence calculated in [Mond 1]. Here the following series have been found:

$$S_k: f(x, y) = (x, y^2, y^3 + x^k + 1y),$$
$$B_k: f(x, y) = (x, y^2, x^2y + y^{2k+1}),$$
$$H_k: f(x, y) = (x, xy + y^{3k-1}, y^3).$$

These series are associated with the map-germs $(x, y^2, y^3)$, $(x, y^2, x^2y)$ and $(x, xy, y^3)$ respectively. They are also discussed in [Mond 2].

The classifications were mostly performed by the brute force approach, classifying r-jets until a sufficient jet was reached. Special methods have been developed for dealing with, for example, weighted homogeneous map-germs (see [Wall 2] for further...
details). But the methods do not generalise, and if the initial choice of map-germ is not finitely determined then no sufficient jet exists. The techniques used to find the series are very individual and do not suggest a general approach. We require different techniques to explain their behaviour.

We observe that the simplest (1-suffix) series (with respect to an equivalence relation $\mathcal{G}$) have the following features:

i) they can be written in the form $f_k = f + r^{k-1}v$,

ii) each $f_k$ is sufficient,

iii) their adjacency relations are

$$\cdots \rightarrow f_{k+1} \rightarrow f_k \rightarrow f_{k-1} \rightarrow \cdots,$$

iv) there exists a value $k_0$ such that if $g$ is a map-germ satisfying $j^{k_0}g = j^{k_0}f$ then $g$ is $\mathcal{G}$-equivalent to $f$ or to $f_k$ for some $k$.

Here $f: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ is not finitely determined for $\mathcal{G}$, $v: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ and $r: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$.

For example, for $A_k$ we put $f(x, y) = y^2$, $v(x, y) = x$ and $r = x$;

for $H_k$: $f(x, y) = (x, xy + y^{3k-1}, y^3)$, we put $f(x, y) = (x, xy, y^3)$, $v(x, y) = (0, y^2, 0)$ and $r = y^3$ (here we specify that $k \geq 2$).

Can we identify those map-germs which might be expected to give rise to series? The minimum condition for a 1-suffix series would seem to be that the map-germ is a stem:

**Definition**: a map germ $f$ is a $\mathcal{G}$-stem if it is not finitely determined for $\mathcal{G}$ and there is an integer $k_0$ such that any map-germ $g$ with the same $k_0$-jet as $f$ is either $\mathcal{G}$-equivalent to $f$ or is finitely determined for $\mathcal{G}$.

In [P 1] Pellikaan has characterised stems for $\mathcal{R}$-equivalence in terms of the transversal types of their singular loci $\Sigma$:

a function is a stem if and only if $\Sigma$ is irreducible with transversal type $A_1$. 
He has also shown that if $\Sigma \neq V(z_0)$ then the 'series' of map-germs $f + z_0^k$ are finitely determined for $k \geq k_0$. However it is not the case that all finitely determined map-germs with the same $k_0$-jet as $f$ are equivalent to one in the 'series', or that each $f + z_0^k$ is sufficient.

This thesis grew from the aim of characterising a stem by the transversal type of its instability locus $\Sigma$ for a map-germ $f:(C^2, 0) \rightarrow (C^3, 0)$ under $\mathcal{A}$-equivalence.

Definition: call $f$ a geometric stem if $\Sigma$ is irreducible and has one of the transversal types $A_2$, $A_3$ or $D_4$.

The conjecture we investigate is

$$f \text{ is a stem if and only if } f \text{ is a geometric stem.}$$

Our second aim is essentially to investigate the question of whether stems give rise to series. Of course, this is not a well-defined question since we don't really know what a series is; however what we shall try to do is to obtain a list of the relevant finitely determined map-germs and see how they behave, with particular reference to the points i)-iv) noted above. This brings us back to the classification problem.

For a finitely determined map-germ a versal deformation can be found by choosing a basis for the module $\Theta(f)/T\mathcal{A}_f$. We also find that, for example, for $f(x, y) = (x, xy, y^3)$ the module $\Theta(f)/T\mathcal{A}_f$ has the basis

$$\left\{ y \frac{\partial}{\partial z}, y^{3k-1} \frac{\partial}{\partial y} : k \geq 1 \right\}$$

(recall that the series associated with $(x, xy, y^3)$ is $(x, xy + y^{3k-1}, y^3)$).

But how do we relate the geometrical type of $f$ to the module $\Theta(f)/T\mathcal{A}_f$?

In fact, we find that there is a strong connection based on the fact that $\Sigma$ is the support of the module $\Theta(f)/T\mathcal{A}_f$; this connection is provided by Samuel's Theory.
of Multiplicity. We have included a summary of this theory in I-1 (I-1:10–I-1:15) along with other useful properties of finite modules. The main point is that

\[ e(\theta(f)/T_{\mathfrak{m}} f) = \sum_{i=1}^{r} \mu(\Sigma_i) \ell_{\Sigma_i}(\theta(f)/T_{\mathfrak{m}} f) \delta(i) \]

where \( \Sigma = \Sigma_1 \cup \ldots \cup \Sigma_r \), \( e \) denotes the multiplicity of the module, \( \ell_{\Sigma_i} \) denotes the length of the localised module \( (\theta(f)/T_{\mathfrak{m}} f)_{\Sigma_i} \) (localise with respect to the prime ideal defining \( \Sigma_i \)) and \( \delta(i) = 1 \) if \( \dim \Sigma_i = \dim \Sigma \) or 0 otherwise (I-1:28).

We therefore follow two main lines of research:

i) the study of the module \( T_{\mathfrak{m}} f \) and its use in finding normal forms,

ii) a study of how the geometrical type of a map-germ may determine whether or not it is a stem,

and link them by looking at the relationship between the transversal type of \( \Sigma \) and \( \ell_{\Sigma}(\theta(f)/T_{\mathfrak{m}} f) \).

We give a brief outline here of the contents of the thesis:

With regard to the aims outlined above the main result from chapter I is the following theorem (I-3:3):

if \( f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) is weighted homogeneous and the Krull dimension and multiplicity of \( \theta(f)/T_{\mathfrak{m}} f \) are both 1 then we can obtain a value \( k_0 \) and a list \( \{ f_k \} \) of finitely determined map-germs whose \( k_0 \)-jet is \( f \);

the list has the following properties:

i) the \( f_k \) can be written in the form \( f_k = f + p_k \), where the \( p_k \) are monomials and \( T_{\mathfrak{m}} f_k = T_{\mathfrak{m}} f + \mathbb{C}\{ p_k : k \geq k' \} \), which has finite codimension.

Examples suggest that at least for simple cases \( p_{k+1} = r p_k \), for some \( r \in \mathbb{O}_2 \) although relations between the \( p_k \) appear to be more complicated in general.

ii) Each \( f_k \) is sufficient,

iii) their adjacency relations are

\[ \ldots \to f_{k+1} \to f_k \to f_{k-1} \to \ldots \]
iv) if \( g \) has the \( k_0 \)-jet \( f \) then \( g \) is either \( \mathcal{A} \)-equivalent to one of the \( f_k \) or is \( k-\mathcal{A} \)-equivalent to \( f \) for all \( k \) (i.e. their \( k \)-jets are \( \mathcal{A}^{(k)} \)-equivalent for all \( k \)).

This theorem suggests that it is enough to define a series of singularities to be the list of finitely determined map-germs associated with a stem. All the other desirable properties seem to come naturally from this though more work needs to be done before we can make a formal definition of a series.

We propose the conjecture

if \( g \) is \( k-\mathcal{G} \)-equivalent to \( f \) for all \( k \) then \( g \) is \( \mathcal{G} \)-equivalent to \( f \).

We have succeeded in proving this conjecture generally for \( \mathcal{G} = \mathcal{R} \) and \( \mathcal{G} = \mathcal{K} \) and for \( \mathcal{G} = \mathcal{A} \) in the special case that \( f \) has 2-jet \((x, y^2, 0)\) (theorems I-1:18 and II-2:5).

We make the definition: \( f \) is a weak stem (for \( \mathcal{G} \)) if there is an integer \( k_0 \) such that any map-germ \( g \) with the same \( k_0 \)-jet as \( f \) is either \( k-\mathcal{G} \)-equivalent to \( f \) for all \( k \) or is finitely determined for \( \mathcal{G} \).

A map-germ satisfying the hypotheses of I-3:3 is thus a weak stem. The rest of Chapter I was developed principally to prove this theorem. In I-1 we summarise various properties of the finite module \( T\mathcal{A}_f \). We also extend the theory of trivial unfoldings of map-germs to \( k \)-jets.

In I-2 we use the Artin-Rees lemma to show that certain submodules of \( T\mathcal{A}_f \) have basically the same structure as \( T\mathcal{A}_f \); these submodules are then used to find normal forms. The theory developed here is also of use in general classification, as we demonstrate in Chapter II. Details of the results are outlined in the introduction to Chapter I.

Chapter III has three sections:

In III-1 we establish the crucial link between the structure of \( T\mathcal{A}_f \) and the geometry (III-1:1):

\( f \) is a geometric stem if and only if \( \Sigma \) is irreducible and \( l_\Sigma(\theta(f)/T\mathcal{A}_f) = 1 \).
We also obtain other links between the transversal type of $\Sigma$ and $\ell_{\Sigma}(\theta(f)/T_{\Sigma}(f))$ which we use to calculate the transversal types for many of the examples given in Chapter II.

In III-2 we investigate part of the original conjecture:

$$f \text{ is a stem } \Rightarrow f \text{ is a geometric stem.}$$

We show that if $f$ is a stem then $\Sigma$ is irreducible and obtain a restricted list for its possible transversal type. However further reduction is still necessary to prove the conjecture.

In III-3 we look again at some of the families of singularities identified in Chapter II: we calculate the numbers $C$, $T$ and $\mu(D_2(f)/\mathbb{Z}_2)$ for the singularities in each family $\{f+p_s\}$ associated with the weak stem $f$. We observe that for each of these families the integer

$$C(f+p_s) + T(f+p_s) + \mu(D_2(f+p_s)/\mathbb{Z}_2) - \text{cod}(\mathbb{F}, f+p_s)$$

appears to be a constant. The details of these calculations form Appendix B.

Finally, we note that progress would not have been possible without looking at a large number of examples. These are presented in Chapter II. The results are collected in section II-1 and the calculations form sections II-2–II-5. Supplementary calculations for the classification form Appendix A. A major contribution to these calculations was the writing of programs to calculate the relevant modules by computer. The programs are described briefly in Chapter II and in more detail in Appendix C.
Acknowledgements.

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Declaration.

The material included in this thesis is original, except where indicated otherwise.
Chapter I: Submodules of $T\mathcal{E}f$.

Introduction.

In this chapter we study submodules of $T\mathcal{E}f$ and their use in the classification of map-germs whose $k$-jet is $f$. The chapter contains three sections.

Section I-1 contains most of the preliminaries and is divided into three subsections.


The major part is a summary of Samuel's Multiplicity Theory.


We define the relations $\mathcal{E}$-equivalence, $k-\mathcal{E}$-equivalence and $\mathcal{E}^{(k)}$-equivalence (I-1:16) and apply the multiplicity theory to the modules of interest.

I-1-3: Criteria for equivalence of map-germs and jets. p. 17.

A commonly used technique is the construction of a trivial unfolding of a map-germ, using the Thom-Levine criterion. We make the obvious extension to a $k$-trivial unfolding of a $k$-jet and prove the corresponding criterion. This criterion is used extensively in section I-2.

In I-2 we look at the use of specific submodules of $T\mathcal{E}f$ in the classification of map-germs with the $k$-jet $f$. The main aim in this section is to find a list of possible normal forms for $j^{k+1}g$ when $j^k f = j^k g$ by inspection of the submodule

$$tf(m_n^2 \cdot \theta(n)) + \omega f(m_p^2 \cdot \theta(p)) + m_n^{k+2} \cdot \theta(f).$$

It is then a relatively simple task to calculate $T\mathcal{E}^{k+1}g$ so as to further restrict the list of normal forms. We also prove a sufficient condition for a normal form to be $(k+1)$-determined. The proofs are based on the following facts (I-2:2 and I-2:4):
if $h \in m_n^{k+1}$ then

$$tf(m_n^2 \cdot \theta(n)) + m_n^{k+2} \cdot \theta(f) = t(f + h)(m_n^2 \cdot \theta(n)) + m_n^{k+2} \cdot \theta(f)$$

and $\omega f(m_p^2 \cdot \theta(p)) + m_n^{k+2} \cdot \theta(f) = \omega(f + h)(m_p^2 \cdot \theta(p)) + m_n^{k+2} \cdot \theta(f)$

and if $f$ is finite there exist integers $k_0$, $k_1$ such that

$$k \geq k_0 \Rightarrow m_n^{k+1} \cdot \theta(f) \cap T \leq f(m_n^2 \cdot \theta(n)) + \omega f(m_p^2 \cdot \theta(p))$$

and $k \geq k_1 \Rightarrow m_n^{k+1} \cdot \theta(f) \cap T \leq f(m_p^2 \cdot \theta(f)).$

Theoretically this means that if $f$ is finite then as we see from the theorems below, for large $k$ we can find the normal forms by looking only at $T \leq f$.

We find normal forms from the following theorem:

(I:1) Theorem: suppose that $\{v_1, \ldots, v_s\} \subseteq m_n^{k+1} \cdot \theta(f)$ and that either

a) $m_n^{k+1} \cdot \theta(f) \subseteq tf(m_n^2 \cdot \theta(n)) + \omega f(m_p^2 \cdot \theta(p)) + m_n^{k+2} \cdot \theta(f) + C\{v_1, \ldots, v_s\}$

or b) $f$ is finite, $k \geq k_0$ and $T \leq f + m_n^{k+1} \cdot \theta(f) = T \leq f + m_n^{k+2} \cdot \theta(f) + C\{v_1, \ldots, v_s\}$.

Then $g \sim f \Rightarrow g \sim f + \sum \lambda_i v_i$ for some $\lambda_1, \ldots, \lambda_s \in C$.

If $f$ is finite, let $\rho$ be the least integer such that $m_n^\rho \subseteq f^\ast m_p^\rho \cdot \theta_n$.

We have the following test for finite determinacy:

(I:2) Theorem:

a) Suppose that $f$ is finite, $k \geq \rho$, and $\{v_1, \ldots, v_s\}$ are as in I:1 b.

If i) - iii) hold then $f + \sum \lambda_i v_i$ is $(k+1)$-determined;
i) $f^*\mathcal{m}_p(T_x f + \mathbb{C}\{v_1, \ldots, v_l\}) \supseteq \mathcal{m}_n^{k+2} \cdot \theta(f)$.

ii) $T_x f + f^*\mathcal{m}_p \cdot \theta(f) \supseteq T_x f + \mathbb{C}\{v_1, \ldots, v_l\}$

iii) for each $\lambda_i \neq 0$ and $v_i = ((v_i)_1, \ldots, (v_i)_p)$

$$f(\mathcal{m}_n^2 \cdot \theta(n)) + \omega f(\mathcal{m}_p^2 \cdot \theta(p)) + f^*\mathcal{m}_p \cdot \{v_1, \ldots, v_l\} + \mathcal{m}_n^{k+2} \cdot \theta(f) \supseteq \mathbb{C}\{(v_i)_t \frac{\partial}{\partial y_j} : 1 \leq t, j \leq p\}.$$ 

b) if $k \geq k_1$ and $T_x f + f^*\mathcal{m}_p \cdot \{v_1, \ldots, v_l\} \supseteq \mathcal{m}_n^{k+2} \cdot \theta(f)$ then i) holds.

c) if $tf(\mathcal{m}_n^2 \cdot \theta(n)) + \omega f(\mathcal{m}_p^2 \cdot \theta(p)) + f^*\mathcal{m}_p \cdot \{v_1, \ldots, v_l\} + \mathcal{m}_n^{k+2} \cdot \theta(f) \supseteq \mathcal{m}_n^{k+1} \cdot \theta(f)$ then iii) holds.

Here $(Y_1, \ldots, Y_p)$ are local coordinates for $\mathbb{CP}$ at 0 and $(x_1, \ldots, x_n)$ are coordinates for $\mathbb{C}^n$.

Hypothesis iii) says that if for example $v_i = ((v_i)_1, 0, \ldots, 0)$ then

$$f(\mathcal{m}_n^2 \cdot \theta(n)) + \omega f(\mathcal{m}_p^2 \cdot \theta(p)) + f^*\mathcal{m}_p \cdot \{v_1, \ldots, v_l\} + \mathcal{m}_n^{k+2} \cdot \theta(f) \supseteq ((v_i)_1, 0, \ldots, 0), (0, (v_i)_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, (v_i)_1).$$

These theorems are proved in I.2 (I.2:6 and I.2:9 respectively). In fact we shall obtain more general, but more technical theorems, of which these are special cases. However, these are the cases we shall apply most often in Chapter II and when studying stems.

In the case $s = 1$ ($v_1 = v \in \mathcal{m}_n^{k+1} \cdot \theta(f)$) and $f$ is weighted homogeneous, we get

(I:3) Theorem: suppose that $v$ is a monomial, i.e. $v = x_1^{i_1} \cdots x_n^{i_n} \frac{\partial}{\partial y_j}$ for some $i_1, \ldots, i_n$, $j$, and that $v$ is not of the same degree as $f_j$. 

if $\nu_n^{k+1} \cdot \theta(f) \leq t(f) + o(f) + m_n^{k+2} \cdot \theta(f) + C(v)$ then

$$T_{\mathcal{A}}(f + \lambda v) + m_n^{k+2} \cdot \theta(f) = T_{\mathcal{A}}(f + C\{v\} + m_n^{k+2} \cdot \theta(f)).$$

Consequently, the submanifold $\{f + \lambda v : \lambda \in \mathbb{C}\{0\}\}$ lies in a single $(k+1)$-orbit. For the proof see I-2:18.

There may be a choice of weights for $f$; in this case the theorem follows if the degrees of $v$ and $f_j$ are different for some choice of weights.

For example consider $(x, xy, y^3)$. For this example we obtain $v = (0, 0, x^k y)$ (reference II-3:4). This is not of degree 3 with respect to the weights $(1,1)$. So

$$T_{\mathcal{A}}(x, xy, y^3 + \lambda x^k y) + m_n^{k+2} \cdot \theta(f) = T_{\mathcal{A}}(x, xy, y^3) + C(\lambda x^k y \frac{\partial}{\partial y_3}) + m_n^{k+2} \cdot \theta(f).$$

We also prove a more general theorem for the construction of $k+1$-trivial unfoldings of a map-germ of the form $f + \sum \lambda_i v_i$ (I-2:19).

In I-3 we try to make use of the information gained by knowing the multiplicity and dimension of the module $\theta(f)/T_{\mathcal{A}} f$.

In particular, we prove theorem (I-3:3) (outlined in the main introduction) and discuss how it may be extended to the general case. We give a list of families of singularities obtained using this theorem and note their properties.
I-1 : Preliminaries.

I-1-1 : Properties of finite modules.

(I-1:1) Definition: A holomorphic map \( \varphi : X \rightarrow Y \) is finite if it is proper and every point \( x \in X \) is an isolated point in the fibre \( \varphi^{-1}(\varphi(x)) \).

(I-1:2) Theorem: If \( x \) is an isolated point of its fibre then there is a neighbourhood of \( x \) on which \( \varphi \) is finite.
Proof: see [F], p. 132.

In particular, the germ \( f:(X,x) \rightarrow Y \) of \( \varphi \) at \( x \) is finite, i.e. for each representative \( f' \) of \( f \) there is a neighbourhood of \( x \) on which \( f' \) is finite.

(I-1:3) Theorem: If \( \varphi : X \rightarrow Y \) is a holomorphic map, \( x \in X \) and \( y = \varphi(x) \), then the following are equivalent:

i) \( \mathcal{O}_{X,x} \) is a finitely generated \( \mathcal{O}_{Y,y} \)-module,

ii) \( \dim_c(\mathcal{O}_{X,x}/m_{Y,y} \mathcal{O}_{X,x}) < \infty \),

iii) \( x \) is an isolated point of its fibre \( X_y \).

Proof: see [F], p. 57.

This implies that the map-germ \( f:(\mathbb{C}^n,x) \rightarrow (\mathbb{C}^p,y) \) is finite if and only if \( \mathcal{O}_{n,x} \) is a finitely generated \( \mathcal{O}_{p,y} \)-module. The action of \( \mathcal{O}_{p,y} \) on \( \mathcal{O}_{n,x} \) is defined by

\[ \chi \circ m = f^*(\chi) \circ m = (\chi \circ f) \circ m, \]

where \( \chi \in \mathcal{O}_{p,y} \) and \( m \in \mathcal{O}_{n,x} \).

If we choose a different map-germ then the \( \mathcal{O}_{p,y} \)-module structure of \( \mathcal{O}_{n,x} \) is different.

Our aim is to study modules related to the module \( \mathcal{O}_{n,x} \) when \( f \) is finite. We start by presenting some well-known properties of finite (i.e. finitely generated) modules. Chief amongst these is Samuel's Theory of Multiplicity, which provides us with two
invariants for describing the (finite) module $M$: the Krull dimension $\dim(M)$ and the multiplicity $e(M)$. The main reference for this material is [Mats]. Another useful reference is [Mum], pp. 116–126.

In what follows, assume that $A$ is a local Noetherian ring of dimension $d$ with maximal ideal $\mathfrak{m}$, $k = A/\mathfrak{m}$ is the residue field and $M$ is an $A$-module.

(I·1:4) Theorem: $M$ is finite if and only if $M/\mathfrak{m} \cdot M$ is a finite-dimensional vector space over $k$. If $M$ is finite then \{u_1, \ldots, u_s\} is an $A$-basis for $M$ if and only if \{\bar{u}_1, \ldots, \bar{u}_s\} is a $k$-basis for $M/\mathfrak{m} \cdot M$, where $\bar{u}_i$ is the image of $u_i$ in the quotient. 

Proof: see [Mats], p. 8.

(I·1:5). Nakayama's Lemma: Let $M$ be finite and suppose that $\mathfrak{m} \cdot M = M$. Then $M = 0$.

Proof: see [Mats], p. 8.

(I·1:6) Corollary: Let $N$, $P$ be submodules of $M$ with $N$ finite.

If $N \subseteq P + \mathfrak{m} \cdot N$ then $N \subseteq P$.

Proof: We have $\mathfrak{m} \cdot (N + P)/P = (\mathfrak{m} \cdot N + P)/P = (N + P)/P$. It follows from I·1:6 that $(N + P)/P = 0$, i.e. $N \subseteq P$. 

We shall frequently need the Artin-Rees lemma for looking at submodules:

(I·1:7) Lemma (Artin-Rees): Let $N$ be a submodule of a finite module $M$ and $I$ an ideal of $A$. Then there exists a positive integer $m$ such that $N \cap I^m \cdot M \subseteq I \cdot N$.

Proof: see [Mats], p. 59.

(I·1:8) Theorem (the Krull Intersection Theorem):

i) for any finite module $M$ and ideal $I \subseteq \mathfrak{m}$, $\bigcap_{m>0} I^m \cdot M = 0$;
if \( N \subseteq M \) is a submodule then \( \bigcap_{m>0} (N + IM) = N. \)

ii) If \( A \) is also an integral domain, \( I \subseteq A \) proper, then \( \bigcap_{m>0} IM = 0. \)

Proof: see [Mats], p. 60.

(I-1:9) Definition: We define \( \dim(M) \) by setting \( \dim(M) = \dim(A/\text{Ann}(M)) \). This is the Krull dimension of \( M \). It is the dimension of the variety defined by the ideal \( \text{Ann}(M) \).

Lemma: If \( M \) is finite then \( \text{Supp}(M) = V(\text{Ann}(M)). \)

Proof: see [Mats], p. 25.

(I-1:10) Definition: If \( I \) is an ideal of \( A \) satisfying \( \mu^\nu I \subseteq I \subseteq \mu I \) for some \( \nu > 0 \) then we shall call \( I \) an ideal of definition. Let \( M \) be a finite \( A \)-module. Then \( M/\mu^nM \) has finite length. We set

\[
\chi_M^I(n) = \ell\left(\frac{M}{\mu^nM}\right)
\]

[Mats], p. 97.

(I-1:11) Theorem: For sufficiently large values of \( n \), \( \chi_M^I \) is a polynomial and is given by

\[
\chi_M^I(n) = \frac{e}{d!} n^d + \text{(lower order terms)}, \text{ where } e \in \mathbb{N}.
\]

Recall that here, \( d = \dim A \), not \( \dim M \).

Proof: see [Mats], p. 97.

We abbreviate \( \chi_M^\mu \) to \( \chi_M \) and call it the Samuel polynomial of \( M \) (over \( A \)). We shall denote the integer \( e \) by \( e(I,M) \).

Definition: The multiplicity of \( M \) over \( A \), \( e_A(M) \), is the integer \( e(\mu, M) \).
Theorem: The polynomial $\chi_M^I$ has the following properties:

i). The degree is independent of $I$. Denote it by $d(M)$. Then $e(I, M) > 0$ if $d(M) = d$ and $e(I, M) = 0$ if $d(M) < d$.

ii) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of finite modules then

$$d(M) = \max\left(d(M'), d(M'')\right),$$

and $\chi_M^I$, $\chi_M^{I'}$ have the same leading coefficient.

iii) $d(M) = \delta(M) = \dim(M)$.

Proof: see [Mats], p. 98.

Here $\delta(M)$ is the least value of $n$ for which there exist $x_1, \ldots, x_n \in m$ such that $\ell(M/x_1M + \ldots + x_nM)$ is finite.

Theorem: The integer $e(I, M)$ has the following properties:

i) $e(I, M) = \lim_{n \to \infty} \frac{d!}{d} \ell\left(\frac{M}{I^nM}\right)$ and if $d = 0$, $e(I, M) = l(M)$.

ii) if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of finite modules then

$$e(I, M) = e(I, M') + e(I, M''),$$

iii) if $\{p_1, \ldots, p_t\}$ are the minimal prime ideals $P$ of $A$ with $\dim(A/p) = d$ then

$$e(I, M) = \sum_{i=1}^{t} e(I', A/p_i) \cdot \ell(M_{p_i}),$$

where $\ell(M_p)$ is the length of $M_p$ as an $A_p$-module and $I' = (I + p)/p$.

iv) if $A$ is an integral domain then $e(I, M) = e(I, A)s$ where $s = \text{rank } M$.

Proof: see [Mats], pp. 107-108.

Usually we will not have $\dim(M) = \dim(A)$. To inspect the leading coefficient of $\chi_M^I$ in this case we consider $M$ as a module over the ring $A_M = A/\text{Ann}(M)$.

Let $I_M = I + \text{Ann}(M)/\text{Ann}(M)$. We then have
dim(M) = dim\left(\frac{A}{\text{Ann}(M)}\right) \quad \text{and} \quad \chi_M^n(n) = \ell\left(\frac{M}{I^nM}\right) = \chi_M(n).

The minimal prime ideals \( \{p_1,M,...,p_t,M \} \subseteq A_M \) \( \dim(A_M/p_i,M) = \dim(M) \) correspond to minimal primes \( \{p_1,...,p_t : p_i \nmid \text{Ann}(M) : \dim(A/p_i) = \dim(M) \} \)

(for each \( i, p_i,M = p_i / \text{Ann}(M) \)).

So \( e(I_M, M) = \sum_{i=1}^{t} e(I'M, A_M/p_i,M) \cdot \ell(M_{p_i,M}) = \sum_{i=1}^{t} e(I’, A/p_i) \cdot \ell(M_{p_i}) \)

since \( I'_M = (I_M + p_i,M)/p_i,M \sim (I + p_i)/p_i = I’, A_M/p_i,M \sim A/p_i \) and \( M_{p_i,M} = M_{p_i} \).

In particular, \( e(\mathfrak{m}_M, M) = \sum_{i=1}^{t} e(\mathfrak{m}_{p_i}, A/p_i) \cdot \ell(M_{p_i}) \).

Note that for any \( p \nmid \text{Ann}(M) \), \( e(\mathfrak{m}_p, A/p) = 0 \) if \( \dim(A/p) < \dim(M) \).

Definition: The multiplicity \( e(M) \) of \( M \) is the multiplicity \( e(\mathfrak{m}_M, M) \) of \( M \) over \( A/\text{Ann}(M) \).

Then \( e(M) = 0 \iff M = 0 \).

For \( e(M) \) to have any meaning the Krull dimension of \( M \) should also be quoted.

Locally, the integer \( e(\mathfrak{m}_p, A/p) \) has geometrical meaning:

(I-1:14) Theorem: Let \( X \) be a variety, \( x \in X \). Then

\[ \text{mult}_x(X) = e_x(\mathfrak{m}_{X,x}, \mathcal{O}_{X,x}). \]

Proof: see [Mum], p. 123.

(I-1:15) Corollary: If \( M_x \) is an \( \mathcal{O}_{X,x} \)-module then

\[ e(\mathfrak{m}_{X,x}, M_x) = \sum_{i=1}^{t} \text{mult}_x(X_{p_i}) \cdot \ell(M_{p_i}). \]

where \( X_{p_i} \) is the variety defined by the prime ideal \( p_i \), \( \dim(X_{p_i}) = \dim(M_x) \) and
$I(M_{p_i})$ is independent of $x \in X_{p_i}$. The $X_{p_i}$ are the branches of $\text{Supp}(M_\lambda)$ which have (maximum) dimension $\dim \left(\text{Supp}(M_\lambda)\right)$.

This covers all the properties of finite modules that we shall need. We next look at equivalence relations between map-germs and construct our modules.

I.1.2 : Equivalence of map-germs and jets.

(I.1:16) Definition: Let $\mathcal{A} = \text{Aut}(\mathbb{C}^n,0) \times \text{Aut}(\mathbb{C}P,0)$. Then $\mathcal{A}$ acts on $\mathcal{O}_n^p$ by

$$(\psi, \varphi) \circ f = \psi \circ \varphi \circ f$$

for $(\psi, \varphi) \in \mathcal{A}$ and $f \in \mathcal{O}_n^p$.

We say that $f$, $g$ are $\mathcal{A}$-equivalent and write $f \sim g$, or just $f \sim g$, if there is some $(\psi, \varphi) \in \mathcal{A}$ such that $f = (\psi, \varphi) \circ g$.

The action of $\mathcal{A}$ on $\mathcal{O}_n^p$ induces an action on $J^k(n,p)$:

let $\mathcal{A}^{(k)} = \{(j^k \psi, j^k \varphi) : (\psi, \varphi) \in \mathcal{A}\}$. Then $\mathcal{A}^{(k)}$ acts on $J^k(n,p)$ by

$$(j^k \psi, j^k \varphi) \circ j^k f = j^k(\psi \circ \varphi \circ f).$$

Note that $j^k f, j^k g$ are $\mathcal{A}^{(k)}$-equivalent if and only if there is some $h \in m_n^{k+1} \mathcal{O}_n^p$ such that $f - h$, $g$ are $\mathcal{A}$-equivalent. Call this second equivalence relation on $\mathcal{O}_n^p$ k-$\mathcal{A}$-equivalence and write $f \sim^k g$ to indicate it. Clearly $f \sim g \Rightarrow f \sim^k g$ for all $k$.

Conjecture: $f \sim^k g$ for all $k \Rightarrow f \sim g$.

We note that the conjecture is true if we replace $\mathcal{A}$-equivalence with $\mathcal{A}$-equivalence or with $\mathcal{R}$-equivalence in the case that $p = 1$. To prove this we need a strengthening of Artin approximation which is due to J. Wavrik:
Theorem (Artin approximation).

Let \( f_i(x, y) \in \mathcal{O}_{n+N} \), \( i = 1, \ldots, m \), where \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_N) \). Suppose that for each \( k \) we have \( y_{k,1}(x), \ldots, y_{k,N}(x) \in \mathcal{M}_n \) such that

\[
f_i(x, y_k(x)) \in \mathcal{M}_n^{k+1}
\]

for each \( i \). Then for any \( c \) there are \( y_1(x), \ldots, y_N(x) \in \mathcal{M}_n \) such that \( f(x, y(x)) = 0 \) and \( y_{k,\ell}(x) - y_\ell(x) \in \mathcal{M}_n \).

See [Wavrik], theorems 2.2 and 2.3.

Theorem:

i) if function-germs \( f, g \) are \( k-\mathcal{R} \)-equivalent for each \( k \) then they are \( \mathcal{R} \)-equivalent,

ii) if map-germs \( f, g \) are \( k-\mathcal{K} \)-equivalent for each \( k \) then they are \( \mathcal{K} \)-equivalent.

Proof: for i) observe that \( \mathcal{R} \)-equivalence follows if and only if the equation

\[
F(x, \varphi) = f(x) - g(\varphi) = 0
\]

has a convergent solution \( \varphi(x) \in (\mathcal{M}_n)^n \) which is an isomorphism. By the above theorem this follows if for each \( k \) there exists an isomorphism \( \varphi_k \) such that

\[
F_i(x, \varphi_k(x)) \in \mathcal{M}_n^{k+1} \quad \text{for} \quad 1 \leq i \leq p
\]

since \( \varphi_{k,\ell}(x) - \varphi_\ell(x) \in \mathcal{M}_n \) ensures that \( \varphi \) is also an isomorphism. But this is precisely the condition \( f, g \) are \( k-\mathcal{R} \)-equivalent for each \( k \).

For part ii) we must find convergent solutions \( A(x), \varphi(x) \) for the equation

\[
F(x, \varphi, A) = Af(x) - g(\varphi) = 0
\]

where \( A \) is an invertible \( p \times p \) matrix. The proof is otherwise the same.

The approximation theorem cannot be used directly to show that

'if map-germs \( f, g \) are \( k-\mathcal{K} \)-equivalent for each \( k \) then they are \( \mathcal{K} \)-equivalent' since we need to find isomorphisms \( \varphi(x) \) and \( \psi(z_1, \ldots, z_p) \) such that \( f(\varphi) - \psi(g) = 0 \) and there is no way of writing this as finding the solution to an equation \( F(x, y(x)) = 0 \).
We shall, however, prove the conjecture in the case that $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ satisfies $j^2 f = (x, y^2, 0)$; see II-2:5 for this proof.

A similar definition of $\mathcal{A}$-equivalence is used for map-germs $f: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, where $\mathcal{A} = \text{Aut}(\mathbb{R}^n, 0) \times \text{Aut}(\mathbb{R}^p, 0)$; this is the definition used in the paper [Mond 1].

Where necessary we will distinguish between these by calling them $\mathcal{A}_{\mathbb{R}}$-equivalence and $\mathcal{A}_{\mathbb{C}}$-equivalence. The main differences between the classifications occur due to the connectivity of submanifolds; e.g. $\mathbb{C}\setminus\{0\}$ is connected but $\mathbb{R}\setminus\{0\}$ is not. Where not stated otherwise we mean $\mathcal{A}_{\mathbb{C}}$-equivalence.

Some types of map-germs.

(I-1:19) Definition: A map-germ $f \in \mathcal{O}_n^p$ is $k$-$\mathcal{A}$-determined (or just $k$-determined) if $f \sim g \Rightarrow f \sim g$, all $g \in \mathcal{O}_n^p$. Call the least $k$ for which this holds the determinacy degree of $f$.

A map-germ $f \in \mathcal{O}_n^p$ is finitely determined (for $\mathcal{A}$) if it is $k$-determined for some $k$.

(I-1:20) Definition: A map-germ $f \in \mathcal{O}_n^p$ is a $k$-stem (for $\mathcal{A}$) if $f$ is not finitely determined but for all $g \in \mathcal{O}_n^p$, $f \sim g \Rightarrow$ either $f \sim g$ or $g$ is finitely determined. Call the least $k$ for which this holds the degree of the stem $f$.

A map-germ $f \in \mathcal{O}_n^p$ is a stem (for $\mathcal{A}$) if it is a $k$-stem for some $k$.

This definition is analogous to that made by R. Pellikaan in [P 1], p. 1. It was proposed by D. Mond.

Example: consider $f(x, y) = (x, y^2, y^3)$. Then $f$ is a 3-stem and
Each $f + p_k$ is $k+1$-determined: see II:2:10.

Throughout this thesis we have frequently found map-germs $f$ which have the following property:

there exists a value $k$ such that if $g$ satisfies $f^k \sim g$ then either $g$ is finitely determined or $f^k \sim g$ for all $k$.

**Definition:** call such a map-germ a weak stem.

If we can prove the conjecture $f^k \sim g$ for all $k \Rightarrow f \sim g$ then such map-germs will be stems. In most cases we suspect to be stems then, we content ourselves with proving that the weaker condition holds. I personally feel that it is unlikely that the conjecture fails for any of the relatively simple examples calculated in this thesis. We leave further investigation of the conjecture for a future research problem.

(I·1:21) **Modules associated with map-germs.**

We choose local coordinates at $x$ and $y$ and consider only map-germs $f : (\mathbb{C}^n,0) \rightarrow (\mathbb{C}^p,0)$. Denote the set of vector fields along $f$ by $\theta(f)$. If $\{Y_1, ..., Y_p\}$ are coordinates for $\mathbb{C}^p$ at $0$, then the vector fields $\{\partial / \partial Y_1, ..., \partial / \partial Y_p\}$ form a free basis for $\theta(f)$ as an $\mathcal{O}_n$-module. We shall write $e_i = \partial / \partial Y_i$ when we need to save space.

A **monomial** in $\theta(f)$ is a term of the form $\psi \partial / \partial Y_i$, where $\psi \in \mathcal{O}_n$ is a monomial.

As well as being an $\mathcal{O}_n$-module, $\theta(f)$ is also a (finitely generated) $\mathcal{O}_p$-module via $f^*$ (if $f$ is finite). If $f$ is the identity map $(\mathbb{C}^n,0) \rightarrow (\mathbb{C}^n,0)$ then denote $\theta(f)$ by $\theta(n)$. 

$$g^3 \sim f \Rightarrow g \sim f \text{ or } g \sim (x,y^2,y^3+x^k y) \text{ for some } k \geq 3.$$
Similarly define $\theta(p)$.

(I·1:22) Lemma: If $f$ is finite, $D \subseteq \theta(f)$ an $\mathcal{O}_p$-module, $C \subseteq \theta(f)$ a subset which satisfies $C \subseteq D + m_n^{k+1} \cdot \theta(f)$, $\forall k$, then $C \subseteq D$.

Proof: $\theta(f)$ is a finite $\mathcal{O}_p$-module and $C \subseteq \bigcap_{k \geq 0} (D + m_n^{k+1} \cdot \theta(f)) = D$ (th. 1.9). \qed

There are ($\mathcal{O}_p$-module) homomorphisms $tf : \theta(n) \rightarrow \theta(f)$ and $\omega : \theta(p) \rightarrow \theta(f)$ defined by

$$tf(\xi) = df \circ \xi, \quad \text{and} \quad \omega(\eta) = \eta \circ f.$$ 

If $I_i \subseteq \mathcal{O}_n, 1 \leq i \leq n$ and $J_i \subseteq \mathcal{O}_p, 1 \leq j \leq p$ are ideals then

$$tf\left( \sum_{i=1}^n I_i \frac{\partial}{\partial x_i} \right) = I_1 \left\{ \frac{\partial f}{\partial x_1} \right\} + \cdots + I_n \left\{ \frac{\partial f}{\partial x_n} \right\} \quad \text{and} \quad \omega \left( \sum_{j=1}^p J_j \frac{\partial}{\partial y_j} \right) = \sum_{j=1}^p f^*(J_j) \frac{\partial}{\partial y_j}.$$ 

(I·1:23) Lemma [Math IV], p. 227:

$f$ is stable if and only if $tf(\theta(n)) + \omega(\theta(p)) = \theta(f)$.

(I·1:24) Lemma: The tangent space $T\mathcal{O}^{(k)}f$ to the $\mathcal{O}^{(k)}$-orbit of $j^kf$ at $j^kf$ is given by

$$T\mathcal{O}^{(k)}f = \left( tf(m_n \cdot \theta(n)) + \omega(m_p \cdot \theta(p)) + m_n^{k+1} \cdot \theta(f) \right) / m_n^{k+1} \cdot \theta(f).$$

Proof: see [Math III], proposition 7·4.

(I·1:25) Notation: Since we want to work in $\theta(f)$ where possible we define

$$T\mathcal{O}^{k}f = tf(m_n \cdot \theta(n)) + \omega(m_p \cdot \theta(p)) + m_n^{k+1} \cdot \theta(f).$$

So

$$T\mathcal{O}^{k}f / m_n^{k+1} \cdot \theta(f) = T\mathcal{O}^{(k)}f \quad \text{and} \quad T\mathcal{O}^{k}f = T\mathcal{O}^{k}g \quad \text{if and only if} \quad T\mathcal{O}^{(k)}f = T\mathcal{O}^{(k)}g.$$ 

Note that $T\mathcal{O}^{k}f \supseteq T\mathcal{O}^{k+1}f$ for all $k$ while it is difficult to compare $T\mathcal{O}^{(k)}f$ and $T\mathcal{O}^{(k+1)}f$. 
Analogously we define the 'tangent space' to the $\mathcal{A}$-orbit of $f$ at $f$ to be

$$T_{\mathcal{A}}f = t\left(\mathcal{m}_n\cdot\theta(n)\right) + \omega f\left(\mathcal{m}_p\cdot\theta(p)\right).$$

Note that by (I-1:22) $T_{\mathcal{A}}^k f = T_{\mathcal{A}}^k g \forall k \Rightarrow T_{\mathcal{A}} f = T_{\mathcal{A}} g$ (and therefore $T_{\mathcal{A}} f$ is an $\mathcal{O}_p$-module via $g^*$). The spaces $T_{\mathcal{A}} f$, $T_{\mathcal{A}}(k) f$, $T_{\mathcal{A}}^k f$ are not in general $\mathcal{O}_n$-modules but they are all $\mathcal{O}_p$-modules via $f^*$.

We have the following criterion for finite determinacy:

(I-1:26) Characterisation Theorem (Mather): The following are equivalent:

i) $f$ is finitely determined,

ii) for some integer $k$, $\mathcal{m}_n\cdot\theta(f) \subseteq t \left(\mathcal{m}_n\cdot\theta(n)\right) + \omega f\left(\mathcal{m}_p\cdot\theta(p)\right)$,

iii) $\dim_\mathbb{C}\left\{\mathcal{m}_n\cdot\theta(f)/t \left(\mathcal{m}_n\cdot\theta(n)\right) + \omega f\left(\mathcal{m}_p\cdot\theta(p)\right)\right\} < \infty$,

iv) for some integer $k'$, $\mathcal{m}_n\cdot\theta(f) \subseteq t \left(\theta(n)\right) + \omega f\left(\theta(p)\right)$,

v) $\dim_\mathbb{C}\left\{\theta(f)/t \left(\theta(n)\right) + \omega f\left(\theta(p)\right)\right\} < \infty$.

Proof-see [du P], p. 121.

We denote by $T_{\mathcal{A}} f$ the module $t \left(\theta(n)\right) + \omega f\left(\theta(p)\right)$.

(I-1:28) Multiplicity Theory and map-germs.

Let $f: (\mathbb{C}^n, x_1, x_2, \ldots, x_l) \to (\mathbb{C}^p, y)$ be a (finite) multi-germ. Then $\theta(f)$ is a finite $\mathcal{O}_{\mathbb{C}^p,y}$-module via $f^* (\theta(f) \sim \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{C}^n,x_i})$, as are its sub-modules and quotient modules.

We will consider the finite $\mathcal{O}_{\mathbb{C}^p,y}$-module $\theta(f)/T_{\mathcal{A}} f$.

Let $\text{Supp}(\theta(f)/T_{\mathcal{A}} f) = \Sigma$.

Now

i) $y \in \Sigma$ if and only if $f$ fails to be stable at $y$ (I-1:23),

ii) $y$ is an isolated point of $\Sigma$ if and only if there is a neighbourhood $U$ of $y$ such that $f$ is stable on $U \setminus \{y\}$,

iii) $y$ is an isolated point of $\Sigma$ if and only if $f$ is finitely determined at
Let $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n$ and consider $\theta(f)_y / T\mathcal{A}f_y$ as a module over

$$A_y = \mathcal{O}_{C^p,y} / \text{Ann}(\theta(f)_y / T\mathcal{A}f_y).$$

Then $\dim(\theta(f)_y / T\mathcal{A}f_y) = \dim(\Sigma) = d,$ say, and the minimal prime ideals $\mathfrak{p}_i$ of $A_y$ with $d = \dim A_y = \dim A_y / \mathfrak{p}_i$ correspond precisely to the prime ideals $\mathfrak{p}_i$ of $\mathcal{O}_{C^p,y},$ where $\mathfrak{p}_i \ni \text{Ann}(\theta(f)_y / T\mathcal{A}f_y)$ and $\Sigma_i = V(\mathfrak{p}_i)$ has dimension $d.$ Then

$$e(\theta(f)_y / T\mathcal{A}f_y) = \sum_{i=1}^r \text{mult}_{\Sigma_i}(\Sigma_i)(\theta(f)_y / T\mathcal{A}f_y) \delta(i),$$

where $\delta(i) = 1$ if $\dim(\Sigma_i) = d$ and 0 if $\dim(\Sigma_i) < d.$

Sheaf Theory: $\theta(f)_y / T\mathcal{A}f_y$ is a coherent sheaf of $\mathcal{O}_{C^p}$-modules so $e(\theta(f)_y / T\mathcal{A}f_y)$ is well-defined as a function of $y \in C^p.$

Proof: Write $\mathcal{O}_{C^n}$ for the sheaf of (germs of) holomorphic functions on $C^n,$ $\mathcal{O}_{C^p}$ for the sheaf on $C^p,$ $\mathcal{V}(f)$ for the sheaf of germs of holomorphic vector fields along $f,$ $\mathcal{V}(n)$ and $\mathcal{V}(p)$ the corresponding sheaves for the identities on $C^n, C^p.$ These are all coherent sheaves. Then the sheaf $\mathcal{V}(f)/ tf(\mathcal{V}(n))$ is also coherent.

Since $f$ is finite, $f_*\left(\mathcal{V}(f)/ tf(\mathcal{V}(n))\right)$ is coherent as an $\mathcal{O}_{C^p}$-module and so is $\mathcal{J}_A(f) = f_*\left(\mathcal{V}(f)/ tf(\mathcal{V}(n))\right)/wf(\mathcal{V}(p)).$ But $\mathcal{J}_{A,y}(f) = \theta(f)_y / T\mathcal{A}f_y.$

See [Wall 1], p. 493.

We remark here that if Krull dim $\theta(f)_y / T\mathcal{A}f_y \geq 1$ then the modules

$$\theta(f)_y / T\mathcal{A}f_y$$

have the same multiplicity; in fact every module

$$\frac{\mathcal{M}_n^k \theta(f) + tf(\mathcal{M}_n^c \theta(n)) + \omega f(\mathcal{M}_p^d \theta(p))}{tf(\mathcal{M}_n^c \theta(n)) + \omega f(\mathcal{M}_p^d \theta(p))}$$

has the same multiplicity for all finite values of $k, c$ and $d$ (provided that the Krull dimension is at least 1).

Proof: Claim: for every pair $(c, d), (c', d')$ there exists a value $k_0$ such that
Proof of claim: this follows from 1.2.4, where we show that for any \((c, d)\) there exists \(k\) such that \(m_n^* \theta(f) \cap T_S f \subseteq tf(m_n^* \theta(n)) + \omega f(m_p^* \theta(p))\) so that

\[
\begin{align*}
m_n^* \theta(f) &\cap \left[ tf(m_n^* \theta(n)) + \omega f(m_p^* \theta(p)) \right] \\
&\subseteq m_n^* \theta(f) \cap tf(m_n^* \theta(n)) + \omega f(m_p^* \theta(p)).
\end{align*}
\]

Conversely, there exists \(k'\) such that

\[
\begin{align*}
m_n^* \theta(f) &\cap \left[ tf(m_n^* \theta(n)) + \omega f(m_p^* \theta(p)) \right] \\
&\subseteq m_n^* \theta(f) \cap tf(m_n^* \theta(n)) + \omega f(m_p^* \theta(p)).
\end{align*}
\]

Choose \(k_0\) to be the maximum of the two values.

I-1.3: Criteria for equivalence of map-germs and jets.

The first criterion is comparison of \(\mathcal{A}\)-tangent spaces:

(I-1.26) Mather's lemma: Let the Lie group \(G\) act smoothly on the manifold \(M\), and suppose that the connected submanifold \(S\) satisfies:

i) for all \(x \in S\), \(T_x G \cdot x \supseteq T_x S\),

ii) the dimension of \(G \cdot x\) is independent of the choice of \(x \in S\).

Then \(S\) is contained in a single \(G\) orbit.

Proof: see [Math IV], lemma 3.1.

We use this to prove

(I-1.27) Theorem: Suppose that map-germs \(f, g \in \mathcal{O}_n^d\) satisfy \(T_{\mathcal{A}^k} f = T_{\mathcal{A}^k} g\).

Then \(f \sim g\).

Proof: this is equivalent to proving

\[T_{\mathcal{A}^k} f = T_{\mathcal{A}^k} g \Rightarrow j^k f, j^k g \text{ are } \mathcal{A}^k\text{-equivalent.}\]

Let \(S = \{j^k f_t \in J^k(n,p); t \in \mathbb{C}\}\) where \(f_t = f + t(g - f)\).

Then \(T_{\mathcal{A}^k} f_t \subseteq T_{\mathcal{A}^k} f\) for all \(t\) since
and any element in $j^k\omega f_t(\mathfrak{m}_p, \theta(p))$ is a sum of terms of the form

$$j^k((\ell)_1 \ldots (\ell)_p, (g)_1 \ldots (g)_p)\partial/\partial Y_i.$$ 

These are elements of $j^k[f^*\mathfrak{m}_p, \omega g(\mathfrak{m}_p, \theta(p))]$. Since $j^k\omega g(\mathfrak{m}_p, \theta(p)) \subseteq T\mathfrak{z}(k)f,$ which is an $\mathfrak{o}_p$-module via $f^*$, the inclusion follows.

Now $T\mathfrak{z}(k)f$ has finite dimension as a vector space over $\mathbb{C}$. So we can choose a basis $\{v^0_1, \ldots, v^0_r\}$ for $T\mathfrak{z}(k)f = T\mathfrak{z}(k)f_0$ over $\mathbb{C}$. We get corresponding spanning sets $\{v^t_1, \ldots, v^t_r\}$ for $T\mathfrak{z}(k)f_t$ and corresponding matrices $A_t = (a_{ij}(t))$ where $v^t_i = \sum_{j=1}^r a_{ij}(t)v^0_j$.

Then $T\mathfrak{z}(k)f_t = T\mathfrak{z}(k)f$ if and only if $A_t$ is invertible, i.e. if and only if $\det(A_t) \neq 0$. Now $\det(A_t)$ is a polynomial in $t$ and is non-zero at $t = 0$, so it has only a finite number of roots. Let $T \subseteq \mathbb{C}$ be given by $T = \{t \in \mathbb{C}: \det(A_t) \neq 0\}$. Then $T$ is open and connected in $\mathbb{C}$ so the submanifold $S_T = \{j^k f_t \in j^k(n,p): t \in T\}$ is open and connected in $j^k(n,p)$.

So the dimension of $\mathfrak{z}(k)$, $j^k f_t$ is independent of the choice of $j^k f_t \in S_T$.

Finally $j^k f - j^k g \in T\mathfrak{z}(k)f_t = T\mathfrak{z}(k)f$ for all $j^k f_t \in S_T$, so $T_s T_T \subseteq T\mathfrak{z}(k)f_t$ for all $j^k f_t \in S_T$. Thus $S_T$ is contained in a single $\mathfrak{z}(k)$-orbit and $j^k f$ and $j^k g$ are $\mathfrak{z}(k)$-equivalent as required. 

Corollary 1: if $f$ is finitely determined then $T\mathfrak{z} f = T\mathfrak{z} g \Rightarrow f \sim g$.

Proof: if $f$ is $k$-determined then

$$T\mathfrak{z} f = T\mathfrak{z} g \Rightarrow T\mathfrak{z} k f = T\mathfrak{z} k g \Rightarrow f \sim^k g \Rightarrow f \sim g.$$

Corollary 2: $T\mathfrak{z} f = T\mathfrak{z} g \Rightarrow T\mathfrak{z} k f = T\mathfrak{z} k g \forall k \Rightarrow f \sim^k g \forall k$. 

More generally,

(I.1:28) Proposition: If $\mathcal{M}_{n}^{\varphi}(f) \subseteq \mathcal{O}(\mathcal{M}_{n}^{\varphi}(n)) + f^{*}\mathcal{M}_{p}^{\varphi}(f)$ for some $\varphi$ then

$$T\mathcal{S}f = T\mathcal{S}g \Rightarrow f \sim g.$$ 

Proof: see [G-H].

We hope that I.1:27 together with the conjecture $f \sim g \quad \forall k \Rightarrow f \sim g$ will lead to a more elementary proof of this theorem.

Remark: these theorems do not hold for $\mathcal{S}_{\mathbb{R}}$-equivalence since they rely on the fact that $\mathbb{C}\setminus\{\text{isolated points}\}$ is connected, which is not true for $\mathbb{R}$. In fact

$$f = (x, y^{3}, x^{2}y + xy^{2} + y^{4}), \quad g = (x, y^{3}-x^{2}y, xy^{2} + y^{4})$$

are examples of map-germs which satisfy $T\mathcal{S}f = T\mathcal{S}g$ but which are not $\mathcal{S}_{\mathbb{R}}$-equivalent (though they are $\mathcal{S}_{\mathbb{C}}$-equivalent): see [Mond 1], p. 360.

The second criterion comes from the theory of unfoldings.

(I.1:29) Definition: A $\mathcal{C}^{s}$-level preserving map-germ is a map-germ

$$F:((\mathbb{C}^{n} \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^{p} \times \mathbb{C}^{s}, 0))$$

of the form $F(x,u) = (f(x,u), u)$. If $f = f(x, 0)$ has some special significance then $F$ is called an unfolding of $f$.

The map $F:((\mathbb{C}^{n} \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^{p} \times \mathbb{C}^{s}, 0))$ is trivial in a region $B \subseteq \mathbb{C}^{s}$ (where $B$ is connected) if there exist $\mathcal{C}^{s}$-level preserving diffeomorphism-germs $H$ of $(\mathbb{C}^{n} \times \mathbb{C}^{s}, 0 \times B)$, $K$ of $(\mathbb{C}^{p} \times \mathbb{C}^{s}, 0 \times B)$ such that

$$K^{-1}F_{*}H = f \times 1_{(\mathcal{C}^{s}, B)}$$

where $f$ is some map-germ $f:((\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}^{p}, 0))$.

The map $F:((\mathbb{C}^{n} \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^{p} \times \mathbb{C}^{s}, 0))$ is $k$-trivial if there exist $\mathcal{C}^{s}$-level preserving diffeomorphism-germs $H$ of $(\mathbb{C}^{n} \times \mathbb{C}^{s}, 0 \times B)$, $K$ of $(\mathbb{C}^{p} \times \mathbb{C}^{s}, 0 \times B)$ such that
where \( f \) is some map-germ \( f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0) \). Here \( j_x f \) is the truncated Taylor series with respect to \( x \) only. There may be higher degree terms involving \( u \).

Rather than \( \Theta(F) \), it is convenient to consider only those vector fields along \( F \) which have zero \( C^s \)-component in \( T\mathbb{C}^p \oplus C^s \mathbb{T}(\mathbb{C}^p \times \mathbb{C}^s) \). The set of all such vector fields is written \( \psi(F) \); it is a free \( \mathcal{O}_{n+s} \)-module of rank \( p \). The corresponding sets of vector fields when \( F \) is the identity map \( (\mathbb{C}^n \times \mathbb{C}^s, 0 \times B) \rightarrow (\mathbb{C}^n \times \mathbb{C}^s, 0 \times B) \) or \( (\mathbb{C}^p \times \mathbb{C}^s, 0 \times B) \rightarrow (\mathbb{C}^p \times \mathbb{C}^s, 0 \times B) \) will be denoted by \( \psi(n+s), \psi(p+s) \) respectively.

There is a well-known criterion for triviality:

(I.1:30) Proposition (Thom-Levine): A \( C \)-level preserving map-germ \( F: (\mathbb{C}^n \times \mathbb{C}, 0 \times [0,1]) \rightarrow (\mathbb{C}^p \times \mathbb{C}, 0 \times [0,1]) \) is trivial if and only if

\[
\partial F^a \in tF^a (\tau n \psi(n+1)) + \omega F^a (\tau p \psi(p+1)), \forall a \in [0,1]
\]

(where \( F^a \) is the germ-restriction of \( F \)).

Proof: see [du P], p 12.

This suggests an obvious analogue for \( k \)-triviality:

(I.1:31) Proposition: A \( C \)-level preserving map-germ \( F: (\mathbb{C}^n \times \mathbb{C}, 0 \times [0,1]) \rightarrow (\mathbb{C}^p \times \mathbb{C}, 0 \times [0,1]) \) is \( k \)-trivial if and only if

\[
\partial F^a \in tF^a (\tau n \psi(n+1)) + \omega F^a (\tau p \psi(p+1)) + \tau n \psi(F), \forall a \in [0,1]
\]

(where \( F^a \) is the germ-restriction of \( F \)).

Proof: (The proof of I.1:30 given in [du P] omits all references to \( \tau n \psi(F) \)).

\( F \) is \( k \)-trivial

\[\Leftrightarrow \exists \ \text{C-level preserving diffeomorphisms} \ H, K \ \text{such that} \ j_x (K^{-1} H) = j^k (f \times 1_{\mathbb{C}}),\]
\[ \Leftrightarrow \partial \cdot (K^{-1} \circ F \circ H)^a \in \mathcal{M}_n^{k+1} \psi(F), \ \forall a \in [0,1] \]

(for this says that \( j_x(K^{-1} \circ F \circ H) \) is independent of \( t \in [0,1] \)).

Differentiation gives that this is equivalent to

\[ \partial \cdot F^a + tF^a(\partial \cdot H^a(\mathcal{H}^a)) + \omega F^a(\partial \cdot K^a(\mathcal{K}^a)) \in \mathcal{M}_n^{k+1} \psi(F), \ \forall a \in [0,1]. \]

Observing that \( \partial \cdot H^a(\mathcal{H}^a) \) and \( \partial \cdot K^a(\mathcal{K}^a) \) vanish on \( 0 \times (\mathcal{C},a) \) gives that \( \partial \cdot H^a(\mathcal{H}^a) \in \mathcal{M}_n \psi(n+1) \) and \( \partial \cdot K^a(\mathcal{K}^a) \in \mathcal{M}_p \psi(p+1) \) and so proves \( \Rightarrow \).

For the 'if' part:

for each \( a \in [0,1] \) there exist, by hypothesis, \( \xi^a \in \mathcal{M}_n \psi(n+1), \eta^a \in \mathcal{M}_p \psi(p+1), \)

\[ V^a \in \mathcal{M}_n^{k+1} \psi(F) \text{ such that } \partial \cdot F^a = tF^a(\xi^a) + \omega F^a(\eta^a) + V^a. \]

Representatives \( \xi^a, \eta^a, V^a \) may be chosen to give germs of vector fields on \( (\mathcal{C}^n \times \mathcal{C}, 0 \times U_a) \) and \( (\mathcal{C}^p \times \mathcal{C}, 0 \times U_a) \), where \( U_a \) is an open neighbourhood of \( a \) in \( \mathcal{C} \), and

\[ \partial \cdot F^t = tF^t((\xi^a)^t) + \omega F^t((\eta^a)^t) + (V^a)^t, \ t \in U_a \]

with \( (\xi^a)^t \in \mathcal{M}_n \psi(n+1), (\eta^a)^t \in \mathcal{M}_p \psi(p+1) \) and \( (V^a)^t \in \mathcal{M}_n^{k+1} \psi(F) \).

Let \( U_1, \ldots, U_r \) be a finite subcover of the open cover \( \{ U_a : a \in [0,1] \} \) of \( [0,1] \) and let \( \phi_1, \ldots, \phi_r \) be a subordinate partition of unity. Let \( \xi = \sum_{i=1}^r \phi_i \xi^a U_i, \ \eta = \sum_{i=1}^r \phi_i \eta^a U_i \)

\[ V = \sum_{i=1}^r \phi_i V^a U_i. \text{ Clearly } \partial \cdot F^a = tF^a(\xi^a) + \omega F^a(\eta^a) + (V^a). \]

A variant on the fundamental existence theorem for solutions of ordinary differential equations states that each germ of a \( \mathcal{C} \)-level preserving vector field on

\( (\mathcal{C}^n \times \mathcal{C}, 0 \times [0,1]) \) which vanishes on \( 0 \times [0,1] \) may be expressed as \( \frac{d}{dt}H \circ H^{-1} \), where \( H \) is a \( \mathcal{C} \)-level preserving diffeomorphism of \( (\mathcal{C}^n \times \mathcal{C}, 0 \times [0,1]) \). Choosing \( H, K \) to give
representatives of $\xi, \eta$, respectively, gives a relation of the form (*) so that $F$ is $k$-trivial.
I.2 : Classification Theory.

The aim is, for large integers $k$ and a fixed integer $\lambda$, to find a normal form for $j^{k+\lambda}g$ when $j^k = j^\lambda g$. We also prove a condition for a normal forms to be $(\lambda + k)$-determined. The key to finding these is that if we compare $T_n^{k+\lambda}f$ and $T_n^{k+\lambda}g$ for $f - g \in \mathcal{M}_n^{k+1} \cdot \mathcal{O}_n^p$ then only a small number of terms are altered and the new terms are relatively easy to calculate. We can then construct $k$-trivial unfoldings of $g$.

If $f$ is finite then let $p$ denote the least integer satisfying $\mathcal{M}_n^{p} \subseteq f^* \mathcal{M}_p \cdot \mathcal{O}_n^p$.

First we determine some properties of $f \in \mathcal{M}_n \cdot \mathcal{O}_n^p$.

(I.2:1) Lemma : If $f \in \mathcal{M}_n \cdot \mathcal{O}_n^p$ then

$$t(f(\mathcal{M}_n \cdot \theta(n))) \subseteq \mathcal{M}_n^{d+R-1} \cdot \theta(f) \quad \text{and} \quad \omega(f(\mathcal{M}_p \cdot \theta(p))) \subseteq (\mathcal{M}_n^d \cdot S).$$

Proof : This is obvious.

(I.2:2) Lemma : Given $\kappa \geq 1$ and $f \in \mathcal{M}_n^d - \mathcal{O}_n^p$, if $R \geq \kappa + 1$ and $S$ satisfies $d.(S-1) \geq \kappa$

then for any $k > \kappa - 1$ and $h \in \mathcal{M}_n^{k+1} \cdot \mathcal{O}_n^p$,

$$t(f+h)(\mathcal{M}_n^R \cdot \theta(n)) + \omega(f+h)(\mathcal{M}_p^S \cdot \theta(p)) = tf(\mathcal{M}_n^R \cdot \theta(n)) + \omega(\mathcal{M}_p^S \cdot \theta(p))$$

modulo $\mathcal{M}_n^{k+\kappa+1} \cdot \theta(f)$.

For convenience we shall write $T_n^{R,S}f$ for the module $tf(\mathcal{M}_n^R \cdot \theta(n)) + \omega(\mathcal{M}_p^S \cdot \theta(p))$.

The lemma then says that given $\kappa$ and $f$, if $R$ and $S$ are as above, then for any $k$
and \( h \in \mathbb{m}_n^{k+1} \cdot \mathbf{p} \)

\[
T_{\mathbf{R,R}'}(f+h) + \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f+h) = T_{\mathbf{R,R}'} \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f).
\]

Further, if \( f \) is finite and \( k \geq \max(\kappa + 1, \rho) \) then

\[
(f+h)^* \mathbb{m}_p^{R} \left( T_{\mathbf{R,R}'}(f+h) + \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f+h) \right) = (f+h)^* \mathbb{m}_p^{R} \left( T_{\mathbf{R,R}'} \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f) \right)
\]

\[
= f^* \mathbb{m}_p^{R} \left( T_{\mathbf{R,R}'} \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f) \right).
\]

Proof of Lemma: If \( R \geq \kappa + 1 \) then

\[
\theta(h) \mathbb{m}_n^k \subseteq \mathbb{m}_n^{k+R} \cdot \theta(h) + \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f) = \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f).
\]

If \( f \) is finite and \( k \geq \rho \) then \((f+h)^* \mathbb{m}_p^{R} \mathbb{m}_n^k = f^* \mathbb{m}_p^{R} \mathbb{m}_n^k \) and

\[
(f+h)^* \mathbb{m}_p^{R} \left( \theta(h) \mathbb{m}_n^k \right) = (f+h)^* \mathbb{m}_p^{R} \mathbb{m}_n^k \cdot \theta(h) = f^* \mathbb{m}_p^{R} \mathbb{m}_n^k \cdot \theta(h).
\]

For \( \omega^f \), consider for example \((f+h)^S = f^S + S^h(f + h^2 \mathbb{p}(f, h))\);

we see that

\[
\omega(f+h)(\mathbb{m}_p^k \cdot \theta(p)) + \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f) = \omega^f(\mathbb{m}_p^k \cdot \theta(p)) + \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f)
\]

provided that \( k+1 \geq \kappa \) and \( \mathbb{m}_n^{k+\kappa+1} \cdot \omega^f(\mathbb{m}_p^k \cdot \theta(p)) \subseteq \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f) \). If \( S \) satisfies

\[
d(S^{-1}) \geq \kappa \]

then

\[
\omega^f(\mathbb{m}_p^k \cdot \theta(p)) \subseteq \mathbb{m}_n^k \cdot \theta(f).
\]

To see that

\[
(f+h)^* \mathbb{m}_p^{S} \left( \omega^f(\mathbb{m}_p^k \cdot \theta(p)) + \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f) \right) = f^* \mathbb{m}_p^{S} \left( \omega^f(\mathbb{m}_p^k \cdot \theta(p)) + \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f) \right)
\]

observe that \((f+h)^* \mathbb{m}_p^k \cdot \omega^f(\mathbb{m}_p^k \cdot \theta(p)) = (f+h)^* \mathbb{m}_p^k \cdot f^* \mathbb{m}_p^k \cdot \omega^f(\mathbb{m}_p^k \cdot \theta(p)) \).

Then \((f+h)^* \mathbb{m}_p^{S^{-1}} \cdot \omega^f(\mathbb{m}_p^k \cdot \theta(p)) + \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f) = f^* \mathbb{m}_p^{S^{-1}} \cdot \omega^f(\mathbb{m}_p^k \cdot \theta(p)) + \mathbb{m}_n^{k+\kappa+1} \cdot \theta(f) \)

by choice of \( S \). \( \square \)
(I.2:3) Corollary: If \( R, S \) are chosen as in I.2:2 then

\[
T \mathcal{A}^{k+1}(f+h) = T \mathcal{A}_{R,S} f + \mathcal{A}^{k+1} \cdot \theta(f) + \mathcal{C}\{u_1(f+h), \ldots, u_\ell(f+h)\},
\]

where \( \{u_1(g), \ldots, u_\ell(g)\} \) denotes the set

\[
\left\{ \alpha \frac{\partial}{\partial x_j}(g), \ g^* (\beta), \ \alpha \in \mathcal{A}_n \setminus \mathcal{A}_n^R, \ \beta \in \mathcal{A}_p \setminus \mathcal{A}_p^S \right\}.
\]

For the case \( R = S = 1 \) we put

\[
u_1(g) = x_1 \frac{\partial}{\partial x_1}(g), \ u_2(g) = x_2 \frac{\partial}{\partial x_1}(g), \ldots, u_n(g) = x_n \frac{\partial}{\partial x_n}(g),
\]

\[
u_{n+1}(g) = (g)_1 \cdot \partial/\partial Y_1, \ u_{n+2}(g) = (g)_1 \cdot \partial/\partial Y_2, \ldots, u_{n+p^2}(g) = (g)_p \cdot \partial/\partial Y_p.
\]

Then \( T \mathcal{A}^{k+1}(f+h) = T \mathcal{A}_{2,2} f + \mathcal{A}^{k+2} \cdot \theta(f) + \mathcal{C}\{u_i(f+h) \mid 1 \leq i \leq n^2 + p^2\} \).

For the general case we take the obvious extension. The numbering does not matter: we only use this for ease of notation.

(I.2:4) Lemma: For any finite map-germ \( f \) and any positive integers \( R, S \) there exist (minimum) integers \( k_0, k_1 \) such that

\[
T \mathcal{A} f \cap \mathcal{A}_n^{k_0+1} \cdot \theta(f) \subseteq T \mathcal{A}_{R,S} f
\]

and \( T \mathcal{A} f \cap \mathcal{A}_n^{k_1+2} \cdot \theta(f) \subseteq f^* \mathcal{A}_p \cdot T \mathcal{A} f \).

Proof: Since \( f \) is finite, \( \theta(f) \) is a finitely generated \( \mathcal{O}_p \)-module so we can apply the Artin-Rees lemma (I.1:7) to \( M = \theta(f), N = T \mathcal{A} f, \) and \( I = f^* \mathcal{A}_p^c \). This says that for any \( c \), there exists an \( m(c) \) such that

\[
T \mathcal{A} f \cap f^* \mathcal{A}_p^{cm(c)} \cdot \theta(f) \subseteq f^* \mathcal{A}_p^c \cdot T \mathcal{A} f.
\]

Now for any positive integers \( R \) and \( S \), there exists a \( k \) such that

\[
\mathcal{A}_n^{k+1} \cdot \theta(f) \subseteq f^* \mathcal{A}_p^{cm} \cdot \theta(f)
\]
(since $\mathcal{m}_n^p \subseteq f^*\mathcal{m}_p \cdot \mathcal{O}_n$). Put $c = 1$ to find $k_1$.

There also exists a $c$ such that

$$f^*\mathcal{m}_p^c \cdot T\mathcal{S} f \subseteq T\mathcal{S}_R S f$$

(since $f^*\mathcal{m}_p^c \cdot \omega f(\mathcal{m}_p \cdot \theta(p)) \subseteq \omega f(\mathcal{m}_p^{c+1} \cdot \theta(p))$ and $f^*\mathcal{m}_p \cdot \mathcal{O}_n \subseteq \mathcal{m}_n$).

Thus

$$T\mathcal{S} f \cap \mathcal{m}_n^{k+1} \cdot \theta(f) \subseteq T\mathcal{S}_R S f.$$

Then let $k_0$ be the least integer for which this inclusion holds. This concludes the proof of lemma 1.2.4.

We shall usually put $\kappa = 1$, when we can choose $R = S = 2$. Then there exists a $k_0$ such that $k \geq k_0$ implies that

$$T\mathcal{S} f \cap \mathcal{m}_n^{k+1} \cdot \theta(f) \subseteq T\mathcal{S}_{2,2} f$$

and if $h \in \mathcal{m}_n^{k+1} \cdot \mathcal{O}_n^p$ then

$$T\mathcal{S}^{k+1}(f+h) = T\mathcal{S}_{2,2} f + \mathcal{m}_n^{k+2} \cdot \theta(f) + C\{x_i \frac{\partial}{\partial x_j}(f+h); 1 \leq i, j \leq n\}$$

$$+ C\{(f+h)_i \partial Y_j; 1 \leq i, j \leq p\}.$$ 

The terms $x_i \frac{\partial}{\partial x_j}(h)$, $(h)_i \partial Y_j$ are in $\mathcal{m}_n^{k+1} \cdot \theta(f)$. If they also lie in $T\mathcal{S}^{k+1} f$ then $T\mathcal{S}^{k+1}(f+h) = T\mathcal{S}^{k+1} f$ and $f+h \sim f$. The elements of interest are those which lie in $T\mathcal{S}^{k} f \setminus T\mathcal{S}^{k+1} f$.

Another possible choice of $\kappa$ is $\kappa = p$ where $\mathcal{m}_n^p \subseteq f^*\mathcal{m}_p \cdot \mathcal{O}_n$. Then
(I-2:5) Definition: Call \( \{v_1, \ldots, v_s\} \subseteq \mathfrak{m}_n^{k+1} \cdot \theta(f) \) a **transverse section** for \( T_x^{k}f \) over \( T_x^{k+\kappa}f \) (or just a **transverse section**) if

\[
T_x^{k}f = T_x^{k+\kappa}f + \mathbb{C}\{v_1, \ldots, v_s\}
\]

and none of the \( v_i \) may be omitted. Note that \( \{v_1, \ldots, v_s\} \) is a transverse section if and only if it is a \( \mathbb{C} \)-basis for \( T_x^{k}f \) over \( T_x^{k+\kappa}f \). We will usually choose a basis of monomials.

If \( R \) and \( S \) are chosen as in I-2:2 and \( k_0 \) is chosen as in I-2:4 then

(I-2:6) Theorem: Suppose that \( \{v_1, \ldots, v_s\} \subseteq \mathfrak{m}_n^{k+1} \cdot \theta(f) \) and that either

\[
a) \ \mathfrak{m}_n^{k+1} \cdot \theta(f) \subseteq T_x^{k} R_s f + \mathfrak{m}_n^{k+1+\kappa} \cdot \theta(f) + \mathbb{C}\{v_1, \ldots, v_s\}
\]

or b) \( f \) is finite, \( k \geq k_0 \) and \( T_x^{k}f = T_x^{k+\kappa}f + \mathbb{C}\{v_1, \ldots, v_s\} \).

Then \( g_k f \Rightarrow g \) is \( (k+\kappa) \)-equivalent to a map-germ of the form

\[
f + \sum_{i=1}^{s} \lambda_i v_i, \text{ some } \lambda_i \in \mathbb{C}.
\]

Proof: After a change of coordinates we may assume that \( j^k g = j^k f \), i.e. that \( g - f \in \mathfrak{m}_n^{k+1} \cdot \theta(f) \). Let \( g - f = h \) and consider the germ \( f+h \).

If a) holds then there exist \( \lambda_i \in \mathbb{C} \) such that

\[
h = \sum_{i=1}^{s} \lambda_i v_i \in T_x^{k} R_s f + \mathfrak{m}_n^{k+1+\kappa} \cdot \theta(f).
\]

If b) holds then there exist \( \lambda_i \in \mathbb{C} \) such that
$$h = \sum_{i=1}^{\ell} \lambda_i v_i \in \mathcal{M}_n^{k+1} \cdot \theta(f) \cap \mathcal{R}^k \mathcal{S}_f \subseteq \mathcal{R}^k \mathcal{S}_f + \mathcal{M}_n^{k+1+k} \cdot \theta(f)$$

Our aim is to show that the family $f + \sum_{i=1}^{\ell} \lambda_i v_i + a(h - \sum_{i=1}^{\ell} \lambda_i v_i)$ is $(k + \kappa)$-trivial.

It is enough to prove

(I-2:7) Lemma: The family $F = (f + h_1 + ah_2, a)$ is $(k + \kappa)$-trivial, where

$$h_1 \in \mathcal{M}_n^{k+1} \cdot \theta(f) \text{ and } h_2 \in \mathcal{M}_n^{k+1} \cdot \theta(f) \cap \mathcal{R}^k \mathcal{S}_f + \mathcal{M}_n^{k+1+k} \cdot \theta(f).$$

Proof of lemma: By Thom-Levine (I-1:31) it is enough to show that

$$h_2 \in t^2a(\mathcal{M}_n \psi(n+1)) + \omega^a(\mathcal{M}_n \psi(p+1)) + \mathcal{M}_n^{k+1} \cdot \psi(F), \forall a \in [0,1].$$

Now $h_2 \in \mathcal{M}_n^{k+1} \cdot \theta(f) \cap \mathcal{R}^k \mathcal{S}_f + \mathcal{M}_n^{k+1+k} \cdot \theta(f)$ and

$$\mathcal{R}^k \mathcal{S}_f + \mathcal{M}_n^{k+1+k} \cdot \theta(f) = \mathcal{R}^k (f + h_1) + \mathcal{M}_n^{k+1+k} \cdot \theta(f)$$

so there exist $\alpha_j \in \mathcal{M}_n, \beta \in \mathcal{M}_p$ and $h_0 \in \mathcal{M}_n \cdot \theta(f)$ such that

$$h_2 = \sum_{j=1}^{\ell} \alpha_j \frac{\partial}{\partial x_j} (f + h_1) + (f + h_1)^*(\beta) + h_0.$$

Then

$$h_2 = \sum_{j=1}^{\ell} \alpha_j \frac{\partial}{\partial x_j} (f + h_1 + ah_2) + (f + h_1 + ah_2)^*(\beta)$$

$$+ \left( h_0 - a \sum_{j=1}^{\ell} \alpha_j \frac{\partial}{\partial x_j} h_2 + (f + h_1)^*(\beta) - (f + h_1 + ah_2)^*(\beta) \right).$$

The term on the second line is an element of $\mathcal{M}_n^{k+1+k} \cdot \theta(f)$ and the result follows.

This concludes the proof of the lemma and hence of the theorem. Theorem I:1 follows by putting $\kappa = 1$. 

Note: since the proof depends on constructing a $k$-trivial unfolding the theorem also
is valid for $\mathcal{R}$-equivalence.

(I-2:8) Example: Let $f(x,y) = (xy, x^2, y^2)$ and $\kappa = 1$ (II-5:2). Then

$$T\mathcal{R}_{2,2}f = \left\{ x^{2i-j-1} \partial / \partial X, x^{2i-j-1} \partial / \partial Y, x^{2i-j-1} \partial / \partial Z : i \geq 2, 0 \leq j \leq 2i \right\} + m_2^2 \begin{pmatrix} x \\ 0 \\ 2x \\ 0 \end{pmatrix}.$$

For $s \geq 1$,

$$g \in \mathcal{R}^{2s}f \Rightarrow g \in \mathcal{R}^{2s+1}(xy + \sum \lambda_j x^{2s+1-j} y^j, x^2 + \alpha y^{2s+1}, y^2 + bx^{2s+1}).$$

and $g \in \mathcal{R}^{2s+1}f \Rightarrow g \in \mathcal{R}^{2s+2}(xy, x^2, y^2)$.

In particular $g \in \mathcal{R}^2f \Rightarrow g \in \mathcal{R}^3(xy + bx^3 + cx^2y + dxy^2 + ey^3, x^2 + fy^3, y^2 + ax^3)$.

The determinacy degree comes from

(I-2:9) Theorem (proof of I:2): Suppose that $f$ is finite, $k \geq p$, \{v_1, ..., v_s\} as in I-2:6 and \{v_1, ..., v_s\} $\sim$ \{v_1, ..., v_s\}.

a) If i)-iii) hold then $f + \sum_{i=1}^s \lambda_i v_i$ is $(k+1)$-determined:

i) $f^* m_{p} \left( T\mathcal{R} f + C\{v_1, ..., v_s\} \right) \supseteq m_{n}^{k+2} \cdot \theta(f)$,

ii) $T\mathcal{R}^{k+1} \left( f + \sum_{i=1}^s \lambda_i v_i \right) \supseteq T\mathcal{R}^{k+2} \cdot \theta(f) + C\{v_1, ..., v_s\}$,

iii) for values $\lambda_i \neq 0, v_i = ((v_i)_1, ..., (v_i)_p)$

$$T\mathcal{R}_{2,2} f + f^* \Theta_p \{v_1, ..., v_s\} + m_{n}^{k+2} \cdot \theta(f) \supseteq C\{(v_i)_j \partial / \partial Y_{l} : 1 \leq l, j \leq p\}.$$

b) If $k \geq k_1$ and $T\mathcal{R}^{k+2} f + f^* m_{p} \{v_1, ..., v_s\} \supseteq m_{n}^{k+2} \cdot \theta(f)$ then i) holds.

c) If $T\mathcal{R}_{2,2} f + f^* \Theta_p \{v_1, ..., v_s\} + m_{n}^{k+2} \cdot \theta(f) \supseteq m_{n}^{k+1} \cdot \theta(f)$ then iii) holds.

Proof: We show that $T\mathcal{R}^{s} \left( f + \sum_{i=1}^s \lambda_i v_i + h \right) = T\mathcal{R}^{s} \left( f + \sum_{i=1}^s \lambda_i v_i \right)$ for all $h \in m_{n}^{k+2} \cdot \theta(f)$. 

Then equivalence follows from I-1:28.

Hypothesis b) implies i) \( f^* m_p \cdot T \phi \, f + f^* m_p \cdot \{v_1, ..., v_i\} \supseteq m_n^{k+2} \cdot \theta(f) \) since for all \( c \in m_n^{k+2} \cdot \theta(f) \) there exist \( b \in f^* m_p \cdot \{v_1, ..., v_i\} \), \( a \in T \phi \, f \) such that \( c = a + b \).

Then \( a = c - b \in m_n^{k+2} \cdot \theta(f) \cap T \phi \, f \subseteq f^* m_p \cdot T \phi \, f \).

By ii) \( T \phi \, (f + \sum_{i=1}^{s} \lambda_i v_i + h) + m_n^{k+2} \cdot \theta(f) = T \phi \, (f + \sum_{i=1}^{s} \lambda_i v_i) + m_n^{k+2} \cdot \theta(f) \)

\( \supseteq T \phi \, f + m_n^{k+2} \cdot \theta(f) + C \cdot \{v_1, ..., v_i\} \),

so \( (f + \sum_{i=1}^{s} \lambda_i v_i + h)^* m_p \cdot (T \phi \, (f + \sum_{i=1}^{s} \lambda_i v_i + h) + m_n^{k+2} \cdot \theta(f)) \)

\( \supseteq (f + \sum_{i=1}^{s} \lambda_i v_i + h)^* m_p \cdot (T \phi \, f + m_n^{k+2} \cdot \theta(f) + C \cdot \{v_1, ..., v_i\}) \)

\( = (f + \sum_{i=1}^{s} \lambda_i v_i + h)^* m_p \cdot \{\frac{\partial f}{\partial x_j} : 1 \leq i \leq n\} + \{f_r \partial / \partial Y_s : 1 \leq r, s \leq p\} \)

\( + m_n^{k+2} \cdot \theta(f) + C \cdot \{v_1, ..., v_i\} \)

\( = f^* m_p \cdot T_{2,2} \phi \cdot f + f^* m_p \cdot m_n \cdot \{\frac{\partial f}{\partial x_j} : 1 \leq i \leq n\} + f^* m_p \cdot m_n^{k+2} \cdot \theta(f) + f^* m_p \cdot \{v_1, ..., v_i\} \)

\( + (f + \sum_{i=1}^{s} \lambda_i v_i + h)^* m_p \cdot \{f_r \partial / \partial Y_s : 1 \leq r, s \leq p\} \)

\( = f^* m_p \cdot T_{2,2} \phi \cdot f + f^* m_p \cdot m_n \cdot \{\frac{\partial f}{\partial x_j} : 1 \leq i \leq n\} + f^* m_p \cdot m_n^{k+2} \cdot \theta(f) + f^* m_p \cdot \{v_1, ..., v_i\} \)

\( + \{(f + \sum_{i=1}^{s} \lambda_i v_i + h) f_r \partial / \partial Y_s : 1 \leq q, r, s \leq p\} \).

But \( (h) q_r \partial / \partial Y_s \in f^* m_p \cdot m_n^{k+2} \cdot \theta(f) \),

and by iii) for \( \lambda_i \neq 0 (v_i) q_r \partial / \partial Y_s \in f^* m_p \cdot T_{2,2} \phi \cdot f + f^* m_p \cdot \{v_1, ..., v_i\} \).
so that \( f^*\nu_p \cdot \omega(f) = \left( f + \sum \lambda_i v_i + h \right) \cdot \omega(f) \)
modulo \( f^*\nu_p \cdot T\mathcal{X}_f \).

and

\[
(f + \sum \lambda_i v_i + h)^* \nu_p \left( T\mathcal{X}_f + \nu_n^{k+2} \cdot \theta(f) + C\{v_1, \ldots, v_t\} \right) =
\]

\[
f^*\nu_p \left( T\mathcal{X}_f + \nu_n^{k+2} \cdot \theta(f) + C\{v_1, \ldots, v_t\} \right).
\]

By i) \( f^*\nu_p \left( T\mathcal{X}_f + C\{v_1, \ldots, v_t\} \right) \geq \nu_n^{k+2} \cdot \theta(f) \) so

\[
T\mathcal{X}_f (f + \sum \lambda_i v_i + h) + f^*\nu_p \cdot \nu_n^{k+2} \cdot \theta(f) \geq \nu_n^{k+2} \cdot \theta(f)
\]

and hence \( T\mathcal{X}_f (f + \sum \lambda_i v_i + h) \geq \nu_n^{k+2} \cdot \theta(f) \) and \( T\mathcal{X}_f (f + \sum \lambda_i v_i + h) = T\mathcal{X}_f (f + \sum \lambda_i v_i) \).

This concludes the proof.

\[
\square
\]

Note: since \( T\mathcal{X}_f (f + \sum \lambda_i v_i) \geq \nu_n^{k+2} \cdot \theta(f) \), if \( f + \sum \lambda_i v_i \) is finitely determined and since \( T\mathcal{X}_f (f + \sum \lambda_i v_i + h) = T\mathcal{X}_f (f + \sum \lambda_i v_i) \), we can apply Mather's lemma to each connected submanifold \( \{f + \sum \lambda_i v_i + ah: a \in \mathbb{R}\} \), that is \( f + \sum \lambda_i v_i \) also has determinacy degree \( k + \kappa \) with respect to \( \mathcal{X}_\mathbb{R} \)-equivalence.

For the rest of this section we calculate the new tangent space with the aim of using \( k \)-triviality to reduce the number of parameters.

Recall that
where the \( u_i \) are of the form given in I-2.3.

(I-2:10) Definition: Let \( \lambda = (\lambda_1, \ldots, \lambda_s) \). Define \( d(\lambda) \) and \( n(\lambda) \) by

\[
d(\lambda) = \dim \mathcal{R}_S \mathcal{R}_S \mathcal{R}_S f + \tau \nu_n^{k+1} \cdot \theta(f) = t - n(\lambda) + \dim \mathcal{C} \left( \mathcal{R}_S \mathcal{R}_S f + \tau \nu_n^{k+1} \cdot \theta(f) \right)
\]

where \( n(\lambda) \) is the number of relations of the form

\[
\sum_{m=1}^{1} \mu_m u_m (f + \sum_{i=1}^{s} \lambda_i \nu_i) \in \mathcal{R}_S \mathcal{R}_S f + \tau \nu_n^{k+1} \cdot \theta(f), \quad \mu_i \in \mathcal{C}.
\]

(I-2:11) Lemma: If \( \mu_m \in \mathcal{C} \) satisfy

\[
\sum_{m=1}^{t} \mu_m u_m (f + \sum_{i=1}^{s} \lambda_i \nu_i) \in \mathcal{R}_S \mathcal{R}_S f + \tau \nu_n^{k+1} \cdot \theta(f)
\]

then

\[
\sum_{m=1}^{t} \mu_m u_m (f + \sum_{i=1}^{s} \lambda_i \nu_i) \in \mathcal{R}_S \mathcal{R}_S f + \tau \nu_n^{k+1} \cdot \theta(f).
\]

Corollary: \( n(\lambda) \leq n(0) \) and \( d(\lambda) \geq d(0) \).

Proof: If

\[
\sum_{m=1}^{t} \mu_m u_m (f + \sum_{i=1}^{s} \lambda_i \nu_i) \in \mathcal{R}_S \mathcal{R}_S f + \tau \nu_n^{k+1} \cdot \theta(f)
\]

then

\[
\sum_{m=1}^{t} \mu_m u_m (f + \sum_{i=1}^{s} \lambda_i \nu_i) \in \mathcal{R}_S^{k+1} f,
\]

so

\[
\sum_{m=1}^{t} \mu_m u_m (f + \sum_{i=1}^{s} \lambda_i \nu_i) - \sum_{m=1}^{t} \mu_m u_m (f) \in \mathcal{R}_S^{k+1} f.
\]

But
\[
\sum_{m=1}^{t} \mu_m u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - \sum_{m=1}^{t} \mu_m u_m(f) \in \mathcal{R}_n^k \cdot \theta(f) \cap T_{k+1}^{k+1},
\]

so

\[
\sum_{m=1}^{t} \mu_m u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - \sum_{m=1}^{t} \mu_m u_m(f) \in T_{k+1}^{k+1}(f + \sum_{i=1}^{s} \lambda_i v_i).
\]

Hence

\[
\sum_{m=1}^{t} \mu_m u_m(f) \in T_{k+1}^{k+1}(f + \sum_{i=1}^{s} \lambda_i v_i).
\]

as required. \(\Box\)

Note: \(\sum_{m=1}^{t} \mu_m u_m(f) \in T_{k+1}^{k+1}(f + \sum_{i=1}^{s} \lambda_i v_i)\) implies that

\[
\sum_{m=1}^{t} \mu_m u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - \sum_{m=1}^{t} \mu_m u_m(f) \in T_{k+1}^{k+1}(f + \sum_{i=1}^{s} \lambda_i v_i).
\]

There are \(n(0)\) such relations.

(I-2:12) Lemma: Suppose that \(k+1 \geq \kappa\). Then for each \(m\) there exist \(a^m_{ij} \in \mathcal{C}\), \(\alpha^m_{ij} \in \mathcal{R}_n^k\), \(\beta_i \in \mathcal{R}_p^s\) such that

\[
u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - \nu_m(f) = \sum_{j=1}^{n} \alpha^m_{ij} \frac{\partial}{\partial x_j} (f + \sum_{i=1}^{s} \lambda_i v_i) + (f + \sum_{i=1}^{s} \lambda_i v_i)^*(\beta)
\]

\[
+ \sum_{i=1}^{s} \lambda_i (h_i - \sum_{j=1}^{n} \alpha^m_{ij} \frac{\partial}{\partial x_j} \sum_{i=1}^{s} \lambda_i v_i + f^*(\beta) - (f + \sum_{i=1}^{s} \lambda_i v_i)^*(\beta)).
\]

The term on the second line is an element of \(\mathcal{R}_n^{k+1} \cdot \theta(f)\).

Proof: If \(u_m \in \text{tf}(\mathcal{R}_n^k \cdot \theta(n))\) then \(u_m(f + \sum_{i=1}^{s} \lambda_i v_i)\) is of the form
\[ \frac{\partial}{\partial x_j} (f + \sum_{i=1}^{s} \lambda_i v_i), \ \alpha \in \mathcal{M}_n, \]

so \( u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - u_m(f) = \alpha \frac{\partial}{\partial x_j} \sum_{i=1}^{s} \lambda_i v_i = \sum_{i=1}^{s} \lambda_i \alpha \frac{\partial}{\partial x_j} v_i \) and \( \alpha \frac{\partial}{\partial x_j} v_i \in \mathcal{M}_n^{k+1} \cdot \theta(f) \).

If \( u_m \in \omega f(\mathcal{M}_n \cdot \theta(p)) \) then \( u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - u_m(f) \) is of the form

\[ \prod_{j=1}^{p} (f + \sum_{i=1}^{s} \lambda_i v_i)_j \cdot \partial / \partial Y_L - \prod_{j=1}^{p} (f)_j \cdot \partial / \partial Y_L. \]

Expand the polynomial, bearing in mind that \( \mathcal{M}_n^{2k+2} \cdot \theta(f) \subseteq \mathcal{M}_n^{k+1} \cdot \theta(f) \);

we get a polynomial of the form

\[ \sum_{i=1}^{s} \lambda_i \sum_{j=1}^{p} \beta_j(v_i)_j \cdot \partial / \partial Y_L, \] where each \( \beta_j(v_i)_j \in \mathcal{M}_n^{k+1} \cdot \theta(f) \).

For example

\[ (f + \sum_{i=1}^{s} \lambda_i v_i)_j (f + \sum_{i=1}^{s} \lambda_i v_i)_j - (f)_j (f)_j = (f)_j \sum_{i=1}^{s} \lambda_i v_i_j + (f)_j \sum_{i=1}^{s} \lambda_i v_i_j + (\sum_{i=1}^{s} \lambda_i v_i)_j (\sum_{i=1}^{s} \lambda_i v_i)_j \]

and \( (\sum_{i=1}^{s} \lambda_i v_i)_j (\sum_{i=1}^{s} \lambda_i v_i)_j \in \mathcal{M}_n^{2k+2} \cdot \theta(f) \).

The sum is \( \sum_{i=1}^{s} \lambda_i [(f)_j (v_i)_j + (f)_j (v_i)_j] \).

For each element \( \chi_t \in \mathcal{M}_n^{k+1} \cdot \theta(f) \) there exist \( c_{i,t} \in \mathbb{C} \) such that

\[ \chi_t - \sum_{i=1}^{s} c_{i,t} v_i \in \mathcal{M}_n^{k+1} \cdot \theta(f) \cap \mathcal{T} \mathcal{S} \mathcal{R}^{k+1} f \subseteq \mathcal{T} \mathcal{R} \mathcal{S} f. \]

so for each \( \chi_t \) there exist \( \alpha_{i,j} \in \mathcal{M}_n, \beta_t \in \mathcal{M}_P \) such that

\[ \chi_t = \sum_{i=1}^{s} c_{i,t} v_i + \sum_{i=1}^{s} \lambda_i \left( \sum_{j=1}^{n} \alpha_{i,j} \frac{\partial}{\partial x_j} f + f^*(\beta_t) \right) \]
The result follows by substitution and collecting terms.

\[\sum_{i=1}^{s} c_{ij}(h)v_j + \sum_{i=1}^{s} \lambda_i \left( \sum_{j=1}^{n} \alpha_{ij} \frac{\partial}{\partial x_j} (f + \sum_{i=1}^{s} \lambda_i v_i) + (f + \sum_{i=1}^{s} \lambda_i v_i)^*(\beta_j) \right) = \sum_{i=1}^{s} \lambda_i \left( \sum_{j=1}^{n} \alpha_{ij} \frac{\partial}{\partial x_j} \sum_{i=1}^{s} \lambda_i v_i + (f + \sum_{i=1}^{s} \lambda_i v_i)^*(\beta_j) - f^*(\beta_j) \right).\]

(I-2:13) Theorem: If \(k + 1 \geq \kappa\) we can write

\[T \mathcal{F}_{k+\kappa}^k(f + \sum_{i=1}^{s} \lambda_i v_i) = T \mathcal{F}_{R, S^f + m_n}^{k+\kappa+1} \theta(f) + \sum_{i=1}^{s} \lambda_i a_{ij}^iv_j + \sum_{i,j=1}^{s} \lambda_i a_{ij}^{t}v_j\]

where \(d = d(\omega) - d(\emptyset), a_{ij}, a_{ij}^t, b_{ij}^c \in \mathbb{C}\) for all \(i, j, c\) and \(\{u_1(f), \ldots, u_t(f)\}\) is linearly independent over \(T \mathcal{F}_{R, S^f + m_n}^{k+\kappa+1} \theta(f)\).

Then \(d(\lambda) = \dim_{\mathbb{C}} T \mathcal{F}_{R, S^f + m_n}^{k+\kappa+1} \theta(f)/m_n^{k+\kappa+1} \theta(f) + t + d\).

Proof: Recall that

\[T \mathcal{F}_{k+\kappa}^k(f + \sum_{i=1}^{s} \lambda_i v_i) = T \mathcal{F}_{R, S^f + m_n}^{k+\kappa+1} \theta(f) + \sum_{i=1}^{s} \lambda_i a_{ij}^iv_j\]

By (2-11) for each \(m,\)

\[u_m(f + \sum_{i=1}^{s} \lambda_i v_i) = u_m(f) + \sum_{i,j=1}^{s} \lambda_i a_{ij}^mv_j\]

modulo \(T \mathcal{F}_{R, S^f + m_n}^{k+\kappa+1} \theta(f)\).

If \(\sum_{m=1}^{t} \mu_m u_m(f) \in T \mathcal{F}_{R, S^f + m_n}^{k+\kappa+1} \theta(f)\) write \(b_{ij}^c = \sum_{m=1}^{t} \mu_m a_{ij}^m\) so that

\[\sum_{m=1}^{t} \mu_m \sum_{i,j=1}^{s} \lambda_i a_{ij}^mv_j = \sum_{i,j=1}^{s} \lambda_i b_{ij}^cv_j.\]

\(\square\).
Example: Let 
\[ f(x, y) = (x^2 + f y^3, y^2 + ax^3, xy + bx^3 + cx^2 y + dxy^2 + ey^3). \]
Write \( p(x, y) = x^2 + f y^3, q(x, y) = y^2 + ax^3, r(x, y) = xy + bx^3 + cx^2 y + dxy^2 + ey^3. \)
Then \( T(f^3) = C[p.\partial/\partial x, q.\partial/\partial y, r.\partial/\partial z] + C((2x^3, 0, x^2 y), (2x^2 y, 0, xy^2), (-2xy^2, 0, y^3), (0, 2x^2 y, x^3), (0, 2xy^2, x^2 y), (0, 2y^3, xy^2), (-2fy^3, 3ax^3, 2bx^3 + cx^2 y - ey^3), (3fy^3, -2ax^3, -bx^3 + dxy^2 + 2ey^3), (-4ey^3, 0, -5ax^3 + 8bx^2 y + 6cxy^2 + 4dy^3), (0, -4bx^3, 4cx^3 + 6dx^2 y + 8exy^2 - 5fy^3) \).

Here \( d(f + q) = 15 + \text{rank } M(\lambda), \) where \( M(\lambda) \) is the matrix whose rows are the last 4 elements and \( \lambda = (a, b, c, d, e, f). \)

We may not always have \( d(\lambda) > d(0) \) but for a large class of map-germs the strict inequality always holds. These are weighted homogeneous map-germs:

(1-2:15) Definition: Suppose \((w_1, ..., w_n)\) is an \(n\)-tuple of positive integers.
If \( f: (C^n, 0) \to C \) is a (finite) sum of monomials \( x_1^{i_1} ... x_n^{i_n} \) where \( w_1i_1 + ... + w_ni_n = d \) then \( f \) is weighted homogeneous (of degree \( d \) with respect to the weights \((w_1, ..., w_n)\)).

If \( f = (f_1, ..., f_p): (C^n, 0) \to C^p \) then \( f \) is weighted homogeneous with respect to weights \((w_1, ..., w_n)\) if each coordinate function \( f_i \) is weighted homogeneous with respect to these weights. The coordinate functions need not have the same degrees.

We give the well-known result:

(1-2:16) Lemma: Suppose \( f \) is weighted homogeneous with respect to weights \((w_1, ..., w_n)\). If \( d_i \) is the degree of \( f_i \) then
\[ w_1 x_1^{\partial} f + ... + w_n x_n^{\partial} f = d_1 f_1 \partial/\partial Y_1 + ... + d_p f_p \partial/\partial Y_p. \]
Proof: The result follows from the observation that

$$w_1 x_1 \frac{\partial}{\partial x_1} x_1 \ldots x_n + \ldots + w_n x_n \frac{\partial}{\partial x_n} x_1 \ldots x_n = (w_1 i_1 + \ldots + w_p i_p) x_1 \ldots x_n = dx_1 \ldots x_n$$

if $x_1 \ldots x_n$ is of degree $d$ w.r.t. $(w_1, \ldots, w_n)$.

(I.2:17) Proposition: Suppose $f$ is as above, $\{v_1, \ldots, v_s\}$ is a transverse section for $T_a^{k+f}$ over $T_a^{k+1}$ and each $v_i = \nu_i \partial / \partial Y_j$ is a monomial not of degree $d_j$.

Then $d(f + \sum \lambda_i v_i) > d(f)$.

Proof: Let $f_\lambda = f + \sum \lambda_i v_i$. Then clearly

$$T_a^{k+f} \ni w_1 x_1 \frac{\partial}{\partial x_1} f_{\lambda} + \ldots + w_n x_n \frac{\partial}{\partial x_n} f_{\lambda} - \left( d_1 (f_{\lambda})_1 \partial / \partial Y_1 + \ldots + d_p (f_{\lambda})_p \partial / \partial Y_p \right).$$

By (I.2:16) $w_1 x_1 \frac{\partial}{\partial x_1} f + \ldots + w_n x_n \frac{\partial}{\partial x_n} f - \left( d_1 f_1 \partial / \partial Y_1 + \ldots + d_p f_p \partial / \partial Y_p \right) = 0$.

Hence

$$T_a^{k+f} \ni w_1 x_1 \frac{\partial}{\partial x_1} \left( \sum \lambda_i v_i \right) + \ldots + w_n x_n \frac{\partial}{\partial x_n} \left( \sum \lambda_i v_i \right)$$

$$- \left( d_1 \left( \sum \lambda_i v_i \right)_1 \partial / \partial Y_1 + \ldots + d_p \left( \sum \lambda_i v_i \right)_p \partial / \partial Y_p \right).$$

If $v_i = x_1 i_1 \ldots x_n i_n \partial / \partial Y_{ji}$ then the sum is $\sum \lambda_i (w_1 i_1 + \ldots + w_n i_n - d_j) v_i$.

Since the $v_i$ are linearly independent over $T_a^{k+f}$, this is not an element of

$$T_a^{k+k+1} + m_n \theta (f) \subseteq T_a^{k+f} + m_n \theta (f)$$
except when $\lambda_i \left( w_{1,i,1} + ... + w_{n,i} - d_{ji} \right) = 0$ for each $i$. But \( w_{1,i,1} + ... + w_{n,i} - d_{ji} \neq 0 \) for each $i$ so this holds only when $\lambda_i = 0$ for all $i$. $\square$

(I-2:18) Proof of I:3: if $s=1$, $f$ is weighted homogeneous and $v = v_1$ is as in I-1:17 then

\[ TS^{k+1}(f + \lambda v) = TS^{k+1}f + C(\lambda v). \]

The submanifold $\{ f + \lambda v : \lambda \in C \setminus \{0\} \}$ lies in a single $S^k$-orbit.

Proof: by (2-12) and (2-16)

\[ TS^{k+1}(f + \lambda v) = TS^{k+2}f + T_n^{k+2} \theta(f) + C\left[ u_1(f) + \lambda a_1, v,..., u_t(f) + \lambda a_t, v, \lambda v \right]. \]

where $a \neq 0$.

It follows that $TS^{k+1}(f + \lambda v) = TS^{k+1}f + C(\lambda v)$.

The submanifold $\{ f + \lambda v : \lambda \in C \setminus \{0\} \}$ satisfies the conditions of Mather's lemma so lies in a single orbit so without loss of generality, $\lambda = 1$. $\square$

Note: for $S_R$-equivalence there are at most two orbits and without loss of generality $\lambda = \pm 1$.

More generally, if $d(f + \sum_{i=1}^s \lambda_i v_i) > d(f)$ then we can construct a $k$-trivial unfolding of the map-germ $f + \sum_{i=1}^s \lambda_i v_i$.

(I-2:19) Theorem: let $\lambda = (\lambda_1,...,\lambda_s)$ and $\sum_{i,j=1}^s \lambda_i b_{i,j}^c v_j \in TS^{k+k}(f + \sum_{i=1}^s \lambda_i v_i)$.

a) If $F_\lambda(x, t) = (f(x) + \sum_{i=1}^s \phi_i(t)v_i(x), t)$ is an unfolding satisfying

\[ \partial_t F^a = \sum_{i,j=1}^s \phi_i(a)b_{i,j}^c v_j \]

and $\phi_i(0) = \lambda_i$
then $F_\lambda$ is $k^+\kappa$-trivial,

b) there exists an unfolding satisfying the above condition, which is hence $(k^+\kappa)$-trivial.

**Proof:**

We first show that if $F$ is an unfolding with $\partial_t F^a = \sum_{i,j=1}^s \phi_i(a) b_{i,j}^c v_j$ then it is $k$-trivial.

By I-1:31 this follows if

$$\sum_{i,j=1}^s \phi_i(a) b_{i,j}^c v_j \in tF^a(\tau_n \cdot \psi(n+1)) + \omega F^a(\tau_p \cdot \psi(p+1)) + \tau_n \cdot \varphi(F).$$

Recall that in I-2:13, there exist $\mu_m^c$ such that $\sum_{m=1}^t \mu_m^c u_m(f) = \tau^R_{S^m} \tau^S_{\varphi(F)} + \varphi(F) \cdot \theta(f)$. 

Now for each $m$ (I-2:12) there exist $a_{ij}^m \in \mathbb{C}$, $\alpha_{ij}^m \in \tau_n$, $\beta_m \in \tau_p$ such that 

$$u_m(f + \sum_{i=1}^s \phi_i(a) v_j) - u_m(f) = \sum_{i,j=1}^s \lambda_i^m a_{ij}^m v_j +$$

$$S \sum_{i=1}^s \lambda_i^m \left( \sum_{j=1}^n \alpha_{ij}^m \gamma_{x_j} (f + \sum_{i=1}^s \lambda_i^m v_j) + (f + \sum_{i=1}^s \phi_i(a) v_j)^* (\beta_m) \right)$$

$$+ \sum_{i=1}^s \lambda_i^m \left[ h_i - \sum_{j=1}^n \alpha_{ij}^m \gamma_{x_j} \sum_{i=1}^s \lambda_i^m v_j + \left( \gamma^*(\beta_m) - (f + \sum_{i=1}^s \phi_i(a) v_j)^* (\beta_m) \right) \right].$$

It follows that

$$\sum_{i,j=1}^s \lambda_i^m a_{ij}^m v_j + u_m(f) \in tF^a(\tau_n \cdot \psi(n+1)) + \omega F^a(\tau_p \cdot \psi(p+1)) + \tau_n \cdot \varphi(F).$$

We use the same technique as in I-2:12 to show that

$$\sum_{m=1}^t \mu_m^c u_m(f) \in tF^a(\tau_n \cdot \psi(n+1)) + \omega F^a(\tau_p \cdot \psi(p+1)) + \tau_n \cdot \varphi(F)$$

so that $\sum_{m=1}^t \mu_m^c \sum_{i,j=1}^s \lambda_i^m a_{ij}^m v_j \in tF^a(\tau_n \cdot \psi(n+1)) + \omega F^a(\tau_p \cdot \psi(p+1)) + \tau_n \cdot \varphi(F)$. 

It remains to show that such a deformation exists.

We can perform a change of basis so that $B(c) = (b_{ij}^c)$ is transformed to a matrix $A(c)$ which is in Jordan Normal Form,

$$A(c) = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ \vdots & \vdots & \ddots \\ 0 & 0 & A_r \end{bmatrix}$$

where each matrix $A_i$ is of the form

$$A_i = \begin{bmatrix} \epsilon_i & 0 & 0 & 0 \\ 0 & \epsilon_i & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \epsilon_i \end{bmatrix}$$

and $\epsilon_i$ is an eigenvalue of $A(c)$.

(I.2.20) Lemma: If $A(c)$ is in Jordan Normal Form at $\lambda$ then the vector field $\sum_{i=1}^s \lambda_i b_{ij}^c v_j$ has integral curves $f_i(\lambda) = f + \sum_{i=1}^s \phi_i(t) v_i$, where $\mu_j : \mathbb{C} \to \mathbb{C}$ is of the form $e^{\sum_{j=1}^r \frac{\lambda_j p_j(\lambda, t)}{m_j}}$, $\epsilon_i$ are the eigenvalues of $A$ and $p(\lambda, t)$ are polynomials in $t$ satisfying $p_j(\lambda, 0) = \lambda_j$.

Proof: Suppose that $A$ is the first block, being $j \times j$ with eigenvalue $\epsilon$.

We get a system of equations

$$\frac{d}{dt} \phi_i(t) = \epsilon \phi_i(t) + \phi_{i+1}(t), \quad 1 \leq i \leq j-1,$$

$$\frac{d}{dt} \phi_j(t) = \epsilon \phi_j(t).$$

The equation $\frac{d}{dt} \phi_j(t) = \epsilon \phi_j(t), \quad \phi_j(0) = \lambda_j$ has solution $\phi_j(t) = \lambda_j e^{\epsilon t}$.

If $\phi_{i+1}(t) = e^{\epsilon t} \left( \frac{\lambda_j}{(m-1)!} t^{m-1} + \frac{\lambda_{j-1}}{(m-2)!} t^{m-2} + \ldots + \lambda_{i+1} \right)$ then the equation
\[
\frac{d}{dt} \varphi_i(t) = \epsilon \varphi_i(t) + \varphi_{i+1}(t), \quad \varphi_i(0) = \lambda_i
\]

has solution

\[
\varphi_i(t) = e^{\epsilon t} \left( \frac{\lambda_j}{m!} t^m + \frac{\lambda_{j-1}}{(m-1)!} t^{m-1} + \ldots + \lambda_{i+1} t + \lambda_i \right), \quad \text{where} \ m = j - i.
\]

A similar solution exists for each remaining block. \(\square\)

Definition: the submanifold \(S \subseteq \mathbb{C}^s\) is a transversal submanifold if for every \(\lambda\) there is a \(k+\kappa\)-trivial unfolding \(F_\lambda(x, t) = \left( f(x) + \sum_{i=1}^{s} \varphi_i(t)v_i(x), t \right)\) and a point \(t_\lambda\) such that

\[(\varphi_1(t_\lambda), \ldots, \varphi_s(t_\lambda)) \in S.\]

Then every map-germ of the form \(f + \sum_{i=1}^{s} \lambda_i v_i : \lambda \in \mathbb{C}^s\) is \(k+\kappa\)-equivalent to one of the form \(f + \sum_{i=1}^{s} \lambda_i v_i : \lambda \in S.\)

We can always find a transversal submanifold:

consider \(\varphi_j(t) = e^{\epsilon t} p_j(t)\) from the unfolding of the case of B in Jordan Normal form;

a) if \(p_j \neq \lambda_j\) then a solution \(\varphi_j(t) = 0\) exists and every unfolding intersects \(\varphi_j = 0,\)

b) if \(p_j = \lambda_j \neq 0,\) then \(\varphi_j(t) = \lambda_j e^{\epsilon t}\) and the unfolding intersects \(\varphi_j = 1.\) Hence every unfolding intersects \(S = (\varphi_j = 1) \cup (\varphi_j = 0).\)

There are many other possible choices for a transversal submanifold. When we perform the inverse transformation to return to the original matrix B these transversal submanifolds transform to transversal submanifolds for B.

(I.2:20) Example: Let \(f(x, y) = (x^2 + fy^3, y^2 + ax^3, xy + bx^3 + cx^2y + dxy^2 + ey^3).\)

Write \(p(x,y) = x^2 + fy^3, \ q(x,y) = y^2 + ax^3, \ r(x,y) = xy + bx^3 + cx^2y + dxy^2 + ey^3.\)

Then \(T\mathcal{A}^3 f \ni (-4ey^3, 0, -5ax^3 + 8bx^2y + 6cxy^2 + 4dy^3).\)
Let $F(x, y, t) = (x^2 + f(t)y^3, y^2 + a(t)x^3, xy + b(t)x^3 + c(t)x^2y + d(t)xy^2 + e(t)y^3)$
where

- $a(t) = a_0$,
- $b(t) = -5a_0t + b_0$,
- $c(t) = -20a_0t^2 + 8b_0t + c_0$,
- $d(t) = -40a_0t^3 + 24b_0t^2 + 6c_0t + d_0$,
- $e(t) = -40a_0t^4 + 32b_0t^3 + 12c_0t^2 + 4d_0t + e_0$,
- $f(t) = 32a_0t^5 - 32b_0t^4 - 16c_0t^3 - 8d_0t^2 - 4e_0t + f_0$.

By I-2:19 $F$ is a 3-trivial unfolding. Further $f(t) = 0$ has roots in $\mathbb{C}$ except when

- $0 = a_0 = b_0 = c_0 = d_0 = e_0$ (this is not the case for $\mathcal{R}$-equivalence).

Consequently $(x^2 + fy^3, y^2 + ax^3, xy + bx^3 + cx^2y + dxy^2 + ey^3)$ is $\mathcal{R}$-equivalent to one of

- $(x^2, y^2 + ax^3, xy + bx^3 + cx^2y + dxy^2 + ey^3)$ or $(x^2 + fy^3, y^2, xy)$.

Without the $k$-triviality theorem we have to use Mather's lemma and check that the dimension of $T\mathcal{R}^3(x^2 + f(t)y^3, y^2 + a(t)x^3, xy + b(t)x^3 + c(t)x^2y + d(t)xy^2 + e(t)y^3)$ is independent of $t$ (an unpleasant task).

Also $T\mathcal{R}^3(x^2, y^2 + ax^3, xy + bx^3 + cx^2y + dxy^2 + ey^3) \ni (0, 3ax^3, 2bx^3 + cx^2y - ey^3)$,

- $(0, -2ax^3, -bx^3 + dxy^2 + 2ey^3)$,
- $(0, -4bx^3, 4cx^3 + 6dx^2y + 8exy^2)$.

Note that these are tangent to $(f = 0)$. Hence we can apply the theorem again by integrating $(0, -4bx^3, 4cx^3 + 6dx^2y + 8exy^2)$ say – see II-5: for further details.

Consider the set $\{ \sum_{i,j=1}^{s} \lambda_i b_{i,j}^c v_j : 1 \leq c \leq d \}$. By I-2:19 applied to $c = 1$ there exists a transversal submanifold $S$, i.e $g \sim f \Rightarrow g$ is $(k+k)$-equivalent to a map-germ

$$f + \sum_{i=1}^{s} \lambda_i v_i, \lambda \in S.$$

To proceed further we need to find vector fields tangent to $S$. If any of the $\sum_{i,j=1}^{s} \lambda_i b_{i,j}^c$ have components tangent to $S$ which are of the same form
(i.e. \( \sum_{ij=1}^{s} \lambda_i b_{ij}^c v_j = \lambda Bv \) for some matrix B) then we can apply I·2:19 to one of these components and find a new transversal submanifold \( S' \subseteq S \). We therefore ask:

Can we find a transversal manifold \( S \) so that the components of the remaining vector fields which are tangent to \( S \) are still of the form \( \sum_{ij=1}^{s} \lambda_i b_{ij}^c v_j \)?

If the answer is yes then we can apply the theorem as long as we have any vector fields left.

Note: the unfolding is also k-trivial for \( \sim_R \)-equivalence but there may not be transversal submanifolds since these depend on the existance of solutions to equations which may not necessarily have real roots.
I-3 : Map-germs with a 1-dimensional instability locus.

We look at what we may deduce about the structure of $\theta(f)/T\mathcal{A}f$ from its Krull dimension and multiplicity.

If $\theta(f)/T\mathcal{A}f$ has Krull dimension 1 then $\Sigma = \text{Sup}(\theta(f)/T\mathcal{A}f)$ is 1-dimensional and for large $c$

$$\dim C \theta(f)/T\mathcal{A}f + f^s \mathcal{A}_{p}\theta(f) = ec + \text{constant},$$

where $e$ is the multiplicity.

Denote by $o(h)$ the order of an element of $\theta(f)$, i.e. the largest value of $k$ satisfying

$$h \in \mathcal{A}_{n}\theta(f).$$

(I.3:1) Lemma: there exists a value $c$ and a sequence of polynomials

$$\{p_1, p_2, \ldots, p_c, s \geq 0\}$$

satisfying the following:

i) for each $\sigma \geq 0$, $T\mathcal{A}f + f^s \mathcal{A}_{p}^c \theta(f) = T\mathcal{A}f + C[p_1, p_2, \ldots, p_c, s \geq 0],$

ii) for each $j, s$, $p_j^{s+1} \in f^s \mathcal{A}_{p}^c[p_1, p_2, \ldots, p_c] + T\mathcal{A}f,$

iii) for each $\sigma$ and $\lambda_1, \ldots, \lambda_c$,

$$\sum_{j=1}^{c} \lambda_j p_j \in T\mathcal{A}f + f^s \mathcal{A}_{p}^c \theta(f) + \mathcal{A}_n^k \theta(f) \iff p_j^\sigma \in \mathcal{A}_n^k \theta(f) \text{ if } \lambda_j \neq 0.$$

Proof: for sufficiently large values of $c$,

$$\dim C \theta(f)/T\mathcal{A}f + f^c \mathcal{A}_{p}\theta(f) = ec + \text{constant}.$$ 

Choose the least value for which this holds.

Then for $\sigma \geq 0$, $\dim C \left( T\mathcal{A}f + f^c \mathcal{A}_{p}^{c+\sigma} \theta(f) \right)/\left( T\mathcal{A}f + f^c \mathcal{A}_{p}^{c+\sigma+1} \theta(f) \right) = e.$

We choose $\{p_1, p_2, \ldots, p_c\}$ so that
a) \( \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^c \cdot \theta(f) = \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + \mathcal{C}\{p_1^0, p_2^0, \ldots, p_e^0\} \),

b) \( \sum_{j=1}^e \lambda_j p_j^0 \in \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + \mathcal{M}_n^k \theta(f) \iff p_j^0 \in \mathcal{M}_n^k \theta(f) \text{ if } \lambda_j \neq 0. \)

To find elements satisfying b) consider the fact that there exists a value \( k_1 \) such that

\[
\mathcal{M}_n^k \cdot \theta(f) \subseteq \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f). \]

Then the set of elements

\[
\{u : u \in \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^c \cdot \theta(f), u \notin \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + \mathcal{C}\{p_1^0, p_2^0, \ldots, p_e^0\}\}
\]

has at least one member of maximal order. Choose one and call it \( p_1^0 \).

The set \( \{u : u \in \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^c \cdot \theta(f), u \notin \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + \mathcal{C}\{p_1^0\}\} \) has a member of maximal order. Choose one and call it \( p_2^0 \). Continue until we have \( e \) elements. These elements satisfy b) since if

\[
\sum_{j=1}^e \lambda_j p_j^0 \in \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + \mathcal{M}_n^k \theta(f)
\]

let \( i \) be the largest value of \( j \) such that \( \lambda_j \neq 0. \) Then

\[
p_i^0 \in \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + \mathcal{M}_n^k \theta(f) + \mathcal{C}\{p_j^0 : j < i\}.
\]

Then there exists \( h \in \mathcal{M}_n^k \theta(f) \) such that

\[
h - p_i^0 \in \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + \mathcal{C}\{p_j^0 : j < i\}.
\]

Consequently, \( h \in \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^c \cdot \theta(f) \) but \( h \notin \mathcal{T} \mathcal{S} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + \mathcal{C}\{p_j^0 : j < i\} \).

Since \( p_i^0 \) is of maximal order in this set we must have \( p_i^0 \in \mathcal{M}_n^k \theta(f) \). Also, for each \( p_j^0 \) with \( j < i, o(p_j^0) \geq o(p_i^0) \), so \( j < i \) implies that \( p_j^0 \in \mathcal{M}_n^k \theta(f) \).
Then $T \mathcal{A} f + f^* \mathcal{M}_p^c \cdot \theta(f) = T \mathcal{A} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + f^* \mathcal{O}_p \{0, 0, 0, 0\}$

so by Nakayama's lemma, $T \mathcal{A} f + f^* \mathcal{M}_p^c \cdot \theta(f) = T \mathcal{A} f + f^* \mathcal{O}_p \{0, 0, 0, 0\}$.

It follows that $T \mathcal{A} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) = T \mathcal{A} f + f^* \mathcal{M}_p \cdot \{0, 0, 0, 0\}$. 

Now suppose that we have found $\{0, 0, 0, 0\}$ satisfying conditions ii) and iii) with

$$T \mathcal{A} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) = T \mathcal{A} f + f^* \mathcal{M}_p \cdot \{0, 0, 0, 0\}.$$ 

We find new elements $\{\sigma, \sigma, \sigma, \sigma\}$ using the above procedure.

Then $T \mathcal{A} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) = T \mathcal{A} f + f^* \mathcal{M}_p^{c+1} \cdot \theta(f) + C \{\sigma, \sigma, \sigma, \sigma\}$

$$= T \mathcal{A} f + f^* \mathcal{M}_p^{c+2} \cdot \theta(f) + C \{\sigma, \sigma, \sigma, \sigma, \sigma, \sigma, \sigma, \sigma\}$$

$$= T \mathcal{A} f + f^* \mathcal{M}_p^{c+3} \cdot \theta(f) + C \{\sigma, \sigma, \sigma, \sigma, \sigma, \sigma, \sigma, \sigma\} : 0 \leq j \leq i \}. 

(1.3.2) Lemma: if $\{p_1, p_2, ..., p_c : s \geq 0\}$ is chosen as in 1.3.1 then there exists a value $k_0$ such that for $k \geq k_0$,

$$T \mathcal{A} f + \mathcal{M}_n^s \cdot \theta(f) = T \mathcal{A} f + C \{p_1, p_s, ..., p_c : o(p_j) \geq k\}.$$  

Proof: Let $k_0$ be the least value such that $\mathcal{M}_n^s \cdot \theta(f) \leq T \mathcal{A} f + f^* \mathcal{M}_p^c \cdot \theta(f)$.

Clearly, $T \mathcal{A} f + C \{p_1, p_s, ..., p_c : o(p_j) \geq k\} \leq T \mathcal{A} f + \mathcal{M}_n^s \cdot \theta(f)$.

To prove the converse, observe that

$$\mathcal{M}_n^s \cdot \theta(f) \leq T \mathcal{A} f + f^* \mathcal{M}_p^c \cdot \theta(f) = T \mathcal{A} f + C \{p_1, p_s, ..., p_c : s \geq \sigma\}.$$
so for each \( h \in \mathcal{m}_n^k \), there exist \( \lambda_j^s \) such that \( h = \sum_{s \geq 0, j = 1}^c \lambda_j^s p_j^s \in \mathcal{T} \). 

Let \( \sigma \) be the least value such that \( \lambda_j^\sigma \neq 0 \) for some \( j \) \((1 \leq j \leq c)\).

Then \( h + \sum_{j=1}^c \lambda_j^\sigma p_j^\sigma \in \mathcal{T} \) so 

\[
\sum_{j=1}^c \lambda_j^\sigma p_j^\sigma \in \mathcal{T} \] 

So \( \sum_{j=1}^c \lambda_j^\sigma p_j^\sigma \in \mathcal{m}_n^k \), i.e., \( p_j^\sigma \in \mathcal{m}_n^k \) if \( \lambda_j^\sigma \neq 0 \).

It follows that \( \mathcal{m}_n^k \) is a particularly simple result in the case that \( f \) is weighted homogeneous and \( e = 1 \). We let \( p_1^s = p_s \).

(I-3:3) Proposition: if \( e = 1 \) and \( f \) is weighted homogeneous then there exists a value \( k_0 \) and a sequence of monomials \( \{p_s : s \geq 0\} \) as in I-3:1 such that

i) if \( k \geq k_0 \), then \( \mathcal{T} f + \mathcal{m}_n^k \theta(f) = \mathcal{T} f + \mathcal{C} \{p_s : o(p_s) \geq k_0\} \),

ii) if \( g \preceq f \) then either \( g \preceq f \), or \( g \preceq f + p_s \) for some \( s \geq 0 \),

iii) each \( f + p_s \) is \( o(p_s) \)-determined (so is sufficient).
Hence if \( g \sim^k 0 f \) then either \( g \sim^k f, \forall k \), or \( g \) is finitely determined so \( f \) is a weak stem.

Proof: we choose a sequence \( \{ p_s : s \geq 0 \} \) as in I:3:1. Then by I:3:2 there exists \( k_0 \) such that

\[
k \geq k_0 \Rightarrow T_{\mathcal{S}} f + m_n^k \theta(f) = T_{\mathcal{S}} f + C\{ p_s : o(p_s) \geq k \} = T_{\mathcal{S}} f + f^* m_p^{c+\sigma} \theta(f),
\]

where \( \sigma \) is the least value of \( s \) such that \( o(p^s) \geq k \).

Then without loss of generality, the \( p^s \) are monomials, since if \( h = h_1 + h_2 \), where \( h, h_1, h_2 \in m_n^k \theta(f) \) then \( h, h_1, h_2 \in T_{\mathcal{S}} f + f^* m_p^{c+\sigma} \theta(f) \), so we can replace \( p_s \) by a suitable component.

Let \( o(s) = o(p_s) \).

Then

\[
T_{\mathcal{S}} f + m_n^{o(s)} \cdot \theta(f) = T_{\mathcal{S}} f + m_n^{o(s)+1} \cdot \theta(f) + C\{ p_s \}
\]

so by I:1, if \( o(s)-1 \) then \( o(s) = o(s)+1 \) for some \( \lambda \in C \).

Now \( f \) is weighted homogeneous, and we can choose \( k_0 \) so that \( o(s) \geq k_0 \) implies that \( p_s \) cannot have the same degree as \( f_j \) for any \( j \).

Thus, by I:3,

\[
T_{\mathcal{S}} (f + \lambda p_s) + m_n^{o(s)+1} \cdot \theta(f) = T_{\mathcal{S}} f + m_n^{o(s)+1} \cdot \theta(f) + C\{ \lambda p_s \}
\]

so that, without loss of generality we may assume that \( \lambda = 1 \).

Finally, we check the hypotheses of I:2 for \( \lambda = 1 \). It is clear that hypothesis ii) holds. By I:2:4 there exists a value \( k_1 \) such that

\[
m_n^{k_1} \cdot \theta(f) \cap T_{\mathcal{S}} f \leq f^* m_p T_{\mathcal{S}} f \leq T_{\mathcal{S}} 2^2 f.
\]

Increase \( k_0 \) if necessary, so that \( k_0 \geq k_1 \).

Then
\[ T \mathcal{S} f + \mathcal{C}(p^f) + \mathcal{M}_n \cdot \theta(f) \geq \mathcal{M}_n \cdot \theta(f) \cap (T \mathcal{S} f + \mathcal{C}(p_s) + \mathcal{M}_n \cdot \theta(f) + T \mathcal{S} f). \]

so c), and hence iii) holds, and

\[ T \mathcal{S} f + f^* \mathcal{M}_p \cdot \{ p_s \} = T \mathcal{S} f + \mathcal{C}(p^f: j > s) = \mathcal{M}_n \cdot \theta(f) + T \mathcal{S} f. \]

Thus b), and hence i) holds. It follows that \( f + p_s \) is \( o(s) \)-determined.

We note the obvious result

(I-3:4) Corollary: if \( f \) also satisfies the condition \( g^k f, \forall k \Rightarrow g \sim f \) then \( f \) is a stem.

This is the case when \( j^2 f = (x, y^2, 0) \) - this is proved in II-2:5.

We also note the obvious adjacency relations

\[ \cdots \rightarrow f + p_{s+1} \rightarrow f + p_s \rightarrow f + p_{s-1} \rightarrow \cdots \]

We use this proposition to obtain families for weak stems, which are presented in the following table. A striking feature of these is the relations which occur between the terms \( p_s \). The calculations for these examples will be found in chapter II.

<table>
<thead>
<tr>
<th>weak stem</th>
<th>family</th>
<th>relation</th>
<th>( f^4(\Sigma) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x, y^2, y^3))</td>
<td>((x, y^2, y^3 + x^k y))</td>
<td>( p_s = x p_{s-1} )</td>
<td>( y = 0 )</td>
</tr>
<tr>
<td>((x, y^2, x^2 y))</td>
<td>((x, y^2, x^2 y + y^{2k+1}))</td>
<td>( p_s = y^2 p_{s-1} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>((x, y^2, x y^3))</td>
<td>((x, y^2, x y^3 + x^k y))</td>
<td>( p_s = x p_{s-1} )</td>
<td>( y = 0 )</td>
</tr>
<tr>
<td>* ((x, y^2, x^3 y + x^2 y^3))</td>
<td>((x, y^2, x^3 y + x^2 y^3 + y^{2k+1}))</td>
<td>( p_s = y^2 p_{s-1} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>* ((x, y^2, x^4 y + 2 x^2 y^3 + y^5))</td>
<td>((x, y^2, x^4 y + 2 x^2 y^3 + y^5 + x^k y))</td>
<td>( p_s = x p_{s-1} )</td>
<td>( x^2 + y^2 = 0 )</td>
</tr>
<tr>
<td>* ((x, y^2, x^4 y + x^2 y^3))</td>
<td>((x, y^2, x^4 y + x^2 y^3 + y^{2k+1}))</td>
<td>( p_s = y^2 p_{s-1} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>* ((x, y^2, x^2 y^3 + y^5))</td>
<td>((x, y^2, x^2 y^3 + y^5 + x^k y))</td>
<td>( p_s = x p_{s-1} )</td>
<td>( y = 0 )</td>
</tr>
<tr>
<td>* ((x, x y, y^3))</td>
<td>((x, x y + y^{3k+2}, y^3))</td>
<td>( p_s = y^3 p_{s-1} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>* ((x, x y + y^3, x y^2))</td>
<td>((x, x y + y^3, x y^2 + y^{3k+1}))</td>
<td>( p_s = y^3 p_{s-1} )</td>
<td>( x = 0 )</td>
</tr>
</tbody>
</table>
weak stem | family | relation | $f^i(\Sigma)$
---|---|---|---
(x, $xy + y^3$, $xy^2 + y^4$) | (x, $xy + y^3$, $xy^2 + y^4 + y^{2k}$) | $p_s = y^2p_{s-1}$ | $y(x+y^2) = 0$
(x, $xy + y^3$, $xy^4$) | (x, $xy + y^3$, $xy^4 + y^{3k+2}$) | $p_s = y^2p_{s-1}$ | $x = 0$
(x, $xy - y^4$, $y^6$) | (x, $xy - y^4$, $y^6 + y^{3k+1}$) | $p_s = y^3p_{s-1}$ | $x-y^3 = 0$
(x, $x^2y + y^3$, $xy^2$) | (x, $x^2y + y^3$, $xy^2 + y^{3k+1}$) | $p_s = y^3p_{s-1}$ | $x = 0$
(x, $x^3$, $x^2y + y^4$) | (x, $x^3$, $x^2y + y^4 + y^{3k+2}$) | $p_s = y^3p_{s-2}$ | $x^2 + y^3 = 0$
(x, $x^3$, $x^2y + y^4 + xy^{3k+2}$) | $p_s \sim xp_{s-1}$ | $x^2y - y^3 = 0$
(x, $x^2y - y^3$, $x^2y^2 - y^4$) | (x, $x^2y - y^3$, $x^2y^2 - y^4 + x^ky^2$) | $p_s = xp_{s-1}$ | $x^2y - y^3 = 0$

* note that $x\cdot xy^{3k+2} = x^2y^{3k+2}$; since $(x^2+y^3)y^{3k+2} \in T_0 f$ this is equivalent to $y^{3k+4}$.
* these families are listed in Table 7.

All the weak stems are listed in Tables 2–5.

The existence of the relations should not be surprising since for each $s$

\[
p_{s+1} \in f^i \mathcal{M}_p \{p_s\} + T_0 f,
\]

so that $p_{s+1} \sim f_ip_s$ for some $i$, say $i = 1$; this is the same $i$ for each $s$ since $e = 1$ implies that $\Sigma$ is smooth so that without loss of generality, $\Sigma = V(Y_2, \ldots, Y_p)$ and hence $f_ip_s \in T_0 f$ for $2 \leq i \leq p$. The example $(x, y^3, x^2y + y^4)$ indicates that this relation is not in general the simple case $p_{s+1} \sim rp_s$ for some $r \in \mathcal{O}_n$. However, note that this is also the only example where $f^i(\Sigma)$ is not smooth. It seems likely from the data above that when $f^i(\Sigma)$ is smooth the simple relation occurs.

There is some evidence to suggest that some sort of relation always occurs and that it is related to the ideal which defines $f^i(\Sigma)$:

for example $f = (x, xy + y^3, xy^2 + y^4)$. Then $f^i(\Sigma) = V(xy + y^3)$.

It is apparent (II-3.7) that $(xy + y^3)\theta(f) \subseteq T_0 f$. Now $x^k\partial/\partial Y_j \in T_0 f$ for all $k, j$ so by necessity we choose $p_s \in y\theta(f)$. Since

\[(xy + y^3)\theta(f) \subseteq T_0 f \text{ and } (xy^2 + y^4)\theta(f) \subseteq T_0 f\]

we are left with $p_{s+1} \sim xp_s \sim y^2p_s$ since $(x+y^2)p_s \in T_0 f$. But $p_{s+1} = y^2p_s$ is the relation observed. Similar behaviour can be observed in other cases.
A future research problem might be to try to determine whether such relations always occur, and what form they might take if they do.

We note that

$$\text{T}x(f+p_s) = \text{T}x f + C\{ p_j; j \geq s \} = \text{T}x(f+p_{s+1}) + C\{ p_s \}$$

and hence $\text{cod} \text{T}x(f+p_s) = \text{cod} \text{T}x(f+p_{s+1}) + 1$.

Proposition I-3:3 no longer holds if we drop the hypothesis that $f$ is weighted homogeneous:

for example let $f = (x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1})$ (II-2:16).

Then $f$ has $e = 1$ and if $j^{12}h = f$ then

$$h \in x^{2k+1}(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1})$$

but $\text{T}x(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1}) \neq (0, 0, y^{2k+3})$ so is not $(2k+1)$-determined.

However, $(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1})$ is $(2k+6)$-determined.

The next step would be to try and show that

$$h \sim (x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1} + by^{2k+3})$$

for some $a \neq 0$, some $b \in C$.

We conjecture that the new normal form should be

$$g \sim f, \forall k, \text{ or } g \sim f + \lambda_1 p_s + \lambda_2 p_{s+1} + \ldots + \lambda_r p_{s+r} \text{ for some } s, r \geq 0, \lambda_1 \neq 0.$$

If we drop the hypothesis that $e = 1$ the behaviour is more complicated:

for example let

$$f = (x, xy + y^3, 2xy^2 + 3y^4) \text{ and } g = (x, xy + y^3, 2xy^2 + y^4) \quad (\text{II-2:8 and II-2:10}).$$

Both of these have $e = 2$;

if $j^4h = f$ then either $h \in \bigcap_{k \geq 0} x^k f$ or $h \sim (x, xy + y^3, 2xy^2 + 3y^4 + y^5)$ and is $k$-determined,

if $j^4h = g$ then it is possible to have $h \sim (x, xy + y^3, (x+y^2)^2 + (x+y^2)y^{2k+1})$. 
which is neither finitely determined nor $\mathcal{A}$-equivalent to $g$.

Here $\mu_0(\Sigma_f) = 2$ and $\mu_0(\Sigma_g) = 1$.

The fundamental difference between these map-germs is that
\[ l^\Sigma(\theta(f)/\mathcal{T}f) = 1 \quad \text{and} \quad l^\Sigma(\theta(f)/\mathcal{T}g) = 2. \]

In chapter III we shall show that $l^\Sigma(\theta(f)/\mathcal{T}f) = 1$ corresponds to a particular geometric condition for $\Sigma$ and in view of this we make the conjecture that in Proposition I-3:3 we can replace the hypothesis $e = 1$ by the hypothesis \[ e = \mu_0(\Sigma_f). \]

The condition $l^\Sigma(\theta(f)/\mathcal{T}f) = 1$ means that the localised module $(\theta(f)/\mathcal{T}f)p(\Sigma)$ has only one generator, so that all the generators of $\theta(f)/\mathcal{T}f$ must collapse onto this one (here $p(\Sigma)$ denotes the prime ideal which defines $\Sigma$). It is not clear how to use this information, though.

The general conjecture is

i) if $e = \mu_0(\Sigma)$ and $\Sigma$ is irreducible then $f$ is a stem,

ii) there is a sequence of polynomials \( \{ p_1, p_2, \ldots, p_e \} \), and values $k, r$ such that

if $g \sim f$ then either $g \sim f$ or

\[
g \sim f + \lambda_{1,s}p_1^s + \lambda_{2,s}p_2^s + \ldots + \lambda_{e,s}p_e^s + \lambda_{1,s+r}p_1^{s+r} + \lambda_{2,s+r}p_2^{s+r} + \ldots + \lambda_{e,s+r}p_e^{s+r}
\]

for some $s, r \geq 0$, (some) $\lambda_{i,s} \neq 0$.

This is a problem for further research.
Chapter II: Map-germs \((\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\).

II·1: Introduction and Summary of Results.

The main aim of this chapter is to inspect map-germs with \(\dim \Sigma = 1\) so as to improve our understanding of such germs and their classifications and to suggest further areas of research. There are two main questions which interest us both for stems and for map-germs which fail to be stems:

i) how is the classification related to the multiplicity of \(T\mathcal{X}f\)?

ii) how are these related to the geometry; in particular what are \(\Sigma\) and \(f^!(\Sigma)\) like?

There are two ways in which we may proceed. The first is to construct map-germs with particular properties that we wish to examine; the second is to perform a systematic classification. The second method has the advantage that we may find map-germs with properties we did not expect but has the disadvantages that there is no obvious stopping point and that it may not yield map-germs having a particular property that we may wish to investigate. The bulk of this chapter (II·2–II·5) is taken up in extending the classification started in [Mond 1] and in the inspection of a few extra map-germs. The aim of looking at many different types of map-germ and the aim of achieving a systematic classification have proved somewhat incompatible with the aim of restricting the length of the chapter; we hope that we have reached an acceptable compromise.

(II·1·1) Results.

We extend the \(\mathcal{X}\)-invariant stratification of \(\Sigma_{J^r}(2,3)\), (the set of \(r\)-jets in \(\Sigma_{J^r}(2,3)\) of rank 1) for \(r \geq 2\), calculated by Mond ([Mond 1]). In [Mond 1] the map-germs are classified over \(\mathbb{R}\), i.e. \(\mathcal{X}_\mathbb{R} = \text{Aut}(\mathbb{R}^2, 0) \times \text{Aut}(\mathbb{R}^3, 0)\). We classify over \(\mathcal{X}_\mathbb{C} = \text{Aut}(\mathbb{C}^2, 0) \times \text{Aut}(\mathbb{C}^3, 0)\). We also include a stratification of some map-germs having corank 2. The move from \(\mathcal{X}_\mathbb{R}\) to \(\mathcal{X}_\mathbb{C}\) results in the collapsing of some of
Mond's strata, for example the map-germs \((x, y^3 - x^2y, xy^2)\) and \((x, y^3, x^2y + xy^2)\) are \(\mathcal{C}\)-equivalent but not \(\mathcal{R}\)-equivalent.

Since one of the aims is to investigate stems we generally only try to identify non-stems rather than completing the stratification over them. We note that inspection of the tangent space is only sometimes useful in showing a map-germ is not a stem. Other methods, suggested in Chapter III can be more useful.

We note, however, that as yet we have failed to determine whether or not the map-germ \((x, xy + xy^3, y^4)\) is a stem.

(II:1) Theorem:

Figures 1 - 3 give an \(\mathcal{C}\)-invariant stratification of \(\Sigma^1 r(2,3)\) for \(r \geq 2\).

Figure 4 gives an \(\mathcal{C}\)-invariant stratification of \(\Sigma^2 r(2,3)\) for \(r \geq 2\).

These figures will be found inside the back cover.

Notation: i) \(\underline{f}\) (underlined) indicates that \(f\) is sufficient,

ii) Mond's stratification lies above and to the left of the red line,

iii) frequently we can give a general form for a finitely determined map-germ with a given \(k\)-jet: these general forms are given in the bottom row. We give only the sufficient forms.

In Tables 2-5 we present what we consider to be the important properties of our map-germs. These are summarised below.

Recall that if Krull dim \(\theta(f)/T\mathcal{C}f = 1\) then there exists a value \(c_0\) such that for \(c \geq c_0\)

\[
\dim_C \frac{f^c \cdot \mathfrak{m}_3^c \cdot \theta(f) + T\mathcal{C}f}{f^{c+1} \cdot \mathfrak{m}_3^{c+1} \cdot \theta(f) + T\mathcal{C}f} = c.
\]
(II:2) Theorem: Tables 2–5 give the following information for several map-germs having Krull dim $\dim \theta(f)/T \mathcal{A} f = 1$:

i) the value $c_0$,

$$f^* \mathfrak{m}_3^c \theta(f) + T \mathcal{A} f$$

for $c \geq c_0$; if the multiplicity is not clear

$$f^* \mathfrak{m}_3^{c+1} \cdot \theta(f) + T \mathcal{A} f$$

from this we sometimes give a second, equivalent basis.

ii) a basis for

iii) The transporter ideal $T(c_0) = \left( T \mathcal{A} f, f^* \mathfrak{m}_3^{c_0} \theta(f) \right)$; since $f^* \mathfrak{m}_3^{c_0} \theta(f)$ is an ideal in $\mathcal{O}_2$, $T(c_0)$ is also an ideal in $\mathcal{O}_2$.

iv) The multiplicity $e$,

v) the multiple curves and cuspidal edges appearing in the image $f(\mathbb{C}^2)$ together with their transversal types:– transversal types are defined below,

vi) whether there is a diagram of the image for the real map-germ,

vi) a stratification over the germ and the determinacy degrees where calculated.

To save space we write the basis elements in the form $h \cdot e_1$, $h \in \mathcal{O}_2$, $e_1 = \partial/\partial X$ etc.

We draw $f(\mathbb{R}^2) \subseteq \mathbb{R}^3$ for some of the map-germs. In several cases a picture is of no help since the important features cannot be indicated: for example, $(x, xy, y^3)$ is 1–1 at all points of $\mathbb{R}^2$ but has a triple curve in $\mathbb{C}^2$. Other cases are very similar to those already drawn. The diagrams are on pages 56–58.

Strata are given by $f$ if $f$ is sufficient or by $\mathcal{A}^k f$ if $f$ is not $k$-determined. We do not calculate determinacy degrees for map-germs which are not sufficient except to indicate if they are not finitely determined. We omit the stratum $\mathcal{A}^{\infty} f$ in each case.
Table 2: \(j^2f = (x, y^2, 0)\).

\(f\) is of the form \((x, y^2, f_3)\). Reference II-2:n

<table>
<thead>
<tr>
<th>(f_3)</th>
<th>(c_0)</th>
<th>basis</th>
<th>(T(c_0))</th>
<th>(e)</th>
<th>curve</th>
<th>(\mu)-type</th>
<th>diagram</th>
<th>stratification</th>
<th>det</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y^3)*</td>
<td>1</td>
<td>(x^2y_{c3})</td>
<td>(y^2)</td>
<td>1</td>
<td>((x, 0) \rightarrow (x, 0, 0))</td>
<td>(A_2)</td>
<td>yes</td>
<td>(y^3 + x^2y)</td>
<td>(c+1)</td>
<td>10</td>
</tr>
<tr>
<td>(x^2y)*</td>
<td>1</td>
<td>(y^{2c+1}_{c3})</td>
<td>(x)</td>
<td>1</td>
<td>((0, y) \rightarrow (0, y^2, 0))</td>
<td>(A_3)</td>
<td>yes</td>
<td>(x^2y + y^{2c+1})</td>
<td>(2c+1)</td>
<td>11</td>
</tr>
<tr>
<td>(xy^3)*</td>
<td>2</td>
<td>(x^2y_{c3})</td>
<td>(y^2)</td>
<td>1</td>
<td>((x, 0) \rightarrow (x, 0, 0)) (\text{or} (0, y) \rightarrow (0, y^2, 0))</td>
<td>(A_2), (A_1)</td>
<td>yes</td>
<td>(xy^3 + x^3y)</td>
<td>(c+1)</td>
<td>12</td>
</tr>
<tr>
<td>(x^3y)</td>
<td>3</td>
<td>(y^{2c+1}<em>{c3}) (\text{or} \ xy^{2c+1}</em>{c3})</td>
<td>(x^2)</td>
<td>2</td>
<td>((0, y) \rightarrow (0, y^2, 0))</td>
<td>(A_3)</td>
<td>no</td>
<td>(\xrightarrow{4c+1} (x^3y + x^2y^{2c+1})) (\text{or} \ x^3y + x^2y^{2c+1})</td>
<td>(2c+1)</td>
<td>13</td>
</tr>
<tr>
<td>(x^3y + x^2y^3)**</td>
<td>3</td>
<td>(y^{2c+1}_{c3})</td>
<td>(x)</td>
<td>1</td>
<td>((0, y) \rightarrow (0, y^2, 0)) (\text{or} (-y^2, y) \rightarrow (-y^2, y^2, 0))</td>
<td>(A_3), (A_1)</td>
<td>yes</td>
<td>(x^3y + x^2y^3 + y^{2c+1})</td>
<td>(2c+1)</td>
<td>15</td>
</tr>
<tr>
<td>((x^2 + y^2)^2 y)**</td>
<td>4</td>
<td>(x^2y_{c3})</td>
<td>(x^2 + y^2)</td>
<td>1</td>
<td>((x, \pm ix) \rightarrow (x, x^2, 0))</td>
<td>(A_3)</td>
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<td>((x^2 + y^2)^2 y + x^3y)</td>
<td>(c+1)</td>
<td>16</td>
</tr>
</tbody>
</table>

* these map-germs are listed in Table 1.

* these map-germs are listed in Table 7.
<table>
<thead>
<tr>
<th>$f_3$</th>
<th>$c_0$</th>
<th>basis</th>
<th>$T(c_0)$</th>
<th>$c$</th>
<th>curve</th>
<th>$\mu$-type</th>
<th>diagram</th>
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<tbody>
<tr>
<td>$x^4y + x^2y^3$</td>
<td>4</td>
<td>$y^{2c+1}c_3$</td>
<td>$x$</td>
<td>1</td>
<td>$(x, \pm ix) \to (x, x^2, 0)$ $(0, y) \to (0, y^2, 0)$</td>
<td>$A_1$</td>
<td>A3</td>
<td>no</td>
<td>$x^4y + x^2y^3 + y^{2c+1}$</td>
<td>$2c+1$</td>
</tr>
<tr>
<td>$x^2y^3 + y^5$</td>
<td>3</td>
<td>$x^5yc_3$</td>
<td>$y^2$</td>
<td>1</td>
<td>$(x, \pm ix) \to (x, x^2, 0)$ $(x, 0) \to (x, 0, 0)$</td>
<td>$A_1$</td>
<td>A3</td>
<td>no</td>
<td>$x^2y^3 + y^5 + x^5y$</td>
<td>$c+1$</td>
</tr>
<tr>
<td>$x^4y$</td>
<td>4</td>
<td>$x^2y^{2c-1}c_3$, $xy^{2c-1}c_3$, $y^{2c+1}c_3$</td>
<td>$x^3$</td>
<td>3</td>
<td>$(0, y) \to (0, y^2, 0)$</td>
<td>$A_2$</td>
<td>A3</td>
<td>no</td>
<td>$x^4y + y^{2c+1}$ $x^4y + x^2y^{2c-1} + y^{2c+1}$ $\asymp x^{2c+1}(x^4y + x^2y^{2c-1})$ $\asymp x^{2c+1}(x^4y + xy^{2c+1})$</td>
<td>$2c+1$</td>
</tr>
<tr>
<td>$x^2y^3$</td>
<td>1</td>
<td>$x^2y^3c_3$, $y^{2c+1}c_3$</td>
<td>$xy^2$</td>
<td>2</td>
<td>$(x, 0)+(x, 0, 0)$ $(0, y)+(0, y^2, 0)$</td>
<td>$A_2$</td>
<td>A3</td>
<td>no</td>
<td>$x^2y^3 + y^{2c+1}$ $\asymp x^{2c+1}(x^2y^3 + x^5y)$ $\asymp x^{2c+1}(x^2y^3 + y^{2c+1})$</td>
<td>$c+1$</td>
</tr>
<tr>
<td>$y^5$</td>
<td>1</td>
<td>$x^6yc_3$, $x^{2c+1}c_3$, $x^{2c+3}c_3$</td>
<td>$y^4$</td>
<td>2</td>
<td>$(x, 0) \to (x, 0, 0)$</td>
<td>$A_4$</td>
<td>no</td>
<td></td>
<td>$y^5 + x^5y$ $y^5 + x^6y + x^{c-2}y^3$ $\asymp x^{c+1}(y^5 + x^{c-2}y^3)$</td>
<td>$c+1$</td>
</tr>
<tr>
<td>$x^8y + x^7y^3 + x^5y^2 + x^2y^9$</td>
<td>10</td>
<td>$y^{2c+1}c_3$</td>
<td>$x$</td>
<td>1</td>
<td>$(0, y) \to (0, y^2, 0)$</td>
<td>$A_3$</td>
<td>A1</td>
<td>no</td>
<td>$\asymp x^{2c+1}(f_3 + y^{2c+1})$</td>
<td>between $2c+3$ and $2c+6$</td>
</tr>
</tbody>
</table>
Table 3: \( f = (x, xy + y^3, f_3) \). Reference II-3:n

<table>
<thead>
<tr>
<th>( f_3 )</th>
<th>( c_0 )</th>
<th>basis</th>
<th>( T(c_0) )</th>
<th>( e )</th>
<th>curves</th>
<th>( \mu )-type</th>
<th>diagram</th>
<th>stratification</th>
<th>det</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>( xy^2 )**</td>
<td>2</td>
<td>( y^{3e+1} \cdot e_3 )</td>
<td>( x )</td>
<td>1</td>
<td>((0, y) \rightarrow (0, y^3, 0)) ((y^3, y) \rightarrow (y^3, 0, -y^3))</td>
<td>( D_4 )</td>
<td>no</td>
<td>( f + y^{3e+1} \cdot e_3 )</td>
<td>3c+1</td>
<td>5</td>
</tr>
<tr>
<td>( xy^2 + y^4 )**</td>
<td>2</td>
<td>( y^{2e+2} \cdot e_3 ) ( \text{or} \ x^2 y^2 \cdot e_3 )</td>
<td>( y(x+y^2) )</td>
<td>1</td>
<td>((x, 0) \rightarrow (x, 0, 0)) ((-y^2, y) \rightarrow (-y^2, 0, 0))</td>
<td>( D_4 )</td>
<td>yes</td>
<td>( f + y^{2e+2} \cdot e_3 )</td>
<td>2c+2</td>
<td>7</td>
</tr>
<tr>
<td>( xy^2 + \frac{3}{2} y^4 )**</td>
<td>2</td>
<td>( y^{2e} \cdot e_3 ), ( y^{2e+1} \cdot e_3 ) ( \text{or} \ x^2 y \cdot e_3 ) ( x^2 y^2 \cdot e_3 )</td>
<td>((x+3y^2)^2 )</td>
<td>2</td>
<td>((-3y^2, y) \rightarrow (-3y^2, -2y^3, \frac{3}{2} y^4))</td>
<td>( A_2 )</td>
<td>yes</td>
<td>( f + y^{2e} \cdot e_3 )</td>
<td>2c</td>
<td>8, 9</td>
</tr>
<tr>
<td>( xy^2 + \frac{1}{2} y^4 )**</td>
<td>2</td>
<td>( y^{2e} \cdot e_3 ), ( y^{2e+1} \cdot e_3 ) ( \text{or} \ x^2 y \cdot e_3 ) ( x^2 y^2 \cdot e_3 )</td>
<td>((x+y^2)^2 )</td>
<td>2</td>
<td>((-y^2, y) \rightarrow (-y^2, 0, -\frac{1}{2} y^4))</td>
<td>( A_5 )</td>
<td>yes</td>
<td>( y^{2c}(f + y^{2c} \cdot e_3) )</td>
<td>-</td>
<td>10, 11, 12</td>
</tr>
</tbody>
</table>

* these map-germs are listed in Table 1.

* these map-germs are listed in Table 7.
<table>
<thead>
<tr>
<th>$f_3$</th>
<th>$c_0$ basis</th>
<th>$T(c_0)$</th>
<th>$c$ curves</th>
<th>$\mu$-type diagram stratification</th>
<th>det</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$xy^3$</td>
<td>$1$ $y^{3c+e_3}, y^{3c+1+e_3}$ $x^3$</td>
<td>$4$ $(0, y) \to (0, y^3, 0)$</td>
<td>$J_{10}$ no $\mathcal{O}^{3c+3}(f + axy^{3c} + by^{3c+3}, e_3)$</td>
<td>$\infty$</td>
<td>$17$</td>
<td></td>
</tr>
<tr>
<td>$xy^4$ **</td>
<td>$1$ $y^{3c+2+e_3}$</td>
<td>$1$ $(0, y) \to (0, y^3, 0)$</td>
<td>$D_4$ no $f + y^{3c+2+e_2}$</td>
<td>$3c+2$</td>
<td>$23$</td>
<td></td>
</tr>
</tbody>
</table>

* these map-germs are listed in Table 1.

* these map-germs are listed in Table 7.
Table 4: \( f = (x, f_2, f_3) \).

<table>
<thead>
<tr>
<th>((f_2, f_3))</th>
<th>(c_0)</th>
<th>basis</th>
<th>(T(c_0))</th>
<th>(e)</th>
<th>curves</th>
<th>(\mu)-type</th>
<th>diagram</th>
<th>stratification</th>
<th>det</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>((xy, y^3)) x</td>
<td>1</td>
<td>(y^{3c+2}.c_2)</td>
<td>x</td>
<td>1</td>
<td>((0, y) \to (0, 0, y^3))</td>
<td>(D_4)</td>
<td>no</td>
<td>(f + y^{3c+2}.c_2)</td>
<td>3c+2</td>
<td>3:4</td>
</tr>
<tr>
<td>((xy, y^4)) 2</td>
<td>(y^{4c+2}.c_2)</td>
<td>(x^2)</td>
<td>3</td>
<td>((0, y) \to (0, 0, y^4))</td>
<td>(X_9)</td>
<td>no</td>
<td>not known</td>
<td></td>
<td>3:24</td>
<td></td>
</tr>
<tr>
<td>((xy + xy^2, y^4))</td>
<td>1</td>
<td>(y^{4c+2}.c_2, y^{4c+3}.c_2)</td>
<td>x</td>
<td>2</td>
<td>((0, y) \to (0, 0, y^4))</td>
<td>(X_9)</td>
<td>no</td>
<td>not known</td>
<td></td>
<td>3:25</td>
</tr>
<tr>
<td>((xy - y^4, y^6)) x 4</td>
<td>(y^{3c+1}.c_3)</td>
<td>(x - y^3)</td>
<td>1</td>
<td>((y^3, y) \to (y^3, 0, y^6))</td>
<td>(D_4)</td>
<td>no</td>
<td>(f + y^{3c+1}.c_3)</td>
<td>3c+1</td>
<td>3:27</td>
<td></td>
</tr>
<tr>
<td>((x^2y + y^3, xy^2)) x 1</td>
<td>(y^{3c+1}.c_3)</td>
<td>x</td>
<td>1</td>
<td>((0, y) \to (0, y^3, 0))</td>
<td>(D_4)</td>
<td>no</td>
<td>(f + y^{3c+1}.c_3)</td>
<td>3c+1</td>
<td>4:3</td>
<td></td>
</tr>
<tr>
<td>((y^3, xy^2)) x 2</td>
<td>(y^{3c+1}.c_3, x^2y.c_2)</td>
<td>(xy^2)</td>
<td>2</td>
<td>((0, y) \to (0, y^3, 0))</td>
<td>((x, 0) \to (x, 0, 0))</td>
<td>(D_4)</td>
<td>no</td>
<td>(\mathcal{A}^{3c+1}(f + y^{3c+1}.c_3))</td>
<td>(\infty)</td>
<td>4:4</td>
</tr>
<tr>
<td>((y^3, x^2y)) x 2</td>
<td>(y^{3c+1}.c_3, y^{3c+2}.c_3, xy^{3c+2}.c_2, x^2y^{3c+2}.c_3)</td>
<td>(x^3)</td>
<td>1</td>
<td>((0, y) \to (0, y^3, 0))</td>
<td>(J_{10})</td>
<td>no</td>
<td>(\mathcal{A}^{3c+1}(f + y^{3c+1}.c_3))</td>
<td>(\infty)</td>
<td>4:8</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\mathcal{A}^{3c+2}(f + y^{3c+2}.c_3))</td>
<td>(\infty)</td>
<td>4:8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\mathcal{A}^{3c+3}(f + y^{3c+3}.c_3))</td>
<td>(\infty)</td>
<td>4:8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\mathcal{A}^{3c+3}(f + y^{3c+3}.c_3))</td>
<td>(\infty)</td>
<td>4:8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(ax^{3}y^{3c+1}.c_3) + (by^{3c+3}.c_3)</td>
<td>(\infty)</td>
<td>4:8</td>
</tr>
<tr>
<td>$(l_2, l_3)$</td>
<td>$c_0$</td>
<td>basis</td>
<td>$^\circ T(c_0)$</td>
<td>$e$</td>
<td>curves</td>
<td>$\mu$-type</td>
<td>diagram</td>
<td>stratification</td>
<td>det</td>
<td>Ref</td>
</tr>
<tr>
<td>-------------</td>
<td>-------</td>
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<td>----------------</td>
<td>---</td>
<td>--------</td>
<td>------------</td>
<td>----------</td>
<td>----------------</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>$(y^3, x^2y+y^4)$ **</td>
<td>2</td>
<td>$x^c y^2 \cdot c_3$</td>
<td>$x^2 + y^3$</td>
<td>1</td>
<td>$(x, y) \rightarrow (x, y^3, 0)$</td>
<td>$D_4$</td>
<td>no</td>
<td>$f + y^{3c+2} \cdot c_3$</td>
<td>3c+2</td>
<td>4:9</td>
</tr>
<tr>
<td>$(x^2y-y^3, x^2y^2-y^4)$ **</td>
<td>1</td>
<td>$y^{3c+2} \cdot c_2$</td>
<td>$y(x^2-y^2)$</td>
<td>1</td>
<td>$(0, y) \rightarrow (0, 0, y^3)$</td>
<td>$D_4$</td>
<td>yes</td>
<td>$f + (0, y^{3c+2}, 0)$</td>
<td>3c+2</td>
<td>4:13</td>
</tr>
<tr>
<td>$(xy + x^3 + x^2y + cxy^2, 2$</td>
<td>$y^{3c+2} \cdot c_2$</td>
<td>$x$</td>
<td>1</td>
<td>$(0, y) \rightarrow (0, 0, y^2)$</td>
<td>$A_3$</td>
<td>no</td>
<td>$f + y^{2c+1} \cdot c_2$</td>
<td>2c+1</td>
<td>5:4</td>
<td></td>
</tr>
</tbody>
</table>

** Table 5: $f$ has corank 2. 

$^*$ these map-germs are listed in Table 1.

$^*$ these map-germs are listed in Table 7.
Diagram 1. $f(x, y) = (x, y^2, y^3)$.

Diagram 2. $f(x, y) = (x, y^2, x^2y)$.

Diagram 3. $f(x, y) = (x, y^2, xy^3)$. 
Diagram 4. $f(x, y) = (x, y^2, x^3y + x^2y^3)$.

Diagram 5. $f(x, y) = (x, xy + y^3, xy^2 + \frac{3}{2}y^4)$.

Diagram 6. $f(x, y) = (x, xy + y^3, xy^2 + y^4)$. 
Diagram 7. $f(x, y) = (x, xy + y^3, xy^2 + \frac{1}{2}y^4)$.  

Diagram 8. $f(x, y) = (x, x^2y - y^3, x^2y^2 - y^4)$.  

(II·1·2) The Transversal Type of $\Sigma$.  

(II·1·2:1) Definition ([P 3], p. 98): suppose that $f, g: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ have isolated singularities at 0. If there exists a family $\{f_t\}$, $t \in [0, 1]$ of germs $f_t: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ of analytic functions with an isolated singular point at 0, such that the Milnor number $\mu(f_t, 0)$ is constant, and $f_0 = f$, $f_1 = g$ and $F: \mathbb{C}^{n+1} \times [0, 1] \rightarrow \mathbb{C}$ defined by $F(x, t) = f_t(x)$ is continuous, then $f$ and $g$ are called $\mu$-homotopic.  

(II·1·2:2) Definition ([P 3], p. 98): let $f: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of an analytic function with a singular locus $(\Sigma, 0)$, such that $\Sigma$ is a germ of a non-singular curve. We say that the transversal $\mu$-type of $f$ along $\Sigma$ at 0 is constant if for every choice of local coordinates $x, y_1, ..., y_n$ such that $\Sigma = V(y_1, ..., y_n)$ the $\mu$-type of $f_t: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ with $f_t(y) = f(t, y)$ is constant for all $t$ sufficiently small.
(II-1-2:3) Proposition ([P 3], p.98) : let \( f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0) \) be a germ of an analytic function with a one dimensional singular locus \((\Sigma, 0)\). Let \((\Sigma_1, 0), \ldots, (\Sigma_r, 0)\) be the branches of \((\Sigma, 0)\). The transversal \(\mu\)-type of \( f \) along \( \Sigma_i \) is constant at all points of a punctured neighbourhood of 0 in \( \Sigma_i \).

**Transversal \(\mu\)-types for map-germs \( f:(\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) [A], p. 73 and p. 76.**

If local coordinates \((X, Y, Z)\) on \( \mathbb{C}^3 \) are such that \( \Sigma = V(Y, Z) \) the transversal type of \( \Sigma \) at 0 is the \(\mu\)-type of \( f_t(Y, Z) = f(t, Y, Z) \) for (sufficiently) small \( t \).

The \(\mu\)-types which occur in this thesis are the following:

**Table 6. Some transversal types.**

<table>
<thead>
<tr>
<th>name</th>
<th>map</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>( f(Y, Z) = Y^2 + Z^2 )</td>
<td></td>
</tr>
<tr>
<td>( A_{2n}, n \geq 1, )</td>
<td>( f(Y, Z) = Y^{2n+1} + Z^2 )</td>
<td></td>
</tr>
<tr>
<td>( A_{2n+1}, n \geq 1, )</td>
<td>( f(Y, Z) = Y^{2n+2} + Z^2 )</td>
<td></td>
</tr>
<tr>
<td>( D_4 )</td>
<td>( f(Y, Z) = Y^2Z + Z^3 )</td>
<td></td>
</tr>
<tr>
<td>( D_k, k \geq 4 )</td>
<td>( f(Y, Z) = Y^2Z + Z^{k-1} )</td>
<td></td>
</tr>
<tr>
<td>( J_{10} )</td>
<td>( f(Y, Z) = Y^3 + \varepsilon Y^2Z^2 + aZ^6: )</td>
<td>( a(4\varepsilon + 27a) \neq 0, \varepsilon(\varepsilon^2 - 1) = 0 )</td>
</tr>
<tr>
<td>( X_9 )</td>
<td>( f(Y, Z) = Y^4 + \varepsilon Y^2Z^2 + aZ^4: )</td>
<td>( a(4\varepsilon + 27a) \neq 0, \varepsilon(\varepsilon^2 - 1) = 0 )</td>
</tr>
</tbody>
</table>
Calculation: the calculations of the transversal types for the map-germs in the tables are performed in III·1. Frequently, it can be found simply by knowing the multiplicity. Note, however, that the relation between the multiplicity and the transversal type can be very complicated: for example, \((x, xy, y^4)\) and \((x, xy + xy^3, y^4)\) both have transversal type \(X_6\) but while for \((x, xy, y^4)\) \(\epsilon = 3\), for \((x, xy + xy^3, y^4)\) \(\epsilon = 2\). One of the differences between these is that for \((x, xy, y^4)\) \(e = 0\) and for \((x, xy + xy^3, y^4)\) \(\epsilon \neq 0\). We expect that as the type becomes more complicated this behaviour will become more common, i.e. 'one' transversal type will correspond to several possible multiplicities.

\[(II·1·3)\] Calculation of Modules.

The map-germs included in this thesis fall into one of the categories
\[f^*m_3 \cdot 0_2 = \langle x, y^2 \rangle \cdot 0_2, \quad f^*m_3 \cdot 0_2 = \langle x, y^3 \rangle \cdot 0_2, \quad f^*m_3 \cdot 0_2 = \langle x, y^4 \rangle \cdot 0_2, \quad f \text{ has corank 2.}\]

For map-germs satisfying \(f^*m_3 \cdot 0_2 = \langle x, y^2 \rangle \cdot 0_2\) we calculate \(T\mathcal{A}f\) according to the procedure developed in [Mond 1].

In certain cases (e.g. \((x, y^3, 0)\)) it is easy to calculate \(T\mathcal{A}f\) essentially by inspection.

For finitely determined map-germs we use the fact that
\[\text{if } f^*m_3 \cdot 0_2 \subseteq m_2 \text{ and } m_2\theta(f) \subseteq T\mathcal{A}^{k+l}f \text{ then } m_2\theta(f) \subseteq T\mathcal{A}^{k+l}f.\]

We calculate \(T\mathcal{A}^{k+l}f\) by computer for a sufficiently large \(k\) that \(m_2\theta(f) \subseteq T\mathcal{A}^{k+l}f\).

For the remaining corank 1 cases we start by calculating \(T\mathcal{A}^k f\) for a large value of \(k\) by computer. From this we choose a module \(C\) which we hope will be equal to \(T\mathcal{A}^k f\). We can check this using Nakayama's lemma (I·1·6)
\[C = T\mathcal{A} f \iff C = T\mathcal{A} f + f^*m_3 \cdot C.\]

These calculations are performed in appendix A.
Corank 2 cases are dealt on an individual basis.

The remaining work consists of calculating \( T_{\mathcal{S}}^k(f+p(\lambda, x, y)) \). There are two ways of proceeding:

i) specific calculation of \( T_{\mathcal{S}}^k(f+p(\lambda, x, y)) \) for given values of \( k \) by computer,

ii) for general \( k \) use of the fact that

\[
T_{\mathcal{S}}^k(f+p(\lambda, x, y)) = T_{\mathcal{S}}^{k+1} 2^k f + m_2 \cdot \theta(f) + \mathcal{C} \left[ x \partial(f+p)/\partial x, x \partial(f+p)/\partial y, y \partial(f+p)/\partial x, y \partial(f+p)/\partial y, (f+p)_{e_j}; 1 \leq i, j \leq 3 \right].
\]

The method of calculation is indicated for each example.

\[(II.1-4) \text{ Computer Programs.}\]

The program currently exists in two forms, \textit{module} and \textit{param} and runs on two computers, the Sun Workstation and the Apple Macintosh.

The first version, module, is designed to calculate a set of generators for the module

\[
T_{\mathcal{S}}^{R,S} f \mod m_2^{k+1} \cdot \theta(f) = tf(m_2 \theta(2)) + \omega f(m_3 \theta(3)) \mod m_2^{k+1} \cdot \theta(f),
\]

for integers \( R, S, k \) and \( f(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y)) \), where each \( f_i \) is a polynomial in \( x \) and \( y \) with integer coefficients.

The generators are simplified as far as possible before being displayed. The data may be saved to an external file. On the Macintosh only, the data can be printed directly from inside the program.

The second version, \textit{param} (for parameters) has the option that the polynomials may take parameter as well as integer coefficients. This version is primarily designed to calculate the modules \( T_{\mathcal{S}}^{f+p(\lambda, x, y)} \mod m_2^{k+1} \cdot \theta(f) \), where \( p \in m_2^k \theta(f) \) and \( \lambda = (\lambda_1, ..., \lambda_r) \) are parameters. It has no facilities for storing products of coefficients, so care must be taken that terms involving products lie in \( m_2^{k+1} \cdot \theta(f) \).
For example, let \( f_\lambda = (xy + ax^3 + bx^2y + cxy^2 + dy^3, x^2, y^2) \) where \( \lambda = (a, b, c, d) \).

We can choose the terms \( x_i y^j \frac{\partial f_\lambda}{\partial x}, x_i y^j \frac{\partial f_\lambda}{\partial y} \) \((i+j \geq R)\) as generators for \( \text{tf}(\mathfrak{m}_2 \Theta(2)) \) so we can choose any value of \( R \) or \( k \), but generators of \( \omega \text{tf}(\mathfrak{m}_3 \Theta(3)) \) are of the form \( \beta(f_1, f_2, f_3) \) where \( \beta \in \mathfrak{m}_3 \Theta(3) \).

Terms involving parameters are of degree 3 so

for \( \omega \text{tf}(\mathfrak{m}_3 \Theta(3)) \) or \( \omega \text{tf}(\mathfrak{m}_3 \Theta(3)) \) we must have \( \mathfrak{m}_2 \leq \mathfrak{m}_2 \cdot \theta(f) \),

for \( \omega \text{tf}(\mathfrak{m}_3 \Theta(3)) \) we must have \( \mathfrak{m}_2 \leq \mathfrak{m}_2 \cdot \theta(f) \),

for \( \omega \text{tf}(\mathfrak{m}_3 \Theta(3)) \) we must have \( \mathfrak{m}_2 \leq \mathfrak{m}_2 \cdot \theta(f) \), etc.

More details on the programs will be found in Appendix C.
We first state some result from [Mond 1].

(II-2:1) Theorem (4.1:1): If $g \in \mathfrak{G}^2(x, y^2, 0)$ then $g$ is equivalent to a germ of the form

$$(x, y) \mapsto (x, y^2, yp(x, y^2)).$$

(II-2:2) Definition: the map-germs $p, q \in \mathcal{O}_2$ are $\mathcal{V}^T$-equivalent if there exists a function $r \in \mathcal{O}_2$ and a diffeomorphism $\varphi = (\varphi_1, \varphi_2) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that

- a) $r(0, 0) \neq 0$,
- b) $\varphi_1(x, -y) = \varphi_1(x, y)$ and $\varphi_2(x, -y) = -\varphi_2(x, y)$,
- c) $r(x, y^2)p(x, y^2) = q(\varphi_1(x, y), \varphi_2(x, y))$.

We prove a new result using the Artin approximation theorem:

(II-2:3) Theorem: if $p, q$ are $k-$\mathcal{V}^T-equivalent for all $k$ then they are $\mathcal{V}^T$-equivalent.

Proof: This is equivalent to finding $r$ and $\psi = (\psi_1, \psi_2)$, where

$$\varphi = (\psi_1(x, y^2), y\psi_2(x, y^2)),$$

which satisfy $r(x, y^2)p(x, y^2) = q(\psi_1(x, y^2), y^2\psi_2(x, y^2))$ and $(\psi_1(x, y^2), y\psi_2(x, y^2))$ is a diffeomorphism.

Thus we need to find $r, \psi$ satisfying

$$r(x, y)p(x, y) = q(\psi_1(x, y), y\psi_2(x, y))$$

and

$$\det \begin{pmatrix} \frac{\partial}{\partial x} \psi_1 & 2y\frac{\partial}{\partial y} \psi_1 \\ y\frac{\partial}{\partial x} \psi_2 & \psi_2 + 2y^2\frac{\partial}{\partial y} \psi_2 \end{pmatrix} \neq \mathfrak{m}_2,$$

i.e. $\psi_2(0, 0)\partial_x \psi_1(0, 0) \neq 0$.

This follows if we can find a solution $(X(x, y), Y(x, y), Z(x, y))$ to the equation

$$0 = F(x, y, X, Y, Z) = Xp(x, y) - q(Y, yZ^2)$$

which satisfies $X(0, 0) \neq 0$ and $Z\partial_x Y \neq 0$ at $(0, 0)$. 

The hypothesis says that this has solutions modulo $\mathfrak{m}_2^k$ for each $k$. Hence by I-1:17 a convergent solution exists.

(II-2:4) Theorem (4.1:1). The map-germs

$$(x, y) \mapsto (x, y^2, yp(x, y^2)), (x, y) \mapsto (x, y^2, yq(x, y^2))$$

are $\L$-equivalent if and only if the function-germs $p, q$ are $\L^T$-equivalent.

(II-2:5) Corollary to II-2:3: if $j^2f = (x, y^2, 0)$ and $f$ and $g$ are $k$-$\L$-equivalent for all $k$ then they are $\L$-equivalent. In particular, if $f$ is a weak stem then it is a stem.

Proof: this follows immediately from the preceding two theorems.

Let $h: \mathbb{C}^2 \to \mathbb{C}^2$ be the fold map $(x, y) \mapsto (x, y^2)$.

(II-2:6) Theorem (4.1:16): suppose that $p(0, 0) = 0$. Then

$$T_\mathfrak{L}^{T \text{fold}} = h^*\left(\langle x\partial_x p, y\partial_x p, y\partial_y p, p \rangle_{0^2} \right) \leq \mathfrak{m}_2^T$$

and

$$T_\mathfrak{L}(x, y^2, yp^h) = \begin{pmatrix} \mathfrak{m}_2^T \\ \mathfrak{m}_2 \{y\} \\ 0 \end{pmatrix} + \left(\mathfrak{m}_2^T + yT_\mathfrak{L}^{T \text{fold}}\right)e_3$$

where $\mathfrak{m}_2^T = h^*\mathfrak{m}_2 = \{x^\ell y^j; \ell + j \geq 1\}$.

(II-2:7) General form for $T_\mathfrak{L}^{2,2}f$ and $f^*\mathfrak{m}_3^T T_\mathfrak{L} f$ where $f(x, y) = (x, y^2, yp(x, y^2))$;

Let $A = \begin{pmatrix} \mathfrak{m}_2 \\ \mathfrak{m}_2 \{y\} \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} \mathfrak{m}_2^2 \{y^2\} \\ \mathfrak{m}_2^2 \{xy, y^3\} \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} \mathfrak{m}_2^2 \{y^2\} \\ \mathfrak{m}_2^2 \{xy, y^2\} \\ 0 \end{pmatrix}$, $D = \begin{pmatrix} \mathfrak{m}_2^2 \\ \mathfrak{m}_2^2 \{xy, y^2\} \\ 0 \end{pmatrix}$.

Then

$$T_\mathfrak{L} f = A + \left(\mathfrak{m}_2^T + yT_\mathfrak{L}^{T \text{fold}}\right)e_3.$$
\[ f^*m_3 \cdot T \mathcal{A} f = B + \left( m_2 \right)^T e_3 + \left( xy, y^3 \right) T \mathcal{X} T \mathcal{P}oh e_3, \]

and

\[ T \mathcal{A}_{2,2} f = C + \left( m_2 \right)^T e_3 + \left( xy, y^3 \right) T \mathcal{X} T \mathcal{P}oh e_3 + C \begin{pmatrix} y^2 \\ 0 \\ y^3 \partial_x \mathcal{P}oh \end{pmatrix}. \]

If \( \partial \mathcal{P} = 0 \) then \( T \mathcal{A}_{2,2} f = D + \left( m_2 \right)^T e_3 + \left( xy, y^3 \right) T \mathcal{X} T \mathcal{P}oh e_3. \)

**Proof:** \( f^*m_3 \cdot T \mathcal{A} f = f^*m_3 \left( A + \left( m_2 + y T \mathcal{X} T \mathcal{P}oh \right) e_3 \right) \)

\[ = B + \left( m_2 \right)^T e_3 + \left( xy, y^3 \right) T \mathcal{X} T \mathcal{P}oh e_3 \]

since

\[ y_p(x, y^2) y T \mathcal{X} T \mathcal{P}oh \leq \left( m_2 \right)^T \]

and \( y_p(x, y^2) m_2 \leq y h^*(p m_2) \leq y T \mathcal{X} T \mathcal{P}oh \)

by definition of \( T \mathcal{X} T \mathcal{P}oh. \)

Now \( T \mathcal{A}_{2,2} f = f^*m_3 \cdot T \mathcal{A} f + C \left[ y^2 \partial f / \partial x, y^2 \partial f / \partial y \right] \)

\[ = f^*m_3 \cdot T \mathcal{A} f + C \begin{pmatrix} y^2 \\ 0 \\ y^3 \partial_x \mathcal{P}oh \end{pmatrix} \begin{pmatrix} 0 \\ 2y^3 \\ y_p \mathcal{P}oh + 2y^4 \partial_y \mathcal{P}oh \end{pmatrix} \]

\[ = C + \left( m_2 \right)^T e_3 + \left( xy, y^3 \right) T \mathcal{X} T \mathcal{P}oh e_3 + C \begin{pmatrix} y^2 \\ 0 \\ y^3 \partial_x \mathcal{P}oh \end{pmatrix}. \]

The final statement is obvious \( \square. \)

We now start the classification of map-germs satisfying \( j^2 f = (x, y^2, 0) \). We proceed by calculating the appropriate modules for each case and applying the theorems I:1, I:2 and I:3 as appropriate. We also use I:2:19 in II:2:9.
\[(\Pi.2:8)\]

\[p = 0\]

\[\mathcal{T}_{\mathcal{K}_{2,2}} f\]

hence \[g \in \mathcal{K}_{2} f \Rightarrow \]

\[g \in \mathcal{K}_{3} f \Rightarrow \]

\[g \in \mathcal{K}_{4} f \Rightarrow \]

and in general for \(k \geq 1\)

\[g \in \mathcal{K}_{2k} f \Rightarrow \]

\[g \in \mathcal{K}_{2k+1} f \Rightarrow \]

\[f = (x, y^2, 0).\]

\[T_{\mathcal{K}^T} T_p h = 0,\]

\[D + \left(\frac{T}{m_2}\right) \cdot e_3,\]

\[g \in \mathcal{K}_{3}(x, y^2, a_0x^2y + a_1y^3),\]

\[g \in \mathcal{K}_{4}(x, y^2, a_0x^3y + a_1xy^3),\]

\[g \in \mathcal{K}_{5}(x, y^2, a_0x^4y + a_1x^2y^3 + a_2y^5),\]

(\Pi.2:9)

\[\text{Let } f_k = (x, y^2, p_k)\]

where

\[p_{2k+1} = a_0x^{2k}y + a_1x^{2k-2}y^3 + \ldots + a_ky^{2k+1},\]

and \[p_{2k+2} = a_0x^{2k+1}y + a_1x^{2k-1}y^3 + \ldots + a_kxy^{2k+1}.\]

\[T_{\mathcal{K}_{2k+1} f_{k,\lambda}} = T_{\mathcal{K}_{2,2} f} + m_2^{k+2} \cdot \theta(f) + \mathcal{C}\left\{\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} p_k \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}\]

\[+ \mathcal{C}\left\{\begin{pmatrix} x \partial p_k / \partial x \\ y \partial p_k / \partial x \end{pmatrix}, \begin{pmatrix} 2xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2y^2 \end{pmatrix}\right\}.\]

so \[T_{\mathcal{K}_{2k+1} f_{k,\lambda}} = A + \left(\frac{T}{m_2}\right) \cdot e_3 + m_2^{2k+2} \cdot \theta(f)\]

\[+ \mathcal{C}\left\{\begin{pmatrix} 0 \\ 0 \\ 2ka_1x^{2k}y + \ldots + 2a_{k-1}x^{2k-1}y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ a_1x^{2k}y + \ldots + (2k+1)a_ky^{2k+1} \end{pmatrix}\right\}.\]

and \[T_{\mathcal{K}_{2k+2} f_{k,\lambda}} = A + \left(\frac{T}{m_2}\right) \cdot e_3 + m_2^{2k+3} \cdot \theta(f)\]
\[
+C \left( \left( \begin{array}{cc}
0 & 0 \\
(2k+1)a_1y^{2k+1} & \ldots + a_ky^{2k+1}
\end{array} \right) \left( \begin{array}{c}
0 \\
a_1x^{2k+1} \\
\ldots + (2k+1)a_ky^{2k+1}
\end{array} \right) \right)
\]

The unfoldings
\[(x, y^2, \Sigma a_j e^{2(k-j)}x^{2(k-j)}y^{2j+1}, t), (x, y^2, \Sigma a_j e^{2j}x^{2(k-j)}y^{2j+1}, t)\]
are \((2k+1)\)-trivial and the unfoldings
\[(x, y^2, \Sigma a_j e^{2(k-j)}x^{2(k-j)+1}y^{2j+1}, t), (x, y^2, \Sigma a_j e^{2j}x^{2(k-j)+1}y^{2j+1}, t)\]
are \((2k+2)\)-trivial.

Examples
\[(x, y^2, a_0x^2y + a_1y^3, t), (x, y^2, a_0x^2y + a_1e^2y^3, t),
(x, y^2, a_0e^2x^2y + a_1xy^3, t), (x, y^2, a_0x^2y + a_1e^2xy^3, t),
(x, y^2, a_0e^4x^4y + a_1e^2x^2y^3 + a_2y^5, t),
(x, y^2, a_0x^4y + a_1e^2x^2y^3 + a_2e^4y^5, t).
\]

In general the strata consist of the sets \((x, y^2, p_k)\) where any two non-zero coefficients of the \(p_k\) may be assumed to be 1 (see below for an example).

Corollary: \(\mathcal{A}^2(x, y^2, 0)\) is a union of the following \(\mathcal{A}\)-invariant strata:
\[
\mathcal{A}^3(x, y^2, x^2y + y^3) \quad \mathcal{A}^3(x, y^2, y^3) \quad \mathcal{A}^3(x, y^2, x^2y),
\mathcal{A}^4(x, y^2, x^3y + xy^3) \quad \mathcal{A}^4(x, y^2, xy^3) \quad \mathcal{A}^4(x, y^2, x^3y)
\mathcal{A}^5(x, y^2, x^4y + bx^2y^3 + y^5) \quad \mathcal{A}^5(x, y^2, x^4y + x^2y^3) \quad \mathcal{A}^5(x, y^2, x^2y^3 + y^5)
\mathcal{A}^5(x, y^2, x^4y) \quad \mathcal{A}^5(x, y^2, x^2y^3) \quad \mathcal{A}^5(x, y^2, 0)
\]

Proof: we demonstrate this by calculating the strata of \(\mathcal{A}^4(x, y^2, 0)\).

For any map-germ \((x, y^2, a_0x^4y + a_1x^2y^3 + a_2y^5)\) we have two 5-trivial unfoldings
\[(x, y^2, a_0e^4t x^4y + a_1e^2t x^2y^3 + a_2y^5, t) \text{ and } (x, y^2, a_0x^4y + a_1e^2t x^2y^3 + a_2e^4ty^5, t).
\]
Suppose that \(a_0 \neq 0\). Then there is a value of \(t\) so that \(a_0e^4t = 1\).

So \((x, y^2, a_0y + a_1x^2y^3 + a_2y^5)\) is 5-equivalent to \((x, y^2, x^4y + a_1x^2y^3 + a_2y^5)\).

Similarly \((x, y^2, x^4y + a_1e^2t x^2y^3 + a_2e^4ty^5, t)\) is 5-trivial;

if \(a_2 \neq 0\) there is a value of \(t\) so that \(a_2e^4t = 1\) and \((x, y^2, x^4y + a_1x^2y^3 + a_2y^5)\) is 5-equivalent to \((x, y^2, x^4y + a_1x^2y^3 + y^5)\), otherwise \(a_1 = 0\) or \(a_1 \neq 0\) and
(x, y^2, x^4y + a_1x^2y^3) is 5-equivalent to (x, y^2, x^4y + x^2y^3).

We repeat this for the case a_0 = 0 to obtain the strata \( \mathcal{A}^5(x, y^2, x^2y^3 + y^5) \), \( \mathcal{A}^5(x, y^2, y^5) \), \( \mathcal{A}^5(x, y^2, x^2y^3) \) and \( \mathcal{A}^5(x, y^2, 0) \).

The other strata are obtained in a similar manner.

Determinacy degrees:

- \((x, y^2, x^2y + y^3)\) is 3-determined, \((x, y^2, x^3y + xy^3)\) is 4-determined and \((x, y^2, x^4y + bx^2y^3 + y^5)\) is 5-determined provided that \(b^2 \neq 4\).

The determinacy degrees are given in [Mond 1] (p 335).

\[(\text{II-2:10})\]

\[f = (x, y^2, y^3).\]

\[p(x, y) = y\]

\[T\mathcal{A}^T_{\text{pol}} = m_{y^2}\{x^j : j \geq 0\},\]

\[T\mathcal{A} f\]

\[A + \left(m_{y^2}\{x^j y : j \geq 0\}\right) \cdot e_3,\]

\[\{x^j y \cdot e_3\},\]

\[1,\]

\[f^* m_2 \cdot T\mathcal{A} f\]

\[B + \left(m_2^2\{y^2, y^3, x^j y : j \geq 1\}\right) \cdot e_3,\]

\[f^* m_3 \cdot \left(T\mathcal{A} f + \{x^j y \cdot e_3\}\right) \supseteq m_{y^2} \cdot \theta(f),\]

provided that \(j \geq 2\).

\[T\mathcal{A}^{2,2} f\]

\[D + \left(m_2^2\{y^2, y^3, x^j y : j \geq 1\}\right) \cdot e_3,\]

hence for \(k \geq 3\) \(g \in \mathcal{A}^k f \Rightarrow g \in \mathcal{A}^{k+1}(x, y^2, y^3 + ax^k y)\).

Weighted homogeneous weights \((1, 1)\),

degree\((y^3) = 3\), degree\((x^k y) = k+1\),

hence for \(k \geq 3\)

\[T\mathcal{A}^{k+1}(x, y^2, y^3 + ax^k y) = T\mathcal{A}^{k+1} f + C\{ax^k y \cdot e_3\}.\]

Thus by I:2 \((x, y^2, y^3 + x^k y)\) is \(k+1\)-determined for \(k \geq 3\).

Hence \(g \in \mathcal{A}^3 f \Rightarrow \) either \(g \sim f\) or \(g \sim (x, y^2, y^3 + x^k y)\); thus \(f\) is a stem.
(II·2:11)  

\[ f = (x, y^2, x^2y). \]

\[ p(x, y) = x^2 \]

\[ T_{\mathcal{A}} f = T_{\mathcal{A}} \mathcal{T}_{\mathcal{P}oh} = \mathcal{m}_2 \{ x, y^2; j \geq 0 \}, \]

\[ \mathcal{A} + \left( \mathcal{m}_2 \{ xy, y^{2j+1}; j \geq 0 \} \right) e_3, \]

\[ \{ y^{2j+1}, e_3 \}, \]

\[ 1, \]

\[ f^* m_3 T_{\mathcal{A}} f \]

\[ B + \left( \mathcal{m}_2 \{ xy, x^2y, y^3, xy^3, y^{2j+1}; j \geq 1 \} \right) e_3, \]

\[ f^* m_3 \left( T_{\mathcal{A}} f + \{ y^{2j+1}, e_3 \} \right) \geq m_2 \{ y^{2j+2}\theta(f) \} \text{ provided that } j \geq 1. \]

\[ T_{\mathcal{A}2,2} f \]

\[ C + \left( \mathcal{m}_2 \{ xy, x^2y, xy^3, y^{2j+1}; j \geq 1 \} \right) + C \left( \begin{pmatrix} 2 \\ 0 \\ 2xy^3 \end{pmatrix} \right), \]

hence \[ g \in \mathcal{A}^3 f \Rightarrow \]

\[ g \in \mathcal{A}^4 (x, y^2, x^2y + ay^3), \]

for \[ k \geq 2 \ g \in \mathcal{A}^{2k} f \Rightarrow \]

\[ g \in \mathcal{A}^{2k+1}(x, y^2, x^2y + ay^{2k+1}), \]

and \[ g \in \mathcal{A}^{2k+1} f \Rightarrow \]

\[ g \in \mathcal{A}^{2k+1}(x, y^2, x^2y). \]

Weighted homogeneous weights \((1, 1)\),

degree\((x^2y) = 3\), degree\((xy^3) = 4\), degree\((y^{2k+1}) = 2k+1\),

hence \[ T_{\mathcal{A}^4}(x, y^2, x^2y + ay^3) = T_{\mathcal{A}^4} f + C\{ axy^3, e_3 \} \text{ and } T_{\mathcal{A}^4} f \ni C\{ axy^3, e_3 \}, \]

hence \{\(x, y^2, x^2y + ay^3\); \(a \in C\)\} lies in a single orbit.

For \[ k \geq 2 \]

\[ T_{\mathcal{A}^{2k+1}}(x, y^2, x^2y + ay^{2k+1}) = T_{\mathcal{A}^{2k+1}} f + C\{ ay^{2k+1}, e_3 \}, \]

thus by I:2 \((x, y^2, x^2y + y^{2k+1})\) is \((2k+1)\)-determined.

Hence \[ g \in \mathcal{A}^3 f \Rightarrow \text{either } g \sim f \text{ or } g \sim (x, y^2, x^2y + y^{2k+1}); \text{ thus } f \text{ is a stem.} \]

(II·2:12)  

\[ f = (x, y^2, xy^3). \]

\[ p(x, y) = xy \]

\[ T_{\mathcal{A}} T_{\mathcal{P}oh} = \mathcal{m}_2 \{ y, x^j; j \geq 0 \}, \]

\[ A + \left( \mathcal{m}_2 \{ y^3, xy^j; j \geq 0 \} \right) e_3, \]

\[ \{ xy^j, e_3 \}, \]

\[ 1, \]
\[ f \cdot m_3 \cdot T \mathcal{S} f \quad B + \left( m_2^{2} \setminus \{ y^2, y^3, x y^3, y^5, x y^j : j \geq 1 \} \right). \]
\[ f \cdot m_3 \left( T \mathcal{S} f + \{ x y^j e_3 \} \right) \geq m_2^{j+2} \cdot \theta(f) \text{ provided that } j \geq 2. \]

\[ T \mathcal{S}_{2,2} f \quad C + \left( m_2^2 \setminus \{ y^2, x y^3, x y^5, x y^j : j \geq 1 \} \right) + C \left\{ \begin{pmatrix} y^2 \\ 0 \\ y^5 \end{pmatrix} \right\}, \]

\[
\text{hence } g \in \mathcal{S}^4 f \Rightarrow g \in \mathcal{S}^5(x, y^2, x y^3 + a x^4 y + b y^5),
\]

\[
\text{for } k \geq 5 \quad g \in \mathcal{S}^k f \Rightarrow g \in \mathcal{S}^{k+1}(x, y^2, x y^3 + a x^k y).
\]

Weighted homogeneous weights \((1, 1),\)

\[
\deg(x y^3) = 4, \deg(x^k y) = k+1.
\]

Calculation by computer gives

\[
T \mathcal{S}^3(x, y^2, x, y^2, x y^3 + a x^4 y + b y^5) = T \mathcal{S}^{k+1} f + C \{ a x^k y : e_3 \}
\]

and by I:3, for \( k \geq 5 \)

\[
T \mathcal{S}^{k+1}(x, y^2, x y^3 + a x^k y) = T \mathcal{S}^{k+1} f + C \{ a x^k y : e_3 \}.
\]

Thus by I:2 \((x, y^2, x y^3 + x^k y)\) is \((k+1)\)-determined for \( k \geq 4.\)

Hence \( g \in \mathcal{S}^4 f \Rightarrow \) either \( g \sim f \) or \( g \sim (x, y^2, x y^3 + x^k y)\); thus \( f \) is a stem.

\[
\text{(II:2:13)} \quad f = (x, y^2, x^2 y).
\]

\[
p(x, y) = x^3 \quad \text{and } \quad T \mathcal{S}^T p = h = m_2 \left\{ x^2, y^2 \right\} = \left\{ x^2, y^2 \right\}, \quad j \geq 0,\]

\[
T \mathcal{S} f \quad A + \left( m_2 \setminus \{ x^2 y, y^{2j+1} \right) \setminus \{ x^{2j+1} \} : j \geq 0 \} e_3,\]

basis \( C \{ y^{2j+1} e_3, x y^{2j+1} e_3 \} \),

multiplicity \( 2 \),

\[
f \cdot m_3 \cdot T \mathcal{S} f \quad B + \left( m_2^2 \setminus \{ y^2, x^2 y, x^3 y, x y^3, y^{2j+1}, x y^{2j+1} : j \geq 1 \} \right),\]

\[
f \cdot m_3 \left( T \mathcal{S} f + \{ y^{2j+1} e_3 \} \right) \geq m_2^{2j+2} \cdot \theta(f) \text{ provided that } j \geq 2.\]
hence \( g \in \mathcal{S}^4(x, y^2, x^3y) \Rightarrow g \in \mathcal{S}^5(x, y^2, x^3y + ax^2y^3 + by^5), \)
for \( k \geq 2 \) \( g \in \mathcal{S}^{2k+2}f \Rightarrow g \in \mathcal{S}^{2k+2}(x, y^2, x^3y + axy^{2k+1}) \)
and \( g \in \mathcal{S}^{2k+1}f \Rightarrow g \in \mathcal{S}^{2k+1}(x, y^2, x^3y + ay^{2k+1}). \)

Computer calculation shows that \( T\mathcal{S}^3(x, y^2, x^3y + ax^2y^3 + by^5) = T\mathcal{S}^3f + \mathcal{C}\{by^5, e_3\}. \)

Weighted homogeneous weights \((1, 1),\)

\[
\text{degree}(x^3y) = 4, \text{degree}(y^{2k+1}) = 2k+1, \text{degree}(xy^{2k+1}) = 2k+2, 
\]

hence for \( k \geq 2 \)
\[
T\mathcal{S}^{2k+2}(x, y^2, x^3y + axy^{2k+1}) = T\mathcal{S}^{2k+2}f + \mathcal{C}\{axy^{2k+1}, e_3\}
\]
and
\[
T\mathcal{S}^{2k+1}(x, y^2, x^3y + ay^{2k+1}) = T\mathcal{S}^{2k+1}f + \mathcal{C}\{ay^{2k+1}, e_3\}.
\]

Thus by I:2 \((x, y^2, x^3y + y^{2k+1})\) is \((2k+1)\)-determined for \( k \geq 2.\)

Hence \( g \in \mathcal{S}^4f \Rightarrow \)
either \( g \sim f \) or \( g \sim (x, y^2, x^3y + y^{2k+1})\) or \( g \in \mathcal{S}^{2k+2}(x, y^2, x^3y + axy^{2k+1}).\)

\[\text{II}2:14\]
\[f = (x, y^2, x^3y + xy^{4k+1}).\]

We show that \( \mathcal{S}^{4k+2}(x, y^2, x^3y + axy^{4k+1}) = \mathcal{S}^{4k+2}(x, y^2, x^3y + x^2y^{2k+1}).\)

Let \( x = x' - \frac{1}{3}y^{2k} \) and \( X' = X + \frac{1}{3}Y^k. \) Then \((x, y^2, x^3y + x^2y^{2k+1})\) is equivalent to
\[
\left( x', y^2, (x' - \frac{1}{3}y^{2k})^3y + (x' - \frac{1}{3}y^{2k})^2y^{2k+1} \right)
\]
\[
= \left( x, y^2, (x')^3y - 3(x')^2\frac{1}{3}y^{2k+1} + 3x'\frac{1}{9}y^{4k+1} - \frac{1}{27}y^{6k+1} + (x')^2y^{2k+1} - 2x'\frac{1}{3}y^{4k+1} + \frac{1}{9}y^{6k+1} \right)
\]
\[
= \left( x, y^2, (x')^3y - \frac{1}{3}x'y^{4k+1} + \frac{2}{27}y^{6k+1} \right).
\]

so
\[
\mathcal{S}^{4k+2}(x, y^2, x^3y + x^2y^{2k+1}) = \mathcal{S}^{4k+2} \left( x, y^2, x^3y - \frac{1}{3}xy^{4k+1} \right)
\]
\[
= \mathcal{S}^{4k+2}(x, y^2, x^3y + xy^{4k+1}).
\]

Now \((x, y^2, x^3y + x^2y^{2k+1}) = (x, y^2, x^2y(x + y^{2k})), \) which has two double curves,
\[
(0, y) \mapsto (0, y^2, 0) \text{ and } (-y^{2k}, y) \mapsto (-y^{2k}, y^2, 0).
\]
The curve \( f(0, y) \) is a double curve with non-transverse self-intersection.

The map-germ \((x, y^2, x^3y)\) has only one double curve. Hence \((x, y^2, x^3y)\) is not a stem.
(II·2:15) \[ f = (x, y^2, x^2 y + x^2 y^3). \]

\[ p(x, y) = x^3 + x^2 y \]

\[ T_x T_p h = m_2 \{ x, x^2, y^2 j: j \geq 1 \}, \]

\[ T_x f \]

\[ A + \left( m_2 \{ x y, x^2 y, y^2 j: j \geq 0 \} \right) e_3, \]

\[ \{ y^{2 j+1} e_3 \}, \]

\[ 1, \]

\[ f^* m_3 T_x f \]

\[ B + \left( m_2 \{ x y, y^2, x^2 y, x^3 y, y^5, y^{2 j+1}: j \geq 1 \} \right), \]

\[ f^* m_3 \left( T_x f + \{ y^{2 j+1} e_3 \} \right) \geq m_2^{2 j+2} T_{\phi(f)} \text{ provided that } j \geq 2. \]

(II·2:16) \[ f = (x, y^2, x^4 y + 2 x^2 y^3 + y^5). \]

\[ p(x, y) = x^4 + 2 x^2 y + y^2 \]

\[ T_x T_p h = \{ x y^{2 t} (x^2 + y^2): j + 2 t \geq 2 \}, \]

\[ T_x f \]

\[ A + \left( m_2^T \cap (x y^{2 t}(x^2 + y^2): j + 2 t \geq 2) \right) e_3, \]

\[ \{ x^c y e_3 \}, \]

\[ 1, \]

\[ f^* m_3 T_x f \]

\[ B + \left( m_2^T \right) + \mathbb{C} \{ x y^{2 t+1}(x^2 + y^2): j + 2 t + 1 \geq 4 \}, \]

\[ f^* m_3 \left( T_x f + \{ x y e_3 \} \right) \geq m_2^{j+2} \cdot T_{\phi(f)} \text{ provided that } j \geq 5. \]
hence for \( k \geq 5 \) \( g \in \mathcal{S}^{k+1f} \Rightarrow g \in \mathcal{S}^{k+1}(x, y^2, x^2y + 2x^2y^3 + y^5 + ax^ky). 

Weighted homogeneous weights \((1, 1)\),

\[
\text{degree}(x^4y + 2x^2y^3 + y^5) = 5, \text{ degree}(x^ky) = k+1,
\]

so for \( k \geq 5 \) \( T_{\mathcal{S}}^{k+1}(x, y^2, x^4y + 2x^2y^3 + y^5 + ax^ky) = T_{\mathcal{S}}^{k+1f} + C\{ax^ky. e_3\}. \)

By I:2 \((x, y^2, x^4y + 2x^2y^3 + y^5 + x^ky)\) is \((k+1)\)-determined for \( k \geq 5. \)

Hence \( g \in \mathcal{S}^{5f} \Rightarrow \) either \( g \sim f \) or \( g \sim (x, y^2, x^4y + 2x^2y^3 + y^5 + x^ky) \) and \( f \) is a stem.

\((II\cdot2:17)\)

\[
f = (x, y^2, x^4y + x^2y^3).
\]

\[
p(x, y) = x^4 + x^2y
\]

\[
T_{\mathcal{S}}^{T_{po}h} = T_{\mathcal{S}}^{2}\{x, x^2, x^3, xy, y^2; j \geq 0\},
\]

\[
T_{\mathcal{S}}^{f} = A + \left( m_2 \{xy, x^2y, x^3y, y^2; j \geq 0\}\right)e_3,
\]

basis

\[
\text{multiplicity} = 1,
\]

\[
f^*m_3.T_{\mathcal{S}}^{f} = B + \left( m_2 \{xy, x^2y, x^3y, x^4y, x^2y^3, xy^5; j \geq 1\}\right)e_3,
\]

\[
f^*m_3\left(T_{\mathcal{S}}^{f} + \{y^{2j+1}. e_3\}\right) \cong m_2^{2j+2}. 0(f) \text{ provided that } j \geq 2.
\]

\[
T_{\mathcal{S}}^{2,2f} = C + \left( m_2^2 \{xy, x^2y, x^3y, x^4y, x^2y^3, xy^5; j \geq 1\}\right)e_3 + C\left( \begin{array}{c} y^2 \\ 0 \\ xy^5 \end{array} \right),
\]

hence \( g \in \mathcal{S}^{5f} \Rightarrow \)

for \( k \geq 3 \) \( g \in \mathcal{S}^{2k} \Rightarrow \)

and \( g \in \mathcal{S}^{2k+1f} \Rightarrow \)

Weighted homogeneous weights \((1, 1)\),

\[
\text{degree}(x^4y + x^2y^3) = 5, \text{ degree}(xy^5) = 6, \text{ degree}(y^{2k+1}) = 2k+1.
\]

\[
T_{\mathcal{S}}^{6}(x, y^2, x^4y + x^2y^3 + axy^5) = T_{\mathcal{S}}^{2k+1f} + C\{axy^5. e_3\} \cong xy^5. e_3
\]

and without loss of generality \( a = 0. \)

For \( k \geq 3 \) \( T_{\mathcal{S}}^{2k+1}(x, y^2, x^4y + x^2y^3 + ay^{2k+1}) = T_{\mathcal{S}}^{2k+1f} + C\{ay^{2k+1}. e_3\}. \)
Hence by I.2 (x, y^2, x^4 y + x^2 y^3 + y^{2k+1}) is (2k+1)-determined for k ≥ 3.

Hence g ∈ S_f either g ∼ f or g ∼ (x, y^2, x^4 y + x^2 y^3 + y^{2k+1}) and f is a stem.

(II-2:18)

\[ f = (x, y^2, x^2 y^3 + y^5). \]

\[ p(x, y) = x^2 y + y^2 \]

\[ T_\mathcal{A}^T p \cdot h = m_2^T \{ x y^2, y^2, x^j : j ≥ 0 \}, \]

\[ \text{basis} \quad A + \left( m_2 \{ y, y^3, x y^3, x^j y : j ≥ 0 \} \right) e_3, \]

\[ \text{multiplicity} \quad \{ x y e_3 \}, \]

\[ f^* m_3 : T_\mathcal{A} f \]

\[ \text{B + } \left( m_2^2 \{ y^2, y^3, y^5, x^2 y^3, x y^3, x^j y : j ≥ 1 \} \right) e_3, \]

\[ f^* m_3 \left( T_\mathcal{A} f + \{ x y e_3 \} \right) \geq m_2^{j+2} \cdot \theta(f) \text{ provided that } j ≥ 4. \]

\[ T_\mathcal{A}_{2,2} f \]

\[ \text{D + } \left( m_2^2 \{ y^2, y^3, x y^3, y^5, x^2 y^3, x y^3 : j ≥ 1 \} \right) e_3, \]

\[ \text{hence for } k ≥ 5 \ g ∈ S^{k} \Rightarrow \ g ∈ S^{k+1}(x, y^2, x^2 y^3 + y^5 + a x^k y). \]

\[ \text{Weighted homogeneous weights } (1, 1), \]

\[ \text{degree}(x^2 y^3 + y^5) = 5, \text{degree}(x^k y) = k+1, \]

\[ \text{hence for } k ≥ 5 \ T_\mathcal{A}^{k+1}(x, y^2, x^2 y^3 + y^5 + a x^k y) = T_\mathcal{A}^{k+1} f + C \{ a x^k y e_3 \}. \]

By I.2 (x, y^2, x^2 y^3 + y^5 + x^k y) is (k+1)-determined for k ≥ 5.

Hence g ∈ S_f either g ∼ f or g ∼ (x, y^2, x^2 y^3 + y^5 + x^k y) and f is a stem.

(II-2:19)

\[ f = (x, y^2, y^5). \]

\[ p(x, y) = y^2 \]

\[ T_\mathcal{A}^T p \cdot h = m_2^T \{ x^j, x^j y^2 : j ≥ 0 \}, \]

\[ \text{basis} \quad A + \left( m_2 \{ x^j, x^j y^3 : j ≥ 0 \} \right) e_3, \]

\[ \text{multiplicity} \quad \{ x y e_3, x^j y e_3 \}, \]

\[ f^* m_3 : T_\mathcal{A} f \]

\[ \text{B + } \left( m_2^2 \{ y^2, y^3, y^5, x^j y, x^j y^3 : j ≥ 1 \} \right) e_3, \]

\[ f^* m_3 \left( T_\mathcal{A} f + \{ x y e_3 \} \right) \geq m_2^{j+2} \cdot \theta(f) \text{ provided that } j ≥ 4. \]
for $k \geq 5$ $g \in \mathfrak{S}_k f$.

Now $C \left\{ \begin{pmatrix} 0 \\ y + ax^k y + bx^{k-2} y^3 \\ x \end{pmatrix} \right\}$

$\sim C \left\{ \begin{pmatrix} 0 \\ 0 \\ y + ax^k y + bx^{k-2} y^3 \\ x \end{pmatrix} \right\} \subseteq T_0(x, y^2, y^5 + ax^k y + bx^{k-2} y^3)$.

Hence

$$T_0^k (x, y^2, y^5 + ax^k y + bx^{k-2} y^3) = T_0^k f + C \{ ax^k y \cdot e_3, bx^{k-2} y^3 \cdot e_3 \}.$$  

The $(k+1)$-orbits are

$$(x, y^2, y^5 + x^k y + x^{k-2} y^3)$$

$$(x, y^2, y^5 + x^k y - x^{k-2} y^3)$$

$$(x, y^5 + x^k y, x^{k-2} y^3)$$

By I:2 $(x, y^2, y^5 + x^k y)$ and $(x, y^2, y^5 + x^k y + x^{k-2} y^3)$ are $(k+1)$-determined for $k \geq 5$.

The map-germ $(x, y^2, y^5 + x^k y + x^{k-2} y^3)$ is not finitely determined since $(x, 0) \mapsto (x, 0, 0)$ is a cuspidal edge. Also $\{(x, \pm y) : y^2 + x^{k-2} = 0\} \mapsto (x, y^2, 0)$ is a double curve, while

$(x, y^2, y^5)$ has no double curves so $(x, y^2, y^5 + x^{k-2} y^3)$ and $(x, y^2, y^5)$ are not equivalent.

Hence $f$ is not a stem and $g \in \mathfrak{S}_k f \Rightarrow g$ is either equivalent to one of $f,$

$(x, y^2, y^5 + x^k y)$ or $(x, y^2, y^5 + x^k y + x^{k-2} y^3)$ or $g \in \mathfrak{S}_k^+(x, y^2, y^5 + x^{k-2} y^3)$.

\[ p(x, y) = x^2 y \]

\[ (\text{II-2:20}) \]

\[ f = (x, y^2, x^2 y^3). \]
\[ T \mathcal{S}_{2,2} f = C + \left( m_2 \{ y^2, x y^5, x^2 y^3, y^{2j+1}, x y^j : j \geq 1 \} \right) e_3 + C \left( \begin{array}{c} y^2 \\ 0 \\ 2 x y^5 \end{array} \right), \]

hence for \( k \geq 3 \) \( g \in \mathcal{S}^{2k} \Rightarrow g \in \mathcal{S}^{2k+1}(x, y^2, x^2 y^3 + ax^{2k}y + by^{2k+1}), \)
and \( g \in \mathcal{S}^{2k-1} \Rightarrow g \in \mathcal{S}^{2k}(x, y^2, x^2 y^3 + ax^{2k-1}y). \)

Weighted homogeneous weights \((1, 1),\)

degree \((x^2 y^3 + y^5) = 5\), degree \((x^k y) = k+1,\)
so for \( k \geq 5 \)

\[ T \mathcal{S}^{k+1}(x, y^2, x^2 y^3 + ax^k y) = T \mathcal{S}^{k+1} f + C\{ax^k y . e_3\}, \text{ while} \]

\[ C \left( \begin{array}{c} 0 \\ 2 x^3 + ax y + by \\ 2k + 1 \end{array} \right), \left( \begin{array}{c} 0 \\ 2 y^2 \\ 2x y + (2k+1)by \end{array} \right), \left( \begin{array}{c} x \\ 2 x^3 + 2k ax y \end{array} \right) \]

\[ \sim C \left( \begin{array}{c} 0 \\ 0 \\ 2 x y \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ 2k \end{array} \right), \left( \begin{array}{c} 0 \\ by^{2k+1} \end{array} \right) \subseteq T \mathcal{S}(x, y^2, x^2 y^3 + ax^{2k}y + by^{2k+1}) \]

so that \( T \mathcal{S}^{k+1}(x, y^2, x^2 y^3 + ax^{2k}y + by^{2k+1}) = T \mathcal{S}^{k+1} f + C\{ax^{2k} y . e_3, \text{ by}^{2k+1} . e_3\}. \)

By I:2 \((x, y^2, x^2 y^3 + ax^{2k}y + by^{2k+1}) \) is \( 2k+1 \)-determined for \( k \geq 3. \)

The map-germs \((x, y^2, x^2 y^3 + x^k y)\) and \((x, y^2, x^2 y^3 + y^{2k+1})\) are neither finitely determined nor equivalent to \((x, y^2, x^2 y^3)\) and hence \( f \) is not a stem.

If \( g \in \mathcal{S}^5 f \) then either \( g \sim f \) or \( g \sim (x, y^2, x^2 y^3 + x^{2k}y + y^{2k+1}) \)
or \( g \in \{ \mathcal{S}^{k+1}(x, y^2, x^2 y^3 + x^k y), \mathcal{S}^{2k+1}(x, y^2, x^2 y^3 + y^{2k+1}) \\}. \)

\[(II-2:21) \]

\[ f = (x, y^2, x^4 y). \]

\[ p(x, y) = x^4 \]

\[ T \mathcal{S} T_{p \circ h} = m_2 \{ x^3, y^2 j, x y^2 j, x^2 y^2 j : j \geq 0 \}, \]

\[ T \mathcal{S} f \]

\[ A + m_2 \{ x y^2 j+1, x y^2 j+1, x^2 y^2 j+1 : j \geq 0 \} e_3, \]

basis \( \{ y^{2j+1} . e_3, x y^{2j+1} . e_3, x^2 y^{2j+1} . e_3 \} \),

multiplicity \(3,\)

\[ f^* m_2 T \mathcal{S} f \]

\[ B + m_2 \{ x y, x^2 y, x^3 y, x^4 y, x y^3, x^2 y^3, x^3 y^3, y^{2j+1}, x y^{2j+1}, x^2 y^{2j+1} : j \geq 1 \} e_3, \]
\[ f^* m_3 \left( T \varphi f + \{ y^{2j+1}, e_3 \} \right) \supseteq m_2^{2j+2} \cdot \theta(f) \text{ provided that } j \geq 3. \]

\[ T \varphi f \quad C + m_2^2 \left\{ xy, y^2, x^3y, x^4y, x^3y^3, y^{2j+1}, xy^{2j+1}, x^2y^{2j+1}; j \geq 1 \right\} e_3 + \mathbb{C} \left\{ \begin{pmatrix} y^2 \\ 0 \\ 3x^3y^3 \end{pmatrix} \right\}, \]

therefore \( g \in \mathcal{Q}^3 f \Rightarrow \)

for \( k \geq 6 \) \( g \in \mathcal{Q}^k f \Rightarrow \)

or

\[ T \varphi f \quad C + m_2^2 \left\{ xy, y^2, x^3y, x^4y, x^3y^3, y^{2j+1}, xy^{2j+1}, x^2y^{2j+1}; j \geq 1 \right\} e_3 + \mathbb{C} \left\{ \begin{pmatrix} y^2 \\ 0 \\ 3x^3y^3 \end{pmatrix} \right\}, \]

hence \( g \in \mathcal{Q}^3 f \Rightarrow \)

"g \in \mathcal{Q}^6(x, y^2, x^4y + ax^3y^3 + by^5),

for \( k \geq 6 \) \( g \in \mathcal{Q}^k f \Rightarrow \)

or

Weighted homogeneous

\[ T \varphi f \quad C + m_2^2 \left\{ xy, y^2, x^3y, x^4y, x^3y^3, y^{2j+1}, xy^{2j+1}, x^2y^{2j+1}; j \geq 1 \right\} e_3 + \mathbb{C} \left\{ \begin{pmatrix} y^2 \\ 0 \\ 3x^3y^3 \end{pmatrix} \right\}, \]

so for \( k \geq 5 \)

Now

\[ C \left\{ \begin{pmatrix} 0 \\ 0 \\ x^4y + ax^2y^{2s-1} + by^{2s+1} \end{pmatrix}, \begin{pmatrix} 0 \\ 2y^2 \\ x^4y + (2s-1)ax^2y^{2s-1} + (2s+1)by^{2s+1} \end{pmatrix}, \begin{pmatrix} x \\ 0 \\ 4x^4y + 2ax^2y^{2s-1} \end{pmatrix} \right\} \]

so \( T \varphi f \quad C + m_2^2 \left\{ xy, y^2, x^3y, x^4y, x^3y^3, y^{2j+1}, xy^{2j+1}, x^2y^{2j+1}; j \geq 1 \right\} e_3 + \mathbb{C} \left\{ \begin{pmatrix} y^2 \\ 0 \\ 3x^3y^3 \end{pmatrix} \right\}, \]

so \( T \varphi f \quad C + m_2^2 \left\{ xy, y^2, x^3y, x^4y, x^3y^3, y^{2j+1}, xy^{2j+1}, x^2y^{2j+1}; j \geq 1 \right\} e_3 + \mathbb{C} \left\{ \begin{pmatrix} y^2 \\ 0 \\ 3x^3y^3 \end{pmatrix} \right\}, \]

By 1:2 \( (x, y^2, x^4y + x^{2s-1}y^{2s+1} + y^{2s+1}) \) and \( (x, y^2, x^4y + y^{2s+1}) \) are \( (2s+1) \)-determined for \( s \geq 3. \)

The map-germ \((x, y^2, x^4y + x^{2s-1}y^{2s+1})\) is neither finitely determined nor equivalent to \((x, y^2, x^4y)\).

Hence \( g \in \mathcal{Q}^3 f \Rightarrow \) either \( g \sim \) one of

\[ f, (x, y^2, x^4y + x^{2s-1}y^{2s+1} + y^{2s+1}) \text{ or } (x, y^2, x^4y + y^{2s+1}) \]

or \( g \) lies in one of...
\[ \mathcal{A}^{2s}(x, y^2, x^4y + xy^{2s-1}) \text{ or } \mathcal{A}^{2s+1}(x, y^2, x^4y + x^2y^{2s-1}) \]

The map-germ \((x, y^2, x^4y + x^2y^{2s-1})\) shows that \(f\) is not a stem.

\[(\text{II-2:22})\]

\[ f = (x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9) \]

Note that \(f\) is not weighted homogeneous.

\[ p(x, y) = y \]

\[ T.\mathcal{A}Tpoh = x^{T(m_2)^7} \setminus \{ x^8, x^9, x^6y^4, \}
\]

\[ +C\{ x^2y^{10}, xy^{12}, x^9 - x^3y^8, x^8 - x^2y^8, x^5y^4 + 2x^2y^8, x^6y^4 + 2x^3y^8, 5x^4y^6 + 2xy^{10} - 14x^3y^8 \}, \]

\[ T.\mathcal{A}f \]

\[ A + (m_2 + yT.\mathcal{A}Tpoh)e_3, \]

basis \[ \{ y^{2j+1}, e_3 \}, \]

multiplicity \[ 1, \]

By I-2:4 and I-1 there is a value \(k_0\) such that for \(k \geq k_0\),

\[ g \in \mathcal{A}f \Rightarrow \]

\[ g \in \mathcal{A}^{k+1}(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2s+1}) \text{ if } k = 2s \]

or

\[ g \in \mathcal{A}^{k+1}(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9) \text{ if } k = 2s+1. \]

\[(\text{II-2:23})\]

\[ f = (x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1}). \]

\[ p(x, y) = y \]

\[ T.\mathcal{A}Tpoh = x^{T(m_2)^7} \setminus \{ x^8, x^9, x^{10}, x^7y^2, x^6y^4, x^5y^6, x^4y^{12}, x^4y^{12}, x^7y^4 \}
\]

\[ +C\{ ay^{2k+2j+4}; j \geq 0, x^2y^{10} + (2k-9)ay^{2k+2}, 5x^4y^8 + 2xy^{12} + 8(16-3k)ay^{2k+2}, x^9 - x^3y^8, x^8 - x^2y^{8} - (2k-9)ay^{2k}, x^5y^4 + 2x^2y^{8} + (8-k)ay^{2k}, x^6y^4 + 2x^3y^{8}, 5x^4y^6 + 2xy^{10} - 14x^3y^8, 8(16-3k)ay^{2k}, x^7y^2 + (3k-16)ay^{2k}, x^7y^4 + (3k-16)ay^{2k+2}, x^5y^6 + 2x^2y^{10} + (8-k)ay^{2k+2}, \}, \]

\[ T.\mathcal{A}f \]

\[ A + (m_2 + yT.\mathcal{A}Tpoh)e_3, \]

Note that \(T.\mathcal{A}f \ni (0, 0, y^{2k+5})\) but \(T.\mathcal{A}f \not\ni (0, 0, y^{2k+3})\) so \(f\) is at worst \((2k+6)\)-determined but is not \((2k+3)\)-determined.

The next step for this map-germ would be to either calculate \(T.\mathcal{A}_{2,2}f\) for

\((x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1})\) or to calculate \(T.\mathcal{A}_{4,3}f \) \((k = 3)\) for

\((x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9)\) with the aim of trying to find out if the family
\[(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1} + by^{2k+3})\]
is 2k+3-determined for all b and all \(a \neq 0\). If this is the case then
\[(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9)\]
is a stem.
\[ \Pi \cdot 3 : j^2 f = (x, xy, 0). \]

(II-3:1) \[ f = (x, xy, 0). \]

\[ T\mathcal{A}_{2,2} f = m_2^2 \begin{pmatrix} 1 & 0 \\ y & x \\ 0 & 0 \end{pmatrix} + \{ x^{i+j} y^i : i+j \geq 2, 1 \leq r \leq 3 \} \]

\[ \begin{pmatrix} m_2 \{ y^k \} \\ m_2 \{ xy, y^k : k \geq 2 \} \\ \{ x^{i+j} y^i : i+j \geq 2 \} \end{pmatrix} + \mathbb{C} \begin{pmatrix} y^k \\ y^{k+1} \\ 0 \end{pmatrix} : k \geq 2 \]

Hence \( g \in \mathcal{A}^2 f \Rightarrow \)

\( g \in \mathcal{A}^3(x, xy + ay^3, bxy^2 + cy^3), \)

\( g \in \mathcal{A}^4(x, xy + ay^4, bxy^3 + cy^4), \)

\( g \in \mathcal{A}^5(x, xy + ay^5, bx^2y^3 + cxy^4 + dy^5), \) etc.

(II-3:2) \[ f = (x, xy + ay^3, bxy^2 + cy^3). \]

\[ T\mathcal{A}^3 f = \begin{pmatrix} m_2 \{ y, y^2 \} \\ m_2 \{ y, y^2, y^3 \} \\ m_2 \{ y, xy, y^2, xy^3 \} \end{pmatrix} \]

\[ + \mathbb{C} \begin{pmatrix} y^2 \\ y^2 \\ 0 \\ 0 \\ 0 \\ ay^3 \\ cy^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

The calculation of \( T\mathcal{A}^3 f \) was performed by computer.

3-orbits :

\( \{ (x, xy + ay^3, bx^2y^3 + cy^3) : a, b \in \mathbb{C}, c \in \mathbb{C}\backslash\{0\} \} \ni (x, xy, y^3), \)

\( \{ (x, xy + ay^3, bxy^2) : a, b \in \mathbb{C}\backslash\{0\} \} \ni (x, xy + y^3, xy^2), \)

\( \{ (x, xy + ay^3, 0) : a \in \mathbb{C}\backslash\{0\} \} \ni (x, xy + y^2, 0), \)

\( \{ (x, xy, bxy^2) : b \in \mathbb{C}\backslash\{0\} \} \ni (x, xy, xy^2). \)

Hence \( g \in \mathcal{A}^2 f \Rightarrow \)

\( g \in \{ \mathcal{A}^3(x, xy, y^3), \mathcal{A}^3(x, xy + y^3, xy^2), \mathcal{A}^3(x, xy + y^3, 0), \)
\[ \mathcal{S}^3(x, xy, xy^2), \mathcal{S}^3(x, xy, 0). \]

(II-3:3) \[ f = (x, xy + ay^4, bxy^3 + cy^4). \]

\[ T\mathcal{S}^4 f \]

\[ \begin{pmatrix}
    \mathcal{M}_2 \{y, y^2, y^3 \} \\
    \mathcal{M}_2 \{y, y^2, y^3, y^4 \} \\
    \mathcal{M}_2 \{y, xy, y^2, xy^2, y^3, xy^3, y^4 \}
\end{pmatrix} + \mathcal{C} \begin{pmatrix}
    \begin{pmatrix}
        y^2 \\
        y^3 \\
        y^4 \\
        y^2 \\
        y^3 \\
    \end{pmatrix} \begin{pmatrix}
        0 \\
        0 \\
        0 \\
        0 \\
        0 \\
    \end{pmatrix}
    \begin{pmatrix}
        0 \\
        0 \\
        ay^4 \\
        cy^4 \\
        0 \\
    \end{pmatrix} \begin{pmatrix}
        0 \\
        0 \\
        0 \\
        0 \\
        0 \\
    \end{pmatrix}
\end{pmatrix}. \]

The calculation of \( T\mathcal{S}^4 f \) was performed by computer.

3-orbits:

\[ \{(x, xy + ay^4, bxy^3 + cy^4) : a, b \in \mathcal{C}, c \in \mathcal{C} \setminus \{0\} \} \ni (x, xy, y^4), \]

\[ \{(x, xy + ay^4, bxy^3) : a, b \in \mathcal{C} \setminus \{0\} \} \ni (x, xy + y^4, xy^3), \]

\[ \{(x, xy + ay^4, 0) : a \in \mathcal{C} \setminus \{0\} \} \ni (x, xy + y^4, 0), \]

\[ \{(x, xy, bxy^3) : b \in \mathcal{C} \setminus \{0\} \} \ni (x, xy, xy^3). \]

Hence \( g \in \mathcal{S}^3 f \Rightarrow g \in \{ \mathcal{S}^4(x, xy, y^4), \mathcal{S}^4(x, xy + y^4, xy^3), \mathcal{S}^4(x, xy + y^4, 0), \]

\[ \mathcal{S}^4(x, xy, xy^3), \mathcal{S}^4(x, xy, 0) \}. \]

(II-3:4) \[ f = (x, xy, y^3). \]

\[ T\mathcal{S} f \]

\[ \begin{pmatrix}
    \mathcal{M}_2 \{y^{3j+1} : j \geq 0 \} \\
    \mathcal{M}_2 \{y^{3j+2} : j \geq 0 \} \\
    \mathcal{M}_2 \{y, y^2 \}
\end{pmatrix} + \mathcal{C} \begin{pmatrix}
    \begin{pmatrix}
        y^{3j+1} \\
        y^{3j+2} \\
        y^3 \\
    \end{pmatrix} \begin{pmatrix}
        0 \\
        0 \\
        0 \\
    \end{pmatrix}
\end{pmatrix}. \]

The calculation of \( T\mathcal{S} f \) will be found at A:1.

Basis \[ \{ y^{3j+2} e_2 : j \geq 1 \}, \]

multiplicity \[ 1, \]
\[ f^* \mathcal{M}_3^* \mathcal{T} \mathcal{S}_f \mathcal{F} \]
\[
\begin{pmatrix}
\mathcal{M}_2^{\{xy, y^3, y^3j+1 : j \geq 1\}} \\
\mathcal{M}_2^{\{xy, xy, y, y, y : j \geq 0\}} \\
\mathcal{M}_2^{\{xy, y^2, y, y, y : j \geq 0\}}
\end{pmatrix}
+ C \begin{pmatrix}
\begin{pmatrix}
xy \\
y^2 \\
y^3
\end{pmatrix} \\
\begin{pmatrix}
3j+1 \\
y^3j+2 \\
y_0
\end{pmatrix}
\end{pmatrix} : j \geq 1\)
\]

\[ f^* \mathcal{M}_3^* \left( \mathcal{T} \mathcal{S}_f \mathcal{F} + C(y^{3j+2}x_2) \right) \geq \mathcal{M}_2^{3j+3} \mathcal{F}(f) \text{ provided that } j \geq 1. \]

\[ T \mathcal{S}_{2,2} f \]
\[
\begin{pmatrix}
\mathcal{M}_2^{\{xy, y, y, y : j \geq 1\}} \\
\mathcal{M}_2^{\{xy, xy, y, y, y : j \geq 0\}} \\
\mathcal{M}_2^{\{xy, y^2, y, y, y : j \geq 0\}}
\end{pmatrix}
+ C \begin{pmatrix}
\begin{pmatrix}
xy \\
y^2 \\
y^3
\end{pmatrix} \\
\begin{pmatrix}
3j+1 \\
y^3j+2 \\
y_0
\end{pmatrix}
\end{pmatrix} : j \geq 1\)
\]

hence \( g \in \mathcal{S}^3 f \Rightarrow \) \( g \in \mathcal{S}^4(x, xy + ay^4, y^3 + by^4) \), for \( k \geq 4 \) \( g \in \mathcal{S}^k f \Rightarrow \) \( g \in \mathcal{S}^{k+1}(x, xy + ay^{3s+2}, y^3) \) if \( k = 3s+1 \)
or \( g \in \mathcal{S}^{k+1}(x, xy, y^3) \) if \( k \neq 3s+1 \).

Weighted homogeneous, weights \((1, 1)\).

Hence \( T \mathcal{S}^{3s+2}(x, xy + ay^{3s+2}, y^3) = T \mathcal{S}^{3s+2}(x, xy, y^3) + C\{ay^{3s+2}\} \)

By I:2 \((x, xy + y^{3s+2}, y^3)\) is \((3s+2)\)-determined for \( s \geq 1 \).

By computer calculation \( T \mathcal{S}^4(x, xy + ay^4, y^3 + by^4) = T \mathcal{S}^4(x, xy, y^3) \).

Hence \( g \in \mathcal{S}^3 f \Rightarrow g \sim (x, xy + y^{3s+2}, y^3) \) for some \( s \geq 1 \) or \( g \in \bigcap_{k \geq 0} \mathcal{S}^k f \) and \( f \) is a weak stem.

(II-3:5) \( f = (x, xy + y^3, xy^2) \).

\[ T \mathcal{S} f \]
\[
\begin{pmatrix}
\mathcal{M}_2^{\{y, y^3j+2 : j \geq 0\}} \\
\mathcal{M}_2^{\{y, y^2\}} \\
\mathcal{M}_2^{\{xy, y^2, y^3j+1 : j \geq 0\}}
\end{pmatrix}
+ C \begin{pmatrix}
\begin{pmatrix}
y^3j+2 \\
y^2 \\
y_0
\end{pmatrix} \\
\begin{pmatrix}
y^3 \\
y_0
\end{pmatrix}
\end{pmatrix} : j \geq 0\}
\]
The calculation of $T_{\mathcal{A}} f$ will be found at A:2.

Basis
\[ \{ y^{3j+1}e_3 : j \geq 1 \} , \]

multiplicity
\[ 1 , \]

\[ f^* m_3^* T_{\mathcal{A}} f \]
\[ \begin{pmatrix}
\mathcal{M}_2 \{ xy, y, xy, y, y, y : j \geq 0 \} \\
\mathcal{M}_2 \{ xy, y, xy, y, y, y \} \\
\mathcal{M}_2 \{ xy, y, x, y, xy, y, xy, y, y, y : j \geq 1 \}
\end{pmatrix} \]

\[ + \mathcal{C} \begin{pmatrix}
0 \\
0 \\
x^2 + xy^3
\end{pmatrix} \begin{pmatrix}
x^2 \\
x^4 \\
x^4 + y^6
\end{pmatrix} \begin{pmatrix}
xy^2 \\
xy^3 \\
xy^3 + y^6
\end{pmatrix} \begin{pmatrix}
y^2 \\
y^4 \\
y^4 + y^6
\end{pmatrix} \begin{pmatrix}
y^3 \\
y^6 \\
y^6 + y^{10}
\end{pmatrix} \begin{pmatrix}
y^{3j+2} \\
y^{3j+4} \\
y^{3j+4}
\end{pmatrix} : j \geq 1 , \]

\[ f^* m_3^* ( T_{\mathcal{A}} f + \mathcal{C} \{ y^{3j+1} e_3 \} ) \equiv m_2^{3j+2} \theta ( f ) \text{ provided that } j \geq 2 . \]

\[ T_{\mathcal{A}} f \]
\[ \begin{pmatrix}
\mathcal{M}_2 \{ xy, xy, y, y, y : j \geq 0 \} \\
\mathcal{M}_2 \{ xy, y, xy, y, y, y \} \\
\mathcal{M}_2 \{ xy, y, x, y, xy, y, xy, y, y, y : j \geq 1 \}
\end{pmatrix} \]

\[ + \mathcal{C} \begin{pmatrix}
8xy^2 + 3y^4 \\
x^2 + xy^3 \\
x^2 + xy^3
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
x^2 + xy^3
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
x^2 + xy^3
\end{pmatrix} \begin{pmatrix}
y^2 \\
y^3 \\
y^4
\end{pmatrix} \begin{pmatrix}
y^3 \\
y^6 \\
y^6 + y^{10}
\end{pmatrix} \begin{pmatrix}
y^4 \\
y^5 \\
y^5 + y^9
\end{pmatrix} \begin{pmatrix}
y^{3j+2} \\
y^{3j+4} \\
y^{3j+4}
\end{pmatrix} : j \geq 1 , \]

hence $g \in \mathcal{A}^3 f \Rightarrow$ $g \in \mathcal{A}^4 ( x, xy + y^3 + ay^4 , xy^2 + bxy^3 + cy^4 )$, $g \in \mathcal{A}^5 ( x, xy + y^3 , xy^2 + ay^5 )$, $g \in \mathcal{A}^6 ( x, xy + y^3 , xy^2 + ay^6 )$. for $k \geq 6$ $g \in \mathcal{A}^k f \Rightarrow$ $g \in \mathcal{A}^{k+1} ( x, xy + y^3 , xy^2 + ay^{3s+1} )$ if $k = 3s$ or $\mathcal{A}^{k+1} ( x, xy + y^3 , xy^2 )$ if $k \neq 3s$. Weighted homogeneous

\[ \text{weights (2, 1),} \]

\[ \text{degree} ( xy^2 ) = 4, \text{ degree} ( y^k ) = k . \]
Hence 
\[ T_x^{3s+1}(x, xy + y^3, xy^2 + ay^{3s+1}) = T_x^{3s+1}f + C\{ay^{3s+1}e_3\} \] if \( s \geq 2 \),
\[ T_x^5(x, xy + y^3, xy^2 + ay^5) = T_x^5f + C\{ay^5e_3\} \]
and 
\[ T_x^6(x, xy + y^3, xy^2 + ay^6) = T_x^6f + C\{ay^6e_3\}. \]

Note that 
\[ T_x^5f \ni y^5e_3 \] and \( T_x^6f \ni y^6e_3 \) so that for these two cases, without loss of generality, \( a = 0 \). Otherwise \( a = 1 \) or \( a = 0 \).

By I:2 \( (x, xy + y^3, xy^2 + y^{3s+1}) \) is \((3s+1)\)-determined provided that \( s \geq 2 \).

Hence \( g \in \mathcal{A}_4^f \Rightarrow g \sim (x, xy + y^3, xy^2 + y^{3s+1}) \) for some \( s \geq 2 \) or \( g \in \bigcap_{k \geq 0} \mathcal{A}_k^f \) and \( f \) is a weak stem.

(II.3.6) \[ f = (x, xy + y^3 + ay^4, xy^2 + bxy^3 + cy^4). \]

Proposition (see [Mond 1], p. 354): if \( g \in \mathcal{A}_3^3(x, xy + y^3, xy^2) \) then \( g \) lies in 
\[ \mathcal{A}_4^4(x, xy + y^3, xy^2 + cy^4). \]

If \( c \neq 0, \frac{1}{2}, 1, \frac{3}{2} \) then \( (x, xy + y^3, xy^2 + cy^4) \) is 4-determined.

(II.3.7) \[ f = (x, xy + y^3, xy^2 + y^4). \]

The calculation of \( T_x f \) will be found at A:3.

Basis \[ \left\{ y^{2j}e_3; j \geq 1 \right\}, \]

alternatively : observe that
\[ (x^jy^{2t+1}+x^{j-1}y^{2t+2})e_2 \in T_x f \Rightarrow (x^jy^{2t+1}+(-1)^{j+1}y^{2(t+j)+3})e_2 \in T_x f, \]
\[ (x^jy^{2t+1}+x^{j-1}y^{2t+2})e_3 \in T_x f \Rightarrow (x^jy^{2t+1}+(-1)^{j+1}y^{2(t+j)+2})e_3 \in T_x f \]
and \( (x^jy^{2t+1}+(-1)^{j+1}x^{j-1}y^{2t+2})e_3 \in T_x f. \)

so we can also choose the basis \[ \left\{ x^jy^2e_3; j \geq 1 \right\}, \]
multiplicity 1;

\[
f^*\mathcal{M}_2^T \mathcal{S} f = \left( \begin{array}{c}
\mathcal{M}_2^T \{xy, y^3, y^4\} \\
\{x, xy : j+l \geq 3\} + \{x y (x+y) : j+l \geq 2\} \\
\{x, x, xy : j+l \geq 3\} + \{xy(x+y), x y (x+y) : j+l \geq 1\}
\end{array} \right)
\]

\[\begin{pmatrix}
\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} xy \\ xy \\ xy \\ xy \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} x y + xy \\ x y + xy \end{bmatrix} & \begin{bmatrix} y^2 \\ y^3 \\ y^4 \\ y^5 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\end{array} \right] + c \begin{pmatrix}
\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} xy \\ xy \\ xy \\ xy \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} x y + xy \\ x y + xy \end{bmatrix} & \begin{bmatrix} y^2 \\ y^3 \\ y^4 \\ y^5 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\end{array} \right] \begin{pmatrix} j \geq 3 \end{pmatrix},
\]

\[f^*\mathcal{M}_2^T \mathcal{S} f + c \{y^2, e_3\} \geq \mathcal{M}_2^{2j+1} \cdot \theta(f) \text{ provided that } j \geq 3.
\]

hence \( g \in \mathcal{S}^4 \Rightarrow \ g \in \mathcal{S}^5(x, xy + y^3, xy^2 + y^4 + ay^5), \)

for \( k \geq 5 \) \( g \in \mathcal{S}^{k} \Rightarrow \ g \in \mathcal{S}^{k+1}(x, xy + y^3, xy^2 + y^4 + ay^{2s}) \text{ if } k = 2s-1 \)

or \( g \in \mathcal{S}^{k+1}(x, xy + y^3, xy^2 + y^4) \) if \( k = 2s. \)

Weighted homogeneous weights (2, 1),

\( \text{degree}(xy^2 + y^4) = 4, \text{degree}(y^k) = k. \)

Hence \( \mathcal{S}^{k+1}(x, xy + y^3, xy^2 + y^4 + ay^{k+1}) = T\mathcal{S}^{k+1} + c\{ay^{k+1}, e_3\}. \)

For \( k = 4, T\mathcal{S}^5 f \ni \{y^5, e_3\} \) so without loss of generality \( a = 0. \) For \( k = 2s, a = 1 \) or \( a = 0 \) and by I:2 \( (x, xy + y^3, xy^2 + y^4 + y^{2s}) \) is 2s-determined for \( s \geq 3. \)

Hence \( g \in \mathcal{S}^4 f \Rightarrow g \sim (x, xy + y^3, xy^2 + y^4 + y^{2s}) \) for some \( s \geq 3 \) or \( g \in \bigcap_{k \geq 0} \mathcal{S}^k f \)

and \( f \) is a weak stem.
(II.3:8) \((x, xy + y^3, xy^2 + \frac{3}{2} y^4) \sim f = (x, xy + y^3, 2xy^2 + 3y^4)\).

\[ T_\mathcal{S} f \]
\[ \mathcal{C} \{(j + 2)x^ly^j + 3jx^{l-1}y^{j+2} \}_{\mathcal{E}} : l \geq 1, j \geq 0, 1 \leq i \leq 3 \}
\[ + \mathcal{C} \left\{ \begin{pmatrix} y^2 \\ y^2 \\ 2y \\ 2y \\ 2y \\ 0 \end{pmatrix}, \begin{pmatrix} -2y \\ 0 \\ jy^{j+1} \\ 2y^{j+2} \\ 0 \end{pmatrix} : j \geq 2 \right\} \]

\( j = 0 \) \( \begin{pmatrix} (j + 2)x^ly^j + 3jx^{l-1}y^{j+2} \}_{\mathcal{E}} \}_{\mathcal{E}} = 2x^le^j \),
\( j = 1 \) \( \begin{pmatrix} (j + 2)x^ly^j + 3jx^{l-1}y^{j+2} \}_{\mathcal{E}} \}_{\mathcal{E}} = 3x^{l-1}(xy + y^3)e^j \),
\( j = 2 \) \( \begin{pmatrix} (j + 2)x^ly^j + 3jx^{l-1}y^{j+2} \}_{\mathcal{E}} \}_{\mathcal{E}} = x^{l-1}(2xy^2 + 3y^4)e^j \),
\( j = 3 \) \( \begin{pmatrix} (j + 2)x^ly^j + 3jx^{l-1}y^{j+2} \}_{\mathcal{E}} \}_{\mathcal{E}} = x^{l-1}(5xy^3 + 9y^5)e^j \),
\( j = 4 \) \( \begin{pmatrix} (j + 2)x^ly^j + 3jx^{l-1}y^{j+2} \}_{\mathcal{E}} \}_{\mathcal{E}} = x^{l-1}(6xy^4 + 12y^6)e^j \), etc.

The calculation of \( T_\mathcal{S} f \) will be found at A.4.

Basis for each term \( (x^ly^j)e^i \) there are constants \( a(l,j), b(l,j) \) such that
\( \begin{pmatrix} ax^ly^j + by^{j+2} \}_{\mathcal{E}} \}_{\mathcal{E}} = T_\mathcal{S} f, \)

for each term \( (x^ly^{2j+2})e^i \) there are constants \( c(l,j), d(l,j) \) such that
\( \begin{pmatrix} c^x^ly^{2j+2} + dx^{l+1}y^{j} \}_{\mathcal{E}} \}_{\mathcal{E}} = T_\mathcal{S} f, \)

for each term \( (x^l y^{2j+1})e^i \) there are constants \( c'(l,j), d'(l,j) \) such that
\( \begin{pmatrix} c^x^ly^{2j+1} + d'x^{l+1}y^{j} \}_{\mathcal{E}} \}_{\mathcal{E}} = T_\mathcal{S} f. \)

Basis is
\[ \{ y^le^j : j \geq 1 \} \]

or equivalently \( \{ x^ly^3, x^ly^2e^3 : j \geq 1 \}, \)

multiplicity
\[ 2. \]

\[ T_\mathcal{S} 2_2^f \]
\[ \mathcal{C} \{(2xy^2 + 3y^4)e_1, (5xy^3 + 9y^5)e_1, (5xy^3 + 9y^5)e_2, \}
\[ (j + 2)x^ly^j + 3jx^{l-1}y^{j+2} \}_{\mathcal{E}} : l \geq 2, \text{ or } l \geq 1 \text{ and } j \geq 4, 1 \leq i \leq 3 \}
\[ + \mathcal{C} \left\{ \begin{pmatrix} xy \\ x^2 \\ 2xy^2 \\ 2xy^3 \\ y \\ y^2 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 2y^5 \\ 2y^4 \\ y \\ y^4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2(j + 1)y^j \\ jy^{j+1} \\ 2y^{j+2} \end{pmatrix} : j \geq 4 \right\} \]

hence for \( k \geq 4 \) \( g \in \mathcal{S}^k f \Rightarrow g \in \mathcal{S}^{k+1}(x, xy + y^3, 2xy^2 + 3y^4 + ay^{k+1}). \)

Weighted homogeneous
\( \text{weights } (2, 1), \)
\( \text{degree}(2xy^2 + 3y^4) = 4, \text{ degree}(y^{k+1}) = k+1. \)
so $T\mathcal{S}^{k+1}(x, xy + y^3, 2xy^2 + 3y^4 + ay^{k+1}) = T\mathcal{S}^{k+1}f + C\{ay^{k+1}, e_3\}$.

We may choose $a = 2$, then

$$(x, xy + y^3, 2xy^2 + 3y^4 + 2y^{k+1}) \sim (x, xy + y^3, xy^2 + \frac{3}{2}y^4 + y^{k+1}).$$

Hence $g \in \mathcal{S}^4 f \Rightarrow$

$$g \in \mathcal{S}^k(x, xy + y^3, xy^2 + \frac{3}{2}y^4 + y^k) \text{ for some } k \geq 5 \text{ or } g \in \bigcap_{k \geq 0} \mathcal{S}^k f.$$

Note: $f$ does not satisfy hypothesis i) of I:2; for determinacy degree see II:3:9.

\[(II\cdot3:9)\] Let $f_k = (x, xy + y^3, 2xy^2 + 3y^4 + 2y^k)$. We show that $f_k$ is $k$-determined for $k \geq 6$.

Let $h \in \mathcal{m}_2^{k+1} \cdot \theta(f)$ and $f = (x, xy + y^3, 2xy^2 + 3y^4)$. Then $T\mathcal{S}(f_k + h) \equiv \mathcal{m}_2^k \cdot \theta(f)$ iff $T\mathcal{S}(f_k + h) + f^* \mathcal{m}_2^k \cdot \theta(f) \equiv \mathcal{m}_2^k \cdot \theta(f)$.

Now $f^* \mathcal{m}_2^k \cdot \theta(f) = \langle x, y^3 \rangle \cdot \mathcal{m}_2^k \cdot \theta(f) = \mathcal{m}_2^k \cdot \theta(f) + x \cdot \mathcal{m}_2^k \cdot \theta(f)$ and

$$T\mathcal{S}_{4,2}f + \mathcal{m}_2^{k+1} \cdot \theta(f) + x \cdot \mathcal{m}_2^k \cdot \theta(f) = T\mathcal{S}_{4,2}(f_k + h) + \mathcal{m}_2^{k+1} \cdot \theta(f) + x \cdot \mathcal{m}_2^k \cdot \theta(f),$$

that is

$$t(f \mathcal{m}_2^k \cdot \theta(2)) + \mathcal{m}_2^{k+1} \cdot \theta(f) = t(f+h)(\mathcal{m}_2^k \cdot \theta(2)) + \mathcal{m}_2^{k+1} \cdot \theta(f)$$

and

$$\omega(f \mathcal{m}_3^2 \cdot \theta(3)) + \mathcal{m}_2^{k+3} \cdot \theta(f) + x \cdot \mathcal{m}_2^k \cdot \theta(f) = \omega((f+h) \mathcal{m}_3^2 \cdot \theta(3)) + \mathcal{m}_2^{k+3} \cdot \theta(f) + x \cdot \mathcal{m}_2^k \cdot \theta(f).$$

Calculation by computer yields

$$T\mathcal{S}_{4,2}f = C \sum \left\{ \left( \begin{array}{c} y^4 \ y^3 \ y^6 \\ -x^5 \ 0 \ 2y^6 \\ \end{array} \right) \left( \begin{array}{cccc} 2xy^3 + 6y^5 \\ 0 \\ 0 \\ \end{array} \right) \left( \begin{array}{c} -j+2 \ y^j \\ 0 \\ 0 \\ \end{array} \right) + \left( \begin{array}{ccc} y^4 \ y^6 \ 4xy^3 + 10y^7 \\ \end{array} \right) \left( \begin{array}{c} 2y^3 + 6y^5 \\ 0 \\ 0 \\ \end{array} \right) \left( \begin{array}{c} 2(j+1)y^j \\ 2jy^{j+2} \\ 0 \\ \end{array} \right) : j \geq 6 \right\}.$$
so for $k \geq 6$,

\[
T_{\text{xi}} 4, 2 f + m^{k+3} \cdot \theta(f) + x \cdot m^{k} \cdot \theta(f) \\
\geq m^{k+2} \cdot \theta(f) + x^2 \cdot m^{k-2} \cdot \theta(f) + m^{k} \cdot \epsilon_1 + m^{k} \cdot \epsilon_2 + x \cdot m^{k-1} \cdot \epsilon_2,
\]

so

\[
T_{\text{xi}} 4, 2 f + m^{k+3} \cdot \theta(f) + x \cdot m^{k} \cdot \theta(f) \supseteq m^{k} \cdot \theta(f) \setminus \{ y^{k+1}, xy^{k-1} e_3, y^{k+1} e_2, y^{k+1} e_3 \},
\]

so

\[
T_{\text{xi}}(f_k + h) \supseteq m^{k} \cdot \theta(f) \setminus \{ y^{k+1}, xy^{k-1} e_3, y^{k+1} e_3 \}.
\]

Let $f_k + h = (x + h_1, xy + y^3 + h_2, 2xy^2 + 3y^4 + 2y^3 + h_3)$.

Then $T_{\text{xi}}(f_k + h)$ contains

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(x + h_1)(xy + y^3 + h_2) & (x + h_1)^2 & x & y^2 \\
0 & 0 & 4xy + 12y^3 + 2ky^{k-1} + \partial_y h_3 & x + 3y^2 + \partial_y h_2 \\
0 & 0 & 4xy + 12y^3 + 2ky^{k-1} + \partial_y h_3 & x + 3y^2 + \partial_y h_2 \\
0 & 0 & 4xy + 12y^3 + 2ky^{k-1} + \partial_y h_3 & x + 3y^2 + \partial_y h_2
\end{pmatrix}
\]

so $T_{\text{xi}}(f_k + h) + m_3 \cdot m_2^{k} \cdot \theta(f)$ contains

\[
\begin{pmatrix}
xy + y^3 & xy^2 & y^4 & 2xy^2 + 3y^4 + 2y^k & 0 \\
0 & 2xy^2 + 3y^4 + 2y^k & 0 & x^2 y + xy^3 & 0 \\
0 & 0 & 2ky^{k-1} + \partial_y h_3 & 4xy + 12y^3 + 2ky^{k-1} + \partial_y h_3 & 0 \\
x^2 + 3xy^2 & 4xy + 12xy^3 + 2kxy^{k-1} & xy^2 + 3y^4 & 4xy^3 + 12y^5 + 2ky^{k+1} & 0
\end{pmatrix}
\]

We can eliminate the low degree terms to show that
\[
\begin{pmatrix}
0 \\
y^k \\
kxy^{k-1} + ky^{k+1}
\end{pmatrix} \in T_{\mathfrak{S}} (f_k + h) + f^* \mathfrak{m}_3 \cdot \mathfrak{m}_2^k \theta(f).
\]

But

\[
\begin{pmatrix}
0 \\
(k+1)y^k \\
(k+1)xy^{k-1} + 3(k-1)y^{k+1}
\end{pmatrix} \in T_{\mathfrak{S}} (f_k + h) + f^* \mathfrak{m}_3 \cdot \mathfrak{m}_2^k \theta(f) \text{ for } k \geq 6 \text{ so } \mathcal{C} \{y^ke_2, xy^{k-1}e_3, y^{k+1}e_3\} \subseteq T_{\mathfrak{S}} (f_k + h) + f^* \mathfrak{m}_3 \cdot \mathfrak{m}_2^k \theta(f), \text{ since }
\]

\[
\det \begin{bmatrix}
1 & k & k \\
k+1 & 0 & 4k \\
0 & k+1 & 3(k-1)
\end{bmatrix} = 2k^2(k+1) \neq 0 \text{ for } k \geq 1.
\]

Then \( \mathfrak{m}_2^{k+1} \cdot \theta(f) \subseteq T_{\mathfrak{S}} (f_k + h) + f^* \mathfrak{m}_3 \cdot \mathfrak{m}_2^k \theta(f). \)

The result follows since we know that \( \mathfrak{m}_2^k \theta(f) \subseteq T_{\mathfrak{S}} f_k + \mathfrak{m}_2^{k+1} \cdot \theta(f). \)

It is shown in [Mond 1] that \((x, xy + y^3, xy^2 + \frac{3}{2} y^4 + y^5)\) is S-determined.

Hence \(g \in \mathfrak{S}^4 f \Rightarrow \)

\[g \sim (x, xy + y^3, xy^2 + \frac{3}{2} y^4 + y^k)\text{ for some } k \geq 5 \text{ or } g \in \bigcap_{k \geq 0} \mathfrak{S}^k f.\]

Hence \((x, xy + y^3, xy^2 + \frac{3}{2} y^4)\) is a weak stem.
j = 0 \quad (x+y^2)x^j y^{2j+1}e_i = (xy+y^3)x^j e_i \quad \text{and} \quad ((j+1)xy^2j+jy^{2j+2})x^j e_i = x^{j+1}e_i,

j = 1 \quad (x+y^2)x^j y^{2j+1}e_i = (xy^3+y^5)x^j e_i \quad \text{and} \quad ((j+1)xy^2j+jy^{2j+2})x^j e_i = (2xy^2+y^4)x^j e_i,

j = 2 \quad (x+y^2)x^j y^{2j+1}e_i = (xy^5+y^7)x^j e_i \quad \text{and} \quad ((j+1)xy^2j+jy^{2j+2})x^j e_i = (3xy^4+2y^6)x^j e_i,

\text{etc.}

The calculation of \( T \mathcal{A} \mathcal{S} f \) will be found at A:5.

\text{Basis} \quad \begin{aligned} &\text{for each term } (x^j y^l)e_i \text{ there are constants } a(l, j), b(l, j) \text{ such that} \\
&\quad (ax^j y^l + by^j + 2l)e_i \in T \mathcal{A} \mathcal{S} f, \\
&\text{for each term } (x^j y^{2j+2})e_i \text{ there are constants } c(l, j), d(l, j) \text{ such that} \\
&\quad (cx^j y^{2j+2} + dx^j l + y^j)e_i \in T \mathcal{A} \mathcal{S} f, \\
&\text{for each term } (x^j y^{2j+1})e_i \text{ there are constants } c'(l, j), d'(l, j) \text{ such that} \\
&\quad (c'x^j y^{2j+1} + d'x^j l + y^j)e_i \in T \mathcal{A} \mathcal{S} f: \\
&\quad \{y^j e_i : j \geq 1\} \\
&\quad \text{or equivalently } \{x^j y e_3, x^j y^2 e_3 : j \geq 1\}, \end{aligned}

\text{basis is} \quad 2.

\text{Multiplicity} \quad 2.

\text{T} \mathcal{A}_{2, 2} f \quad C\{ (x+y^2)x^{l-1} y^{2j+1}e_i , ((j+1)xy^{2j}+jy^{2j+2})x^j e_i \\
\quad : \ l \geq 2, j \geq 0 \text{ or } l = 1, j \geq 2 \text{ or } l = 1 \text{ and } j \geq 2 \} \\
\quad + C\{ (x+y^2)x^{l-1} y^{2j+1}e_2, x^j y^{2j} e_2 : l \geq 2, j \geq 0 \text{ or } l = 1 \text{ and } j \geq 2 \}

\text{hence for } k \geq 4 \ g \in \mathcal{A}^{k+1} \Rightarrow \quad g \in \mathcal{A}^{k+1}(x, xy + y^3, 2xy^2 + y^4 + ay^{k+1}).

\text{Weighted homogeneous} \quad \begin{aligned} &\text{weights } (2, 1), \\
&\text{degree}(2xy^2 + y^4) = 4, \text{ degree}(ay^{k+1}) = k+1 \\
&\text{so} \quad T \mathcal{A}^{k+1}(x, xy + y^3, 2xy^2 + y^4 + ay^{k+1}) = T \mathcal{A}^{k+1} f + C\{ ay^{k+1} e_3 \}.
\end{aligned}

\text{We may choose } a = 2, \text{ then} \quad (x, xy + y^3, 2xy^2 + y^4 + 2y^{k+1}) \sim (x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^{k+1}).

\text{Hence } g \in \mathcal{A}^4 f \Rightarrow \quad g \in \mathcal{A}^{k+1}(x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^{k+1}) \quad \text{for some } k \geq 4 \text{ or } g \in \bigcap_{k \geq 0} \mathcal{A}^k f.
We do not expect $f$ to be a stem since it does not have the right transversal type. However we shall repeat the calculation II·3:9 to see how $k$-determinacy fails.

\[ (II·3:11) \quad f_k = (x, xy + y^3, 2xy^2 + y^4 + 2y^{k+1}). \]

Let $h \in \mathfrak{m}_2^{k+1} \cdot \theta(f)$ and $f = (x, xy + y^3, 2xy^2 + y^4)$. Then

\[ T \mathfrak{S}(f_k + h) \supseteq \mathfrak{m}_2^k \cdot \theta(f) \iff T \mathfrak{S}(f_k + h) + f^* \mathfrak{m}_3 \mathfrak{m}_2^k \theta(f) \supseteq \mathfrak{m}_2^k \theta(f). \]

Now $f^* \mathfrak{m}_3 \mathfrak{m}_2^k \theta(f) = \langle x, y^3 \rangle \mathfrak{m}_2^k \theta(f) = \mathfrak{m}_2^{k+3} \cdot \theta(f) + x \cdot \mathfrak{m}_2^k \theta(f)$

and

\[ T \mathfrak{S}_4 f + \mathfrak{m}_2^k \cdot \theta(f) + x \cdot \mathfrak{m}_2^k \theta(f) = T \mathfrak{S}_4(f_k + h) + \mathfrak{m}_2^k \cdot \theta(f) + x \cdot \mathfrak{m}_2^k \theta(f) \]

as in II·3:9.

Calculation by computer yields

\[ T \mathfrak{S}_4 f = \mathcal{C}[(x+y^2)x^{l-1}y_{2j+1}; l+j \geq 3]e_1 + \mathcal{C}[x^2, ((j+1)xy^2j+jy^2j+2)x^l; l+j \geq 2]e_1 \]

\[ + \mathcal{C}[(x+y^2)xy, 3xy^4+2y^6, 3x^2y^2-y^6] \quad (x+y^2)x^{l-1}y_{2j+1}; l+j \geq 3]e_2 \]

\[ + \mathcal{C}[x^2, x^3, x^l y_{2j}; l+j \geq 4]e_2 \]

\[ \mathcal{C}[(x+y^2)x^{l-1}y_{2j+1}e_1; l+j \geq 4]e_3 + \mathcal{C}[x^2, ((j+1)xy^2j+jy^2j+2)x^l; l+j \geq 2]e_3 \]

\[ + \mathcal{C}[(x+y^2)xy, 2x^3y-3xy^5-y^7, 2x^2y^3+3xy^5+y^7]e_3 \]

so for $k \geq 7$

\[ T \mathfrak{S}_4 f + \mathfrak{m}_2^{k+3} \cdot \theta(f) + x \cdot \mathfrak{m}_2^k \theta(f) \]

\[ \supseteq \mathfrak{m}_2^{k+2} \cdot \theta(f) + x^2 \cdot \mathfrak{m}_2^{k-2} \cdot \theta(f) + \mathfrak{m}_2^k e_1 + \mathfrak{m}_2^k e_2 + x \cdot \mathfrak{m}_2^k e_2, \]

so

\[ T \mathfrak{S}_4 f + \mathfrak{m}_2^{k+3} \cdot \theta(f) + x \cdot \mathfrak{m}_2^k \theta(f) \supseteq \mathfrak{m}_2^k \theta(f) \setminus \{ y^{k+1}e_3, xy^{k-1}e_3, y^k e_2, y^{k+1} e_3 \}, \]

so

\[ T \mathfrak{S}(f_k + h) \supseteq \mathfrak{m}_2^k \theta(f) \setminus \{ y^{k+1}e_3, y^k e_2, xy^{k-1}e_3, y^{k+1} e_3 \}. \]
Let $f_k + h = (x + h_1, xy + y^3 + h_2, 2xy^2 + y^4 + 2y^k + h_3)$.

Then $T \mathcal{A}(f_k + h)$ contains

\[
\left( \begin{array}{c}
xy + y^3 + h_2 \\
0 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
y + xh_2 \\
2y^2 + xh_3 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
y + xh_2 \\
2y^2 + xh_3 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
1 + xh_1 \\
y + xh_2 \\
2xy^2 + y^4 + 2y^k + h_2 \\
\end{array} \right)
\left( \begin{array}{c}
1 + xh_1 \\
y + xh_2 \\
2xy^2 + y^4 + 2y^k + h_2 \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \right).
\]

Then $T \mathcal{A}(f_k + h)$ contains

\[
\left( \begin{array}{c}
x + y \\
0 \\
0 \\
(x + h_1)(xy + y^3 + h_2) \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
(x + h_1)^2 \\
x \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
\partial_y h_1 \\
x + 3y^2 + \partial_y h_2 \\
4xy + 4y^3 + 2ky^{k-1} + \partial_y h_3 \\
\end{array} \right)
\left( \begin{array}{c}
y^2 \\
x + 3y^2 + \partial_y h_2 \\
4xy + 4y^3 + 2ky^{k-1} + \partial_y h_3 \\
\end{array} \right)
\]

so $T \mathcal{A}(f_k + h) + f^*m_3^k m_2^k \theta(f)$ contains

\[
\left( \begin{array}{c}
xy + y \\
0 \\
0 \\
2xy^3 + 2y^5 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
y \\
y \\
2y^4 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
x + xy^2 \\
4xy + 4y^3 + 2kxy^{k-1} \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
x^2 + 3xy^2 \\
4x^2y + 4xy^3 + 2kxy^{k+1} \\
\end{array} \right)
\]

We can eliminate the low degree terms to show that

\[
\left( \begin{array}{c}
0 \\
y^k \\
kxy^{k-1} + ky^{k+1} \\
\end{array} \right) \in T \mathcal{A}(f_k + h) + f^*m_3^k m_2^k \theta(f).
\]

If $k = 2s$ then

\[
\left( \begin{array}{c}
0 \\
0 \\
xy^{2s-1} + y^{2s+1} \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
0 \\
2s \\
\end{array} \right) \in T \mathcal{A}(f_k + h) + f^*m_3^k m_2^k \theta(f).
\]

Consequently $f_{2s}$ is not $2s$-determined.
If \( k = 2s+1 \) then
\[
\begin{pmatrix}
0 \\
0 \\
(s+1)x y + sy
\end{pmatrix} \begin{pmatrix}
0 \\
(s+1)y^{2s+1} \\
2s+2
\end{pmatrix} \in T\mathcal{X}(f_k+h) + f^*m_3^sm_2^k\theta(f)
\]
so \( C[y_2e_2, xy^{k-1}e_3, y^{k+1}e_3] \subseteq T\mathcal{X}(f_k+h) + f^*m_3^sm_2^k\theta(f) \), since
\[
\begin{vmatrix}
s+1 & 0 & 1 \\
0 & s+1 & s \\
-1 & 2s+1 & 2s+1
\end{vmatrix} = 2(s+1)^2 \neq 0 \text{ for } s \geq 0.
\]
Hence \( T\mathcal{X}(f_k+h) + f^*m_3^sm_2^k\theta(f) \supseteq m_2^{k+1}\theta(f) \)
and \( T\mathcal{X}(f_k+h) + m_2^{k+1}\theta(f) = T\mathcal{X}f_k + m_2^{k+1}\theta(f) \supseteq m_2^k\theta(f) \).
Consequently \( f_{2s+1} \) is \( 2s+1 \)-determined.

Hence \( g \in \mathcal{X}^3f \Rightarrow g \sim (x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^{2s+1}) \) for some \( s \geq 3 \) or
\[
g \in \mathcal{X}^{2k}(x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^{2s}) \text{ for some } s \geq 3 \text{ or } g \in \bigcap_{k \geq 0} \mathcal{X}^{kf}.
\]

(II:3:12) The map-germ \( (x, xy + y^3, xy^2 + \frac{1}{2}y^4) \) is not a stem.

Proof: first note that \( (x, xy + y^3, xy^2 + \frac{1}{2}y^4) \sim (x, xy + y^3, (x+y^2)^2) \).
We prove that the map-germ \( (x, xy + y^3, (x+y^2)^2 + y^{2k+1}(x+y^2)) \) is neither finitely determined nor equivalent to \( (x, xy + y^3, (x+y^2)^2) \).
The curve \( \{x+y^2=0\} \rightarrow (x, 0, 0) \) is a double curve for each map-germ and
\[
\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = \begin{pmatrix}
1 \\
y \\
2(x+y^2)+y^{2k+1}
\end{pmatrix} \begin{pmatrix}
0 \\
x+3y^2 \\
2y^{2k+2}
\end{pmatrix}.
\]
On \{x+y^2=0\} this becomes \( \begin{pmatrix} 1 & 0 \\ y & 2y \\ 2k+1 & 2y^{2k+2} \end{pmatrix} \) and it is clear that the curve does not have transversal type \( A_1 \), so \((x, xy + y^3, (x+y^2)^2 + y^{2k+1}(x+y^2)) \) is not finitely determined.

We identify the double point set of each map-germ. For a map-germ of the form \((x, b(x, y), c(x, y)) \) this is the zero set of the ideal

\[
\langle x-x', x+y^2+yy'+(y')^2, (x+y^2)^2 \rangle \]

For \((x, xy + y^3, (x+y^2)^2) \) this ideal becomes

\[
\langle x-x', x+y^2+yy'+(y')^2, (x+y^2)^2 \rangle = \langle x-x', x+y^2+yy'+(y')^2 \rangle \]

So \((x, xy + y^3, (x+y^2)^2) \) has a single double curve \( y = -y' \) and \( x+y^2 = 0 \) with image \((X, 0, 0)\).

For \((x, xy + y^3, (x+y^2)^2 + y^{2k+1}(x+y^2)) \) the ideal becomes

\[
\langle x-x', x+y^2+yy'+(y')^2, 2x(y+y') + y^3 + y^2y' + (y')^2 + (y+y')^3 \rangle = \langle x-x', x+y^2+yy'+(y')^2 \rangle \]

This has a 2-1 double curve \( y = -y' \) and \( x+y^2 = 0 \) with image \((X, 0, 0)\). Note that \( f^4(X, 0, 0) = \{(y+(x+y^2) = 0) \cap \{(x+y^2)(x+y^2+ y^{2k+1}) = 0\} \) is the irreducible curve \( x+y^2 = 0 \).

Now \((y+y')^3+yy'(y+y')(y^{2k-1}+y^{2k-2}y'+...+(y')^{2k-1}) \)

\[
= (y+y')^2(y+y'+yy'(y^{2k-2}+y^{2k-4}(y')^2+...+(y')^{2k-2}))
\]

so the map-germ has a second curve defined by

\[
(y+y')+yy'(y^{2k-2}+y^{2k-4}(y')^2+...+(y')^{2k-2}) = 0.
\]

Let \( y+y' = 0 \) then \((y+y')+yy'(y^{2k-2}+y^{2k-4}(y')^2+...+(y')^{2k-2}) = kyy'y^{2k-2} = 0
\)

so the curves \((y+y')+yy'(y^{2k-2}+y^{2k-4}(y')^2+...+(y')^{2k-2}) = 0 \) and \( y+y' = 0 \) intersect only at \( y = y' = 0 \). If \((y+y')+yy'(y^{2k-2}+y^{2k-4}(y')^2+...+(y')^{2k-2}) \) also maps onto \((X, 0, 0)\) then \((X, 0, 0)\) is at least a triple curve, otherwise we have found a second double curve. This shows that \((x, xy + y^3, (x+y^2)^2 + y^{2k+1}(x+y^2)) \) cannot be equivalent to...
(x, xy + y^3, (x+y^2)^2) and concludes the proof.

\(f = (x, xy + y^3, 0)\).

\[
\begin{pmatrix}
\mathfrak{m}_2 \{xy, y^2, y^3\} \\
\mathfrak{m}_2 \{xy, y^2, x^2 y^3, y^4\} \\
\{i(xy + y^3)^j : i + j \geq 2\}
\end{pmatrix}
+ \mathbb{C}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
x^2 y + x^3 & bxy^3 & cxy^4 & cy^4
\end{pmatrix}
\]

The calculation of \(T \mathcal{S}_2 f + \mathfrak{m}_2 \mathfrak{m}_2^\theta(f)\) performed by computer established that

\[
\mathfrak{m}_2^4 e_1 + \mathfrak{m}_2^4 e_2 \subseteq T \mathcal{S}_2 f.
\]

Hence \(g \in \mathcal{S}^3 f \Rightarrow g \in \mathcal{S}^4(x, xy + y^3 + ay^4, bxy^3 + cy^4),\)

\(g \in \mathcal{S}^4 f \Rightarrow g \in \mathcal{S}^5(x, xy + y^3, ax^2 y^3 + bxy^4 + cy^5)\) etc.

\(f = (x, xy + y^3 + ay^4, bxy^3 + cy^4)\).

\[
\begin{pmatrix}
\mathfrak{m}_2 \{y\} \\
\mathfrak{m}_2 \{y, y^2\} \\
\mathfrak{m}_2 \{y, y, x, x y, x y^2, x y^3, x y^4\}
\end{pmatrix}
+ \mathbb{C}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
x^2 y + x^3 & bxy^3 & cxy^4 & cy^4
\end{pmatrix}
\]

The calculation of \(T \mathcal{S}_4 f\) was performed by computer.

4-orbits: \(\{(x, xy + y^3 + ay^4, bxy^3 + cy^4) : a, b \in \mathbb{C}, c \in \mathbb{C}\{0\}\} \ni (x, xy + y^3, y^4),\)

\(\{(x, xy + y^3 + ay^4, bxy^3) : a \in \mathbb{C}, b \in \mathbb{C}\{0\}\} \ni (x, xy + y^3, xy^3),\)

\(\{(x, xy + y^3 + ay^4, 0) : a \in \mathbb{C}\} \ni (x, xy + y^3, 0),\)

hence \(g \in \mathcal{S}^3 f \Rightarrow g \in \{\mathcal{S}^4(x, xy + y^3, y^4), \mathcal{S}^4(x, xy + y^3, xy^3), \mathcal{S}^4(x, xy + y^3, 0)\}.\)
(II·3:15) \[ f = (x, xy + y^3, ax^2y^3 + bxy^4 + cy^5). \]

\[
T\mathcal{S}^5f = \left( \begin{array}{c}
\mathcal{m}_2 \setminus \{y\} \\
\mathcal{m}_2 \setminus \{y, y^2\} \\
\mathcal{m}_2 \setminus \{y, xy, y^2, x, y, x^2y, x^3y, x^2, x^3, x^4, x^4, y^5\}
\end{array} \right)
\]

\[
+ C \left( \begin{array}{c}
0 \\
0 \\
x y + y^3
\end{array} \right) \left( \begin{array}{c}
0 \\
0 \\
x y + xy^3
\end{array} \right) \left( \begin{array}{c}
0 \\
0 \\
x y + 2xy
\end{array} \right) \left( \begin{array}{c}
0 \\
0 \\
x y^2 + xy
\end{array} \right) \left( \begin{array}{c}
y \\
y^2 \\
x y^4 + by^5
\end{array} \right)
\]

The calculation of \( T\mathcal{S}^5f \) was performed by computer.

The unfolding \( F(x, y, t) = (x, xy + y^3, a(t)x^2y^3 + b(t)xy^4 + c(t)y^5) \), where

\[
a(t) = 10c_0t^2 + 4b_0t + a_0, \quad b(t) = 5c_0t + b_0, \quad c(t) = c_0,
\]

is 5-trivial so if \( b_0 \neq 0 \) or \( c_0 \neq 0 \) then without loss of generality, \( a = 0 \).

The 5-orbits are:

\[
\{(x, xy + y^3, bxy^4 + cy^5): b, c \in \mathbb{C} \setminus \{0\}\} \ni (x, xy + y^3, xy^4 + y^5),
\]

\[
\{(x, xy + y^3, bxy^4): b \in \mathbb{C} \setminus \{0\}\} \ni (x, xy + y^3, xy^4),
\]

\[
\{(x, xy + y^3, cy^5): c \in \mathbb{C} \setminus \{0\}\} \ni (x, xy + y^3, y^5),
\]

\[
\{(x, xy + y^3, ax^2y^3): a \in \mathbb{C} \setminus \{0\}\} \ni (x, xy + y^3, x^2y^3),
\]

hence \( g \in \mathcal{S}^4f \Rightarrow g \in \{ \mathcal{S}^5(x, xy + y^3, xy^4 + y^5), \mathcal{S}^5(x, xy + y^3, xy^4), \mathcal{S}^5(x, xy + y^3, y^5), \mathcal{S}^5(x, xy + y^3, x^2y^3), \mathcal{S}^5(x, xy + y^3, 0) \}. \)
Calculations of $T^{7}\mathcal{S}f + \mathcal{M}_2 \theta(f)$ and $T^{10}\mathcal{S}_{2,2}f + \mathcal{M}_2 \cdot \theta(f)$ were performed by computer.

Hence $g \in \mathcal{S}^4 f \Rightarrow g \in \mathcal{S}^5 (x, xy + y^3, y^4 + ay^5),
g \in \mathcal{S}^5 f \Rightarrow g \in \mathcal{S}^6 (x, xy + y^3, y^4 + ay^6),
g \in \mathcal{S}^k f, k \geq 6 \Rightarrow g \in \mathcal{S} (x, xy + y^3, y^4).

Since $f$ is weighted homogeneous,

$T^{5}\mathcal{S}(x, xy + y^3, y^4 + ay^5) = T^{5}\mathcal{S}f + \mathcal{C}\{ay^5 \cdot e_3\} \equiv y^5 \cdot e_3,
\ T^{6}\mathcal{S}(x, xy + y^3, y^4 + ay^6) = T^{6}\mathcal{S}f + \mathcal{C}\{ay^6 \cdot e_3\} \equiv y^6 \cdot e_3.$

It follows that $f$ is 4-determined.
(II.3:17) \[ f = (x, xy + y^3, xy^3). \]

\[
\begin{pmatrix}
\mathfrak{m}_2 \{ y^{j+1}, xy^{3j+1}, xy^{3j+2}, x y^{3j+2}, j \geq 0 \} \\
\mathfrak{m}_2 \{ y^{j+1}, xy^{3j+1}, xy^{3j+2}, j \geq 0 \} \\
\mathfrak{m}_2 \{ y^{j+1}, xy^{3j+1}, xy^{3j+2}, 2, 3j+2, j \geq 0 \}
\end{pmatrix}
\]

\[ T_{\mathfrak{m}} f \]

\[ + \mathcal{C} \begin{pmatrix}
(6j+11)x^2 y^{2j-1} + 6xy^{3j+1} & \frac{1}{2} j(j-1)x y^{2j-4} + jxy^{3j-2} + y^3j \\
0 & 0 \\
0 & 0 
\end{pmatrix}, \]

\[
\begin{pmatrix}
\mathfrak{m}_2 \{ y, xy, xy^3, 2, 3j+2 \} \\
\mathfrak{m}_2 \{ y, xy, xy^3, 2, 3j+2 \} \\
\mathfrak{m}_2 \{ y, xy, xy^3, 2, 3j+2 \}
\end{pmatrix}
\]

The calculation of \( T_{\mathfrak{m}} f \) will be found at A:6.

Basis

\[ \{ xy^{3j+1} e_3, x y^{3j+1} e_3, y^{3j+2} e_3, y^{3j+3} e_3 : j \geq 3 \} \]

Multiplicity

4.

\[ T_{\mathfrak{m}} f_{2,2} \]

\[
\begin{pmatrix}
\mathfrak{m}_2 \{ y^{j+2}, xy^{3j+1}, xy^{3j+2}, x y^{3j+2}, j \geq 0 \} \\
\mathfrak{m}_2 \{ xy, y^{3j+1}, y^{3j+2}, xy^{3j+2}, j \geq 0 \} \\
\mathfrak{m}_2 \{ xy, y^{3j+1}, y^{3j+2}, 2, 3j+2, j \geq 0 \}
\end{pmatrix}
\]

\[ + \mathcal{C} \begin{pmatrix}
(6j+11)x^2 y^{3j-1} + 6xy^{3j+1} & -xy^{3j-1} \\
0 & xy^{3j-1} \\
0 & 0 
\end{pmatrix}, \]

\[
\begin{pmatrix}
\mathfrak{m}_2 \{ y^{j+1}, xy^{3j+1}, xy^{3j+2}, x y^{3j+2}, j \geq 0 \} \\
\mathfrak{m}_2 \{ xy, y^{3j+1}, y^{3j+2}, xy^{3j+2}, j \geq 0 \} \\
\mathfrak{m}_2 \{ xy, y^{3j+1}, y^{3j+2}, 2, 3j+2, j \geq 0 \}
\end{pmatrix}
\]
\[
\begin{vmatrix}
1 & j(j-1)x^2 & jx^2y & \frac{3j}{2} & \frac{3j-4}{2} + jxy & \frac{3j-2}{2} + y^3 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
(6j+1)x^2 & y^{3j-2} & \frac{(3j^2+2j-1)y^{3j}}{2} & \frac{y^{3j-1}}{2} & \frac{y^{3j-2}}{2} & \frac{3j^2}{2} + 8j \\
\end{vmatrix} + C
\]

\[
\begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
(6j+1)x^2 & y^{3j-1} & \frac{y^{3j-2}}{2} + 6y^{3j+1} & \frac{y^{3j-1}}{2} & \frac{y^{3j-2}}{2} & \frac{3j^2}{2} + 8j \\
\end{vmatrix} + C
\]

\[
\begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
2xy^2 - 3y^4 & 3xy^2 & \frac{y^2}{2} & \frac{y^3}{2} & y^4 & 0 \\
x^2y + 2xy^2 + y^6 & 0 & \frac{y^2}{2} & \frac{y^3}{2} & y^4 & \frac{x}{2} + xy^3 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
(3) & (3j^2 - 4j)xy^{3j-2} + (6j-7)y^{3j} & (3j^2 + 8j) \\
\end{vmatrix}
\]

Hence \( g \in \mathcal{A}^4f \Rightarrow g \in \mathcal{A}^5(x, xy + y^3, xy^3 + axy^4 + by^5), \)

\( g \in \mathcal{A}^5f \Rightarrow g \in \mathcal{A}^6(x, xy + y^3, xy^3 + axy^5 + by^6), \)

for \( k \geq 6 \) \( g \in \mathcal{A}^k_f \Rightarrow \)

\( g \in \mathcal{A}^{k+1}(x, xy + y^3, xy^3 + ay^{3s+1}) \) if \( k = 3s, \)

or \( g \in \mathcal{A}^{k+1}(x, xy + y^3, xy^3 + ay^{3s+2}) \) if \( k = 3s+1, \)

or \( g \in \mathcal{A}^{k+1}(x, xy + y^3, xy^3 + ay^{3s+2} + by^{3s+3}) \) if \( k = 3s+2. \)

Each map-germ \((x, xy + y^3, xy^3 + axy^{3s-1} + by^{3s})\) has a triple curve

\((0, y) \rightarrow (0, y^3, by^{3s}).\)

The terms

\[
\begin{vmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
x^2y + 3y^3 & xy + 3y^3 & xy + 3y^3 \\
x^2y + x^3y + axy^3 + by^{3s} & x^2y + x^3y + axy^3 + by^{3s} & x^2y + x^3y + axy^3 + by^{3s} \\
\end{vmatrix}
\]

imply that
so \((x, xy + y^3, xy^3 + a xy^{3s-1} + by^{3s})\) is not equivalent to \((x, xy + y^3, xy^3)\) and hence \((x, xy + y^3, xy^3)\) is not a stem.

(II-3:18) \[ f = (x, xy + y^3, xy^3 + axy^4 + by^5). \]

\[
\begin{pmatrix}
0 \\
0 \\
(3s-4)axy^{3s-1} + (3s-5)by^{3s}
\end{pmatrix} \in T_{x,y}^{3s+1}(x, xy + y^3, xy^3 + axy^{3s+2} + by^{3s+3})
\]

Calculation of \(T_{x,y}^5 f\) is performed by computer.

Orbits— if \(5b-6 \neq 0\) then without loss of generality \(a=0\).

Thus orbits are represented by \((x, xy + y^3, xy^3 + by^5)\),

\((x, xy + y^3, xy^3 + xy^4 + \frac{6}{2}y^5)\) and \((x, xy + y^3, xy^3)\).

(II-3:19) \[ f = (x, xy + y^3, xy^3 + axy^5 + by^6). \]

Here \(H^j = \mathbb{C}\{x_i\cdot y^j: \ 0 \leq i \leq j\}\).

\[
\begin{pmatrix}
\mathfrak{m}_2 \{y, xy, x^2, y^2, y^3, y^4\} \\
\mathfrak{m}_2 \{y, xy, x^2, y^2, x^3, y^4, y^5\} \\
\{x, x^2, x^3, x^4, x^5, x^6, y, y^2, y^3, y^4, y^5\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{xy+y^3\} \\
\{xy+y^3\} \\
\{xy+y^3\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]
Calculation of $T \alpha f$ is performed by computer.

The relevant term for the unfolding is

$$\begin{pmatrix}
0 \\
0 \\
by + 2axy
\end{pmatrix}.$$  

If $b \neq 0$ then $b = 1$.

Otherwise $b = 0$ and $a = 0$ or $a = 1$.

Thus orbits are represented by $(x, xy + y^3, xy^3 + axy^5 + y^6)$,

$(x, xy + y^3, xy^3 + xy^5)$ and $(x, xy + y^3, xy^3)$.

Clearly $(x, xy + y^3, xy^3 + xy^5)$ and $(x, xy + y^3, xy^3 + axy^5 + y^6)$ are not finitely determined since they have triple curves $(0, y) \rightarrow (0, y^3, 0)$ and $(0, y^3, y^6)$ respectively.

$$f = (x, xy + y^3, xy^4 + y^5).$$

$T \alpha f + m^2 \cdot \theta(f)$
\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
-x^2 + 2x + 2 & -x^2 + 2x + 2 \\
\end{pmatrix}
\begin{pmatrix}
y_2 \\
y_2 \\
-y^3 + 15y^6 \\
0 \\
5y^6 \\
0 \\
\end{pmatrix}
\] 

\[
T \mathcal{A}_{2,2} \mathcal{A} 
\begin{pmatrix}
\mathcal{M}_2 \setminus \{xy, y, y, y, y\} \\
\mathcal{M}_2 \setminus \{xy, y, y, y, y, y\} \\
\mathcal{M}_2 \setminus \{xy, y, xy, y, x, y, xy, y, y, y\} \\
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
xy \\
xy \\
y^2 \\
0 \\
y^2 \\
y^2 \\
0 \\
x^2 + xy^3 \\
x^2 + xy^3 \\
\end{pmatrix}
\]

Calculations of \(T \mathcal{A} \mathcal{A} \mathcal{A} + \mathcal{M}_2 \cdot 6(f)\) and \(T \mathcal{A}_{2,2} \mathcal{A} + \mathcal{M}_2 \cdot 6(f)\) were performed by computer.

\[
g \in \mathcal{A}^5 \Rightarrow \quad g \in \mathcal{A}^6(x, xy + y^3, xy^4 + y^5 + ay^6),
\]

\[
g \in \mathcal{A}^6 \Rightarrow \quad g \in \mathcal{A}^7(x, xy + y^3, xy^4 + y^5 + ay^7),
\]

\[
g \in \mathcal{A}^k, k \geq 7 \Rightarrow \quad g \in \mathcal{A}(x, xy + y^3, xy^4 + y^5).
\]

So \(f\) is at worst 7-determined and is at least 6-determined since \((0, 0, y^6) \notin T \mathcal{A} \mathcal{A} \mathcal{A} \).

Now \(T \mathcal{A} \mathcal{A} \mathcal{A}^7(x, xy + y^3, xy^4 + y^5 + ay^7) = T \mathcal{A} \mathcal{A} \mathcal{A}\)
(calculation of \(T \mathcal{A} \mathcal{A} \mathcal{A}^7(x, xy + y^3, xy^4 + y^5 + ay^7)\) is performed by computer).

Orbits- \(\{(x, xy + y^3, xy^4 + y^5 + ay^7) : a \in \mathbb{C}\}\) lies in a single 7-orbit, hence

\(x, xy + y^3, xy^4 + y^5\) is 6-determined.
The calculation of $T\mathcal{A}_{2,2}f + \mathcal{M}_2 \cdot \theta(f)$ was performed by computer.

By computer we calculate that

$$T\mathcal{A}_{2,2}f + \mathcal{M}_2 \cdot \theta(f) = T\mathcal{A}_{2,2}f + \mathcal{M}_2 \cdot \theta(f).$$

Hence $(x, xy + y^3, xy^4 + y^5 + ay^6 + by^7)$ and $(x, xy + y^3, xy^4 + y^5 + ay^6 + by^8)$ are 6-detennined for all $a.$
\[(II-3:22)\]

\[f = (x, xy + y^3, y^5).\]

\[T \mathcal{A} f + m_2^{12} \cdot \theta(f)\]

\[
\begin{pmatrix}
\mathcal{M}_2 \{y, xy, y^3 \} \\
\mathcal{M}_2 \{y, y^2, xy^2, y^4 \} \\
\mathcal{M}_2 \{y, xy, x^2, y^2, xy^2, y^3, x^3, xy^3, y^4, xy^4, y^5, xy^5, y^6 \}
\end{pmatrix}
\]

\[+ C \begin{pmatrix}
\begin{pmatrix}
y^2 \\
y^3 \\
0 \\
0 \\
5y^6 \\
y + y^3 \\
x^2 + xy^3 \\
x y + x^2 y^3
\end{pmatrix} \\
\begin{pmatrix}
y^3 \\
y^4 \\
0 \\
0 \\
0 \\
x^2 + xy^3 \\
x y + x^2 y^3 \\
x^2 + xy^3
\end{pmatrix} \\
\begin{pmatrix}
y^2 \\
y^3 \\
0 \\
0 \\
2xy + 3y^6 \\
x^2 + xy^3 \\
x y + x^2 y^3 \\
x^2 + xy^3
\end{pmatrix}
\end{pmatrix}.
\]

\[T \mathcal{A}_{12} f + m_2^{12} \cdot \theta(f)\]

\[
\begin{pmatrix}
\mathcal{M}_2 \{xy, y^2, x^2, y^3, xy^3, y^4, y^5 \} \\
\mathcal{M}_2 \{xy, y^2, xy^2, y^3, x^2, y^3, xy^3, y^4, y^5, xy^5, y^6 \} \\
\mathcal{C} \{x^2, x^3, x^4, x^5, xy, y^6 \} + m_2^{12} \{x^2, y^3, xy^3, y^4, y^5, xy^5, y^6 \}
\end{pmatrix}
\]

\[+ C \begin{pmatrix}
\begin{pmatrix}
x^2 y - y^5 \\
x^2 y - y^5 \\
0 \\
0 \\
x^2 y + 2xy^5 + 6y^6 \\
x^2 y + 2xy^5 + 6y^6 \\
x^2 y + 2xy^5 + 6y^6 \\
x^2 y + 2xy^5 + 6y^6 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \\
\begin{pmatrix}
x^5 + y^5 \\
x^5 + y^5 \\
0 \\
0 \\
x^3 y + 2xy^7 + y^8 \\
x^3 y + 2xy^7 + y^8 \\
x^3 y + 2xy^7 + y^8 \\
x^3 y + 2xy^7 + y^8 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \\
\begin{pmatrix}
x^3 y + 5y^6 \\
x^3 y + 5y^6 \\
0 \\
0 \\
x^5 + 2y^8 \\
x^5 + 2y^8 \\
x^5 + 2y^8 \\
x^5 + 2y^8 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\end{pmatrix}.
\]

Calculations of \(T \mathcal{A} f + m_2^{12} \cdot \theta(f)\) and \(T \mathcal{A}_{12} f + m_2^{12} \cdot \theta(f)\) performed by computer.

Hence \(g \in \mathcal{A}^5 f \Rightarrow \)

\(g \in \mathcal{A}^6 f \Rightarrow \)

\(g \in \mathcal{A}^7 f \Rightarrow \)
g ∈ S_kf, k ≥ 8 ⇒

\[ g \in S(x, xy + y^3, y^5). \]

f is at worst 8-detennined.

Since f is weighted homogeneous and degree(y^5) ≠ degree(y^k) unless k = 5 we have

\[ T \mathcal{S}^k(x, xy + y^3, y^5 + ay^k) = T \mathcal{S}^k(x, xy + y^3, y^5) + C(ay^k) \text{.} \]

Since \( T \mathcal{S} (x, xy + y^3, y^5) \supseteq \{y^7 \cdot e_3, y^8 \cdot e_3\} \) it follows that \( (x, xy + y^3, y^5) \) is 6-detennined. It is not 5-detennined since \( T \mathcal{S} (x, xy + y^3, y^5) \not\supseteq \{y^6 \cdot e_3\} \).

By computer we calculate that \( (x, xy + y^3, y^5 + y^6) \supseteq m_2 \cdot \theta(f) \setminus \{y^7 \cdot e_3\} \) and that

\[ T \mathcal{S}^7(x, xy + y^3, y^5 + y^6 + ay^7) \supseteq m_2 \cdot \theta(f) \text{ for all } a \text{ so that } (x, xy + y^3, y^5 + y^6) \text{ is} \]

6-detennined.

\[
(I1\cdot3:23) \quad f = (x, xy + y^3, xy^4).
\]

\[
T \mathcal{S} f = \begin{pmatrix}
\mathcal{M}_2 \setminus \{xy, y^2, xy^2, y^3, x^2y, y^5, y^7, y^3j+1 : j \geq 0\} \\
\mathcal{M}_2 \setminus \{x, xy, y^2, xy^2, y^3, x^2y, y^5, y^7\} \\
\mathcal{M}_2 \setminus \{y, xy, y^2, xy^2, y^3, x^2y, y^5, y^7\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
&
\mathcal{C}\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} &
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
&
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} \\
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} &
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} \\
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} &
\begin{pmatrix}
y_2 \\
y_3 \\
y_4
\end{pmatrix} &
\begin{pmatrix}
y_2 \\
y_3 \\
y_4
\end{pmatrix} &
\begin{pmatrix}
y_2 \\
y_3 \\
y_4
\end{pmatrix} &
\begin{pmatrix}
y_2 \\
y_3 \\
y_4
\end{pmatrix}
\end{pmatrix}
\]

The calculation of \( T \mathcal{S} f \) will be found at A:7.

Basis \( \{y^{3j+2} \cdot e_3 : j \geq 3\} \),

multiplicity \( 1 \).

At A:7 it is shown that
\[ f \mathcal{m}_3(T \mathcal{A} f) \equiv \mathcal{m}_2^{11} \cdot \theta(f) \{ y^{3j+13}, y^{3j+11}, j > 0 \} + \mathcal{C} \left( \begin{pmatrix} y^{3j+1} \\ y^{3j+5} \end{pmatrix} : j \geq 2 \right) . \]

\[ T \mathcal{A}_{2,2} f \]

\[ + \mathcal{C} \left( \begin{pmatrix} y^{3j+2} \\ x y + y^3 \end{pmatrix} \right) \]

\[ g \in \mathcal{A}_5 f \Rightarrow \]
\[ g \in \mathcal{A}_6 f \Rightarrow \]
\[ g \in \mathcal{A}_7 f \Rightarrow \]
\[ g \in \mathcal{A}_8 f \Rightarrow \]
\[ g \in \mathcal{A}_9 f \Rightarrow \]
\[ for \ k \geq 10 \ g \in \mathcal{A}_k f \Rightarrow \]

\[ g \in \mathcal{A}_6(x, y + y^3, x y^4 + ax y^5 + by^6), \]
\[ g \in \mathcal{A}_7(x, y + y^3, x y^4 + ax y^6 + by^7), \]
\[ g \in \mathcal{A}_8(x, x y + y^3, x y^4 + ax^2 y^6 + by^8), \]
\[ g \in \mathcal{A}_9(x, x y + y^3, x y^4 + ax y^8 + by^9), \]
\[ g \in \mathcal{A}_{10}(x, x y + y^3, x y^4 + ay^{10}), \]
\[ g \in \mathcal{A}_{k+1}(x, x y + y^3, x y^4) \ if \ k \neq 3s+1, \]
FIGS 1-4 FOR SECTION II.1
or \( g \in \mathfrak{A}^{k+1}(x, xy + y^3, xy^4 + ay^{3s+2}) \) if \( k = 3s+1 \).

Weighted homogeneous weights \((2, 1)\),

\[ \text{degree}(xy^4) = 6, \text{degree}(y^{3s+2}) = 3s+2. \]

Hence for \( s \geq 3 \)

\[ T_5^3s+2(x, xy + y^3, xy^4 + ay^{3s+2}) = T_5^3s+2f + C\{ay^{3s+2}, c_3\}. \]

By I:2 \((x, xy + y^3, xy^4 + y^{3s+2})\) is \((3s+2)\)-determined for \( s \geq 3 \).

Hence \( g \in \mathfrak{A}^{10f} \Rightarrow g \sim (x, xy + y^3, xy^4 + y^{3s+2}) \) for some \( s \geq 3 \) or \( g \in \bigcap_{k \geq 0} \mathfrak{A}^k f \) and \( f \) is a weak stem.

\[
\text{(I:3:24)}
\]

\[ f = (x, xy, y^4). \]

\[
T_5 \mathfrak{A} f = \begin{pmatrix}
\mathfrak{m}_2 \setminus \{(4j+1, 4j+2, xy^{4j+2} : j \geq 0)\} \\
\mathfrak{m}_2 \setminus \{y, 4j+2, xy^{4j+3} : j \geq 0\} \\
\mathfrak{m}_2 \setminus \{(x, y, y, xy, y^{4j+2} : j \geq 0)\}
\end{pmatrix}
\]

\[ + C \begin{pmatrix}
y^{4j+1} \\
y^{4j+2} \\
y^{4j+3} \\
0
\end{pmatrix}
\begin{pmatrix}
(y) \\
(x) \\
(y) \\
(0)
\end{pmatrix}
\begin{pmatrix}
0 \\
x^{4j+2} \\
x^{4j+3} \\
x^{4j+6}
\end{pmatrix}
\]

The calculation of \( T_5 \mathfrak{A} f \) will be found at A:8.

Basis \( \{y^{4s+2}, e_2, y^{4s+3}, e_2, y^{3j+2}, c_3 : j \geq 3\} \),

multiplicity 3.

Let \( \kappa = 4 \).

If \( h \in \mathfrak{m}_2^{k+1} \cdot \theta(f) \) then by I:2:2, \( T_5 \mathfrak{A} h + \mathfrak{m}_2^{k+5} \cdot \theta(f) = T_5 \mathfrak{A} h + \mathfrak{m}_2^{k+5} \cdot \theta(f) \).

Since \( f \) is finite, there exists a value \( k_0 \) such that \( T_5 \mathfrak{A} f \cap \mathfrak{m}_2^{k_0} \cdot \theta(f) \subseteq T_5 \mathfrak{A} f \)

so in particular, \( T_5 \mathfrak{A} f \supseteq \)

\[
\begin{pmatrix}
\mathfrak{m}_2 \setminus \{(4j+1, 4j+2, xy^{4j+2} : j \geq j_0)\} \\
\mathfrak{m}_2 \setminus \{(4j+2, 4j+3, xy^{4j+3} : j \geq j_0)\} \\
\mathfrak{m}_2 \setminus \{y^{4j+2} : j \geq j_0\}
\end{pmatrix}
\]

\[ \mathfrak{m}_2 \setminus \{y^{4j+2}, y^{4j+3}, xy^{4j+2} : j \geq j_0\}. \]
Let \( k = 4s-1 \). Then

\[
\mathcal{m}_2 \cdot \theta(f) \subseteq T \mathcal{S} 5f + \mathcal{m}_2 \cdot \theta(f) + \mathbb{C}\{y^{4s+2}e_2, y^{4s+3}e_2, xy^{4s-1}e_2\}.
\]

Then by 1-2.6, \( g \in \mathcal{S}^{4s+4}(x, xy + ay^{4s+2} + by^{4s+3} + cxy^{4s-1}, y^4) \).

Now \( (x, xy + cxy^{4s-1}, y^4) \) is not finitely determined, since \( (0, y) \rightarrow (0, 0, y^4) \) is a curve of quadruple points.

Also, \( f \) is weighted homogeneous so

\[
T \mathcal{S}^{4s+4}(x, xy + cxy^{4s-1}, y^4) = T \mathcal{S}^{4s+4}(x, xy, y^4) + \mathbb{C}\{cxy^{4s-1}e_2\},
\]

and \( (x, xy + cxy^{4s-1}, y^4) \sim (x, xy + xy^{4s-1}, y^4) \) is not \((4s+4)\)-equivalent to \( (x, xy, y^4) \) and is therefore not equivalent to \( (x, xy, y^4) \).

Therefore \( (x, xy, y^4) \) is not a stem.

\[
\text{(II-3:25)} \quad f = (x, xy + xy^3, y^4).
\]

\[
T \mathcal{S} f \begin{pmatrix} m_2 \{y^{4j+1}, y^{4j+2}, y^{4j+3} : j \geq 0\} \\ m_2 \{y^{4j+1}, y^{4j+2}, y^{4j+3} : j \geq 0\} \\ m_2 \{y, y, y, xy^2\} \end{pmatrix} + \mathbb{C}\begin{pmatrix} y^{4j+3}, y^{4j+5} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ y^{4j+5}, y^{4j+7} \\ y^{4j+3}, y^{4j+7} \end{pmatrix} \begin{pmatrix} y^{4j+2} \\ y^{4j+1} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} : j \geq 0.
\]

The calculation of \( T \mathcal{S} f \) will be found at A:9.

Basis \( \{y^{4s+2}e_2, y^{4s+3}e_2, j \geq 3\} \),

multiplicity \( k \)

For sufficiently large \( k \), \( \mathcal{m}_2 \cdot \theta(f) \cap T \mathcal{S} f \subseteq T \mathcal{S}^{2,2} f \)

so that \( g \in \mathcal{S}^k f \Rightarrow \quad g \in \mathcal{S}^{k+1}(x, xy + xy^3 + ay^{4s+3}, y^4) \) if \( k = 4s+2 \)

or \( g \in \mathcal{S}^{k+1}(x, xy + xy^3 + ay^{4s+2}, y^4) \) if \( k = 4s+1 \)
or \( g \in \mathcal{X}^{k+1}(x, xy + xy^3, y^4) \) otherwise.

Now \( T\mathcal{X}(x, xy + xy^3 + ay^k, y^4) \ni x \begin{pmatrix} 1 \\ y + y^3 \\ 0 \end{pmatrix} \) so that

\[
(0, ay^k, 0) \in T\mathcal{X}(x, xy + xy^3 + ay^k, y^4)
\]

and there are (at most) two \( k \)-orbits:

\( \mathcal{X}^{k+1}(x, xy + xy^3 + y^k, y^4) \) and \( \mathcal{X}^{k+1}(x, xy + xy^3, y^4) \).

By computer we find that

\[
T\mathcal{X}_{2,2}(x, xy + xy^3 + y^6, y^4) \ni m_2 \cdot \theta(f) \setminus \{y^7, e_2\}
\]

\[
T\mathcal{X}_{2,2}(x, xy + xy^3 + y^7, y^4) \ni m_2 \cdot \theta(f) \setminus \{y^{10}, e_2\}
\]

\[
T\mathcal{X}_{2,2}(x, xy + xy^3 + y^{10}, y^4) \ni m_2 \cdot \theta(f) \setminus \{y^{11}, e_2, y^{13}, e_2, y^{15}, e_2\} + \mathcal{C}(\{y^{13} + y^{15}\} \cdot e_2)
\]

\[
T\mathcal{X}_{2,2}(x, xy + xy^3 + y^{11}, y^4) \ni m_2 \cdot \theta(f) \setminus \{y^{14}, e_2, y^{18}, e_2\}
\]

\[
T\mathcal{X}_{2,2}(x, xy + xy^3 + y^{14}, y^4) \ni m_2 \cdot \theta(f) \setminus \{y^{15}, e_2, y^{17}, e_2, y^{19}, e_2, y^{21}, e_2, y^{23}, e_2\}
\]

\[
+ \mathcal{C}(\{y^{17} + y^{19}\} \cdot e_2, (y^{21} + y^{23}) \cdot e_2)
\]

\[
T\mathcal{X}_{2,2}(x, xy + xy^3 + y^{15}, y^4) \ni m_2 \cdot \theta(f) \setminus \{y^{18}, e_2, y^{22}, e_2, y^{26}, e_2\}.
\]

Further attempts to complete the classification have failed. It is not known whether or not \( f \) is a stem.

\[f = (x, xy - y^4, y^6)\].

\[
T\mathcal{X} f = \begin{pmatrix} m_2 \backslash \{ y^2, xy, x y, y, x y \} : j, l \geq 0 \\ m_2 \backslash \{ y, y^3, xy, x y, y, x y \} : j, l \geq 0 \\ m_2 \backslash \{ y, xy, xy, y, x y, y, x y \} : j, l \geq 0 \\ m_2 \backslash \{ y^8, xy, xy, y, x y, x y, y, x y \} : j, l \geq 0 \end{pmatrix}
\]
The calculation of $T_{\mathcal{S}_{2f}}$ will be found at A:10.

At A:10 it is shown that $f^*m_3(T_{\mathcal{S}_{2f}}) \geq 1$.
\[
\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
2x +y & 1 & 0 \\
4xy & 8 & -y
\end{array}\right)
\]

\text{Weighted homogeneous}

\text{Hence}

By I:2 \((x, xy - y^4, y^6 + y^{3s+1})\) is \((3s+1)\)-determined for \(s \geq 4\).

Hence \(g \in \mathcal{A}^{11f} \Rightarrow g \sim (x, xy - y^4, y^6 + y^{3s+1})\) for some \(s \geq 4\) or \(g \in \bigcap_{k \geq 0} \mathcal{A}^{kf}\) and \(f\) is a weak stem.
II·4: \( j^2 f = (x, 0, 0) \).

\[(II·4:1)\]

\( f = (x, 0, 0) \).

\[\mathcal{T}\mathcal{A}_2 f\]

\[
\begin{pmatrix}
\mathcal{M}_2^2 \\
\{x^j : j \geq 2\}
\end{pmatrix}
\]

hence \( g \in \mathcal{S}_2 f \Rightarrow g \in \mathcal{S}_3 (x, ax^2 y + bxy^2 + cy^3, dx^2 y + exy^2 + fy^3) \),

\( g \in \mathcal{S}_3 f \Rightarrow g \in \mathcal{S}_4 (x, ax^3 y + bx^2 y^2 + cxy^3 + dy^4, ex^3 y + fx^2 y^2 + gxy^3 + hy^4) \),

etc.

\[(II·4:2)\]

\( f = (x, ax^2 y + bxy^2 + cy^3, dx^2 y + exy^2 + fy^3) \).

We show the 3-orbits are as follows:

\[
(x, x^2 y + y^3, x^2 y), \quad (x, y^3, x^2 y), \quad (x, y^3, x^2 y), \quad (x, y^3, 0), \quad (x, y^3 - x^2 y, 0),
\]

\[
(x, xy^2, x^2 y), \quad (x, xy^2, 0), \quad (x, x^2 y, 0), \quad (x, 0, 0).
\]

This agrees with \([\text{Mond 1}]\).

Proof: we shall make extensive use of the unfolding criterion I·2:19.

\[
\mathcal{T}\mathcal{A}_3 f
\]

\[
\begin{pmatrix}
\mathcal{M}_2 \setminus \{y\} \\
\{x, x^2, x^3\}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 \\
ax^2 y + bxy^2 + cy^3 & dx^2 y + exy^2 + fy^3
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
ax^2 y + bxy^2 + cy^3 & bx^2 y + 3cxy^2 & 2axy^2 + by^3 \\
0 & bx^2 y + 3cxy^2 & 2axy^2 + by^3
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0\end{pmatrix}
\]

The calculation of \(\mathcal{T}\mathcal{A}_3 f\) was performed by computer.

For each point \((a_0, b_0, c_0, d_0, e_0, f_0)\) the unfolding

\[
(x, a_0 x^2 y + b_0 xy^2 + c_0 y^3, (a_0 t + d_0)x^2 y + (b_0 t + e_0)xy^2 + (c_0 t + f_0)y^3)
\]

is 3-trivial. If \(c_0 \neq 0\) then without loss of generality \(f_0 = 0\). If \(c_0 = 0\) and \(f_0 \neq 0\) then interchange \(f_2\) and \(f_3\). Without loss of generality
Then $T \mathcal{A}^3 f$ contains

$$C \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & bxy^2 + cy^3 & bx^2 + 3cy^2 \\
0 & dx^2 + exy^2 & 0 & exy^2 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$}

Note that these are all tangent to $\{ f = 0 \}$. For each point $(a, b, c, d, e)$ the unfolding $(x - ax^2 y + bxy^2 + cy^3, exy^2)$ is 3-trivial. If $e_0 \neq 0$ then w.l.o.g. $d_0 = 0$. Without loss of generality $f = (x, ax^2 y + bxy^2 + cy^3, exy^2)$ or $(x, ax^2 y + bxy^2 + cy^3, dx^2 y)$.

In the first case we consider $C \begin{pmatrix}
0 \\
0 & 0 & bxy^2 + cy^3 & bx^2 + 3cy^2 \\
0 & dx^2 + exy^2 & 0 & exy^2 \\
0 & 0 & 0 & 0
\end{pmatrix}.$

Note that these are all tangent to $\{ d = 0 \}$. For each point $(a, b, c, e)$ the unfolding $(x, a_0 x^2 y + b_0 xy^2 + c_0 y^3, e_0 xy^2)$ is 3-trivial (recall $e_0 \neq 0$). Without loss of generality $e_0 = 1$.

We consider $C \begin{pmatrix}
0 \\
0 & 0 & bxy^2 + cy^3 & bx^2 + 3cy^2 \\
0 & dx^2 + exy^2 & 0 & exy^2 \\
0 & 0 & 0 & 0
\end{pmatrix}.$

It is clear that without loss of generality, $f$ is one of $(x, x^2 y + y^3, xy^2), (x, x^2 y, xy^2), (x, y^3, xy^2), (x, 0, xy^2)$.

In the second case we consider

$$C \begin{pmatrix}
0 & 0 & bxy^2 + cy^3 & bx^2 + 3cy^2 \\
0 & dx^2 + exy^2 & 0 & exy^2 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$

f = (x, ax^2 y + bxy^2 + cy^3, dx^2 y + exy^2)
Note that these are all tangent to \{e = 0\}.

If \( d_0 \neq 0 \) then the unfolding \((x, a_0 x^2 y + b_0 x y^2 + c_0 y^3, e^1 d_0 x^2 y)\) is 3-trivial. Without loss of generality \( d_0 = 1 \). We consider \( \mathcal{C} \left( \begin{pmatrix} 0 \\ x y \\ b x y \\ c x y \\ c y^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \).

It is clear that without loss of generality, \( f \) is one of
\[(x, x y^2, x^2 y), (x, y^3, x^2 y), (x, 0, x^2 y)\].

If \( d_0 = 0 \) we consider
\[\mathcal{C} \left( \begin{pmatrix} 0 \\ a x^2 y + b x y^2 + c y^3 \\ b x y + 2 c y^3 \\ b x^2 y + 3 c x y^2 \\ 0 \end{pmatrix} \right) \].

The unfolding \((x, (1/2) c_0^2 + b_0 t + a_0)x^2 y + (3 c_0 t + b_0) x y^2 + c_0 y^3, 0)\) is 3-trivial. We may assume that \( b_0 = 0 \) unless \( c_0 = 0 \).

It is clear that without loss of generality, \( f \) is one of
\[(x, x^2 y, 0), (x, x^2 y - y^3, 0), (x, x y^2, 0), (x, y^3, 0), (x, 0, 0)\].

Hence \( g \in \mathcal{A}^2(x, 0, 0) \Rightarrow \)
\[ g \in \{ \mathcal{A}^3(x, x^2 y + y^3, x y^2), \mathcal{A}^3(x, y^3, x^2 y), \mathcal{A}^3(x, y^2, x^2 y), \mathcal{A}^3(x, y^3, x^2 y), \mathcal{A}^3(x, x^2 y - y^3, 0), \mathcal{A}^3(x, y^3, 0), \mathcal{A}^3(x, x^2 y, 0), \mathcal{A}^3(x, x y^2, 0), (x, 0, 0)\} \]

\[(\text{II} \cdot 4 \cdot 3) \quad f = (x, x^2 y + y^3, x y^2) \]

\[\mathcal{A} f \]
\[\mathcal{T} \mathcal{A} f \]

The calculation of \( \mathcal{T} \mathcal{A} f \) will be found at A:11.

Basis \( \left\{ y^{3k+1} e_3 \right\} \),

Multiplicity 1.
\[ f^* \mathfrak{m}_3 \cdot T \mathcal{A} f \]
\[ \left( \begin{array}{c}
m_2^2 \{ y^2, y^3, xy^2, y^{3j+2} : j \geq 1 \} \\
m_2^2 \{ xy, y^2, xy, y^3, y^4, y^5 \} \\
m_2^2 \{ xy, y^2, xy, y^3, y^4, y^5, y^{3j+1} : j \geq 1 \} \\
+ \mathbb{C} \left( \begin{array}{c}
xy^2 \\
y^{3j+2} \\
y^3 \\
x^2y \\
y^{3j+4} \\
y^5
\end{array} \right) : j \geq 1 \right)
\]

\[ \left( \begin{array}{c}
m_2^2 \{ y^2, y^3, xy^2, y^{3j+2} : j \geq 1 \} \\
m_2^2 \{ xy, y^2, xy, y^3, y^4, y^5 \} \\
m_2^2 \{ xy, y^2, xy, y^3, y^4, y^5, y^{3j+1} : j \geq 1 \} \\
+ \mathbb{C} \left( \begin{array}{c}
0 \\
xy^2 \\
y^{3j+2} \\
0 \\
x^2y \\
y^{3j+4} \\
y^5
\end{array} \right) : j \geq 0 \right)
\]

\[ g \in \mathcal{A}^3f \Rightarrow \]
\[ g \in \mathcal{A}^4(x, x^2y + y^3, xy^2 + ay^4), \]
\[ g \in \mathcal{A}^4f \Rightarrow \]
\[ g \in \mathcal{A}^5(x, x^2y + y^3, xy^2 + axy^4 + by^5), \]
and for \( k \geq 5 \) \( g \in \mathcal{A}^kf \Rightarrow \)
\[ g \in \mathcal{A}^{k+1}(x, x^2y + y^3, xy^2 + ay^{3s+1}) \text{ if } k = 3s, \]
or \( \mathcal{A}^{k+1}(x, x^2y + y^3, xy^2) \text{ if } k \neq 3s. \)

Weighted homogeneous weights \((1,1),\)
\[ \text{degree}(xy^2) = 3, \text{degree}(y^{3s+1}) = 3s+1. \]

Hence \( \mathcal{A}^{3s+1}(x, x^2y + y^3, xy^2 + ay^{3s+1}) = \mathcal{A}^{3s+1}f + \mathbb{C}\{ ay^{3s+1}, e_3 \}. \)

For \( s \geq 2, \) by I:2 \( (x, x^2y + y^3, xy^2 + y^{3s+1}) \) is \((3s+1)-determined.\)

In [Mond 1] the determinacy degree of \((x, x^2y + y^3, xy^2 + y^4)\) is shown to be 4.

By computer we calculate that
\[
T\mathcal{S}^5(x, x^2y + y^3, xy^2 + axy^4 + by^5) = \\
\begin{pmatrix}
\mathcal{M}_2 \setminus \{y^2\} \\
\mathcal{M}_2 \setminus \{y, xy, y^2\} \\
\mathcal{M}_2 \setminus \{y, xy, y^2, y^4\}
\end{pmatrix} + C\begin{pmatrix}
y^2 \\
0 \\
y^4
\end{pmatrix}
\]
so that \(\{(x, x^2y + y^3, xy^2 + axy^4 + by^5) : a, b \in \mathbb{C}\}\) lies in a single 5-orbit.

Hence \(g \in \mathcal{S}^3 \Rightarrow g \sim (x, x^2y + y^3, xy^2 + y^{3s+1})\) for some \(s \geq 1\) or \(g \in \bigcap_{k \geq 0} \mathcal{S}^k\) and \(f\) is a weak stem.

\[(\Pi.4:4)\]
\[f = (x, y^3, xy^2).\]

\[
T\mathcal{S}f = \\
\begin{pmatrix}
\mathcal{M}_2 \setminus \{y^{3j+2} : j \geq 0\} \\
\mathcal{M}_2 \setminus \{y^2, xy^2 : j \geq 0\} \\
\mathcal{M}_2 \setminus \{xy, y^2, y^{3j+1} : j \geq 0\}
\end{pmatrix} + C\begin{pmatrix}
y^{3j+2} \\
0 \\
y^{3j+4}
\end{pmatrix}.
\]

The calculation of \(T\mathcal{S}f\) will be found at A:12.

Basis \(\{x^j y^e_2, y^{3j+1} e_3\}\),

multiplicity \(2\),

\[
f^*\mathcal{M}_3 T\mathcal{S}f = \\
\begin{pmatrix}
\mathcal{M}_2 \setminus \{y^3, xy^2, y^{3j+2} : j \geq 0\} \\
\mathcal{M}_2 \setminus \{y^2, xy^2, y^3, y^4, y, xy : j \geq 1\} \\
\mathcal{M}_2 \setminus \{xy, y^2, xy^2, y^3, y^4, y, y^5, xy : j \geq 1\}
\end{pmatrix} + C\begin{pmatrix}
xy^2 \\
0 \\
xy^4
\end{pmatrix} : j \geq 1.
\]

\[f^*\mathcal{M}_3 \left(T\mathcal{S}f + C\{x^{3j} y^e_2, y^{3j+1} e_3\}\right) \cong \mathcal{M}_2^{3j+2} \cdot \Theta(f)\] provided \(j \geq 2\).
\[
\begin{align*}
&\begin{cases}
\mathfrak{m}_2^2 \{ y^3, xy^2, y^{3j+2} : j \geq 0 \} \\
\mathfrak{m}_2^2 \{ y^2, xy^2, y^3, y^j : j \geq 1 \} \\
\mathfrak{m}_2^2 \{ xy, y^2, x^2 y, xy^2, y^3, y^j : j \geq 1 \}
\end{cases} \\
+ \mathbb{C} \begin{pmatrix}
0 & 0 & y^3 & 0 \\
3y & 0 & 0 & 0 \\
2xy^4 & 0 & 0 & 0 \\
0 & xy^3 & 0 & 0
\end{pmatrix}, \quad j \geq 0,
\end{align*}
\]

hence $g \in \mathcal{F}_f \Rightarrow$

\[
\begin{align*}
g &\in \mathcal{F}_{4}(x, y^3 + ax^3 y, xy^2 + by^4), \\
g &\in \mathcal{F}_{5}(x, y^3 + ax^4 y, xy^2 + by^4 + cy^5), \\
\text{and for } k \geq 5 \quad g \in \mathcal{F}_{k}(x, y^3 + ax^3 y, xy^2 + by^3 + cy^s) : k = 3s,
\end{align*}
\]

or $\mathcal{F}_{k+1}(x, y^3 + ax^3 y, xy^2) : k \neq 3s.$

Weighted homogeneous weights $(1, 1)$.

Hence

\[
T\mathcal{F}_{k+1}(x, y^3 + ax^3 y, xy^2) = T\mathcal{F}_{k+1} + \mathbb{C}\{ ax^3 y, e_2 \}
\]

and

\[
T\mathcal{F}_{3s+1}(x, y, xy^2 + by^3) = T\mathcal{F}_{3s+1} + \mathbb{C}\{ by^3 + e_3 \}.
\]

The germs $(x, y^3, xy^2 + y^{3s+1})$ and $(x, y^3 + x^4 y, xy^2)$ are not finitely determined (having a cuspidal edge and a triple curve respectively). Neither are they equivalent to $(x, y^3, xy^2)$ (which has both) and which is hence not a stem.

(II-4:5)

\[
f = (x, y^3 + ax^3 y, xy^2 + by^{3s+1}).
\]

\[
T\mathcal{F}_{3s+1} = T\mathcal{F}_{2,2}(x, y^3, xy^2) + \mathbb{C} \begin{pmatrix}
\begin{pmatrix}
x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & y^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & y^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & y^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & y^3
\end{pmatrix} \\
\begin{pmatrix}
x^3 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x^3 & x & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x^3 & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x^3 & x & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x^3 & x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x^3 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x^3 & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3
\end{pmatrix}
\end{pmatrix}.
\]
The 3s+1-orbits are
\[ \{(x, y^3 + ax^{3s}y, xy^2 + by^{3s+1}) : a, b \in \mathbb{C} \setminus \{0\} \} \]
\[ \{(x, y^3, xy^2 + b y^{3s+1}) : b \in \mathbb{C} \setminus \{0\} \}. \]

Then \((x, y^3 + ax^{3s}y, xy^2 + by^{3s+1}) \) is \( (3s+1) \)-determined provided that \( s \geq 2 \), since the hypotheses of \( 1:2 \) are satisfied.

\[ f = (x, y^3 + ax^4y, xy^2 + bxy^4 + cy^5). \]

The 5-orbits are
\[ \{(x, y^3 + ax^4y, xy^2 + bxy^4 + cy^5) : a \in \mathbb{C} \setminus \{0\} ; b, c \in \mathbb{C} \} \]
and
\[ \{(x, y^3, xy^2 + bxy^4 + cy^5) : b, c \in \mathbb{C} \}. \]

The calculation of \( T \mathcal{S}^f \) was performed by computer.

Hence the 5-orbits are
\[ \{(x, y^3 + ax^4y, xy^2 + bxy^4 + cy^5) : a \in \mathbb{C} \setminus \{0\} ; b, c \in \mathbb{C} \} \]
and
\[ \{(x, y^3, xy^2 + bxy^4 + cy^5) : b, c \in \mathbb{C} \}. \]

The two orbits are represented by the map-germs \((x, y^3, xy^2), (x, y^3 + x^4y, xy^2)\).

Hence \( g \in \mathcal{S}^3(x, y^3, xy^2) \Rightarrow g \sim (x, y^3 + x^{3s}y, xy^2 + y^{3s+1}) \) or
\[ g \in \mathcal{S}^{k+1}(x, y^3 + x^ky, xy^2) \text{ or } g \in \mathcal{S}^{3s+1}(x, y^3, xy^2 + y^{3s+1}) \text{ or } g \in \bigcap_{k \geq 0} \mathcal{S}^k f. \]

\[ f = (x, y^3, x^2y). \]
\[ + \mathbb{C} \left( \begin{pmatrix} 0 \\ 3y^{3j+4} \\ 2 \end{pmatrix} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \cdots \end{pmatrix} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \cdots \end{pmatrix} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} : j \geq 0 \right) \]

The calculation of \( T_{\mathcal{A}_f} \) will be found at A:13.

Basis \[ \{ xy^{3k+2}, x^2 y^{3k+2}, x^3 y^{k+1}, x^3 y^{k+2} \} \]

multiplicity 4,

\[
T_{\mathcal{A}_{2,2} f} = \left( \begin{array}{c}
\mathbb{C}^{\mathbb{C}}_{m_2} \{ x, y, x^{3j+1}, y^{3j+1} : j \geq 0 \} \\
\mathbb{C}^{\mathbb{C}}_{m_2} \{ x, y, x, y, x^{3j+1}, y^{3j+1} : j \geq 1 \} \\
\mathbb{C}^{\mathbb{C}}_{m_2} \{ x, y, x, y, x, y, x, y, x, y, x, y, x, y, x, y : j \geq 0 \} \\
\end{array} \right) + \mathbb{C} \left( \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 3xy^4 \\ 2xy^4 \end{pmatrix} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 3xy^4 \\ 2xy^4 \end{pmatrix} : j \geq 0 \right) \]

hence \( g \in \mathcal{A}_f^3 \Rightarrow \quad g \in \mathcal{A}_f^4 (x, y^3, x^2 y + ax^2 y^2 + by^4) \),

\( g \in \mathcal{A}_f^4 \Rightarrow \quad g \in \mathcal{A}_f^5 (x, y^3, x^2 y + ax^2 y^2 + bxy^4 + cy^5) \),

and for \( k \geq 5 \) \( g \in \mathcal{A}_f^k \Rightarrow \quad g \in \mathcal{A}_f^{k+1} (x, y^3, x^2 y + ax^2 y^s - 1 + by^{3s+1}) : k = 3s \),

or \( \mathcal{A}_f^{k+1} (x, y^3, x^2 y + ay^{3s+2}) : k = 3s + 1 \),

or \( \mathcal{A}_f^{k+1} (x, y^3, x^2 y + axy^{3s+2}) : k = 3s + 2 \).

The germs \[ (x, y^3, x^2 y + x^2 y^{3j+2}), (x, y^3, x^2 y + xy^{3j+2}), (x, y^3, x^2 y + y^{3j+1}) \]

are not finitely determined;

the map-germ \( (x, y^3, x^2 y + y^{3j+1}) = (x, y^3, y(x^2 + y^{3j})) \) has two triple curves with smooth pre-image along \( x^2 + y^{3j} = 0 \) if \( j \) is even, or one triple curve with non-smooth pre-image if \( j \) is odd.

The map-germ \( (x, y^3, x^2 y + xy^{3j+2}) \) has a triple curve along \( x = 0 \), which has transversal type \( D_4 \).

Consequently, \( f \) is not a stem, since it has a single, smooth curve of triple points.
which has transversal type $J_{10}$ (the image satisfies $Z^3 = YX^6$ where $Y$ is a constant).

(II-4:8) \[ f = (x, y^3, x^2y + y^4). \]

The curve $(x^2 + y^3) = 0 \rightarrow (x, y^3, 0)$ is a triple curve of transversal type $D_4$. Note that the image of this curve is smooth, but not the pre-image.

\[
\begin{align*}
T \varphi f &= \begin{pmatrix}
\tau_2 \{(x, y)^{3j+1} : l, j \geq 0\} + \mathbb{C}\{(x, y)^{3j+1} (x^2 + y^3) : l+j \geq 0\} \\
\tau_2 \{(x, y)^{3j+1} : l, j \geq 0\} + \mathbb{C}\{(x, y)^{3j+1} (x^2 + y^3) : l+j \geq 0\} \\
\tau_2 \{y, xy, x y^{3j+2} : l, j \geq 0\} + \mathbb{C}\{(x, y)^{3j+2} (x^2 + y^3) : l+j \geq 1\}
\end{pmatrix} \\
&+ \mathbb{C}\begin{pmatrix}
0 \\
-3y^4 \\
2x^2y^4 + 4y^5
\end{pmatrix}
\begin{pmatrix}
l^{3j+4} \\
x y^{3j+4} \\
2x^2y^3 + 4y^5
\end{pmatrix}
\begin{pmatrix}
l^{3j+4} \\
xy^{3j+4} \\
2x^2y^{3j+2}
\end{pmatrix}
\begin{pmatrix}
l^{3j+4} \\
x y^{3j+1} \\
2x^2y^{3j+1}
\end{pmatrix}
\begin{pmatrix}
l^{3j+4} \\
x y^{3j+1} \\
2x^2y^{3j+2}
\end{pmatrix}
\begin{pmatrix}
l^{3j+4} \\
x y^{3j+1} \\
2x^2y^{3j+2}
\end{pmatrix}
: j \geq 0
\end{align*}
\]

The calculation of $T \varphi f$ will be found at A:14.

Basis \{x^jy^{2j}e_3\}, multiplicity 1.

Note that $(x^{2j}y^2 + (-1)^{j+1}y^{3j+2})e_3$ and $(x^{2j+1}y^2 + (-1)^{j+1}xy^{3j+2})e_3 \in T \varphi f$ so that we can rewrite the basis as \{y^{3j+2}e_3\} or \{xy^{3j+2}e_3\}.

\[
\begin{align*}
f \tau_3 T \varphi f &= \begin{pmatrix}
\tau_2 \{y, x^2y, x y^{3j+1} : l+j \geq 1\} + \mathbb{C}\{(x, y)^{3j+1} (x^2 + y^3) : l+j \geq 1\} \\
\tau_2 \{y, x^2y, x y^{3j+1} : l+j \geq 1\} + \mathbb{C}\{(x, y)^{3j+1} (x^2 + y^3) : l+j \geq 1\} \\
\tau_2 \{x, y, x^2y, x y^{3j+2} : l, j \geq 0\} + \mathbb{C}\{(x, y)^{3j+2} (x^2 + y^3) : l \geq 2 \text{ or } j \geq 1\}
\end{pmatrix} \\
&+ \mathbb{C}\begin{pmatrix}
0 \\
3x^2y^4 \\
x y^{3j+2} + 4xy^5
\end{pmatrix}
\begin{pmatrix}
x y^{3j+2} \\
2x^2y^3 + 4y^5
\end{pmatrix}
\begin{pmatrix}
x y^{3j+2} \\
2x^2y^{3j+2}
\end{pmatrix}
\begin{pmatrix}
x y^{3j+2} \\
2x^2y^{3j+2}
\end{pmatrix}
\begin{pmatrix}
x y^{3j+2} \\
2x^2y^{3j+2}
\end{pmatrix}
\begin{pmatrix}
x y^{3j+2} \\
2x^2y^{3j+2}
\end{pmatrix}
: j \geq 0
\end{align*}
\]

$f \tau_3 (T \varphi f + \mathbb{C}\{y^{3j+2}e_3\}) \supseteq \tau_2^{3j+3} \cdot \theta(f)$ provided $j \geq 1$

and $f \tau_3 (T \varphi f + \mathbb{C}\{xy^{3j+2}e_3\}) \supseteq \tau_2^{3j+4} \cdot \theta(f)$ provided $j \geq 1$. 
The calculation of $T \mathcal{K}^5f$ performed by computer.
The 5-orbits are
\[ \{(x, y^3, x^2 y + y^4 + ax^3 y^2 + by^5): b \in \mathbb{C}\setminus\{0\}; \ a \in \mathbb{C}\}, \]
\[ \{(x, y^3, x^2 y + y^4 + ax^3 y^2): a \in \mathbb{C}\}. \]

The two orbits are represented by the map-germs
\[ (x, y^3, x^2 y + y^4), (x, y^3, x^2 y + y^4 + y^5). \]

By I:2 \( (x, y^3, x^2 y + y^4 + y^5) \) is 5-determined.

Hence \( g \in \mathcal{A}^3f \Rightarrow \)
\[ g \sim (x, y^3, x^2 y + y^4 + y^{3s+2}) \text{ or } g \sim (x, y^3, x^2 y + y^4 + xy^{3s+2}) \text{ for some } s \geq 1 \]
\[ \text{or } g \in \bigcap_{k \geq 0} \mathcal{A}^k f; \]

thus \( f \) is a weak stem.

\((\text{II}-4:10)\)
\[ f = (x, x^2 y - y^3, 0). \]

\[ \mathcal{T}\mathcal{A}_{2,2}^2 f \]
\[ \begin{pmatrix} m_2^4 + \mathbb{C}\{ x^2, x^3 \} \\ \{ x^j (x^2 y - y^3)^l : l + j \geq 2 \} \end{pmatrix} \]

The calculation of \( \mathcal{T}\mathcal{A}_{2,2}^2 f \) was performed by computer (calculation modulo \( \mathcal{A}_1^2 \theta(f) \) is sufficient to establish that \( m_2^4 \frac{\partial}{\partial x} + m_2^4 \frac{\partial}{\partial y} \subseteq \mathcal{T}\mathcal{A}_{2,2}^2 f. \)

Then
\[ g \in \mathcal{A}^3 f \Rightarrow \]
\[ g \in \mathcal{A}^4(x, x^2 y - y^3, ax^2 y^2 + bxy^3 + cy^4), \]
\[ g \in \mathcal{A}^4 f \Rightarrow \]
\[ g \in \mathcal{A}^5(x, x^2 y - y^3, ax^3 y^2 + bx^2 y^3 + cxy^4 + dy^5) \text{ etc.} \]

We can obviously choose any one non-zero coefficient to be 1. In fact this is the only simplification which can be made, giving the orbits
\[ (x, x^2 y - y^3, ax^2 y^2 + bxy^3 + y^4), (x, x^2 y - y^3, ax^2 y^2 + xy^3), (x, x^2 y - y^3, x^2 y^2) \]

Worse still, none of the orbits \( (x, x^2 y - y^3, ax^2 y^2 + bxy^3 + y^4) \) are 4-determined.

We look at just one of these 4-jets:
The calculation of $T_{\mathfrak{m}} f$ will be found at A:15.

Basis

\[ \{ x^j y^k e_3 \}, \]

multiplicity

1,

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
x^2 y^2 & -y^4 & 0 & 0 & 0 \\
x^2 y^3 & y x^3 y^3 & x^4 y^2 & 5 x^4 y - 6 y^5 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
y \\
x y^2 \\
-x y^2 \\
0 \\
0 \\
2 x y^{3} \\
4 y^5 \\
\end{pmatrix}
\]
\[
T_{\mathcal{S}_2,2f} \begin{pmatrix}
\mathcal{M}_2 \{xy, x^2y, y^3\}, \\
\mathcal{C} \{x^2, x^3, x^4, x^5, xy, x^2y : l + 2j \geq 6\} + \mathcal{C} \{x^{2j+1}y : l + j \geq 1\} \\
\mathcal{C} \{x^2, x^3, x^4, x^5, x^6, x^2y^{2j+1} : l + 2j \geq 6\} + \mathcal{C} \{x^{2j+2}y^2 : l + j \geq 1\}
\end{pmatrix}
+ \begin{pmatrix}
2y^3 & 0 & 0 & 0 & 0 \\
0 & 3y^2 & 0 & 0 & 0 \\
0 & 0 & x^2y & -2y^2 & 2y^3 \\
\end{pmatrix}
\begin{pmatrix}
x^2y & -xy & 3xy & 5y^3 & 0 \\
0 & x^2y & 2y^4 & 2y^5 & x^2y^2 \\
0 & 0 & 2xy & 2xy^2 & 2xy^3 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
\end{pmatrix}
\]

hence \( g \in \mathcal{S}_4^f \Rightarrow g \in \mathcal{S}_5^f(x, x^2y - y^3, x^2y^2 - y^4 + ax^4y + bx^3y^2 + cy^5), \)

\( g \in \mathcal{S}_5^f \Rightarrow g \in \mathcal{S}_6^f(x, x^2y - y^3, x^2y^2 - y^4 + ax^4y^2 + bx^5y), \)

and for \( k \geq 6 \) \( g \in \mathcal{S}_k^f \Rightarrow g \in \mathcal{S}_{k+1}^f(x, x^2y - y^3, x^2y^2 - y^4 + ax^{k-1}y^2). \)

Homogeneous degree \((x^2y^2 - y^4) = 4, \) degree \((x^{k-1}y^2) = k+1, \)

hence \( T_{\mathcal{S}_{k+1}^f(x, x^2y - y^3, x^2y^2 - y^4 + ax^{k-1}y^2)} = T_{\mathcal{S}_k^f} + \mathcal{C}(ax^{k-1}y^2). \)

By I.2 \((x, x^2y - y^3, x^2y^2 - y^4 + x^{k-1}y^2)\) is \( k+1- \)determined for \( k \geq 6. \)

Hence \( g \in \mathcal{S}_6^f \Rightarrow \)

\( g \sim (x, x^2y - y^3, x^2y^2 - y^4 + x^{k-1}y^2) \) for some \( k \geq 6 \) or \( g \in \bigcap_{k \geq 0} \mathcal{S}_k^f \)

and \( f \) is a weak stem.
II-5 : Map-germs having corank 2.

We prove the results presented in table 4 and figure 4.

(II-5-1) Proposition: if \( f \) has corank 2 then \( f \) lies in one of the following \( \mathcal{A} \)-invariant strata:

\[
\mathcal{A}^2(xy, x^2, y^2), \quad \mathcal{A}^2(x^2 + y^2, xy, 0), \quad \mathcal{A}^2(x^2, y^2, 0), \\
\mathcal{A}^2(x^2, xy, 0), \quad \mathcal{A}^2(xy, 0, 0), \quad \mathcal{A}^2(x^2, 0, 0), \quad \mathcal{A}^2(0, 0, 0).
\]

Proof: if \( f \) has corank 2 then

\[
j^2f = (a_1x^2 + a_2xy + a_3y^2, b_1x^2 + b_2xy + b_3y^2, c_1x^2 + c_2xy + c_3y^2)
\]

for some \( a_i, b_i, c_i \in \mathbb{C} \).

If \( \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \neq 0 \) then we can find a change of basis so that

\[
j^2f = (xy, x^2, y^2)
\]

otherwise we may assume that \( c_1 = c_2 = c_3 = 0 \), i.e.

\[
j^2f = (a_1x^2 + a_2xy + a_3y^2, b_1x^2 + b_2xy + b_3y^2, 0).
\]

If \( a_1 \neq 0 \) then we can subtract \( b_1X/a_1 \) from \( Y \) so that without loss of generality, \( b_1 = 0 \). This leads to the following:

if \( b_2 \neq 0 \) then let \( x' = (b_2x + b_3y) \). Then \( j^2f \) becomes \( (\alpha_1x^2 + \alpha_2xy + \alpha_3y^2, b_2xy, 0) \).

We get \((x^2 + y^2, xy, 0)\) or \((x^2, xy, 0)\) depending on \( \alpha_3 \) (note \((x^2 + y^2, xy, 0)\) and \((x^2 - y^2, xy, 0)\) are equivalent under the coordinate changes \( y' = iy, Y' = -iY \)).

If \( b_2 = 0 \) then let \( x' = x + \frac{a_2}{2a_1}y \). Then \( j^2f \) becomes \((a_1(x')^2 + \alpha_3y^2, b_3y^2, 0)\).

This gives the forms \((x^2, y^2, 0)\) if \( b_3 \neq 0 \) and \((x^2 + y^2, 0, 0)\) or \((x^2, 0, 0)\) if \( b_3 = 0 \). Then \((x^2 + y^2, 0, 0) = ((x + iy)(x - iy), 0, 0)\) is equivalent to \((xy, 0, 0)\) under the coordinate change \( x' = x + iy, y' = x - iy \).

Finally, we suppose that \( a_1 = b_1 = 0 \). If \( \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} \neq 0 \) then without loss of
generality \( j^2 f = (xy, y^2, 0) \). Let \( x' = y, y' = x \) to get \((xy, x^2, 0)\).

Otherwise without loss of generality \( b_2 = b_3 = 0 \) and \( j^2 f \) is one of

\((xy + y^2, 0, 0), (xy, 0, 0) \) or \((y^2, 0, 0)\).

We let \( x' = x + y \) to show that \((xy + y^2, 0, 0) \sim (xy, 0, 0) \) and exchange \( x, y \) to show that \((y^2, 0, 0) \sim (x^2, 0, 0)\).

This concludes the proof of the proposition.

Note that none of these are finitely determined since they are all at least 2-1 everywhere. (They have 2-dimensional instability loci.)

We look at \((xy, x^2, y^2)\) and \((x^2 + y^2, xy, 0)\) in more detail:

(II.5.2) Proposition: if \( j^2 f = (xy, x^2, y^2) \) then \( f \) lies in one of

\[ \mathcal{A}^{2s+1}(xy + \Sigma \lambda_j x^{2s+1-j}y^j, x^2, y^2), \]

\[ \mathcal{A}^{2s+1}(xy, x^2, y^2 + x^{2s+1}) \] or \( \bigcap_{s \geq 0} \mathcal{A}^s(xy, x^2, y^2) \) for some \( s > 0 \)

(this is the example used in I.2—see 8, 14 and 20 in that section for the 3-jets).

Proof:

\[ T_{\mathcal{A}^{2s+1}} f = \left\{ x^{2i} - jy^i e_1, x^{2i} - jy^i e_2, x^{2i} - jy^i e_3 : i \geq 2, 0 \leq j \leq 2i \right\} + m_2^2 \begin{pmatrix} x \\ 0 \\ 2x \\ 2y \\ 0 \end{pmatrix} \]

so for \( s \geq 1 \), \( g \in \mathcal{A}^{2s} \Rightarrow g \in \mathcal{A}^{2s+1}(xy + \Sigma \lambda_j x^{2s+1-j}y^j, x^2 + \alpha y^{2s+1}, y^2 + bx^{2s+1}) \)

and \( g \in \mathcal{A}^{2s+1} f \Rightarrow g \in \mathcal{A}^{2s+2}(xy, x^2, y^2) \).

For \( f = (xy + \Sigma \lambda_j x^{2s+1-j}y^j, x^2 + \alpha y^{2s+1}, y^2 + bx^{2s+1}) \)

\[ T_{\mathcal{A}^{2s+1}} f = C \{ x^{2i} - jy^i e_1, x^{2i} - jy^i e_2, x^{2i} - jy^i e_3 : i \geq 2, 0 \leq j \leq 2i \} + m_2^2 \begin{pmatrix} x \\ 0 \\ 2x \\ 2y \\ 0 \end{pmatrix} \]

+ \( m_2^{2s+2} \cdot \Theta(f) + C \{ (xy + \Sigma \lambda_j x^{2s+1-j}y^j) e_4, (x^2 + \alpha y^{2s+1}) e_5, (y^2 + bx^{2s+1}) e_6 : 1 \leq \ell \leq 3 \} \)
\[
+ C \begin{pmatrix}
    xy + \sum \lambda_j(2s+1-j)x^{2s+1-j}y^j \\
    2x^2 \\
    (2s+1)\beta x^{2s+1}
\end{pmatrix} + C \begin{pmatrix}
    xy + \sum \lambda_j x^{2s+1-j}y^j \\
    (2s+1)\alpha y^{2s+1} \\
    2y^2
\end{pmatrix} + C \begin{pmatrix}
    y^2 + \sum \lambda_j(2s+1-j)x^{2s-j}y^{j+1} \\
    2xy \\
    (2s+1)\beta x^{2s+1} y
\end{pmatrix}
\]

We rearrange this so that the last 4 terms become
\[
C \begin{pmatrix}
    -\frac{1}{2} (2s+3)\beta x^{2s+1} + \sum \lambda_j(2s-j)x^{2s-j}y^{j+1} \\
    -2\lambda_2 x^{2s+1} \\
    0
\end{pmatrix} + C \begin{pmatrix}
    -\frac{1}{2} (2s+3)\alpha y^{2s+1} + \sum \lambda_j(j+1)x^{2s+2-j}y^{j-1} \\
    0 \\
    -2\lambda_0 x^{2s+1}
\end{pmatrix}
\]

We shall use these to construct \((2s+1)\)-trivial unfoldings.

Consider
\[
\begin{pmatrix}
    -\frac{1}{2} (2s+3)\alpha y^{2s+1} + \sum \lambda_j(j+1)x^{2s+2-j}y^{j-1} \\
    0 \\
    -2\lambda_0 x^{2s+1}
\end{pmatrix}
\]

Let \(F(x, y, t) = (xy + \sum \lambda_j(t)x^{2s+1-j}y^j, x^2 + \alpha(t)y^{2s+1}, y^2 + \beta(t)x^{2s+1})\)

where \(\alpha(t) = \alpha_0\).

\[
\lambda_{2s+1}(t) = -\frac{1}{2} (2s+3)\alpha_0 + \lambda_{2s+1}^0,
\]

\[
\lambda_{2s}(t) = (2s+2)(-\frac{1}{2} (2s+3)\alpha_0 \frac{1}{2} t^2 + \lambda_{2s+1}^0) + \lambda_{2s}^0,
\]

\[
\lambda_{2s-1}(t) = (2s+1)(2s+2)(\frac{1}{2} (2s+3)\alpha_0 \frac{1}{3!} t^3 + \lambda_{2s+1}^0 \frac{1}{2} t^2) + (2s+1)\lambda_{2s}^0 + \lambda_{2s-1}^0.
\]
\[ \lambda_0(t) = (2s+2)! \left( \frac{1}{2} (2s+3) \alpha_0 \right) + \frac{1}{(2s+2)!} t^{2s+2} + \frac{1}{(2s+1)!} \frac{1}{(2s+1)!} t^{2s+1} + \ldots + 2\lambda_1 t + \lambda_0. \]

\[ \beta(t) = -2(2s+2)! \left( \frac{1}{2} (2s+3) \alpha_0 \right) \frac{1}{(2s+3)!} t^{2s+3} + \frac{1}{(2s+1)!} \frac{1}{(2s+2)!} t^{2s+2} + \ldots + -2\lambda_0 t + \beta_0. \]

Then \[ \partial_t F_a = \begin{pmatrix} 2s+1 + \sum \lambda_j(a) \cdot (j+1) x^{2s+2-j} y^{j-1} \\ 0 \\ -2\lambda_0(a) x^{2s+1} \end{pmatrix} \]
and by I-2:19 \( F \) is a 

(2s+1)-trivial unfolding. If any of \( (\alpha_0, \lambda_0, \ldots, \lambda_{2s+1}) \) are non-zero then there exists a value of \( t \) such that \( \beta(t) = 0. \)

So \( (xy + \lambda_j x^{2s+1-j} y^j, x^2 + \alpha y^{2s+1}, y^2 + \beta x^{2s+1}) \) is (2s+1)-equivalent to either

\( (xy + \lambda_j x^{2s+1-j} y^j, x^2 + \alpha y^{2s+1}, y^2) \) or \( (xy, x^2, y^2 + \beta x^{2s+1}). \)

On \( \{xy + \sum \lambda_j x^{2s+1-j} y^j, x^2 + \alpha y^{2s+1}, y^2\} \) the remaining terms become

\[
\begin{pmatrix}
\sum \lambda_j(2s-j)x^{2s-j}y^{j+1} \\
-2\lambda_{2s+1}y^{2s+1} \\
0
\end{pmatrix},
\begin{pmatrix}
\sum \lambda_j(j+1)x^{2s+1-j}y^j \\
(2s+1)\alpha y^{2s+1} \\
0
\end{pmatrix},
\begin{pmatrix}
\sum \lambda_j(2s-j)x^{2s+1-j}y^j \\
-2\alpha y^{2s+1} \\
0
\end{pmatrix};
\]

note that these are tangent to \( (\beta=0). \)

Consider 
\[
\begin{pmatrix}
\sum \lambda_j(2s-j)x^{2s-j}y^{j+1} \\
-2\lambda_{2s+1}y^{2s+1} \\
0
\end{pmatrix}.
\]

Let 
\[ F(x, y, t) = (xy + \sum \lambda_j(t)x^{2s+1-j} y^j, x^2 + \alpha(t)y^{2s+1}, y^2) \]

where 
\[ \lambda_0(t) = \lambda_0, \]
\[ \lambda_1(t) = 2s\lambda_0 + \lambda_1, \]
\[ \lambda_2(t) = 2s(2s-1)\lambda_0 + \lambda_2, \]
\[ \ldots. \]
\[ \lambda_{2s+1}(t) = 2s! \frac{1}{(2s+1)!} t^{2s+1} + \ldots + \lambda_{2s} t + \lambda_{2s+1}, \]

\[ \alpha(t) = -2 \left( 2s! \frac{1}{(2s+2)!} t^{2s+2} + \ldots + \lambda_{2s} \frac{1}{2s} t^{2s} + \lambda_{2s+1} t \right) + \alpha_0. \]

Then \[ \partial_t F^a = \begin{pmatrix}
\sum_{j=0}^{2s-j} (2s-j)x^{2s-j} y^{j+1} \\
-2\sum_{j=1}^{2s+1} (a)y^{2s+1} \\
0
\end{pmatrix} \] and \( F \) is a \((2s+1)\)-trivial unfolding.

If any of \( \lambda_{0}, \ldots, \lambda_{2s+1} \) are non-zero then there exists a value of \( t \) such that \( \alpha(t) = 0 \). So \((xy + \sum \lambda_j x^{2s+1-j} y^j, x^2 + \alpha y^{2s+1}, y^2)\) is \((2s+1)\)-equivalent to either

\[(xy + \sum \lambda_j x^{2s+1-j} y^j, x^2, y^2)\) or \((xy, x^2 + \alpha y^{2s+1}, y^2)\).

Under the change of coordinates \( x \leftrightarrow y, Y \leftrightarrow Z, (xy, x^2 + \alpha y^{2s+1}, y^2) \) is equivalent to \((xy, x^2, y^2 + \alpha x^{2s+1})\).

On \( \{(xy + \sum \lambda_j x^{2s+1-j} y^j, x^2, y^2)\} \) the remaining terms become

\[ \begin{pmatrix}
\left( \sum_{j=0}^{2s-j} \lambda_j x^{2s+1-j} y^j \right) \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
\left( \sum_{j=1}^{2s+1} \lambda_j y^{2s+1} \right) \\
0 \\
0
\end{pmatrix} \]

Note these are tangent to \((\alpha = 0, \beta = 0)\).

Let \( F_1(x, y, t) = (xy + \sum \lambda_j(t) x^{2s+1-j} y^j, x^2, y^2) \) where \( \lambda_j(t) = \lambda_j e((s-1)t) \)

\( F_2(x, y, t) = (xy + \sum \lambda_j(t) x^{2s+1-j} y^j, x^2, y^2) \) where \( \lambda_j(t) = \lambda_j e(2s-j)t \).

These are both \((2s+1)\)-trivial unfoldings. For any two values \( i, j \) with \( \lambda_i^0 \neq 0, \lambda_j^0 \neq 0 \), we may assume that \( \lambda_i^0 = 1 \) and \( \lambda_j^0 = 1 \).

Similarly, for \((xy, x^2, y^2 + \beta x^{2s+1})\), if \( \beta \neq 0 \) we may assume that \( \beta = 1 \).

The map-germ \((xy, x^2, y^2 + x^{2s+1})\) is clearly not finitely determined since
Thus $f$ is 3-determined if and only if the matrix
\[
\begin{bmatrix}
a & 0 & -c & -2d & 0 & 0 \\
0 & a & 0 & -c & -2d & 0 \\
0 & b & 2c & 3d & 0 & 0 \\
0 & 0 & a & 0 & -c & -2d \\
0 & 0 & b & 2c & 3d & 0 \\
0 & 0 & 0 & b & 2c & 3d \\
\end{bmatrix}
\]
has non-zero determinant and $a, d$ are both non-zero.

The determinant is $ad(4ac^3+4db^3+27a^2d^2-b^2c^2-18abcd)$.

Thus we get

$(xy + x^3 + bx^2y + cxy^2 + y^3, x^2, y^2)$ is 3-determined iff

$$4c^3 + 4b^3 + 27a^2d^2 - b^2c^2 - 18abcd \neq 0,$$

and $(xy + ax^3 + bx^2y + cxy^2, x^2, y^2)$ is not 3-determined.

Note that $(xy + bx^2y + cxy^2 + dy^3, x^2, y^2) \sim (xy + dx^3 + cx^2y + bxy^2, x^2, y^2)$ under the coordinate change $x \leftrightarrow y, Y \leftrightarrow Z$.

Conclusion: $\mathcal{S}^2(xy, x^2, y^2)$ is a union of the following $\mathcal{S}$-invariant strata:

$(xy + x^3 + bx^2y + cxy^2 + y^3, x^2, y^2)$ if $4c^3 + 4b^3 + 27a^2d^2 - b^2c^2 - 18abcd \neq 0$,

$\mathcal{S}^3(xy + x^3 + bx^2y + cxy^2 + y^3, x^2, y^2)$ if $4c^3 + 4b^3 + 27a^2d^2 - b^2c^2 - 18abcd = 0$,

$\mathcal{S}^3(xy + x^3 + bx^2y + cxy^2, x^2, y^2), \quad \mathcal{S}^3(xy + x^3 + bx^2y, x^2, y^2), \quad \mathcal{S}^3(xy + x^3, x^2, y^2), \quad \mathcal{S}^3(xy + x^2y + cxy^2, x^2, y^2), \quad \mathcal{S}^3(xy + x^2y, x^2, y^2), \quad \mathcal{S}^3(xy + x^2y, x^2, y^2), \quad \mathcal{S}^3(xy + x^2y, x^2, y^2), \quad \mathcal{S}^3(xy, x^2, y^2)$,
We note that \((xy + x^3 + x^2y + cxy^2, x^2, y^2)\) has a double curve \((0, ±y)\) → \((0, 0, y^2)\) which does not have transversal type \(A_1\).

\((II-5:4)\) Proposition: if \(j^3f = (xy + x^3 + x^2y + cxy^2, x^2, y^2)\) and \(c(c-4)\neq 0\) then \(f\) lies in one of

\[
\bigcap_{s \geq 0} \mathcal{A}^s(xy + x^3 + x^2y + cxy^2, x^2, y^2)
\]

or \((xy + x^3 + x^2y + cxy^2, x^2 + \alpha y^{2s+1}, y^2)\) for some \(s \geq 2, \alpha \neq 0\).

In the second case \(f\) is \((2s+1)\)-determined. Hence \((xy + x^3 + x^2y + cxy^2, x^2, y^2)\) is a weak stem.

Proof: Let \(\kappa = 3, R = 4\) and \(S = 3\).

Claim: \(T_A^{4,3}f \supseteq m_2^6 \cdot \Theta(f) \setminus \{(0, y^{2j+1}, 0): j \geq 3\}\) and

\(T_A^{2,2}f \supseteq m_2^4 \cdot \Theta(f) \setminus \{(0, y^{2j+1}, 0): j \geq 2\}\).

Then for \(k \geq 4\)

\[j^kf = (xy + x^3 + x^2y + cxy^2, x^2, y^2) \Rightarrow f \in \mathcal{A}^{2k+3}(xy + x^3 + x^2y + cxy^2, x^2 + \alpha y^{2k+1}, y^2) = f_\alpha \text{ say.}
\]

Then

\[T_A^{4,3}f_\alpha + m_2^{2k+4} \cdot \Theta(f) = T_A^{4,3}f + m_2^{2k+4} \cdot \Theta(f) \supseteq m_2^6 \cdot \Theta(f) \setminus \{(0, y^{2j+1}, 0): j \geq 3\}\]

Since \(T_A^{4,3}f_\alpha = f^* m_3 \cdot T_A^{2,2}f_\alpha\) it follows that \(f_\alpha\) is \(2k+1\)-determined if

\[(0, y^{2k+3}, 0) \in T_A^{2,2}f_\alpha + m_2^{2k+4} \cdot \Theta(f)\]

Proof of claim: Note that

\[m_2^6 \cdot \Theta(f) \setminus \{(0, y^{2j+1}, 0): j \geq 3\}\]

is an \(O_3\)-module via \((xy + x^3 + x^2y + cxy^2, x^2, y^2)\) and that

\[f^* m_3 \cdot m_2^2 \cdot \Theta(f) \setminus \{(0, y^{2j+1}, 0): j \geq 3\} = m_2^8 \cdot \Theta(f) \setminus \{(0, y^{2j+1}, 0): j \geq 4\}\]

Therefore \(T_A^{4,3}f \supseteq m_2^6 \cdot \Theta(f) \setminus \{(0, y^{2j+1}, 0): j \geq 3\}\) if and only if
This follows if we show that $T \mathcal{A}_{4.3} f + m_2^8 \theta(f) \supseteq m_2^6 \theta(f) \setminus \{(0, y^{j+1}, 0) : j \geq 3\}$ since

$$m_2^8 \theta(f) \cap m_2^6 \theta(f) \setminus \{(0, y^{j+1}, 0) : j \geq 3\} = m_2^8 \theta(f) \setminus \{(0, y^{j+1}, 0) : j \geq 4\}.$$ 

In $\omega(m_3^3 \theta(3))$ terms involving the square of the parameters are in $m_2^8 \theta(f)$ so we can calculate $T \mathcal{A}_{4.3} f + m_2^8 \theta(f)$ by computer: we find that

$$T \mathcal{A}_{4.3} f + m_2^8 \theta(f) = m_2^8 \theta(f) + C\left\{x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, 16x^7 - cxy^6, 8x^6y + cxy^6, 4x^5y^2 - cxy^6, 2x^4y^3 + cxy^6, x^3y^4 - cxy^6, x^2y^5 + 2cxy^6, 2cy^7 + cxy^6, (3-8c)xy^6 + 2cy^7\right\}e_1 + C\left\{x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, y^6, xy^5 + (1-2c)x^2y^5 + cxy^6, 8x^7 + cxy^6, -4x^6y + cxy^6, 2x^5y^2 + cxy^6, -x^4y^3 + cxy^6, x^3y^4 + 2cxy^6, 16x^6y - cy^7, 8x^5y^2 + cy^7, 4x^4y^3 - cy^7, 2x^3y^4 + cy^7, x^2y^5 + 2cxy^6, x^2y^5 - cy^7, xy^6 + 2cxy^6, x^2y^5 + cy^7\right\}e_3$$

$$+ C\left\{(x^5, 4x, 3y, 2x, 2y, 0, 0, 0, 0, 0), (x^4, x, y, 0, 0, 0, 0), (x^3, 2x, y, 2y, 0, 0), (x^2, 3x, y, 2x, 4y, 0, 0), (x, 4, y, 0, 0), (y, 5, 0, 0)\right\} + C\left\{(2xy^6 - cy^7, -16cy^6 - cxy^6, 4xy^6, -cy^7, 7y, 0, 2xy^6), (3x^2y^6, 16x^7 + 16cy^6, 3y^7, 0)\right\}.$$ 

This contains $m_2^6 \theta(f) \setminus \{(0, y^{j+1}, 0) : j \geq 3\}$ provided that $c(c-4) \neq 0$; the relevant terms are underlined in red.
We obtain $T\varpi_{2,2}f$ in a similar way (by calculating $T\varpi_{2,2}f + m_2^6 \cdot \theta(f)$ by computer).

We next show that $(0, y^{2k+3}, 0) \in T\varpi_{2,2}f + m_2^{2k+4} \cdot \theta(f)$, recalling that

$$m_2^6 \cdot \theta(f) \setminus \{(0, y^{2j+1}, 0): j \geq 3\} \subseteq T\varpi_{2,2}f.$$  

The important terms are:

$$C\{x^4, x^3y, y^4, x^2y^2 + \alpha y^{2k+3}, x^2y^2 + 2x^4y + 2x^3y^2 + 2x^2y^3, xy^3 + x^3y^2 + x^2y^3 + cxy^4\} \cdot e_1$$

$$+ C\left\{\begin{pmatrix} x^3 + 3x^2y^2 + 2x^4y + cx^3y^2 & x^3 + 2x^2y^3 + 2cy^4 \\ 2x^2y & (2k+1)x^3y \\ 0 & 0 \end{pmatrix}\right\}$$

Since $C\{3x^5 + 2x^4y + cx^3y^2, 3x^4y + 2x^3y^2 + cx^2y^3, x^5 + 2cx^4y, x^4y + 2cx^3y^2\} \cdot e_1$

$$= C\{x^5, x^4y, x^3y^2, x^2y^3\} \cdot e_1$$

we see that $x^2y^2 \cdot e_2 \in T\varpi_{2,2}f + m_2^{2k+4} \cdot \theta(f)$ and hence if $\alpha \neq 0$,

$$(0, y^{2k+3}, 0) \in T\varpi_{2,2}f + m_2^{2k+4} \cdot \theta(f)$$

as required.

This concludes the proof of the proposition.

We perform a similar analysis for $(xy, x^2 + y^2, 0)$.

(II-5:5) Let $f = (xy, x^2 + y^2, 0)$. Then $T\varpi_{2,2}f =$

$$\begin{cases} \{x^{2l-1-j}y^j: \ell \geq 2, 0 \leq j \leq 2\ell\} \cup \{x^{2t+1-2j}y^{2j} - xy^{2t}, x^{2t-2j}y^{2j+1} - y^{2t+1}: \ell \geq 0, 0 \leq j \leq \ell\} \\
\{x^{2l-1-j}y^j: \ell \geq 2, 0 \leq j \leq 2\ell\} \cup \{x^{2t+1-2j}y^{2j} - xy^{2t}, x^{2t-2j}y^{2j+1} - y^{2t+1}: \ell \geq 0, 0 \leq j \leq \ell\} \\
\{x^{2t-j}y^j + xy^j: \ell \geq 2, 0 \leq j \leq 2\ell\} \end{cases}$$
For $s \geq 1$,
\[
g \in \mathcal{S}_{2s} \Rightarrow g \in \mathcal{S}_{2s+1}(xy, x^2 + y^2 + \alpha xy + \beta y_{2s+1}, \Sigma \lambda_j x^{2s+1-j} y^j)
\]
and $g \in \mathcal{S}_{2s+1} \Rightarrow g \in \mathcal{S}_{2s+2}(xy, x^2 + y^2, \Sigma \lambda_j x^{2s+1-j} y_{s+1+j})$.

For $f = (xy, x^2 + y^2 + \alpha xy + \beta y_{2s+1}, \Sigma \lambda_j x^{2s+1-j} y^j)$, $T_{\mathcal{S}_{2s+1}} f =$
\[
\begin{pmatrix}
\mathcal{S}_{2s+1} f \\
\end{pmatrix}
\]
We see that the following are in $T\mathcal{L}^{2s+1}f$: 

\[
\begin{pmatrix}
0 \\
\lambda_0+\lambda_2+...+\lambda_{2s}xy^{2s}+(\lambda_1+\lambda_3+...+\lambda_{2s+1})y^{2s+1} \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
2x^2+2y^2+(2s+1)axy^{2s}+(2s+1)by^{2s+1} \\
\Sigma_l(2s+1)x^{2s+1-j}y^l
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 \\
\lambda_0+\lambda_2+...+\lambda_{2s}xy^{2s}+(\lambda_1+\lambda_3+...+\lambda_{2s+1})y^{2s+1} \\
0
\end{pmatrix}
\begin{pmatrix}
2sax^2y^{2s-1}+(2s+1)byx^{2s}+\alpha y^{2s+1} \\
\Sigma\left(\lambda_j(2s+1-j)+\lambda_{j+2}(j+2)\right)x^{2s-j}y^{j+1}
\end{pmatrix},
\]

These give

\[
\begin{pmatrix}
0 \\
\lambda_0+\lambda_2+...+\lambda_{2s}xy^{2s}+(\lambda_1+\lambda_3+...+\lambda_{2s+1})y^{2s+1} \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
\alpha xy^s+\beta y^{2s+1} \\
0
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 \\
\lambda_1+\lambda_3+...+\lambda_{2s+1}xy^{2s}+(\lambda_0+\lambda_2+...+\lambda_{2s})y^{2s+1} \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
(2s+3)byx^{2s}+(2s+3)\alpha y^{2s+1} \\
\Sigma\left(\lambda_j(2s+1-j)+\lambda_{j+2}(j+2)\right)x^{2s-j}y^{j+1}
\end{pmatrix}.
\]

from which to construct our $(2s+1)$-trivial unfoldings.

Let $A = (\lambda_0+\lambda_2+...+\lambda_{2s})$, $B = (\lambda_1+\lambda_3+...+\lambda_{2s+1})$.

a) If $A^2-B^2 \neq 0$ then

\[
T\mathcal{L}^{2s+1}f \ni \begin{pmatrix} y^{2s+1} \\ xy^{2s} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \text{It is clear that the codimension of } T\mathcal{L}^{2s+1}f \text{ is independent of } \alpha \text{ and } \beta, \text{ so without loss of generality, } \alpha = \beta = 0.
\]

The remaining terms to consider are

\[
\begin{pmatrix}
0 \\
\Sigma\lambda_jx^{2s+1-j}y^l \\
\lambda_1x^{2s+1}+\Sigma(\lambda_j(2s+1-j)+\lambda_{j+2}(j+2))x^{2s-j}y^{j+1}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

For the second term, we try to solve the equations
\[
\frac{d}{dt} \lambda_0 = \lambda_1, \quad \frac{d}{dt} \lambda_1 = 2\lambda_2 + (2s+1)\lambda_0, \quad \frac{d}{dt} \lambda_2 = 3\lambda_3 + 2s\lambda_1.
\]

\[
\frac{d}{dt} \lambda_3 = 4\lambda_4 + (2s-1)\lambda_2, \quad \frac{d}{dt} \lambda_j = (j+1)\lambda_{j+1} + (2s+2-j)\lambda_{j-1}, \quad \frac{d}{dt} \lambda_{2s+1} = \lambda_{2s}.
\]

Note that in this system

\[
\frac{d}{dt} \lambda_0 + \frac{d}{dt} \lambda_1 + \cdots + \frac{d}{dt} \lambda_{2s} = (2s+2)(\lambda_1 + \lambda_3 + \cdots + \lambda_{2s+1})
\]

and

\[
\frac{d}{dt} \lambda_1 + \frac{d}{dt} \lambda_3 + \cdots + \frac{d}{dt} \lambda_{2s+1} = (2s+2)(\lambda_0 + \lambda_2 + \cdots + \lambda_{2s})
\]

i.e. \( \frac{d}{dt} A = (2s+2)B \) and \( \frac{d}{dt} B = (2s+2)A \).

So \( \frac{d^2}{dt^2} A - (2s+2)^2 A = 0 \) and the solution is

\[
A(t) = A_0 e^{(2s+2)t} + A_1 e^{-(2s+2)t}, \quad B(t) = (2s+2)A_0 e^{(2s+2)t} - (2s+2)A_1 e^{-(2s+2)t}
\]

for some constants \( A_0, A_1 \).

We conclude that the unfolding

\[
F(x, y, t) = (xy, x^2 + y^2, \sum \lambda_j(t)x^{2s+1-j}y^j),
\]

where

\[
(\lambda_0(t) + \lambda_2(t) + \cdots + \lambda_{2s}(t)) = A_0 e^{(2s+2)t} + A_1 e^{-(2s+2)t}
\]

and

\[
(\lambda_1(t) + \lambda_3(t) + \cdots + \lambda_{2s+1}(t)) = (2s+2)A_0 e^{(2s+2)t} - (2s+2)A_1 e^{-(2s+2)t}
\]

is \((2s+1)\)-trivial.

Further, if \( A_0 \neq 0 \) and \( A_1 \neq 0 \) then there exists a value of \( t \) for which

\[
\lambda_0(t) + \lambda_2(t) + \cdots + \lambda_{2s}(t) = 0.
\]

The condition \( A_0 = 0 \) or \( A_1 = 0 \) is equivalent to \( A^2 - B^2 = 0 \).

The remaining term

\[
\begin{pmatrix}
0 \\
\sum \lambda_j x^{2s+1-j}y^j
\end{pmatrix}
\]

implies that any non-zero value of \( \lambda_j \) may be assumed to be equal to 1.

Hence if \((xy, x^2 + y^2 + \alpha xy^{2s} + \beta y^{2s+1}, \sum \lambda_j x^{2s+1-j}y^j)\) satisfies \( A^2 - B^2 \neq 0 \) then it is
equivalent to a map-germ of the form

\[(xy, x^2 + y^2, \sum \lambda_jx^{2s+1-jy})\]

where \(\lambda_0 + \lambda_2 + \ldots + \lambda_{2s} = 0\), \(\lambda_1 + \lambda_3 + \ldots + \lambda_{2s+1} \neq 0\) and some non-zero \(\lambda_j\) may be assumed to be equal to 1. For example this gives the three jets

\[(xy, x^2 + y^2, x^3 + \lambda_1x^2y - xy^2 + \lambda_3y^3)\text{ if } \lambda_0 \neq 0, \lambda_1 + \lambda_3 \neq 0\]

or \((xy, x^2 + y^2, x^2y + \lambda_3y^3)\text{ if } \lambda_0 = 0, \lambda_3 \neq -1, \text{ or } (xy, x^2 + y^2, y^3)\).

b) If \(A^2 - B^2 = 0\) but \(A \neq 0\) then under the transformation \(y \leftrightarrow -y\),

\[(xy, x^2 + y^2 + \alpha xy^{2s} + \beta y^{2s+1}, \sum \lambda_jx^{2s+1-jy})\]

\[\leftrightarrow (-xy, x^2 + y^2 + \alpha xy^{2s} - \beta y^{2s+1}, \sum (-1)^j\lambda_jx^{2s+1-jy})\]

so \(A \leftrightarrow A\) and \(B \leftrightarrow -B\) so without loss of generality, we assume that \(A = B\).

Consideration of

\[
\begin{pmatrix}
0 \\
\alpha xy^{2s} + \beta y^{2s+1} \\
0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 \\
Axy^{2s} + Ay^{2s+1} \\
0
\end{pmatrix}
\]

shows that if \(\alpha \neq \beta\) then

\[T \mathcal{A}^{2s+1} \ni \begin{pmatrix}
0 \\
\alpha xy^{2s} + \beta y^{2s+1} \\
0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 \\
x^{2s+1} - xy^{2s} \\
0
\end{pmatrix}
\text{ or }
\begin{pmatrix}
0 \\
\alpha xy^{2s} \\
0
\end{pmatrix}
\text{ or }
\begin{pmatrix}
0 \\
Axy^{2s} \\
0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 \\
\beta y^{2s+1} \\
0
\end{pmatrix}
\]

so that without loss of generality, \(\alpha = 1\) and \(\beta = 0\) or \(\alpha = 0\) and \(\beta = 1\).

Suppose that \(\alpha = 1\) and \(\beta = 0\). Since

\[\begin{pmatrix}
0 \\
x^{2s+1} - xy^{2s} \\
0
\end{pmatrix} \in T \mathcal{A}^{2s+1}
\]

the unfolding \((xy, x^2 + y^2 + xy^{2s} + ax^{2s+1} - xy^{2s}, \sum \lambda_jx^{2s+1-jy})\) is \((2s+1)\)-trivial (lemma I.2.7). Hence

\[(xy, x^2 + y^2 + \alpha xy^{2s}, \sum \lambda_jx^{2s+1-jy})\text{ is } (2s+1)\)-equivalent to

\[(xy, x^2 + y^2 + x^{2s+1}, \sum \lambda_jx^{2s+1-jy})\]

Now apply the transformation \(x \leftrightarrow y\) to see that without loss of generality we may assume that \(\alpha = 0\) and \(\beta = 1\).
If $\alpha = \beta$ then \[ \{(xy, x^2 + y^2 + \alpha xy y^2s + \alpha y y^2s+1, \Sigma \lambda_j x^{2s+1-j}y^j) : \alpha \in \mathbb{C} \} \] lies in a single orbit, and without loss of generality, $\alpha = 0$.

In either case we are left with the terms

\[
\begin{pmatrix}
0 \\
0 \\
\Sigma \lambda_j x^{2s+1-j}y^j
\end{pmatrix}
\] and

\[
\begin{pmatrix}
0 \\
0 \\
\lambda_1 x^{2s+1} + \Sigma \left( \lambda_j (2s+1-j) + \lambda_{j+2}(j+2) \right) x^{2s-j}y^{j+1}
\end{pmatrix}
\]

For the second term, we again try to solve the equations

\[
\frac{d}{dt} \lambda_0 = \lambda_1, \quad \frac{d}{dt} \lambda_1 = 2\lambda_2 + (2s+1)\lambda_0, \quad \frac{d}{dt} \lambda_2 = 3\lambda_3 + 2s\lambda_1,
\]

\[
\frac{d}{dt} \lambda_3 = 4\lambda_4 + (2s-1)\lambda_2, ..., \quad \frac{d}{dt} \lambda_j = (j+1)\lambda_{j+1} + (2s+2-j)\lambda_{j-1}, ..., \quad \frac{d}{dt} \lambda_{2s+1} = \lambda_{2s}.
\]

Now we have

\[
\frac{d}{dt} \lambda_0 + \frac{d}{dt} \lambda_2 + ... + \frac{d}{dt} \lambda_{2s} = (2s+2)(\lambda_1 + \lambda_3 + ... + \lambda_{2s+1}) = (2s+2)(\lambda_0 + \lambda_2 + ... + \lambda_{2s})
\]

and

\[
\frac{d}{dt} \lambda_1 + \frac{d}{dt} \lambda_3 + ... + \frac{d}{dt} \lambda_{2s+1} = (2s+2)(\lambda_0 + \lambda_2 + ... + \lambda_{2s}) = (2s+2)(\lambda_1 + \lambda_3 + ... + \lambda_{2s+1})
\]

i.e. $\frac{d}{dt} A = (2s+2)A$.

The solution is $A(t) = A_0 e^{(2s+2)t}$ for some constant $A_0$ (which, by hypothesis, is non-zero). There exists a solution to $A(t) = 1$.

The remaining term

\[
\begin{pmatrix}
0 \\
0 \\
\Sigma \lambda_j x^{2s+1-j}y^j
\end{pmatrix}
\]

implies that any non-zero value of $\lambda_j$ may be assumed to be equal to 1.

Hence if $(xy, x^2 + y^2 + \alpha xy y^2s + \beta y y^2s+1, \Sigma \lambda_j x^{2s+1-j}y^j)$ satisfies $A^2 - B^2 = 0$ then it is equivalent to a map-germ of the form

$$(xy, x^2 + y^2, \Sigma \lambda_j x^{2s+1-j}y^j) \text{ or } (xy, x^2 + y^2 + y 2s+1, \Sigma \lambda_j x^{2s+1-j}y^j)$$

where $\lambda_0 + \lambda_2 + ... + \lambda_{2s} = \lambda_1 + \lambda_3 + ... + \lambda_{2s+1} = 1$ and some non-zero $\lambda_j$ may be assumed to
(0, y) \rightarrow (0, 0, y^2)

is a double curve with non-transverse self-intersection:
\[ \frac{\partial f}{\partial x} = (y, 2x, (2s+1)x^{2s}) \text{ and } \frac{\partial f}{\partial y} = (x, 0, 2y). \]

Consequently, we have
\[ j^2f = (xy, x^2, y^2) \Rightarrow \]
\[ j^3f \sim (xy + \lambda_0x^3 + \lambda_1x^2y + \lambda_2xy^2 + \lambda_3y^3, x^2, y^2) \text{ or } (xy, x^2, y^2 + x^3), \]
\[ j^3f = (xy, x^2, y^2) \Rightarrow \]
\[ j^3f \sim (xy + \lambda_0x^5 + \lambda_1x^4y + \lambda_2x^3y^2 + \lambda_3x^2y^3 + \lambda_4xy^4 + \lambda_5y^5, x^2, y^2) \]
\[ \text{ or } (xy, x^2, y^2 + x^5) \]
\[ j^3f = (xy, x^2, y^2) \Rightarrow \]
\[ j^3f \sim (xy + \lambda_0x^7 + \lambda_1x^6y + \lambda_2x^5y^2 + \lambda_3x^4y^3 + \lambda_4x^3y^4 + \lambda_5x^2y^5 + \lambda_6xy^6 + \lambda_7y^7, x^2, y^2) \]
\[ \text{ or } (xy, x^2, y^2 + x^7), \]

etc.

We find conditions for \( f \) to be 3-determined. This is equivalent to finding conditions

for \( m^4_{\text{for}}(f) \subseteq T \mathcal{S}_{2,2}f \), which holds if and only if

\[ m^4_{\text{for}}(f) \subseteq T \mathcal{S}_{2,2}f + m^6_{\text{for}}(f). \]

Calculation using the program gives

\[ T \mathcal{S}_{2,2}f + m^6_{\text{for}}(f) = m^4_{\text{for}}(f) + \mathcal{C}[ x^4, x^3y, x^2y^2, xy^3, y^4, ax^5-cx^3y^2-2dx^2y^3, \]
\[ -ax^4y + cx^2y^3 + 2dxy^4, bx^4y + 2cx^3y^2 + 3dx^2y^3, ax^3y^2-cxy^4-2dy^5, \]
\[ bx^3y^2 + 2cx^2y^3 + 3dxy^4, bx^2y^2 + 2cxy^4 + 3dy^5 ] \cdot e_1 \]
\[ + \mathcal{C}[ x^4, x^2y^2, y^4, x^3y + ax^5 + bx^4y + cx^3y^2 + dx^2y^3, 4xy^3 + bx^2y^3 + 2cxy^4 + 3dy^5, \]
\[ ax^4y + bx^3y^2 + cx^2y^3 + dxy^4, 4ax^3y^2 + 2bx^2y^3 + 2cxy^4 + dy^5 ] \cdot e_2 \]
\[ + \mathcal{C}[ x^4, x^2y^2, y^4, x^3y - 4bx^4y - 2cx^3y^2 - 3dx^2y^3, xy^3 + ax^3y^2 + bx^2y^3 + cxy^4 + dy^5, \]
\[ ax^4y + bx^3y^2 + cx^2y^3 + dxy^4, ax^5 + 2bx^4y + 3cx^3y^2 + 4dx^2y^3 ] \cdot e_3 \]
be equal to 1. For example, this gives the three jets

\[(xy, x^2 + y^2, x^3 + \lambda_1 x^2 y + (1-\lambda_1) y^3)\] or \[(xy, x^2 + y^2 + y^3, x^3 + \lambda_1 x^2 y + (1-\lambda_1) y^3)\]

if \( \lambda_0 \neq 0, \)

\[(xy, x^2 + y^2 + xy^2 + (1-\lambda_1) y^3)\] or \[(xy, x^2 + y^2 + y^3 \lambda_1 x^2 y + xy^2 + (1-\lambda_1) y^3)\]

if \( \lambda_0 = 0. \)

c) If \( A = 0 = B \) we consider

\[\begin{pmatrix}
0 & 0 \\
\alpha xy 2s + \beta y 2s+1 & (2s+3)\beta xy 2s + (2s+3)\alpha y 2s+1 \\
\Sigma (\lambda_j (2s+1-j) + \lambda_{j+2} (j+2)) x^{2s-j-j+1} & 0
\end{pmatrix}\]

For the second term, we try to solve the equations

\[\frac{d}{dt} \alpha = (2s+3)\beta, \quad \frac{d}{dt} \beta = (2s+3)\alpha, \quad \frac{d}{dt} \lambda_0 = \lambda_1, \quad \frac{d}{dt} \lambda_1 = 2\lambda_2 + (2s+1)\lambda_0, \quad \frac{d}{dt} \lambda_2 = 3\lambda_3 + 2s\lambda_1, \]

\[\frac{d}{dt} \lambda_3 = 4\lambda_4 + (2s-1)\lambda_2, \ldots, \quad \frac{d}{dt} \lambda_j = (j+1)\lambda_{j+1} + (2s+2-j)\lambda_{j-1}, \ldots, \quad \frac{d}{dt} \lambda_{2s+1} = \lambda_{2s}. \]

Now we have \(\frac{d^2}{dt^2} \alpha - (2s+2)^2 \alpha = 0\)

\[\frac{d}{dt} \lambda_0 + \frac{d}{dt} \lambda_2 + \ldots + \frac{d}{dt} \lambda_{2s} = (2s+2)(\lambda_1 + \lambda_3 + \ldots + \lambda_{2s+1}) = 0\]

and \(\frac{d}{dt} \lambda_1 + \frac{d}{dt} \lambda_3 + \ldots + \frac{d}{dt} \lambda_{2s+1} = (2s+2)(\lambda_0 + \lambda_2 + \ldots + \lambda_{2s}) = 0\)

and the solution is

\[\alpha(t) = \alpha_0 e^{(2s+2)t} + \alpha_1 e^{-(2s+2)t}, \quad \beta(t) = (2s+2)\alpha_0 e^{(2s+2)t} - (2s+2)\alpha_1 e^{-(2s+2)t}, \]

\[A = B = 0\]

for some constants \(\alpha_0, \alpha_1.\)

If \(\alpha_0 \neq \alpha_1\) then there exists a solution to \(\alpha(t) = 0,\) and
\[
T \mathcal{A}^{2s+1} \ni \begin{pmatrix}
0 \\
\beta y^{2s+1} \\
0
\end{pmatrix}
\]

so without loss of generality, \( \beta = 1 \).

Otherwise \( \alpha = \pm \beta \) and there either exists a solution to \( \alpha(t) = 1 \) or \( \alpha_0 = \alpha_1 = 0 \).

Hence, if \((xy, x^2 + y^2 + \alpha xy^{2s} + \beta y^{2s+1}, \Sigma \lambda_j x^{2s+1-ij})\) satisfies \( A = B = 0 \) then it is equivalent to a map-germ of the form

\[
(xy, x^2 + y^2, \Sigma \lambda_j x^{2s+1-ij}) \text{ or } (xy, x^2 + y^2 + y^{2s+1}, \Sigma \lambda_j x^{2s+1-ij})
\]

or \((xy, x^2 + y^2 + xy^2 + y^{2s+1}, \Sigma \lambda_j x^{2s+1-ij})\)

where \( \lambda_0 + \lambda_2 + \cdots + \lambda_{2s} = \lambda_1 + \lambda_3 + \cdots + \lambda_{2s+1} = 0 \).

The remaining term \( \begin{pmatrix}
0 \\
0 \\
\Sigma \lambda_j x^{2s+1-ij}
y
\end{pmatrix} \) implies that any non-zero value of \( \lambda_j \) may be assumed to be equal to 1.

We conclude that the orbits are:

\[\mathcal{A}^{2s+1}(xy, x^2 + y^2, \Sigma \lambda_j x^{2s+1-ij}) : \lambda_0 + \lambda_2 + \cdots + \lambda_{2s} = 0, \lambda_1 + \lambda_3 + \cdots + \lambda_{2s+1} \neq 0, \text{ some } \lambda_j = 1,\]

\[\mathcal{A}^{2s+1}(xy, x^2 + y^2, \Sigma \lambda_j x^{2s+1-ij}) : \lambda_0 + \lambda_2 + \cdots + \lambda_{2s} = \lambda_1 + \lambda_3 + \cdots + \lambda_{2s+1} = 1, \text{ some } \lambda_j = 1,\]

\[\mathcal{A}^{2s+1}(xy, x^2 + y^2 + x^2y + y^{2s+1}, \Sigma \lambda_j x^{2s+1-ij}) : \lambda_0 + \lambda_2 + \cdots + \lambda_{2s} = \lambda_1 + \lambda_3 + \cdots + \lambda_{2s+1} = 0, \text{ some } \lambda_j = 1,\]

\[\mathcal{A}^{2s+1}(xy, x^2 + y^2 + xy^2 + y^{2s+1}, \Sigma \lambda_j x^{2s+1-ij}) : \lambda_0 + \lambda_2 + \cdots + \lambda_{2s} = \lambda_1 + \lambda_3 + \cdots + \lambda_{2s+1} = 0, \text{ some } \lambda_j = 1.\]

In particular the three jets are

\((xy, x^2 + y^2, x^3 + \lambda_1 x^2 y - xy^2 + \lambda_3 y^3) : \lambda_1 + \lambda_3 \neq 0, \quad (xy, x^2 + y^2, x^2 y + \lambda_3 y^3) : \lambda_3 \neq -1,\)

\((xy, x^2 + y^2, y^3) \quad \text{ and } \quad (xy, x^2 + y^2, x^3 + \lambda_1 x^2 y + (1-\lambda_1)y^3)\)
\[(xy, x^2 + y^2 + y^3, x^3 + \lambda_1 x^2 y + (1-\lambda_1)y^3)\quad (xy, x^2 + y^2, \lambda_1 x^2 y + xy^2 + (1-\lambda_1)y^3)\]
\[(xy, x^2 + y^2 + y^3, \lambda_1 x^2 y + xy^2 + (1-\lambda_1)y^3)\quad (xy, x^2 + y^2, x^3 + \lambda_1 x^2 y -xy^2 - \lambda_1 y^3)\]
\[(xy, x^2 + y^2, x^2 y -y^3)\quad (xy, x^2 + y^2, y^3)\]
\[(xy, x^2 + y^2 + y^3, \lambda_1 x^2 y -xy^2 - \lambda_1 y^3)\quad (xy, x^2 + y^2 + y^3, x^3 + \lambda_1 x^2 y -xy^2 - \lambda_1 y^3)\]
\[(xy, x^2 + y^2 + y^3, 0)\quad (xy, x^2 + y^2 + y^3, x^3 + \lambda_1 x^2 y -xy^2 - \lambda_1 y^3)\]
\[(xy, x^2 + y^2 + xy^2 + y^3, x^2 y -y^3)\quad (xy, x^2 + y^2 + xy^2 + y^3, 0)\].

None of these are 3-determined, though it is to be expected that many (particularly in the families defined by parameters) are finitely determined.
Chapter III: Multiplicity, Geometry and Stems.

Introduction.

If \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) is a finite map-germ and \( \Sigma = \text{Supp}(\theta(f)/\mathcal{I}\mathcal{S} f) \) is 1-dimensional then \( f \) is not finitely determined and \( \Sigma \) contains at least one of a cuspidal edge, a double curve with transversal type \( A_{2n+3} (n \geq 0) \) or a curve of \( m-1 \) points for some \( m \geq 3 \) [Mond 2]. The simplest cases are the transversal types \( A_2, A_3 \) and \( D_4 \).

Definition. The curve \( \Sigma_1 \subseteq \Sigma \) is of \textit{stem type} if it is one of

i) a curve of triple points with transversal type \( D_4 \),

ii) a double curve with transversal type \( A_3 \),

iii) a cuspidal edge with transversal type \( A_2 \).

Definition. The map-germ \( f \) is a \textit{geometric stem} if \( \Sigma \) is an irreducible curve of stem type.

D. Mond put forward the conjecture for \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \):

\[ f \text{ is a stem } \iff f \text{ is a geometric stem.} \]

In III.1 we relate the multiplicity to the transversal type; in particular we find

\[ f \text{ is a geometric stem } \iff \Sigma \text{ is irreducible and } e = \mu_0(\Sigma). \]

Mond's conjecture then becomes

\[ f \text{ is a stem } \iff \Sigma \text{ is irreducible and } e = \mu_0(\Sigma). \]

This means that we can apply the theory in Chapter I to the problem; in particular

\[ f \text{ is a stem } \iff \Sigma \text{ is irreducible and } e = \mu_0(\Sigma) \]

is the conjecture proposed in I.3.

We also find direct relationships between the multiplicity and the transversal types \( A_n \) \((n \geq 2)\). At the end of the section we calculate the transversal types for the map-germs listed in Tables 2-5 of Chapter II.
In III-2 we investigate the conjecture

\[ f \text{ is a stem} \Rightarrow f \text{ is a geometric stem.} \]

In particular we show that if \( f \) is a stem then it is finite and \( \Sigma \) is irreducible and is one of

a) a 1-1 cuspidal edge,

b) a double curve with transversal type \( A_{2n+3} \) for some \( n \geq 0 \),

c) an \( m-1 \) curve (\( m \geq 3 \)) with transversal type consisting of immersed sheets intersecting pairwise transversely.

Further reduction of this list is a future research problem. The third case is particularly interesting in that it has proved difficult to determine whether or not the conjecture holds for some simple cases.

In III-3 we look at the geometry of the families for some weak stems. We calculate some invariants for each map-germ in the family; these invariants are the numbers \( C, T \) and \( \mu(\hat{D}_2(f)/\mathbb{Z}_2) \) which are defined in [Mond 2] and [M-M].
III.1: Geometry and Multiplicity.

In this section we shall try to see how the multiplicity of $\theta(f)/T\mathscr{X}f$ and the transversal types of the branches of $\Sigma$ are related. In this section denote $\theta(f)/T\mathscr{X}f$ by $N\mathscr{X}f$.

From the multiplicity theory (I.1:15) we know that if $\Sigma = \Sigma_1 \cup ... \cup \Sigma_r$ then

$$e(N\mathscr{X}f) = \Sigma \mu_0(\Sigma) \cdot l(N\mathscr{X}f_{\Sigma_i})$$

We shall look at how $l(N\mathscr{X}f_{\Sigma_i})$ is related to the transversal type of $\Sigma_i$. In particular we shall identify those cases for which $l(N\mathscr{X}f_{\Sigma_i}) = 1$. To do this we need to look at multi-germs.

In general if $f : \mathbb{C}^2, p_1, p_2, ..., p_l \rightarrow \mathbb{C}^3, y$ is a multi-germ then

$$e(N\mathscr{X}f_y) = \Sigma \mu_y(\Sigma) \cdot l(N\mathscr{X}f_{\Sigma_i})$$

The number $l(N\mathscr{X}f_{\Sigma_i})$ depends only on $\Sigma_i$ and not on the point $y \in \Sigma_i$ where $N\mathscr{X}f_y$ is calculated so we can calculate $l(N\mathscr{X}f_{\Sigma_i})$ at any point of $\Sigma_i$ and in particular we may assume that

i) $\Sigma = \Sigma_i$ is smooth and irreducible at $y = 0$ (i.e. $\mu_0(\Sigma) = 1$

and ii) no double curves (of transversal type $A_1$) intersect $\Sigma$ at 0.

Then $l(N\mathscr{X}f_{\Sigma_i}) = e(N\mathscr{X}f_0)$.

These assumptions make it much easier to find normal forms for map-germs having a particular transversal type along $\Sigma$ and hence to estimate the multiplicity.

The main result here is:

(III.1:1) Theorem: $l(N\mathscr{X}f_{\Sigma_i}) = 1 \iff f$ is of stem type along $\Sigma_i$.

Consequently $f$ is a geometric stem if and only if $\Sigma$ is irreducible and

$$l(N\mathscr{X}f_{\Sigma}) = 1$$

equivalently $e = \mu_0(\Sigma)$.

We prove this in a series of lemmas which give precise values for $l$ for transversal types $A_k$ and minimum values for the remaining types.
If $f$ is a multi-germ, we choose local coordinates $(x_i, y_i)$ at $p_i = (0,0)$ for each $i$ and give the local maps $f_i : (\mathbb{C}^2, p_i) \to (\mathbb{C}^3,0)$. The coordinates on $\mathbb{C}^3$ are $(X, Y, Z)$.

The first observation to make is

(III·1:2) Lemma: if $t \geq 2$ and we consider $f$ as two maps

$$ f_1 : (\mathbb{C}^2, p_1, ..., p_t) \to (\mathbb{C}^3,0) \quad \text{and} \quad f_2 : (\mathbb{C}^2, p_{t+1}, ..., p_t) \to (\mathbb{C}^3,0) \quad (1 \leq i < t) $$

then

$$ e(N_{\mathbb{C}^2} f_0) \geq e(N_{\mathbb{C}^2} f_{1,0}) + e(N_{\mathbb{C}^2} f_{2,0}). $$

Proof. This is clear since $T_{\mathbb{C}^2} f_0 \subseteq T_{\mathbb{C}^2} f_{1,0} \oplus T_{\mathbb{C}^2} f_{2,0}$.

We conjecture that

$f_1$ and $f_2$ intersect transversely if and only if $e(N_{\mathbb{C}^2} f_0) = e(N_{\mathbb{C}^2} f_{1,0}) + e(N_{\mathbb{C}^2} f_{2,0})$.

(III·1:3) Case 1: $t=1$.

In this case $f$ must have a cuspidal edge. We choose coordinates $(x, y)$ on $(\mathbb{C}^2, 0)$ and $(X, Y, Z)$ on $(\mathbb{C}^3,0)$ so that the cuspidal edge $\Sigma$ is given by $Y = 0, Z = 0$ and $f^t(\Sigma)$ is given by $y = 0$.

Suppose $f(x, y) = (a(x, y), b(x, y), c(x, y))$. Then $b(x, 0) = c(x, 0) = 0$. Putting $b = yb'$ and $c = yc'$ we have

$$ \frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial a}{\partial x} \\ \frac{\partial b'}{\partial x} \\ \frac{\partial c'}{\partial x} \end{pmatrix}, \quad \frac{\partial f}{\partial y} = \begin{pmatrix} \frac{\partial a}{\partial y} \\ b' + y \frac{\partial b'}{\partial y} \\ c' + y \frac{\partial c'}{\partial y} \end{pmatrix}. $$

It follows that $b'(x,0) = c'(x,0) = 0$ and that $\partial_2 a(0,0) \neq 0$ so replacing $x$ by $x' = a(x, y)$ we may assume that $f$ is of the form $(x, y^2b(x,y), y^2c(x,y))$.

Case 1: If $b \notin m_2$ then we may assume $c \in m_2$ (subtract $b$ if it is not). Then $j^2f = (x, y^2, 0)$ and by II·2:1 $f$ is of the form $(x, y^2, y^3q(x, y^2))$. It also follows that $\Sigma$ has transversal type $A_{2n}$ for some $n \geq 1$.

i) If $q \notin m_2$ then $f$ is isomorphic to $(x, y^2, y^3)$ and $e(N_{\mathbb{C}^2} f_0) = 1$. 
This follows from II·2:9.

ii) If \( q \) is of the form \( y^k \phi \), for some \( k \geq 0 \) and some \( \phi \in \mathfrak{m}_2 \), then \( q(x,y) = 0 \) is a double curve intersecting \( \Sigma \) at \((0,0,0)\); this contradicts the choice of \( o \in \Sigma \).

iii) If \( y^3 q = y^{2n+1} \phi(x, y^2) \) and \( \phi \notin \mathfrak{m}_2 \) then by II·2:5
\[
<xy^n \partial_x \phi, y^{n+1} \partial_x \phi, y^{n+1} \partial_y \phi + ny^n \phi, y^n \phi>_{\mathfrak{m}_2} = y^n_{\mathfrak{o}_2}
\]
and \( T \mathcal{A}(x, y^2, y^{2n+1} \phi(x, y^2)) = \left( \begin{array}{c} \mathfrak{m}_2 \\ \mathfrak{m}_2 \setminus \{y\} \\ 2n \\ y \setminus \{y: j \leq n\} \end{array} \right) \)

It is clear that \( T \mathcal{A} \neq \begin{pmatrix} 0 \\ 0 \\ s \\ x \end{pmatrix} \) for \( r \leq n-1 \) and hence \( e(N \mathcal{A}f_0) = n \).

We note that for \((x, y^2, y^{2n+1} \phi(x, y^2))\), the image is defined by the equation
\[ Z^2 - Y^{2n+1} \phi^2(X, Y) = 0. \]
Since \( \phi \notin \mathfrak{m}_2 \) the transversal type of \( \Sigma = V(Y, Z) \) is that of \( Z^2 - Y^{2n+1} = 0 \), i.e. in this case \( \Sigma \) has transversal type \( A_{2n} \).

Case 2: if \( b, c \in \mathfrak{m}_2 \) but \( b(x,0) \neq 0 \) then write \( b(x,y) = p(x) + yq(x,y) \). Consider the map at a point \((e,0)\) on \( f^*(\Sigma) \). Making the change of coordinates \( x \to x + e, y \to y \) we get
\[
( x, y^2 p(x + e) + y^3 q(x + e, y), y^2 c(x + e, y))
\]
which has \( f^2 = (x, y^2, 0) \) and behaves as case 1. Since 0 was assumed to be a generic point of \( \Sigma \) it must also have this property, i.e. \( f \) has \( e(N \mathcal{A}f_0) = n \) if and only if it has transversal type \( A_{2n} \).

Case 3: if \( f \) is of the form \((x, y^3 b(x,y), y^3 c(x,y))\) then
\[
T \mathcal{A} f = \left\{ \begin{array}{c} \{ x \} \\ \{ y \} \\ \{ b \} : s + t \geq 1 \end{array} \right\} + \mathfrak{m}_2 \left( \begin{array}{c} 1 \\ y \partial_x b \\ 3y^2 b + y^3 \partial_y b \\ y \partial_x c \\ 3y^2 c + y^3 \partial_y c \end{array} \right)
\]
and $T \not\approx f \not\approx \begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}$ for any $s$ so $e(N \not\approx f_0) \geq 2$. Note that $\Sigma$ does not have transversal type $A_{2n}$ in this case.

We can distinguish cases 1 and 2 and case 3 by looking at the $O_2$-module

$$Q_f = O_2/'\partial_x f_1 \partial_y f_2 - \partial_x f_2 \partial_y f_1, \partial_x f_2 \partial_y f_3 - \partial_x f_3 \partial_y f_1, \partial_x f_2 \partial_y f_3 - \partial_x f_3 \partial_y f_2.$$

In the case $(x, y^2, y^{2n+1} \varphi(x,y^2))$ this becomes

$$O_2/'\partial y$$

which has multiplicity 1 and in the case $(x, y^3 b(x,y), y^3 c(x,y))$ it becomes

$$'\partial_x f_2 \partial_y f_3 - \partial_x f_2 \partial_y f_1, \partial_x f_2 \partial_y f_3 - \partial_x f_3 \partial_y f_2 \leq 'y$$

so that $Q_f$ has multiplicity $\geq 2$.

This can be calculated at any point $p$ of the (irreducible) cuspidal edge $C$ since $e(Q_f) = \mu_p(C) e(Q_f)$.

In other words, $\mu_p(Q_f) = 1$ if and only the transversal type of $\Sigma = f(C)$ is $A_{2n}$ for some $n \geq 1$.

(III.1.4) Case 2: $t \geq 2 : f$ has at least 1 cuspidal edge.

The transversal type of $\Sigma$ cannot be $A_{2n}$.

Let $f : (C^2, p_1, p_2) \rightarrow (C^3, 0)$. Suppose $f_1 : (C^2, p_1) \rightarrow (C^3, 0)$ and $f_2 : (C^2, p_2) \rightarrow (C^3, 0)$.

If $f_1$ and $f_2$ both have a cuspidal edge then

$$e(N \not\approx f_0) \geq e(N \not\approx f_1,0) + e(N \not\approx f_2,0) \geq 2$$

so we may assume that $f_1$ is an immersion.

If $f_2$ has a cuspidal edge then we may assume that $f_2$ is $(x, y^2, y^3)$. Otherwise $e(N \not\approx f_2,0) \geq 2$. We may also assume that $f_1$ is of the form $(x, \psi(x,y), \varphi(x,y))$, and since $f_1(C^2)$ intersects $f_2(C^2)$ along $Y = 0, Z = 0$ we must have $\psi(x,0) = \varphi(x,0) = 0$.

Then $f_1$ is of the form $(x, y \psi(x,y), y \varphi(x,y))$ and
This module does not contain any elements of the type

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
x^s & x^s
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
x^s & x^s
\end{pmatrix}
\]. Hence \( e(N^f_0) \geq 2 \).

If \( f \) is 3-1 or more onto a cuspidal edge \( \Sigma \) then \( e(N^f_0) \geq 2 \) since it has a map which is 2-1 onto a cuspidal edge embedded in it so we can apply lemma III.1.1. This is enough to establish

(III·1·5) Lemma: suppose that \( \Sigma_i \) is a cuspidal edge and that \( C = f^i(\Sigma_i) \).

Then i) \( f(\Sigma_i) = 1 \) if and only if \( \Sigma_i \) has transversal type \( A_2 \)

and ii) \( \Sigma \) has transversal type \( A_{2n} \Rightarrow f(\Sigma) = n \).

(III·1·6) Case 3: \( f \) has \( t+1 \) immersed sheets.

Let \( f: (\mathbb{C}^2, P_1, ..., P_{t+1}) \to (\mathbb{C}^3, 0) \).

Without loss of generality we may assume that the \( t+1 \) local maps are of the form

\[
(x, y, 0), (x, y, y^{k_1}\varphi_1(x, y)), \ldots, (x, y, y^{k_t}\varphi_t(x, y))
\]

where \( \Sigma \) is the curve \( Y = 0, Z = 0 \) and \( 1 \leq k_1 \leq \ldots \leq k_t \) (if \( t = 1 \) then \( k_1 \geq 2 \) so that the intersection is non-transverse). Since no other double curves pass through \( (0,0,0) \) we must have \( \varphi_i(0,0) \neq 0 \). Then

\[
T^f \subseteq \left( \begin{array}{cccc}
m_2 & m_2 & m_2 & \ldots & m_2 \\
m_2 & y m_2 & y m_2 & \ldots & y m_2 \\
k_1 & k_1 & k_2 & \ldots & k_t \\
y m_2 & y m_2 & y m_2 & \ldots & y m_2
\end{array} \right)
\]
The missing terms are of the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & x^s & 0 & \ldots & 0 \\
0 & 0 & 0 & x^s & \ldots & x^s \\
0 & 0 & 0 & 0 & \ldots & 0 \\
\end{pmatrix}
\]

There are \( k_1 + k_2 + \ldots + k_t + k_{t+1} + (t-1) \) missing terms for each \( s \), with \( k_t + (t-1) + 1 \) equations between them. This means that

\[
e(\mathcal{N}, f_0) \geq \left( \sum_{i=1}^{t} k_i \right) - 1.
\]

We see that

i) if \( t \geq 3 \) then \( e(\mathcal{N}, f_0) \geq 2 \),

ii) if \( t = 2 \) and \( k_2 \geq 2 \) then \( e(\mathcal{N}, f_0) \geq 2 \),

iii) if \( t = 1 \) and \( k_1 \geq 3 \) then \( e(\mathcal{N}, f_0) \geq 2 \).

This means that \( e \) is an 'upper limit' for the transversal type of \( \Sigma \).

In particular, if \( t = 1 \) then

\((x, y, 0), (x, y, y^k \varphi(x,y))\)

is equivalent to

\((x, y, -y^k \varphi(x,y)), (x, y, y^k \varphi(x,y))\)

and \( Z^2 - Y^{2k} \varphi(X, Y) = 0 \) so that \( \Sigma \) has transversal type \( A_{2k-1} \) where \( k \leq e+1 \) so that

\( e = 1 \Rightarrow A_3, \quad e = 2 \Rightarrow A_3 \text{ or } A_5, \text{ etc.} \)

Note that this estimate is a lower limit. For example it does not distinguish between the cases indicated in diagram 9:
Diagram 9. Inequivalent Map-germs with the same lower estimate for \( l \).

We next look at the two cases

iv) \((x, y, 0), (x, y, y^k \varphi(x, y))\) (transversal type \( A_{2k-1} \)),

and v) \((x, y, 0), (x, y, y \varphi_1(x, y)), (x, y, y \varphi_2(x, y))\) (transversal type \( D_4 \)).

For iv) note that \( y^{k-1}(k \varphi + y \partial_y \varphi) \cdot m_2 = y^{k-1} \cdot m_2 \) since \( k \varphi + y \partial_y \varphi \notin m_2 \) so

\[
T_{\mathcal{A} f} = \begin{pmatrix}
m_2 & m_2 \\ m_2 & m_2 \\ y^{k-1} m_2 & y^{k-1} m_2 \\
\end{pmatrix} + C \begin{pmatrix}
0 & 0 \\
0 & 0 \\
x^r y^r & x^r y^r \\
\end{pmatrix} : r < k \quad \text{and} \quad \mathcal{L}(N_{\mathcal{A} f_0}) = k.
\]

Hence a double curve \( \Sigma_1 \) has transversal type \( A_{2k-1} \) if and only if \( \mathcal{L}(N_{\mathcal{A} f_1}) = k \).

For v) assume that \( \varphi_1(x, 0) \neq \varphi_2(x, 0) \) so that \( f_2 \) and \( f_3 \) are transverse. If this is not the case then we choose coordinates so that \( f_2 \), say, is given by \((x, y, 0)\) and use case ii) above. We calculate \( T_{\mathcal{A} f} \):

now

\[
\mathcal{L}_2 \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \subset T_{\mathcal{A} f},
\]

consideration of the terms

\[
\begin{pmatrix}
0 & x y^j & x y^j \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
0 & x y^j \varphi_1 & x y^j \varphi_2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} 
\]

imply that \( y \mathcal{L}_2 \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \subset T_{\mathcal{A} f}, \)
consideration of the terms
\[
yM_2^2 \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 1 & 0 \\
\phi_1 + y\partial_y \varphi_1 & 0 \end{array} \right) \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & 1 \\
0 & \partial_y \varphi_2 \end{array} \right) \right\}
\]
implies that \( yM_2^2 \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 \end{array} \right) \right\} \subset \mathcal{TSf}, \)

consideration of the terms
\[
\left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & 0 \\
x^s y^s i 
i \end{array} \right) \right\} \text{ and } \left\{ \left( \begin{array}{ccc} 0 & 1 & 0 \\
0 & 0 & 0 \\
y\partial_x \varphi_1 & 0 \end{array} \right) \right\}
\]

imply that \( yM_2^2 \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 \end{array} \right) \right\}, yM_2 \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 \end{array} \right) \right\} \subset \mathcal{TSf}, \) so
\[
\mathcal{TSf} = \left( \begin{array}{ccc} M_2 & M_2 & M_2 \\
M_2 & yM_2 & yM_2 \\
yM_2 & yM_2 & yM_2 \end{array} \right) + \\
\left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & x^s \\
x^s \end{array} \right) \cdot \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & x^s \\
x^s \end{array} \right) \cdot \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & x^s \\
x^s \end{array} \right)
\]

and \( e(N \mathcal{Sf}_0) = 1. \) Thus

(III-1.7) Lemma: suppose that \( \Sigma_i \) is not a cuspidal edge. Then
i) \( I(N \mathcal{Sf}_{\Sigma_i}) = 1 \iff \) the transversal type of \( \Sigma_i \) is either \( A_3 \) or \( D_4. \)
ii) if \( \Sigma_i \) is a double curve then
\[
I(N \mathcal{Sf}_{\Sigma_i}) = k \iff \text{the transversal type of } \Sigma_i \text{ is } A_{2k-1}.
\]

This concludes the proof of Theorem III-1.1.

We finish this section by calculating the transversal types of the map-germs given in Tables 2-5.
In the case that the map-germ has $e = 1$ this is just a case of assigning the transversal types $A_2, A_3$ or $D_4$ according to whether $\Sigma$ is a cuspidal edge, a double curve or a triple curve respectively. These map-germs are not listed below. We use Arnold's lists (Table 6) for the more complicated transversal types.

i) $(x, y^2, x^3 y)$: this has $e = 2$. The curve $\Sigma = V(X, Z)$ is a smooth, irreducible double curve. By III:2:7 $e = 2$ implies the transversal type must be $A_5$.

ii) $(x, y^2, x^4 y)$: this has $e = 3$ but is otherwise like i). Thus the transversal type must be $A_7$.

iii) $(x, y^2, x^2 y^3)$: this has $e = 2$. The curve $\Sigma = V(XY, Z)$ has two smooth components, hence $l(N_{s(\Sigma_1)}) = 1$ on each curve. The curve $V(X, Z)$ is a double curve and is hence $A_3$ while the curve $V(Y, Z)$ is a cuspidal edge and is hence $A_2$.

iv) $(x, y^2, y^5)$: this has $e = 2$. The curve $\Sigma = V(Y, Z)$ is smooth and irreducible hence $l(N_{s(\Sigma)}) = 2$. It is a 1-1 cuspidal edge with $e(Q_1) = 1$ and is hence $A_4$.

v) $(x, xy + y^3, x y^2 + 3 y^4)$: this has $e = 2$. The curve $\Sigma = V(X^2 + 6Z, 4X^3 + 27Y^2)$ is not smooth, hence $l(N_{s(\Sigma)}) = 1$ and $\mu_0(\Sigma) = 2$. It is a cuspidal edge and is hence $A_2$.

vi) $(x, xy + y^3, xy^2 + \frac{1}{2} y^4)$: this has $e = 2$. The curve $\Sigma = V(X^2 + 2Z, Y)$ is smooth, hence $l(N_{s(\Sigma)}) = 2$. It is a double curve and is hence $A_5$.

vii) $(x, xy + y^3, xy^3)$: this has $e = 4$. The image satisfies $(XY - Z)^3 - X^5 Z = 0$ which is equivalent under $Z \to XY - Z$ to $Z^3 + X^5 Z - X^6 Y = 0$. The curve $\Sigma$ is $V(X, Z)$ and $Z^3 + X^5 Z - X^6 Y$ has multiplicity 10 so $\Sigma$ has transversal type $J_{10}$.

vii) $(x, xy, y^4)$: this has $e = 3$. The image satisfies $Y^4 - X^4 Z = 0$. The curve $\Sigma$ is $V(X, Y)$ so has transversal type $X_9$ (with $e = 0$ and $a = -Z$).

viii) $(x, xy + xy^3, y^4)$: this has $e = 2$. The image satisfies $Y^4 - 4ZX^2 Y^2 - ZX^4 (Z - 1)^2 = 0$. The curve $\Sigma$ is $V(X, Y)$ so has transversal type $X_9$ (with $e \neq 0$).

ix) $(x, y^3, xy^2)$: this has $e = 2$. The curve $\Sigma = V(XY, Z)$ has two smooth components,
hence \( \ell(\text{N}_\text{f}_2) = 1 \) on each curve. The curve \( V(X, Z) \) is a triple curve and is hence \( D_4 \) while the curve \( V(Y, Z) \) is a cuspidal edge and is hence \( A_2 \).

x) \( (x, y^3, x^2y) \): this has \( e = 4 \). The image satisfies \( Z^3 - X^6Y = 0 \). The curve \( \Sigma \) is \( V(X, Z) \) so has transversal type \( J_{10} \) (with \( e = 0 \)).
III·2: Geometry of Stems.

If \( f \) fails to be a stem then for each \( k \) there exists a map-germ \( f_k \) satisfying 
\[ j^k f = j^k f_k \]
which is neither finitely determined nor equivalent to \( f \). We use this fact to find restrictions on the geometry of a stem. Throughout this section \( \Sigma \) denotes the set of points in \( \mathbb{C}^3 \) at which \( f \) fails to be stable and \( \Sigma_k \) denotes the corresponding set for \( f_k \).

The principal result is

(III·2:1) Theorem: Suppose that \( f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) and that \( C \subseteq \mathbb{C}^2 \) is a (not necessarily irreducible) curve, with \( 0 \in C \). Then for any \( k \) there exists a map-germ \( f_k:(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) satisfying

i) \( j^k f = j^k f_k \),

ii) if \( (x, y) \in C \) then 
\[ f(x, y) = f_k(x, y), \ \partial_x f(x, y) = \partial_x f_k(x, y) \] and \( \partial_y f(x, y) = \partial_y f_k(x, y) \),

iii) the set \( \{(x, y) \in \mathbb{C}^2\setminus C : f_k(x, y) \in \Sigma_k\} \) consists of isolated points of \( \mathbb{C}^2\setminus C \);

consequently there is a neighbourhood of \((0, 0)\) on which \( f_k^{-1}(\Sigma_k) = C \).

For example, suppose \( f \) has a cuspidal edge with an immersed sheet intersecting along the edge as in diagram 10:

![Diagram 10. A map-germ which is not a stem.](image-url)
We expect to find an $f_k$ as in diagram 11:

![Diagram 11. A deformation of the map-germ in diagram 10.](image)

Notice that $f_k^{-1}(\Sigma_k) = C \cup \{p_1, p_2, \ldots; p_i \notin C\}$.

Condition ii) ensures that

a) $f_k$ is an immersion at $(x, y) \in C$ if and only if $f$ is an immersion at $(x, y) \in C$,

b) if $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in C$ with $f(z_1) = f(z_2)$ then $f_k(z_1) = f_k(z_2)$ and $f_k$ has a transverse double point at $z_1, z_2 \in C$ if and only if $f$ has a transverse double point at $z_1, z_2 \in C$,

c) if $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in C$ with $f(x_1, y_1) = f(x_2, y_2) = f(x_3, y_3)$ then $f_k(x_1, y_1) = f_k(x_2, y_2) = f_k(x_3, y_3)$ so $f_k$ has a triple point at $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in C$ if and only if $f$ has a triple point at $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in C$.

The proof of Theorem III-2:1 does not assume that $f$ is finite but the following corollaries do. To see that $f$ cannot be a stem if it is not finite see III-2:9.

(III-2:2) Corollary: if $f$ is a stem then $\Sigma$ is an irreducible curve.

Proof: suppose that $\Sigma$ is not irreducible. Let $\Sigma_1 \subseteq \Sigma$. Let $C = f^{-1}(\Sigma_1)$. By III-2:1, for each $k$ there exists $f_k$ with the properties.
\[ \Sigma_k \subseteq f_k(C) = \Sigma_1 \subseteq \Sigma \text{ so } f_k \text{ is not } \not\sim \text{-equivalent to } f, \]

if \( y \in \Sigma_1 \) then \( f^1(y) \subseteq C \).

If \( f \) is finite then \( \Sigma_1 \) is at least one of a triple curve, a double curve with transversal type not \( A_1 \) or a cuspidal edge. This is preserved for \( f_k \) so \( y \in \Sigma_k \), i.e. \( \Sigma_k = \Sigma_1 \) and \( f_k \) is not finitely determined. Thus \( f \) is not a stem.

(III-2:3) Corollary: if \( f \) is a stem and \( \Sigma \) is a cuspidal edge then \( f^1(\Sigma) \) is irreducible.

Proof: Suppose that \( f^1(\Sigma) \) is reducible. Let \( C \subseteq f^1(\Sigma) \) satisfy \( x \in C \Rightarrow f \) is not an immersion at \( x \).

Apply III-2:1 to this curve; for each \( k \) there exists \( f_k \) with the properties

\[ f_k^{-1}(\Sigma_k) \subseteq C \cup \{ \text{isolated points} \} \text{ so } f_k \text{ is not equivalent to } f, \text{ and} \]

\[ x \in C \Rightarrow f_k \text{ fails to be an immersion at } x. \]

So \( f_k(C) \) is a cuspidal edge and \( f_k \) is not finitely determined, hence \( f \) is not a stem.

(III-2:4) Corollary: if \( f \) is a stem and \( \Sigma \) is a non-transverse double curve then \( f^1(\Sigma) \) has at most two branches.

Proof: Let \( C \) be either

one curve (\( f \) not 1-1 on \( C \)) with non-transverse self-intersection or

two curves whose images intersect non-transversely.

Apply III-2:1 to this curve; for each \( k \) there exists \( f_k \) with the properties

\[ f_k^{-1}(\Sigma_k) \subseteq C \cup \{ \text{isolated points} \} \text{ so } f_k \text{ is not equivalent to } f \text{ if } C \not= f^1(\Sigma), \text{ and} \]

\[ x_1, x_2 \in C, \text{ with } f(x_1) = f(x_2) \Rightarrow f_k \text{ fails to be transverse at } x_1, x_2. \]

So \( f_k(C) \) is a non-transverse double curve and \( f_k \) is not finitely determined, hence \( f \) is not a stem.
(III.2:5) Corollary: if \( f \) is a stem and \( \Sigma \) is a triple curve then \( f^4(\Sigma) \) has at most three branches.

Proof: Let \( C \) be either
- one curve on which \( f \) is \( n-1, n \geq 3 \) or
- two curves \( C_1, C_2: f \) is 2-1 on \( C_1 \) and either 1-1 or 2-1 on \( C_2 \)
- three curves \( C_1, C_2, C_3: f \) is 1-1 on each \( C_i \).

Apply III.2:1 to this curve: for each \( k \) there exists \( f_k \) with the properties

\[
f_k^{-1}(\Sigma) \subseteq C \cup \{\text{isolated points}\} \quad \text{so} \quad f_k \text{ is not equivalent to } f \text{ if } C \neq f^4(\Sigma), \text{ and}
\]

\[
x_1 x_2 x_3 \in C, \text{ with } f(x_1) = f(x_2) = f(x_3) \Rightarrow f_k \text{ has a triple point at } x_1, x_2, x_3.
\]

So \( f_k(C) \) is a triple curve and \( f_k \) is not finitely determined, hence \( f \) is not a stem.

This shows that

(III.2:6) Proposition: if \( f \) is a stem then

i) \( \Sigma \) is irreducible,

ii) \( f^4(\Sigma) \) has at most three irreducible components,

iii) if \( f^4(\Sigma) \) has three components then \( \Sigma \) has transversal type \( D_4 \),

iv) if \( f^4(\Sigma) \) has two components then on each component \( f \) is either a 1-1 immersion or is 2-1 with transversal type \( A_1 \).

The principal technique in the proof of III.2:1 is the use of transversality.

(III.2:7) Theorem [Math IV, Proposition 1.6]. Let \( S = \{z_1, \ldots, z_m\} \subseteq \mathbb{C}^n \), and suppose that \( f(z_1) = \ldots = f(z_m) \). Let \( A_i \) be the germ at \( z_i \) of the set

\[
\{z \in \mathbb{C}^n \mid \text{the germ of } f \text{ at } z \text{ is } \simeq \text{-equivalent to the germ of } f \text{ at } z_i\}.
\]

Then the multi-germ \( f:(\mathbb{C}^n, S) \to (\mathbb{C}^p, f(z_i)) \) is stable if and only if

i) each germ \( f:(\mathbb{C}^n, z_i) \to (\mathbb{C}^p, f(z_i)) \) is stable, for \( 1 \leq i \leq m \),

ii) the map-germ \( f\times\ldots\times f:(A_1\times\ldots\times A_m, (z_1, \ldots, z_m)) \to (\mathbb{C}^p)^m \) is transverse to the set \( \Delta_{m,p} := \{(z_1, \ldots, z_m) \mid z_i = z_j \text{ for all } i, j\} \).
(III·2:8) Transversality Lemma:

If \( F: X \times Y \to Z \) is transverse to the submanifold \( W \) of \( Z \) then there is an open, dense set \( U \) in \( Y \) so that \( f_y: X \to Z \) is transverse to \( W \) if \( y \in U \). Here \( f_y(x) = F(x, y) \).

The following result demonstrates the use of the transversality lemma.

(III·2:9) Lemma: if \( f \) is a stem then it is finite.

Proof: If \( f \) is not finite then the ideal \( f^*\mathcal{O}_3 \cap \mathcal{O}_2 \) defines a curve \( C \) unless \( f \) is the zero map. After a change of coordinates we may assume that \( x = 0 \) is not one of its branches. Then in a neighbourhood of \( 0 \) \( f(0, y) \neq 0 \) if \( y \neq 0 \). Consider the unfolding \( F_k: \mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C}^3 \) given by

\[
F_k(x, y, \alpha, \beta, \gamma) = (f_1(x, y) + \alpha x^k, f_2(x, y) + \beta x^k, f_3(x, y) + \gamma x^k)
\]

On \( x = 0 \) \( F_k(0, y, \alpha, \beta, \gamma) \neq f(0, y) \) except at \( y = 0 \).

Otherwise \( \partial F_k/\partial \alpha = (x^k, 0, 0), \partial F_k/\partial \beta = (0, x^k, 0), \partial F_k/\partial \gamma = (0, 0, x^k) \), and \( F_k \) is a submersion. Hence \( F_k \) intersects \( (0, 0, 0) \) transversely. Then by the transversality lemma there is an open dense set \( U \) in \( \mathbb{C}^3 \) such that \( (\alpha, \beta, \gamma) \in U \) implies that

\[
f_{\alpha, \beta, \gamma}(x, y) = (f_1(x, y) + \alpha x^k, f_2(x, y) + \beta x^k, f_3(x, y) + \gamma x^k)
\]

is transverse to \( (0, 0, 0) \) and hence \( f_{\alpha, \beta, \gamma} \) is a finite map-germ. It is not finitely determined since if \( (x, y) \in C \) then \( f_{\alpha, \beta, \gamma}(x, y) = (\alpha x^k, \beta x^k, \gamma x^k) \) which is a \( k-1 \) curve.

We use exactly the same technique to prove III·2:1 except that here we look at three maps derived from the unfolding, restricting to the open dense sets as we proceed.

The calculations to prove transversality are also rather more complicated.

Proof of III·2:1;

Suppose that \( C = V(\varphi) \) and \( \varphi(0, 0) = 0 \).

Let \( N_k = \{(i, j) : k \leq i+j \leq k+1 \} \) and let
$A_k = \{\sum_{i=0}^{k} \alpha_{i,j} x^i y^j, \sum_{i=0}^{k} \beta_{i,j} x^i y^j, \sum_{i=0}^{k} \gamma_{i,j} x^i y^j : \alpha_{i,j}, \beta_{i,j}, \gamma_{i,j} \in \mathbb{C}, (i,j) \in N_k\}.$

Let $f_\sigma(z) = f(z) + \varphi^2(z)\sigma(z)$ where $\sigma \in A_k$.

Claim: there is an open dense set $U_k \subseteq A_k$ such that for each $\sigma \in U_k$ and $z \in \mathbb{C}^2 \setminus \mathbb{C}$ then either $f_\sigma(z)$ is an isolated $m$-1 point for some $m \geq 3$ or an isolated point in $f_\sigma^{-1}(f_\sigma(\mathbb{C}))$ or $f_\sigma$ is stable at $f_\sigma(z)$.

Any such $f_\sigma$ can be chosen for the map-germ $f_k$ since it also satisfies $j^k f = j^k f_\sigma$

$(\varphi^2 \sigma \in m_2 \cdot \theta(f))$ and $f(z) = f_\sigma(z)$, $\partial_x f(z) = \partial_x f_\sigma(z)$, $\partial_y f(z) = \partial_y f_\sigma(z)$ if $\varphi(z) = 0$.

Claim 1: the map $F_1 : \mathbb{C}^2 \times A_k \to J^1(2,3)$ given by

$$F_1 (z, \sigma) = \begin{pmatrix} \partial_x f_1(z) + \varphi^2(z)\sigma_1(z) & \partial_y f_1(z) + \varphi^2(z)\sigma_1(z) \\ \partial_x f_2(z) + \varphi^2(z)\sigma_2(z) & \partial_y f_2(z) + \varphi^2(z)\sigma_2(z) \\ \partial_x f_3(z) + \varphi^2(z)\sigma_3(z) & \partial_y f_3(z) + \varphi^2(z)\sigma_3(z) \end{pmatrix}$$

is a submersion if $z \notin \mathbb{C}$.

Hence by III-2:8 there is an open dense set $U_1$ such that for $\sigma \in U_1$,

$$j^1 (f + \varphi^2 \sigma)$$

is transverse to $\Sigma^1 \subseteq J^1(2,3)$ at $z \notin \mathbb{C}$

where $\Sigma^1 = \{\text{elements in } J^1(2,3) \text{ of rank } \leq 1\}$.

Hence the germ $f_\sigma : (\mathbb{C}^2, z) \to (\mathbb{C}^3, f_\sigma(z))$ is stable and hypothesis (i) of III-2:7 holds.

Claim 2: the map $F_2 : (\mathbb{C}^2)^{(2)} \setminus C^{(2)} \times U_1 \to (\mathbb{C}^3)^{(2)}$ given by

$$F_2(z_1, z_2, \sigma) = ((f + \varphi^2 \sigma)(z_1), (f + \varphi^2 \sigma)(z_2))$$

is transverse to $\Delta_{2,3} = \{(\gamma, \hat{\gamma}) : \hat{\gamma} \in \mathbb{C}^3\}$.

Hence by III-2:8 there is an open dense set $U_2$ such that for $\sigma \in U_2$,

$$((f + \varphi^2 \sigma)(z_1), (f + \varphi^2 \sigma)(z_2))$$

is transverse to $\Delta_{2,3}$.

Suppose that $z \notin \mathbb{C}$ and that $f_\sigma^{-1}(f_\sigma(z)) = \{z, z\}$, say. Then
\[ f_\sigma \times f_\sigma : (A_1 \times A_2, (z, z')) \to (\mathbb{C}^3)^2 \]
is transverse to the set \( \Delta_{2,3} \). If \( z' \notin \mathbb{C} \) then by III.2:7 the multi-germ is stable.

Otherwise suppose that \( z \notin \mathbb{C} \) and that \( f_\sigma^{-1}(f_\sigma(z)) \cap \mathbb{C} \neq \emptyset \).

Now \( f_\sigma \times f_\sigma((\mathbb{C}^2 \setminus \mathbb{C}) \times \mathbb{C}) \) and \( \Delta_{2,3} \) have codimension 3 so \( f_\sigma \times f_\sigma((\mathbb{C}^2 \setminus \mathbb{C}) \times \mathbb{C}) \cap \Delta_{2,3} \)
has codimension 6 and is hence an isolated point, i.e. \( z \) is an isolated point in \( f_\sigma^{-1}(f_\sigma(\mathbb{C})) \).

Claim 3: the map \( F_3 : (\mathbb{C}^2 \setminus \mathbb{C})^{(3)} \times U_3 \to (\mathbb{C}^3)^3 \) given by

\[
F_3(z_1, z_2, z_3, \sigma) = ((f + \varphi^2 \sigma)(z_1), (f + \varphi^2 \sigma)(z_2), (f + \varphi^2 \sigma)(z_3))
\]
is transverse to \( \Delta_{3,3} = \{(\bar{y}, \bar{y}, \bar{y}) : \bar{y} \in \mathbb{C}^3 \} \).

Now \( ((f + \varphi^2 \sigma)(z_1), (f + \varphi^2 \sigma)(z_2), (f + \varphi^2 \sigma)(z_3))((\mathbb{C}^2)^{(3)} \setminus \mathbb{C}^3) \) has dimension 6, \( \Delta_{3,3} \) has dimension 3 and \( J^1(2, 3) \) has dimension 9. Thus the intersection of

\( (f + \varphi^2 \sigma)^{(3)}((\mathbb{C}^2 \setminus \mathbb{C})^{(3)}) \) and \( \Delta_{3,3} \), if transverse, has dimension 0, i.e. it consists of isolated points. Note that any of these points may be 4–1 or greater so are not necessarily stable.

Proof of claim 1:

\[
T_{F_1(z, \sigma)} F_1(\mathbb{C}^2 \times A_k) \cong C \left[ \frac{\partial}{\partial a_{i,j}} j^1(f + \varphi^2 \sigma) : (i, j) \in N_k \right]
\]

\[
= C \begin{pmatrix}
\partial_x(x y \varphi^2)(z) & \partial_y(x y \varphi^2)(z) & 0 & 0 \\
0 & 0 & \partial_x(x y \varphi^2)(z) & \partial_y(x y \varphi^2)(z) \\
0 & 0 & 0 & 0 \\
\partial_x(x y \varphi^2)(z) & \partial_y(x y \varphi^2)(z) & \partial_{x'}(x y \varphi^2)(z) & \partial_{y'}(x y \varphi^2)(z)
\end{pmatrix}
\]

Then \( F_1 \) is a submersion unless for every pair \((i, j), (i', j') \in N_k\) the matrix
has zero determinant.

Each determinant is \( \phi^3 x^{i+i'-1} y^{j+j'-1} \left( \phi(ij'-i'j) + 2(i-i') y \partial_y \phi + 2(j'-j) x \partial_x \phi \right) \).

If \( i+j = i' + j' = k \) then this becomes

\[
(i-i') \phi^3 x^{i+i'-1} y^{j+j'-1} (k \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)
\]

and if \( i+j = i' + j' = k+1 \) then it becomes

\[
(i-i') \phi^3 x^{i+i'-1} y^{j+j'-1} (k+1) \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)
\]

In particular, if \( F_1 \) fails to be a submersion then all of the following equations are satisfied:

\[
\begin{align*}
\phi^3 y^{2k-2} (k \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)(z) &= 0, \\
\phi^3 x^{2k-2} (k \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)(z) &= 0, \\
\phi^3 x^{2k-1} ((k+1) \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)(z) &= 0 \\
\text{and } \phi^3 y^{2k-1} ((k+1) \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)(z) &= 0.
\end{align*}
\]

Now \( z = (x, y) \neq (0, 0) \) and \( z \notin C \) so \( \phi(z) \neq 0 \)

so the above equations are satisfied if and only if

\[
(k \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)(z) = 0 \text{ and } ((k+1) \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)(z) = 0.
\]

But then \( ((k+1) \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)(z) - (k \phi + 2 y \partial_y \phi + 2 x \partial_x \phi)(z) = \phi(z) = 0 \), which contradicts the choice of \( z \notin C \).

Proof of claim 2: Suppose that \( F_2(z_1, z_2, \sigma) \in \Delta_{2, 3} \).

Now \( T_{F_2}(z_1, z_2, \sigma) \Delta^2 = C \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \)

and \( T_{F_2}(z_1, z_2, \sigma) F_2((C^2)^2 \setminus C^2) \times U_1) \cong C \left\{ \frac{\partial}{\partial z_{ij}} \left( f + \phi^2 \sigma(z_1), f + \phi^2 \sigma(z_2) \right) : (i, j) \in N_k \right\} \)
Here \( z_1 = (x_1, y_1) \) and \( z_2 = (x_2, y_2) \).

Then \( F_2 \) is transverse to \( \Delta_{2, \bar{3}} \) unless

\[
\varphi^2(x_2, y_2)x_2^i y_2^j - \varphi^2(x_1, y_1)x_1^i y_1^j = 0 \text{ for all } (i, j) \in \mathbb{N}_k,
\]

i.e. \( \varphi^2(x_2, y_2)x_2^i y_2^j = \varphi^2(x_1, y_1)x_1^i y_1^j \) for all \( (i, j) \in \mathbb{N}_k \).

In particular, \( \varphi^2(x_2, y_2)x_2^k = \varphi^2(x_1, y_1)x_1^k \), \( \varphi^2(x_2, y_2)x_2^{k+1} = \varphi^2(x_1, y_1)x_1^{k+1} \).

\[
\varphi^2(x_2, y_2)y_2^k = \varphi^2(x_1, y_1)y_1^k \text{ and } \varphi^2(x_2, y_2)y_2^{k+1} = \varphi^2(x_1, y_1)y_1^{k+1}.
\]

If \( \varphi^2(x_2, y_2) = 0 \) then \( \varphi^2(x_1, y_1) \neq 0 \) and \( (x_1, y_1) \neq (0, 0) \) since \( (x_1, y_1) \notin \mathbb{C} \) so one of \( \varphi^2(x_1, y_1)x_1^k, \varphi^2(x_1, y_1)y_1^k \) is non-zero.

If \( \varphi^2(x_2, y_2) \neq 0 \neq \varphi^2(x_1, y_1) \) then \( x \neq 0 \) implies \( \frac{x_1}{x} = \frac{x_2}{x} \) so that \( x_1 = x_2 \)

and \( y \neq 0 \) implies \( \frac{y_1}{y} = \frac{y_2}{y} \) so that \( y_1 = y_2 \).

But we cannot have \( (x_1, y_1) = (x_2, y_2) \) so \( F_2 \) is transverse to \( \Delta_{2, \bar{3}} \) as required.

Proof of claim 3: Suppose that \( F_3(z_1, z_2, z_3, \sigma) \in \Delta_{3, \bar{3}} \).

Now \( T_{F_3(z_1, z_2, z_3, \sigma)} \Delta_{3, \bar{3}} = \mathbb{C} \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \)

and \( T_{F_3(z_1, z_2, z_3, \sigma)} F_3(\mathbb{C}^2 \setminus \mathbb{C})^{(3)} \times U_2) \)

\[
= \mathbb{C} \left\{ \partial_x(f + \varphi^2 \sigma)(z_1) \partial_0 \), \partial_y(f + \varphi^2 \sigma)(z_1) \partial_0 \), \partial_0 \partial_x(f + \varphi^2 \sigma)(z_2) \partial_0 \right\}.
\]
\[
\left( \partial_y(f + \varphi^2 \sigma)(z_2), 0 \right), \left( \partial_x(f + \varphi^2 \sigma)(z_2), 0 \right), \left( \partial_y(f + \varphi^2 \sigma)(z_3), 0 \right), \left( \partial_x(f + \varphi^2 \sigma)(z_3), 0 \right), \left( \partial_y(f + \varphi^2 \sigma)(z_3), 0 \right), \left( \partial_x(f + \varphi^2 \sigma)(z_3), 0 \right),
\]
\[
\begin{pmatrix}
\varphi x_1 & \varphi x_2 & \varphi x_3 & \varphi x_4 & \varphi x_5 & \varphi x_6 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\[
: (i, j) \in N_k
\]

Subtracting the first column in each of these matrices using the elements of \( T \Delta_{3,3} \)
gives the vector space spanned by
\[
\left\{ (0 \partial_x(f + \varphi^2 \sigma)(z_1), 0 \partial_x(f + \varphi^2 \sigma)(z_1)), (0 \partial_y(f + \varphi^2 \sigma)(z_1), 0 \partial_y(f + \varphi^2 \sigma)(z_1)) \right\},
\]
\[
\left\{ (0 \partial_x(f + \varphi^2 \sigma)(z_2), 0), (0 \partial_y(f + \varphi^2 \sigma)(z_2), 0), (0 \partial_x(f + \varphi^2 \sigma)(z_3), 0), (0 \partial_y(f + \varphi^2 \sigma)(z_3)) \right\},
\]
\[
\left\{ (0 \alpha_{i,j} \beta_{i,j} R_1, (0 \alpha_{i,j} \beta_{i,j} R_2, (0 \alpha_{i,j} \beta_{i,j} R_3: (i, j) \in N_k,}
\]
\]
\[
\text{where } \alpha_{i,j} = \varphi^2(x_2, y_2) x_2^i j - \varphi^2(x_1, y_1) x_1^i j, \beta_{i,j} = \varphi^2(x_3, y_3) x_3^i j - \varphi^2(x_1, y_1) x_1^i j
\]
\[
\text{and } (a, b, c) R_i \text{ denotes the matrix with } (a, b, c) \text{ for the } i \text{th row and zeros elsewhere.}
\]

The claim follows if this has dimension 6.

Suppose that \( \alpha_{i,j} \neq 0 \) for some \((i, j)\).

Subtract the second column from the third to get the space spanned by
\[
\left\{ (0 \alpha_{i,j} \partial_x(f + \varphi^2 \sigma)(z_1) - \beta_{i,j} \partial_x(f + \varphi^2 \sigma)(z_1)), (0 \alpha_{i,j} \partial_y(f + \varphi^2 \sigma)(z_1) - \beta_{i,j} \partial_y(f + \varphi^2 \sigma)(z_1)) \right\},
\]
\[
\left\{ (0 \beta_{i,j} \partial_x(f + \varphi^2 \sigma)(z_2), 0 \beta_{i,j} \partial_y(f + \varphi^2 \sigma)(z_2)) \right\},
\]
\[
\left\{ (0 \partial_x(f + \varphi^2 \sigma)(z_3), 0 \partial_y(f + \varphi^2 \sigma)(z_3)) \right\}.
\]

The claim follows if this has dimension 3.

Now \( \det \left( \beta_{i,j} \partial_x(f + \varphi^2 \sigma)(z_2), \partial_x(f + \varphi^2 \sigma)(z_3), \partial_y(f + \varphi^2 \sigma)(z_3) \right) = \beta_{i,j} \det \left( \partial_x(f + \varphi^2 \sigma)(z_2), \partial_x(f + \varphi^2 \sigma)(z_3), \partial_y(f + \varphi^2 \sigma)(z_3) \right) \)

and \( \det \left( \beta_{i,j} \partial_y(f + \varphi^2 \sigma)(z_2), \partial_x(f + \varphi^2 \sigma)(z_3), \partial_y(f + \varphi^2 \sigma)(z_3) \right) = \beta_{i,j} \det \left( \partial_y(f + \varphi^2 \sigma)(z_2), \partial_x(f + \varphi^2 \sigma)(z_3), \partial_y(f + \varphi^2 \sigma)(z_3) \right) \)

and one of
\[
\det \left( \partial_y(f + \varphi^2 \sigma)(z_2), \partial_x(f + \varphi^2 \sigma)(z_3), \partial_y(f + \varphi^2 \sigma)(z_3) \right),
\]
\[
\det \left( \partial_x(f + \varphi^2 \sigma)(z_2), \partial_x(f + \varphi^2 \sigma)(z_3), \partial_y(f + \varphi^2 \sigma)(z_3) \right)
\]

is non-zero by choice of \( \sigma \) so transversality holds unless \( \beta_{i,j} = 0 \).
But then we get
\[
\det(\alpha_{ij}\partial_x(f + \varphi^2\sigma)(z_1) \partial_x(f + \varphi^2\sigma)(z_3) \partial_y(f + \varphi^2\sigma)(z_3)) = \alpha_{ij}\det(\partial_x(f + \varphi^2\sigma)(z_1) \partial_x(f + \varphi^2\sigma)(z_3) \partial_y(f + \varphi^2\sigma)(z_3))
\]
and
\[
\det\left(\alpha_{ij}\partial_y(f + \varphi^2\sigma)(z_1) \partial_x(f + \varphi^2\sigma)(z_3) \partial_y(f + \varphi^2\sigma)(z_3)\right) = \alpha_{ij}\det\left(\partial_y(f + \varphi^2\sigma)(z_1) \partial_x(f + \varphi^2\sigma)(z_3) \partial_y(f + \varphi^2\sigma)(z_3)\right)
\]
and one of
\[
\det\left(\partial_y(f + \varphi^2\sigma)(z_1) \partial_x(f + \varphi^2\sigma)(z_3) \partial_y(f + \varphi^2\sigma)(z_3)\right)
\]
is non-zero by choice of \(\sigma\).

What happens if \(\alpha_{ij} = 0\) for all \((i, j) \in \mathbb{N}_k\)?

Then for all \((i, j) \in \mathbb{N}_k\) \(\varphi^2(x_2, y_2)x_2y_2^j - \varphi^2(x_1, y_1)x_1^iy_1^j = 0\),

i.e. \(\varphi^2(x_2, y_2)x_2y_2^j = \varphi^2(x_1, y_1)x_1^iy_1^j\).

In particular, as in claim 2,

\[
\frac{y_1}{y_2} = \frac{x_1}{x_2} = \frac{k+1}{k}
\]

so that \(x_1 = x_2\) and \(y_1 = y_2\). But this contradicts the hypothesis.

This concludes the proof of theorem III-2:1.

If, for example, \(f^4(\Sigma)\) is irreducible then this theorem gives us no information on the transversal type of \(\Sigma\). We now inspect this problem. For this it is only necessary to find map-germs \(f_k\) which have a different transversal type along \(f_k(C)\); we do not mind introducing new branches to \(\Sigma_k\).

We first show that a stem is 1-1 along a cuspidal edge. In the case \(f\) is \(m-1\) along a cuspidal edge we make the following assumptions:

by III-2:2 we may assume that \(C = f^4(\Sigma)\) is irreducible so \(\varphi\) is prime. Let \(I = (\varphi) = I(f^4(\Sigma))\) be the prime ideal defining \(C\). Since \(C\) is a curve, without loss of
generality there is no value of $k$ satisfying $x^k \in I$ (and hence $\phi \neq x$).

(III-2:10) Hypothesis: we make the assumption that for some $i$ (say $i = 1$) the matrix

$$
\begin{pmatrix}
\partial_x f_1 & \partial_y f_1 \\
\partial_x \phi & \partial_y \phi
\end{pmatrix}
$$

has non-zero determinant along $C$,

i.e. $\partial_x f_1 \partial_y \phi - \partial_y f_1 \partial_x \phi \notin I$.

(III-2:11) Proposition: if $f$ is $3$-1 or more along a cuspidal edge and satisfies hypothesis III-2:10 then $f$ is not a stem. In fact for each $k$ there is a map-germ $f_k$ with an $m$-1 curve which has only isolated non-immersive points.

Proof:

Let $A = \{ \sigma = (a_1 x^k + a_2 x^{k+1}, a_3 x^k + a_4 x^{k+1}, a_5 x^k + a_6 x^{k+1}) : a_i \in \mathbb{C} \}$.

Claim: under these conditions the map $F(x, y, \sigma) = j^1(f(x, y) + \phi(x, y)\sigma)$ is transverse to $\Sigma^1 = \{2\times3$ matrices of rank $\leq 1\}$ except at isolated points. The result follows.

Proof of claim:

We identify $\Sigma^1$ with the manifold $\left\{ \left( \begin{array}{c} \lambda a \\ \mu a \\ \lambda b \\ \mu b \\ \lambda c \\ \mu c \end{array} \right) : a, b, c, \lambda, \mu \in \mathbb{C} \right\}$. Note that if, say $\lambda \neq 0$ then without loss of generality, $\lambda = 1$ so there should really be only 4 variables. Then the tangent space of $\Sigma^1$ is spanned by

$$
\left\{ \left( \begin{array}{c} \lambda \\ \mu \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ \lambda \\ \mu \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \lambda \\ \mu \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ b \\ c \end{array} \right) : a, b, c, \lambda, \mu \in \mathbb{C} \right\}
$$

(note that at any point this space has dimension 4).

Now for example

$$
\frac{\partial F}{\partial a_1} = \begin{pmatrix}
x^k \partial_x \phi + kx^{k-1} \phi & x^k \partial_x \phi \\
0 & 0 \\
0 & 0
\end{pmatrix}
$$
If $F(x, y, \sigma) \in \Sigma^1$ then $(\lambda: \mu)$ is in the ratio

$$(\partial_x(f_1 + \phi \sigma) : \partial_y(f_1 + \phi \sigma))$$

so transversality follows if the matrix

$$
\begin{pmatrix}
  x^k \partial_x \phi + kx^{k-1} \phi & x^k \partial_y \phi \\
  x^{k+1} \partial_x \phi + (k+1)x^k \phi & x^{k+1} \partial_y \phi \\
  \partial_x(f_1 + \phi \sigma) & \partial_y(f_1 + \phi \sigma)
\end{pmatrix}
$$

has rank 2.

If this fails then

$$x^k(\partial_x f_1 \partial_y \phi - \partial_y f_1 \partial_x \phi) + kx^{k-1} \phi \partial_y f_1 = 0$$

and

$$x^{k+1}(\partial_x f_1 \partial_y \phi - \partial_y f_1 \partial_x \phi) + (k+1)x^k \phi \partial_y f_1 = 0$$

so that we also have $x^k \phi \partial_y f_1 = 0$.

The condition fails when $x(\partial_x f_1 \partial_y \phi - \partial_y f_1 \partial_x \phi) = 0$ and $x \phi \partial_y f_1 = 0$.

Now the only curve which intersects $\Sigma^1$ at other than isolated points is $\phi = 0$, hence $x(\partial_x f_1 \partial_y \phi - \partial_y f_1 \partial_x \phi) = 0$ intersects $\Sigma^1$ only at isolated points.

This proves the claim and hence the proposition.

---

(III-2:12) Proposition: if $f$ has a double cuspidal edge and hypothesis III-2:10 holds then $f$ is not a stem.

Proof:

Let $f_k = (f_1, f_2, f_3 + x^k \phi)$ and $(x, y) \in C$.

Then

$$j^1 f_k(x, y) = \begin{pmatrix}
  \partial_x f_1 & \partial_y f_1 \\
  \partial_x f_2 & \partial_y f_2 \\
  \partial_x f_3 + x^k \partial_x \phi & \partial_y f_3 + x^k \partial_y \phi
\end{pmatrix}$$

Then $f_k$ is an immersion at all but isolated points of $C$ since

$$x^k \text{det} \begin{pmatrix}
  \partial_x f_1 & \partial_y f_1 \\
  \partial_x \phi & \partial_y \phi
\end{pmatrix} \notin (\phi).$$

Now $f_k$ has a double curve along $C$ and
has rank 2 since the tangent space of \( f \) along \( f(C) \) is tangent to \( f(C) \), i.e. the

\[
\begin{pmatrix}
\frac{\partial}{\partial x} f_1(z_1) & \frac{\partial}{\partial y} f_1(z_1) & \frac{\partial}{\partial x} f_1(z_2) & \frac{\partial}{\partial y} f_1(z_2) \\
\frac{\partial}{\partial x} f_2(z_1) & \frac{\partial}{\partial y} f_2(z_1) & \frac{\partial}{\partial x} f_2(z_2) & \frac{\partial}{\partial y} f_2(z_2) \\
\frac{\partial}{\partial x} (f_3+x^k \phi)(z_1) & \frac{\partial}{\partial y} (f_3+x^k \phi)(z_1) & \frac{\partial}{\partial x} (f_3+x^k \phi)(z_2) & \frac{\partial}{\partial y} (f_3+x^k \phi)(z_2)
\end{pmatrix}
\]

has rank 1 when \( f(z_1) = f(z_2) \). This proves the proposition.  \( \Box \)

Note: we cannot guarantee that the map-germ \((f_1, f_2, f_3 + x^k \phi)\) does not have a new
cuspidal edge off \( C \).

What happens if hypothesis III.2:10 fails, i.e. if \( \det \begin{pmatrix} \frac{\partial}{\partial x} f_i & \frac{\partial}{\partial y} f_i \end{pmatrix} \in (\phi) \) for all \( i \)? At a
point where \( C \) is smooth we may choose local coordinates so that \( \phi = y \). Then the
condition becomes \( \frac{\partial}{\partial x} f_i \in (y) \), and hence \( f_i \in (y) \) for all \( i \). Then \( f \) fails to be finite at
a smooth point of \( C \). Since \( C \) is smooth almost everywhere and \( f \) is finite at \( 0 \) this
gives a contradiction. Hence this hypothesis holds for all finite map-germs.

(III.2:13) Proposition: if \( f \) is a stem with an \( m-1 \) curve \((m \geq 3)\) then \( \Sigma \) has
transversal type consisting of immersed sheets which are pairwise transverse.

Proof: we assume that \( I = (\phi) = I(f^i(\Sigma)) \) is the reduced ideal defining \( C = f^i(\Sigma) \).
Since \( C \) is a curve, we can also assume that \( x=0 \) is not a branch of \( C \) so that
\( C \cap \{x=0\} \) consists of isolated points only. If \( y^k \in I \) for any \( k \) then \( C \subseteq V(y^k) \) and
hence \( \phi = y \) (since \( \phi \) defines a reduced curve). Then without loss of generality, either
\( \phi = y \) or \( x \) is not a factor of \( \phi \) and for each \( k \) \( y^k \notin I \).

We may also assume that
\[
\begin{pmatrix}
\partial_x f_1(z_1) & \partial_y f_1(z_1) & \partial_x f_1(z_2) & \partial_y f_1(z_2) \\
\partial_x f_2(z_1) & \partial_y f_2(z_1) & \partial_x f_2(z_2) & \partial_y f_2(z_2) \\
\partial_x f_3(z_1) & \partial_y f_3(z_1) & \partial_x f_3(z_2) & \partial_y f_3(z_2)
\end{pmatrix}
\]

has rank 2 when \(f(z_1) = f(z_2)\).

Let \(A = \{ \sigma = (a_1 x^k + a_2 x^{k+1}, a_3 x^k + a_4 x^{k+1}, a_5 x^k + a_6 x^{k+1}) : a_i \in \mathbb{C} \}, \)
\(B = \{ \tau = (b_1 y^k + b_2 y^{k+1}, b_3 y^k + b_4 y^{k+1}, b_5 y^k + b_6 y^{k+1}) : b_i \in \mathbb{C} \}. \)

If \(\varphi = y\) then we consider the unfolding \(f = f + \varphi \sigma\) where \(\sigma \in A\), otherwise we consider the unfolding \(f = f + \varphi (\sigma + \tau)\) where \(\sigma \in A\) and \(\tau \in B\).

It is enough to show that
\[
\begin{pmatrix}
\partial_x f_{\varphi,1}(z_1) & \partial_y f_{\varphi,1}(z_1) & \partial_x f_{\varphi,1}(z_2) \\
\partial_x f_{\varphi,2}(z_1) & \partial_y f_{\varphi,2}(z_1) & \partial_x f_{\varphi,2}(z_2) \\
\partial_x f_{\varphi,3}(z_1) & \partial_y f_{\varphi,3}(z_1) & \partial_x f_{\varphi,3}(z_2)
\end{pmatrix}
\]

has rank \(\leq 2\) at only isolated points of \(C\). If \(\partial_x \varphi \in I\) then replace the third column with the \(y\)-derivative; they cannot both vanish. Hence we may assume that \(\partial_x \varphi \notin I\).

We prove that \(F(z, z, \sigma, \tau) = \begin{pmatrix}
\partial_x f_{\varphi,1}(z_1) & \partial_y f_{\varphi,1}(z_1) & \partial_x f_{\varphi,1}(z_2) \\
\partial_x f_{\varphi,2}(z_1) & \partial_y f_{\varphi,2}(z_1) & \partial_x f_{\varphi,2}(z_2) \\
\partial_x f_{\varphi,3}(z_1) & \partial_y f_{\varphi,3}(z_1) & \partial_x f_{\varphi,3}(z_2)
\end{pmatrix}\)
is transverse to the submanifold \(S = \{ 3 \times 3 \text{ matrices of rank } \leq 2 \}\).

We identify \(S\) with the manifold
\[
\left\{ \begin{pmatrix}
\alpha a & \lambda b & \lambda c \\
\beta d & \beta e & \beta f \\
\gamma a + \delta d \gamma b + \delta e \gamma c + \delta f
\end{pmatrix} : \alpha, \beta, \gamma, \delta, a, b, c, d, e, f \in \mathbb{C} \right\}.
\]

To show that this is the correct set suppose that \(A\) is an element of \(S\). If the first row of \(A\) is \((0 \ 0 \ 0)\) then put \(\lambda = 0\) otherwise put \(\lambda = 1\) and \((a \ b \ c)\) equal to the first row of \(A\). Repeat this with the second row. If neither of these rows are zero then \(\gamma\) and \(\delta\) are uniquely defined so that \(A\) is of the above form. If \(\lambda = 0\) then let \(\gamma = 1\) and if \(\beta = 0\) let \(\delta = 1\). Choose \(a, \ldots, f\) appropriately to get the required form. Note it is never necessary to choose \((\alpha, \gamma)\) or \((\beta, \delta)\) = \((0, 0)\).
Then \( TS = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ a & b & c \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \alpha & 0 & 0 \\ \gamma & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix} \right\} \).

Now for example (differentiating with respect to \( a, a, \) and \( a \))

\[
TF \in \mathcal{C} \left( \begin{pmatrix} k \partial_x \varphi & x_1 \partial_y \varphi & x_2 \partial_x \varphi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 & 0 & 0 \\ k \partial_x \varphi & x_1 \partial_y \varphi & x_2 \partial_x \varphi \\ 0 & 0 & 0 \end{pmatrix} \right),
\]

\[
\begin{pmatrix} k \partial_x \varphi & x_1 \partial_y \varphi & x_2 \partial_x \varphi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1 \partial_x \varphi & x_1 \partial_y \varphi & x_2 \partial_x \varphi \end{pmatrix}
\]

If \( \varphi = y \) then

\[
\text{rank} \begin{pmatrix} k \partial_x \varphi & x_1 \partial_y \varphi & x_2 \partial_x \varphi \\ 0 & 0 & 0 \\ 0 \end{pmatrix} = \text{rank} \begin{pmatrix} k \partial_x \varphi & x_1 \partial_y \varphi & x_2 \partial_x \varphi \\ 0 & 0 & 0 \\ 0 \end{pmatrix}
\]

otherwise we add the rows

\[
(y_1 \partial_x \varphi, y_1 \partial_y \varphi, y_2 \partial_x \varphi), (y_1 \partial_x \varphi, y_1 \partial_y \varphi, y_2 \partial_x \varphi)
\]

to each matrix and the same result follows.

Now \( \begin{pmatrix} \partial_x f_1 & \partial_y f_1 & \partial_x f_1' \\ x_1 \partial_x \varphi & x_1 \partial_y \varphi & x_2 \partial_x \varphi \\ x_1^{k+1} \partial_x \varphi & x_1^{k+1} \partial_y \varphi & x_2^{k+1} \partial_x \varphi \end{pmatrix} \) has determinant \( x_1 x_2 (\partial_x f_1 \partial_y \varphi - \partial_y f_1 \partial_x \varphi)(x_1-x_2) \)
We have seen that without loss of generality \( \frac{\partial}{\partial x} f_1 \neq \frac{\partial}{\partial y} f_1 \neq \frac{\partial}{\partial x} f'_1 \) has determinant \( y_1 y_2 (\frac{\partial}{\partial x} f_1 \frac{\partial}{\partial y} \varphi - \frac{\partial}{\partial y} f_1 \frac{\partial}{\partial x} \varphi)(y_1 - y_2) \).

The sets

\[ \{(0, y_1, x_2, y_2) : f(0, y_1) = f(x_2, y_2)\}, \quad \{(x_1, y_1, 0, y_2) : f(x_1, y_1) = f(0, y_2)\} \]

consist of isolated points and we have either \( (x_1 - x_2) = 0 \) and \( y = 0 \) or \( (x_1 - x_2) \) and \( (y_1 - y_2) = 0 \) which both imply that \( (x_1, y_1) = (x_2, y_2) \).

This concludes the proof. \( \square \)

We have proved

(III-2:12) Theorem: if \( f \) is a stem then it is finite, \( \Sigma \) is an irreducible curve and exactly one of the following holds:

1) \( \Sigma \) has transversal type \( D_4 \).
2) \( \Sigma \) is a 1-1 cuspidal edge.
3) \( \Sigma \) is a double curve with either one or two pre-images; The transversal type is \( A_{2m+1} \) for some \( m \geq 1 \).
4) \( \Sigma \) is an \( m \)-1 curve (\( m > 3 \)) with irreducible preimage and transversal type consisting of immersed sheets which are pairwise transverse.
5) \( \Sigma \) is a 4-1 curve with two pre-images, \( f \) being 2-1 on each pre-image. The transversal type of \( \Sigma \) consists of immersed sheets which are pairwise transverse.

(III-2:13) Conjecture: if \( f \) is a stem and (2) or (3) holds then the transversal type is \( A_2 \) or \( A_3 \).

Example: \( (x, y^2, y^5) \) has transversal type \( A_4 \):

for each \( k \) \( (x, y^2, y^5 + x^k y^3) \) has transversal type \( A_2 \).

Example: \( (x, y^2, x^3 y) \) has transversal type \( A_5 \)

for each \( k \) \( (x, y^2, x^3 y + x^2 y^{2s+1}) \) has transversal type \( A_3 \).
For a given example we suggest looking at deformations of the form

\[ f + \varphi p \text{ or } f + \varphi^2 p \]

for some suitably chosen \( p \in \Theta(f) \).

Progress on this problem is limited by the way that the transversal type is defined: it is found by looking at a smooth point of \( \Sigma \) where we can assume that \( \Sigma = \{ (X, 0, 0): X \in \mathbb{C} \} \). We must then study the function \( p(X, Y, Z) \) which defines the image and how it changes as we change \( f \). Alternatively we can study the module \( \Theta(f)/T^\infty f \) to see how its generators are related.

(III.2:14) Conjecture: if \( f \) satisfies (4) or (5) then it is not a stem.

Example: \((x, xy + xy^3, y^4)\) is 4-1 along \((0, y)\).

Example: \((xy, x^2 + y^2, x^2 y + xy^4)\) has 2-1 curves

\[ (0, y) \rightarrow (0, y^2, 0) \text{ and } (x, 0) \rightarrow (0, x^2, 0). \]

In the case \( m > 4 \) is not a prime we have for example

\((x, xy, y^6)\) and the family \((x, xy + y^{6k+3}, y^6)\).

We expect it to be simple to reduce the problem to the case that \( m = 4 \) or \( m \) is prime. However from this point progress has halted. We have tried to gain insight into it by looking at the above examples. We have not been able to find map-germs with triple curves which have these 5-jets, nor have we been able to prove that no such map-germs exist. We have no suggestions as to how to find a deformation in a given case.
We calculate the numbers defined in [M–M] and [Mond 1] for various families \{f_k: k \geq k_0\} of map-germs of corank 1 calculated in the previous sections. We shall assume that each map-germ is of the form \(f = (x, f_2(x, y), f_3(x, y))\). These numbers are

\[C = \dim \mathcal{O}_2/Q_f,\]
\[T = \dim \mathcal{O}_3/\mathcal{F}_2,\]
\[\mu(\tilde{D}^2(f)/\mathbb{Z}_2),\]

where \(Q_f\) is the ramification ideal \(Q_f = \langle \partial f_2/\partial y, \partial f_3/\partial y \rangle\), \(\mathcal{F}_2\) is the second Fitting ideal of the presentation matrix of \(f\), \(\tilde{D}^2(f) \subseteq \mathbb{C} \times \mathbb{C}^2\) is the double point scheme defined by the ideal

\[\langle f_2(x, y) - f_2(x, y'), f_3(x, y) - f_3(x, y') \rangle/\langle y - y', y - y' \rangle\]

and \(\tilde{D}^2(f)/\mathbb{Z}_2\) is the quotient scheme under the symmetric group \(\mathbb{Z}_2\) which permutes \(y\) and \(y'\); if \(h(x, y, y') = (x, yy', y+y')\), \(\tilde{D}^2(f) = V(H)\) and \(\tilde{D}^2(f)/\mathbb{Z}_2 = V(H')\) then

\[H = H' \circ h.\]

These numbers are related to the geometry although the relation may not be obvious:

- the number \(C\) is the cross-cap number, i.e. the number of cross-caps which appear in a generic deformation of \(f\). It is infinite if and only if \(f\) has a cuspidal edge.

- the number \(T\) is the triple point number. We expect that an infinite value for \(T\) is connected with the presence of a triple curve.

For example, \(T\) is infinite for \((x, y^3, y^4)\) and \((x, y^3, y^4 + x^k y)\) has a triple curve for every \(k\).

We expect that an infinite value for \(\mu(\tilde{D}^2(f)/\mathbb{Z}_2)\) is related to the presence of a double curve whose transversal type is \(A_{2n+1}\) for some \(n \geq 1\).

For example, \(\mu(\tilde{D}^2(f)/\mathbb{Z}_2)\) is infinite for \((x, y^2, y^5)\) and \((x, y^2, (x^{2k} - y^2)^2 y)\) has such a curve.
This last number is related to \( \mu(\tilde{D}^2(f)) \) by the formula
\[
\mu(\tilde{D}^2(f)) = 2\mu(\tilde{B}^2(f)/\mathbb{Z}_2) + C - 1.
\]
Consequently, \( \mu(\tilde{D}^2(f)) \) is finite if and only if \( \mu(\tilde{B}^2(f)/\mathbb{Z}_2) \) and \( C \) are both finite.

We aim to test the conjecture
\[
\text{cod} (\mathcal{A}, f) - \left( C + T + \mu(\tilde{B}^2(f)/\mathbb{Z}_2) \right)
\]
is a constant for the family of map-germs related to a weak stem.

For map-germs which are weighted homogeneous R. Pellikaan and T. de Jong have recently announced a proof of the formula
\[
\text{cod} (\mathcal{A}, f) - \left( C + T + \mu(\tilde{D}^2(f)/\mathbb{Z}_2) \right) = -1
\]
so the conjecture is true for weighted homogeneous families, eg. \((x, y^2, y^3 + x^ky)\).

We shall make use of the fact that for these families \( \{f_k : k \geq k_0\} \) some of the numbers are independent of \( k \).

(III-3:1) lemma: if \( C(f) \) is finite then there is a value \( k \) such that
\[
j^k g = j^k f \Rightarrow C(g) = C(f).
\]
Proof: if \( C \) is finite then
\[
Q_f \supseteq \mathfrak{m}_2 \text{ for some } r.
\]
Let \( k = r \) and write \( g \) in the form \( f + h, h \in \mathfrak{m}_2^{r+1} \), then
\[
Q_g = \langle \partial g_2/\partial y, \partial g_3/\partial y \rangle \\
= \langle \partial f_2/\partial y + \partial h_2/\partial y, \partial f_3/\partial y + \partial h_2/\partial y \rangle \\
= \langle \partial f_2/\partial y, \partial f_3/\partial y \rangle = Q_f
\]
since \( h \in \mathfrak{m}_2^{r+1} \).

Hence \( C(f) = C(g) \).

(III-3:2) lemma: if \( \mu(\tilde{D}^2(f)/\mathbb{Z}_2) \) is finite then there is a value \( k \) such that
\[
j^k g = j^k f \Rightarrow \mu(\tilde{D}^2(g)/\mathbb{Z}_2) = \mu(\tilde{D}^2(f)/\mathbb{Z}_2).
\]
Proof: \( \mu(\mathcal{D}^2(f)/\mathbb{Z}_2) \) is finite \( \iff \hat{\mathcal{D}}^2(f)/\mathbb{Z}_2 \) has an isolated singularity
\[ \iff H' \text{ is finitely determined with respect to } \mathcal{H}\text{-equivalence.} \]

But then \( \hat{\mathcal{D}}^2(f)/\mathbb{Z}_2 \) (and hence \( \mu(\mathcal{D}^2(f)/\mathbb{Z}_2) \)) is determined by the \( r \)-jet of \( H' \). Then \( H = H' \circ h \) so \( H' \) is determined by the \( 2r \)-jet of \( H \), which is determined by the \( (2r+1) \)-jet of \( f \).

(III·3·3) lemma: if \( T(f) \) is finite then there is a value \( k \) such that
\[ j^k g = j^k f \Rightarrow T(g) = T(f). \]

Proof: \( T = \frac{1}{6} \dim_{\mathbb{C}} \mathcal{O}_{\mathcal{D}^3 f} \) so \( T \) is finite \( \iff \mathcal{D}^3 f \) has an isolated singularity
\[ \iff \text{the map defining } \mathcal{D}^3 f \text{ is finitely determined with respect to } \mathcal{E}\text{-equivalence.} \]

But then \( \mathcal{D}^3 f \) (and hence \( T \)) is determined by the \( r \)-jet of the map, which is determined by a \( k \)-jet of \( f \) for some \( k \).

The results are probably also true for map-germs having corank 2.

For a geometric stem the numbers are quite well-behaved:

(III·3·4) lemma: if \( f \) is a geometric stem then exactly two of the three numbers \( C(f), T(f) \) and \( \mu(\mathcal{D}^2(f)/\mathbb{Z}_2) \) are finite.

Proof: suppose first that \( \Sigma \) has transversal type \( A_2 \). We show that \( \hat{\mathcal{D}}^2(f)/\mathbb{Z}_2 \) has at worst an isolated singularity and that the triple point set is at worst an isolated point. It is sufficient to look at a generic point of \( \Sigma \); then we can assume that \( f \) is equivalent to \( (x, y^2, y^3) \).

This map-germ has the presentation matrix
\[ \begin{bmatrix} -Z & Y \\ Y^2 & -Z \end{bmatrix} \]so \( T = 0 \).

Also \( \mathcal{D}^2(f) = V(y+y', y^2 + yy' + y'^2) \).

Let \( \mathcal{D}^2(f)/\mathbb{Z}_2 = V(H') \), where \( H' = H'(\sigma, \tau) \) and \( H'(y+y', yy') = (y+y', y^2 + yy' + y'^2) \)
then \( H'(\sigma, \tau) = (\sigma, \sigma^2 - \tau) \), so \( \mathcal{D}^2(f)/\mathbb{Z}_2 \) is smooth and \( \mu(\mathcal{D}^2(f)/\mathbb{Z}_2) = 0 \).

This proves the assertion for the transversal type \( A_2 \).

It is clear that for the remaining cases (\( A_3 \) and \( D_4 \)) \( C \) must be finite since the map-
germ obviously can't have a cuspidal edge.

The case $A_3$ at a generic point of $\Sigma$ is identical to the case for $(x, y^2, x^2y)$ at a generic point of $\Sigma$, i.e. $T = 0$ except at $(0, 0)$ [Mond 2].

Similarly, by comparison with $(x, xy, y^3)$, $\mu(\tilde{D}^2(f)/\mathbb{Z}_2) = 0$ at a generic point of $\Sigma$ when it has transversal type $D_4$.

Finally, we note that for each family $f + p_s$ $\text{cod} (\mathcal{S}, f + p_{s+1}) = \text{cod} (\mathcal{S}, f + p_s) + 1$ (section I-3, p. 55) so that $\text{cod} (\mathcal{S}, f + p_s) = s + c(f)$, where $c(f)$ is a constant depending only on $f$. This means that we need only check that

$$C(f+ p_s) + T(f+ p_s) + \mu(\tilde{D}^2(f+ p_s)/\mathbb{Z}_2) = s + d(f)$$

for some constant $d(f)$.

The table below shows that this is satisfied by several families (the reference refers to the calculation in Appendix B. One calculation is included here to demonstrate the method.

**Table 7.** $\mathcal{S}$-invariants for the families associated with some weak stems.

<table>
<thead>
<tr>
<th>families</th>
<th>C</th>
<th>T</th>
<th>$\mu(\tilde{D}^2(f)/\mathbb{Z}_2)$</th>
<th>ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x, y^2, x^3y + x^2y^3 + y^{2k+1})$</td>
<td>3</td>
<td>0</td>
<td>$k+1$</td>
<td>B:1</td>
</tr>
<tr>
<td>$(x, y^2, x^4y + 2x^2y^3 + y^5 + x^ky)$</td>
<td>4</td>
<td>0</td>
<td>$k-1$</td>
<td>B:2</td>
</tr>
<tr>
<td>$(x, y^2, x^4y + x^2y^3 + y^{2k+1})$</td>
<td>4</td>
<td>0</td>
<td>$k+1$</td>
<td>B:3</td>
</tr>
<tr>
<td>$(x, y^2, x^2y^3 + y^5 + x^ky)$</td>
<td>$k$</td>
<td>0</td>
<td>$3$</td>
<td>B:4</td>
</tr>
<tr>
<td>$(x, xy+y^3, xy^2 + y^{3k+1})$</td>
<td>3</td>
<td>$k$</td>
<td>1</td>
<td>B:5</td>
</tr>
<tr>
<td>$(x, xy+y^3, xy^2 + y^4 + y^{2k})$</td>
<td>3</td>
<td>$k-1$</td>
<td>1</td>
<td>B:6</td>
</tr>
<tr>
<td>$(x, xy+y^3, 2xy^2 + 3y^4 + y^k)$</td>
<td>$k$</td>
<td>1</td>
<td>1</td>
<td>B:7</td>
</tr>
<tr>
<td>$(x, xy+y^3, xy^4 + y^5)$</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>B:8</td>
</tr>
<tr>
<td>$(x, xy+y^3, xy^4 + y^{3k+2})$</td>
<td>5</td>
<td>$k+2$</td>
<td>6</td>
<td>B:9</td>
</tr>
<tr>
<td>$(x, x^2y + y^3, xy^2 + y^{3k+1})$</td>
<td>5</td>
<td>$k$</td>
<td>1</td>
<td>B:10</td>
</tr>
<tr>
<td>$(x, y^3, x^2y + y^4 + y^{3k+2})$</td>
<td>4</td>
<td>$2k$</td>
<td>2</td>
<td>see below</td>
</tr>
<tr>
<td>$(x, y^3, x^2y + y^4 + xy^{3k+2})$</td>
<td>4</td>
<td>$2k+1$</td>
<td>2</td>
<td>see below</td>
</tr>
<tr>
<td>$(x, x^2y - y^3, x^2y^2 - y^4 + x^ky^2)$</td>
<td>6</td>
<td>$k$</td>
<td>2</td>
<td>B:11</td>
</tr>
</tbody>
</table>
For example \((x, y^3, x^2 y + y^4 + y^{3k+2})\), \((x, y^3, x^2 y + y^4 + xy^{3k+2})\).

\[ Q_f = \langle y^2, x^2 \rangle \text{ therefore } C = 4. \]

\[ \mathcal{D}^2(x, y^3, x^2 y + y^4) = V(y^2 + y y' + y^2, x^2 + y^3 + y^2 y' + y y'^2 + y'^3) = V(y^2 + y y' + y^2, x^2 + y^3). \]

We show that \(h_f = (y^2 + y y' + y^2, x^2 + y^3)\) is \(\mathcal{C}\)-finite (hence \(h_f\) has an isolated singularity so that \(\mu(\mathcal{D}^2(x, y^3, x^2 y + y^4))\) is finite—then

\[ \mu(\mathcal{D}^2(x, y^3, x^2 y + y^4)) = \mu(\mathcal{D}^2(x, y^3, x^2 y + y^4 + (x)y^{3k+2}) \text{ as above}). \]

Since \(h_f\) is weighted homogeneous \(\mu(h_f) = \dim \mathfrak{g}_3 / \langle h_f, \mathfrak{g}_h \rangle\) which is finite if and only if \(h_f\) is \(\mathcal{C}\)-finite. Then \(\mu(\mathcal{D}^2(x, y^3, x^2 y + y^4)) = \dim \mathfrak{g}_3 / \mathfrak{g} \) where

\[ I = \langle y^2 + y y' + y^2, x^2 + y^3, x(2y + y'), x(y + 2y'), y(2y + 2y') \rangle \]

\[ = \langle x^3, xy, xy', x^2 + y^3, y^2 + y y' + y^2, y^3 + 2 y^2 y' \rangle \]

\[ \supseteq \{ y^4, y^3 y', y^2 y^2, y y^3, y^4 \}. \]

Hence \(\mu(\mathcal{D}^2(f)) = 7 = 2\mu(\mathcal{D}^2(f)/\mathbb{Z}_2) + C - 1 = \mu(\mathcal{D}^2(f)/\mathbb{Z}_2) = 2. \]

We calculate a presentation for \((x, y^3, x^2 y + y^4 + y^{3k+2})\).

\[
\begin{array}{ccc}
1 & y & y^2 \\
x^2 y + y^4 + y^{3k+2} & -Z & X^2 + Y \\
x^2 y^2 + y^5 + y^{3k+3} & Y^{k+1} & -Z \\
x^2 y^3 + y^6 + y^{3k+4} & X^2 Y + Y^2 & Y^{k+1} \\
\end{array}
\]

Then \(\mathcal{A}_2 = \{ Z, Y + X^2, Y^{k} \} \mathfrak{g}_3\). Let \(Y' = Y + X^2\), then \(\mathcal{A}_2 = \{ Z, Y', X^{2k} \} \mathfrak{g}_3\) and \(T = 2k\). 

We calculate a presentation for \((x, y^3, x^2 y + y^4 + xy^{3k+2})\).

\[
\begin{array}{ccc}
1 & y & y^2 \\
x^2 y + y^4 + y^{3k+2} & -Z & X^2 + Y \\
x^2 y^2 + y^5 + y^{3k+3} & XY^{k+1} & -Z \\
x^2 y^3 + y^6 + y^{3k+4} & X^2 Y + Y^2 & XY^{k+1} \\
\end{array}
\]

Then \(\mathcal{A}_2 = \{ Z, Y + X^2, XY^{k} \} \mathfrak{g}_3\). Let \(Y' = Y + X^2\), then \(\mathcal{A}_2 = \{ Z, Y', X^{2k+1} \} \mathfrak{g}_3\) and \(T = 2k+1\). 

Appendix A: Calculations for Chapter II.

Calculations of $T \not\propto f$ for non-finitely determined map-germs in sections II-3 and II-4.

\[(A-1)\]

Let $C = \left\{ \begin{array}{l}
\mathbb{A}_2 \backslash \{(y, y, y^3) : j \geq 0\} \\
\mathbb{A}_2 \backslash \{(y, y^3, y^5, y^7) : j \geq 0\}
\end{array} \right\} + \mathbb{C} \left\{ \begin{array}{l}
y^{3j+1} \\
y^{3j+2} \\
y^{3j+3}
\end{array} \right\} : j \geq 0 \right\}

Then $f \ast m_3C = \left\langle x, x^2, x^3, y^3 \right\rangle + \mathbb{C} \left\langle x, x^2, x^3, y^3 \right\rangle \left( \frac{\partial}{\partial x} \right) + \mathbb{C} \left\langle x, x^2, x^3, y^3 \right\rangle \left( \frac{\partial}{\partial y} \right) + \mathbb{C} \left\langle x, x^2, x^3, y^3 \right\rangle \left( \frac{\partial}{\partial z} \right)

\begin{align*}
\left( \frac{\partial}{\partial x} \right) + \mathbb{C} \left\langle x, x^2, x^3, y^3 \right\rangle \left( \frac{\partial}{\partial y} \right) + \mathbb{C} \left\langle x, x^2, x^3, y^3 \right\rangle \left( \frac{\partial}{\partial z} \right)
\end{align*}

$C$ is spanned by:

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{bmatrix}
\]

$T \not\propto f$ is spanned by:

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{bmatrix}
\]

Hence $C = T \not\propto f + f \ast m_3C$ so that $C = T \not\propto f$. 

(A.2) \[ f = (x, xy + y^3, xy^2) \]

Let \( C = \left( m_2 \{ y, y^3 \} : j \geq 0 \right) + C \left( y^3 \right) \begin{pmatrix} \begin{array}{c} 0 \\ y^2 \\ y^3 \end{array} \end{pmatrix} : j \geq 0 \)

\[ = \left( \langle x \rangle + C \{ y^3, y^{3j+4} : j \geq 0 \} \right) \frac{\partial}{\partial x} + \left( \langle x, y^3 \rangle \right) \frac{\partial}{\partial y} \]

\[ = (x, xy^2) + C \{ x, y^{3j+5}, y^{3j+6} : j \geq 0 \} \frac{\partial}{\partial z} + C \left( y^{3j+4} \right) \begin{pmatrix} \begin{array}{c} 0 \\ y^2 \\ y^3 \end{array} \end{pmatrix} : j \geq 0 \]

Then \( f^* m_3 \cdot C = \left( \langle x, y^3 \rangle + C \{ y^{3j+6}, y^{3j+7} : j \geq 0 \} \right) \frac{\partial}{\partial x} + \left( \langle x^2, y^3 \rangle \right) \frac{\partial}{\partial y} \]

\[ + \left( \langle x^2, x^2 y^2, xy^5 \rangle + C \{ x^2, x^2 y + x y^3, y^{3j+8}, y^{3j+9} : j \geq 0 \} \right) \frac{\partial}{\partial z} \]

\[ = \left( \begin{array}{c} 0 \\ xy \\ xy^2 \\ y^3 \end{array} \right) \begin{pmatrix} \begin{array}{c} 0 \\ 0 \\ y^2 \\ y^3 \end{array} \end{pmatrix} : j \geq 1 \]

\( C \) is spanned by: \( x \frac{\partial}{\partial x}, xy \frac{\partial}{\partial x}, y \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, xy \frac{\partial}{\partial y}, y^4 \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial y}, y^5 \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial z}, y^6 \frac{\partial}{\partial z}, y^7 \frac{\partial}{\partial z} \)

\( T:= f \) is spanned by: \( x \frac{\partial}{\partial x}, (xy+y^3) \frac{\partial}{\partial x}, xy^2 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, (xy+y^3) \frac{\partial}{\partial y}, y^2 \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial z}, y^3 \frac{\partial}{\partial z}, y^4 \frac{\partial}{\partial z}, (xy+y^3) \frac{\partial}{\partial z} \)

\[ \left( \begin{array}{c} x \\ xy_2 \\ y^2 \\ y^3 \\ y^4 \\ y^5 \\ y^6 \\ y^7 \\ xy^2 \\ xy^3 \\ xy^4 \\ xy^5 \\ xy^6 \\ xy^7 \end{array} \right) \]
Working modulo $f^*m_3'C$ we can replace the last 4 elements by

$$\begin{pmatrix}0 & 0 & -2xy^2 \\ 3xy^2 & xy+3y^3 & 0 \\ 2x^2y & 2xy^2 & 3y^5 \\ -2x^2y & 0 & 0 \end{pmatrix}$$

Hence $C = T \not\subseteq f + f^*m_3'C$ so that $C = T \not\subseteq f$.

\[(A.3)\quad f = (x, xy + y^3, xy^2 + y^4)\]

Let $C = \left\{ \begin{array}{c} j2t+1 \{x, x y : j+\ell \geq 2\} + \{x y : (x+y):j, j \geq 0\} \\
\{x, x, x y : j+\ell \geq 2\} + \{y(x+y), x y (x+y):j, j \geq 0\} \end{array} \right\} + C \left\{ \begin{array}{c} j2t+1 \{y\} \\
0 \end{array} \right\}$

$$+ C \left\{ \begin{array}{c} j2t+1 \{y\} \\
0 \end{array} \right\} = \left( \begin{array}{c} x, y^2 \end{array} \right) \frac{\partial}{\partial x} + \left( \begin{array}{c} x^2 y, y^3 \end{array} \right) \frac{\partial}{\partial y}$$

Then $f^*m_3'C = \left( \begin{array}{c} x^2, xy^2, y^5 \end{array} \right) \frac{\partial}{\partial x} + C \left\{ \begin{array}{c} x^2, x^2 y + xy^3, x j + 1 y^{2t}, x y^{2t+4} : j + \ell \geq 2 \end{array} \right\} \frac{\partial}{\partial y}$

$$+ C \left\{ \begin{array}{c} y^6, x j + 1 y^{2t+1}(x+y^2), y^{2t+5}(x+y^2):j, j \geq 0 \end{array} \right\} \frac{\partial}{\partial y} + \left( \begin{array}{c} x^2, x j + 1, x j + 1 y^{2t+1}, x y(x+y^2), y^{2t+5}:j + \ell \geq 2 \end{array} \right\} \frac{\partial}{\partial y}$$

$$+ C \left\{ \begin{array}{c} j2t+1 \{y\} \\
0 \end{array} \right\} = \left\{ \begin{array}{c} x, y^2, y^3, y^4 \end{array} \right\} = \left\{ \begin{array}{c} \{x, y, y, y \} \\
\{x, x y : j+\ell \geq 3\} + \{x y : (x+y):j+\ell \geq 2\} \\
\{x, x, x y : j+\ell \geq 3\} + \{x y(x+y), x y (x+y):j+\ell \geq 1\} \end{array} \right\} + C \left\{ \begin{array}{c} j2t+1 \{y\} \\
0 \end{array} \right\}$$

$$+ C \left\{ \begin{array}{c} j2t+1 \{y\} \\
0 \end{array} \right\} = \left\{ \begin{array}{c} x + y^3 \end{array} \right\} \frac{\partial}{\partial x} + \left\{ \begin{array}{c} x y + x y^3 \end{array} \right\} \frac{\partial}{\partial y}$$

$C$ is spanned by: $x \frac{\partial}{\partial x}, xy \frac{\partial}{\partial x}, y^2 \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, y^4 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, xy^2 \frac{\partial}{\partial y}, (x+y^3) \frac{\partial}{\partial y}$, $(x^2+y^4) \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial y}, xy^3 \frac{\partial}{\partial y}, y^4 \frac{\partial}{\partial y}$,
\[ T \mathrel{\not\subset} f \text{ is spanned by: } x \frac{\partial}{\partial x}, (xy+y^3) \frac{\partial}{\partial x}, (xy^2+y^4) \frac{\partial}{\partial x}, (xy+y^3) \frac{\partial}{\partial y}, (xy^2+y^4) \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, (xy+y^3) \frac{\partial}{\partial z}, (xy^2+y^4) \frac{\partial}{\partial z}. \]

\[
\begin{pmatrix}
  x \\
  xy \\
  xy^2
\end{pmatrix}, \begin{pmatrix}
  y \\
  y^2 \\
  y^3 \\
\end{pmatrix}, \begin{pmatrix}
  y^2 \\
  y^4 \\
  y^5 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  2x+3xy^2 \\
  2x^2+4y^3 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  x+y+3y^3 \\
  2xy^2+4y^3 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  2xy^3 \\
  2xy^4+4y^6 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  x^2 + y^5 \\
  2xy^4+4y^6 \\
\end{pmatrix}
\]

Hence \( C = T \mathrel{\not\subset} f + f^* m_3 C \) so that \( C = T \mathrel{\not\subset} f \).

(A.4)

\[ f = (x, xy + y^3, 2xy^2 + 3y^4) \]

Let \( C = C[\{(j+2)x^iy^j + 3jx^jy^{j+2} : \ell \geq 1, j \geq 0\} \frac{\partial}{\partial x} + C[\{(j+2)x^iy^j + 3jx^jy^{j+2} : \ell \geq 1, j \geq 0\} \frac{\partial}{\partial y} + C[\{(j+2)x^iy^j + 3jx^jy^{j+2} : \ell \geq 1, j \geq 0\} \frac{\partial}{\partial z} + C\{(j+2)x^iy^j + 3jx^jy^{j+2} : \ell \geq 1, j \geq 0\} \frac{\partial}{\partial y} + C\{(j+2)x^iy^j + 3jx^jy^{j+2} : \ell \geq 1, j \geq 0\} \frac{\partial}{\partial z} \]

Clearly \( f^* m_3 C \supseteq C[\{(j+2)x^iy^j + 3jx^jy^{j+2} : \ell \geq 1, j \geq 0\} (c = \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})] \]

Then \( (j+3)(xy + y^3)(j+2)x^iy^j + 3jx^jy^{j+2} \]

\[ = (j+2)(j+3)x^iy^jy^{j+1} + 3jx^jy^{j+1} + 3(j+1)x^jy^{j+3} + 3(j+3)x^jy^{j+5} \]

since \( (j+3)(j+2)+3j(j+3) = 4j^2 + 14j + 6 = 3(j+1)(j+2)+j(j+5) \).

hence \( f^* m_3 C \supseteq C[\{(j+2)x^iy^j + 3jy^{j+2} : \ell \geq 1, j \geq 0\} (c = \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})] \]

We check that

\[ (j+4)(2xy^2 + 3y^4)(j+2)x^iy^j + 3jx^jy^{j+2} = 2(j+2)(j+4)x^iy^jy^{j+2} + 3(j+2)x^jy^{j+4} + 3(j+4)x^jy^{j+6} \]

since \( 3(j+4)(j+2)+6j(j+4) = 9j^2 + 42j + 24 = 3(j+6)(j+6)+6(j+2)^2 \).

We also have \( f^* m_3 C \supseteq C\left\{ \begin{pmatrix}
  x \\
  xy \\
  xy^2
\end{pmatrix}, \begin{pmatrix}
  y \\
  y^2 \\
  y^3 \\
\end{pmatrix}, \begin{pmatrix}
  y^2 \\
  y^4 \\
  y^5 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  2x+3xy^2 \\
  2x^2+4y^3 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  x+y+3y^3 \\
  2xy^2+4y^3 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  x^2 + y^5 \\
  2xy^4+4y^6 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  x^2 + y^5 \\
  2xy^4+4y^6 \\
\end{pmatrix} : j \geq 2 \right\} \]

For \( j \geq 3, 2(j+3)(j+1)(j+2)x^iy^jy^{j+2} \frac{\partial}{\partial x} \] and \( 2j \frac{\partial}{\partial y} (j+4)x^iy^jy^{j+2} + 3(j+2)x^jy^{j+4} \frac{\partial}{\partial z} \)

as are \( (j+4)(j+2)x^iy^jy^{j+2} \frac{\partial}{\partial x} \) and \( 2j \frac{\partial}{\partial y} (j+4)x^iy^jy^{j+2} + 3(j+2)x^jy^{j+4} \frac{\partial}{\partial z} \)

hence \( f^* m_3 C \supseteq C\left\{ \begin{pmatrix}
  x \\
  xy \\
  xy^2
\end{pmatrix}, \begin{pmatrix}
  y \\
  y^2 \\
  y^3 \\
\end{pmatrix}, \begin{pmatrix}
  y^2 \\
  y^4 \\
  y^5 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  2x+3xy^2 \\
  2x^2+4y^3 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  x+y+3y^3 \\
  2xy^2+4y^3 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  x^2 + y^5 \\
  2xy^4+4y^6 \\
\end{pmatrix}, \begin{pmatrix}
  0 \\
  x^2 + y^5 \\
  2xy^4+4y^6 \\
\end{pmatrix} : j \geq 5 \right\} \]

Finally, to establish that \( C \) is indeed an \( 3 \)-module via \( f^* \) we show that
\[
\begin{align*}
(j+3)(j+5)(xy + y^3) & \begin{pmatrix} -(j + 2)y^j \\ 0 \\ 2jy^{j+2} \end{pmatrix} = -(j+2)(j+5)(j + 3)xy^{j+1} + 3(j+1)y^{j+3} \frac{\partial}{\partial x} \\
+2j(j+3)(j + 5)xy^{j+3} + 3(j+3)y^{j+5}) \frac{\partial}{\partial z} - 2j(j+2) & \begin{pmatrix} -(j + 5)y^{j+3} \\ 0 \\ 2(j+3)y^{j+5} \end{pmatrix} \\
(j+4)(j+6)(2xy^2+3y^4) & \begin{pmatrix} -(j + 2)y^j \\ 0 \\ 2jy^{j+2} \end{pmatrix} = -2(j+2)(j+6)(j + 4)xy^{j+2} + 3(j+2)y^{j+4}) \frac{\partial}{\partial x} \\
+4j(j+4)(j + 6)xy^{j+4} + 3(j+4)y^{j+6}) \frac{\partial}{\partial z} - 3j(j+2) & \begin{pmatrix} -(j + 6)y^{j+4} \\ 0 \\ 2(j+4)y^{j+6} \end{pmatrix} \\
(j+4)(j+3)(xy + y^3) & \begin{pmatrix} 2(j + 1)y^j \\ jy^{j+1} \\ 0 \end{pmatrix} = 2(j+4)(j+1)(j + 3)xy^{j+1} + 3(j+1)y^{j+3} \frac{\partial}{\partial x} \\
+j(j+3)(j+4)xy^{j+2} + 3(j+2)y^{j+4}) \frac{\partial}{\partial y} - 2j(j+1) & \begin{pmatrix} 2(j + 4)y^{j+3} \\ 0 \\ (j + 3)y^{j+4} \end{pmatrix} \\
(j+5)(j+4)(2xy^2+3y^4) & \begin{pmatrix} 2(j + 1)y^j \\ jy^{j+1} \\ 0 \end{pmatrix} = 4(j+5)(j+1)(j+4)xy^{j+2} + 3(j+2)y^{j+4}) \frac{\partial}{\partial x} \\
+2j(j+4)(j+5)xy^{j+3} + 3(j+3)y^{j+5}) \frac{\partial}{\partial y} - 3j(j+1) & \begin{pmatrix} 2(j + 5)y^{j+4} \\ 0 \\ (j + 4)y^{j+5} \end{pmatrix} 
\end{align*}
\]

C is spanned by: \((j+2)x \, y^j + 3jy^{j+2}e\) for \(0 \leq j \leq 2\), i.e.

\[
\begin{align*}
x & \frac{\partial}{\partial x}, (xy+y^3) \frac{\partial}{\partial x}, (2xy^2+3y^4) \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, (xy+y^3) \frac{\partial}{\partial y}, (2xy^2+3y^4) \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, \\
(xy+y^3) \frac{\partial}{\partial z}, (2xy^2+3y^4) \frac{\partial}{\partial z}, \\
\begin{pmatrix} y \\ y^2 \\ 2y^3 \end{pmatrix}, \begin{pmatrix} -y^2 \\ -5y^3 \\ -6y^4 \end{pmatrix}, \begin{pmatrix} -5y^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6y^4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (6y^2)^2 \\ (2y^3) \\ (3y^4) \end{pmatrix}, \begin{pmatrix} (8y^3) \\ (8y^6) \\ (10y^5) \end{pmatrix}, \begin{pmatrix} (10y^4) \\ (4y^5) \end{pmatrix}
\end{align*}
\]
is spanned by: 

$$(xy+y^3)\frac{\partial}{\partial x}, (2xy^2+3y^4)\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, (xy+y^3)\frac{\partial}{\partial y}, (2xy^2+3y^4)\frac{\partial}{\partial y}, x\frac{\partial}{\partial z},$$

$$(xy+y^3)\frac{\partial}{\partial z}, (2xy^2+y^4)\frac{\partial}{\partial z},$$

\[
\begin{pmatrix}
    x & y^2 & y^3 & 0 & 0 & 0 & 0 \\
    xy & y^2 & y^3 & x^2 + 2xy^2 & xy + 3y^3 & 2xy^3 + 3y^4 & 3xy^3 + 3y^5 \\
    2xy & y^2 & y^3 & 2x^2 y + 12xy^3 & 4xy^3 + 12y^4 & 4xy^3 + 12y^5 & 4xy^3 + 12y^6
\end{pmatrix}
\]

Rewrite the last 3 columns modulo $f \star \mathbb{m}_3 \cdot C$ to get

\[
\begin{pmatrix}
    0 & 0 & 0 \\
    y^3 & 5y^4 & 3y^5 \\
    3y^4 & 16y^5 & 10y^6
\end{pmatrix}
\]

bearing in mind that $f \star \mathbb{m}_3 \cdot C \ni 5x y^3 + 9y^5 e, 6x y^4 + 12y^6 e.$
Hence $C = T \not\subseteq f + f \star \mathbb{m}_3 \cdot C$ so that $C = T \not\subseteq f.$

(A.5) 

$$f = (x, xy + y^3, 2xy^2 + y^4)$$

Let $C = C \{(x+y^2)x^l, ((j+1)x+xy+j^2 + 3)xy^2, 0 \geq l \geq 0\} \frac{\partial}{\partial x} + C \{(x+y^2)x^l, l \geq 0\} \frac{\partial}{\partial y} + C \{(x+y^2)x, x^l, 0 \geq l \geq 0\} \frac{\partial}{\partial z}.$

It is clear that

$$f \star \mathbb{m}_3 \cdot C \ni C \{(x+y^2)x^l, ((j+1)x+xy+j^2 + 3)xy^2, 0 \geq l \geq 0\} \frac{\partial}{\partial x} + C \{(x+y^2)x^l, l \geq 0\} \frac{\partial}{\partial y} + C \{(x+y^2)x^l, 0 \geq l \geq 0\} \frac{\partial}{\partial y}$$

Now $(xy+y^3)y^2 \frac{\partial}{\partial y} = (xy+y^3)y^2 \frac{\partial}{\partial y}$ and $(2xy^2+y^4)y^2 \frac{\partial}{\partial y} \sim y^2 \frac{\partial}{\partial y} \in f \star \mathbb{m}_3 \cdot C$ for $j \geq 2,$

$$(j+2)(xy+y^3)x^l \frac{\partial}{\partial x} y^2j+1 = (j+2)(x^l+y^2j+2+2x^l+1y^2j+4+xy^2j+6) = (j+2)x^l+2y^2j+2+(j+1)x^l+1y^2j+4 + (j+2)x^l+2j+6$$

so that

$$((j+1)x+xy+j^2 + 3)xy^2 \frac{\partial}{\partial x} y^2j+1 \in f \star \mathbb{m}_3 \cdot C$$

and $(2xy^2 + y^4)(x+y^2)x^l \frac{\partial}{\partial x} y^2j+1 = 2(x+y^2)x^l+y^2j+3 + (x+y^2)x^l+y^2j+5$

so that

$$(x+y^2)x^l \frac{\partial}{\partial x} y^2j+1, (x+y^2)x^l \frac{\partial}{\partial y} y^2j+1, (x+y^2)x^l \frac{\partial}{\partial z} y^2j+1 \in f \star \mathbb{m}_3 \cdot C$$

Also $(j+2)(2xy^2 + y^4)(x+y^2)x^l \frac{\partial}{\partial x} y^2j+1 = (j+2)(2x^l+y^2j+2 + (3+1)x^l+y^2j+6)$
so that \((j+1)xy^2i+jy^2j+2)x^j \frac{\partial}{\partial x}, ((j+1)xy^2i+jy^2j+2)x^j \frac{\partial}{\partial z} \in f* m_3 C\) for \(j \geq 3\)

and \((xy+y)(j+1)xy^2i+jy^2j+1) = (j+1)(x^2y^2j+1 + xy^2j+3) + j(xy^2j+3 + y^2j+5)\)

so that \((x+y^2)x^jy^2j+1 \frac{\partial}{\partial x}, (x+y^2)x^jy^2j+1 \frac{\partial}{\partial z} \in f* m_3 C\) for \(j \geq 3\).

Hence \(f* m_3 C \supseteq \mathcal{C} \left\{ (xy+y^2)x^jy^2j+1, ((j+1)xy^2i+jy^2j+2)x^j; j \geq 2 \text{ or } \ell \geq 1 \text{ and } j \geq 0 \right\} \frac{\partial}{\partial x}\)

+ \(\mathcal{C} \left\{ (xy+y^2)x^jy^2j+1, j \geq 2 \text{ or } \ell \geq 1 \text{ and } j \geq 0 \right\} \frac{\partial}{\partial y} + \mathcal{C} \left\{ x^2, x^jy^2j; \ell + j \geq 3 \right\} \frac{\partial}{\partial y}\)

+ \(\mathcal{C} \left\{ (xy+y^2)x^jy^2j+1, ((j+1)xy^2i+jy^2j+2)x^j; j \geq 2 \text{ or } \ell \geq 1 \text{ and } j \geq 0 \right\} \frac{\partial}{\partial z}\)

\(f* m_3 C \supseteq \mathcal{C} \left\{ \begin{pmatrix} xy \\ y^2j+1 \\ 2xy^3 \\ 2jxy^{2j+2} \end{pmatrix}, \begin{pmatrix} j+1 \end{pmatrix} \right\} \begin{pmatrix} x^2j+1 \\ y^2j+3 \\ 2x^jy^{2j+3} \\ 0 \end{pmatrix}; j \geq 1 \right\} \frac{\partial}{\partial z}\)

Now

\(\left( j+1 \right) \begin{pmatrix} x^jy^{2j} \\ 0 \\ 2jxy^{2j+2} \end{pmatrix} = \left( j+1 \right) \begin{pmatrix} (j+1)xy^2i+jy^2j+2 \frac{\partial}{\partial x} + 2j ((j+1)xy^2i+jy^2j+2) \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} x^2j+1 \\ y^2j+3 \\ 2x^jy^{2j+3} \\ 0 \end{pmatrix}\)

\(\left( j+1 \right) \begin{pmatrix} x^jy^{2j+1} \\ 0 \\ 2jxy^{2j+2} \end{pmatrix} = \left( j+1 \right) \begin{pmatrix} (j+1)xy^2i+jy^2j+2 \frac{\partial}{\partial x} + j(j+1)(xy^2i+jy^2j+2) \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} -y^{2j+2} \\ -j \end{pmatrix}\)

\(\begin{pmatrix} (j+1)xy^2i+jy^2j+2 \frac{\partial}{\partial x} + j(x+y^2)y^2j+3 \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} x^jy^{2j+1} \\ 0 \\ 2jxy^{2j+2} \end{pmatrix}\)

so that \(f * m_3 C\)

\(\supseteq \mathcal{C} \left\{ \begin{pmatrix} xy \\ y^2j+1 \\ 2xy^3 \\ 2jxy^{2j+2} \end{pmatrix}, \begin{pmatrix} j+1 \end{pmatrix} \right\} \begin{pmatrix} x^2j+1 \\ y^2j+3 \\ 2x^jy^{2j+3} \\ 0 \end{pmatrix}; j \geq 3 \right\} \frac{\partial}{\partial z}\)

\(\mathcal{C}\) is spanned by: \(x \frac{\partial}{\partial x}, (x+y^2) \frac{\partial}{\partial x}, 2x^2 + y^2 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, (x+y^2) \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial y}, y^4 \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}\).
\[(xy+y^3)\frac{\partial}{\partial Z}, (2xy^2+y^4)\frac{\partial}{\partial Z}, (xy^3+y^5)\frac{\partial}{\partial Z}, \begin{pmatrix} y^2 \\ y \\ y^4 \\ y^6 \\ 2y^5 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^3 \\ y^3 \\ y^3 \\ 4y^3 \end{pmatrix}, \begin{pmatrix} y^4 \\ y^4 \\ y^4 \\ y^4 \\ y^4 \end{pmatrix}, \begin{pmatrix} y^5 \\ y^5 \\ y^5 \\ y^5 \\ 0 \end{pmatrix}, \begin{pmatrix} y^6 \\ y^6 \\ y^6 \\ y^6 \\ 0 \end{pmatrix} \]

\(T \not\subseteq f\) is spanned by: \(x \frac{\partial}{\partial X}(xy+y^3)\frac{\partial}{\partial X}, (2xy^2+y^4)\frac{\partial}{\partial X}, x \frac{\partial}{\partial Y}(xy+y^3)\frac{\partial}{\partial Y}, (2xy^2+y^4)\frac{\partial}{\partial Y}, x \frac{\partial}{\partial Z}(xy+y^3)\frac{\partial}{\partial Z}, (2xy^2+y^4)\frac{\partial}{\partial Z}, x \frac{\partial}{\partial Z}\),

\[(xy+y^3)\frac{\partial}{\partial Z}, (2xy^2+y^4)\frac{\partial}{\partial Z}, \begin{pmatrix} x \\ xy \\ 2xy^2 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 2y^4 \end{pmatrix}, \begin{pmatrix} y^4 \\ y^5 \\ 2y^6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4x^2 + 3xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4x^2 + 4y^4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4xy^2 + 4y^5 \end{pmatrix} \]

Hence \(C = T \not\subseteq f + f^* m_3' C\) so that \(C = T \not\subseteq f\).

\[(A \cdot 6) \quad (x, xy + y^3, xy^3) \]

Let \(C = \begin{pmatrix} \mathbb{R} \setminus \{y^{j+1}, xy^{3j+1}, xy^{3j+2}, 2 \leq 3j+2 : j \geq 0\} \\ \mathbb{R} \setminus \{y^{3j+1}, y^{3j+2}, xy^{3j+2} : j \geq 0\} \\ \mathbb{R} \setminus \{y^{j+1}, xy^{3j+1}, xy^{3j+2}, 2 \leq 3j+2 : j \geq 0\} \end{pmatrix} \)

\[+ C \left\{ \begin{pmatrix} (6j+11)x^2 y^3j-1 + 6xy^3j+1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} j(j-1)x^2 y^3j-4 + jxy^3j-2 + y^j \end{pmatrix} \right\} \]

\[+ C \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

\[+ C \begin{pmatrix} 0 \\ (6j+1)xy^3j-2 + 6y^{3j+1} \\ 0 \end{pmatrix} \]

\[+ C \begin{pmatrix} 0 \\ (6j+1)xy^3j-2 + 6y^{3j+1} \\ 0 \end{pmatrix} \]

\[+ C \begin{pmatrix} 0 \\ (6j+1)y^{3j} \\ (3j+8j)y^{3j+1} \end{pmatrix} \]

\[+ C \begin{pmatrix} 0 \\ (3j+8j)y^{3j+1} \\ 0 \end{pmatrix} \]

\[C = \langle x^2 \rangle + C \{ xy^3, x^2 y^3, x^2 y^{3j+1}, (6j+17)x^2 y^3j+2 + 6xy^3j+4, 1(j+1)x^2 y^3j-1 + (j+1)xy^3j+1 + y^3j+3, j \geq 0 \} \frac{\partial}{\partial X} \]

\[+ \langle x^2 \rangle + C \{ y^{3j+3}, xy^3j, xy^3j+1, (6j+7)xy^3j+2 + 6y^{3j+4}, j \geq 0 \} \frac{\partial}{\partial Y} \]

\[+ \langle x^2 \rangle + C \{ xy^3, x^2 y^3, x^2 y^{3j+1}, \frac{1}{2} (j+1)x^2 y^3j-1 + (j+1)xy^3j+1 + y^3j+3, (6j+5)x^2 y^3j-2 + 6xy^3j+4, j \geq 0 \} \frac{\partial}{\partial Z} \]
\[ + C \begin{pmatrix} y \\ y^2 \\ y^3 \\ y^4 \\ y^5 \end{pmatrix} = \begin{pmatrix} (6j+1)y^{3j} \\ (3j^2 + 8j)y^{3j+1} \\ -xy^{3j-1} \\ 0 \\ 0 \end{pmatrix}, + \begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ y^{3j+3} \\ y^{3j+4} \end{pmatrix} = \begin{pmatrix} (3j^2 + 2j - 1)y^{3j} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} y^{3j-1} \\ y^{3j+1} \\ y^{3j+2} \end{pmatrix} : j \geq 1 \]

Note that \((6j+11)(\frac{1}{2}(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3})-\frac{1}{2}(j+1)((6j+11)x^2y^{3j-1}+6xy^{3j+1})\)

\[= (j+1)(6j+11)xy^{3j+1}+(6j+11)y^{3j+3} - \frac{1}{2}(j+1)(6j+11)xy^{3j+1} = (j+1)(3j+11)xy^{3j+1}+(6j+11)y^{3j+3} \text{ where } j \geq 1 \]

and \((6j-1)(\frac{1}{2}(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3})-\frac{1}{2}(j+1)((6j-1)x^2y^{3j-1}+6xy^{3j+1})\)

\[= (j+1)(6j-1)xy^{3j+1}+(6j-1)y^{3j+3} - \frac{1}{2}(j+1)(6j-1)xy^{3j+1} = (j+1)(3j-1)xy^{3j+1}+(6j-1)y^{3j+3} \text{ where } j \geq 1 \]

so that \((j+1)(3j+11)xy^{3j+1}+(6j+11)y^{3j+3} \frac{\partial}{\partial x}, (j+1)(3j-1)xy^{3j+1}+(6j-1)y^{3j+3} \frac{\partial}{\partial z} \in C \text{ for } j \geq 0\).

\[f^*\mathfrak{m}_3 C \ni \langle x^3 \rangle + C\{x^2, x^2y+xy^3, xy^{3j_1}, x^2y^{3j_1}, xy^{3j_1} + y^{3j_1} + x^2y^{3j_1} + y^{3j_1} + 6xy^{3j_1+4}, \]
\[\frac{1}{2}(j+1)x^2y^{3j_1}+(j+1)xy^{3j_1}+y^{3j_1}+y^{3j_1}+y^{3j_1}, j \geq 1 \} \frac{\partial}{\partial x} \]

\[+ \langle x^2 \rangle + C\{ y^{3j_3}, xy^{3j_1}, xy^{3j_1} + 6y^{3j_1+4}, j \geq 1 \} \frac{\partial}{\partial y} \]

\[+ \langle x^3 \rangle + C\{ x^2, x^2y+xy^3, xy^{3j_3}, x^2y^{3j_3}, x^2y^{3j_3} \}, \frac{1}{2}(j+1)x^2y^{3j_1}+(j+1)xy^{3j_1}+y^{3j_1}, j \geq 1 \} \frac{\partial}{\partial z} \]

\[(6j+5)x^2y^{3j_1+2}+6xy^{3j_1+4}; j \geq 1 \} \frac{\partial}{\partial z} \]

and \(f^*\mathfrak{m}_3 C \ni (j+1)(3j+11)xy^{3j+1}+(6j+11)y^{3j+3} \frac{\partial}{\partial x}, (j+1)(3j-1)xy^{3j+1}+(6j-1)y^{3j+3} \frac{\partial}{\partial z} \text{ for } j \geq 2\).

Now \[x \begin{pmatrix} (6j+1)y^{3j} \\ (3j^2 + 8j)y^{3j+1} \\ 0 \end{pmatrix} = (6j+1)xy^{3j} \frac{\partial}{\partial x} + (3j^2 + 8j)xy^{3j+1} \frac{\partial}{\partial y} \text{ so } xy^{3j} \frac{\partial}{\partial x} \in f^*\mathfrak{m}_3 C \]

\[x \begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} = \begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} \in f^*\mathfrak{m}_3 C \text{ for } j \geq 1, \]

\[x \begin{pmatrix} y^{3j-1} \\ y^{3j+1} \\ 0 \end{pmatrix} = xy^{3j} \frac{\partial}{\partial x} + xy^{3j+3} \frac{\partial}{\partial z} + y^{3} \begin{pmatrix} 3j-1 \\ 3j+1 \\ 3j+2 \end{pmatrix} \]

\[\begin{pmatrix} 0 \\ y^{3j+2} \end{pmatrix} \in f^*\mathfrak{m}_3 C \text{ for } j \geq 2, \]
\[
\begin{align*}
(x+y)^3 \begin{pmatrix}
-xy^{3j-1} \\
y^{3j+2} \\
0
\end{pmatrix} &= -x^2y^{3j+3} \frac{\partial}{\partial x} + xy^{3j+2} \frac{\partial}{\partial y} + y^3 \begin{pmatrix}
-xy^{3j-1} \\
y^{3j+2} \\
0
\end{pmatrix} \Rightarrow \begin{pmatrix}
-xy^{3j-1} \\
y^{3j+2} \\
0
\end{pmatrix} \in f*\gamma_3^*C \text{ for } j \geq 2,
\end{align*}
\]
\[
(x+y)^3 \begin{pmatrix}
y^{3j+1} \\
0 \\
-xy^{3j+2} + y^{3j+4}
\end{pmatrix} = -x^2y^{3j+3} \frac{\partial}{\partial z} + \begin{pmatrix}
y^{3j+2} \\
0 \\
y^{3j+5}
\end{pmatrix} + y^3 \begin{pmatrix}
y^{3j+1} \\
0 \\
-xy^{3j+2} + y^{3j+4}
\end{pmatrix}
\]
\[
\Rightarrow \begin{pmatrix}
0 \\
-xy^{3j+2} + y^{3j+4} \\
y^{3j+1}
\end{pmatrix} \in f*\gamma_3^*C \text{ for } j \geq 2,
\]
\[
(6j+5)(3j+11)(j+1)x^{(6j+1)}((3j+11)(j+1)xy^{3j+1}+(6j+11)y^{3j+3}) \frac{\partial}{\partial x}
\]
\[
-(j+1)(3j+11)((6j+5)x^2y^{3j+2}+6xy^{3j+4}) \frac{\partial}{\partial z} + (6j+11)\begin{pmatrix}
-xy^{3j+1} \\
y^{3j+2} \\
0
\end{pmatrix}
\]
\[
(6j+11)(6j+7)x^{(6j+1)}((6j+7)x^2y^{3j+1}+6xy^{3j+4}) \frac{\partial}{\partial x}
\]
\[
(6j+7)(6j+11)y^{3j+4} \frac{\partial}{\partial y} - 6 \begin{pmatrix}
(6j+7)xy^{3j+4} \\
0
\end{pmatrix} \in f*\gamma_3^*C \text{ for } j \geq 2,
\]
\[
(j+1)(3j+11)(6j+7)(xy+y^3)\begin{pmatrix}
(6j+1)(3j^2+8j)y^{3j+1} \\
0
\end{pmatrix} = (6j+7)(6j+1)(j+1)(3j+11)y^{3j+1}+(6j+11)y^{3j+3}) \frac{\partial}{\partial x}
\]
\[
+(j+1)(3j+11)(3j^2+8j)((6j+7)x^2y^{3j+4}) \frac{\partial}{\partial y} + (3j^2+8j)(j+1)\begin{pmatrix}
(6j+7)y^{3j+3} \\
0
\end{pmatrix}
\]
\[
\begin{align*}
&\left(7(xy^4 + y^7) \right) + \left(11(xy^5 + y^7) \right) \in f^*\mathfrak{m}_3^j C \text{ and } \left(6j+7\right)y^{3j+3} + \left(3j+11\right)(j+1)y^{3j+4} \in f^*\mathfrak{m}_3^j C \text{ for } j \geq 2, \\
&\text{Hence } f^*\mathfrak{m}_3^j C \supseteq \left< x^3 \right> + C^j \{x^2, x^2 y, xy^3, x^2 y^3, x^2 y^3j+1, \ (6j+17)x^2 y^3j+2 + 6xy^3j+4, \\
&\quad \left(\frac{1}{2}(j+1)2y^{3j+1} + (j+1)xy^{3j+1} + y^{3j+3} : j \geq 1 \right) \frac{\partial}{\partial x} \\
&+ \left< x^2 \right> + C \{y^{3j+3}, xy^3j, xy^{3j+1}, (6j+7)xy^{3j+2} + 6y^{3j+4} : j \geq 1 \} \frac{\partial}{\partial y} \\
&+ \left< x^3 \right> + C \{x^2, x^2 y + xy^3, xy^{3j+3}, x^2 y^3j, x^2 y^3j+1, \ \frac{1}{2}(j+1)2y^{3j+1} + (j+1)xy^{3j+1} + y^{3j+3}, \\
&\quad (6j+5)x^2 y^{3j+2} + 6xy^{3j+4} : j \geq 1 \} \frac{\partial}{\partial z} + C \left\{ \left( x^2 \right) \left( \begin{array}{c}
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{array} \right) \right\} \left( \begin{array}{c}
(6j+7)y^{3j+3} \\
(3j+11)(j+1)y^{3j+4}
\end{array} \right) + \left\{ \left( x^2 \right) \left( \begin{array}{c}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{array} \right) \right\} \left( \begin{array}{c}
y^{3j+1} \\
0
\end{array} \right) \\
&\quad \left\{ \left( x^2 \right) \left( \begin{array}{c}
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{array} \right) \right\} \left( \begin{array}{c}
(3j+11)(3j+1) \frac{\partial}{\partial y} \left( \begin{array}{c}
y^{3j+1} \\
0
\end{array} \right) \\
(3j+11)(3j+3) \frac{\partial}{\partial y} \left( \begin{array}{c}
y^{3j+1} \\
0
\end{array} \right) : j \geq 2 \right\}
\end{align*}
\]

\(C\) is spanned by: \(x^2 \frac{\partial}{\partial x}, xy^3 \frac{\partial}{\partial x}, (xy + y^3) \frac{\partial}{\partial x}, (17xy^2 + 6y^4) \frac{\partial}{\partial y}, y^2 \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial y}, y^3 \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial z}, \left(7xy^2 + 6y^4\right) \frac{\partial}{\partial y}, (xy + y^3) \frac{\partial}{\partial z}, (5x^2 y^2 + 6xy^4) \frac{\partial}{\partial z}, xy^3 \frac{\partial}{\partial z}, \left( \begin{array}{c}
y \\
y^2
\end{array} \right) \), other terms.

\(\mathfrak{T} \mathcal{S} f\) is spanned by: \(x^2 \frac{\partial}{\partial x}, (xy + y^3) \frac{\partial}{\partial x}, xy^3 \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial y}, (xy + y^3) \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial z}, (xy + y^3) \frac{\partial}{\partial z}, \) \(xy^3 \frac{\partial}{\partial z}, \left( \begin{array}{c}
x \\
y
\end{array} \right), \left( \begin{array}{c}
y \\
y^2
\end{array} \right), \left( \begin{array}{c}
y^3 \\
y^4
\end{array} \right), \left( \begin{array}{c}
x^2 + 3xy^2 \\
3xy^2
\end{array} \right), \left( \begin{array}{c}
0 \\
xy + y^3
\end{array} \right), \left( \begin{array}{c}
0 \\
x^2 + 3y^4
\end{array} \right), \left( \begin{array}{c}
2y^5 \\
x^2 + 3y^5
\end{array} \right) \). Long and somewhat tedious calculations yield that \(C \subseteq \mathfrak{T} \mathcal{S} f + f^*\mathfrak{m}_3^j C \) so that \(C = \mathfrak{T} \mathcal{S} f\).

\((A.7)\)

\((x, xy + y^3, xy^4)\)

\[\begin{pmatrix}
\mathfrak{m}_2 \{xy, y^2, xy^2, xy, y^3, y^4, y^5, y^6, y^7 : j \geq 0\} \\
\mathfrak{m}_2 \{y, y^2, xy^3, xy^4, xy^5, y^6, y^7, y^8 : j \geq 0\} \\
\mathfrak{m}_2 \{xy, y^2, xy, xy^2, xy^3, xy^4, xy^5, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8 : j \geq 0\}
\end{pmatrix} \]
\[
\begin{pmatrix}
3 + y & xy^4 & 0 \\
0 & y + 5 & 4xy + 3y^7 \\
0 & y + 3y^4 & 2xy + 3y^8 \\
\end{pmatrix}
\begin{pmatrix}
x^2 + 3y^4 \\
x + 2x^2 + xy^3 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
y^8 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
y^{3j+1} \\
y^{3j+5} \\
\end{pmatrix} : j \geq 2
\]

\[
= \left\langle x^3, x^2y^3, xy^5 \right\rangle + C\left\langle x^2, x^2y + xy^3, x^2y^2 + 2xy^4 + y^6, y^{3j+8}, y^{3j+9} : j \geq 0 \right\rangle \frac{\partial}{\partial x}
\]

\[
+ \left\langle x^3, x^2y^3, xy^6, y^8 \right\rangle + C\left\langle x^2, x^2y, xy^3, x^2y^2 + 2xy^4, xy^4 + y^6 \right\rangle \frac{\partial}{\partial y}
\]

\[
+ \left\langle x^5, x^4, x^2y^7, x^3y^4, x^4y^9 \right\rangle + C\left\langle x^2, x^4, x^2y^4, x^2y + xy^3, x^3y - xy^7, x^4y + y^8, x^3y^2 + xy^6, x^2y^5 + xy^7, x^2y^2 + 2xy^4 + y^6, x^3y^3 + y^9, 11x^2y - 6xy^8 - 9y^10, x^4y^2 + 2x^2y^6, 3xy^7 + 2y^9, y^{3j+12}, y^{3j+13} : j \geq 0 \right\rangle \frac{\partial}{\partial z}
\]

\[
+ C\left\langle \begin{pmatrix}
xy^2 \\
x^2y \\
x^3 \\
\end{pmatrix}, \begin{pmatrix}
0 \\
-xy^4 \\
3xy^9 \\
\end{pmatrix}, \begin{pmatrix}
x^2y \\
-xy^4 \\
-3xy^10 \\
\end{pmatrix}, \begin{pmatrix}
y^6 + 4xy^4 \\
3y^7 \\
0 \\
\end{pmatrix} \right\rangle \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix} : j \geq 2
\]

We note that \( f \ast m_3 \cdot C \supseteq m_2 \cdot \{y^{3j+13} \frac{\partial}{\partial x}, y^{3j+11} \frac{\partial}{\partial z} : j \geq 0 \} + C\left\langle \begin{pmatrix}
y^{3j+1} \\
y^{3j+5} \\
\end{pmatrix} : j \geq 2 \right\rangle \).

\( C \) is spanned by: 
\[
x \frac{\partial}{\partial x}, (xy+y^3) \frac{\partial}{\partial x}, xy^4 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, y^3 \frac{\partial}{\partial y}, xy^4 \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, (xy+y^3) \frac{\partial}{\partial z},
\]

\[
xy^4 \frac{\partial}{\partial z}, (8x^2y^3 + 3xy^5 - 9y^7) \frac{\partial}{\partial z}, \left( \begin{pmatrix}
x^2 \\
x^3 \\
x^4 \\
\end{pmatrix}, \begin{pmatrix}
y^2 \\
y^3 \\
y^4 \\
\end{pmatrix}, \begin{pmatrix}
y^3 \\
y^4 \\
y^5 \\
\end{pmatrix} \right) \begin{pmatrix}
0 \\
-2xy^2 \\
-4x^2y^6 + 3y^8 \\
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix} : j \geq 2
\]

\( T \otimes f \) is spanned by: 
\[
x \frac{\partial}{\partial x}, (xy+y^3) \frac{\partial}{\partial x}, xy^4 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, (xy+y^3) \frac{\partial}{\partial y}, xy^4 \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, (xy+y^3) \frac{\partial}{\partial z},
\]

\[
xy^4 \frac{\partial}{\partial z}, \left( \begin{pmatrix}
x \\
x^2y^4 \\
x^3y \\
x^4y^6 \\
\end{pmatrix}, \begin{pmatrix}
y^2 \\
y^3 \\
y^4 \\
y^5 \\
\end{pmatrix}, \begin{pmatrix}
y^3 \\
y^4 \\
y^5 \\
y^6 \\
\end{pmatrix} \right) \begin{pmatrix}
0 \\
x^2 + 3xy^2 \\
4x^2y^4 \\
4x^2y^6 \\
\end{pmatrix} + \begin{pmatrix}
0 \\
x^2 + 3xy^2 \\
4x^2y^4 \\
4x^2y^6 \\
\end{pmatrix} : j \geq 2
\]

Hence \( C = T \otimes f + f \ast m_3 \cdot C \) so that \( C = T \otimes f \).
\[
C = \begin{pmatrix}
\mathcal{m}_2 \{ y^4j+1, y^4j+2, y^4j+3 : j \geq 0 \} \\
\mathcal{m}_2 \{ y, y^4j+2, y^4j+3, y^4j+4 : j \geq 0 \} \\
\mathcal{m}_2 \{ y, y^3, y^2, y^4j+2 : j \geq 0 \}
\end{pmatrix} + C \left( \begin{pmatrix}
y^4j+1 \\
y^4j+2 \\
y^4j+3 \\
0
\end{pmatrix}, \begin{pmatrix}
y^4j+2 \\
y^4j+3 \\
y^4j+3 \\
y^4j+6
\end{pmatrix} : j \geq 0 \right)
\]

\[
= \left( \langle x^2 \rangle + C \{ y^4j+3, y^4j+4, xy^4j, xy^4j+1, xy^4j+3 : j \geq 0 \} \right) \frac{\partial}{\partial x} 
\]

\[
+ \left( \langle x^2 \rangle + C \{ y^4j+4, y^4j+5, xy^4j, xy^4j+1, xy^4j+2 : j \geq 0 \} \right) \frac{\partial}{\partial y}
\]

\[
+ \left( \langle x^2, xy^3 \rangle + C \{ x, y, y^4j+4, y^4j+5, y^4j+6, y^4j+7, j = 0 \} \right) \frac{\partial}{\partial z}
\]

Then \( f^* \mathcal{m}_3 \cdot C = \left( \langle x^2 \rangle + C \{ y^4j+7, y^4j+8, xy^4j+4, xy^4j+5, xy^4j+3 : j \geq 0 \} \right) \frac{\partial}{\partial x}
\]

\[
+ \left( \langle x^2 \rangle + C \{ y^4j+8, xy^4j+4, xy^4j+5, xy^4j+6, j \geq 0 \} \right) \frac{\partial}{\partial y}
\]

\[
+ \left( \langle x^2, xy^4 \rangle + C \{ y^4j+8, y^4j+9, y^4j+11, j \geq 0 \} \right) \frac{\partial}{\partial z}
\]

\[
C \text{ is spanned by: } \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{x}{\partial x}, \frac{y}{\partial y}, \frac{z}{\partial z}, \frac{y^2}{\partial x}, \frac{z^2}{\partial z}, \frac{y^4}{\partial x}, \frac{y^4}{\partial y}, \frac{y^4}{\partial z}, \frac{y^4}{\partial x}, \frac{y^4}{\partial y}, \frac{y^4}{\partial z}, \frac{z^6}{\partial x}, \frac{z^6}{\partial y}, \frac{z^6}{\partial z}
\]

\[
T \mathcal{X} f \text{ is spanned by: } \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{y^4}{\partial x}, \frac{y^4}{\partial y}, \frac{y^4}{\partial z}, \frac{y^4}{\partial x}, \frac{y^4}{\partial y}, \frac{y^4}{\partial z}, \frac{z^6}{\partial x}, \frac{z^6}{\partial y}, \frac{z^6}{\partial z}
\]

Hence \( C = T \mathcal{X} f + f^* \mathcal{m}_3 \cdot C \) so that \( C = T \mathcal{X} f \).

(A.9) \[
(x, xy+xy^3, y^4)
\]

\[
T \mathcal{X} f = \begin{pmatrix}
\mathcal{m}_2 \{ y^4j+1, y^4j+2, y^4j+3 : j \geq 0 \} \\
\mathcal{m}_2 \{ y^4j+1, y^4j+2, y^4j+3 : j \geq 0 \} \\
\mathcal{m}_2 \{ y, y^3, y^2 \}
\end{pmatrix}
\]
\[
\begin{aligned}
&+c \left[
\begin{pmatrix}
y^{4j+3} & y^{4j+5} \\
y^{4j+5} & y^{4j+7} \\
y^{4j+3} & y^{4j+7} \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
y^{4j+2} \\
y^{4j+3} \\
y^{4j+2} \\
0
\end{pmatrix}
\right]_{j \geq 0}
\end{aligned}
\]

\[C = \left( x + c \{ y^{4j+4}, y^{4j}(y^3 - y^5) : j \geq 0 \} \right) \frac{\partial}{\partial x} + \left( x + c \{ y^{4j+4}, y^{4j}(y^3 - y^5) : j \geq 0 \} \right) \frac{\partial}{\partial y}
\]

\[+ \left( x^2, xy^3, y^4 \right) c(x, xy) \frac{\partial}{\partial z} + c \left[
\begin{pmatrix}
xy_2 \\
xy_3 \\
0
\end{pmatrix}
\begin{pmatrix}
y^{4j+2} \\
y^{4j+3} \\
0
\end{pmatrix}
\begin{pmatrix}
y^{4j+1} \\
y^{4j+2} \\
0
\end{pmatrix}
\right]_{j \geq 0}
\]

Then \(f^* m_3 \cdot C = \left( x^2, xy^3 \right) c \{ y^{4j+8}, y^{4j+4}(y^3 - y^5) : j \geq 0 \} \frac{\partial}{\partial x} + \left( x^2, xy^4 \right) c \{ y^{4j+8}, y^{4j+4}(y^3 - y^5) : j \geq 0 \} \frac{\partial}{\partial y}
\]

\[C \text{ is spanned by: } x \frac{\partial}{\partial x}, xy \frac{\partial}{\partial x}, y^4 \frac{\partial}{\partial x}, y^4(y^3 - y^5) \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, xy \frac{\partial}{\partial y}, xy \frac{\partial}{\partial y}, y^4 \frac{\partial}{\partial y} y^6 \frac{\partial}{\partial y}, y^7 \frac{\partial}{\partial y},
\]

\( (x^2, xy^3, y^4) \frac{\partial}{\partial z}, y^4 \frac{\partial}{\partial z}, x \frac{\partial}{\partial z}, (xy+xy^3) \frac{\partial}{\partial z}, y^4 \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, (xy+xy^3) \frac{\partial}{\partial z},
\]

\[y^4 \frac{\partial}{\partial z}, \left( x \begin{pmatrix}
y_2 \\
y_3 \\
y_4 \\
y_5
\end{pmatrix}, \begin{pmatrix}
y_2 \\
y_3 \\
y_4 \\
y_5
\end{pmatrix}, \begin{pmatrix}
y_2 + 3xy^2 \\
y_2 + 3xy^2 \\
y_5 \\
y_5
\end{pmatrix}, \begin{pmatrix}
y_2 + 3xy^2 \\
y_2 + 3xy^2 \\
y_5 \\
y_5
\end{pmatrix}, \begin{pmatrix}
y_2 + 3xy^2 \\
y_2 + 3xy^2 \\
y_5 \\
y_5
\end{pmatrix}
\right)_{j \geq 0}
\]

Hence \(C = T \Leftrightarrow f + f^* m_3 \cdot C \) so that \(C = T \Leftrightarrow f\).

(A-10) 
\[
(x, xy-y^4, y^6)
\]

\[
\left(?\begin{array}{@{}c@{}}
\text{m}_2 \{y, xy, x y, y, x y : j, l \geq 0\} \\
\text{m}_2 \{y, y, xy, x y, x y : j, l \geq 0\} \\
\text{m}_2 \{y, xy, x y, y, x y : j, l \geq 0\} \\
\text{m}_2 \{y, xy, x y, y, x y : j, l \geq 0\}
\end{array}\right)
\]
\[
\begin{align*}
C &= \left\{ x^{y3j+1} : l+j > 0 \right\} + \left\{ x^{y3j+1} : l+j \geq 0 \right\} \frac{\partial}{\partial x} \\
+ &\left\{ x^{y3j+1} : l+j > 0 \right\} + \left\{ x^{y3j+1} : l+j \geq 0 \right\} \frac{\partial}{\partial y} \\
+ &\left\{ x^{y3j+1} : l+j > 0 \right\} + \left\{ x^{y3j+1} : l+j \geq 0 \right\} \frac{\partial}{\partial z} \\
\end{align*}
\]
Note that $f^*\mathfrak{m}_3 \cdot C \supseteq \left\{ \begin{array}{c} \mathfrak{m}_2 \{ x \ y \ j : 3j + \ell \geq 11 \} \\ \mathfrak{m}_2 \{ x \ y \ j : 3j + \ell \geq 10 \} \\ \mathfrak{m}_2 \{ x \ y \ j : 3j + \ell \geq 11 \} \end{array} \right\} + C \left( \begin{array}{c} y_{3j+1} \\ y_{3j+2} \\ 0 \\ 2y_{3j+4} \end{array} \right) : j \geq 2 \right\}

C$ is spanned by:

\[
\left( \begin{array}{c} \frac{\partial}{\partial x}, (xy-y^4) \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, y^6 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, xy \frac{\partial}{\partial y}, (xy^2-y^5) \frac{\partial}{\partial y}, y^4 \frac{\partial}{\partial y}, y^6 \frac{\partial}{\partial y}, \end{array} \right)
\]

\[
\left( \begin{array}{c} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{array} \right)
\]

$T \mathfrak{f}$ is spanned by:

\[
\left( \begin{array}{c} (x, y^2, y^3, xy^2) \\ y^6 \frac{\partial}{\partial z} \\ (x, y^2, y^3, xy^2) \\ (x, y^2, y^3, xy^2) \\ (x, y^2, y^3, xy^2) \\ (x, y^2, y^3, xy^2) \\ (x, y^2, y^3, xy^2) \\ (x, y^2, y^3, xy^2) \end{array} \right)
\]

Hence $C = T \mathfrak{f} + f^*\mathfrak{m}_3 \cdot C$ so that $C = T \mathfrak{f}.$

(A-11)

\[
(x, x^2 y + y^3, xy^2)
\]

Let $C = \left\{ \begin{array}{c} \mathfrak{m}_2 \{ y^{3j+2} : j \geq 0 \} \\ \mathfrak{m}_2 \{ y, y^2, y^3 : j \geq 0 \} \\ \mathfrak{m}_2 \{ x, y^2, y^{3j+1} : j \geq 0 \} \end{array} \right\} + C \left( \begin{array}{c} y^{3j+2} \\ y^3 \\ y^4 \\ y^5 \\ y^6 \\ y^7 \end{array} \right) : j \geq 0 \right\}

\]

\[
= (\langle x \rangle + C \langle y^{3j+1}, y^{3j+2} : j \geq 0 \rangle) \frac{\partial}{\partial x} + (\langle x^2, xy^2, y^3 \rangle + C \langle x \rangle) \frac{\partial}{\partial y}
\]

\[
+ (\langle x^2, xy^2 \rangle + C \langle x, y^{3j+3}, y^{3j+5} : j \geq 0 \rangle) \frac{\partial}{\partial z} + C \left( \begin{array}{c} y^{3j+2} \\ y^{3j+3} \\ y^{3j+4} \end{array} \right) : j \geq 0 \right\}
\]

Then $f^*\mathfrak{m}_3 \cdot C = (\langle x^2, xy^3 \rangle + C \langle x, y^{3j+4}, y^{3j+6} : j \geq 0 \rangle) \frac{\partial}{\partial x} + (\langle x^3, x^2 y^2, xy^3, y^6 \rangle + C \langle x^2 \rangle) \frac{\partial}{\partial y}

\]

\[
+ (\langle x^3, x^2 y^2, xy^5 \rangle + C \langle x^2, xy^3, y^{3j+8}, y^{3j+8} : j \geq 0 \rangle) \frac{\partial}{\partial z} + C \left( \begin{array}{c} xy^2 \\ y^{3j+2} \\ y^4 \\ 0 \\ 0 \end{array} \right) : j \geq 1 \right\}
\]
\[
C \text{ is spanned by: } x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, x y, x^{2} \frac{\partial}{\partial x}, \ldots, x^{3} \frac{\partial}{\partial y}, x^{4} \frac{\partial}{\partial x}, x^{2} y \frac{\partial}{\partial y}, y^{3} \frac{\partial}{\partial y}, y^{4} \frac{\partial}{\partial x}, \ldots
\]

\[
T \mathcal{S} f \text{ is spanned by: } x \frac{\partial}{\partial x}, (x^{2} y + y^{3}) \frac{\partial}{\partial x}, x^{2} \frac{\partial}{\partial y}, x \frac{\partial}{\partial y}, (x^{2} y + y^{3}) \frac{\partial}{\partial y}, x^{2} \frac{\partial}{\partial x}, (x^{2} y + y^{3}) \frac{\partial}{\partial x},
\]

\[
\begin{pmatrix}
x^{2} y \frac{\partial}{\partial z} & x y \frac{\partial}{\partial z} & y^{3} \frac{\partial}{\partial z} & y^{5} \frac{\partial}{\partial z}
\end{pmatrix}
\]

Rewriting these modulo \( f \ast m_{2} \cdot C \) we get: \( x \frac{\partial}{\partial x}, y^{3} \frac{\partial}{\partial x}, x y^{2} \frac{\partial}{\partial y}, x \frac{\partial}{\partial y}, (x^{2} y + y^{3}) \frac{\partial}{\partial y}, x y^{2} \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y},
\]

\[
\begin{pmatrix}
x \frac{\partial}{\partial y}, (x^{2} y + y^{3}) \frac{\partial}{\partial y}, x y^{2} \frac{\partial}{\partial y}, x^{2} y \frac{\partial}{\partial y}, y^{3} \frac{\partial}{\partial x}, y \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y},
\end{pmatrix}
\]

Hence \( C = T \mathcal{S} f + f \ast m_{3} \cdot C \) so that \( C = T \mathcal{S} f \).

(A.12) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
\[ + (x^3, x^2y^2, xy^5) + C \{ x^2, xy^3, y^{3j+6}, y^{3j+8}; j \geq 0 \} \frac{\partial}{\partial z} + C \begin{pmatrix} x^2 \\ y^{3j+2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} y^{3j+2} \\ 0 \\ 0 \\ 3j+4 \end{pmatrix} : j \geq 1 \]

\[ \begin{pmatrix} \mathfrak{m}_2 \{ y^3, xy, y^{3j+2}; j \geq 0 \} \\ \mathfrak{m}_2 \{ y^2, xy^2, y^3, y^4, y^5, xy; j \geq 1 \} \\ \mathfrak{m}_2 \{ xy, y^2, x, x, y, x, y, y, y, y, 3j+1; j \geq 1 \} \end{pmatrix} + C \begin{pmatrix} y^2 \\ y^{3j+2} \\ 0 \\ 0 \\ 0 \end{pmatrix} : j \geq 1 \]

C is spanned by: \( x \frac{\partial}{\partial x}, y \frac{\partial}{\partial x}, xy^2 \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, xy^2 \frac{\partial}{\partial y}, y^3 \frac{\partial}{\partial y}, y^4 \frac{\partial}{\partial y}, y^5 \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, xy^2 \frac{\partial}{\partial z}, y^3 \frac{\partial}{\partial z}, y^5 \frac{\partial}{\partial z}, \begin{pmatrix} y^2 \\ 0 \\ 0 \end{pmatrix} \]

T\( \mathfrak{f} \) is spanned by: \( x \frac{\partial}{\partial x}, xy^2 \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, xy^2 \frac{\partial}{\partial y}, y^3 \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, xy^2 \frac{\partial}{\partial z}, y^3 \frac{\partial}{\partial z}, y^5 \frac{\partial}{\partial z}, \begin{pmatrix} y^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \)

Hence \( C = T \mathfrak{f} + f^* \mathfrak{m}_3 \mathcal{C} \) so that \( C = T \mathfrak{f} \)

(A.13) \( f = (x, y^3, x^2y) \)

Let \( C = \)

\[ \begin{pmatrix} \mathfrak{m}_2 \{ xy^{3j+1}, y^{3j+1}; j \geq 0 \} \\ \mathfrak{m}_2 \{ xy, y^{3j+1}; j \geq 0 \} \\ \mathfrak{m}_2 \{ xy, xy^{2}, y^{3j+2}, 2y^{3j+2}, 3j+2; j \geq 0 \} \end{pmatrix} \]

\[ + C \begin{pmatrix} 0 \\ xy^{3j+1} \\ 3j+4 \end{pmatrix} \begin{pmatrix} y^{3j+1} \\ 0 \\ 0 \end{pmatrix} : j \geq 0 \]

\[ = (x^2) + C \{ y^{3j+2}, y^{3j+3}, xy^{3j+2}, y^{3j+4}; j \geq 0 \} \frac{\partial}{\partial x} + (y^2) + C \{ x, y^{3j+3}, y^{3j+5}; j \geq 0 \} \frac{\partial}{\partial y} \]

\[ + (x^2) + C \{ y^{3j+3}, xy^{3j+4}, xy^{3j+4}, y^{3j+1}; j \geq 0 \} \frac{\partial}{\partial z} + C \begin{pmatrix} 0 \\ xy^{3j+1} \\ 3y^{3j+4} \end{pmatrix} \begin{pmatrix} y^{3j+1} \\ 0 \\ 2y^{3j+4} \end{pmatrix} : j \geq 0 \]
\[ f \mathfrak{m}_3 C = (\langle x^3, x^2 y^3 \rangle + C(x^2, x^2 y^2, y^{3j+5}, y^{3j+6}, x y^{3j+3}, x y^{3j+2}; j \geq 0) \frac{\partial}{\partial x} + \\
(\langle x^3, x^2 y^2, xy^5 \rangle + C(x^2, xy^3, y^{3j+6}, y^{3j+8}; j \geq 0) \frac{\partial}{\partial y} \\
+ \langle x^4, x^3 y^3 \rangle + C(x^3, x^3 y, y^{3j+6}, xy^{3j+3}, y^{3j+7}, x^2 y^3, x^2 y^{3j+4}; j \geq 0) \frac{\partial}{\partial z} \]

\[ + C \left( \begin{array}{ccc}
0 & -x^2 y & 0 \\
3 x y^4 & 6 x y & x y^{3j+1} \\
x^2 y & 2 x y & y^{3j+4}
\end{array} \right) \left( \begin{array}{c}
y^{3j+4} \\
0 \\
x^2 y^{3j+4}
\end{array} \right) : j \geq 0 \}

C \text{ is spanned by: } x \frac{\partial}{\partial x}, x^2 y \frac{\partial}{\partial x}, y^2 \frac{\partial}{\partial x}, x^3 \frac{\partial}{\partial x}, x^2 y \frac{\partial}{\partial y}, x y \frac{\partial}{\partial y}, y^3 \frac{\partial}{\partial y}, y^5 \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, \]

\[ x^2 y \frac{\partial}{\partial z}, x y^2 \frac{\partial}{\partial z}, y \frac{\partial}{\partial z}, y^{3j+1} \frac{\partial}{\partial z}, y^{3j+2} \frac{\partial}{\partial z}, y^{3j+3} \frac{\partial}{\partial z}, y^{3j+4} \frac{\partial}{\partial z}, \left( \begin{array}{c}
0 \\
0 \\
x^2 y
\end{array} \right) \left( \begin{array}{c}
y \\
0 \\
2 x y
\end{array} \right) \left( \begin{array}{c}
y^4 \\
x^2 y \\
y^{3j+1}
\end{array} \right)
\]

T \mathscr{f} f \text{ is spanned by: } x \frac{\partial}{\partial x}, x^2 y \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, x^2 y \frac{\partial}{\partial y}, y^3 \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, x^2 y \frac{\partial}{\partial z}, \]

\[ y^3 \frac{\partial}{\partial z}, \left( \begin{array}{c}
x \\
0 \\
x^2 y
\end{array} \right) \left( \begin{array}{c}
y \\
0 \\
x^2 y
\end{array} \right) \left( \begin{array}{c}
0 \\
3 x y^2 \\
x^3
\end{array} \right) \left( \begin{array}{c}
0 \\
3 y^3 \\
3 y^4
\end{array} \right) \left( \begin{array}{c}
0 \\
0 \\
x^2 y
\end{array} \right) \left( \begin{array}{c}
0 \\
0 \\
x^2 y
\end{array} \right) \left( \begin{array}{c}
0 \\
0 \\
x^2 y
\end{array} \right)
\]

Hence \( C = \mathfrak{T} \mathscr{f} f + f \mathfrak{m}_3 C \) so that \( C = \mathfrak{T} \mathscr{f} f \).

(A.14) \[ f = (x, y^3, x^2 y + y^4) \]

Let \( C = \)

\[ \left( \begin{array}{c}
\mathfrak{m}_2 [\{ x y^{3j+1} : l, j \geq 0 \} + C(x^3, y^{3j+1} (x^2+y^3), l+j \geq 0) \\
\mathfrak{m}_2 [\{ x y^{3j+1} : l, j \geq 0 \} + C(x^3, y^{3j+1} (x^2+y^3), l+j \geq 0) \\
\mathfrak{m}_2 [\{ y, xy, xy^{3j+2} : l, j \geq 0 \} + C(x^3, y^{3j+1} (x^2+y^3), l+j \geq 1) \\
\end{array} \right)
\]

\[ + C \left( \begin{array}{ccc}
0 & -x^{3j+4} & 0 \\
3 y^4 & 2 x y^{3j+4} & 0 \\
x y + 4 & 2 x y & 0
\end{array} \right) \left( \begin{array}{c}
xy^{3j+1} \\
0 \\
x y^{3j+1}
\end{array} \right) : j \geq 0 \}
\]

\[ = (C[x^3 y^3, x y^{3j+2}, x^3 y^{3j+1} (x^2+y^3), y(x^2+y^3), y^2; l+j > 0]) \frac{\partial}{\partial x} + \\
+C[x^3 y^3, x y^{3j+2}, x^3 y^{3j+1} (x^2+y^3), y(x^2+y^3); l+j > 0]) \frac{\partial}{\partial y} \]
\[
+ \left( \mathcal{C}(x^3 y, x^4 y^3 + 1, x^3 y^3 + 1) \right) \frac{\partial}{\partial z}
+ \mathcal{C}
\left(
\begin{pmatrix}
0 & 2y^{-1} & -y^{3j+4} & xy^{-1} & y^{3j+4} & y^{3j+1} \\
3y^{-3} & 2xy & 2y^{3j+7} & 0 & 2y^{3j+7} & 0 \\
x^{-1} & 0 & 0 & 0 & 2x^{3j+2} & 2xy^{3j+2} \\
x^{2} & 0 & 0 & 0 & 2x^{3j+2} & 2xy^{3j+2}
\end{pmatrix}
\right) : j \geq 0
\]

Then \( f^* \mathcal{M}_2^3 \mathcal{C} = \left( \mathcal{C}(x^3 y, x^4 y^3 + 1, x^3 y^3 + 1) \right) \frac{\partial}{\partial x}
+ \left( \mathcal{C}(x^3 y, x^4 y^3 + 1, x^3 y^3 + 1) \right) \frac{\partial}{\partial y}
+ \left( \mathcal{C}(x^3 y, x^4 y^3 + 1, x^3 y^3 + 1) \right) \frac{\partial}{\partial z}
\]

\[
= \mathcal{M}_2^3 \left\{ y^2, y^3, \ldots, y^{3j+1} : \ell \geq 0 \right\} + \mathcal{C}(x^3 y, x^4 y^3 + 1, x^3 y^3 + 1) \frac{\partial}{\partial z}
\]

\[
+ \mathcal{C}
\left(
\begin{pmatrix}
0 & 2y^{-1} & -y^{3j+7} & xy^{-1} & y^{3j+4} & y^{3j+1} \\
xy^{-4} & 2xy & 2y^{3j+7} & 0 & 2y^{3j+7} & 0 \\
x^{-1} & 0 & 0 & 0 & 2x^{3j+2} & 2xy^{3j+2} \\
x^{2} & 0 & 0 & 0 & 2x^{3j+2} & 2xy^{3j+2}
\end{pmatrix}
\right) : j \geq 0
\]

C is spanned by \( y \cdot \frac{\partial}{\partial x}, (x^2 y^2 y^4) \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial y}, (x^2 y^2 y^4) \frac{\partial}{\partial y}, y^2 \frac{\partial}{\partial y}, y \frac{\partial}{\partial z} \).

\[
T \mathcal{M}_2 \mathcal{C} \text{ is spanned by } x \frac{\partial}{\partial x}, (x^2 y^2 y^4) \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, (x^2 y^2 y^4) \frac{\partial}{\partial y}, y^3 \frac{\partial}{\partial y}, (x^2 y^2 y^4) \frac{\partial}{\partial z},
\]

\[
\begin{pmatrix}
x & y^2 & y^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 3y^2 & 3y^3 & 3y^4 \\
x^2 y & 2xy^2 & 2x^3 y & 2x^4 y & 2y^4 + 4y^5 & 2y^4 + 4y^5 \\
2x^2 y & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Hence \( C = T \mathcal{M}_2 \mathcal{C} + f^* \mathcal{M}_2^3 \mathcal{C} \) so that \( C = T \mathcal{M}_2 \mathcal{C} \).
Let $C = \{ x, x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \}$.

Then $f \ast m_2 \{ x, y \} = \begin{pmatrix}
\mathcal{C} \{ x, x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x, x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x, x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x, x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \}
\end{pmatrix}$

where

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
x^2 y - y^4 & 0 & 0 & 0 \\
0 & x^2 y - y^3 & 3y - xy^3 & x^4 - y^2 y^3 \\
0 & -xy^2 & -y^4 & 0 \\
x y^2 - 6y^5 & 0 & 0 & 2xy^3 & 4y^5 & xy^2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
y & xy & 3y & 5xy & 0 \\
-xy^2 & -y^4 & 0 & x^4 & 0 \\
0 & 0 & 2xy^3 & 4y^5 & xy^2
\end{pmatrix}
\]

Then $f \ast m_3 \cdot C = \begin{pmatrix}
\mathcal{C} \{ x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \}
\end{pmatrix}$

where

\[
\begin{pmatrix}
xy & 2y & 3xy & 5y & 0 \\
x^2 & 2 & 4 & 0 & x^4 \\
x^2 & 2 & 4 & 0 & xy^2 \\
x^2 & 2 & 4 & 0 & xy^2 \\
x^2 & 2 & 4 & 0 & xy^2
\end{pmatrix}
\]

Then $f \ast m_3 \cdot C = \begin{pmatrix}
\mathcal{C} \{ x, x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x, x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x, x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x, x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \}
\end{pmatrix}$

where

\[
\begin{pmatrix}
\mathcal{C} \{ x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \} \\
\mathcal{C} \{ x^2, x^3, x^4, x^5, x^6, y, xy, y^2, xy^2, y^3, y^4 \}
\end{pmatrix}
\]

where $\mathcal{C}(x, y, z) = x^2 + y^2 + z^2$.
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
x^2y - x^2y^3 & 5x^2y^3 - 6xy^5 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
xy & 3xy & x^2y & 3xy \\
-x^2y & 0 & -x^3y & 0 \\
x^4y^2 & x^5y^2 & x^6y^2 & x^7y^2 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 \\
xy \\
x^2y & 2x^2y \\
\end{pmatrix}
\]

\[
C \text{ is spanned by: } x \frac{\partial}{\partial x}, x^2y \frac{\partial}{\partial x}, y^3 \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, xy^4 \frac{\partial}{\partial y}, x^3y^2 \frac{\partial}{\partial y}, y(x^2-y^2) \frac{\partial}{\partial y}, y^2(x^2-y^2) \frac{\partial}{\partial y}, y^2(x^2-y^2) \frac{\partial}{\partial y}, y(5x^4-y^4) \frac{\partial}{\partial y},
\]
\[
(\begin{array}{cc}
2x^2y^3 & -2xy^3 \\
0 & 2x^2y^3 \\
x^4y^2 & 2x^5y^2 \\
\end{array})
\]

\[
T \varphi f \text{ is spanned by: } x \frac{\partial}{\partial x}, (x^2y-y^3) \frac{\partial}{\partial x}, (x^2y^2-y^4) \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, (x^2y^3-y^4) \frac{\partial}{\partial y}, (x^2y^4-y^5) \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial z},
\]
\[
(\begin{array}{cc}
2x^2y^3 & -2xy^3 \\
0 & 2x^2y^3 \\
x^4y^2 & 2x^5y^2 \\
\end{array})
\]

\[
\begin{pmatrix}
0 \\
x^2y-3y^3 \\
2x^2y^4-3y^4 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 \\
x^2y-3y^3 \\
2x^2y^4-3y^4 \\
\end{pmatrix}
\]

\[
Hence \ C = T \varphi f + f^* m_{\varphi}^3 C \text{ so that } C = T \varphi f.
\]
Appendix B: Calculations of $\mathcal{A}$ -invariants for III·3.

For map-germs of the form $(x, y^2, y_p(x, y^2))$ the presentation matrix is

$$
\begin{bmatrix}
-Z & p(X, Y) \\
Y_p(X, Y) & -Z
\end{bmatrix}
$$

and $T = 0$.

B:1. $(x, y^2, x^3y + x^2y^3 + y^{2k+1})$.

$Q_f = \langle y, x^3 \rangle$ therefore $C = 3$.

$\tilde{D}^2(f) = V(y+y', x^3 + x^2(y^2+yy'+y'^2) + y^{2k+1}y^{2k-1}y' + ... + y^{2k}) = V(y+y', x^3 + x^2y^2 + y^{2k})$.

Then $\mu(\tilde{D}^2(f)) = \dim \mu(V(x^3 + x^2y^2 + y^{2k})) = \dim \mathcal{O}_2/\langle 3x^2 + 2xy^2, 2x^2y + 2ky^{2k-1} \rangle$.

Let $I = \langle 3x^2 + 2xy^2, 2x^2y + 2ky^{2k-1} \rangle$ and suppose that $xy^5 m_2 \subseteq I$.

Then $x^3y + kxy^{2k-1} \sim x^3y \mod xy^5 m_2$, $3x^3y + 2x^2y^3 \sim 2x^2y^3$, $3x^3y + 2xy^5 \sim 2xy^5$,

hence $xy^5 \mathcal{O}_2 \subseteq I$, hence $\langle x^4, x^3y, x^2y^3, xy^5, y^{2k+1} \rangle \subseteq I$, hence $\mu(\tilde{D}^2(f)) = 2k+4 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1$. Hence $\mu(\tilde{D}^2(f)/\mathbb{Z}_2) = k+1$.

B:2. $(x, y^2, x^4y + 2x^2y^3 + y^5 + x^k y)$.

$Q_f = \langle y, x^4 \rangle$ therefore $C = 4$.

$\tilde{D}^2(f) = V(y+y', x^4 + 2x^2(y^2+yy'+y'^2) + y^4 + y^3y' + ... + y^4 + y^k)$

$= V(y+y', x^4 + 2x^2 + y^4 + x^k) = V(y+y', (x^2 + y^2)^2 + x^k)$

Then $\mu(\tilde{D}^2(f)) = \dim \mu(V((x^2 + y^2)^2 + x^{k-1})) = \dim \mathcal{O}_2/\langle 4x(x^2 + y^2) + kx^{k-1}, 4y(x^2 + y^2) \rangle$.

Let $I = \langle 4x(x^2 + y^2) + kx^{k-1}, y(x^2 + y^2) \rangle$. Then

$xy(x^2 + y^2) \in I \Rightarrow x^{k-1}y \in I \Rightarrow x^{k-3}y^3 \in I \Rightarrow ... \Rightarrow (x)y^{k(-1)} \in I$

(depending on whether $k$ is even or odd)

and $xy^2(x^2 + y^2) \in I \Rightarrow x^{k-1}y^2 \in I \Rightarrow x^{k-3}y^4 \in I \Rightarrow ... \Rightarrow (x)y^{k(+1)} \in I$.

Then $x^{k+1} \in I$ and $\mathcal{O}_2/I$ has a basis $\{1, x, ..., x^k, y, xy, ..., x^{k-2}y, y^2\}$ so $\mu(\tilde{D}^2(f)) = 2k+1 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1$. Hence $\mu(\tilde{D}^2(f)/\mathbb{Z}_2) = k+1$.

B:3. $(x, y^2, x^4y + x^2y^3 + y^{2k+1})$.

$Q_f = \langle y, x^4 \rangle$ therefore $C = 4$. 
\[ \tilde{D}^2(f) = V(y+y', x^4 + x^2(y^2 + y'y + y^2) + y^{2k} + y^{2k-1}y' + \ldots + y^{2k}) = V(y+y', x^4 + x^2y^2 + y^{2k}) \]

Then \( \mu(\tilde{D}^2(f)) = \text{cod} \mu(V(x^4 + x^2y^2 + y^{2k})) = \dim \mathfrak{o}_2/\langle 4x^3 + 2xy^2, 2x^2y + 2ky^{2k-1} \rangle \)

Let \( I = \langle 2x^3 + xy^2, x^2y + ky^{2k-1} \rangle \) and suppose that \( xy^3 \mathfrak{m}_2 \subseteq I \).

Then \( x^3y + kxy^{2k-1} \sim x^3y \mod xy^3 \mathfrak{m}_2 \) and \( x^3y + xy^3 \sim xy^3 \in I + xy^3 \mathfrak{m}_2 \)

hence \( xy^3 \mathfrak{o}_2 \subseteq I \), hence \( \langle x^5, x^3y, xy^3, y^{2k+1} \rangle \subseteq I \), hence \( \mu(\tilde{D}^2(f)) = 2k+5 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1 \). Hence \( \mu(\tilde{D}^2(f)/\mathbb{Z}_2) = k+1 \).

**B:4.** \((x, y^2, x^2y^3 + y^5 + x^ky)\).

\( Q_f = \langle y, x^k \rangle \) therefore \( C = k \).

\[ \tilde{D}^2(f) = V(y+y', x^2(y^2 + y'y + y^2) + y^4 + y^3y' + \ldots + y^4 + x^k) = V(y+y', x^2y^2 + y^4 + x^k) \]

Then \( \mu(\tilde{D}^2(f)) = \text{cod} \mu(V(x^2y^2 + y^4 + x^k)) = \dim \mathfrak{o}_2/\langle 2xy^2 + kx^{k-1}, 2x^2y + 4y^3 \rangle \)

Let \( I = \langle 2xy^2 + kx^{k-1}, x^2y + 2y^3 \rangle \) and suppose that \( x^3y \mathfrak{m}_2 \subseteq I \).

Then \( 2xy^3 + kxy^{2k-1} \sim 2xy^3 \mod x^3y \mathfrak{m}_2 \) and \( x^3y + 2xy^3 \sim x^3y \in I + x^3y \mathfrak{m}_2 \)

hence \( x^3y \mathfrak{o}_2 \subseteq I \), hence \( \langle x^5, x^3y, xy^3, x^{k+1} \rangle \subseteq I \), hence \( \mu(\tilde{D}^2(f)) = k+5 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1 \). Hence \( \mu(\tilde{D}^2(f)/\mathbb{Z}_2) = 3 \).

For map-germs of the form \((x, xy + y^3, p(x, y))\) we note the following:

Let \( X = x, Y = xy + y^3 \). Then

\[ y^4 = yY - XY \]
\[ y^5 = y^2Y - XY + X^2y \]
\[ y^6 = Y^2 - 2XYy + X^2y^2 \]
\[ y^7 = yY^2 - 2XYy^2 + X^2Y - X^3y \]
\[ y^8 = y^2Y^2 - 2XY^2 + 3X^2Yy - X^3y^2 \]
\[ y^9 = Y^3 - 3yXY^2 + 3X^2Yy^2 - X^3Y + X^4y \] etc.

We see that

i) \( y^{3j+i} \in \mathfrak{m}_2^j \{1, y, y^2\} \),

ii) \( y^{3j+i} = y^iY^j \text{ modulo } X \cdot \mathfrak{m}_2^j \) for \( i \in \{0, 1, 2\} \),

iii) \( y^{2j+i} = (-1)^j y^iX^j \text{ modulo } Y \cdot \mathfrak{m}_2 \) for \( i \in \{1, 2\} \).
B:5. \((x, xy + y^3, xy^2 + y^{3k+1})\).

\[ Q_f = \langle x + 3y^2, 2xy + (3k+1)y^{3k} \rangle = \langle x', -6y^3 + (3j+1)y^{3k} \rangle \] under the coordinate change \(x' = x + 3y^2\) so \(C = 3\).

Then \(\hat{D}^2((x, xy + y^3, xy^2)) = V(x+y^2+yy'+y^2, x(y+y')) = V(x+y^2+yy'+y^2, y^3+2y^2y'+2yy'^2+y^3)\)

Then \(\mu(\hat{D}^2(f)) = \mu(\hat{D}^2(x, xy + y^3, xy^2))\) provided that \(\mu(\hat{D}^2(x, xy + y^3, xy^2))\) is finite.

Then \(\mu(\hat{D}^2(x, xy + y^3, xy^2)) = \mu(V(y^3+2y^2y'+2yy'^2+y^3))\)

This gives 2 equations between the 3 terms \(\{y^2, yy', y'^2\}\).

Multiplying each equation by \(x, y\) gives 4 equations between the 4 terms \(\{y^3, y^2y', yy'^2, y'^3\}\). Hence \(\dim = 4\) since the matrix \[
\begin{bmatrix}
3 & 4 & 2 & 0 \\
0 & 3 & 4 & 2 \\
2 & 4 & 3 & 0 \\
0 & 2 & 4 & 3
\end{bmatrix}
\]
has non-zero determinant.

Then \(\mu(\hat{D}^2(f)) = 4 = 2\mu(\hat{D}^2(f)/\mathbb{Z}_2) + C - 1 \Rightarrow \mu(\hat{D}^2(f)/\mathbb{Z}_2) = 1\).

Calculate a presentation for \((x, xy + y^3, xy^2 + y^{3k+1})\) mod \(X.Y.Z\)

\[
\begin{array}{cccc}
1 & y & y^2 & y^3k+3 \\
xy^2+y^{3k+1} & -Z & y^k & X \\
xy^3+y^{3k+2} & 0 & -Z & y^k \\
xy^4+y^{3k+3} & y^{k+1} & 0 & -Z
\end{array}
\]

Then \(S_2 + X.Y.Z = \{X, Z, Y^k\}0_3\) so that \(S_2 = \{X, Z, Y^k\}0_3\) and \(T = k\).

B:6. \((x, xy + y^3, xy^2 + y^4 + y^{2k})\).

\[ Q_f = \langle x + 3y^2, 2xy + 4y^3 + 2ky^{2k-1} \rangle = \langle x', -2y^3 + 2ky^{2k-1} \rangle \] under the coordinate change \(x' = x + 3y^2\) so \(C = 3\).

Then \(\mu(\hat{D}^2(f)) = \mu(\hat{D}^2(x, xy + y^3, xy^2 + y^4))\) provided that \(\mu(\hat{D}^2(x, xy + y^3, xy^2 + y^4))\) is finite. Then \(\mu(\hat{D}^2(f)) = \mu(V(y^3+y^4+y'^2)) = \dim \langle 2yy', y^2, y^2 + 2yy' \rangle\)

This gives 2 equations between the 3 terms \(\{y^2, yy', y'^2\}\).
Multiplying each equation by $x, y$ gives 4 equations between the 4 terms

$$\{ y^3, y^2y', yy'^2, y'^3 \}.$$ Hence dim $= 4$ since

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

has non-zero determinant.

Then $\mu(D^2(f)) = 4 = 2\mu(D^2(f)/\mathbb{Z}_2) + C - 1 \Rightarrow \mu(D^2(f)/\mathbb{Z}_2) = 1.$

Calculate a presentation for $(x, xy + y^3, xy^2 + y^4 + y^{2k})$ mod $Y \cdot m_3$

$$\begin{bmatrix} 1 & y & y^2 \\ xy^2 + y^4 + y^{2k} & -Z & Y \\ xy^3 + y^5 + y^{2k+1} & 0 & -Z + (-1)^k X^k \\ xy^3 + y^6 + y^{2k+2} & 0 & 0 \end{bmatrix}$$

Then $\mathcal{S}_2 + Y \cdot m_3 = \{ Y, Z, X^{k-1} \} \mathcal{O}_3$ so that $\mathcal{S}_2 = \{ Y, Z, X^{k-1} \} \mathcal{O}_3$ and $T = k-1.$

B:7. $(x, xy + y^3, 2xy^2 + 3y^4 + y^k)$.

$Q_f = <x + 3y^2, 4xy + 12y^3 + ky^{k-1}> = <x', ky^{k-1}>$ under the coordinate change $x' = x + 3y^2$ so $C = k.$

$\hat{D}^2((x, xy + y^3, 2xy^2 + 3y^4 + y^k)) =$

$V(x+y^2 + yy'+ y^2, 2(x+y') + 3(y^3+y^2y'+yy'^2+y^3)+y^k+...+y^{k'})$

$= V(x+y^2 + yy'+ y^2, 3(y^3+y^2y'+yy'^2+y^3)+y^k+...+y^{k} - 2(y+y')(y^2+yy'+y^2))$

$= V(x+y^2 + yy'+ y^2, y^3-y^2y'-yy'^2+y^3+y^{k}+...+y^{k'}).$

Then $\mu(D^2(f)) = \text{cod} \mu(V(y^3-y^2y'-yy'^2+y^3+y^{k}+...+y^{k'}))$

$= \dim \mathcal{O}_2/<3y^2 - 2yy' - y^2+ky^{k-1}+...+y^{k-1}+...+y^{k'}/> - y^2 - 2yy'+ 3y^2+ky^{k-1}+...+ky^{k-1}>$

Let $I = <3y^2 - 2yy' - y^2+ky^{k-1}+...+yk^{k-1}, -y^2 - 2yy'+ 3y^2+yk^{k-1}+...+ky^{k-1}>$

Assume that $\{ y^3, y^2y', yy'^2 \} \mathcal{O}_2 \subseteq I.$ Then $I + \{ y^3, y^2y', yy'^2 \} \mathcal{O}_2$

$= <3y^2 - 2yy' - y^2+ky^{k-1}, -y^2 - 2yy'+ 3y^2+ky^{k-1}> + \{ y^3, y^2y', yy'^2 \} \mathcal{O}_2$

$\supseteq \{ 3y^3 - 2y^2y' - yy'^2, -y^3 - 2y^2y' + 3yy'^2, 3ky^2y' - 2kyy'^2 - ky^3+ky'^2y' - 2yy'^2 + 3y^3+ky^k \}.$ Hence $\{ y^3, y^2y', yy'^2, y^k \} \mathcal{O}_2 \subseteq I$ and $\mu(D^2(f)) = k + 1 = 2\mu(D^2(f)/\mathbb{Z}_2) + C - 1 \Rightarrow \mu(D^2(f)/\mathbb{Z}_2)=1.$
Calculate a presentation for \((x, xy + y^3, 2xy^2 + 3y^4 + y^5)\) mod \(\mathfrak{m}_3^2\) assuming that \(k \geq 6\).

\[
\begin{array}{ccc}
1 & y & y^2 \\
2xy^2 + 3y^4 & -Z & 3Y \\
2xy^3 + 3y^5 & 0 & -Z \\
2xy^4 + 3y^6 & 0 & 0
\end{array}
\]

Then \(\mathcal{S}_2 + \mathfrak{m}_3^2 = \{Y, Z, X\} \mathfrak{O}_3\) so that \(\mathcal{S}_2 = \{Y, Z, X\} \mathfrak{O}_3\) and \(T = 1\).

B:8. \((x, xy + y^3, xy^4 + y^5)\).

\[
Q_f = \langle x + 3y^2, 4xy^3 + 5y^4 \rangle = \langle x', 5y^4 - 12y^5 \rangle
\]
under the coordinate change \(x' = x + 3y^2\) so \(C = 4\).

\[
\mathcal{D}^2((x, xy + y^3, xy^4 + y^5)) = V(x + y^2 + y'y^2, x(y^3 + y^2y' + yy'^2 + y'^2) + y^4 + \ldots + y^4)
\]
\[
= V(x + y^2 + y'y^2, y^4 + \ldots + y^4 - 2(y^3 + y^2y' + yy'^2 + y'^2)(y^2 + y'y + y^2))
\]
\[
= V(x + y^2 + y'y^2, y^4 + \ldots + y^4 - 2y^5 - 4y^4y' - 6y^3y^2 - 6y^2y'^3 - 4yy'^4 - 2y'^5)
\]
Then \(\mu(\mathcal{D}^2(f)) = \text{cod} \mu(V(y^4 + \ldots + y^4 - 2y^5 - 4y^4y' - 6y^3y^2 - 6y^2y'^3 - 4yy'^4 - 2y'^5))
\]
\[
= \dim \mathfrak{O}_2/I, \text{ where } I = \langle 4y^3 + 3y^2y' + 2yy'^2 + y'^3 - 10y^4 - 16y^3y' - 18y^2y'^2 - 12yy'^3 - 4y'y^4, \\
y^3 + 2y^2y' + 3yy'^2 + 4y'^3 - 4y^4 - 12y^3y' - 18y^2y'^2 - 16yy'^3 - 10y'^4 \rangle.
\]

Calculate this modulo \(\mathfrak{m}_2^2\). Then there are 6 equations between the 6 monomials of degree 5, so \(I \supseteq \mathfrak{m}_2^5\) since

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 0 \\
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 & 3 \\
4 & 3 & 2 & 1 & 0 \\
0 & 4 & 3 & 2 & 1 \\
0 & 0 & 4 & 3 & 2 & 1
\end{bmatrix}
\]

has non-zero determinant.

Hence \(\mu(\mathcal{D}^2(f)) = 9 = 2\mu(\mathcal{D}^2(f)/\mathfrak{Z}_2) + C - 1 \Rightarrow \mu(\mathcal{D}^2(f)/\mathfrak{Z}_2) = 3\).

Calculate a presentation for \((x, xy + y^3, xy^4 + y^5)\).
Then $\mathcal{S}_2 = \{Z+XY, Y-X^2, X^2+XY\} \mathfrak{o}_3$. Let $Y' = Y-X^2$, then $\mathcal{S}_2 = \{Z, Y', X^2+X^3\} \mathfrak{o}_3$ and $T = 2$.

B:9. $(x, xy+y^3, xy^4+y^{3k+2})$.

$Q_f = \langle x + 3y^2, 4xy^3 + (3k+2)y^{3k+1} \rangle = \langle x', -12y^5 + (3k+2)y^{3k+1} \rangle$ under the coordinate change $x' = x + 3y^2$ so $C = 5$.

$\mathcal{D}^2((x, xy + y^3, xy^4)) = V(x+y^2+yy'+y^2, x(y^3+y^2y'+yy^2+y^3))$

$= V(x+y^2+yy'+y^2, (y^3+y^2y'+yy^2+y^3)(y^2+yy'+y^2))$

$= V(x+y^2+yy'+y^2, y^5+2y^4y'+3y^3y^2+3y^2y^3+2yy^4+y^5)$

Then $\mu(\mathcal{D}^2(x, xy + y^3, xy^4)) = \text{cod} \mu(V(y^5+2y^4y'+3y^3y^2+3y^2y^3+2yy^4+y^5))$

$= \dim \mathfrak{o}_2/I$ where $I = \langle 5y^4+8y^3y'+9y^2y^2+6yy^3+2y^4, 2y^4+6y^3y'+9y^2y^2+8yy^3+5y^4 \rangle$. Then $\mu(\mathcal{D}^2(f)) = \mu(\mathcal{D}^2(x, xy + y^3, xy^4))$ provided that $\mu(\mathcal{D}^2(x, xy + y^3, xy^4))$ is finite.

Calculate this modulo $\mathfrak{m}_2$. Then there are 8 equations between the 8 monomials of degree 7, so $I \supseteq \mathfrak{m}_2$ since the relevant matrix has non-zero determinant.

Hence $\mu(\mathcal{D}^2(f)) = 16 = 2\mu(\mathcal{D}^2(f)/\mathbb{Z}_2)+C-1 \Rightarrow \mu(\mathcal{D}^2(f)/\mathbb{Z}_2) = 6$.

Calculate a presentation for $(x, xy+y^3, xy^4+y^{3k+2})$ modulo $X\cdot \mathfrak{m}_2^3$.

$xy^4+y^{3k+2}$ $-Z$ $XY$ $-X^2+Y^k$

$xy^5+y^{3k+3}$ $Y^{k+1}-X^2Y$ $-Z+X^3$ $XY$

$xy^6+y^{3k+4}$ $XY^2$ $Y^{k+1}-2X^2Y$ $-Z+X^3$

Then $\mathcal{S}_2 + X\cdot \mathfrak{m}_3^3 = \{Z, XY, -X^2+Y^k\} \mathfrak{o}_3$. Then $\mathcal{S}_2 + X\cdot \mathfrak{m}_3^3 = \{Z, XY, X^3, Y^{k+1}\} \mathfrak{o}_3$ and $T = k+2$. 
B:10. \((x, x^2y + y^3, xy^2 + y^3k+1)\).

\[ Q_f = \langle x^2 + 3y^2, 2xy + (3k+1)y^3 \rangle \cong \langle x^3, y^3, xy \rangle \text{ so } C = 4. \]

\[ \overline{D}^2((x, x^2y + y^3, xy^2)) = V(x^2+y^2+yy'+y^2, x(y+y')). \]

We show that

\[ h_f = (x^2+y^2+yy'+y^2, x(y+y')) \]

is \(\mathcal{X}\)-finite. Since \(h_f\) is weighted homogeneous \(\mu(h_f) = \dim \mathcal{O}_3/\langle h_f, J_h \rangle\) which is finite if and only if \(h_f\) is \(\mathcal{X}\)-finite.

Then \(\mu(\overline{D}^2(x, x^2y + y^3, xy^2)) = \dim \mathcal{O}_3/ I\)

where \( I = \langle x^2+y^2+yy'+y^2, x(y+y'), 2x^2-2y^2-3yy'-2y^2, x(y-y') \rangle \)

= \( \langle x^3, xy, xy', x^2+y^2+yy'+y^2, 2x^2-2y^2-3yy'-2y^2 \rangle \)

\( \cong \{ y^3, y^2y', yy'^2, y^3 \}. \)

Hence \( \mu(\overline{D}^2(f)) = 5 = 2\mu(\overline{D}^2(f)/\mathbb{Z}_2) + C - 1 \Rightarrow \)

\( \mu(\overline{D}^2(f)/\mathbb{Z}_2) = 1. \)

Calculate a presentation for \((x, x^2y + y^3, xy^2 + y^3k+1)\) modulo \(X\cdot\mathcal{M}_3: \)

\[
\begin{array}{ccc}
1 & y & y^2 \\
xy^2 + y^3k+1 & -Z & Y^k & X \\
xy^3 + y^3k+2 & 0 & -Z & Y^k \\
xy^4 + y^3k+3 & Y^{k+1} & 0 & -Z \\
\end{array}
\]

since \(y^3k+i = y^iY^k \) modulo \(X\cdot\mathcal{M}_3.\)

Then \(\mathcal{S}_2 + X\cdot\mathcal{M}_3 = \{ Z, X, Y^k \}\mathcal{O}_3. \) Then \(\mathcal{S}_2 = \{ Z, X, Y^k \}\mathcal{O}_3\) and \(T = k. \)

B:11. \((x, x^2y - y^3, x^2y^2 - y^4 + x^ky^2).\)

\[ Q_f = \langle x^2-3y^2, 2x^2y-4y^4+x^ky \rangle = \langle x^2-3y^2, -10y^4+x^ky \rangle \text{ therefore } C = 6. \]

\[ \overline{D}(x, x^2y - y^3, x^2y^2 - y^4) = V(x^2-y^2-yy'-y^2, x^2(y+y')-y^2y'-yy^2-y^3) \]

= \( V(x^2-y^2-yy'-y^2, (y+y')(y^2+yy'+y^2) - y^3-y^2y'-yy^2-y^3) \)

= \( V(x^2-y^2-yy'-y^2, y^2y'+yy^2). \)

We show that \(h_f = (x^2-y^2-yy'-y^2, y^2y'+yy^2)\) is \(\mathcal{X}\)-finite.

Since \(h_f\) is weighted homogeneous \(\mu(h_f) = \dim \mathcal{O}_3/\langle h_f, J_h \rangle\) which is finite if and only if \(h_f\) is \(\mathcal{X}\)-finite. Then \(\mu(D^2(x, x^2y - y^3, x^2y^2 - y^4)) = \dim \mathcal{O}_3/ I\)

where \( I = \langle x^2-y^2-yy'-y^2, y^2y'+yy^2, x(2yy'+y^2), x(y^2+2yy'), (y-y')(2y+y')(y+2y') \rangle \)

= \( \langle x^2-y^2-yy'-y^2, y^2y'+yy^2, x(2yy'+y^2), x(y^2+2yy'), y^3+3y^2y'-y^3 \rangle \)
\[\{y^5, y'^4y', y^3y^2, y^2y^3, yy^4, y'^5\}\] since the matrix \[
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
1 & 3 & 0 & -1 & 0 & 0 \\
0 & 1 & 3 & 0 & -1 & 0 \\
0 & 0 & 1 & 3 & 0 & -1 
\end{bmatrix}
\] has non-zero determinant. Hence \(\mu(\tilde{D}^2(f)) = 9 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1 \Rightarrow \mu(\tilde{D}^2(f)/\mathbb{Z}_2) = 2.\)

Calculate a presentation for \((x, x^2y - y^3, x^2y^2 - y^4 + x^k y^2).\)

\[
\begin{array}{ccc}
1 & y & y^2 \\
x^2y^2 - y^4 + x^k y^2 & -Z & Y - X^k \\
x^2y^3 - y^5 + x^k y^3 & -X^k Y & -Z + X^k + 2 Y \\
x^2y^4 - y^6 + x^k y^4 & -Y^2 & -X^k Y + X^2 Y - Z + X^k + 2 \\
\end{array}
\]

Then \(\mathcal{S}_2 = \{Z, Y, X^k\}_{3} \) and \( T = k. \)
Appendix C : The Computer Programs.

Contents.

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>C1</td>
</tr>
<tr>
<td>Language and Computers</td>
<td>C1</td>
</tr>
<tr>
<td>Presentation and Saving Data</td>
<td>C2</td>
</tr>
<tr>
<td>Basic Algorithms</td>
<td>C6</td>
</tr>
</tbody>
</table>

Introduction.

There are two versions of the program, module and param, which calculate a set of generators for the module

$$T_{R,S,f}^R = tf(\tau_2^R \cdot \theta(2)) + \omega f(\tau_3^S \cdot \theta(3)) \mod \tau_2^{k+1} \theta(f)$$

for integers $R, S, K$ and $f(x, y) = (f_1, f_2, f_3)$, where each $f_i$ is a polynomial with integer coefficients. In the version param the polynomials may also take parameter coefficients.

Language and Computers.

Versions exist to run on the Sun Workstation and the Apple Macintosh, with different versions for each computer, making use of their different capabilities. The programs are written in Pascal (since this is the only language I know). To save the data in an external file it has been necessary to use language which is specific to these computers, namely Berkely Pascal (on the Sun) and MacPascal. I believe that this is only necessary when saving or printing the data; however, without this facility the programs are of limited use. A lot of problems were found in transporting the programs from the Mac to the Sun and it is likely that similar problems would be found in transporting the program to other systems.

Apple Macintosh.

Advantages: it is relatively easy to write and debug programs in MacPascal. There are also features of the language which make it easy to send data to an external file without overwriting anything and to send data directly to the printer. The system does not crash when it receives unexpected input (a character when it expects an integer).

Disadvantages: it is very slow and there is not much memory space (one of
the main differences in these versions is the use of numerous prompts to let the user know that things are happening). The range of integers is small (±(2^{15}−1) or ±32,767) so we have to resort to the use of alternative methods of storing numbers. The use of non-standard Pascal makes the program difficult to transport to other computers.

Note: a manual for use of the package **Lightspeed Pascal** has recently become available at Warwick. This includes a compiler so it should become possible to provide compiled versions of the programs, which will be much faster and have more memory space available.

**Sun Workstation.**

Advantages: it is far faster and far more memory available. The integer range is larger (at least ±2,000,000,000) and the program is closer to standard Pascal so is likely to be less difficult to transport.

Disadvantages: it is more difficult to save data and more errors can be made doing so (the chief being overwriting a file that was wanted). It is more difficult to debug the programs.

**Presentation of data.**

Every effort has been made to make the programs easy to use. There are numerous prompts for the user and the user can make small alterations or corrections to the example quickly.

The module can be expressed in the form

\[ I_1 \partial / \partial X + I_2 \partial / \partial Y + I_3 \partial / \partial Z + \mathbb{C} \{ v_1, \ldots, v_r \} \]

where each \( I_j \) is an \( \mathcal{O}_3 \)-module via \( f^* \) and the supplementary generators \( v_j \) are of the form \( (p_1, p_2, p_3) \) where at least two of the \( p_i \)’s are non-zero.

We give generators for each row \( I_j \) in turn, followed by a list of supplementary generators. If there are none then this is indicated. Generators for \( I \) are given in ascending order, starting with monomials; if there are a large number of monomial generators then we give them in the form \( H_n \backslash \{ \text{missing monomials} \} \), where \( H_n \) indicates the set \( \{ x^{n-j} y^j : 0 \leq j \leq n \} \).
Running the programs (an example).

For example, suppose we are calculating

\[ T_R,S f \text{ modulo } m_2^8(f) \text{ for } f = (x, xy + y^3, 2xy^2 + y^4) \]

with the program param.

The prompt

This program calculates the terms in \( tf(M^R) + wf(N^S) \) for

\[ f(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y)) \]

It will only run if \( f_1, f_2 \) and \( f_3 \) have no constant terms or if \( f_2 \) or \( f_3 \) are zero.

In this version \( f \) may have parameter values.

Program written by D. Ratcliffe.

please enter R

should appear. In the programs we denote \( m_2^2(2) \) by M and \( m_3^3(3) \) by N.

Powers are indicated by a circumflex before the integer. The program the waits for

the data to be given. The procedure for this example is as follows:

Please enter R 1
Please enter S 1
Please enter k 7

Type in number of terms in \( f_1 \) 1

Do you wish to add terms with moduli ? n

Type in scalar 1
Type in power of x 1
Type in power of y 0

Type in number of terms in \( f_2 \) 2

Do you wish to add terms with moduli ? n

Type in scalar 1
Type in power of x 1
Type in power of y 1

Type in scalar 1
Type in power of x 0
Type in power of y 3

Type in number of terms in \( f_3 \) 2

Do you wish to add terms with moduli ? n

Type in scalar 1
Type in power of x 1
Type in power of y 2

Type in scalar 1
Type in power of x 0
Type in power of y 4
Calculating $\text{tf}(M^1) + \text{wf}(N^1) \mod M^2 \cdot 8$
for $f = (x, xy + y^3, xy^2 + y^4)$

Is this correct? y

The program then proceeds with the calculation, giving prompts as it completes certain stages, eg. 'calculated $\omega f$', etc. The prompt 'comparing supplementary generators' means that the program has started to compare supplementary generators with supplementary generators. This is frequently the slowest part of the program for a complicated map-germ. The phrase 'place $= n$' indicates which list of supplementary generators is being studied. There are four lists:

- place $= 0$: this indicates that there are no zeros on any row of the generator,
- place $= 1$: this indicates that the first row is zero,
- place $= 2$: this indicates that the second row is zero,
- place $= 3$: this indicates that the third row is zero.

The indicative phrase will be repeated every time the program runs through that list applying the particular algorithm.

A typical display will be as follows:

$\text{tf}(M^1) + \text{wf}(N^1) \mod M^2 \cdot 8$ for $f = (x, xy + y^3, xy^2 + y^4)$ is as follows:

generators in row(1) are
$H_1\{ y \}$ $H_2, H_3, H_4, H_5, H_6, H_7,$

generators in row(2) are
$H_1\{ y \} 2: C\{ x^2, xy + y^3 \} H_3\{ y^3, x^2y \} + C\{ x^2y - y^5 \} H_4\{ xy^3 \} + C\{ xy^3 + y^5 \} H_5\{ y^5 \} H_6, H_7,$

generators in row(3) are
$H_1\{ y \} 2: C\{ x^2, xy + y^3 \} H_3\{ y^3, xy^2 \} + C\{ xy^2 + y^4 \} H_4\{ y^4, x^2y^2 \} + C\{ x^2y^2 - y^6 \} H_5\{ xy^4 \} + C\{ xy^4 + y^6 \} H_6\{ y^6 \} H_7,$

supplementary generators are
$(0, y^3, y^4), (0, y^5, y^6)$
$(y, y^2, y^3)$

cod $= 7$

cod is the codimension of $\mathfrak{m}_2^0(f) / T \mathcal{S}_{R, S}^f$. Note that if simplifications have been missed or parameters are involved, this number is frequently wrong.

The display continues as follows:

Is the display satisfactory? y
Do you wish to save the data? y

enter name of file (10 characters):
IF THERE IS ALREADY A FILE WITH THIS NAME IT WILL BE OVERWRITTEN!
examplel

filename is examplel

Do you wish to continue? n

The display is controlled by counting the number of terms or supplementary
generators printed on a line and putting in a return at fixed intervals. This is
essential since otherwise the entire display would be printed on one line. On the
screen this does not matter, but it results in the loss of data when it sent to a
printer. This does not take account of the length of the terms, so if the coefficients
are large, the preset values may be too big. If it is suspected that this may occur,
the values may be altered.

Note that the name may have blanks (or returns) at the end (but not in the
middle). If the program pauses at this point, it is usually because not enough
characters have been entered; press returns until it continues.

The allowed input is as follows:

k must lie between 1 and 40. This is an arbitrarily chosen maximum; it is
the size of certain arrays used in the program.

f_1 must be non-zero and have no constant term. This is because the
program starts by calculating powers of f_1 until some power lies in \( \mathfrak{m}_2^{k+1} \).

The other polynomials may be zero, but may not have a constant term.

Expansion A future plan is to expand this to a package of programs to calculate
submodules of \( T \mathfrak{A} f, T \mathfrak{K} f \) and \( T \mathfrak{R} f \) for map-germs \( (\mathbb{C}^p, 0) \rightarrow (\mathbb{C}^p, 0) \) for small
values of \( n \) and \( p \).
Basic Algorithms.

For storing data we use the same technique throughout; we form lists of records using pointers, creating a record at the end of the list when we have new data to store and destroying the record when it is no longer needed. This keeps the data stored to a minimum. The disadvantage of this is that bugs related to pointers are difficult to locate and eradicate.

1) The polynomial.
Polynomials are formed term by term and ordered. The ordering system is
\[ x^i y^j < y^{i'} y^{j'} \] if \( i+j < i'+j' \).
If \( i+j = i'+j' \) then the order becomes
\[ x^i y^j < y^{i'} y^{j'} \] if \( i > i' \).
If there are parameters then the overriding ordering is 'constants first'. Thus the first term in a polynomial is a constant unless all its terms have parameters.

2) Simplification procedure for polynomials.
When inspecting a term \( p \) in the polynomial \( P \) being simplified, if there is a polynomial already stored having \( p \) as its lead term either with constant coefficient or the same coefficient as \( p \) then \( p \) is eliminated from \( P \).
For example, if the polynomial \( x^2 y^2 + y^6 \) is inspected and \( x^2 y^2 + xy^4 \) is already stored then \( x^2 y^2 + y^6 \) is replaced by \( -xy^4 + y^6 \). If, later \( 2xy^4 + y^6 \) is added then this is stored as \( y^6 \). On following checks the set \( \{x^2 y^2 + xy^4, x^2 y^2 + y^6, 2xy^4 + y^6\} \) initially calculated will be stored as \( \{x^2 y^2, xy^4, y^6\} \).
This simplification process is also applied to the supplementary generators.

The disadvantage with this is that simple patterns in the generators may be obscured. This is a common problem with generators of the form \( x^k p \) when \( f \) has corank 1.

3) Simplification procedure for supplementary generators.
The problem is how, for example, do we recognise that \( (xy, x^2y, 0) \) and \( (xy, 2x^2y, 0) \) should be replaced with generators \( (xy, 0, 0) \) and \( (0, x^2y, 0) \) while at the same time avoiding constantly cycling through the loop
\[ \{(xy, x^2y, 0), (xy, 0, x^2y^2)\} \leftrightarrow \{(xy, x^2y, 0), (0, -x^2y, x^2y^2)\}? \]
For each row of the map being inspected, we select terms which may cancel and form a list. We then attempt to simplify all the maps in the list by elimination of polynomials.

For example, given \((x^2y^2 + xy^4 + y^6, p_2, p_3)\), we search for all map-germs of the form \((ax^2y^2 + bxy^4 + cy^6, p_2, p_3)\), where \(a, b, c \in \mathbb{Z}\). We then try to eliminate the terms \(x^2y^2, xy^4, y^6\) from the first row of each map-germ in the list. The simplification procedure is as follows:

1. Increase the 'place' of the map—i.e. try to reduce rows to zero. This also avoids looping, since in the example

   \[
   \{(xy, x^2y, 0), (xy, 0, x^2y^2)\} \leftrightarrow \{(xy, x^2y, 0), (0, -x^2y, x^2y^2)\}
   \]

   \((xy, 0, x^2y^2)\) has place 2 so is preferred to \((0, -x^2y, x^2y^2)\), which has place 1.

2. Shorten the polynomial if none of the other rows is made longer.

We do not guarantee that the procedures will give a complete simplification in every case, but this is safer than getting into infinite loops through trying to be too clever.
program Parameters(input, output);  

const
LIMIT = 40;  
Cp = 3;  
DEFTLT = 12;  
DEFTVL = 3;  
MAXVALUE = 2000000000;  
NOQUERY = 10;
ADDMD2 = true;
DEBUG = 4;  
ACTDIS = true;  
DEFDIS = false;
DOMAP = false;
DOPOLY = true;
ADD = true;
STRINGLENGTH = 45;
DELETE = true;
KEEP = false;
CHARTOLINE = 80;

Parameters

const
LIMIT = 40;  
Cp = 3;  
DEFTLT = 12;  
DEFTVL = 3;  
MAXVALUE = 2000000000;  
NOQUERY = 10;
ADDMD2 = true;
DEBUG = 4;  
ACTDIS = true;  
DEFDIS = false;
DOMAP = false;
DOPOLY = true;
ADD = true;
STRINGLENGTH = 45;
DELETE = true;
KEEP = false;
CHARTOLINE = 80;

Parameters
var
continue, ch: char;
k, R, S, Odim, forder, Printno, Lno, Vno: integer;
one: polynomial;
f: map;
Df: Dmap;
wf: module;
tf: vectorstore;
rerun: boolean;
name, changef: charlist;
fdatas: mapdata;
rangesubrange: inset;
dialogue: record
query: array[1..NOQUERY] of string;
response: set of char;
end;

{A: ERASURE PROCEDURES}

procedure ruboutpoly (var a: polynomial);
begin
while a <> nil do
begin
item := a;
a := a^.next;
dispose(item);
end;
end;

procedure ruboutmap (var f: map);
begin
for c := 1 to Cp do
if f[c] <> nil then ruboutpoly(f[c]);
end;

procedure ruboutdata (var tf: vectorstore);
begin
for j := 0 to 3 do
while tf[j] <> nil do
begin
v := tf[j];
tf[j] := tf[j]^next;
if v^.f <> nil then
begin
ruboutmap(v^.f);
dispose(v^.f);
dispose(v);
end;
end;
end;
end;

procedure ruboutlist (var list: polylist);
var
v: polylist;
begin
while list <> nil do
begin
ruboutpoly
ruboutmap
ruboutdata
rubouttf
ruboutlist
v:= list;
list:= list^.next;
if v^.poly <> nil then ruboutpoly(v^.poly);
dispose(v);
end
end; {ruboulist}

begin
for coord:= 1 to Cp do
for grade:= 0 to LIMIT do
begin
wf[coord][grade].single:= [];
if wf[coord][grade].sum <> nil then ruboutlist(wf[coord][grade].sum);
end;
ruboutlist(tf);
end; {ruboutdata}

{B: MISCELLANEOUS PROCEDURES}

procedure makequery (number : integer; ref : char; var reply : char);
begin
repeat
writeln;
write(dialogue.query[number].ref,' ? ');
readln(reply);
until (reply in dialogue.response);
if reply = 'Y' then reply := 'y';
if reply = 'N' then reply := 'n';
end; {makequery}

procedure getpolydata (q : polynomial; var qdata : polydata);
begin
qdata.maxcoeff := 0;
qdata.mincoeff := 0;
qdata.moduli:=false;
qdata.length :=0;
if q<> nil then qdata.moduli := (q^.m <> ' ');
while q <> nil do
begin
{for each term}
if (q^.s <> 0) then
begin
{get data}
if qdata.maxcoeff < q^.s then qdata.maxcoeff := q^.s;
qdata.length := qdata.length + 1;
if qdata.mincoeff > q^.s then qdata.mincoeff := q^.s;
qdata.moduli := (q^.m <> ' ') or qdata.moduli;
end; {of get data}
q := q^.next;
end;
end; {getpolydata}

procedure getmapdata (v : map; var vdata : mapdata);
var
c : integer;
begin
for c := 1 to Cp do getpolydata(v[c], vdata[c])
end; {getmapdata}

procedure getlimits (var data : mapdata; var max, min : integer);
var
i : integer;
begin
max := data[1].maxcoeff;
min := data[1].mincoeff;
for i := 2 to Cp do
begin
if max < data[i].maxcoeff then max := data[i].maxcoeff;
if min > data[i].mincoeff then min := data[i].mincoeff;
end;
end;

function polycopy (old : polynomial; ix, iy, factor : integer; deleteold : boolean) : polynomial;
{ calculate factor*x^ix*y^iy*old }
var
newp, oldterm, newterm : polynomial;
begin
newp := nil;
if factor = 0 then write(‘error in polycopy’);
while old <> nil do
begin
if (old.x + old.y + ix + iy <= k) and (old.s <> 0) then
begin
{ copy term; first add new term to polynomial }
if newp = nil then
begin newp := new(newterm); newterm := newterm.next end;
else begin newp := new(newterm.next); newterm := newterm.next end;
with newterm do
begin
s := old.s * factor;
x := old.x + ix;
y := old.y + iy;
m := old.m;
next := nil;
end;
end;
{ of save term }
oldterm := old;
old := old.next;
if deleteold then dispose(oldterm);
end;
{ go on to next term }
polycopy := newp
end;

{ C: DISPLAY PROCEDURES }

procedure show (power : integer; ch : char); { display ch^power }
begin
if power <> 0 then
begin
write(ch : 1);
if power <> 1 then write(‘‘, power : 1)
end;
end;

procedure wln (var counter : integer); { note: counter usually indicates the number of objects printed on a line }
begin wln; counter := 0 end;

procedure displaypoly (p : polynomial; var printno : integer; ln0 : integer);
var
first : boolean;
begin
first := true;
while p <> nil do
begin
{ go on to next term }
printno := printno + 1;
if not first and (p.s >= 0) then write(‘+’ : 2);
if p.s < 0 then write(‘-’ : 2);
if first then first := not first;
if (p.x + p.y = 0) or (abs(p.s) <= 1) then write(abs(p.s) : 1);
if p.m <> then write(p.m);
show(p.x, ‘x’);
show(p.y, ‘y’);
if (printno mod lno = 0) then writeln(printno);
    p := p^.next;
end; (of go on to next term)
if first then writel('0');  \{displaypoly\}
end;

procedure displaymap (v : map; var printno : integer; lno : integer);
var
c : integer;
begin
    writeln('');
    for c := 1 to Cp do
        begin
            displaypoly(v[c], printno, lno);
            if c <> Cp then writeln(', ')
        end;
    writeln('');
end;
\{displaymap\}

procedure displayTAF (var tf : vectorstore; \{subprocedures:-\} displayTAF
var
wF : module;
realdisplay : boolean); \{displayTAF\}
\begin{verbatim}
var
    lno, vno, printno, i, dim, j : integer;
    name : shortstring; \{nb : realdisplay false indicates that this\}
    ch, ok : char; \{is a display for debugging purposes and does\}
    defaultis, savedis : boolean; \{not include options to improve layout or save to a file\}
procedure Displaywf (var wF : module; \{subprocedure showsingle\}
        var n, lno : integer; var dim, printno : integer);
cons showH = true;
var
    i, j, count1, count2 : integer;
    list : polylist;
inrange : intset;
procedure showsingle (terms : intset; count1, count2, i : integer;
        var printno : integer; showH : boolean); \{display (monomials) or HN-monomials, depending on showH\}
var
    j, c, n : integer;
begin
    if (i + 3 - count1 + count2 + printno > lno) then writeln(printno);
        c:=0;
    if showH then
        begin
            write('H', i : 1);
            n := i+1-count1;
            printno := printno + 1;
            if (count1 < i + 1) then write('
')
                else write('');
        end
    else
        begin
            write(' ', i : 1, ': C(', n := count1; printno := printno+1 end;
            for j := 0 to i do
                if (j in terms) then
                    begin
                        write(' '); show(j, 'x'); show(i-j, 'y');
                        printno := printno + 1;
                        c:=c+1;
                        if (c <> n) or (count2>0) then write('*')
                    end;
        if (showH and (count1 < i + 1)) or (not showH and (count2=0)) then write('?')
\end{verbatim}
\{showsingle\}
\begin{verbatim}
end;
begin
  writeln(printno);
  writeln(' generators in row(' , n : 1 , ') are ');
for i := 0 to k do
  begin
    count1 := 0;
    count2 := 0;
    intrange := [0..i];
    list := w[n][i].sum;
    while list <> nil do
      begin
        count := 0;
        count2 := 0;
        intrange := [0..i];
        list := w[n][i].sum;
        while list <> nil do
          begin
            if list'.poly <> nil then count2 := count2 + 1
            else list := list'.next;
          end;
          for j := 0 to i do
            begin
              count1 := count1 + 1;
              dim := dim + count1 + count2;
            end;
        end;
        if count1 > (i div 2) then
          showsingle(intrange - w[n][i].single, count1, count2, i, printno, showH)
        else if count1 > 0 then
          showsingle(w[n][i].single, count1, count2, i, printno, not showH);
        if (count2 > 0) then
          displaypolynomials
          begin
            if (count1 = 0) and not (i = 1) then writeln(printno);
            if (count1 > (i div 2)) then writeln('+');
            if (count1=0) or (count1>(i div 2)) then writeln('C');
            printno := printno + 1;
            list := w[n][i].sum;
            while list <> nil do
              begin
                count := 0;
                v := tf[ij];
                writeln(printno);
                if printedvectors and (count mod vno <> 0) then writeln(printno);
                while v <> nil do
                  begin
                    if not printedvectors then
                      begin
                        printedvectors := true;
                        writeln('supplementary generators are ');
                      end;
                    count := count + 1;
                    dim := dim + 1;
                    displaymap(v'.r, printno, ln0);
                    end;
              end;
          end;
        end;
      end;
    end;
end;
{displaywvf}
procedure displaytf (var tf : vectorstore; var dim, printno : integer;
                    ln0, vno : integer);
var
  v : vectorlist;
  count, j : integer;
  printedvectors : boolean;
begin
  printedvectors := false;
  for j := 3 downto 0 do
    begin
      count:=0;
      v := tf[ij];
      writeln(printno);
      if printedvectors and (count mod vno <> 0) then writeln(printno);
      while v <> nil do
        begin
          if not printedvectors then
            begin
              printedvectors := true;
              writeln('supplementary generators are ');
            end;
          count := count + 1;
          dim := dim + 1;
          displaymap(v'.r, printno, ln0);
        end;
    end;
end;
{showsingle}
{displaywvf}
\end{verbatim}
if v'.next <> nil then write(',
if count mod vno = 0 then writeln(printno);
v := v'.next;
end;
{display terms}
end;
if not printed vectors then writeln(' no supplementary generators ');
end;
{display)}

procedure getname(var name:shortstring);
var n:integer;
begin
writeln(' enter name of file (10 characters):');
writeln('IF THERE IS ALREADY A FILE WITH THIS NAME IT WILL BE OVERWRITTEN!');
for n:=1 to 9 do read(name[n]);
readln(name[10]);
writeln(' filename is ' , name)
end;

begin
savedis := false;
ok:= 'y';
for j = 1 to 2 do {on j=1 send data to terminal screen : on j=2 send to a file}
if (j = 1) or savedis then {send data to relevant output}
begin
repeat {until the display is ok}
ch := 'n';
if j = 1 then
begin
{define the display parameters}
if realdisplay and (ok='n') then makequery(6, ', ', ch);
defauldis := (ch = 'n');
if not defaultdis then {get new printout parameters}
begin
writeln('The printer prints approximately ',CHARTOLINE:1,' characters on a line. ');
write('How many supplementary generators to a line? (currently ',vno:1,) ')
readln(vno);
writeln('How many generators to a line? (currently ',lno:1,) ')
readln(lno);
end
else {use preset parameters}
begin vno := DEFV1L; lno := DEFTTL end;
{set display parameters}
end;
{write a new line and puts the input value to zero}
writeln(printno);
wln(printno);
if savedis then begin getname(name); rewrite(output, name); end; {data is sent to file called name}
write(' tf(M''R:1,) + w(N'',S:1,) mod M'', k + 1 : 1, ' for f = ');
displaymap(f, printno, lno);
writeln(' is as follows:-');
for i := 1 to Cp do begin
Displaywf(wf, i, lno, dim, printno);
writeln;
end;
end;
displaywf(if, dim, printno, lno, vno);
writeln;
writeln(', cod =', Odim - dim : 1);
if savedis then rewrite(output,'/dev/tty'); {data output returned to terminal}
if realdisplay and (j = 1) then makequery(4, ', ', ok)
else ok := 'y'; {check display is satisfactory before saving}
until ok = 'y';
if not savedis then makequery(7, ', ch); {is the data to be saved?}
savedis=(ch = 'y');
end; {for j = 1 to 2}
function HCF (m, n : integer) : integer;
var
rem, high, low, absm, absn : integer;
begin
ABSM := abs(m);
ABSN := abs(n);
if absm > absn then
begin high := absm; low := absn; end
else
begin high := absn; low := absm end;
if low > 0 then
repeat
rem := high mod low;
high := low;
low := rem
until low = 0
else if low < 0 then write(" bad data in HCF:low");
if (n <= 0) and (m <= 0) then high := -high;
HCF := high;
end;

function HCFpoly (p : polynomial) : integer;
var
d : integer;
begin
d := 0;
while p <> nil do
begin
  d := HCF(d, p'.S);
p := p'.next;
end;
HCFpoly := d
end;

procedure dividepoly (p : polynomial; factor :integer);
begin
if (factor <> 1) and (factor <> 0) then
while p <> nil do
begin
  if p'.S mod abs(factor) <> 0 then write(" bad data in dividepoly");
  else p'.S := p'.S div factor;
p := p'.next;
end;
end;

procedure dividemap (var f : map);
var
  coord, d: integer;
function HCFmap (f : map) : integer;
var
di : integer;
begin
d := HCFpoly(f[1]);
for i := 2 to Cp do
  d := HCF(d, HCFpoly(f[i]));
HCFmap := d;
end;
begin
  d := HCFmap(f);
  for coord := 1 to Cp do
    dividepoly(f[coord], d)
procedure  getmultipliers (scalar1, scalar2, max1, min1, max2, min2 : integer;  
                   var m1, m2 : integer; var multiOK : boolean);
var  
   d : integer;  
   {check that m1*max1, m1*min1, m2*max2, m2*min2 are in range}
begin
   d := HCF(scalar1, scalar2);
   m1 := scalar2 div d;
   m2 := scalar1 div d;
   multiOK := (max1 < MAXVALUE div abs(m1)) and (-min1 < MAXVALUE div abs(m1))
                   and (max2 < MAXVALUE div abs(m2)) and (-min2 < MAXVALUE div abs(m2))
end;

{E: ADDITION AND SCALAR MULTIPLICATION OF POLYNOMIALS}

procedure search (p : polynomial; x, y : integer; var m:char;  
                   var scalar :integer; var isin : boolean);
begin
   {find the coefficients of the term with powers x, y if it is in p}
   scalar := 0;  
   {write(' search ')}
   m:= ' ';  
   isin := false;
   while (p<> nil) and not isin do
       begin
          isin := (p,x = x). and (p,y = y);
          if isin then begin scalar := p,s; m:=p,m end;
          p := p.next;
       end;
end;

function addtotal (var total : polynomial; newterm : polynomial) : polynomial;  
                   (add the monomial newterm to the polynomial total) 
                   { in the correct order}
var
   inalready : boolean;
   t,last : polynomial;
begin(addtotal)
   inalready := false;
   if newterm <> nil then if newterm.next <> nil then writer error in addtotal ;
   if total = nil then total:=newterm
   else if newterm <> nil then
       begin
          t := total;
          last:=nil;
          while (t<> nil) and not inalready do
              begin
                 if (newterm,x + newterm,y <= t,x + t,y) and ((t,m<>' ') or (newterm,m = ' ')) then
                     begin
                        inalready := true;
                        if (newterm,x = t,x) and (newterm,y = t,y) and (newterm,m = t,m) then
                            begin
                                t,s:= t,s + newterm,s;
                                dispose(newterm)
                            end
                        else
                            begin
                                if last = nil then total:= newterm else last.next:=newterm;
                                newterm.next:= t
                            end;
                     end
              end;
       end;
end;
if not inalready then last^.next := newterm;
end;  \{end of putin\}
addiototal := total
end;  \{addiototal\}

procedure scalarproduct (p : polynomial; scalar : integer);
begin
if (scalar <> 1) then
while p <> nil do
begin
p^.s := p^.s * scalar;
p := p^.next;
end
end;  \{scalarproduct\}

procedure doaddn (var p, q : polynomial; m1, m2 : integer; m : char);
var
g ,temp: polynomial;
terms : integer;
begin
scalarproduct(p, m1);
if q<>nil then g := polycopy(q, 0, 0, m2,KEEP) else g:=nil;
terms := 0;
if DEBUG <= 1 then
begin
writer(' addpq ');
dispalyopoly(p, terms, DEFITL);
dispalyopoly(g, terms, DEFITL);
end;
while g <> nil do
begin
  temp:=g;
  if m <> '•' then begin ir temp^.m<> then writer(' warning : product of parameters ')
else temp^.m:=m end;
  g := g^.next;
  temp^.next := nil:
  p := addiototal(p, temp)
  end;
end;  \{doaddn\}

procedure testaddn (p, q : polynomial; m, n : integer; var addnOK : boolean);
var
scalar : integer;
isin : boolean;
ch : char;
begin
addnOK := true;
write(' testaddn ');
while (p <> nil) and addnOK do
begin
  search(q, p^.x, p^.y, ch, scalar,isin);
isin:=isin and (ch = p^.m);
  ifisin then
  begin
    if ((p^.s >= 0) and (m >= 0)) or ((p^.s <= 0) and (m <= 0)) then
    addnOK := (n * scalar <= MAXVALUE - m * p^.s)
else if ((p^.s >= 0) and (m <= 0)) or ((p^.s <= 0) and (m >= 0)) then
    addnOK := (n * scalar >= -MAXVALUE - m * p^.s)
  end;
p := p^.next;
end;
end;  \{testaddn\}
{F: STORING DATA}

procedure addtowf (var wf : module; {DEBUG })
  coord : integer; var p : polynomial; var pdata : polydata);
var
tersp : integer;
list,last : polylist;
pisin : boolean;
begin
  list := nil;
tersp := 0;
pisin := false;
  writeln(’ addtowf ’, coord: i, ’ ’);
  if DEBUG <= 2 then displaypoly(p, tersp, DEFTTL);
  if (p>nil) then
begin
  {if p is a monomial without a parameter then add p to set of monomials}
  if (pdata.length = 1 and (p.m = ’ ’)) then
  else if (p.x + p.y > 0) and (p.x + p.y <= k) then
begin
  {otherwise add to list of polynomials}
  {find p if it is already stored or add a new record to the list if not}
  if wf[coord][p.x + p.y].sum = nil then
begin
  new(list);
  wf[coord][p.x + p.y].sum := list;
  wf[coord][p.x + p.y].sum.next := nil;
end {if wf.sum=nil}
else
begin
  list := wf[coord][p.x + p.y].sum;
  while (list <> nil) and not pisin do
begin
    last := list;
    pisin := ( p = last.poly); {find end of list or p}
    list := list.next;
end {while list<>nil}
  list := last;
  if not pisin then
begin
  new(last.next);
  list := last.next;
  list.next := nil;
end {if not pisin}
end;{of create new record}
  {now put (new) data into (new) record}
  list.poly := p;
  list.data := pdata
end;{if length>1}
end;{if length>0}
if DEBUG <= 1 then
  displayTAF(tf, wf, DEFDIS);
end;

procedure getplace ( var newplace : coord : integer; var vdata : mapdata;
  vzeroed : boolean);
var
count, i : integer;
begin
  count := 0;
  for i := 1 to 3 do
    if vdata[i].length > 0 then
    begin
      count := count + i;
      if i = 3 then
        begin
          count := count + 1
end;
end;
zeroed := false;
coord := 0;
case count of
  0:
    begin
      newplace := 4;
      zeroed := true;
    end;
  1, 2, 4:
    begin
      if count = 4 then
        begin
          count := 3;
          coord := 4;
        end;
      else
        begin
          newplace := 4;
          coord := count;
        end;
    end;
  3, 5, 6, 7:
    begin
      if count = 3 then
        begin
          count := 4;
          newplace := 4;
          coord := count;
        end;
      else
        begin
          count := 7 - count;
          newplace := 0;
        end;
    end;
end;
end;

•

\[ \{ v \text{ is the zero map} \} \]

\[ \{ v \text{ has one non-zero row} \} \]

\[ \{ \text{coord locates the non-zero row} \} \]

\[ \{ v \text{ has two or more non-zero rows} \} \]

\[ \{ \text{newplace locates the zero row if there is one} \} \]

\[ \{ \text{getplace} \} \]

\[ \text{procedure storemap (var } v : \text{ map; place, coord : integer; var } vdata : \text{ mapdata; storemap) \} \]

\[ \text{var } \text{altwf : blist; var } \text{wf : module; var } \text{tf : vectorstore; } \]

\[ \text{max, min : integer}; \]

\[ \{ \text{adds the map } v \text{ to the data store if or wf as appropriate} \} \]

\[ \text{procedure addtof (var } h : \text{ map; var } hdata : \text{ mapdata; var } \text{ tf : vectorstore; } \]

\[ \text{place } \text{, max, min : integer }; \]

\[ \{ \text{adds } h, \text{ hdata, max and min to tf} \} \]

\[ \text{var } v : \text{ vectorlist; } \]

\[ \text{order, i, terms : integer; } \]

\[ \begin{align*}
\text{begin} \\
\text{terms} & := 0; \\
\text{if } \text{DEBUG} <= 3 \text{ then begin write(' addtof- ', place : 1, \'); displaymap(h, terms, DEFITL); end; } \\
v & := \text{tf}[\text{place}]; \\
\text{order} & := k + 1; \\
\text{for } i := 1 \text{ to } 3 \text{ do } \\
\text{if } h[i] <> \text{nil then if order > h[i].x + h[i].y then order := h[i].x + h[i].y; } \\
\text{if } (\text{order} <= k) \text{ then } \\
\text{begin } \\
f \text{ if } v <> \text{ nil then } \\
\text{begin} \\
\text{while } v'.\text{next} <> \text{nil do } v := v'.\text{next; } \\
\text{new(v'.next); } \\
v' := v'.\text{next; } \\
\text{end } \\
\text{else } \\
\text{begin new(v); tf[place] := v; end; } \\
\text{add data to new record} \\
\text{new(v'.f); } \\
v'.f := h; \\
v'.\text{good} := \text{true; } \\
v'.\text{maxcoeff} := \text{max;} \\
v'.\text{mincoeff} := \text{min;} \\
v'.\text{data} := hdata; \\
v'.\text{next} := \text{nil } \\
\text{end; } \\
\text{addtof} \\
\end{align*} \]

\[ \text{\( \cdot 12 \)} \]
begin \{storemap\}
if (place = 4) and (coord <> 0) then
begin
begin
addtowf(wf, coord, v[coord], vdata[coord]);
if vdata[coord].length = 1 then
if v[coord].m <> 0 then ruboutpoly(v[coord]);
alwt[coord] := true;
end
else if place <> 4 then addtov(v, vdata, tf, place, max, min);
if (DEBUG <= 2) and (place <> 4) then displayTAf(tf, wf, DEFDIS);
end;
\{storemap\}

procedure tidypoly (var p : polynomial; \{removes terms with zero coefficient\}
var pdata : polydata);
\{and divides out by HCF\}
begin
\{updates data\}
p:= polycopy(p,0,0,1,DELETE); \{purpose : to remove terms with zero coefficient\}
dividemap(p, HCFpoly(p));
getpolydata(p, pdata);
end;

procedure tidymap (var h : map; var hdata : mapdata; var max, min : integer);
\{as tidypoly\}
begin
\{updates data\}
for i := 1 to Cp do
h[i]:= polycopy(h[i],0,0,1,DELETE);
dividemap(h);
getmapdata(h, hdata);
getlimits(hdata, max, min);
end;

\{G: SIMPLIFYING GENERATORS\}

procedure comparewf (var h : map; var p : polynomial; var hdata : mapdata; var pdata : polydata; var alth, altp, doingpoly : boolean; coord : integer; var max, min : integer; place : integer);
\{remove terms in \(w_f\) from \(h\) or polynomial \(p\) if possible\}
begin
\{write(' enter comparewf ');
printno := 0;
altp := false;
altp := false;
if doingpoly then begin length := pdata.length; poly := p end
else begin length := hdata[coord].length; poly := h[coord]; end;
if (DEBUG <= 2) and (place <> 4) then
begin
write(' compare ');
displaypoly(poly, printno, DEFTTL); end;

while newterm <> nil do
\{inspect poly term by term\}
begin
foundpoly := false;
if (newterm.x in w[coord] and (newterm.x + newterm.y).single then
begin
newterm.s := 0;
altp := doingpoly;
length := length - 1;
alth := not doingpoly;
end
else if (newterm.s <> 0) and ((length > 1) or (place <> 4)) then
begin  {search terms in list for a match with the current term of p}
list := w[f[coord] [newterm'.x + newterm'.y].sum;
while (list <> nil) and not foundpoly do
begin
if list'.poly <> nil then
if (list'.poly'.x = newterm'.x) then
begin  {a match has been found: get multipliers and check the numbers are not too large for the compute}
foundpoly := true;
getmultipliers(list'.poly'.s, newterm'.s, list'.data.maxcoeff, list'.data.mincoeff,
max, min, coeff1, coeff2, multnOK);
addnOK := multnOK and (abs(list'.data.maxcoeff * coeff1) < MAXVALUE - abs(max * coeff2))
and (abs(list'.data.mincoeff * coeff1) < MAXVALUE - abs(min * coeff2));
addnOK := addnOK and (abs(list'.data.maxcoeff * coeff1) < MAXVALUE - abs(max * coeff2))
and (abs(list'.data.mincoeff * coeff1) < MAXVALUE - abs(min * coeff2));
if multnOK and not addnOK then testaddn(poly, list'.poly, coeff1, --coeff1, addnOK);
if addnOK then
begin
if list'.poly'.m = newterm'.m then ch := '
else ch := newterm'.m;
doaddn(poly, list'.poly, coeff2, -coeff1, ch);
{add -ch*coeff1*list'.poly to coeff2*poly}
if not doingpoly then
for c := 1 to Cp do if c <> coord then scalarproduct(h[c], coeff2);
lalt := doingpoly;
lalt := not doingpoly;
end;
end;
list := list'.next;
end;
end;
newterm := newterm'.next;
end;
if altl then tidymap(h, hdata, max, min);
if altl then tidypoly(p, pdata);
{write(" exit comparewf ");}
end;

procedure comparemap (var h : map; var hdata : mapdata; var altl : boolean; comparemap
checkwith : blist; var max, min, place, co : integer);
(feed the map to comparewf)
var
coord, terms : integer;
p : polynomial;
ndata : polydata;
alp, altmap, zeroed : boolean;
begin
p := nil;
terms := 0;
alh := false;
if DEBUG <= 2 then displaymap(h, terms, DEFTTL);
for coord := 1 to Cp do
if checkwith[coord] then
begin
getplace( place, co, hdata, zeroed);
if (place <> 4) or (coord = co) then
comparewfh, p, hdata, ndata, altmap, alp, DOMAP, coord, max, min, place);
alh := alh or altmap
end;
getplace( place, co, hdata, zeroed);
{comparemap}
end;

procedure sortwf (var w : module; var altw : boolean; coord : integer);
sortw
var
{define variables}

h : map; {input: h, output: h}
hd : mapdata;
last, store, current: poly list;
i, max, min: integer;
al, qal, halt: boolean;

begin

altwf := false;
for i := 1 to 3 do h[i] := nil;
max := 0;
min := 0;
repeat
{until no further simplification is possible}

alt := false;
{write("crosschecking row[', coord : 1, ']";)
for i := 0 to k do
begin
{simplify list i for each i}
store := wf[coord][i].sum;
current:=nil;
while store <> nil do
begin
last:=current;
current := store;
store := store.next;
if last <> nil then last.next := store
else wf[coord][i].sum:=store; {remove current poly from list}
comparewf(h, current'.poly, hdata, current'.data, halt, qal, DOPOLY, coord, max, min, 4);
if qal then
begin
if current'.poly<>nil then
addtowf(wf, coord, current'.poly, current'.data);
if (current'.data.length = 1) and not current'.data.moduli then ruboutpoly(current'.poly);
alt := true; {further simplification may be possible}
dispose(current);
current:=last;
end
else begin
{replace current poly in list}
if last <> nil then last.next := current
else wf[coord][i].sum:=current;end;
altwf := alt or altwf; {record if wI has been simplified}
end;
end;
until (not alt);
end;

{sortw}

procedure sortTAf (var wf : module; {subprocedures: sortf, comparef})
var tf : vectorstore;
const DEBUG = 6;
{define variables}
var
i : integer;
al, init : blist;
rec : blist4;

procedure sortf (var tf : vectorstore; var wf : module; var compcoord, altwf : blist; var altf : blist4);
var
elt, last, current: vectorlist; {remove maps marked as unwanted or zero}
altmap: boolean;
j, place, coord : integer;

begin
{write("crosschecking if");}
for j := 3 downto 0 do
begin
elt := tf[j];
current=nil;
while elt <> nil do
begin

last := current;
current := elt;
etl := elt'.next;
if last <= nil then last'.next := elt else tf[j] := elt;
if current'.good and (current'.f <= nil) then
  comparemap(current'.f, current'.data, altmap, compcoord, current'.maxcoeff, current'.mincoeff, place, coord);
if altmap or not current'.good then
  begin
    if current'.good and (current'.f <= nil) then
      storemap(current'.f);
    dispose(current'.f);
  end;
dispose(current); current := last;
if (place >= 0) and (place < 4) then
  altmap[place] := true;
end
else {replace map in list}
  begin
    if last <= nil then last'.next := current else tf[j] := current end;
  end; {while elt <> nil}
end; {for j = 3 to 0}
end;

{sort}
procedure comparetf (var tf : vectorstore; {subprocedures - display list}
  var wf : module;
  var altwf : blist;
  var alt : blist(4);
  var c, place, listno, pno : integer;
  list : vectorlist;
  maps : maplist;

procedure displaylist (list : maplist);
  var temp : integer;
begin
  temp := 0;
  writeln;
  write(‘present maps are:’);
  while list <= nil do
    begin
      if list'.newmap <= nil then displaymap(list'.newmap', temp, DEFTTL);
      if not list'.save then write(‘ not ’);
      write(‘ save ’);
      if not list'.done then write(‘ not ’);
      write(‘ done ’);
      list := list'.next;
    end;
end;
{displaylist}

procedure placeinmaps (f : mappointer; fdata : mapdata;
  max, min, place : integer; var maps : maplist);

  var
    newelt : maplist;
    c : integer;
begin {write(' enter placeinmaps :)');}
if maps = nil then {create a new record},
begin
new(maps);
maps,.next := nil;
newelt := maps;
end
else begin
newelt := maps;
while newelt,.next <> nil do
newelt := newelt,.next;
new(newelt,.next);
newelt := newelt,.next;
end; {of create record}
with newelt do {put data in record}
begin
new(newmap);
for c := 1 to Cp do newmap[c] := polycopy(f[c], 0, 0, 1,KEEP);
oldmap := f;
olddata := fdata;
newmax := max;
coord := 0;
oldplace := place;
newplace := place;
newmin := min;
save := false;
done := false;
next := nil;
end; {writer'
exit placeinmaps :) end;
procedure constructlist (f : mappointer; {sub procedure checkifin} list : vectorlist; coord, place : integer; var maps : maplist);
var
{decide which maps to compare}
occurring : boolean;
procedure checkifin (p, q : polynomial; var occurs : boolean);
var
scalar : integer;
ch : char;
begin
occurs := (p <> nil);
while (q <> nil) and occurs do
begin
search(p, q,.x, q,.y, ch, scalar, occurs);
qu := q,.next;
end;
end; {checkifin}
begin {write(' enter constructlist :)');}
while list <> nil do
begin {go through each map in list and decide whether or not to compare it with the map indicated by f}
if list,.good then
begin
checkifin(f[coord], list,.f[coord], occurs);
if occurs then placeinmaps(list,.f, list,.data, list,.maxcoeff, list,.mincoeff, place, maps);
end;
list := list,.next;
end; {write(' exit constructlist :)');}
end;
procedure sortlist (var maps : maplist; c : integer);
var
{compare maps in list 'maps'}
alldone, isin, multiOK, addnOK, zeroed, notlonger, moduli: boolean;
list : maplist;
sh : mappointer;
shdata : mapdata;
ch, m : char;

begin {write(' enter placeinmaps :)');}
if maps = nil then {create a new record},
begin
new(maps);
maps,.next := nil;
newelt := maps;
end
else begin
newelt := maps;
while newelt,.next <> nil do
newelt := newelt,.next;
new(newelt,.next);
newelt := newelt,.next;
end; {of create record}
with newelt do {put data in record}
begin
new(newmap);
for c := 1 to Cp do newmap[c] := polycopy(f[c], 0, 0, 1,KEEP);
oldmap := f;
olddata := fdata;
newmax := max;
coord := 0;
oldplace := place;
newplace := place;
newmin := min;
save := false;
done := false;
next := nil;
end; {writer'
exit placeinmaps :) end;
procedure constructlist (f : mappointer; {sub procedure checkifin} list : vectorlist; coord, place : integer; var maps : maplist);
var
{decide which maps to compare}
occurring : boolean;
procedure checkifin (p, q : polynomial; var occurs : boolean);
var
scalar : integer;
ch : char;
begin
occurs := (p <> nil);
while (q <> nil) and occurs do
begin
search(p, q,.x, q,.y, ch, scalar, occurs);
qu := q,.next;
end;
end; {checkifin}
begin {write(' enter constructlist :)');}
while list <> nil do
begin {go through each map in list and decide whether or not to compare it with the map indicated by f}
if list,.good then
begin
checkifin(f[coord], list,.f[coord], occurs);
if occurs then placeinmaps(list,.f, list,.data, list,.maxcoeff, list,.mincoeff, place, maps);
end;
list := list,.next;
end; {write(' exit constructlist :)');}
end;
procedure sortlist (var maps : maplist; c : integer);
var
{compare maps in list 'maps'}
alldone, isin, multiOK, addnOK, zeroed, notlonger, moduli: boolean;
list : maplist;
sh : mappointer;
shdata : mapdata;
ch, m : char;
max, min, m1, m2, scalar, x, y, co, s : integer;
begin \textit{write(' enter sort list ');} \]
repeat
  list := maps;
  s:=0;
  alldone := true;
  shdata[c].length := 20;
  if list <> nil then \textit{\{find shortest poly not yet done\}}
    begin
      if (not list'.done) and (list'.newdata[c].length > 0) and (alldone or (shdata[c].length > list'.newdata[c].length))
        begin
          alldone := false;
          sh := list'.newmap;
          shdata := list'.newdata;
          max := list'.newmax;
          min := list'.newmin;
        end;
      list := list'.next;
    \end;
end; \textit{\{of find shortest\}}
if not alldone and (sh <> nil) then if sh[c]<>nil then
begin
  list := maps;
  moduli:=false;
  for co:=1 to Cp do moduli:=moduli or sh'data[co].moduli;
  x := sh[c].x;
  y := sh[c].y;
  while list <> nil do \textit{\{run through list\}}
    begin
      with list'.do do
        if not done then
          begin
            if (newmap = sh) or (newdata[c].length = 0) then done := true
            else if newmap <> nil then \textit{\{attempt to subtract shortest from newmap\}}
              begin
                isin:=false;
                if newmap'[c]<> nil then search(newmap'[c].x, y, m, scalar, isin);
                isin:=isin and (not moduli or ( m = sh'[c].m));
                if isin then \textit{\{get multipliers and check size of numbers\}}
                  begin
                    getmultipliers(sh'[c].s, scalar, max, min, newmax, newmin, m1, m2, multiOK);
                    addnOK := multiOK;
                    if DEBUG <= 2 then write(' m1,m2 ', m1 : 1, ', ', m2 : 1, ', ');
                    for co := 1 to Cp do
                      begin
                        addnOK := addnOK and
                          (abs(shdata[co].maxcoeff * m1) < MAXVALUE - abs(newdata[co].maxcoeff * m2)) and
                          (abs(shdata[co].mincoeff * m1) < MAXVALUE - abs(newdata[co].mincoeff * m2));
                        addnOK := addnOK and
                          (abs(shdata[co].maxcoeff * m1) < MAXVALUE - abs(newdata[co].mincoeff * m2)) and
                          (abs(shdata[co].mincoeff * m1) < MAXVALUE - abs(newdata[co].mincoeff * m2));
                      end;
                    if multiOK and not addnOK then
                      begin
                        addnOK := true;
                        for co := 1 to Cp do if addnOK then testaddn(newmap'[co], sh'[co], m2, -m1, addnOK);
                      end;
                    if addnOK then \textit{\{add -m*m1*sh to m2*newmap\}}
                      begin
                        if (m = sh'[c].m) then m:='';
                        for co := 1 to Cp do doaddn(newmap'[co], sh'[co], m2, -m1, m);
                        tidymap(newmap', newdata, newmax, newmin);
                        getplace( newplace, coord, newdata, zeroed);
notlonger := true; \quad \text{(decide whether newmap is a simplification)}
for co := 1 to Cp do notlonger := notlonger and (newdata[co].length <= olddata[co].length);
list/save := (newplace > oldplace) or ((newplace = oldplace) and notlonger and (newdata[c].length < olddata[c].length))
end;
end;
end;
list := list*.next;
end;
until alldone;\{write(' exit sortlist ');
end;
procedure save_list (var maps : maplist; var tf : vectorstore; var wf : module; var altwf : blist; var altf : blist4);
var list : maplist;
foundmap : boolean;
store : vectorlist;
begind\{write(' enter save_list ');
while maps <> nil do
begin
list := maps;
maps := maps*.next;
with list do
begin
if save then
begin
foundmap := false;
store := tf[oldplace];
while (store <> nil) and not foundmap do
begin
foundmap := (store*.f = oldmap);
if not foundmap then store := store*.next;
end;
if foundmap then
begin
if newmap <> nil then
storemap(newmap*, newplace, coord, newdata, altwf, wf, tf, newmax, newmin);
store*.good := false; \quad \{mark old map to be removed\}
if newplace <> 4 then
altf[newplace] := true; \quad \{record that a simplification of tf has occurred\}
end
end
else if newmap <> nil then ruboutmap(newmap*); \{if new is not simpler\}
end;
dispose(list);
list := maps;
end;\{write(' exit save_list ');
end;
begin
\{compare\}
repeat
\{until no further simplification occurs\}
\{write(' loop 1 ');
for place := 0 to 3 do
if altf[place] then
begin
write(' place= ', place : 1, ' '); list := tf[place];
altf[place] := false;
while list <> nil do
begin
pno:=0;
maps := nil;
for c := Cp downto 1 do
if (c <> place) and (list*.f <> nil) and list*.good then
begin
  {
    write("element is ");
    displaymap(list'.f,pno,10);} 
for i := 3 downto place do {decide which lists should be searched}
begin
  if (place = 2) and (i = 3) then listno := 1
  else listno := i;
  if (listno <> 0) then constructlist(list'.f, tf[listno], c, listno, maps);
end;
{displaylist(maps);} 
sortlist(maps, c);
{displaylist(maps);} 
savelist(maps, tf, wf, altwf, alttf);
end;
list:=list'.next;
end;
end;
until not (alt[1] or alt[2] or alt[3] or alt[0])
end;
{compareTf}
begin
  {sortTf}
rec[0] := true;
for i := 1 to 3 do begin alt[i] := true; rec[i] := true; end;
repeat {until no further simplification occurs}
repeat {until no further simplification by comparison with polynomials occurs}
if DEBUG <= 2 then displayTf(tf, wf, DEFDIS);
write(" crosschecking ");
for i := 1 to 3 do init[i] := alt[i];
for i := 1 to Cp do if alt[i] then sortwf(wf, alt[i], i);
if init[1] or init[2] or init[3] then sort(tf, wf, init, alt, rec);
until not (alt[1] or alt[2] or alt[3]);
if DEBUG <= 2 then displayTf(tf, wf, DEFDIS);
write(" comparing supplementary generators ");
comparetf(tf, wf, all, rec);
end;
{sortTf}
{II: CALCULATING GENERATORS}

procedure getwf (var wf : module; f : map);
var
  
  {subprocedures: polyproduct, newlist}
  
  
  
  {
    addto, copylist, eraselist}
  
  
  {
    calculate wf given f}
  
  
  
begin
  total := nil;
  s := q;
  while s <> nil do begin
    r := p;
    if s'.s <> 0 then begin
      while r <> nil do begin
        if (r'.s <> 0) and (r'.x + s'.x + r'.y + s'.y <= k) then begin
          new(item);
          item'.x := r'.x + s'.x;
          item'.y := r'.y + s'.y;
          item'.s := r'.s * s'.s;
          if r'.m <> -1 then item'.m := r'.m else item'.m := s'.m;
          item'.next := nil;
          total := addtototal(total, item);
        end;
      end;
    end;
  end;
end;
\begin{verbatim}
end;
\end{verbatim}
while v^.next <> nil do v := v^.next;
if newlist <> nil then v^.next := newlist^.next
end;
end;

{addto}
function copylist (list : polylist) : polylist;
var
  newlist, newelt : polylist;
begin
  if list=nil then newlist:=new(newelt) else new(newelt);
  newlist := newlist;
  while list <> nil do
    begin
      newelt^.poly := polycopy(list^.poly, 0, 0, 1,KEEP);
      newelt^.data := list^.data;
      newelt^.next := nil;
      list := list^.next;
      if list<>nil then
        begin
          new(newelt^.next);
          newelt := newelt^.next;
        end;
    end;
  copylist := newlist;
end;

{copylist}
procedure eraselist (var list : polylist);
var
  listmarker : polylist;
begin
  listmarker := list;
  while list <> nil do
    begin
      list := list^.next;
      dispose(listmarker);
      listmarker := list;
    end;
end;
{eraselist}

begin
  new(listA);
  listA^.poly := nil;
  new(listA^.next);
  A ^= listA^.next;
  A^.next := nil;
  A^.poly := one;
  getpolydata(A^.poly, A^.data);
  order := 0;
  A^.no:=0;
  repeat {get powers of f1}
    previous := A^.poly;
    new(A^.next);
    A^.next^.no:=A^.no +1;
    A := A^.next;
    A^.next := nil;
    A^.poly := polyproduct(previous, f[1]);
    tidypoly(A^.poly, A^.data);
    getpolydata(A^.poly, A^.data);
    if A^.no>=S then addtowf(wf, 1, A^.poly, A^.data);
  until order >= k;
  { writeln('powers of f1 calculated');}
  store := listA;    {=list of powers of f1}
  for i := 2 to Cp do {get rest of wf}
    if f[i]<nil then
      begin
        order := 0;
        order := 0;
    end;
listA := store;  \{= list of products of powers of f1 to f(i-1)\}
if f[i].x + f[i].y > 0 then
    repeat
        listB := newlist(listA, f[i].x + f[i].y, wf);
        order := order + f[i].x + f[i].y;
        addto(store, listB);  \{add powers involving f1 to accumulated list\}
        listA := listB
    until order >= k;
    \{ writeln(’Powers of f’, name[i], ’ calculated.’); \}
end;
listB := nil; listA := nil; eraselist(store);
sortwf(wf, alteredwf, 1);
write(’calculated wf’);
for i := 2 to Cp do
    for j := 0 to k do
        begin
            wf[i][j].single := wf[1][j].single;
            wf[i][j].sum := copylist(wf[1][j].sum);
        end;
end;

procedure gettf (var Df : Dmap; var tf : vectorstore; var wf : module);
    \{calculate generators for if given Df\}
    i, j, newplace, coord, tp, max, min : integer;
    ch : char;
    checkwith, altwf : blist;
    g : Dmap;
    gdata : mapdata;
    altmap : boolean;
begin \{gettf\}
    for i := 1 to Cp do checkwith[i] := true;
    max := 0;
    min := 0;
    tp := 0;
    for i := (k + 1 - order) downto R do
        for j := 0 to i do \{calculate x'(i-j)y*Df for each i and j\}
            begin
                \{if (j = 0) then \}
                write(’order=’, i, ’ : ’);
                for ch := ’x’ to ’y’ do
                    begin
                        for coord := 1 to Cp do g[ch][coord] := polycopy(Df[ch][coord], i - j, j, 1,KEEP);
                        tidymap(g[ch], gdata, max, min);
                        comparemap(g[ch], gdata, altmap, checkwith, max, min, newplace, coord);
                        storemap(g[ch], newplace, coord, gdata, altwf, wf, tf, max, min);
                    end; \{for i, j\}
                end;
                \{gettf\}
            end;
write(’calculated tf’);
end; \{gettf\}

\{getvaliddata\}

procedure getvaliddata; \{subprocedures : checkpoly, readpoly, diff\}
var
    gooddata, altif : boolean;
    i, printno(pno) : integer;
    reply, ch, addtof : char;
begin \{checkpoly\}
    if p = nil then gooddata := (c<>1)
    else gooddata := (p.x + p.y > 0) and (p.x + p.y < LIMIT)
end; \{checkpoly\}

procedure readpoly (var f : polynomial; ch : char; addtof : boolean);
var

i, {r} n, scalar : integer;
moduli : char;
item : polynomial;

begin
writeln; { r := 0; }
if addtof then write('How many terms do you wish to add to \( f \), ch, '?' );
else write('Type in number of terms in \( f \), ch, ' );
readln(n);
makequery(10,' ',moduli);

for i := 1 to n do
begin
writeln;
scalar := MAXVALUE + 1;
while (scalar < -MAXVALUE) or (scalar > MAXVALUE) do
begin
write(' Type in scalar ');
readln(scalar);
if (scalar <= -MAXVALUE) or (scalar > MAXVALUE) then
write(' scalar is out of range. Please re-enter a new value. ');
end;
if scalar <> 0 then
begin
new(item);
item.s := scalar;
if moduli = 'y' then begin
write(' Type in modulus, ');
readln(item.m) end
else item.m := ' ';
write(' Type in power of x ');
readln(item.x);
write(' Type in power of y ');
readln(item.y);
item.next := nil;
f := addtototal(f, item)
end;
end;
writeln;

{remove terms with zero coefficient}
f:= polycopy(f,0,0,1,DELETE);
{displaypoly(f, r.DEFTIL);}

begin
printno := 0;

repeat
begin
if rerun then
begin
writeln(' You may leave \( f(i) \) unchanged or add terms to it. Otherwise it will be replaced by your new input. ');
for i := 0 to Cp do
if i > 0 then makequery(1, name[i], change[i])
else makequery(2, name[i], change[i]);
ediff
end;
end;
writepoly(f, r.DEFTIL);

function diff (p : polynomial; w : variablename) : polynomial; {differentiate p}
var
item, v : polynomial;

begin
v := polycopy(p, 0, 0, 1,KEEP);
item := v;
while v <> nil do
begin

end;
writeln;
f:= polycopy(f,0,0,1,DELETE); {remove terms with zero coefficient}
diff := item
end;
begin
printno := 0;

repeat
begin
if rerun then
begin
writeln(' You may leave \( f(i) \) unchanged or add terms to it. Otherwise it will be replaced by your new input. ');
for i := 0 to Cp do
if i > 0 then makequery(1, name[i], change[i])
else makequery(2, name[i], change[i]);
ediff
end;
end;
Please enter R:
readln(R);
write(' Please enter S:
readln(S);
if changef[0] = 'y' then
 repeat
 write(' Please enter k:
readln(k);
 if not (k in range) then
 write(' k is out of the permitted range. Please reenter it:
until k in range;
for i := 1 to Cp do {read fii]
if changef[i] = 'y' then
 begin
 if rerun then
 makequery(9, name[i], addtof);
alft := true;
 repeat
 if addtof = 'n' then ruboutpoly(fii);
 readpoly(fii, name[i], (addtof = 'y'));
 checkpoly(fii, i, gooddata);
 if not gooddata then
 begin
 addtof := 'n';
 write('I', name[i], ' is invalid. Please reenter it.
end;
 until gooddata;
 end;
{of read fii]}
writeln('Calculating t(M' + R' 1) + w(N' + S'1) mod M2', k + 1 : 1);
write('For f=', f = ');
displaymap(f, prinno, DEFTTL);
makequery(3, '*' reply);
{verify data]
rerun := true;
 until reply = 'y';
for i := 1 to Cp do dividepoly(fii, HCFpoly(fii)));
getmapdata(f, fdata);
if alft then
 for ch := 'x' to 'y' do
 begin
 ruboutmap(Df[ch])
 for i := 1 to 3 do Df[ch][i] := diff(f[i], ch);
 dividemap(Df[ch])
 end;
{displaymap(Df'x', prinno, 10);}
{displaymap(Df'y', prinno, 10);}
forder := ([1]'x + [1]'y;
for i := 2 to Cp do
 if f[i]<> nil then if ([i]'x + [i]'y < forder) and ([i]'x + [i]'y > 0) then
 forder := [i]'x + [i]'y;
 subrange := [1..k];
 Odim := 3 * ((k + 1) * (k + 2) div 2) - 3;
end;
{getvaliddata]}
procedure initialisevar;
 var
 i, j: integer;
 ch:char;
begin
 name[1] := '1';
 name[2] := '2';
 name[3] := '3';
 name[0] := 'k';
 for i := 1 to 3 do f[i] := nil;
 for i := 1 to 3 do for ch := 'x' to 'y' do Df[ch][i] := nil;

for i := 0 to Cp do
    change[i] := 'y';
rerun := false;
dialogue.query[1] := 'Do you wish to alter f?';
dialogue.query[2] := 'Do you wish to alter \%'f;'
     Is this correct?';
dialogue.query[4] := 'Is the display satisfactory?';
dialogue.query[5] := 'Do you wish to continue?';
dialogue.query[6] := 'Do you wish to alter the display parameters?';
dialogue.query[7] := 'Do you wish to save the data?';
dialogue.query[8] := 'Do you wish to print the data?';
dialogue.query[9] := 'Do you wish to add terms to f?';
dialogue.query[10] := 'Do you wish to alter the display parameters?';
dialogue.response := ['n', 'y', 'N', 'Y'];
for i := 1 to Cp do
    for j := 1 to LIMIT do
        begin
            w[i][j].sum := nil;
            w[i][j].single := [];
        end;
    for i := 0 to 3 do tf[i] := nil;
    range := [1..LIMIT];
    new(one);
with one do
    begin
        next := nil;
        x := 0;
        y := 0;
        s := 1;
        m := 1;
    end;
end;

procedure informuser;
begin
write('This program calculates the terms in tf(M'R) + wz(N'S) for ');
write('f(x,y)= (f1(x,y),f2(x,y),f3(x,y)).');
write('It will only run if f1,f2 and f3 have no constant terms or if f2 or f3 are zero. ');
write('The calculation is performed modulo M2k+1, where k < LIMIT+1:1');
write('This version f may have parameter values. ');
write('Program written by D. Ratcliffe.');
end;

begin
{MAIN PROGRAM }
initialisevar;{name, changef, rerun, wf, if, one, range , dialogue}
informuser;
repeat
    getvaliddata{f, Df, Odim, forder, subrange}
    getwf(wf, f); {calculate wf given f : store data in wf}
    getdf(Df, if, wf); {calculate if given Df : store data in if or wf}
    sortTAl(tf, wf); {simplify the generators of TAf}
    displayTAl(tf, wf, ACTDIS); {display/save the data}
    ruboutdata(tf, wf); {dispose of the data}
    makequery(5, '""; continue);
    until continue = 'n';
    ruboutmap(f);
    for ch := 'x' to 'y' do
        ruboutmap(Df[ch]);
end.
R1

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We give a complete list of normal forms for 3-jets whose 2-jet is \((x^2 + y^2, xy, 0)\):

\[
(xy, x^2 + y^2, x^3 + \lambda_1 x^2 y - xy^2 + \lambda_3 y^3) : \lambda_1 \neq 0,
(xy, x^2 + y^2, x^3 + \lambda_1 x^2 y + (1-\lambda_1)y^3)
(xy, x^2 + y^2 + y^3, \lambda_1 x^2 y + xy^2 + (1-\lambda_1)y^3)
(xy, x^2 + y^2, 0)
(xy, x^2 + y^2 + y^3, 0)
(xy, x^2 + y^2 + xy^2 + y^3, 0).
\]

\[
(xy, x^2 + y^2, x^2 y + \lambda_3 y^3) : \lambda_3 \neq -1,
(xy, x^2 + y^2 + y^3, x^3 + \lambda_1 x^2 y + (1-\lambda_1)y^3)
(xy, x^2 + y^2, x^3 + \lambda_1 x^2 y - xy^2 - \lambda_1 y^3)
(xy, x^2 + y^2 + y^3, x^3 + \lambda_1 x^2 y - xy^2 - \lambda_1 y^3)
(xy, x^2 + y^2 + xy^2 + y^3, x^3 + \lambda_1 x^2 y - xy^2 - \lambda_1 y^3)
(xy, x^2 + y^2 + xy^2 + y^3, x^2 y - y^3)
(xy, x^2 + y^2 + y^3, x^2 y - y^3)
(xy, x^2 + y^2 + xy^2 + y^3, x^2 y - y^3).
\]
\( (x, y^2, y_3) \)
Figure 3.

The diagram illustrates the relationships between various expressions involving terms like $x$, $y$, $x^2$, $y^3$, $xy$, and constants $a$, $b$, and $c$. The expressions are plotted along the x-axis, which represents $x$, and the y-axis, which represents $y$. Different colors are used to distinguish between different sets of expressions. The diagram shows how these expressions change with respect to each other and how they are interconnected.

Key points:
- The diagram includes expressions such as $x^n y^m$, where $n$ and $m$ are integers.
- Various cases are highlighted, such as when $x$ and $y$ are multiplied together or raised to different powers.
- The diagram indicates how these expressions behave under different conditions, such as when $x$ or $y$ is equal to zero or a constant.

This visual representation is useful for understanding the relationships between these algebraic expressions and how they evolve as variables change.
\[ \begin{align*} 
& (xy, x^2, y^2) \\
& (x^2 + y^2, xy, 0) \\
& (x^2, y^2, 0) \\
& (x^2, xy, 0) \\
& (xy, 0, 0) \\
& (x^2, 0, 0) \\
& (0, 0, 0) \\
\end{align*} \]