On the Classification and Geometry of Finite Map-germs.

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Summary.

Summary of Chapter I:

1. We use Samuel's theory of multiplicity to describe the structure of $\theta(f)/T\mathcal{A}f$ in terms of the multiplicity $e(\theta(f)/T\mathcal{A}f)$ and the dimension of the instability locus Σ (this dimension is also the Krull dimension of $\theta(f)/T\mathcal{A}f$).

2. We extend the theory of trivial unfoldings of map-germs to the theory of k-trivial unfoldings of k-jets.

3. We develop a method for obtaining normal forms for the (k+1)-jets having a given k-jet f by inspection of certain submodules of $T\mathcal{A}f$. We give a test for sufficiency of a normal form and a method for constructing a (k+1)-trivial unfolding of the normal form.

4. We show that if f is weighted homogeneous and the Krull dimension and multiplicity of $\theta(f)/T\mathcal{A}f$ are both 1 then f is a weak stem and we can find an integer k and a complete list of normal forms for finitely determined map-germs whose k-jet is f. The list has many similarities with the series found by Arnold and Mond.

Summary of Chapter II:

We extend Mond's classification of map-germs $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ under $\mathcal{A}$-equivalence. This chapter also demonstrates the use of the classification theory developed in I·2 and provides a large number of examples for use in the rest of the thesis.

Summary of Chapter III:

1. We prove that f is a geometric stem if and only if Σ is irreducible and the localised module $(\theta(f)/T\mathcal{A}f)$ has length $\ell = 1$ (localise with respect to the prime ideal defining Σ).

2. We also prove that if Σ has transversal type A_{2n} or A_{2n+1} for some $n \geq 1$ then $\ell = n$. This makes it easier to determine n if Σ has transversal type of A_{2n} or A_{2n+1} since we shall calculate ℓ anyway.

3. If f is a stem we show that Σ is irreducible and give a list of transversal types that Σ may have (although having one of these transversal types does not necessarily indicate that f is a stem).

4. We look at how the numbers C, T and $\mu(D_2(f)/\mathbb{Z}_2)$ behave for the families of map-germs associated with some weak stems. We observe that for a given family $\{f+p_s\}$ the integer $C(f+p_s)+T(f+p_s)+\mu(D_2(f+p_s)/\mathbb{Z}_2)-\mathrm{cod}(\mathcal{A}, f+p_s)$ appears to be a constant.

Appendices A and B contain supplementary calculations.
Appendix C is a description of the computer programs written to calculate the modules used in classifying the map-germs of Chapter II.

Contents.

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Tables.

Figures.

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Diagrams.

Introduction.

The most extensive lists of singularities known so far were produced by Arnold for functions up to $\mathcal{R}$-equivalence ([A.G.V] : section 15). A striking feature of these lists is the occurrence of series.

For example,

$$A_k\colon\ f(x, y) = x^{k+1} + y^2,$$

$$D_k\colon\ f(x, y) = x^2y + y^{k-1},$$

$$T_{k,\ell,m}\colon\ f(x, y, z) = axyz + x^k + y^\ell + z^m.$$

We associate these series with the functions $f(x, y) = y^2$, $f(x, y) = x^2y$ and $f(x, y, z) = xyz$ respectively, all of which fail to be finitely determined for $\mathcal{R}$-equivalence.

We quote

> 'Although the series undoubtedly exist, it is not at all clear what a series of singularities is. ' ([A.G.V], p. 243)

and

> 'However, a general definition of a series of singularities is not known. It is only clear that the series are associated with singularities of infinite multiplicities.'

([A.G.V], p. 244)

Series also occur in the classification of map-germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ up to $\mathcal{A}$-equivalence calculated in [Mond 1]. Here the following series have been found:

$$S_k\colon\ f(x, y) = (x, y^2, y^3 + x^{k+1}y),$$

$$B_k\colon\ f(x, y) = (x, y^2, x^2y + y^{2k+1}),$$

$$H_k\colon\ f(x, y) = (x, xy + y^{3k-1}, y^3).$$

These series are associated with the map-germs (x, y^2, y^3), (x, y^2, x^2y) and (x, xy, y^3) respectively. They are also discussed in [Mond 2].

The classifications were mostly performed by the brute force approach, classifying r-jets until a sufficient jet was reached. Special methods have been developed for dealing with, for example, weighted homogeneous map-germs (see [Wall 2] for further

details). But the methods do not generalise, and if the initial choice of map-germ is not finitely determined then no sufficient jet exists. The techniques used to find the series are very individual and do not suggest a general approach. We require different techniques to explain their behaviour.

We observe that the simplest (1-suffix) series (with respect to an equivalence relation $\mathcal{G}$) have the following features:

i) they can be written in the form $f_k = f + r^{k-i}v$,

ii) each f_k is sufficient,

iii) their adjacency relations are

$$\cdots \to f_{k+1} \to f_k \to f_{k-1} \to \cdots,$$

iv) there exists a value k_0 such that if g is a map-germ satisfying $j^{k_0}g = j^{k_0}f$ then g is $\mathcal{G}$-equivalent to f or to f_k for some k.

Here $f:(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is not finitely determined for $\mathcal{G}$, $v:(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ and $r:(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$.

For example, for A_k we put $f(x, y) = y^2$, $v(x, y) = x$ and $r = x$;

for H_k: $f(x, y) = (x, xy + y^{3k-1}, y^3)$, we put $f(x, y) = (x, xy, y^3)$, $v(x, y) = (0, y^2, 0)$ and $r = y^3$ (here we specify that $k \geq 2$).

Can we identify those map-germs which might be expected to give rise to series? The minimum condition for a 1-suffix series would seem to be that the map-germ is a stem:

<u>Definition</u>: a map germ f is a $\mathcal{G}$-stem if it is not finitely determined for $\mathcal{G}$ and there is an integer k_0 such that any map-germ g with the same k_0-jet as f is either $\mathcal{G}$-equivalent to f or is finitely determined for $\mathcal{G}$.

In [P 1] Pellikaan has characterised stems for $\mathcal{R}$-equivalence in terms of the transversal types of their singular loci Σ:

a function is a stem if and only if Σ is irreducible with transversal type A_1.

He has also shown that if $\Sigma \neq V(z_0)$ then the 'series' of map-germs $f + z_0^k$ are finitely determined for $k \geq k_0$. However it is not the case that all finitely determined map-germs with the same k_0-jet as f are equivalent to one in the 'series', or that each $f + z_0^k$ is sufficient.

This thesis grew from the aim of characterising a stem by the transversal type of its instability locus Σ for a map-germ $f:(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ under $\mathcal{A}$-equivalence.

Definition: call f a <u>geometric stem</u> if Σ is irreducible and has one of the transversal types A_2, A_3 or D_4.

The conjecture we investigate is

f is a stem if and only if f is a geometric stem.

Our second aim is essentially to investigate the question of whether stems give rise to series. Of course, this is not a well-defined question since we don't really know what a series is; however what we shall try to do is to obtain a list of the relevant finitely determined map-germs and see how they behave, with particular reference to the points i)-iv) noted above. This brings us back to the classification problem.

For a finitely determined map-germ a versal deformation can be found by choosing a basis for the module $\theta(f)/T\mathcal{A}_e f$. We also find that, for example, for $f(x, y) = (x, xy, y^3)$ the module $\theta(f)/T\mathcal{A}_e f$ has the basis

$$\left\{ y\frac{\partial}{\partial Z}, y^{3k-1}\frac{\partial}{\partial Y} : k \geq 1 \right\}$$

(recall that the series associated with (x, xy, y^3) is $(x, xy + y^{3k-1}, y^3)$).
But how do we relate the geometrical type of f to the module $\theta(f)/T\mathcal{A}_e f$?
In fact, we find that there is a strong connection based on the fact that Σ is the support of the module $\theta(f)/T\mathcal{A}_e f$; this connection is provided by <u>Samuel's Theory</u>

of Multiplicity. We have included a summary of this theory in I·1 (I·1:10–I·1:15) along with other useful properties of finite modules. The main point is that

$$e(\theta(f)/T\mathscr{A}_e f) = \sum_{i=1}^{r} \mu(\Sigma_i)\, \ell_{\Sigma_i}(\theta(f)/T\mathscr{A}_e f)\delta(i)$$

where $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$, e denotes the multiplicity of the module, ℓ_{Σ_i} denotes the length of the localised module $\left(\theta(f)/T\mathscr{A}_e f\right)_{\Sigma_i}$ (localise with respect to the prime ideal defining Σ_i) and $\delta(i) = 1$ if $\dim \Sigma_i = \dim \Sigma$ or 0 otherwise (I·1:28).

We therefore follow two main lines of research:

i) the study of the module $T\mathscr{A}_e f$ and its use in finding normal forms,

ii) a study of how the geometrical type of a map-germ may determine whether or not it is a stem,

and link them by looking at the relationship between the transversal type of Σ and $\ell_\Sigma(\theta(f)/T\mathscr{A}_e f)$.

We give a brief outline here of the contents of the thesis:

With regard to the aims outlined above the main result from chapter I is the following theorem (I·3:3):

if $f:(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is weighted homogeneous and the Krull dimension and multiplicity of $\theta(f)/T\mathscr{A}f$ are both 1 then we can obtain a value k_0 and a list $\{f_k\}$ of finitely determined map-germs whose k_0-jet is f;

the list has the following properties:

i) the f_k can be written in the form $f_k = f + p_k$, where the p_k are monomials and $T\mathscr{A}f_{k'} = T\mathscr{A}f + \mathbb{C}\{p_k : k \geq k'\}$, which has finite codimension.

Examples suggest that at least for simple cases $p_{k+1} = r \cdot p_k$, for some $r \in \mathcal{O}_2$ although relations between the p_k appear to be more complicated in general.

ii) Each f_k is sufficient,

iii) their adjacency relations are

$$\cdots \to f_{k+1} \to f_k \to f_{k-1} \to \cdots,$$

iv) if g has the k_0-jet f then g is either $\mathscr{A}$- equivalent to one of the f_k or is k-$\mathscr{A}$- equivalent to f for all k (i.e. their k-jets are $\mathscr{A}^{(k)}$- equivalent for all k).

This theorem suggests that it is enough to define a series of singularities to be the list of finitely determined map-germs associated with a stem. All the other desirable properties seem to come naturally from this though more work needs to be done before we can make a formal definition of a series.

We propose the conjecture

if g is k-$\mathscr{G}$- equivalent to f for all k then g is $\mathscr{G}$- equivalent to f.

We have succeeded in proving this conjecture generally for $\mathscr{G} = \mathscr{R}$ and $\mathscr{G} = \mathscr{K}$ and for $\mathscr{G} = \mathscr{A}$ in the special case that f has 2-jet $(x, y^2, 0)$ (theorems I·1:18 and II·2:5).

We make the definition: f is a <u>weak stem</u> (for $\mathscr{G}$) if there is an integer k_0 such that any map-germ g with the same k_0-jet as f is either k-$\mathscr{G}$-equivalent to f for all k or is finitely determined for $\mathscr{G}$.

A map-germ satisfying the hypotheses of I·3:3 is thus a weak stem. The rest of Chapter I was developed principally to prove this theorem. In I·1 we summarise various properties of the finite module $T\mathscr{A}_e f$. We also extend the theory of trivial unfoldings of map-germs to k-jets.

In I·2 we use the Artin-Rees lemma to show that certain submodules of $T\mathscr{A}_e f$ have basically the same structure as $T\mathscr{A}_e f$; these submodules are then used to find normal forms. The theory developed here is also of use in general classification, as we demonstrate in Chapter II. Details of the results are outlined in the introduction to Chapter I.

Chapter III has three sections:

In III·1 we establish the crucial link between the structure of $T\mathscr{A}_e f$ and the geometry (III·1:1):

f is a geometric stem if and only if Σ is irreducible and $\ell_\Sigma(\theta(f)/T\mathscr{A}_e f) = 1$.

We also obtain other links between the transversal type of Σ and $\ell_\Sigma(\theta(f)/T\mathscr{A}_e f)$ which we use to calculate the transversal types for many of the examples given in Chapter II.

In III·2 we investigate part of the original conjecture:

$$f \text{ is a stem} \Rightarrow f \text{ is a geometric stem.}$$

We show that if f is a stem then Σ is irreducible and obtain a restricted list for its possible transversal type. However further reduction is still necessary to prove the conjecture.

In III·3 we look again at some of the families of singularities identified in Chapter II: we calculate the numbers C, T and $\mu(D_2(f)/\mathbb{Z}_2)$ for the singularities in each family $\{f+p_s\}$ associated with the weak stem f. We observe that for each of these families the integer

$$C(f+p_s)+T(f+p_s)+\mu(D_2(f+p_s)/\mathbb{Z}_2)-\operatorname{cod}(\mathscr{A}, f+p_s)$$

appears to be a constant. The details of these calculations form Appendix B.

Finally, we note that progress would not have been possible without looking at a large number of examples. These are presented in Chapter II. The results are collected in section II·1 and the calculations form sections II·2–II·5. Supplementary calculations for the classification form Appendix A. A major contribution to these calculations was the writing of programs to calculate the relevant modules by computer. The programs are described briefly in Chapter II and in more detail in Appendix C.

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Declaration.

The material incuded in this thesis is original, except where indicated otherwise.

Chapter I : Submodules of $T\mathcal{A}f$.

Introduction.

In this chapter we study submodules of $T\mathcal{A}f$ and their use in the classification of map-germs whose k-jet is f. The chapter contains three sections.

Section I·1 contains most of the preliminaries and is divided into three subsections.

I·1·1 : Properties of finite modules. p. 5.

The major part is a summary of Samuel's Multiplicity Theory.

I·1·2 : Equivalence of map-germs and jets. p. 10.

We define the relations $\mathcal{A}$-equivalence, k-$\mathcal{A}$-equivalence and $\mathcal{A}^{(k)}$-equivalence (I·1:16) and apply the multiplicity theory to the modules of interest.

I·1·3 : Criteria for equivalence of map-germs and jets. p. 17.

A commonly used technique is the construction of a trivial unfolding of a map-germ, using the Thom-Levine criterion. We make the obvious extension to a k-trivial unfolding of a k-jet and prove the corresponding criterion. This criterion is used extensively in section I·2.

In I·2 we look at the use of specific submodules of $T\mathcal{A}f$ in the classification of map-germs with the k-jet f. The main aim in this section is to find a list of possible normal forms for $j^{k+1}g$ when $j^kf = j^kg$ by inspection of the submodule

$$tf\left(m_n^2\cdot\theta(n)\right) + \omega f\left(m_p^2\cdot\theta(p)\right) + m_n^{k+2}\cdot\theta(f).$$

It is then a relatively simple task to calculate $T\mathcal{A}^{k+1}g$ so as to further restrict the list of normal forms. We also prove a sufficent condition for a normal form to be (k+1)-determined. The proofs are based on the following facts (I·2:2 and I·2:4):

if $h \in \mathfrak{m}_n^{k+1} \cdot \theta(f)$ then

$$tf\left(\mathfrak{m}_n^2 \cdot \theta(n)\right) + \mathfrak{m}_n^{k+2} \cdot \theta(f) = t(f+h)\left(\mathfrak{m}_n^2 \cdot \theta(n)\right) + \mathfrak{m}_n^{k+2} \cdot \theta(f)$$

$$\text{and } \omega f\left(\mathfrak{m}_p^2 \cdot \theta(p)\right) + \mathfrak{m}_n^{k+2} \cdot \theta(f) = \omega(f+h)\left(\mathfrak{m}_p^2 \cdot \theta(p)\right) + \mathfrak{m}_n^{k+2} \cdot \theta(f)$$

and if f is finite there exist integers k_0, k_1 such that

$$k \geq k_0 \Rightarrow \mathfrak{m}_n^{k+1} \cdot \theta(f) \cap T\mathcal{A}f \subseteq tf\left(\mathfrak{m}_n^2 \cdot \theta(n)\right) + \omega f\left(\mathfrak{m}_p^2 \cdot \theta(p)\right)$$

$$\text{and } k \geq k_1 \Rightarrow \mathfrak{m}_n^{k+1} \cdot \theta(f) \cap T\mathcal{A}f \subseteq f^*\mathfrak{m}_p \cdot \left(T\mathcal{A}f\right).$$

Theoretically this means that if f is finite then as we see from the theorems below, for large k we can find the normal forms by looking only at $T\mathcal{A}f$.

We find normal forms from the following theorem:

(I:1) Theorem: suppose that $\{v_1,\ldots,v_s\} \subseteq \mathfrak{m}_n^{k+1} \cdot \theta(f)$ and that either

$$\text{a) } \mathfrak{m}_n^{k+1} \cdot \theta(f) \subseteq tf\left(\mathfrak{m}_n^2 \cdot \theta(n)\right) + \omega f\left(\mathfrak{m}_p^2 \cdot \theta(p)\right) + \mathfrak{m}_n^{k+2} \cdot \theta(f) + \mathbb{C}\{v_1,\ldots,v_s\}$$

$$\text{or b) } f \text{ is finite, } k \geq k_0 \text{ and } T\mathcal{A}f + \mathfrak{m}_n^{k+1} \cdot \theta(f) = T\mathcal{A}f + \mathfrak{m}_n^{k+2} \cdot \theta(f) + \mathbb{C}\{v_1,\ldots,v_s\}.$$

$$\text{Then } g \overset{k}{\sim} f \Rightarrow g \overset{k+1}{\sim} f + \Sigma\lambda_i v_i \text{ for some } \lambda_1,\ldots,\lambda_s \in \mathbb{C}.$$

If f is finite, let ρ be the least integer such that $\mathfrak{m}_n^{\rho} \subseteq f^*\mathfrak{m}_p \cdot \mathcal{O}_n$.

We have the following test for finite determinacy:

(I:2) Theorem:

a) Suppose that f is finite, $k \geq \rho$, and $\{v_1,\ldots,v_s\}$ are as in I·1 b.

If i) – iii) hold then $f + \Sigma\lambda_i v_i$ is (k+1)-determined;

i) $f^*\mathfrak{m}_p\cdot\left(T\mathcal{A}f+\mathbb{C}\{v_1,\dots,v_t\}\right)\supseteq \mathfrak{m}_n^{k+2}\cdot\theta(f)$,

ii) $T\mathcal{A}\left(f+\Sigma\lambda_i v_i\right)+\mathfrak{m}_n^{k+2}\cdot\theta(f)\supseteq T\mathcal{A}f+\mathbb{C}\{v_1,\dots,v_t\}$

iii) for each $\lambda_i\neq 0$ and $v_i=((v_i)_1,\dots,(v_i)_p)$

$$f\left(\mathfrak{m}_n^2\cdot\theta(n)\right)+\omega f\left(\mathfrak{m}_p^2\cdot\theta(p)\right)+f^*\mathcal{O}_p\cdot\{v_1,\dots,v_t\}+\mathfrak{m}_n^{k+2}\cdot\theta(f)\supseteq\mathbb{C}\{(v_i)_\ell\cdot\frac{\partial}{\partial Y_j}:1\leq\ell,j\leq p\}.$$

b) if $k\geq k_1$ and $T\mathcal{A}f+f^*\mathfrak{m}_p\cdot\{v_1,\dots,v_t\}\supseteq\mathfrak{m}_n^{k+2}\cdot\theta(f)$ then i) holds.

c) if $tf\left(\mathfrak{m}_n^2\cdot\theta(n)\right)+\omega f\left(\mathfrak{m}_p^2\cdot\theta(p)\right)+f^*\mathcal{O}_p\cdot\{v_1,\dots,v_t\}+\mathfrak{m}_n^{k+2}\cdot\theta(f)\supseteq\mathfrak{m}_n^{k+1}\cdot\theta(f)$ then iii) holds.

Here $(Y_1,\dots,Y_p)$ are local coordinates for $\mathbb{C}^p$ at 0 and $(x_1,\dots,x_n)$ are coordinates for $\mathbb{C}^n$.

Hypothesis iii) says that if for example $v_i=((v_i)_1,0,\dots,0)$ then

$$f\left(\mathfrak{m}_n^2\cdot\theta(n)\right)+\omega f\left(\mathfrak{m}_p^2\cdot\theta(p)\right)+f^*\mathcal{O}_p\cdot\{v_1,\dots,v_t\}+\mathfrak{m}_n^{k+2}\cdot\theta(f)$$
$$\ni((v_i)_1,0,\dots,0),(0,(v_i)_1,\dots,0),\dots,(0,\dots,0,(v_i)_1).$$

These theorems are proved in I·2 (I·2:6 and I·2:9 respectively). In fact we shall obtain more general, but more technical theorems, of which these are special cases. However, these are the cases we shall apply most often in Chapter II and when studying stems.

In the case $s=1$ ($v_1=v\in\mathfrak{m}_n^{k+1}\cdot\theta(f)$) and f is weighted homogeneous, we get

(I:3) Theorem: suppose that v is a monomial, i.e. $v=x_1^{i_1}\dots x_n^{i_n}\cdot\frac{\partial}{\partial Y_j}$ for some $i_1,\dots,i_n$, j, and that v is not of the same degree as f_j;

if $\mathfrak{m}_n^{k+1}\cdot\theta(f) \subseteq tf\left(\mathfrak{m}_n^2\cdot\theta(n)\right) + \omega f\left(\mathfrak{m}_p^2\cdot\theta(p)\right) + \mathfrak{m}_n^{k+2}\cdot\theta(f) + \mathbb{C}\{v\}$ then

$$T\mathcal{A}\left(f+\lambda v\right) + \mathfrak{m}_n^{k+2}\cdot\theta(f) = T\mathcal{A}f + \mathbb{C}\{\lambda v\} + \mathfrak{m}_n^{k+2}\cdot\theta(f).$$

Consequently, the submanifold $\left\{f+\lambda v : \lambda \in \mathbb{C}\backslash\{0\}\right\}$ lies in a single $(k+1)$-orbit. For the proof see I·2:18.

There may be a choice of weights for f; in this case the theorem follows if the degrees of v and f_j are different for some choice of weights.
For example consider (x, xy, y^3). For this example we obtain $v = (0, 0, x^k y)$ (reference II·3:4). This is not of degree 3 with respect to the weights (1,1). So

$$T\mathcal{A}(x, xy, y^3 + \lambda x^k y) + \mathfrak{m}_n^{k+2}\cdot\theta(f) = T\mathcal{A}(x, xy, y^3) + \mathbb{C}\{\lambda x^k y \frac{\partial}{\partial Y_3}\} + \mathfrak{m}_n^{k+2}\cdot\theta(f).$$

We also prove a more general theorem for the construction of k+1-trivial unfoldings of a map-germ of the form $f + \Sigma\lambda_i v_i$ (I·2:19).

In I·3 we try to make use of the information gained by knowing the multiplicity and dimension of the module $\theta(f)/T\mathcal{A}f$.
In particular, we prove theorem (I·3:3) (outlined in the main introduction) and discuss how it may be extended to the general case. We give a list of families of singularities obtained using this theorem and note their properties.

I·1 : Preliminaries.

I·1·1 : Properties of finite modules.

(I·1:1) Definition : A holomorphic map $\varphi:X \to Y$ is finite if it is proper and every point $x \in X$ is an isolated point in the fibre $\varphi^{-1}(\varphi(x))$.

(I·1:2) Theorem : If x is an isolated point of its fibre then there is a neighbourhood of x on which φ is finite.

Proof : see [F], p. 132.

In particular, the germ $f:(X,x) \to Y$ of φ at x is finite, i.e. for each representative f' of f there is a neighbourhood of x on which f' is finite.

(I·1:3) Theorem : If $\varphi:X \to Y$ is a holomorphic map, $x \in X$ and $y = \varphi(x)$, then the following are equivalent :

i) $\mathcal{O}_{X,x}$ is a finitely generated $\mathcal{O}_{Y,y}$-module,

ii) $\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\cdot\mathcal{O}_{X,x}) < \infty$,

iii) x is an isolated point of its fibre X_y.

Proof : see [F], p. 57.

This implies that the map-germ $f:(\mathbb{C}^n,x) \to (\mathbb{C}^p,y)$ is finite if and only if $\mathcal{O}_{n,x}$ is a finitely generated $\mathcal{O}_{p,y}$-module. The action of $\mathcal{O}_{p,y}$ on $\mathcal{O}_{n,x}$ is defined by

$$\chi\circ m = f^*(\chi)\circ m = (\chi\circ f)\circ m, \text{ where } \chi\in\mathcal{O}_{p,y} \text{ and } m \in \mathcal{O}_{n,x}.$$

If we choose a different map-germ then the $\mathcal{O}_{p,y}$-module structure of $\mathcal{O}_{n,x}$ is different.

Our aim is to study modules related to the module $\mathcal{O}_{n,x}$ when f is finite. We start by presenting some well-known properties of finite (i.e. finitely generated) modules. Chief amongst these is Samuel's Theory of Multiplicity, which provides us with two

invariants for describing the (finite) module M: the Krull dimension dim(M) and the multiplicity e(M). The main reference for this material is [Mats]. Another useful reference is [Mum], pp. 116-126.

In what follows, assume that A is a local Noetherian ring of dimension d with maximal ideal $\mathfrak{m}$, $k=A/\mathfrak{m}$ is the residue field and M is an A-module.

(I·1:4) Theorem: M is finite if and only if $M/\mathfrak{m}\cdot M$ is a finite-dimensional vector space over k. If M is finite then $\{u_1,\dots,u_s\}$ is an A-basis for M if and only if $\{\bar{u}_1,\dots,\bar{u}_s\}$ is a k-basis for $M/\mathfrak{m}\cdot M$, where $\bar{u}_i$ is the image of u_i in the quotient.
Proof: see [Mats], p. 8.

(I·1:5). Nakayama's Lemma: Let M be finite and suppose that $\mathfrak{m}\cdot M = M$. Then $M = 0$.
Proof: see [Mats], p. 8.

(I·1:6) Corollary: Let N, P be submodules of M with N finite.

$$\text{If } N \subseteq P + \mathfrak{m}\cdot N \text{ then } N \subseteq P.$$

Proof: We have $\mathfrak{m}\cdot(N+P)/P = (\mathfrak{m}\cdot N+P)/P = (N+P)/P$. It follows from I·1:6 that $(N+P)/P = 0$, i.e. $N \subseteq P$. [Mats], p. 8.

We shall frequently need the Artin-Rees lemma for looking at submodules:
(I·1:7) Lemma (Artin-Rees): Let N be a submodule of a finite module M and I an ideal of A. Then there exists a positive integer m such that

$$N \cap I^m\cdot M \subseteq I\cdot N.$$

Proof: see [Mats], p. 59.

(I·1:8) Theorem (the Krull Intersection Theorem):

i) for any finite module M and ideal $I \subseteq \mathfrak{m}$, $\bigcap_{m>0} I^m\cdot M = 0$;

if $N \subseteq M$ is a submodule then $\bigcap_{m>0}\left(N + I^m \cdot M\right) = N$.

ii) If A is also an integral domain, $I \subseteq A$ proper, then $\bigcap_{m>0} I^m = 0$.

Proof : see [Mats], p. 60.

(I·1:9) Definition : We define dim(M) by setting $\dim(M) = \dim\left(A/\mathrm{Ann}(M)\right)$. This is the <u>Krull dimension</u> of M. It is the dimension of the variety defined by the ideal Ann(M).

Lemma : If M is finite then $\mathrm{Supp}(M) = V\left(\mathrm{Ann}(M)\right)$.

Proof : see [Mats], p. 25.

(I·1:10) Definition : If I is an ideal of A satisfying $\mathfrak{m}^\nu \subseteq I \subseteq \mathfrak{m}$ for some $\nu > 0$ then we shall call I an ideal of definition. Let M be a finite A-module. Then $M/I^n \cdot M$ has finite length. We set

$$\chi_M^I(n) = \ell\left(\frac{M}{I^n \cdot M}\right) \qquad \text{[Mats], p. 97.}$$

(I·1:11) Theorem : For sufficiently large values of n, χ_M^I is a polynomial and is given by

$$\chi_M^I(n) = \frac{e}{d!}\, n^d + (\text{lower order terms}), \text{ where } e \in \mathbb{N}.$$

Recall that here, $d = \dim A$, not dim M.

Proof : see [Mats], p. 97.

We abbreviate $\chi_M^{\mathfrak{m}}$ to χ_M and call it the <u>Samuel polynomial of M</u> (over A). We shall denote the integer e by e(I,M).

Definition : The <u>multiplicity of M over A</u>, $e_A(M)$, is the integer $e(\mathfrak{m}, M)$.

(I·1:12) Theorem : The polynomial χ_M^I has the following properties :

i). The degree is independent of I. Denote it by d(M). Then $e(I,M) > 0$ if $d(M) = d$ and $e(I,M) = 0$ if $d(M) < d$.

ii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finite modules then

$d(M) = \max\left(d(M'), d(M'')\right)$, and if I is any ideal of definition then $\chi_M^I - \chi_{M''}^I$

and $\chi_{M'}^I$ have the same leading coefficient.

iii) $d(M) = \delta(M) = \dim(M)$.

Proof : see [Mats], p. 98.

Here $\delta(M)$ is the least value of n for which there exist $x_1,\ldots,x_n \in \mathfrak{m}$ such that $\ell(M/x_1M+\ldots+x_nM)$ is finite.

(I·1:13) Theorem : The integer e(I, M) has the following properties :

i) $e(I,M) = \lim\limits_{n\to\infty} \frac{d!}{n^d}\ell\left(\frac{M}{I^n\cdot M}\right)$ and if $d = 0$, $e(I,M) = \ell(M)$,

ii) if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finite modules then

$$e(I,M) = e(I,M') + e(I,M''),$$

iii) if $\{p_1,\ldots,p_t\}$ are the minimal prime ideals p of A with $\dim(A/p) = d$ then

$$e(I,M) = \sum_{i=1}^{t} e\left(I', A/p_i\right)\cdot \ell(M_{p_i}),$$

where $\ell(M_p)$ is the length of M_p as an A_p-module and $I' = (I+p)/p$,

iv) if A is an integral domain then $e(I,M) = e(I,A)\cdot s$ where $s = \operatorname{rank} M$.

Proof : see [Mats], pp. 107-108.

Usually we will not have $\dim(M) = \dim(A)$. To inspect the leading coefficient of χ_M^I in this case we consider M as a module over the ring $A_M = A/\mathrm{Ann}(M)$.

Let $I_M = I + \mathrm{Ann}(M)/\mathrm{Ann}(M)$. We then have

$$\dim(M) = \dim\big(A/\mathrm{Ann}(M)\big) \text{ and } \chi_M^{I_M}(n) = \ell\Big(\frac{M}{I_M^n \cdot M}\Big) = \ell\Big(\frac{M}{I^n \cdot M}\Big) = \chi_M^{I}(n).$$

The minimal prime ideals $\{p_{1,M},\dots,p_{t,M} \subseteq A_M : \dim(A_M/p_{i,M}) = \dim(M)\}$ correspond to minimal primes $\{p_1,\dots,p_t : p_i \supseteq \mathrm{Ann}(M) : \dim(A/p_i) = \dim(M)\}$
(for each i, $p_{i,M} = p_i/\mathrm{Ann}(M)$).

$$\text{So } e(I_M, M) = \sum_{i=1}^{t} e\big(I_M', A_M/p_{i,M}\big)\cdot \ell(M_{p_{i,M}}), = \sum_{i=1}^{t} e\big(I', A/p_i\big)\cdot \ell(M_{p_i})$$

since $I_M' = (I_M + p_{i,M})/p_{i,M} \sim (I + p_i)/p_i = I'$, $A_M/p_{i,M} \sim A/p_i$ and $M_{p_{i,M}} = M_{p_i}$.

$$\text{In particular, } e(\mathfrak{m}_M, M) = \sum_{i=1}^{t} e\big(\mathfrak{m}_{p_i}, A/p_i\big)\cdot \ell(M_{p_i}).$$

Note that for any $p \supseteq \mathrm{Ann}(M)$, $e\big(\mathfrak{m}_p, A/p\big) = 0$ if $\dim(A/p) < \dim(M)$.

Definition: The <u>multiplicity e(M) of M</u> is the multiplicity $e(\mathfrak{m}_M, M)$ of M over $A/\mathrm{Ann}(M)$.

Then $$e(M) = 0 \Leftrightarrow M = 0.$$

For e(M) to have any meaning the Krull dimension of M should also be quoted.

Locally, the integer $e\big(\mathfrak{m}_p, A/p\big)$ has geometrical meaning:

(I·1:14) Theorem: Let X be a variety, $x \in X$. Then

$$\mathrm{mult}_x(X) = e_x(\mathfrak{m}_{X,x}, \mathcal{O}_{X,x}).$$

Proof: see [Mum], p. 123.

(I·1:15) Corollary: If M_x is an $\mathcal{O}_{X,x}$-module then

$$e(\mathfrak{m}_{X,x}, M_x) = \sum_{i=1}^{t} \mathrm{mult}_x\big(X_{p_i}\big)\cdot \ell(M_{p_i}),$$

where X_{p_i} is the variety defined by the prime ideal p_i, $\dim\big(X_{p_i}\big) = \dim(M_x)$ and

$\ell(M_{p_i})$ is independent of $x \in X_{p_i}$. The X_{p_i} are the branches of $\mathrm{Supp}(M_x)$ which have (maximum) dimension $\dim\left(\mathrm{Supp}(M_x)\right)$.

This covers all the properties of finite modules that we shall need. We next look at equivalence relations between map-germs and construct our modules.

I·1·2 : Equivalence of map-germs and jets.

(I·1:16) Definition : Let $\mathcal{A} = \mathrm{Aut}(\mathbb{C}^n,0) \times \mathrm{Aut}(\mathbb{C}^p,0)$. Then $\mathcal{A}$ acts on $\mathcal{O}_n^p$ by

$$(\psi,\varphi)\circ f = \psi\circ f\circ\varphi \text{ for } (\psi,\varphi) \in \mathcal{A} \text{ and } f \in \mathcal{O}_n^p.$$

We say that f, g are $\mathcal{A}$-equivalent and write $f \overset{\mathcal{A}}{\sim} g$, or just $f \sim g$, if there is some $(\psi,\varphi) \in \mathcal{A}$ such that $f = (\psi,\varphi)\circ g$.

The action of $\mathcal{A}$ on $\mathcal{O}_n^p$ induces an action on $J^k(n,p)$:
let $\mathcal{A}^{(k)} = \{(j^k\psi, j^k\varphi) : (\psi,\varphi) \in \mathcal{A}\}$. Then $\mathcal{A}^{(k)}$ acts on $J^k(n,p)$ by

$$(j^k\psi, j^k\varphi)\circ j^k f = j^k\left(\psi\circ f\circ\varphi\right).$$

Note that $j^k f, j^k g$ are $\mathcal{A}^{(k)}$-equivalent if and only if there is some $h \in \mathfrak{m}_n^{k+1}\cdot\mathcal{O}_n^p$ such that $f - h$, g are $\mathcal{A}$-equivalent. Call this second equivalence relation on $\mathcal{O}_n^p$ <u>k-$\mathcal{A}$-equivalence</u> and write $f \overset{k}{\sim} g$ to indicate it. Clearly $f \sim g \Rightarrow f \overset{k}{\sim} g$ for all k.

Conjecture: $f \overset{k}{\sim} g$ for all $k \Rightarrow f \sim g$.

We note that the conjecture is true if we replace $\mathcal{A}$-equivalence with $\mathcal{K}$-equivalence or with $\mathcal{R}$-equivalence in the case that $p = 1$. To prove this we need a strengthening of Artin approximation which is due to J. Wavrik:

(I·1:17) Theorem (Artin approximation).

Let $f_i(x, y) \in \mathcal{O}_{n+N}$, $i = 1,\dots, m$, where $x = (x_1,\dots, x_n)$ and $y = (y_1,\dots, y_N)$. Suppose that for each k we have $y_{k,1}(x),\dots, y_{k,N}(x) \in \mathfrak{m}_n$ such that

$$f_i(x, y_k(x)) \in \mathfrak{m}_n^{k+1}$$

for each i. Then for any c there are $y_1(x),\dots, y_N(x) \in \mathfrak{m}_n$ such that $f(x, y(x)) = 0$

and $y_{k,\nu}(x) - y_\nu(x) \in \mathfrak{m}_n^c$.

See [Wavrik], theorems 2.2 and 2.3.

(I·1:18) Theorem :

i) if function-germs f, g are k-$\mathcal{R}$-equivalent for each k then they are $\mathcal{R}$-equivalent,

ii) if map-germs f, g are k-$\mathcal{K}$-equivalent for each k then they are $\mathcal{K}$-equivalent.

Proof : for i) observe that $\mathcal{R}$-equivalence follows if and only if the equation

$$F(x, \varphi) = f(x) - g(\varphi) = 0$$

has a convergent solution $\varphi(x) \in (\mathfrak{m}_n)^n$ which is an isomorphism. By the above theorem this follows if for each k there exists an isomorphism φ_k such that

$F_i(x, \varphi_k(x)) \in \mathfrak{m}_n^{k+1}$ for $1 \le i \le p$ since $\varphi_{k,\nu}(x) - \varphi_\nu(x) \in \mathfrak{m}_n^2$ ensures that φ is also an isomorphism. But this is precisely the condition f, g are k-$\mathcal{R}$-equivalent for each k. For part ii) we must find convergent solutions A(x), $\varphi(x)$ for the equation

$$F(x, \varphi, A) = Af(x) - g(\varphi) = 0$$

where A is an invertible p×p matrix. The proof is otherwise the same.

The approximation theorem cannot be used directly to show that

'if map-germs f, g are k-$\mathcal{A}$-equivalent for each k then they are $\mathcal{A}$-equivalent'

since we need to find isomorphisms $\varphi(x)$ and $\psi(z_1,\dots z_p)$ such that $f(\varphi) - \psi(g) = 0$ and there is no way of writing this as finding the solution to an equation $F(x, y(x)) = 0$.

We shall, however, prove the conjecture in the case that $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ satisfies $j^2f = (x, y^2, 0)$; see II·2:5 for this proof.

A similar definition of $\mathcal{A}$-equivalence is used for map-germs $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, where $\mathcal{A} = \mathrm{Aut}(\mathbb{R}^n,0)\times\mathrm{Aut}(\mathbb{R}^p,0)$; this is the definition used in the paper [Mond 1]. Where necessary we will distinguish between these by calling them $\mathcal{A}_{\mathbb{R}}$-equivalence and $\mathcal{A}_{\mathbb{C}}$-equivalence. The main differences between the classifications occur due to the connectivity of submanifolds; e.g. $\mathbb{C}\setminus\{0\}$ is connected but $\mathbb{R}\setminus\{0\}$ is not. Where not stated otherwise we mean $\mathcal{A}_{\mathbb{C}}$-equivalence.

<u>Some types of map-germs.</u>

(I·1:19) Definition : A map-germ $f\in\mathcal{O}_n^p$ is <u>k-$\mathcal{A}$-determined</u> (or just k-determined) if $f \overset{k}{\sim} g \Rightarrow f\sim g$, all $g\in\mathcal{O}_n^p$. Call the least k for which this holds the <u>determinacy degree</u> of f.

A map-germ $f\in\mathcal{O}_n^p$ is <u>finitely determined</u> (for $\mathcal{A}$) if it is k-determined for some k.

(I·1:20) Definition : A map-germ $f\in\mathcal{O}_n^p$ is a <u>k-stem</u> (for $\mathcal{A}$) if f is not finitely determined but for all $g\in\mathcal{O}_n^p$, $f \overset{k}{\sim} g \Rightarrow$ either $f\sim g$ or g is finitely determined. Call the least k for which this holds the <u>degree</u> of the stem f.

A map-germ $f\in\mathcal{O}_n^p$ is a <u>stem</u> (for $\mathcal{A}$) if it is a k-stem for some k.
This definition is analagous to that made by R. Pellikaan in [P 1], p. 1. It was proposed by D. Mond.

Example : consider $f(x, y) = (x, y^2, y^3)$. Then f is a 3-stem and

$$g \overset{3}{\sim} f \Rightarrow g \sim f \text{ or } g \sim (x, y^2, y^3+x^k y) \text{ for some } k \geq 3.$$

Each $f + p_k$ is k+1-determined: see II·2:10.

Throughout this thesis we have frequently found map-germs f which have the following property:

there exists a value k such that if g satifies $f \overset{k}{\sim} g$ then either g is finitely determined or $f \overset{k}{\sim} g$ for all k.

<u>Definition</u>: call such a map-germ a <u>weak stem</u>.

If we can prove the conjecture $f \overset{k}{\sim} g$ for all $k \Rightarrow f \sim g$ then such map-germs will be stems. In most cases we suspect to be stems then, we content ourselves with proving that the weaker condition holds. I personally feel that it is unlikely that the conjecture fails for any of the relatively simple examples calculated in this thesis. We leave further investigation of the conjecture for a future research problem.

(I·1:21) <u>Modules associated with map-germs.</u>

We choose local coordinates at x and y and consider only map-germs $f:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$. Denote the set of vector fields along f by $\theta(f)$. If $\{Y_1,\ldots, Y_p\}$ are coordinates for $\mathbb{C}^p$ at 0, then the vector fields $\{\partial/\partial Y_1,\ldots,\partial/\partial Y_p\}$ form a free basis for $\theta(f)$ as an $\mathcal{O}_n$-module. We shall write $e_i = \partial/\partial Y_i$ when we need to save space.

A <u>monomial in $\theta(f)$</u> is a term of the form $\nu\cdot\partial/\partial Y_i$, where $\nu \in \mathcal{O}_n$ is a monomial.

As well as being an $\mathcal{O}_n$-module, $\theta(f)$ is also a (finitely generated) $\mathcal{O}_p$-module via f^* (if f is finite). If f is the identity map $(\mathbb{C}^n,0) \longrightarrow (\mathbb{C}^n,0)$ then denote $\theta(f)$ by $\theta(n)$.

Similarly define $\theta(p)$.

(I·1:22) Lemma: If f is finite, $D \subseteq \theta(f)$ an $\mathcal{O}_p$-module, $C \subseteq \theta(f)$ a subset which satisfies $C \subseteq D + \mathfrak{m}_n^{k+1} \cdot \theta(f)$, $\forall k$, then $C \subseteq D$.

Proof: $\theta(f)$ is a finite $\mathcal{O}_p$-module and $C \subseteq \bigcap_{k \geq 0} \left(D + \mathfrak{m}_n^{k+1} \cdot \theta(f)\right) = D$ (th. 1.9). □.

There are ($\mathcal{O}_p$-module) homomorphisms $tf:\theta(n) \to \theta(f)$ and $\omega f:\theta(p) \to \theta(f)$ defined by

$$tf(\xi) = df \circ \xi, \quad \text{and} \quad \omega f(\eta) = \eta \circ f.$$

If $I_i \subseteq \mathcal{O}_n$, $1 \leq i \leq n$ and $J_i \subseteq \mathcal{O}_p$, $1 \leq j \leq p$ are ideals then

$$tf\left(\bigoplus_{i=1}^{n} I_i \cdot \partial/\partial x_i\right) = I_1 \cdot \left\{\partial f/\partial x_1\right\} + \ldots + I_n \cdot \left\{\partial f/\partial x_n\right\} \text{ and } \omega f\left(\bigoplus_{j=1}^{p} J_j \cdot \partial/\partial Y_j\right) = \bigoplus_{j=1}^{p} f^*(J_j) \cdot \partial/\partial Y_j.$$

(I·1:23) Lemma [Math IV], p. 227:

f is stable if and only if $tf(\theta(n)) + \omega f(\theta(p)) = \theta(f)$.

(I·1:24) Lemma: The tangent space $T\mathcal{A}^{(k)}f$ to the $\mathcal{A}^{(k)}$-orbit of j^kf at j^kf is given by

$$T\mathcal{A}^{(k)}f = \left(tf(\mathfrak{m}_n \cdot \theta(n)) + \omega f(\mathfrak{m}_p \cdot \theta(p)) + \mathfrak{m}_n^{k+1} \cdot \theta(f)\right) / \mathfrak{m}_n^{k+1} \cdot \theta(f).$$

Proof: see [Math III], proposition 7·4.

(I·1:25) Notation: Since we want to work in $\theta(f)$ where possible we define

$$T\mathcal{A}^{k}f = tf(\mathfrak{m}_n \cdot \theta(n)) + \omega f(\mathfrak{m}_p \cdot \theta(p)) + \mathfrak{m}_n^{k+1} \cdot \theta(f).$$

So

$T\mathcal{A}^{k}f / \mathfrak{m}_n^{k+1} \cdot \theta(f) = T\mathcal{A}^{(k)}f$ and $T\mathcal{A}^{k}f = T\mathcal{A}^{k}g$ if and only if $T\mathcal{A}^{(k)}f = T\mathcal{A}^{(k)}g$.

Note that $T\mathcal{A}^{k}f \supseteq T\mathcal{A}^{k+1}f$ for all k while it is difficult to compare $T\mathcal{A}^{(k)}f$ and $T\mathcal{A}^{(k+1)}f$.

Analogously we define the 'tangent space' to the $\mathcal{A}$-orbit of f at f to be

$$T\mathcal{A}f = tf\left(\mathfrak{m}_n\cdot\theta(n)\right) + \omega f\left(\mathfrak{m}_p\cdot\theta(p)\right).$$

Note that by (I·1:22) $T\mathcal{A}^k f = T\mathcal{A}^k g \ \forall k \Rightarrow T\mathcal{A}f = T\mathcal{A}g$ (and therefore $T\mathcal{A}f$ is an $\mathcal{O}_p$-module via g^*). The spaces $T\mathcal{A}f$, $T\mathcal{A}^{(k)}f$, $T\mathcal{A}^k f$ are not in general $\mathcal{O}_n$-modules but they are all $\mathcal{O}_p$-modules via f^*.

We have the following criterion for finite determinacy:

(I·1:26) Characterisation Theorem (Mather): The following are equivalent:

i) f is finitely determined,

ii) for some integer k, $\mathfrak{m}_n^k\cdot\theta(f) \subseteq tf(\mathfrak{m}_n\cdot\theta(n)) + \omega f(\mathfrak{m}_p\cdot\theta(p))$,

iii) $\dim_{\mathbb{C}}\left\{\mathfrak{m}_n\cdot\theta(f)/tf(\mathfrak{m}_n\cdot\theta(n)) + \omega f(\mathfrak{m}_p\cdot\theta(p))\right\} < \infty$,

iv) for some integer k', $\mathfrak{m}_n^{k'}\cdot\theta(f) \subseteq tf(\theta(n)) + \omega f(\theta(p))$,

v) $\dim_{\mathbb{C}}\left\{\theta(f)/tf(\theta(n)) + \omega f(\theta(p))\right\} < \infty$.

Proof-see [du P], p. 121.

We denote by $T\mathcal{A}_e f$ the module $tf(\theta(n)) + \omega f(\theta(p))$.

(I·1:28) <u>Multiplicity Theory and map-germs</u>.

Let $f : (\mathbb{C}^n, x_1, x_2, \ldots, x_t) \to (\mathbb{C}^p, y)$ be a (finite) multi-germ. Then $\theta(f)$ is a finite $\mathcal{O}_{\mathbb{C}^p,y}$-module via f^* $(\theta(f) \sim \bigoplus_{i=1}^{t} \mathcal{O}^p_{\mathbb{C}^n,x_i})$, as are its sub-modules and quotient modules.

We will consider the finite $\mathcal{O}_{\mathbb{C}^p,y}$-module $\theta(f)_y / T\mathcal{A}_e f_y$.

Let $\mathrm{Supp}(\theta(f)_y / T\mathcal{A}_e f_y) = \Sigma$.

Now i) $y \in \Sigma$ if and only if f fails to be stable at y (I·1:23),

ii) y is an isolated point of Σ if and only if there is a neighbourhood U of y such that f is stable on $U\backslash\{y\}$,

iii) y is an isolated point of Σ if and only if f is finitely determined at

y ([Wall 1], p. 492).

Let $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_r$, and consider $\theta(f)_y / T\mathscr{A}_e f_y$ as a module over

$$A_y = \mathcal{O}_{\mathbb{C}^p,y} / \operatorname{Ann}(\theta(f)_y / T\mathscr{A}_e f_y).$$

Then $\dim(\theta(f)_y / T\mathscr{A}_e f_y) = \dim(\Sigma)$ (= d, say), and the minimal prime ideals $\bar{p}_i$ of A_y with $d = \dim A_y = \dim A_y/\bar{p}_i$ correspond precisely to the prime ideals p_i of $\mathcal{O}_{\mathbb{C}^p,y}$, where $p_i \supseteq \operatorname{Ann}(\theta(f)_y / T\mathscr{A}_e f_y)$ and $\Sigma_i = V(p_i)$ has dimension d. Then

$$e(\theta(f)_y / T\mathscr{A}_e f_y) = \sum_{i=1}^{r} \operatorname{mult}_y(\Sigma_i) \cdot \ell_{\Sigma_i}(\theta(f)_y / T\mathscr{A}_e f_y) \cdot \delta(i),$$

where $\delta(i) = 1$ if $\dim(\Sigma_i) = d$ and 0 if $\dim(\Sigma_i) < d$.

Sheaf Theory: $\theta(f)_y / T\mathscr{A}_e f_y$ is a coherent sheaf of $\mathcal{O}_{\mathbb{C}^p}$-modules so $e(\theta(f)_y / T\mathscr{A}_e f_y)$ is well-defined as a function of $y \in \mathbb{C}^p$.

Proof: Write $\mathcal{O}_{\mathbb{C}^n}$ for the sheaf of (germs of) holomorphic functions on $\mathbb{C}^n$, $\mathcal{O}_{\mathbb{C}^p}$ for the sheaf on $\mathbb{C}^p$, $\mathscr{V}(f)$ for the sheaf of germs of holomorphic vector fields along f, $\mathscr{V}(n)$ and $\mathscr{V}(p)$ the corresponding sheaves for the identities on $\mathbb{C}^n$, $\mathbb{C}^p$. These are all coherent sheaves. Then the sheaf $\mathscr{V}(f)/\, tf(\mathscr{V}(n))$ is also coherent.

Since f is finite, $f_*\big(\mathscr{V}(f)/tf(\mathscr{V}(n))\big)$ is coherent as an $\mathcal{O}_{\mathbb{C}^p}$-module and so is $\mathscr{S}_A(f) = f_*\big(\mathscr{V}(f)/\, tf(\mathscr{V}(n))\big)/wf(\mathscr{V}(p))$. But $\mathscr{S}_{A,y}(f) = \theta(f)_y / T\mathscr{A}_e f_y$.

See [Wall 1], p. 493. □.

We remark here that if Krull dim $\theta(f)_y / T\mathscr{A}_e f_y \geq 1$ then the modules

$$\theta(f)_y / T\mathscr{A}_e f_y \text{ and } \theta(f)_y / T\mathscr{A} f_y$$

have the same multiplicity; in fact every module

$$\frac{\mathfrak{m}_n^k \cdot \theta(f) + tf(\mathfrak{m}_n^c \cdot \theta(n)) + \omega f(\mathfrak{m}_p^d \cdot \theta(p))}{tf(\mathfrak{m}_n^c \cdot \theta(n)) + \omega f(\mathfrak{m}_p^d \cdot \theta(p))}$$

has the same multiplicity for all finite values of k, c and d (provided that the Krull dimension is at least 1).

Proof: Claim: for every pair (c, d), (c′, d′) there exists a value k_0 such that

$$\mathfrak{m}_n^{k_0}\cdot\theta(f)\cap\left\{tf(\mathfrak{m}_n^{c'}\cdot\theta(n))+\omega f(\mathfrak{m}_p^{d'}\cdot\theta(p))\right\} = \mathfrak{m}_n^{k_0}\cdot\theta(f)\cap\left\{tf(\mathfrak{m}_n^{c}\cdot\theta(n))+\omega f(\mathfrak{m}_p^{d}\cdot\theta(p))\right\}.$$

Proof of claim: this follows from I·2:4, where we show that for any (c, d) there exists k such that $\mathfrak{m}_n^{k}\cdot\theta(f)\cap T\mathcal{A}f \subseteq tf(\mathfrak{m}_n^{c}\cdot\theta(n))+\omega f(\mathfrak{m}_p^{d}\cdot\theta(p))$ so that

$$\mathfrak{m}_n^{k}\cdot\theta(f)\cap\left\{tf(\mathfrak{m}_n^{c'}\cdot\theta(n))+\omega f(\mathfrak{m}_p^{d'}\cdot\theta(p))\right\} \subseteq \mathfrak{m}_n^{k}\cdot\theta(f)\cap tf(\mathfrak{m}_n^{c}\cdot\theta(n))+\omega f(\mathfrak{m}_p^{d}\cdot\theta(p)).$$

Conversly, there exists k′ such that

$$\mathfrak{m}_n^{k'}\cdot\theta(f)\cap\left\{tf(\mathfrak{m}_n^{c'}\cdot\theta(n))+\omega f(\mathfrak{m}_p^{d'}\cdot\theta(p))\right\} \subseteq \mathfrak{m}_n^{k'}\cdot\theta(f)\cap tf(\mathfrak{m}_n^{c}\cdot\theta(n))+\omega f(\mathfrak{m}_p^{d}\cdot\theta(p)).$$

Choose k_0 to be the maximum of the two values.

I·1·3 : <u>Criteria for equivalence of map-germs and jets</u>.

The first criterion is comparison of $\mathcal{A}$-tangent spaces:

(I·1:26) Mather's lemma: Let the Lie group G act smoothly on the manifold M, and suppose that the connected submanifold S satisfies:

i) for all $x \in S$, $T_xG\cdot x \supseteq T_xS$,

ii) the dimension of $G\cdot x$ is independent of the choice of $x \in S$.

Then S is contained in a single G orbit.

Proof: see [Math IV], lemma 3·1.

We use this to prove

(I·1:27) Theorem: Suppose that map-germs $f, g \in \mathcal{O}_n^p$ satisfy $T\mathcal{A}^k f = T\mathcal{A}^k g$.

$$\text{Then } f \overset{k}{\sim} g.$$

Proof: this is equivalent to proving

$$T\mathcal{A}^{(k)}f = T\mathcal{A}^{(k)}g \Rightarrow j^kf, j^kg \text{ are } \mathcal{A}^{(k)}\text{-equivalent.}$$

Let $S = \{j^kf_t \in J^k(n,p) : t \in \mathbb{C}\}$ where $f_t = f + t(g - f)$.

Then $T\mathcal{A}^{(k)}f_t \subseteq T\mathcal{A}^{(k)}f$ for all t since

$$j^k\mathfrak{m}_n\frac{\partial f_t}{\partial x_i} = j^k\mathfrak{m}_n\left(t\frac{\partial g}{\partial x_i} + (1-t)\frac{\partial f}{\partial x_i}\right) \subseteq j^k\mathfrak{m}_n\frac{\partial f}{\partial x_i} + j^k\mathfrak{m}_n\frac{\partial g}{\partial x_i} \subseteq T\mathcal{A}^{(k)}f$$

and any element in $j^k\omega f_t(\mathfrak{m}_p.\theta(p))$ is a sum of terms of the form

$$j^k((f)_1^{i_1}\ldots(f)_p^{i_p}(g)_1^{j_1}\ldots(g)_p^{j_p})\partial/\partial Y_i.$$

These are elements of $j^k[f^*\mathfrak{m}_p.\omega g(\mathfrak{m}_p.\theta(p))]$. Since

$$j^k\omega g(\mathfrak{m}_p.\theta(p)) \subseteq T\mathcal{A}^{(k)}f,$$

which is an $\mathcal{O}_p$-module via f^*, the inclusion follows.

Now $T\mathcal{A}^{(k)}f$ has finite dimension as a vector space over $\mathbb{C}$. So we can choose a basis $\{v_1^0,\ldots,v_r^0\}$ for $T\mathcal{A}^{(k)}f = T\mathcal{A}^{(k)}f_0$ over $\mathbb{C}$. We get corresponding spanning sets $\{v_1^t,\ldots,v_r^t\}$ for $T\mathcal{A}^{(k)}f_t$ and $r\times r$ matrices $A_t = (a_{i,j}(t))$ where $v_i^t = \sum_{j=1}^{r} a_{i,j}(t)v_j^0$.

Then $T\mathcal{A}^{(k)}f_t = T\mathcal{A}^{(k)}f$ if and only if A_t is invertible, i.e. if and only if $\det(A_t) \neq 0$. Now $\det(A_t)$ is a polynomial in t and is non-zero at $t = 0$, so it has only a finite number of roots. Let $T \subseteq \mathbb{C}$ be given by $T = \{t \in \mathbb{C}: \det(A_t) \neq 0\}$. Then T is open and connected in $\mathbb{C}$ so the submanifold $S_T = \{j^kf_t \in J^k(n,p): t \in T\}$ is open and connected in $J^k(n,p)$.

So the dimension of $\mathcal{A}^{(k)}.j^kf_t$ is independent of the choice of $j^kf_t \in S_T$.

Finally $j^kf - j^kg \in T\mathcal{A}^{(k)}f_t = T\mathcal{A}^{(k)}f$ for all $j^kf_t \in S_T$, so $T_tS_T \subseteq T\mathcal{A}^{(k)}f_t$ for all $j^kf_t \in S_T$. Thus S_T is contained in a single $\mathcal{A}^{(k)}$-orbit and j^kf and j^kg are $\mathcal{A}^{(k)}$-equivalent as required. □.

Corollary 1: if f is finitely determined then $T\mathcal{A}f = T\mathcal{A}g \Rightarrow f \sim g$.

Proof: if f is k-determined then

$$T\mathcal{A}f = T\mathcal{A}g \Rightarrow T\mathcal{A}^kf = T\mathcal{A}^kg \Rightarrow f \overset{k}{\sim} g \Rightarrow f \sim g.$$

Corollary 2: $\quad T\mathcal{A}f = T\mathcal{A}g \Rightarrow T\mathcal{A}^kf = T\mathcal{A}^kg \ \forall k \Rightarrow f \overset{k}{\sim} g \ \forall k.$

More generally,

(I·1:28) Proposition : If $\mathfrak{m}_n^{\rho}\cdot\theta(f) \subseteq tf(\mathfrak{m}_n\cdot\theta(n)) + f^*\mathfrak{m}_p\cdot\theta(f)$ for some ρ then

$$T\mathcal{A}f = T\mathcal{A}g \Rightarrow f\sim g.$$

Proof : see [G-H].

We hope that I·1:27 together with the conjecture $f \overset{k}{\sim} g\ \forall k \Rightarrow f\sim g$ will lead to a more elementary proof of this theorem.

Remark : these theorems do not hold for $\mathcal{A}_{\mathbb{R}}$-equivalence since they rely on the fact that $\mathbb{C}\backslash\{\text{isolated points}\}$ is connected, which is not true for $\mathbb{R}$. In fact

$$f = (x, y^3, x^2y + xy^2 + y^4),\ \ g = (x, y^3 - x^2y, xy^2 + y^4)$$

are examples of map-germs which satisfy $T\mathcal{A}f = T\mathcal{A}g$ but which are not $\mathcal{A}_{\mathbb{R}}$-equivalent (though they are $\mathcal{A}_{\mathbb{C}}$-equivalent) : see [Mond 1], p. 360.

The second criterion comes from the theory of <u>unfoldings.</u>

(I·1:29) Definition : A <u>$\mathbb{C}^s$-level preserving map-germ</u> is a map-germ

$$F:(\mathbb{C}^n\times\mathbb{C}^s,0) \to (\mathbb{C}^p\times\mathbb{C}^s,0)$$

of the form $F(x,u) = (\underline{f}(x,u),u)$. If $f = \underline{f}(x,0)$ has some special significance then F is called an <u>unfolding</u> of f.

The map $F:(\mathbb{C}^n\times\mathbb{C}^s, 0\times B) \to (\mathbb{C}^p\times\mathbb{C}^s, 0\times B)$ is <u>trivial</u> in a region $B \subseteq \mathbb{C}^s$ (where B is connected) if there exist $\mathbb{C}^s$-level preserving diffeomorphism-germs H of $(\mathbb{C}^n\times\mathbb{C}^s, 0\times B)$, K of $(\mathbb{C}^p\times\mathbb{C}^s, 0\times B)$ such that

$$K^{-1}\circ F\circ H = f\times 1_{(\mathbb{C}^s,B)}$$

where f is some map-germ $f:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$.

The map $F:(\mathbb{C}^n\times\mathbb{C}^s, 0\times B) \to (\mathbb{C}^p\times\mathbb{C}^s, 0\times B)$ is <u>k-trivial</u> if there exist $\mathbb{C}^s$-level preserving diffeomorphism-germs H of $(\mathbb{C}^n\times\mathbb{C}^s, 0\times B)$, K of $(\mathbb{C}^p\times\mathbb{C}^s, 0\times B)$ such that

$$j^k_x(K^{-1}\circ F\circ H) = j^k(f \times 1_{(\mathbb{C}^s,B)}),$$

where f is some map-germ $f:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$. Here $j^k_x F$ is the truncated Taylor series <u>with respect to x only</u>. There may be higher degree terms involving u.

Rather than $\theta(F)$, it is convenient to consider only those vector fields along F which have zero $\mathbb{C}^s$-component in $T\mathbb{C}^p \oplus \mathbb{C}^s \equiv T(\mathbb{C}^p \times \mathbb{C}^s)$. The set of all such vector fields is written $\psi(F)$; it is a free $\mathcal{O}_{n+s}$-module of rank p. The corresponding sets of vector fields when F is the identity map $(\mathbb{C}^n \times \mathbb{C}^s, 0 \times B) \to (\mathbb{C}^n \times \mathbb{C}^s, 0 \times B)$ or $(\mathbb{C}^p \times \mathbb{C}^s, 0 \times B) \to (\mathbb{C}^p \times \mathbb{C}^s, 0 \times B)$ will be denoted by $\psi(n+s)$, $\psi(p+s)$ respectively.

There is a well-known criterion for triviality:

(I·1:30) Proposition (Thom-Levine): A $\mathbb{C}$-level preserving map-germ $F:(\mathbb{C}^n \times \mathbb{C}, 0 \times [0,1]) \to (\mathbb{C}^p \times \mathbb{C}, 0 \times [0,1])$ is trivial if and only if

$$\partial\cdot F^a \in tF^a(\mathfrak{m}_n\psi(n+1)) + \omega F^a(\mathfrak{m}_p\psi(p+1)), \forall a \in [0,1]$$

(where F^a is the germ-restriction of F).

Proof: see [du P], p 12.

This suggests an obvious analogue for k-triviality:

(I·1:31) Proposition: A $\mathbb{C}$-level preserving map-germ $F:(\mathbb{C}^n \times \mathbb{C}, 0 \times [0,1]) \to (\mathbb{C}^p \times \mathbb{C}, 0 \times [0,1])$ is k-trivial if and only if

$$\partial\cdot F^a \in tF^a(\mathfrak{m}_n\psi(n+1)) + \omega F^a(\mathfrak{m}_p\psi(p+1)) + \mathfrak{m}_n^{k+1}\psi(F), \ \forall a \in [0,1]$$

(where F^a is the germ-restriction of F).

Proof: (The proof of I·1:30 given in [du P] omits all references to $\mathfrak{m}_n^{k+1}\psi(F)$).

F is k-trivial

$\Leftrightarrow \exists$ $\mathbb{C}$-level preserving diffeomorphisms H, K such that $j^k_x(K^{-1}\circ F\circ H) = j^k(f \times 1_{\mathbb{C}})$,

$$\Leftrightarrow \partial\cdot(K^{-1}\circ F\circ H)^a \in \mathfrak{m}_n^{k+1}\psi(F), \quad \forall a \in [0,1]$$

(for this says that $j_x^k(K^{-1}\circ F\circ H)$ is independent of $t \in [0,1]$).

Differentiation gives that this is equivalent to

$$(*) \qquad \partial\cdot F^a + tF^a(\partial\cdot H^a\circ(H^a)^{-1}) + \omega F^a(\partial\cdot K^a\circ(K^a)^{-1}) \in \mathfrak{m}_n^{k+1}\psi(F), \quad \forall a \in [0,1].$$

Observing that $\partial\cdot H^a\circ(H^a)^{-1}$ and $\partial\cdot K^a\circ(K^a)^{-1}$ vanish on $0\times(\mathbb{C},a)$ gives that $\partial\cdot H^a\circ(H^a)^{-1} \in \mathfrak{m}_n\psi(n+1)$ and $\partial\cdot K^a\circ(K^a)^{-1} \in \mathfrak{m}_p\psi(p+1)$ and so proves $\Rightarrow$.

For the 'if' part:

for each $a \in [0,1]$ there exist, by hypothesis, $\xi^a \in \mathfrak{m}_n\psi(n+1), \eta^a \in \mathfrak{m}_p\psi(p+1)$,

$V^a \in \mathfrak{m}_n^{k+1}\psi(F)$ such that $\partial F^a = tF^a(\xi^a) + \omega F^a(\eta^a) + V^a$.

Representatives $\xi^{U_a}, \eta^{U_a}, V^{U_a}$ of ξ^a, η^a, V^a may be chosen to give germs of vector fields on $(\mathbb{C}^n\times\mathbb{C}, 0\times U_a)$ and $(\mathbb{C}^p\times\mathbb{C}, 0\times U_a)$, where U_a is an open neighbourhood of a in $\mathbb{C}$, and

$$\partial\cdot F^t = tF^t((\xi^{U_a})^t) + \omega F^t((\eta^{U_a})^t) + (V^{U_a})^t, \; t \in U_a$$

with $(\xi^{U_a})^t \in \mathfrak{m}_n\psi(n+1)$, $(\eta^{U_a})^t \in \mathfrak{m}_p\psi(p+1)$ and $(V^{U_a})^t \in \mathfrak{m}_n^{k+1}\psi(F)$.

Let $U_1,\ldots,U_r$ be a finite subcover of the open cover $\{U_a : a \in [0,1]\}$ of $[0,1]$ and let

$\varphi_1,\ldots,\varphi_r$ be a subordinate partition of unity. Let $\xi = \sum_{i=1}^{r}\varphi_i\xi^{U_i}$, $\eta = \sum_{i=1}^{r}\varphi_i\eta^{U_i}$

$V = \sum_{i=1}^{r}\varphi_i V^{U_i}$. Clearly $\partial\cdot F^a = tF^a(\xi^a) + \omega F^a(\eta^a) + (V^a)$.

A variant on the fundamental existance theorem for solutions of ordinary differential equations states that each germ of a $\mathbb{C}$-level preserving vector field on

$(\mathbb{C}^n\times\mathbb{C}, 0\times[0,1])$ which vanishes on $0\times[0,1]$ may be expressed as $\frac{d}{dt}H\circ H^{-1}$, where H is a $\mathbb{C}$-level preserving diffeomorphism of $(\mathbb{C}^n\times\mathbb{C}, 0\times[0,1])$. Choosing H, K to give

representatives of ξ, η, respectively, gives a relation of the form $(*)$ so that F is k-trivial. $\square$

I·2 : Classification Theory.

The aim is, for large integers k and a fixed integer κ, to find a normal form for $j^{k+\kappa}g$ when $j^k f = j^k g$. We also prove a condition for a normal forms to be $(k+\kappa)$-determined. The key to finding these is that if we compare $T\mathscr{A}^{k+\kappa}f$ and $T\mathscr{A}^{k+\kappa}g$ for $f-g \in \mathfrak{m}_n^{k+1}.\mathcal{O}_n^p$ then only a small number of terms are altered and the new terms are relatively easy to calculate. We can then construct k-trivial unfoldings of g.

If f is finite then let ρ denote the least integer satisfying $\mathfrak{m}_n^{\rho} \subseteq f^*\mathfrak{m}_p.\mathcal{O}_n$.

First we determine some properties of $f \in \mathfrak{m}_n.\mathcal{O}_n^p$.

(I·2:1) Lemma : If $f \in \mathfrak{m}_n^d.\mathcal{O}_n^p$ then

$$tf\left(\mathfrak{m}_n^R.\theta(n)\right) \subseteq \mathfrak{m}_n^{d+R-1}.\theta(f) \text{ and } \omega f\left(\mathfrak{m}_p^S.\theta(p)\right) \subseteq (\mathfrak{m}_n^d)^S.\theta(f).$$

Proof : This is obvious.

(I·2:2) Lemma : Given $\kappa \geq 1$ and $f \in \mathfrak{m}_n^d.\mathcal{O}_n^p$, if $R \geq \kappa+1$ and S satisfies $d.(S-1) \geq \kappa$ then for any $k > \kappa-1$ and $h \in \mathfrak{m}_n^{k+1}.\mathcal{O}_n^p$,

$$t(f+h)\left(\mathfrak{m}_n^R.\theta(n)\right) + \omega(f+h)\left(\mathfrak{m}_p^S.\theta(p)\right) = tf\left(\mathfrak{m}_n^R.\theta(n)\right) + \omega f\left(\mathfrak{m}_p^S.\theta(p)\right)$$

$$\text{modulo } \mathfrak{m}_n^{k+\kappa+1}.\theta(f).$$

For convenience we shall write $T\mathscr{A}_{R,S}f$ for the module $tf\left(\mathfrak{m}_n^R.\theta(n)\right) + \omega f\left(\mathfrak{m}_p^S.\theta(p)\right)$. The lemma then says that given κ and f, if R and S are as above, then for any k

and $h \in \mathfrak{m}_n^{k+1} \cdot \mathcal{O}_n^p$

$$T\mathcal{A}_{R,S}(f+h) + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f+h) = T\mathcal{A}_{R,S} f + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f).$$

Further, if f is finite and $k \geq \max(\kappa+1, \rho)$ then

$$(f+h)^* \mathfrak{m}_p \cdot \left(T\mathcal{A}_{R,S}(f+h) + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f+h)\right) = (f+h)^* \mathfrak{m}_p \cdot \left(T\mathcal{A}_{R,S} f + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)\right)$$

$$= f^* \mathfrak{m}_p \cdot \left(T\mathcal{A}_{R,S} f + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)\right).$$

Proof of Lemma : If $R \geq \kappa + 1$ then

$$th(\mathfrak{m}_n^R \cdot \theta(n)) \subseteq \mathfrak{m}_n^{k+R} \cdot \theta(h) = \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(h) = \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f).$$

If f is finite and $k \geq \rho$ then $(f+h)^* \mathfrak{m}_p \cdot \mathfrak{m}_n^R = f^* \mathfrak{m}_p \cdot \mathfrak{m}_n^R$ and

$$(f+h)^* \mathfrak{m}_p \cdot \left(tf(\mathfrak{m}_n^R \cdot \theta(n))\right) = (f+h)^* \mathfrak{m}_p \cdot \mathfrak{m}_n^R \cdot tf(\theta(n)) = f^* \mathfrak{m}_p \cdot \mathfrak{m}_n^R \cdot tf(\theta(n)).$$

For ωf, consider for example $(f+h)^S = f^S + Sf^{S-1}h + h^2 p(f, h)$;

we see that

$$\omega(f+h)(\mathfrak{m}_p^S \cdot \theta(p)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f) = \omega f(\mathfrak{m}_p^S \cdot \theta(p)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)$$

provided that $k+1 \geq \kappa$ and $\mathfrak{m}_n^{k+1} \cdot \omega f(\mathfrak{m}_p^{S-1} \cdot \theta(p)) \subseteq \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)$. If S satisfies $d.(S-1) \geq \kappa$ then

$$\omega f(\mathfrak{m}_p^{S-1} \cdot \theta(p)) \subseteq \mathfrak{m}_n^{\kappa} \cdot \theta(f).$$

To see that

$$(f+h)^* \mathfrak{m}_p \cdot \left(\omega f(\mathfrak{m}_p^S \cdot \theta(p)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)\right) = f^* \mathfrak{m}_p \cdot \left(\omega f(\mathfrak{m}_p^S \cdot \theta(p)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)\right)$$

observe that $(f+h)^* \mathfrak{m}_p \cdot \omega f(\mathfrak{m}_p^S \cdot \theta(p)) = (f+h)^* \mathfrak{m}_p \cdot f^* \mathfrak{m}_p \cdot \omega f(\mathfrak{m}_p^{S-1} \cdot \theta(p))$.

Then $(f+h)^* \mathfrak{m}_p \cdot \omega f(\mathfrak{m}_p^{S-1} \cdot \theta(p)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f) = f^* \mathfrak{m}_p \cdot \omega f(\mathfrak{m}_p^{S-1} \cdot \theta(p)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)$

by choice of S. □

(I·2:3) Corollary : If R, S are chosen as in I·2:2 then

$$T\mathscr{A}^{k+\kappa}(f+h) = T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f) + \mathbb{C}\{u_1(f+h),\ldots, u_t(f+h)\},$$

where $\{u_1(g),\ldots, u_t(g)\}$ denotes the set

$$\left\{\alpha\frac{\partial}{\partial x_j}(g),\ g^*(\beta),\ \alpha\in \mathfrak{m}_n\backslash\mathfrak{m}_n^R, \beta\in \mathfrak{m}_p\backslash\mathfrak{m}_p^S\right\}.$$

For the case $R = S = 1$ we put

$$u_1(g) = x_1\frac{\partial}{\partial x_1}(g),\ u_2(g) = x_2\frac{\partial}{\partial x_1}(g),\ldots, u_{n^2}(g) = x_n\frac{\partial}{\partial x_n}(g),$$

$$u_{n^2+1}(g) = (g)_1\cdot\partial/\partial Y_1,\ u_{n^2+2}(g) = (g)_1\cdot\partial/\partial Y_2,\ldots, u_{n^2+p^2}(g) = (g)_p\cdot\partial/\partial Y_p.$$

Then $T\mathscr{A}^{k+1}(f+h) = T\mathscr{A}_{2,2}f + \mathfrak{m}_n^{k+2}\cdot\theta(f) + \mathbb{C}\{u_i(f+h): 1\le i\le n^2+p^2\}$.

For the general case we take the obvious extension. The numbering does not matter: we only use this for ease of notation.

(I·2:4) Lemma : For any finite map-germ f and any positive integers R, S there exist (minimum) integers k_0, k_1 such that

$$T\mathscr{A}f \cap \mathfrak{m}_n^{k_0+1}\cdot\theta(f) \subseteq T\mathscr{A}_{R,S}f$$

$$\text{and } T\mathscr{A}f \cap \mathfrak{m}_n^{k_1+2}\cdot\theta(f) \subseteq f^*\mathfrak{m}_p\cdot T\mathscr{A}f.$$

Proof : Since f is finite, $\theta(f)$ is a finitely generated $\mathcal{O}_p$-module so we can apply the Artin-Rees lemma (I·1:7) to $M = \theta(f)$, $N = T\mathscr{A}f$, and $I = f^*\mathfrak{m}_p^c$. This says that for any c, there exists an m(c) such that

$$T\mathscr{A}f \cap f^*\mathfrak{m}_p^{cm(c)}\cdot\theta(f) \subseteq f^*\mathfrak{m}_p^c\cdot T\mathscr{A}f.$$

Now for any positive integers R and S, there exists a k such that

$$\mathfrak{m}_n^{k+1}\cdot\theta(f) \subseteq f^*\mathfrak{m}_p^{cm}\cdot\theta(f)$$

(since $\mathfrak{m}_n^\rho \subseteq f^*\mathfrak{m}_p\cdot\mathcal{O}_n$). Put $c = 1$ to find k_1.

There also exists a c such that

$$f^*\mathfrak{m}_p^c\cdot T\mathscr{A}f \subseteq T\mathscr{A}_{R,S}f$$

(since $f^*\mathfrak{m}_p^c\cdot\omega f\big(\mathfrak{m}_p\cdot\theta(p)\big) \subseteq \omega f\big(\mathfrak{m}_p^{c+1}\cdot\theta(p)\big)$ and $f^*\mathfrak{m}_p\cdot\mathcal{O}_n \subseteq \mathfrak{m}_n$).

Thus

$$T\mathscr{A}f \cap \mathfrak{m}_n^{k+1}\cdot\theta(f) \subseteq T\mathscr{A}_{R,S}f.$$

Then let k_0 be the least integer for which this inclusion holds. This concludes the proof of lemma I·2:4. □

We shall usually put $\kappa = 1$, when we can choose $R = S = 2$. Then there exists a k_0 such that $k \geq k_0$ implies that

$$T\mathscr{A}f \cap \mathfrak{m}_n^{k+1}\cdot\theta(f) \subseteq T\mathscr{A}_{2,2}f$$

and if $h \in \mathfrak{m}_n^{k+1}\cdot\mathcal{O}_n^p$ then

$$T\mathscr{A}^{k+1}(f+h) = T\mathscr{A}_{2,2}f + \mathfrak{m}_n^{k+2}\cdot\theta(f) + \mathbb{C}\{x_i\frac{\partial}{\partial x_j}(f+h) : 1 \leq i, j \leq n\}$$
$$+ \mathbb{C}\{(f+h)_i\cdot\partial/\partial Y_j : 1 \leq i, j \leq p\}.$$

The terms $x_i\frac{\partial}{\partial x_j}(h)$, $(h)_i\cdot\partial/\partial Y_j$ are in $\mathfrak{m}_n^{k+1}\cdot\theta(f)$. If they also lie in $T\mathscr{A}^{k+1}f$ then

$T\mathscr{A}^{k+1}(f+h) = T\mathscr{A}^{k+1}f$ and $f+h \overset{k+1}{\sim} f$. The elements of interest are those which lie in $T\mathscr{A}^k f \setminus T\mathscr{A}^{k+1}f$.

Another possible choice of κ is $\kappa = \rho$ where $\mathfrak{m}_n^\rho \subseteq f^*\mathfrak{m}_p\cdot\mathcal{O}_n$. Then

$$T\mathcal{A}^{k+\rho}(f+h) \supseteq \mathfrak{m}_n^{k+1}\cdot\theta(f) \Rightarrow T\mathcal{A}(f+h) \supseteq \mathfrak{m}_n^{k+1}\cdot\theta(f)$$

and f+h is finitely determined.

(I·2:5) Definition : Call $\{v_1,\dots, v_s\} \subseteq \mathfrak{m}_n^{k+1}\cdot\theta(f)$ a <u>transverse section for $T\mathcal{A}^k f$ over $T\mathcal{A}^{k+\kappa}f$</u> (or just a <u>transverse section</u>) if

$$T\mathcal{A}^k f = T\mathcal{A}^{k+\kappa}f + \mathbb{C}\{v_1,\dots, v_s\}$$

and none of the v_i may be omitted. Note that $\{v_1,\dots, v_s\}$ is a transverse section if and only if it is a $\mathbb{C}$-basis for $T\mathcal{A}^k f$ over $T\mathcal{A}^{k+\kappa}f$. We will usually choose a basis of monomials.

If R and S are chosen as in I·2:2 and k_0 is chosen as in I·2:4 then

(I·2:6) Theorem : Suppose that $\{v_1,\dots, v_s\} \subseteq \mathfrak{m}_n^{k+1}\cdot\theta(f)$ and that either

$$\text{a) } \mathfrak{m}_n^{k+1}\cdot\theta(f) \subseteq T\mathcal{A}_{R,S}f + \mathfrak{m}_n^{k+1+\kappa}\cdot\theta(f) + \mathbb{C}\{v_1,\dots, v_s\}$$

$$\text{or b) } f \text{ is finite, } k \geq k_0 \text{ and } T\mathcal{A}^k f = T\mathcal{A}^{k+\kappa}f + \mathbb{C}\{v_1,\dots, v_s\}.$$

Then $g \overset{k}{\sim} f \Rightarrow g$ is $(k+\kappa)$-equivalent to a map-germ of the form

$$f + \sum_{i=1}^{s} \lambda_i v_i, \text{ some } \lambda_i \in \mathbb{C}.$$

Proof : After a change of coordinates we may assume that $j^k g = j^k f$, i.e. that $g - f \in \mathfrak{m}_n^{k+1}\cdot\theta(f)$. Let $g - f = h$ and consider the germ f+h.

If a) holds then there exist $\lambda_i \in \mathbb{C}$ such that

$$h - \sum_{i=1}^{s} \lambda_i v_i \in T\mathcal{A}_{R,S}f + \mathfrak{m}_n^{k+1+\kappa}\cdot\theta(f).$$

If b) holds then there exist $\lambda_i \in \mathbb{C}$ such that

$$h - \sum_{i=1}^{s} \lambda_i v_i \in \mathfrak{m}_n^{k+1} \cdot \theta(f) \cap T\mathscr{A}^{k+\kappa} f \subseteq T\mathscr{A}_{R,S} f + \mathfrak{m}_n^{k+1+\kappa} \cdot \theta(f).$$

Our aim is to show that the family $f + \sum_{i=1}^{s} \lambda_i v_i + a(h - \sum_{i=1}^{s} \lambda_i v_i)$ is $(k+\kappa)$-trivial.

It is enough to prove

(I·2:7) Lemma: The family $F=(f+h_1+ah_2, a)$ is $(k+\kappa)$-trivial, where

$$h_1 \in \mathfrak{m}_n^{k+1} \cdot \theta(f) \text{ and } h_2 \in \mathfrak{m}_n^{k+1} \cdot \theta(f) \cap T\mathscr{A}_{R,S} f + \mathfrak{m}_n^{k+1+\kappa} \cdot \theta(f).$$

Proof of lemma: By Thom-Levine (I·1:31) it is enough to show that

$$h_2 \in tF^a(\mathfrak{m}_n \psi(n+1)) + \omega f^a(\mathfrak{m}_p \psi(p+1)) + \mathfrak{m}_n^{k+1} \cdot \psi(F), \forall\, a \in [0,1].$$

Now $h_2 \in \mathfrak{m}_n^{k+1} \cdot \theta(f) \cap T\mathscr{A}_{R,S} f + \mathfrak{m}_n^{k+1+\kappa} \cdot \theta(f)$ and

$$T\mathscr{A}_{R,S} f + \mathfrak{m}_n^{k+1+\kappa} \cdot \theta(f) = T\mathscr{A}_{R,S}(f+h_1) + \mathfrak{m}_n^{k+1+\kappa} \cdot \theta(f)$$

so there exist $\alpha_j \in \mathfrak{m}_n^R$, $\beta \in \mathfrak{m}_p^S$ and $\bar{h}_0 \in \mathfrak{m}_n^{k+1+\kappa} \cdot \theta(f)$ such that

$$h_2 = \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial x_j}(f+h_1) + (f+h_1)^*(\beta) + \bar{h}_0.$$

Then $h_2 = \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial x_j}(f+h_1+ah_2) + (f+h_1+ah_2)^*(\beta)$

$$+ \left(\bar{h}_0 - a \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial x_j} h_2 + (f+h_1)^*(\beta) - (f+h_1+ah_2)^*(\beta)\right).$$

The term on the second line is an element of $\mathfrak{m}_n^{k+1+\kappa} \cdot \theta(f)$ and the result follows. This concludes the proof of the lemma and hence of the theorem. Theorem I:1 follows by putting $\kappa = 1$. □

Note: since the proof depends on constructing a k-trivial unfolding the theorem also

is valid for $\mathcal{A}_{\mathbb{R}}$-equivalence.

(I·2:8) Example : Let $f(x,y) = (xy, x^2, y^2)$ and $\kappa = 1$ (II·5:2). Then

$$T\mathcal{A}_{2,2}f=\left\{x^{2i-j}y^{j}.\partial/\partial X, x^{2i-j}y^{j}.\partial/\partial Y, x^{2i-j}y^{j}.\partial/\partial Z : i \geq 2, 0 \leq j \leq 2i\right\}+ \mathfrak{m}_2^2\left\{\begin{pmatrix} x \\ 0 \\ 2y \end{pmatrix}, \begin{pmatrix} y \\ 2x \\ 0 \end{pmatrix}\right\}.$$

For $s \geq 1$,

$$g \in \mathcal{A}^{2s}f \Rightarrow g \in \mathcal{A}^{2s+1}(xy + \Sigma\lambda_j x^{2s+1-j}y^{j}., x^2 + \alpha y^{2s+1}, y^2 + bx^{2s+1})$$

$$\text{and } g \in \mathcal{A}^{2s+1}f \Rightarrow g \in \mathcal{A}^{2s+2}(xy, x^2, y^2).$$

In particular $g \in \mathcal{A}^2 f \Rightarrow g \in \mathcal{A}^3(xy + bx^3 + cx^2y + dxy^2 + ey^3, x^2 + fy^3, y^2 + ax^3)$.

The determinacy degree comes from

(I·2:9) Theorem (proof of I:2) : Suppose that f is finite, $k \geq \rho$, $\{v_1,..., v_s\}$ as in I·2:6 and $\{v_1,..., v_t\} \subseteq \{v_1,..., v_s\}$.

a) If i)-iii) hold then $f+\sum_{i=1}^{s}\lambda_i v_i$ is (k+1)-determined:

i) $f^*\mathfrak{m}_p\cdot\left(T\mathcal{A}f + \mathbb{C}\{v_1,..., v_t\}\right) \supseteq \mathfrak{m}_n^{k+2}\cdot\theta(f)$,

ii) $T\mathcal{A}^{k+1}(f+\sum_{i=1}^{s}\lambda_i v_i) \supseteq T\mathcal{A}f + \mathfrak{m}_n^{k+2}\cdot\theta(f) + \mathbb{C}\cdot\{v_1,..., v_t\}$,

iii) for values $\lambda_i \neq 0$, $v_i = ((v_i)_1,..., (v_i)_p)$

$$T\mathcal{A}_{2,2}f + f^*\mathcal{O}_p\cdot\{v_1,..., v_t\} + \mathfrak{m}_n^{k+2}\cdot\theta(f) \supseteq \mathbb{C}\{(v_i)_j\cdot\partial/\partial Y_\ell : 1 \leq \ell, j \leq p\}.$$

b) If $k \geq k_1$ and $T\mathcal{A}f + f^*\mathfrak{m}_p\cdot\{v_1,..., v_t\} \supseteq \mathfrak{m}_n^{k+2}\cdot\theta(f)$ then i) holds.

c) If $T\mathcal{A}_{2,2}f + f^*\mathcal{O}_p\cdot\{v_1,..., v_t\} + \mathfrak{m}_n^{k+2}\cdot\theta(f) \supseteq \mathfrak{m}_n^{k+1}\cdot\theta(f)$ then iii) holds.

Proof : We show that $T\mathcal{A}(f+\sum_{i=1}^{s}\lambda_i v_i + h) = T\mathcal{A}(f+\sum_{i=1}^{s}\lambda_i v_i)$ for all $h \in \mathfrak{m}_n^{k+2}\cdot\theta(f)$.

Then equivalence follows from I·1:28.

Hypothesis b) implies i) $f^*\mathfrak{m}_p \cdot T\mathscr{A} f + f^*\mathfrak{m}_p \cdot \{v_1, \ldots, v_t\} \supseteq \mathfrak{m}_n^{k+2} \cdot \theta(f)$ since for all

$c \in \mathfrak{m}_n^{k+2} \cdot \theta(f)$ there exist $b \in f^*\mathfrak{m}_p \cdot \{v_1, \ldots, v_t\}$, $a \in T\mathscr{A} f$ such that $c = a + b$.

Then $a = c - b \in \mathfrak{m}_n^{k+2} \cdot \theta(f) \cap T\mathscr{A} f \subseteq f^*\mathfrak{m}_p \cdot T\mathscr{A} f$.

By ii) $T\mathscr{A}(f + \sum_{i=1}^{s} \lambda_i v_i + h) + \mathfrak{m}_n^{k+2} \cdot \theta(f) \quad = T\mathscr{A}(f + \sum_{i=1}^{s} \lambda_i v_i) + \mathfrak{m}_n^{k+2} \cdot \theta(f)$

$$\supseteq T\mathscr{A} f + \mathfrak{m}_n^{k+2} \cdot \theta(f) + \mathbb{C} \cdot \{v_1, \ldots, v_t\},$$

so $(f + \sum_{i=1}^{s} \lambda_i v_i + h)^* \mathfrak{m}_p \cdot \left(T\mathscr{A}(f + \sum_{i=1}^{s} \lambda_i v_i + h) + \mathfrak{m}_n^{k+2} \cdot \theta(f)\right)$

$$\supseteq (f + \sum_{i=1}^{s} \lambda_i v_i + h)^* \mathfrak{m}_p \cdot \left(T\mathscr{A} f + \mathfrak{m}_n^{k+2} \cdot \theta(f) + \mathbb{C} \cdot \{v_1, \ldots, v_t\}\right)$$

$$= (f + \sum_{i=1}^{s} \lambda_i v_i + h)^* \mathfrak{m}_p \cdot \left(T\mathscr{A}_{2,2} f + \mathfrak{m}_n \cdot \left\{\partial f/\partial x_j : 1 \le i \le n\right\} + \left\{f_r \cdot \partial/\partial Y_s : 1 \le r, s \le p\right\}\right.$$

$$\left. + \mathfrak{m}_n^{k+2} \cdot \theta(f) + \mathbb{C} \cdot \{v_1, \ldots, v_t\}\right)$$

$$= f^*\mathfrak{m}_p \cdot T\mathscr{A}_{2,2} f + f^*\mathfrak{m}_p \cdot \mathfrak{m}_n \cdot \left\{\partial f/\partial x_i : 1 \le i \le n\right\} + f^*\mathfrak{m}_p \cdot \mathfrak{m}_n^{k+2} \cdot \theta(f) + f^*\mathfrak{m}_p \cdot \{v_1, \ldots, v_t\}$$

$$+ (f + \sum_{i=1}^{s} \lambda_i v_i + h)^* \mathfrak{m}_p \left\{f_r \cdot \partial/\partial Y_s : 1 \le r, s \le p\right\}$$

$$= f^*\mathfrak{m}_p \cdot T\mathscr{A}_{2,2} f + f^*\mathfrak{m}_p \cdot \mathfrak{m}_n \cdot \left\{\partial f/\partial x_i : 1 \le i \le n\right\} + f^*\mathfrak{m}_p \cdot \mathfrak{m}_n^{k+2} \cdot \theta(f) + f^*\mathfrak{m}_p \cdot \{v_1, \ldots, v_t\}$$

$$+ \left\{(f + \sum_{i=1}^{s} \lambda_i v_i + h)_q f_r \cdot \partial/\partial Y_s : 1 \le q, r, s \le p\right\}.$$

But $(h)_q f_r \cdot \partial/\partial Y_s \in f^*\mathfrak{m}_p \cdot \mathfrak{m}_n^{k+2} \cdot \theta(f)$,

and by iii) for $\lambda_i \neq 0$ $(v_i)_q f_r \cdot \partial/\partial Y_s \in f^*\mathfrak{m}_p \cdot T\mathscr{A}_{2,2} f + f^*\mathfrak{m}_p \cdot \{v_1, \ldots, v_t\}$,

so that $f^*\mathfrak{m}_p\cdot\omega f(\mathfrak{m}_p\cdot\theta(p)) \equiv (f+\sum_{i=1}^{s}\lambda_i v_i + h)\cdot\omega f(\mathfrak{m}_p\cdot\theta(p))$

$$\text{modulo } f^*\mathfrak{m}_p\cdot T\mathcal{A}_{2,2}f + f^*\mathfrak{m}_p\cdot\{v_1,\dots,v_t\} + f^*\mathfrak{m}_p\cdot\mathfrak{m}_n^{k+2}\cdot\theta(f)$$

and

$$(f+\sum_{i=1}^{s}\lambda_i v_i + h)^*\mathfrak{m}_p\cdot\left(T\mathcal{A}f + \mathfrak{m}_n^{k+2}\cdot\theta(f) + \mathbb{C}\{v_1,\dots,v_t\}\right) =$$

$$f^*\mathfrak{m}_p\cdot\left(T\mathcal{A}f + \mathfrak{m}_n^{k+2}\cdot\theta(f) + \mathbb{C}\cdot\{v_1,\dots,v_t\}\right).$$

By i) $f^*\mathfrak{m}_p\cdot\left(T\mathcal{A}f + \mathbb{C}\cdot\{v_1,\dots,v_t\}\right) \supseteq \mathfrak{m}_n^{k+2}\cdot\theta(f)$ so

$$T\mathcal{A}(f+\sum_{i=1}^{s}\lambda_i v_i + h) + f^*\mathfrak{m}_p\cdot\mathfrak{m}_n^{k+2}\cdot\theta(f) \supseteq \mathfrak{m}_n^{k+2}\cdot\theta(f)$$

and hence $T\mathcal{A}(f+\sum_{i=1}^{s}\lambda_i v_i + h) \supseteq \mathfrak{m}_n^{k+2}\cdot\theta(f)$ and $T\mathcal{A}(f+\sum_{i=1}^{s}\lambda_i v_i + h) = T\mathcal{A}(f+\sum_{i=1}^{s}\lambda_i v_i)$.

This concludes the proof. □.

Note: since $T\mathcal{A}(f+\sum_{i=1}^{s}\lambda_i v_i) \supseteq \mathfrak{m}_n^{k+2}\cdot\theta(f)$, $f+\sum_{i=1}^{s}\lambda_i v_i$ is finitely determined and since $T\mathcal{A}(f+\sum_{i=1}^{s}\lambda_i v_i + h) = T\mathcal{A}(f+\sum_{i=1}^{s}\lambda_i v_i)$, we can apply Mather's lemma to each connected submanifold $\{f+\sum_{i=1}^{s}\lambda_i v_i + ah : a \in \mathbb{R}\}$, that is $f+\sum_{i=1}^{s}\lambda_i v_i$ also has determinacy degree $k+\kappa$ with respect to $\mathcal{A}_{\mathbb{R}}$-equivalence.

For the rest of this section we calculate the new tangent space with the aim of using k-triviality to reduce the number of parameters.

Recall that

$$T\mathscr{A}^{k+\kappa}(f+\sum_{i=1}^{s}\lambda_i v_i) = T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f) + \mathbb{C}\left\{u_1(f+\sum_{i=1}^{s}\lambda_i v_i),\ldots,u_t(f+\sum_{i=1}^{s}\lambda_i v_i)\right\}$$

where the u_i are of the form given in I·2:3.

(I·2:10) Definition : Let $\underline{\lambda} = (\lambda_1,\ldots,\lambda_s)$. Define $d(\underline{\lambda})$ and $n(\underline{\lambda})$ by

$$d(\underline{\lambda}) = \dim_{\mathbb{C}} T\mathscr{A}^{(k+\kappa)}(f+q),$$

$$= t - n(\underline{\lambda}) + \dim_{\mathbb{C}}\left(T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)\right)/\mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)$$

where $n(\underline{\lambda})$ is the number of relations of the form

$$\sum_{m=1}^{t}\mu_m u_m(f+\sum_{i=1}^{s}\lambda_i v_i) \in T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f),\ \mu_i \in \mathbb{C}.$$

(I·2:11) Lemma : If $\mu_m \in \mathbb{C}$ satisfy

$$\sum_{m=1}^{t}\mu_m u_m(f+\sum_{i=1}^{s}\lambda_i v_i) \in T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)$$

then
$$\sum_{m=1}^{t}\mu_m u_m(f) \in T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f).$$

Corollary : $n(\underline{\lambda}) \le n(0)$ and $d(\underline{\lambda}) \ge d(0)$.

Proof : If

$$\sum_{m=1}^{t}\mu_m u_m(f+\sum_{i=1}^{s}\lambda_i v_i) \in T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)$$

then

$$\sum_{m=1}^{t}\mu_m u_m(f+\sum_{i=1}^{s}\lambda_i v_i) \in T\mathscr{A}^{k+\kappa}f,$$

so

$$\sum_{m=1}^{t}\mu_m u_m(f+\sum_{i=1}^{s}\lambda_i v_i) - \sum_{m=1}^{t}\mu_m u_m(f) \in T\mathscr{A}^{k+\kappa}f.$$

But

$$\sum_{m=1}^{t} \mu_m u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - \sum_{m=1}^{t} \mu_m u_m(f) \in \mathfrak{m}_n^{k+1} \cdot \theta(f) \cap T\mathcal{A}^{k+\kappa} f,$$

so

$$\sum_{m=1}^{t} \mu_m u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - \sum_{m=1}^{t} \mu_m u_m(f) \in T\mathcal{A}_{R,S} f + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f).$$

Hence

$$\sum_{m=1}^{t} \mu_m u_m(f) \in T\mathcal{A}_{R,S} f + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)$$

as required. □

Note: $\sum_{m=1}^{t} \mu_m u_m(f) \in T\mathcal{A}_{R,S} f + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f) \subseteq T\mathcal{A}^{k+\kappa}(f + \sum_{i=1}^{s} \lambda_i v_i)$ implies that

$$\sum_{m=1}^{t} \mu_m u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - \sum_{m=1}^{t} \mu_m u_m(f) \in T\mathcal{A}^{k+\kappa}(f + \sum_{i=1}^{s} \lambda_i v_i).$$

There are $n(\underline{0})$ such relations.

(I·2:12) Lemma : Suppose that $k+1 \geq \kappa$. Then for each m there exist $a_{ij}^m \in \mathbb{C}$, $\alpha_{ij}^m \in \mathfrak{m}_n^R$, $\beta_i \in \mathfrak{m}_p^S$ such that

$$u_m(f + \sum_{i=1}^{s} \lambda_i v_i) - u_m(f) = \sum_{i,j=1}^{s} \lambda_i a_{ij}^m(h) v_j + \sum_{i=1}^{s} \lambda_i \Big(\sum_{j=1}^{n} \alpha_{ij}^m \frac{\partial}{\partial x_j}(f + \sum_{i=1}^{s} \lambda_i v_i) + (f + \sum_{i=1}^{s} \lambda_i v_i)^*(\beta) \Big)$$

$$+ \sum_{i=1}^{s} \lambda_i \Big(\bar{h}_i - \sum_{j=1}^{n} \alpha_{ij}^m \frac{\partial}{\partial x_j} \sum_{i=1}^{s} \lambda_i v_i + f^*(\beta) - (f + \sum_{i=1}^{s} \lambda_i v_i)^*(\beta) \Big).$$

The term on the second line is an element of $\mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)$.

Proof: If $u_m \in tf(\mathfrak{m}_n \cdot \theta(n))$ then $u_m(f + \sum_{i=1}^{s} \lambda_i v_i)$ is of the form

$$\alpha\frac{\partial}{\partial x_j}(f+\sum_{i=1}^{s}\lambda_i v_i),\ \alpha\in \mathfrak{m}_n,$$

so $u_m(f+\sum_{i=1}^{s}\lambda_i v_i)-u_m(f)=\alpha\frac{\partial}{\partial x_j}\sum_{i=1}^{s}\lambda_i v_i=\sum_{i=1}^{s}\lambda_i\alpha\frac{\partial}{\partial x_j}v_i$ and $\alpha\frac{\partial}{\partial x_j}v_i\in \mathfrak{m}_n^{k+1}\cdot\theta(f)$.

If $u_m\in\omega f(\mathfrak{m}_p\cdot\theta(p))$ then $u_m(f+\sum_{i=1}^{s}\lambda_i v_i)-u_m(f)$ is of the form

$$\prod_{j=1}^{p}(f+\sum_{i=1}^{s}\lambda_i v_i)_j^{r_j}\cdot\partial/\partial Y_\ell-\prod_{j=1}^{p}(f)_j^{r_j}\cdot\partial/\partial Y_\ell.$$

Expand the polynomial, bearing in mind that $\mathfrak{m}_n^{2k+2}\cdot\theta(f)\subseteq\mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)$;

we get a polynomial of the form

$$\sum_{i=1}^{s}\lambda_i\sum_{j=1}^{p}\beta_j(f)(v_i)_j\cdot\partial/\partial Y_\ell,\ \text{where each } \beta_j(f)(v_t)_j\in\mathfrak{m}_n^{k+1}\cdot\theta(f).$$

For example

$$(f+\sum_{i=1}^{s}\lambda_i v_i)_j(f+\sum_{i=1}^{s}\lambda_i v_i)_{j'}-(f)_j(f)_{j'}=(f)_j(\sum_{i=1}^{s}\lambda_i v_i)_{j'}+(f)_{j'}(\sum_{i=1}^{s}\lambda_i v_i)_j+(\sum_{i=1}^{s}\lambda_i v_i)_j(\sum_{i=1}^{s}\lambda_i v_i)_{j'}$$

$$\text{and}\quad(\sum_{i=1}^{s}\lambda_i v_i)_j(\sum_{i=1}^{s}\lambda_i v_i)_{j'}\in\mathfrak{m}_n^{2k+2}\cdot\theta(f).$$

The sum is $\sum_{i=1}^{s}\lambda_i[(f)_j(v_i)_{j'}+(f)_{j'}(v_i)_j]$.

For each element $\chi_t\in\mathfrak{m}_n^{k+1}\cdot\theta(f)$ there exist $c_{i,t}\in\mathbb{C}$ such that

$$\chi_t-\sum_{i=1}^{s}c_{i,t}v_i\in\mathfrak{m}_n^{k+1}\cdot\theta(f)\cap T\mathscr{A}^{k+\kappa}f\subseteq T\mathscr{A}_{R,S}f$$

so for each χ_t there exist $\alpha_{i,j}^{t}\in\mathfrak{m}_n^{R}$, $\beta_t\in\mathfrak{m}_p^{S}$ such that

$$\chi_t=\sum_{i=1}^{s}c_{i,t}v_i+\sum_{i=1}^{s}\lambda_i\Big(\sum_{j=1}^{n}\alpha_{i,j}^{t}\frac{\partial}{\partial x_j}f+f^*(\beta_t)\Big)$$

$$= \sum_{i=1}^{s} c_{i,j}(h)v_j + \sum_{i=1}^{s}\lambda_i\Big(\sum_{j=1}^{n} \alpha_{i,j}^{t}\frac{\partial}{\partial x_j}(f+\sum_{i=1}^{s}\lambda_i v_i) + (f+\sum_{i=1}^{s}\lambda_i v_i)^*(\beta_t)\Big)$$

$$- \sum_{i=1}^{s}\lambda_i\Big(\sum_{j=1}^{n} \alpha_{i,j}^{t}\frac{\partial}{\partial x_j}\sum_{i=1}^{s}\lambda_i v_i + (f+\sum_{i=1}^{s}\lambda_i v_i)^*(\beta_t) - f^*(\beta_t)\Big).$$

The result follows by substitution and collecting terms. □.

(I·2:13) Theorem : If $k+1 \geq \kappa$ we can write

$$T\mathscr{A}^{k+\kappa}(f+\sum_{i=1}^{s}\lambda_i v_i) = T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)$$

$$+ \mathbb{C}\Big\{u_1(f) + \sum_{i,j=1}^{s}\lambda_i a_{i,j}^{1} v_j, \ldots, u_{t'}(f) + \sum_{i,j=1}^{s}\lambda_i a_{i,j}^{t'} v_j\Big\} + \mathbb{C}\{\sum_{i,j=1}^{s}\lambda_i b_{i,j}^{c} v_j : \ 1 \leq c \leq d\},$$

where $d = d(\underline{\lambda}) - d(\underline{0})$, $a_{i,j}^{1}, \ldots, a_{i,j}^{t'}, b_{i,j}^{c} \in \mathbb{C}$ for all i, j, c and $\{u_1(f), \ldots, u_{t'}(f)\}$ is linearly independent over $T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)$.

Then $d(\underline{\lambda}) = \dim_{\mathbb{C}}[T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\theta(f)]/\mathfrak{m}_n^{k+\kappa+1}\theta(f) + t' + d$.

Proof : Recall that

$$T\mathscr{A}^{k+\kappa}(f+\sum_{i=1}^{s}\lambda_i v_i) = T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f) + \mathbb{C}\Big\{u_1(f+\sum_{i=1}^{s}\lambda_i v_i), \ldots, u_t(f+\sum_{i=1}^{s}\lambda_i v_i)\Big\}.$$

By (2·11) for each m,

$$u_m(f+\sum_{i=1}^{s}\lambda_i v_i) = u_m(f) + \sum_{i,j=1}^{s}\lambda_i a_{i,j}^{m} v_j. \text{ modulo } T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)$$

If $\sum_{m=1}^{t}\mu_m^c u_m(f) \in T\mathscr{A}_{R,S}f + \mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)$ write $b_{i,j}^{c} = \sum_{m=1}^{t}\mu_m^c a_{i,j}^{m}$ so that

$$\sum_{m=1}^{t}\mu_m^c \sum_{i,j=1}^{s}\lambda_i a_{i,j}^{m} v_j, = \sum_{i,j=1}^{s}\lambda_i b_{i,j}^{c} v_j.$$ □.

(I·2:14) Example : Let

$$f(x, y) = (x^2 + fy^3, y^2 + ax^3, xy + bx^3 + cx^2y + dxy^2 + ey^3).$$

Write $p(x,y) = x^2 + fy^3$, $q(x,y) = y^2 + ax^3$, $r(x,y) = xy + bx^3 + cx^2y + dxy^2 + ey^3$.

Then $T\mathscr{A}^3 f = \mathbb{C}\{p\cdot\partial/\partial X, q\cdot\partial/\partial Y, r\cdot\partial/\partial Z\} + \mathbb{C}\{(2x^3, 0, x^2y), (2x^2y, 0, xy^2),$

$(2xy^2, 0, y^3), (0, 2x^2y, x^3), (0, 2xy^2, x^2y), (0, 2y^3, xy^2),$

$(-2fy^3, 3ax^3, 2bx^3 + cx^2y - ey^3),$

$(3fy^3, -2ax^3, -bx^3 + dxy^2 + 2ey^3),$

$(-4ey^3, 0, -5ax^3 + 8bx^2y + 6cxy^2 + 4dy^3),$

$(0, -4bx^3, 4cx^3 + 6dx^2y + 8exy^2 - 5fy^3)\}.$

Here $d(f + q) = 15 + \operatorname{rank} M(\lambda)$, where $M(\lambda)$ is the matrix whose rows are the last 4 elements and $\lambda = (a, b, c, d, e, f)$.

We may not always have $d(\underline{\lambda}) > d(0)$ but for a large class of map-germs the strict inequality always holds. These are weighted homogeneous map-germs:

(I·2:15) Definition : Suppose $(w_1,\ldots,w_n)$ is an n-tuple of positive integers.

If $f:(\mathbb{C}^n,0) \to \mathbb{C}$ is a (finite) sum of monomials $x_1^{i_1}\ldots x_n^{i_n}$ where $w_1 i_1 + \ldots + w_n i_n = d$ then f is <u>weighted homogeneous</u> $\big($of degree d with respect to the weights $(w_1,\ldots,w_n)\big)$.

If $f = (f_1,\ldots, f_p) : (\mathbb{C}^n,0) \to \mathbb{C}^p$ then f is <u>weighted homogeneous</u> with respect to weights $(w_1,\ldots,w_n)$ if each coordinate function f_i is weighted homogeneous with respect to these weights. The coordinate functions need not have the same degrees.

We give the well-known result :

(I·2:16) Lemma : Suppose f is weighted homogeneous with respect to weights $(w_1,\ldots,w_n)$. If d_i is the degree of f_i then

$$w_1x_1\frac{\partial}{\partial x_1}f + \ldots + w_nx_n\frac{\partial}{\partial x_n}f = d_1f_1\partial/\partial Y_1 + \ldots + d_pf_p\partial/\partial Y_p.$$

Proof: The result follows from the observation that

$$w_1x_1\frac{\partial}{\partial x_1}x_1^{i_1}\dots x_n^{i_n}+\dots+w_nx_n\frac{\partial}{\partial x_n}x_1^{i_1}\dots x_n^{i_n}=(w_1i_1+\dots+w_pi_p)x_1^{i_1}\dots x_n^{i_n}=dx_1^{i_1}\dots x_n^{i_n}$$

if $x_1^{i_1}\dots x_n^{i_n}$ is of degree d w.r.t. $(w_1,\dots,w_n)$.

(I·2:17) Proposition : Suppose f is as above, $\{v_1,\dots,v_s\}$ is a transverse section for $T\mathscr{A}^k f$ over $T\mathscr{A}^{k+\kappa}f$ and each $v_i=\nu_i\partial/\partial Y_j$ is a monomial not of degree d_j.

Then $d(f+\sum_{i=1}^{s}\lambda_iv_i)>d(f)$.

Proof: Let $f_\lambda=f+\sum_{i=1}^{s}\lambda_iv_i$. Then clearly

$$T\mathscr{A}^kf_\lambda\ni w_1x_1\frac{\partial}{\partial x_1}f_\lambda+\dots+w_nx_n\frac{\partial}{\partial x_n}f_\lambda-\left(d_1(f_\lambda)_1\partial/\partial Y_1+\dots+d_p(f_\lambda)_p\partial/\partial Y_p\right).$$

By (I·2:16) $w_1x_1\frac{\partial}{\partial x_1}f+\dots+w_nx_n\frac{\partial}{\partial x_n}f-\left(d_1f_1\partial/\partial Y_1+\dots+d_pf_p\partial/\partial Y_p\right)=0.$

Hence

$$T\mathscr{A}^kf_\lambda\ni w_1x_1\frac{\partial}{\partial x_1}(\sum_{i=1}^{s}\lambda_iv_i)+\dots+w_nx_n\frac{\partial}{\partial x_n}(\sum_{i=1}^{s}\lambda_iv_i)$$

$$-\left(d_1(\sum_{i=1}^{s}\lambda_iv_i)_1\partial/\partial Y_1+\dots+d_p(\sum_{i=1}^{s}\lambda_iv_i)_p\partial/\partial Y_p\right)$$

$$=\sum_{i=1}^{s}\lambda_i\left(w_1x_1\frac{\partial}{\partial x_1}(v_i)+\dots+w_nx_n\frac{\partial}{\partial x_n}(v_i)-\left(d_1(v_i)_1\partial/\partial Y_1+\dots+d_p(v_i)_p\partial/\partial Y_p\right)\right).$$

If $v_i=x_1^{i_{1,i}}\dots x_n^{i_{n,i}}\partial/\partial Y_{j_i}$ then the sum is $\sum_{i=1}^{s}\lambda_i\left(w_1i_{1,i}+\dots+w_ni_{n,i}-d_{j_i}\right)v_i$.

Since the v_i are linearly independent over $T\mathscr{A}^kf$, this is not an element of

$$T\mathscr{A}_{R,S}f+\mathfrak{m}_n^{k+\kappa+1}\theta(f)\subseteq T\mathscr{A}f+\mathfrak{m}_n^{k+\kappa+1}\cdot\theta(f)$$

except when $\lambda_i\left(w_1 i_{1,i} + \dots + w_n i_{n,i} - d_{j_i}\right) = 0$ for each i.
But $\left(w_1 i_{1,i} + \dots + w_n i_{n,i} - d_{j_i}\right) \neq 0$ for each i so this holds only when $\lambda_i = 0$ for all i.□.

(I·2:18) Proof of I:3 : if s= 1, f is weighted homogeneous and $v = v_1$ is as in I·1:17 then

$$T\mathcal{A}^{k+1}(f+\lambda v) = T\mathcal{A}^{k+1}f + \mathbb{C}\{\lambda v\}.$$

The submanifold $\{f+\lambda v : \lambda \in \mathbb{C}\backslash\{0\}\}$ lies in a single $\mathcal{A}^k$-orbit.
Proof : by (2·12) and (2·16)

$$T\mathcal{A}^{k+1}(f+\lambda v) = T\mathcal{A}_{2,2}f + \mathfrak{m}_n^{k+2}\cdot\theta(f) + \mathbb{C}\left\{u_1(f)+\lambda a_1 v,\dots,u_{t'}(f)+\lambda a_{t'} v, \lambda a v\right\}.$$

where $a \neq 0$.
It follows that $T\mathcal{A}^{k+1}(f+\lambda v) = T\mathcal{A}^{k+1}f + \mathbb{C}\{\lambda v\}$.
The submanifold $\{f+\lambda v : \lambda \in \mathbb{C}\backslash\{0\}\}$ satisfies the conditions of Mather's lemma so lies in a single orbit so without loss of generality, $\lambda = 1$. □

Note: for $\mathcal{A}_{\mathbb{R}}$-equivalence there are at most two orbits and without loss of generality $\lambda = \pm 1$.

More generally, if $d(f+\sum_{i=1}^{s}\lambda_i v_i) > d(f)$ then we can construct a k-trivial unfolding of the map-germ $f+\sum_{i=1}^{s}\lambda_i v_i$.

(I·2:19) Theorem : let $\lambda = (\lambda_1,\dots,\lambda_s)$ and $\sum_{i,j=1}^{s}\lambda_i b_{i,j}^c v_j \in T\mathcal{A}^{k+\kappa}(f+\sum_{i=1}^{s}\lambda_i v_i)$.

a) If $F_\lambda(\underline{x},t) = \left(f(\underline{x}) + \sum_{i=1}^{s}\varphi_i(t)v_i(\underline{x}), t\right)$ is an unfolding satisfying

$$\partial_t F^a = \sum_{i,j=1}^{s}\varphi_i(a)b_{i,j}^c v_j \text{ and } \varphi_i(0) = \lambda_i$$

then F_λ is $k+\kappa$-trivial,

b) there exists an unfolding satisfying the above condition, which is hence $(k+\kappa)$-trivial.

Proof:

We first show that if F is an unfolding with $\partial_t F^a = \sum_{i,j=1}^{s} \varphi_i(a) b^c_{i,j} v_j$ then it is k-trivial.

By I·1:31 this follows if

$$\sum_{i,j=1}^{s} \varphi_i(a) b^c_{i,j} v_j \in tF^a(\mathfrak{m}_n \cdot \psi(n+1)) + \omega F^a(\mathfrak{m}_p \cdot \psi(p+1)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \varphi(F).$$

Recall that in I·2:13, there exist μ^c_m such that $\sum_{m=1}^{t} \mu^c_m u_m(f) \in T\mathcal{A}_{R,S} f + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)$.

Now for each m (I·2:12) there exist $a^m_{i,j} \in \mathbb{C}$, $\alpha^m_{i,j} \in \mathfrak{m}_n^R$, $\beta_m \in \mathfrak{m}_p^S$ such that

$$u_m(f + \sum_{i=1}^{s} \varphi_i(a) v_i) - u_m(f) = \sum_{i,j=1}^{s} \lambda_i a^m_{i,j} v_j +$$

$$\sum_{i=1}^{s} \lambda_i \Big(\sum_{j=1}^{n} \alpha^m_{i,j} \frac{\partial}{\partial x_j} (f + \sum_{i=1}^{s} \lambda_i v_i) + (f + \sum_{i=1}^{s} \varphi_i(a) v_i)^*(\beta_m) \Big)$$

$$+ \sum_{i=1}^{s} \lambda_i \Big(\bar{h}_i - \sum_{j=1}^{n} \alpha^m_{i,j} \frac{\partial}{\partial x_j} \sum_{i=1}^{s} \lambda_i v_i + \Big(f^*(\beta_m) - (f + \sum_{i=1}^{s} \varphi_i(a) v_i)^*(\beta_m) \Big) \Big).$$

It follows that

$$\sum_{i,j=1}^{s} \lambda_i a^m_{i,j} v_j + u_m(f) \in tF^a(\mathfrak{m}_n \cdot \psi(n+1)) + \omega F^a(\mathfrak{m}_p \cdot \psi(p+1)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \varphi(F)$$

We use the same technique as in I·2:12 to show that

$$\sum_{m=1}^{t} \mu^c_m u_m(f) \in tF^a(\mathfrak{m}_n \cdot \psi(n+1)) + \omega F^a(\mathfrak{m}_p \cdot \psi(p+1)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \varphi(F)$$

so that $\sum_{m=1}^{t} \mu^c_m \sum_{i,j=1}^{s} \lambda_i a^m_{i,j} v_j \in tF^a(\mathfrak{m}_n \cdot \psi(n+1)) + \omega F^a(\mathfrak{m}_p \cdot \psi(p+1)) + \mathfrak{m}_n^{k+\kappa+1} \cdot \varphi(F)$

but $\sum_{m=1}^{t} \mu_m^c \sum_{i,j=1}^{s} \lambda_i a_{i,j}^m v_j = \sum_{i,j=1}^{s} \lambda_i b_{i,j}^c v_j$ and k-triviality follows.

It remains to show that such a deformation exists.

We can perform a change of basis so that $B(c) = (b_{i,j}^c)$ is transformed to a matrix $A(c)$ which is in Jordan Normal Form,

i.e. $A(c)=\begin{bmatrix} A_1 & 0 & . & 0 \\ 0 & A_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & A_r \end{bmatrix}$ where each matrix A_i is of the form $\begin{bmatrix} \varepsilon_i & 1 & 0 & . & 0 & 0 \\ 0 & \varepsilon_i & 1 & . & 0 & 0 \\ 0 & 0 & \varepsilon_i & . & 0 & 0 \\ . & . & . & \ddots & . & . \\ 0 & 0 & 0 & . & \varepsilon_i & 1 \\ 0 & 0 & 0 & . & 0 & \varepsilon_i \end{bmatrix}$

and ε_i is an eigenvalue of $A(c)$.

(I·2:20) Lemma: If $A(c)$ is in Jordan Normal Form at $\underline{\lambda}$ then the vector field $\sum_{i,j=1}^{s} \lambda_i b_{i,j}^c v_j$ has integral curves $f_t(\lambda) = f + \sum_{i=1}^{s} \varphi_i(t) v_i$, where $\mu_j : \mathbb{C} \to \mathbb{C}$ is of the form $e^{\varepsilon t} p_j(\lambda,t)$, ε_i are the eigenvalues of A and $p(\lambda,t)$ are polynomials in t satisfying $p_j(\lambda,0) = \lambda_j$.

Proof: Suppose that A is the first block, being $j \times j$ with eigenvalue ε.

We get a system of equations

$$\frac{d}{dt}\varphi_i(t) = \varepsilon\varphi_i(t) + \varphi_{i+1}(t),\ 1 \le i \le j-1,\ \frac{d}{dt}\varphi_j(t) = \varepsilon\varphi_j(t).$$

The equation $\frac{d}{dt}\varphi_j(t) = \varepsilon\varphi_j(t)$, $\varphi_j(0) = \lambda_j$ has solution $\varphi_j(t) = \lambda_j e^{\varepsilon t}$.

If $\varphi_{i+1}(t) = e^{\varepsilon t}\left(\frac{\lambda_j}{(m-1)!} t^{m-1} + \frac{\lambda_{j-1}}{(m-2)!} t^{m-2} + \ldots + \lambda_{i+1}\right)$ then the equation

$$\frac{d}{dt}\varphi_i(t) = \varepsilon\varphi_i(t) + \varphi_{i+1}(t), \quad \varphi_i(0) = \lambda_i$$

has solution

$$\varphi_i(t) = e^{\varepsilon t}\left(\frac{\lambda_j}{m!}t^m + \frac{\lambda_{j-1}}{(m-1)!}t^{m-1} + \ldots + \lambda_{i+1}t + \lambda_i\right), \text{ where } m = j - i.$$

A similar solution exists for each remaining block. □.

Definition: the submanifold $S \subseteq \mathbb{C}^s$ is a <u>transversal submanifold</u> if for every λ there is a k+κ-trivial unfolding $F_\lambda(\underline{x}, t) = \left(f(\underline{x}) + \sum_{i=1}^{s} \varphi_i(t)v_i(\underline{x}), t\right)$ and a point t_λ such that $(\varphi_1(t_\lambda),\ldots, \varphi_s(t_\lambda)) \in S$. Then every map-germ of the form $f + \sum_{i=1}^{s} \lambda_i v_i : \lambda \in \mathbb{C}^s$ is k+κ-equivalent to one of the form $f + \sum_{i=1}^{s} \lambda_i v_i : \lambda \in S$.

We can always find a transversal submanifold:
consider $\varphi_j(t) = e^{\varepsilon_i t}p_j(t)$ from the unfolding of the case of B in Jordan Normal form;

a) if $p_j \neq \lambda_j$ then a solution $p_j(t) = 0$ exists and every unfolding intersects $\varphi_j = 0$,

b) if $p_j = \lambda_j \neq 0$, then $\varphi_j(t) = \lambda_j e^{\varepsilon_i t}$ and the unfolding intersects $\varphi_j = 1$. Hence every unfolding intersects $S = \left(\varphi_j = 1\right) \cup \left(\varphi_j = 0\right)$.

There are many other possible choices for a transversal submanifold. When we perform the inverse transformation to return to the original matrix B these transversal submanifolds transform to transversal submanifolds for B.

(I·2:20) Example: Let $f(x, y) = (x^2 + fy^3, y^2 + ax^3, xy + bx^3 + cx^2y + dxy^2 + ey^3)$.
Write $p(x,y) = x^2 + fy^3$, $q(x,y) = y^2 + ax^3$, $r(x,y) = xy + bx^3 + cx^2y + dxy^2 + ey^3$.
Then $T\mathscr{A}^3 f \ni (-4ey^3, 0, -5ax^3 + 8bx^2y + 6cxy^2 + 4dy^3)$.

Let $F(x, y, t) = (x^2 + f(t)y^3, y^2 + a(t)x^3, xy + b(t)x^3 + c(t)x^2y + d(t)xy^2 + e(t)y^3)$

where
$$a(t) = a_0,$$
$$b(t) = -5a_0t + b_0,$$
$$c(t) = -20a_0t^2 + 8b_0t + c_0,$$
$$d(t) = -40a_0t^3 + 24b_0t^2 + 6c_0t + d_0,$$
$$e(t) = -40a_0t^4 + 32b_0t^3 + 12c_0t^2 + 4d_0t + e_0,$$
$$f(t) = 32a_0t^5 - 32b_0t^4 - 16c_0t^3 - 8d_0t^2 - 4e_0t + f_0.$$

By I·2:19 F is a 3-trivial unfolding. Further $f(t) = 0$ has roots in $\mathbb{C}$ except when $0 = a_0 = b_0 = c_0 = d_0 = e_0$ (this is not the case for $\mathcal{A}_{\mathbb{R}}$-equivalence).

Consequently $(x^2 + fy^3, y^2 + ax^3, xy + bx^3 + cx^2y + dxy^2 + ey^3)$ is $\mathcal{A}$-equivalent to one of

$$(x^2, y^2 + ax^3, xy + bx^3 + cx^2y + dxy^2 + ey^3) \text{ or } (x^2 + fy^3, y^2, xy).$$

Without the k-triviality theorem we have to use Mather's lemma and check that the dimension of $T\mathcal{A}^3(x^2 + f(t)y^3, y^2 + a(t)x^3, xy + b(t)x^3 + c(t)x^2y + d(t)xy^2 + e(t)y^3)$ is independent of t (an unpleasant task).

Also $T\mathcal{A}^3(x^2, y^2 + ax^3, xy + bx^3 + cx^2y + dxy^2 + ey^3) \ni (0, 3ax^3, 2bx^3 + cx^2y - ey^3)$,

$$(0, -2ax^3, -bx^3 + dxy^2 + 2ey^3), (0, -4bx^3, 4cx^3 + 6dx^2y + 8exy^2).$$

Note that these are tangent to $(f = 0)$. Hence we can apply the theorem again by integrating $(0, -4bx^3, 4cx^3 + 6dx^2y + 8exy^2)$ say - see II·5: for further details.

Consider the set $\{\sum_{i,j=1}^{s} \lambda_i b_{ij}^c v_j : 1 \le c \le d\}$. By I·2:19 applied to $c = 1$ there exists a transversal submanifold S, i.e $g \overset{k}{\sim} f \Rightarrow g$ is $(k+\kappa)$-equivalent to a map-germ

$$f + \sum_{i=1}^{s} \lambda_i v_i, \ \lambda \in S.$$

To proceed further we need to find vector fields tangent to S. If any of the $\sum_{ij=1}^{s} \lambda_i b_{ij}^c$ have components tangent to S which are of the same form

(i.e. $\sum_{i,j=1}^{s} \lambda_i b^c_{i,j} v_j = \underline{\lambda} B \underline{v}$ for some matrix B) then we can apply I·2:19 to one of these components and find a new transversal submanifold $S' \subseteq S$. We therefore ask:

Can we find a transversal manifold S so that the components of the remaining vector fields which are tangent to S are still of the form $\sum_{i,j=1}^{s} \lambda_i b^c_{i,j} v_j$?

If the answer is yes then we can apply the theorem as long as we have any vector fields left.

Note:. the unfolding is also k-trivial for $\mathcal{A}_{\mathbb{R}}$-equivalence but there may not be transversal submanifolds since these depend on the existance of solutions to equations which may not necessarily have real roots.

I·3 : Map-germs with a 1-dimensional instability locus.

We look at what we may deduce about the structure of $\theta(f)/T\mathcal{A}f$ from its Krull dimension and multiplicity.

If $\theta(f)/T\mathcal{A}f$ has Krull dimension 1 then $\Sigma = \mathrm{Supp}(\theta(f)/T\mathcal{A}_e f)$ is 1-dimensional and for large c

$$\dim_{\mathbb{C}} \theta(f)/T\mathcal{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f) = ec + \text{constant},$$

where e is the multplicity.

Denote by o(h) the <u>order</u> of an element of $\theta(f)$, i.e. the largest value of k satisfying $h \in \mathfrak{m}_n^k\cdot\theta(f)$.

(I·3:1) Lemma : there exists a value c and a sequence of polynomials $\{p_1^s, p_2^s, \ldots, p_e^s : s \geq 0\}$ satisfying the following:

i) for each $\sigma \geq 0$, $T\mathcal{A}f + f^*\mathfrak{m}_p^{c+\sigma}\cdot\theta(f) = T\mathcal{A}f + \mathbb{C}\{p_1^s, p_2^s, \ldots, p_e^s : s \geq \sigma\}$,

ii) for each j, s, $\quad p_j^{s+1} \in f^*\mathfrak{m}_p\cdot\{p_1^s, p_2^s, \ldots, p_e^s\} + T\mathcal{A}f$,

iii) for each σ and $\lambda_1, \ldots, \lambda_e$,

$$\sum_{j=1}^{e} \lambda_j p_j^\sigma \in T\mathcal{A}f + f^*\mathfrak{m}_p^{c+\sigma+1}\cdot\theta(f) + \mathfrak{m}_n^k\cdot\theta(f) \iff p_j^\sigma \in \mathfrak{m}_n^k\cdot\theta(f) \text{ if } \lambda_j \neq 0.$$

Proof : for sufficiently large values of c,

$$\dim_{\mathbb{C}} \theta(f)/T\mathcal{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f) = ec + \text{constant}.$$

Choose the least value for which this holds.

Then for $\sigma \geq 0$, $\dim_{\mathbb{C}} \left(T\mathcal{A}f + f^*\mathfrak{m}_p^{c+\sigma}\cdot\theta(f)\right)/\left(T\mathcal{A}f + f^*\mathfrak{m}_p^{c+\sigma+1}\cdot\theta(f)\right) = e.$

We choose $\{p_1^0, p_2^0, \ldots, p_e^0\}$ so that

$$\text{a) } T\mathscr{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f) = T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f) + \mathbb{C}\{p_1^0, p_2^0, \ldots, p_e^0\},$$

$$\text{b) } \sum_{j=1}^{e}\lambda_j p_j^0 \in T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f) + \mathfrak{m}_n^k\cdot\theta(f) \Leftrightarrow p_j^0 \in \mathfrak{m}_n^k\cdot\theta(f) \text{ if } \lambda_j \neq 0.$$

To find elements satisfying b) consider the fact that there exists a value k_1 such that $\mathfrak{m}_n^{k_1}\cdot\theta(f) \subseteq T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f)$. Then the set of elements

$$\{u : u \in T\mathscr{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f),\ u \notin T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f)\}$$

has at least one member of maximal order. Choose one and call it p_1^0.

The set $\{u : u \in T\mathscr{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f),\ u \notin T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f) + \mathbb{C}\{p_1^0\}\}$ has a member of maximal order. Choose one and call it p_2^0. Continue until we have e elements. These elements satisfy b) since if

$$\sum_{j=1}^{e}\lambda_j p_j^0 \in T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f) + \mathfrak{m}_n^k\cdot\theta(f)$$

let i be the largest value of j such that $\lambda_j \neq 0$. Then

$$p_i^0 \in T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f) + \mathfrak{m}_n^k\cdot\theta(f) + \mathbb{C}\{p_j^0 : j < i\}.$$

Then there exists $h \in \mathfrak{m}_n^k\cdot\theta(f)$ such that

$$h - p_i^0 \in T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f) + \mathbb{C}\{p_j^0 : j < i\}.$$

Consequently, $h \in T\mathscr{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f)$ but $h \notin T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f) + \mathbb{C}\{p_j^0 : j < i\}$.

Since p_i^0 is of maximal order in this set we must have $p_i^0 \in \mathfrak{m}_n^k\cdot\theta(f)$. Also, for each p_j^0 with $j < i$, $o(p_j^0) \geq o(p_i^0)$, so $j < i$ implies that $p_j^0 \in \mathfrak{m}_n^k\cdot\theta(f)$.

Then $T\mathscr{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f) = T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f) + f^*\mathcal{O}_p\cdot\{p_1^0, p_2^0, \ldots, p_e^0\}$

so by Nakayama's lemma, $T\mathscr{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f) = T\mathscr{A}f + f^*\mathcal{O}_p\cdot\{p_1^0, p_2^0, \ldots, p_e^0\}$.

It follows that $T\mathscr{A}f + f^*\mathfrak{m}_p^{c+1}\cdot\theta(f) = T\mathscr{A}f + f^*\mathfrak{m}_p\cdot\{p_1^0, p_2^0, \ldots, p_e^0\}$.

Now suppose that we have found $\{p_1^0, p_2^0, \ldots, p_e^0, \ldots, p_1^\sigma, p_2^\sigma, \ldots, p_e^\sigma\}$ satisfying conditions ii) and iii) with

$$T\mathscr{A}f + f^*\mathfrak{m}_p^{c+\sigma+1}\cdot\theta(f) = T\mathscr{A}f + f^*\mathfrak{m}_p\cdot\{p_1^\sigma, p_2^\sigma, \ldots, p_e^\sigma\}.$$

We find new elements $\{p_1^{\sigma+1}, p_2^{\sigma+1}, \ldots, p_e^{\sigma+1}\}$ using the above procedure.

$$\begin{aligned}
\text{Then } T\mathscr{A}f + f^*\mathfrak{m}_p^{c+\sigma}\cdot\theta(f) &= T\mathscr{A}f + f^*\mathfrak{m}_p^{c+\sigma+1}\cdot\theta(f) + \mathbb{C}\{p_1^\sigma, p_2^\sigma, \ldots, p_e^\sigma\} \\
&= T\mathscr{A}f + f^*\mathfrak{m}_p^{c+\sigma+2}\cdot\theta(f) + \mathbb{C}\{p_1^\sigma, p_2^\sigma, \ldots, p_e^\sigma, p_1^{\sigma+1}, p_2^{\sigma+1}, \ldots, p_e^{\sigma+1}\} \\
&= T\mathscr{A}f + f^*\mathfrak{m}_p^{c+\sigma+i}\cdot\theta(f) + \mathbb{C}\{p_1^{\sigma+j}, p_2^{\sigma+j}, \ldots, p_e^{\sigma+j} : 0 \le j \le i\}.
\end{aligned}$$

□.

(I·3:2) Lemma : if $\{p_1^s, p_2^s, \ldots, p_e^s : s \ge 0\}$ is chosen as in I·3:1 then there exists a value k_0 such that for $k \ge k_0$,

$$T\mathscr{A}f + \mathfrak{m}_n^k\cdot\theta(f) = T\mathscr{A}f + \mathbb{C}\{p_1^s, p_2^s, \ldots, p_e^s : o(p_j^s) \ge k\}.$$

Proof : Let k_0 be the least value such that $\mathfrak{m}_n^{k_0}\cdot\theta(f) \subseteq T\mathscr{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f)$.

Clearly, $T\mathscr{A}f + \mathbb{C}\{p_1^s, p_2^s, \ldots, p_e^s : o(p_j^s) \ge k\} \subseteq T\mathscr{A}f + \mathfrak{m}_n^k\cdot\theta(f)$.

To prove the converse, observe that

$$\mathfrak{m}_n^k\cdot\theta(f) \subseteq T\mathscr{A}f + f^*\mathfrak{m}_p^c\cdot\theta(f) = T\mathscr{A}f + \mathbb{C}\{p_1^s, p_2^s, \ldots, p_e^s : s \ge \sigma\}$$

so for each $h \in \mathfrak{m}_n^k \cdot \theta(f)$, there exist λ_j^s such that $h - \sum_{s \geq 0, j=1}^{e} \lambda_j^s p_j^s \in T\mathcal{A}f$.

Let σ be the least value such that $\lambda_j^\sigma \neq 0$ for some j $(1 \leq j \leq e)$.

Then $h - \sum_{j=1}^{e} \lambda_j^\sigma p_j^\sigma \in T\mathcal{A}f + f^* \mathfrak{m}_p^{c+\sigma+1} \cdot \theta(f)$ so

$$\sum_{j=1}^{e} \lambda_j^\sigma p_j^\sigma \in T\mathcal{A}f + f^* \mathfrak{m}_p^{c+\sigma+1} \cdot \theta(f) + \mathfrak{m}_n^k \cdot \theta(f).$$

So $\sum_{j=1}^{e} \lambda_j^\sigma p_j^\sigma \in \mathfrak{m}_n^k \cdot \theta(f)$, i.e., $p_j^\sigma \in \mathfrak{m}_n^k \cdot \theta(f)$ if $\lambda_j^\sigma \neq 0$.

It follows that $\mathfrak{m}_n^k \cdot \theta(f) \subseteq T\mathcal{A}f + \mathbb{C}\{p_1^s, p_2^s, \ldots, p_e^s : o(p_j^s) \geq k\}$. □.

It follows from this that

$$T\mathcal{A}f + \mathfrak{m}_n^{k+1} \cdot \theta(f) = T\mathcal{A}f + \mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f) + \mathbb{C}\{p_1^s, p_2^s, \ldots, p_e^s : k + \kappa \geq o(p_j^s) > k\}$$

and that $g \overset{k}{\sim} f \Rightarrow g \sim f + \Sigma \lambda_{j,s} p_j^s$ modulo $\mathfrak{m}_n^{k+\kappa+1} \cdot \theta(f)$ for some $\lambda_{j,s}$ and

$k + \kappa \geq o(p_j^s) > k$.

This gives a particularly simple result in the case that f is weighted homogeneous and $e = 1$. We let $p_1^s = p_s$.

(I·3:3) Proposition : if $e = 1$ and f is weighted homogeneous then there exists a value k_0 and a sequence of monomials $\{p_s : s \geq 0\}$ as in I·3:1 such that

i) if $k \geq k_0$, then $T\mathcal{A}f + \mathfrak{m}_n^k \cdot \theta(f) = T\mathcal{A}f + \mathbb{C}\{p_s : o(p_s) \geq k\}$,

ii) if $g \overset{k}{\sim} 0\, f$ then either $g \overset{k}{\sim} f, \forall k$, or $g \sim f + p_s$ for some $s \geq 0$,

iii) each $f + p_s$ is $o(p_s)$-determined (so is sufficient).

Hence if $g \overset{k}{\sim} 0 f$ then either $g \overset{k}{\sim} f, \forall k$, or g is finitely determined so f is a weak stem.

Proof: we choose a sequence $\{p_s : s \geq 0\}$ as in I·3:1. Then by I·3:2 there exists k_0 such that

$$k \geq k_0 \Rightarrow T\mathcal{A}f + \mathfrak{m}_n^k \cdot \theta(f) = T\mathcal{A}f + \mathbb{C}\{p_s : o(p_s) \geq k\} = T\mathcal{A}f + f^*\mathfrak{m}_p^{c+\sigma} \cdot \theta(f),$$

where σ is the least value of s such that $o(p^s) \geq k$.

Then without loss of generality, the p^s are monomials, since if $h = h_1 + h_2$, where $h, h_1, h_2 \in \mathfrak{m}_n^k \cdot \theta(f)$ then $h, h_1, h_2 \in T\mathcal{A}f + f^*\mathfrak{m}_p^{c+\sigma} \cdot \theta(f)$, so we can replace p_s by a suitable component.

Let $o(s) = o(p_s)$.

Then
$$T\mathcal{A}f + \mathfrak{m}_n^{o(s)} \cdot \theta(f) = T\mathcal{A}f + \mathfrak{m}_n^{o(s)+1} \cdot \theta(f) + \mathbb{C}\{p_s\}$$

so by I:1, if $g \overset{o(s)-1}{\sim} f$ then $g \overset{o(s)}{\sim} f + \lambda p_s$ for some $\lambda \in \mathbb{C}$.

Now f is weighted homogeneous, and we can choose k_0 so that $o(s) \geq k_0$ implies that p_s cannot have the same degree as f_j for any j.

Thus, by I:3,

$$T\mathcal{A}(f + \lambda p_s) + \mathfrak{m}_n^{o(s)+1} \cdot \theta(f) = T\mathcal{A}f + \mathfrak{m}_n^{o(s)+1} \cdot \theta(f) + \mathbb{C}\{\lambda p_s\}$$

so that, without loss of generality we may assume that $\lambda = 1$.

Finally, we check the hypotheses of I:2 for $\lambda = 1$. It is clear that hypothesis ii) holds. By I·2:4 there exists a value k_1 such that

$$\mathfrak{m}_n^{k_1} \cdot \theta(f) \cap T\mathcal{A}f \subseteq f^*\mathfrak{m}_p \cdot T\mathcal{A}f \subseteq T\mathcal{A}_{2,2}f.$$

Increase k_0 if necessary, so that $k_0 \geq k_1$.

Then

$$T\mathscr{A}_{2,2}f+\mathbb{C}\{p^s\}+m_n^{o(s)+1}\cdot\theta(f) \supseteq m_n^{o(s)}\cdot\theta(f)\cap\left(T\mathscr{A}f+\mathbb{C}\{p_s\}+m_n^{o(s)+1}\cdot\theta(f)\right) \supseteq m_n^{o(s)}\cdot\theta(f)$$

so c), and hence iii) holds, and

$$T\mathscr{A}f+f^*m_p\cdot\{p_s\} = T\mathscr{A}f+\mathbb{C}\{p_j: j>s\} = m_n^{o(s)+1}\cdot\theta(f)+T\mathscr{A}f.$$

Thus b), and hence i) holds. It follows that $f+p_s$ is o(s)-determined. □.

We note the obvious result

(I·3:4) Corollary: if f also satisfies the condition $g \overset{k}{\sim} f, \forall k \Rightarrow g \sim f$ then f is a stem.

This is the case when $j^2f = (x, y^2, 0)$ - this is proved in II·2:5.

We also note the obvious adjacency relations

$$\cdots \to f+p_{s+1} \to f+p_s \to f+p_{s-1} \to \cdots$$

We use this proposition to obtain families for weak stems, which are presented in the following table. A striking feature of these is the relations which occur between the terms p_s. The calculations for these examples will be found in chapter II.

Table 1. Some weak stems and their families.

	weak stem	family	relation	$f^{-1}(\Sigma)$
	(x, y^2, y^3)	(x, y^2, y^3+x^ky)	$p_s = xp_{s-1}$	$y = 0$
	(x, y^2, x^2y)	(x, y^2, x^2y+y^{2k+1})	$p_s = y^2p_{s-1}$	$x = 0$
	(x, y^2, xy^3)	(x, y^2, xy^3+x^ky)	$p_s = xp_{s-1}$	$y = 0$
*	$(x, y^2, x^3y+x^2y^3)$	$(x, y^2, x^3y+x^2y^3+y^{2k+1})$	$p_s = y^2p_{s-1}$	$x = 0$
*	$(x, y^2, x^4y+2x^2y^3+y^5)$	$(x, y^2, x^4y+2x^2y^3+y^5+x^ky)$	$p_s = xp_{s-1}$	$x^2+y^2 = 0$
*	$(x, y^2, x^4y+x^2y^3)$	$(x, y^2, x^4y+x^2y^3+y^{2k+1})$	$p_s = y^2p_{s-1}$	$x = 0$
*	$(x, y^2, x^2y^3+y^5)$	$(x, y^2, x^2y^3+y^5+x^ky)$	$p_s = xp_{s-1}$	$y = 0$
	(x, xy, y^3)	$(x, xy+y^{3k+2}, y^3)$	$p_s = y^3p_{s-1}$	$x = 0$
*	$(x, xy+y^3, xy^2)$	$(x, xy+y^3, xy^2+y^{3k+1})$	$p_s = y^3p_{s-1}$	$x = 0$

	weak stem	family	relation	$f^{-1}(\Sigma)$
*	$(x, xy+y^3, xy^2+y^4)$	$(x, xy+y^3, xy^2+y^4+y^{2k})$	$p_s = y^2 p_{s-1}$	$y(x+y^2)=0$
*	$(x, xy+y^3, xy^4)$	$(x, xy+y^3, xy^4+y^{3k+2})$	$p_s = y^3 p_{s-1}$	$x = 0$
	$(x, xy-y^4, y^6)$	$(x, xy-y^4, y^6+y^{3k+1})$	$p_s = y^3 p_{s-1}$	$x-y^3 = 0$
*	(x, x^2y+y^3, xy^2)	$(x, x^2y+y^3, xy^2+y^{3k+1})$	$p_s = y^3 p_{s-1}$	$x = 0$
*	(x, y^3, x^2y+y^4)	$(x, y^3, x^2y+y^4+y^{3k+2})$	$p_s = y^3 p_{s-2}$	$x^2+y^3 = 0$
		$(x, y^3, x^2y+y^4+xy^{3k+2})$	$p_s \sim xp_{s-1}$ ×	
*	$(x, x^2y-y^3, x^2y^2-y^4)$	$(x, x^2y-y^3, x^2y^2-y^4+x^ky^2)$	$p_s = xp_{s-1}$	$x^2y-y^3=0$

× note that $x \cdot xy^{3k+2} = x^2y^{3k+2}$; since $(x^2+y^3)y^{3k+2} \in T\mathscr{A}f$ this is equivalent to y^{3k+4}.

* these families are listed in Table 7.

All the weak stems are listed in Tables 2-5.

The existance of the relations should not be surprising since for each s

$$p_{s+1} \in f^*\mathfrak{m}_p \cdot \{p_s\} + T\mathscr{A}f,$$

so that $p_{s+1} \sim f_i p_s$ for some i, say $i = 1$; this is the same i for each s since $e = 1$ implies that Σ is smooth so that without loss of generality, $\Sigma = V(Y_2, \ldots Y_p)$ and hence $f_i p_s \in T\mathscr{A}f$ for $2 \le i \le p$. The example (x, y^3, x^2y+y^4) indicates that this relation is not in general the simple case $p_{s+1} \sim rp_s$ for some $r \in \mathcal{O}_n$. However, note that this is also the only example where $f^{-1}(\Sigma)$ is not smooth. It seems likely from the data above that when $f^{-1}(\Sigma)$ is smooth the simple relation occurs.

There is some evidence to suggest that some sort of relation always occurs and that it is related to the ideal which defines $f^{-1}(\Sigma)$:

for example $f = (x, xy+y^3, xy^2+y^4)$. Then $f^{-1}(\Sigma) = V(xy+y^3)$.

It is apparent (II·3:7) that $(xy+y^3)\theta(f) \subseteq T\mathscr{A}f$. Now $x^k\partial/\partial Y_j \in T\mathscr{A}f$ for all k, j so by necessity we choose $p_s \in y\theta(f)$. Since

$$(xy+y^3)\theta(f) \subseteq T\mathscr{A}f \text{ and } (xy^2+y^4)\theta(f) \subseteq T\mathscr{A}f$$

we are left with $p_{s+1} \sim xp_s \sim y^2p_s$ since $(x+y^2)p_s \in T\mathscr{A}f$. But $p_{s+1} = y^2p_s$ is the relation observed. Similar behaviour can be observed in other cases.

A future research problem might be to try to determine whether such relations always occur, and what form they might take if they do.

We note that

$$T\mathscr{A}(f+p_s) = T\mathscr{A}f + \mathbb{C}\{p_j : j \geq s\} = T\mathscr{A}(f+p_{s+1}) + \mathbb{C}\{p_s\}$$

and hence $\operatorname{cod} T\mathscr{A}(f+p_s) = \operatorname{cod} T\mathscr{A}(f+p_{s+1}) + 1$.

Proposition I·3:3 no longer holds if we drop the hypothesis that f is weighted homogeneous:

for example let $f = (x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1})$ (II·2:16).

Then f has $e = 1$ and if $j^{12}h = f$ then

$$h \in \mathscr{A}^{2k+1}(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1})$$

but $T\mathscr{A}(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1}) \not\ni (0, 0, y^{2k+3})$ so is not (2k+1)-determined.

However, $(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1})$ is (2k+6)-determined.

The next step would be to try and show that

$$h \sim (x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1} + by^{2k+3})$$

for some $a \neq 0$, some $b \in \mathbb{C}$.

We conjecture that the new normal form should be

$$g \overset{k}{\sim} f, \forall k, \text{ or } g \sim f + \lambda_1 p_s + \lambda_2 p_{s+1} + \ldots + \lambda_r p_{s+r} \text{ for some } s, r \geq 0,\ \lambda_1 \neq 0.$$

If we drop the hypothesis that $e = 1$ the behaviour is more complicated:

for example let

$$f = (x, xy + y^3, 2xy^2 + 3y^4) \text{ and } g = (x, xy + y^3, 2xy^2 + y^4) \qquad \text{(II·2:8 and II·2:10).}$$

Both of these have $e = 2$;

if $j^4h = f$ then either $h \in \bigcap_{k \geq 0} \mathscr{A}^k f$ or $h \sim (x, xy + y^3, 2xy^2 + 3y^4 + y^k)$ and is k-determined,

if $j^4h = g$ then it is possible to have $h \sim (x, xy + y^3, (x+y^2)^2 + (x+y^2)y^{2k+1})$,

which is neither finitely determined nor $\mathcal{A}$-equivalent to g.

Here $\mu_0(\Sigma_f) = 2$ and $\mu_0(\Sigma_g) = 1$.

The fundamental difference between these map-germs is that

$$\ell_\Sigma(\theta(f)/T\mathcal{A}f) = 1 \text{ and } \ell_\Sigma(\theta(f)/T\mathcal{A}g) = 2.$$

In chapter III we shall show that $\ell_\Sigma(\theta(f)/T\mathcal{A}f) = 1$ corresponds to a particular geometric condition for Σ and in view of this we make the conjecture that in Proposition I·3:3 we can replace the hypothesis $e = 1$ by the hypothesis

$$e = \mu_0(\Sigma_f).$$

The condition $\ell_\Sigma(\theta(f)/T\mathcal{A}f) = 1$ means that the localised module $(\theta(f)/T\mathcal{A}f)_{p(\Sigma)}$ has only one generator, so that all the generators of $\theta(f)/T\mathcal{A}f$ must collapse onto this one (here $p(\Sigma)$ denotes the prime ideal which defines Σ). It is not clear how to use this information, though.

The general conjecture is

i) if $e = \mu_0(\Sigma)$ and Σ is irreducible then f is a stem,

ii) there is a sequence of polynomials $\{p_1^s, p_2^s, \ldots, p_e^s : s \geq 0\}$, and values k, r such that

if $g \overset{k}{\sim} f$ then either $g \sim f$ or

$$g \sim f + \lambda_{1,s}p_1^s + \lambda_{2,s}p_2^s + \ldots + \lambda_{e,s}p_e^s{}^r + \ldots + \lambda_{1,s+r}p_1^{s+r} + \lambda_{2,s+r}p_2^{s+r} + \ldots + \lambda_{e,s+r}p_e^{s+r}$$

$$\text{for some } s, r \geq 0, \text{ (some) } \lambda_{i,s} \neq 0.$$

This is a problem for further research.

Chapter II : Map-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$.

II·1 : Introduction and Summary of Results.

The main aim of this chapter is to inspect map-germs with dim $\Sigma = 1$ so as to improve our understanding of such germs and their classifications and to suggest further areas of research. There are two main questions which interest us both for stems and for map-germs which fail to be stems:

i) how is the classification related to the multiplicity of $T\mathscr{A}f$?

ii) how are these related to the geometry; in particular what are Σ and $f^{-1}(\Sigma)$ like?

There are two ways in which we may proceed. The first is to construct map-germs with particular properties that we wish to examine; the second is to perform a systematic classification. The second method has the advantage that we may find map-germs with properties we did not expect but has the disadvantages that there is no obvious stopping point and that it may not yield map-germs having a particular property that we may wish to investigate. The bulk of this chapter (II·2-II·5) is taken up in extending the classification started in [Mond 1] and in the inspection of a few extra map-germs. The aim of looking at many different types of map-germ and the aim of achieving a systematic classification have proved somewhat incompatible with the aim of restricting the length of the chapter; we hope that we have reached an acceptable compromise.

(II·1:1) <u>Results.</u>

We extend the $\mathscr{A}$-invariant stratification of $\Sigma^1 J^r(2,3)$, (the set of r-jets in $\Sigma^1 J^r(2,3)$ of rank 1) for $r \geq 2$, calculated by Mond ([Mond 1]). In [Mond 1] the map-germs are classified over $\mathbb{R}$, i.e. $\mathscr{A}_{\mathbb{R}} = \mathrm{Aut}(\mathbb{R}^2, 0) \times \mathrm{Aut}(\mathbb{R}^3, 0)$. We classify over $\mathscr{A}_{\mathbb{C}} = \mathrm{Aut}(\mathbb{C}^2, 0) \times \mathrm{Aut}(\mathbb{C}^3, 0)$. We also include a stratification of some map-germs having corank 2. The move from $\mathscr{A}_{\mathbb{R}}$ to $\mathscr{A}_{\mathbb{C}}$ results in the collapsing of some of

Mond's strata, for example the map-germs (x, y^3-x^2y, xy^2) and (x, y^3, x^2y+xy^2) are $\mathcal{A}_{\mathbb{C}}$-equivalent but not $\mathcal{A}_{\mathbb{R}}$-equivalent.

Since one of the aims is to investigate stems we generally only try to identify non-stems rather than completing the stratification over them. We note that inspection of the tangent space is only sometimes useful in showing a map-germ is not a stem. Other methods, suggested in Chapter III can be more useful.

We note, however, that as yet we have failed to determine whether or not the map-germ $(x, xy+xy^3, y^4)$ is a stem.

(II:1) Theorem :

Figures 1 -3 give an $\mathcal{A}$-invariant stratification of $\Sigma^1 J^r(2,3)$ for $r \geq 2$.

Figure 4 gives an $\mathcal{A}$-invariant stratification of $\Sigma^2 J^r(2,3)$ for $r \geq 2$.

These figures will be found inside the back cover.

Notation: i) $\underline{f}$ (underlined) indicates that f is sufficient,

ii) Mond's stratification lies above and to the left of the red line,

iii) frequently we can give a general form for a finitely determined map-germ with a given k-jet: these general forms are given in the bottom row. We give only the sufficient forms.

In Tables 2-5 we present what we consider to be the important properties of our map-germs. These are summarised below.

Recall that if Krull $\dim \theta(f)/T\mathcal{A}f = 1$ then there exists a value c_0 such that for $c \geq c_0$

$$\dim_{\mathbb{C}} \frac{f^* m_3^c \cdot \theta(f) + T\mathcal{A}f}{f^* m_3^{c+1} \cdot \theta(f) + T\mathcal{A}f} = e.$$

(II:2) Theorem: Tables 2-5 give the following information for several map-germs having Krull $\dim \theta(f)/T\mathcal{A}f = 1$:

i) the value c_0,

ii) a basis for $\dfrac{f^*\mathfrak{m}_3^c\cdot\theta(f)+T\mathcal{A}f}{f^*\mathfrak{m}_3^{c+1}\cdot\theta(f)+T\mathcal{A}f}$ for $c \geq c_0$; if the multiplicity is not clear from this we sometimes give a second, equivalent basis.

iii) The transporter ideal $T(c_0) = \left(T\mathcal{A}f, f^*\mathfrak{m}_3^{c_0}\cdot\theta(f)\right)$; since $f^*\mathfrak{m}_3^{c_0}\cdot\theta(f)$ is an ideal in $\mathcal{O}_2$ $T(c_0)$ is also an ideal in $\mathcal{O}_2$.

iv) The multiplicity e,

iv) the multiple curves and cuspidal edges appearing in the image $f(\mathbb{C}^2)$ together with their transversal types:- transversal types are defined below,

v) whether there is a diagram of the image for the real map-germ,

vi) a stratification over the germ and the determinacy degrees where calculated.

To save space we write the basis elements in the form $h\cdot e_i$, $h \in \mathcal{O}_2$, $e_1 = \partial/\partial X$ etc.

We draw $f(\mathbb{R}^2) \subseteq \mathbb{R}^3$ for some of the map-germs. In several cases a picture is of no help since the important features cannot be indicated: for example, (x, xy, y^3) is 1-1 at all points of $\mathbb{R}^2$ but has a triple curve in $\mathbb{C}^2$. Other cases are very similar to those already drawn. The diagrams are on pages 56-58.

Strata are given by f if f is sufficient or by $\mathcal{A}^k f$ if f is not k-determined. We do not calculate determinacy degrees for map-germs which are not sufficient except to indicate if they are not finitely determined. We omit the stratum $\mathcal{A}^\infty f$ in each case.

Table 2 : $j^2f = (x, y^2, 0)$.

f is of the form (x, y^2, f_3). Reference II·2:n

f_3	c_0	basis	$T(c_0)$	e	curve	μ-type	diagram	stratification	det	n
y^3 *	1	$x^c y \cdot e_3$	y^2	1	$(x, 0) \to (x, 0, 0)$	A_2	yes	$y^3 + x^c y$	c+1	10
x^2y *	1	$y^{2c+1} \cdot e_3$	x	1	$(0, y) \to (0, y^2, 0)$	A_3	yes	$x^2y + y^{2c+1}$	2c+1	11
xy^3 *	2	$x^c y \cdot e_3$	y^2	1	$(x, 0) \to (x, 0, 0)$ $(0, y) \to (0, y^2, 0)$	A_2 A_1	yes	$xy^3 + x^c y$	c+1	12
x^3y	3	$y^{2c+1} \cdot e_3$, $xy^{2c+1} \cdot e_3$	x^2	2	$(0, y) \to (0, y^2, 0)$	A_5	no	$x^3y + y^{2c+1}$ $\mathcal{S}^{4c+1}(x^3y + x^2y^{2c+1})$ $\mathcal{S}^{4c+3}(x^3y + xy^{4c+3})$	2c+1 ∞ –	13
$x^3y + x^2y^3$ **	3	$y^{2c+1} \cdot e_3$	x	1	$(0, y) \to (0, y^2, 0)$ $(-y^2, y) \to (-y^2, y^2, 0)$	A_3 A_1	yes	$x^3y + x^2y^3 + y^{2c+1}$	2c+1	15
$(x^2+y^2)^2y$ **	4	$x^c y \cdot e_3$	x^2+y^2	1	$(x, \pm ix) \to (x, x^2, 0)$	A_3	no	$(x^2+y^2)^2y + x^c y$	c+1	16

* these map-germs are listed in Table 1.

* these map-germs are listed in Table 7.

f_3	c_0	basis	$T(c_0)$	e	curve	μ-type	diagram	stratification	det	n
$x^4y+x^2y^3$ **	4	$y^{2c+1}\cdot e_3$	x	1	$(x,\pm ix)\to(x,x^2,0)$ $(0,y)\to(0,y^2,0)$	A_1 A_3	no	$x^4y+x^2y^3+y^{2c+1}$	2c+1	17
$x^2y^3+y^5$ **	3	$x^c y\cdot e_3$	y^2	1	$(x,\pm ix)\to(x,x^2,0)$ $(x,0)\to(x,0,0)$	A_1 A_2	no	$x^2y^3+y^5+x^cy$	c+1	18
x^4y	4	$x^2y^{2c-1}\cdot e_3$, $xy^{2c-1}\cdot e_3$, $y^{2c+1}\cdot e_3$	x^3	3	$(0,y)\to(0,y^2,0)$	A_7	no	x^4y+y^{2c+1} $x^4y+x^2y^{2c-1}+y^{2c+1}$ $\mathscr{A}^{2c+1}(x^4y+x^2y^{2c-1})$ $\mathscr{A}^{2c+2}(x^4y+xy^{2c+1})$	2c+1 2c+1 ∞ -	21
x^2y^3	1	$x^{c+2}y\cdot e_3$, $y^{2c+1}\cdot e_3$	xy^2	2	$(x,0)\to(x,0,0)$ $(0,y)\to(0,y^2,0)$	A_2 A_3	no	$x^2y^3+x^{2c}y+y^{2c+1}$ $\mathscr{A}^{c+1}(x^2y^3+x^cy)$ $\mathscr{A}^{2c+1}(x^2y^3+y^{2c+1})$	c+1 ∞ ∞	20
y^5	1	$x^cy\cdot e_3$, $x^{c-2}y^3\cdot e_3$	y^4	2	$(x,0)\to(x,0,0)$	A_4	no	y^5+x^cy $y^5+x^cy+x^{c-2}y^3$ $\mathscr{A}^{c+1}(y^5+x^{c-2}y^3)$	c+1 c+1 ∞	19
$x^8y+x^7y^3$ $+x^5y^5+x^2y^9$	10	$y^{2c+1}\cdot e_3$	x	1	$(0,y)\to(0,y^2,0)$ other curves	A_3 A_1	no	$\mathscr{A}^{2c+1}(f_3+y^{2c+1})$	between 2c+3 and 2c+6	22

Table 3 : $f = (x, xy + y^3, f_3)$. Reference II·3:n

f_3	c_0	basis	$T(c_0)$	e	curves	μ-type	diagram	stratification	det	n
xy^2 **	2	$y^{3c+1}.e_3$	x	1	$(0, y) \to (0, y^3, 0)$ $(-y^2,y) \to (-y^2,0,-y^4)$	D_4 A_1	no	$f + y^{3c+1}.e_3$	3c+1	5
$xy^2 + y^4$ **	2	$y^{2c+2}e_3$ or $x^c y^2.e_3$	$y(x+y^2)$	1	$(x, 0) \to (x, 0, 0)$ $(-y^2,y) \to (-y^2,0,0)$	D_4	yes	$f + y^{2c+2}.e_3$	2c+2	7
$xy^2+\frac{3}{2}y^4$ *	2	$y^{2c}.e_3, y^{2c+1}.e_3$ or $x^c y.e_3, x^c y^2.e_3$	$(x+3y^2)^2$	2	$(-3y^2,y) \to (-3y^2,-2y^3,-\frac{3}{2}y^4)$	A_2	yes	$f + y^{2c}.e_3$ $f + y^{2c+1}.e_3$	2c 2c+1	8, 9
$xy^2+\frac{1}{2}y^4$	2	$y^{2c}.e_3, y^{2c+1}.e_3$ or $x^c y.e_3, x^c y^2.e_3$	$(x+y^2)^2$	2	$(-y^2,y) \to (-y^2, 0,-\frac{1}{2}y^4)$	A_5	yes	$\mathcal{S}^{2c}(f + y^{2c}.e_3)$ $f + y^{2c+1}.e_3$ $f + (x+y^2)y^{2c-1}$	- 2c+1 ∞	10, 11, 12

** these map-germs are listed in Table 1.

* these map-germs are listed in Table 7.

f_3	c_0	basis	$T(c_0)$	e	curves	μ-type	diagram	stratification	det	n
xy^3	1	$y^{3c}.e_3$, $y^{3c+1}.e_3$ $y^{3c+2}.e_3$, $xy^{3c+2}.e_3$	x^3	4	$(0, y) \to (0, y^3, 0)$	J_{10}	no	$\mathscr{A}^{3c+3}(f + axy^{3c} + by^{3c+3}).e_3)$ $\mathscr{A}^{3c+1}(f + y^{3c+1}.e_3)$ $\mathscr{A}^{3c+2}(f + y^{3c+2}.e_3)$	∞ - -	17
xy^4 **	1	$y^{3c+2}.e_3$	x	1	$(0, y) \to (0, y^3, 0)$	D_4	no	$f + y^{3c+2}.e_2$	3c+2	23

* these map-germs are listed in Table 1.

* these map-germs are listed in Table 7.

Table 4 : $f = (x, f_2, f_3)$.

(f_2, f_3)	c_0	basis	$T(c_0)$	e	curves	μ-type	diagram	stratification	dct	Ref
(xy, y^3) *	1	$y^{3c+2}\cdot e_2$	x	1	$(0, y) \to (0, 0, y^3)$	D_4	no	$f + y^{3c+2}\cdot e_2$	3c+2	3:4
(xy, y^4)	2	$y^{4c+2}\cdot e_2$ $y^{4c+3}\cdot e_2, xy^{4c+3}\cdot e_2$	x^2	3	$(0, y) \to (0, 0, y^4)$	X_9	no	not known		3:24
$(xy + xy^3, y^4)$	1	$y^{4c+2}\cdot e_2, y^{4c+3}\cdot e_2$	x	2	$(0, y) \to (0, 0, y^4)$	X_9	no	not known		3:25
$(xy - y^4, y^6)$ *	4	$y^{3c+1}\cdot e_3$	$x - y^3$	1	$(y^3, y) \to (y^3, 0, y^6)$	D_4	no	$f + y^{3c+1}\cdot e_3$	3c+1	3:27
$(x^2y + y^3,$ $xy^2)$ **	1	$y^{3c+1}\cdot e_3$	x	1	$(0, y) \to (0, y^3, 0)$	D_4	no	$f + y^{3c+1}\cdot e_3$	3c+1	4:3
(y^3, xy^2)	2	$y^{3c+1}\cdot e_3,$ $x^c y\cdot e_2$	xy^2	2	$(0, y) \to (0, y^3, 0)$ $(x, 0) \to (x, 0, 0)$	D_4 A_2	no	$\mathscr{A}^{3c+1}(f + y^{3c+1}\cdot e_3)$ $f + (0, x^{3c}y, y^{3c+1})$ $\mathscr{A}^{c+1} f + x^c y\cdot e_2$	∞ 3c+1 ∞	4:4
(y^3, x^2y)	2	$y^{3c+1}\cdot e_3, y^{3c+2}\cdot e_3$ $xy^{3c+2}\cdot e_3, x^2y^{3c+2}\cdot e_3$	x^3	1	$(0, y) \to (0, y^3, 0)$	J_{10}	no	$\mathscr{A}^{3c+1}(f + y^{3c+1}\cdot e_3)$ $\mathscr{A}^{3c+2}(f + y^{3c+2}\cdot e_3)$ $\mathscr{A}^{3c+3}(f + ax^2y^{3c+1}\cdot e_3)$ $+ by^{3c+3}\cdot e_3$	- - ∞	4:8

(f_2, f_3)	c_0	basis	$^{\circ}T(c_0)$	e	curves	μ-type	diagram	stratification	det	Ref
(y^3, x^2y+y^4) **	2	$x^c y^2 \cdot e_3$ or $y^{3c+2} \cdot e_3$ if c even $xy^{3c+2} \cdot e_3$ if c odd	x^2+y^3 $x^2+y^3 = 0$	1	$(x, y) \to (x, y^3, 0)$	D_4	no $f + xy^{3c+2} \cdot e_3$	$f + y^{3c+2} \cdot e_3$ 3c+3	3c+2	4:9
$(x^2y-y^3,$ $x^2y^2-y^4)$ **	1	$y^{3c+2} \cdot e_2$	$y(x^2-y^2)$	1	$(0, y) \to (0, 0, y^3)$	D_4	yes	$f + (0, y^{3c+2}, 0)$	3c+2	4:13

Table 5 : f has corank 2.

f	c_0	basis	$T(c_0)$	e	curves	μ-type	diagram	stratification	det	Ref
$(xy + x^3 + x^2y + cxy^2,$ $x^2, y^2) : c(c+4) \neq 0.$	2	$y^{3c+2} \cdot e_2$	x	1	$(0, y) \to (0, 0, y^2)$ other curves	A_3 A_1	no	$f + y^{2c+1} \cdot e_2$	2c+1	5:4

* these map-germs are listed in Table 1.

* these map-germs are listed in Table 7.

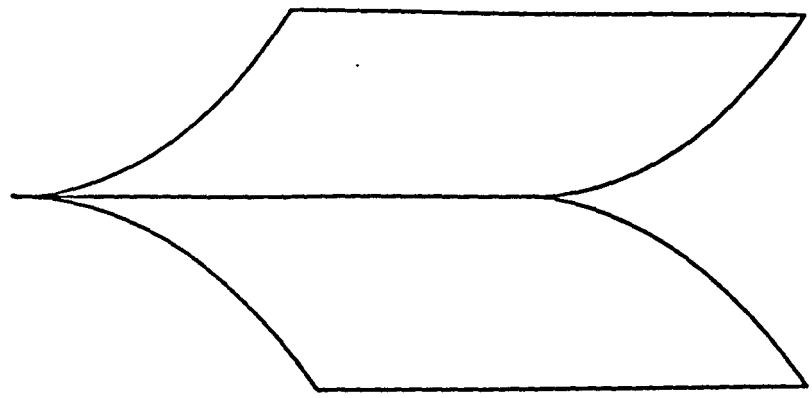


Diagram 1. $f(x, y) = (x, y^2, y^3)$.

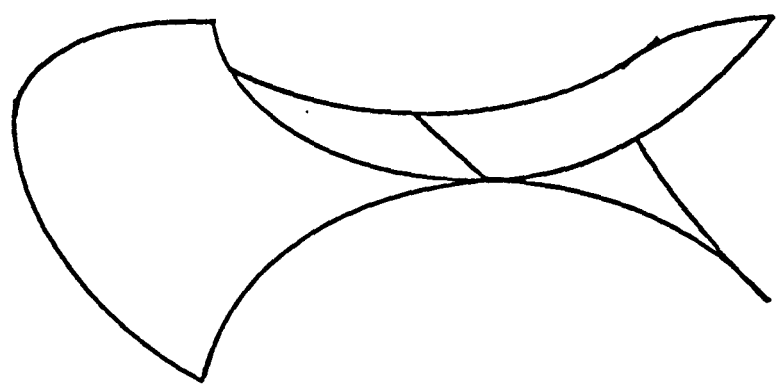


Diagram 2. $f(x, y) = (x, y^2, x^2y)$.

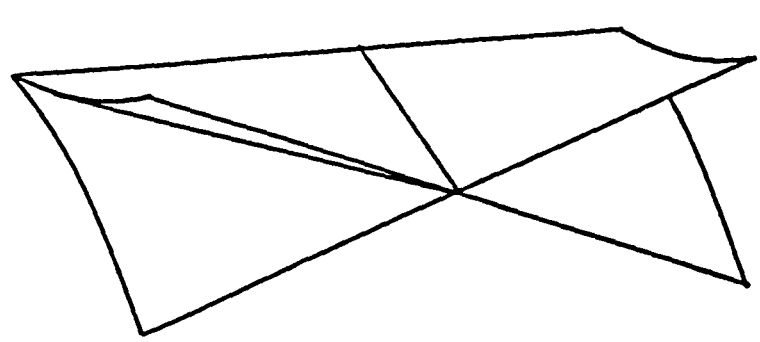


Diagram 3. $f(x, y) = (x, y^2, xy^3)$.

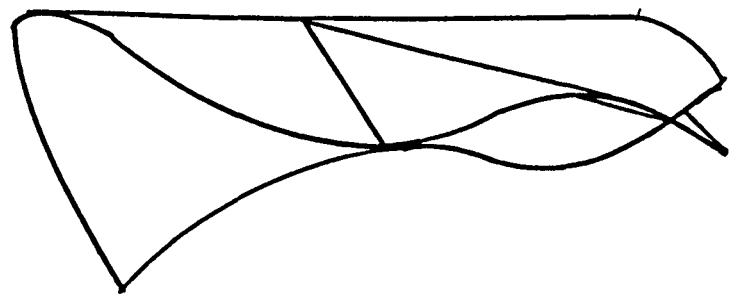


Diagram 4. $f(x, y) = (x, y^2, x^3y + x^2y^3)$.

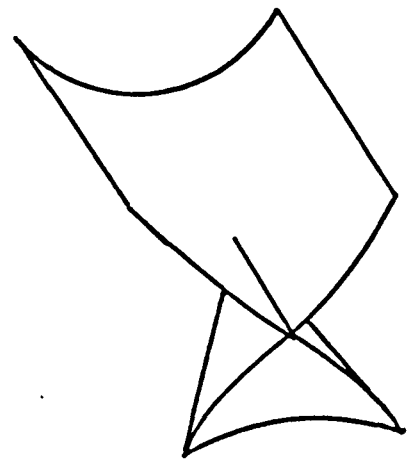


Diagram 5. $f(x, y) = (x, xy + y^3, xy^2 + \frac{3}{2}y^4)$.

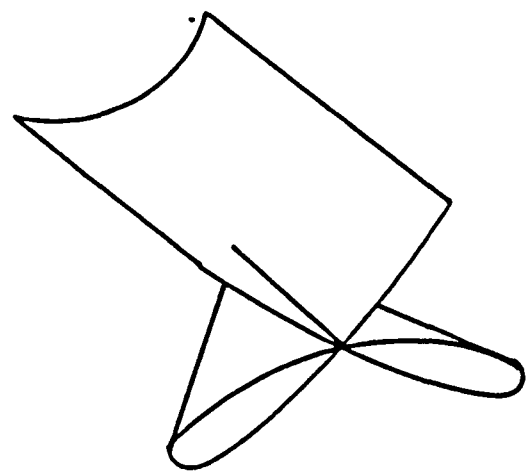


Diagram 6. $f(x, y) = (x, xy + y^3, xy^2 + y^4)$.

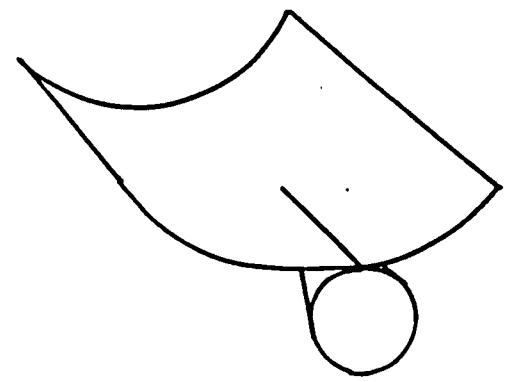


Diagram 7. $f(x, y) = (x, xy + y^3, xy^2 + \frac{1}{2}y^4)$.

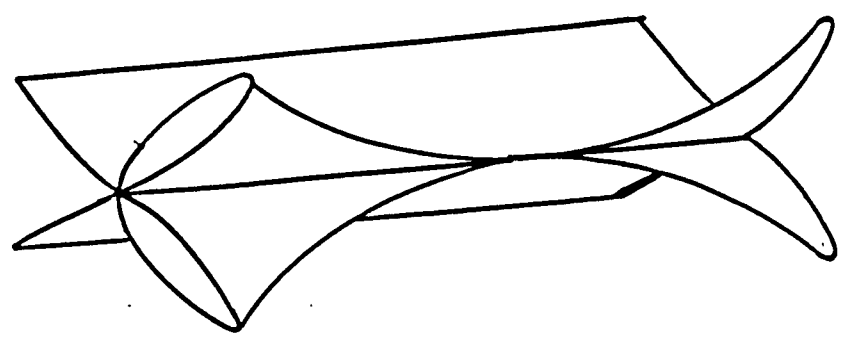


Diagram 8. $f(x, y) = (x, x^2y - y^3, x^2y^2 - y^4)$.

(II·1·2) <u>The Transversal Type of Σ</u>.

(II·1·2:1) Definition ([P 3], p. 98) : suppose that $f, g:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ have isolated singularities at 0. If there exists a family $\{f_t\}$, $t \in [0, 1]$ of germs $f_t:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ of analytic functions with an isolated singular point at 0, such that the Milnor number $\mu(f_t,0)$ is constant, and $f_0 = f$, $f_1 = g$ and $F:\mathbb{C}^{n+1}\times[0, 1] \to \mathbb{C}$ defined by $F(x, t) = f_t(x)$ is continuous, then f and g are called <u>μ-homotopic</u>.

(II·1·2:2) Definition ([P 3], p. 98) : let $f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a singular locus $(\Sigma,0)$, such that Σ is a germ of a non-singular curve. We say that the <u>transversal μ-type</u> of f along Σ at 0 is constant if for every choice of local coordinates $x, y_1,..., y_n$ such that $\Sigma = V(y_1,..., y_n)$ the μ-type of $f_t:(\mathbb{C}^n,0) \to (\mathbb{C},0)$ with $f_t(y) = f(t, y)$ is constant for all t sufficiently small.

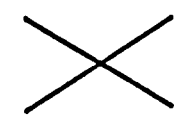
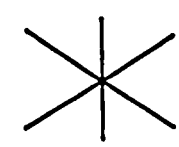
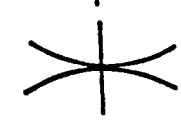
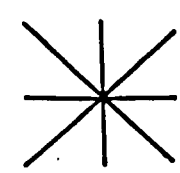
(II·1·2:3) Proposition ([P 3], p.98) : let $f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus $(\Sigma, 0)$. Let $(\Sigma_1, 0),\ldots,(\Sigma_r, 0)$ be the branches of $(\Sigma, 0)$. The transversal μ-type of f along Σ_i is constant at all points of a punctured neighbourhood of 0 in Σ_i.

<u>Transversal μ-types for map-germs $f:(\mathbb{C}^3,0) \to (\mathbb{C},0)$</u> [A], p. 73 and p. 76.

If local coordinates (X, Y, Z) on $\mathbb{C}^3$ are such that $\Sigma = V(Y, Z)$ the transversal type of Σ at 0 is the μ-type of $f_t(Y, Z) = f(t, Y, Z)$ for (sufficiently) small t.

The μ-types which occur in this thesis are the following:

Table 6. Some transversal types.

name	map	description
A_1	$f(Y, Z) = Y^2 + Z^2$	
$A_{2n}, n \geq 1,$	$f(Y, Z) = Y^{2n+1} + Z^2$	
$A_{2n+1}, n \geq 1,$	$f(Y, Z) = Y^{2n+2} + Z^2$	
D_4	$f(Y, Z) = Y^2Z + Z^3$	
$D_k, k > 4$	$f(Y, Z) = Y^2Z + Z^{k-1}$	
J_{10}	$f(Y, Z) = Y^3 + \varepsilon Y^2Z^2 + aZ^6$: $a(4\varepsilon + 27a) \neq 0, \varepsilon(\varepsilon^2 - 1) = 0$	
X_9	$f(Y, Z) = Y^4 + \varepsilon Y^2Z^2 + aZ^4$: $a(4\varepsilon + 27a) \neq 0, \varepsilon(\varepsilon^2 - 1) = 0$	

Calculation : the calculations of the transversal types for the map-germs in the tables are performed in III·1. Frequently, it can be found simply by knowing the multiplicity. Note, however, that the relation between the multiplicity and the transversal type can be very complicated : for example, (x, xy, y^4) and $(x, xy+xy^3, y^4)$ both have tranversal type X_9 but while for (x, xy, y^4) $e = 3$, for $(x, xy+xy^3, y^4)$ $e = 2$. One of the differences between these is that for (x, xy, y^4) $\varepsilon = 0$ and for $(x, xy+xy^3, y^4)$ $\varepsilon \neq 0$. We expect that as the type becomes more complicated this behaviour will become more common, i.e. 'one' transversal type will correspond to several possible multiplicities.

(II·1·3) Calculation of Modules.

The map-germs included in this thesis fall into one of the categories $f^*\mathfrak{m}_3\cdot\mathcal{O}_2 = \langle x, y^2\rangle\cdot\mathcal{O}_2$, $f^*\mathfrak{m}_3\cdot\mathcal{O}_2 = \langle x, y^3\rangle\cdot\mathcal{O}_2$, $f^*\mathfrak{m}_3\cdot\mathcal{O}_2 = \langle x, y^4\rangle\cdot\mathcal{O}_2$, f has corank 2.

For map-germs satisfying $f^*\mathfrak{m}_3\cdot\mathcal{O}_2 = \langle x,y^2\rangle\cdot\mathcal{O}_2$ we calculate $T\mathcal{A}f$ according to the procedure developed in [Mond 1].

In certain cases (e.g. $(x, y^3, 0)$) it is easy to calculate $T\mathcal{A}f$ essentially by inspection.

For finitely determined map-germs we use the fact that

$$\text{if } f^*\mathfrak{m}_3\mathcal{O}_2 \subseteq \mathfrak{m}_2^{\ell} \text{ and } \mathfrak{m}_2^k\theta(f) \subseteq T\mathcal{A}^{k+\ell}f \text{ then } \mathfrak{m}_2^k\theta(f) \subseteq T\mathcal{A}f$$

We calculate $T\mathcal{A}^{k+\ell}f$ by computer for a sufficiently large k that $\mathfrak{m}_2^k\theta(f) \subseteq T\mathcal{A}^{k+\ell}f$.

For the remaining corank 1 cases we start by calculating $T\mathcal{A}^kf$ for a large value of k by computer. From this we choose a module C which we hope will be equal to $T\mathcal{A}f$. We can check this using Nakayama's lemma (I·1:6)

$$C = T\mathcal{A}f \Leftrightarrow C = T\mathcal{A}f + f^*\mathfrak{m}_3\cdot C.$$

These calculations are performed in appendix A.

Corank 2 cases are dealt on an individual basis.

The remaining work consists of calculating $T\mathcal{A}^k(f+p(\lambda, x, y))$. There are two ways of proceeding:

i) specific calculation of $T\mathcal{A}^k(f+p(\lambda, x, y))$ for given values of k by computer,

ii) for general k use of the fact that

$$T\mathcal{A}^k(f+p(\lambda, x, y)) = T\mathcal{A}_{2,2}f + \mathfrak{m}_2^{k+1}\cdot\theta(f) + \mathbb{C}\{x\partial(f+p)/\partial x, x\partial(f+p)/\partial y, y\partial(f+p)/\partial x, y\partial(f+p)/\partial y, (f+p)_i\cdot e_j: 1 \le i, j \le 3\}.$$

The method of calculation is indicated for each example.

(II·1·4) Computer Programs.

The program currently exists in two forms, module and param and runs on two computers, the Sun Workstation and the Apple Macintosh.

The first version, module, is designed to calculate a set of generators for the module

$$T\mathcal{A}_{R,S}f \bmod \mathfrak{m}_2^{k+1}\cdot\theta(f) = tf(\mathfrak{m}_2^R\theta(2)) + \omega f(\mathfrak{m}_3^S\theta(3)) \bmod \mathfrak{m}_2^{k+1}\cdot\theta(f),$$

for integers R, S, k and $f(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y))$, where each f_i is a polynomial in x and y with integer coefficients.

The generators are simplified as far as possible before being displayed. The data may be saved to an external file. On the Macintosh only, the data can be printed directly from inside the program.

The second version, param (for parameters) has the option that the polynomials may take parameter as well as integer coefficients. This version is primarily designed to calculate the modules $T\mathcal{A}(f+p(\lambda, x, y)) \bmod \mathfrak{m}_2^{k+1}\cdot\theta(f)$, where $p \in \mathfrak{m}_2^k\cdot\theta(f)$ and $\lambda = (\lambda_1, ..., \lambda_r)$ are parameters. It has no facilities for storing products of coefficients, so care must be taken that terms involving products lie in $\mathfrak{m}_2^{k+1}\cdot\theta(f)$.

For example, let $f_\lambda = (xy + ax^3 + bx^2y + cxy^2 + dy^3, x^2, y^2)$ where $\lambda = (a, b, c, d)$. We can choose the terms $x^iy^j\partial f_\lambda/\partial x$, $x^iy^j\partial f_\lambda/\partial y$ $(i+j \geq R)$ as generators for $tf(\mathfrak{m}_2^R\theta(2))$ so we can choose any value of R or k, but generators of $\omega f(\mathfrak{m}_3^S\theta(3))$ are of the form $\beta(f_1, f_2, f_3)$ where $\beta \in \mathfrak{m}_3^S\theta(3)$.

Terms involving parameters are of degree 3 so

$$\text{for } \omega f(\mathfrak{m}_3\theta(3)) \text{ or } \omega f(\mathfrak{m}_3^2\theta(3)) \text{ we must have } \mathfrak{m}_2^6 \subseteq \mathfrak{m}_2^{k+1}\cdot\theta(f),$$

$$\text{for } \omega f(\mathfrak{m}_3^3\theta(3)) \text{ we must have } \mathfrak{m}_2^8 \subseteq \mathfrak{m}_2^{k+1}\cdot\theta(f),$$

$$\text{for } \omega f(\mathfrak{m}_3^4\theta(3)) \text{ we must have } \mathfrak{m}_2^{10} \subseteq \mathfrak{m}_2^{k+1}\cdot\theta(f), \text{ etc.}$$

More details on the programs will be found in Appendix C.

II·2 : $j^2f = (x, y^2, 0)$.

We first state some result from [Mond 1].

(II·2:1) Theorem (4.1:1) : If $g \in \mathscr{A}^2(x, y^2, 0)$ then g is equivalent to a germ of the form

$$(x, y) \mapsto (x, y^2, yp(x, y^2)).$$

(II·2:2) Definition : the map-germs $p, q \in \mathcal{O}_2$ are <u>$\mathscr{K}^T$-equivalent</u> if there exists a function $r \in \mathcal{O}_2$ and a diffeomorphism $\varphi = (\varphi_1, \varphi_2): (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that

a) $r(0, 0) \neq 0$,

b) $\varphi_1(x,-y) = \varphi_1(x, y)$ and $\varphi_2(x,-y) = -\varphi_2(x, y)$,

and c) $r(x, y^2)p(x, y^2) = q(\varphi_1(x, y), \varphi_2^2(x, y))$.

We prove a new result using the Artin approximation theorem:

(II·2:3) Theorem : if p, q are k-$\mathscr{K}^T$- equivalent for all k then they are $\mathscr{K}^T$- equivalent.

Proof: This is equivalent to finding r and $\psi = (\psi_1, \psi_2)$, where

$$\varphi = (\psi_1(x, y^2), y\psi_2(x, y^2)),$$

which satisfy $r(x, y^2)p(x, y^2) = q(\psi_1(x, y^2), y^2\psi_2^2(x, y^2))$ and $(\psi_1(x, y^2), y\psi_2(x, y^2))$ is a diffeomorphism.

Thus we need to find r, ψ satisfying

$$r(x, y)p(x, y) = q(\psi_1(x, y), y\psi_2^2(x, y)) \text{ and } \det \begin{pmatrix} \partial_x\psi_1 & 2y\partial_y\psi_1 \\ y\partial_x\psi_2 & \psi_2+2y^2\partial_y\psi_2 \end{pmatrix} \neq \mathfrak{m}_2,$$

$$\text{i.e. } \psi_2(0, 0)\partial_x\psi_1(0,0) \neq 0.$$

This follows if we can find a solution $(X(x, y), Y(x, y), Z(x, y))$ to the equation

$$0 = F(x, y, X, Y, Z) = Xp(x, y) - q(Y, yZ^2)$$

which satisfies $X(0, 0) \neq 0$ and $Z\partial_x Y \neq 0$ at $(0, 0)$.

The hypothesis says that this has solutions modulo $\mathfrak{m}_2^k$ for each k. Hence by I·1:17 a convergent solution exists.

(II·2:4) Theorem (4.1:1). The map-germs

$$(x, y) \mapsto (x, y^2, yp(x, y^2)),\ (x, y) \mapsto (x, y^2, yq(x, y^2))$$

are $\mathcal{A}$-equivalent if and only if the function-germs p, q are $\mathcal{K}^T$-equivalent.

(II·2:5) Corollary to II·2:3: if $j^2f = (x, y^2, 0)$ and f and g are k-$\mathcal{A}$-equivalent for all k then they are $\mathcal{A}$-equivalent. In particular, if f is a weak stem then it is a stem.

Proof: this follows immediately from the preceeding two theorems.

Let $h: \mathbb{C}^2 \to \mathbb{C}^2$ be the fold map $(x, y) \to (x, y^2)$.

(II·2:6) Theorem (4.1:16): suppose that $p(0, 0) = 0$. Then

$$T\mathcal{K}^T p\circ h = h^*\left(\langle x\partial_x p, y\partial_x p, y\partial_y p, p\rangle \cdot \mathcal{O}_2\right) \subseteq \mathfrak{m}_2^T$$

$$\text{and } T\mathcal{A}(x, y^2, yp\circ h) = \begin{pmatrix} \mathfrak{m}_2 \\ \mathfrak{m}_2\backslash\{y\} \\ 0 \end{pmatrix} + \left(\mathfrak{m}_2^T + yT\mathcal{K}^T p\circ h\right)\cdot e_3$$

where $\mathfrak{m}_2^T = h^*\mathfrak{m}_2 = \{x^\ell y^{2j} : \ell + j \geq 1\}$.

(II·2:7) General form for $T\mathcal{A}_{2,2}f$ and $f^*\mathfrak{m}_3 \cdot T\mathcal{A}f$ where $f(x, y) = (x, y^2, yp(x, y^2))$;

$$\text{Let } A = \begin{pmatrix} \mathfrak{m}_2 \\ \mathfrak{m}_2\backslash\{y\} \\ 0 \end{pmatrix},\ B = \begin{pmatrix} \mathfrak{m}_2^2\backslash\{y^2\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, y^3\} \\ 0 \end{pmatrix},\ C = \begin{pmatrix} \mathfrak{m}_2^2\backslash\{y^2\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2\} \\ 0 \end{pmatrix},\ D = \begin{pmatrix} \mathfrak{m}_2^2 \\ \mathfrak{m}_2^2\backslash\{xy, y^2\} \\ 0 \end{pmatrix}.$$

Then $\quad T\mathcal{A}f = A + \left(\mathfrak{m}_2^T + yT\mathcal{K}^T p\circ h\right)\cdot e_3,$

$$f^*\mathfrak{m}_3\cdot T\mathscr{A}f = B + \left(\mathfrak{m}_2^T\right)^2\cdot e_3 + \left(\langle xy, y^3\rangle T\mathscr{K}^T p\circ h\right)\cdot e_3,$$

and $$T\mathscr{A}_{2,2}f = C + \left(\mathfrak{m}_2^T\right)^2\cdot e_3 + \left(\langle xy, y^3\rangle T\mathscr{K}^T p\circ h\right)\cdot e_3 + \mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ y^3\partial_x p\circ h\end{pmatrix}\right\}.$$

If $\partial_x p = 0$ then $T\mathscr{A}_{2,2}f = D + \left(\mathfrak{m}_2^T\right)^2\cdot e_3 + \left(\langle xy, y^3\rangle T\mathscr{K}^T p\circ h\right)\cdot e_3$.

Proof: $$f^*\mathfrak{m}_3\cdot T\mathscr{A}f = f^*\mathfrak{m}_3\cdot\left(A + \left(\mathfrak{m}_2^T + yT\mathscr{K}^T p\circ h\right)\cdot e_3\right)$$

$$= B + \left(\mathfrak{m}_2^T\right)^2\cdot e_3 + \left(\langle xy, y^3\rangle T\mathscr{K}^T p\circ h\right)\cdot e_3$$

since $$yp(x, y^2)yT\mathscr{K}^T p\circ h \subseteq \left(\mathfrak{m}_2^T\right)^2$$

and $$yp(x, y^2)\mathfrak{m}_2^T \subseteq yh^*(p\mathfrak{m}_2) \subseteq yT\mathscr{K}^T p\circ h$$

by definition of $T\mathscr{K}^T p\circ h$.

Now $$T\mathscr{A}_{2,2}f = f^*\mathfrak{m}_3\cdot T\mathscr{A}f + \mathbb{C}\left\{y^2\partial f/\partial x, y^2\partial f/\partial y\right\}$$

$$= f^*\mathfrak{m}_3\cdot T\mathscr{A}f + \mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ y^3\partial_x p\circ h\end{pmatrix}, \begin{pmatrix} 0 \\ 2y^3 \\ y^2 p\circ h + 2y^4\partial_y p\circ h\end{pmatrix}\right\}$$

$$= C + \left(\mathfrak{m}_2^T\right)^2\cdot e_3 + \left(\langle xy, y^3\rangle T\mathscr{K}^T p\circ h\right)\cdot e_3 + \mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ y^3\partial_x p\circ h\end{pmatrix}\right\}$$

The final statement is obvious □.

We now start the classification of map-germs satisfying $j^2f = (x, y^2, 0)$. We proceed by calculating the appropriate modules for each case and applying the theorems I:1, I:2 and I:3 as appropriate. We also use I·2:19 in II·2:9.

(II·2:8) $\qquad f = (x, y^2, 0).$

$p = 0 \qquad T\mathscr{X}^T p\circ h = 0,$

$T\mathscr{A}_{2,2}f \qquad D + \left(\mathfrak{m}_2^T\right)^2 \cdot e_3,$

hence $g \in \mathscr{A}^2 f \Rightarrow \qquad g \in \mathscr{A}^3(x, y^2, a_0x^2y + a_1y^3),$

$g \in \mathscr{A}^3 f \Rightarrow \qquad g \in \mathscr{A}^4(x, y^2, a_0x^3y + a_1xy^3),$

$g \in \mathscr{A}^4 f \Rightarrow \qquad g \in \mathscr{A}^5(x, y^2, a_0x^4y + a_1x^2y^3 + a_2y^5),$

and in general for $k \geq 1$

$g \in \mathscr{A}^{2k} f \Rightarrow \qquad g \in \mathscr{A}^{2k+1}(x, y^2, a_0x^{2k}y + a_1x^{2k-2}y^3 + \ldots + a_ky^{2k+1}),$

and $g \in \mathscr{A}^{2k+1} f \Rightarrow \qquad g \in \mathscr{A}^{2k+2}(x, y^2, a_0x^{2k+1}y + a_1x^{2k-1}y^3 + \ldots + a_kxy^{2k+1}).$

(II·2:9) $\qquad$ Let $f_k = (x, y^2, p_k)$

where

$$p_{2k+1} = a_0x^{2k}y + a_1x^{2k-2}y^3 + \ldots + a_ky^{2k+1},$$
$$\text{and } p_{2k+2} = a_0x^{2k+1}y + a_1x^{2k-1}y^3 + \ldots + a_kxy^{2k+1}.$$

$$T\mathscr{A}^{k+1}f_{k,\lambda} = T\mathscr{A}_{2,2}f + \mathfrak{m}_2^{k+2}\cdot\theta(f) + \mathbb{C}\left\{\begin{pmatrix}x\\0\\0\end{pmatrix}, \begin{pmatrix}0\\x\\0\end{pmatrix}, \begin{pmatrix}0\\0\\x\end{pmatrix}, \begin{pmatrix}y^2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\y^2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\y^2\end{pmatrix}, \begin{pmatrix}p_k\\0\\0\end{pmatrix}, \begin{pmatrix}0\\p_k\\0\end{pmatrix}, \begin{pmatrix}0\\0\\p_k\end{pmatrix}\right\}$$

$$+ \mathbb{C}\left\{\begin{pmatrix}x\\0\\x\,\partial p_k/\partial x\end{pmatrix}, \begin{pmatrix}y\\0\\y\,\partial p_k/\partial x\end{pmatrix}, \begin{pmatrix}0\\2xy\\x\,\partial p_k/\partial y\end{pmatrix}, \begin{pmatrix}0\\2y^2\\x\,\partial p_k/\partial y\end{pmatrix}\right\}.$$

so $T\mathscr{A}^{2k+1}f_{k,\lambda} = A + \left(\mathfrak{m}_2^T\right)^2\cdot e_3 + \mathfrak{m}_2^{2k+2}\cdot\theta(f)$

$$+ \mathbb{C}\left\{\begin{pmatrix}0\\0\\2ka_1x^{2k}y + \ldots + 2a_{k-1}x^2y^{2k-1}\end{pmatrix}, \begin{pmatrix}0\\0\\a_1x^{2k}y + \ldots + (2k+1)a_ky^{2k+1}\end{pmatrix}\right\}.$$

and $T\mathscr{A}^{2k+2}f_{k,\lambda} = A + \left(\mathfrak{m}_2^T\right)^2\cdot e_3 + \mathfrak{m}_2^{2k+3}\cdot\theta(f)$

$$+\mathbb{C}\left\{\begin{pmatrix}0\\0\\(2k+1)a_1x^{2k+1}y+\ldots+a_kxy^{2k+1}\end{pmatrix},\begin{pmatrix}0\\0\\a_1x^{2k+1}y+\ldots+(2k+1)a_kxy^{2k+1}\end{pmatrix}\right\}.$$

The unfoldings

$$(x, y^2, \Sigma a_j e^{2(k-j)t}x^{2(k-j)}y^{2j+1}, t),\ (x, y^2, \Sigma a_j e^{2jt}x^{2(k-j)}y^{2j+1}, t)$$

are (2k+1)-trivial and the unfoldings

$$(x, y^2, \Sigma a_j e^{2(k-j)t}x^{2(k-j)+1}y^{2j+1}, t)\ (x, y^2, \Sigma a_j e^{2jt}x^{2(k-j)+1}y^{2j+1}, t).$$

are (2k+2)-trivial.

Examples $(x, y^2, a_0e^{2t}x^2y + a_1y^3, t), (x, y^2, a_0x^2y + a_1e^{2t}y^3, t),$

$(x, y^2, a_0e^{3t}x^3y + a_1xy^3, t), (x, y^2, a_0x^3y + a_1e^{2t}xy^3, t),$

$(x, y^2, a_0e^{4t}x^4y + a_1e^{2t}x^2y^3 + a_2y^5, t),$

$(x, y^2, a_0x^4y + a_1e^{2t}x^2y^3 + a_2e^{4t}y^5, t).$

In general the strata consist of the sets (x, y^2, p_k) where any two non-zero coefficients of the p_k may be assumed to be 1 (see below for an example).

Corollary : $\mathscr{A}^2(x, y^2, 0)$ is a union of the following $\mathscr{A}$-invariant strata:

$\mathscr{A}^3(x, y^2, x^2y + y^3)$	$\mathscr{A}^3(x, y^2, y^3)$	$\mathscr{A}^3(x, y^2, x^2y)$,
$\mathscr{A}^4(x, y^2, x^3y + xy^3)$	$\mathscr{A}^4(x, y^2, xy^3)$	$\mathscr{A}^4(x, y^2, x^3y)$
$\mathscr{A}^5(x, y^2, x^4y + bx^2y^3 + y^5)$	$\mathscr{A}^5(x, y^2, x^4y + x^2y^3)$	$\mathscr{A}^5(x, y^2, x^2y^3 + y^5)$
$\mathscr{A}^5(x, y^2, x^4y)$	$\mathscr{A}^5(x, y^2, x^2y^3)$	$\mathscr{A}^5(x, y^2, y^5)$
	$\mathscr{A}^5(x, y^2, 0)$	

Proof: we demonstrate this by calculating the strata of $\mathscr{A}^4(x, y^2, 0)$.

For any map-germ $(x, y^2, a_0x^4y + a_1x^2y^3 + a_2y^5)$ we have two 5-trivial unfoldings

$$(x, y^2, a_0e^{4t}x^4y + a_1e^{2t}x^2y^3 + a_2y^5, t) \text{ and } (x, y^2, a_0x^4y + a_1e^{2t}x^2y^3 + a_2e^{4t}y^5, t).$$

Suppose that $a_0 \neq 0$. Then there is a value of t so that $a_0e^{4t} = 1$.

So $(x, y^2, a_0y + a_1x^2y^3 + a_2y^5)$ is 5-equivalent to $(x, y^2, x^4y + \bar{a}_1x^2y^3 + a_2y^5)$.

Similarly $(x, y^2, x^4y + a_1e^{2t}x^2y^3 + a_2e^{4t}y^5, t)$ is 5-trivial;

if $a_2 \neq 0$ there is a value of t so that $a_2e^{4t} = 1$ and $(x, y^2, x^4y + a_1x^2y^3 + a_2y^5)$ is 5-equivalent to $(x, y^2, x^4y + \bar{a}_1x^2y^3 + y^5)$, otherwise $a_1 = 0$ or $a_1 \neq 0$ and

$(x, y^2, x^4y + a_1x^2y^3)$ is 5-equivalent to $(x, y^2, x^4y + x^2y^3)$.

We repeat this for the case $a_0 = 0$ to obtain the strata $\mathscr{A}^5(x, y^2, x^2y^3 + y^5)$, $\mathscr{A}^5(x, y^2, y^5)$, $\mathscr{A}^5(x, y^2, x^2y^3)$ and $\mathscr{A}^5(x, y^2, 0)$.

The other strata are obtained in a similar manner.

Determinacy degrees:

$(x, y^2, x^2y + y^3)$ is 3-determined, $(x, y^2, x^3y + xy^3)$ is 4-determined and $(x, y^2, x^4y + bx^2y^3 + y^5)$ is 5-determined provided that $b^2 \neq 4$.

The determinacy degrees are given in [Mond 1] (p 335).

(II·2:10) $\qquad f = (x, y^2, y^3)$.

$p(x, y) = y$	$T\mathscr{X}^T p\circ h = \mathfrak{m}_2^T\backslash\{x^j : j \geq 0\}$,
$T\mathscr{A}f$	$A + \left(\mathfrak{m}_2\backslash\{x^jy : j \geq 0\}\right)\cdot e_3$,
basis	$\left\{x^jy\cdot e_3\right\}$,
multiplicity	1,
$f^*\mathfrak{m}_3\cdot T\mathscr{A}f$	$B + \left(\mathfrak{m}_2^2\backslash\{y^2, y^3, x^jy : j \geq 1\}\right)\cdot e_3$,

$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \left\{x^jy\cdot e_3\right\}\right) \supseteq \mathfrak{m}_2^{j+2}\cdot\theta(f)$ provided that $j \geq 2$.

$T\mathscr{A}_{2,2}f$	$D + \left(\mathfrak{m}_2^2\backslash\{y^2, y^3, x^jy : j \geq 1\}\right)\cdot e_3$,
hence for $k \geq 3$ $g \in \mathscr{A}^k f \Rightarrow$	$g \in \mathscr{A}^{k+1}(x, y^2, y^3 + ax^ky)$.
Weighted homogeneous	weights (1, 1),
	$\text{degree}(y^3) = 3$, $\text{degree}(x^ky) = k+1$,
hence for $k \geq 3$	$T\mathscr{A}^{k+1}(x, y^2, y^3 + ax^ky) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ax^ky\cdot e_3\}$.

Thus by I:2 $(x, y^2, y^3 + x^ky)$ is k+1-determined for $k \geq 3$.

Hence $g \in \mathscr{A}^3 f \Rightarrow$ either $g \sim f$ or $g \sim (x, y^2, y^3 + x^ky)$; thus f is a stem.

(II·2:11) $f = (x, y^2, x^2y)$.

$p(x, y) = x^2$ $\qquad T\mathscr{K}^T p\circ h = \mathfrak{m}_2^T\backslash\{x, y^{2j}: j \geq 0\}$,

$T\mathscr{A}f$ $\qquad A + \left(\mathfrak{m}_2\backslash\{xy, y^{2j+1}: j \geq 0\}\right)\cdot e_3$,

basis $\qquad \left\{y^{2j+1}.e_3\right\}$,

multiplicity $\qquad 1$,

$f^*\mathfrak{m}_3\cdot T\mathscr{A}f$ $\qquad B + \left(\mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, y^3, xy^3, y^{2j+1}: j \geq 1\}\right)\cdot e_3$,

$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \left\{y^{2j+1}.e_3\right\}\right) \supseteq \mathfrak{m}_2^{2j+2}\cdot\theta(f)$ provided that $j \geq 1$.

$T\mathscr{A}_{2,2}f$ $\qquad C + \left(\mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^3, y^{2j+1}: j \geq 1\}\right) + \mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ 2xy^3 \end{pmatrix}\right\}$,

hence $g \in \mathscr{A}^3 f \Rightarrow$ $\qquad g \in \mathscr{A}^4(x, y^2, x^2y + axy^3)$,

for $k \geq 2$ $g \in \mathscr{A}^{2k}f \Rightarrow$ $\qquad g \in \mathscr{A}^{2k+1}(x, y^2, x^2y + ay^{2k+1})$,

and $g \in \mathscr{A}^{2k+1}f \Rightarrow$ $\qquad g \in \mathscr{A}^{2k+1}(x, y^2, x^2y)$.

Weighted homogeneous $\qquad$ weights $(1, 1)$,

$\text{degree}(x^2y) = 3, \text{degree}(xy^3) = 4, \text{degree}(y^{2k+1}) = 2k+1$,

hence $\quad T\mathscr{A}^4(x, y^2, x^2y + axy^3) = T\mathscr{A}^4 f + \mathbb{C}\{axy^3\cdot e_3\}$ and $T\mathscr{A}^4 f \ni \mathbb{C}\{xy^3\cdot e_3\}$,

hence $\{(x, y^2, x^2y + axy^3): a \in \mathbb{C}\}$ lies in a single orbit.

For $k \geq 2$ $\qquad T\mathscr{A}^{2k+1}(x, y^2, x^2y + ay^{2k+1}) = T\mathscr{A}^{2k+1}f + \mathbb{C}\{ay^{2k+1}.e_3\}$,

thus by I:2 $(x, y^2, x^2y + y^{2k+1})$ is $(2k+1)$-determined.

Hence $g \in \mathscr{A}^3 f \Rightarrow$ either $g \sim f$ or $g \sim (x, y^2, x^2y + y^{2k+1})$; thus f is a stem.

(II·2:12) $f = (x, y^2, xy^3)$.

$p(x, y) = xy$ $\qquad T\mathscr{K}^T p\circ h = \mathfrak{m}_2^T\backslash\{y, x^j: j \geq 0\}$,

$T\mathscr{A}f$ $\qquad A + \left(\mathfrak{m}_2\backslash\{y^3, x^jy,: j \geq 0\}\right)\cdot e_3$,

basis $\qquad \left\{x^jy\cdot e_3\right\}$,

multiplicity $\qquad 1$,

$f^*\mathfrak{m}_3 \cdot T\mathscr{A}f$ $\qquad B + \left(\mathfrak{m}_2^2 \backslash \{y^2, y^3, xy^3, y^5, x^jy : j \geq 1\}\right)$,

$f^*\mathfrak{m}_3 \cdot \left(T\mathscr{A}f + \{x^jy \cdot e_3\}\right) \supseteq \mathfrak{m}_2^{j+2} \cdot \theta(f)$ provided that $j \geq 4$.

$T\mathscr{A}_{2,2}f$ $\qquad C + \left(\mathfrak{m}_2^2 \backslash \{y^2, y^3, xy^3, y^5, x^jy : j \geq 1\}\right) + \mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ y^5 \end{pmatrix}\right\}$,

hence $g \in \mathscr{A}^4 f \Rightarrow$ $\qquad g \in \mathscr{A}^5(x, y^2, xy^3 + ax^4y + by^5)$,

for $k \geq 5$ $g \in \mathscr{A}^k f \Rightarrow$ $\qquad g \in \mathscr{A}^{k+1}(x, y^2, xy^3 + ax^ky)$.

Weighted homogeneous $\qquad$ weights $(1, 1)$,

$\text{degree}(xy^3) = 4, \text{degree}(x^ky) = k+1$.

Calculation by computer gives

$$T\mathscr{A}^5(x, y^2, x, y^2, xy^3 + ax^4y + by^5) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ax^4y \cdot e_3\}$$

and by I:3, for $k \geq 5$ $\quad T\mathscr{A}^{k+1}(x, y^2, xy^3 + ax^ky) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ax^ky \cdot e_3\}$.

Thus by I:2 $(x, y^2, xy^3 + x^ky)$ is $(k+1)$-determined for $k \geq 4$.

Hence $g \in \mathscr{A}^4 f \Rightarrow$ either $g \sim f$ or $g \sim (x, y^2, xy^3 + x^ky)$; thus f is a stem.

(II·2:13) $\qquad f = (x, y^2, x^3y)$.

$p(x, y) = x^3$ $\qquad T\mathscr{K}^T p \circ h = \mathfrak{m}_2^T \backslash \{x^2, y^{2j}, xy^{2j} : j \geq 0\}$,

$T\mathscr{A}f$ $\qquad A + \left(\mathfrak{m}_2 \backslash \{x^2y, y^{2j+1}, xy^{2j+1} : j \geq 0\}\right) \cdot e_3$,

basis $\qquad \mathbb{C}\left\{y^{2j+1} \cdot e_3, xy^{2j+1} \cdot e_3\right\}$,

multiplicity $\qquad 2$,

$f^*\mathfrak{m}_3 \cdot T\mathscr{A}f$ $\qquad B + \left(\mathfrak{m}_2^2 \backslash \{y^2, x^2y, x^3y, x^2y^3, y^{2j+1}, xy^{2j+1} : j \geq 1\}\right)$,

$f^*\mathfrak{m}_3 \cdot \left(T\mathscr{A}f + \{y^{2j+1} \cdot e_3\}\right) \supseteq \mathfrak{m}_2^{2j+2} \cdot \theta(f)$ provided that $j \geq 2$.

$T\mathscr{A}_{2,2}f$ $\qquad C + \left(\mathfrak{m}_2^2 \backslash \{y^2, x^2y, x^3y, x^2y^3, y^{2j+1}, xy^{2j+1} : j \geq 1\}\right) + \mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ 3x^2y^3 \end{pmatrix}\right\}$,

hence $g \in \mathscr{A}^4(x, y^2, x^3y) \Rightarrow$ $\quad g \in \mathscr{A}^5(x, y^2, x^3y + ax^2y^3 + by^5)$,

for $k \geq 2$ $g \in \mathscr{A}^{2k+2}f \Rightarrow$ $\quad g \in \mathscr{A}^{2k+2}(x, y^2, x^3y + axy^{2k+1})$

and $g \in \mathscr{A}^{2k+1}f \Rightarrow$ $\quad g \in \mathscr{A}^{2k+1}(x, y^2, x^3y + ay^{2k+1})$.

Computer calculation shows that $T\mathscr{A}^5(x, y^2, x^3y + ax^2y^3 + by^5) = T\mathscr{A}^5f + \mathbb{C}\{by^5.e_3\}$.

Weighted homogeneous weights (1, 1),

$$\text{degree}(x^3y) = 4,\ \text{degree}(y^{2k+1}) = 2k+1,\ \text{degree}(xy^{2k+1}) = 2k+2,$$

hence for $k \geq 2$ $\quad T\mathscr{A}^{2k+2}(x, y^2, x^3y + axy^{2k+1}) = T\mathscr{A}^{2s+2}f + \mathbb{C}\{axy^{2k+1}.e_3\}$

and $\quad T\mathscr{A}^{2k+1}(x, y^2, x^3y + ay^{2k+1}) = T\mathscr{A}^{2k+1}f + \mathbb{C}\{ay^{2k+1}.e_3\}$.

Thus by I:2 $(x, y^2, x^3y + y^{2k+1})$ is $(2k+1)$-determined for $k \geq 2$.

Hence $g \in \mathscr{A}^4f \Rightarrow$

either $g \sim f$ or $g \sim (x, y^2, x^3y + y^{2k+1})$ or $g \in \mathscr{A}^{2k+2}(x, y^2, x^3y + axy^{2k+1})$.

(II·2:14) $\qquad f = (x, y^2, x^3y + xy^{4k+1})$.

We show that $\mathscr{A}^{4k+2}(x, y^2, x^3y + axy^{4k+1}) = \mathscr{A}^{4k+2}(x, y^2, x^3y + x^2y^{2k+1})$.

Let $x = x' - \frac{1}{3}y^{2k}$ and $X' = X + \frac{1}{3}Y^k$. Then $(x, y^2, x^3y + x^2y^{2k+1})$ is equivalent to

$$\left(x', y^2, (x' - \tfrac{1}{3}y^{2k})^3y + (x' - \tfrac{1}{3}y^{2k})^2y^{2k+1}\right)$$

$$= \left(x, y^2, (x')^3y - 3(x')^2\tfrac{1}{3}y^{2k+1} + 3x'\tfrac{1}{9}y^{4k+1} - \tfrac{1}{27}y^{6k+1} + (x')^2y^{2k+1} - 2x'\tfrac{1}{3}y^{4k+1} + \tfrac{1}{9}y^{6k+1}\right)$$

$$= \left(x, y^2, (x')^3y - \tfrac{1}{3}x'y^{4k+1} + \tfrac{2}{27}y^{6k+1}\right),$$

so

$$\mathscr{A}^{4k+2}(x, y^2, x^3y + x^2y^{2k+1}) = \mathscr{A}^{4k+2}\left(x, y^2, x^3y - \tfrac{1}{3}xy^{4k+1}\right)$$

$$= \mathscr{A}^{4k+2}(x, y^2, x^3y + xy^{4k+1}).$$

Now $(x, y^2, x^3y + x^2y^{2k+1}) = (x, y^2, x^2y(x + y^{2k}))$, which has two double curves,

$$(0, y) \mapsto (0, y^2, 0) \text{ and } (-y^{2k}, y) \mapsto (-y^{2k}, y^2, 0).$$

The curve $f(0, y)$ is a double curve with non-transverse self-intersection.

The map-germ (x, y^2, x^3y) has only one double curve. Hence (x, y^2, x^3y) is not a stem.

(II·2:15) $\qquad f = (x, y^2, x^3y + x^2y^3).$

$p(x, y) = x^3 + x^2y \qquad T\mathscr{K}^T p\circ h = \mathfrak{m}_2^T\backslash\{x, x^2, xy^2, y^{2j}: j \geq 1\},$

$T\mathscr{A}f \qquad A + \left(\mathfrak{m}_2\backslash\{xy, x^2y, xy^3, y^{2j+1}: j \geq 0\}\right)\cdot e_3,$

basis $\qquad \left\{y^{2j+1}\cdot e_3\right\},$

multiplicity $\qquad 1,$

$f^*\mathfrak{m}_3\cdot T\mathscr{A}f \qquad B + \left(\mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, x^3y, xy^3, x^2y^3, xy^5, y^{2j+1}: j \geq 1\}\right),$

$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \left\{y^{2j+1}\cdot e_3\right\}\right) \supseteq \mathfrak{m}_2^{2j+2}\cdot\theta(f)$ provided that $j \geq 2$.

$$T\mathscr{A}_{2,2}f \qquad C+\left(\mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, x^3y, xy^3, x^2y^3, xy^5, y^{2j+1}: j \geq 1\}\right)+\mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ 3x^2y^3 + 2xy^5 \end{pmatrix}\right\},$$

for $k \geq 3$ $\; g \in \mathscr{A}^{2k}f \Rightarrow \qquad g \in \mathscr{A}^{2k+1}(x, y^2, x^3y + x^2y^3 + ay^{2k+1})$

and $g \in \mathscr{A}^{2k+1}f \Rightarrow \qquad g \in \mathscr{A}^{2k+2}(x, y^2, x^3y + x^2y^3).$

Weighted homogeneous $\qquad$ weights (2, 1),

$\qquad \text{degree}(x^3y + x^2y^3) = 7, \; \text{degree}(y^{2k+1}) = 2k+1,$

hence for $k \geq 4 \qquad T\mathscr{A}^{2k+1}(x, y^2, x^3y + x^2y^3 + ay^{2k+1}) = T\mathscr{A}^{2k+1}f + \mathbb{C}\{ay^{2k+1}\cdot e_3\}.$

By I:2 $(x, y^2, x^3y + x^2y^3 + y^{2k+1})$ is $(2k+1)$-determined for $k \geq 4$.

Hence $g \in \mathscr{A}^9 f \Rightarrow$ either $g \sim f$ or $g \sim (x, y^2, x^3y + x^2y^3 + y^{2k+1})$ and f is a stem.

(II·2:16) $\qquad f = (x, y^2, x^4y + 2x^2y^3 + y^5).$

$p(x, y) = x^4 + 2x^2y + y^2 \qquad T\mathscr{K}^T p\circ h = \{x^jy^{2\ell}(x^2+y^2): j + 2\ell \geq 2\},$

$T\mathscr{A}f \qquad A + \left(\mathfrak{m}_2^T \cup \mathbb{C}\{x^jy^{2\ell+1}(x^2+y^2): j + 2\ell \geq 2\}\right)\cdot e_3,$

basis $\qquad \left\{x^cy\cdot e_3\right\},$

multiplicity $\qquad 1,$

$f^*\mathfrak{m}_3\cdot T\mathscr{A}f \qquad B + \left(\mathfrak{m}_2^T\right)^2 + \mathbb{C}\{x^jy^{2\ell+1}(x^2+y^2): j + 2\ell + 1 \geq 4\},$

$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \left\{x^jy\cdot e_3\right\}\right) \supseteq \mathfrak{m}_2^{j+2}\cdot\theta(f)$ provided that $j \geq 5$.

$T\mathscr{A}_{2,2}f$ $\qquad D + \left(\mathfrak{m}_2^T\right)^2 + \mathbb{C}\{x^j y^{2\ell+1}(x^2+y^2) : j + 2\ell + 1 \geq 4\}\Big)$,

hence for $k \geq 5$ $g \in \mathscr{A}^k f \Rightarrow$ $\qquad g \in \mathscr{A}^{k+1}(x, y^2, x^4y + 2x^2y^3 + y^5 + ax^ky)$.

Weighted homogeneous $\qquad$ weights $(1, 1)$,

$$\text{degree}(x^4y + 2x^2y^3 + y^5) = 5, \text{degree}(x^ky) = k+1,$$

so for $k \geq 5$ $\qquad T\mathscr{A}^{k+1}(x, y^2, x^4y + 2x^2y^3 + y^5 + ax^ky) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ax^ky \cdot e_3\}$.

By I:2 $(x, y^2, x^4y + 2x^2y^3 + y^5 + x^ky)$ is $(k+1)$-determined for $k \geq 5$.

Hence $g \in \mathscr{A}^5 f \Rightarrow$ either $g \sim f$ or $g \sim (x, y^2, x^4y + 2x^2y^3 + y^5 + x^ky)$ and f is a stem.

(II·2:17) $\qquad f = (x, y^2, x^4y + x^2y^3)$.

$p(x, y) = x^4 + x^2y$ $\qquad T\mathscr{K}^T p \circ h = \mathfrak{m}_2^T \backslash \{x, x^2, x^3, xy^2, y^{2j} : j \geq 0\}$,

$T\mathscr{A}f$ $\qquad A + \left(\mathfrak{m}_2 \backslash \{xy, x^2y, x^3y, xy^3, y^{2j+1} : j \geq 0\}\right) \cdot e_3$,

basis $\qquad \left\{y^{2j+1} \cdot e_3\right\}$,

multiplicity $\qquad 1$,

$f^*\mathfrak{m}_3 \cdot T\mathscr{A}f$ $\qquad B + \left(\mathfrak{m}_2^2 \backslash \{xy, y^2, x^2y, x^3y, xy^3, x^4y, x^2y^3, xy^5, y^{2j+1} : j \geq 1\}\right) \cdot e_3$,

$f^*\mathfrak{m}_3 \cdot \left(T\mathscr{A}f + \left\{y^{2j+1} \cdot e_3\right\}\right) \supseteq \mathfrak{m}_2^{2j+2} \cdot \theta(f)$ provided that $j \geq 2$.

$T\mathscr{A}_{2,2}f$ $\qquad C + \left(\mathfrak{m}_2^2 \backslash \{xy, y^2, x^2y, x^3y, xy^3, x^4y, x^2y^3, xy^5, y^{2j+1} : j \geq 1\}\right) \cdot e_3 + \mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ xy^5 \end{pmatrix}\right\}$,

hence $g \in \mathscr{A}^5 f \Rightarrow$ $\qquad g \in \mathscr{A}^6(x, y^2, x^4y + x^2y^3 + axy^5)$,

for $k \geq 3$ $g \in \mathscr{A}^{2k} f \Rightarrow$ $\qquad g \in \mathscr{A}^{2k+1}(x, y^2, x^4y + x^2y^3 + ay^{2k+1})$

and $g \in \mathscr{A}^{2k+1} f \Rightarrow$ $\qquad g \in \mathscr{A}^{2k+2}(x, y^2, x^4y + x^2y^3)$.

Weighted homogeneous $\qquad$ weights $(1, 1)$,

$$\text{degree}(x^4y + x^2y^3) = 5, \text{degree}(xy^5) = 6, \text{degree}(y^{2k+1}) = 2k+1,$$

hence $\qquad T\mathscr{A}^6(x, y^2, x^4y + x^2y^3 + axy^5) = T\mathscr{A}^{2k+1}f + \mathbb{C}\{axy^5 \cdot e_3\} \ni xy^5 \cdot e_3$

and without loss of generality $a = 0$.

For $k \geq 3$ $\qquad T\mathscr{A}^{2k+1}(x, y^2, x^4y + x^2y^3 + ay^{2k+1}) = T\mathscr{A}^{2k+1}f + \mathbb{C}\{ay^{2k+1} \cdot e_3\}$.

Hence by I·2 $(x, y^2, x^4y + x^2y^3 + y^{2k+1})$ is $(2k+1)$-determined for $k \geq 3$. Hence $g \in \mathscr{A}^5 f \Rightarrow$ either $g \sim f$ or $g \sim (x, y^2, x^4y + x^2y^3 + y^{2k+1})$ and f is a stem.

(II·2:18) $\qquad f = (x, y^2, x^2y^3 + y^5).$

$p(x, y) = x^2y + y^2$	$T\mathscr{K}^T p\circ h = \mathfrak{m}_2^T \backslash \{xy^2, y^2, x^j : j \geq 0\},$
$T\mathscr{A}f$	$A + \left(\mathfrak{m}_2 \backslash \{y, y^3, xy^3, x^jy : j \geq 0\}\right)\cdot e_3,$
basis	$\left\{x^jy\cdot e_3\right\},$
multiplicity	$1,$
$f^*\mathfrak{m}_3\cdot T\mathscr{A}f$	$B + \left(\mathfrak{m}_2^2 \backslash \{y^2, y^3, xy^3, y^5, x^2y^3, x^jy : j \geq 1\}\right)\cdot e_3,$

$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \left\{x^jy\cdot e_3\right\}\right) \supseteq \mathfrak{m}_2^{j+2}\cdot\theta(f)$ provided that $j \geq 4$.

$T\mathscr{A}_{2,2}f$	$D + \left(\mathfrak{m}_2^2 \backslash \{y^2, y^3, xy^3, y^5, x^2y^3, x^jy : j \geq 1\}\right)\cdot e_3,$
hence for $k \geq 5$ $g \in \mathscr{A}^k f \Rightarrow$	$g \in \mathscr{A}^{k+1}(x, y^2, x^2y^3 + y^5 + ax^ky).$
Weighted homogeneous	weights $(1, 1),$
	$\text{degree}(x^2y^3 + y^5) = 5, \text{degree}(x^ky) = k+1,$

hence for $k \geq 5$ $\quad T\mathscr{A}^{k+1}(x, y^2, x^2y^3 + y^5 + ax^ky) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ax^ky\cdot e_3\}.$

By I·2 $(x, y^2, x^2y^3 + y^5 + x^ky)$ is $(k+1)$-determined for $k \geq 5$. Hence $g \in \mathscr{A}^5 f \Rightarrow$ either $g \sim f$ or $g \sim (x, y^2, x^2y^3 + y^5 + x^ky)$ and f is a stem.

(II·2:19) $\qquad f = (x, y^2, y^5).$

$p(x, y) = y^2$	$T\mathscr{K}^T p\circ h = \mathfrak{m}_2^T \backslash \{x^j, x^jy^2 : j \geq 0\},$
$T\mathscr{A}f$	$A + \left(\mathfrak{m}_2 \backslash \{x^jy, x^jy^3 : j \geq 0\}\right)\cdot e_3,$
basis	$\left\{x^jy\cdot e_3, x^jy^3\cdot e_3\right\},$
multiplicity	$2,$
$f^*\mathfrak{m}_3\cdot T\mathscr{A}f$	$B + \left(\mathfrak{m}_2^2 \backslash \{y^2, y^3, y^5, x^jy, x^jy^3 : j \geq 1\}\right)\cdot e_3,$

$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \left\{x^jy\cdot e_3\right\}\right) \supseteq \mathfrak{m}_2^{j+2}\cdot\theta(f)$ provided that $j \geq 4$.

$T\mathscr{A}_{2,2}f$ $\qquad D + \left(\mathfrak{m}_2^2\backslash\{y^2, y^3, y^5, x^jy, x^jy^3 : j \geq 1\}\right)\cdot e_3,$

hence for $k \geq 5$ $g \in \mathscr{A}^k f \Rightarrow$ $\qquad g \in \mathscr{A}^{k+1}(x, y^2, y^5 + ax^ky + bx^{k-2}y^3).$

$$\text{Now } \mathbb{C}\left\{\begin{pmatrix} 0 \\ 0 \\ y^5 + ax^ky + bx^{k-2}y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \\ 5y^5 + ax^ky + 3bx^{k-2}y^3 \end{pmatrix}, \begin{pmatrix} x \\ 0 \\ kax^ky + (k-2)bx^{k-2}y^3 \end{pmatrix}\right\}$$

$$\sim \mathbb{C}\left\{\begin{pmatrix} 0 \\ 0 \\ y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ ax^ky \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ bx^{k-2}y^3 \end{pmatrix}\right\} \subseteq T\mathscr{A}(x, y^2, y^5 + ax^ky + bx^{k-2}y^3).$$

Hence $\qquad T\mathscr{A}^{k+1}(x, y^2, y^5 + ax^ky + bx^{k-2}y^3) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ax^ky\cdot e_3, bx^{k-2}y^3\cdot e_3\}.$

The (k+1)-orbits are $\qquad (x, y^2, y^5 + x^ky + x^{k-2}y^3)$

$(x, y^2, y^5 + x^{k-2}y^3)$, $(x, y^2, y^5 + x^ky)$ and (x, y^2, y^5).

By I:2 (x, y^2, y^5+x^ky) and $(x, y^2, y^5+x^ky+x^{k-2}y^3)$ are (k+1)-determined for $k \geq 5$. The map-germ $(x, y^2, y^5+ x^{k-2}y^3)$ is not finitely determined since $(x, 0) \mapsto (x, 0, 0)$ is a cuspidal edge. Also $\{(x, \pm y) : y^2+ x^{k-2}= 0\} \mapsto (x, y^2,0)$ is a double curve, while (x, y^2, y^5) has no double curves so $(x, y^2, y^5+ x^{k-2}y^3)$ and (x, y^2, y^5) are not equivalent.

Hence f is not a stem and $g \in \mathscr{A}^5 f \Rightarrow g$ is either equivalent to one of f, (x, y^2, y^5+x^ky) or $(x, y^2, y^5+x^ky+x^{k-2}y^3)$ or $g \in \mathscr{A}^{k+1}(x, y^2, y^5+ x^{k-2}y^3)$.

(II·2:20) $\qquad f = (x, y^2, x^2y^3).$

$p(x, y) = x^2y$ $\qquad T\mathscr{K}^T p\circ h = \mathfrak{m}_2^T\backslash\{x^j, xy^2, y^{2j} : j \geq 0\},$

$T\mathscr{A}f$ $\qquad A + \left(\mathfrak{m}_2\backslash\{xy^3, y^{2j+1}, x^jy : j \geq 0\}\right)\cdot e_3,$

basis $\qquad \left\{y^{2j+1}\cdot e_3, x^jy\cdot e_3\right\},$

multiplicity $\qquad 2,$

$f^*\mathfrak{m}_3\cdot T\mathscr{A}f$ $\qquad B + \mathfrak{m}_2^2\backslash\{y^2, xy^5, x^2y^3, y^{2j+1}, x^jy : j \geq 1\}\cdot e_3,$

$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \left\{y^{2j+1}\cdot e_3, x^{2j}y\cdot e_3\right\}\right) \supseteq \mathfrak{m}_2^{2j+2}\cdot\theta(f)$ provided that $j \geq 2$.

$T\mathscr{A}_{2,2}f$ $\qquad C + \left(m_2^2 \backslash \{y^2, xy^5, x^2y^3, y^{2j+1}, x^jy : j \geq 1\}\right)\cdot e_3 + \mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ 2xy^5 \end{pmatrix}\right\}$,

hence for $k \geq 3$ $g \in \mathscr{A}^{2k}f \Rightarrow$ $g \in \mathscr{A}^{2k+1}(x, y^2, x^2y^3 + ax^{2k}y + by^{2k+1})$,

and $g \in \mathscr{A}^{2k-1}f \Rightarrow$ $g \in \mathscr{A}^{2k}(x, y^2, x^2y^3 + ax^{2k-1}y)$.

Weighted homogeneous weights $(1, 1)$,

$\text{degree}(x^2y^3 + y^5) = 5, \text{degree}(x^ky) = k+1$,

so for $k \geq 5$ $\quad T\mathscr{A}^{k+1}(x, y^2, x^2y^3 + ax^ky) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ax^ky\cdot e_3\}$, while

$$\mathbb{C}\left\{\begin{pmatrix} 0 \\ 0 \\ x^2y^3 + ax^{2k}y + by^{2k+1} \end{pmatrix}, \begin{pmatrix} 0 \\ 2y^2 \\ 3x^2y^3 + ax^{2k}y + (2k+1)by^{2k+1} \end{pmatrix}, \begin{pmatrix} x \\ 0 \\ 2x^2y^3 + 2kax^{2k}y \end{pmatrix}\right\}$$

$$\sim \mathbb{C}\left\{\begin{pmatrix} 0 \\ 0 \\ x^2y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ ax^{2k}y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ by^{2k+1} \end{pmatrix}\right\} \subseteq T\mathscr{A}(x, y^2, x^2y^3 + ax^{2k}y + by^{2k+1})$$

so that $\quad T\mathscr{A}^{k+1}(x, y^2, x^2y^3 + ax^{2k}y + by^{2k+1}) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ax^{2k}y\cdot e_3, by^{2k+1}\cdot e_3\}$.

By I:2 $(x, y^2, x^2y^3 + x^{2k}y + y^{2k+1})$ is $2k+1$-determined for $k \geq 3$.

The map-germs $(x, y^2, x^2y^3 + x^ky)$ and $(x, y^2, x^2y^3 + y^{2k+1})$ are neither finitely determined nor equivalent to (x, y^2, x^2y^3) and hence f is not a stem.

If $g \in \mathscr{A}^5f$ then either $g \sim f$ or $g \sim (x, y^2, x^2y^3 + x^{2k}y + y^{2k+1})$

or $g \in \left\{\mathscr{A}^{k+1}(x, y^2, x^2y^3 + x^ky), \mathscr{A}^{2k+1}(x, y^2, x^2y^3 + y^{2k+1})\right\}$.

(II·2:21) $\qquad f = (x, y^2, x^4y)$.

$p(x, y) = x^4$ $\qquad T\mathscr{K}^T p\circ h = m_2^T \backslash \{x^3, y^{2j}, xy^{2j}, x^2y^{2j} : j \geq 0\}$,

$T\mathscr{A}f$ $\qquad A + \left(m_2 \backslash \{x^3y, y^{2j+1}, xy^{2j+1}, x^2y^{2j+1} : j \geq 0\}\right)\cdot e_3$,

basis $\qquad \left\{y^{2j+1}\cdot e_3, xy^{2j+1}\cdot e_3, x^2y^{2j+1}\cdot e_3\right\}$,

multiplicity $\qquad 3$,

$f^*m_3\cdot T\mathscr{A}f$ $\qquad B + m_2^2 \backslash \{xy, y^2, x^3y, x^4y, x^3y^3, y^{2j+1}, xy^{2j+1}, x^2y^{2j+1} : j \geq 1\}\cdot e_3$,

$f^*\mathfrak{m}_3\cdot\left(T\mathcal{A}f+\{y^{2j+1}\cdot e_3\}\right)\supseteq \mathfrak{m}_2^{2j+2}\cdot\theta(f)$ provided that $j\geq 3$.

$$T\mathcal{A}_{2,2}f \quad C+\mathfrak{m}_2^2\backslash\{xy, y^2, x^3y, x^4y, x^3y^3, y^{2j+1}, xy^{2j+1}, x^2y^{2j+1}: j\geq 1\}\big)\cdot e_3+\mathbb{C}\left\{\begin{pmatrix} y^2\\ 0\\ 3x^3y^3\end{pmatrix}\right\},$$

hence $g\in\mathcal{A}^5f\Rightarrow$ $\quad g\in\mathcal{A}^6(x, y^2, x^4y+ax^3y^3+bxy^5)$,

for $k\geq 6$ $g\in\mathcal{A}^kf\Rightarrow$ $\quad g\in\mathcal{A}^{k+1}(x, y^2, x^4y+ax^2y^{2s-1}+by^{2s+1})$ if $k=2s$

or $\quad g\in\mathcal{A}^{k+1}(x, y^2, x^4y+axy^{2s+1})$ if $k=2s+1$.

Weighted homogeneous $\quad$ weights $(1, 1)$,

$\text{degree}(x^2y^3+y^5)=5, \text{degree}(x^ky)=k+1$,

so for $k\geq 5$ $\quad T\mathcal{A}^{k+1}(x, y^2, x^4y+axy^{2s+1})=T\mathcal{A}^{k+1}f+\mathbb{C}\{axy^{2s+1}\cdot e_3\}$.

Now

$$\mathbb{C}\left\{\begin{pmatrix}0\\0\\x^4y+ax^2y^{2s-1}+by^{2s+1}\end{pmatrix},\begin{pmatrix}0\\2y^2\\x^4y+(2s-1)ax^2y^{2s-1}+(2s+1)by^{2s+1}\end{pmatrix},\begin{pmatrix}x\\0\\4x^4y+2ax^2y^{2s-1}\end{pmatrix}\right\}$$

$$\sim\mathbb{C}\left\{\begin{pmatrix}0\\0\\x^4y\end{pmatrix},\begin{pmatrix}0\\0\\axy^{2s-1}\end{pmatrix},\begin{pmatrix}0\\0\\by^{2s+1}\end{pmatrix}\right\}\subseteq T\mathcal{A}(x, y^2, x^4y+ax^2y^{2s-1}+by^{2s+1})$$

so $\quad T\mathcal{A}^{k+1}(x, y^2, x^4y+ax^2y^{2s-1}+by^{2s+1})=T\mathcal{A}^{k+1}f+\mathbb{C}\{ax^2y^{2s-1}\cdot e_3, by^{2k+1}\cdot e_3\}$.

By I:2 $(x, y^2, x^4y+x^2y^{2s-1}+y^{2s+1})$ and (x, y^2, x^4y+y^{2s+1}) are $(2s+1)$-determined for $s\geq 3$.

The map-germ $(x, y^2, x^4y+x^2y^{2s-1})$ is neither finitely determined nor equivalent to (x, y^2, x^4y).

Hence $g\in\mathcal{A}^5f\Rightarrow$ either $g\sim$ one of

$$f, (x, y^2, x^4y+x^2y^{2s-1}+y^{2s+1}) \text{ or } (x, y^2, x^4y+y^{2s+1})$$

or g lies in one of

$$\mathscr{A}^{2s}(x, y^2, x^4y + xy^{2s-1}) \text{ or } \mathscr{A}^{2s+1}(x, y^2, x^4y + x^2y^{2s-1})$$

The map-germ $(x, y^2, x^4y + x^2y^{2s-1})$ shows that f is not a stem.

(II·2:22) $$f = (x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9)$$

Note that f is not weighted homogeneous.

$p(x, y) = y$ $$T\mathscr{T}^T p\circ h = x\cdot\left(\mathfrak{m}_2^T\right)^7 \backslash \{x^8, x^9, x^6y^4\},$$
$$+\mathbb{C}\{x^2y^{10}, xy^{12}, x^9 - x^3y^8, x^8 - x^2y^8, x^5y^4 + 2x^2y^8, x^6y^4 - + 2x^3y^8,$$
$$5x^4y^6 + 2xy^{10} - 14x^3y^8\},$$

$T\mathscr{A}f$ $$A + \left(\mathfrak{m}_2^T + yT\mathscr{T}^T p\circ h\right)\cdot e_3,$$

basis $$\left\{y^{2j+1}.e_3\right\},$$

multiplicity $$1,$$

By I·2:4 and I:1 there is a value k_0 such that for $k \geq k_0$,

$g \in \mathscr{A}^k f \Rightarrow$ $$g \in \mathscr{A}^{k+1}(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2s+1}) \text{ if } k = 2s$$

or $$g \in \mathscr{A}^{k+1}(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9) \text{ if } k = 2s+1.$$

(II·2:23) $$f = (x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1}).$$

$p(x, y) = y$ $$T\mathscr{T}^T p\circ h = x\cdot\left(\mathfrak{m}_2^T\right)^7 \backslash \{x^8, x^9, x^{10}, x^7y^2, x^6y^4, x^5y^6, x^4y^{12}, x^4y^{12}, x^7y^4\}$$
$$+\mathbb{C}\{ay^{2k+2j+4}: j \geq 0, x^2y^{10} + (2k-9)ay^{2k+2}, 5x^4y^8 + 2xy^{12} + 8(16-3k)ay^{2k+2}, x^9 - x^3y^8,$$
$$x^8 - x^2y^8 - (2k-9)ay^{2k}, x^5y^4 + 2x^2y^8 + (8-k)ay^{2k}, x^6y^4 + 2x^3y^8, 5x^4y^6 + 2xy^{10} - 14x^3y^8 +$$
$$8(16-3k)ay^{2k}, x^7y^2 + (3k-16)ay^{2k}, x^7y^4 + (3k-16)ay^{2k+2}, x^5y^6 + 2x^2y^{10} + (8-k)ay^{2k+2}\},$$

$T\mathscr{A}f$ $$A + \left(\mathfrak{m}_2^T + yT\mathscr{T}^T p\circ h\right)\cdot e_3,$$

Note that $T\mathscr{A}f \ni (0, 0, y^{2k+5})$ but $T\mathscr{A}f \not\ni (0, 0, y^{2k+3})$ so f is at worst (2k+6)-determined but is not (2k+3)-determined.

The next step for this map-germ would be to either calculate $T\mathscr{A}_{2,2}f$ for $(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1})$ or to calculate $T\mathscr{A}_{4,3}f$ ($\kappa = 3$) for $(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9)$ with the aim of trying to find out if the family

$$(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9 + ay^{2k+1} + by^{2k+3})$$

is 2k+3-determined for all b and all $a \neq 0$. If this is the case then

$$(x, y^2, x^8y + x^7y^3 + x^5y^5 + x^2y^9)$$

is a stem.

II·3 : $j^2f = (x, xy, 0)$.

(II·3:1) $$f = (x, xy, 0).$$

$$T\mathscr{A}_{2,2}f = \mathfrak{m}_2^2\left\{\begin{pmatrix}1\\y\\0\end{pmatrix}, \begin{pmatrix}0\\x\\0\end{pmatrix}\right\} + \{x^{i+j}y^j \cdot e_r : i+j \geq 2,\ 1 \leq r \leq 3\}$$

$$= \begin{pmatrix} \mathfrak{m}_2^2 \setminus \{y^k\} \\ \mathfrak{m}_2^2 \setminus \{xy, y^k : k \geq 2\} \\ \{x^{i+j}y^j : i+j \geq 2\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix}y^k\\y^{k+1}\\0\end{pmatrix} : k \geq 2\right\}.$$

Hence $g \in \mathscr{A}^2 f \Rightarrow$ $$g \in \mathscr{A}^3(x, xy + ay^3, bxy^2 + cy^3),$$

$g \in \mathscr{A}^3 f \Rightarrow$ $$g \in \mathscr{A}^4(x, xy + ay^4, bxy^3 + cy^4),$$

$g \in \mathscr{A}^4 f \Rightarrow$ $$g \in \mathscr{A}^5(x, xy + ay^5, bx^2y^3 + cxy^4 + dy^5), \text{ etc.}$$

(II·3:2) $$f = (x, xy + ay^3, bxy^2 + cy^3).$$

$$T\mathscr{A}^3 f = \begin{pmatrix} \mathfrak{m}_2 \setminus \{y, y^2\} \\ \mathfrak{m}_2 \setminus \{y, y^2, y^3\} \\ \mathfrak{m}_2 \setminus \{y, xy, y^2, xy^2, y^3\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{\begin{pmatrix}y^2\\y^3\\0\end{pmatrix}, \begin{pmatrix}y\\y^2\\by^3\end{pmatrix}, \begin{pmatrix}0\\0\\xy+ay^3\end{pmatrix}, \begin{pmatrix}0\\ay^3\\0\end{pmatrix}, \begin{pmatrix}0\\cy^3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\bxy^2\end{pmatrix}, \begin{pmatrix}0\\0\\cxy^2\end{pmatrix}, \begin{pmatrix}0\\0\\cy^3\end{pmatrix}\right\}.$$

The calculation of $T\mathscr{A}^3 f$ was performed by computer.

3-orbits : $$\left\{(x, xy + ay^3, bxy^2 + cy^3) : a, b \in \mathbb{C}, c \in \mathbb{C}\setminus\{0\}\right\} \ni (x, xy, y^3),$$
$$\left\{(x, xy + ay^3, bxy^2) : a, b \in \mathbb{C}\setminus\{0\}\right\} \ni (x, xy + y^3, xy^2),$$
$$\left\{(x, xy + ay^3, 0) : a \in \mathbb{C}\setminus\{0\}\right\} \ni (x, xy + y^3, 0),$$
$$\left\{(x, xy, bxy^2) : b \in \mathbb{C}\setminus\{0\}\right\} \ni (x, xy, xy^2).$$

Hence $g \in \mathscr{A}^2 f \Rightarrow$ $$g \in \left\{\mathscr{A}^3(x, xy, y^3), \mathscr{A}^3(x, xy + y^3, xy^2), \mathscr{A}^3(x, xy + y^3, 0),\right.$$

$$\mathscr{A}^3(x, xy, xy^2), \mathscr{A}^3(x, xy, 0)\}.$$

(II·3:3) $$f = (x, xy + ay^4, bxy^3 + cy^4).$$

$$T\mathscr{A}^4 f \quad \begin{pmatrix} \mathfrak{m}_2\backslash\{y, y^2, y^3\} \\ \mathfrak{m}_2\backslash\{y, y^2, y^3, y^4\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, xy^2, y^3, xy^3, y^4\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{\begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix} \begin{pmatrix} y \\ y^2 \\ by^4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy + ay^4 \end{pmatrix}, \begin{pmatrix} 0 \\ ay^4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ cy^4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ bxy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ cxy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ cy^4 \end{pmatrix}\right\}.$$

The calculation of $T\mathscr{A}^4 f$ was performed by computer.

3-orbits : $$\{(x, xy + ay^4, bxy^3 + cy^4) : a, b \in \mathbb{C}, c \in \mathbb{C}\backslash\{0\}\} \ni (x, xy, y^4),$$
$$\{(x, xy + ay^4, bxy^3) : a, b \in \mathbb{C}\backslash\{0\}\} \ni (x, xy + y^4, xy^3),$$
$$\{(x, xy + ay^4, 0) : a \in \mathbb{C}\backslash\{0\}\} \ni (x, xy + y^4, 0),$$
$$\{(x, xy, bxy^3) : b \in \mathbb{C}\backslash\{0\}\} \ni (x, xy, xy^3).$$

Hence $g \in \mathscr{A}^3 f \Rightarrow$ $$g \in \{\mathscr{A}^4(x, xy, y^4), \mathscr{A}^4(x, xy + y^4, xy^3), \mathscr{A}^4(x, xy + y^4, 0), \mathscr{A}^4(x, xy, xy^3), \mathscr{A}^4(x, xy, 0)\}.$$

(II·3:4) $$f = (x, xy, y^3).$$

$$T\mathscr{A} f \quad \begin{pmatrix} \mathfrak{m}_2\backslash\{y^{3j+1} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^2\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix} : j \geq 0\right\}.$$

The calculation of $T\mathscr{A} f$ will be found at A:1.

Basis $$\{y^{3j+2}e_2 : j \geq 1\},$$

multiplicity $$1,$$

$$f^*\mathfrak{m}_3\cdot T\mathscr{A}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{xy, y^2, y^3, y^{3j+1} : j\geq 1\} \\ \mathfrak{m}_2^2\backslash\{xy, xy^2, y^3, y^4, y^{3j+2} : j\geq 0\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, xy^2, y^3, y^4, y^5\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix} : j\geq 1\right\},$$

$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \mathbb{C}\{y^{3j+2}.e_2\}\right) \supseteq \mathfrak{m}_2^{3j+3}\cdot\theta(f)$ provided that $j\geq 1$.

$$T\mathscr{A}_{2,2}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{xy, y^2, y^3, y^{3j+1} : j\geq 1\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, xy^2, y^3, y^4, y^{3j+2} : j\geq 0\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, xy^2, y^3, y^4\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 \\ 3y^4 \end{pmatrix} : j\geq 1\right\},$$

hence $g\in\mathscr{A}^3f \Rightarrow$ $\qquad g\in\mathscr{A}^4(x, xy+ay^4, y^3+by^4)$,

for $k\geq 4$ $g\in\mathscr{A}^kf \Rightarrow$ $\qquad g\in\mathscr{A}^{k+1}(x, xy+ay^{3s+2}, y^3)$ if $k = 3s+1$

or $g\in\mathscr{A}^{k+1}(x, xy, y^3)$ if $k\neq 3s+1$.

Weighted homogeneous, $\qquad$ weights (1, 1).

Hence $T\mathscr{A}^{3s+2}(x, xy+ay^{3s+2}, y^3) = T\mathscr{A}^{3s+2}(x, xy, y^3) + \mathbb{C}\{ay^{3s+2}\}$

By I:2 $(x, xy+y^{3s+2}, y^3)$ is $(3s+2)$-determined for $s\geq 1$.

By computer calculation $T\mathscr{A}^4(x, xy+ay^4, y^3+by^4) = T\mathscr{A}^4(x, xy, y^3)$.

Hence $g\in\mathscr{A}^3f \Rightarrow g\sim(x, xy+y^{3s+2}, y^3)$ for some $s\geq 1$ or $g\in\bigcap_{k\geq 0}\mathscr{A}^kf$ and f is a weak stem.

(II·3:5) $\qquad f = (x, xy+y^3, xy^2)$.

$$T\mathscr{A}f \qquad \begin{pmatrix} \mathfrak{m}_2\backslash\{y, y^{3j+2} : j\geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^2\} \\ \mathfrak{m}_2\backslash\{xy, y^2, y^3, y^{3j+1} : j\geq 0\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix} : j\geq 0\right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:2.

Basis $\{y^{3j+1}e_3 : j \geq 1\}$,

multiplicity 1,

$$f^*\mathfrak{m}_3 \cdot T\mathscr{A}f \quad \begin{pmatrix} \mathfrak{m}_2^2 \backslash \{xy, y^2, xy^2, y^3, y^4, y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2^2 \backslash \{xy, y^2, xy^2, y^3, y^4, y^5\} \\ \mathfrak{m}_2^2 \backslash \{xy, y^2, x^2y, xy^2, y^3, xy^3, xy^4, y^5, y^6, y^{3j+1} : j \geq 1\} \end{pmatrix}$$

$$+\mathbb{C}\left\{ \begin{pmatrix} 0 \\ 0 \\ x^2y + xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2xy^4 + y^6 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}, \begin{pmatrix} xy \\ xy^2 \\ xy^3 \end{pmatrix}, \begin{pmatrix} y^4 \\ y^5 \\ y^6 \end{pmatrix}, \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1 \right\},$$

$f^*\mathfrak{m}_3 \cdot \left(T\mathscr{A}f + \mathbb{C}\{y^{3j+1} \cdot e_3\}\right) \supseteq \mathfrak{m}_2^{3j+2} \cdot \theta(f)$ provided that $j \geq 2$.

$$T\mathscr{A}_{2,2}f \quad \begin{pmatrix} \mathfrak{m}_2^2 \backslash \{xy, xy^2, y^3, y^4, y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2^2 \backslash \{xy, y^2, xy^2, y^3, y^4, y^5\} \\ \mathfrak{m}_2^2 \backslash \{xy, y^2, x^2y, xy^2, y^3, xy^3\ xy^4, y^5, y^6, y^{3j+1} : j \geq 1\} \end{pmatrix}$$

$$+\mathbb{C}\left\{ \begin{pmatrix} 8xy^2 + 3y^4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y + xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2xy^4 + y^6 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ y^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 + 3y^4 \\ 2xy^3 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 \\ 3y^5 \\ 2xy^4 \end{pmatrix}, \begin{pmatrix} xy \\ xy^2 \\ xy^3 \end{pmatrix}, \begin{pmatrix} y^4 \\ y^5 \\ y^6 \end{pmatrix}, \begin{pmatrix} y^4 \\ 4y^5 \\ 0 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}, \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1 \right\},$$

hence $g \in \mathscr{A}^3 f \Rightarrow$ $g \in \mathscr{A}^4(x, xy + y^3 + ay^4, xy^2 + bxy^3 + cy^4)$,

$g \in \mathscr{A}^4 f \Rightarrow$ $g \in \mathscr{A}^5(x, xy + y^3, xy^2 + ay^5)$,

$g \in \mathscr{A}^5 f \Rightarrow$ $g \in \mathscr{A}^6(x, xy + y^3, xy^2 + ay^6)$,

for $k \geq 6$ $g \in \mathscr{A}^k f \Rightarrow$ $g \in \mathscr{A}^{k+1}(x, xy + y^3, xy^2 + ay^{3s+1})$ if $k = 3s$

or $\mathscr{A}^{k+1}(x, xy + y^3, xy^2)$ if $k \neq 3s$.

Weighted homogeneous weights (2, 1),

degree$(xy^2) = 4$, degree$(y^k) = k$.

Hence $$T\mathscr{A}^{3s+1}(x, xy+y^3, xy^2+ay^{3s+1}) = T\mathscr{A}^{3s+1}f + \mathbb{C}\{ay^{3j+1}e_3\} \text{ if } s \geq 2,$$
$$T\mathscr{A}^{5}(x, xy+y^3, xy^2+ay^5) = T\mathscr{A}^{5}f + \mathbb{C}\{ay^5e_3\}$$
$$\text{and } T\mathscr{A}^{6}(x, xy+y^3, xy^2+ay^6) = T\mathscr{A}^{6}f + \mathbb{C}\{ay^6e_3\}.$$

Note that $T\mathscr{A}^5 f \ni y^5e_3$ and $T\mathscr{A}^6 f \ni y^6e_3$ so that for these two cases, without loss of generality, $a = 0$. Otherwise $a = 1$ or $a = 0$.

By I:2 $(x, xy+y^3, xy^2+y^{3s+1})$ is $(3s+1)$-determined provided that $s \geq 2$.

Hence $g \in \mathscr{A}^4 f \Rightarrow g \sim (x, xy+y^3, xy^2+y^{3s+1})$ for some $s \geq 2$ or $g \in \bigcap_{k\geq 0} \mathscr{A}^k f$ and f is a weak stem.

(II·3:6) $$f = (x, xy+y^3+ay^4, xy^2+bxy^3+cy^4).$$

Proposition (see [Mond 1], p. 354): if $g \in \mathscr{A}^3(x, xy+y^3, xy^2)$ then g lies in
$$\mathscr{A}^4(x, xy+y^3, xy^2+cy^4).$$

If $c \neq 0, \frac{1}{2}, 1, \frac{3}{2}$ then $(x, xy+y^3, xy^2+cy^4)$ is 4-determined.

(II·3:7) $$f = (x, xy+y^3, xy^2+y^4).$$

$$T\mathscr{A}f \quad \begin{pmatrix} \mathfrak{m}_2\backslash\{y\} \\ \{x, x^jy^{2\ell} : j+\ell \geq 2\} + \{x^jy^{2\ell+1}(x+y^2) : \ell, j \geq 0\} \\ \{x, x^j, x^jy^{2\ell+1} : j+\ell \geq 2\} + \{y(x+y^2), x^jy^{2\ell+2}(x+y^2) : \ell, j \geq 0\} \end{pmatrix}$$
$$+\mathbb{C}\left\{\begin{pmatrix} y \\ y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ y^{2j-1} \\ y^{2j} \end{pmatrix} : j \geq 2\right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:3.

Basis $$\{y^{2j}e_3 : j \geq 1\},$$

alternatively: observe that

$$(x^jy^{2\ell+1}+x^{j-1}y^{2\ell+3})e_2 \in T\mathscr{A}f \Rightarrow \left(x^jy^{2\ell+1}+(-1)^{j+1}y^{2(\ell+j)+3}\right)e_2 \in T\mathscr{A}f,$$
$$(x^jy^{2\ell}+x^{j-1}y^{2\ell+2})e_3 \in T\mathscr{A}f \Rightarrow \left(x^jy^{2\ell}+(-1)^{j+1}y^{2(\ell+j)+2}\right)e_3 \in T\mathscr{A}f$$
$$\text{and } \left(x^jy^{2\ell}+(-1)^{\ell+1}x^{(j+\ell)}y^2\right)e_3 \in T\mathscr{A}f.$$

so we can also choose the basis $$\{x^jy^2e_3 : j \geq 1\},$$

multiplicity 1;

$$f^*\mathfrak{m}_3\cdot T\mathscr{A}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{xy, y^2, y^3, y^4\} \\ \{x^2, x^j y^{2\ell} : j+\ell \geq 3\} + \{x^j y^{2\ell+1}(x+y^2) : j+\ell \geq 2\} \\ \{x^2, x^j, x^j y^{2\ell+1} : j+\ell \geq 3\} + \{xy(x+y^2), x^j y^{2\ell+2}(x+y^2) : j+\ell \geq 1\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} 0 \\ x^2y+xy^3 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix}, \begin{pmatrix} xy \\ xy^2 \\ xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ -xy^3 \\ y^6 \end{pmatrix}\begin{pmatrix} y^4 \\ y^5 \\ y^6 \end{pmatrix}, \begin{pmatrix} 0 \\ y^{2j-1} \\ y^{2j} \end{pmatrix} : j \geq 4\right\},$$

$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \mathbb{C}\{y^{2j}\cdot e_3\}\right) \supseteq \mathfrak{m}_2^{2j+1}\cdot\theta(f)$ provided that $j \geq 3$.

$$T\mathscr{A}_{2,2}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{xy, y^2, y^3\} \\ \{x^2, x^j y^{2\ell} : j+\ell \geq 3\} + \{x^j y^{2\ell+1}(x+y^2) : j+\ell \geq 1\} \\ \{x^2, x^j, x^j y^{2\ell+1} : j+\ell \geq 3\} + \{xy(x+y^2), x^j y^{2\ell+2}(x+y^2) : j+\ell \geq 1\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix}, \begin{pmatrix} xy \\ xy^2 \\ xy^3 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ y^4 \end{pmatrix}\begin{pmatrix} y^3 \\ y^4 \\ y^5 \end{pmatrix}\begin{pmatrix} 0 \\ xy^2+3y^4 \\ 2xy^3+4y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ y^{2j-1} \\ y^{2j} \end{pmatrix} : j \geq 3\right\},$$

hence $g \in \mathscr{A}^4 f \Rightarrow$ $\quad g \in \mathscr{A}^5(x, xy+y^3, xy^2+y^4+ay^5)$,

for $k \geq 5$ $g \in \mathscr{A}^k f \Rightarrow$ $\quad g \in \mathscr{A}^{k+1}(x, xy+y^3, xy^2+y^4+ay^{2s})$ if $k = 2s-1$

or $g \in \mathscr{A}^{k+1}(x, xy+y^3, xy^2+y^4)$ if $k = 2s$.

Weighted homogeneous weights (2, 1),

degree$(xy^2+y^4) = 4$, degree$(y^k) = k$.

Hence $T\mathscr{A}^{k+1}(x, xy+y^3, xy^2+y^4+ay^{k+1}) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ay^{k+1}\cdot e_3\}$.

For $k = 4$, $T\mathscr{A}^5 f \ni \{y^5\cdot e_3\}$ so without loss of generality $a = 0$. For $k = 2s$, $a = 1$ or $a = 0$ and by I:2 $(x, xy+y^3, xy^2+y^4+y^{2s})$ is 2s-determined for $s \geq 3$.

Hence $g \in \mathscr{A}^4 f \Rightarrow g \sim (x, xy+y^3, xy^2+y^4+y^{2s})$ for some $s \geq 3$ or $g \in \bigcap_{k\geq 0}\mathscr{A}^k f$ and f is a weak stem.

(II·3:8) $\quad (x, xy + y^3, xy^2 + \frac{3}{2}y^4) \sim f = (x, xy + y^3, 2xy^2 + 3y^4).$

$T\mathscr{A}f$
$$\mathbb{C}\left\{\left((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}\right)e_i : \ell \geq 1, j \geq 0, 1 \leq i \leq 3\right\}$$
$$+\,\mathbb{C}\left\{\begin{pmatrix} y \\ y^2 \\ 2y^3 \end{pmatrix}, \begin{pmatrix} -(j+2)y^j \\ 0 \\ 2jy^{j+2} \end{pmatrix}, \begin{pmatrix} 2(j+1)y^j \\ jy^{j+1} \\ 0 \end{pmatrix} : j \geq 2\right\};$$

$j = 0$ $\quad \left((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}\right)e_i = 2x^\ell e_i,$

$j = 1$ $\quad \left((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}\right)e_i = 3x^{\ell-1}\left(xy + y^3\right)e_i,$

$j = 2$ $\quad \left((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}\right)e_i = 2x^{\ell-1}\left(2xy^2 + 3y^4\right)e_i,$

$j = 3$ $\quad \left((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}\right)e_i = x^{\ell-1}\left(5xy^3 + 9y^5\right)e_i,$

$j = 4$ $\quad \left((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}\right)e_i = x^{\ell-1}\left(6xy^4 + 12y^6\right)e_i,$ etc.

The calculation of $T\mathscr{A}f$ will be found at A:4.

Basis for each term $\left(x^\ell y^j\right)e_i$ there are constants $a(\ell, j)$, $b(\ell, j)$ such that
$$\left(ax^\ell y^j + by^{j+2\ell}\right)e_i \in T\mathscr{A}f,$$
for each term $\left(x^\ell y^{2j+2}\right)e_i$ there are constants $c(\ell, j)$, $d(\ell, j)$ such that
$$\left(cx^\ell y^{2j+2} + dx^{\ell+j}y^2\right)e_i \in T\mathscr{A}f,$$
for each term $\left(x^\ell y^{2j+1}\right)e_i$ there are constants $c'(\ell, j)$, $d'(\ell, j)$ such that
$$\left(c'x^\ell y^{2j+1} + d'x^{\ell+j}y\right)e_i \in T\mathscr{A}f.$$

Basis is $\quad \left\{y^j e_3 : j \geq 1\right\}$

or equivalently $\left\{x^j y e_3, x^j y^2 e_3 : j \geq 1\right\}$,

multiplicity $\quad 2.$

$T\mathscr{A}_{2,2}f$
$$\mathbb{C}\left\{\left(2xy^2 + 3y^4\right)e_1, \left(5xy^3 + 9y^5\right)e_1, \left(5xy^3 + 9y^5\right)e_2,\right.$$
$$\left.\left((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}\right)e_i : \ell \geq 2, \text{ or } \ell \geq 1 \text{ and } j \geq 4, 1 \leq i \leq 3\right\}$$
$$+\,\mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 2y^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 2y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 + 3y^4 \\ 4xy^3 + 12y^5 \end{pmatrix}, \begin{pmatrix} -(j+2)y^j \\ 0 \\ 2jy^{j+2} \end{pmatrix}, \begin{pmatrix} 2(j+1)y^j \\ jy^{j+1} \\ 0 \end{pmatrix} : j \geq 4\right\},$$

hence for $k \geq 4$ $\; g \in \mathscr{A}^k f \Rightarrow \quad g \in \mathscr{A}^{k+1}(x, xy + y^3, 2xy^2 + 3y^4 + ay^{k+1}).$

Weighted homogeneous $\quad$ weights $(2, 1)$,

$\text{degree}(2xy^2 + 3y^4) = 4$, $\text{degree}(y^{k+1}) = k+1$,

$$\text{so } T\mathscr{A}^{k+1}(x, xy+y^3, 2xy^2+3y^4+ay^{k+1}) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ay^{k+1}\cdot e_3\}.$$

We may choose $a = 2$, then

$$(x, xy+y^3, 2xy^2+3y^4+2y^{k+1}) \sim (x, xy+y^3, xy^2+\tfrac{3}{2}y^4+y^{k+1}).$$

Hence $g \in \mathscr{A}^4 f \Rightarrow$

$$g \in \mathscr{A}^k(x, xy+y^3, xy^2+\tfrac{3}{2}y^4+y^k) \text{ for some } k \geq 5 \text{ or } g \in \bigcap_{k\geq 0}\mathscr{A}^k f.$$

Note: f does not satisfy hypothesis i) of I:2; for determinacy degree see II·3:9.

(II·3:9) Let $f_k = (x, xy+y^3, 2xy^2+3y^4+2y^k)$.

We show that f_k is k-determined for $k \geq 6$.

Let $h \in m_2^{k+1}\cdot\theta(f)$ and $f = (x, xy+y^3, 2xy^2+3y^4)$.

Then $T\mathscr{A}(f_k+h) \supseteq m_2^k\cdot\theta(f)$ iff $T\mathscr{A}(f_k+h) + f^*m_3\cdot m_2^k\cdot\theta(f) \supseteq m_2^k\cdot\theta(f)$.

Now $f^*m_3\cdot m_2^k\cdot\theta(f) = \langle x, y^3\rangle\cdot m_2^k\cdot\theta(f) = m_2^{k+3}\cdot\theta(f) + x\cdot m_2^k\cdot\theta(f)$

and $$T\mathscr{A}_{4,2}f + m_2^{k+3}\cdot\theta(f) + x\cdot m_2^k\cdot\theta(f) = T\mathscr{A}_{4,2}(f_k+h) + m_2^{k+3}\cdot\theta(f) + x\cdot m_2^k\cdot\theta(f),$$

that is $$tf\left(m_2^4\cdot\theta(2)\right) + m_2^{k+3}\cdot\theta(f) = t(f+h)\left(m_2^4\cdot\theta(2)\right) + m_2^{k+3}\cdot\theta(f)$$

and

$$\omega f\left(m_3^2\cdot\theta(3)\right) + m_2^{k+3}\cdot\theta(f) + x\cdot m_2^k\cdot\theta(f) = \omega(f+h)\left(m_3^2\cdot\theta(3)\right) + m_2^{k+3}\cdot\theta(f) + x\cdot m_2^k\cdot\theta(f).$$

Calculation by computer yields

$$\begin{aligned} T\mathscr{A}_{4,2}f = \ & \mathbb{C}\Big\{\left(7xy^5+15y^7\right)e_1, \left(7xy^5+15y^7\right)e_2, \left(2x^2y^3+5xy^5+3y^7\right)e_i \\ & \left((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}\right)e_i : \ell \geq 3 \text{ or } \ell = 2 \text{ and } j \neq 3 \\ & \text{or } \ell = 1 \text{ and } j \neq 0, 1, 2, 3, 5, 1 \leq i \leq 3\Big\} \end{aligned}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y^4 \\ y^5 \\ 2y^6 \end{pmatrix}, \begin{pmatrix} -y^5 \\ 0 \\ 4xy^3+10y^7 \end{pmatrix} \begin{pmatrix} xy^3 \\ 0 \\ 10xy^3+24y^7 \end{pmatrix} \begin{pmatrix} 2xy^3+6y^5 \\ y^6 \\ 0 \end{pmatrix}, \begin{pmatrix} -(j+2)y^j \\ 0 \\ 2jy^{j+2} \end{pmatrix}, \begin{pmatrix} 2(j+1)y^j \\ jy^{j+1} \\ 0 \end{pmatrix} : j \geq 6\right\}$$

so for $k \geq 6$,

$$T\mathscr{A}_{4,2}f + m_2^{k+3}\cdot\theta(f) + x\cdot m_2^{k}\cdot\theta(f)$$

$$\supseteq m_2^{k+2}\cdot\theta(f) + x^2\cdot m_2^{k-2}\cdot\theta(f) + m_2^{k}\cdot e_1 + m_2^{k+1}\cdot e_2 + x\cdot m_2^{k-1}\cdot e_2,$$

so
$$T\mathscr{A}_{4,2}f + m_2^{k+3}\cdot\theta(f) + x\cdot m_2^{k}\cdot\theta(f) \supseteq m_2^{k}\cdot\theta(f) \setminus \{y^k e_3, xy^{k-1}e_3, y^k e_2, y^{k+1}e_3\},$$

so
$$T\mathscr{A}(f_k + h) \supseteq m_2^{k}\cdot\theta(f) \setminus \{y^k e_2, y^k e_3, xy^{k-1}e_3, y^{k+1}e_3\}.$$

Let $f_k + h = (x + h_1, xy + y^3 + h_2, 2xy^2 + 3y^4 + 2y^k + h_3)$.

Then $T\mathscr{A}(f_k + h)$ contains

$$\left\{\begin{pmatrix} xy + y^3 + h_2 \\ 0 \\ 0 \end{pmatrix}, xy\cdot\begin{pmatrix} 1 + \partial_x h_1 \\ y + \partial_x h_2 \\ 2y^2 + \partial_x h_3 \end{pmatrix}, y^3\cdot\begin{pmatrix} 1 + \partial_x h_1 \\ y + \partial_x h_2 \\ 2y^2 + \partial_x h_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2xy^2 + 3y^4 + 2y^k + h_2 \\ 0 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0 \\ 0 \\ (x + h_1)(xy + y^3 + h_2) \end{pmatrix}, \begin{pmatrix} 0 \\ (x + h_1)^2 \\ 0 \end{pmatrix}, x\cdot\begin{pmatrix} \partial_y h_1 \\ x + 3y^2 + \partial_y h_2 \\ 4xy + 12y^3 + 2ky^{k-1} + \partial_y h_3 \end{pmatrix},$$

$$\left. y^2\cdot\begin{pmatrix} \partial_y h_1 \\ x + 3y^2 + \partial_y h_2 \\ 4xy + 12y^3 + 2ky^{k-1} + \partial_y h_3 \end{pmatrix}\right\}$$

so $T\mathscr{A}(f_k + h) + f^* m_3\cdot m_2^{k}\cdot\theta(f)$ contains

$$\left\{\begin{pmatrix} xy + y^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ xy^2 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 2y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ 2xy^2 + 3y^4 + 2y^k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y + xy^3 \end{pmatrix}\begin{pmatrix} 0 \\ x^2 \\ 0 \end{pmatrix},\right.$$

$$\left. \begin{pmatrix} 0 \\ x^2 + 3xy^2 \\ 4x^2y + 12xy^3 + 2kxy^{k-1} \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 + 3y^4 \\ 4xy^3 + 12y^5 + 2ky^{k+1} \end{pmatrix}\right\}.$$

We can eliminate the low degree terms to show that

$$\begin{pmatrix} 0 \\ y^k \\ kxy^{k-1} + ky^{k+1} \end{pmatrix} \in T\mathscr{A}(f_k + h) + f^*\mathfrak{m}_3 \cdot \mathfrak{m}_2^k \cdot \theta(f).$$

But $\begin{pmatrix} 0 \\ 0 \\ (k+1)xy^{k-1} + 3(k-1)y^{k+1} \end{pmatrix}, \begin{pmatrix} 0 \\ (k+1)y^k \\ 4ky^{k+1} \end{pmatrix} \in T\mathscr{A}(f_k + h) + f^*\mathfrak{m}_3 \cdot \mathfrak{m}_2^k \cdot \theta(f)$ for

$k \geq 6$ so $\mathbb{C}\{y^k e_2, xy^{k-1}e_3, y^{k+1}e_3\} \subseteq T\mathscr{A}(f_k + h) + f^*\mathfrak{m}_3 \cdot \mathfrak{m}_2^k \cdot \theta(f)$, since

$$\det \begin{bmatrix} 1 & k & k \\ k+1 & 0 & 4k \\ 0 & k+1 & 3(k-1) \end{bmatrix} = 2k^2(k+1) \neq 0 \text{ for } k \geq 1.$$

Then $\mathfrak{m}_2^{k+1} \cdot \theta(f) \subseteq T\mathscr{A}(f_k + h) + f^*\mathfrak{m}_3 \cdot \mathfrak{m}_2^k \cdot \theta(f)$.

The result follows since we know that $\mathfrak{m}_2^k \cdot \theta(f) \subseteq T\mathscr{A} f_k + \mathfrak{m}_2^{k+1} \cdot \theta(f)$.

It is shown in [Mond 1] that $(x, xy + y^3, xy^2 + \frac{3}{2}y^4 + y^5)$ is 5-determined.

Hence $g \in \mathscr{A}^4 f \Rightarrow$

$$g \sim (x, xy + y^3, xy^2 + \tfrac{3}{2}y^4 + y^k) \text{ for some } k \geq 5 \text{ or } g \in \bigcap_{k \geq 0} \mathscr{A}^k f.$$

Hence $(x, xy + y^3, xy^2 + \frac{3}{2}y^4)$ is a weak stem.

(II·3:10) $\quad (x, xy + y^3, xy^2 + \frac{1}{2}y^4) \sim f = (x, xy + y^3, 2xy^2 + y^4)$.

$$\begin{aligned} T\mathscr{A} f \quad & \mathbb{C}\left\{(x+y^2)x^\ell y^{2j+1}e_i, ((j+1)xy^{2j} + jy^{2j+2})x^\ell e_i : \ell, j \geq 0, i = 1, 3\right\} \\ & + \mathbb{C}\{(x+y^2)x^\ell y^{2j+1}e_2 : \ell, j \geq 0\} + \mathbb{C}\{x, x^\ell y^{2j} e_2 : \ell + j \geq 2\} \\ & + \mathbb{C}\left\{\begin{pmatrix} y \\ y^2 \\ 2y^3 \end{pmatrix}, \begin{pmatrix} (j+1)y^{2j} \\ 0 \\ jy^{2j+2} \end{pmatrix}, \begin{pmatrix} y^{2j+1} \\ 0 \\ 2y^{2j+3} \end{pmatrix}, \begin{pmatrix} -y^{2j} \\ jy^{2j+1} \\ 0 \end{pmatrix} : j \geq 1\right\}; \end{aligned}$$

$j = 0$ $\quad (x+y^2)x^\ell y^{2j+1}e_i = (xy+y^3)x^\ell e_i$ and $((j+1)xy^{2j}+jy^{2j+2})x^\ell e_i = x^{\ell+1}e_i$,

$j = 1$ $\quad (x+y^2)x^\ell y^{2j+1}e_i = (xy^3+y^5)x^\ell e_i$ and $((j+1)xy^{2j}+jy^{2j+2})x^\ell e_i = (2xy^2+y^4)x^\ell e_i$,

$j = 2$ $\quad (x+y^2)x^\ell y^{2j+1}e_i = (xy^5+y^7)x^\ell e_i$ and $((j+1)xy^{2j}+jy^{2j+2})x^\ell e_i = (3xy^4+2y^6)x^\ell e_i$,

etc.

The calculation of $T\mathscr{A}f$ will be found at A:5.

Basis for each term $\left(x^\ell y^j\right)e_i$ there are constants $a(\ell, j)$, $b(\ell, j)$ such that

$$\left(ax^\ell y^j + by^{j+2\ell}\right)e_i \in T\mathscr{A}f,$$

for each term $\left(x^\ell y^{2j+2}\right)e_i$ there are constants $c(\ell, j)$, $d(\ell, j)$ such that

$$\left(cx^\ell y^{2j+2} + dx^{\ell+j}y^2\right)e_i \in T\mathscr{A}f,$$

for each term $\left(x^\ell y^{2j+1}\right)e_i$ there are constants $c'(\ell, j)$, $d'(\ell, j)$ such that

$$\left(c'x^\ell y^{2j+1} + d'x^{\ell+j}y\right)e_i \in T\mathscr{A}f:$$

basis is

$$\left\{y^j e_3 : j \geq 1\right\}$$

or equivalently $\left\{x^j y e_3, x^j y^2 e_3 : j \geq 1\right\}$,

multiplicity 2.

$T\mathscr{A}_{2,2}f$

$$\mathbb{C}\Big\{(x+y^2)x^{\ell-1}y^{2j+1}e_i, ((j+1)xy^{2j}+jy^{2j+2})x^\ell e_i$$
$$: \ell \geq 2, j \geq 0 \text{ or } \ell = 1, j \geq 2\, i = 1, 3\Big\}$$
$$+ \mathbb{C}\{(x+y^2)x^{\ell-1}y^{2j+1}e_2, x^\ell y^{2j}e_2 : \ell \geq 2, j \geq 0 \text{ or } \ell = 1 \text{ and } j \geq 2\}$$

$$+ \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 2y^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 2y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 + 3y^4 \\ 4xy^3 + 4y^5 \end{pmatrix}, \begin{pmatrix} (j+1)y^{2j} \\ 0 \\ jy^{2j+2} \end{pmatrix}, \begin{pmatrix} y^{2j+1} \\ 0 \\ 2y^{2j+3} \end{pmatrix}, \begin{pmatrix} -y^{2j} \\ jy^{2j+1} \\ 0 \end{pmatrix} : j \geq 2\right\},$$

hence for $k \geq 4$ $g \in \mathscr{A}^k f \Rightarrow$ $\quad g \in \mathscr{A}^{k+1}(x, xy + y^3, 2xy^2 + y^4 + ay^{k+1})$.

Weighted homogeneous weights $(2, 1)$,

$$\text{degree}(2xy^2 + y^4) = 4, \ \text{degree}(y^{k+1}) = k+1$$

so $\quad T\mathscr{A}^{k+1}(x, xy + y^3, 2xy^2 + y^4 + ay^{k+1}) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ay^{k+1}.e_3\}$.

We may choose $a = 2$, then

$$(x, xy + y^3, 2xy^2 + y^4 + 2y^{k+1}) \sim (x, xy + y^3, xy^2 + \tfrac{1}{2}y^4 + y^{k+1}).$$

Hence $g \in \mathscr{A}^4 f \Rightarrow$

$$g \in \mathscr{A}^{k+1}(x, xy + y^3, xy^2 + \tfrac{1}{2}y^4 + y^{k+1}) \text{ for some } k \geq 4 \text{ or } g \in \bigcap_{k \geq 0} \mathscr{A}^k f.$$

We do not expect f to be a stem since it does not have the right transversal type. However we shall repeat the calculation II·3:9 to see how k-determinacy fails.

(II·3:11) $$f_k = (x, xy + y^3, 2xy^2 + y^4 + 2y^{k+1}).$$

Let $h \in \mathfrak{m}_2^{k+1}\cdot\theta(f)$ and $f = (x, xy + y^3, 2xy^2 + y^4)$. Then

$$T\mathcal{A}(f_k+h) \supseteq \mathfrak{m}_2^k\cdot\theta(f) \text{ iff } T\mathcal{A}(f_k+h) + f^*\mathfrak{m}_3\cdot\mathfrak{m}_2^k\cdot\theta(f) \supseteq \mathfrak{m}_2^k\cdot\theta(f).$$

Now $f^*\mathfrak{m}_3\cdot\mathfrak{m}_2^k\cdot\theta(f) = \langle x, y^3\rangle\cdot\mathfrak{m}_2^k\cdot\theta(f) = \mathfrak{m}_2^{k+3}\cdot\theta(f) + x\cdot\mathfrak{m}_2^k\cdot\theta(f)$

and $$T\mathcal{A}_{4,2}f + \mathfrak{m}_2^{k+3}\cdot\theta(f) + x\cdot\mathfrak{m}_2^k\cdot\theta(f) = T\mathcal{A}_{4,2}(f_k + h) + \mathfrak{m}_2^{k+3}\cdot\theta(f) + x\cdot\mathfrak{m}_2^k\cdot\theta(f)$$

as in II·3:9.

Calculation by computer yields

$$\begin{aligned} T\mathcal{A}_{4,2}f = \quad & \mathbb{C}\{(x+y^2)x^{\ell-1}y^{2j+1} : \ell+j \geq 3\}e_1 + \mathbb{C}\{x^2, ((j+1)xy^{2j}+jy^{2j+2})x^\ell : \ell+j \geq 2\}e_1 \\ & + \mathbb{C}\{(x+y^2)xy, 3xy^4+2y^6, 3x^2y^2-y^6, (x+y^2)x^{\ell-1}y^{2j+1} : \ell+j \geq 3\}e_2 \\ & + \mathbb{C}\{x^2, x^3, x^\ell y^{2j} : \ell+j \geq 4\}e_2 \\ & \mathbb{C}\{(x+y^2)x^{\ell-1}y^{2j+1}e_1 : \ell+j \geq 4\}e_3 + \mathbb{C}\{x^2, ((j+1)xy^{2j}+jy^{2j+2})x^\ell : \ell+j \geq 2\}e_3 \\ & + \mathbb{C}\{(x+y^2)xy, 2x^3y-3xy^5-y^7, 2x^2y^3+3xy^5+y^7\}e_3 \end{aligned}$$

$$+ \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 2y^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 2y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 + 3y^4 \\ 4xy^3 + 4y^5 \end{pmatrix}, \begin{pmatrix} (j+1)y^{2j} \\ 0 \\ jy^{2j+2} \end{pmatrix}, \begin{pmatrix} y^{2j+1} \\ 0 \\ 2y^{2j+3} \end{pmatrix}, \begin{pmatrix} -y^{2j} \\ jy^{2j+1} \\ 0 \end{pmatrix} : j \geq 3\right\}$$

so for $k \geq 7$

$$T\mathcal{A}_{4,2}f + \mathfrak{m}_2^{k+3}\cdot\theta(f) + x\cdot\mathfrak{m}_2^k\cdot\theta(f)$$

$$\supseteq \mathfrak{m}_2^{k+2}\cdot\theta(f) + x^2\cdot\mathfrak{m}_2^{k-2}\cdot\theta(f) + \mathfrak{m}_2^k\cdot e_1 + \mathfrak{m}_2^{k+1}\cdot e_2 + x\cdot\mathfrak{m}_2^{k-1}\cdot e_2,$$

so $$T\mathcal{A}_{4,2}f + \mathfrak{m}_2^{k+3}\cdot\theta(f) + x\cdot\mathfrak{m}_2^k\cdot\theta(f) \supseteq \mathfrak{m}_2^k\cdot\theta(f) \setminus \{y^k e_3, xy^{k-1}e_3, y^k e_2, y^{k+1}e_3\},$$

so $$T\mathcal{A}(f_k + h) \supseteq \mathfrak{m}_2^k\cdot\theta(f) \setminus \{y^k e_2, y^k e_3, xy^{k-1}e_3, y^{k+1}e_3\}.$$

Let $f_k + h = (x + h_1, xy + y^3 + h_2, 2xy^2 + y^4 + 2y^k + h_3)$.

Then $T\mathscr{A}(f_k + h)$ contains

$$\left\{\begin{pmatrix} xy + y^3 + h_2 \\ 0 \\ 0 \end{pmatrix}, xy\cdot\begin{pmatrix} 1 + \partial_x h_1 \\ y + \partial_x h_2 \\ 2y^2 + \partial_x h_3 \end{pmatrix}, y^3\cdot\begin{pmatrix} 1 + \partial_x h_1 \\ y + \partial_x h_2 \\ 2y^2 + \partial_x h_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2xy^2 + y^4 + 2y^k + h_2 \\ 0 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0 \\ 0 \\ (x + h_1)(xy + y^3 + h_2) \end{pmatrix}, \begin{pmatrix} 0 \\ (x + h_1)^2 \\ 0 \end{pmatrix}, x\cdot\begin{pmatrix} \partial_y h_1 \\ x + 3y^2 + \partial_y h_2 \\ 4xy + 4y^3 + 2ky^{k-1} + \partial_y h_3 \end{pmatrix},$$

$$\left. y^2\cdot\begin{pmatrix} \partial_y h_1 \\ x + 3y^2 + \partial_y h_2 \\ 4xy + 4y^3 + 2ky^{k-1} + \partial_y h_3 \end{pmatrix}\right\}$$

so $T\mathscr{A}(f_k + h) + f^*\mathfrak{m}_3\cdot\mathfrak{m}_2^k\cdot\theta(f)$ contains

$$\left\{\begin{pmatrix} xy + y^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ xy^2 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 2y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ 2xy^2 + y^4 + 2y^k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y + xy^3 \end{pmatrix}\begin{pmatrix} 0 \\ x^2 \\ 0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 \\ x^2 + 3xy^2 \\ 4x^2y + 4xy^3 + 2kxy^{k-1} \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 + 3y^4 \\ 4xy^3 + 4y^5 + 2ky^{k+1} \end{pmatrix}\right\}.$$

We can eliminate the low degree terms to show that

$$\begin{pmatrix} 0 \\ -y^k \\ kxy^{k-1} + ky^{k+1} \end{pmatrix} \in T\mathscr{A}(f_k + h) + f^*\mathfrak{m}_3\cdot\mathfrak{m}_2^k\cdot\theta(f).$$

If $k = 2s$ then $\begin{pmatrix} 0 \\ 0 \\ xy^{2s-1} + y^{2s+1} \end{pmatrix}, \begin{pmatrix} 0 \\ y^{2s} \\ 0 \end{pmatrix} \in T\mathscr{A}(f_k + h) + f^*\mathfrak{m}_3\cdot\mathfrak{m}_2^k\cdot\theta(f)$.

Consequently f_{2s} is not 2s-determined.

If $k = 2s+1$ then

$$\begin{pmatrix} 0 \\ 0 \\ (s+1)xy^{2s} + sy^{2s+2} \end{pmatrix}, \begin{pmatrix} 0 \\ (s+1)\,y^{2s+1} \\ y^{2s+2} \end{pmatrix} \in T\mathcal{A}(f_k + h) + f^*\mathfrak{m}_3 \cdot \mathfrak{m}_2^k \cdot \theta(f)$$

so $\mathbb{C}\{y^k e_2, xy^{k-1}e_3, y^{k+1}e_3\} \subseteq T\mathcal{A}(f_k + h) + f^*\mathfrak{m}_3 \cdot \mathfrak{m}_2^k \cdot \theta(f)$, since

$$\det \begin{bmatrix} s+1 & 0 & 1 \\ 0 & s+1 & s \\ -1 & 2s+1 & 2s+1 \end{bmatrix} = 2(s+1)^2 \neq 0 \text{ for } s \geq 0.$$

Hence $T\mathcal{A}(f_k + h) + f^*\mathfrak{m}_3 \cdot \mathfrak{m}_2^k \cdot \theta(f) \supseteq \mathfrak{m}_2^{k+1} \cdot \theta(f)$

and $T\mathcal{A}(f_k + h) + \mathfrak{m}_2^{k+1} \cdot \theta(f) = T\mathcal{A} f_k + \mathfrak{m}_2^{k+1} \cdot \theta(f) \supseteq \mathfrak{m}_2^k \cdot \theta(f)$.

Consequently f_{2s+1} is $2s+1$-determined.

Hence $g \in \mathcal{A}^3 f \Rightarrow g \sim (x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^{2s+1})$ for some $s \geq 3$ or

$$g \in \mathcal{A}^{2k}(x, xy + y^3, xy^2 + \tfrac{1}{2}y^4 + y^{2s}) \text{ for some } s \geq 3 \text{ or } g \in \bigcap_{k \geq 0} \mathcal{A}^k f.$$

(II·3:12) The map-germ $(x, xy + y^3, xy^2 + \frac{1}{2}y^4)$ is not a stem.

Proof: first note that $(x, xy + y^3, xy^2 + \frac{1}{2}y^4) \sim (x, xy + y^3, (x+y^2)^2)$.

We prove that the map-germ $(x, xy + y^3, (x+y^2)^2 + y^{2k+1}(x+y^2))$ is neither finitely determined nor equivalent to $(x, xy + y^3, (x+y^2)^2)$.

The curve $\{(x+y^2) = 0\} \to (x, 0, 0)$ is a double curve for each map-germ and

$$(\partial f/\partial x, \partial f/\partial y) = \begin{pmatrix} 1 & 0 \\ y & x+3y^2 \\ 2(x+y^2)+y^{2k+1} & 4y(x+y^2)+(2k+1)y^{2k}(x+y^2)+2y^{2k+2} \end{pmatrix}.$$

On $\{(x+y^2)=0\}$ this becomes $\begin{pmatrix} 1 & 0 \\ y & 2y^2 \\ y^{2k+1} & 2y^{2k+2} \end{pmatrix}$ and it is clear that the curve does not have transversal type A_1, so $(x, xy+y^3, (x+y^2)^2+y^{2k+1}(x+y^2))$ is not finitely determined.

We identify the double point set of each map-germ. For a map-germ of the form $(x, b(x,y), c(x,y))$ this is the zero set of the ideal

$$\langle x-x', \frac{b(x,y)-b(x,y')}{y-y'}, \frac{c(x,y)-c(x,y')}{y-y'} \rangle.$$

For $(x, xy+y^3, (x+y^2)^2)$ this ideal becomes

$$\langle x-x', x+y^2+yy'+(y')^2, 2x(y+y')+y^3+y^2y'+y(y')^2+(y')^3 \rangle$$
$$= \langle x-x', x+y^2+yy'+(y')^2, (y+y')^3 \rangle$$

So $(x, xy+y^3, (x+y^2)^2)$ has a single double curve $y=-y'$ and $x+y^2=0$ with image $(X,0,0)$.

For $(x, xy+y^3, (x+y^2)^2+y^{2k+1}(x+y^2))$ the ideal becomes

$$\langle x-x', x+y^2+yy'+(y')^2,$$
$$2x(y+y')+y^3+y^2y'+y(y')^2+(y')^3+x(y^{2k}+y^{2k-1}y'+\ldots+(y')^{2k})+y^{2k+2}+y^{2k+1}y'+\ldots+(y')^{2k+2} \rangle$$
$$= \langle x-x', x+y^2+yy'+(y')^2, (y+y')^3+yy'(y+y')(y^{2k-1}+y^{2k-2}y'+\ldots+(y')^{2k-1}) \rangle.$$

This has a 2-1 double curve $y=-y'$ and $x+y^2=0$ with image $(X,0,0)$. Note that $f^{-1}(X,0,0) = \{y(x+y^2)=0\} \cap \{(x+y^2)(x+y^2+y^{2k+1})=0\}$ is the irreducible curve $x+y^2=0$. Now $(y+y')^3+yy'(y+y')(y^{2k-1}+y^{2k-2}y'+\ldots+(y')^{2k-1})$

$$= (y+y')^2(y+y'+yy'(y^{2k-2}+y^{2k-4}(y')^2+\ldots+(y')^{2k-2}))$$

so the map-germ has a second curve defined by

$$(y+y')+yy'(y^{2k-2}+y^{2k-4}(y')^2+\ldots+(y')^{2k-2})=0.$$

Let $y+y'=0$: then $(y+y')+yy'(y^{2k-2}+y^{2k-4}(y')^2+\ldots+(y')^{2k-2}) = kyy'y^{2k-2}=0$ so the curves $(y+y')+yy'(y^{2k-2}+y^{2k-4}(y')^2+\ldots+(y')^{2k-2})=0$ and $y+y'=0$ intersect only ay $y=y'=0$. If $(y+y')+yy'(y^{2k-2}+y^{2k-4}(y')^2+\ldots+(y')^{2k-2})$ also maps onto $(X,0,0)$ then $(X,0,0)$ is at least a triple curve, otherwise we have found a second double curve. This shows that $(x, xy+y^3, (x+y^2)^2+y^{2k+1}(x+y^2))$ cannot be equivalent to

$(x, xy+y^3, (x+y^2)^2)$ and concludes the proof.

(II·3:13) $$f = (x, xy+y^3, 0).$$

$$T\mathscr{A}_{2,2}f \quad \begin{pmatrix} \mathfrak{m}_2^2\setminus\{xy, y^2, y^3\} \\ \mathfrak{m}_2^2\setminus\{xy, y^2, xy^2, y^3, y^4\} \\ \{x^i(xy+y^3)^j : i+j \geq 2\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} xy+3y^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2+3y^4 \\ 0 \end{pmatrix}\begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix}\right\}.$$

The calculation of $T\mathscr{A}_{2,2}f + \mathfrak{m}_2^7\cdot\theta(f)$ performed by computer established that

$$\mathfrak{m}_2^4\cdot e_1 + \mathfrak{m}_2^4\cdot e_2 \subseteq T\mathscr{A}_{2,2}f.$$

Hence $g \in \mathscr{A}^3f \Rightarrow$ $\quad g \in \mathscr{A}^4(x, xy+y^3+ay^4, bxy^3+cy^4)$,

$g \in \mathscr{A}^4f \Rightarrow$ $\quad g \in \mathscr{A}^5(x, xy+y^3, ax^2y^3+bxy^4+cy^5)$ etc.

(II·3:14) $$f = (x, xy+y^3+ay^4, bxy^3+cy^4).$$

$$T\mathscr{A}^4f \quad \begin{pmatrix} \mathfrak{m}_2\setminus\{y\} \\ \mathfrak{m}_2\setminus\{y, y^2\} \\ \mathfrak{m}_2\setminus\{y, xy, y^2, x^2y, xy^2, y^3, xy^3, y^4\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} 0 \\ 0 \\ xy+y^3+ay^4 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ bxy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ cxy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ cy^4 \end{pmatrix}\right\}$$

The calculation of $T\mathscr{A}^4f$ was performed by computer.

4-orbits: $\left\{(x, xy+y^3+ay^4, bxy^3+cy^4) : a, b \in \mathbb{C}, c \in \mathbb{C}\setminus\{0\}\right\} \ni (x, xy+y^3, y^4)$,

$\left\{(x, xy+y^3+ay^4, bxy^3) : a \in \mathbb{C}, b \in \mathbb{C}\setminus\{0\}\right\} \ni (x, xy+y^3, xy^3)$,

$\left\{(x, xy+y^3+ay^4, 0) : a \in \mathbb{C}\right\} \ni (x, xy+y^3, 0)$,

hence $g \in \mathscr{A}^3f \Rightarrow$ $\quad g \in \left\{\mathscr{A}^4(x, xy+y^3, y^4), \mathscr{A}^4(x, xy+y^3, xy^3), \mathscr{A}^4(x, xy+y^3, 0)\right\}$.

(II·3:15) $$f = (x, xy + y^3, ax^2y^3 + bxy^4 + cy^5).$$

$$T\mathscr{A}^5 f \quad \begin{pmatrix} \mathfrak{m}_2\backslash\{y\} \\ \mathfrak{m}_2\backslash\{y, y^2\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, y^4, x^2y^3, xy^4, y^5\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{ \begin{pmatrix} 0 \\ 0 \\ xy + y^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x^2y + xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y^2 + 2xy^4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x^3y + x^2y^3 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 \\ 0 \\ bxy^4 + 2cy^5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ ax^2y^3 - cy^5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4bx^2y^3 + 5cxy^4 \end{pmatrix} \begin{pmatrix} y \\ y^2 \\ 2axy^4 + by^5 \end{pmatrix} \right\}.$$

The calculation of $T\mathscr{A}^5 f$ was performed by computer.

The unfolding $F(x, y, t) = (x, xy + y^3, a(t)x^2y^3 + b(t)xy^4 + c(t)y^5)$, where

$$a(t) = 10c_0t^2 + 4b_0t + a_0,\ b(t) = 5c_0t + b_0,\ c(t) = c_0,$$

is 5-trivial so if $b_0 \neq 0$ or $c_0 \neq 0$ then without loss of generality, $a = 0$.

The 5-orbits are:
$$\left\{(x, xy + y^3, bxy^4 + cy^5) : b, c \in \mathbb{C}\backslash\{0\}\right\} \ni (x, xy + y^3, xy^4 + y^5),$$
$$\left\{(x, xy + y^3, bxy^4) : b \in \mathbb{C}\backslash\{0\}\right\} \ni (x, xy + y^3, xy^4),$$
$$\left\{(x, xy + y^3, cy^5) : c \in \mathbb{C}\backslash\{0\}\right\} \ni (x, xy + y^3, y^5),$$
$$\left\{(x, xy + y^3, ax^2y^3) : a \in \mathbb{C}\backslash\{0\}\right\} \ni (x, xy + y^3, x^2y^3),$$

hence $g \in \mathscr{A}^4 f \Rightarrow$
$$g \in \left\{\mathscr{A}^5(x, xy + y^3, xy^4 + y^5), \mathscr{A}^5(x, xy + y^3, xy^4), \mathscr{A}^5(x, xy + y^3, y^5), \mathscr{A}^5(x, xy + y^3, x^2y^3), \mathscr{A}^5(x, xy + y^3, 0)\right\}.$$

(II·3:16) $\qquad f = (x, xy + y^3, y^4).$

$$T\mathscr{A}f + \mathfrak{m}_2^7\cdot\theta(f) \qquad \begin{pmatrix} \mathfrak{m}_2\backslash\{y\} \\ \mathfrak{m}_2\backslash\{y, y^2\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, xy^2, y^3\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}\right\},$$

$$T\mathscr{A}_{2,2}f + \mathfrak{m}_2^{10}\cdot\theta(f) \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{xy, y^2, y^3, y^4\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, xy^2, y^3, y^4, y^5\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, x^2y^2, xy^3, y^4, y^5, y^6\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix}, \begin{pmatrix} y^4 \\ y^5 \\ 0 \end{pmatrix}, \begin{pmatrix} -3y^4 \\ 0 \\ 4y^6 \end{pmatrix}, \begin{pmatrix} -xy-3y^3 \\ 0 \\ 4y^5 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 \\ xy^2+3y^4 \\ 4y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ 3y^5 \\ 4y^6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y^2+y^6 \end{pmatrix}\right\}.$$

Calculations of $T\mathscr{A}f + \mathfrak{m}_2^7\cdot\theta(f)$ and $T\mathscr{A}_{2,2}f + \mathfrak{m}_2^{10}\cdot\theta(f)$ were performed by computer.

Hence $g \in \mathscr{A}^4 f \Rightarrow \qquad g \in \mathscr{A}^5(x, xy + y^3, y^4 + ay^5),$

$g \in \mathscr{A}^5 f \Rightarrow \qquad g \in \mathscr{A}^6(x, xy + y^3, y^4 + ay^6),$

$g \in \mathscr{A}^k f, k \geq 6 \Rightarrow \qquad g \in \mathscr{A}(x, xy + y^3, y^4).$

Since f is weighted homogeneous,

$$T\mathscr{A}^5(x, xy + y^3, y^4 + ay^5) = T\mathscr{A}^5 f + \mathbb{C}\{ay^5\cdot e_3\} \ni y^5\cdot e_3,$$

$$T\mathscr{A}^6(x, xy + y^3, y^4 + ay^6) = T\mathscr{A}^6 f + \mathbb{C}\{ay^6\cdot e_3\} \ni y^6\cdot e_3.$$

It follows that f is 4-determined.

(II·3:17) $f = (x, xy + y^3, xy^3)$.

$$T\mathscr{A}f \qquad \begin{pmatrix} \mathfrak{m}_2\backslash\{y^{j+1}, xy^{3j+1}, xy^{3j+2}, x^2y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y^{3j+1}, y^{3j+2}, xy^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y^{j+1}, xy^{3j+1}, xy^{3j+2}, x^2y^{3j+2} : j \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} (6j+11)x^2y^{3j-1}+6xy^{3j+1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}j(j-1)x^2y^{3j-4}+jxy^{3j-2}+y^{3j} \\ 0 \\ 0 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0 \\ (6j+1)xy^{3j-2}+6y^{3j+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ (6j-1)x^2y^{3j-1}+6xy^{3j+1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ (3j^2-4j)xy^{3j-2}+(6j-7)y^{3j} \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^4 \end{pmatrix},$$

$$\left.\begin{pmatrix} (6j+1)y^{3j} \\ (3j^2+8j)y^{3j+1} \\ 0 \end{pmatrix}, \begin{pmatrix} -xy^{3j-1} \\ y^{3j+2} \\ 0 \end{pmatrix}, \begin{pmatrix} y^{3j-2} \\ 0 \\ -xy^{3j-1}+y^{3j+1} \end{pmatrix}, \begin{pmatrix} (3j^2+2j-1)y^{3j} \\ 0 \\ (3j^2+8j)y^{3j+3} \end{pmatrix}, \begin{pmatrix} xy^{3j-1} \\ 0 \\ xy^{3j+2} \end{pmatrix}, \begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} : j \geq 1\right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:6.

Basis $\{xy^{3j+1}.e_3, y^{3j+1}.e_3, y^{3j+2}.e_3, y^{3j+3}.e_3 : j \geq 3\}$,

multiplicity 4,

$$T\mathscr{A}_{2,2}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{y^{j+2}, xy^{3j+1}, xy^{3j+2}, x^2y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2^2\backslash\{xy, y^3, y^{3j+1}, y^{3j+2}, xy^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2^2\backslash\{x^2y, xy^3, y^{j+2}, xy^{3j+1}, xy^{3j+2}, x^2y^{3j+2} : j \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} (6j+11)x^2y^{3j-1}+6xy^{3j+1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -xy^{3j-1} \\ y^{3j+2} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{3j-1} \\ 0 \\ xy^{3j+2} \end{pmatrix} : j \geq 1\right\}$$

$$+\mathbb{C}\left\{\begin{pmatrix}\frac{1}{2}j(j-1)x^2y^{3j-4}+jxy^{3j-2}+y^{3j}\\0\\0\end{pmatrix},\begin{pmatrix}0\\(6j+1)xy^{3j-2}+6y^{3j+1}\\0\end{pmatrix},\begin{pmatrix}(6j+1)y^{3j}\\(3j^2+8j)y^{3j+1}\\0\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0\\0\\(6j+11)x^2y^{3j-1}+6xy^{3j+1}\end{pmatrix},\begin{pmatrix}y^{3j-2}\\0\\-xy^{3j-1}+y^{3j+1}\end{pmatrix},\begin{pmatrix}(3j^2+2j-1)y^{3j}\\0\\(3j^2+8j)y^{3j+3}\end{pmatrix},\begin{pmatrix}y^{3j-1}\\0\\y^{3j+2}\end{pmatrix}:j\geq 2\right\}$$

$$+\mathbb{C}\left\{\begin{pmatrix}0\\0\\x^2y^2+2xy^4+y^6\end{pmatrix},\begin{pmatrix}3xy\\2xy^2-3y^4\\0\end{pmatrix},\begin{pmatrix}xy\\xy^2\\xy^4\end{pmatrix},\begin{pmatrix}y^2\\y^3\\y^5\end{pmatrix},\begin{pmatrix}y^3\\y^4\\y^6\end{pmatrix},\begin{pmatrix}0\\0\\x^2y+xy^3\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0\\0\\(3j^2-4j)xy^{3j-2}+(6j-7)y^{3j}\end{pmatrix}:j\geq 3\right\}.$$

Hence $g\in\mathscr{A}^4f\Rightarrow$ $\quad g\in\mathscr{A}^5(x,xy+y^3,xy^3+axy^4+by^5)$,

$g\in\mathscr{A}^5f\Rightarrow$ $\quad g\in\mathscr{A}^6(x,xy+y^3,xy^3+axy^5+by^6)$,

for $k\geq 6$ $g\in\mathscr{A}^kf\Rightarrow$ $\quad g\in\mathscr{A}^{k+1}(x,xy+y^3,xy^3+ay^{3s+1})$ if $k=3s$,

or $g\in\mathscr{A}^{k+1}(x,xy+y^3,xy^3+ay^{3s+2})$ if $k=3s+1$,

or $g\in\mathscr{A}^{k+1}(x,xy+y^3,xy^3+axy^{3s+2}+by^{3s+3})$ if $k=3s+2$.

Each map-germ $(x,xy+y^3,xy^3+axy^{3s-1}+by^{3s})$ has a triple curve

$$(0,y)\to(0,y^3,by^{3s}).$$

The terms

$$\begin{pmatrix}0\\0\\xy^3+axy^{3s-1}+by^{3s}\end{pmatrix},\begin{pmatrix}0\\xy+3y^3\\3xy^3+(3s-1)axy^{3s-1}+3sby^{3s}\end{pmatrix},\begin{pmatrix}0\\xy\\xy^3+axy^{3s-1}\end{pmatrix},\begin{pmatrix}0\\xy+y^3\\0\end{pmatrix}$$

imply that

$$\begin{pmatrix} 0 \\ 0 \\ (3s-4)axy^{3s-1}+(3s-5)by^{3s} \end{pmatrix} \in T\mathscr{A}^{3s+1}(x, xy+y^3, xy^3+axy^{3s+2}+by^{3s+3})$$

so $(x, xy+y^3, xy^3+axy^{3s-1}+by^{3s})$ is not equivalent to $(x, xy+y^3, xy^3)$ and hence $(x, xy+y^3, xy^3)$ is not a stem.

(II·3:18) $\qquad f = (x, xy+y^3, xy^3+axy^4+by^5).$

$$T\mathscr{A}^5 f = \begin{pmatrix} \mathfrak{m}_2\backslash\{y, xy, y^2, y^3\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, y^3, y^4\} \\ \{x, x^2, x^3, x^4, x^3y, x^5, x^4y, x^3y^2, x^2y^3, axy^4, (5b-6)xy^4\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy+y^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy+y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y-by^5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3x^2y^2+5bxy^4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy^3+by^5 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} y \\ y^2 \\ y^4+ay^5 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \\ y^5-by^5 \end{pmatrix}, \begin{pmatrix} 0 \\ y^3 \\ by^5 \end{pmatrix}, \begin{pmatrix} 6y^3 \\ 0 \\ -bxy^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix}\right\}.$$

Calculation of $T\mathscr{A}^5 f$ is performed by computer.

Orbits- if $5b-6 \neq 0$ then without loss of generality $a=0$.

Thus orbits are represented by $(x, xy+y^3, xy^3+by^5)$,

$$(x, xy+y^3, xy^3+xy^4+\tfrac{6}{5}y^5) \text{ and } (x, xy+y^3, xy^3).$$

(II·3:19) $\qquad f = (x, xy+y^3, xy^3+axy^5+by^6).$

Here $H^j = \mathbb{C}\{x^{j-i}y^i : 0 \leq i \leq j\}$.

$$T\mathscr{A}^6 f = \begin{pmatrix} \mathfrak{m}_2\backslash\{y, xy, y^2, y^3, xy^2, y^4\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, y^3, xy^2, y^4, y^5\} \\ \{x, x^2, x^3, x^4, x^3y, H^5\backslash\{y^5, xy^4\}, H^6\backslash\{y^6, xy^5\}\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy+y^3 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} xy^2+y^4 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ xy+y^3 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 7xy^2+6y^4-6by^5 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 2x^2y-by^6 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0 \\ 0 \\ 2x^2y^2-3y^6+4bxy^5 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 4xy^4+5y^6-4bxy^5 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ by^6+2axy^5 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 2xy^3+by^6 \end{pmatrix},$$

$$\left.\begin{pmatrix} y \\ y^2 \\ y^4+ay^6 \end{pmatrix},\begin{pmatrix} y^2 \\ 0 \\ y^5-by^6 \end{pmatrix},\begin{pmatrix} 0 \\ y^3 \\ by^6 \end{pmatrix},\begin{pmatrix} 4y^3 \\ 0 \\ 11y^6+4by^5 \end{pmatrix},\begin{pmatrix} 7y^3 \\ 11y^4-4by^5 \\ 0 \end{pmatrix},\begin{pmatrix} y^4 \\ y^5 \\ 0 \end{pmatrix},\begin{pmatrix} -y^4 \\ 0 \\ xy^5 \end{pmatrix}\right\}.$$

Calculation of $T\mathcal{A}^6 f$ is performed by computer.

The relevant term for the unfolding is $\begin{pmatrix} 0 \\ 0 \\ by^6+2axy^5 \end{pmatrix}$. If $b \neq 0$ then $b = 1$.

Otherwise $b = 0$ and $a = 0$ or $a = 1$.

Thus orbits are represented by $(x, xy+y^3, xy^3+axy^5+y^6)$,

$$(x, xy+y^3, xy^3+xy^5) \text{ and } (x, xy+y^3, xy^3).$$

Clearly $(x, xy+y^3, xy^3+xy^5)$ and $(x, xy+y^3, xy^3+axy^5+y^6)$ are not finitely determined since they have triple curves $(0,y)\to(0,y^3,0)$ and $(0,y^3,y^6)$ respectively.

(II·3:20) $$f = (x, xy+y^3, xy^4+y^5).$$

$$T\mathcal{A}f + m_2^{10}\cdot\theta(f) \qquad \begin{pmatrix} m_2\backslash\{y, xy, y^2, y^3\} \\ m_2\backslash\{y, xy, xy^2, y^4\} \\ \mathbb{C}\{x, x^2, x^3, x^4, H^5\backslash\{x^2y^3\}, H^6\backslash\{y^6\}\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy+y^3 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 5y^2+2y^3 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ xy^2+y^4 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ x^2y^2+y^6 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0\\0\\x^3y+x^2y^3\end{pmatrix},\begin{pmatrix}0\\0\\-8x^3y+15y^6\end{pmatrix},\begin{pmatrix}y\\y^2\\0\end{pmatrix},\begin{pmatrix}-2y^3\\0\\5y^6\end{pmatrix},\begin{pmatrix}y^3\\y^4\\0\end{pmatrix}\right\},$$

$$T\mathscr{A}_{2,2}f \quad \begin{pmatrix} m_2^2\backslash\{xy, y^2, y^3, y^4\} \\ m_2^2\backslash\{xy, y^2, xy^2, y^3, y^4, y^5\} \\ m_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, y^4, x^2y^3, xy^4, y^5, y^6, y^7\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix}xy\\xy^2\\0\end{pmatrix},\begin{pmatrix}y^2\\y^3\\y^6\end{pmatrix},\begin{pmatrix}y^3\\y^4\\y^7\end{pmatrix},\begin{pmatrix}5y^3\\5y^4-3y^5\\0\end{pmatrix},\begin{pmatrix}y^4\\y^5\\0\end{pmatrix},\begin{pmatrix}-3y^4\\0\\5y^7\end{pmatrix},\begin{pmatrix}0\\xy^2+3y^4\\5y^6\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0\\0\\x^2y+xy^3\end{pmatrix},\begin{pmatrix}0\\0\\x^3y+x^2y^3\end{pmatrix},\begin{pmatrix}0\\0\\x^2y^2+2xy^4+y^6\end{pmatrix}\right\}.$$

Calculations of $T\mathscr{A}f+m_2^{10}\cdot\theta(f)$ and $T\mathscr{A}_{2,2}f+m_2^{10}\cdot\theta(f)$ were performed by computer.

$g\in\mathscr{A}^5f \Rightarrow$ $\qquad g\in\mathscr{A}^6(x, xy+y^3, xy^4+y^5+ay^6)$,

$g\in\mathscr{A}^6f \Rightarrow$ $\qquad g\in\mathscr{A}^7(x, xy+y^3, xy^4+y^5+ay^7)$,

$g\in\mathscr{A}^kf, k\geq 7 \Rightarrow$ $\qquad g\in\mathscr{A}(x, xy+y^3, xy^4+y^5)$.

So f is at worst 7-determined and is at least 6-determined since $(0,0,y^6)\notin T\mathscr{A}f$.

Now $T\mathscr{A}^7(x, xy+y^3, xy^4+y^5+ay^7) = T\mathscr{A}f$

(calculation of $T\mathscr{A}^7(x, xy+y^3, xy^4+y^5+ay^7)$ is performed by computer).

Orbits- $\{(x, xy+y^3, xy^4+y^5+ay^7): a\in\mathbb{C}\}$ lies in a single 7-orbit, hence $(x, xy+y^3, xy^4+y^5)$ is 6-determined.

(II·3:21) $$f = (x, xy + y^3, xy^4 + y^5 + ay^6).$$

$$T\mathscr{A}_{2,2}f + \mathfrak{m}_2^{12}\cdot\theta(f) \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, xy^3, y^4, y^5\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, x^2y^2, xy^3, y^4, y^5, xy^4, y^6\} \\ \mathbb{C}\{x^2, x^3, x^4, x^4y, x^5\} + \mathfrak{m}_2^6\backslash\{x^2y^4, y^6, xy^6, y^7, y^8\}\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix}5xy^2+y^5\\0\\0\end{pmatrix},\begin{pmatrix}x^2y-y^5\\0\\0\end{pmatrix},\begin{pmatrix}xy^3+y^5\\0\\0\end{pmatrix},\begin{pmatrix}0\\x^2y+xy^3\\0\end{pmatrix},\begin{pmatrix}0\\x^2y^2-y^6\\0\end{pmatrix},\begin{pmatrix}0\\5x^2y-3y^6\\0\end{pmatrix},\right.$$

$$\begin{pmatrix}0\\xy^4+y^6\\0\end{pmatrix},\begin{pmatrix}0\\0\\x^2y+xy^3\end{pmatrix},\begin{pmatrix}0\\0\\x^3y+x^2y^3\end{pmatrix},\begin{pmatrix}0\\0\\x^2y^2+2xy^5+y^6\end{pmatrix},\begin{pmatrix}0\\0\\3x^3y^2+4ay^8\end{pmatrix},$$

$$\begin{pmatrix}0\\0\\3x^2y^4-ay^8\end{pmatrix},\begin{pmatrix}0\\0\\3xy^6+2ay^8\end{pmatrix},\begin{pmatrix}0\\0\\3y^8-2ay^8\end{pmatrix},\begin{pmatrix}0\\0\\2xy^3+by^6\end{pmatrix},\begin{pmatrix}xy\\xy^2\\0\end{pmatrix},$$

$$\left.\begin{pmatrix}y^2\\y^3\\y^6\end{pmatrix},\begin{pmatrix}y^3\\y^4\\y^7\end{pmatrix},\begin{pmatrix}y^5\\y^6\\0\end{pmatrix},\begin{pmatrix}-xy-3y^3\\0\\5y^6+(6a-3)y^7\end{pmatrix},\begin{pmatrix}-15y^4+7y^5\\0\\25y^7\end{pmatrix},\begin{pmatrix}5y^4+2y^5\\5y^5\\0\end{pmatrix}\right\}.$$

The calculation of $T\mathscr{A}_{2,2}f + \mathfrak{m}_2^{12}\cdot\theta(f)$ was performed by computer.

$g \in \mathscr{A}^6 f \Rightarrow$ $\qquad g \in \mathscr{A}^7(x, xy + y^3, xy^4 + y^5 + ay^6 + by^7)$,

$g \in \mathscr{A}^7 f \Rightarrow$ $\qquad g \in \mathscr{A}^8(x, xy + y^3, xy^4 + y^5 + ay^6 + by^8)$,

$g \in \mathscr{A}^k f, k \geq 8 \Rightarrow$ $\qquad g \in \mathscr{A}(x, xy + y^3, xy^4 + y^5 + ay^6)$.

By computer we calculate that

$$T\mathscr{A}^7(x, xy + y^3, xy^4 + y^5 + ay^6 + by^7) = T\mathscr{A}^7(x, xy + y^3, xy^4 + y^5 + ay^6)$$

$$\text{and } T\mathscr{A}^8(x, xy + y^3, xy^4 + y^5 + ay^6 + by^8) = T\mathscr{A}^8(x, xy + y^3, xy^4 + y^5 + ay^6).$$

$$\text{Hence } (x, xy + y^3, xy^4 + y^5 + ay^6 + by^7) \overset{7}{\sim} (x, xy + y^3, xy^4 + y^5 + ay^6)$$

$$\text{and } (x, xy + y^3, xy^4 + y^5 + ay^6 + by^8) \overset{8}{\sim} (x, xy + y^3, xy^4 + y^5 + ay^6)$$

so that $(x, xy + y^3, xy^4 + y^5 + ay^6)$ is 6-determined for all a.

(II·3:22)

$$f = (x, xy + y^3, y^5).$$

$$T\mathscr{A}f + m_2^{12}\cdot\theta(f) \quad \begin{pmatrix} m_2\backslash\{y, xy, y^3\} \\ m_2\backslash\{y, y^2, xy^2, y^4\} \\ m_2\backslash\{y, xy, y^2, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, y^4, x^2y^3, xy^4, y^6\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix}, \begin{pmatrix} -2y^3 \\ 0 \\ 5y^6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^3y+x^2y^3 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 \\ 0 \\ x^2y^2-2y^6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2xy^4+3y^6 \end{pmatrix}, \begin{pmatrix} xy+y^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2+y^4 \\ 0 \end{pmatrix}\right\}.$$

$$T\mathscr{A}_{2,2}f + m_2^{12}\cdot\theta(f) \quad \begin{pmatrix} m_2^2\backslash\{xy, y^2, x^2y, y^3, xy^3, y^4, y^5\} \\ m_2^2\backslash\{xy, y^2, xy^2, y^3, x^2y^2, y^4, y^5, xy^4, y^6\} \\ \mathbb{C}\{x^2, x^3, x^4, x^4y, x^5\} + m_2^6\backslash\{x^2y^4, y^6, xy^6, y^7, y^8\}\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} x^2y-y^5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} xy^3+y^5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2y^2-y^6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^4+y^6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^3y+x^2y^3 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0 \\ 0 \\ x^2y^2+2xy^5+y^6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^3y^2+2y^8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2x^2y^4-3y^8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy^6+y^8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3xy^6+2ay^8 \end{pmatrix},$$

$$\left.\begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix}, \begin{pmatrix} y^4 \\ y^5 \\ 0 \end{pmatrix}, \begin{pmatrix} y^5 \\ y^6 \\ 0 \end{pmatrix}, \begin{pmatrix} -2y^5 \\ 0 \\ 5y^8 \end{pmatrix}, \begin{pmatrix} -3y^4 \\ 0 \\ 5y^7 \end{pmatrix}, \begin{pmatrix} -xy-3y^3 \\ 0 \\ 5y^6 \end{pmatrix}\right\}.$$

Calculations of $T\mathscr{A}f + m_2^{12}\cdot\theta(f)$ and $T\mathscr{A}_{2,2}f + m_2^{12}\cdot\theta(f)$ performed by computer.

Hence $g \in \mathscr{A}^5 f \Rightarrow$ $\quad g \in \mathscr{A}^6(x, xy + y^3, y^5 + ay^6)$,

$g \in \mathscr{A}^6 f \Rightarrow$ $\quad g \in \mathscr{A}^7(x, xy + y^3, y^5 + ay^7)$,

$g \in \mathscr{A}^7 f \Rightarrow$ $\quad g \in \mathscr{A}^8(x, xy + y^3, y^5 + ay^8)$,

$g \in \mathscr{A}^k f, k \geq 8 \Rightarrow$ $\qquad g \in \mathscr{A}(x, xy + y^3, y^5)$.

f is at worst 8-determined.

Since f is weighted homogeneous and degree(y^5) $\neq$ degree(y^k) unless $k = 5$ we have

$$T\mathscr{A}^k(x, xy + y^3,\ y^5 + ay^k) = T\mathscr{A}^k(x, xy + y^3,\ y^5) + \mathbb{C}\{ay^k \cdot e_3\}.$$

Since $T\mathscr{A}(x, xy + y^3,\ y^5) \ni \{y^7 \cdot e_3, y^8 \cdot e_3\}$ it follows that $(x, xy + y^3,\ y^5)$ is 6-determined. It is not 5-determined since $T\mathscr{A}(x, xy + y^3,\ y^5) \not\ni \{y^6 \cdot e_3\}$.

By computer we calculate that $(x, xy + y^3,\ y^5 + y^6) \supseteq \mathfrak{m}_2^7 \cdot \theta(f) \backslash \{y^7 \cdot e_3\}$ and that

$T\mathscr{A}^7(x, xy + y^3,\ y^5 + y^6 + ay^7) \supseteq \mathfrak{m}_2^6 \cdot \theta(f)$ for all a so that $(x, xy + y^3,\ y^5 + y^6)$ is 6-determined.

(II·3:23) $\qquad f = (x, xy + y^3, xy^4)$.

$$T\mathscr{A}f \quad \begin{pmatrix} \mathfrak{m}_2\backslash\{xy, y^2, xy^2, y^3, x^2y^2, y^6, y^7, y^{3j+1} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^2, xy^2, y^4, y^5, xy^5, y^7\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, y^4, x^3y^2, x^2y^3, x^4y^2, xy^5, y^6 \\ xy^6, y^7, x^2y^6, xy^8, y^{10}, y^{3j+2} : j \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy+y^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2y^2+y^6 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -xy^5+y^7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^3y+x^2y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y^2+y^6 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0 \\ 0 \\ 8x^2y^3+3xy^5-9y^7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^3y^2+xy^6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^4y^2+x^2y^6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y^6-3y^{10} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -xy^8+4y^{10} \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^5 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ y^7 \end{pmatrix},$$

$$\left.\begin{pmatrix} 0 \\ -2xy^2 \\ xy^5-3y^7 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2+3y^4 \\ 4xy^5 \end{pmatrix}, \begin{pmatrix} y^6 \\ 0 \\ -3y^{10} \end{pmatrix}, \begin{pmatrix} 3y^4 \\ 0 \\ -4xy^6+3y^8 \end{pmatrix}, \begin{pmatrix} -4xy^2 \\ 3y^5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4y^6 \\ 3y^7 \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \\ y^6 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \\ xy^6 \end{pmatrix}, \begin{pmatrix} y^{3j+1} \\ 0 \\ y^{3j+5} \end{pmatrix} : j \geq 2\right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:7.

Basis $\qquad \{y^{3j+2} \cdot e_3 : j \geq 3\}$,

multiplicity $\qquad$ 1.

At A:7 it is shown that

$$f^*\mathfrak{m}_3\cdot(T\mathscr{A}f) \supseteq \mathfrak{m}_2^{11}\cdot\theta(f)\setminus\{y^{3j+13}.e_3,\ y^{3j+11}.e_3 : j>0\} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+1}\\ 0\\ y^{3j+5}\end{pmatrix} : j\geq 2\right\}.$$

$$T\mathscr{A}_{2,2}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\setminus\{xy, y^2, x^2y, xy^2, y^3, x^2y^2, xy^3, xy^4, y^5, y^6, y^7, y^{3j+1} : j\geq 1\} \\ \mathfrak{m}_2^2\setminus\{xy, y^2, xy^2, y^3, x^2y^2, y^4, xy^4, y^5, xy^5, y^6, y^7\} \\ \mathfrak{m}_2^7\setminus\{y^7, xy^6, x^2y^5, x^2y^6, xy^7, y^8, xy^8, y^9, y^{10}, y^{3j+2} : j\geq 0\} \\ +\mathbb{C}\{x^2, x^3, x^4, x^5, x^6, x^5y, x^2y^4\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} x^2y+xy^3\\ 0\\ 0\end{pmatrix}, \begin{pmatrix} x^2y^2+2xy^4+y^6\\ 0\\ 0\end{pmatrix}, \begin{pmatrix} 7xy^3+6y^5\\ 0\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ x^2y^2-y^6\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ xy^4+y^6\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ x^2y+xy^3\end{pmatrix},\right.$$

$$\begin{pmatrix} 0\\ 0\\ x^3y+x^2y^3\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ x^2y^2+2xy^4+y^6\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ x^4y-y^9\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ x^3y^2+xy^6\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ x^4y^2+x^2y^6\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ x^3y^3+y^9\end{pmatrix},$$

$$\begin{pmatrix} 0\\ 0\\ 3x^2y^5-2y^9\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ 3xy^7+2y^9\end{pmatrix}, \begin{pmatrix} y^2\\ y^3\\ y^6\end{pmatrix}, \begin{pmatrix} y^3\\ y^4\\ y^7\end{pmatrix}, \begin{pmatrix} y^6\\ y^7\\ y^{10}\end{pmatrix}, \begin{pmatrix} 0\\ -2xy^2\\ xy^5-3y^7\end{pmatrix}, \begin{pmatrix} 0\\ xy^2+3y^4\\ 4xy^5\end{pmatrix}, \begin{pmatrix} y^6\\ 0\\ -3y^{10}\end{pmatrix},$$

$$\begin{pmatrix} 3y^4\\ 0\\ -4xy^6+3y^8\end{pmatrix}, \begin{pmatrix} 3xy^4\\ 0\\ -4x^2y^6+3xy^8\end{pmatrix}, \begin{pmatrix} 4xy\\ 3xy^2-3y^4\\ 0\end{pmatrix}, \begin{pmatrix} -4xy^2\\ 3y^5\\ 0\end{pmatrix}, \begin{pmatrix} -4x^2y^2\\ 3xy^5\\ 0\end{pmatrix}, \begin{pmatrix} 4y^5\\ 7y^6\\ 0\end{pmatrix}, \begin{pmatrix} 4y^6\\ 3y^7\\ 0\end{pmatrix},$$

$$\left.\begin{pmatrix} 3y^5\\ 0\\ 7y^9\end{pmatrix}, \begin{pmatrix} xy^2\\ 0\\ xy^6\end{pmatrix}, \begin{pmatrix} x^2y^2\\ 0\\ x^2y^6\end{pmatrix}, \begin{pmatrix} y^{3j+1}\\ 0\\ y^{3j+5}\end{pmatrix} : j\geq 2\right\}.$$

Hence $g\in\mathscr{A}^5f \Rightarrow$ $\quad g\in\mathscr{A}^6(x, xy+y^3, xy^4+axy^5+by^6)$,

$g\in\mathscr{A}^6f \Rightarrow$ $\quad g\in\mathscr{A}^7(x, xy+y^3, xy^4+axy^6+by^7)$,

$g\in\mathscr{A}^7f \Rightarrow$ $\quad g\in\mathscr{A}^8(x, xy+y^3, xy^4+ax^2y^6+by^8)$,

$g\in\mathscr{A}^8f \Rightarrow$ $\quad g\in\mathscr{A}^9(x, xy+y^3, xy^4+axy^8+by^9)$,

$g\in\mathscr{A}^9f \Rightarrow$ $\quad g\in\mathscr{A}^{10}(x, xy+y^3, xy^4+ay^{10})$,

for $k\geq 10$ $g\in\mathscr{A}^kf \Rightarrow$ $\quad g\in\mathscr{A}^{k+1}(x, xy+y^3, xy^4)$ if $k\neq 3s+1$,

FIGS 1 - 4 for Section ii.1

EXT

or $g \in \mathscr{A}^{k+1}(x, xy+y^3, xy^4+ay^{3s+2})$ if $k = 3s+1$.

Weighted homogeneous weights (2, 1),

degree$(xy^4) = 6$, degree$(y^{3s+2}) = 3s+2$.

Hence for $s \geq 3$ $\quad T\mathscr{A}^{3s+2}(x, xy+y^3, xy^4+ay^{3s+2}) = T\mathscr{A}^{3s+2}f + \mathbb{C}\{ay^{3s+2}.e_3\}$.

By I:2 $(x, xy+y^3, xy^4+y^{3s+2})$ is $(3s+2)$-determined for $s \geq 3$.

Hence $g \in \mathscr{A}^{10}f \Rightarrow g \sim (x, xy+y^3, xy^4+y^{3s+2})$ for some $s \geq 3$ or $g \in \bigcap_{k \geq 0} \mathscr{A}^k f$ and f is a weak stem.

(II·3:24) $\quad f = (x, xy, y^4)$.

$$T\mathscr{A}f \quad \begin{pmatrix} \mathfrak{m}_2 \backslash \{y^{4j+1}, y^{4j+2}, xy^{4j+2} : j \geq 0\} \\ \mathfrak{m}_2 \backslash \{y, y^{4j+2}, y^{4j+3}, xy^{4j+3} : j \geq 0\} \\ \mathfrak{m}_2 \backslash \{y, y^3, xy^2, y^{4j+2} : j \geq 0\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{ \begin{pmatrix} y^{4j+1} \\ y^{4j+2} \\ 0 \end{pmatrix}, \begin{pmatrix} y^{4j+2} \\ y^{4j+3} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{4j+2} \\ xy^{4j+3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^{4j+3} \\ 4y^{4j+6} \end{pmatrix} : j \geq 1 \right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:8.

Basis $\quad \{y^{4s+2}.e_2, y^{4s+3}.e_2, y^{3j+2}.e_3 : j \geq 3\}$,

multiplicity $\quad 3$.

Let $\kappa = 4$.

If $h \in \mathfrak{m}_2^{k+1}.\theta(f)$ then by I·2:2, $T\mathscr{A}_{5,5}f + \mathfrak{m}_2^{k+5}.\theta(f) = T\mathscr{A}_{5,5}(f+h) + \mathfrak{m}_2^{k+5}.\theta(f)$.

Since f is finite, there exists a value k_0 such that $T\mathscr{A}f \cap \mathfrak{m}_2^{k_0}.\theta(f) \subseteq T\mathscr{A}_{5,5}f$

so in particular, $T\mathscr{A}_{5,5}f \supseteq \begin{pmatrix} \mathfrak{m}_2^{k_0} \backslash \{y^{4j+1}, y^{4j+2}, xy^{4j+2} : j \geq j_0\} \\ \mathfrak{m}_2^{k_0} \backslash \{y^{4j+2}, y^{4j+3}, xy^{4j+3} : j \geq j_0\} \\ \mathfrak{m}_2^{k_0} \backslash \{y^{4j+2} : j \geq j_0\} \end{pmatrix}$

$$+\mathbb{C}\left\{\begin{pmatrix} y^{4j+1} \\ y^{4j+2} \\ 0 \end{pmatrix}, \begin{pmatrix} y^{4j+2} \\ y^{4j+3} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{4j+2} \\ xy^{4j+3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^{4j+3} \\ 4y^{4j+6} \end{pmatrix} : j \geq j_0 \right\}.$$

Let $k = 4s-1$. Then

$$\mathfrak{m}_2^{4s} \cdot \theta(f) \subseteq T\mathscr{A}_{5,5}f + \mathfrak{m}_2^{4s+4} \cdot \theta(f) + \mathbb{C}\{y^{4s+2}.e_2, y^{4s+3}.e_2, xy^{4s-1}.e_2\}.$$

Then by I·2:6, $g \in \mathscr{A}^{4s}f \Rightarrow g \in \mathscr{A}^{4s+4}(x, xy + ay^{4s+2} + by^{4s+3} + cxy^{4s-1}, y^4)$.

Now $(x, xy + cxy^{4s-1}, y^4)$ is not finitely determined, since $(0, y) \to (0, 0, y^4)$ is a curve of quadruple points.

Also, f is weighted homogeneous so

$$T\mathscr{A}^{4s+4}(x, xy + cxy^{4s-1}, y^4) = T\mathscr{A}^{4s+4}(x, xy, y^4) + \mathbb{C}\{cxy^{4s-1}.e_2\},$$

and $(x, xy + cxy^{4s-1}, y^4) \sim (x, xy + xy^{4s-1}, y^4)$ is not $(4s+4)$-equivalent to (x, xy, y^4) and is therefore not equivalent to (x, xy, y^4).

Therefore (x, xy, y^4) is not a stem.

(II·3:25) $\qquad f = (x, xy + xy^3, y^4)$.

$$T\mathscr{A}f \qquad \begin{pmatrix} \mathfrak{m}_2 \backslash \{y^{4j+1}, y^{4j+2}, y^{4j+3} : j \geq 0\} \\ \mathfrak{m}_2 \backslash \{y^{4j+1}, y^{4j+2}, y^{4j+3} : j \geq 0\} \\ \mathfrak{m}_2 \backslash \{y, y^4, y^3, xy^2\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y^{4j+3} - y^{4j+5} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ y^{4j+5} + y^{4j+7} \\ 0 \end{pmatrix} \begin{pmatrix} y^{4j+2} \\ y^{4j+3} - y^{4j+7} \\ 0 \end{pmatrix} \begin{pmatrix} y^{4j+1} \\ y^{4j+2} \\ 0 \end{pmatrix} : j \geq 0 \right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:9.

Basis	$\{y^{4s+2}.e_2, y^{4s+3}.e_2 : j \geq 3\}$,
multiplicity	2,

For sufficiently large k, $\mathfrak{m}_2^k \cdot \theta(f) \cap T\mathscr{A}f \subseteq T\mathscr{A}_{2,2}f$

so that $g \in \mathscr{A}^k f \Rightarrow$ $\quad g \in \mathscr{A}^{k+1}(x, xy + xy^3 + ay^{4s+3}, y^4)$ if $k = 4s+2$

or $g \in \mathscr{A}^{k+1}(x, xy + xy^3 + ay^{4s+2}, y^4)$ if $k = 4s+1$

or $g \in \mathcal{A}^{k+1}(x, xy + xy^3, y^4)$ otherwise.

Now $T\mathcal{A}(x, xy + xy^3 + ay^k, y^4) \ni x \cdot \begin{pmatrix} 1 \\ y + y^3 \\ 0 \end{pmatrix}$ so that

$$(0, ay^k, 0) \in T\mathcal{A}(x, xy + xy^3 + ay^k, y^4)$$

and there are (at most) two k-orbits:

$$\mathcal{A}^{k+1}(x, xy + xy^3 + y^k, y^4) \text{ and } \mathcal{A}^{k+1}(x, xy + xy^3, y^4).$$

By computer we find that

$$T\mathcal{A}_{2,2}(x, xy + xy^3 + y^6, y^4) \supseteq m_2^7 \cdot \theta(f) \setminus \{y^7 \cdot e_2\}$$

$$T\mathcal{A}_{2,2}(x, xy + xy^3 + y^7, y^4) \supseteq m_2^8 \cdot \theta(f) \setminus \{y^{10} \cdot e_2\}$$

$$T\mathcal{A}_{2,2}(x, xy + xy^3 + y^{10}, y^4) \supseteq m_2^{11} \cdot \theta(f) \setminus \{y^{11} \cdot e_2, y^{13} \cdot e_2, y^{15} \cdot e_2\} + \mathbb{C}\{(y^{13} + y^{15}) \cdot e_2\}$$

$$T\mathcal{A}_{2,2}(x, xy + xy^3 + y^{11}, y^4) \supseteq m_2^{12} \cdot \theta(f) \setminus \{y^{14} \cdot e_2, y^{18} \cdot e_2\}$$

$$T\mathcal{A}_{2,2}(x, xy + xy^3 + y^{14}, y^4) \supseteq m_2^{15} \cdot \theta(f) \setminus \{y^{15} \cdot e_2, y^{17} \cdot e_2, y^{19} \cdot e_2, y^{21} \cdot e_2, y^{23} \cdot e_2\}$$
$$+ \mathbb{C}\{(y^{17} + y^{19}) \cdot e_2, (y^{21} + y^{23}) \cdot e_2\}$$

$$T\mathcal{A}_{2,2}(x, xy + xy^3 + y^{15}, y^4) \supseteq m_2^{16} \cdot \theta(f) \setminus \{y^{18} \cdot e_2, y^{22} \cdot e_2, y^{26} \cdot e_2\}.$$

Further attempts to complete the classification have failed. It is not known whether or not f is a stem.

(II·3:27) $\qquad f = (x, xy - y^4, y^6).$

$$T\mathcal{A}f \qquad \begin{pmatrix} m_2 \setminus \{y^2, xy^2, x^2y^2, y^8, x^\ell y^{3j+1} : j, \ell \geq 0\} \\ m_2 \setminus \{y, y^3, xy^3, x^2y^3, y^9, x^\ell y^{3j+2} : j, \ell \geq 0\} \\ m_2 \setminus \{y^2, xy^2, y^3, x^2y^2, xy^3, x^2y^3, y^5, xy^5, x^2y^5 \\ y^8, xy^8, y^{11}, x^\ell y^{3j+1} : j, \ell \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} x^{\ell}y^{3j+1}(x-y^3) \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} x^2y^2+y^8 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^{\ell}y^{3j+2}(x-y^3) \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^2y^3+y^9 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^{\ell}y^{3j+7}(x-y^3) \end{pmatrix}\right.$$

$$\begin{pmatrix} 0 \\ 0 \\ x^{\ell+3}y^4(x-y^3) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^{\ell}y(x-y^3) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y^2+9y^8 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ xy^5+4y^8 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 4x^3y^2+9y^{11} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y^5+y^{11} \end{pmatrix}$$

$$\left.\begin{pmatrix} 0 \\ 0 \\ 4xy^8-y^{11} \end{pmatrix}\begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}\begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}\begin{pmatrix} xy^2 \\ xy^3 \\ 0 \end{pmatrix}\begin{pmatrix} y^8 \\ y^9 \\ 0 \end{pmatrix}\begin{pmatrix} 2y^8 \\ 0 \\ 3y^{11} \end{pmatrix}\begin{pmatrix} -xy^2 \\ 0 \\ 6y^8 \end{pmatrix}\begin{pmatrix} y^{3j+4} \\ y^{3j+5} \\ 0 \end{pmatrix}\begin{pmatrix} y^{3j+4} \\ 0 \\ 2y^{3j+7} \end{pmatrix} : \ell, j \geq 0\right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:10.

basis $\{y^{3j+1}\cdot e_3 : j \geq 4\}$,

multiplicity 1

At A:10 it is shown that $f^*\mathfrak{m}_3\cdot(T\mathscr{A}f) \supseteq$

$$\begin{pmatrix} \mathfrak{m}_2^{12}\setminus\{x^{\ell}y^{3j+1} : 3j+\ell \geq 11\} \\ \mathfrak{m}_2^{12}\setminus\{x^{\ell}y^{3j+2} : 3j+\ell \geq 10\} \\ \mathfrak{m}_2^{12}\setminus\{x^{\ell}y^{3j+1} : 3j+\ell \geq 11\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix}, \begin{pmatrix} y^{3j+1} \\ 0 \\ 2y^{3j+4} \end{pmatrix} : j \geq 2\right\}$$

$$T\mathscr{A}_{2,2}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\setminus\{y^2, xy^2, y^3, x^2y^2, y^5, y^6, y^8, x^{\ell}y^{3j+1} : j+\ell \geq 1\} \\ \mathfrak{m}_2^2\setminus\{xy, y^3, xy^3, y^4, x^2y^3, y^6, y^7, y^9, x^{\ell}y^{3j+2} : j, \ell \geq 0\} \\ \mathfrak{m}_2^2\setminus\{y^2, xy^2, y^3, x^2y^2, xy^3, x^3y^2, x^2y^3, y^5, xy^5, y^6, x^2y^5 \\ y^8, y^9, xy^8, y^{11}, x^{\ell}y^{3j+1} : j, \ell \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} x^{\ell}y^{3j+1}(x-y^3) \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} x^2y^2+y^8 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^{\ell}y^{3j+2}(x-y^3) \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^2y^3+y^9 \\ 0 \end{pmatrix}\right.$$

$$\begin{pmatrix} 0 \\ 0 \\ x^{\ell}y^{3j+7}(x-y^3) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^{\ell+3}y^4(x-y^3) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^{\ell}y(x-y^3) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y^2+9y^8 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ xy^5+4y^8 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 4x^3y^2+9y^{11} \end{pmatrix}$$

$$\left.\begin{pmatrix}0\\0\\x^2y^5+y^{11}\end{pmatrix}\begin{pmatrix}0\\0\\4xy^8-y^{11}\end{pmatrix}\begin{pmatrix}y\\y^2\\0\end{pmatrix}\begin{pmatrix}y^2\\y^3\\0\end{pmatrix}\begin{pmatrix}xy^2\\xy^3\\0\end{pmatrix}\begin{pmatrix}y^8\\y^9\\0\end{pmatrix}\begin{pmatrix}2y^8\\0\\3y^{11}\end{pmatrix}\begin{pmatrix}-xy^2\\0\\6y^8\end{pmatrix}\begin{pmatrix}y^{3j+4}\\y^{3j+5}\\0\end{pmatrix}\begin{pmatrix}y^{3j+4}\\0\\2y^{3j+7}\end{pmatrix}:\ell, j\geq 0\right\}$$

$g \in \mathscr{A}^6 f \Rightarrow$ $\quad g \in \mathscr{A}^7(x, xy - y^4, y^6 + ax^3y^4 + by^7)$,

$g \in \mathscr{A}^7 f \Rightarrow$ $\quad g \in \mathscr{A}^8(x, xy - y^4, y^6 + ay^8)$,

$g \in \mathscr{A}^8 f \Rightarrow$ $\quad g \in \mathscr{A}^9(x, xy - y^4, y^6 + ay^9)$,

$g \in \mathscr{A}^9 f \Rightarrow$ $\quad g \in \mathscr{A}^{10}(x, xy - y^4, y^6 + ay^{10})$,

$g \in \mathscr{A}^{10} f \Rightarrow$ $\quad g \in \mathscr{A}^{11}(x, xy - y^4, y^6 + ay^{11})$,

$g \in \mathscr{A}^k f, k \geq 11 \Rightarrow$ $\quad g \in \mathscr{A}^{k+1}(x, xy - y^4, y^6 + ay^{3s+1})$ if $k = 3s$,

or $g \in \mathscr{A}^{k+1}(x, xy - y^4, y^6)$ if $k \neq 3s$.

Weighted homogeneous $\quad$ weights $(3, 1)$

$$\text{degree}(y^6) = 6, \text{degree}(y^{3s+1}) = 3s+1$$

Hence $\quad T\mathscr{A}^{3s+1}(x, xy - y^4, y^6 + ay^{3s+1}) = T\mathscr{A}^{3s+1}f + \mathbb{C}\{ay^{3s+1}.e_3\}$.

By I:2 $(x, xy - y^4, y^6 + y^{3s+1})$ is $(3s+1)$-determined for $s \geq 4$.

Hence $g \in \mathscr{A}^{11} f \Rightarrow g \sim (x, xy - y^4, y^6 + y^{3s+1})$ for some $s \geq 4$ or $g \in \bigcap_{k \geq 0} \mathscr{A}^k f$ and f is a weak stem.

II·4 : $j^2f = (x, 0, 0)$.

(II·4:1) $$f = (x, 0, 0).$$

$$T\mathscr{A}_{2,2}f \qquad \begin{pmatrix} m_2^2 \\ \{x^j : j \geq 2\} \\ \{x^j : j \geq 2\} \end{pmatrix}$$

hence $g \in \mathscr{A}^2 f \Rightarrow \quad g \in \mathscr{A}^3(x, ax^2y + bxy^2 + cy^3, dx^2y + exy^2 + fy^3)$,

$g \in \mathscr{A}^3 f \Rightarrow \quad g \in \mathscr{A}^4(x, ax^3y + bx^2y^2 + cxy^3 + dy^4,\ ex^3y + fx^2y^2 + gxy^3 + hy^4)$,

etc.

(II·4:2) $$f = (x, ax^2y + bxy^2 + cy^3, dx^2y + exy^2 + fy^3).$$

We show the 3-orbits are as follows:

$$(x, x^2y + y^3, xy^2), (x, y^3, xy^2), (x, y^3, x^2y), (x, y^3, 0), (x, y^3 - x^2y, 0),$$
$$(x, xy^2, x^2y), (x, xy^2, 0), (x, x^2y, 0),\ (x, 0, 0).$$

This agrees with [Mond 1].

Proof: we shall make extensive use of the unfolding criterion I·2:19.

$$T\mathscr{A}^3 f \begin{pmatrix} m_2\backslash\{y\} \\ \{x, x^2, x^3\} \\ \{x, x^2, x^3\} \end{pmatrix} + \mathbb{C}\left\{ \begin{pmatrix} 0 \\ ax^2y + bxy^2 + cy^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ dx^2y + exy^2 + fy^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ ax^2y + bxy^2 + cy^3 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ dx^2y + exy^2 + fy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ bxy^2 + 2cy^3 \\ exy^2 + 2fy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ bx^2y + 3cxy^2 \\ ex^2y + 3fxy^2 \end{pmatrix}, \begin{pmatrix} y \\ 2axy^2 + by^3 \\ 2dxy^2 + ey^3 \end{pmatrix} \right\}.$$

The calculation of $T\mathscr{A}^3 f$ was performed by computer.

For each point $(a_0, b_0, c_0, d_0, e_0, f_0)$ the unfolding

$$(x, a_0x^2y + b_0xy^2 + c_0y^3, (a_0t + d_0)x^2y + (b_0t + e_0)xy^2 + (c_0t + f_0)y^3)$$

is 3-trivial. If $c_0 \neq 0$ then without loss of generality $f_0 = 0$. If $c_0 = 0$ and $f_0 \neq 0$ then interchange f_2 and f_3. Without loss of generality

$$f = (x, ax^2y + bxy^2 + cy^3, dx^2y + exy^2).$$

Then $T\mathscr{A}^3 f$ contains

$$\mathbb{C}\left\{\begin{pmatrix} 0 \\ ax^2y + bxy^2 + cy^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ dx^2y + exy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ dx^2y + exy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ bxy^2 + 2cy^3 \\ exy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ bx^2y + 3cxy^2 \\ ex^2y \end{pmatrix}\right\}.$$

Note that these are all tangent to $\{f = 0\}$.

For each point $(a_0, b_0, c_0, d_0, e_0)$ the unfolding

$$(x, (\tfrac{3}{2} c_0t^2 + b_0t + a_0)x^2y + (3c_0t + b_0)xy^2 + c_0y^3, (e_0t + d_0)x^2y + e_0xy^2)$$

is 3-trivial. If If $e_0 \neq 0$ then w.l.o.g. $d_0 = 0$. Without loss of generality

$$f = (x, ax^2y + bxy^2 + cy^3, exy^2) \text{ or } (x, ax^2y + bxy^2 + cy^3, dx^2y).$$

In the first case we consider $\mathbb{C}\left\{\begin{pmatrix} 0 \\ ax^2y + bxy^2 + cy^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ exy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ exy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ bxy^2 + 2cy^3 \\ 0 \end{pmatrix}\right\}.$

Note that these are all tangent to $\{d = 0\}$.

For each point (a_0, b_0, c_0, e_0) the unfolding $(x, a_0x^2y + b_0xy^2 + c_0y^3, e^te_0xy^2)$ is 3-trivial (recall $e_0 \neq 0$). Without loss of generality $e_0 = 1$.

We consider $\mathbb{C}\left\{\begin{pmatrix} 0 \\ ax^2y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ cy^3 \\ 0 \end{pmatrix}\right\}.$

It is clear that without loss of generality, f is one of

$$(x, x^2y + y^3, xy^2), (x, x^2y, xy^2), (x, y^3, xy^2), (x, 0, xy^2)$$

In the second case we consider

$$\mathbb{C}\left\{\begin{pmatrix} 0 \\ ax^2y + bxy^2 + cy^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ dx^2y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ dx^2y \end{pmatrix}, \begin{pmatrix} 0 \\ bxy^2 + 2cy^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ bx^2y + 3cxy^2 \\ 0 \end{pmatrix}\right\}$$

Note that these are all tangent to $\{e = 0\}$.

If $d_0 \neq 0$ then the unfolding $(x, a_0x^2y + b_0xy^2 + c_0y^3, e^t d_0x^2y)$ is 3-trivial. Without loss of generality $d_0 = 1$. We consider $\mathbb{C}\left\{\begin{pmatrix}0\\x^2y\\0\end{pmatrix}, \begin{pmatrix}0\\bxy^2\\0\end{pmatrix}, \begin{pmatrix}0\\cxy^2\\0\end{pmatrix}, \begin{pmatrix}0\\cy^3\\0\end{pmatrix}\right\}$.

It is clear that without loss of generality, f is one of

$$(x, xy^2, x^2y), (x, y^3, x^2y), (x, 0, x^2y).$$

If $d_0 = 0$ we consider

$$\mathbb{C}\left\{\begin{pmatrix}0\\ax^2y + bxy^2 + cy^3\\0\end{pmatrix}, \begin{pmatrix}0\\bxy^2 + 2cy^3\\0\end{pmatrix}, \begin{pmatrix}0\\bx^2y + 3cxy^2\\0\end{pmatrix}\right\}.$$

The unfolding $(x, (\frac{3}{2}c_0t^2 + b_0t + a_0)x^2y + (3c_0t + b_0)xy^2 + c_0y^3, 0)$ is 3-trivial. We may assume that $b_0 = 0$ unless $c_0 = 0$.

It is clear that without loss of generality, f is one of

$$(x, x^2y, 0), (x, x^2y - y^3, 0), (x, xy^2, 0), (x, y^3, 0), (x, 0, 0).$$

Hence $g \in \mathscr{A}^2(x, 0, 0) \Rightarrow$

$$g \in \{\mathscr{A}^3(x, x^2y + y^3, xy^2), \mathscr{A}^3(x, y^3, xy^2), \mathscr{A}^3(x, xy^2, x^2y), \mathscr{A}^3(x, y^3, x^2y),$$
$$\mathscr{A}^3(x, x^2y - y^3, 0), \mathscr{A}^3(x, y^3, 0), \mathscr{A}^3(x, x^2y, 0), \mathscr{A}^3(x, xy^2, 0), (x, 0, 0)\}$$

(II·4:3) $\qquad f = (x, x^2y + y^3, xy^2).$

$$T\mathscr{A}f \qquad \begin{pmatrix}\mathfrak{m}_2\backslash\{y^{3j+2} : j \geq 0\}\\ \mathfrak{m}_2\backslash\{y, xy, y^2\}\\ \mathfrak{m}_2\backslash\{xy, y^2, y^{3j+1} : j \geq 0\}\end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix}y^{3j+2}\\0\\y^{3j+4}\end{pmatrix} : j \geq 0\right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:11.

Basis $\qquad \{y^{3k+1}.e_3\}$,

multiplicity $\qquad 1$,

$$f^*\mathfrak{m}_3\cdot T\mathcal{A}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{y^2, y^3, xy^2, y^{3j+2} : j \geq 1\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, y^4, y^5\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, y^4, xy^4, y^5, y^{3j+1} : j \geq 1\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}, \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1\right\},$$

$f^*\mathfrak{m}_3\cdot\left(T\mathcal{A}f + \mathbb{C}\{y^{3j+1}.e_3\}\right) \supseteq \mathfrak{m}_2^{3j+2}\cdot\theta(f)$ provided that $j \geq 2$.

$$T\mathcal{A}_{2,2}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{y^2, y^3, xy^2, y^{3j+2} : j \geq 1\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, y^5\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, xy^4, y^5, y^{3j+1} : j \geq 1\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} 0 \\ 3y^5 \\ 2xy^4 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}, \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix}, \begin{pmatrix} y^3 \\ 0 \\ y^5 \end{pmatrix} : j \geq 0\right\},$$

hence $g \in \mathcal{A}^3 f \Rightarrow$ $\quad g \in \mathcal{A}^4(x, x^2y + y^3, xy^2 + ay^4)$,

$g \in \mathcal{A}^4 f \Rightarrow$ $\quad g \in \mathcal{A}^5(x, x^2y + y^3, xy^2 + axy^4 + by^5)$,

and for $k \geq 5$ $g \in \mathcal{A}^k f \Rightarrow$ $\quad g \in \mathcal{A}^{k+1}(x, x^2y + y^3, xy^2 + ay^{3s+1})$ if $k = 3s$,

or $\mathcal{A}^{k+1}(x, x^2y + y^3, xy^2)$ if $k \neq 3s$.

Weighted homogeneous $\quad$ weights $(1,1)$,

$\text{degree}(xy^2) = 3, \text{degree}(y^{3s+1}) = 3s+1$.

Hence $\quad T\mathcal{A}^{3s+1}(x, x^2y + y^3, xy^2 + ay^{3s+1}) = T\mathcal{A}^{3s+1}f + \mathbb{C}\{ay^{3s+1}.e_3\}$.

For $s \geq 2$, by I:2 $(x, x^2y + y^3, xy^2 + y^{3s+1})$ is $(3s+1)$-determined.

In [Mond 1] the determinacy degree of $(x, x^2y + y^3, xy^2 + y^4)$ is shown to be 4.

By computer we calculate that

$$T\mathscr{A}^5(x, x^2y+y^3, xy^2+axy^4+by^5) = \begin{pmatrix} \mathfrak{m}_2\backslash\{y^2\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, y^4\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} y^2 \\ 0 \\ y^4 \end{pmatrix}\right\}$$

so that $\left\{(x, x^2y+y^3, xy^2+axy^4+by^5) : a, b \in \mathbb{C}\right\}$ lies in a single 5-orbit.

Hence $g \in \mathscr{A}^3 f \Rightarrow g \sim (x, x^2y+y^3, xy^2+y^{3s+1})$ for some $s \geq 1$ or $g \in \bigcap_{k\geq 0} \mathscr{A}^k f$ and f is a weak stem.

(II·4:4) $\qquad f = (x, y^3, xy^2).$

$$T\mathscr{A}f \qquad \begin{pmatrix} \mathfrak{m}_2\backslash\{y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y^2, x^jy : j \geq 0\} \\ \mathfrak{m}_2\backslash\{xy, y^2, y^{3j+1} : j \geq 0\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 0\right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:12.

Basis $\qquad \left\{x^jy\cdot e_2, y^{3j+1}.e_3\right\},$

multiplicity $\qquad 2,$

$$f^*\mathfrak{m}_3\cdot T\mathscr{A}f \qquad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{y^3, xy^2, y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2^2\backslash\{y^2, xy^2, y^3, y^4, y^5, x^jy : j \geq 1\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, xy^4, y^5, y^{3j+1} : j \geq 1\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{\begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}\begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1\right\},$$

$$f^*\mathfrak{m}_3\cdot\left(T\mathscr{A}f + \mathbb{C}\{x^{3j}y\cdot e_2, y^{3j+1}.e_3\}\right) \supseteq \mathfrak{m}_2^{3j+2}\cdot\theta(f) \text{ provided } j \geq 2.$$

$$T\mathcal{A}_{2,2}f \qquad \begin{pmatrix} m_2^2\backslash\{y^3, xy^2, y^{3j+2} : j \geq 0\} \\ m_2^2\backslash\{y^2, xy^2, y^3, y^5, x^j y : j \geq 1\} \\ m_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, xy^4, y^5, y^{3j+1} : j \geq 1\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{ \begin{pmatrix} 0 \\ 3y^5 \\ 2xy^4 \end{pmatrix} \begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix} \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} \begin{pmatrix} y^3 \\ 0 \\ y^5 \end{pmatrix} : j \geq 0 \right\},$$

hence $g \in \mathcal{A}^3 f \Rightarrow$ $\quad g \in \mathcal{A}^4(x, y^3 + ax^3y, xy^2 + by^4)$,

$g \in \mathcal{A}^4 f \Rightarrow$ $\quad g \in \mathcal{A}^5(x, y^3 + ax^4y, xy^2 + bxy^4 + cy^5)$,

and for $k \geq 5$ $g \in \mathcal{A}^k f \Rightarrow$ $\quad g \in \mathcal{A}^{k+1}(x, y^3 + ax^{3s}y, xy^2 + by^{3s+1}) : k = 3s$,

or $\mathcal{A}^{k+1}(x, y^3 + ax^k y, xy^2) : k \neq 3s$.

Weighted homogeneous $\quad$ weights (1, 1).

Hence $\quad T\mathcal{A}^{k+1}(x, y^3 + ax^k y, xy^2) = T\mathcal{A}^{k+1}f + \mathbb{C}\{ax^k y \cdot e_2\}$

and $T\mathcal{A}^{3s+1}(x, y, xy^2 + by^{3s+1}) = T\mathcal{A}^{3s+1}f + \mathbb{C}\{by^{3s+1} \cdot e_3\}$.

The germs $(x, y^3, xy^2 + y^{3s+1})$ and $(x, y^3 + x^k y, xy^2)$ are not finitely determined (having a cuspidal edge and a triple curve respectively). Neither are they equivalent to (x, y^3, xy^2) (which has both) and which is hence not a stem..

(II·4:5) $\qquad f = (x, y^3 + ax^{3s}y, xy^2 + by^{3s+1})$.

$$T\mathcal{A}^{3s+1}f = T\mathcal{A}_{2,2}(x, y^3, xy^2) + \mathbb{C}\left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}, \begin{pmatrix} y^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y^3 + ax^{3s}y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ y^3 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 \\ 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 \\ 0 \\ xy^2 + by^{3s+1} \end{pmatrix}, \begin{pmatrix} x \\ 3sax^{3s}y \\ xy^2 \end{pmatrix}, \begin{pmatrix} y \\ 3sax^{3s-1}y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3xy^2 + ax^{3s+1} \\ 2x^2y + (3s+1)bxy^{3s} \end{pmatrix}, \begin{pmatrix} 0 \\ 3y^3 + ax^{3s}y \\ 2xy^2 + (3s+1)by^{3s+1} \end{pmatrix} \right\}$$

$$= \qquad T\mathscr{A}^{3s+1}(x, y^3, xy^2) + \mathbb{C}\left\{\begin{pmatrix} 0 \\ ax^{3s}y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ by^{3s+1} \end{pmatrix}\right\}.$$

The 3s+1-orbits orbits are $\left\{(x, y^3 + ax^{3s}y, xy^2 + by^{3s+1}) : a, b \in \mathbb{C}\backslash\{0\}\right\}$

$\left\{(x, y^3 + ax^{3s}y, xy^2) : a \in \mathbb{C}\backslash\{0\}\right\}, \left\{(x, y^3, xy^2 + by^{3s+1}) : b \in \mathbb{C}\backslash\{0\}\right\}.$

Then $(x, y^3 + ax^{3s}y, xy^2 + by^{3s+1}) \sim (x, y^3 + x^{3s}y, xy^2 + y^{3s+1})$ is (3s+1)-determined provided that $s \geq 2$, since the hypotheses of I:2 are satisfied.

(II·4:6) $\qquad f = (x, y^3 + ax^4y, xy^2 + bxy^4 + cy^5).$

$$T\mathscr{A}^5 f \qquad \begin{pmatrix} \mathfrak{m}_2\backslash\{y^2\} \\ \mathfrak{m}_2\backslash\{y^2, x^jy : 4 \geq j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, y^4\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} 0 \\ ax^4y \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \\ y^4 \end{pmatrix}\right\}.$$

The calculation of $T\mathscr{A}^5 f$ was performed by computer.

Hence the 5-orbits are $\left\{(x, y^3 + ax^4y, xy^2 + bxy^4 + cy^5) : a \in \mathbb{C}\backslash\{0\};\ b, c \in \mathbb{C}\right\}$

and $\qquad \left\{(x, y^3, xy^2 + bxy^4 + cy^5) : b, c \in \mathbb{C}\right\}.$

The two orbits are represented by the map-germs $(x, y^3, xy^2), (x, y^3 + x^4y, xy^2)$.

Hence $g \in \mathscr{A}^3(x, y^3, xy^2) \Rightarrow g \sim (x, y^3 + x^{3s}y, xy^2 + y^{3s+1})$ or

$$g \in \mathscr{A}^{k+1}(x, y^3 + x^ky, xy^2) \text{ or } g \in \mathscr{A}^{3s+1}(x, y^3, xy^2 + y^{3s+1}) \text{ or } g \in \bigcap_{k \geq 0} \mathscr{A}^k f.$$

(II·4:7) $\qquad f = (x, y^3, x^2y).$

$$T\mathscr{A} f \qquad \begin{pmatrix} \mathfrak{m}_2\backslash\{xy^{3j+1}, y^{3j+1} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{xy, y^2, y^{3j+1} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{xy, xy^{3j+2}, x^2y^{3j+2}, y^{3j+1}, y^{3j+2} : j \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix}0\\3y^{3j+4}\\x^2y^{3j+2}\end{pmatrix},\begin{pmatrix}xy^{3j+1}\\0\\2x^2y^{3j+2}\end{pmatrix},\begin{pmatrix}y^{3j+1}\\0\\2xy^{3j+2}\end{pmatrix}:j\geq 0\right\}$$

The calculation of $T\mathscr{A}f$ will be found at A:13.

Basis $\{xy^{3k+2}.e_3, x^2y^{3k+2}.e_3, y^{3k+1}.e_3, y^{3k+2}.e_3\}$,

multiplicity 4,

$$T\mathscr{A}_{2,2}f \quad \begin{pmatrix}\mathfrak{m}_2^2\backslash\{x^2y, y^{3j+1}, xy^{3j+1}:j\geq 0\}\\ \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, xy^4, y^{3j+1}:j\geq 1\}\\ \mathfrak{m}_2^2\backslash\{xy, x^2y, y^3, xy^4, x^3y^2, y^{3j+1}, y^{3j+2}, xy^{3j+2}, x^2y^{3j+2}:j\geq 0\}\end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix}y^3\\0\\2xy^4\end{pmatrix},\begin{pmatrix}0\\3xy^4\\x^3y^2\end{pmatrix},\begin{pmatrix}-x^2y\\6xy^4\\0\end{pmatrix},\begin{pmatrix}y^{3j+4}\\0\\2xy^{3j+5}\end{pmatrix},\begin{pmatrix}xy^{3j+1}\\0\\2x^2y^{3j+2}\end{pmatrix},\begin{pmatrix}0\\3y^{3j+4}\\x^2y^{3j+2}\end{pmatrix}:j\geq 0\right\},$$

hence $g\in\mathscr{A}^3f \Rightarrow$ $g\in\mathscr{A}^4(x, y^3, x^2y+ax^2y^2+by^4)$,

$g\in\mathscr{A}^4f \Rightarrow$ $g\in\mathscr{A}^5(x, y^3, x^2y+ax^3y^2+bxy^4+cy^5)$,

and for $k\geq 5$ $g\in\mathscr{A}^kf \Rightarrow$ $g\in\mathscr{A}^{k+1}(x, y^3, x^2y+ax^2y^{3s-1}+by^{3s+1}) : k=3s$,

or $\mathscr{A}^{k+1}(x, y^3, x^2y+ay^{3s+2}) : k=3s+1$,

or $\mathscr{A}^{k+1}(x, y^3, x^2y+axy^{3s+2}) : k=3s+2$.

The germs

$$(x, y^3, x^2y+x^2y^{3j+2}), (x, y^3, x^2y+xy^{3j+2}), (x, y^3, x^2y+y^{3j+1})$$

are not finitely determined;

the map-germ $(x, y^3, x^2y+y^{3j+1}) = (x, y^3, y(x^2+y^{3j}))$ has two triple curves with smooth pre-image along $x^2+y^{3j}=0$ if j is even, or one triple curve with non-smooth pre-image if j is odd.

The map-germ (x, y^3, x^2y+xy^{3j+2}) has a triple curve along $x=0$, which has transversal type D_4.

Consequently, f is not a stem, since it has a single, smooth curve of triple points

which has transversal type J_{10} (the image satisfies $Z^3 = YX^6$ where Y is a constant).

(II·4:8) $$f = (x, y^3, x^2y + y^4).$$

The curve $(x^2 + y^3) = 0 \to (x, y^3, 0)$ is a triple curve of transversal type D_4. Note that the image of this curve is smooth, but not the pre-image.

$$T\mathscr{A}f \quad \begin{pmatrix} \mathfrak{m}_2\backslash\{x^\ell y^{3j+1} : \ell, j \geq 0\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2+y^3) : \ell+j \geq 0\} \\ \mathfrak{m}_2\backslash\{x^\ell y^{3j+1} : \ell, j \geq 0\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2+y^3) : \ell+j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, xy, x^\ell y^{3j+2} : \ell, j \geq 0\} + \mathbb{C}\{x^\ell y^{3j+2}(x^2+y^3) : \ell+j \geq 1\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{ \begin{pmatrix} 0 \\ 3y^4 \\ x^2y^2+4y^5 \end{pmatrix}, \begin{pmatrix} -y^{3j+4} \\ 2xy^{3j+4} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{3j+4} \\ 2y^{3j+7} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{3j+1} \\ 0 \\ 2x^2y^{3j+2} \end{pmatrix}, \begin{pmatrix} y^{3j+1} \\ 0 \\ 2xy^{3j+2} \end{pmatrix} : j \geq 0 \right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:14.

Basis $\{x^jy^2 \cdot e_3\}$,

multiplicity 1.

Note that $(x^{2j}y^2 + (-1)^{j+1}y^{3j+2}) \cdot e_3$ and $(x^{2j+1}y^2 + (-1)^{j+1}xy^{3j+2}) \cdot e_3 \in T\mathscr{A}f$ so that we can rewrite the basis as $\{y^{3j+2} \cdot e_3\}$ or $\{xy^{3j+2} \cdot e_3\}$.

$$f^*\mathfrak{m}_3 \cdot T\mathscr{A}f \quad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{y^2, y^3, x^\ell y^{3j+1} : \ell+j \geq 1\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2+y^3) : \ell+j \geq 1\} \\ \mathfrak{m}_2\backslash\{y^2, xy^2, y^3, y^5, x^\ell y^{3j+1} : \ell+j \geq 1\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2+y^3) : \ell+j \geq 1\} \\ \mathfrak{m}_2\backslash\{xy, x^2y, y^3, y^4, x^\ell y^{3j+2} : \ell, j \geq 0\} + \mathbb{C}\{x^\ell y^{3j+2}(x^2+y^3) : \ell \geq 2 \text{ or } j \geq 1\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{ \begin{pmatrix} 0 \\ 3xy^4 \\ x^3y^2+4xy^5 \end{pmatrix}, \begin{pmatrix} x^2y \\ 0 \\ 2x^3y^2 \end{pmatrix}, \begin{pmatrix} -y^{3j+7} \\ 2xy^{3j+7} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{3j+4} \\ 2y^{3j+7} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{3j+1} \\ 0 \\ 2x^2y^{3j+2} \end{pmatrix}, \begin{pmatrix} y^{3j+4} \\ 0 \\ 2xy^{3j+5} \end{pmatrix} : j \geq 0 \right\},$$

$f^*\mathfrak{m}_3 \cdot \left(T\mathscr{A}f + \mathbb{C}\{y^{3j+2}) \cdot e_3\}\right) \supseteq \mathfrak{m}_2^{3j+3} \cdot \theta(f)$ provided $j \geq 1$

and $f^*\mathfrak{m}_3 \cdot \left(T\mathscr{A}f + \mathbb{C}\{xy^{3j+2}) \cdot e_3\}\right) \supseteq \mathfrak{m}_2^{3j+4} \cdot \theta(f)$ provided $j \geq 1$.

$$T\mathscr{A}_{2,2}f \quad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{x^\ell y^{3j+1} : \ell+j \geq 1\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2+y^3) : \ell+j \geq 1\} \\ \mathfrak{m}_2\backslash\{y^2, xy^2, y^3, x^\ell y^{3j+1} : \ell+j \geq 1\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2+y^3) : \ell+j \geq 1\} \\ \mathfrak{m}_2\backslash\{xy, x^2y, y^3, y^4, x^\ell y^{3j+2} : \ell, j \geq 0\} + \mathbb{C}\{x^\ell y^{3j+2}(x^2+y^3) : \ell \geq 2 \text{ or } j \geq 1\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{\begin{pmatrix} x^2y+4y^4 \\ -6xy^4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3y^4 \\ 2x^2y^2+4y^5 \end{pmatrix}, \begin{pmatrix} x^2y \\ 0 \\ 2x^3y^2 \end{pmatrix}, \begin{pmatrix} -y^{3j+7} \\ 2xy^{3j+7} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{3j+4} \\ 2y^{3j+7} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{3j+1} \\ 0 \\ 2x^2y^{3j+2} \end{pmatrix}, \begin{pmatrix} y^{3j+4} \\ 0 \\ 2xy^{3j+5} \end{pmatrix} : j \geq 0\right\},$$

hence $g \in \mathscr{A}^4 f \Rightarrow$ $\quad g \in \mathscr{A}^5(x, y^3, x^2y + y^4 + ax^3y^2 + by^5)$,

and for $k \geq 5$ $g \in \mathscr{A}^k f \Rightarrow$ $\quad g \in \mathscr{A}^{k+1}(x, y^3, x^2y + y^4 + axy^{3s+2})$ if $k = 3s+2$,

or $\mathscr{A}^{k+1}(x, y^3, x^2y + y^4)$ if $k = 3s$,

or $\mathscr{A}^{k+1}(x, y^3, x^2y + y^4 + ay^{3s+2})$ if $k = 3s+1$.

Weighted homogeneous $\quad$ weights $(3, 2)$,

$$\text{degree}(x^2y + y^4) = 8, \text{degree}(y^{3s+2}) = 6s+4, \text{ degree}(xy^{3s+2}) = 6s+7.$$

Hence $\quad T\mathscr{A}^{3s+3}(x, y^3, x^2y + y^4 + axy^{3s+2}) = T\mathscr{A}^{3s+3}f + \mathbb{C}\{axy^{3j+2}.e_3\}$

and $\quad T\mathscr{A}^{3s+2}(x, y^3, x^2y + y^4 + ay^{3s+2}) = T\mathscr{A}^{3s+2}f + \mathbb{C}\{ay^{3j+2}.e_3\}$.

By I:2 $(x, y^3, x^2y + y^4 + xy^{3s+2})$ is $(3s+3)$-determined and $(x, y^3, x^2y + y^4 + xy^{3s+2})$ is $(3s+2)$-determined.

(II·4:9) $\quad f = (x, y^3, x^2y + y^4 + ax^3y^2 + by^5)$.

$$T\mathscr{A}^5 f \quad \begin{pmatrix} \mathfrak{m}_2\backslash\{y, xy\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, x^2y, y^4\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, xy^2, x^2y^2, y^5\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{\begin{pmatrix} 0 \\ x^2y + y^4 \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \\ 2xy^2 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \\ 2x^2y^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3y^4 \\ x^2y^2 + 4y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ by^5 \end{pmatrix}\right\}.$$

The calculation of $T\mathscr{A}^5 f$ performed by computer.

The 5-orbits are $\{(x, y^3, x^2y + y^4 + ax^3y^2 + by^5): b \in \mathbb{C}\setminus\{0\};\ a \in \mathbb{C}\}$,

$\{(x, y^3, x^2y + y^4 + ax^3y^2): a \in \mathbb{C}\}$.

The two orbits are represented by the map-germs

$$(x, y^3, x^2y + y^4), (x, y^3, x^2y + y^4 + y^5).$$

By I:2 $(x, y^3, x^2y + y^4 + y^5)$ is 5-determined.

Hence $g \in \mathcal{A}^3 f \Rightarrow$

$$g \sim (x, y^3, x^2y + y^4 + y^{3s+2}) \text{ or } g \sim (x, y^3, x^2y + y^4 + xy^{3s+2}) \text{ for some } s \geq 1$$

$$\text{or } g \in \bigcap_{k \geq 0} \mathcal{A}^k f;$$

thus f is a weak stem.

(II·4:10) $\qquad f = (x, x^2y - y^3, 0).$

$$T\mathcal{A}_{2,2}f \qquad \begin{pmatrix} \mathfrak{m}_2^2 \\ \mathfrak{m}_2^4 + \mathbb{C}\{x^2, x^3\} \\ \{x^j(x^2y - y^3)^\ell : \ell + j \geq 2\} \end{pmatrix}$$

The calculation of $T\mathcal{A}_{2,2}f$ was performed by computer (calculation modulo $\mathfrak{m}_2^7 \cdot \theta(f)$ is sufficient to estabish that $\mathfrak{m}_2^4 \cdot \frac{\partial}{\partial X} + \mathfrak{m}_2^4 \cdot \frac{\partial}{\partial Y} \subseteq T\mathcal{A}_{2,2}f$. Then

$g \in \mathcal{A}^3 f \Rightarrow \qquad g \in \mathcal{A}^4(x, x^2y - y^3, ax^2y^2 + bxy^3 + cy^4),$

$g \in \mathcal{A}^4 f \Rightarrow \qquad g \in \mathcal{A}^5(x, x^2y - y^3, ax^3y^2 + bx^2y^3 + cxy^4 + dy^5)$ etc.

We can obviously choose any one non-zero coefficient to be 1. In fact this is the only simplification which can be made, giving the orbits

$$(x, x^2y - y^3, ax^2y^2 + bxy^3 + y^4), (x, x^2y - y^3, ax^2y^2 + xy^3), (x, x^2y - y^3, x^2y^2)$$

Worse still, none of the orbits $(x, x^2y - y^3, ax^2y^2 + bxy^3 + y^4)$ are 4-determined.

We look at just one of these 4-jets:

(II·4:11) $$f = (x, x^2y - y^3, x^2y^2 - y^4).$$

$$T\mathscr{A}f \quad \begin{pmatrix} \mathfrak{m}_2\backslash\{y, xy\} \\ \mathbb{C}\{x, x^2, x^3, x^4, x^\ell y^{2j} : \ell + 2j \geq 5\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2 - y^2) : \ell, j \geq 0\} \\ \mathbb{C}\{x, x^2, x^3, x^4, x^5, x^\ell y^{2j+1} : \ell + 2j \geq 5\} + \mathbb{C}\{x^\ell y^{3j+2}(x^2 + y^3) : \ell, j \geq 0\} \end{pmatrix}$$

$$+ \left\{ \begin{pmatrix} 0 \\ x^2y^2 - y^4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y - y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^3y - xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^4y - x^2y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 5x^4y - 6y^5 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} y \\ -xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ -y^4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3y \\ 0 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} 5xy \\ 0 \\ 4y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ x^jy \\ x^jy^2 \end{pmatrix} : j \geq 2 \right\}.$$

The calculation of $T\mathscr{A}f$ will be found at A:15.

Basis $$\left\{ x^jy^2 \cdot e_3 \right\},$$

multiplicity $$1,$$

$$f^*\mathfrak{m}_3 \cdot T\mathscr{A}f \quad \begin{pmatrix} \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, y^3\} \\ \mathbb{C}\{x^2, x^3, x^4, x^5, x^\ell y^{2j} : \ell + 2j \geq 6\} + \mathbb{C}\{x^\ell y^{2j+1}(x^2 - y^2) : 2\ell + j \geq 2\} \\ \mathbb{C}\{x^2, x^3, x^4, x^5, x^6, x^\ell y^{2j+1} : \ell + 2j \geq 6\} + \mathbb{C}\{x^\ell y^{2j+2}(x^2 - y^2) : \ell + j \geq 1\} \end{pmatrix}$$

$$+ \left\{ \begin{pmatrix} 0 \\ 0 \\ x^4y - x^2y^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 5x^3y^3 - 6xy^5 \end{pmatrix} \begin{pmatrix} xy \\ -x^2y^2 \\ 0 \end{pmatrix} \begin{pmatrix} 3xy \\ 0 \\ 2x^2y^3 \end{pmatrix} \begin{pmatrix} x^2y \\ -x^3y^2 \\ 0 \end{pmatrix} \begin{pmatrix} 3x^2y \\ 0 \\ 2x^3y^3 \end{pmatrix} \begin{pmatrix} 0 \\ x^jy \\ x^jy^2 \end{pmatrix} : j \geq 3 \right\},$$

$f^*\mathfrak{m}_3 \cdot \left(T\mathscr{A}f + \mathbb{C}\{x^jy^2 \cdot e_3\} \right) \supseteq \mathfrak{m}_2^{j+3} \cdot \theta(f)$ provided that $j \geq 4$.

$$T\mathscr{A}_{2,2}f \quad \begin{pmatrix} \mathfrak{m}_2\backslash\{xy, x^2y, y^3,\} \\ \mathbb{C}\{x^2, x^3, x^4, x^5, x^\ell y^{2j} : \ell + 2j \geq 6\} + \mathbb{C}\{x^\ell y^{2j+1}(x^2 - y^2) : \ell + j \geq 1\} \\ \mathbb{C}\{x^2, x^3, x^4, x^5, x^6, x^\ell y^{2j+1} : \ell + 2j \geq 6\} + \mathbb{C}\{x^\ell y^{2j+2}(x^2 - y^2) : \ell + j \geq 1\} \end{pmatrix}$$

$$+ \left\{ \begin{pmatrix} x^2y - 2y^3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x^3y^2 - xy^4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x^3y - xy^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x^4y - x^2y^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 5x^5y - 6xy^5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x^5y - x^3y^3 \end{pmatrix} \right.$$

$$\left. \begin{pmatrix} 2y^3 \\ -xy^4 \\ 0 \end{pmatrix} \begin{pmatrix} xy \\ -x^2y^2 \\ 0 \end{pmatrix} \begin{pmatrix} 3xy \\ 0 \\ 2x^2y^3 \end{pmatrix} \begin{pmatrix} 5y^3 \\ 0 \\ 2xy^5 \end{pmatrix} \begin{pmatrix} 0 \\ x^2y^2 - 3y^4 \\ 2x^2y^3 - 4y^5 \end{pmatrix} \begin{pmatrix} 0 \\ x^jy \\ x^jy^2 \end{pmatrix} : j \geq 3 \right\},$$

hence $g \in \mathscr{A}^4 f \Rightarrow$ $\quad g \in \mathscr{A}^5(x, x^2y - y^3, x^2y^2 - y^4 + ax^4y + bx^3y^2 + cy^5)$,

$g \in \mathscr{A}^5 f \Rightarrow$ $\quad g \in \mathscr{A}^6(x, x^2y - y^3, x^2y^2 - y^4 + ax^4y^2 + bx^5y)$,

and for $k \geq 6$ $g \in \mathscr{A}^k f \Rightarrow$ $\quad g \in \mathscr{A}^{k+1}(x, x^2y - y^3, x^2y^2 - y^4 + ax^{k-1}y^2)$.

Homogeneous $\quad$ degree$(x^2y^2 - y^4) = 4$, degree$(x^{k-1}y^2) = k+1$,

hence $\quad T\mathscr{A}^{k+1}(x, x^2y - y^3, x^2y^2 - y^4 + ax^{k-1}y^2) = T\mathscr{A}^{k+1}f + \mathbb{C}\{ax^{k-1}y^2.e_3\}$.

By I:2 $(x, x^2y - y^3, x^2y^2 - y^4 + x^{k-1}y^2)$ is k+1-determined for $k \geq 6$.

Hence $g \in \mathscr{A}^6 f \Rightarrow$

$$g \sim (x, x^2y - y^3, x^2y^2 - y^4 + x^{k-1}y^2) \text{ for some } k \geq 6 \text{ or } g \in \bigcap_{k \geq 0} \mathscr{A}^k f$$

and f is a weak stem.

II·5 : Map-germs having corank 2.

We prove the results presented in table 4 and figure 4.

(II·5·1) Proposition : if f has corank 2 then f lies in one of the following $\mathscr{A}$-invariant strata:

$$\mathscr{A}^2(xy, x^2, y^2), \quad \mathscr{A}^2(x^2+y^2, xy, 0), \quad \mathscr{A}^2(x^2, y^2, 0),$$
$$\mathscr{A}^2(x^2, xy, 0), \quad \mathscr{A}^2(xy, 0, 0), \quad \mathscr{A}^2(x^2, 0, 0), \quad \mathscr{A}^2(0, 0, 0).$$

Proof : if f has corank 2 then

$$j^2f = (a_1x^2 + a_2xy + a_3y^2, b_1x^2 + b_2xy + b_3y^2, c_1x^2 + c_2xy + c_3y^2)$$

for some $a_i, b_i, c_i \in \mathbb{C}$.

If $\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \neq 0$ then we can find a change of basis so that

$$j^2f = (xy, x^2, y^2)$$

otherwise we may assume that $c_1 = c_2 = c_3 = 0$, i.e.

$$j^2f = (a_1x^2 + a_2xy + a_3y^2, b_1x^2 + b_2xy + b_3y^2, 0).$$

If $a_1 \neq 0$ then we can subtract b_1X/a_1 from Y so that without loss of generality, $b_1 = 0$. This leads to the following:

if $b_2 \neq 0$ then let $x' = \left(b_2x + b_3y\right)$. Then j^2f becomes $(\alpha_1x^2 + \alpha_2xy + \alpha_3y^2, b_2xy, 0)$. We get $(x^2+y^2, xy, 0)$ or $(x^2, xy, 0)$ depending on α_3 (note $(x^2+y^2, xy, 0)$ and $(x^2-y^2, xy, 0)$ are equivalent under the coordinate changes $y' = iy$, $Y' = -iY$).

If $b_2 = 0$ then let $x' = x + \dfrac{a_2}{2a_1}y$. Then j^2f becomes $(a_1(x')^2 + \alpha_3y^2, b_3y^2, 0)$.

This gives the forms $(x^2, y^2, 0)$ if $b_3 \neq 0$ and $(x^2+y^2, 0, 0)$ or $(x^2, 0, 0)$ if $b_3 = 0$. Then $(x^2+y^2, 0, 0) = ((x+iy)(x-iy), 0, 0)$ is equivalent to $(xy, 0, 0)$ under the coordinate change $x' = x+iy$, $y' = x-iy$.

Finally, we suppose that $a_1 = b_1 = 0$. If $\det\begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} \neq 0$ then without loss of

generality $j^2f = (xy, y^2, 0)$. Let $x' = y$, $y' = x$ to get $(xy, x^2, 0)$.

Otherwise without loss of generality $b_2 = b_3 = 0$ and j^2f is one of

$$(xy + y^2, 0, 0), (xy, 0, 0) \text{ or } (y^2, 0, 0).$$

We let $x' = x + y$ to show that $(xy + y^2, 0, 0) \sim (xy, 0, 0)$ and exchange x, y to show that $(y^2, 0, 0) \sim (x^2, 0, 0)$.

This concludes the proof of the proposition. □.

Note that none of these are finitely determined since they are all at least 2-1 everywhere. (They have 2-dimensional instability loci!)

We look at (xy, x^2, y^2) and $(x^2 + y^2, xy, 0)$ in more detail:

(II·5·2) Proposition : if $j^2f = (xy, x^2, y^2)$ then f lies in one of

$$\mathscr{A}^{2s+1}(xy + \Sigma\lambda_j x^{2s+1-j}y^j, x^2, y^2),$$

$$\mathscr{A}^{2s+1}(xy, x^2, y^2 + x^{2s+1}) \text{ or } \bigcap_{s \geq 0} \mathscr{A}^s(xy, x^2, y^2) \text{ for some } s > 0$$

(this is the example used in I·2-see 8, 14 and 20 in that section for the 3-jets).

Proof :

$$T\mathscr{A}_{2,2}f = \left\{x^{2i-j}y^j.e_1, x^{2i-j}y^j.e_2, x^{2i-j}y^j.e_3 : i \geq 2, 0 \leq j \leq 2i\right\} + \mathfrak{m}_2^2.\left\{\begin{pmatrix} x \\ 0 \\ 2y \end{pmatrix}, \begin{pmatrix} y \\ 2x \\ 0 \end{pmatrix}\right\}$$

so for $s \geq 1$, $g \in \mathscr{A}^{2s}f \Rightarrow g \in \mathscr{A}^{2s+1}(xy + \Sigma\lambda_j x^{2s+1-j}y^j, x^2 + \alpha y^{2s+1}, y^2 + bx^{2s+1})$

and $\qquad g \in \mathscr{A}^{2s+1}f \Rightarrow g \in \mathscr{A}^{2s+2}(xy, x^2, y^2)$.

For $f = (xy + \Sigma\lambda_j x^{2s+1-j}y^j, x^2 + \alpha y^{2s+1}, y^2 + \beta x^{2s+1})$

$$T\mathscr{A}^{2s+1}f = \mathbb{C}\left\{x^{2i-j}y^j.e_1, x^{2i-j}y^j.e_2, x^{2i-j}y^j.e_3 : i \geq 2, 0 \leq j \leq 2i\right\} + \mathfrak{m}_2^2.\left\{\begin{pmatrix} x \\ 0 \\ 2y \end{pmatrix}, \begin{pmatrix} y \\ 2x \\ 0 \end{pmatrix}\right\}$$

$$+ \mathfrak{m}_2^{2s+2}.\theta(f) + \mathbb{C}\left\{(xy + \Sigma\lambda_j x^{2s+1-j}y^j).e_\ell, (x^2 + \alpha y^{2s+1}).e_\ell, (y^2 + \beta x^{2s+1}).e_\ell : 1 \leq \ell \leq 3\right\}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy+\Sigma\lambda_j(2s+1-j)x^{2s+1-j}y^j \\ 2x^2 \\ (2s+1)\beta x^{2s+1} \end{pmatrix}, \begin{pmatrix} xy+\Sigma\lambda_j jx^{2s+1-j}y^j \\ (2s+1)\alpha y^{2s+1} \\ 2y^2 \end{pmatrix}, \begin{pmatrix} y^2+\Sigma\lambda_i(2s+1-j)x^{2s-j}y^{j+1} \\ 2xy \\ (2s+1)\beta x^{2s}y \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} x^2+\Sigma\lambda_j jx^{2s+2-j}y^{j-1} \\ (2s+1)\alpha xy^{2s} \\ 2xy \end{pmatrix}\right\}$$

We rearrange this so that the last 4 terms become

$$\mathbb{C}\left\{\begin{pmatrix} -\frac{1}{2}(2s+3)\beta x^{2s+1}+\Sigma\lambda_j(2s-j)x^{2s-j}y^{j+1} \\ -2\lambda_{2s+1}y^{2s+1} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}(2s+3)\alpha y^{2s+1}+\Sigma\lambda_j(j+1)x^{2s+2-j}y^{j-1} \\ 0 \\ -2\lambda_0 x^{2s+1} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \Sigma\lambda_j(j-1)x^{2s+1-j}y^j \\ (2s+1)\alpha y^{2s+1} \\ -2\beta x^{2s+1} \end{pmatrix}, \begin{pmatrix} \Sigma\lambda_j(2s-j)x^{2s+1-j}y^j \\ -2\alpha y^{2s+1} \\ (2s+1)\beta x^{2s+1} \end{pmatrix}\right\}.$$

We shall use these to construct (2s+1)-trivial unfoldings.

Consider $\begin{pmatrix} -\frac{1}{2}(2s+3)\alpha y^{2s+1}+\Sigma\lambda_j(j+1)x^{2s+2-j}y^{j-1} \\ 0 \\ -2\lambda_0 x^{2s+1} \end{pmatrix}$.

Let $F(x, y, t) = (xy+\Sigma\lambda_j(t)x^{2s+1-j}y^j, x^2+\alpha(t)y^{2s+1}, y^2+\beta(t)x^{2s+1})$

where $\alpha(t) = \alpha_0$,

$$\lambda_{2s+1}(t) = -\tfrac{1}{2}(2s+3)\alpha_0 t+\lambda^0_{2s+1},$$

$$\lambda_{2s}(t) = (2s+2)\left(-\tfrac{1}{2}(2s+3)\alpha_0\tfrac{1}{2}t^2+\lambda^0_{2s+1}t\right)+\lambda^0_{2s},$$

$$\lambda_{2s-1}(t) = (2s+1)(2s+2)\left(-\tfrac{1}{2}(2s+3)\alpha_0\tfrac{1}{3!}t^3+\lambda^0_{2s+1}\tfrac{1}{2}t^2\right)+(2s+1)\lambda^0_{2s}t+\lambda^0_{2s-1},$$

...,

$$\lambda_0(t) = (2s+2)!\left(-\tfrac{1}{2}(2s+3)\alpha_0 \tfrac{1}{(2s+2)!}t^{2s+2} + \lambda^0_{2s+1}\tfrac{1}{(2s+1)!}t^{2s+1}\right) + \ldots + 2\lambda^0_1 t + \lambda^0_0,$$

$$\beta(t) = -2(2s+2)!\left(-\tfrac{1}{2}(2s+3)\alpha_0 \tfrac{1}{(2s+3)!}t^{2s+3} + \lambda^0_{2s+1}\tfrac{1}{(2s+2)!}t^{2s+2}\right) + \ldots + -2\lambda^0_0 t + \beta_0.$$

Then $\partial_t F^a = \begin{pmatrix} -\frac{1}{2}(2s+3)\alpha(a)y^{2s+1} + \Sigma\lambda_j(a)(j+1)x^{2s+2-j}y^{j-1} \\ 0 \\ -2\lambda_0(a)x^{2s+1} \end{pmatrix}$ and by I·2:19 F is a

(2s+1)-trivial unfolding. If any of $(\alpha_0, \lambda^0_0, \ldots, \lambda^0_{2s+1})$ are non-zero then there exists a value of t such that $\beta(t) = 0$.

So $(xy + \Sigma\lambda_j x^{2s+1-j}y^j, x^2 + \alpha y^{2s+1}, y^2 + \beta x^{2s+1})$ is (2s+1)-equivalent to either

$$(xy + \Sigma\lambda_j x^{2s+1-j}y^j, x^2 + \alpha y^{2s+1}, y^2) \text{ or } (xy, x^2, y^2 + \beta x^{2s+1}).$$

On $\left\{(xy + \Sigma\lambda_j x^{2s+1-j}y^j, x^2 + \alpha y^{2s+1}, y^2)\right\}$ the remaining terms become

$$\left\{\begin{pmatrix} \Sigma\lambda_j(2s-j)x^{2s-j}y^{j+1} \\ -2\lambda_{2s+1}y^{2s+1} \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma\lambda_j(j-1)x^{2s+1-j}y^j \\ (2s+1)\alpha y^{2s+1} \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma\lambda_j(2s-j)x^{2s+1-j}y^j \\ -2\alpha y^{2s+1} \\ 0 \end{pmatrix}\right\};$$

note that these are tangent to $(\beta=0)$.

Consider $\begin{pmatrix} \Sigma\lambda_j(2s-j)x^{2s-j}y^{j+1} \\ -2\lambda_{2s+1}y^{2s+1} \\ 0 \end{pmatrix}$.

Let $$F(x, y, t) = (xy + \Sigma\lambda_j(t)x^{2s+1-j}y^j, x^2 + \alpha(t)y^{2s+1}, y^2)$$

where $\lambda_0(t) = \lambda^0_0$,

$$\lambda_1(t) = 2s\lambda^0_0 t + \lambda^0_1,$$

$$\lambda_2(t) = 2s(2s-1)\lambda^0_0\tfrac{1}{2}t^2 + (2s-1)\lambda^0_1 t + \lambda^0_2, \ldots,$$

$$\lambda_{2s+1}(t) = 2s!\lambda_0^0 \frac{1}{(2s+1)!} t^{2s+1} + \ldots + \lambda_{2s}^0 t + \lambda_{2s+1}^0,$$

$$\alpha(t) = -2\left(2s!\lambda_0^0 \frac{1}{(2s+2)!} t^{2s+2} + \ldots + \lambda_{2s}^0 \frac{1}{2} t^2 + \lambda_{2s+1}^0 t\right) + \alpha_0.$$

Then $\partial_t F^a = \begin{pmatrix} \Sigma\lambda_j(a)(2s-j)x^{2s-j}y^{j+1} \\ -2\lambda_{2s+1}(a)y^{2s+1} \\ 0 \end{pmatrix}$ and F is a (2s+1)-trivial unfolding.

If any of $(\lambda_0^0, \ldots, \lambda_{2s+1}^0)$ are non-zero then there exists a value of t such that $\alpha(t) = 0$. So $(xy + \Sigma\lambda_j x^{2s+1-j}y^j, x^2 + \alpha y^{2s+1}, y^2)$ is (2s+1)-equivalent to either

$$(xy + \Sigma\lambda_j x^{2s+1-j}y^j, x^2, y^2) \text{ or } (xy, x^2 + \alpha y^{2s+1}, y^2).$$

Under the change of coordinates $x \longleftrightarrow y$, $Y \longleftrightarrow Z$, $(xy, x^2 + \alpha y^{2s+1}, y^2)$ is equivalent to $(xy, x^2, y^2 + \alpha x^{2s+1})$.

On $\left\{(xy + \Sigma\lambda_j x^{2s+1-j}y^j, x^2, y^2)\right\}$ the remaining terms become

$$\left\{\begin{pmatrix} \Sigma\lambda_j(j-1)x^{2s+1-j}y^j \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma\lambda_j(2s-j)x^{2s+1-j}y^j \\ 0 \\ 0 \end{pmatrix}\right\}$$

Note these are tangent to $(\alpha = 0, \beta = 0)$.

Let $F_1(x, y, t) = (xy + \Sigma\lambda_j(t)x^{2s+1-j}y^j, x^2, y^2)$ where $\lambda_j(t) = \lambda_j^0 e^{(j-1)t}$

$F_2(x, y, t) = (xy + \Sigma\lambda_j(t)x^{2s+1-j}y^j, x^2, y^2)$ where $\lambda_j(t) = \lambda_j^0 e^{(2s-j)t}$.

These are both (2s+1)-trivial unfoldings. For any two values i, j with $\lambda_i^0 \neq 0$, $\lambda_j^0 \neq 0$,

we may assume that $\lambda_i^0 = 1$ and $\lambda_j^0 = 1$.

Similarly, for $(xy, x^2, y^2 + \beta x^{2s+1})$, if $\beta \neq 0$ we may assume that $\beta = 1$.

The map-germ $(xy, x^2, y^2 + x^{2s+1})$ is clearly not finitely determined since

$$+\mathbb{C}\left\{\begin{pmatrix} x^3 \\ 0 \\ 2x^2y \end{pmatrix}, \begin{pmatrix} x^2y \\ 0 \\ 2xy^2 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \\ 2y^3 \end{pmatrix}, \begin{pmatrix} x^5 \\ 0 \\ 2x^4y \end{pmatrix}, \begin{pmatrix} x^4y \\ 0 \\ 2x^3y^2 \end{pmatrix}, \begin{pmatrix} x^3y^2 \\ 0 \\ 2x^2y^3 \end{pmatrix}, \begin{pmatrix} x^2y^3 \\ 0 \\ 2xy^4 \end{pmatrix}, \begin{pmatrix} xy^4 \\ 0 \\ 2y^5 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} x^2y \\ 2x^3 \\ 0 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 2x^2y \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ 2xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x^4y \\ 2x^5 \\ 0 \end{pmatrix}, \begin{pmatrix} x^3y^2 \\ 2x^4y \\ 0 \end{pmatrix}, \begin{pmatrix} x^2y^3 \\ 2x^3y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy^4 \\ 2x^2y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y^5 \\ 2xy^4 \\ 0 \end{pmatrix}\right\}$$

Thus f is 3-determined if and only if the matrix $\begin{bmatrix} a & 0 & -c & -2d & 0 & 0 \\ 0 & a & 0 & -c & -2d & 0 \\ 0 & b & 2c & 3d & 0 & 0 \\ 0 & 0 & a & 0 & -c & -2d \\ 0 & 0 & b & 2c & 3d & 0 \\ 0 & 0 & 0 & b & 2c & 3d \end{bmatrix}$

has non-zero determinant and a, d are both non-zero.

The determinant is $ad(4ac^3+4db^3+27a^2d^2-b^2c^2-18abcd)$.

Thus we get

$(xy+x^3+bx^2y+cxy^2+y^3, x^2, y^2)$ is 3-determined iff

$$4c^3+4b^3+27-b^2c^2-18bc \neq 0,$$

and $(xy+ax^3+bx^2y+cxy^2, x^2, y^2)$ is not 3-determined.

Note that $(xy+bx^2y+cxy^2+dy^3, x^2, y^2) \sim (xy+dx^3+cx^2y+bxy^2, x^2, y^2)$ under the coordinate change $x \leftrightarrow y$, $Y \leftrightarrow Z$.

Conclusion: $\mathscr{A}^2(xy, x^2, y^2)$ is a union of the following $\mathscr{A}$-invariant strata:

$$(xy+x^3+bx^2y+cxy^2+y^3, x^2, y^2) \text{ if } 4c^3+4b^3+27-b^2c^2-18bc \neq 0,$$

$$\mathscr{A}^3(xy+x^3+bx^2y+cxy^2+y^3, x^2, y^2) \text{ if } 4c^3+4b^3+27-b^2c^2-18bc = 0,$$

$$\mathscr{A}^3(xy+x^3+x^2y+cxy^2, x^2, y^2), \quad \mathscr{A}^3(xy+x^3+xy^2, x^2, y^2), \quad \mathscr{A}^3(xy+x^3, x^2, y^2),$$

$$\mathscr{A}^3(xy+x^2y+xy^2, x^2, y^2), \quad \mathscr{A}^3(xy+xy^2, x^2, y^2), \quad \mathscr{A}^3(xy+x^2y, x^2, y^2),$$

$$\mathscr{A}^3(xy, x^2, y^2+x^3), \quad \mathscr{A}^3(xy, x^2, y^2).$$

We note that $(xy+x^3+x^2y+cxy^2, x^2, y^2)$ has a double curve $(0,\pm y)\to(0,0,y^2)$ which does not have transversal type A_1.

(II·5:4) Proposition : if $j^3f = (xy+x^3+x^2y+cxy^2, x^2, y^2)$ and $c(c-4)\neq 0$ then f lies in one of

$$\bigcap_{s\geq 0} \mathscr{A}^s(xy+x^3+x^2y+cxy^2, x^2, y^2)$$

$$\text{or } (xy+x^3+x^2y+cxy^2, x^2+\alpha y^{2s+1}, y^2) \text{ for some } s\geq 2, \alpha\neq 0.$$

In the second case f is (2s+1) - determined. Hence $(xy+x^3+x^2y+cxy^2, x^2, y^2)$ is a weak stem.

Proof : Let $\kappa = 3$, $R = 4$ and $S = 3$.

Claim : $T\mathscr{A}_{4,3}f \supseteq m_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j\geq 3\}$ and

$$T\mathscr{A}_{2,2}f \supseteq m_2^4\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j\geq 2\}.$$

Then for $k\geq 4$ $j^{2k}f = (xy+x^3+x^2y+cxy^2, x^2, y^2) \Rightarrow$

$$f\in \mathscr{A}^{2k+3}(xy+x^3+x^2y+cxy^2, x^2+\alpha y^{2k+1}, y^2) = f_\alpha \text{ say.}$$

Then

$$T\mathscr{A}_{4,3}f_\alpha + m_2^{2k+4}\cdot\theta(f) = T\mathscr{A}_{4,3}f + m_2^{2k+4}\cdot\theta(f) \supseteq m_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j\geq 3\}$$

Since $T\mathscr{A}_{4,3}f_\alpha = f^*m_3\cdot T\mathscr{A}_{2,2}f_\alpha$ it follows that f_α is 2k+1-determined if

$$(0, y^{2k+3}, 0) \in T\mathscr{A}_{2,2}f_\alpha + m_2^{2k+4}\cdot\theta(f)$$

Proof of claim : Note that

$$m_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j\geq 3\}$$

is an $\mathcal{O}_3$-module via $(xy+x^3+x^2y+cxy^2, x^2, y^2)$ and that

$$f^*m_3\cdot m_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j\geq 3\} = m_2^8\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j\geq 4\}.$$

Therefore $T\mathscr{A}_{4,3}f \supseteq m_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j\geq 3\}$ if and only if

$$T\mathscr{A}_{4,3}f + \mathfrak{m}_2^8\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j \geq 4\} \supseteq \mathfrak{m}_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j \geq 3\}.$$

This follows if we show that $T\mathscr{A}_{4,3}f + \mathfrak{m}_2^8\cdot\theta(f) \supseteq \mathfrak{m}_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j \geq 3\}$ since

$$\mathfrak{m}_2^8\cdot\theta(f) \cap \mathfrak{m}_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j \geq 3\} = \mathfrak{m}_2^8\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j \geq 4\}.$$

In $\omega f(\mathfrak{m}_3^3\cdot\theta(3))$ terms involving the square of the parameters are in $\mathfrak{m}_2^8\cdot\theta(f)$ so we can calculate $T\mathscr{A}_{4,3}f + \mathfrak{m}_2^8\cdot\theta(f)$ by computer: we find that .

$$\begin{aligned}
T\mathscr{A}_{4,3}f + \mathfrak{m}_2^8\cdot\theta(f) = \mathfrak{m}_2^8\cdot\theta(f) &+ \mathbb{C}\{x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, 16x^7-cxy^6, \\
&8x^6y+cxy^6, 4x^5y^2-cxy^6, 2x^4y^3+cxy^6, x^3y^4-cxy^6, \underline{x^2y^5+2cxy^6}, \underline{2cx^2y^5+cxy^6}, \\
&(3-8c)xy^6+2cy^7\}\cdot e_1 \\
&+ \mathbb{C}\{x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, y^6, xy^5 + (1-2c)x^2y^5 + cxy^6, 8x^7+cx^2y^5, \\
&-4x^6y+cx^2y^5, 2x^5y^2+cx^2y^5, -x^4y^3+cx^2y^5, x^3y^4+2cx^2y^5\}\cdot e_2 \\
&+ \mathbb{C}\{x^6, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, 16x^5y + 16x^7 + 16cx^5y^2 + cy^7, \\
&16x^6y-cy^7, 8x^5y^2+cy^7, 4x^4y^3-cy^7, 2x^3y^4+cy^7, x^2y^5+2cxy^6, x^2y^5-cy^7, xy^6+2cy^7\}\cdot e_3
\end{aligned}$$

$$+\mathbb{C}\left\{\begin{pmatrix} x^5 \\ 0 \\ 2x^4y \end{pmatrix}, \begin{pmatrix} x^4y \\ 0 \\ 2x^3y^2 \end{pmatrix}, \begin{pmatrix} x^3y^2 \\ 0 \\ 2x^2y^3 \end{pmatrix}, \begin{pmatrix} x^2y^3 \\ 0 \\ 2xy^4 \end{pmatrix}, \begin{pmatrix} xy^4 \\ 0 \\ 2y^5 \end{pmatrix}, \begin{pmatrix} x^4y \\ 2x^5 \\ 0 \end{pmatrix}, \begin{pmatrix} x^3y^2 \\ 2x^4y \\ 0 \end{pmatrix}, \begin{pmatrix} x^2y^3 \\ 2x^3y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy^4 \\ 2x^2y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y^5 \\ 2xy^4 \\ 0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 2cxy^6-cy^7 \\ 3x^2y^6 \\ 0 \end{pmatrix}, \begin{pmatrix} -16cx^6y-cxy^6 \\ 0 \\ 16x^7+16cx^5y^2 \end{pmatrix}, \begin{pmatrix} 4cxy^6-cy^7 \\ 0 \\ 3y^7 \end{pmatrix}, \begin{pmatrix} y^7 \\ 2xy^6 \\ 0 \end{pmatrix}\right\}$$

This contains $\mathfrak{m}_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j \geq 3\}$ provided that $c(c-4) \neq 0$; the relevant terms are underlined in red.

We obtain $T\mathscr{A}_{2,2}f$ in a similar way (by calculating $T\mathscr{A}_{2,2}f + \mathfrak{m}_2^6\cdot\theta(f)$ by computer).

We next show that $(0, y^{2k+3}, 0) \in T\mathscr{A}_{2,2}f_\alpha + \mathfrak{m}_2^{2k+4}\cdot\theta(f)$, recalling that

$$\mathfrak{m}_2^6\cdot\theta(f)\backslash\{(0, y^{2j+1}, 0): j \geq 3\} \subseteq T\mathscr{A}_{2,2}f_\alpha.$$

The important terms are:

$$\mathbb{C}\left\{x^4, x^3y, y^4, x^2y^2 + \alpha y^{2k+3}, x^2y^2 + 2x^4y + 2x^3y^2 + 2x^2y^3, xy^3 + x^3y^2 + x^2y^3 + cxy^4\right\}\cdot e_i$$
$$+ \mathbb{C}\left\{3x^5 + 2x^4y + cx^3y^2, 3x^4y + 2x^3y^2 + cx^2y^3, x^5 + 2cx^4y, x^4y + 2cx^3y^2,\right\}\cdot e_1$$

$$+ \mathbb{C}\left\{\begin{pmatrix} xy^3 + 3x^3y^2 + 2x^2y^3 + cxy^4 \\ 2x^2y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy^3 + x^2y^3 + 2cxy^4 \\ (2k+1)\alpha y^{2k+3} \\ 0 \end{pmatrix}\right\}$$

Since $\mathbb{C}\left\{3x^5 + 2x^4y + cx^3y^2, 3x^4y + 2x^3y^2 + cx^2y^3, x^5 + 2cx^4y, x^4y + 2cx^3y^2\right\}\cdot e_1$
$$= \mathbb{C}\left\{x^5, x^4y, x^3y^2, x^2y^3\right\}\cdot e_1$$

we see that $x^2y^2\cdot e_2 \in T\mathscr{A}_{2,2}f_\alpha + \mathfrak{m}_2^{2k+4}\cdot\theta(f)$ and hence if $\alpha \neq 0$,

$$(0, y^{2k+3}, 0) \in T\mathscr{A}_{2,2}f_\alpha + \mathfrak{m}_2^{2k+4}\cdot\theta(f) \text{ as required.}$$

This concludes the proof of the proposition. □.

We perform a similar analysis for $(xy, x^2 + y^2, 0)$.

(II·5:5) Let $f = (xy, x^2 + y^2, 0)$. Then $T\mathscr{A}_{2,2}f =$

$$\begin{pmatrix} \{x^{2\ell-j}y^j : \ell \geq 2, 0 \leq j \leq 2\ell\} \cup \{x^{2\ell+1-2j}y^{2j} - xy^{2\ell}, x^{2\ell-2j}y^{2j+1} - y^{2\ell+1} : \ell \geq 0, 0 \leq j \leq \ell\} \\ \{x^{2\ell-j}y^j : \ell \geq 2, 0 \leq j \leq 2\ell\} \cup \{x^{2\ell+1-2j}y^{2j} - xy^{2\ell}, x^{2\ell-2j}y^{2j+1} - y^{2\ell+1} : \ell \geq 0, 0 \leq j \leq \ell\} \\ \{x^{2\ell-j}y^j + x^jy^{2\ell-j} : \ell \geq 2, 0 \leq j \leq 2\ell\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y^{s\ell+1} \\ 2xy^{2\ell} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{s\ell} \\ 2y^{2\ell+1} \\ 0 \end{pmatrix} : \ell \geq 1\right\}$$

For $s \geq 1$,

$g \in \mathscr{A}^{2s}f \Rightarrow \quad g \in \mathscr{A}^{2s+1}(xy, x^2+y^2+\alpha xy^{2s}+\beta y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j)$

and $g \in \mathscr{A}^{2s+1}f \Rightarrow g \in \mathscr{A}^{2s+2}(xy, x^2+y^2, \Sigma\lambda_j x^{s-1-j}y^{s+1+j})$.

For $f = (xy, x^2+y^2+\alpha xy^{2s}+\beta y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j)$ $T\mathscr{A}^{2s+1}f =$

$$\begin{pmatrix} \{x^{2\ell-j}y^j : \ell \geq 2, 0 \leq j \leq 2\ell\} \cup \{x^{2\ell+1-2j}y^{2j}-xy^{2\ell}, x^{2\ell-2j}y^{2j+1}-y^{2\ell+1} : \ell \geq 0, 0 \leq j \leq \ell\} \\ \{x^{2\ell-j}y^j : \ell \geq 2, 0 \leq j \leq 2\ell\} \cup \{x^{2\ell+1-2j}y^{2j}-xy^{2\ell}, x^{2\ell-2j}y^{2j+1}-y^{2\ell+1} : \ell \geq 0, 0 \leq j \leq \ell\} \\ \{x^{2\ell-j}y^j + x^j y^{2\ell-j} : \ell \geq 2, 0 \leq j \leq 2\ell\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y^{s\ell+1} \\ 2xy^{2\ell} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{s\ell} \\ 2y^{2\ell+1} \\ 0 \end{pmatrix} : \ell \geq 1\right\} + \mathbb{C}\left\{\begin{pmatrix} xy \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy \end{pmatrix}, \begin{pmatrix} x^2+y^2+\alpha xy^{2s}+\beta y^{2s+1} \\ 0 \\ 0 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0 \\ x^2+y^2+\alpha xy^{2s}+\beta y^{2s+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2+y^2+\alpha xy^{2s}+\beta y^{2s+1} \end{pmatrix}, \begin{pmatrix} \Sigma\lambda_j x^{2s+1-j}y^j \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \Sigma\lambda_j x^{2s+1-j}y^j \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ \Sigma\lambda_j x^{2s+1-j}y^j \end{pmatrix}, \begin{pmatrix} xy \\ 2x^2+\alpha xy^{2s} \\ \Sigma\lambda_j(2s+1-j)x^{2s+1-j}y^j \end{pmatrix}, \begin{pmatrix} xy \\ 2y^2+2s\alpha xy^{2s}+(2s+1)\beta y^{2s+1} \\ \Sigma\lambda_j j x^{2s+1-j}y^j \end{pmatrix}$$

$$\left.\begin{pmatrix} y^2 \\ 2xy+2s\alpha y^{2s+1} \\ \Sigma\lambda_j(2s+1-j)x^{2s-j}y^{j+1} \end{pmatrix}, \begin{pmatrix} x^2 \\ 2xy+2s\alpha x^2y^{2s-1}+(2s+1)\beta xy^{2s} \\ \Sigma\lambda_j j x^{2s+2-j}y^{j-1} \end{pmatrix}\right\}$$

We see that the following are in $T\mathscr{A}^{2s+1}f$:

$$\begin{pmatrix} (\lambda_0+\lambda_2+\ldots+\lambda_{2s})xy^{2s}+(\lambda_1+\lambda_3+\ldots+\lambda_{2s+1})y^{2s+1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2x^2+2y^2+(2s+1)\alpha xy^{2s}+(2s+1)\beta y^{2s+1} \\ \Sigma\lambda_j(2s+1)x^{2s+1-j}y^j \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ (\lambda_0+\lambda_2+\ldots+\lambda_{2s})xy^{2s}+(\lambda_1+\lambda_3+\ldots+\lambda_{2s+1})y^{2s+1} \\ 0 \end{pmatrix}, \begin{pmatrix} x^2+y^2 \\ 2s\alpha x^2y^{2s-1}+(2s+1)\beta xy^{2s}+\alpha y^{2s+1} \\ \Sigma\left(\lambda_j(2s+1-j)+\lambda_{j+2}(j+2)\right)x^{2s-j}y^{j+1} \end{pmatrix}$$

These give

$$\begin{pmatrix} 0 \\ (\lambda_0+\lambda_2+\ldots+\lambda_{2s})xy^{2s}+(\lambda_1+\lambda_3+\ldots+\lambda_{2s+1})y^{2s+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha xy^{2s}+\beta y^{2s+1} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ (\lambda_1+\lambda_3+\ldots+\lambda_{2s+1})xy^{2s}+(\lambda_0+\lambda_2+\ldots+\lambda_{2s})y^{2s+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ (2s+3)\beta xy^{2s}+(2s+3)\alpha y^{2s+1} \\ \Sigma\left(\lambda_j(2s+1-j)+\lambda_{j+2}(j+2)\right)x^{2s-j}y^{j+1} \end{pmatrix}$$

from which to construct our (2s+1)-trivial unfoldings.

Let $A = (\lambda_0+\lambda_2+\ldots\lambda_{2s})$, $B = (\lambda_1+\lambda_3+\ldots\lambda_{2s+1})$.

a) If $A^2-B^2 \neq 0$ then

$T\mathscr{A}^{2s+1}f \ni \begin{pmatrix} 0 \\ y^{2s+1} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ xy^{2s} \\ 0 \end{pmatrix}$. It is clear that that the codimension of $T\mathscr{A}^{2s+1}f$ is then independent of α and β, so without loss of generality, $\alpha = \beta = 0$.

The remaining terms to consider are

$$\begin{pmatrix} 0 \\ 0 \\ \Sigma\lambda_j x^{2s+1-j}y^j \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ \lambda_1 x^{2s+1}+\Sigma\left(\lambda_j(2s+1-j)+\lambda_{j+2}(j+2)\right)x^{2s-j}y^{j+1} \end{pmatrix}$$

For the second term, we try to solve the equations

$\frac{d}{dt}\lambda_0 = \lambda_1$, $\frac{d}{dt}\lambda_1 = 2\lambda_2 + (2s+1)\lambda_0$, $\frac{d}{dt}\lambda_2 = 3\lambda_3 + 2s\lambda_1$,

$\frac{d}{dt}\lambda_3 = 4\lambda_4 + (2s-1)\lambda_2, \ldots, \frac{d}{dt}\lambda_j = (j+1)\lambda_{j+1} + (2s+2-j)\lambda_{j-1}, \ldots, \frac{d}{dt}\lambda_{2s+1} = \lambda_{2s}$.

Note that in this system

$$\frac{d}{dt}\lambda_0 + \frac{d}{dt}\lambda_2 + \ldots + \frac{d}{dt}\lambda_{2s} = (2s+2)(\lambda_1+\lambda_3+\ldots\lambda_{2s+1})$$

$$\text{and } \frac{d}{dt}\lambda_1 + \frac{d}{dt}\lambda_3 + \ldots + \frac{d}{dt}\lambda_{2s+1} = (2s+2)(\lambda_0+\lambda_2+\ldots\lambda_{2s})$$

$$\text{i.e. } \frac{d}{dt}A = (2s+2)B \text{ and } \frac{d}{dt}B = (2s+2)A.$$

so $\frac{d^2}{dt^2}A - (2s+2)^2A = 0$ and the solution is

$$A(t) = A_0e^{(2s+2)t} + A_1e^{-(2s+2)t},\ B(t) = (2s+2)A_0e^{(2s+2)t} - (2s+2)A_1e^{-(2s+2)t}$$

for some constants A_0, A_1.

We conclude that the unfolding

$F(x, y, t) = (xy, x^2 + y^2, \Sigma\lambda_j(t)x^{2s+1-j}y^j)$, where

$$(\lambda_0(t)+\lambda_2(t)+\ldots\lambda_{2s}(t)) = A_0e^{(2s+2)t} + A_1e^{-(2s+2)t}$$

$$\text{and } (\lambda_1(t)+\lambda_3(t)+\ldots\lambda_{2s+1}(t)) = (2s+2)A_0e^{(2s+2)t} - (2s+2)A_1e^{-(2s+2)t}$$

is (2s+1)-trivial.

Further, if $A_0 \neq 0$ and $A_1 \neq 0$ then there exists a value of t for which $\lambda_0(t)+\lambda_2(t)+\ldots\lambda_{2s}(t)) = 0$. The condition $A_0 = 0$ or $A_1 = 0$ is equivalent to $A^2-B^2 = 0$.

The remaining term $\begin{pmatrix} 0 \\ 0 \\ \Sigma\lambda_j x^{2s+1-j}y^j \end{pmatrix}$ implies that any non-zero value of λ_j may be assumed to be equal to 1.

Hence if $(xy, x^2 + y^2 + \alpha xy^{2s} + \beta y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j)$ satisfies $A^2-B^2 \neq 0$ then it is

equivalent to a map-germ of the form

$$(xy, x^2+y^2, \Sigma\lambda_j x^{2s+1-j}y^j)$$

where $\lambda_0+\lambda_2+\ldots\lambda_{2s} = 0$, $\lambda_1+\lambda_3+\ldots\lambda_{2s+1} \neq 0$ and some non-zero λ_j may be assumed to be equal to 1. For example this gives the three jets

$$(xy, x^2+y^2, x^3+\lambda_1 x^2y - xy^2 + \lambda_3 y^3) \text{ if } \lambda_0 \neq 0,\ \lambda_1+\lambda_3 \neq 0$$

$$\text{or } (xy, x^2+y^2, x^2y+\lambda_3 y^3) \text{ if } \lambda_0 = 0,\ \lambda_3 \neq -1, \text{ or } (xy, x^2+y^2, y^3).$$

b) If $A^2-B^2 = 0$ but $A \neq 0$ then under the transformation $y \longleftrightarrow -y$,

$(xy, x^2+y^2+\alpha xy^{2s}+\beta y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j)$

$$\longleftrightarrow (-xy, x^2+y^2+\alpha xy^{2s}+-\beta y^{2s+1}, \Sigma(-1)^j\lambda_j x^{2s+1-j}y^j)$$

so $A \longleftrightarrow A$ and $B \longleftrightarrow -B$ so without loss of generality, we assume that $A = B$. Consideration of

$$\begin{pmatrix} 0 \\ \alpha xy^{2s}+\beta y^{2s+1} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ Axy^{2s}+Ay^{2s+1} \\ 0 \end{pmatrix}$$

shows that if $\alpha \neq \beta$ then

$$T\mathscr{A}^{2s+1}f \ni \begin{pmatrix} 0 \\ Ay^{2s+1} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ x^{2s+1}-xy^{2s} \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \alpha xy^{2s} \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ Axy^{2s} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \beta y^{2s+1} \\ 0 \end{pmatrix}$$

so that without loss of generality, $\alpha = 1$ and $\beta = 0$ or $\alpha = 0$ and $\beta = 1$.

Suppose that $\alpha = 1$ and $\beta = 0$. Since $\begin{pmatrix} 0 \\ x^{2s+1}-xy^{2s} \\ 0 \end{pmatrix} \in T\mathscr{A}_{2,2}f$

the unfolding $(xy, x^2+y^2+xy^{2s}+a(x^{2s+1}-xy^{2s}), \Sigma\lambda_j x^{2s+1-j}y^j)$ is (2s+1)-trivial (lemma I·2:7). Hence

$(xy, x^2+y^2+\alpha xy^{2s}, \Sigma\lambda_j x^{2s+1-j}y^j)$ is (2s+1)-equivalent to

$$(xy, x^2+y^2+x^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j).$$

Now apply the transfofmation $x \longleftrightarrow y$ to see that without loss of generality we may assume that $\alpha = 0$ and $\beta = 1$.

If $\alpha = \beta$ then $\left\{(xy, x^2+y^2+\alpha xy^{2s}+\alpha y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j) : \alpha \in \mathbb{C}\right\}$ lies in a single orbit, and without loss of generality, $\alpha = 0$.

In either case we are left with the terms

$$\begin{pmatrix} 0 \\ 0 \\ \Sigma\lambda_j x^{2s+1-j}y^j \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ \lambda_1 x^{2s+1}+\Sigma\left(\lambda_j(2s+1-j)+\lambda_{j+2}(j+2)\right)x^{2s-j}y^{j+1} \end{pmatrix}$$

For the second term, we again try to solve the equations

$$\frac{d}{dt}\lambda_0 = \lambda_1,\ \frac{d}{dt}\lambda_1 = 2\lambda_2+(2s+1)\lambda_0,\ \frac{d}{dt}\lambda_2 = 3\lambda_3+2s\lambda_1,$$

$$\frac{d}{dt}\lambda_3 = 4\lambda_4+(2s-1)\lambda_2,\ldots,\ \frac{d}{dt}\lambda_j = (j+1)\lambda_{j+1}+(2s+2-j)\lambda_{j-1},\ldots,\ \frac{d}{dt}\lambda_{2s+1} = \lambda_{2s}.$$

Now we have

$$\frac{d}{dt}\lambda_0+\frac{d}{dt}\lambda_2+\ldots+\frac{d}{dt}\lambda_{2s} = (2s+2)(\lambda_1+\lambda_3+\ldots\lambda_{2s+1}) = (2s+2)(\lambda_0+\lambda_2+\ldots\lambda_{2s})$$

$$\text{and } \frac{d}{dt}\lambda_1+\frac{d}{dt}\lambda_3+\ldots+\frac{d}{dt}\lambda_{2s+1} = (2s+2)(\lambda_0+\lambda_2+\ldots\lambda_{2s}) = (2s+2)(\lambda_1+\lambda_3+\ldots\lambda_{2s+1})$$

$$\text{i.e. } \frac{d}{dt}A = (2s+2)A.$$

The solution is $A(t) = A_0e^{(2s+2)t}$ for some constant A_0 (which, by hypothesis, is non-zero). There exists a solution to $A(t) = 1$.

The remaining term $\begin{pmatrix} 0 \\ 0 \\ \Sigma\lambda_j x^{2s+1-j}y^j \end{pmatrix}$ implies that any non-zero value of λ_j may be assumed to be equal to 1.

Hence if $(xy, x^2+y^2+\alpha xy^{2s}+\beta y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j)$ satisfies $A^2-B^2 = 0$ then it is equivalent to a map-germ of the form

$$(xy, x^2+y^2, \Sigma\lambda_j x^{2s+1-j}y^j) \text{ or } (xy, x^2+y^2+y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j)$$

where $\lambda_0+\lambda_2+\ldots\lambda_{2s} = \lambda_1+\lambda_3+\ldots\lambda_{2s+1} = 1$ and some non-zero λ_j may be assumed to

$$(0, y) \to (0, 0, y^2)$$

is a double curve with non-transverse self-intersection:

$$\partial f/\partial x = (y, 2x, (2s+1)x^{2s}) \text{ and } \partial f/\partial y = (x, 0, 2y).$$

Consequently, we have

$j^2f = (xy, x^2, y^2) \Rightarrow$

$$j^3f \sim (xy + \lambda_0x^3 + \lambda_1x^2y + \lambda_2xy^2 + \lambda_3y^3, x^2, y^2) \text{ or } (xy, x^2, y^2 + x^3),$$

$j^3f = (xy, x^2, y^2) \Rightarrow$

$$j^3f \sim (xy + \lambda_0x^5 + \lambda_1x^4y + \lambda_2x^3y^2 + \lambda_3x^2y^3 + \lambda_4xy^4 + \lambda_5y^5, x^2, y^2)$$
$$\text{or } (xy, x^2, y^2 + x^5)$$

$j^3f = (xy, x^2, y^2) \Rightarrow$

$$j^3f \sim (xy + \lambda_0x^7 + \lambda_1x^6y + \lambda_2x^5y^2 + \lambda_3x^4y^3 + \lambda_4x^3y^4 + \lambda_5x^2y^5 + \lambda_6xy^6 + \lambda_7y^7, x^2, y^2)$$
$$\text{or } (xy, x^2, y^2 + x^7),$$

etc.

(II·5:3) $\qquad f = (xy + ax^3 + bx^2y + cxy^2 + dy^3, x^2, y^2).$

We find conditions for f to be 3-determined. This is equivalent to finding conditions for $\mathfrak{m}_2^4\cdot\theta(f) \subseteq T\mathcal{A}_{2,2}f$, which holds if and only if

$$\mathfrak{m}_2^4\cdot\theta(f) \subseteq T\mathcal{A}_{2,2}f + \mathfrak{m}_2^6\cdot\theta(f).$$

Calculation using the program gives

$$\begin{aligned}
T\mathcal{A}_{2,2}f + \mathfrak{m}_2^6\cdot\theta(f) = \mathfrak{m}_2^6\cdot\theta(f) + \mathbb{C}\{ & x^4, x^3y, x^2y^2, xy^3, y^4, ax^5-cx^3y^2-2dx^2y^3, \\
& -ax^4y + cx^2y^3 + 2dxy^4, bx^4y + 2cx^3y^2 + 3dx^2y^3, ax^3y^2-cxy^4-2dy^5, \\
& bx^3y^2 + 2cx^2y^3 + 3dxy^4, bx^2y^3 + 2cxy^4 + 3dy^5\}\cdot e_1 \\
+ \mathbb{C}\{ & x^4, x^2y^2, y^4, x^3y + ax^5 + bx^4y + cx^3y^2 + dx^2y^3, 4xy^3 + bx^2y^3 + 2cxy^4 + 3dy^5, \\
& ax^4y + bx^3y^2 + cx^2y^3 + dxy^4, 4ax^3y^2 + 3bx^2y^3 + 2cxy^4 + dy^5\}\cdot e_2 \\
+ \mathbb{C}\{ & x^4, x^2y^2, y^4, x^3y - 4bx^4y - 2cx^3y^2-3dx^2y^3, xy^3 + ax^3y^2 + bx^2y^3 + cxy^4 + dy^5, \\
& ax^4y + bx^3y^2 + cx^2y^3 + dxy^4, ax^5 + 2bx^4y + 3cx^3y^2 + 4dx^2y^3\}\cdot e_3
\end{aligned}$$

be equal to 1. For example, this gives the three jets

$$(xy, x^2+y^2, x^3+\lambda_1 x^2y+(1-\lambda_1)y^3) \text{ or } (xy, x^2+y^2+y^3, x^3+\lambda_1 x^2y+(1-\lambda_1)y^3)$$
$$\text{if } \lambda_0 \neq 0,$$
$$(xy, x^2+y^2, \lambda_1 x^2y+xy^2+(1-\lambda_1)y^3) \text{ or } (xy, x^2+y^2+y^3, \lambda_1 x^2y+xy^2+(1-\lambda_1)y^3)$$
$$\text{if } \lambda_0 = 0.$$

c) If $A = 0 = B$ we consider

$$\begin{pmatrix} 0 \\ \alpha xy^{2s}+\beta y^{2s+1} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ (2s+3)\beta xy^{2s}+(2s+3)\alpha y^{2s+1} \\ \Sigma\left(\lambda_j(2s+1-j)+\lambda_{j+2}(j+2)\right)x^{2s-j}y^{j+1} \end{pmatrix}$$

For the second term, we try to solve the equations

$$\frac{d}{dt}\alpha = (2s+3)\beta, \frac{d}{dt}\beta = (2s+3)\alpha, \frac{d}{dt}\lambda_0 = \lambda_1, \frac{d}{dt}\lambda_1 = 2\lambda_2+(2s+1)\lambda_0, \frac{d}{dt}\lambda_2 = 3\lambda_3+2s\lambda_1,$$

$$\frac{d}{dt}\lambda_3 = 4\lambda_4+(2s-1)\lambda_2, \ldots, \frac{d}{dt}\lambda_j = (j+1)\lambda_{j+1}+(2s+2-j)\lambda_{j-1}, \ldots, \frac{d}{dt}\lambda_{2s+1} = \lambda_{2s}.$$

Now we have $\frac{d^2}{dt^2}\alpha - (2s+2)^2\alpha = 0$

$$\frac{d}{dt}\lambda_0+\frac{d}{dt}\lambda_2+\ldots+\frac{d}{dt}\lambda_{2s} = (2s+2)(\lambda_1+\lambda_3+\ldots\lambda_{2s+1}) = 0$$

$$\text{and } \frac{d}{dt}\lambda_1+\frac{d}{dt}\lambda_3+\ldots+\frac{d}{dt}\lambda_{2s+1} = (2s+2)(\lambda_0+\lambda_2+\ldots\lambda_{2s}) = 0$$

and the solution is

$$\alpha(t) = \alpha_0 e^{(2s+2)t}+\alpha_1 e^{-(2s+2)t}, \ \beta(t) = (2s+2)\alpha_0 e^{(2s+2)t}-(2s+2)\alpha_1 e^{-(2s+2)t},$$
$$A = B = 0$$

for some constants α_0, α_1.

If $\alpha_0 \neq \alpha_1$ then there exists a solution to $\alpha(t) = 0$, and

$$T\mathscr{A}^{2s+1}f \ni \begin{pmatrix} 0 \\ \beta y^{2s+1} \\ 0 \end{pmatrix}$$

so without loss of generality, $\beta = 1$.

Otherwise $\alpha = \pm\beta$ and there either exists a solution to $\alpha(t) = 1$ or $\alpha_0 = \alpha_1 = 0$. Hence, if $(xy, x^2 + y^2 + \alpha xy^{2s} + \beta y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j)$ satisfies $A = B = 0$ then it is equivalent to a map-germ of the form

$$(xy, x^2 + y^2, \Sigma\lambda_j x^{2s+1-j}y^j) \text{ or } (xy, x^2 + y^2 + y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j)$$
$$\text{or } (xy, x^2 + y^2 + xy^2 + y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j)$$

where $\lambda_0+\lambda_2+\ldots\lambda_{2s} = \lambda_1+\lambda_3+\ldots\lambda_{2s+1} = 0$.

The remaining term $\begin{pmatrix} 0 \\ 0 \\ \Sigma\lambda_j x^{2s+1-j}y^j \end{pmatrix}$ implies that any non-zero value of λ_j may be assumed to be equal to 1.

We conclude that the orbits are:

$\mathscr{A}^{2s+1}(xy, x^2 + y^2, \Sigma\lambda_j x^{2s+1-j}y^j) : \lambda_0+\lambda_2+\ldots\lambda_{2s} = 0,\ \lambda_1+\lambda_3+\ldots\lambda_{2s+1} \neq 0$, some $\lambda_j = 1$,

$\mathscr{A}^{2s+1}(xy, x^2 + y^2, \Sigma\lambda_j x^{2s+1-j}y^j) : \lambda_0+\lambda_2+\ldots\lambda_{2s} = \lambda_1+\lambda_3+\ldots\lambda_{2s+1} = 1$, some $\lambda_j = 1$,

$\mathscr{A}^{2s+1}(xy, x^2 + y^2 + y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j) : \lambda_0+\lambda_2+\ldots\lambda_{2s} = \lambda_1+\lambda_3+\ldots\lambda_{2s+1} = 1$,
some $\lambda_j = 1$,

$\mathscr{A}^{2s+1}(xy, x^2 + y^2, \Sigma\lambda_j x^{2s+1-j}y^j) : \lambda_0+\lambda_2+\ldots\lambda_{2s} = \lambda_1+\lambda_3+\ldots\lambda_{2s+1} = 0$, some $\lambda_j = 1$,

$\mathscr{A}^{2s+1}(xy, x^2 + y^2 + y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j) : \lambda_0+\lambda_2+\ldots\lambda_{2s} = \lambda_1+\lambda_3+\ldots\lambda_{2s+1} = 0$,
some $\lambda_j = 1$,

$\mathscr{A}^{2s+1}(xy, x^2 + y^2 + xy^2 + y^{2s+1}, \Sigma\lambda_j x^{2s+1-j}y^j) : \lambda_0+\lambda_2+\ldots\lambda_{2s} = \lambda_1+\lambda_3+\ldots\lambda_{2s+1} = 0$,
some $\lambda_j = 1$.

In particular the three jets are

$(xy, x^2 + y^2, x^3 + \lambda_1 x^2y - xy^2 + \lambda_3 y^3) : \lambda_1+\lambda_3 \neq 0$, $\quad (xy, x^2 + y^2, x^2y + \lambda_3 y^3) : \lambda_3 \neq -1$,

$(xy, x^2 + y^2, y^3)$ $\quad (xy, x^2 + y^2, x^3 + \lambda_1 x^2y + (1-\lambda_1)y^3)$

$(xy, x^2+y^2+y^3, x^3+\lambda_1x^2y+(1-\lambda_1)y^3)$ $(xy, x^2+y^2, \lambda_1x^2y+xy^2+(1-\lambda_1)y^3)$

$(xy, x^2+y^2+y^3, \lambda_1x^2y+xy^2+(1-\lambda_1)y^3)$ $(xy, x^2+y^2, x^3+\lambda_1x^2y-xy^2-\lambda_1y^3)$

(xy, x^2+y^2, x^2y-y^3) $(xy, x^2+y^2, 0)$

$(xy, x^2+y^2+y^3, x^3+\lambda_1x^2y-xy^2-\lambda_1y^3)$ $(xy, x^2+y^2+y^3, x^2y-y^3)$

$(xy, x^2+y^2+y^3, 0)$ $(xy, x^2+y^2+xy^2+y^3, x^3+\lambda_1x^2y-xy^2-\lambda_1y^3)$

$(xy, x^2+y^2+xy^2+y^3, x^2y-y^3)$ $(xy, x^2+y^2+xy^2+y^3, 0)$.

None of these are 3-determined, though it is to be expected that many (particularly in the families defined by parameters) are finitely determined.

Chapter III : Multiplicity, Geometry and Stems.

Introduction.

If $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ is a finite map-germ and $\Sigma = \mathrm{Supp}(\theta(f)/T\mathcal{A}f)$ is 1-dimensional then f is not finitely determined and Σ contains at least one of a cuspidal edge, a double curve with transversal type A_{2n+3} $(n\geq 0)$ or a curve of m-1 points for some $m\geq 3$ [Mond 2]. The simplest cases are the tranversal types A_2, A_3 and D_4.

Definition. The curve $\Sigma_i \subseteq \Sigma$ is of <u>stem type</u> if it is one of

i) a curve of triple points with transversal type D_4,

ii) a double curve with transversal type A_3,

iii) a cuspidal edge with transversal type A_2.

Definition. The map-germ f is a <u>geometric stem</u> if Σ is an irreducible curve of stem type.

D. Mond put forward the conjecture for $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$:

$$f \text{ is a stem} \Leftrightarrow f \text{ is a geometric stem.}$$

In III·1 we relate the multiplicity to the transversal type; in particular we find

$$f \text{ is a geometric stem} \Leftrightarrow \Sigma \text{ is irreducible and } e=\mu_0(\Sigma).$$

Mond's conjecture then becomes

$$f \text{ is a stem} \Leftrightarrow \Sigma \text{ is irreducible and } e=\mu_0(\Sigma).$$

This means that we can apply the theory in Chapter I to the problem; in particular

$$f \text{ is a stem} \Leftarrow \Sigma \text{ is irreducible and } e=\mu_0(\Sigma)$$

is the conjecture proposed in I·3.

We also find direct relationships between the multiplicity and the transversal types A_n $(n\geq 2)$. At the end of the section we calculate the transversal types for the map-germs listed in Tables 2-5 of Chapter II.

In III·2 we investigate the conjecture

$$f \text{ is a stem} \Rightarrow f \text{ is a geometric stem.}$$

In particular we show that if f is a stem then it is finite and Σ is irreducible and is one of

a) a 1-1 cuspidal edge,

b) a double curve with transversal type A_{2n+3} for some $n \geq 0$,

c) an m-1 curve ($m \geq 3$) with transversal type consisting of immersed sheets intersecting pairwise transversly.

Further reduction of this list is a future research problem. The third case is particularly interesting in that it has proved difficult to determine whether or not the conjecture holds for some simple cases.

In III·3 we look at the geometry of the families for some weak stems. We calculate some invariants for each map-germ in the family; these invariants are the numbers C, T and $\mu(\widetilde{D}_2(f)/\mathbb{Z}_2)$ which are defined in [Mond 2] and [M-M].

III·1 : Geometry and Multiplicity.

In this section we shall try to see how the multiplicty of $\theta(f)/T\mathcal{A}f$ and the transversal types of the branches of Σ are related. In this section denote $\theta(f)/T\mathcal{A}f$ by $N\mathcal{A}f$.

From the multiplicity theory (I·1:15) we know that if $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_r$ then

$$e(N\mathcal{A}f) = \Sigma\mu_0(\Sigma_i)\cdot\ell(N\mathcal{A}f_{\Sigma_i}).$$

We shall look at how $\ell(N\mathcal{A}f_{\Sigma_i})$ is related to the transversal type of Σ_i. In particular we shall identify those cases for which $\ell(N\mathcal{A}f_{\Sigma_i}) = 1$. To do this we need to look at multi-germs.

In general if $f:(\mathbb{C}^2, p_1, p_2, \ldots, p_l) \to (\mathbb{C}^3, y)$ is a multi-germ then

$$e(N\mathcal{A}f_y) = \Sigma\mu_y(\Sigma_i)\cdot\ell(N\mathcal{A}f_{\Sigma_i}).$$

The number $\ell(N\mathcal{A}f_{\Sigma_i})$ depends only on Σ_i and not on the point $y \in \Sigma_i$ where $N\mathcal{A}f_y$ is calculated so we can calculate $\ell(N\mathcal{A}f_{\Sigma_i})$ at any point of Σ_i and in particular we may assume that

i) $\Sigma = \Sigma_i$ is smooth and irreducible at $y = 0$ (i.e. $\mu_0(\Sigma) = 1$)

and ii) no double curves (of transversal type A_1) intersect Σ at 0.

Then $\ell(N\mathcal{A}f_\Sigma) = e(N\mathcal{A}f_0)$.

These assumptions make it much easier to find normal forms for map-germs having a particular tranversal type along Σ and hence to estimate the multiplicity.

The main result here is:

(III·1:1) Theorem: $\ell(N\mathcal{A}f_{\Sigma_i}) = 1 \Leftrightarrow f$ is of stem type along Σ_i.

Consequently f is a geometric stem if and only if Σ is irreducible and

$$\ell(N\mathcal{A}f_\Sigma) = 1, \text{ equivalently } e = \mu_0(\Sigma).$$

We prove this in a series of lemmas which give precise values for ℓ for transversal types A_k and minimum values for the remaining types.

If f is a multi-germ, we choose local coordinates (x_i, y_i) at $p_i = (0,0)$ for each i and give the local maps $f_i : (\mathbb{C}^2, p_i) \to (\mathbb{C}^3, 0)$. The coordinates on $\mathbb{C}^3$ are (X, Y, Z).

The first observation to make is

(III·1:2) Lemma : if $t \geq 2$ and we consider f as two maps

$$f_1 : (\mathbb{C}^2, p_1, \ldots, p_i) \to (\mathbb{C}^3, 0) \text{ and } f_2 : (\mathbb{C}^2, p_{i+1}, \ldots, p_t) \to (\mathbb{C}^3, 0) \quad (1 \leq i < t)$$

then

$$e(N\mathscr{A} f_0) \geq e(N\mathscr{A} f_{1,0}) + e(N\mathscr{A} f_{2,0}).$$

Proof. This is clear since $T\mathscr{A} f_0 \subseteq T\mathscr{A} f_{1,0} \oplus T\mathscr{A} f_{2,0}$.

We conjecture that

f_1 and f_2 intersect transversly if and only if $e(N\mathscr{A} f_0) = e(N\mathscr{A} f_{1,0}) + e(N\mathscr{A} f_{2,0})$.

(III·1:3) Case 1 : t=1.

In this case f must have a cuspidal edge. We choose coordinates (x, y) on $(\mathbb{C}^2, 0)$ and (X, Y, Z) on $(\mathbb{C}^3, 0)$ so that the cuspidal edge Σ is given by $Y = 0, Z = 0$ and $f^{-1}(\Sigma)$ is given by $y = 0$.

Suppose $f(x, y) = (a(x, y), b(x, y), c(x, y))$. Then $b(x, 0) = c(x, 0) = 0$. Putting $b = yb'$ and $c = yc'$ we have

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \partial_x a \\ \partial_x b' \\ \partial_x c' \end{pmatrix}, \quad \frac{\partial f}{\partial y} = \begin{pmatrix} \partial_y a \\ b' + y\partial_y b' \\ c' + y\,\partial_y c' \end{pmatrix}.$$

It follows that $b'(x,0) = c'(x,0) = 0$ and that $\partial_x a(0,0) \neq 0$ so replacing x by $x' = a(x,y)$ we may assume that f is of the form $(x, y^2 b(x,y), y^2 c(x,y))$.

<u>Case 1</u>: If $b \notin \mathfrak{m}_2$ then we may assume $c \in \mathfrak{m}_2$ (subtract b if it is not). Then $j^2 f = (x, y^2, 0)$ and by II·2:1 f is of the form $(x, y^2, y^3 q(x, y^2))$. It also follows that Σ has transversal type A_{2n} for some $n \geq 1$.

i) If $q \notin \mathfrak{m}_2$ then f is isomorphic to (x, y^2, y^3) and $e(N\mathscr{A} f_0) = 1$.

This follows from II·2:9.

ii) If q is of the form $y^k\varphi$, for some $k \geq 0$ and some $\varphi \in \mathfrak{m}_2$, then $q(x,y) = 0$ is a double curve intersecting Σ at $(0,0,0)$; this contradicts the choice of $0 \in \Sigma$.

iii) If $y^3q = y^{2n+1}\varphi(x, y^2)$ and $\varphi \notin \mathfrak{m}_2$ then by II·2:5

$$\langle xy^n\partial_x\varphi, y^{n+1}\partial_x\varphi, y^{n+1}\partial_y\varphi + ny^n\varphi, y^n\varphi\rangle\mathcal{O}_2 = y^n\mathcal{O}_2$$

$$\text{and } T\mathcal{A}(x, y^2, y^{2n+1}\varphi(x, y^2)) = \begin{pmatrix} \mathfrak{m}_2 \\ \mathfrak{m}_2\backslash\{y\} \\ y^{2n}\mathfrak{m}_2 + \{x^s y^{2j} : j \leq n\} \end{pmatrix}$$

It is clear that $T\mathcal{A}f \not\ni \begin{pmatrix} 0 \\ 0 \\ x^s y^{2r+1} \end{pmatrix}$ for $r \leq n-1$ and hence $e(N\mathcal{A}f_0) = n$.

We note that for $(x, y^2, y^{2n+1}\varphi(x, y^2))$, the image is defined by the equation

$$Z^2 - Y^{2n+1}\varphi^2(X, Y) = 0.$$

Since $\varphi \notin \mathfrak{m}_2$ the transversal type of $\Sigma = V(Y, Z)$ is that of $Z^2 - Y^{2n+1} = 0$, i.e. in this case Σ has transversal type A_{2n}.

<u>Case 2</u>: if $b, c \in \mathfrak{m}_2$ but $b(x,0) \neq 0$ then write $b(x,y) = p(x) + yq(x,y)$. Consider the map at a point $(\varepsilon, 0)$ on $f^{-1}(\Sigma)$. Making the change of coordinates $x \to x + \varepsilon, y \to y$ we get

$$(x, y^2p(x+\varepsilon) + y^3q(x+\varepsilon, y), y^2c(x+\varepsilon, y))$$

which has $j^2f = (x, y^2, 0)$ and behaves as case 1. Since 0 was assumed to be a generic point of Σ it must also have this property, i.e. f has $e(N\mathcal{A}f_0) = n$ if and only if it has transversal type A_{2n}.

<u>Case 3</u>: if f is of the form $(x, y^3b(x,y), y^3c(x,y))$ then

$$T\mathcal{A}f = \begin{pmatrix} \{x^r y^{3(s+t)} b^s c^t : r+s+t \geq 1\} \\ \{x^r y^{3(s+t)} b^s c^t : r+s+t \geq 1\} \\ \{x^r y^{3(s+t)} b^s c^t : r+s+t \geq 1\} \end{pmatrix} + \mathfrak{m}_2\left\{\begin{pmatrix} 1 \\ y^3\partial_x b \\ y^3\partial_x c \end{pmatrix}, \begin{pmatrix} 0 \\ 3y^2b + y^3\partial_y b \\ 3y^2c + y^3\partial_y c \end{pmatrix}\right\}$$

and $T\mathscr{A}f \not\ni \begin{pmatrix} 0 \\ x^s y \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ x^s y \end{pmatrix}$ for any s so $e(N\mathscr{A}f_0) \geq 2$. Note that Σ does not have transversl type A_{2n} in this case.

We can distinguish cases 1 and 2 and case 3 by looking at the $\mathcal{O}_2$-module

$$Q_f = \mathcal{O}_2/\langle \partial_x f_1 \partial_y f_2 - \partial_x f_2 \partial_y f_1,\ \partial_x f_1 \partial_y f_3 - \partial_x f_3 \partial_y f_1,\ \partial_x f_2 \partial_y f_3 - \partial_x f_3 \partial_y f_2 \rangle.$$

In the case $(x, y^2, y^{2n+1}\varphi(x, y^2))$ this becomes

$$\mathcal{O}_2/\langle y \rangle$$

which has multiplicity 1 and in the case $(x, y^3 b(x,y), y^3 c(x,y))$ it becomes

$$\langle \partial_x f_1 \partial_y f_2 - \partial_x f_2 \partial_y f_1,\ \partial_x f_1 \partial_y f_3 - \partial_x f_3 \partial_y f_1,\ \partial_x f_2 \partial_y f_3 - \partial_x f_3 \partial_y f_2 \rangle \subseteq \langle y^2 \rangle$$

so that Q_f has multiplicity ≥ 2.

This can be calculated at any point p of the (irreducible) cuspidal edge C since

$$e(Q_f) = \mu_p(C)\ell_C(Q_f).$$

In other words, $\ell_C(Q_f)=1$ if and only the transversal type of $\Sigma = f(C)$ is A_{2n} for some $n \geq 1$.

(III·1:4) Case 2 : $t \geq 2$: f has at least 1 cuspidal edge.

The transversal type of Σ cannot be A_{2n}.

Let $f:(\mathbb{C}^2, p_1, p_2) \to (\mathbb{C}^3, 0)$. Suppose $f_1 : (\mathbb{C}^2, p_1) \to (\mathbb{C}^3, 0)$ and $f_2 : (\mathbb{C}^2, p_2) \to (\mathbb{C}^3, 0)$.

If f_1 and f_2 both have a cuspidal edge then

$$e(N\mathscr{A}f_0) \geq e(N\mathscr{A}f_{1,0}) + e(N\mathscr{A}f_{2,0}) \geq 2$$

so we may assume that f_1 is an immersion.

If f_2 has a cuspidal edge then we may assume that f_2 is (x, y^2, y^3). Otherwise $e(N\mathscr{A}f_{2,0}) \geq 2$. We may also assume that f_1 is of the form $(x, \psi(x,y), \varphi(x,y))$, and since $f_1(\mathbb{C}^2)$ intersects $f_2(\mathbb{C}^2)$ along $Y = 0, Z = 0$ we must have $\psi(x,0) = \varphi(x,0) = 0$. Then f_1 is of the form $(x, y\psi(x,y), y\varphi(x,y))$ and

$$T\mathscr{A}f \subseteq \begin{pmatrix} m_2 & m_2 \\ ym_2 & y^2 m_2 \\ ym_2 & y^2 m_2 \end{pmatrix} + \left\{ \begin{pmatrix} 0 & 0 \\ x^s & x^s \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ x^s & x^s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2x^s y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x^s \psi & 0 \\ x^s \varphi & 0 \end{pmatrix} \right\}$$

This module does not contain any elements of the type

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & x^s \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & x^s y \end{pmatrix}. \text{ Hence } e(N\mathscr{A}f_0) \geq 2.$$

If f is 3-1 or more onto a cuspidal edge Σ then $e(N\mathscr{A}_e f_0) \geq 2$ since it has a map which is 2-1 onto a cuspidal edge embedded in it so we can apply lemma III·1:1. This is enough to establish

(III·1:5) Lemma: suppose that Σ_i is a cuspidal edge and that $C = f^{-1}(\Sigma_i)$.
Then i) $\ell(N\mathscr{A}f_{\Sigma_i}) = 1$ if and only if Σ_i has transversal type A_2
and ii) Σ has transversal type $A_{2n} \Rightarrow \ell(N\mathscr{A}f_{\Sigma_i}) = n$.

(III·1:6) Case 3 : f has t+1 immersed sheets.

Let $f : (\mathbb{C}^2, p_1, \ldots, p_{t+1}) \to (\mathbb{C}^3, 0)$.

Without loss of generality we may assume that the t+1 local maps are of the form

$$(x, y, 0)\ , (x, y, y^{k_1}\varphi_1(x,y)), \ldots, (x, y, y^{k_t}\varphi_t(x,y))$$

where Σ is the curve $Y = 0, Z = 0$ and $1 \leq k_1 \leq \ldots \leq k_t$ (if $t = 1$ then $k_1 \geq 2$ so that the intersection is non-transverse). Since no other double curves pass through (0,0,0) we must have $\varphi_i(0,0) \neq 0$. Then

$$T\mathscr{A}f \subseteq \begin{pmatrix} m_2 & m_2 & m_2 & \cdots & m_2 \\ m_2 & ym_2 & ym_2 & \cdots & ym_2 \\ y^{k_t} m_2 & y^{k_1} m_2 & y^{k_2} m_2 & \cdots & y^{k_t} m_2 \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ x^s y^j & 0 & \dots & 0 & x^s y^j & \dots & x^s y^j \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & x^s & 0 & \dots & 0 \\ 0 & \dots & 0 & k_i x^s y^{k_i - 1} & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & x^s & x^s & \dots & x^s \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}\right\}.$$

The missing terms are of the form

$$\begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & x^s y^j & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & x^s & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

There are $k_1 + k_2 + \dots k_t + k_t + (t-1)$ missing terms for each s, with $k_t + (t-1) + 1$ equations between them. This means that

$$e(N\mathscr{A} f_0) \geq \left(\sum_{i=1}^{t} k_i\right) - 1.$$

We see that

i) if $t \geq 3$ then $e(N\mathscr{A} f_0) \geq 2$,

ii) if $t = 2$ and $k_2 \geq 2$ then $e(N\mathscr{A} f_0) \geq 2$,

iii) if $t = 1$ and $k_1 \geq 3$ then $e(N\mathscr{A} f_0) \geq 2$.

This means that e is an 'upper limit' for the transversal type of Σ.

In particular, if $t = 1$ then

$$(x, y, 0)\ ,\ (x, y, y^k\varphi(x,y))$$

is equivalent to

$$(x, y, -y^k\varphi(x,y))\ ,\ (x, y, y^k\varphi(x,y))$$

and $Z^2 - Y^{2k}\varphi(X, Y) = 0$ so that Σ has transversal type A_{2k-1} where $k \leq e+1$ so that $e = 1 \Rightarrow A_3$, $e = 2 \Rightarrow A_3$ or A_5, etc.

Note that this estimate is a lower limit. For example it does not distinguish between the cases indicated in diagram 9:

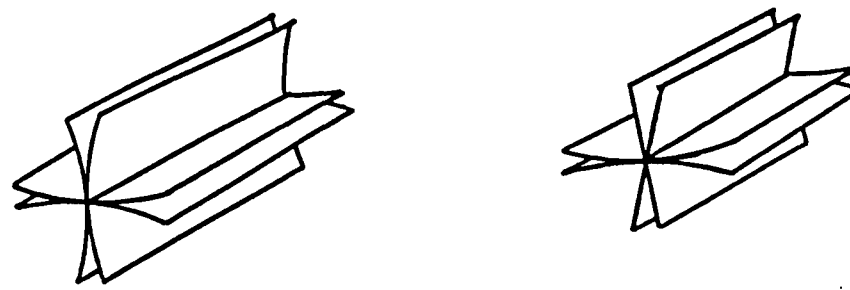


Diagram 9. Inequivalent Map-germs with the same lower estimate for ℓ.

We next look at the two cases

iv) $(x, y, 0), (x, y, y^k\varphi(x, y))$ (transversal type A_{2k-1}),

and v) $(x, y, 0), (x, y, y\varphi_1(x, y)), (x, y, y\,\varphi_2(x, y))$ (transversal type D_4).

For iv) note that $y^{k-1}(k\varphi + y\partial_y\varphi)\cdot\mathfrak{m}_2 = y^{k-1}\cdot\mathfrak{m}_2$ since $k\varphi + y\partial_y\varphi \notin \mathfrak{m}_2$ so

$$T\mathcal{A}f = \begin{pmatrix} \mathfrak{m}_2 & \mathfrak{m}_2 \\ \mathfrak{m}_2 & \mathfrak{m}_2 \\ y^{k-1}\mathfrak{m}_2 & y^{k-1}\mathfrak{m}_2 \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ x^s y^r & x^s y^r \end{pmatrix} : r < k\right\} \text{ and } e(N\mathcal{A}f_0) = k.$$

Hence a double curve Σ_i has transversal type A_{2k-1} if and only if $\ell(N\mathcal{A}f_{\Sigma_i}) = k$.

For v) assume that $\varphi_1(x,0) \neq \varphi_2(x,0)$ so that f_2 and f_3 are transverse. If this is not the case then we choose coordinates so that f_2, say, is given by $(x, y, 0)$ and use case ii) above. We calculate $T\mathcal{A}f$:

now $$\mathfrak{m}_2\cdot\left\{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} \subset T\mathcal{A}f,$$

consideration of the terms

$$\begin{pmatrix} 0 & x^s y^j & x^s y^j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x^s y^j \varphi_1^j & x^s y^j \varphi_2^j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & x^s y^j & x^s y^j \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & x^s y^j \varphi_1^j & x^s y^j \varphi_2^j \\ 0 & 0 & 0 \end{pmatrix}$$

imply that $$y\mathfrak{m}_2\cdot\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right\} \subset T\mathcal{A}f,$$

consideration of the terms

$$y\mathfrak{m}_2\cdot\left\{\begin{pmatrix}0&0&0\\0&1&0\\0&\varphi_1+y\partial_y\varphi_1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&\varphi_2+y\partial_y\varphi_2\end{pmatrix}\right\}$$

implies that $y\mathfrak{m}_2\cdot\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}\subset T\mathcal{A}f$,

consideration of the terms

$$\left\{\begin{pmatrix}0&0&0\\0&0&0\\x^sy^j&x^sy^j&x^sy^j\end{pmatrix}\right\} \text{ and } \mathfrak{m}_2\cdot\left\{\begin{pmatrix}0&1&0\\0&0&0\\0&y\partial_x\varphi_1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\\0&0&y\partial_x\varphi_2\end{pmatrix}\right\}$$

imply that $y\mathfrak{m}_2\cdot\left\{\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\right\},\mathfrak{m}_2\cdot\left\{\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\right\}\subset T\mathcal{A}f$, so

$$T\mathcal{A}f=\begin{pmatrix}\mathfrak{m}_2&\mathfrak{m}_2&\mathfrak{m}_2\\\mathfrak{m}_2&y\mathfrak{m}_2&y\mathfrak{m}_2\\y\mathfrak{m}_2&y\mathfrak{m}_2&y\mathfrak{m}_2\end{pmatrix}+$$

$$\mathbb{C}\left\{\begin{pmatrix}0&0&0\\0&0&0\\x^s&x^s&x^s\end{pmatrix},\begin{pmatrix}0&0&0\\0&x^s&x^s\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&x^s&0\\0&x^s\varphi_1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&x^s\\0&0&x^s\varphi_2\end{pmatrix}\right\}$$

and $e(N\mathcal{A}f_0)=1$. Thus

(III·1:7) Lemma: suppose that Σ_i is not a cuspidal edge. Then

i) $\ell(N\mathcal{A}f_{\Sigma_i})=1 \Leftrightarrow$ the transversal type of Σ_i is either A_3 or D_4.

ii) if Σ_i is a double curve then

$$\ell(N\mathcal{A}f_{\Sigma_i})=k \Leftrightarrow \text{the transversal type of } \Sigma_i \text{ is } A_{2k-1}.$$

This concludes the proof of Theorem III·1:1.

We finish this section by calculating the transversal types of the map-germs given in Tables 2-5.

In the case that the map-germ has $e = 1$ this is just a case of assigning the transversal types A_2, A_3 or D_4 according to whether Σ is a cuspidal edge, a double curve or a triple curve respectively. These map-germs are not listed below. We use Arnold's lists (Table 6) for the more complicated transversal types.

i) (x, y^2, x^3y): this has $e = 2$. The curve $\Sigma = V(X, Z)$ is a smooth, irreducible double curve. By III·2:7 $e = 2$ implies the transversal type must be A_5.

ii) (x, y^2, x^4y): this has $e = 3$ but is otherwise like i). Thus the transversal type must be A_7.

iii) (x, y^2, x^2y^3): this has $e = 2$. The curve $\Sigma = V(XY, Z)$ has two smooth components, hence $\ell(N\mathscr{A}f_{\Sigma_i}) = 1$ on each curve. The curve $V(X, Z)$ is a double curve and is hence A_3 while the curve $V(Y, Z)$ is a cuspidal edge and is hence A_2.

iv) (x, y^2, y^5): this has $e = 2$ The curve $\Sigma = V(Y, Z)$ is smooth and irreducible hence $\ell(N\mathscr{A}f_{\Sigma}) = 2$. It is a 1-1 cuspidal edge with $e(Q_f) = 1$ and is hence A_4.

v) $(x, xy + y^3, xy^2 + \frac{3}{2}y^4)$: this has $e = 2$ The curve $\Sigma = V(X^2+6Z, 4X^3+27Y^2)$ is not smooth, hence $\ell(N\mathscr{A}f_{\Sigma}) = 1$ and $\mu_0(\Sigma) = 2$. It is a cuspidal edge and is hence A_2.

vi) $(x, xy + y^3, xy^2 + \frac{1}{2}y^4)$: this has $e = 2$ The curve $\Sigma = V(X^2+2Z, Y)$ is smooth, hence $\ell(N\mathscr{A}f_{\Sigma}) = 2$. It is a double curve and is hence A_5.

vii) $(x, xy + y^3, xy^3)$: this has $e = 4$. The image satisfies

$(XY-Z)^3-X^5Z = 0$ which is equivalent under $Z \to XY-Z$ to $Z^3+X^5Z-X^6Y = 0$.

The curve Σ is $V(X, Z)$ and $Z^3+X^5Z-X^6Y$ has multiplicity 10 so Σ has transversal type J_{10}.

vii) (x, xy, y^4): this has $e = 3$. The image satisfies $Y^4-X^4Z = 0$. The curve Σ is $V(X, Y)$ so has transversal type X_9 (with $\varepsilon=0$ and $a = -Z$).

viii) $(x, xy + xy^3, y^4)$: this has $e = 2$. The image satisfies

$$Y^4-4ZX^2Y^2-ZX^4(Z-1)^2 = 0.$$

The curve Σ is $V(X, Y)$ so has transversal type X_9 (with $\varepsilon \neq 0$).

ix) (x, y^3, xy^2): this has $e = 2$. The curve $\Sigma = V(XY, Z)$ has two smooth components,

hence $\ell(N\mathscr{S}f_{\Sigma_i}) = 1$ on each curve. The curve $V(X, Z)$ is a triple curve and is hence D_4 while the curve $V(Y, Z)$ is a cuspidal edge and is hence A_2.

x) (x, y^3, x^2y): this has $e = 4$. The image satisfies $Z^3 - X^6Y = 0$. The curve Σ is $V(X, Z)$ so has transversal type J_{10} (with $\varepsilon = 0$).

III·2 : Geometry of Stems.

If f fails to be a stem then for each k there exists a map-germ f_k satisfying $j^k f = j^k f_k$ which is neither finitely determined nor equivalent to f. We use this fact to find restrictions on the geometry of a stem. Throughout this section Σ denotes the set of points in $\mathbb{C}^3$ at which f fails to be stable and Σ_k denotes the corresponding set for f_k.

The principal result is

(III·2:1) Theorem : suppose that $f:(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ and that $C \subseteq \mathbb{C}^2$ is a (not necessarily irreducible) curve, with $0 \in C$. Then for any k there exists a map-germ $f_k:(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ satisfying

i) $j^k f = j^k f_k$,

ii) if $(x, y) \in C$ then

$$f(x, y) = f_k(x, y),\ \partial_x f(x, y) = \partial_x f_k(x, y) \text{ and } \partial_y f(x, y) = \partial_y f_k(x, y),$$

iii) the set $\{(x, y) \in \mathbb{C}^2 \backslash C : f_k(x, y) \in \Sigma_k\}$ consists of isolated points of $\mathbb{C}^2 \backslash C$;

consequently there is a neighbourhood of (0, 0) on which $f_k^{-1}(\Sigma_k) = C$.

For example, suppose f has a cuspidal edge with an immersed sheet intersecting along the edge as in diagram 10:

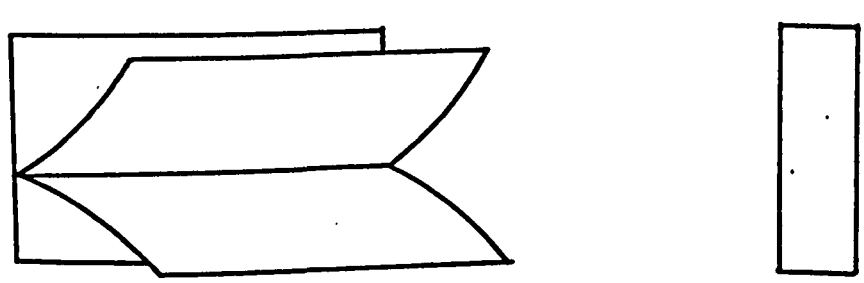


Diagram 10. A map-germ which is not a stem.

We expect to find an f_k as in diagram 11:

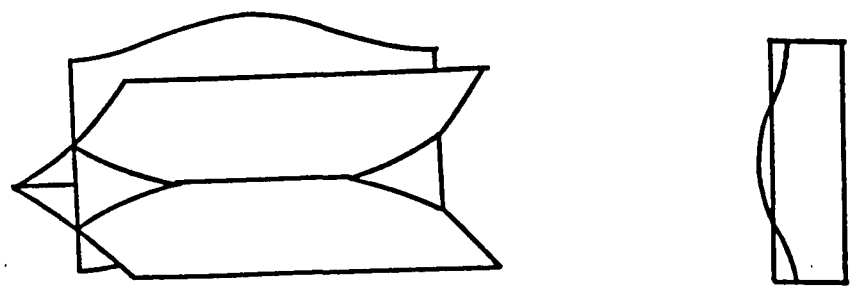


Diagram 11. A deformation of the map-germ in diagram 10.

Notice that $f_k^{-1}(\Sigma_k) = C \cup \{p_1, p_2, \ldots : p_i \notin C\}$.

Condition ii) ensures that

a) f_k is an immersion at $(x, y) \in C$ if and only if f is an immersion at $(x, y) \in C$,

b) if $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in C$ with $f(z_1) = f(z_2)$ then $f_k(z_1) = f_k(z_2)$ and f_k has a transverse double point at $z_1, z_2 \in C$ if and only if f has a transverse double point at $z_1, z_2 \in C$,

c) if $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in C$ with $f(x_1, y_1) = f(x_2, y_2) = f(x_3, y_3)$ then

$$f_k(x_1, y_1) = f_k(x_2, y_2) = f_k(x_3, y_3)$$

so f_k has a triple point at $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in C$ if and only if f has a triple point at $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in C$.

The proof of Theorem III·2:1 does not assume that f is finite but the followinng corollaries do. To see that f cannot be a stem if it is not finite see III·2:9.

(III·2:2) Corollary: if f is a stem then Σ is an irreducible curve.

Proof: suppose that Σ is not irreducible. Let $\Sigma_i \subsetneq \Sigma$. Let $C = f^{-1}(\Sigma_i)$. By III·2:1, for each k there exists f_k with the properties.

$$\Sigma_k \subseteq f_k(C) = \Sigma_i \subsetneq \Sigma \text{ so } f_k \text{ is not } \mathscr{A}\text{-equivalent to } f,$$
$$\text{if } y \in \Sigma_i \text{ then } f^{-1}(y) \subseteq C.$$

If f is finite then Σ_i is at least one of a triple curve, a double curve with transversal type not A_1 or a cuspidal edge. This is preserved for f_k so $y \in \Sigma_k$, i.e. $\Sigma_k = \Sigma_i$ and f_k is not finitely determined. Thus f is not a stem.

(III·2:3) Corollary: if f is a stem and Σ is a cuspidal edge then $f^{-1}(\Sigma)$ is irreducible.

Proof: Suppose that $f^{-1}(\Sigma)$ is reducible.

Let $C \subseteq f^{-1}(\Sigma)$ satisfy $x \in C \Rightarrow f$ is not an immersion at x.

Apply III·2:1 to this curve; for each k there exists f_k with the properties

$$f_k^{-1}(\Sigma_k) \subseteq C \cup \{\text{isolated points}\} \text{ so } f_k \text{ is not equivalent to } f, \text{ and}$$
$$x \in C \Rightarrow f_k \text{ fails to be an immersion at } x.$$

So $f_k(C)$ is a cuspidal edge and f_k is not finitely determined, hence f is not a stem.

(III·2:4) Corollary: if f is a stem and Σ is a non-transverse double curve then $f^{-1}(\Sigma)$ has at most two branches.

Proof: Let C be either

one curve (f not 1-1 on C) with non-transverse self-intersection or

two curves whose images intersect non-transversely.

Apply III·2:1 to this curve; for each k there exists f_k with the properties

$$f_k^{-1}(\Sigma_k) \subseteq C \cup \{\text{isolated points}\} \text{ so } f_k \text{ is not equivalent to } f \text{ if } C \neq f^{-1}(\Sigma), \text{ and}$$
$$\underline{x}_1, \underline{x}_2 \in C, \text{ with } f(\underline{x}_1) = f(\underline{x}_2) \Rightarrow f_k \text{ fails to be transverse at } \underline{x}_1, \underline{x}_2.$$

So $f_k(C)$ is a non-transverse double curve and f_k is not finitely determined, hence f is not a stem.

(III·2:5) Corollary: if f is a stem and Σ is a triple curve then $f^{-1}(\Sigma)$ has at most three branches.

Proof: Let C be either

one curve on which f is $n-1, n \geq 3$ or

two curves C_1, C_2: f is 2-1 on C_1 and either 1-1 or 2-1 on C_2

three curves C_1, C_2, C_3: f is 1-1 on each C_i.

Apply III·2:1 to this curve: for each k there exists f_k with the properties

$f_k^{-1}(\Sigma_k) \subseteq C \cup \{\text{isolated points}\}$ so f_k is not equivalent to f if $C \neq f^{-1}(\Sigma)$, and

$\underline{x}_1, \underline{x}_2, \underline{x}_3 \in C$, with $f(\underline{x}_1) = f(\underline{x}_2) = f(\underline{x}_3) \Rightarrow f_k$ has a triple point at $\underline{x}_1, \underline{x}_2, \underline{x}_3$.

So $f_k(C)$ is a triple curve and f_k is not finitely determined, hence f is not a stem.

This shows that

(III·2:6) Proposition: if f is a stem then

i) Σ is irreducible,

ii) $f^{-1}(\Sigma)$ has at most three irreducible components,

iii) if $f^{-1}(\Sigma)$ has three components then Σ has transversal type D_4,

iv) if $f^{-1}(\Sigma)$ has two components then on each component f is either a 1-1 immersion or is 2-1 with transversal type A_1.

The principal technique in the proof of III·2:1 is the use of <u>transversality</u>.

(III·2:7) Theorem [Math IV, Proposition 1·6]. Let $S = \{z_1, \ldots, z_m\} \in \mathbb{C}^n$, and suppose that $f(z_1) = \ldots = f(z_m)$. Let A_i be the germ at z_i of the set

$\{z \in \mathbb{C}^n \mid$ the germ of f at z is $\mathcal{A}$-equivalent to the germ of f at $z_i\}$.

Then the multi-germ $f:(\mathbb{C}^n, S) \to (\mathbb{C}^p, f(z_i))$ is stable if and only if

i) each germ $f:(\mathbb{C}^n, z_i) \to (\mathbb{C}^p, f(z_i))$ is stable, for $1 \leq i \leq m$,

ii) the map-germ $f \times \ldots \times f:(A_1 \times \ldots \times A_m, (z_1, \ldots, z_m)) \to (\mathbb{C}^p)^m$ is transverse to the set $\Delta_{m,p} := \{(z_1, \ldots, z_m) \mid z_i = z_j$ for all $i, j\}$.

(III·2:8) Transversality Lemma :

If $F: X \times Y \to Z$ is transverse to the submanifold W of Z then there is an open, dense set U in Y so that $f_y : X \to Z$ is transverse to W if $y \in U$. Here $f_y(x) = F(x, y)$.

The following result demonstrates the use of the transversality lemma.

(III·2:9) Lemma : if f is a stem then it is finite.

Proof: If f is not finite then the ideal $f^* m_3 \cdot \mathcal{O}_2$ defines a curve C unless f is the zero map. After a change of coordinates we may assume that $x = 0$ is not one of its branches. Then in a neighbourhood of 0 $f(0, y) \neq 0$ if $y \neq 0$. Consider the unfolding $F_k: \mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$F_k(x, y, \alpha, \beta, \gamma) = (f_1(x, y) + \alpha x^k, f_2(x, y) + \beta x^k, f_3(x, y) + \gamma x^k)$$

On $x = 0$ $F_k(0, y, \alpha, \beta, \gamma) = f(0, y) \neq 0$ except at $y = 0$.

Otherwise $\partial F_k/\partial\alpha = (x^k, 0, 0)$, $\partial F_k/\partial\beta = (0, x^k, 0)$, $\partial F_k/\partial\gamma = (0, 0, x^k)$, and F_k is a submersion. Hence F_k intersects $(0, 0, 0)$ transversly. Then by the transversality lemma there is an open dense set U in $\mathbb{C}^3$ such that $(\alpha, \beta, \gamma) \in U$ implies that

$$f_{\alpha,\beta,\gamma} = (f_1(x, y) + \alpha x^k, f_2(x, y) + \beta x^k, f_3(x, y) + \gamma x^k)$$

is transverse to $(0, 0, 0)$ and hence $f_{\alpha,\beta,\gamma}$ is a finite map-germ. It is not finitely determined since if $(x, y) \in C$ then $f_{\alpha,\beta,\gamma}(x, y) = (\alpha x^k, \beta x^k, \gamma x^k)$ which is a k-1 curve.

We use exactly the same technique to prove III·2:1 except that here we look at three maps derived from the unfolding, restricting to the open dense sets as we proceed. The calculations to prove transversality are also rather more complicated.

Proof of III·2:1;

Suppose that $C = V(\varphi)$ and $\varphi(0, 0) = 0$.

Let $N_k = \{(i, j) : k \leq i+j \leq k+1\}$ and let

$$A_k = \{(\sum_{i=0}^{k} \alpha_{i,j}x^iy^j, \sum_{i=0}^{k} \beta_{i,j}x^iy^j, \sum_{i=0}^{k} \gamma_{i,j}x^iy^j) : \alpha_{i,j}, \beta_{i,j}, \gamma_{i,j} \in \mathbb{C},\ (i,j) \in N_k\}.$$

Let $f_\sigma(z) = f(z) + \varphi^2(z)\sigma(z)$ where $\sigma \in A_k$.

Claim: there is an open dense set $U_k \subseteq A_k$ such that for each $\sigma \in U_k$ and $z \in \mathbb{C}^2 \backslash C$ then either $f_\sigma(z)$ is an isolated m-1 point for some $m \geq 3$ or an isolated point in $f_\sigma^{-1}(f_\sigma(C))$ or f_σ is stable at $f_\sigma(z)$.

Any such f_σ can be chosen for the map-germ f_k since it also satisfies $j^kf = j^kf_\sigma$ $(\varphi^2\sigma \in \mathfrak{m}_2^{k+1} \cdot \theta(f))$ and $f(z) = f_\sigma(z)$, $\partial_x f(z) = \partial_x f_\sigma(z)$, $\partial_y f(z) = \partial_y f_\sigma(z)$ if $\varphi(z) = 0$.

Claim 1: the map $F_1 : \mathbb{C}^2 \times A_k \to J^1(2,3)$ given by

$$F_1(z,\sigma) = \begin{pmatrix} \partial_x\left(f_1(z) + \varphi^2(z)\sigma_1(z)\right) & \partial_y\left(f_1(z) + \varphi^2(z)\sigma_1(z)\right) \\ \partial_x\left(f_2(z) + \varphi^2(z)\sigma_2(z)\right) & \partial_y\left(f_2(z) + \varphi^2(z)\sigma_2(z)\right) \\ \partial_x\left(f_3(z) + \varphi^2(z)\sigma_3(z)\right) & \partial_y\left(f_3(z) + \varphi^2(z)\sigma_3(z)\right) \end{pmatrix}$$

is a submersion if $z \notin C$.

Hence by III·2:8 there is an open dense set U_1 such that for $\sigma \in U_1$,

$$j^1(f + \varphi^2\sigma) \text{ is transverse to } \Sigma^1 \subseteq J^1(2,3) \text{ at } z \notin C$$

where $\Sigma^1 = \{\text{elements in } J^1(2,3) \text{ of rank} \leq 1\}$.

Hence the germ $f_\sigma : (\mathbb{C}^2, z) \to (\mathbb{C}^3, f_\sigma(z))$ is stable and hypothesis (i) of III·2:7 holds.

Claim 2: the map $F_2 : (\mathbb{C}^2)^{(2)} \backslash C^{(2)} \times U_1 \to (\mathbb{C}^3)^{(2)}$ given by

$$F_2(z_1, z_2, \sigma) = ((f + \varphi^2\sigma)(z_1), (f + \varphi^2\sigma)(z_2))$$

is transverse to $\Delta_{2,3} = \{(\hat{y}, \hat{y}) : \hat{y} \in \mathbb{C}^3\}$.

Hence by III·2:8 there is an open dense set U_2 such that for $\sigma \in U_2$,

$$((f + \varphi^2\sigma)(z_1), (f + \varphi^2\sigma)(z_2)) \text{ is transverse to } \Delta_{2,3}.$$

Suppose that $z \notin C$ and that $f_\sigma^{-1}\left(f_\sigma(z)\right) = \{z, z'\}$, say. Then

$$f_\sigma \times f_\sigma : (A_1 \times A_2, (z, z')) \to (\mathbb{C}^3)^2$$

is transverse to the set $\Delta_{2,3}$. If $z' \notin C$ then by III·2:7 the multi-germ is stable.

Otherwise suppose that $z \notin C$ and that $f_\sigma^{-1}\left(f_\sigma(z)\right) \cap C \neq \emptyset$.

Now $f_\sigma \times f_\sigma((\mathbb{C}^2 \backslash C) \times C)$ and $\Delta_{2,3}$ have codimension 3 so $f_\sigma \times f_\sigma((\mathbb{C}^2 \backslash C) \times C) \cap \Delta_{2,3}$ has codimension 6 and is hence an isolated point, i.e. z is an isolated point in

$f_\sigma^{-1}\left(f_\sigma(C)\right)$.

Claim 3 : the map $F_3 : \left(\mathbb{C}^2 \backslash C\right)^{(3)} \times U_3 \to \left(\mathbb{C}^3\right)^3$ given by

$$F_3(z_1, z_2, z_3, \sigma) = ((f + \varphi^2\sigma)(z_1), (f + \varphi^2\sigma)(z_2), (f + \varphi^2\sigma)(z_3))$$

is transverse to $\Delta_{3,3} = \{(\hat{y}, \hat{y}, \hat{y}) : \hat{y} \in \mathbb{C}^3\}$.

Now $((f + \varphi^2\sigma)(z_1), (f + \varphi^2\sigma)(z_2), (f + \varphi^2\sigma)(z_3))(\left(\mathbb{C}^2\right)^{(3)} \backslash C^3)$ has dimension 6, $\Delta_{3,3}$ has dimension 3 and $J^1(2, 3)$ has dimension 9. Thus the intersection of $(f + \varphi^2\sigma)^{(3)}\left(\mathbb{C}^2 \backslash C\right)^{(3)}$ and $\Delta_{3,3}$, if transverse, has dimension 0, i.e. it consists of isolated points. Note that any of these points may be 4-1 or greater so are not necessarily stable.

Proof of claim 1:

$$T_{F_1(z,\sigma)}F_1(\mathbb{C}^2 \times A_k) \supseteq \mathbb{C}\left\{\frac{\partial}{\partial a_{i,j}} j^1(f + \varphi^2\sigma) : (i, j) \in N_k\right\}$$

$$= \mathbb{C}\left\{\begin{pmatrix} \partial_x(x^i y^j \varphi^2)(z) & \partial_y(x^i y^j \varphi^2)(z) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \partial_x(x^i y^j \varphi^2)(z) & \partial_y(x^i y^j \varphi^2)(z) \\ 0 & 0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_x(x^i y^j \varphi^2)(z) & \partial_y(x^i y^j \varphi^2)(z) \end{pmatrix} : (i, j) \in N_k\right\}$$

Then F_1 is a submersion unless for every pair $(i, j), (i', j') \in N_k$ the matrix

$$\begin{pmatrix} \partial_x(x^i y^j \varphi^2)(z) & \partial_y(x^i y^j \varphi^2)(z) \\ \partial_x(x^{i'} y^{j'} \varphi^2)(z) & \partial_y(x^{i'} y^{j'} \varphi^2)(z) \end{pmatrix}$$

has zero determinant.

Each determinant is $\varphi^3 x^{i+i'-1} y^{j+j'-1}(\varphi(ij'-i'j) + 2(i-i')y\partial_y\varphi + 2(j'-j)x\partial_x\varphi)$.

If $i+j = i'+j' = k$ then this becomes

$$(i-i')\varphi^3 x^{i+i'-1} y^{j+j'-1}(k\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)$$

and if $i+j = i'+j' = k+1$ then it becomes

$$(i-i')\varphi^3 x^{i+i'-1} y^{j+j'-1}((k+1)\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)$$

In particular, if F_1 fails to be a submersion then all of the following equations are satisfied:

$$\varphi^3 y^{2k-2}(k\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)(z) = 0,$$
$$\varphi^3 x^{2k-2}(k\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)(z) = 0,$$
$$\varphi^3 x^{2k-1}((k+1)\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)(z) = 0$$
$$\text{and } \varphi^3 y^{2k-1}((k+1)\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)(z) = 0.$$

Now $z = (x, y) \neq (0, 0)$ and $z \notin C$ so $\varphi(z) \neq 0$

so the above equations are satisfied if and only if

$$(k\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)(z) = 0 \text{ and } ((k+1)\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)(z) = 0.$$

But then $((k+1)\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)(z) - (k\varphi + 2y\partial_y\varphi + 2x\partial_x\varphi)(z) = \varphi(z) = 0$, which contradicts the choice of $z \notin C$.

Proof of claim 2: Suppose that $F_2(z_1, z_2, \sigma) \in \Delta_{2,3}$.

Now $T_{F_2(z_1,z_2,\sigma)}\Delta^2 = \mathbb{C}\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}\right\}$

and $T_{F_2(z_1,z_2,\sigma)} F_2((\mathbb{C}^2)^{(2)} \setminus C^{(2)} \times U_1) \supseteq \mathbb{C}\left\{\frac{\partial}{\partial a_{ij}}\left(f + \varphi^2\sigma(z_1), f + \varphi^2\sigma(z_2)\right) : (i, j) \in N_k\right\}$

$$= \mathbb{C}\left\{\begin{pmatrix} \varphi^2 x_1^i y_1^j & \varphi^2 x_2^i y_2^j \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \varphi^2 x_1^i y_1^j & \varphi^2 x_2^i y_2^j \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \varphi^2 x_1^i y_1^j & \varphi^2 x_2^i y_2^j \end{pmatrix} : (i,j) \in N_k \right\}.$$

Here $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

Then F_2 is transverse to $\Delta_{2,3}$ unless

$$\varphi^2(x_2, y_2)x_2^i y_2^j - \varphi^2(x_1, y_1)x_1^i y_1^j = 0 \text{ for all } (i,j) \in N_k,$$

$$\text{i.e. } \varphi^2(x_2, y_2)x_2^i y_2^j = \varphi^2(x_1, y_1)x_1^i y_1^j \text{ for all } (i,j) \in N_k.$$

In particular, $\varphi^2(x_2, y_2)x_2^k = \varphi^2(x_1, y_1)x_1^k$, $\varphi^2(x_2, y_2)x_2^{k+1} = \varphi^2(x_1, y_1)x_1^{k+1}$,

$$\varphi^2(x_2, y_2)y_2^k = \varphi^2(x_1, y_1)y_1^k \text{ and } \varphi^2(x_2, y_2)y_2^{k+1} = \varphi^2(x_1, y_1)y_1^{k+1}.$$

If $\varphi^2(x_2, y_2) = 0$ then $\varphi^2(x_1, y_1) \neq 0$ and $(x_1; y_1) \neq (0,0)$ since $(x_1, y_1) \notin C$ so one of $\varphi^2(x_1, y_1)x_1^k$, $\varphi^2(x_1, y_1)y_1^k$ is non-zero.

If $\varphi^2(x_2, y_2) \neq 0 \neq \varphi^2(x_1, y_1)$ then $x \neq 0$ implies $\dfrac{x_1^{k+1}}{x_1^k} = \dfrac{x_2^{k+1}}{x_2^k}$ so that $x_1 = x_2$

and $y \neq 0$ implies $\dfrac{y_1^{k+1}}{y_1^k} = \dfrac{y_2^{k+1}}{y_2^k}$ so that $y_1 = y_2$.

But we cannot have $(x_1, y_1) = (x_2, y_2)$ so F_2 is transverse to $\Delta_{2,3}$ as required.

Proof of claim 3: Suppose that $F_3(z_1, z_2, z_3, \sigma) \in \Delta_{3,3}$.

Now $T_{F_3(z_1,z_2,z_3,\sigma)}\Delta_{3,3} = \mathbb{C}\left\{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\right\}$

and $T_{F_3(z_1,z_2,z_3,\sigma)} F_3\left(\left(\mathbb{C}^2 \setminus C\right)^{(3)} \times U_2\right)$

$$= \mathbb{C}\left\{\left(\partial_x(f + \varphi^2\sigma)(z_1)\ \underline{0}\ \underline{0}\right), \left(\partial_y(f + \varphi^2\sigma)(z_1)\ \underline{0}\ \underline{0}\right), \left(\underline{0}\ \partial_x(f + \varphi^2\sigma)(z_2)\ \underline{0}\right),\right.$$

$$\left(\underline{0}\,\partial_y(f+\varphi^2\sigma)(z_2)\,\underline{0}\right), \left(\underline{0}\,\underline{0}\,\partial_x(f+\varphi^2\sigma)(z_3)\right), \left(\underline{0}\,\underline{0}\,\partial_y(f+\varphi^2\sigma)(z_3)\right),$$

$$\begin{pmatrix} \varphi^2 x_1^i y_1^j & \varphi^2 x_2^i y_2^j & \varphi^2 x_3^i y_3^j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \varphi^2 x_1^i y_1^j & \varphi^2 x_2^i y_2^j & \varphi^2 x_3^i y_3^j \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varphi^2 x_1^i y_1^j & \varphi^2 x_2^i y_2^j & \varphi^2 x_3^i y_3^j \end{pmatrix}$$

$$: (i,j) \in N_k\Big\}.$$

Subtracting the first column in each of these matrices using the elements of $T\Delta_{3,3}$ gives the vector space spanned by

$$\Big\{\left(\underline{0}\,\partial_x(f+\varphi^2\sigma)(z_1)\,\partial_x(f+\varphi^2\sigma)(z_1)\right), \left(\underline{0}\,\partial_x(f+\varphi^2\sigma)(z_1)\,\partial_x(f+\varphi^2\sigma)(z_1)\right),$$
$$\left(\underline{0}\,\partial_x(f+\varphi^2\sigma)(z_2)\,\underline{0}\right), \left(\underline{0}\,\partial_y(f+\varphi^2\sigma)(z_2)\,\underline{0}\right), \left(\underline{0}\,\underline{0}\,\partial_x(f+\varphi^2\sigma)(z_3)\right), \left(\underline{0}\,\underline{0}\,\partial_y(f+\varphi^2\sigma)(z_3)\right)$$
$$(0\ \alpha_{i,j}\ \beta_{i,j})R_1, (0\ \alpha_{i,j}\ \beta_{i,j})R_2, (0\ \alpha_{i,j}\ \beta_{i,j})R_3 : (i,j) \in N_k\Big\},$$

where $\alpha_{i,j} = \varphi^2(x_2, y_2)x_2^i y_2^j - \varphi^2(x_1, y_1)x_1^i y_1^j$, $\beta_{i,j} = \varphi^2(x_3, y_3)x_3^i y_3^j - \varphi^2(x_1, y_1)x_1^i y_1^j$ and $(a, b, c)R_i$ denotes the matrix with (a, b, c) for the ith row and zeros elsewhere. The claim follows if this has dimension 6.

Suppose that $\alpha_{i,j} \neq 0$ for some (i, j).

Subtract the second column from the third to get the space spanned by

$$\Big\{\left(\underline{0}\,\underline{0}\,\alpha_{i,j}\partial_x(f+\varphi^2\sigma)(z_1) - \beta_{i,j}\partial_x(f+\varphi^2\sigma)(z_1)\right),$$
$$\left(\underline{0}\,\underline{0}\,\alpha_{i,j}\partial_y(f+\varphi^2\sigma)(z_1) - \beta_{i,j}\partial_y(f+\varphi^2\sigma)(z_1)\right),$$
$$\left(\underline{0}\,\underline{0}\,\beta_{i,j}\partial_x(f+\varphi^2\sigma)(z_2)\right), \left(\underline{0}\,\underline{0}\,\beta_{i,j}\partial_x(f+\varphi^2\sigma)(z_2)\right),$$
$$\left(\underline{0}\,\underline{0}\,\partial_x(f+\varphi^2\sigma)(z_3)\right), \left(\underline{0}\,\underline{0}\,\partial_y(f+\varphi^2\sigma)(z_3)\right)\Big\}.$$

The claim follows if this has dimension 3.

Now $\det\left(\beta_{i,j}\partial_x(f+\varphi^2\sigma)(z_2)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$

$$= \beta_{i,j}\det\left(\partial_x(f+\varphi^2\sigma)(z_2)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$$

and $\det\left(\beta_{i,j}\partial_y(f+\varphi^2\sigma)(z_2)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$

$$= \beta_{i,j}\det\left(\partial_y(f+\varphi^2\sigma)(z_2)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$$

and one of

$$\det\left(\partial_y(f+\varphi^2\sigma)(z_2)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right),$$
$$\det\left(\partial_x(f+\varphi^2\sigma)(z_2)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$$

is non-zero by choice of σ so transversality holds unless $\beta_{i,j} = 0$.

But then we get

$$\det\left(\alpha_{i,j}\partial_x(f+\varphi^2\sigma)(z_1)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$$
$$= \alpha_{i,j}\det\left(\partial_x(f+\varphi^2\sigma)(z_1)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$$

and $\det\left(\alpha_{i,j}\partial_y(f+\varphi^2\sigma)(z_1)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$
$$= \alpha_{i,j}\det\left(\partial_y(f+\varphi^2\sigma)(z_1)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$$

and one of

$$\det\left(\partial_y(f+\varphi^2\sigma)(z_1)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right),$$
$$\det\left(\partial_x(f+\varphi^2\sigma)(z_1)\ \partial_x(f+\varphi^2\sigma)(z_3)\ \partial_y(f+\varphi^2\sigma)(z_3)\right)$$

is non-zero by choice of σ.

What happens if $\alpha_{i,j}= 0$ for all $(i,j) \in N_k$?

Then for all $(i,j) \in N_k$ $\varphi^2(x_2, y_2)x_2^i y_2^j - \varphi^2(x_1, y_1)x_1^i y_1^j = 0$,

$$\text{i.e. } \varphi^2(x_2, y_2)x_2^i y_2^j = \varphi^2(x_1, y_1)x_1^i y_1^j.$$

In particular, as in claim 2,

$$\frac{y_1^{k+1}}{y_1^k} = \frac{y_2^{k+1}}{y_2^k} \text{ and } \frac{x_1^{k+1}}{x_1^k} = \frac{x_2^{k+1}}{x_2^k}$$

so that $x_1 = x_2$ and $y_1 = y_2$. But this contradicts the hypothesis.

This concludes the proof of theorem III·2:1 □.

If, for example, $f^{-1}(\Sigma)$ is irreducible then this theorem gives us no information on the transversal type of Σ. We now inspect this problem. For this it is only necessary to find map-germs f_k which have a different transversal type along $f_k(C) \subseteq \Sigma_k$; we do not mind introducing new branches to Σ_k.

We first show that a stem is 1-1 along a cuspidal edge. In the case f is m-1 along a cuspidal edge we make the following assumptions:

by III·2:2 we may assume that $C = f^{-1}(\Sigma)$ is irreducible so φ is prime. Let $I = (\varphi) = I(f^{-1}(\Sigma))$ be the prime ideal defining C. Since C is a curve, without loss of

generality there is no value of k satisfying $x^k \in I$ (and hence $\varphi \neq x$).

(III·2:10) Hypothesis : we make the assumption that for some i (say i = 1) the matrix

$$\begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x \varphi & \partial_y \varphi \end{pmatrix}$$

has non-zero determinant along C,

$$\text{i.e. } \partial_x f_i \partial_y \varphi - \partial_y f_i \partial_x \varphi \notin I.$$

(III·2:11) Proposition : if f is 3-1 or more along a cuspidal edge and satisfies hypothesis III·2:10 then f is not a stem. In fact for each k there is a map-germ f_k with an m-1 curve which has only isolated non-immersive points.

Proof :

$$\text{Let } A = \{\, \sigma = (a_1x^k + a_2x^{k+1},\ a_3x^k + a_4x^{k+1},\ a_5x^k + a_6x^{k+1}) : a_i \in \mathbb{C}\}.$$

Claim : under these conditions the map $F(x, y, \sigma) = j^1\big(f(x, y) + \varphi(x, y)\sigma\big)$ is transverse to $\Sigma^1 = \{2\times 3 \text{ matrices of rank} \leq 1\}$ except at isolated points. The result follows.

Proof of claim:

We identify Σ^1 with the manifold $\left\{\begin{pmatrix} \lambda a & \mu a \\ \lambda b & \mu b \\ \lambda c & \mu c \end{pmatrix} : a, b, c, d, \lambda, \mu \in \mathbb{C}\right\}$. Note that if, say $\lambda \neq 0$ then without loss of generality, $\lambda = 1$ so there should really be only 4 variables. Then the tangent space of Σ^1 is spanned by

$$\left\{\begin{pmatrix} \lambda & \mu \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \lambda & \mu \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \lambda & \mu \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \end{pmatrix} : a, b, c, d, \lambda, \mu \in \mathbb{C}\right\}$$

(note that at any point this space has dimension 4).

Now for example $\dfrac{\partial F}{\partial a_1} = \begin{pmatrix} x^k\partial_x\varphi + kx^{k-1}\varphi & x^k\partial_x\varphi \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

If $F(x, y, \sigma) \in \Sigma^1$ then $(\lambda:\mu)$ is in the ratio

$$(\partial_x(f_1 + \varphi\sigma_1) : \partial_y(f_1 + \varphi\sigma_1))$$

so transversality follows if the matrix

$$\begin{pmatrix} x^k\partial_x\varphi + kx^{k-1}\varphi & x^k\partial_y\varphi \\ x^{k+1}\partial_x\varphi + (k+1)x^k\varphi & x^{k+1}\partial_y\varphi \\ \partial_x(f_1+\varphi\sigma) & \partial_y(f_1+\varphi\sigma) \end{pmatrix}$$

has rank 2.

If this fails then

$$x^k(\partial_x f_1\partial_y\varphi - \partial_y f_1\partial_x\varphi) + kx^{k-1}\varphi\partial_y f_1 = 0$$

$$\text{and } x^{k+1}(\partial_x f_1\partial_y\varphi - \partial_y f_1\partial_x\varphi) + (k+1)x^k\varphi\partial_y f_1 = 0$$

$$\text{so that we also have } x^k\varphi\partial_y f_1 = 0.$$

The condition fails when $x(\partial_x f_1\partial_y\varphi - \partial_y f_1\partial_x\varphi) = 0$ and $x\varphi\partial_y f_1 = 0$.

Now the only curve which intersects Σ^1 at other than isolated points is $\varphi = 0$, hence $x(\partial_x f_1\partial_y\varphi - \partial_y f_1\partial_x\varphi) = 0$ intersects Σ^1 only at isolated points.

This proves the claim and hence the proposition. □.

(III·2:12) Proposition : if f has a double cuspidal edge and hypothesis III·2:10 holds then f is not a stem.

Proof:

Let $f_k = (f_1, f_2, f_3 + x^k\varphi)$ and $(x, y) \in C$.

$$\text{Then } j^1 f_k(x, y) = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \\ \partial_x f_3 + x^k\partial_x\varphi & \partial_y f_3 + x^k\partial_x\varphi \end{pmatrix}.$$

Then f_k is an immersion at all but isolated points of C since

$$x^k \det \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x\varphi & \partial_y\varphi \end{pmatrix} \notin (\varphi).$$

Now f_k has a double curve along C and

$$\begin{pmatrix} \partial_x f_1(z_1) & \partial_y f_1(z_1) & \partial_x f_1(z_2) & \partial_y f_1(z_2) \\ \partial_x f_2(z_1) & \partial_y f_2(z_1) & \partial_x f_2(z_2) & \partial_y f_2(z_2) \\ \partial_x\left(f_3+x^k\varphi\right)(z_1) & \partial_y\left(f_3+x^k\varphi\right)(z_1) & \partial_x\left(f_3+x^k\varphi\right)(z_2) & \partial_y\left(f_3+x^k\varphi\right)(z_2) \end{pmatrix}$$

has rank 2 since the tangent space of f along f(C) is tangent to f(C), i.e. the

matrix $\begin{pmatrix} \partial_x f_1(z_1) & \partial_y f_1(z_1) & \partial_x f_1(z_2) & \partial_y f_1(z_2) \\ \partial_x f_2(z_1) & \partial_y f_2(z_1) & \partial_x f_2(z_2) & \partial_y f_2(z_2) \\ \partial_x f_3(z_1) & \partial_y f_3(z_1) & \partial_x f_3(z_2) & \partial_y f_3(z_2) \end{pmatrix}$

has rank 1 when $f(z_1) = f(z_2)$. This proves the proposition. □.

Note: we cannot guarantee that the map-germ $(f_1, f_2, f_3 + x^k\varphi)$ does not have a new cuspidal edge off C.

What happens if hypothesis III·2:10 fails, i.e. if $\det\begin{pmatrix} \partial_x f_i & \partial_y f_i \\ \partial_x\varphi & \partial_y\varphi \end{pmatrix} \in (\varphi)$ for all i? At a point where C is smooth we may choose local coordinates so that $\varphi = y$. Then the condition becomes $\partial_x f_i \in (y)$, and hence $f_i \in (y)$ for all i. Then f fails to be finite at a smooth point of C. Since C is smooth almost everywhere and f is finite at 0 this gives a contradiction. Hence this hypothesis holds for all finite map-germs.

(III·2:13) Proposition: if f is a stem with an m-1 curve ($m \geq 3$) then Σ has transversal type consisting of immersed sheets which are pairwise transverse.

Proof: we assume that $I = (\varphi) = I(f^{-1}(\Sigma))$ is the reduced ideal defining $C = f^{-1}(\Sigma)$. Since C is a curve, we can also assume that x=0 is not a branch of C so that $C \cap \{x=0\}$ consists of isolated points only. If $y^k \in I$ for any k then $C \subseteq V(y^k)$ and hence $\varphi = y$ (since φ defines a reduced curve). Then without loss of generality, either $\varphi = y$ or x is not a factor of φ and for each k $y^k \notin I$.

We may also assume that

$$\begin{pmatrix} \partial_x f_1(z_1) & \partial_y f_1(z_1) & \partial_x f_1(z_2) & \partial_y f_1(z_2) \\ \partial_x f_2(z_1) & \partial_y f_2(z_1) & \partial_x f_2(z_2) & \partial_y f_2(z_2) \\ \partial_x f_3(z_1) & \partial_y f_3(z_1) & \partial_x f_3(z_2) & \partial_y f_3(z_2) \end{pmatrix}$$

has rank 2 when $f(z_1) = f(z_2)$.

Let $A = \{\sigma = (a_1x^k+a_2x^{k+1}, a_3x^k+a_4x^{k+1}, a_5x^k+a_6x^{k+1}) : a_i \in \mathbb{C}\}$,

$B = \{\tau = (b_1y^k+b_2y^{k+1}, b_3y^k+b_4y^{k+1}, b_5y^k+b_6y^{k+1}) : b_i \in \mathbb{C}\}$.

If $\varphi = y$ then we consider the unfolding $f_\varphi = f + \varphi\sigma$ where $\sigma \in A$, otherwise we consider the unfolding $f_\varphi = f+\varphi(\sigma+\tau)$ where $\sigma \in A$ and $\tau \in B$.

It is enough to show that

$$\begin{pmatrix} \partial_x f_{\varphi,1}(z_1) & \partial_y f_{\varphi,1}(z_1) & \partial_x f_{\varphi,1}(z_2) \\ \partial_x f_{\varphi,2}(z_1) & \partial_y f_{\varphi,2}(z_1) & \partial_x f_{\varphi,2}(z_2) \\ \partial_x f_{\varphi,3}(z_1) & \partial_y f_{\varphi,3}(z_1) & \partial_x f_{\varphi,3}(z_2) \end{pmatrix}$$

has rank ≤ 2 at only isolated points of C. If $\partial_x\varphi \in I$ then replace the third column with the y-derivative; they cannot both vanish. Hence we may assume that $\partial_x\varphi \notin I$.

We prove that $F(z, z, \sigma, \tau) = \begin{pmatrix} \partial_x f_{\varphi,1}(z_1) & \partial_y f_{\varphi,1}(z_1) & \partial_x f_{\varphi,1}(z_2) \\ \partial_x f_{\varphi,2}(z_1) & \partial_y f_{\varphi,2}(z_1) & \partial_x f_{\varphi,2}(z_2) \\ \partial_x f_{\varphi,3}(z_1) & \partial_y f_{\varphi,3}(z_1) & \partial_x f_{\varphi,3}(z_2) \end{pmatrix}$

is transverse to the submanifold $S = \{3\times 3$ matrices of rank $\leq 2\}$.

We identify S with the manifold

$$\left\{ \begin{pmatrix} \lambda a & \lambda b & \lambda c \\ \beta d & \beta e & \beta f \\ \gamma a+\delta d & \gamma b+\delta e & \gamma c+\delta f \end{pmatrix} : \alpha, \beta, \gamma, \delta, a, b, c, d, e, f \in \mathbb{C} \right\}.$$

To show that this is the correct set suppose that A is an element of S. If the first row of A is (0 0 0) then put $\lambda = 0$ otherwise put $\lambda = 1$ and (a b c) equal to the first row of A. Repeat this with the second row. If neither of these rows are zero then γ and δ are uniquely defined so that A is of the above form. If $\lambda = 0$ then let $\gamma = 1$ and if $\beta = 0$ let $\delta = 1$. Choose a,..., f appropriately to get the required form. <u>Note</u> it is never necessary to choose (α, γ) or $(\beta, \delta) = (0, 0)$.

$$\text{Then } TS = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ d & e & f \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & c \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & e & f \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & \delta & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & 0 & \delta \end{pmatrix} \right\}.$$

Now for example (differentiating with respect to a, a, and a)

$$TF \ni \mathbb{C}\left\{ \begin{pmatrix} x_1^k \partial_x \varphi & x_1^k \partial_y \varphi & x_2^k \partial_x \varphi' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ x_1^k \partial_x \varphi & x_1^k \partial_y \varphi & x_2^k \partial_x \varphi' \\ 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1^k \partial_x \varphi & x_1^k \partial_y \varphi & x_2^k \partial_x \varphi' \end{pmatrix} \right\}$$

If $\varphi = y$ then

$$\operatorname{rank} \begin{pmatrix} \partial_x f_{\varphi,1} & \partial_y f_{\varphi,1} & \partial_x f'_{\varphi,1} \\ x_1^k \partial_x \varphi & x_1^k \partial_y \varphi & x_2^k \partial_x \varphi' \\ x_1^{k+1} \partial_x \varphi & x_1^{k+1} \partial_y \varphi & x_2^{k+1} \partial_x \varphi' \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \partial_x f_1 & \partial_y f_1 & \partial_x f'_1 \\ x_1^k \partial_x \varphi & x_1^k \partial_y \varphi & x_2^k \partial_x \varphi' \\ x_1^{k+1} \partial_x \varphi & x_1^{k+1} \partial_y \varphi & x_2^{k+1} \partial_x \varphi' \end{pmatrix}$$

otherwise we add the rows

$$(y_1^k \partial_x \varphi \quad y_1^k \partial_y \varphi \quad y_2^k \partial_x \varphi'),\ (y_1^{k+1} \partial_x \varphi \quad y_1^{k+1} \partial_y \varphi \quad y_2^{k+1} \partial_x \varphi')$$

to each matrix and the same result follows.

Now $\begin{pmatrix} \partial_x f_1 & \partial_y f_1 & \partial_x f'_1 \\ x_1^k \partial_x \varphi & x_1^k \partial_y \varphi & x_2^k \partial_x \varphi' \\ x_1^{k+1} \partial_x \varphi & x_1^{k+1} \partial_y \varphi & x_2^{k+1} \partial_x \varphi' \end{pmatrix}$ has determinant $x_1^k x_2^k (\partial_x f_1 \partial_y \varphi - \partial_y f_1 \partial_x \varphi)(x_1 - x_2)$

and $\begin{pmatrix} \partial_x f_1 & \partial_y f_1 & \partial_x f'_1 \\ y_1^k \partial_x \varphi & y_1^k \partial_y \varphi & y_2^k \partial_x \varphi' \\ y_1^{k+1} \partial_x \varphi & y_1^{k+1} \partial_y \varphi & y_2^{k+1} \partial_x \varphi' \end{pmatrix}$ has determinant $y_1^k y_2^k (\partial_x f_1 \partial_y \varphi - \partial_y f_1 \partial_x \varphi)(y_1 - y_2)$.

We have seen that without loss of generality $\partial_x f_1 \partial_y \varphi - \partial_y f_1 \partial_x \varphi \notin I$. The sets

$$\{(0, y_1, x_2, y_2): f(0, y_1) = f(x_2, y_2)\}, \ \{(x_1, y_1, 0, y_2): f(x_1, y_1) = f(0, y_2)\}$$

consist of isolated points and we have either $(x_1 - x_2) = 0$ and $y = 0$ or $(x_1 - x_2)$ and $(y_1 - y_2) = 0$ which both imply that $(x_1, y_1) = (x_2, y_2)$.

This concludes the proof. □.

We have proved

(III·2:12) Theorem: if f is a stem then it is finite, Σ is an irreducible curve and exactly one of the following holds:

1) Σ has transversal type D_4.

2) Σ is a 1-1 cuspidal edge.

3) Σ is a double curve with either one or two pre-images; The transversal type is A_{2m+1} for some $m \geq 1$.

4) Σ is an m-1 curve ($m > 3$) with irreducible preimage and transversal type consisting of immersed sheets which are pairwise transverse.

5) Σ is a 4-1 curve with two pre-images, f being 2-1 on each pre-image. The transversal type of Σ consists of immersed sheets which are pairwise transverse.

(III·2:13) Conjecture: if f is a stem and (2) or (3) holds then the transversal type is A_2 or A_3.

Example: (x, y^2, y^5) has transversal type A_4:

for each k $(x, y^2, y^5 + x^k y^3)$ has transversal type A_2.

Example: $(x, y^2, x^3 y)$ has transversal type A_5

for each k $(x, y^2, x^3 y + x^2 y^{2s+1})$ has transversal type A_3

For a given example we suggest looking at deformations of the form

$$f + \varphi p \text{ or } f + \varphi^2 p$$

for some suitably chosen $p \in \theta(f)$.

Progress on this problem is limited by the way that the transversal type is defined: it is found by looking at a smooth point of Σ where we can assume that $\Sigma = \{(X, 0, 0) : X \in \mathbb{C}\}$. We must then study the function $p(X, Y, Z)$ which defines the image and how it changes as we change f. Alternatively we can study the module $\theta(f)/T\mathcal{A}f$ to see how its generators are related.

(III·2:14) Conjecture: if f satisfies (4) or (5) then it is not a stem.

Example: $(x, xy + xy^3, y^4)$ is 4-1 along $(0, y)$.

Example: $(xy, x^2 + y^2, x^2y + xy^4)$ has 2-1 curves

$$(0, y) \to (0, y^2, 0) \text{ and } (x, 0) \to (0, x^2, 0).$$

In the case $m > 4$ is not a prime we have for example

$$(x, xy, y^6) \text{ and the family } (x, xy + y^{6k+3}, y^6).$$

We expect it to be simple to reduce the problem to the case that $m = 4$ or m is prime. However from this point progress has halted. We have tried to gain insight into it by looking at the above examples. We have not been able to find map-germs with triple curves which have these 5-jets, nor have we been able to prove that no such map-germs exist. We have no suggestions as to how to find a deformation in a given case .

III·3 $\mathcal{A}$-invariant numbers.

We calculate the numbers defined in [M-M] and [Mond 1] for various families $\{f_k : k \geq k_0\}$ of map-germs of corank 1 calculated in the previous sections. We shall assume that each map-germ is of the form $f = (x, f_2(x,y), f_3(x,y))$. These numbers are

$$C = \dim \mathcal{O}_2/Q_f,$$
$$T = \dim \mathcal{O}_3/\mathcal{F}_2,$$
$$\mu(\tilde{D}^2(f)/\mathbb{Z}_2),$$

where Q_f is the ramification ideal $Q_f = \langle \partial f_2/\partial y, \partial f_3/\partial y \rangle$,
$\mathcal{F}_2$ is the second Fitting ideal of the presentation matrix of f,
$\tilde{D}^2(f) \subseteq \mathbb{C}\times\mathbb{C}^2$ is the double point scheme defined by the ideal

$$\left\langle \frac{f_2(x,y)-f_2(x,y')}{y-y'}, \frac{f_3(x,y)-f_3(x,y')}{y-y'} \right\rangle$$

and $\tilde{D}^2(f)/\mathbb{Z}_2$ is the quotient scheme under the symmetric group $\mathbb{Z}_2$ which permutes y and y'; if $h(x,y,y') = (x, yy', y+y')$, $\tilde{D}^2(f) = V(H)$ and $\tilde{D}^2(f)/\mathbb{Z}_2 = V(H')$ then

$$H = H' \circ h.$$

These numbers are related to the geometry although the relation may not be obvious:

the number C is the cross-cap number, i.e. the number of cross-caps which appear in a generic deformation of f. It is infinite if and only if f has a cuspidal edge.

the number T is the triple point number. We expect that an infinite value for T is connected with the presence of a triple curve.
For example, T is infinite for (x, y^3, y^4) and $(x, y^3, y^4 + x^k y)$ has a triple curve for every k.

We expect that an infinite value for $\mu(\tilde{D}^2(f)/\mathbb{Z}_2)$ is related to the presence of a double curve whose transversal type is A_{2n+1} for some $n \geq 1$.
For example, $\mu(\tilde{D}^2(f)/\mathbb{Z}_2)$ is infinite for (x, y^2, y^5) and $(x, y^2, (x^{2k} - y^2)^2 y)$ has such a curve.

This last number is related to $\mu(\tilde{D}^2(f))$ by the formula

$$\mu(\tilde{D}^2(f)) = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1.$$

Consequently, $\mu(\tilde{D}^2(f))$ is finite if and only if $\mu(\tilde{D}^2(f)/\mathbb{Z}_2)$ and C are both finite.

We aim to test the conjecture

$$\operatorname{cod}(\mathscr{A}_e, f) - \left(C + T + \mu(\tilde{D}^2(f)/\mathbb{Z}_2)\right)$$

is a constant for the family of map-germs related to a weak stem.

For map-germs which are weighted homogeneous R. Pellikaan and T. de Jong have recently announced a proof of the formula

$$\operatorname{cod}(\mathscr{A}_e, f) - \left(C + T + \mu(\tilde{D}^2(f)/\mathbb{Z}_2)\right) = -1$$

so the conjecture is true for weighted homogeneous families, eg. $\{(x, y^2, y^3 + x^k y)\}$.

We shall make use of the fact that for these families $\{f_k : k \geq k_0\}$ some of the numbers are independent of k.

(III·3:1) lemma: if C(f) is finite then there is a value k such that

$$j^k g = j^k f \Rightarrow C(g) = C(f).$$

Proof: if C is finite then

$$Q_f \supseteq \mathfrak{m}_2^r \text{ for some } r.$$

Let $k = r$ and write g in the form f+h, $h \in \mathfrak{m}_2^{r+1}$, then

$$\begin{aligned} Q_g &= \langle \partial g_2/\partial y, \partial g_3/\partial y \rangle \\ &= \langle \partial f_2/\partial y + \partial h_2/\partial y, \partial f_3/\partial y + \partial h_2/\partial y \rangle \\ &= \langle \partial f_2/\partial y, \partial f_3/\partial y \rangle = Q_f \end{aligned}$$

since $h \in \mathfrak{m}_2^{r+1}$.

Hence $C(f) = C(g)$. □.

(III·3:2) lemma: if $\mu(\tilde{D}^2(f)/\mathbb{Z}_2)$ is finite then there is a value k such that

$$j^k g = j^k f \Rightarrow \mu(\tilde{D}^2(g)/\mathbb{Z}_2) = \mu(\tilde{D}^2(f)/\mathbb{Z}_2).$$

Proof: $\mu(\tilde{D}^2(f)/\mathbb{Z}_2)$ is finite $\Leftrightarrow$ $\tilde{D}^2(f)/\mathbb{Z}_2$ has an isolated singularity

$\Leftrightarrow$ H′ is finitely determined with respect to $\mathscr{K}$-equivalence.

But then $\tilde{D}^2(f)/\mathbb{Z}_2$ (and hence $\mu(\tilde{D}^2(f)/\mathbb{Z}_2)$) is determined by the r-jet of H′. Then $H = H' \circ h$ so H′ is determined by the 2r-jet of H, which is determined by the (2r+1)-jet of f. □

(III·3:3) lemma: if T(f) is finite then there is a value k such that

$$j^k g = j^k f \Rightarrow T(g) = T(f).$$

Proof: $T = \frac{1}{6}\dim_{\mathbb{C}} \mathcal{O}_{\tilde{D}^3 f}$ so T is finite $\Leftrightarrow$ $\tilde{D}^3 f$ has an isolated singularity

$\Leftrightarrow$ the map defining $\tilde{D}^3 f$ is finitely determined with respect to $\mathscr{C}$-equivalence.

But then $\tilde{D}^3 f$ (and hence T) is determined by the r-jet of the map, which is determined by a k-jet of f for some k. □

The results are probably also true for map-germs having corank 2.

For a geometric stem the numbers are quite well-behaved:

(III·3:4) lemma: if f is a geometric stem then exactly two of the three numbers C(f), T(f) and $\mu(\tilde{D}^2(f)/\mathbb{Z}_2)$ are finite.

Proof: suppose first that Σ has transversal type A_2. We show that $\tilde{D}^2(f)/\mathbb{Z}_2$ has at worst an isolated singularity and that the triple point set is at worst an isolated point. It is sufficient to look at a generic point of Σ; then we can assume that f is equivalent to (x, y^2, y^3).

This map-germ has the presentation matrix $\begin{bmatrix} -Z & Y \\ Y^2 & -Z \end{bmatrix}$ so $T = 0$.

Also $\tilde{D}^2(f) = V(y+y', y^2 + yy' + y'^2)$.

Let $\tilde{D}^2(f)/\mathbb{Z}_2 = V(H')$, where $H' = H'(\sigma, \tau)$ and $H' \circ (y+y', yy') = (y+y', y^2 + yy' + y'^2)$ then $H'(\sigma, \tau) = (\sigma, \sigma^2 - \tau)$, so $\tilde{D}^2(f)/\mathbb{Z}_2$ is smooth and $\mu(\tilde{D}^2(f)/\mathbb{Z}_2) = 0$.

This proves the assertion for the transversal type A_2.

It is clear that for the remaining cases (A_3 and D_4) C must be finite since the map-

germ obviously can't have a cuspidal edge.

The case A_3 at a generic point of Σ is identical to the case for (x, y^2, x^2y) at a generic point of Σ, i.e. $T = 0$ except at $(0, 0)$ [Mond 2].

Similarly, by comparison with (x, xy, y^3), $\mu(\tilde{D}^2(f)/\mathbb{Z}_2) = 0$ at a generic poit of Σ when it has transversal type D_4.

Finally, we note that for each family $f + p_s$ $\mathrm{cod}(\mathscr{A}, f + p_{s+1}) = \mathrm{cod}(\mathscr{A}, f + p_s) + 1$ (section I·3, p. 55) so that $\mathrm{cod}(\mathscr{A}, f + p_s) = s + c(f)$, where $c(f)$ is a constant depending only on f. This means that we need only check that

$$C(f + p_s) + T(f + p_s) + \mu(\tilde{D}^2(f + p_s)/\mathbb{Z}_2) = s + d(f)$$

for some constant $d(f)$.

The table below shows that this is satified by several families (the reference refers to the calculation in Appendix B. One calculation is included here to demonstrate the method.

Table 7. $\mathscr{A}$-invariants for the families associated with some weak stems.

families	C	T	$\mu(\tilde{D}^2(f)/\mathbb{Z}_2)$	ref
$(x, y^2, x^3y + x^2y^3 + y^{2k+1})$	3	0	k+1	B:1
$(x, y^2, x^4y + 2x^2y^3 + y^5 + x^ky)$	4	0	k-1	B:2
$(x, y^2, x^4y + x^2y^3 + y^{2k+1})$	4	0	k+1	B:3
$(x, y^2, x^2y^3 + y^5 + x^ky)$	k	0	3	B:4
$(x, xy + y^3, xy^2 + y^{3k+1})$	3	k	1	B:5
$(x, xy + y^3, xy^2 + y^4 + y^{2k})$	3	k-1	1	B:6
$(x, xy + y^3, 2xy^2 + 3y^4 + y^k)$	k	1	1	B:7
$(x, xy + y^3, xy^4 + y^5)$	4	2	3	B:8
$(x, xy + y^3, xy^4 + y^{3k+2})$	5	k+2	6	B:9
$(x, x^2y + y^3, xy^2 + y^{3k+1})$	5	k	1	B:10
$(x, y^3, x^2y + y^4 + y^{3k+2})$	4	2k	2	see below
$(x, y^3, x^2y + y^4 + xy^{3k+2})$	4	2k+1	2	see below
$(x, x^2y - y^3, x^2y^2 - y^4 + x^ky^2)$	6	k	2	B:11

For example $(x, y^3, x^2y + y^4 + y^{3k+2})$, $(x, y^3, x^2y + y^4 + xy^{3k+2})$.

$$Q_f = \langle y^2, x^2 \rangle \text{ therefore } C = 4.$$

$\tilde{D}^2(x, y^3, x^2y + y^4) = V(y^2+yy'+y'^2, x^2+y^3+y^2y'+yy'^2+y'^3) = V(y^2+yy'+y'^2, x^2+y^3)$.

We show that $h_f = (y^2+yy'+y'^2, x^2+y^3)$ is $\mathscr{K}$-finite (hence h_f has an isolated singularity so that $\mu(\tilde{D}^2(x, y^3, x^2y + y^4))$ is finite-then

$$\mu(\tilde{D}^2(x, y^3, x^2y + y^4)) = \mu(\tilde{D}^2(x, y^3, x^2y + y^4 + (x)y^{3k+2}) \text{ as above}).$$

Since h_f is weighted homogeneous $\mu(h_f) = \dim \mathcal{O}_3/\langle h_f, J_h \rangle$ which is finite if and only if h_f is $\mathscr{K}$-finite. Then $\mu(\tilde{D}^2(x, y^3, x^2y + y^4)) = \dim \mathcal{O}_3/I$ where

$$I = \langle y^2+yy'+y'^2, x^2+y^3, x(2y+y'), x(y+2y'), y^2(y+2y') \rangle$$
$$= \langle x^3, xy, xy', x^2+y^3, y^2+yy'+y'^2, y^3+2y^2y' \rangle$$
$$\supseteq \{y^4, y^3y', y^2y'^2, yy'^3, y'^4\}.$$

Hence $\mu(\tilde{D}^2(f)) = 7 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2)+C-1 \Rightarrow \mu(\tilde{D}^2(f)/\mathbb{Z}_2)=2$.

We calculate a presentation for $(x, y^3, x^2y + y^4 + y^{3k+2})$.

	1	y	y^2
$x^2y + y^4 + y^{3k+2}$	$-Z$	X^2+Y	Y^k
$x^2y^2 + y^5 + y^{3k+3}$	Y^{k+1}	$-Z$	X^2+Y
$x^2y^3 + y^6 + y^{3k+4}$	X^2Y+Y^2	Y^{k+1}	$-Z$

Then $\mathscr{F}_2 = \{Z, Y+X^2, Y^k\}\mathcal{O}_3$. Let $Y' = Y+X^2$, then $\mathscr{F}_2 = \{Z, Y', X^{2k}\}\mathcal{O}_3$ and $T = 2k$.

We calculate a presentation for $(x, y^3, x^2y + y^4 + xy^{3k+2})$.

	1	y	y^2
$x^2y + y^4 + y^{3k+2}$	$-Z$	X^2+Y	XY^k
$x^2y^2 + y^5 + y^{3k+3}$	XY^{k+1}	$-Z$	X^2+Y
$x^2y^3 + y^6 + y^{3k+4}$	X^2Y+Y^2	XY^{k+1}	$-Z$

Then $\mathscr{F}_2 = \{Z, Y+X^2, XY^k\}\mathcal{O}_3$. Let $Y' = Y+X^2$, then $\mathscr{F}_2 = \{Z, Y', X^{2k+1}\}\mathcal{O}_3$ and $T = 2k+1$.

Appendix A : Calculations for Chapter II.

Calculations of $T\mathscr{A}f$ for non-finitely determined map-germs in sections II·3 and II·4.

(A·1) $$f = (x, xy, y^3)$$

Let $$C = \begin{pmatrix} \mathfrak{m}_2\backslash\{y^{3j+1} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^2\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix} : j \geq 0\right\}$$

$$= \left(\langle x\rangle + \mathbb{C}\{y^{3j+2}, y^{3j+3} : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle x\rangle + \mathbb{C}\{y^{3j+3}, y^{3j+4} : j \geq 0\}\right)\frac{\partial}{\partial Y} + \left(\langle x, y^3\rangle\right)\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix} : j \geq 0\right\}$$

Then $f^*\mathfrak{m}_3 \cdot C = \left(\langle x^2, xy^3\rangle + \mathbb{C}\{xy^2,\ y^{3j+5}, y^{3j+6} : j \geq 0\}\right)\frac{\partial}{\partial X} +$

$$\left(\langle x^2, xy^3\rangle + \mathbb{C}\{y^{3j+6}, y^{3j+7} : j \geq 0\}\right)\frac{\partial}{\partial Y} + \left(\langle x^2, xy^3, y^6\rangle\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix} : j \geq 1\right\}$$

$$= \begin{pmatrix} \mathfrak{m}_2^2\backslash\{xy, y^2, y^3, y^{3j+1} : j \geq 1\} \\ \mathfrak{m}_2^2\backslash\{xy, xy^2, y^3, y^4, y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2^2\backslash\{xy, y^2, xy^2, y^3, y^4, y^5\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix} : j \geq 1\right\}$$

C is spanned by: $x\frac{\partial}{\partial X}, xy\frac{\partial}{\partial X}, y^2\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, xy\frac{\partial}{\partial Y}, y^3\frac{\partial}{\partial Y}, y^4\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, xy\frac{\partial}{\partial Z}, xy^2\frac{\partial}{\partial Z},$

$$y^3\frac{\partial}{\partial Z}, y^4\frac{\partial}{\partial Z}, y^5\frac{\partial}{\partial Z}, \begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}.$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, xy\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, xy\frac{\partial}{\partial Y}, y^3\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, xy\frac{\partial}{\partial Z}, y^3\frac{\partial}{\partial Z},$

$$\begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \\ 3xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 \\ 3y^4 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^3 \\ 3y^5 \end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3 \cdot C$ so that $C = T\mathscr{A}f$.

(A·2) $$f = (x, xy + y^3, xy^2)$$

$$\text{Let } C = \begin{pmatrix} \mathfrak{m}_2 \backslash \{y, y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2 \backslash \{y, y^2\} \\ \mathfrak{m}_2 \backslash \{xy, y^2, y^3, y^{3j+1} : j \geq 0\} \end{pmatrix} + \mathbb{C}\left\{ \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy + y^3 \end{pmatrix} : j \geq 0 \right\}$$

$$= \left(\langle x \rangle + \mathbb{C}\{y^{3j+3}, y^{3j+4} : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle x, y^3 \rangle\right)\frac{\partial}{\partial Y} +$$

$$\left(\langle x^2, xy^2 \rangle + \mathbb{C}\{x, y^{3j+5}, y^{3j+6} : j \geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{ \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy + y^3 \end{pmatrix} : j \geq 0 \right\}$$

$$\text{Then } f^* \mathfrak{m}_3 \cdot C = \left(\langle x^2, xy^3 \rangle + \mathbb{C}\{y^{3j+6}, y^{3j+7} : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle x^2, xy^3, y^6 \rangle\right)\frac{\partial}{\partial Y}$$

$$+ \left(\langle x^3, x^2y^2, xy^5 \rangle + \mathbb{C}\{x^2, x^2y + xy^3, y^{3j+8}, y^{3j+9} : j \geq 0\}\right)\frac{\partial}{\partial Z}$$

$$+ \mathbb{C}\left\{ \begin{pmatrix} xy \\ xy^2 \\ xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2xy^4 + y^6 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}, \begin{pmatrix} y^4 \\ y^5 \\ y^6 \end{pmatrix}, \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1 \right\}$$

$$= \begin{pmatrix} \mathfrak{m}_2^2 \backslash \{xy, y^2, xy^2, y^3, y^4, y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2^2 \backslash \{xy, y^2, xy^2, y^3, y^4, y^5\} \\ \mathfrak{m}_2^2 \backslash \{xy, y^2, x^2y, xy^2, y^3, xy^3, xy^4, y^5, y^6, y^{3j+1} : j \geq 1\} \end{pmatrix}$$

$$+ \mathbb{C}\left\{ \begin{pmatrix} 0 \\ 0 \\ x^2y + xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2xy^4 + y^6 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}, \begin{pmatrix} xy \\ xy^2 \\ xy^3 \end{pmatrix}, \begin{pmatrix} y^4 \\ y^5 \\ y^6 \end{pmatrix}, \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1 \right\}$$

$$C \text{ is spanned by}: x\frac{\partial}{\partial X}, xy\frac{\partial}{\partial X}, xy^2\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, y^4\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, xy\frac{\partial}{\partial Y}, xy^2\frac{\partial}{\partial Y}, y^3\frac{\partial}{\partial Y}, y^4\frac{\partial}{\partial Y}, y^5\frac{\partial}{\partial Y}$$

$$, x\frac{\partial}{\partial Z}, x^2y\frac{\partial}{\partial Z}, xy^2\frac{\partial}{\partial Z}, y^5\frac{\partial}{\partial Z}, \begin{pmatrix} y^2 \\ 0 \\ y^4 \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xy + y^3 \end{pmatrix}$$

$$T\mathscr{A}f \text{ is spanned by}: x\frac{\partial}{\partial X}, (xy+y^3)\frac{\partial}{\partial X}, xy^2\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (xy+y^3)\frac{\partial}{\partial Y}, xy^2\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, (xy+y^3)\frac{\partial}{\partial Z},$$

$$xy^2\frac{\partial}{\partial Z}, \begin{pmatrix} x \\ xy \\ xy^2 \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ y^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2+3xy^2 \\ 2x^2y \end{pmatrix}, \begin{pmatrix} 0 \\ xy+3y^3 \\ 2xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2+3y^4 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^3+3y^5 \\ 2xy^4 \end{pmatrix}$$

Working modulo $f^*\mathfrak{m}_3\cdot C$ we can replace the last 4 elements by

$$\begin{pmatrix}0\\3xy^2\\2x^2y\end{pmatrix},\ \begin{pmatrix}0\\xy+3y^3\\2xy^2\end{pmatrix},\ \begin{pmatrix}0\\xy^2+3y^4\\-2x^2y\end{pmatrix},\ \begin{pmatrix}-2xy^2\\3y^5\\0\end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3\cdot C$ so that $C = T\mathscr{A}f$.

(A·3) $$f = (x, xy+y^3, xy^2+y^4)$$

$$\text{Let } C = \begin{pmatrix}\mathfrak{m}_2\backslash\{y\}\\ \{x, x^jy^{2\ell}: j+\ell\geq 2\} + \{x^jy^{2\ell+1}(x+y^2): \ell, j\geq 0\}\\ \{x, x^j, x^jy^{2\ell+1}: j+\ell\geq 2\} + \{y(x+y^2), x^jy^{2\ell+2}(x+y^2): \ell, j\geq 0\}\end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix}y\\y^2\\y^3\end{pmatrix}, \begin{pmatrix}0\\y^{2j-1}\\y^{2j}\end{pmatrix}: j\geq 2\right\}$$

$$= \left(\langle x, y^2\rangle\right)\frac{\partial}{\partial X} + \left(\mathbb{C}\{x, x^jy^{2\ell}: j+\ell\geq 2\} + \mathbb{C}\{x^jy^{2\ell+1}(x+y^2): j,\ell\geq 0\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\mathbb{C}\{x, x^j, x^jy^{2\ell+1}: j+\ell\geq 2\} + \mathbb{C}\{y(x+y^2), x^jy^{2\ell+2}(x+y^2): j,\ell\geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix}y\\y^2\\y^3\end{pmatrix}, \begin{pmatrix}0\\y^{2j-1}\\y^{2j}\end{pmatrix}: j\geq 2\right\}$$

$$\text{Then } f^*\mathfrak{m}_3\cdot C = \left(\langle x^2, xy^2, y^5\rangle\right)\frac{\partial}{\partial X} + \left(\mathbb{C}\{x^2, x^2y+xy^3, x^{j+1}y^{2\ell}, x^jy^{2\ell+4}: j+\ell\geq 2\}\right.\frac{\partial}{\partial Y}$$

$$\left.+\mathbb{C}\{y^6, x^{j+1}y^{2\ell+1}(x+y^2), y^{2\ell+5}(x+y^2): j,\ell\geq 0\}\right)\frac{\partial}{\partial Y} + \left(\mathbb{C}\{x^2, x^{j+1}, x^{j+1}y^{2\ell+1}, xy(x+y^2), y^{2\ell+5}: j+\ell\geq 2\}\right.\frac{\partial}{\partial Z}$$

$$\left.+\mathbb{C}\{y^7, xy(x+y^2), x^{j+1}y^{2\ell+2}(x+y^2), y^{2\ell+4}(x+y^2): j,\ell\geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix}xy\\xy^2\\xy^3\end{pmatrix}, \begin{pmatrix}y^4\\y^5\\y^6\end{pmatrix}, \begin{pmatrix}0\\xy^{2j-1}\\xy^{2j}\end{pmatrix}: j\geq 2\right\}$$

$$= \begin{pmatrix}\mathfrak{m}_2^2\backslash\{xy, y^2, y^3, y^4\}\\ \{x^2, x^jy^{2\ell}: j+\ell\geq 3\} + \{x^jy^{2\ell+1}(x+y^2): j+\ell\geq 2\}\\ \{x^2, x^j, x^jy^{2\ell+1}: j+\ell\geq 3\} + \{xy(x+y^2), x^jy^{2\ell+2}(x+y^2): j+\ell\geq 1\}\end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix}0\\x^2y+xy^3\\0\end{pmatrix}\begin{pmatrix}0\\0\\x^2y+xy^3\end{pmatrix}, \begin{pmatrix}xy\\xy^2\\xy^3\end{pmatrix}, \begin{pmatrix}0\\-xy^3\\y^6\end{pmatrix}\begin{pmatrix}y^4\\y^5\\y^6\end{pmatrix}, \begin{pmatrix}0\\y^{2j-1}\\y^{2j}\end{pmatrix}: j\geq 4\right\}$$

$$\text{C is spanned by}: x\frac{\partial}{\partial X}, xy\frac{\partial}{\partial X}, y^2\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, y^4\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, xy^2\frac{\partial}{\partial Y}, (xy+y^3)\frac{\partial}{\partial Y}, y^4\frac{\partial}{\partial Y},$$

$$(xy^3+y^5)\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, (xy+y^3)\frac{\partial}{\partial Z}, (xy^2+y^4)\frac{\partial}{\partial Z}, x^2y\frac{\partial}{\partial Z}, xy^3\frac{\partial}{\partial Z}, y^5\frac{\partial}{\partial Z}, \begin{pmatrix}y\\y^2\\y^3\end{pmatrix}, \begin{pmatrix}0\\y^3\\y^4\end{pmatrix}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (xy+y^3)\frac{\partial}{\partial X}, (xy^2+y^4)\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (xy+y^3)\frac{\partial}{\partial Y}, (xy^2+y^4)\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z},$

$$(xy+y^3)\frac{\partial}{\partial Z}, (xy^2+y^4)\frac{\partial}{\partial Z},$$

$$\begin{pmatrix} x \\ xy \\ xy^2 \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ y^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2+3xy^2 \\ 2x^2y+4xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ xy+3y^3 \\ 2xy^2+4y^4 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2+3y^4 \\ 2xy^3+4y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^3+3y^5 \\ 2xy^4+4y^6 \end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3 \cdot C$ so that $C = T\mathscr{A}f$.

(A·4) $$f = (x, xy + y^3, 2xy^2 + 3y^4)$$

Let $C = \mathbb{C}\{(j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2} : \ell \geq 1, j \geq 0\}\frac{\partial}{\partial X} + \mathbb{C}\{((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}) : \ell \geq 1, j \geq 0\}\frac{\partial}{\partial Y}$

$$+ \mathbb{C}\{((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}) : \ell \geq 1, j \geq 0\}\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} y \\ y^2 \\ 2y^3 \end{pmatrix}, \begin{pmatrix} -(j+2)y^j \\ 0 \\ 2jy^{j+2} \end{pmatrix}, \begin{pmatrix} 2(j+1)y^j \\ jy^{j+1} \\ 0 \end{pmatrix} : j \geq 2\right\}$$

Clearly $f^*\mathfrak{m}_3 \cdot C \supseteq \mathbb{C}\{((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}) : \ell \geq 2, j \geq 0\}e$ $(e = \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z})$

Then $(j+3)(xy + y^3)((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2})$

$$= (j+2)((j+3)x^{\ell+1}y^j + 1 + 3(j+1)x^\ell y^{j+3}) + j((j+5)x^\ell y^{j+3} + 1 + 3(j+3)x^{\ell-1}y^{j+5})$$

since $(j+3)(j+2)+3j(j+3) = 4j^2+14j+6 = 3(j+1)(j+2)+j(j+5)$,

hence $f^*\mathfrak{m}_3 \cdot C \supseteq \mathbb{C}\{((j+2)xy^j + 3jy^{j+2})e : j \geq 3\}e$ $(e = \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z})$.

We check that

$(j+4)(2xy^2 + 3y^4)((j+2)x^\ell y^j + 3jx^{\ell-1}y^{j+2}) =$

$$2(j+2)((j+4)x^{\ell+1}y^{j+2} + 3(j+2)x^\ell y^{j+4}) + 3j((j+6)x^\ell y^{j+4} + 3(j+4)x^{\ell-1}y^{j+6})$$

since $3(j+4)(j+2)+6j(j+4) = 9j^2+42j+24 = 3j(j+6)+6(j+2)^2$.

We also have $f^*\mathfrak{m}_3 \cdot C \supseteq \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} -(j+2)xy^j \\ 0 \\ 2jxy^{j+2} \end{pmatrix}, \begin{pmatrix} 2(j+1)xy^j \\ jxy^{j+1} \\ 0 \end{pmatrix} : j \geq 2\right\}$.

For $j \geq 3$, $2(j+3)(j+1)((j+2)xy^j + 3jy^{j+2})\frac{\partial}{\partial X}, j(j+2)((j+3)xy^{j+1} + 3(j+1)y^{j+3})\frac{\partial}{\partial Y} \in f^*\mathfrak{m}_3 \cdot C$

as are $(j+4)((j+2)xy^j + 3jy^{j+2})\frac{\partial}{\partial X}$ and $2j((j+4)xy^{j+2} + 3(j+2)y^{j+4})\frac{\partial}{\partial Z}$

hence $f^*\mathfrak{m}_3 \cdot C \supseteq \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} -xy^2 \\ 0 \\ xy^4 \end{pmatrix}, \begin{pmatrix} 6xy^2 \\ 2xy^3 \\ 0 \end{pmatrix}, \begin{pmatrix} -(j+2)y^j \\ 0 \\ 2jy^{j+2} \end{pmatrix}, \begin{pmatrix} 2(j+1)y^j \\ jy^{j+1} \\ 0 \end{pmatrix} : j \geq 5\right\}$.

Finally, to establish that C is indeed an $\mathcal{O}_3$-module via f^* we show that

$$(j+3)(j+5)(xy+y^3)\begin{pmatrix}-(j+2)y^j\\0\\2jy^{j+2}\end{pmatrix} = -(j+2)(j+5)\left((j+3)xy^{j+1}+3(j+1)y^{j+3}\right)\frac{\partial}{\partial X}$$

$$+2j(j+3)\left((j+5)xy^{j+3}+3(j+3)y^{j+5}\right)\frac{\partial}{\partial Z} - 2j(j+2)\begin{pmatrix}-(j+5)y^{j+3}\\0\\2(j+3)y^{j+5}\end{pmatrix}$$

$$(j+4)(j+6)(2xy^2+3y^4)\begin{pmatrix}-(j+2)y^j\\0\\2jy^{j+2}\end{pmatrix} = -2(j+2)(j+6)\left((j+4)xy^{j+2}+3(j+2)y^{j+4}\right)\frac{\partial}{\partial X}$$

$$+4j(j+4)\left((j+6)xy^{j+4}+3(j+4)y^{j+6}\right)\frac{\partial}{\partial Z} - 3j(j+2)\begin{pmatrix}-(j+6)y^{j+4}\\0\\2(j+4)y^{j+6}\end{pmatrix}$$

$$(j+4)(j+3)(xy+y^3)\begin{pmatrix}2(j+1)y^j\\jy^{j+1}\\0\end{pmatrix} = 2(j+4)(j+1)\left((j+3)xy^{j+1}+3(j+1)y^{j+3}\right)\frac{\partial}{\partial X}$$

$$+j(j+3)\left((j+4)xy^{j+2}+3(j+2)y^{j+4}\right)\frac{\partial}{\partial Y} - 2j(j+1)\begin{pmatrix}2(j+4)y^{j+3}\\(j+3)y^{j+4}\\0\end{pmatrix}$$

$$(j+5)(j+4)(2xy^2+3y^4)\begin{pmatrix}2(j+1)y^j\\jy^{j+1}\\0\end{pmatrix} = 4(j+5)(j+1)\left((j+4)xy^{j+2}+3(j+2)y^{j+4}\right)\frac{\partial}{\partial X}$$

$$+2j(j+4)\left((j+5)xy^{j+3}+3(j+3)y^{j+5}\right)\frac{\partial}{\partial Y} - 3j(j+1)\begin{pmatrix}2(j+5)y^{j+4}\\(j+4)y^{j+5}\\0\end{pmatrix}$$

C is spanned by: $(j+2)x\ y^j+3jy^{j+2}e$ for $0\le j\le 2$, i.e.

$$x\frac{\partial}{\partial X},\ (xy+y^3)\frac{\partial}{\partial X},\ (2xy^2+3y^4)\frac{\partial}{\partial X},\ x\frac{\partial}{\partial Y},\ (xy+y^3)\frac{\partial}{\partial Y},\ (2xy^2+3y^4)\frac{\partial}{\partial Y},\ x\frac{\partial}{\partial Z},$$

$$(xy+y^3)\frac{\partial}{\partial Z},\ (2xy^2+3y^4)\frac{\partial}{\partial Z},\ \begin{pmatrix}y\\y^2\\2y^3\end{pmatrix},\ \begin{pmatrix}-y^2\\0\\y^4\end{pmatrix}\begin{pmatrix}-5y^3\\0\\6y^5\end{pmatrix}\begin{pmatrix}-6y^4\\0\\8y^6\end{pmatrix}\begin{pmatrix}6y^2\\2y^3\\0\end{pmatrix}\begin{pmatrix}8y^3\\3y^4\\0\end{pmatrix}\begin{pmatrix}10y^4\\4y^5\\0\end{pmatrix}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (xy+y^3)\frac{\partial}{\partial X}, (2xy^2+3y^4)\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (xy+y^3)\frac{\partial}{\partial Y}, (2xy^2+3y^4)\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z},$

$$(xy+y^3)\frac{\partial}{\partial Z}, (2xy^2+y^4)\frac{\partial}{\partial Z},$$

$$\begin{pmatrix} x \\ xy \\ 2xy^2 \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ 2y^3 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 2y^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 2y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2+3xy^2 \\ 4x^2y+12xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ xy+3y^3 \\ 4xy^2 12y^4 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2+3y^4 \\ 4xy^3+12y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^3+3y^5 \\ 4xy^4+12y^6 \end{pmatrix}$$

Rewrite the last 3 columns modulo $f^*\mathfrak{m}_3\cdot C$ to get $\begin{pmatrix} 0 \\ y^3 \\ 3y^4 \end{pmatrix}\begin{pmatrix} 0 \\ 5y^4 \\ 16y^5 \end{pmatrix}\begin{pmatrix} 0 \\ 3y^5 \\ 10y^6 \end{pmatrix}$

bearing in mind that $f^*\mathfrak{m}_3\cdot C \ni 5xy^3+9y^5e,\ 6xy^4+12y^6e$.
Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3\cdot C$ so that $C = T\mathscr{A}f$.

(A·5) $$f = (x, xy+y^3, 2xy^2+y^4)$$

Let $C =$ $$\mathbb{C}\left\{(x+y^2)x^\ell y^{2j+1}, ((j+1)xy^{2j}+jy^{2j+2})x^\ell : \ell, j \geq 0\right\}\frac{\partial}{\partial X}$$

$$+\ \mathbb{C}\{(x+y^2)x^\ell y^{2j+1} : \ell, j \geq 0\}\frac{\partial}{\partial Y} + \mathbb{C}\{x, x^\ell y^{2j} : \ell+j \geq 2\}\frac{\partial}{\partial Y} +$$

$$\mathbb{C}\left\{(x+y^2)x^\ell y^{2j+1}, ((j+1)xy^{2j}+jy^{2j+2})x^\ell : \ell, j \geq 0\right\}\frac{\partial}{\partial Z} + \mathbb{C}\left\langle \begin{pmatrix} y \\ y^2 \\ 2y^3 \end{pmatrix}, \begin{pmatrix} (j+1)y^{2j} \\ 0 \\ 2jy^{2j+2} \end{pmatrix}, \begin{pmatrix} y^{2j+1} \\ 0 \\ 2y^{2j+3} \end{pmatrix}, \begin{pmatrix} -y^{2j} \\ jy^{2j+1} \\ 0 \end{pmatrix} : j \geq 1 \right\}$$

It is clear that

$$f^*\mathfrak{m}_3\cdot C \supseteq \mathbb{C}\left\{(x+y^2)x^\ell y^{2j+1}, ((j+1)xy^{2j}+jy^{2j+2})x^\ell : \ell \geq 1, j \geq 0\right\}\frac{\partial}{\partial X}$$

$$+\ \mathbb{C}\{(x+y^2)x^\ell y^{2j+1} : \ell \geq 1, j \geq 0\}\frac{\partial}{\partial Y} + \mathbb{C}\{x^2, x^\ell y^{2j} : \ell+j \geq 3, \ell \geq 1\}\frac{\partial}{\partial Y}$$

$$+\ \mathbb{C}\left\{(x+y^2)x^\ell y^{2j+1}, ((j+1)xy^{2j}+jy^{2j+2})x^\ell : \ell \geq 1, j \geq 0\right\}\frac{\partial}{\partial Z}$$

Now $(xy+y^3)y^{2j}\frac{\partial}{\partial Y} = (x+y^2)y^{2j+1}\frac{\partial}{\partial Y}$ and $(2xy^2+y^4)y^{2j}\frac{\partial}{\partial Y} \sim y^{2j+4}\frac{\partial}{\partial Y} \in f^*\mathfrak{m}_3\cdot C$ for $j \geq 2$,

$(j+2)(xy+y^3)(x+y^2)x^\ell y^{2j+1} = (j+2)(x^{\ell+2}y^{2j+2}+2x^{\ell+1}y^{2j+4}+x^\ell y^{2j+6}) =$
$(j+2)x^{\ell+2}y^{2j+2}+(j+1)x^{\ell+1}y^{2j+4}+(j+3)x^{\ell+1}y^{2j+4}+(j+2)x^\ell y^{2j+6}$

so that $((j+1)xy^{2i}+jy^{2j+2})x^\ell\frac{\partial}{\partial X}, y^{2j+2}\frac{\partial}{\partial Y}, ((j+1)xy^{2j}+jy^{2j+2})x^\ell\frac{\partial}{\partial Z} \in f^*\mathfrak{m}_3\cdot C$ for $j \geq 2$

and $(2xy^2+y^4)(x+y^2)x^\ell y^{2j+1} = 2(x+y^2)x^{\ell+1}y^{2j+3} + (x+y^2)x^\ell y^{2j+5}$

so that $(x+y^2)x^\ell y^{2j+1}\frac{\partial}{\partial X}, (x+y^2)x^\ell y^{2j+1}\frac{\partial}{\partial Y}, (x+y^2)x^\ell y^{2j+1}\frac{\partial}{\partial Z} \in f^*\mathfrak{m}_3\cdot C$ for $j \geq 2$.

Also $(j+2)(2xy^2+y^4)((j+1)xy^{2i}+jy^{2j+1}) = (j+2)(2(j+1)x^2y^{2j+2}+(3j+1)xy^{2j+4}+jy^{2j+6}$

$$= 2(j+1)((j+2)x^2y^{2j+2} + (j+1)xy^{2j+4}) + j((j+3)xy^{2j+4} + (j+2)y^{2j+6})$$

so that $((j+1)xy^{2j}+jy^{2j+2})x^\ell \frac{\partial}{\partial X}, ((j+1)xy^{2j}+jy^{2j+2})x^\ell \frac{\partial}{\partial Z} \in f^*\mathfrak{m}_3\cdot C$ for $j \geq 3$

and $(xy + y^3)((j+1)xy^{2j}+jy^{2j+1}) = (j+1)(x^2y^{2j+1} + xy^{2j+3}) + j(xy^{2j+3} + y^{2j+5})$

so that $(x+y^2)x^\ell y^{2j+1}\frac{\partial}{\partial X}, (x+y^2)x^\ell y^{2j+1} \frac{\partial}{\partial Z} \in f^*\mathfrak{m}_3\cdot C$ for $j \geq 3$.

Hence $f^*\mathfrak{m}_3\cdot C \supseteq \mathbb{C}\left\{(x+y^2)x^\ell y^{2j+1}, ((j+1)xy^{2j}+jy^{2j+2})x^\ell : j \geq 2 \text{ or } \ell \geq 1 \text{ and } j \geq 0\right\} \frac{\partial}{\partial X}$

$$+ \mathbb{C}\{(x+y^2)x^\ell y^{2j+1} : j \geq 2 \text{ or } \ell \geq 1 \text{ and } j \geq 0\} \frac{\partial}{\partial Y} + \mathbb{C}\{x^2, x^\ell y^{2j} : \ell + j \geq 3\} \frac{\partial}{\partial Y}$$

$$+ \mathbb{C}\left\{(x+y^2)x^\ell y^{2j+1}, ((j+1)xy^{2j}+jy^{2j+2})x^\ell : j \geq 2 \text{ or } \ell \geq 1 \text{ and } j \geq 0\right\} \frac{\partial}{\partial Z}$$

$$f^*\mathfrak{m}_3\cdot C \supseteq \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} (j+1)xy^{2j} \\ 0 \\ 2jxy^{2j+2} \end{pmatrix}, \begin{pmatrix} xy^{2j+1} \\ 0 \\ 2xy^{2j+3} \end{pmatrix}, \begin{pmatrix} -xy^{2j} \\ jxy^{2j+1} \\ 0 \end{pmatrix} : j \geq 1 \right\}$$

Now

$$(j+3)\begin{pmatrix} (j+1)xy^{2j} \\ 0 \\ 2jxy^{2j+2} \end{pmatrix} = (j+3)((j+1)xy^{2j}+jy^{2j+2})\frac{\partial}{\partial X} + 2j((j+3)xy^{2j+2}+(j+2)y^{2j+4})\frac{\partial}{\partial Z} - j\begin{pmatrix} (j+3)y^{2j+2} \\ 0 \\ 2(j+2)y^{2j+4} \end{pmatrix},$$

$$\begin{pmatrix} xy^{2j+1} \\ 0 \\ 2xy^{2j+3} \end{pmatrix} = (xy^{2j+1}+y^{2j+3})\frac{\partial}{\partial X} + 2(xy^{2j+3}+y^{2j+5})\frac{\partial}{\partial Z} - \begin{pmatrix} y^{2j+3} \\ 0 \\ 2y^{2j+5} \end{pmatrix},$$

$$(j+1)\begin{pmatrix} -xy^{2j} \\ jxy^{2j+1} \\ 0 \end{pmatrix} = ((j+1)xy^{2j}+jy^{2j+2})\frac{\partial}{\partial X} + j(j+1)(xy^{2j+1}+y^{2j+3})\frac{\partial}{\partial Y} - \begin{pmatrix} -y^{2j+2} \\ (j+1)y^{2j+3} \\ 0 \end{pmatrix},$$

$$\text{and } (xy+y^3)\begin{pmatrix} (j+1)y^{2j} \\ 0 \\ 2jy^{2j+2} \end{pmatrix} = (j+1)(x+y^2)y^{2j+1}\frac{\partial}{\partial X} + j(x+y^2)y^{2j+3}\frac{\partial}{\partial Z}$$

so that $f^*\mathfrak{m}_3\cdot C$

$$\supseteq \mathbb{C}\left\{(x+y^2)y^3\frac{\partial}{\partial X}, \begin{pmatrix} xy \\ xy^2 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix} \begin{pmatrix} -xy^2 \\ xy^3 \\ 0 \end{pmatrix} \begin{pmatrix} xy^2+y^4 \\ xy^3+y^5 \\ 2xy^4+2y^6 \end{pmatrix}, \begin{pmatrix} (j+1)y^{2j} \\ 0 \\ 2jy^{2j+2} \end{pmatrix}, \begin{pmatrix} y^{2j-1} \\ 0 \\ 2y^{2j+1} \end{pmatrix}, \begin{pmatrix} -y^{2j} \\ jy^{2j+1} \\ 0 \end{pmatrix} : j \geq 3 \right\}.$$

C is spanned by: $x\frac{\partial}{\partial X}, (xy+y^3)\frac{\partial}{\partial X}, (2xy^2+y^4)\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (xy+y^3)\frac{\partial}{\partial Y}, xy^2\frac{\partial}{\partial Y}, y^4\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z},$

$$(xy+y^3)\frac{\partial}{\partial Z},\ (2xy^2+y^4)\frac{\partial}{\partial Z},\ (xy^3+y^5)\frac{\partial}{\partial Z},\ \begin{pmatrix} y \\ y^2 \\ 2y^3 \end{pmatrix},\ \begin{pmatrix} y^2 \\ 0 \\ y^4 \end{pmatrix}\begin{pmatrix} -y^2 \\ y^3 \\ 0 \end{pmatrix}\begin{pmatrix} y^3 \\ 0 \\ 2y^5 \end{pmatrix}\begin{pmatrix} 3y^4 \\ 0 \\ 4y^6 \end{pmatrix}\begin{pmatrix} -y^4 \\ 2y^5 \\ 0 \end{pmatrix}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (xy+y^3)\frac{\partial}{\partial X}, (2xy^2+y^4)\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (xy+y^3)\frac{\partial}{\partial Y}, (2xy^2+y^4)\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z},$

$$(xy+y^3)\frac{\partial}{\partial Z},\ (2xy^2+y^4)\frac{\partial}{\partial Z},\ \begin{pmatrix} x \\ xy \\ 2xy^2 \end{pmatrix},\ \begin{pmatrix} y \\ y^2 \\ 2y^3 \end{pmatrix},\ \begin{pmatrix} y^2 \\ y^3 \\ 2y^4 \end{pmatrix},\ \begin{pmatrix} y^3 \\ y^4 \\ 2y^5 \end{pmatrix},\ \begin{pmatrix} 0 \\ x^2+3xy^2 \\ 4x^2y+4xy^3 \end{pmatrix},\ \begin{pmatrix} 0 \\ xy+3y^3 \\ 4xy^2+4y^4 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ xy^2+3y^4 \\ 4xy^3+4y^5 \end{pmatrix},\ \begin{pmatrix} 0 \\ xy^3+3y^5 \\ 4xy^4+4y^6 \end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3 \cdot C$ so that $C = T\mathscr{A}f$.

(A·6) $(x, xy+y^3, xy^3)$

Let $C =$

$$\begin{pmatrix} \mathfrak{m}_2\backslash\{y^{j+1},\ xy^{3j+1}, xy^{3j+2},\ x^2y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y^{3j+1}, y^{3j+2},\ xy^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y^{j+1},\ xy^{3j+1}, xy^{3j+2},\ x^2y^{3j+2} : j \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} (6j+11)x^2y^{3j-1}+6xy^{3j+1} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}j(j-1)x^2y^{3j-4}+jxy^{3j-2}+y^{3j} \\ 0 \\ 0 \end{pmatrix}\right.$$

$$\begin{pmatrix} 0 \\ (6j+1)xy^{3j-2}+6y^{3j+1} \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ (6j-1)x^2y^{3j-1}+6xy^{3j+1} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ (3j^2-4j)xy^{3j-2}+(6j-7)y^{3j} \end{pmatrix}\begin{pmatrix} y \\ y^2 \\ y^4 \end{pmatrix}$$

$$\left.\begin{pmatrix} (6j+1)y^{3j} \\ (3j^2+8j)y^{3j+1} \\ 0 \end{pmatrix}\begin{pmatrix} -xy^{3j-1} \\ y^{3j+2} \\ 0 \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ -xy^{3j+2}+y^{3j+4} \end{pmatrix}\begin{pmatrix} (3j^2+2j-1)y^{3j} \\ 0 \\ (3j^2+8j)y^{3j+3} \end{pmatrix}\begin{pmatrix} xy^{3j-1} \\ 0 \\ xy^{3j+2} \end{pmatrix}\begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} : j \geq 1\right\}$$

$$C = \left(\langle x^3\rangle + \mathbb{C}\{xy^{3j}, x^2y^{3j}, x^2y^{3j+1}, (6j+17)x^2y^{3j+2}+6xy^{3j+4}, \tfrac{1}{2}j(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3} : j \geq 0\}\right)\frac{\partial}{\partial X}$$

$$+\left(\langle x^2\rangle + \mathbb{C}\{y^{3j+3}, xy^{3j}, xy^{3j+1}, (6j+7)xy^{3j+2}+6y^{3j+4} : j \geq 0\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\langle x^3\rangle + \mathbb{C}\{xy^{3j}, x^2y^{3j}, x^2y^{3j+1}, \tfrac{1}{2}j(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3}, (6j+5)x^2y^{3j+2}+6xy^{3j+4} : j \geq 0\}\right)\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y \\ y^2 \\ y^4 \end{pmatrix}, \begin{pmatrix} (6j+1)y^{3j} \\ (3j^2+8j)y^{3j+1} \\ 0 \end{pmatrix}, \begin{pmatrix} -xy^{3j-1} \\ y^{3j+2} \\ 0 \end{pmatrix}, \begin{pmatrix} y^{3j+1} \\ 0 \\ -xy^{3j+2}+y^{3j+4} \end{pmatrix}, \begin{pmatrix} (3j^2+2j-1)y^{3j} \\ 0 \\ (3j^2+8j)y^{3j+3} \end{pmatrix}, \begin{pmatrix} xy^{3j-1} \\ 0 \\ xy^{3j+2} \end{pmatrix}, \begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} : j\geq 1\right\}$$

Note that $(6j+11)(\frac{1}{2}j(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3})-\frac{1}{2}j(j+1)((6j+11)x^2y^{3j-1}+6xy^{3j+1})$

$= (j+1)(6j+11)xy^{3j+1}+(6j+11)y^{3j+3}-\frac{1}{2}j(j+1)6xy^{3j+1} = (j+1)(3j+11)xy^{3j+1}+(6j+11)y^{3j+3}$ where $j\geq 1$

and $(6j-1)(\frac{1}{2}j(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3})-\frac{1}{2}j(j+1)((6j-1)x^2y^{3j-1}+6xy^{3j+1})$

$= (j+1)(6j-1)xy^{3j+1}+(6j-1)y^{3j+3}-\frac{1}{2}j(j+1)6xy^{3j+1} = (j+1)(3j-1)xy^{3j+1}+(6j-1)y^{3j+3}$ where $j\geq 1$

so that $(j+1)(3j+11)xy^{3j+1}+(6j+11)y^{3j+3}\frac{\partial}{\partial X}$, $(j+1)(3j-1)xy^{3j+1}+(6j-1)y^{3j+3}\frac{\partial}{\partial Z} \in C$ for $j\geq 0$.

$$\begin{aligned} f^*\mathfrak{m}_3\cdot C \supseteq & \Big(\langle x^3\rangle + \mathbb{C}\{x^2, x^2y+xy^3, xy^{3j+3}, x^2y^{3j}, x^2y^{3j+1}, (6j+17)x^2y^{3j+2}+6xy^{3j+4}, \\ & \tfrac{1}{2}j(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3} : j\geq 1\Big)\frac{\partial}{\partial X} \\ & + \Big(\langle x^2\rangle + \mathbb{C}\{y^{3j+3}, xy^{3j}, xy^{3j+1}, (6j+7)xy^{3j+2}+6y^{3j+4} : j\geq 1\}\Big)\frac{\partial}{\partial Y} \\ & + \Big(\langle x^3\rangle + \mathbb{C}\{x^2, x^2y+xy^3, xy^{3j+3}, x^2y^{3j}, x^2y^{3j+1}, \tfrac{1}{2}j(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3}, \\ & (6j+5)x^2y^{3j+2}+6xy^{3j+4} : j\geq 1\}\Big)\frac{\partial}{\partial Z} \end{aligned}$$

and $f^*\mathfrak{m}_3\cdot C \ni (j+1)(3j+11)xy^{3j+1}+(6j+11)y^{3j+3}\frac{\partial}{\partial X}$, $(j+1)(3j-1)xy^{3j+1}+(6j-1)y^{3j+3}\frac{\partial}{\partial Z}$ for $j\geq 2$.

Now $x\begin{pmatrix} (6j+1)y^{3j} \\ (3j^2+8j)y^{3j+1} \\ 0 \end{pmatrix} = (6j+1)xy^{3j}\frac{\partial}{\partial X} + (3j^2+8j)xy^{3j+1}\frac{\partial}{\partial Y}$ so $xy^{3j}\frac{\partial}{\partial X} \in f^*\mathfrak{m}_3\cdot C$

$$x\begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} = \begin{pmatrix} xy^{3j-1} \\ 0 \\ xy^{3j+2} \end{pmatrix} \in f^*\mathfrak{m}_3\cdot C \text{ for } j\geq 1,$$

$$(xy+y^3)\begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} = xy^{3j}\frac{\partial}{\partial X} + xy^{3j+3}\frac{\partial}{\partial Z} + y^3\begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} \Rightarrow \begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} \in f^*\mathfrak{m}_3\cdot C \text{ for } j\geq 2,$$

$$(xy+y^3)\begin{pmatrix} -xy^{3j-1} \\ y^{3j+2} \\ 0 \end{pmatrix} = -x^2y^{3j}\frac{\partial}{\partial X}+xy^{3j+3}\frac{\partial}{\partial Y}+y^3\begin{pmatrix} -xy^{3j-1} \\ y^{3j+2} \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -xy^{3j-1} \\ y^{3j+2} \\ 0 \end{pmatrix} \in f^*\mathfrak{m}_3\cdot C \text{ for } j \geq 2,$$

$$(xy+y^3)\begin{pmatrix} y^{3j+1} \\ 0 \\ -xy^{3j+2}+y^{3j+4} \end{pmatrix} = -x^2y^{3j+3}\frac{\partial}{\partial Z}+\begin{pmatrix} xy^{3j+2} \\ 0 \\ xy^{3j+5} \end{pmatrix}+y^3\begin{pmatrix} y^{3j+1} \\ 0 \\ -xy^{3j+2}+y^{3j+4} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y^{3j+1} \\ 0 \\ -xy^{3j+2}+y^{3j+4} \end{pmatrix} \in f^*\mathfrak{m}_3\cdot C \text{ for } j \geq 2,$$

$$(6j+5)(3j+11)(j+1)x\begin{pmatrix} y^{3j+1} \\ 0 \\ -xy^{3j+2}+y^{3j+4} \end{pmatrix} = (6j+5)\left((3j+11)(j+1)xy^{3j+1}+(6j+11)y^{3j+3}\right)\frac{\partial}{\partial X}$$

$$-(j+1)(3j+11)\left((6j+5)x^2y^{3j+2}+6xy^{3j+4}\right)\frac{\partial}{\partial Z}+(6j+11)\begin{pmatrix} -(6j+5)y^{3j+3} \\ 0 \\ (j+1)(3j+11)xy^{3j+4} \end{pmatrix}$$

$$(6j+11)(6j+7)x\begin{pmatrix} -xy^{3j-1} \\ y^{3j+2} \\ 0 \end{pmatrix} = -(6j+7)\left((6j+11)x^2y^{3j-1}+6xy^{3j+1}\right)\frac{\partial}{\partial X}$$

$$+(6j+11)\left((6j+7)xy^{3j-1}+6y^{3j+4}\right)\frac{\partial}{\partial Y}-6\begin{pmatrix} (6j+7)xy^{3j+1} \\ (6j+11))y^{3j+4} \\ 0 \end{pmatrix} \in f^*\mathfrak{m}_3\cdot C \text{ for } j \geq 2,$$

$$(j+1)(3j+11)(6j+7)(xy+y^3)\begin{pmatrix} (6j+1)y^{3j} \\ (3j^2+8j)y^{3j+1} \\ 0 \end{pmatrix} = (6j+7)(6j+1)\left((j+1)(3j+11)xy^{3j+1}+(6j+11)y^{3j+3}\right)\frac{\partial}{\partial X}$$

$$+(j+1)(3j+11)(3j^2+8j)\left((6j+7)xy^{3j+2}+6y^{3j+4}\right)\frac{\partial}{\partial Y}+(3j^2+8j)(j+1)\begin{pmatrix} (6j+7)y^{3j+3} \\ (3j+11)(j+1)y^{3j+4} \\ 0 \end{pmatrix}$$

$$\text{so}\ \begin{pmatrix} 7(xy^4+y^6) \\ 11(xy^5+y^7) \\ 0 \end{pmatrix} \in f^*\mathfrak{m}_3\cdot C \text{ and } \begin{pmatrix} (6j+7)y^{3j+3} \\ (3j+11)(j+1)y^{3j+4} \\ 0 \end{pmatrix} \in f^*\mathfrak{m}_3\cdot C \text{ for } j \geq 2,$$

Hence $f^*\mathfrak{m}_3\cdot C \supseteq \Big(\langle x^3\rangle + \mathbb{C}\{x^2, x^2y, xy^{3j}, x^2y^{3j}, x^2y^{3j+1}, (6j+17)x^2y^{3j+2}+6xy^{3j+4},$

$$\tfrac{1}{2}j(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3} : j \geq 1\Big)\frac{\partial}{\partial X}$$

$$+\Big(\langle x^2\rangle + \mathbb{C}\{y^{3j+3}, xy^{3j}, xy^{3j+1}, (6j+7)xy^{3j+2}+6y^{3j+4} : j \geq 1\}\Big)\frac{\partial}{\partial Y}$$

$$+\Big(\langle x^3\rangle + \mathbb{C}\{x^2, x^2y+xy^3, xy^{3j+3}, x^2y^{3j}, x^2y^{3j+1},\ \tfrac{1}{2}j(j+1)x^2y^{3j-1}+(j+1)xy^{3j+1}+y^{3j+3},$$

$$(6j+5)x^2y^{3j+2}+6xy^{3j+4} : j \geq 1\}\Big)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} y \\ y^2 \\ y^4 \end{pmatrix}, \begin{pmatrix} (6j+7)y^{3j+3} \\ (3j+11)(j+1)y^{3j+4} \\ 0 \end{pmatrix}, \begin{pmatrix} -xy^{3j-1} \\ y^{3j+2} \\ 0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} y^{3j+1} \\ 0 \\ -xy^{3j+2}+y^{3j+4} \end{pmatrix}, \begin{pmatrix} (3j^2+2j-1)y^{3j} \\ 0 \\ (3j^2+8j)y^{3j+3} \end{pmatrix}, \begin{pmatrix} xy^{3j-1} \\ 0 \\ xy^{3j+2} \end{pmatrix}, \begin{pmatrix} y^{3j-1} \\ 0 \\ y^{3j+2} \end{pmatrix} : j \geq 2\right\}$$

C is spanned by: $x\frac{\partial}{\partial X}, xy^3\frac{\partial}{\partial X}, (xy+y^3)\frac{\partial}{\partial X}, (17x^2y^2+6xy^4)\frac{\partial}{\partial X}\ x\frac{\partial}{\partial Y}, xy\frac{\partial}{\partial Y}, y^3\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z},$

$$(7xy^2+6y^4)\frac{\partial}{\partial Y}, (xy+y^3)\frac{\partial}{\partial Z}, (5x^2y^2+6xy^4)\frac{\partial}{\partial Z}, xy^3\frac{\partial}{\partial Z}, \begin{pmatrix} y \\ y^2 \\ y^4 \end{pmatrix}, + \text{other terms}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (xy+y^3)\frac{\partial}{\partial X}, xy^3\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (xy+y^3)\frac{\partial}{\partial Y}, xy^3\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, (xy+y^3)\frac{\partial}{\partial Z},$

$$xy^3\frac{\partial}{\partial Z}, \begin{pmatrix} x \\ xy \\ xy^3 \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^4 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ y^5 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ y^6 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2+3xy^2 \\ 3x^2y^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy+3y^3 \\ 3xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2+3y^4 \\ 3xy^4 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^3+3y^5 \\ 2xy^5 \end{pmatrix}$$

Long and somewhat tedious calculations yield that $C \subseteq T\mathscr{A}f + f^*\mathfrak{m}_3\cdot C$ so that $C = T\mathscr{A}f$.

(A.7) $(x, xy+y^3, xy^4)$

$$T\mathscr{A}f \begin{pmatrix} \mathfrak{m}_2\backslash\{xy, y^2, xy^2, y^3, x^2y^2, y^6, y^7, y^{3j+1} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^2, xy^2, y^4, y^5, xy^5, y^7\} \\ \mathfrak{m}_2\backslash\{y, xy, y^2, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, y^4, x^3y^2, x^2y^3, x^4y^2, xy^5, y^6 \\ xy^6, y^7, x^2y^6, xy^8, y^{10}, y^{3j+2} : j \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy+y^3 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} x^2y^2+y^6 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ -xy^5+y^7 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^3y+x^2y^3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y^2+y^6 \end{pmatrix}\right.$$

$$\begin{pmatrix} 0 \\ 0 \\ 8x^2y^3+3xy^5-9y^7 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^3y^2+xy^6 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^4y^2+x^2y^6 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y^6-3y^{10} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -xy^8+4y^{10} \end{pmatrix}\begin{pmatrix} y \\ y^2 \\ y^5 \end{pmatrix}\begin{pmatrix} y^3 \\ y^4 \\ y^7 \end{pmatrix}$$

$$\left.\begin{pmatrix} 0 \\ -2xy^2 \\ xy^5-3y^7 \end{pmatrix}\begin{pmatrix} 0 \\ xy^2+3y^4 \\ 4xy^5 \end{pmatrix}\begin{pmatrix} y^6 \\ 0 \\ -3y^{10} \end{pmatrix}\begin{pmatrix} 3y^4 \\ 0 \\ -4xy^6+3y^8 \end{pmatrix}\begin{pmatrix} -4xy^2 \\ 3y^5 \\ 0 \end{pmatrix}\begin{pmatrix} 4y^6 \\ 3y^7 \\ 0 \end{pmatrix}\begin{pmatrix} y^2 \\ 0 \\ y^6 \end{pmatrix}\begin{pmatrix} xy^2 \\ 0 \\ xy^6 \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ y^{3j+5} \end{pmatrix} : j \geq 2\right\}$$

$$C = \left(\langle x^3,\ xy^3\rangle + \mathbb{C}\{x, x^2, x^2y, xy+y^3, x^2y^2+y^6,\ y^{3j+5}, y^{3j+9} : j \geq 0\}\right)\frac{\partial}{\partial X}$$

$$+\left(\langle x^2,\ xy^6,\ xy^5-y^7\rangle + \mathbb{C}\{x, xy, y^3, xy^3, xy^4, y^6\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\langle x^5, x^3y^3, x^2y^7, x^4y^2+x^2y^6, 4x^2y^6-3xy^8\rangle + \mathbb{C}\{x, x^2, x^3, x^4, x^4y, xy^4, x^2y^4, x^2y^5, xy^7, xy+y^3,\right.$$

$$\left. x^2y+xy^3, x^3y+x^2y^3, x^2y^2+y^6, x^3y^2+xy^6, 8x^2y^3+3xy^5-9y^7, x^2y^6-3y^{10}, y^{3j+9}, y^{3j+13} : j \geq 0\}\right)\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y \\ y^2 \\ y^5 \end{pmatrix}\begin{pmatrix} y^3 \\ y^4 \\ y^7 \end{pmatrix}\begin{pmatrix} 0 \\ -2xy^2 \\ xy^5-3y^7 \end{pmatrix}\begin{pmatrix} 0 \\ xy^2+3y^4 \\ 4xy^5 \end{pmatrix}\begin{pmatrix} y^6 \\ 0 \\ -3y^{10} \end{pmatrix}\begin{pmatrix} 3y^4 \\ 0 \\ -4xy^6+3y^8 \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} -4xy^2 \\ 3y^5 \\ 0 \end{pmatrix}\begin{pmatrix} 4y^6 \\ 3y^7 \\ 0 \end{pmatrix}\begin{pmatrix} y^2 \\ 0 \\ y^6 \end{pmatrix}\begin{pmatrix} xy^2 \\ 0 \\ xy^6 \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ y^{3j+5} \end{pmatrix} : j \geq 2\right\}$$

$$f^*\mathfrak{m}_3 \cdot C = \left(\langle x^4,\ x^2y^3,\ xy^6\rangle + \mathbb{C}\{x^2, x^3, x^3y, xy^5, x^2y+xy^3,\ x^3y^2, x^2y^2+2xy^4+y^6,\ y^{3j+8},\right.$$

$$\left. y^{3j+9} : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle x^3,\ x^2y^3,\ xy^7, y^{10}\rangle + \mathbb{C}\{x^2, x^2y, xy^3, xy^6, x^2y^2+xy^4, xy^4+y^6, y^9\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\langle x^6,\ x^4y^3, x^2y^8,\ x^5y^2,\ x^3y^6, xy^{11}\rangle + \mathbb{C}\{x^2, x^3, x^4, x^5, x^5y, x^2y^4, x^3y^4, x^3y^5, x^2y^7, x^2y+xy^3,\right.$$

$$x^3y+x^2y^3, x^4y+x^3y^3, x^3y^2+xy^6, x^2y^5+xy^7, x^2y^2+2xy^4+y^6, x^3y^3+y^9, 11x^2y^6-6xy^8-9y^{10},$$

$$\left. x^4y^2+x^2y^6, 3xy^7+2y^9, xy^{10}, xy^9,\ y^{3j+12}, y^{3j+13} : j \geq 0\}\right)\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ xy^5 \end{pmatrix}\begin{pmatrix} xy^3 \\ xy^4 \\ xy^7 \end{pmatrix}\begin{pmatrix} xy^2 \\ 0 \\ xy^6 \end{pmatrix}\begin{pmatrix} 3xy^4 \\ 0 \\ -4x^2y^6+3xy^8 \end{pmatrix}\begin{pmatrix} 0 \\ 2x^2y^2 \\ 4xy^7 \end{pmatrix}\begin{pmatrix} 0 \\ 2xy^4 \\ 4x^2y^5 \end{pmatrix}\begin{pmatrix} -4x^2y^2 \\ 3xy^5 \\ 0 \end{pmatrix}\begin{pmatrix} x^2y^2 \\ 0 \\ x^2y^6 \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} xy^3+y^5 \\ 0 \\ xy^7+y^9 \end{pmatrix}\begin{pmatrix} xy^4+y^6 \\ xy^5+y^7 \\ xy^8+y^{10} \end{pmatrix}\begin{pmatrix} 0 \\ 4xy^5+3y^7 \\ 4x^2y^6+4xy^8 \end{pmatrix}\begin{pmatrix} xy^2+y^4 \\ xy^3+y^5 \\ xy^6+y^8 \end{pmatrix}\begin{pmatrix} 0 \\ y^8 \\ 0 \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ y^{3j+5} \end{pmatrix} : j \geq 2 \right\}$$

$$= \left(\langle x^3,\ x^2y^3,\ xy^5\rangle + \mathbb{C}\{x^2, x^2y+xy^3, x^2y^2+2xy^4+y^6,\ y^{3j+8}, y^{3j+9} : j \geq 0\}\right)\frac{\partial}{\partial X}$$

$$+\left(\langle x^3,\ x^2y^3,\ xy^6, y^8\rangle + \mathbb{C}\{x^2, x^2y, xy^3, x^2y^2+xy^4, xy^4+y^6\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\langle x^5, x^4y^3, x^2y^7, x^3y^4, xy^9\rangle + \mathbb{C}\{x^2, x^3, x^4, x^2y^4, x^2y+xy^3, x^3y+x^2y^3, x^4y+x^3y^3, x^3y^2+xy^6, x^2y^5+xy^7,\right.$$

$$\left. x^2y^2+2xy^4+y^6, x^3y^3+y^9, 11x^2y^6-6xy^8-9y^{10}, x^4y^2+x^2y^6, 3xy^7+2y^9, y^{3j+12}, y^{3j+13} : j \geq 0\}\right)\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ xy^5 \end{pmatrix}\begin{pmatrix} 0 \\ -xy^4 \\ 2xy^7 \end{pmatrix}\begin{pmatrix} 2xy^3 \\ 3xy^4 \\ 0 \end{pmatrix}\begin{pmatrix} 4y^5 \\ -7xy^4 \\ 0 \end{pmatrix}\begin{pmatrix} xy^2 \\ 0 \\ xy^6 \end{pmatrix}\begin{pmatrix} 3xy^4 \\ 0 \\ -4x^2y^6+3xy^8 \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} 4y^6+4xy^4 \\ 3y^7 \\ 0 \end{pmatrix}\begin{pmatrix} y^6+4xy^4 \\ 0 \\ -3y^{10} \end{pmatrix}\begin{pmatrix} -4xy^4 \\ 3y^7-3xy^5 \\ 0 \end{pmatrix}\begin{pmatrix} 2xy^4 \\ 0 \\ x^2y^2-3y^{10} \end{pmatrix}\begin{pmatrix} 0 \\ 3xy^4 \\ 4y^9 \end{pmatrix}\begin{pmatrix} 0 \\ 2xy^4 \\ 4x^2y^5 \end{pmatrix}\begin{pmatrix} 0 \\ 3xy^4 \\ -4x^3y^3 \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ y^{3j+5} \end{pmatrix} : j \geq 2 \right\}$$

We note that $f^*\mathfrak{m}_3 \cdot C \supseteq \mathfrak{m}_2^{11} \cdot \theta(f) \setminus \{y^{3j+13}\frac{\partial}{\partial X}, y^{3j+11}\frac{\partial}{\partial Z} : j \geq 0\} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+1} \\ 0 \\ y^{3j+5} \end{pmatrix} : j \geq 2\right\}$.

C is spanned by: $x\frac{\partial}{\partial X}, (xy+y^3)\frac{\partial}{\partial X}, xy^4\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, xy\frac{\partial}{\partial Y}, y^3\frac{\partial}{\partial Y}, xy^4\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, (xy+y^3)\frac{\partial}{\partial Z},$

$$xy^4\frac{\partial}{\partial Z}, (8x^2y^3+3xy^5-9y^7)\frac{\partial}{\partial Z}, \begin{pmatrix} y \\ y^2 \\ y^5 \end{pmatrix}\begin{pmatrix} y^3 \\ y^4 \\ y^7 \end{pmatrix}\begin{pmatrix} 0 \\ -2xy^2 \\ xy^5-3y^7 \end{pmatrix}\begin{pmatrix} 0 \\ xy^2+3y^4 \\ 4xy^5 \end{pmatrix}\begin{pmatrix} 3y^4 \\ 0 \\ -4xy^6+3y^8 \end{pmatrix}\begin{pmatrix} 0 \\ 3y^5 \\ 4xy^6 \end{pmatrix}\begin{pmatrix} y^2 \\ 0 \\ y^6 \end{pmatrix}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (xy+y^3)\frac{\partial}{\partial X}, xy^4\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (xy+y^3)\frac{\partial}{\partial Y}, xy^4\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, (xy+y^3)\frac{\partial}{\partial Z},$

$$xy^4\frac{\partial}{\partial Z}, \begin{pmatrix} x \\ xy \\ xy^4 \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^5 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ y^6 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ y^7 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2+3xy^2 \\ 4x^2y^3 \end{pmatrix}, \begin{pmatrix} 0 \\ xy+3y^3 \\ 4xy^4 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2+3y^4 \\ 4xy^5 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^3+3y^5 \\ 4xy^6 \end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3 \cdot C$ so that $C = T\mathscr{A}f$.

(A·8) (x, xy, y^4)

$$C = \begin{pmatrix} \mathfrak{m}_2\backslash\{y^{4j+1}, y^{4j+2}, xy^{4j+2} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^{4j+2}, y^{4j+3}, xy^{4j+3} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^3, xy^2, y^{4j+2} : j \geq 0\} \end{pmatrix} + \mathbb{C}\left\{ \begin{pmatrix} y^{4j+1} \\ y^{4j+2} \\ 0 \end{pmatrix} \begin{pmatrix} y^{4j+2} \\ y^{4j+3} \\ 0 \end{pmatrix} \begin{pmatrix} xy^{4j+2} \\ xy^{4j+3} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ xy^{4j+3} \\ 4y^{4j+6} \end{pmatrix} : j \geq 0 \right\}$$

$$= \left(\langle x^2\rangle + \mathbb{C}\{y^{4j+3}, y^{4j+4}, xy^{4j}, xy^{4j+1}, xy^{4j+3} : j \geq 0\}\right)\frac{\partial}{\partial X}$$

$$+ \left(\langle x^2\rangle + \mathbb{C}\{y^{4j+4}, y^{4j+5}, xy^{4j}, xy^{4j+1}, xy^{4j+2} : j \geq 0\}\right)\frac{\partial}{\partial Y}$$

$$+ \left(\langle x^2, xy^3\rangle + \mathbb{C}\{x, xy, y^{4j+4}, y^{4j+5}, y^{4j+7} : j \geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{ \begin{pmatrix} y^{4j+1} \\ y^{4j+2} \\ 0 \end{pmatrix} \begin{pmatrix} y^{4j+2} \\ y^{4j+3} \\ 0 \end{pmatrix} \begin{pmatrix} xy^{4j+2} \\ xy^{4j+3} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ xy^{4j+3} \\ 4y^{4j+6} \end{pmatrix} : j \geq 0 \right\}$$

Then $f^*\mathfrak{m}_3 \cdot C = \left(\langle x^2\rangle + \mathbb{C}\{y^{4j+7}, y^{4j+8}, xy^{4j+4}, xy^{4j+5}, xy^{4j+3} : j \geq 0\}\right)\frac{\partial}{\partial X}$

$$+ \left(\langle x^2\rangle + \mathbb{C}\{y^{4j+8}, y^{4j+9}, xy^{4j+4}, xy^{4j+5}, xy^{4j+6} : j \geq 0\}\right)\frac{\partial}{\partial Y} + \left(\langle x^2, xy^4\rangle + \mathbb{C}\{y^{4j+8}, y^{4j+9}, y^{4j+11} : j \geq 0\}\right)\frac{\partial}{\partial Z}$$

$$+ \mathbb{C}\left\{ \begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix} \begin{pmatrix} xy^2 \\ xy^3 \\ 0 \end{pmatrix} \begin{pmatrix} y^{4j+1} \\ y^{4j+2} \\ 0 \end{pmatrix} \begin{pmatrix} y^{4j+2} \\ y^{4j+3} \\ 0 \end{pmatrix} \begin{pmatrix} xy^{4j+2} \\ xy^{4j+3} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ xy^{4j+3} \\ 4y^{4j+6} \end{pmatrix} : j \geq 1 \right\}$$

C is spanned by: $x\frac{\partial}{\partial X}$, $xy\frac{\partial}{\partial X}$, $y^3\frac{\partial}{\partial X}$, $y^4\frac{\partial}{\partial X}$, $x\frac{\partial}{\partial Y}$, $xy\frac{\partial}{\partial Y}$, $xy^2\frac{\partial}{\partial Y}$, $y^4\frac{\partial}{\partial Y}$, $y^5\frac{\partial}{\partial Y}$, $x\frac{\partial}{\partial Z}$,

$$xy\frac{\partial}{\partial Z},\ xy^3\frac{\partial}{\partial Z},\ y^4\frac{\partial}{\partial Z},\ y^5\frac{\partial}{\partial Z},\ y^7\frac{\partial}{\partial Z},\ \begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix} \begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ xy^3 \\ 4y^6 \end{pmatrix}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}$, $xy\frac{\partial}{\partial X}$, $y^4\frac{\partial}{\partial X}$, $x\frac{\partial}{\partial Y}$, $xy\frac{\partial}{\partial Y}$, $y^4\frac{\partial}{\partial Y}$, $x\frac{\partial}{\partial Z}$, $xy\frac{\partial}{\partial Z}$,

$$y^4\frac{\partial}{\partial Z},\ \begin{pmatrix} x \\ xy \\ 0 \end{pmatrix} \begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix} \begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix} \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix} \begin{pmatrix} y^4 \\ y^5 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x^2 \\ 4xy^3 \end{pmatrix} \begin{pmatrix} 0 \\ xy \\ 4y^4 \end{pmatrix} \begin{pmatrix} 0 \\ xy^2 \\ 4y^5 \end{pmatrix} \begin{pmatrix} 0 \\ xy^3 \\ 4y^6 \end{pmatrix} \begin{pmatrix} 0 \\ xy^4 \\ 4y^7 \end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3 \cdot C$ so that $C = T\mathscr{A}f$.

(A·9) $(x, xy+xy^3, y^4)$

$$T\mathscr{A}f \qquad \begin{pmatrix} \mathfrak{m}_2\backslash\{y^{4j+1}, y^{4j+2}, y^{4j+3} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y^{4j+1}, y^{4j+2}, y^{4j+3} : j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^4, y^3, xy^2\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y^{4j+3}-y^{4j+5} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ y^{4j+5}+y^{4j+7} \\ 0 \end{pmatrix}\begin{pmatrix} y^{4j+2} \\ y^{4j+3}-y^{4j+7} \\ 0 \end{pmatrix}\begin{pmatrix} y^{4j+1} \\ y^{4j+2} \\ 0 \end{pmatrix} : j \geq 0\right\}$$

$$C = \left(\langle x\rangle + \mathbb{C}\{y^{4j+4}, y^{4j}(y^3-y^5) : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle x\rangle + \mathbb{C}\{y^{4j+4}, y^{4j}(y^5+y^7) : j \geq 0\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\langle x^2, xy^3, y^4\rangle + \mathbb{C}\{x, xy\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix}\begin{pmatrix} xy^2 \\ xy^3 \\ 0 \end{pmatrix}\begin{pmatrix} y^{4j+2} \\ y^{4j+3}-y^{4j+7} \\ 0 \end{pmatrix}\begin{pmatrix} y^{4j+1} \\ y^{4j+2} \\ 0 \end{pmatrix} : j \geq 0\right\}$$

Then $f^*\mathfrak{m}_3 \cdot C = \left(\langle x^2, xy^3\rangle + \mathbb{C}\{y^{4j+8}, y^{4j+4}(y^3-y^5) : j \geq 0\}\right)\frac{\partial}{\partial X}$

$$+\left(\langle x^2, xy^4\rangle + \mathbb{C}\{y^{4j+8}, y^{4j+4}(y^5+y^7) : j \geq 0\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\langle x^3, x^2y^3, xy^4, y^8\rangle + \mathbb{C}\{x^2, x^2y, x^2y^2\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix}\begin{pmatrix} xy^2 \\ xy^3 \\ 0 \end{pmatrix}\begin{pmatrix} y^{4j+2} \\ y^{4j+3}-y^{4j+7} \\ 0 \end{pmatrix}\begin{pmatrix} y^{4j+1} \\ y^{4j+2} \\ 0 \end{pmatrix} : j \geq 1\right\}$$

C is spanned by: $x\frac{\partial}{\partial X}$, $xy^2\frac{\partial}{\partial X}$, $y^4\frac{\partial}{\partial X}$, $y^4(y^3-y^5)\frac{\partial}{\partial X}$, $x\frac{\partial}{\partial Y}$, $xy\frac{\partial}{\partial Y}$, $xy^2\frac{\partial}{\partial Y}$, $xy^3\frac{\partial}{\partial Y}$,

$y^4\frac{\partial}{\partial Y}$, $y^4(y^5+y^7)\frac{\partial}{\partial Y}$, $x\frac{\partial}{\partial Z}$, $xy\frac{\partial}{\partial Z}$, $xy^3\frac{\partial}{\partial Z}$, $y^4\frac{\partial}{\partial Z}$, $y^5\frac{\partial}{\partial Z}$, $y^6\frac{\partial}{\partial Z}$, $y^7\frac{\partial}{\partial Z}$, $\begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}\begin{pmatrix} y^2 \\ y^3-y^7 \\ 0 \end{pmatrix}$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (xy+xy^3)\frac{\partial}{\partial X}, y^4\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (xy+xy^3)\frac{\partial}{\partial Y}, y^4\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, (xy+xy^3)\frac{\partial}{\partial Z}$,

$y^4\frac{\partial}{\partial Z}, \begin{pmatrix} x \\ xy \\ xy^2 \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ y^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2+3xy^2 \\ 2x^2y+4xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ xy+3y^3 \\ 2xy^2+4y^4 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2+3y^4 \\ 2xy^3+4y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^3+3y^5 \\ 2xy^4+4y^6 \end{pmatrix}$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3 \cdot C$ so that $C = T\mathscr{A}f$.

(A·10) $\qquad (x, xy-y^4, y^6)$

$$T\mathscr{A}f \qquad \begin{pmatrix} \mathfrak{m}_2\backslash\{y^2, xy^2, x^2y^2, y^8, x^\ell y^{3j+1} : j, \ell \geq 0\} \\ \mathfrak{m}_2\backslash\{y, y^3, xy^3, x^2y^3, y^9, x^\ell y^{3j+2} : j, \ell \geq 0\} \\ \mathfrak{m}_2\backslash\{y^2, xy^2, y^3, x^2y^2, xy^3, x^2y^3, y^5, xy^5, x^2y^5 \\ y^8, xy^8, y^{11}, x^\ell y^{3j+1} : j, \ell \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} x^{\ell}y^{3j+1}(x-y^3) \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} x^2y^2+y^8 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^{\ell}y^{3j+2}(x-y^3) \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^2y^3+y^9 \\ 0 \end{pmatrix}\right.$$

$$\begin{pmatrix} 0 \\ 0 \\ x^{\ell}y^{3j+7}(x-y^3) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^{\ell+3}y^4(x-y^3) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^{\ell}y(x-y^3) \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y^2+9y^8 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ xy^5+4y^8 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 4x^3y^2+9y^{11} \end{pmatrix}$$

$$\left.\begin{pmatrix} 0 \\ 0 \\ x^2y^5+y^{11} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 4xy^8-y^{11} \end{pmatrix}\begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}\begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}\begin{pmatrix} xy^2 \\ xy^3 \\ 0 \end{pmatrix}\begin{pmatrix} y^8 \\ y^9 \\ 0 \end{pmatrix}\begin{pmatrix} 2y^8 \\ 0 \\ 3y^{11} \end{pmatrix}\begin{pmatrix} -xy^2 \\ 0 \\ 6y^8 \end{pmatrix}\begin{pmatrix} y^{3j+4} \\ y^{3j+5} \\ 0 \end{pmatrix}\begin{pmatrix} y^{3j+4} \\ 0 \\ 2y^{3j+7} \end{pmatrix} : \ell, j \geq 0\right\}$$

$$C=\left(\mathbb{C}\{x^{\ell}y^{3j}: \ell+j>0\}+\mathbb{C}\{x^2y^2+y^8, x^{\ell}y^{3j+1}(x-y^3), x^{\ell+3}y^2, x^{\ell}y^5, x^{\ell+1}y^8, x^{\ell}y^{3j+11}: \ell, j \geq 0\}\right)\frac{\partial}{\partial X}$$

$$+\left(\mathbb{C}\{x^{\ell}y^{3j+1}:\ell+j>0\}+\mathbb{C}\{x^{\ell+1}, x^{\ell+3}y^3, x^2y^3+y^9, x^{\ell}y^{3j+2}(x-y^3), x^{\ell}y^6, x^{\ell+1}y^9, x^{\ell}y^{3j+12}:\ell, j \geq 0\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\mathbb{C}\{ x^{\ell}y(x-y^3), x^{\ell+1}, x^{\ell+3}y^2, x^{\ell+3}y^3, x^{\ell+3}y^5, x^{\ell}y^{3j+6}, x^{\ell+2}y^8, x^{\ell+1}y^{11}, x^{\ell}y^{3j+14}: \ell, j \geq 0\}\right.$$

$$\left.+\mathbb{C}\{x^2y^2+9y^8, xy^5+4y^8, 4x^3y^2+9y^{11}, x^2y^5+y^{11}, 4xy^8-y^{11}, x^{\ell}y^{3j+1}(x-y^3): \ell+j \geq 4\}\right)\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}\begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}\begin{pmatrix} xy^2 \\ xy^3 \\ 0 \end{pmatrix}\begin{pmatrix} y^8 \\ y^9 \\ 0 \end{pmatrix}\begin{pmatrix} 2y^8 \\ 0 \\ 3y^{11} \end{pmatrix}\begin{pmatrix} -xy^2 \\ 0 \\ 6y^8 \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ 2y^{3j+4} \end{pmatrix} : j \geq 1\right\}$$

$$f^*\mathfrak{m}_3\cdot C = \left(\mathbb{C}\{x^{\ell+1}y^{3j}, x^{\ell}y^{3j+6}: \ell+j>0\}+\mathbb{C}\{x^2y^2+y^8, x^{\ell+1}y^{3j+1}(x-y^3), x^{\ell}y^{3j+4}(x-y^3), x^{\ell+3}y^2,\right.$$

$$\left. x^{\ell+1}y^5, x^{\ell+1}y^8, x^{\ell}y^{3j+11}: \ell, j \geq 0\}\right)\frac{\partial}{\partial X}+$$

$$+\left(\mathbb{C}\{x^{\ell+1}y^{3j+1}, x^{\ell}y^{3j+7}: \ell+j>0\}+\mathbb{C}\{x^{\ell+2}, x^{\ell+3}y^3, x^2y^3+y^9,\right.$$

$$\left. x^{\ell+1}y^{3j+2}(x-y^3), x^{\ell}y^{3j+5}(x-y^3), x^{\ell+1}y^6, x^{\ell+1}y^9, x^{\ell}y^{3j+12}: \ell, j \geq 0\}\right)\frac{\partial}{\partial Y}+$$

$$\mathbb{C}\{x^{\ell+1}y(x-y^3), x^{\ell+3}y^4(x-y^3), x^{\ell}y^{3j+7}(x-y^3), x^{\ell+2}, x^{\ell+4}y^2, x^{\ell+3}y^3, x^{\ell+3}y^5, x^{\ell+1}y^{3j+6}, y^{3j+12}, x^{\ell+2}y^8,$$

$$x^{\ell+1}y^{11}, x^{\ell}y^{3j+14}:\ell, j \geq 0\}+\mathbb{C}\{x^2y^2-2xy^5+y^8, x^3y^2+9xy^8, x^2y^5+4xy^8, 4xy^8-y^{11}\}\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix} xy \\ xy^2 \\ 0 \end{pmatrix}\begin{pmatrix} xy^2 \\ xy^3 \\ 0 \end{pmatrix}\begin{pmatrix} y^4 \\ y^5 \\ 0 \end{pmatrix}\begin{pmatrix} y^5 \\ y^6 \\ 0 \end{pmatrix}\begin{pmatrix} y^6 \\ y^7 \\ 0 \end{pmatrix}\begin{pmatrix} y^8 \\ y^9 \\ 0 \end{pmatrix}\begin{pmatrix} 2y^8 \\ 0 \\ 3y^{11} \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ 2y^{3j+4} \end{pmatrix} : j \geq 2\right\}$$

Note that $f^*\mathfrak{m}_3\cdot C \supseteq \begin{pmatrix} \mathfrak{m}_2^{12}\setminus\{x^\ell y^{3j+1} : 3j+\ell \geq 11\} \\ \mathfrak{m}_2^{12}\setminus\{x^\ell y^{3j+2} : 3j+\ell \geq 10\} \\ \mathfrak{m}_2^{12}\setminus\{x^\ell y^{3j+1} : 3j+\ell \geq 11\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+1} \\ y^{3j+2} \\ 0 \end{pmatrix}, \begin{pmatrix} y^{3j+1} \\ 0 \\ 2y^{3j+4} \end{pmatrix} : j \geq 2\right\}$

C is spanned by: $x\frac{\partial}{\partial X}, (xy-y^4)\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, y^6\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, xy\frac{\partial}{\partial Y}, (xy^2-y^5)\frac{\partial}{\partial Y}, y^4\frac{\partial}{\partial Y}, y^6\frac{\partial}{\partial Y},$

$x\frac{\partial}{\partial Z}, (xy-y^4)\frac{\partial}{\partial Z}, x^3y^2\frac{\partial}{\partial Z}, (xy^5+4y^8)\frac{\partial}{\partial Z}, y^6\frac{\partial}{\partial Z}, y^9\frac{\partial}{\partial Z}, \begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} -xy^2 \\ 0 \\ 6y^8 \end{pmatrix}\begin{pmatrix} y^4 \\ 0 \\ 2y^7 \end{pmatrix}$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (xy-y^4)\frac{\partial}{\partial X}, y^6\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (xy-y^4)\frac{\partial}{\partial Y}, y^6\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, (xy-y^4)\frac{\partial}{\partial Z},$

$y^6\frac{\partial}{\partial Z}, \begin{pmatrix} x \\ xy \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ y^4 \\ 0 \end{pmatrix}, \begin{pmatrix} y^4 \\ y^5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2-4xy^3 \\ 6xy^5 \end{pmatrix}, \begin{pmatrix} 0 \\ xy-4y^4 \\ 6y^6 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2-4y^5 \\ 6y^7 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^3-4y^6 \\ 6y^8 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^4-4y^7 \\ 6y^9 \end{pmatrix}$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3\cdot C$ so that $C = T\mathscr{A}f$.

(A·11) (x, x^2y+y^3, xy^2)

Let $C = \begin{pmatrix} \mathfrak{m}_2\setminus\{y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2\setminus\{y, xy, y^2\} \\ \mathfrak{m}_2\setminus\{xy, y^2, y^{3j+1} : j \geq 0\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 0\right\}$

$$= \left(\langle x\rangle + \mathbb{C}\{y^{3j+1}, y^{3j+3} : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle x^2, xy^2, y^3\rangle + \mathbb{C}\{x\}\right)\frac{\partial}{\partial Y}$$

$$+ \left(\langle x^2, xy^2\rangle + \mathbb{C}\{x, y^{3j+3}, y^{3j+5} : j \geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 0\right\}$$

Then $f^*\mathfrak{m}_3\cdot C = \left(\langle x^2, xy^3\rangle + \mathbb{C}\{xy, y^{3j+4}, y^{3j+6} : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle x^3, x^2y^2, xy^3, y^6\rangle + \mathbb{C}\{x^2\}\right)\frac{\partial}{\partial Y}$

$$+ \left(\langle x^3, x^2y^2, xy^5\rangle + \mathbb{C}\{x^2, xy^3, y^{3j+6}, y^{3j+8} : j \geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}\begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1\right\}$$

$$= \begin{pmatrix} m_2^2\backslash\{y^2, y^3, xy^2, y^{3j+2} : j \geq 1\} \\ m_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, y^4, y^5\} \\ m_2^2\backslash\{xy, y^2, x^2y, xy^2, y^3, y^4, xy^4, y^5, y^{3j+1} : j \geq 1\} \end{pmatrix} + \mathbb{C}\left\{ \begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix} \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1 \right\}$$

C is spanned by: $x\frac{\partial}{\partial X}, y\frac{\partial}{\partial X}, xy^2\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, x^2y\frac{\partial}{\partial Y}, xy^2\frac{\partial}{\partial Y}, y^3\frac{\partial}{\partial Y}, y^4\frac{\partial}{\partial Y}, y^5\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z},$

$$x^2y\frac{\partial}{\partial Z},\ xy^2\frac{\partial}{\partial Z},\ y^3\frac{\partial}{\partial Z},\ y^5\frac{\partial}{\partial Z},\ \begin{pmatrix} y^2 \\ 0 \\ y^4 \end{pmatrix}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (x^2y+y^3)\frac{\partial}{\partial X}, xy^2\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (x^2y+y^3)\frac{\partial}{\partial Y}, xy^2\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, (x^2y+y^3)\frac{\partial}{\partial Z},$

$$xy^2\frac{\partial}{\partial Z}, \begin{pmatrix} x \\ 2x^2y \\ xy^2 \end{pmatrix}, \begin{pmatrix} y \\ 2xy^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} y^2 \\ 2xy^3 \\ y^4 \end{pmatrix}, \begin{pmatrix} y^3 \\ 2xy^4 \\ y^5 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3+3xy^2 \\ 2x^2y \end{pmatrix}, \begin{pmatrix} 0 \\ x^2y+3y^3 \\ 2xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2y^2+3y^4 \\ 2xy^3 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2y^3+3y^5 \\ 2xy^4 \end{pmatrix}$$

Rewriting these modulo $f^*m_3 \cdot C$ we get: $x\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, xy^2\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (x^2y+y^3)\frac{\partial}{\partial Y}, xy^2\frac{\partial}{\partial Y},$

$$x\frac{\partial}{\partial Z},\ (x^2y+y^3)\frac{\partial}{\partial Z}, xy^2\frac{\partial}{\partial Z},\ x^2y\frac{\partial}{\partial Y},\ y\frac{\partial}{\partial X},\ \begin{pmatrix} y^2 \\ 0 \\ y^4 \end{pmatrix},\ y^5\frac{\partial}{\partial Z},\ x^2y\frac{\partial}{\partial Z},\ y^4\frac{\partial}{\partial Y},\ y^5\frac{\partial}{\partial Y}$$

Hence $C = T\mathscr{A}f + f^*m_3 \cdot C$ so that $C = T\mathscr{A}f$.

(A·12) $$f = (x, y^3, xy^2)$$

$$\text{Let } C = \begin{pmatrix} m_2\backslash\{y^{3j+2} : j \geq 0\} \\ m_2\backslash\{y^2, x^jy : j \geq 0\} \\ m_2\backslash\{xy, y^2, y^{3j+1} : j \geq 0\} \end{pmatrix} + \mathbb{C}\left\{ \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 0 \right\}$$

$$= \left(\langle x\rangle + \mathbb{C}\{y^{3j+1}, y^{3j+3} : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle xy^2, y^3\rangle + \mathbb{C}\{x^j : j \geq 1\}\right)\frac{\partial}{\partial Y}$$

$$+ \left(\langle x^2, xy^2\rangle + \mathbb{C}\{x, y^{3j+3}, y^{3j+5} : j \geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{ \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 0 \right\}$$

Then $f^*m_3 \cdot C = \left(\langle x^2, xy^3\rangle + \mathbb{C}\{xy, y^{3j+4}, y^{3j+6} : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle x^2y^2, xy^3, y^6\rangle + \mathbb{C}\{x^j : j \geq 2\}\right)\frac{\partial}{\partial Y}$

$$+\left(\langle x^3, x^2y^2, xy^5\rangle + \mathbb{C}\{x^2, xy^3, y^{3j+6}, y^{3j+8} : j \geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}, \begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1\right\}$$

$$\begin{pmatrix} \mathfrak{m}_2^2 \backslash \{y^3, xy^2, y^{3j+2} : j \geq 0\} \\ \mathfrak{m}_2^2 \backslash \{y^2, xy^2, y^3, y^4, y^5, x^j y : j \geq 1\} \\ \mathfrak{m}_2^2 \backslash \{xy, y^2, x^2y, xy^2, y^3, xy^4, y^5, y^{3j+1} : j \geq 1\} \end{pmatrix} + \mathbb{C}\left\{\begin{pmatrix} xy^2 \\ 0 \\ xy^4 \end{pmatrix}\begin{pmatrix} y^{3j+2} \\ 0 \\ y^{3j+4} \end{pmatrix} : j \geq 1\right\}$$

C is spanned by : $x\frac{\partial}{\partial X}$, $y\frac{\partial}{\partial X}$, $xy^2\frac{\partial}{\partial X}$, $y^3\frac{\partial}{\partial X}$, $x\frac{\partial}{\partial Y}$, $xy^2\frac{\partial}{\partial Y}$, $y^3\frac{\partial}{\partial Y}$, $y^4\frac{\partial}{\partial Y}$, $y^5\frac{\partial}{\partial Y}$, $x\frac{\partial}{\partial Z}$,

$$xy^2\frac{\partial}{\partial Z},\ x^2y\frac{\partial}{\partial Z},\ y^3\frac{\partial}{\partial Z},\ y^5\frac{\partial}{\partial Z},\ \begin{pmatrix} y^2 \\ 0 \\ y^4 \end{pmatrix}$$

$T\mathscr{A}f$ is spanned by : $x\frac{\partial}{\partial X}$, $xy^2\frac{\partial}{\partial X}$, $y^3\frac{\partial}{\partial X}$, $x\frac{\partial}{\partial Y}$, $xy^2\frac{\partial}{\partial Y}$, $y^3\frac{\partial}{\partial Y}$, $x\frac{\partial}{\partial Z}$, $xy^2\frac{\partial}{\partial Z}$,

$$y^3\frac{\partial}{\partial Z},\ \begin{pmatrix} y \\ 0 \\ y^3 \end{pmatrix},\ \begin{pmatrix} y^2 \\ 0 \\ y^4 \end{pmatrix},\ \begin{pmatrix} y^3 \\ 0 \\ y^5 \end{pmatrix},\ \begin{pmatrix} 0 \\ 3xy^2 \\ 2x^2y \end{pmatrix},\ \begin{pmatrix} 0 \\ 3y^4 \\ 2xy^3 \end{pmatrix},\ \begin{pmatrix} 0 \\ 3y^5 \\ 2xy^4 \end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3 \cdot C$ so that $C = T\mathscr{A}f$.

(A·13) $\qquad f = (x, y^3, x^2y)$

Let $C =$

$$\begin{pmatrix} \mathfrak{m}_2 \backslash \{xy^{3j+1}, y^{3j+1} : j \geq 0\} \\ \mathfrak{m}_2 \backslash \{xy, y^2, y^{3j+1} : j \geq 0\} \\ \mathfrak{m}_2 \backslash \{xy, xy^{3j+2}, x^2y^{3j+2}, y^{3j+1}, y^{3j+2} : j \geq 0\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} 0 \\ 3y^{3j+4} \\ x^2y^{3j+2} \end{pmatrix}\begin{pmatrix} xy^{3j+1} \\ 0 \\ 2x^2y^{3j+2} \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ 2xy^{3j+2} \end{pmatrix} : j \geq 0\right\}$$

$$= \left(\langle x^2\rangle + \mathbb{C}\{y^{3j+2}, y^{3j+3}, xy^{3j}, xy^{3j+2} : j \geq 0\}\right)\frac{\partial}{\partial X} + \left(\langle x^2, xy^2\rangle + \mathbb{C}\{x, y^{3j+3}, y^{3j+5} : j \geq 0\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\langle x^3\rangle + \mathbb{C}\{y^{3j+3}, xy^{3j}, xy^{3j+4}, x^2y^{3j}, x^2y^{3j+1} : j \geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} 0 \\ 3y^{3j+4} \\ x^2y^{3j+2} \end{pmatrix}\begin{pmatrix} xy^{3j+1} \\ 0 \\ 2x^2y^{3j+2} \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ 2xy^{3j+2} \end{pmatrix} : j \geq 0\right\}$$

$$f^*\mathfrak{m}_3\cdot C = \left(\langle x^3, x^2y^3\rangle + \mathbb{C}\{x^2, x^2y^2, y^{3j+5}, y^{3j+6}, xy^{3j+3}, xy^{3j+2} : j \geq 0\}\right)\frac{\partial}{\partial X} +$$

$$\left(\langle x^3, x^2y^2, xy^5\rangle + \mathbb{C}\{x^2, xy^3, y^{3j+6}, y^{3j+8} : j \geq 0\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\langle x^4, x^3y^3\rangle + \mathbb{C}\{x^3, x^3y, y^{3j+6}, xy^{3j+3}, xy^{3j+7}, x^2y^{3j}, x^2y^{3j+4} : j \geq 0\}\right)\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix} 0 \\ 3xy^4 \\ x^3y^2 \end{pmatrix}\begin{pmatrix} -x^2y \\ 6xy^4 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 3y^{3j+7} \\ x^2y^{3j+5} \end{pmatrix}\begin{pmatrix} xy^{3j+1} \\ 0 \\ 2x^2y^{3j+2} \end{pmatrix}\begin{pmatrix} y^{3j+4} \\ 0 \\ 2xy^{3j+5} \end{pmatrix} : j \geq 0\right\}$$

C is spanned by: $x\frac{\partial}{\partial X}$, $x^2y\frac{\partial}{\partial X}$, $y^2\frac{\partial}{\partial X}$, $y^3\frac{\partial}{\partial X}$, $x\frac{\partial}{\partial Y}$, $x^2y\frac{\partial}{\partial Y}$, $xy^2\frac{\partial}{\partial Y}$, $y^3\frac{\partial}{\partial Y}$, $y^5\frac{\partial}{\partial Y}$, $x\frac{\partial}{\partial Z}$,

$$x^2y\frac{\partial}{\partial Z},\ xy^2\frac{\partial}{\partial Z},\ y^3\frac{\partial}{\partial Z},\ xy^4\frac{\partial}{\partial Z},\ \begin{pmatrix} 0 \\ 3y^4 \\ x^2y^2 \end{pmatrix},\ \begin{pmatrix} y \\ 0 \\ 2xy^2 \end{pmatrix}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}$, $x^2y\frac{\partial}{\partial X}$, $y^3\frac{\partial}{\partial X}$, $x\frac{\partial}{\partial Y}$, $x^2y\frac{\partial}{\partial Y}$, $y^3\frac{\partial}{\partial Y}$, $x\frac{\partial}{\partial Z}$, $x^2y\frac{\partial}{\partial Z}$,

$$y^3\frac{\partial}{\partial Z},\ \begin{pmatrix} x \\ 0 \\ 2x^2y \end{pmatrix},\ \begin{pmatrix} y \\ 0 \\ 2xy^2 \end{pmatrix},\ \begin{pmatrix} y^2 \\ 0 \\ 2xy^3 \end{pmatrix},\ \begin{pmatrix} y^3 \\ 0 \\ 2xy^4 \end{pmatrix},\ \begin{pmatrix} 0 \\ 3xy^2 \\ x^3 \end{pmatrix},\ \begin{pmatrix} 0 \\ 3y^3 \\ x^2y \end{pmatrix},\ \begin{pmatrix} 0 \\ 3y^4 \\ x^2y^2 \end{pmatrix},\ \begin{pmatrix} 0 \\ 3y^5 \\ x^2y^3 \end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3\cdot C$ so that $C = T\mathscr{A}f$.

(A·14) $$f = (x, y^3, x^2y + y^4)$$

Let $$C = \begin{pmatrix} \mathfrak{m}_2\backslash\{x^\ell y^{3j+1} : \ell, j \geq 0\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2+y^3) : \ell+j \geq 0\} \\ \mathfrak{m}_2\backslash\{x^\ell y^{3j+1} : \ell, j \geq 0\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2+y^3) : \ell+j \geq 0\} \\ \mathfrak{m}_2\backslash\{y, xy, x^\ell y^{3j+2} : \ell, j \geq 0\} + \mathbb{C}\{x^\ell y^{3j+2}(x^2+y^3) : \ell+j \geq 1\} \end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix} 0 \\ 3y^4 \\ x^2y^2+4y^5 \end{pmatrix}\begin{pmatrix} -y^{3j+4} \\ 2xy^{3j+4} \\ 0 \end{pmatrix}\begin{pmatrix} xy^{3j+4} \\ 2y^{3j+7} \\ 0 \end{pmatrix}\begin{pmatrix} xy^{3j+1} \\ 0 \\ 2x^2y^{3j+2} \end{pmatrix}\begin{pmatrix} y^{3j+1} \\ 0 \\ 2xy^{3j+2} \end{pmatrix} : j \geq 0\right\}$$

$$= \left(\mathbb{C}\{x^\ell y^{3j}, x^\ell y^{3j+2}, x^\ell y^{3j+1}(x^2+y^3), y(x^2+y^3), y^2 : \ell+j > 0\}\right)\frac{\partial}{\partial X}$$

$$+\left(\mathbb{C}\{x^\ell y^{3j}, x^\ell y^{3j+2}, x^\ell y^{3j+1}(x^2+y^3), y(x^2+y^3) : \ell+j > 0\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\mathbb{C}\{x^{\ell}y^{3j}, x^{\ell+1}y^{3j+1}, x^{\ell}y^{3j+2}(x^2+y^3): \ell+j>0\} + \mathbb{C}\{y^{3j+4}: j\geq 0\}\right)\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix}0\\3y^4\\x^2y^2+4y^5\end{pmatrix}\begin{pmatrix}-y^{3j+4}\\2xy^{3j+4}\\0\end{pmatrix}\begin{pmatrix}xy^{3j+4}\\2y^{3j+7}\\0\end{pmatrix}\begin{pmatrix}xy^{3j+1}\\0\\2x^2y^{3j+2}\end{pmatrix}\begin{pmatrix}y^{3j+1}\\0\\2xy^{3j+2}\end{pmatrix}: j\geq 0\right\}$$

Then $f^*\mathfrak{m}_3\cdot C=\left(\mathbb{C}\{x^{\ell}y^{3j}, x^{\ell}y^{3j+2}, x^{\ell}y^{3j+1}(x^2+y^3), xy(x^2+y^3), y^4(x^2+y^3), xy^2, y^5: \ell+j>1\}\right)\frac{\partial}{\partial X}$

$$+\left(\mathbb{C}\{x^{\ell}y^{3j}, x^{\ell}y^{3j+2}, x^{\ell}y^{3j+1}(x^2+y^3), xy(x^2+y^3), y^4(x^2+y^3): \ell+j>1\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\mathbb{C}\{x^{\ell}y^{3j}, x^{\ell+1}y^{3j+1}, x^{\ell}y^{3j+2}(x^2+y^3):\ell+j>1\}+\mathbb{C}\{xy^2(x^2+y^3), y^5(x^2+y^3), xy^4, y^{3j+7}:j\geq 0\}\right)\frac{\partial}{\partial Z}$$

$$+\mathbb{C}\left\{\begin{pmatrix}0\\xy^4\\xy^5\end{pmatrix}\begin{pmatrix}x^2y\\0\\2x^3y^2\end{pmatrix}\begin{pmatrix}-y^{3j+7}\\2xy^{3j+7}\\0\end{pmatrix}\begin{pmatrix}xy^{3j+4}\\2y^{3j+7}\\0\end{pmatrix}\begin{pmatrix}xy^{3j+1}\\0\\2x^2y^{3j+2}\end{pmatrix}\begin{pmatrix}y^{3j+4}\\0\\2xy^{3j+5}\end{pmatrix}: j\geq 0\right\}$$

$$=\begin{pmatrix}\mathfrak{m}_2^2\backslash\{y^2, y^3, x^{\ell}y^{3j+1}: \ell+j\geq 1\} + \mathbb{C}\{x^{\ell}y^{3j+1}(x^2+y^3): \ell+j\geq 1\}\\ \mathfrak{m}_2\backslash\{y^2, xy^2, y^3, y^5, x^{\ell}y^{3j+1}: \ell+j\geq 1\} + \mathbb{C}\{x^{\ell}y^{3j+1}(x^2+y^3): \ell+j\geq 1\}\\ \mathfrak{m}_2\backslash\{xy, x^2y, y^3, y^4, x^{\ell}y^{3j+2}:\ell, j\geq 0\}+\mathbb{C}\{x^{\ell}y^{3j+2}(x^2+y^3):\ell\geq 2 \text{ or } j\geq 1\}\end{pmatrix}$$

$$+\mathbb{C}\left\{\begin{pmatrix}0\\xy^4\\xy^5\end{pmatrix}\begin{pmatrix}x^2y\\0\\2x^3y^2\end{pmatrix}\begin{pmatrix}-y^{3j+7}\\2xy^{3j+7}\\0\end{pmatrix}\begin{pmatrix}xy^{3j+4}\\2y^{3j+7}\\0\end{pmatrix}\begin{pmatrix}xy^{3j+1}\\0\\2x^2y^{3j+2}\end{pmatrix}\begin{pmatrix}y^{3j+4}\\0\\2xy^{3j+5}\end{pmatrix}: j\geq 0\right\}$$

C is spanned b y: $x\frac{\partial}{\partial X}, (x^2y+y^4)\frac{\partial}{\partial X}, y^2\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (x^2y+y^4)\frac{\partial}{\partial Y}, xy^2\frac{\partial}{\partial Y}, y^3\frac{\partial}{\partial Y},$

$$y^5\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, x^2y\frac{\partial}{\partial Z} y^3\frac{\partial}{\partial Z}, y^4\frac{\partial}{\partial Z}, \begin{pmatrix}y\\0\\2xy^2\end{pmatrix}, \begin{pmatrix}0\\3y^4\\x^2y^2+4y^5\end{pmatrix}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (x^2y+y^4)\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (x^2y+y^4)\frac{\partial}{\partial Y}, y^3\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, (x^2y+y^4)\frac{\partial}{\partial Z},$

$$y^3\frac{\partial}{\partial Z}, \begin{pmatrix}x\\0\\2x^2y\end{pmatrix}, \begin{pmatrix}y\\0\\2xy^2\end{pmatrix}, \begin{pmatrix}y^2\\0\\2xy^3\end{pmatrix}, \begin{pmatrix}y^3\\0\\2xy^4\end{pmatrix}, \begin{pmatrix}0\\3xy^2\\x^3+4xy^3\end{pmatrix}, \begin{pmatrix}0\\3y^3\\x^2y+4y^4\end{pmatrix}, \begin{pmatrix}0\\3y^4\\x^2y^2+4y^5\end{pmatrix}, \begin{pmatrix}0\\3y^5\\x^2y^3+4y^6\end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3\cdot C$ so that $C = T\mathscr{A}f$.

(A·15) $$f = (x, x^2y-y^3, x^2y^2-y^4)$$

$$\text{Let } C = \begin{pmatrix} \mathfrak{m}_2\backslash\{y, xy\} \\ \mathbb{C}\{x, x^2, x^3, x^4, x^\ell y^{2j} : \ell+2j \geq 5\} + \mathbb{C}\{x^\ell y^{3j+1}(x^2-y^2) : \ell,j \geq 0\} \\ \mathbb{C}\{x, x^2, x^3, x^4, x^5, x^\ell y^{2j+1} : \ell+2j \geq 5\} + \mathbb{C}\{x^\ell y^{3j+2}(x^2-y^2) : \ell,j \geq 0\} \end{pmatrix}$$

$$+\left\{\begin{pmatrix} 0 \\ x^2y^2-y^4 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^2y-y^3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^3y-xy^3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x^4y-x^2y^3 \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} 0 \\ 0 \\ 5x^4y-6y^5 \end{pmatrix}\begin{pmatrix} y \\ -xy^2 \\ 0 \end{pmatrix}\begin{pmatrix} xy \\ -y^4 \\ 0 \end{pmatrix}\begin{pmatrix} 3y \\ 0 \\ 2xy^3 \end{pmatrix}\begin{pmatrix} 5xy \\ 0 \\ 4y^5 \end{pmatrix}\begin{pmatrix} 0 \\ x^jy \\ x^jy^2 \end{pmatrix} : j \geq 2\right\}$$

$$= \left(\langle x^2, y^2\rangle + \mathbb{C}\{x\}\right)\frac{\partial}{\partial X} + \left(\mathbb{C}\{x^\ell y^{2j} : \ell+2j \geq 5\} + \mathbb{C}\{x^{\ell+1}, y^2(x^2-y^2), x^\ell y^{2j+1}(x^2-y^2) : \ell,j \geq 0\}\right)\frac{\partial}{\partial Y}$$

$$+\left(\mathbb{C}\{x^\ell y^{2j+1} : \ell+2j \geq 5\} + \mathbb{C}\{x^\ell, x^2y(x^2-y^2), xy(x^2-y^2), y(5x^4-6y^4), y(x^2-y^2), x^\ell y^{2j+2}(x^2-y^2) :\right.$$

$$\left.\ell,j \geq 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} y \\ -xy^2 \\ 0 \end{pmatrix}\begin{pmatrix} xy \\ -y^4 \\ 0 \end{pmatrix}\begin{pmatrix} 3y \\ 0 \\ 2xy^3 \end{pmatrix}\begin{pmatrix} 5xy \\ 0 \\ 4y^5 \end{pmatrix}\begin{pmatrix} 0 \\ x^jy \\ x^jy^2 \end{pmatrix} : j \geq 2\right\}$$

$$\text{Then } f^*\mathfrak{m}_3 \cdot C = \left(\langle x^3, xy^2, y^4\rangle + \mathbb{C}\{x^2\}\right)\frac{\partial}{\partial X} + \mathbb{C}\{x^\ell y^{2j} : \ell+2j \geq 6\}\frac{\partial}{\partial Y} +$$

$$\mathbb{C}\{x^2, x^3, x^4, x^5, xy(x^2-y^2), xy^2(x^2-y^2), x^\ell y^{2j+1}(x^2-y^2) : \ell+j \geq 2\}\Big)\frac{\partial}{\partial Y}$$

$$+\left(\mathbb{C}\{x^\ell y^{2j+1} : \ell+2j \geq 6\} + \mathbb{C}\{x^\ell, x^3y(x^2-y^2), x^2y(x^2-y^2), xy(x^2-y^2), xy(5x^4-6y^4),\right.$$

$$\left.x^\ell y^{2j+2}(x^2-y^2) : \ell+j > 0\}\right)\frac{\partial}{\partial Z} + \mathbb{C}\left\{\begin{pmatrix} xy \\ -x^2y^2 \\ 0 \end{pmatrix}\begin{pmatrix} x^2y \\ -xy^4 \\ 0 \end{pmatrix}\begin{pmatrix} 3xy \\ 0 \\ 2x^2y^3 \end{pmatrix}\begin{pmatrix} 5x^2y \\ 0 \\ 4xy^5 \end{pmatrix}\begin{pmatrix} 0 \\ x^jy \\ x^jy^2 \end{pmatrix} : j \geq 3\right\}$$

$$\text{Then } f^*\mathfrak{m}_3 \cdot C = \begin{pmatrix} \mathfrak{m}_2^2\backslash\{xy, y^2, x^2y, y^3\} \\ \mathbb{C}\{x^2, x^3, x^4, x^5, x^\ell y^{2j} : \ell+2j \geq 6\} + \mathbb{C}\{x^\ell y^{2j+1}(x^2-y^2) : 2\ell+j \geq 2\} \\ \mathbb{C}\{x^2, x^3, x^4, x^5, x^6, x^\ell y^{2j+1} : \ell+2j \geq 6\} + \mathbb{C}\{x^\ell y^{2j+2}(x^2-y^2) : \ell+j \geq 1\} \end{pmatrix}$$

$$+\left\{\begin{pmatrix}0\\0\\x^4y-x^2y^3\end{pmatrix}\begin{pmatrix}0\\0\\5x^3y^3-6xy^5\end{pmatrix}\begin{pmatrix}xy\\-x^2y^2\\0\end{pmatrix}\begin{pmatrix}3xy\\0\\2x^2y^3\end{pmatrix}\begin{pmatrix}x^2y\\-x^3y^2\\0\end{pmatrix}\begin{pmatrix}3x^2y\\0\\2x^3y^3\end{pmatrix}\begin{pmatrix}0\\x^jy\\x^jy^2\end{pmatrix}:j\geq 3\right\}$$

C is spanned by: $x\frac{\partial}{\partial X}, x^2y\frac{\partial}{\partial X}, y^3\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, xy^4\frac{\partial}{\partial Y}, x^3y^2\frac{\partial}{\partial Y}, y(x^2-y^2)\frac{\partial}{\partial Y}, y^2(x^2-y^2)\frac{\partial}{\partial Y},$

$y^3(x^2-y^2)\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z}, xy^5\frac{\partial}{\partial Z}, x^3y^3\frac{\partial}{\partial Z}, x^5y\frac{\partial}{\partial Z}, y(x^2-y^2)\frac{\partial}{\partial Z}, y^2(x^2-y^2)\frac{\partial}{\partial Z}, y(5x^4-y^4)\frac{\partial}{\partial Z},$

$$\begin{pmatrix}y\\-xy^2\\0\end{pmatrix}\begin{pmatrix}0\\x^2y\\x^2y^2\end{pmatrix}\begin{pmatrix}3y\\0\\2xy^3\end{pmatrix}$$

$T\mathscr{A}f$ is spanned by: $x\frac{\partial}{\partial X}, (x^2y-y^3)\frac{\partial}{\partial X}, (x^2y^2-y^4)\frac{\partial}{\partial X}, x\frac{\partial}{\partial Y}, (x^2y-y^3)\frac{\partial}{\partial Y}, (x^2y^2-y^4)\frac{\partial}{\partial Y}, x\frac{\partial}{\partial Z},$

$$(x^2y-y^3)\frac{\partial}{\partial Z}, (x^2y^2-y^4)\frac{\partial}{\partial Z}, \begin{pmatrix}x\\2x^2y\\2x^2y^2\end{pmatrix}, \begin{pmatrix}y\\2xy^2\\2xy^3\end{pmatrix}, \begin{pmatrix}y^2\\2xy^3\\2xy^4\end{pmatrix}, \begin{pmatrix}y^3\\2xy^4\\2xy^5\end{pmatrix}, \begin{pmatrix}0\\x^3-3xy^2\\2x^3y-3xy^3\end{pmatrix},$$

$$\begin{pmatrix}0\\x^2y-3y^3\\2x^2y^2-3y^4\end{pmatrix}, \begin{pmatrix}0\\x^2y^2-3y^4\\2x^2y^3-3y^5\end{pmatrix}, \begin{pmatrix}0\\x^2y^3-3y^5\\2x^2y^4-3y^6\end{pmatrix}$$

Hence $C = T\mathscr{A}f + f^*\mathfrak{m}_3\cdot C$ so that $C = T\mathscr{A}f$.

Appendix B : Calculations of $\mathscr{A}$ -invariants for III·3.

For map-germs of the form $(x, y^2, yp(x, y^2))$ the presentation matrix is

$$\begin{bmatrix} -Z & p(X, Y) \\ Yp(X, Y) & -Z \end{bmatrix} \text{ and } T = 0.$$

B:1. $(x, y^2, x^3y + x^2y^3 + y^{2k+1})$.

$$Q_f = \langle y, x^3 \rangle \text{ therefore } C = 3.$$

$\tilde{D}^2(f) = V(y+y', x^3 + x^2(y^2+yy'+y'^2) + y^{2k}+y^{2k-1}y'+...+y'^{2k}) = V(y+y', x^3 + x^2y^2 + y^{2k})$.

Then $\mu(D^2(f)) = \operatorname{cod} \mu(V(x^3 + x^2y^2 + y^{2k})) = \dim \mathcal{O}_2/\langle 3x^2 + 2xy^2, 2x^2y + 2ky^{2k-1} \rangle$.

Let $I = \langle 3x^2 + 2xy^2, 2x^2y + 2ky^{2k-1} \rangle$ and suppose that $xy^5\mathfrak{m}_2 \subseteq I$.

Then $x^3y + kxy^{2k-1} \sim x^3y \bmod xy^5\mathfrak{m}_2$, $3x^3y + 2x^2y^3 \sim 2x^2y^3$, $3x^2y^3 + 2xy^5 \sim 2xy^5$,

hence $xy^5\mathcal{O}_2 \subseteq I$, hence $\langle x^4, x^3y, x^2y^3, xy^5, y^{2k+1} \rangle \subseteq I$, hence

$\mu(\tilde{D}^2(f)) = 2k+4 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1$. Hence $\mu(\tilde{D}^2(f)/\mathbb{Z}_2) = k+1$.

B:2. $(x, y^2, x^4y + 2x^2y^3 + y^5 + x^ky)$.

$$Q_f = \langle y, x^4 \rangle \text{ therefore } C = 4.$$

$\tilde{D}^2(f) = V(y+y', x^4 + 2x^2(y^2+yy'+y'^2) + y^4+y^3y'+...+y'^4 + x^k)$

$= V(y+y', x^4 + 2x^2y^2 + y^4 + x^k) = V(y+y', (x^2+ y^2)^2 + x^k)$

Then $\mu(\tilde{D}^2(f)) = \operatorname{cod} \mu(V((x^2+ y^2)^2 + x^{k-1})) = \dim \mathcal{O}_2/\langle 4x(x^2+ y^2) + kx^{k-1}, 4y(x^2+ y^2) \rangle$

Let $I = \langle 4x(x^2+ y^2) + kx^{k-1}, y(x^2+ y^2) \rangle$. Then

$$xy(x^2+ y^2) \in I \Rightarrow x^{k-1}y \in I \Rightarrow x^{k-3}y^3 \in I \Rightarrow ... \Rightarrow (x)y^{k(-1)} \in I$$

(depending on whether k is even or odd)

and $xy^2(x^2+ y^2) \in I \Rightarrow x^{k-1}y^2 \in I \Rightarrow x^{k-3}y^4 \in I \Rightarrow ... \Rightarrow (x)y^{k(+1)} \in I$.

Then $x^{k+1} \in I$ and $\mathcal{O}_2/I$ has a basis $\{1, x, ..., x^k, y, xy, ..., x^{k-2}y, y^2\}$ so

$\mu(\tilde{D}^2(f)) = 2k+1 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1$. Hence $\mu(\tilde{D}^2(f)/\mathbb{Z}_2) = k-1$.

B:3. $(x, y^2, x^4y + x^2y^3 + y^{2k+1})$.

$$Q_f = \langle y, x^4 \rangle \text{ therefore } C = 4.$$

$\tilde{D}^2(f) = V(y+y', x^4 + x^2(y^2+yy'+y'^2) + y^{2k}+y^{2k-1}y'+\ldots+y'^{2k}) = V(y+y', x^4 + x^2y^2 + y^{2k})$

Then $\mu(\tilde{D}^2(f)) = \operatorname{cod}\mu(V(x^4 + x^2y^2 + y^{2k})) = \dim \mathcal{O}_2/\langle 4x^3 + 2xy^2, 2x^2y + 2ky^{2k-1}\rangle$

Let $I = \langle 2x^3 + xy^2, x^2y + ky^{2k-1}\rangle$ and suppose that $xy^3\mathfrak{m}_2 \subseteq I$.

Then $x^3y + kxy^{2k-1} \sim x^3y \mod xy^3\mathfrak{m}_2$ and $x^3y + xy^3 \sim xy^3 \in I + xy^3\mathfrak{m}_2$

hence $xy^3\mathcal{O}_2 \subseteq I$, hence $\langle x^5, x^3y, xy^3, y^{2k+1}\rangle \subseteq I$, hence

$\mu(\tilde{D}^2(f)) = 2k+5 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1$. Hence $\mu(\tilde{D}^2(f)/\mathbb{Z}_2) = k+1$.

B:4. $(x, y^2, x^2y^3 + y^5 + x^ky)$.

$$Q_f = \langle y, x^k\rangle \text{ therefore } C = k.$$

$\tilde{D}^2(f) = V(y+y', x^2(y^2+yy'+y'^2) + y^4+y^3y'+\ldots+y'^4 + x^k) = V(y+y', x^2y^2 + y^4 + x^k)$

Then $\mu(\tilde{D}^2(f)) = \operatorname{cod}\mu(V(x^2y^2 + y^4 + x^k)) = \dim \mathcal{O}_2/\langle 2xy^2 + kx^{k-1}, 2x^2y + 4y^3\rangle$

Let $I = \langle 2xy^2 + kx^{k-1}, x^2y + 2y^3\rangle$ and suppose that $x^3y\mathfrak{m}_2 \subseteq I$.

Then $2xy^3 + kx^{k-1}y \sim 2xy^3 \mod x^3y\mathfrak{m}_2$ and $x^3y + 2xy^3 \sim x^3y \in I + x^3y\mathfrak{m}_2$

hence $x^3y\mathcal{O}_2 \subseteq I$, hence $\langle y^5, x^3y, xy^3, x^{k+1}\rangle \subseteq I$, hence

$\mu(\tilde{D}^2(f)) = k+5 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1$. Hence $\mu(\tilde{D}^2(f)/\mathbb{Z}_2) = 3$.

For map-germs of the form $(x, xy + y^3, p(x, y))$ we note the following:

Let $X = x$, $Y = xy + y^3$. Then

$$y^4 = yY - Xy^2$$
$$y^5 = y^2Y - XY + X^2y$$
$$y^6 = Y^2 - 2XYy + X^2y^2$$
$$y^7 = yY^2 - 2XYy^2 + X^2Y - X^3y$$
$$y^8 = y^2Y^2 - 2XY^2 + 3X^2Yy - X^3y^2$$
$$y^9 = Y^3 - 3yXY^2 + 3X^2Yy^2 - X^3Y + X^4y \text{ etc.}$$

We see that

i) $y^{3j+i} \in \mathfrak{m}_2^j\{1, y, y^2\}$,

ii) $y^{3j+i} = y^iY^j$ modulo $X\cdot\mathfrak{m}_2^{j-1}$ for $i \in \{0, 1, 2\}$,

iii) $y^{2j+i} = (-1)^jy^iX^j$ modulo $Y\cdot\mathfrak{m}_2$ for $i \in \{1, 2\}$.

B:5. $(x, xy+y^3, xy^2+y^{3k+1})$.

$Q_f = \langle x+3y^2, 2xy+(3k+1)y^{3k}\rangle = \langle x', -6y^3+(3j+1)y^{3k}\rangle$ under the coordinate change $x' = x+3y^2$ so $C = 3$.

$\tilde{D}^2((x, xy+y^3, xy^2)) = V(x+y^2+yy'+y'^2, x(y+y'))$

$= V(x+y^2+yy'+y'^2, (y+y')(y^2+yy'+y'^2)) = V(x+y^2+yy'+y'^2, y^3+2y^2y'+2yy'^2+y'^3)$

Then $\mu(\tilde{D}^2(f)) = \mu(\tilde{D}^2(x, xy+y^3, xy^2))$ provided that $\mu(\tilde{D}^2(x, xy+y^3, xy^2))$ is finite.

Then $\mu(\tilde{D}^2(x, xy+y^3, xy^2)) = \operatorname{cod}\mu(V(y^3+2y^2y'+2yy'^2+y'^3))$

$= \dim \mathcal{O}_2/\langle 3y^2+4yy'+2y'^2, 2y^2+4yy'+3y'^2\rangle$

This gives 2 equations between the 3 terms $\{y^2, yy', y'^2\}$.

Multiplying each equation by x, y gives 4 equations between the 4 terms

$\{y^3, y^2y', yy'^2, y'^3\}$. Hence dim = 4 since $\begin{bmatrix} 3 & 4 & 2 & 0 \\ 0 & 3 & 4 & 2 \\ 2 & 4 & 3 & 0 \\ 0 & 2 & 4 & 3 \end{bmatrix}$ has non-zero determinant.

Then $\mu(\tilde{D}^2(f)) = 4 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1 \Rightarrow \mu(\tilde{D}^2(f)/\mathbb{Z}_2) = 1$.

Calculate a presentation for $(x, xy+y^3, xy^2+y^{3k+1})$ mod $X\cdot\mathfrak{m}_2$

	1	y	y^2
xy^2+y^{3k+1}	$-Z$	Y^k	X
xy^3+y^{3k+2}	0	$-Z$	Y^k
xy^4+y^{3k+3}	Y^{k+1}	0	$-Z$

Then $\mathcal{F}_2 + X\cdot\mathfrak{m}_3 = \{X, Z, Y^k\}\mathcal{O}_3$ so that $\mathcal{F}_2 = \{X, Z, Y^k\}\mathcal{O}_3$ and $T = k$.

B:6. $(x, xy+y^3, xy^2+y^4+y^{2k})$.

$Q_f = \langle x+3y^2, 2xy+4y^3+2ky^{2k-1}\rangle = \langle x', -2y^3+2ky^{2k-1}\rangle$ under the coordinate change $x' = x+3y^2$ so $C = 3$.

$\tilde{D}^2((x, xy+y^3, xy^2+y^4)) = V(x+y^2+yy'+y'^2, x(y+y')+y^3+y^2y'+yy'^2+y'^3)$

$= V(x+y^2+yy'+y'^2, y^3+y^2y'+yy'^2+y'^3-(y+y')(y^2+yy'+y'^2)) = V(x+y^2+yy'+y'^2,$
$y^2y'+yy'^2)$.

Then $\mu(\tilde{D}^2(f)) = \mu(\tilde{D}^2(x, xy+y^3, xy^2+y^4))$ provided that $\mu(\tilde{D}^2(x, xy+y^3, xy^2+y^4))$ is finite. Then $\mu(\tilde{D}^2(f)) = \operatorname{cod}\mu(V(y^2y'+yy'^2)) = \dim \mathcal{O}_2/\langle 2yy'+y'^2, y^2+2yy'\rangle$

This gives 2 equations between the 3 terms $\{y^2, yy', y'^2\}$.

Multiplying each equation by x, y gives 4 equations between the 4 terms

$\{y^3, y^2y', yy'^2, y'^3\}$. Hence dim = 4 since $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ has non-zero determinant.

Then $\mu(\tilde{D}^2(f)) = 4 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2) + C - 1 \Rightarrow \mu(\tilde{D}^2(f)/\mathbb{Z}_2) = 1$.

Calculate a presentation for $(x, xy+y^3, xy^2+y^4+y^{2k})$ mod $Y\cdot\mathfrak{m}_3$

	1	y	y^2
$xy^2+y^4+y^{2k}$	$-Z$	Y	$(-1)^{k-1}X^{k-1}$
$xy^3+y^5+y^{2k+1}$	0	$-Z+(-1)^kX^k$	Y
$xy^3+y^6+y^{2k+2}$	0	0	$-Z+(-1)^kX^k$

Then $\mathcal{F}_2 + Y\cdot\mathfrak{m}_3 = \{Y, Z, X^{k-1}\}\mathcal{O}_3$ so that $\mathcal{F}_2 = \{Y, Z, X^{k-1}\}\mathcal{O}_3$ and $T = k-1$.

B:7. $(x, xy+y^3, 2xy^2+3y^4+y^k)$.

$Q_f = \langle x+3y^2, 4xy+12y^3+ky^{k-1}\rangle = \langle x', ky^{k-1}\rangle$ under the coordinate change

$$x' = x+3y^2 \text{ so } C = k.$$

$\tilde{D}^2((x, xy+y^3, 2xy^2+3y^4+y^k)) =$

$V(x+y^2+yy'+y'^2, 2x(y+y') + 3(y^3+y^2y'+yy'^2+y'^3)+y^k+\ldots+y'^k)$

$= V(x+y^2+yy'+y'^2,\ 3(y^3+y^2y'+yy'^2+y'^3) +y^k+\ldots+y'^k - 2(y+y')(y^2+yy'+y'^2))$

$= V(x+y^2+yy'+y'^2,\ y^3-y^2y'-yy'^2+y'^3 +y^k+\ldots+y'^k)$.

Then $\mu(\tilde{D}^2(f)) = \operatorname{cod} \mu(V(y^3-y^2y'-yy'^2+y'^3) +y^k+\ldots+y'^k))$

$= \dim \mathcal{O}_2/\langle 3y^2 - 2yy' - y'^2+ky^{k-1}+\ldots+y'^{k-1}, -y^2 - 2yy'+ 3y'^2+y^{k-1}+\ldots+ky'^{k-1}\rangle$

Let $I = \langle 3y^2 - 2yy' - y'^2+ky^{k-1}+\ldots+y'^{k-1}, -y^2 - 2yy'+ 3y'^2+y^{k-1}+\ldots+ky'^{k-1}\rangle$

Assume that $\{y^3, y^2y', yy'^2\}\mathfrak{m}_2 \subseteq I$. Then $I + \{y^3, y^2y', yy'^2\}\mathfrak{m}_2$

$= \langle 3y^2 - 2yy' - y'^2+y'^{k-1}, -y^2 - 2yy'+ 3y'^2+ky'^{k-1}\rangle + \{y^3, y^2y', yy'^2\}\mathfrak{m}_2$

$\supseteq \{3y^3 - 2y^2y' - yy'^2, -y^3 - 2y^2y'+ 3yy'^2,$

$3ky^2y' - 2kyy'^2 - ky'^3+ky'^k-y^2y' - 2yy'^2+ 3y'^3+ky'^k\}$. Hence $\{y^3, y^2y', yy'^2, y'^k\}\mathcal{O}_2 \subseteq I$

and $\mu(\tilde{D}^2(f)) = k+1 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2)+C-1 \Rightarrow \mu(\tilde{D}^2(f)/\mathbb{Z}_2)=1$.

Calculate a presentation for $(x, xy+y^3, 2xy^2+3y^4+y^k)$ mod $\mathfrak{m}_3^2$ assuming that $k \geq 6$

	1	y	y^2
$2xy^2+3y^4$	-Z	3Y	-X
$2xy^3+3y^5$	0	-Z	3Y
$2xy^4+3y^6$	0	0	-Z

Then $\mathcal{F}_2 + \mathfrak{m}_3^2 = \{Y, Z, X\}\mathcal{O}_3$ so that $\mathcal{F}_2 = \{Y, Z, X\}\mathcal{O}_3$ and $T = 1$.

B:8. $(x, xy+y^3, xy^4+y^5)$.

$Q_f = \langle x+3y^2, 4xy^3+5y^4\rangle = \langle x', 5y^4-12y^5\rangle$ under the coordinate change

$$x' = x+3y^2 \text{ so } C = 4.$$

$\tilde{D}^2((x, xy+y^3, xy^4+y^5)) = V(x+y^2+yy'+y'^2, x(y^3+y^2y'+yy'^2+y'^3)+y^4+\ldots+y'^4)$

$= V(x+y^2+yy'+y'^2,\ y^4+\ldots+y'^4 - 2(y^3+y^2y'+yy'^2+y'^3)(y^2+yy'+y'^2))$

$= V(x+y^2+yy'+y'^2,\ y^4+\ldots+y'^4 - 2y^5-4y^4y'-6y^3y'^2-6y^2y'^3-4yy'^4-2y'^5)$.

Then $\mu(\tilde{D}^2(f)) = \text{cod}\ \mu(V(y^4+\ldots+y'^4 - 2y^5-4y^4y'-6y^3y'^2-6y^2y'^3-4yy'^4-2y'^5)$

$= \dim \mathcal{O}_2/I$, where $I = \langle 4y^3+3y^2y'+2yy'^2+y'^3-10y^4-16y^3y'-18y^2y'^2-12yy'^3-4y'^4,$

$y^3+2y^2y'+3yy'^2+4y'^3-4y^4-12y^3y'-18y^2y'^2-16yy'^3-10y'^4\rangle$.

Calculate this modulo $\mathfrak{m}_2^6$. Then there are 6 equations between the 6 monomials of degree 5, so $I \supseteq \mathfrak{m}_2^5$ since $\begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 \end{bmatrix}$ has non-zero determinant.

Hence $\mu(\tilde{D}^2(f)) = 9 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2)+C-1 \Rightarrow \mu(\tilde{D}^2(f)/\mathbb{Z}_2)=3$.

Calculate a presentation for $(x, xy+y^3, xy^4+y^5)$.

	1	y	y^2
xy^4+y^5	$-Z-XY$	X^2+XY	$Y-X^2$
xy^5+y^6	Y^2-X^2Y	$-Z-2XY+X^3$	X^2+XY
xy^6+y^7	X^2Y+XY^2	$Y^2-2X^2-X^3$	$-Z-2XY+X^3$

Then $\mathcal{F}_2 = \{Z+XY, Y-X^2, X^2+XY\}\mathcal{O}_3$. Let $Y' = Y-X^2$, then $\mathcal{F}_2 = \{Z, Y', X^2+X^3\}\mathcal{O}_3$ and $T = 2$.

B:9. $(x, xy+y^3, xy^4+y^{3k+2})$.

$Q_f = \langle x+3y^2, 4xy^3+(3k+2)y^{3k+1}\rangle = \langle x', -12y^5+(3k+2)y^{3k+1}\rangle$ under the coordinate change $x' = x+3y^2$ so $C = 5$.

$\tilde{D}^2((x, xy+y^3, xy^4)) = V(x+y^2+yy'+y'^2, x(y^3+y^2y'+yy'^2+y'^3))$

$= V(x+y^2+yy'+y'^2, (y^3+y^2y'+yy'^2+y'^3)(y^2+yy'+y'^2))$

$= V(x+y^2+yy'+y'^2, y^5+2y^4y'+3y^3y'^2+3y^2y'^3+2yy'^4+y'^5)$

Then $\mu(\tilde{D}^2(x, xy+y^3, xy^4)) = \operatorname{cod}\mu(V(y^5+2y^4y'+3y^3y'^2+3y^2y'^3+2yy'^4+y'^5)$

$= \dim \mathcal{O}_2/I$ where $I = \langle 5y^4+8y^3y'+9y^2y'^2+6yy'^3+2y'^4, 2y^4+6y^3y'+9y^2y'^2+8yy'^3+5y'^4\rangle$.

Then $\mu(\tilde{D}^2(f)) = \mu(\tilde{D}^2(x, xy+y^3, xy^4))$ provided that $\mu(\tilde{D}^2(x, xy+y^3, xy^4))$ is finite.

Calculate this modulo $\mathfrak{m}_2^8$. Then there are 8 equations between the 8 monomials of degree 7, so $I \supseteq \mathfrak{m}_2^7$ since the relevant matrix has non-zero determinant.

Hence $\mu(\tilde{D}^2(f)) = 16 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2)+C-1 \Rightarrow \mu(\tilde{D}^2(f)/\mathbb{Z}_2)=6$.

Calculate a presentation for $(x, xy+y^3, xy^4+y^{3k+2})$ modulo $X\cdot\mathfrak{m}_3^3$.

	1	y	y^2
xy^4+y^{3k+2}	$-Z$	XY	$-X^2+Y^k$
xy^5+y^{3k+3}	$Y^{k+1}-X^2Y$	$-Z+X^3$	XY
xy^6+y^{3k+4}	XY^2	$Y^{k+1}-2X^2Y$	$-Z+X^3$

Then $\mathcal{F}_2+X\cdot\mathfrak{m}_3^3 = \{Z, XY, -X^2+Y^k\}\mathcal{O}_3$. Then $\mathcal{F}_2+X\cdot\mathfrak{m}_3^3 \supseteq \{Z, XY, X^3, Y^{k+1}\}\mathcal{O}_3$ and $T = k+2$.

B:10. $(x, x^2y + y^3, xy^2 + y^{3k+1})$.

$$Q_f = \langle x^2 + 3y^2, 2xy + (3k+1)y^{3k}\rangle \supseteq \langle x^3, y^3, xy\rangle \text{ so } C = 4.$$

$\tilde{D}^2((x, x^2y + y^3, xy^2)) = V(x^2+y^2+yy'+y'^2, x(y+y'))$. We show that

$$h_f = (x^2+y^2+yy'+y'^2, x(y+y'))$$

is $\mathcal{K}$-finite. Since h_f is weighted homogeneous $\mu(h_f) = \dim \mathcal{O}_3/\langle h_f, J_h\rangle$ which is finite if and only if h_f is $\mathcal{K}$-finite.

Then $\mu(\tilde{D}^2(x, x^2y + y^3, xy^2)) = \dim \mathcal{O}_3/I$

where $I = \langle x^2+y^2+yy'+y'^2, x(y+y'), 2x^2-2y^2-3yy'-y'^2, 2x^2-y^2-3yy'-2y'^2, x(y-y')\rangle$

$= \langle x^3, xy, xy', x^2+y^2+yy'+y'^2, 2x^2-2y^2-3yy'-y'^2, 2x^2-y^2-3yy'-2y'^2\rangle$

$\supseteq \{y^3, y^2y', yy'^2, y'^3\}$. Hence $\mu(\tilde{D}^2(f)) = 5 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2)+C-1 \Rightarrow$ $\mu(\tilde{D}^2(f)/\mathbb{Z}_2)=1$.

Calculate a presentation for $(x, x^2y + y^3, xy^2 + y^{3k+1})$ modulo $X\cdot\mathfrak{m}_3$:

	1	y	y^2
$xy^2 + y^{3k+1}$	$-Z$	Y^k	X
$xy^3 + y^{3k+2}$	0	$-Z$	Y^k
$xy^4 + y^{3k+3}$	Y^{k+1}	0	$-Z$

since $y^{3k+i} = y^iY^k$ modulo $X\cdot\mathfrak{m}_3$.

Then $\mathcal{F}_2 + X\cdot\mathfrak{m}_3 = \{Z, X, Y^k\}\mathcal{O}_3$. Then $\mathcal{F}_2 = \{Z, X, Y^k\}\mathcal{O}_3$ and $T = k$.

B:11. $(x, x^2y - y^3, x^2y^2 - y^4 + x^ky^2)$.

$$Q_f = \langle x^2-3y^2, 2x^2y-4y^4 + x^ky\rangle = \langle x^2-3y^2, -10y^4 + x^ky\rangle \text{ therefore } C = 6.$$

$\tilde{D}^2(x, x^2y - y^3, x^2y^2 - y^4) = V(x^2-y^2-yy'-y'^2, x^2(y+y')-y^3-y^2y'-yy'^2-y'^3)$

$= V(x^2-y^2-yy'-y'^2, (y+y')(y^2+yy'+y'^2) -y^3-y^2y'-yy'^2-y'^3)$

$= V(x^2-y^2-yy'-y'^2, y^2y'+yy'^2)$.

We show that $h_f = (x^2-y^2-yy'-y'^2, y^2y'+yy'^2)$ is $\mathcal{K}$-finite.

Since h_f is weighted homogeneous $\mu(h_f) = \dim \mathcal{O}_3/\langle h_f, J_h\rangle$ which is finite if and only if h_f is $\mathcal{K}$-finite. Then $\mu(D^2(x, x^2y - y^3, x^2y^2 - y^4)) = \dim \mathcal{O}_3/I$

where $I = \langle x^2-y^2-yy'-y'^2, y^2y'+yy'^2, x(2yy'+y'^2), x(y^2+2yy'), (y-y')(2y+y')(y+2y')\rangle$

$= \langle x^2-y^2-yy'-y'^2, y^2y'+yy'^2, x(2yy'+y'^2), x(y^2+2yy'), y^3+3y^2y'-y'^3\rangle$

$$\supseteq \{y^5, y^4y', y^3y'^2, y^2y'^3, yy'^4, y'^5\} \text{ since the matrix } \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 & -1 \end{bmatrix} \text{ has}$$

non-zero determinant. Hence $\mu(\tilde{D}^2(f)) = 9 = 2\mu(\tilde{D}^2(f)/\mathbb{Z}_2)+C-1 \Rightarrow \mu(\tilde{D}^2(f)/\mathbb{Z}_2)=2$.

Calculate a presentation for $(x, x^2y - y^3, x^2y^2 - y^4 + x^ky^2)$.

	1	y	y^2
$x^2y^2 - y^4 + x^ky^2$	$-Z$	Y	X^k
$x^2y^3 - y^5 + x^ky^3$	$-X^kY$	$-Z+X^{k+2}$	Y
$x^2y^4 - y^6 + x^ky^4$	$-Y^2$	$-X^kY+X^2Y$	$-Z+X^{k+2}$

Then $\mathcal{F}_2 = \{Z, Y, X^k\}\mathcal{O}_3$ and $T = k$.

Appendix C : The Computer Programs.

Contents.

Introduction.

There are two versions of the program, module and param, which calculate a set of generators for the module

$$T\mathscr{A}_{R,S}f = tf\left(m_2^R \cdot \theta(2)\right) + \omega f\left(m_3^S \cdot \theta(3)\right) \text{ modulo } m_2^{k+1}\theta(f)$$

for integers R, S, K and $f(x, y) = (f_1, f_2, f_3)$, where each f_i is a polynomial with integer coefficients. In the version param the polynomials may also take parameter coefficients.

Language and Computers.

Versions exist to run on the Sun Workstation and the Apple Macintosh, with different versions for each computer, making use of their different capabilities. The programs are written in Pascal (since this is the only language I know). To save the data in an external file it has been necessary to use language which is specific to these computers, namely Berkely Pascal (on the Sun) and MacPascal. I believe that this is only necessary when saving or printing the data; however, without this facility the programs are of limited use. A lot of problems were found in transporting the programs from the Mac to the Sun and it is likely that similar problems would be found in transporting the program to other systems.

Apple Macintosh.

Advantages: it is relatively easy to write and debug programs in MacPascal. There are also features of the language which make it easy to send data to an external file without overwriting anything and to send data directly to the printer. The system does not crash when it recieves unexpected input (a character when it expects an integer).

Disadvantages: it is very slow and there is not much memory space (one of

the main differences in these versions is the use of numerous prompts to let the user know that things are happening). The range of integers is small ($\pm(2^{15}-1)$ or $\pm 32{,}767$) so we have to resort to the use of alternative methods of storing numbers. The use of non-standard Pascal makes the program difficult to transport to other computers.

Note: a manual for use of the package Lightspeed Pascal has recently become available at Warwick. This includes a compiler so it should become possible to provide compiled versions of the programs, which will be much faster and have more memory space available.

Sun Workstation.

Advantages: it is far faster and far more memory available. The integer range is larger (at least $\pm 2{,}000{,}000{,}000$) and the program is closer to standard Pascal so is likely to be less difficult to transport.

Disadvantages: it is more difficult to save data and more errors can be made doing so (the chief being overwriting a file that was wanted). It is more difficult to debug the programs.

Presentation of data.

Every effort has been made to make the programs easy to use. There are numerous prompts for the user and the user can make small alterations or corrections to the example quickly.

The module can be expressed in the form

$$I_1\partial/\partial X + I_2\partial/\partial Y + I_3\partial/\partial Z + \mathbb{C}\{v_1,\dots v_r\}$$

where each I_j is an $\mathcal{O}_3$-module via f^* and the supplementary generators v_j are of the form (p_1, p_2, p_3) where at least two of the p_i's are non-zero.

We give generators for each row I_j in turn, followed by a list of supplementary generators. If there are none then this is indicated. Generators for I are given in ascending order, starting with monomials; if there are a large number of monomial generators then we give them in the form Hn\{missing monomials}, where Hn indicates the set $\{x^{n-j}y^j : 0 \le j \le n\}$.

Running the programs (an example).

For example, suppose we are calculating

$$T\mathcal{A}_{R,S}f \text{ modulo } \mathfrak{m}_2^8\theta(f) \text{ for } f = (x, xy + y^3, 2xy^2 + y^4)$$

with the program param.

The prompt

```
This program calculates the terms in tf(M^R) + wf(N^S) for
f(x,y)= (f1(x,y),f2(x,y),f3(x,y)).
It will only run if f1,f2 and f3 have no constant terms or if f2 or f3 are ze:
In  this  version  f  may  have  parameter  values.
 Program  written  by  D. Ratcliffe.
 Please enter R
```

should appear. In the programs we denote $\mathfrak{m}_2\cdot\theta(2)$ by M and $\mathfrak{m}_3\cdot\theta(3)$ by N.

Powers are indicated by a circumflex before the integer. The program the waits for the data to be given. The procedure for this example is as follows:

```
Please enter R 1
 Please enter S 1
 Please enter k.7

Type in number of terms in f1  1

        Do you wish to add terms with moduli  ? n

 Type in scalar 1
Type in power of x  1
Type in power of y  0


Type in number of terms in f2  2

        Do you wish to add terms with moduli  ? n

 Type in scalar 1
Type in power of x  1
Type in power of y  1

 Type in scalar 1
Type in power of x  0
Type in power of y  3


Type in number of terms in f3  2

        Do you wish to add terms with moduli  ? n

 Type in scalar 1
Type in power of x  1
Type in power of y  2

 Type in scalar 1
Type in power of x  0
Type in power of y  4
```

```
Calculating tf(M^1) + wf(N^1)  mod M2^8
for  f=(x, xy +y^3, xy^2 +y^4)
                                Is this correct  ? y
```

The program then proceeds with the calculation, giving prompts as it completes certain stages, eg. 'calculated ωf', etc. The prompt 'comparing supplementary generators' means that the program has started to compare supplementary generators with supplementary generators. This is frequently the slowest part of the program for a complicated map-germ. The phrase 'place = n' indicates which list of supplementary generators is being studied. There are four lists:

place = 0 : this indicates that there are no zeros on any row of the generator,

place = 1 : this indicates that the the first row is zero,

place = 2 : this indicates that the second row is zero,

place = 3 : this indicates that the third row is zero.

The indicative phrase will be repeated every time the program runs through that list applying the particular algorithm.

A typical display will be as follows:

```
tf(M^1) + wf(N^1) mod M^8 for f = (x, xy +y^3, xy^2 +y^4) is as follows

generators in row(1) are
H1\{ y} H2, H3, H4, H5, H6, H7,

generators in row(2) are
H1\{ y} 2: C{ x^2, xy +y^3} H3\{ y^3, x^2y,}+C{ x^2y -y^5
} H4\{ xy^3,}+C{ xy^3 +y^5} H5\{ y^5} H6, H7,

generators in row(3) are
H1\{ y} 2: C{ x^2, xy +y^3} H3\{ y^3, xy^2,}+C{ xy^2 +y^4
} H4\{ y^4, x^2y^2,}+C{ x^2y^2 -y^6} H5\{ xy^4,}+C{ xy^4 +y^6}
H6\{ y^6} H7,


supplementary generators are
(0, y^3, y^4), (0, y^5, y^6)
(y, y^2, y^3)
cod = 7
```

cod is the codimension of $m_2\theta(f)/T\mathcal{A}_{R,S}f$. Note that if simplifications have been missed or parameters are involved, this number is frequently wrong.

The display continues as follows:

```
Is the display satisfactory  ? y
```

```
                    Do you wish to save the data  ? y

 enter name of file (10 characters):
IF THERE IS ALREADY A FILE WITH THIS NAME IT WILL BE OVERWRITTEN!
example1

 filename is example1

                    Do you wish to continue  ? n
```

The display is controlled by counting the number of terms or supplementary generators printed on a line and putting in a return at fixed intervals. This is essential since otherwise the entire display would be printed on one line. On the screen this does not matter, but it results in the loss of data when it sent to a printer. This does not take account of the length of the terms, so if the coefficients are large, the preset values may be too big. If it is suspected that this may occur, the values may be altered.

Note that the name may have blanks (or returns) at the end (but not in the middle). If the program pauses at this point, it is usually because not enough characters have been entered; press returns until it continues.

The allowed input is as follows:

k must lie between 1 and 40. This is an arbitrarily chosen maximum; it is the size of certain arrays used in the program.

f_1 must be non-zero and have no constant term. This is because the program starts by calculating powers of f_1 until some power lies in m_2^{k+1}.

The other polynomials may be zero, but may not have a constant term.

<u>Expansion</u> A future plan is to expand this to a package of programs to calculate submodules of $T\mathscr{A}f$, $T\mathscr{K}f$ and $T\mathscr{R}f$ for map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ for small vaues of n and p.

Basic Algorithms.

For storing data we use the same technique throughout; we form lists of records using pointers, creating a record at the end of the list when we have new data to store and destroying the record when it is no longer needed. This keeps the data stored to a minimum. The disadvantage of this is that bugs related to pointers are difficult to locate and eradicate.

1) The polynomial.

Polynomials are formed term by term and ordered. The ordering system is

$$x^i y^j < y^{i'} y^{j'} \text{ if } i+j < i'+j'.$$

If $i+j = i'+j'$ then the order becomes

$$x^i y^j < y^{i'} y^{j'} \text{ if } i > i''.$$

If there are parameters then the overriding ordering is 'constants first'. Thus the first term in a polynomial is a constant unless all its terms have parameters.

2) Simplification procedure for polynomials.

When inspecting a term p in the polynomial P being simplified, if there is a polynomial already stored having p as its lead term either with constant coefficient or the same coefficient as p then p is eliminated from P.

For example, if the polynomial $x^2y^2 + y^6$ is inspected and $x^2y^2 + xy^4$ is already stored then $x^2y^2 + y^6$ is replaced by $-xy^4 + y^6$. If, later $2xy^4 + y^6$ is added then this is stored as y^6. On following cheks the set $\{x^2y^2 + xy^4, x^2y^2 + y^6, 2xy^4 + y^6\}$ initially calculated will be stored as $\{x^2y^2, xy^4, y^6\}$.

This simplification process is also applied to the supplementary generators.

The disadvantage with this is that simple patterns in the generators may be obscured. This is a common problem with generators of the form $x^k p$ when f has corank 1.

3) Simplification procedure for supplementary generators.

The problem is how, for example, do we recognise that $(xy, x^2y, 0)$ and $(xy, 2x^2y, 0)$ should be replaced with generators $(xy, 0, 0)$ and $(0, x^2y, 0)$ while at the same time avoiding constantly cycling through the loop

$$\{(xy, x^2y, 0), (xy, 0, x^2y^2)\} \leftrightarrow \{(xy, x^2y, 0), (0, -x^2y, x^2y^2)\}?$$

For each row of the map being inspected, we select terms which may cancel and form a list. We then attempt to simplify all the maps in the list by elimination of polynomials.

For example, given $(x^2y^2 + xy^4 + y^6, p_2, p_3)$, we search for all map-germs of the form $(ax^2y^2 + bxy^4 + cy^6, p_2, p_3)$, where $a, b, c \in \mathbb{Z}$. We then try to eliminate the terms x^2y^2, xy^4, y^6 from the first row of each map-germ in the list. The simplification procedure is as follows:

1. Increase the 'place' of the map- i.e. try to reduce rows to zero. This also avoids looping, since in the example

$$\{(xy,\ x^2y, 0), (xy, 0, x^2y^2)\} \longleftrightarrow \{(xy,\ x^2y, 0), (0, -x^2y, x^2y^2)\}$$

$(xy, 0, x^2y^2)$ has place 2 so is preferred to $(0, -x^2y, x^2y^2)$, which has place 1.

2. Shorten the poynomial if none of the other rows is made longer.

We do not guarantee that the porocedures will give a complete simplification in every case, but this is safer than getting into infinite loops through trying to be too clever.

```
program Parameters(input, output);   {commence 12-12-89}

const
 LIMIT = 40;                        {maximum value for k}
 Cp = 3;
 DEFTTL = 12;                        {preset display parameter}
 DEFVTL = 3;                         {preset display parameter}
 MAXVALUE = 2000000000;                {maximum integer value handled}
 NOQUERY = 10;
 ADDM2 = true;
 DEBUG = 4;                          {resetting debug accesses checking routines}
 ACTDIS = true; {ACTualDISplay allows options in display procedures}
 DEFDIS = false;{DEFaultDISplay eliminates options and merely displays the data currently stored}
 DOMAP = false;
 DOPOLY = true;
 ADD = true;
 STRINGLENGTH = 45;
 DELETE = true;
 KEEP = false;
 CHARTOLINE = 80;

type
 allowedint = 0..LIMIT;
 charlist = array[0..Cp] of char;
 intset = set of allowedint;
 blist = array[1..Cp] of boolean;
 shortstring = packed array[1..10] of char;
 string = packed array[1..STRINGLENGTH] of char;
 blist4 = array[0..3] of boolean;
 variablename = 'x'..'y';
 polynomial = ^polyterm;
  polyterm = record
    x, y, s  : integer;
    m : char;
    next : polynomial;
  end;
 map = array[1..Cp] of polynomial;
 mappointer = ^map;
 Dmap = array['x'..'y'] of map;
 polydata = record
    length, maxcoeff, mincoeff : integer;
    moduli : boolean;
  end;
 mapdata = array[1..Cp] of polydata;
 polylist = ^listmember;
 listmember = record
   poly : polynomial;
   data : polydata;
   no : integer;
   next : polylist
  end;
 checklist = array[0..LIMIT] of record
   single : intset;
   sum : polylist
  end;
 module = array[1..Cp] of checklist;
 vectorlist = ^vector;
 vector = record
   f : mappointer;
   data : mapdata;
   good : boolean;
   maxcoeff, mincoeff : integer;
   next : vectorlist
  end;
 vectorstore = array[0..3] of vectorlist;
```

```
var
 continue, ch : char;
 k, R, S, Odim, forder, Printno, Lno, Vno : integer;
 one : polynomial;
 f : map;
 Df : Dmap;
 wf : module;
 tf : vectorstore;
 rerun : boolean;
 name, changef : charlist;
 fdata : mapdata;
 range, subrange : intset;
 dialogue : record
   query : array[1..NOQUERY] of string;
   response : set of char;
  end;

{A: ERASURE PROCEDURES}

procedure ruboutpoly (var a : polynomial);                          ruboutpoly
 var  item : polynomial;
 begin
  while a <> nil do
   begin
    item:= a;
    a:= a^.next;
    dispose(item);
   end
 end;                                          {ruboutpoly}

procedure ruboutmap (var f : map);                                  ruboutmap
 var
  c : integer;
 begin
  for c:= 1 to Cp do if f[c]<>nil then ruboutpoly(f[c]);
 end;                                          {ruboutmap}

procedure ruboutdata (var tf : vectorstore; {subprocedures:-ruboutff,}   ruboutdata
                      var wf : module);                      {ruboutlist}
 var
  coord, grade : integer;
 procedure ruboutf (var tf : vectorstore);                          ruboutf
  var
   v : vectorlist;
   j : integer;
  begin
   for j:= 0 to 3 do
    while tf[j]  <> nil do
     begin
      v:= tf[j];
      tf[j]:= tf[j]^.next;
      if v^.f <> nil then
       begin
        ruboutmap(v^.f^);
        dispose(v^.f);
       end;
      dispose(v);
     end;
  end;                                         {ruboutf}
 procedure ruboutlist (var list : polylist);                        ruboutlist
  var
   v : polylist;
  begin
   while list <> nil do
    begin
```

```
      v:= list;
      list:= list^.next;
      if v^.poly <> nil then ruboutpoly(v^.poly);
      dispose(v);
     end
   end;                                          {ruboutlist}
begin                                            {ruboutdata}
 for coord:= 1 to Cp do
  for grade:= 0 to LIMIT do
   begin
    wf[coord][grade].single:= [];
    if wf[coord][grade].sum <> nil then ruboutlist(wf[coord][grade].sum);
   end;
 ruboutf(tf);
end;                                             {ruboutdata}
```

{B: MISCELLANEOUS PROCEDURES}

makequery

```
procedure makequery (number : integer; ref : char; var reply : char);
begin
 repeat
  writeln;
  write(dialogue.query[number], ref, '? ');
  readln(reply);
 until (reply in dialogue.response);
 if reply = 'Y' then reply := 'y';
 if reply = 'N' then reply := 'n';
end;                                             {makequery}
```

getpolydata

```
procedure getpolydata (q : polynomial; var qdata : polydata);
begin
 qdata.maxcoeff := 0;
 qdata.mincoeff := 0;
 qdata.moduli:=false;
 qdata.length :=0;
 if q<> nil  then qdata.moduli := (q^.m <> ' ');
 while q <> nil do
  begin          {for each term}
   if (q^.s <> 0) then
    begin        {get data}
     if qdata.maxcoeff < q^.s then qdata.maxcoeff := q^.s;
     qdata.length := qdata.length + 1;
     if qdata.mincoeff > q^.s then qdata.mincoeff := q^.s;
     qdata.moduli := (q^.m <> ' ') or qdata.moduli;
    end;         {of get data}
   q := q^.next;
  end;
 end;                                            {getpolydata}
```

getmapdata

```
procedure getmapdata (v : map; var vdata : mapdata);
 var
  c : integer;
 begin
  for c := 1 to Cp do getpolydata(v[c], vdata[c])
 end;                                            {getmapdata}
```

getlimits

```
procedure getlimits (var data : mapdata; var max, min : integer);
 var
  i : integer;
 begin
  max := data[1].maxcoeff;
  min := data[1].mincoeff;
  for i := 2 to Cp do
   begin
    if max < data[i].maxcoeff then max := data[i].maxcoeff;
```

```
    if min > data[i].mincoeff then min := data[i].mincoeff;
   end;
  end;                                          {getlimits}

  function polycopy (old : polynomial; ix, iy, factor : integer; deleteold : boolean) : polynomial;   polycopy
   var                                          {calculate factor*x^ix*y^iy*old}
    newp, oldterm, newterm : polynomial;
 {when factor = 1, ix = iy = 0 this is used to remove terms with zero coefficient}
   begin
    newp:=nil;
    if factor = 0 then write(' error in polycopy ');
    while old <> nil do
     begin
      if (old^.x + old^.y + ix + iy <= k) and (old^.s<>0) then
       begin     {copy term:first add new term to polynomial}
        if newp=nil then
         begin new(newp); newterm:=newp end
        else
         begin new(newterm^.next); newterm:=newterm^.next end;
        with newterm^ do
         begin
          s := old^.s * factor;
          x := old^.x + ix;
          y := old^.y + iy;
          m:= old^.m;
          next:=nil;
         end;
       end;     {of save term}
      oldterm:=old;
      old := old^.next;
      if deleteold then dispose(oldterm)
     end;      {go on to next term}
    polycopy := newp
   end;                                         {polycopy}

  {C:DISPLAY PROCEDURES}

  procedure show (power : integer; ch : char);  {display ch^power}                    show
   begin
    if power <> 0 then
     begin
      write(ch : 1);
      if power <> 1 then write('^', power : 1)
     end;
   end;                                         {show}

  procedure wln (var counter : integer);                                              wln
   {note : counter usually indicates the number of objects printed on a line}
   begin writeln; counter := 0 end;             {wln}

  procedure displaypoly (p : polynomial; var printno : integer; lno : integer);       displaypoly
   var
    first : boolean;
   begin
    first:=true;
    while p <> nil do
     begin {go on to next term}
      printno := printno + 1;
      if  not first and (p^.s >= 0) then write('+' : 2);
      if p^.s < 0 then write('-' : 2);
      if first then first:= not first;
      if (p^.x + p^.y = 0) or (abs(p^.s) <> 1) then write(abs(p^.s) : 1);
      if p^.m <> ' ' then  write(p^.m) ;
      show(p^.x, 'x');
      show(p^.y, 'y');
```

```
    if (printno mod lno = 0) then wln(printno);
    p:=p^.next;
   end; {of go on to next term}
 if first then write('0');
end;                                                    {displaypoly}

procedure displaymap (v : map;  var printno : integer; lno : integer);      displaymap
 var
  c : integer;
 begin
  write('(');
  for c := 1 to Cp do
   begin
    displaypoly(v[c], printno, lno);
    if c <> Cp then write(', ')
   end;
  write(')');
 end;                                              {displaymap}

procedure displayTAf (var tf : vectorstore;  {subprocedures:-}              displayTAf
                      var wf : module;            {displaywf}
                      realdisplay : boolean);     {displaytf}
 var                                                {getname}
  lno, vno, printno, i, dim, j : integer;
  name : shortstring;              {nb : realdisplay false indicates that this}
  ch , ok : char;                   {is a display for debugging purposes and does}
  defaultdis, savedis : boolean;  {not include options to improve layout or save to a file}
  procedure Displaywf (var wf : module;       {subprocedure  showsingle}    Displaywf
                       n, lno : integer; var dim, printno : integer);
   const showH = true;
   var
    i, j, count1, count2 : integer;
    list : polylist;
    intrange : intset;
   procedure showsingle (terms : intset; count1, count2, i : integer;        showsingle
                         var printno : integer; showH : boolean);
    var      {display (monomials) or Hi\(monomials), depending on showH}
      j, c, n : integer;
    begin
     if (i + 3 - count1 + count2 + printno > lno) then wln(printno);
      c:=0;
     if showH then
      begin
       write(' H', i : 1);
       n:= i+1-count1;
       printno := printno + 1;
       if (count1 < i + 1) then write('\{')
       else write(',');
      end
     else
     begin write(' ', i : 1, ': C{');n:=count1; printno:=printno+1 end;
     for j := 0 to i do
      if (j in terms) then
       begin
        write(' ');
        show(j, 'x');
        show(i - j, 'y');
        printno := printno + 1;
        c:=c+1;
        if (c <> n) or (count2>0) then write(',')
       end;
      if (showH and (count1 < i + 1)) or (not showH and (count2=0)) then write('}')
```

```
 end;                                                    {showsingle}
begin                                                      {displaywf}
 wln(printno);
 writeln(' generators in row(', n : 1, ') are ');
 for i := 0 to k do
  begin            {run through k}
   count1 := 0;
   count2 := 0;
   intrange := [0..i];
   list := wf[n][i].sum;
   while list <> nil do
    begin  {count polynomials}
     if list^.poly <> nil then count2 := count2 + 1
      else write(' nil poly in list ');
     list := list^.next;
    end;
   for j := 0 to i do {count monomials}
    if j in wf[n][i].single then count1 := count1 + 1;
   dim := dim + count1 + count2;
   if count1 > (i div 2) then
    showsingle(intrange - wf[n][i].single, count1, count2, i, printno, showH)
   else if count1 > 0 then
    showsingle(wf[n][i].single, count1, count2, i, printno, not showH);
   if (count2 > 0) then {display polynomials}
    begin
     if (count1 = 0) and not (i = 1) then wln(printno);
     if (count1 > (i div 2)) then write('+');
     if (count1=0) or (count1>(i div 2)) then write('C{');
     printno := printno + 1;
     list := wf[n][i].sum;
     while list <> nil do
      begin
       write(' ');
       if (list^.poly <> nil) then displaypoly(list^.poly, printno, lno);
       if list^.next <> nil then write(',');
       list := list^.next;
      end;
     write('}')
    end            {display polynomials}
  end;             {run through k}
end;                                                       {displaywf}
procedure displaytf (var tf : vectorstore; var dim, printno : integer;
                     lno, vno : integer);
 var
  v : vectorlist;
  count, j : integer;
  printedvectors : boolean;
 begin
  printedvectors := false;
  for j := 3 downto 0 do
   begin
    count:=0;
    v := tf[j];
    wln(printno);
    if printedvectors and (count mod vno <> 0) then wln(printno);
    while v <> nil do
     begin          {display supplementary generators}
      if not printedvectors then
       begin
        printedvectors := true;
        writeln('supplementary generators are ');
       end;
      count := count + 1;
      dim := dim + 1;
      displaymap(v^.f^, printno, lno);
```

displaytf

```
      if v^.next <> nil then write(', ');
      if count mod vno = 0 then wln(printno);
      v := v^.next;
     end;            {display terms}
    end;
   if not printedvectors then writeln(' no supplementary generators ');
 end;                                                    {displaytf}
procedure getname(var name:shortstring);                                  getname
 var n:integer;
  begin
   writeln(' enter name of file (10 characters):');
   writeln('IF THERE IS ALREADY A FILE WITH THIS NAME IT WILL BE OVERWRITTEN!');
   for n:=1 to 9 do read(name[n]);
   readln(name[10]);
   writeln(' filename is ', name)
  end;                                                     {getname}
begin                                                      {displayTAf}
 savedis := false;
 ok:='y';
 for j := 1 to 2 do  {on j=1 send data to terminal screen : on j=2 send to a file}
  if (j = 1) or savedis then  {send data to relevant output}
   begin
    repeat      {until the display is ok}
     ch := 'n';
     if j = 1 then
      begin        {determine the display parameters}
       if realdisplay and (ok='n') then makequery(6, ' ', ch);
       defaultdis := (ch = 'n');
       if not defaultdis then  {get new printout parameters}
        begin
         writeln('The printer prints approximately ',CHARTOLINE:1,' characters on a line. ');
         write('How many supplementary generators to a line? (currently ',vno:1,') ');
         readln(vno);
         write('How many generators to a line? (currently ',lno:1,') ');
         readln(lno);
        end
       else  {use preset parameters}
        begin vno := DEFVTL; lno := DEFTTL end;
      end;   {set display parameters}
     wln(printno);
     wln(dim);        {wln writes a new line and puts the input value to zero}
     if savedis then begin getname(name); rewrite(output, name); end; {data is sent to file called name}
     write(' tf(M^',R:1,') + wf(N^',S:1,') mod M^', k + 1 : 1, ' for f = ');
     displaymap(f, printno, lno);
     writeln(' is as follows:-');
     for i := 1 to Cp do
      begin
       Displaywf(wf, i, lno, dim, printno);
       writeln;
      end;
     displaytf(tf, dim, printno, lno, vno);
     writeln;
     writeln(' cod = ', Odim - dim : 1);
     if savedis then rewrite(output,'/dev/tty');  {data output returned to terminal}
     if realdisplay and (j = 1) then makequery(4, ' ', ok)
     else ok := 'y';   {check display is satisfactory before saving}
    until ok = 'y';
    if not savedis then makequery(7,' ',ch);  {is the data to be saved?}
    savedis:=(ch = 'y');
   end;    {for j = 1 to 2}
end;                                                       {displayTAf}
```

{D: HCF and DIVISION PROCEDURES}

HCF

```
function HCF (m, n : integer) : integer;
 var
  rem, high, low, absm, absn : integer;
 begin
  absm:=abs(m);
  absn:=abs(n);
  if absm > absn then
   begin high := absm; low := absn; end
  else
   begin high := absn; low := absm end;
  if low > 0 then
   repeat
    rem := high mod low;
    high := low;
    low := rem
   until low = 0
  else if low < 0 then write(' bad data in HCF:low ');
  if (n <= 0) and (m <= 0) then high := -high;
  HCF := high;
 end;                                                    {HCF}
```

HCFpoly

```
function HCFpoly (p : polynomial) : integer;
 var
  d : integer;
 begin
  d := 0;
  while p <> nil do
   begin
    d := HCF(d, p^.s);
    p := p^.next;
   end;{while}
  HCFpoly := d
 end;                                                    {HCFpoly}
```

dividepoly

```
procedure dividepoly (p : polynomial; factor :integer);
 begin
  if (factor <> 1) and (factor <> 0) then
   while p <> nil do
    begin
     if p^.s mod abs(factor) <> 0 then write(' bad data in dividepoly ')
     else p^.s := p^.s div factor;
     p := p^.next;
    end
  end;                                                   {dividepoly}
```

dividemap

```
procedure dividemap (var f : map);
 var                                   {subprocedure:-HCFmap}
  coord ,d: integer;
 function HCFmap (f : map) :integer;
  var
   d,i : integer;
  begin
   d := HCFpoly(f[1]);
   for i := 2 to Cp do
    d := HCF(d, HCFpoly(f[i]));
   HCFmap := d;
  end;                                  {HCFmap}
 begin                                  {dividemap}
  d := HCFmap(f);
  for coord := 1 to Cp do
   dividepoly(f[coord], d)
```

HCFmap

```
end;                                    {dividemap}

procedure getmultipliers (scalar1, scalar2, max1, min1, max2, min2 : integer;   getmultipliers
                          var m1, m2 : integer; var multOK : boolean);
 var              {Find m1 and m2 so that m1*scalar1 = m2*scalar2}
  d :integer;     {check that m1*max1, m1*min1, m2*max2, m2*min2 are in range}
 begin
  d := HCF(scalar1, scalar2);
  m1 := scalar2 div d;
  m2 := scalar1 div d;
  multOK := (max1 < MAXVALUE div abs(m1)) and (-min1 < MAXVALUE div abs(m1))
        and (max2 < MAXVALUE div abs(m2)) and (-min2 < MAXVALUE div abs(m2))
 end;                                   {getmultipliers}

{E: ADDITION AND SCALAR MULTIPLICATION OF POLYNOMIALS}

procedure search (p : polynomial; x, y : integer;var m:char;                    search
                  var scalar :integer; var isin : boolean);
 begin      {find the coefficients of the term with powers x, y if it is in p}
  scalar := 0;  {write(' search ');}
  m:= ' ';
  isin := false;
  while (p<> nil) and not isin do
   begin
    isin := (p^.x = x) and (p^.y = y);
    if isin  then begin  scalar := p^.s; m:=p^.m  end;
    p := p^.next;
   end;
 end;                                   {search}

function addtototal (var total : polynomial;  newterm : polynomial) : polynomial;   addtototal
 var                        {add the monomial newterm to the polynomial total}
  inalready : boolean;                  { in the correct order}
  t,last : polynomial;
 begin{addtototal}
  inalready := false;
  if newterm <> nil then if newterm^.next <> nil then write(' error in addtototal ');
 if total = nil then total:=newterm
 else if newterm <>nil then
  begin
   t := total;
   last:=nil;
   while (t<> nil) and not inalready  do
    begin
     if (newterm^.x + newterm^.y <= t^.x + t^.y) and ((t^.m<>' ') or (newterm^.m = ' ')) then
      if (newterm^.x + newterm^.y < t^.x + t^.y) or (newterm^.x>= t^.x) then
       begin
        inalready := true;
         if (newterm^.x = t^.x) and (newterm^.y = t^.y) and (newterm^.m = t^.m) then
          begin
           t^.s:= t^.s + newterm^.s;
           dispose(newterm)
          end
         else
          begin
           if last = nil then total:= newterm else last^.next:=newterm;
           newterm^.next:= t
          end;
        end;
       if not inalready then
        begin
         last:= t;
         t := t^.next;
        end;
      end;
```

```
    if not inalready then last^.next := newterm;
   end; {end of putin}
  addtototal := total
end;                                                    {addtototal}

procedure scalarproduct (p : polynomial; scalar :  integer);                scalarproduct
begin
 if (scalar <> 1) then
  while p <> nil do
   begin
    p^.s := p^.s * scalar;
    p := p^.next;
   end
end;                                                    {scalarproduct}

procedure doaddn (var p, q : polynomial; m1, m2 : integer; m : char);                doaddn
 var                          {add m*m2*q to m1*p}
  g ,temp: polynomial;
  terms : integer;
begin
 scalarproduct(p, m1);
 if q<>nil then  g := polycopy(q, 0, 0, m2,KEEP) else g:=nil;
 terms := 0;
 if DEBUG <= 1  then
  begin
   write(' addpq ');
   displaypoly(p, terms, DEFTTL);
   displaypoly(g, terms, DEFTTL);
  end;
   while g <> nil do
    begin
     temp:=g;
     if m <>' ' then begin if  temp^.m<>' ' then write(' warning : product of  parameters ')
                                  else temp^.m:=m end;
     g := g^.next;          {add each monomial to p using procedure addtototal}
     temp^.next := nil;
     p := addtototal(p, temp)
    end;
  end;                                                  {doaddn}

procedure testaddn (p, q : polynomial; m, n :  integer; var addnOK : boolean);          testaddn
 var
  scalar :  integer;
  isin : boolean;
  ch : char;
begin                          {check that the coefficients of m*p+n*q are in range}
 addnOK := true;
 write(' testaddn ');
  while (p <> nil) and addnOK do
   begin
   search(q, p^.x, p^.y, ch, scalar, isin);
   isin:= isin and (ch = p^.m);
    if isin then
     begin
      if ((p^.s >= 0) and (m >= 0)) or ((p^.s <= 0) and (m <= 0)) then
       addnOK := (n * scalar <= MAXVALUE - m * p^.s)
      else if ((p^.s >= 0) and (m <= 0)) or ((p^.s <= 0) and (m >= 0)) then
       addnOK := (n * scalar >= -MAXVALUE - m * p^.s)
     end;
    p := p^.next;
   end
 end;                                                   {testaddn}
```

{F: STORING DATA}

addtowf

```
procedure addtowf (var wf : module;                          { DEBUG }
                   coord : integer; var p : polynomial; var pdata : polydata);
 var               {add polynomial p to the array wf[coord] or update data for p}
  tersp : integer;
  list,last : polylist;
  pisin : boolean;
 begin
  list := nil;
  tersp := 0;
  pisin := false;
{write(' addtowf -', coord : 1, ' ');}
  if DEBUG <= 2 then displaypoly(p, tersp, DEFTTL);
  if (p<>nil) then
   begin  {if p is a monomial without a parameter then add p to set of monomials}
    if (pdata.length = 1) and (p^.m = ' ') then
       wf[coord][p^.x + p^.y].single := wf[coord][p^.x + p^.y].single + [p^.x]
    else if (p^.x + p^.y> 0) and (p^.x + p^.y<= k) then
       begin              {otherwise add to list of polynomials}
        {find p if it is already stored or add a new record to the list if not}
         if wf[coord][p^.x + p^.y].sum = nil then
           begin
            new(list);
            wf[coord][p^.x + p^.y].sum:=list;
            wf[coord][p^.x + p^.y].sum^.next := nil;
           end {if wf.sum=nil}
         else
           begin
            list := wf[coord][p^.x + p^.y].sum;
            while (list <> nil) and not pisin do
             begin
              last:=list;
              pisin := ( p = last^.poly);  {find end of list or p}
              list := list^.next;
             end;{while list<>nil}
            list:= last;
            if not pisin then
             begin
              new(last^.next);
              list := last^.next;
              list^.next := nil;
             end;{if not pisin}
           end;{of create new record}
            {now put (new) data into (new) record}
       list^.poly := p;
       list^.data := pdata
      end;{ if length>1}
    end;{if length>0}
   if DEBUG <= 1 then
    displayTAf(tf, wf, DEFDIS);
  end;                                                   {addtowf}
```

getplace

```
  procedure getplace ( var newplace, coord : integer; var vdata : mapdata;
                       var zeroed : boolean);
  var                  {decide where to store the map v whose data is given}
    count, i : integer;
  begin
   count := 0;
   for i := 1 to 3 do
    if vdata[i].length > 0 then
     begin
      count := count + i;
      if i = 3 then
       count := count + 1
```

```
  end;
zeroed := false;
coord := 0;
case count of
 0 :
  begin                          {v is the zero map}
   newplace := 4;
   zeroed := true;
  end;
 1, 2, 4 :
  begin                          {v has one non-zero row}
   if count = 4 then
    count := 3;
   newplace := 4;
   coord := count;               {coord locates the non-zero row}
  end;
 3, 5, 6, 7 :
  begin                          {v has two or more non-zero rows}
   if count = 3 then
    count := 4;
   count := 7 - count;
   newplace := count;            {newplace locates the zero row if there is one}
  end;
 end;
end;                             {getplace}
```

storemap

```
procedure storemap (var v : map; place, coord : integer;  var vdata : mapdata;
                    var altwf : blist; var wf : module; var tf : vectorstore;
                    max, min : integer);
 {adds the map v to the data store tf or wf as appropriate}
```

addtotf

```
 procedure addtotf (var h : map; var hdata : mapdata; var tf : vectorstore;
                    place ,max, min: integer );
  var               {adds h, hdata, max and min to tf}
   v : vectorlist;
   order, i, terms : integer;
 begin
  terms := 0;
  if DEBUG <= 3 then begin write(' addtotf- ', place : 1, ' '); displaymap(h, terms, DEFTTL);end;
  v := tf[place];
  order := k + 1;
  for i := 1 to 3 do
   if h[i]<>nil then if order > h[i]^.x + h[i]^.y then order := h[i]^.x + h[i]^.y;
  if (order <= k) then
   begin
    if v <> nil then          {create a new record}
     begin
      while v^.next <> nil do v := v^.next;
      new(v^.next);
      v := v^.next;
     end
    else
     begin new(v); tf[place] := v; end;
    {add data to new record}
    new(v^.f);
    v^.f^ := h;
    v^.good:=true;
    v^.maxcoeff := max;
    v^.mincoeff := min;
    v^.data := hdata;
    v^.next := nil
   end;
  end;                        {addtotf}
```

```
begin                                   {storemap}
 if (place = 4) and (coord <> 0) then
  begin
   addtowf(wf, coord, v[coord], vdata[coord]);
   if vdata[coord].length = 1 then
   if v[coord]^.m=' ' then ruboutpoly(v[coord]);
   altwf[coord] := true;
  end
 else if place <> 4 then addtotf(v, vdata, tf, place, max, min);
 if (DEBUG <= 2) and (place <> 4) then displayTAf(tf, wf, DEFDIS);
end;                                    {storemap}

procedure tidypoly (var p : polynomial;  {removes terms with zero coefficient}        tidypoly
        var pdata : polydata);           {and divides out by HCF}
begin                                      {updates data}
 p:= polycopy(p,0,0,1,DELETE);{purpose : to remove terms with zero coefficient}
 dividepoly(p, HCFpoly(p));
 getpolydata(p, pdata);
end;

procedure tidymap (var h : map; var hdata : mapdata; var max, min : integer);        tidymap
 var                                       {as tidypoly}
   i : integer;
begin
 for i := 1 to Cp do
   h[i]:= polycopy(h[i],0,0,1,DELETE);
 dividemap(h);
 getmapdata(h, hdata);
 getlimits(hdata, max, min);
end;

{G: SIMPLIFYING GENERATORS}

procedure comparewf (var h : map; var p : polynomial; var hdata : mapdata;        comparewf
                     var pdata : polydata; var alth, altp : boolean;
                    doingpoly : boolean; coord : integer; var max,min : integer;place : integer);
 var                {remove terms in wf from map h or polynomial p if possible}
  printno,  length, c, coeff1, coeff2  : integer;
  newterm, poly : polynomial;
  list : polylist;
  ch : char;
  foundpoly, multnOK, addnOK : boolean;
 begin
  {write(' enter comparewf ');}
  printno := 0;
  altp := false;
  alth := false;
  if doingpoly then begin length := pdata.length; poly := p end
  else begin length := hdata[coord].length; poly := h[coord]; end;
  if DEBUG <= 2 then begin write(' compare '); displaypoly(poly, printno, DEFTTL); end;
  newterm := poly;
  while newterm <> nil do        {inspect poly term by term}
   begin
    foundpoly := false;
    if newterm^.x in wf[coord][newterm^.x + newterm^.y].single then
     begin                {remove monomials by putting the coefficient to zero}
      newterm^.s := 0;
      altp := doingpoly;
      length:= length-1;
      alth := not doingpoly;
     end
    else if (newterm^.s <> 0) and ((length > 1) or (place <> 4)) then
```

```
    begin        {search terms in list for a match with the current term of p}
     list := wf[coord][newterm^.x + newterm^.y].sum;
     while (list <> nil) and not foundpoly do
      begin
       if list^.poly<>nil then
       if (list^.poly^.x = newterm^.x) then
        if (list^.poly^.m=newterm^.m) or not list^.data.moduli then
         begin {a match has been found: get multipliers and check the numbers are not too large for the comput
          foundpoly := true;
          getmultipliers(list^.poly^.s, newterm^.s, list^.data.maxcoeff, list^.data.mincoeff,
                         max, min, coeff1, coeff2, multnOK);
           addnOK := multnOK and (abs(list^.data.maxcoeff * coeff1) < MAXVALUE - abs(max * coeff2))
                  and (abs(list^.data.mincoeff * coeff1) < MAXVALUE - abs(max * coeff2));
           addnOK := addnOK and (abs(list^.data.mincoeff * coeff1) < MAXVALUE - abs(min * coeff2)) and
                   (abs(list^.data.maxcoeff * coeff1) < MAXVALUE - abs(min * coeff2));
           if multnOK and not addnOK then testaddn(poly, list^.poly, coeff2, -coeff1, addnOK);
           if addnOK then
            begin
             if list^.poly^.m = newterm^.m  then  ch:= ' '
                                  else ch:=newterm^.m;
             doaddn(poly, list^.poly, coeff2, -coeff1, ch);
          {add -ch*coeff1*list^.poly to coeff2*poly}
             if not doingpoly then
              for c := 1 to Cp do if c <> coord then scalarproduct(h[c], coeff2);
             altp := doingpoly;
             alth := not doingpoly;
            end
          end;
         list := list^.next;
        end;
      end;
    newterm := newterm^.next;
  end;
 if alth then tidymap(h, hdata, max, min);
 if altp then tidypoly(p, pdata);
{ write(' exit comparewf ');}
 end;                                                    {comparewf}

 procedure comparemap (var h : map; var hdata : mapdata; var alth : boolean;  comparemap
                       checkwith : blist; var max, min ,place, co : integer);
  var                                  {feed the map to comparewf}
   coord, terms : integer;
   p : polynomial;
   pdata : polydata;
   altp, altmap, zeroed : boolean;
 begin
  p := nil;
  terms := 0;
  alth := false;
  if DEBUG <= 2 then displaymap(h, terms, DEFTTL);
  for coord := 1 to Cp do
   if checkwith[coord] then
    begin
     getplace( place, co, hdata, zeroed);
     if (place <> 4) or (coord = co) then
      comparewf(h, p, hdata, pdata, altmap, altp, DOMAP, coord, max, min, place);
     alth := alth or altmap
    end;
  getplace( place, co, hdata, zeroed);
 end;                                                    {comparemap}

 procedure sortwf (var wf : module; var altwf : boolean; coord : integer);          sortwf
```

```
var                                  {simplify wf as much as possible}
 h : map;            {output altwf true indicates simplification has occured}
 hdata : mapdata;
 last, store ,current: polylist;
 i, max, min: integer;
 alt, qalt, halt : boolean;
begin
 altwf := false;
 for i := 1 to 3 do h[i] := nil;
 max := 0;
 min := 0;
 repeat                    {until no further simplification is possible}
  alt := false;
  {write(' crosschecking row[', coord : 1, ']');}
  for i := 0 to k do
   begin      {simplify list i for each i}
    store := wf[coord][i].sum;
    current:=nil;
    while store <> nil do
     begin
      last:=current;
      current := store;
      store := store^.next;
      if last <> nil then last^.next := store
       else    wf[coord][i].sum:=store;          {remove current poly from list}
       comparewf(h, current^.poly, hdata, current^.data, halt, qalt, DOPOLY, coord, max, min, 4);
       if qalt then
        begin
         if current^.poly<>nil then
            addtowf(wf, coord, current^.poly, current^.data);
         if (current^.data.length = 1) and not current^.data.moduli then ruboutpoly(current^.poly);
         alt := true;              {further simplification may be possible}
         dispose(current);
         current:=last;
        end
       else begin          {replace current poly in list}
       if last <> nil then last^.next := current
        else    wf[coord][i].sum:=current;end;
       altwf := alt or altwf;  {record if wf has been simplified}
      end;
    end;
  until (not alt);
 end;                                                {sortwf}

 procedure sortTAf (var wf : module;{subprocedures: sorttf, comparetf}          sortTAf
        var tf : vectorstore);
  const DEBUG = 6;                    {simplify all generators as much possible}
  var
   i : integer;
   alt, init : blist;
   rec : blist4;
  procedure sorttf (var tf : vectorstore; var wf : module; var compcoord, altwf : blist;     sorttf
                    var alttf : blist4);
   var                                {compare maps to polynomials in wf}
    elt, last ,current: vectorlist; {remove maps marked as unwanted or zero}
    altmap: boolean;
    j, place, coord : integer;
   begin
    {write(' crosschecking tf ');}
    for j := 3 downto 0 do
     begin
      elt := tf[j];
      current:=nil;
      while elt <> nil do
       begin
```

```
      last:=current;                    {remove map being inspected}
      current:=elt;
      elt:=elt^.next;
      if last <>nil then last^.next:= elt else tf[j]:= elt;
      if current^.good  and (current^.f <> nil) then
       comparemap(current^.f^, current^.data, altmap, compcoord, current^.maxcoeff, current^.mincoeff, place, coord);
      if altmap or not current^.good then
         begin
          if current^.good and (current^.f <> nil) then
            storemap(current^.f^, place, coord, current^.data, altwf, wf, tf, current^.maxcoeff, current^.mincoeff)
         .else if (current^.f <> nil) then
           begin
            ruboutmap(current^.f^);
            dispose(current^.f)
           end;
          dispose(current);
          current:=last;
          if (place >= 0) and (place < 4) then
             altf[place] := true;
         end
        else        {replace map in list}
          begin if last<>nil then last^.next:= current else tf[j]:= current end;
       end;{while elt <>nil}
    end;{for j=3 to 0}
  end;                                          {sortf}
procedure comparetf (var tf : vectorstore; {subprocedures- displaylist}          comparetf
         var wf : module;                    {placeinmaps, constructlist}
         var altwf : blist;                  {sortlist, savelist}
         var altf : blist4);
  type                                  {simplify maps by comparison with maps}
   maplist = ^mapelt;
   mapelt = record                      {the list is used to store the old map}
     newmap, oldmap : mappointer;       {and its simplification}
     olddata, newdata : mapdata;        {this makes it easier to remove old and replace with new}
     save, done : boolean;
     oldplace, newplace, coord, newmax, newmin : integer;
     next : maplist;
    end;
  var
   c, place, {n,} i, listno, pno : integer;
   list : vectorlist;
   maps : maplist;
  procedure displaylist (list : maplist);                                        displaylist
   var                                  {display list : used in debugging}
    temsp : integer;
  begin
   temsp := 0;
   writeln;
   write(' present maps are:-');
    while list<> nil do
     begin
      if list^.newmap <> nil then displaymap(list^.newmap^, temsp, DEFTTL);
      if not list^.save then write(' not ');
      write(' save ');
      if not list^.done then write(' not ');
      write(' done ');
      list := list^.next;
    end;
  end;                                          {displaylist}
  procedure placeinmaps (f : mappointer; fdata : mapdata;                        placeinmaps
                         max, min, place : integer; var maps : maplist);
   var
    newelt : maplist;
    c : integer;
```

```
begin {write(' enter placeinmaps ');}
 if maps = nil then                {create a new record}
  begin
   new(maps);
   maps^.next := nil;
   newelt := maps;
  end
 else begin
   newelt := maps;
   while newelt^.next <> nil do
     newelt := newelt^.next;
   new(newelt^.next);
   newelt := newelt^.next;
  end;      {of create record}
 with newelt^ do        {put data in record}
  begin
   new(newmap);
   for c := 1 to Cp do newmap^[c] := polycopy(f^[c], 0, 0, 1,KEEP);
   oldmap := f;
   olddata := fdata;
   newdata := fdata;
   newmax := max;
   coord := 0;
   oldplace := place;
   newplace := place;
   newmin := min;
   save := false;
   done := false;
   next := nil;
  end; {write(' exit placeinmaps ');}
 end;
 procedure constructlist (f : mappointer; {subprocedure checkifin}          constructlist
        list : vectorlist; coord, place : integer; var maps : maplist);
  var                          {decide which maps to compare}
   occurs : boolean;
  procedure checkifin (p, q : polynomial; var occurs : boolean);             checkifin
   var
    scalar : integer;
    ch:char;
  begin
   occurs :=  (p<>nil);
   while (q <> nil) and occurs do
    begin
     search(p, q^.x, q^.y, ch, scalar, occurs);
     q := q^.next;
    end;
  end;                                        {checkifin}
 begin { write(' enter constructlist ');}
  while list <> nil do
   begin    {go through each map in list and decide whether or not to compare it with the map indicated by f}
    if list^.good then
     begin
      checkifin(f^[coord], list^.f^[coord], occurs);
      if occurs then placeinmaps(list^.f, list^.data, list^.maxcoeff, list^.mincoeff, place, maps);
     end;
    list := list^.next;
   end;{write(' exit constructlist ');}
 end;
 procedure sortlist (var maps : maplist; c : integer);                       sortlist
  var                          {compare maps in list 'maps'}
   alldone,  isin, multOK, addnOK, zeroed, notlonger, moduli: boolean;
   list : maplist;
   sh : mappointer;
   shdata : mapdata;
   ch,m:char;
```

```
  max, min, m1, m2, scalar, x, y, co, s : integer;
begin {write(' enter sortlist ');}
 repeat
  list := maps;
  s:=0;
  alldone := true;
  shdata[c].length := 20;
  if list <> nil then    {find shortest poly not yet done}
   begin
    if (not list^.done) and (list^.newdata[c].length > 0) and (alldone or (shdata[c].length > list^.newdata[c].length)
     begin
      alldone := false;
      sh := list^.newmap;
      shdata := list^.newdata;
      max := list^.newmax;
      min := list^.newmin;
     end;
    list := list^.next;
  end;  {of find shortest}
  if not alldone  and (sh <>nil) then if sh^[c]<>nil then
   begin
    list := maps;
    moduli:=false;
    for co:=1 to Cp do moduli:=moduli or shdata[co].moduli;
    x := sh^[c]^.x;
    y := sh^[c]^.y;
    while list <> nil do     {run through list}
     begin
      with list^ do
       if not done then
        begin
         if (newmap = sh) or (newdata[c].length = 0) then done := true
         else if newmap <> nil then  {attempt to subtract shortest from newmap}
          begin
           isin:=false;
           if newmap^[c]<> nil then search(newmap^[c],x, y, m, scalar, isin);
           isin:=isin and (not moduli or( m = sh^[c]^.m));
           if isin then    {get multipliers and check size of numbers}
            begin
             getmultipliers(sh^[c]^.s, scalar, max, min, newmax, newmin, m1, m2, multOK);
             addnOK := multOK;
             if DEBUG <= 2 then write(' m1,m2 ', m1 : 1, ' ', m2 : 1, ' ');
             for co := 1 to Cp do
              begin
               addnOK := addnOK and
                  (abs(shdata[co].maxcoeff * m1) < MAXVALUE - abs(newdata[co].maxcoeff * m2)) and
                  (abs(shdata[co].mincoeff * m1) < MAXVALUE - abs(newdata[co].maxcoeff * m2));
              addnOK := addnOK and
                  (abs(shdata[co].mincoeff * m1) < MAXVALUE - abs(newdata[co].mincoeff * m2)) and
                   (abs(shdata[co].maxcoeff * m1) < MAXVALUE - abs(newdata[co].mincoeff * m2));
             end;
            if multOK and not addnOK then
             begin
              addnOK := true;
              for co := 1 to Cp do if addnOK then testaddn(newmap^[co], sh^[co], m2, -m1, addnOK);
             end;
            if addnOK then      {add -m*m1*sh to m2*newmap}
             begin
              if (m = sh^[c]^.m)  then m:=' ';
              for co := 1 to Cp do doaddn(newmap^[co], sh^[co], m2, -m1, m);
              tidymap(newmap^, newdata, newmax, newmin);
              getplace( newplace, coord, newdata, zeroed);
```

```
                notlonger := true;          {decide whether newmap is a simplification}
                for co := 1 to Cp do notlonger := notlonger and (newdata[co].length <= olddata[co].length);
                list^.save := (newplace > oldplace) or
                              ((newplace = oldplace) and notlonger and (newdata[c].length < olddata[c].length))
              end;
            end;
          end;
        end;
      list := list^.next;
    end;
  end;
 until alldone;{write(' exit sortlist ');}
end;
procedure savelist (var maps : maplist; var tf : vectorstore; var wf : module;
                    var altwf : blist; var alttf : blist4);
 var                          {replace old maps with their simplifications}
  list : maplist;
  foundmap : boolean;
  store : vectorlist;
begin {write(' enter savelist ');}
 while maps <> nil do
  begin
   list := maps;
   maps := maps^.next;
   with list^ do
    begin
     if save then              {find old map and replace with newmap}
      begin
       foundmap := false;
       store := tf[oldplace];
       while (store <> nil) and not foundmap do
        begin
         foundmap := (store^.f = oldmap);
         if not foundmap then store := store^.next;
        end;
       if foundmap then
        begin
         if newmap <> nil then
          storemap(newmap^, newplace, coord, newdata, altwf, wf, tf, newmax, newmin);
         store^.good:=false;      {mark old map to be removed}
         if newplace <> 4 then
         alttf[newplace] := true;  {record that a simplification of tf has occurred}
        end
      end
     else if newmap <> nil then ruboutmap(newmap^);  {if new is not simpler}
    end;
   dispose(list);
   list := maps;
  end;{write(' exit savelist ');}
 end;
begin                                                {comparetf}
 repeat                                  {until no further simplification occurs}
{write(' loop 1 ');}
   for place := 0 to 3 do
    if alttf[place] then
     begin
      write(' place- ', place : 1, ' ');
      list := tf[place];
      alttf[place] := false;
      while list <> nil do
       begin
        pno:=0;
        maps := nil;
        for c := Cp downto 1 do
         if (c <> place) and (list^.f <> nil)and list^.good then
```

savelist

```
          begin
          { write(' element is ');
           displaymap(list^.f^,pno,10);}
           for i := 3 downto place do    {decide which lists should be searched}
            begin
             if (place = 2) and (i = 3) then listno := 1
             else listno := i;
             if (listno <> c) then constructlist(list^.f, tf[listno], c, listno, maps);
            end;
            {displaylist(maps);}
           sortlist(maps, c);
           { displaylist(maps);}
           savelist(maps, tf, wf, altwf, alttf);
          end;
         list:=list^.next;
        end;
      end;
  until not (alttf[1] or alttf[2] or alttf[3] or alttf[0])
 end;                                          {comparetf}
begin                                          {sortTAf}
 rec[0] := true;
 for i := 1 to 3 do begin alt[i] := true; rec[i] := true; end;
 repeat  {until no further simplification occurs}
  repeat {until no further simplification by comparison with polynomials occurs}
   if DEBUG <= 2 then displayTAf(tf, wf, DEFDIS);
   write(' crosschecking ');
   for i := 1 to 3 do init[i] := alt[i];
   for i := 1 to Cp do if alt[i] then sortwf(wf, alt[i], i);
   if init[1] or init[2] or init[3] then sorttf(tf, wf, init, alt, rec);
  until not (alt[1] or alt[2] or alt[3]);
  if DEBUG <= 2 then displayTAf(tf, wf, DEFDIS);
  write(' comparing supplementary generators ');
  comparetf(tf, wf, alt, rec);
 until not (rec[0] or rec[1] or rec[2] or rec[3] or alt[2] or alt[3] or alt[1])
end;                                          {sortTAf}
```

{H: CALCULATING GENERATORS}

```
procedure getwf (var wf : module; f : map);                                  getwf
 var                                    { subprocedures: polyproduct, newlist}
  listA, A, listB, store : polylist;   { addto, copylist, eraselist}
  order, i, j : integer;
  previous : polynomial;          {calculate wf given f}
  alteredwf : boolean;
 function polyproduct (p, q : polynomial) : polynomial; {multiplies p and q}   polyproduct
  var
   r, s, item, total : polynomial;
  begin
   total := nil;
   s := q;
   while s <> nil do
     begin
      r := p;
      if s^.s <> 0 then
       while r <> nil do
        begin
         if (r^.s <> 0)and ( r^.x + s^.x + r^.y + s^.y <=k) then
          begin
           new(item);
           item^.x := r^.x + s^.x;
           item^.y := r^.y + s^.y;
           item^.s := r^.s * s^.s;
           if r^.m<>' ' then item^.m:=r^.m else item^.m:=s^.m;
           item^.next := nil;
           total:=addtototal(total,item);
```

```
        end;
      r:=r^.next
     end;{end of multiply by s^.term}
    s := s^.next;
   end; {end of select s^.term}
 polyproduct := total
end;                                        {polyproduct}
function newlist (oldlist : polylist; q : polynomial; ordq : integer;
                  var wf : module) : polylist;
 var          {multiply each polynomial in oldlist by q and store in newlist}
  newl, v, w : polylist;                {store in wf if necessary}
  newpoly : polynomial;
  h : map;
  hdata : mapdata;
  newdata : polydata;
  i, ord,max, min : integer;
  alt, altmap : boolean;
begin  {newlist}
 new(newl);
 for i := 1 to Cp do h[i] := nil;
 max := 0;
 min := 0;
 newl^.next := nil;
 newl^.poly := one;
 newl^.no:=0;
 w := newl;
 v := oldlist;
 if v <> nil then
  while v^.next <> nil do
   begin {multiply v^.poly by q}
    v := v^.next;
    if v^.poly<>nil then
     begin
      ord := v^.poly^.x + v^.poly^.y + ordq;
      if ord <= k then
       begin
        newpoly := polyproduct(v^.poly, q);
        tidypoly(newpoly, newdata);
        comparewf(h, newpoly, hdata, newdata, altmap, alt, DOPOLY, 1, max, min, 4);
        if newpoly<>nil then
         if (newpoly^.x + newpoly^.y<= k) then
          begin
           new(w^.next);
           w := w^.next;
           w^.next := nil;
           w^.poly := newpoly;
           w^.no:=v^.no +1;
           w^.data := newdata;
          if w^.no>=S then addtowf(wf, 1, newpoly, newdata);
         end
      end;{of multiply}
    end;
   end;
  w^.next := nil;
  newlist := newl
end;                                        {newlist}
procedure addto (var oldlist : polylist; newlist : polylist);
 var                {add newlist to the end of oldlist}
  v : polylist;
begin
 v := oldlist;
 if v = nil then oldlist := newlist
 else
  begin
   if v^.next<>nil then
```

newlist

addto

```
    while v^.next <> nil do v := v^.next;
    if newlist <> nil then v^.next := newlist^.next
   end;
end;                                    {addto}
function copylist (list : polylist) : polylist;
 var
  newlist, newelt : polylist;
begin
 if list=nil then newelt:=nil else new(newelt);
 newlist := newelt;
 while list <> nil do
  begin
   newelt^.poly := polycopy(list^.poly, 0, 0, 1,KEEP);
   newelt^.data := list^.data;
   newelt^.next := nil;
   list := list^.next;
   if list<>nil then
     begin
      new(newelt^.next);
      newelt := newelt^.next;
     end;
  end;
 copylist := newlist;
end;                                    {copylist}
procedure eraselist (var list : polylist);
 var
  listmarker : polylist;
begin
 listmarker := list;
 while list <> nil do
  begin
   list := list^.next;
   dispose(listmarker);
   listmarker := list;
  end;
end;                                    {eraselist}
begin                                   {getwf}
 new(listA);
 listA^.poly := nil;
 new(listA^.next);
 A := listA^.next;
 A^.next := nil;
 A^.poly := one;
 getpolydata(A^.poly, A^.data);
 order := 0;
 A^.no:=0;
 repeat {get powers of f1}
  previous := A^.poly;
  new(A^.next);
  A^.next^.no:=A^.no +1;
  A := A^.next;
  A^.next := nil;
  A^.poly := polyproduct(previous, f[1]);
  tidypoly(A^.poly, A^.data);
  order := order + f[1]^.x + f[1]^.y;
  getpolydata(A^.poly, A^.data);
  if A^.no>=S then addtowf(wf, 1, A^.poly, A^.data);
 until order >= k;
{ writeln('powers of f1 calculated');}
 store := listA;        {=list of powers of f1}
 for i := 2 to Cp do {get rest of wf}
  if f[i]<>nil then
   begin
    order := 0;
```

copylist

eraselist

```
    listA := store;   {= list of products of powers of f1 to f(i-1)}
    if f[i]^.x + f[i]^.y> 0 then
     repeat
      listB := newlist(listA, f[i], f[i]^.x + f[i]^.y, wf);
      order := order + f[i]^.x + f[i]^.y;
      addto(store, listB); {add powers involving f1 to accumulated list}
      listA := listB
     until order >= k;
  { writeln(' Powers of f', name[i], ' calculated.');}
   end;
  listB:=nil;listA:=nil; eraselist(store);
  sortwf(wf, alteredwf, 1);
  write(' calculated wf');
  for i := 2 to Cp do
   for j := 0 to k do
    begin
     wf[i][j].single := wf[1][j].single;
     wf[i][j].sum := copylist(wf[1][j].sum);
    end
end;                                        {getwf}

procedure gettf (var Df : Dmap; var tf : vectorstore; var wf : module);        gettf
 var                  {calculate generators for tf given Df}
  i, j, newplace, coord, tp , max, min : integer;
  ch : char;
  checkwith, altwf : blist;
  g : Dmap;
  gdata : mapdata;
  altmap : boolean;
begin  {gettf}
 for i := 1 to Cp do checkwith[i] := true;
 max := 0;
 min := 0;
 tp := 0;
 for i := (k + 1 - forder) downto R do
  for j := 0 to i do  {calculate x^(i-j)y^j*Df for each i and j}
   begin
    {if (j = 0) then
     write(' order=', i : 1);}
     for ch := 'x' to 'y' do
      begin
       for coord := 1 to Cp do g[ch][coord] := polycopy(Df[ch][coord], i - j, j, 1,KEEP);
       tidymap(g[ch], gdata, max, min);
       comparemap(g[ch], gdata, altmap, checkwith, max, min, newplace, coord);
       storemap(g[ch], newplace, coord, gdata, altwf, wf, tf, max, min);
      end;{for i,j}
   end;
 write(' calculated tf');
end;                                        {gettf}

{I : GET INITIAL DATA }

procedure getvaliddata; {subprocedures : checkpoly, readpoly, diff}            getvaliddata
 var
  gooddata, altf : boolean;
  i, printno{,pno} : integer;
  reply, ch, addtof : char;
  procedure checkpoly (p : polynomial; c : integer; var gooddata : boolean);   checkpoly
  begin
   if p = nil then gooddata:= (c<>1)
   else gooddata := ((p^.x + p^.y > 0) and (p^.x + p^.y <= LIMIT))
  end; {checkpoly}
  procedure readpoly (var f : polynomial; ch : char; addtof : boolean);        readpoly
   var
```

```
  i,{r,} n, scalar : integer;
  moduli : char;
  item : polynomial;
begin
 writeln; { r:=0;}
 if addtof then write('How many terms do you wish to add to f', ch, '?')
 else write('Type in number of terms in f', ch, ' ');
 readln(n);
 makequery(10,' ',moduli);
 for i := 1 to n do
  begin
   writeln;
   scalar := MAXVALUE + 1;
   while (scalar < -MAXVALUE) or (scalar > MAXVALUE) do
    begin
     write(' Type in scalar ');
     readln(scalar);
     if (scalar <-MAXVALUE) or (scalar>MAXVALUE) then
       write(' scalar is out of range. Please re-enter a new value. ');;
    end;
   if scalar <> 0 then
    begin
     new(item);
     item^.s := scalar;
     if moduli = 'y' then begin write(' Type  in  modulus. '); readln(item^.m) end
      else item^.m :=' ';
     write('Type in power of x  ');
     readln(item^.x);
     write('Type in power of y  ');
     readln(item^.y);
     item^.next := nil;
     f := addtototal(f, item)
    end;
  end;
  writeln;
  f:= polycopy(f,0,0,1,DELETE);        {remove terms with zero coefficient}
{displaypoly(f, r,DEFTTL);}
 end;                                          {readpoly}
 function diff (p : polynomial; w : variablename) : polynomial;                    diff
  var                    {differentiate p}
   item, v : polynomial;
 begin
  v := polycopy(p, 0, 0, 1,KEEP);
  item := v;
  while v <> nil do
   begin
    case w of
     'x' : begin v^.s := v^.s * v^.x; if v^.x > 0 then v^.x := v^.x - 1 end;
     'y' : begin v^.s := v^.s * v^.y; if v^.y > 0 then v^.y := v^.y - 1 end;
    end;
    v := v^.next;
   end;
  item:= polycopy(item,0,0,1,DELETE); {remove terms with zero coefficient}
 diff := item
end;                                   {diff}
begin                                  {getvaliddata}
 printno := 0;
 repeat            {until reply=y;ie data is correct}
  if rerun then
   begin
writeln(' You may leave f(i) unchanged or add terms to it. Otherwise it will be replaced by your new input. ');
    for i := 0 to Cp do
     if i > 0 then makequery(1, name[i], changef[i])
     else makequery(2, name[i], changef[i]);
```

```
  end;  {of if rerun}
write(' Please enter R ');
readln(R);
write(' Please enter S ');
readln(S);
if changef[0] = 'y' then
  repeat
   write(' Please enter k.');
   readln(k);
   if not (k in range) then
    write(' k is out of the permitted range .Please reenter it');
  until k in range;
for i := 1 to Cp do          {read f[i]}
 if changef[i] = 'y' then
  begin
   if rerun then
    makequery(9, name[i], addtof);
    altf := true;
    repeat
     if addtof = 'n' then ruboutpoly(f[i]);
     readpoly(f[i], name[i], (addtof = 'y'));
     checkpoly(f[i], i, gooddata);
     if not gooddata then
      begin
       addtof:='n';
       write('f', name[i], ' is invalid .Please reenter it.')
      end;
    until gooddata
   end;                  {of read f[i]}
  writeln('Calculating tf(M^',R:1,') + wf(N^',S:1,')  mod M2^', k + 1 : 1);
  write('for  f=');
  displaymap(f, printno, DEFTTL);
  makequery(3, ' ', reply);              {verify data}
  rerun := true;
 until reply = 'y';
 for i := 1 to Cp do dividepoly(f[i], HCFpoly(f[i]));
 getmapdata(f, fdata);
 if altf then
  for ch := 'x' to 'y' do
   begin
    ruboutmap(Df[ch]);
    for i := 1 to 3 do Df[ch][i] := diff(f[i], ch);
    dividemap(Df[ch]);
   end;
{displaymap(Df['x'],printno,10);}
{displaymap(Df['y'],printno,10);}
  forder := f[1]^.x + f[1]^.y;
  for i := 2 to Cp do
   if f[i]<>nil then if (f[i]^.x + f[i]^.y< forder) and (f[i]^.x + f[i]^.y> 0) then
    forder := f[i]^.x + f[i]^.y;
  subrange := [1..k];
  Odim := 3 * ((k + 1) * (k + 2) div 2) - 3;
 end;                                    {getvaliddata}

 procedure initialisevar;                                    initialisevar
  var
   i, j : integer;
   ch:char;
 begin
  name[1] := '1';
  name[2] := '2';
  name[3] := '3';
  name[0] := 'k';
  for i:=1 to 3 do f[i]:=nil;
  for i:=1 to 3 do for ch:='x' to 'y' do Df[ch][i]:=nil;
```

```
for i := 0 to Cp do
 changef[i] := 'y';
rerun := false;
dialogue.query[1] := '                    Do you wish to alter f';
dialogue.query[2] := '                    Do you wish to alter ';
dialogue.query[3] := '                    Is this correct';
dialogue.query[4] := '                    Is the display satisfactory';
dialogue.query[5] := '                    Do you wish to continue';
dialogue.query[6] := ' Do you wish to alter the display parameters';
dialogue.query[7] := '                    Do you wish to save the data';
dialogue.query[8] := '                    Do you wish to print the data';
dialogue.query[9] := '                    Do you wish to add terms to f';
dialogue.query[10] := '                   Do you wish to add terms with moduli';
dialogue.response := ['n', 'y', 'N', 'Y'];
for i := 1 to Cp do
 for j := 1 to LIMIT do
  begin
   wf[i][j].sum := nil;
   wf[i][j].single := [];
  end;
for i := 0 to 3 do tf[i] := nil;
range := [1..LIMIT];
new(one);
with one^ do
 begin
  next := nil;
  x := 0;
  y := 0;
  s := 1;
  m := ' ';
 end;
end;                                        {initialisevar}

procedure informuser;                                          informuser
begin
 writeln(' This program calculates the terms in tf(M^R) + wf(N^S) for ');
 writeln('                f(x,y)= (f1(x,y),f2(x,y),f3(x,y)).');
 write('It will only run if f1,f2 and f3 have no constant terms or if f2 or f3 are zero. ');
 writeln('The calculation is performed modulo M2^k+1,  where  k < ',LIMIT+1:1);
writeln('In this version f may have parameter values.');
writeln(' Program written by D. Ratcliffe.');
 end;                                       {informuser}

begin                                       {MAIN PROGRAM }
 initialisevar;{name, changef, rerun, wf, tf, one, range , dialogue}
 informuser;
 repeat
  getvaliddata;{f, Df, Odim, forder, subrange}
  getwf(wf, f);          {calculate wf given f : store data in wf}
  getif(Df, tf, wf);  {calculate tf given Df : store data in tf or wf}
  sortTAf(wf, tf);     {simplify the generators of TAf}
  displayTAf(tf, wf, ACTDIS);  {display/save the data}
  ruboutdata(tf, wf);            {dispose of the data}
  makequery(5, ' ', continue);
 until continue = 'n';
 ruboutmap(f);
 for ch := 'x' to 'y' do
  ruboutmap(Df[ch]);
end.
```

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We give a complete list of normal forms for 3-jets whose 2-jet is $(x^2 + y^2, xy, 0)$;

$(xy, x^2 + y^2, x^3 + \lambda_1 x^2 y - xy^2 + \lambda_3 y^3) : \lambda_1 + \lambda_3 \neq 0,$

$(xy, x^2 + y^2, x^3 + \lambda_1 x^2 y + (1-\lambda_1) y^3)$

$(xy, x^2 + y^2 + y^3, \lambda_1 x^2 y + xy^2 + (1-\lambda_1) y^3)$

$(xy, x^2 + y^2, 0)$

$(xy, x^2 + y^2 + y^3, 0)$

$(xy, x^2 + y^2 + xy^2 + y^3, 0).$

$(xy, x^2 + y^2, x^2 y + \lambda_3 y^3) : \lambda_3 \neq -1,$

$(xy, x^2 + y^2 + y^3, x^3 + \lambda_1 x^2 y + (1-\lambda_1) y^3)$

$(xy, x^2 + y^2, x^3 + \lambda_1 x^2 y - xy^2 - \lambda_1 y^3)$

$(xy, x^2 + y^2 + y^3, x^3 + \lambda_1 x^2 y - xy^2 - \lambda_1 y^3)$

$(xy, x^2 + y^2 + xy^2 + y^3, x^3 + \lambda_1 x^2 y - xy^2 - \lambda_1 y^3)$

$(xy, x^2 + y^2, y^3)$

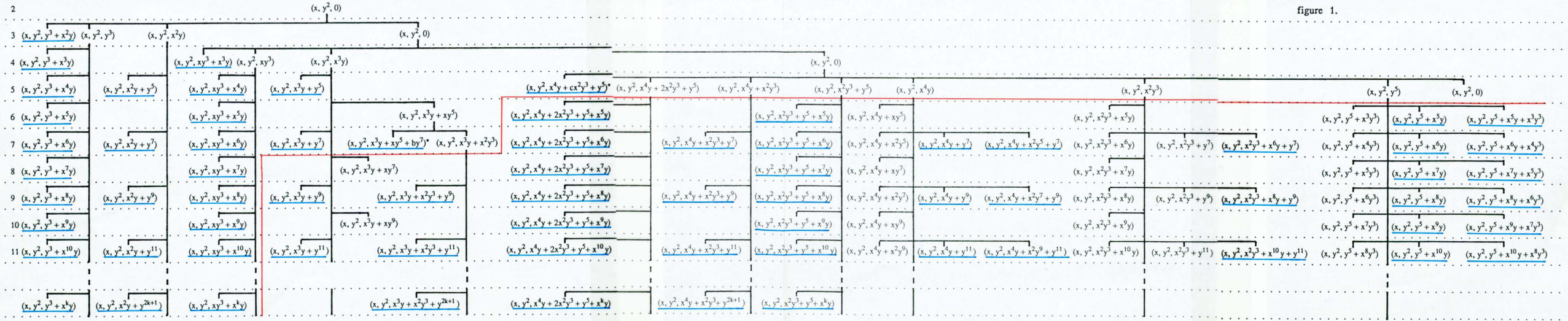
$(xy, x^2 + y^2, \lambda_1 x^2 y + xy^2 + (1-\lambda_1) y^3)$

$(xy, x^2 + y^2, x^2 y - y^3)$

$(xy, x^2 + y^2 + y^3, x^2 y - y^3)$

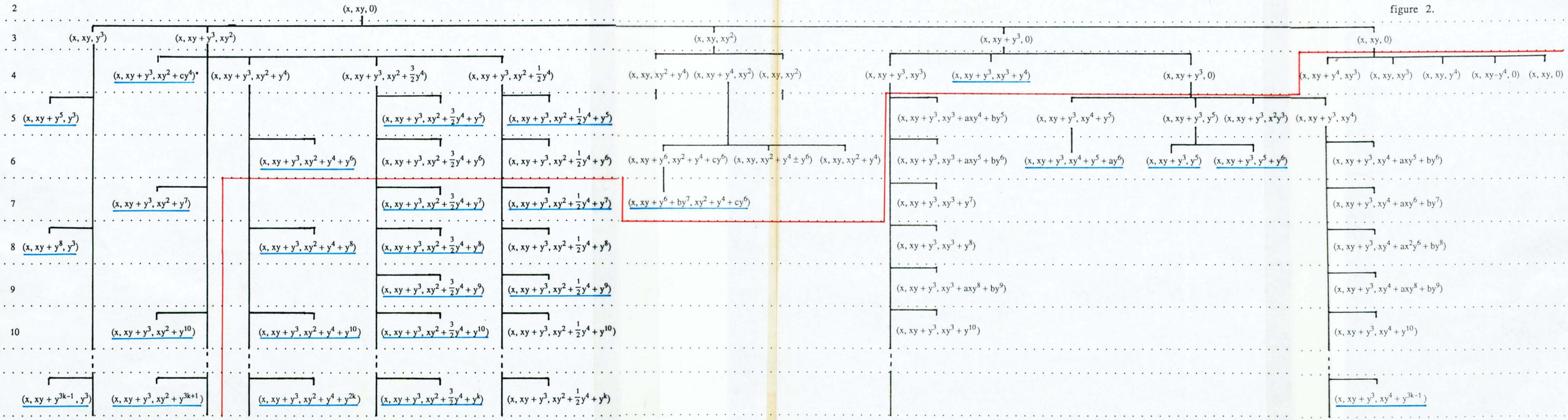
$(xy, x^2 + y^2 + xy^2 + y^3, x^2 y - y^3)$

figure 1.



*$c^2 \neq 4$, $27b^2 + 4 \neq 0$

figure 2.



*$c \neq 0, 1, \frac{1}{2}, \frac{3}{2}$.

figure 3.

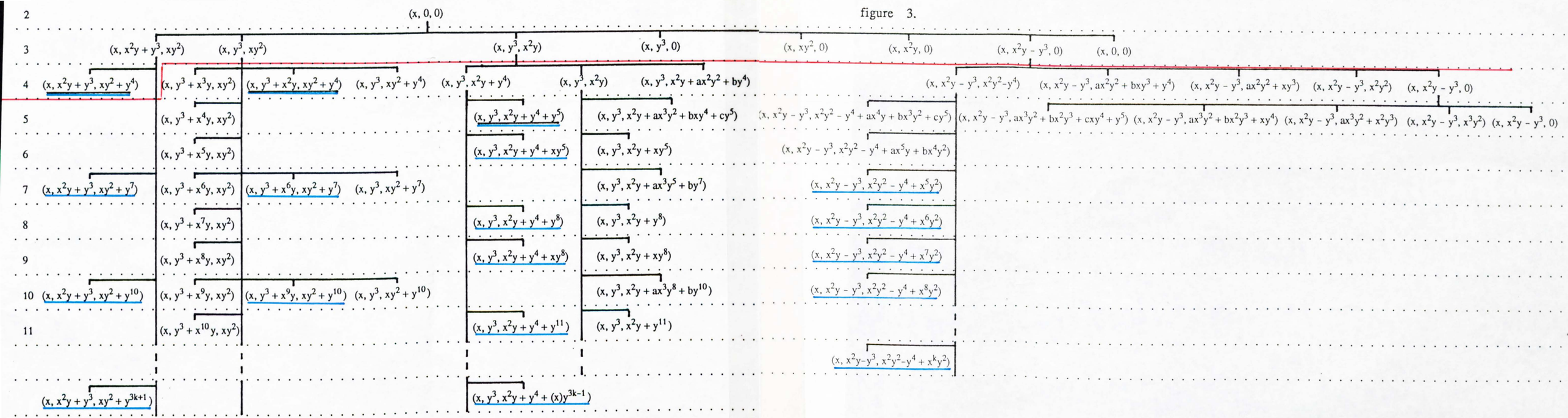
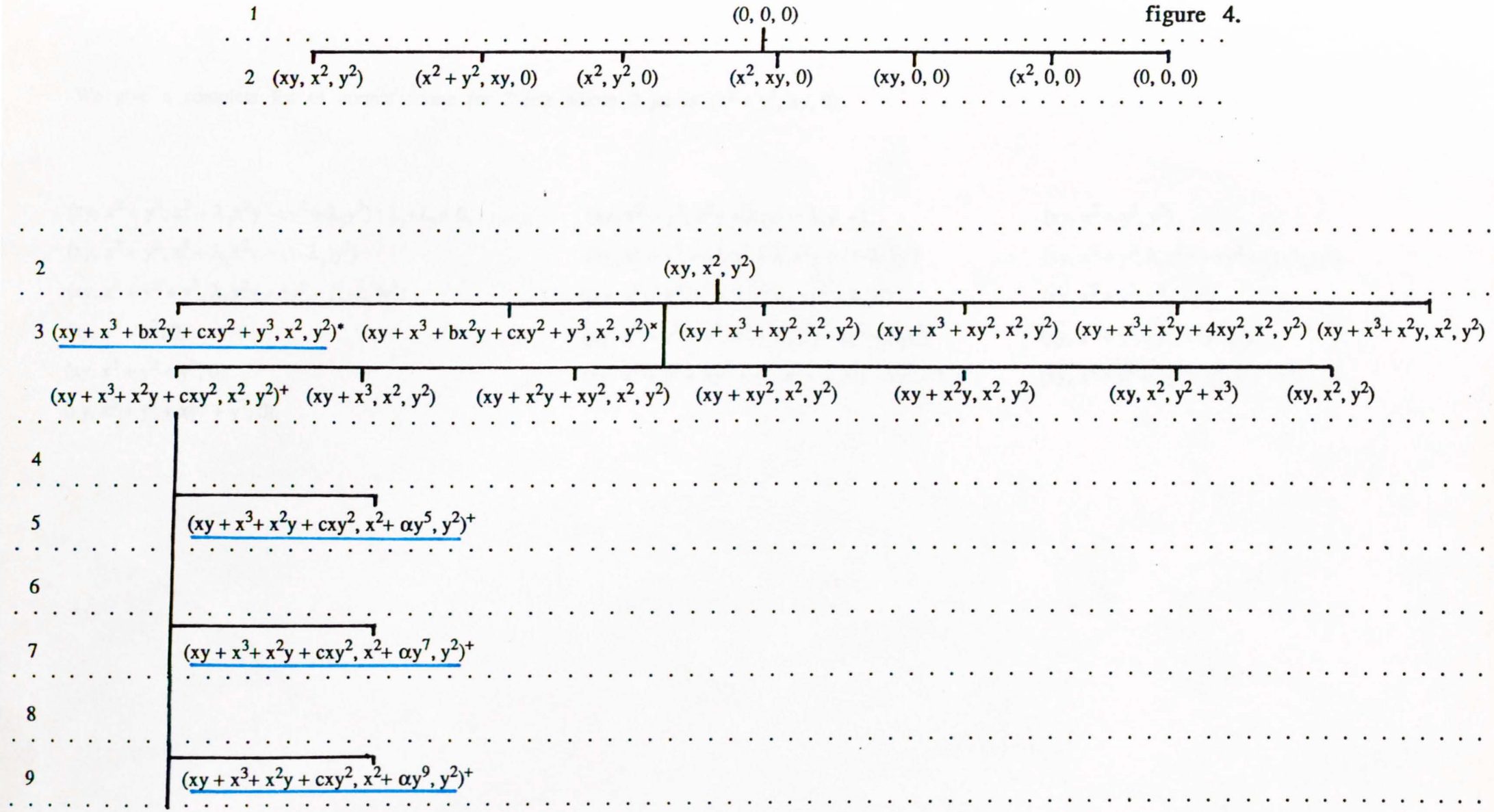

2
(x, 0, 0)
3
(x, x²y + y³, xy²)
(x, y³, xy²)
(x, y³, x²y)
(x, y³, 0)
(x, xy², 0)
(x, x²y, 0)
(x, x²y − y³, 0)
(x, 0, 0)
4
(x, x²y + y³, xy² + y⁴)
(x, y³ + x³y, xy²)
(x, y³ + x²y, xy² + y⁴)
(x, y³, xy² + y⁴)
(x, y³, x²y + y⁴)
(x, y³, x²y)
(x, y³, x²y + ax²y² + by⁴)
(x, x²y − y³, x²y²−y⁴)
(x, x²y − y³, ax²y² + bxy³ + y⁴)
(x, x²y − y³, ax²y² + xy³)
(x, x²y − y³, x²y²)
(x, x²y − y³, 0)
5
(x, y³ + x⁴y, xy²)
(x, y³, x²y + y⁴ + y⁵)
(x, y³, x²y + ax³y² + bxy⁴ + cy⁵)
(x, x²y − y³, x²y² − y⁴ + ax⁴y + bx³y² + cy⁵)
(x, x²y − y³, ax³y² + bx²y³ + cxy⁴ + y⁵)
(x, x²y − y³, ax³y² + bx²y³ + xy⁴)
(x, x²y − y³, ax³y² + x²y³)
(x, x²y − y³, x³y²)
(x, x²y − y³, 0)
6
(x, y³ + x⁵y, xy²)
(x, y³, x²y + y⁴ + xy⁵)
(x, y³, x²y + xy⁵)
(x, x²y − y³, x²y² − y⁴ + ax⁵y + bx⁴y²)
7
(x, x²y + y³, xy² + y⁷)
(x, y³ + x⁶y, xy²)
(x, y³ + x⁶y, xy² + y⁷)
(x, y³, xy² + y⁷)
(x, y³, x²y + ax³y⁵ + by⁷)
(x, x²y − y³, x²y² − y⁴ + x⁵y²)
8
(x, y³ + x⁷y, xy²)
(x, y³, x²y + y⁴ + y⁸)
(x, y³, x²y + y⁸)
(x, x²y − y³, x²y² − y⁴ + x⁶y²)
9
(x, y³ + x⁸y, xy²)
(x, y³, x²y + y⁴ + xy⁸)
(x, y³, x²y + xy⁸)
(x, x²y − y³, x²y² − y⁴ + x⁷y²)
10
(x, x²y + y³, xy² + y¹⁰)
(x, y³ + x⁹y, xy²)
(x, y³ + x⁹y, xy² + y¹⁰)
(x, y³, xy² + y¹⁰)
(x, y³, x²y + ax³y⁸ + by¹⁰)
(x, x²y − y³, x²y² − y⁴ + x⁸y²)
11
(x, y³ + x¹⁰y, xy²)
(x, y³, x²y + y⁴ + y¹¹)
(x, y³, x²y + y¹¹)
(x, x²y−y³, x²y²−y⁴ + xᵏy²)
(x, x²y + y³, xy² + y³ᵏ⁺¹)
(x, y³, x²y + y⁴ + (x)y³ᵏ⁻¹)

figure 4.



$^{*}4c^3+4b^3+27-b^2c^2-18bc \neq 0$, $^{\times}4c^3+4b^3+27-b^2c^2-18bc = 0$ $^{+}c(c+4) \neq 0, \alpha \neq 0$.