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An Investigation
Concerning Some Recent Developments
in Growth Theory

by

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Abstract

In chapter I, we model monopolistic competition in the spirit of Blanchard and Kiyotaki (1987) and we study the implications of this market structure for the existence of dynamically inefficient equilibria. We show that, with free entry, the presence of some pure profit does not rule out dynamic inefficiency, while the assumption of blockaded entry makes dynamic inefficiency impossible, since every firm grows at the aggregate growth rate and all future profits are capitalised in advance.

Chapter II is devoted to the analysis of some recent development of the theory of long-run endogenous growth. We build a model where technology is non rival but partially specific to each firm, and monopolistic competition is modelled assuming that output is produced by means of a fixed measure of intermediate goods. The main result obtained in this chapter is that the growth rate, with infinite lives, is independent from the degree of monopolistic power; with finite lives this relation becomes more complex.

The purpose of chapter III is to show that a negative relationship between capital accumulation and money growth is not incompatible with the common practice of inserting money into the utility function in the fashion of Sidrauski (1967). Adopting an instantaneous utility function allowing for a non-unit elasticity of substitution between real money balances and consumption, we find that the occurrence of the Tobin effect depends on the values of the parameters.

In chapter IV we consider the effect of technological uncertainty on welfare in an endogenous growth model based on positive spillovers. We encompass the presence of non traded labour income by means of contingent claim analysis and we find that a reduction of risk may be welfare damaging in presence of (Hicks-neutral) technological shocks. We consider also the effects of an additional distributive disturbance and we show that a positive correlation of the capital income share with the technological shock reduces the parameters set where the perverse effect occurs.

In chapter V we devote again our attention to the analysis of a Sidrauski-type model, adopting a stochastic framework. We consider the role of monetary uncertainty in the Blanchard perpetual youth framework and we show that, with C.R.R.A. preferences, both the monetary disturbance and its covariance with the real process affect consumption and therefore growth. Hence we highlight a strong "monetary randomness relevance" result.

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Chapter II

Endogenous Growth and Overlapping Generations in a Model of Monopolistic Competition with Blockaded Entry.

Chapter III

Growth Models with Money in the Utility Function: a General Treatment of the Transitional Behaviour

Introduction

During the last decade, growth theory has become one of the central issues in macroeconomics. The sudden reawakening of this research field is related to Romer's (1986) seminal paper, which fostered the outbreak of contributions describing models characterised by sustained economic growth. However, the theoretical and empirical concern with endogenous growth has had an impact also on the traditional neoclassical theory. This variety of interests characterises my research as well: the present Dissertation mainly addresses to a set of issues, such as monopolistic competition, overlapping generations, monetary growth and uncertainty in theoretical endogenous growth models; however it also appraises some aspects of those topics in more traditional frameworks.

The first chapter actually is, from a logical standpoint, antecedent to the attempt to endogenise the growth rate of an economic system. Its aim is the reappraisal of the possibility of dynamically inefficient equilibria in an overlapping generations model characterised by imperfectly competitive markets. Our starting point is the observation that any deviation from perfect competition implies the presence of pure profits (of rents in Tirole's (1985) terminology) and hence of an asset market. Tirole's analysis establishes that, if rents per period increase at the (asymptotic) rate of economic growth and they can be capitalised before their creation, a perfect foresight equilibrium must be efficient. If it were not, the rent per period would grow at a rate exceeding the rate of interest and its market value would be infinite.

In this chapter we model a monopolistic competition framework in the spirit of Blanchard and Kiyotaki (1987), who in turn build on Dixit and Stiglitz (1977). This approach is convenient, since it preserves a remarkable degree of aggregation, while allowing for the presence of many firms, each of which enjoys some market power. We introduce several alternative assumptions concerning the entry possibilities for new firms. It turns out that each of these hypotheses involves different implications for the existence of dynamically inefficient equilibria. In particular, we show that, with free entry, the presence of some pure profit does not rule out dynamic inefficiency. In those cases, the growth rate of each firm is lower than the aggregate one; hence the discount rate adopted by individual firms is not high enough to exclude the possibility of inefficient equilibria. On the contrary, the assumption of blockaded entry makes dynamic inefficiency impossible, since every firm grows at the aggregate growth rate and all future profits are capitalised in advance.

We think that these results are interesting for two different aspects. First, if the market structure excludes dynamic inefficiency, the possibility of running a rational Ponzi game is precluded as well. Hence, the market structure becomes a key characteristic in assessing the role of public debt. Second, our result may be referred to the empirical evidence that the US economy is dynamically efficient.

In chapter II, our interest in the role of imperfectly competitive market structures is closely related to the recent development of the theory of long-run endogenous growth.

In order to understand the role of monopolistic competition in endogenous growth models, we attempt to classify the existing literature according to the reasons provided to justify the use of a productive structure which allows for a sufficiently high and non decreasing marginal product of accumulable factors. We may identify three broad groups. The first one, including Romer (1986), Jones and Manuelli (1990) and Rebelo (1991), assumes constant or increasing returns to the accumulable inputs in (at least) one sector of the economic system, often advocating Marshallian externalities to justify this hypothesis. The second group, following Lucas (1988), makes explicit the possibility of accumulation of human capital. Finally, the third one relies on the effects of an increasing stock of knowledge. In contributions such as Romer (1990), Helpman and Grossman (1991) and Aghion and Howitt (1992), the role of imperfect competition comes into play. Recent research has attempted to incorporate industrial innovations into growth theory starting from the "schumpeterian" idea that entrepreneurs invest resources in the hope of discovering something of commercial value. Hence, potential innovators expect to be able to get a profit from their research: firms must be able to sell their products at a price exceeding unit cost.

Chapter II is motivated by some scepticism about the positive link between the degree of monopolistic rents and the growth rate, via the amount of resources devoted to research, which characterises many leading models of innovation and growth. Grossman and Helpman (1991, ch. 4) consider national product as the output of a competitive industry which uses several different intermediate inputs. Each input has its own "quality ladder", that is a boundless sequence of quality improvements. Potential innovators invest in research to "step up the ladder" for one of the intermediate goods and enjoy a rent determined by the difference between their good and the second best one. In the model there is free entry, in the sense that every firm can undertake the research required to obtain a better variety of any good. Hence, higher profits change into higher R&D expenditure and eventually into faster growth.

Romer (1990) and Helpman and Grossman (1991, ch. 3) take the alternative route of building a model of increasing product variety, which regards again national product as a homogeneous "final" good, obtained in a competitive setting, via a number of intermediate goods, which are imperfect substitutes. Therefore, the producers of these goods enjoy some pure profit. However, production of every intermediate good entails the purchase of a blueprint; the cost for that equals, due to free entry, the discounted stream of subsequent profits. Again, the effect of imperfect competition on growth is positive: the higher the monopolistic profits for the intermediate good producer, the higher the amount of investment in research.

To challenge this view about the link between the growth rate and the degree of monopoly, we build a model where technology is non rival but partially specific to each firm, and monopolistic competition is modelled assuming that output is produced by means of a fixed measure of intermediate goods. In our model, entry is blockaded, since we assume that the acquisition of the "specific knowledge" is costly while Bertrand competition prevails within the market for every intermediate input. The amount of resources devoted to research is the outcome of an optimal intertemporal program.

The main result obtained in this chapter is that the growth rate, with infinite lives, is independent from the degree of monopolistic power; however, with finite lives this relation becomes more complex. It is interesting to remark that, while the positive link between monopolistic competition and growth is cut off, the degree of monopolistic power does not turn out to be hindering for growth because of the implied reduction in the interest rate.

In chapters III to V we abandon the monopolistic competition framework, modelling growth without describing the process of technological improvement. Hence, we often invoke the presence of a Marshallian externality to be able to work in an environment with sustained economic growth. We pay this price in order to be able to investigate a set of issues which can hardly be fitted in a fully specified model of growth with accumulation of knowledge.

The starting point of chapter III is the perception that many empirical studies do not support the Tobin view of a positive relationship between capital accumulation and nominal money growth. The purpose of our contribution is to show that a negative relationship between capital accumulation and money growth is not incompatible with the common practice of inserting money into the utility function in the fashion of Sidrauski (1967).

We first consider a traditional growth model which departs from Fischer's (1979) analysis only insofar as the instantaneous utility function allows for a non-unit elasticity of substitution between real money balances and consumption. This relaxation is justified on the ground that, within Sidrauski's approach, the degree of substitution between the arguments of the utility function should depend on the characteristics of the transaction technology. We study the transitional dynamics of the model and we show that there are situations, depending on parameters values, where the Tobin effect is reversed. This analysis is performed partly analytically and partly by means of numerical techniques.

Since linear-in-capital models have been recently considered a useful simplification to study various issues in a growth framework, we also analyse a model of this type, highlighting its lack of transitional dynamics. The immediate jump of the system to its steady growth rate configuration entails the superneutrality of money, whose growth rate, in an infinitely lived agents framework, may have real effects only during the transition. Therefore, we resort again to numerical techniques to study a model where the production function is such that the marginal productivity of capital is only asymptotically constant, as in Jones and Manuelli (1990). Again, we will be able to highlight situations where the Tobin effect prevails and cases where capital accumulation is reduced by an increase in monetary growth. Interestingly, we will show that the stock of capital is permanently altered by temporary variations in the money growth rate. Hence, we provide an example of the hysteresis effect which characterises endogenous growth models.

Chapters IV and V simplify the production side of the models to linear-in-capital technologies. This further restriction is made in order to be able to indroduce uncertainty by means of geometric brownian motions, without abandoning closed form solutions for the models. However, we do not push this strategy to the point of excluding labour from the production function. Rather, our aggregate models rely on externalities.

This choice is particularly important in chapter IV, where we consider the effect of technological uncertainty on welfare. In an endogenous growth model based on positive spillovers, this effect turns out to be non trivial, since precautionary savings affect not only the level of income but also its growth rate. Hence, a reduction in uncertainty may slow growth and it can therefore be welfare damaging whenever growth is suboptimal, as it is in spillover models. However, insofar as we deal with risk averse individuals, a reduction of the growth rate caused by a decrease in uncertainty does not necessarily imply harmful consequences.

In chapter IV we analyse this point, which is similar to the one made by Devereux and Smith (1994), modelling technological uncertainty by means of geometric brownian motions. The presence of non traded labour income is encompassed by means of contingent claim analysis, since the asset entitling the holder to future labour income has been assumed to be non-traded, due to a moral hazard problem. In this setting a reduction of risk may be welfare damaging in presence of (Hicks-neutral) technological risk; this result, which contrasts to

Devereux and Smith's one, is our main finding. This striking difference is to be ascribed to the chosen stochastic structure: the impact of the standard deviation of the technological process on the expected growth rate is much higher in our model, due to Ito's lemma, which implies first order effects for standard deviations.

We consider also the effects of an additional distributive shock and we show that a positive correlation of the capital income share with the technological disturbance reduces the parameters set where the perverse effect occurs. Hence, a distributive shock proves important in re-establishing the traditional welfare-augmenting effect of a reduction in risk.

In chapter V we devote again our attention to the analysis of a Sidrauski-type model, adopting a stochastic framework, whose production side is similar to the one studied in chapter IV. It seems natural, at this stage, to extend the analysis to consider the role of monetary uncertainty. Since none of the existing contributions encompasses the case of a finite horizon, we investigate the role of "ongoing" monetary policy volatility in the Blanchard (1985) perpetual youth framework. This approach parallels the many existing works which study the role of monetary policy in nonstochastic frameworks, establishing the Tobin effect (e.g. Marini and van der Ploeg, (1988), and van der Ploeg and Alogoskoufis, (1994)). In this context we use the traditional restrictions concerning preferences, discussed in chapter III.

In chapter V we extend the existing literature by abandoning, within the intertemporal C.R.R.A. utility function class, the hypothesis of a unitary degree of relative risk aversion. We are able to solve the model guessing that the maximum value function is age dependent even if the portfolio shares turn out to depend on the level of individual wealth and hence on age. Our analysis allows us to show that the nominal interest rate is affected not only by the money growth rate and by the standard deviation of the monetary disturbance but also by the elasticity of intertemporal substitution and by the covariance between the technological disturbance and the nominal money process. Since the nominal interest rate affects consumption, through the same channels highlighted by the literature about non-stochastic models, we obtain a strong "monetary randomness relevance" result. In fact, both the monetary disturbance and its covariance with the real process affect consumption and therefore growth.

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Chapter I

Agents Heterogeneity, Monopolistic Competition and Dynamic Inefficiency

1. Introduction

Since Diamond (1965) seminal contribution, the robustness of the possibility of dynamic inefficient equilibria in overlapping generations models has been analysed in various alternative frameworks.

In this chapter, we address the issue of the implications of real rents (dividends), including the existence of an asset market. Hence, our work is deeply related to Tirole's (1985) contribution, which studies, in an otherwise standard version of Diamond's model, not only the possibility of bubbles but also the implications of an asset that brings a real rent. It that paper, it is shown that, if rents per period increase at the (asymptotic) rate of economic growth, a perfect foresight equilibrium must be efficient. Were this not true, the rent per period would grow at a rate exceeding the rate of interest. Hence, its market value would be infinite. Tirole (1985, p. 1074) provides several examples of assets bringing a real rent, such as natural resources, land, decreasing return to scale technologies and paintings and jewels for their consumption value.

However, most of his analysis is performed assuming that the aggregate quantity of rent is exogenously fixed in terms of output. Hence, dynamic efficiency remains possible and so does also the presence of asset bubbles. Moreover, Tirole remarks that, since rents are created over time, often they cannot be capitalised before their creation. In his own words (1985, p. 1080): "For example a painting to be created by a 21st century master cannot be sold by the painter's forebears." In such cases, bubbles (and dynamic inefficiency) need not be inconsistent with rents per period growing at the same rate of the economic system, since the flow of rents stemming from a single asset does not grow and must be capitalised using the interest rate.

In contrast to this analysis, McCallum (1987) and Homburg (1991), assign to land an explicit role in production of aggregate output and rule out inefficient equilibria. In fact, in their models, the marginal product of land (the rent) grows at the asymptotical growth rate of output[1]. Clearly, this result depends on the characteristics of the aggregate production function; it does not hold, as suggested by O'Connel and Zeldes (1988, pp. 441-2, fn. 19), whenever the land income share vanishes in the long run. Homburg (1991) and Rhee (1991) provide two different formalisations for this point. Moreover, Rhee notes that the decline in the US land income share, during the post-war period, has not been quick enough to be conclusive.

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[1] A similar point has been made by Muller and Woodford (1988, p. 962) while considering a model where finitely and infinitely lived agents coexist.

The case of a decreasing return to scale technology has been analysed by Dechert and Yamamoto (1992) in an overlapping generations model with no population growth[2]. In their setting, rents are distributed to shareholders in the form of dividends and these agents, who are old, sell (non bubbly) stocks to the young. Since the equities values would approach infinity as the interest rate gets closer and closer to naught, Dechert and Yamamoto conclude that "the stock market serves the same purpose as the transversality condition in an infinite-horizon growth model (1992, p. 399)."

This model suffers from two drawbacks: the number of firms is not determined and, in case of population growth, the share of rents on output would continuously grow over time, an implication which seem rather counterfactual[3].

In this paper, we take a route that is similar to, but different from, the one followed by Dechert and Yamamoto. We study a monopolistic competition framework, built in the spirit of Blanchard and Kiyotaki (1987) and Kiyotaki (1988), where the presence of rents is associated to the deviation from competitive behaviour[4]. We introduce several alternative assumptions concerning the entry possibilities for new firms. It turns out that each of these hypotheses bears

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[2] That model is stochastic, but their point concerning dynamic efficiency can be easily recasted in a deterministic framework.

[3] If we assume that each agent runs his own firm due to some indivisibility in labour, thereby tying the number of firm to the number of young people, we avoid such an implication but we are no longer able to exclude dynamic inefficient equilibria, since we are back to Tirole's case of rents which cannot be capitalised in advance.

[4] These deviations seem empirically relevant. For example, Hall (1988) estimated, for the U.S. economy, the ratio of the difference between price and marginal cost to price in various sectors and found it significantly different from zero.

different implications for the existence of dynamic inefficient equilibria. In particular, we show that, with free entry, the presence of some pure profit does not rule out dynamic inefficiency; on the contrary, if entrance is blockaded, the economic system must be efficient. This is interesting in at least two respects. First, if the market structure excludes dynamic inefficiency, also the possibility of running a rational Ponzi game is precluded. Hence, the role of public debt can be influenced by the market structure. Moreover, our result may be referred to the empirical evidence that the US economy is dynamically efficient: the difference between gross profit and investment, reported by Abel et. al. (1989), could be partly related to the presence of monopolistic distortions.

In what follows, we start summarising the "consumers' side", adopting Buiter's (1988) framework, which allows both for a positive probability of death at the individual level and for agents disconnectedness in the sense of Weil (1989)[5]. In section 3 we present our monopolistic competition framework, we then discuss the role of the asset market under various assumptions concerning the structure of sunk costs and the entry possibility for firms (section 4). We assume away, throughout this chapter, government debt, public expenditure and hence taxation. Section 5 concludes.

2. Aggregate consumption and asset accumulation

Following Blanchard (1985), we assume that each individual agent faces a constant instantaneous probability of death, λ, which also represents, due to the law of large numbers, the fraction of each cohort that dies at each instant. This hypothesis, coupled with the one of a constant birth rate β, as in Buiter (1988), has

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[5] The works by Blanchard (1985) and Weil (1989) imply the possibility of dynamic inefficiency when production is competitive.

the relevant merit of allowing for aggregation.

2.1 Individual consumption

To keep the analysis as simple as possible, we adopt, at the single agent's level, a logarithmic specification for the time separable utility function. Thus, the representative individual born at time s maximises, at time t:

$$U(t,s) = \int_t^{\infty} ln[c(\tau,s)]\, e^{-(\theta+\lambda)(\tau-t)} d\tau$$

$$\text{s.t.} \quad \overset{o}{a}(t,s) = [r(t)+\lambda]a(t,s) + w(t,s) - c(t,s)$$

where $c(t,s)$ is consumption, $a(t,s)$ the stock of assets and $w(t,s)$ labour income, all considered at time t for the individuals born at time s; θ and $r(t)$ are, respectively, the intertemporal time preference rate and the interest rate. Notice that the usual Blanchard-Yaari actuarially fair insurance mechanism is at work. The following "no-Ponzi game" condition holds:

$$\lim_{\tau\to\infty} a(\tau,s) \exp\left(-\int_t^{\tau} [r(z)+\lambda]dz \right) = 0$$

We assume, as in Blanchard, ((1985), p. 235), that the effect of retirement can be stylised letting the labour income decline with age at a constant rate ρ. With logarithmic preferences, a necessary condition for dynamic inefficiency in Blanchard's model is a sufficiently high ρ, namely, $\rho>\theta$, an hypothesis that we maintain[6].

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[6] With C.I.E.S. preferences, the necessary condition becomes $(1-S)\lambda+\rho>S\theta$, where S is the

Following usual methods, it is possible to show that the consumption behaviour, at the individual level, is described by the following equation:

$$c(t,s) = (\theta+\lambda)[a(t,s)+h(t,s)]$$

where $h(t,s)$, human wealth, is defined as:

$$h(t,s) = \int_t^{\infty} w(\tau,s) \exp\left(-\int_t^{\tau} [r(z)+\lambda]dz \right) d\tau$$

2.2 Population dynamics and aggregation

Given our assumptions concerning death and birth rates, the population at time t is:

$$N(t)=N(0)\, e^{nt}; \qquad (1)$$

we set $N(0)=1$ with no loss of generality; $n=\beta-\lambda$ is the constant population growth rate.

The population aggregate corresponding to any individual stock or flow variable $x(t,s)$, is indicated with $X(t)$ and defined as:

$$X(t)= \int_{-\infty}^{t} x(t,s)\beta e^{\beta s} e^{-\lambda t} ds \qquad (2)$$

A possible explanation for the aggregation rule (2) starts from the definition

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elasticity of intertemporal substitution. Hence, the lower S, the less tight becomes this condition.

of the share of people born at s who survive at time t:

$$S(t,s)=\beta N(s)\left(1-\lambda\int_s^t e^{-\lambda(\tau-s)}d\tau\right)=\beta N(s)e^{-\lambda(t-s)} \tag{3}$$

Existing population at time t can obviously be expressed as the integral sum of surviving people:

$$N(t)=\int_{-\infty}^t \beta N(s)e^{-\lambda(t-s)}ds$$

Differentiation of the last equation gives:

$$\overset{o}{N}(t) = \beta N(t) -\lambda \int_{-\infty}^t \beta N(s)e^{-\lambda(t-s)}ds = (\beta-\lambda)N(t)$$

therefore, solving this differential equation for $N(t)$, we verify the correctness of equation (1) and we notice that $n=\beta-\lambda$, as claimed.

Substitution of (1) into (3) gives:

$$S(t,s)=\beta N(0)e^{\beta s}e^{-\lambda t}$$

which clarifies equation (2).

While the aggregation of consumption and assets is simple, we need to elaborate more on human wealth, due to the evolution of labour income. At the individual level it can be expressed as:

$$w(t,s) = \phi\ W(t)/N(t)e^{-\rho(t-s)} \tag{4}$$

where ϕ is a constant to be determined by means of the aggregate constraint:

$$W(t) = \int_{-\infty}^{t} w(t,s)\ \beta e^{\beta s} e^{-\lambda t} ds$$

Substituting equation (4) into the last one we get:

$$W(t) = \int_{-\infty}^{t} \frac{\phi\ W(t) e^{-\rho(t-s)}\ \beta e^{\beta s} e^{-\lambda t}}{N(t)} ds = \phi\ W(t) \int_{-\infty}^{t} e^{-\rho(t-s)}\ \beta e^{\beta(s-t)}\ ds$$

Therefore $\phi = \frac{\beta+\rho}{\beta}$, we may now rewrite the individual-*s* human wealth as follows:

$$h(t,s) = \int_{t}^{\infty} \frac{\beta+\rho}{\beta} \frac{W(\tau)}{N(\tau)} \exp\left(-\int_{t}^{\tau} [r(z)+\lambda+\rho] dz \right) \exp[-\rho(t-s)] d\tau$$

Hence, aggregate human wealth is given by:

$$H(t) = \int_{t}^{\infty} h(t,s)\ \beta e^{\beta s} e^{-\lambda t} ds =$$

$$= \int_{-\infty}^{t} \left[\int_{t}^{\infty} \frac{\beta+\rho}{\beta} \frac{W(\tau)}{N(\tau)} \exp\left(-\int_{t}^{\tau} [r(z)+\lambda+\rho] dz \right) d\tau\ \beta e^{\beta s} e^{-\lambda t} e^{-\rho(t-s)} \right] ds$$

Taking all the terms independent from *s* out of the first integral gives:

$$H(t) = \int_t^{\infty} \frac{W(\tau)}{N(\tau)} exp\left(-\int_t^{\tau} [r(z)+\lambda+\rho]dz \right) d\tau \, (\beta+\rho)e^{-(\lambda+\rho)t} \int_{-\infty}^{t} e^{(\beta+\rho)s}ds =$$

$$= \int_t^{\infty} W(\tau) \, exp\left(-\int_t^{\tau} [r(z)+\beta+\rho]dz \right) d\tau$$

where the second expression is obtained solving the integral in ds and then simplifying.

In differential equation form we have:

$$\overset{o}{H} = (r + \beta + \rho)H - W \tag{5}$$

The presence of the βH term in equation (5) reflects the fact that all the agents, even the newborn, have the same life expectancy (Buiter, (1988), p. 283). Notice that, from equation (5) on, we take as understood the time index t whenever not confusing.

The dynamic equation for A can be obtained, from the individuals' budget constraints, using standard techniques:

$$\overset{o}{A} = rA + W - C \tag{6}$$

where the absence of the λA term is due to the insurance companies' activity, which transfer resources from those who die to those who survive: clearly, this process is not affected by the birth rate.

Finally, we provide the formulation for aggregate consumption:

$$C=(\theta+\lambda)(A+H) \tag{7}$$

Differentiating with respect to time equation (7) and exploiting equations (5) and (6), we get a law of motion for consumption:

$$\overset{o}{C} = (r+n+\rho-\theta)C - (\theta+\lambda)(\beta+\rho)A \tag{8}$$

It is useful to compare this equation with the corresponding one in the Ramsey model with population growth (see e.g. Blanchard and Fisher, (1989)). In that framework, the second addendum on the right hand side is missing and per capita consumption is multiplied by (r-θ). The presence of the n term in our equation reflects the fact that existing agents are not concerned about the unborn people[7].

Equations (6) and (8) summarise the aggregate behaviour of our continuum of disconnected families.

3. A monopolistic competition framework

In what follows, the economy is composed of $v(t)$ firms, each producing an intermediate good, $x_i(t)$, which is an imperfect substitute for the others. National income, $Y(t)$, is regarded as a flow of output obtained by means of the specific goods; we impose, at any time, an equal and constant elasticity of substitution between any pair of the intermediate products, so that each firm has some monopolistic power:

$$Y(t)= v(t)^{(\mu-1)/\mu}\left(\int_0^{v(t)} x(t)_i^{\mu}\, di\right)^{1/\mu} \qquad 0<\mu<1 \tag{9}$$

[7] Notice that, at the per capita level, consumption is not affected by population growth.

The constant outside the big round brackets is a normalization implying that an increase in the number of varieties does not affect the aggregate marginal productivity of primary inputs (for a similar approach, see Blanchard and Kiyotaki, (1987), p. 649 and Kiyotaki, (1988), p. 697).[8] We could also have introduced monopolistic competition assuming a time separable utility function characterised by a sub-utility similar to the one proposed by Dixit and Stiglitz (1977). However, to accept this approach, the investment demand functions for intermediate goods must be forced, for tractability, to have the same elasticity of consumption demands (see Kiyotaki, (1988), p. 700); the production side of the economic system must be accordingly constrained. Similar problems are implied by the presence of government expenditure.

Considering then (9) as a "production function", we determine the demand of every single intermediate good solving a time-separable cost minimisation problem:

$$\min_{\{x_i\}} \int_0^{v(t)} p_i(t)x_i(t)\, di$$

where (9) is the static constraint; $p_i(t)$ is the price of the i-th specific good.

Using standard techniques (for a recent example, see Grossman and Helpman,

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[8] Without this normalization, the model could turn out to be an endogenous growth one. This class of models, insofar we are concerned with dynamic inefficiency, has very different implications, in which we do not wish to be involved. For reference on this specific point, see Kohn and Marion (1993) and especially King and Ferguson (1993). A normalisation allowing for a positive relationship between the productivity of primary inputs and the number of varieties would make interesting the problem of determining the optimal number of intermediate inputs, an issue which we completely ignore.

(1991), pp. 45-47), we obtain the following system of demand functions :

$$x_i = \frac{Y}{v}\left(\frac{p_i}{P}\right)^{1/(\mu-1)} \tag{10}$$

where

$$P = \left(\frac{1}{v}\int_0^v p_i^{\mu/(\mu-1)} di\right)^{(\mu-1)/\mu} \tag{11}$$

is both a price index and the aggregate price level.[9] In a symmetric equilibrium, all the firms produce the same amount of output and charge the same price, hence, from (9) and (11) respectively, $Y=vx$ and $P=p_i$.

We now consider the problem of the representative firm, which, acting in a deterministic environment, maximises the discounted stream of its cash flows. Therefore, the firm solves, at time t, the problem:

$$max\ Q_i(t) = max \int_t^\infty [p_i(\tau)x_i(\tau) - w(\tau)L_i(\tau)]\ exp\left(-\int_t^\tau r(z)dz\right)d\tau \tag{12}$$

where L_i is labour used by the i-th firm, and w is the nominal wage. Notice that we allow the supply of labour to be different from population; this is consistent with the downward sloping profile for individual labour income, if the decline is due to a reduction of individual supply, in efficiency terms, related to age. However, we let the aggregate labour supply to increase at the population growth rate. We have

[9] Notice that $\int_0^v p_i x_i\, di = PY$ if x_i is given by (3).

assumed away capital since this hypothesis allows for relevant simplification without altering our basic results.

The firm faces two static constraints, the first of which is a production function:

$$x_i = \gamma L_i e^{gt} \tag{13}$$

which displays constant returns to scale in labour. Notice that we allow for the possibility of an exogenous rate of productivity growth, g. The firms must also take account of the inverse demand function for x_i, equation (10).

Since there are no dynamic constraints, problem (12) can be reduced to a sequence of static optimizations:

$$\max_{\{L(\tau)_i\}} \frac{v(\tau)^{\mu-1} P(\tau) x(\tau)_i^{\mu}}{Y(\tau)^{\mu-1}} - w(\tau) L(\tau)_i, \qquad \tau \in [t, \infty)$$

for which the first order conditions are:

$$\frac{\mu\gamma e^{g\tau} v(\tau)^{\mu-1} P(\tau) x(\tau)_i^{\mu-1}}{Y(\tau)^{\mu-1}} - w(\tau) = 0, \qquad \tau \in [t, \infty)$$

Hence, in a symmetric equilibrium, $w(\tau)/P(\tau) = \mu\gamma e^{g\tau}$

We may now set, with no loss of generality, $p_i = P = 1$, $i \in [0, v]$ (which is consistent with our analysis of the consumer's problem). In a symmetric equilibrium $L_i = L/v$, hence, by use of the normalization above and of our expression for wages, we transform (12) into:

$$Q_i = \int_t^{\infty} (1-\mu)\gamma \frac{L(\tau)}{v(\tau)} e^{g\tau} \exp\left(-\int_t^{\tau} r(z)dz \right) d\tau$$

which, in differential form, is:

$$\overset{o}{Q}_i = -\ (1-\mu)\gamma\frac{L}{v}e^{gt} + rQ_i \tag{14}$$

Hence, we have found an arbitrage equation which prices our assets, requiring that capital gains plus rents equate what could be obtained investing the value of the firm on the bond market. From (14) it is clear that the interest rate cannot be negative, as in Dechert and Yamamoto (1992); however this is no longer a sufficient condition to rule out dynamic inefficiency.

Notice that the value of each firm depends only on future profits; hence, in a symmetric equilibrium, it is equal for every firm, regardless to the diversity in the dates at which they entered into the market.

4. The role of the asset market

We now tie together the analysis of sections 2 and 3, studing the implications of the firm pricing equation (14) on the aggregate behaviour.

Maintaining our assumption, according to which there is no government debt, the outstanding stock of assets is given by:

$$A = vQ_i$$

Differentiation of this equation with respect to time gives:

$$\overset{o}{A} = \overset{o}{v}Q_i + v\overset{o}{Q}_i$$

Substitution of equation (6) for the left-hand side and of equation (14) in the right hand side yields:

$$rvQ_i + \mu\gamma L\, e^{gt} - C = \overset{o}{v}Q_i + rvQ_i - (1-\mu)\gamma L\, e^{gt}$$

Therefore, we obtain a differential equation for the number of firms:

$$\overset{o}{v}Q_i = \gamma L\, e^{gt} - C \qquad (15)$$

As for consumption, it is useful to reformulate our differential equation in terms of per worker "efficiency" units. Define $c = C/(Le^{gt})$ to get:

$$\overset{o}{c} = (r+\rho-\theta-g)c - (\theta+\lambda)(\beta+\rho)\frac{vQ_i}{Le^{gt}} \qquad (16)$$

The third equation of the model is simply (14). Since we have four unknowns (consumption, interest rate, number of varieties and value of a firm), we need a further relation to close the model. An hypothesis concerning the possibility, for new firms, of entering into the market provides the missing equation. In what follows, we examine several alternative assumptions.

(a) Entry is free but it entails a fixed cost, constant over time

At the level of stylised facts, a cost with these characteristics could be entailed by the need, for the entrant firms, of purchasing an estate, a plant and/or some machinery. In this case, the ratio between the fixed cost F and aggregate output is decreasing over time[10].

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[10] The hypothesis of an "infinitely lived" entry cost may be unappealing but it is not crucial to establish the possibility of dynamic inefficiency with free entry. For example, assume that every firm must bear an initial one-off cost and then an outlay equal to a given share of the fixed sunk cost in every period. In this case, given ρ, β, λ, θ, g, the minimum values for μ compatible with dynamic inefficiency is reduced; hence the parameters set allowing for inefficient equilibria is increased. A similar result can be obtained assuming a (fixed) probability of exit from the market for the single firm, a hypothesis that could be introduced to stylise the possibility of a technology or taste shock. The implications of these assumptions are similar to those of capital depreciation in the standard overlapping generations model.

The free entry assumption implies that the value of every firm must be equal to the fixed cost, $Q_i=F$. Therefore, the differential equation (14) reduces to the following ordinary one:

$$rvF = (1-\mu)\gamma L e^{gt} \tag{17}$$

Equation (15), using our definition for c, becomes:

$$\overset{o}{v}F = \gamma L\, e^{gt} - c\, L\, e^{gt} \tag{18}$$

It is now convenient to introduce an auxiliary variable, u, which is defined as the number of varieties in per worker efficiency terms: $u = v/(Le^{gt})$. Differentiation with respect to time gives

$$\overset{o}{u} = \frac{\overset{o}{v}}{Le^{gt}} - (n+g)u$$

Hence, using (18)

$$\overset{o}{u} = \frac{\gamma - c}{F} - (n+g)u \tag{19}$$

Exploiting our definition of u, we get, from equation (16) and (17), respectively:

$$\overset{o}{c} = (r+\rho-\theta-g)c - (\theta+\lambda)(\beta+\rho)Fu \tag{20}$$

$$r = \frac{(1-\mu)\gamma}{Fu} \tag{21}$$

The simplicity of system (19-21) allows the explicit calculation of the long-run interest rate, which is:

$$r = \frac{\theta + g - \rho + (1-\mu)(g+n) + \sqrt{[(\rho - g - \theta) + (1-\mu)(g+n)]^2 + 4(1-\mu)(\lambda+\theta)(\beta+\rho)}}{2}$$

In a dynamically inefficient situation, the interest rate is lower that the growth rate, which happens if: $(1-\mu)(\lambda+\theta)(\beta+\rho) < \mu(g+n)(n+\rho-\theta)$. Therefore, we notice that dynamic inefficiency is compatible with the market structure generated by a fixed entry cost. However, the higher the degree of monopoly (the lower μ), the less significant is the set of values for β, λ, ρ, n and g, entailing an inefficient situation.

The economic intuition for the possibility of dynamic inefficiency is simple: the structure of the entry cost implies that, in the steady state, the growth rate of the number of firms is equal to the one for output, hence the existing firms do not grow over time; accordingly, in discounting future profits, firms do not modify the interest rate with the growth rate. The fact that, in a perfect foresight framework, rational agents could capitalise in advance the value of still non existing (or simply non producing) firms is not relevant, since, in our framework, the value of these firms, before they sunk the cost, is naught. Hence, the economic system behaves as if it could not capitalise in advance future rents, as in Tirole's example of paintings.

To get some more information about the characteristics of the steady state, we now substitute r out of equations (19-20), obtaining a system composed of two differential equations:

$$\overset{o}{c} = \left(\frac{(1-\mu)\gamma}{Fu} + \rho - \theta - g\right)c - (\theta+\lambda)(\beta+\rho)Fu$$

$$\overset{o}{u} = \frac{\gamma - c}{F} - (n+g)u$$

Figure 1 depicts their qualitative behaviour, suggesting that the equilibrium is unique with a saddlepoint structure.

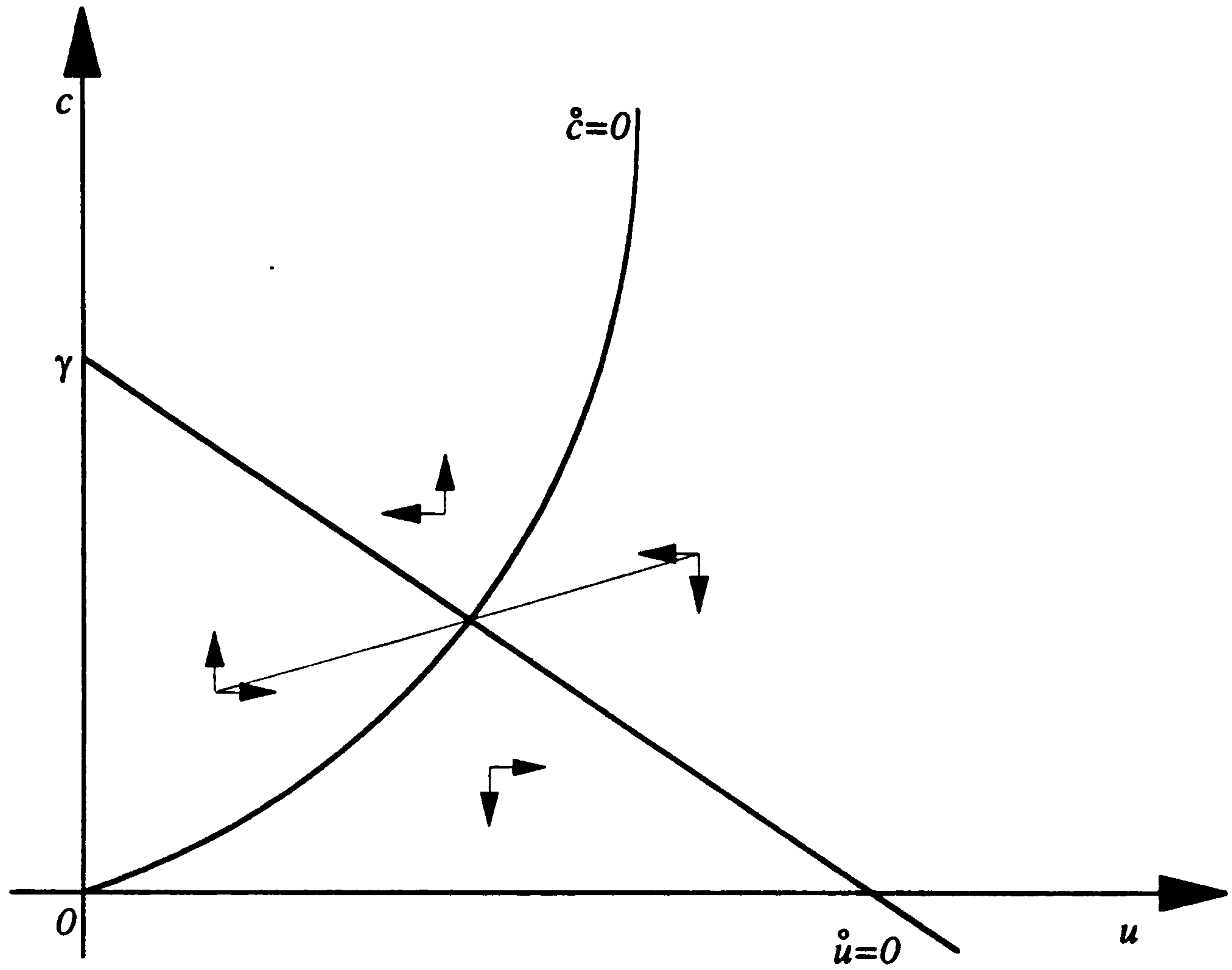

Figure 1. Saddlepath stability in the consumption-per worker varieties space

It is immediate to note that the interest rate, in steady state, is higher than $\max\{0, \theta+g-\rho\}$. From equation (21), it cannot be negative, an implication of the fact that it is used by firms to discount their future cash flow; equation (20) establishes that, if consumption in per worker efficiency units is positive in steady state, then $r > \theta+g-\rho$.

Following Blanchard (1985, pp. 237-38) we show that the long-run interest rate is lower than $\beta+\theta+g$. This can be proved by contradiction. If $r \geq \beta+\theta+g$, it follows that $(r+\rho-\theta-g)c \geq (\beta+\rho)c$, hence, from (20), $(\beta+\rho)c \leq (\theta+\lambda)(\beta+\rho)Fu$, or $c \leq (\theta+\lambda)Fu$. However, from equation (19), we obtain that, in steady state,

$c = \gamma\text{-}(g+n)\ Fu$. (This can also be derived from the aggregate constraint (9) via the production function (13)). Therefore $(\theta+\lambda)Fu \geq \gamma\text{-}(g+n)Fu$ or $\gamma \leq Fu\ (\beta+\theta+g \leq$ $\leq rFu$. Recall from equation (17) that $rFu = (1\text{-}\mu)\gamma$, hence we derived the contradiction. In economic terms, our chain of inequalities implies that output, in per capita efficiency units, is lower that per period dividends, a clear impossibility.

(b) Entry is free but it entails a fixed cost proportional to per capita output

Imagine now an environment, similar to those discussed by Grossman and Helpman (e.g. (1991), chs. 3 and 5), where entrance requires the development of a new "blueprint", which must be achieved by the work of some scientists. If their productivity is constant, their wage increase with output (i.e. with average wages) and the entry cost involves the postulated features.

The free entry assumption implies again that the value of every firm must be equal to the fixed cost, $Q_i(t)=F(t)=\alpha\gamma e^{gt}$. In this case, the differential equation (14) becomes:

$$\alpha g = -\ (1\text{-}\mu)\frac{L}{v} + r\alpha \tag{17'}$$

As before, we introduce an auxiliary variable, u_1, which is now defined as the number of varieties per worker: $u_1 = v/L$. Differentiation of u_1 with respect to time gives

$$\overset{o}{u}_1 = \frac{\overset{o}{v}}{L} - nu_1$$

Hence, using the appropriate version of (18), we get:

$$\overset{o}{u}_1 = \frac{\gamma\text{-}c}{\alpha\gamma} - nu_1 \tag{19'}$$

Exploiting our definition of u_1, we obtain, from equation (16) and (17'), respectively:

$$\overset{o}{c} = (r+\rho-\theta-g)c - (\theta+\lambda)(\beta+\rho)\alpha\gamma u_1 \tag{20'}$$

$$r = g + \frac{(1-\mu)}{\alpha u_1} \tag{21'}$$

In the present case, calculations show that the interest rate is smaller that the growth rate if $(1-\mu)(\lambda+\theta)(\beta+\rho) < \mu n(n+\rho-\theta)$. Hence, dynamic inefficiency is possible, since $\mu\in(0,1)$ and $\rho>\theta$, if $n>0$. In this case, the entry cost increases at the exogenous productivity growth rate and the existing firms grow over time at the same speed. Therefore, they take account of productivity growth, but not of the increase in population, when they discount their profits (equation (17')). Hence, this market structure prevents the existence of dynamic inefficient equilibria only if population does not grow.

As for the characteristics of the steady state, one can easily see that it is unique and saddlepath stable; the interest rate, in steady state, is higher than g. This comes from equations (20') and (21'), for the same reasons spelled out in the previous section, and from the fact that $\theta<\rho$. We demonstrate, again by contradiction, that the long-run interest rate is lower than $\beta+\theta+g$. If $r \geq \beta+\theta+g$, it follows that $(r+\rho-\theta-g)c \geq (\beta+\rho)c$, hence, from (20'), $(\beta+\rho)\, c\leq (\theta+\lambda)(\beta+\rho)\alpha\gamma u_1$, or $c \leq (\theta+\lambda)\alpha\gamma u_1$. From equation (19'), we obtain that, in steady state, $c = \gamma-n\alpha\gamma u_1$, which is the expenditure required to set up the new brands. Therefore $\gamma \leq \alpha\gamma u_1(\beta+\theta) \leq (r-g)\alpha\gamma u_1$. From equation (17') we see that the last chain of (weak) inequalities implies a contradiction. The economic intuition is the same we provided before.

(c) Entry is free but it entails a fixed cost proportional to population

A possible justification for this hypothesis is the need, for the new entrants, of setting up an advertising campaign.

As in the previous cases, the value of every firm is equal to the fixed cost: $Q_i(t)=F(t)=\delta N(t)=\varepsilon e^{nt}$. The differential equation (14) becomes:

$$\varepsilon n e^{nt} = -(1-\mu)\gamma\frac{L}{v}e^{gt} + r\varepsilon e^{nt} \tag{17''}$$

The most convenient definition for an auxiliary variable is now the number of varieties per efficiency units: $u_2 = v/e^{gt}$. Differentiation of u_2 with respect to time gives

$$\overset{o}{u}_2 = \frac{\overset{o}{v}}{e^{gt}} - gu_2$$

Hence, using the appropriate version of (18), we get:

$$\overset{o}{u}_2 = \frac{\gamma - c}{\varepsilon} - gu_2 \tag{19''}$$

Exploiting our definition of u_2, we obtain as before, from equation (16) and (17''), respectively:

$$\overset{o}{c} = (r+\rho-\theta-g)c - (\theta+\lambda)(\beta+\rho)\varepsilon u_2 \tag{20''}$$

$$r = n + \frac{(1-\mu)\gamma}{\varepsilon u_2}$$

In this case, calculations show that the interest rate is smaller that the growth rate if $(1-\mu)(\lambda+\theta)(\beta+\rho) < \mu g(n+\rho-\theta)$. Hence, dynamic inefficiency is possible, since $\mu\in(0,1)$ and $\rho>\theta$, if and only if the exogenous productivity growth rate is positive. This case is symmetric to the previous one: the entry cost increases at the population growth rate and the existing firms raise production over time at the same speed. Therefore, in discounting future profits, firms take account of population growth, but not of the increase in productivity. Hence, this market structure rules out dynamic inefficiency if productivity does not grow.

The characteristics of the steady state are the same we discussed before; the long-run interest rate is higher than $\max\{n, \theta+g-\rho\}$, but it is lower than $\beta+\theta+g$. If $r \geq \beta+\theta+g$, it follows that $(r+\rho-\theta-g)c \geq (\beta+\rho)c$, hence, from (20"), $(\beta+\rho)c \leq$ $\leq (\theta+\lambda)(\beta+\rho)\varepsilon u_2$, or $c \leq (\theta+\lambda)\varepsilon u_2$. From equation (19"), we obtain that, in steady state, $c=\gamma-g\varepsilon u_2$, which is, again, the expenditure required to set up the new brands. Therefore $\gamma \leq \varepsilon u_2(\theta+\lambda+g) \leq (r-n)\varepsilon u_2$. From equation (17") we see that $(r-n)\varepsilon u_2=(1-\mu)\gamma$, we find the desired contradiction.

(d) Entry is blockaded

The fixed number of firms is justified assuming that the varieties are given and accepting Bertrand competition within the market of each differentiated good. In this case, any type of lump-sum entry cost, however small, is sufficient to lock the number of firms.

We, assume, for simplicity, that v is equal to 1; from the aggregate constraint (9) and the production function (13) we have that $c=\gamma$.

As before, it is convenient to introduce an auxiliary variable, u_3, which is the value of a firms in per worker efficiency units, i.e. $u_3 = Q_i/(Le^{gt})$. Differentiation with respect to time gives $\overset{o}{u}_3 = \frac{\overset{o}{Q}_i}{Le^{gt}} - (n+g)u_3$; using equation (14) we obtain:

$$\overset{o}{u}_3 = -(1-\mu)\gamma + (r-n-g)u_3 \tag{22}$$

from which it is apparent that the steady state is dynamically efficient, since u_3 may not be negative. In this case, each firm increases its production at the economic system growth rate. Hence, the discount of the flow of future profits is performed using the interest rate reduced by the aggregate rate of economic growth: dynamic inefficiency becomes impossible. Since all the future rents are capitalised, the presence of the stock market plays the same role as the transversality condition in an infinite-horizon growth model, as it happens in Dechert and Yamamoto model.

Equation (16) and our definition for u_3 imply that $(r+\rho-\theta-g)c = (\theta+\lambda)(\beta+\rho)u_3$ a relation that may be used to explicit the interest rate:

$$r = \frac{(\theta+\lambda)(\beta+\rho)}{\gamma} u_3 + (\theta+g-\rho)$$

Introducing this expression into (22) we obtain the differential equation describing the solution of the system:

$$\overset{o}{u}_3 = -(1-\mu)\gamma - (n+\rho-\theta)u_3 + \frac{(\theta+\lambda)(\beta+\rho)}{\gamma} u_3^2$$

The phase diagram for this equation implies that there is a unique meaningful steady state, which is unstable. Since u_3 is not predetermined, we notice the absence of the transitional dynamics: the system immediately jumps to its perfect foresight equilibrium.

The long-run interest rate is higher than $n+g$, but it is lower than $\beta+\theta+g$. If $r \geq \beta+\theta+g$, it follows that $(r+\rho-\theta-g)c \geq (\beta+\rho)c$. Hence, $(\beta+\rho)c \leq (\theta+\lambda)(\beta+\rho)u_3$, or $c \leq (\theta+\lambda)u_3$. From equation (22) and from our assumption about the long run level of the interest rate, we obtain that, in steady state, $u_3 = \frac{(1-\mu)\gamma}{(r-n-g)} \leq \frac{(1-\mu)\gamma}{(\theta+\lambda)}$.

Therefore $c=\gamma \leq (1-\mu)\gamma$, a contradiction implying that dividends are higher than output.

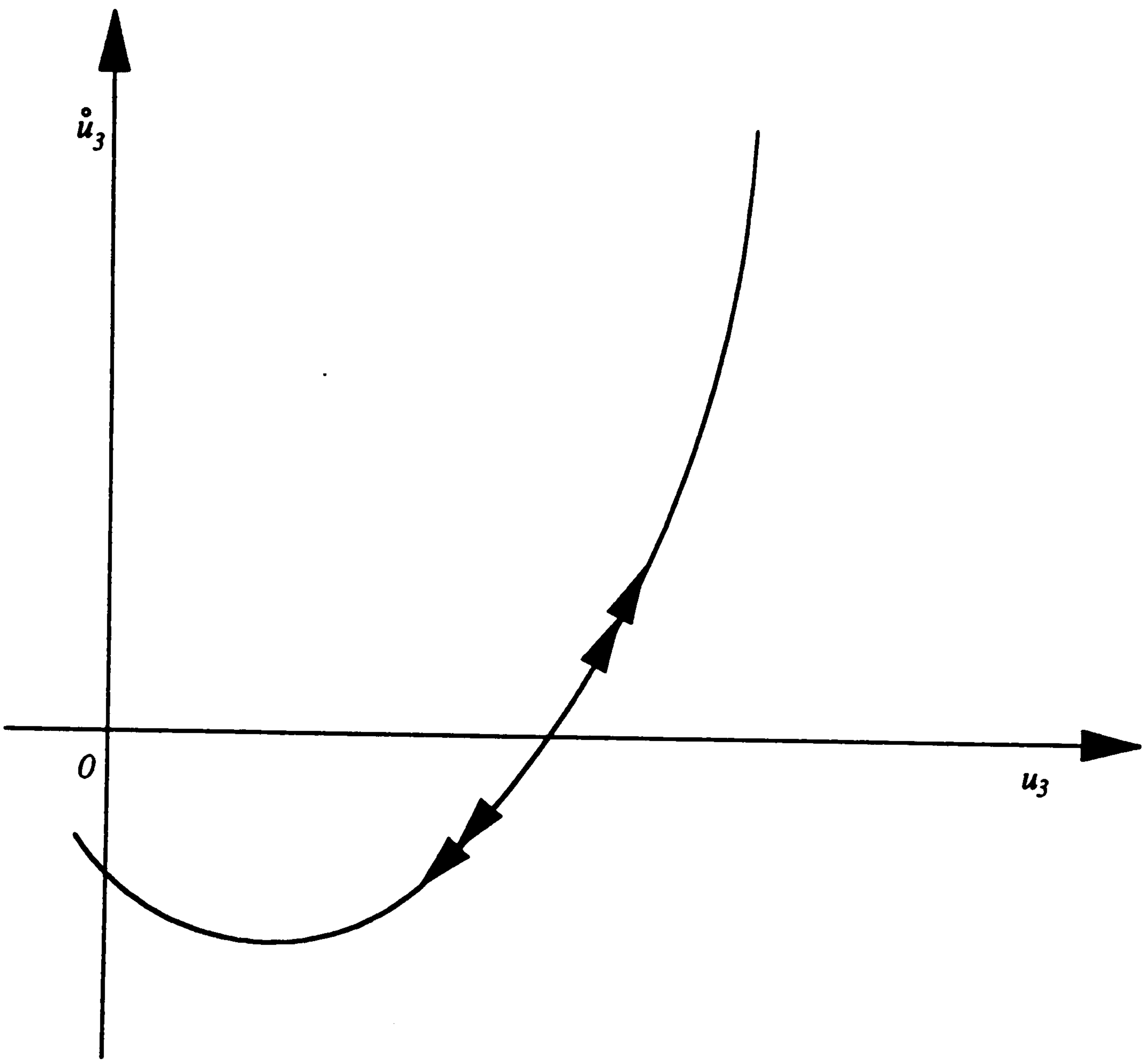

Figure 2. Saddlepath stability of equation (22).

5. Concluding remarks

In the present chapter, we emphasise the importance of the market structure to determine whether dynamic inefficiency is possible in a closed economy with no outstanding public debt. If the deviation from perfect competition implies the existence of some pure profit, the value of each firm is given by the stream of its discounted future cash flow, which adds a forward looking differential equation to the model. The level of the discount rate adopted by individual firms plays a key role in our model. If it takes account of the aggregate growth rate, dynamic inefficiency is ruled out; on the other hand, if the growth rate of each firm is lower than the aggregate one the possibility of inefficient equilibria is restored.

We have analysed market structures characterised by monopolistic competition. We have considered the case of free entry, determining the number of existing firms through the introduction of a fixed cost to be paid at the time of entrance into the market. The importance of the assumptions concerning the form of such cost is remarkable. For example, if the sunk cost is constant over time, existing firms do not grow in the steady state; accordingly, they do not modify the interest rate with the growth rate to discount their future profits. Hence dynamic inefficiency is possible with either population or exogenous productivity growth. Alternatively, if the entry cost increases at the exogenous productivity growth rate, being related for example to wages, existing firms grow, in the long run, at the same speed. Therefore, in discounting future profits, firms take account of productivity growth, but not of the increase in population: this structure for sunk costs prevents the existence of dynamic inefficient equilibria only if population is stationary. A cost increasing with population establishes the symmetric result.

The assumption of blockaded entry makes dynamic inefficiency impossible, since every firm grows at the aggregate growth rate. In this case, all the future rents are capitalised and the presence of the stock market serves the same purpose as the transversality condition in an infinite-horizon growth model.

Notice that, to establish efficiency, it is not necessary that every firm grows at the aggregate rate, but it is sufficient that some firm expands at the aggregate speed. Hence, in a more widely specified model, it should be possible to rule out dynamic inefficiency whenever a sector of the economy grows at the aggregate rate (or faster) and does not allow for new entries.

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Chapter II

Endogenous Growth and Overlapping Generations in a Model of Monopolistic Competition with Blockaded Entry.

1. Introduction

In recent years a considerable number of contributions have endeavoured to relate the rate of growth of an economic system to its "fundamentals": preferences, technology and market structures.

Within this burgeoning literature, research-based growth models are particularly appealing: in contrast to frameworks which account for growth assuming constant or increasing returns to the accumulable inputs in (at least) one sector of the economic system, they provide a detailed microfoundation for the process of technology improvement, which is at the basis of economic growth. In these models, some elements of imperfect competition are important: in a highly competitive environment the whole firm's revenue has to be used to reward productive factors and no room for research funds is left. For example, Grossman and Helpman, in various contributions, ascribe a monopolistic power to each producer; their models are then characterised by the assumption of free entry in the industrial sector, so that the profits accrued to producers exactly balance the outlays in research that they have to bear in order to enter into the market. In these frameworks, a higher monopolistic power resolves into larger research expenditures and, hence, into faster growth. In what follows, we drop the hypothesis of free entry, considering an environment of fixed product variety. A first result consists in the fact that the growth rate becomes independent of the degree of monopolistic competition if the economic system is populated by infinitely lived representative agents. This contrasts with the existing literature and it is due to a "macroeconomic externality". In a Blanchard-Yaari framework, it turns out that the relation between the growth rate and the competition level of the system may even be reversed. As wealth is enhanced by the stream of future profits, agents decrease their saving and this may reduce the growth rate. It will also be shown that the ricardian debt-neutrality proposition does not hold in the sense that an increase in the debt/output ratio reduces the growth rate, as already suggested by various contributions, which used endogenous growth models based on linear-in-capital technologies.

The remainder of this chapter is organised into six sections.

Existing models of monopolistic competition and growth are quickly surveyed in section 2; we then introduce imperfect competition with blockaded entry (section 3) and we find the solution for the case of an infinitely lived representative individual (section 4); Blanchard-Yaari overlapping generations are then fitted into this framework. Section 7 concludes.

2. Leading models of imperfect competition and growth

The links between growth and increasing product variety have been explored by Romer (1987), (1990), Grossman and Helpman (1989), (1991a, chs. 3 and 5) and

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Insofar as the determinants of the growth rate are concerned, the results of the Grossman and Helpman model of increasing product quality (1991a, chs. 4 and 5), (1991b) are identical to those which emerge from the increasing variety model. Also Aghion and Howitt (1992) build a model where an increase in the degree of market power increases the stationary amount of research. In these settings the number of goods is given, but every firm may improve any existing product, obtaining a profit that is proportional to the increase in quality. This profit lasts until some other producer steps up in the "quality ladder". Notice that the leader of each industry, in equilibrium, undertakes no research, since the reward from gaining an additional advantage on the followers does not justify the research cost. Hence, any innovation is obtained from outsiders. Free entry in the product market plays again an important role: at the system level, the volume of profits equals the investment in research. Therefore, the monopoly power, which in this model is acquired via the introduction of innovations, has beneficial effects on growth.

The Schumpeterian taste of these models is perceptible: firms devote resources to research and development in order to secure a stream of monopoly profits; the free entry assumptions imply that this stream just covers the outlays in research.

3. Monopolistic competition with blockaded entry

In what follows, national product is regarded as a homogeneous final good, obtained in a competitive setting, via a production function imposing constant elasticity of substitution between every pair of intermediate goods. The hypothesis about the variety of the differentiated goods, which is taken as given, is the same as

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(where $L(t)$ is labour used for the final good production) In this model the same parameter, α, determines both the monopoly level and labour productivity, and the two effects cannot be disentangled.

in Grossman and Helpman (1991a), (1991b) but we assume that the access to the market of each intermediate is blockaded. As each producer enjoys a partial monopolistic power, he obtains some pure profit, and he determines which share of it invest in research according to an (intertemporally) optimal plan. The result of this effort is a (continuous) stream of process innovations. Hence, this model avoids the unappealing implication according to which technological improvements are obtained by the new entrants. Each firm manages its own research, so that the problem of splitting profits between producers and researchers is avoided.

Hindrance to new competition may be due to the possibility, for each single firm, of appropriating of some "product specific information". Hence, a share of firm's technological know-how remains private, while research provides a "general knowledge" effect. This distinction can be based on the possibility of obtaining patents, as in the models by Grossman and Helpman and by Romer.[3] Notice that, if we introduce the hypothesis of Bertrand competition within each market, a lump sum cost to be paid starting to produce any good is enough to justify the assumed market structure. Such initial outlay could actually be necessary to set up a research laboratory and to let it reach the current "standard of knowledge".

According to the assumptions above, national product is regarded as a flow of output produced, in a competitive setting, by means of a fixed measure of "intermediate" goods. We impose an equal and constant elasticity of substitution between every pair of these goods at any time:

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[3] A more sophisticated reasoning could rely on the distinction between research and imitation (as in Rustichini and Schmitz, (1991)): one could argue that to obtain all the informations about product specific technologies can be very costly. A precise formalization of this point would make the model much more complex.

$$Y(t)=C(t)+I(t)+G(t)=\left(\int_0^1 x(t)_i^{\mu}\,di\right)^{1/\mu} \qquad 0<\mu<1 \tag{1}$$

For simplicity, we normalise to unity the product variety. Hence, we do not need to discuss whether the number of varieties affects the aggregate marginal productivity of primary inputs, as it happens in models of increasing product variety. As noted in the previous chapter, although we could have introduced monopolistic competition assuming a time separable utility function, this approach would have required to force, for tractability, the investment demand functions for intermediate goods to have the same elasticity of consumption ones (see Kiyotaki, (1988), p. 700). Similar problems are implied by the presence of government expenditure.

3.1 The demand for a single intermediate good

Accepting the interpretation according to which (1) is a production function, final goods producers solve the (time-separable) cost minimisation problem:

$$\min \int_0^1 p_i x_i\,di$$

$$\text{s.t.}\ \left(\int_0^1 x_i^{\mu}\,di\right)^{1/\mu}=Y$$

Using standard techniques, (See, for example, Grossman and Helpman, (1991a, pp. 45-47)) we obtain the following system of conditional demand functions:

$$x_i = Y\left(\frac{p_i}{P}\right)^{1/(\mu-1)} \tag{2}$$

where

$$P = \left(\int_0^1 p_i^{\mu/(\mu-1)} di\right)^{(\mu-1)/\mu}$$

is both an index of intermediate inputs prices and the aggregate price level. [4]

3.2 Firm's intertemporal optimisation

The intermediate goods sector of the model is composed of a continuum of firms, which are modelled as lying, equally spaced from one another, on a circle of unit length; they are indexed by their position $i \in [0,1]$ on the circle. We assume that the technologies used to produce the intermediate goods can be characterised by a degree of "similarity". The firms (and hence the goods) are positioned on the circle depending on their technological characteristics: the closer their positions, the more similar their production processes.

The representative firm, managing research, faces an explicit intertemporal problem. Acting in a deterministic environment, it maximises the discounted stream of its cash flows, i.e. it solves, at time t, the problem:

$$max \int_t^\infty [p_i x_i - w(L_{i1} + L_{i2}) - PI] \exp\left(-\int_t^\tau r(z)dz\right) d\tau$$

[4] Notice that $\int_0^1 p_i x_i \, di = PY$ if x_i is given by (2)

where x_i is output of firm i; L_{i1} and L_{i2} are the labour quantities employed to produce goods and in the research activity, respectively, w is the wage paid to them, I is investment and r is the interest rate, all considered at time τ, so that the time suffix can be suppressed.

It is necessary to specify two dynamic constraints; the first one is provided by the technology of acquisition for new knowledge. The simple specification chosen by Lucas for human capital (1988, pp. 18-19), is used here at the single firm level:

$$\overset{o}{S}_i = \kappa S^r{}_i L_{i2}$$

where S_i is the stock of knowledge available for the representative firm at time τ, $S^r{}_i$ is the stock of knowledge relevant for it and κ is the exogenous productivity parameter in research laboratories. A dot over a variable denotes, as usual, its derivative with respect to time.

$S^r{}_i$ can be defined as the integral sum of the publicly available results obtained by all the firms, weighted by a function taking account of the similarities between the technology of firm i and those of the other firms. This function, $\delta(i,j)$, depends negatively on the distance, measured on the unit circle, between firm i and the generic firm j. As the measure of each firm is zero, we can write:

$$S^r{}_i = 2 \int_i^{i+1/2} S^p{}_j \delta(i,j) dj$$

where $S^p{}_j$ is the share of the knowledge obtained by firm j that is publicly available.

To get a more tractable formulation, we need several further assumptions. We introduce, first, the hypothesis that the publicly available fraction of knowledge is exogenous and equal for every firm, i.e. that $S^p{}_j = \omega S_j$, $\omega \in (0,1]$. (Were $\omega=0$, there would not be an externality problem) Therefore, also the possibility that firms try to prevent actively the diffusion of their private knowledge is ruled out[5]. Moreover, we need to specify an "inverse distance" function. Defining $\Delta=j-i$, a simple example is: $\delta(i,j)=1-\psi\Delta$, if $\psi\Delta<1$, $\delta(i,j)=0$ otherwise[6]. With $\psi<2$, we get:

$$S^r_i = 2\int_0^{1/2} \omega S_j (1-\psi\Delta) d\Delta$$

then, using firms symmetry and the unit-measure hypothesis for firms we obtain: $S^r_i = \omega S_J(1-\psi/4)$.

Hence, define $\phi=\kappa\omega(1-\psi/4)$ to get:

$$\overset{o}{S_i} = \phi\, S_J\, L_{i2} \tag{3}$$

The accumulation of knowledge, therefore, depends positively on the productivity in research and on the share of results which cannot be prevented from becoming public; it depends negatively on the technological diversities among firms.[7]

Notice that equation (3) does not necessarily imply that the single enterprise has at its disposal the entire stock of scientific knowledge; notice, moreover, that the hypothesis of barriers to entry caused by firm specific knowledge can coexist with that of a positive externality induced by research.

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[5] However, since each firm has zero measure, it perceives that its research activity has negligible effects on the other agents' stock of knowledge; therefore the rapresentative firm has no incentive in using resources to avoid the diffusion of its private scientific results.

[6] As long as the "inverse distance" function does not depend on S_j, more complex formulations, such as a quadratic one, do not alter our main results.

[7] The parameter ψ can reflect also the communication infrastructure of the economic system as in Schmitz, (1989).

As in Lucas' formulation, constant returns to the accumulation of knowledge are crucial to get an equilibrium with steady per capita growth; with diminishing returns the rate of increase of knowledge would converge to zero, no matter which share of labour is devoted to research [8].

The second dynamic constraint, in absence of adjustment costs, is simply:

$$\overset{o}{K}_i = I_i$$

where K_i is capital.

The firm faces the following production function:

$$x_i = K_i^{\alpha} L_{i1}{}^{1-\alpha} S_i^{\beta} \tag{4}$$

which is assumed to display constant returns to scale in the "rival" factors, capital and labour, as in Romer (1990)[9]. The Cobb-Douglas specification is crucial to get an easy formulation for the economy's common growth rate.

The last constraint that must be taken into account is the inverse demand function for x_i:

$$p_i = x_i^{\mu-1} Y^{1-\mu} P$$

To simplify the problem, we can normalise total labour supply to unity[10]; as

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[8] Non constant returns to labour can, on the contrary, be fitted into the model. In this case, the rate of growth of the economic system would depend on the number of existing firms. Hence, the assumption of free entry would become important for the determination of the growth rate through its effect on the allocation of resources for research.

[9] The hypothesis of increasing returns to scale is not crucial for the model, in the sense that it is possible to obtain steady growth with a production function linear in all the productive factors.

[10] This conceals the dependence of the growth rate on the size of population, which

the total measure of firms is also unity, by imposing equilibrium on the labour market, we get the following current-value hamiltonian for the typical intermediate goods producer:

$$H:\ PY^{1-\mu}(K_i^{\alpha}L_{i1}^{1-\alpha}S_i^{\beta})^{\mu} - w - PI_i + \lambda_1\phi S_J(1-L_{i1}) + \lambda_2 I$$

where λ_1 and λ_2 are the costate variables. Using the final goods as numeraire, therefore normalizing P to unity, we can get some slight further simplifications, obtaining:

$$H:\ Y^{1-\mu}(K_i^{\alpha}L_{i1}^{1-\alpha}S_i^{\beta})^{\mu} - w - I_i + \lambda_1\phi S_J(1-L_{i1}) + \lambda_2 I$$

The first-order (necessary) conditions are:

$$H_{L_{i1}}\ \ \mu(1-\alpha)\, Y^{1-\mu}\, K_i^{\mu\alpha}\, L_{i1}^{\mu(1-\alpha)-1} S_i^{\beta\mu} - \lambda_1\phi\, S_J = 0 \tag{5}$$

$$H_{I_i}\ \ \lambda_2 = 1$$

$$H_{K_i}\ \ \mu\alpha\, Y^{1-\mu} K_i^{\mu\alpha-1} L_{i1}^{\mu(1-\alpha)}\, S_i^{\beta\mu} = -\mathring{\lambda}_2 + r\lambda_2 \tag{6}$$

$$H_{S_i}\ \mu\beta\, Y^{1-\mu} K_i^{\mu\alpha}\, L_{i1}^{\mu(1-\alpha)}\, S_i^{\beta\mu-1} = -\mathring{\lambda}_1 + r\lambda_1 \tag{7}$$

The transversality conditions are: $\lim_{\tau\to\infty} \lambda_1(\tau)S_i(\tau)\exp\left(-\int_t^{\tau} r(z)dz\right) = 0$ and $\lim_{\tau\to\infty} K_i(\tau)\exp\left(-\int_t^{\tau} r(z)dz\right) = 0.$

In a symmetric equilibrium, given the hypothesis concerning the total measure of firms, aggregate output of final goods is equal to the production of a single

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characterizes Romer (1987) and Grossman Helpman (1991a) models.

intermediate firm. Hence, one may verify, using (4), that the set of necessary conditions can be expressed as:

$$\mu(1-\alpha)\, Y/L_{1i} - \lambda_1 \phi\, S_J = 0 \tag{5'}$$

$$\mu\alpha Y/K_i = r \tag{6'}$$

$$\mu\beta Y/S_i = -\overset{o}{\lambda}_1 + r\lambda_1 \tag{7'}$$

It is also possible to show that each firm, at any point of time, acts so as to equate the fraction μ of the marginal productivity of labour to the market wage rate.

The set of necessary conditions can be used to work out the proportional growth solution. Equation (5) can be expressed as:

$$Y^{1-\mu}\, K_i^{\mu\alpha}\, L_{i1}^{\mu(1-\alpha)}\, S_i^{\beta\mu-1} = \frac{\lambda_1\, \phi\, L_{i1}}{\mu(1-\alpha)}$$

hence, substituting it into (7), we get:

$$\frac{S_j\ \beta}{S_i 1-\alpha}\lambda_1\, \phi\, L_{i1} = -\overset{o}{\lambda}_1 + r\lambda_1.$$

Thus,

$$\frac{\overset{o}{\lambda}_1}{\lambda_1} = r - \frac{\beta}{1-\alpha}\frac{S_j}{S_i}\phi\, L_1 \tag{8}$$

Differentiation with respect to time of equation (5) gives:

$$\overset{o}{\lambda}_1/\lambda_1 + \overset{o}{S}_j/S_j = (1-\mu)\, \overset{o}{Y}/Y + \mu\alpha\, \overset{o}{K}_i/K_i + \mu\beta\, \overset{o}{S}_i/S_i \tag{9}$$

(as $\overset{o}{L}_1/L_1 = 0$ in the long run).

From equation (4) we get:

$$\overset{o}{x}_i/x_i = \alpha \overset{o}{K}_i/K_i + \beta\overset{o}{S}_i/S_i \tag{10}$$

Hence, defining $\overset{o}{S}_i/S_i$ as g_{si}, we can obtain, g_i, the rate of growth common to output and capital in the i-th sector:

$$g_i = \frac{\beta}{1-\alpha} g_{si} \tag{11}$$

In a symmetric equilibrium g_i=g and g_{si}=g_s, $i \in [0,1]$, therefore equation (9) becomes:

$$\overset{o}{\lambda}_1/\lambda_1 = (1-\mu)\ \overset{o}{Y}/Y + \mu\ (\alpha \overset{o}{K}_i/K_i + \beta\ \overset{o}{S}_i/S_i) - \overset{o}{S}_j/S_j\ =$$

$$= [1-\mu(1-\alpha)]g - (\mu\beta - 1)g_s \tag{12}$$

Hence, using (11), we obtain that, in a symmetric equilibrium,

$$\frac{\overset{o}{\lambda}_1}{\lambda_1} = \left(1 - \frac{1-\alpha}{\beta}\right) g.$$

Equating this expression to (8) and using (3) to substitute out L_2, we get:

$$\left(1 - \frac{1-\alpha}{\beta}\right) g = r - \frac{\beta}{1-\alpha}\phi\ (1-L_2) = r - \frac{\beta}{1-\alpha}\phi\left(1 - \frac{g_s}{\phi}\right)$$

Hence, substituting out g_s and solving for g:

$$g = \frac{\beta}{1-\alpha}\left(\frac{\beta}{1-\alpha}\phi - r\right) \tag{13}$$

The growth rate is a positive function of ϕ, the exogenous parameter which characterises the research sector, and is negatively related with the interest rate.

It is interesting to note that the partial solution for the growth rate, in contrast with the existing literature, is not affected by the degree of monopolistic competition.

To understand this result, we can reformulate equation (12) by use of (10):

$$\overset{o}{\lambda}_I/\lambda_I = (1-\mu)\ \overset{o}{Y}/Y + \mu\, \overset{o}{x}_i/x_i - \overset{o}{S}_J/S_J$$

The impact of the growth rate of firm i's output on the growth rate of the marginal value of knowledge is reduced by monopolistic competition (since μ<1). However, this market structure implies a relation between λ_I and aggregate output that is not present when markets are perfectly competitive (the case μ=1). In the symmetric setting this "macroeconomic externality" exactly offsets the former negative effect.

Another distinctive feature of this model consists in the fact that the representative firm, in general, does not invest all the profits that it gains by exploiting its (partial) monopolistic power. We compute the value of the firm in a steady state equilibrium, under the simplifying assumption that it borrows the whole of physical capital from families, and we check the existence of pure profit. The hypothesis concerning its financial structure of the firm does not affect its value, since, in a deterministic environment, it is immaterial whether investment is financed from borrowing or from issuing equities. Therefore, the value of the representative firm is the sum of discounted pure profits and of the capital used in production, and it turns out to be:

$$V_i(t) = max \int_t^{\infty} [x_i(\tau)-w(\tau)-r(\tau)K_i(\tau)]exp\left(-\int_t^{\tau} r(z)dz\right)d\tau + K_i(t)$$

Using the first order conditions (6') and the fact that the wage is a fraction μ of the labour marginal productivity, we get:

$$V^*_i(t) = \int_t^{\infty} \left(1 - \frac{(1-\alpha)\mu}{L_1(\tau)} - \alpha\mu \right) Y(\tau) exp\left(- \int_t^{\tau} r(z)dz \right) d\tau + \frac{\mu\alpha Y(t)}{r(t)}$$

In a steady growth equilibrium this expression becomes notably simpler: the interest and the growth rate are constant; the labour share used to produce goods is also constant and equal to $(\phi - g_s)/\phi$, hence integrating we get:

$$V^*_i(t) = \left(1 - \frac{(1-\alpha)\mu\phi}{\phi - g_s} - \alpha\mu \right) \frac{Y(t)}{r-g} + \frac{\mu\alpha Y(t)}{r}. \tag{14}$$

The first addendum in (14) represents the discounted stream of pure profits; it is immediate to see that, in a steady growth equilibrium, it will disappear only if:

$$\left(1 - \mu(1-\alpha)\frac{\phi}{\phi - g_s} \right) = \mu\alpha$$

i.e. if:

$$\mu^* = \frac{\phi - g_s}{\phi - \alpha g_s}$$

Hence, with $\mu < \mu^*$ we expect the emergence of pure profit in the long run. It is possible to show that this condition must hold whenever the first order conditions are sufficient for a maximum, i.e. whenever the transversality conditions are fulfilled and the Hamiltonian for the representative firm is concave[11]. (The converse is not true.) Hence, with $\mu > \mu^*$ the model collapses: we may consider μ^* the maximum level of competition compatible with a decentralised economy and

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[11] See, e.g. Beavis and Dobbs, (1990).

with the growth rate g.

4. Consumer behaviour and optimal growth with infinite lives

4.1 Intertemporal behaviour of the representative agent

We now introduce the standard hypothesis of a time-separable utility function characterised by a constant elasticity of substitution. Therefore, the household's maximisation problem can be written as follows:

$$max \int_t^\infty [c(\tau)^{1-R}/(1-R)]\, exp[-\theta(\tau-t)]d\tau$$

$$s.t.\ \mathring{a}/a = r(t)a(t) + w(t) - l(t) - c(t)$$

where $a(t)$ is the non human wealth at time t, $w(t)$ the labour income, $l(t)$ is the lump-sum tax and R is the reciprocal of the constant elasticity of intertemporal substitution. To avoid an explosive accumulation of debt, the representative consumer is required to take account also of the "no-Ponzi game" condition:

$$\lim_{\tau\to\infty} a(\tau)\, exp\left(\int_t^\tau r(z)dz\right) = 0$$

The necessary conditions for this problem are:

$$c(\tau)^{-R} = \lambda_3$$

$$r\lambda_3 = - \mathring{\lambda}_3/\lambda_3 + \theta\lambda_3$$

where λ_3 is the costate variable associated with individual assets.

The first order conditions can be summarised by:

$$\overset{o}{c}(\tau) = (1/R)(r-\theta)c(\tau) \tag{15}$$

Equation (15) implies that consumption can grow at a steady rate only if:

$$r=\theta+Rg. \tag{16}$$

4.2 Determination of the growth rate

Solving the system composed of equations (13) and (16) we get the steady state growth rate for the economy:

$$g = \frac{\beta^2\,\phi - \beta(1-\alpha)\,\theta}{(1-\alpha)\,(1-\alpha + R\beta)} \tag{17}$$

The growth rate exhibits the usual negative dependence on the intertemporal preference and on the reciprocal of the intertemporal elasticity of substitution: the higher these parameters are, the less willing is the representative consumer to substitute present for future consumption. The positive relation of the growth rate to the exogenous parameter characterising research is quite obvious as well. Using the technique developed in Mulligan and Sala-i-Martin (1993) it is possible to show that the balanced growth path is determined.

More interestingly, equation (17) does not show any relation between the growth rate and the degree of competition. Intuitively, such a relation should exist, since the presence of monopoly reduces, ceteris paribus, the interest rate, and this affects both consumers and firms behaviour.

Formally, this "neutrality" result is due to the fact that the competition parameter and the marginal productivity of capital do not enter separately into equations (13) and (16), which therefore determine only the interest rate; if μ

varies, the capital stock, in the long run, adjusts to keep the interest rate unchanged. Hence, the fact that μ does not explicitly appear into (13) has deep consequences.

Moreover, let π be the pure profit to output ratio and recall, from (14), that, in a steady state equilibrium, $\pi = 1 - \frac{(1-\alpha)\mu\phi}{\phi - g} - \alpha\mu$. Hence, by use of (17) we can compute:

$$\frac{\partial \pi}{\partial \phi} = \frac{(1-\alpha)\mu}{(\phi - g_s)^2}\left(g_s - \phi\frac{\partial g_s}{\partial \phi}\right) = -\frac{(1-\alpha)\mu}{(\phi - g_s)^2}\frac{(1-\alpha)\,\theta}{(1-\alpha + R\beta)} < 0$$

and

$$\frac{\partial \pi}{\partial \theta} = \frac{(1-\alpha)\mu\phi}{(\phi - g_s)^2}\left(-\frac{\partial g_s}{\partial \theta}\right) = \frac{(1-\alpha)\mu\phi}{(\phi - g_s)^2}\frac{(1-\alpha)}{(1-\alpha + R\beta)} > 0$$

Therefore, we conclude that the model unambiguously predicts that the share of pure profit is negatively affected by an increase in the growth rate due to a variation in ϕ or θ.

If we set $\beta = 1-\alpha$ the expression for the growth rate can be simplified to:

$$g = \frac{\phi - \theta}{1+R} \tag{17'}$$

In this case, the growth rate becomes independent from the factors' marginal productivities; therefore the growth rate gains independence also from income distribution. The special case of the above equation (17') also allows the growth rate of knowledge to be equal to the one for physical quantities, simplifying the analysis of steady growth.

4.3 The command optimum

Suppose that a planner aims at maximising welfare at time t: facing a representative consumer, he has to solve the following problem:

$$max \int_{t}^{\infty} [c(\tau)^{1-R}/(1-R)] \exp[-\theta(\tau-t)] d\tau$$

subject to:

$$\overset{o}{K}(\tau) = K(\tau)^{\alpha} L_1(\tau)^{1-\alpha} S_i(\tau)^{\beta} - c(\tau)$$

and

$$\overset{o}{S}_i(\tau) = \phi S_i(\tau) (1 - L_1(\tau))$$

(where, for aggregate output, we use again the hypothesis of unit measure for firms)

After some simplification, and dropping the time indexes, the set of necessary conditions can be expressed as:

$$c^{-R} = \xi_1$$

$$\overset{o}{c} = (1/R)(\alpha Y/K - \theta) c$$

$$(1-\alpha) \frac{Y}{L_1} - \left(\frac{\xi_2}{\xi_1}\right) \phi S_i = 0$$

$$\beta \frac{Y}{S_i} + \left(\frac{\xi_2}{\xi_1}\right) \phi (1-L_1) = - \left(\frac{\overset{o}{\xi_2}}{\xi_1}\right) + \theta \left(\frac{\xi_2}{\xi_1}\right)$$

where ξ_2 is the costate variable associated with the stock of knowledge and ξ_1 is the one associated with that of capital. If we define ξ as the ratio between the two costate variables[12], using the fact that $\overset{o}{\xi}=\overset{o}{\xi}_2/\xi_1-(\overset{o}{\xi}_1/\xi_1)\xi$, the system can be simplified further to get:

$$\overset{o}{c} = (1/R)(\alpha Y/K - \theta)\, c$$

$$(1-\alpha)\; Y/L_1 - \xi\phi\, S_i = 0$$

$$\beta Y/S_i = -\overset{o}{\xi} - (\overset{o}{\xi}_1/\xi_1)\xi + \xi\, [\theta - \phi(1-L_1)] \tag{18}$$

or, as $c^{-R} = \xi_1$,

$$\beta Y/S_i = -\overset{o}{\xi} + \xi\, [\theta + \sigma\overset{o}{c}/c - \phi(1-L_1)]$$

Following the same procedure used above, we can work out the optimal steady state growth rate, which is:

$$g^{*}= \frac{\beta\phi - (1 - \alpha)\,\theta}{(1 - \alpha)\, R} \qquad \left(= \frac{\phi - \theta}{R} \text{ if } \beta = (1 - \alpha)\right)$$

The externality problem embedded in the model, which is apparent in the addendum $-\xi\phi(1-L_1)$ in equation (18), causes the "command" optimum growth rate, g^*, to be higher than the market one. (Compare with equation (17))

Consider, now, the system composed of the two sets of first order conditions, derived from the "decentralised" maximisation pursued by families and firms, acting atomistically. After some simplification, it can be expressed as:

$$\overset{o}{c} = (1/R)(\mu\alpha Y/K - \theta)\, c$$

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[12] ξ can be interpreted as the ratio between the marginal values, at time τ, of capital and knowledge.

$$\mu(1-\alpha)\, Y/L_1 - \lambda_1 \phi\, S_i = 0$$

$$\mu\beta Y/S_i = -\overset{o}{\lambda}_1 + \mu\alpha Y/K\, \lambda_1$$

Inspection of the two sets of conditions shows that, to reach the optimum, factors' marginal productivities must be equalised to their marginal valuations. Thus, a planner wishing to decentralise decisions has to offset the static distortion caused by monopolistic competition; this can be obtained subsidising production of every commodity at the rate $(1/\mu)-1$. More interestingly, it can be shown that a subsidy to investment can increase to the optimum the amount of capital and it can augment the stock of knowledge as well. It turns out that this more feasible policy measure can eliminate the static distortion. This is due to the fact that, in this model, there is not an explicit price for S_i; rather, its valuation can be affected manipulating the level of capital and correcting the distortion due to the presence of the externality in research. As to this point, the planner must reduce the private marginal value of knowledge, λ_1, to ξ. This can be implemented via a subsidy to "research and development" expenditure, for example transferring to firms a share of the wage bill for research staff.

5. Consumer behaviour and optimal growth with finite lives

As we have already pointed out, the market structure considered in this chapter entails, in general, the presence of pure profit. Within the infinite lives case, it is not relevant to consider the way in which such income is distributed to the households, insofar as this distribution is egalitarian. In the Ramsey case, consumption evolution depends only on the difference between the market interest rate and the subjective time preference, so that, to keep everything as simple as possible, one can consider pure profit as a lump-sum transfer from firms to the representative agents. Similarly, in various papers on endogenous growth and overlapping generations it is assumed that profits, arising as a consequence of externalises, are handed over to consumers in a lump-sum fashion. (See e. g. Alogoskoufis and Van der Ploeg, (1990, p. 6))

In Blanchard's framework, which is used in this section, people have a potentially infinite horizon, but face, at each instant of time, a constant probability of death. As shown by Blanchard (1985, pp. 227-9 in particular), this limited uncertainty affects the relation between consumption and wealth. Therefore the way in which income is distributed becomes relevant, as it may influence the assets' total value. For this reason, a modification of the consumer budget constraint seems necessary: we introduce the hypothesis that profits are distributed to shareholders and hence we assume that the private sector's overall assets are equal to the firms' total value.

5.1 The consumer problem: a restricted version

To simplify the analysis, we use a logarithmic specification for the time separable utility function. Thus, the representative individual born at time s maximises, at time t, the functional:

$$max_{\{c(\tau,s)\}}\ E\left(\int_t^{\infty} ln[c(\tau,s)]\ exp[-\theta(\tau-t)]d\tau \,|\Omega_t \right)$$

s.t. $\overset{o}{a}(t,s)= [r(t)+p]a(t,s) + w(t) - l(t) - c(t,s)$

where Ω_t is the information set at time t; a "no-Ponzi" game condition also applies.

Following usual methods, it is possible to show that the aggregate behaviour can be summarised by the system:

$$\overset{o}{C} = (r - \theta)C - p\,(p + \theta)(V+D) \tag{19}$$

$$\overset{o}{K} = Y - C - G \tag{20}$$

where p is the instant probability of death; the substitution of V for K comes from our hypothesis concerning the distribution of profits; D and G are the stock of debt and the government expenditure, respectively. As usual, the time index has been suppressed. As equation (19) and (20) are non autonomous, they must now be "deflated" by using income, so that the system can be rewritten as:

$$\overset{o}{z} + gz = (r - \theta)z - p(p + \theta)(v+d)$$

$$\overset{o}{x} + gx = 1 - z - f$$

where z, x, d and f are the ratio of consumption, capital, public debt and government expenditure to income, respectively; v is the ratio between total value of the firm and income.

5.2 A steady state solution for the model

We now focus again our attention to the proportional growth solution for the system. We introduce also a further limitation: we consider only the case $\beta=1-\alpha$, which allows for a briefer parametrization. In a steady state equilibrium, we can substitute out v[13]; then, recalling from (13) that, in this simplified, case $r=\phi-g$, we

[13] Dividing (18) by $Y(t)$ one gets the long run equilibrium value for v, i.e.:

$$v = \left(1 - \frac{(1-\alpha)\mu\phi}{\phi-g} - \alpha\mu\right)\frac{1}{r-g} + \frac{\mu\alpha}{r}.$$

get:

$$\overset{o}{z} = (r - \theta - g)\, z - p(p+\theta)\left(\frac{\mu\alpha}{\phi - g} + \frac{(1 - \mu)\, \phi - (1 - \mu\alpha)\, g}{(\phi - g)\, (\phi - 2g)} + d\right) \tag{21}$$

$$\overset{o}{x} = 1 - z - \mu\alpha \frac{g}{\phi - g} - f \tag{22}$$

To obtain the possible steady state solutions for the model, we set to zero $\overset{o}{x}$ and $\overset{o}{z}$ in the previous equations (21) and (22) and we combine the resulting expressions, obtaining:

$$(\phi - \theta - 2g)\left(1 - f - \mu \frac{\alpha g}{\phi - g}\right) = p(p + \theta)\left(\frac{[1 - \mu\, (1 - \alpha)]\, \phi - (1 + \mu\alpha)\, g}{(\phi - g)\, (\phi - 2g)} + d\right) \tag{23}$$

We should now solve equation (23) for the growth rate. To help in looking for solutions, we define:

$$A(g) = (\phi - \theta - 2g)\left(1 - f - \mu \frac{\alpha g}{\phi - g}\right)$$

which is the left hand side of (23), and

$$B(g) = p(p + \theta)\left(\frac{(1 - \mu\, (1 - \alpha))\, \phi - (1 + \mu\alpha)\, g}{(\phi - g)\, (\phi - 2g)} + d\right)$$

which is the right hand side of (23), and we study separately these two main addenda.

We now summarise the relevant properties of $A(g)$. It exhibits a vertical asymptote at $g=\phi$; at a zero growth rate it assumes the value: $A(0) = (1-f)(\phi-\theta)>0$; when g approaches r (so that $g=\phi/2$, since $r+g=\phi$), the corresponding value is: $A\left(\frac{\phi}{2}\right) = \theta\, (\alpha\mu + f - 1)$, which is less than zero if:

$$f < 1 - \alpha\mu. \tag{24}$$

This condition has an economic meaning: it requires that the consumption/output ratio is positive even when g reaches $\phi/2$ (see equation (22)).

Moreover, setting $A(g) = 0$, we get: $g_1 = \frac{\phi - \theta}{2}$,which corresponds to the "Ramsey" solution (see equation (17')), and: $g_2 = \frac{\phi\,(1 - f)}{1 - f + \alpha\mu}$. $A(g)$ is positive for $g<g_1$ and $g>g_2$.

If condition (24) holds, it is possible to show that $g_2>\phi/2$, and hence also that $g_1<g_2$. Notice also that, for $g>g_2$, the consumption/income ratio is negative; in this interval no sensible long run solution is therefore possible.

Consider now the first derivative of $A(g)$,

$$\frac{\partial A}{\partial g} = -\frac{2g^2(1 - f + \alpha\mu) - 4\phi g\,(1 - f + \alpha\mu) + \phi\,(\alpha\mu\,(\phi - \theta) - 2\phi\,(f - 1))}{(\phi - g)^2}$$

and notice that it is positive between g_{min} and g_{MAX}, where:
$g_{min} = \phi - \left(\frac{\alpha\mu\phi\,(\phi + \theta)}{2(\alpha\mu + 1 - f)}\right)^{1/2}$ and $g_{MAX} = \phi + \left(\frac{\alpha\mu\phi\,(\phi + \theta)}{2(\alpha\mu + 1 - f)}\right)^{1/2}$.

Finally, notice that g_{min} is bigger than $(\phi - \theta)/2$ if $f < 1 - \alpha\mu\left(\frac{\phi - \theta}{\phi + \theta}\right)$ and that this condition is encompassed by (24).

Focusing our attention on $B(g)$, we notice that it is just a multiple of the ratio between total assets and income. As $p(p+\theta)$ can never be negative, in the interval where A(g) is negative, i.e. for $g \in (g_1, g_2)$, no sensible long run equilibrium is possible, since $B(g)<0$ is not acceptable. Thus, it is sufficient to study $B(g)$ in the interval $[0,\phi/2)$. The value for $B(g)$ at a zero growth rate is:

$$B(0) = p(p+\theta)\left(\frac{1 - \mu\,(1 - \alpha)}{\phi} + d\right)$$

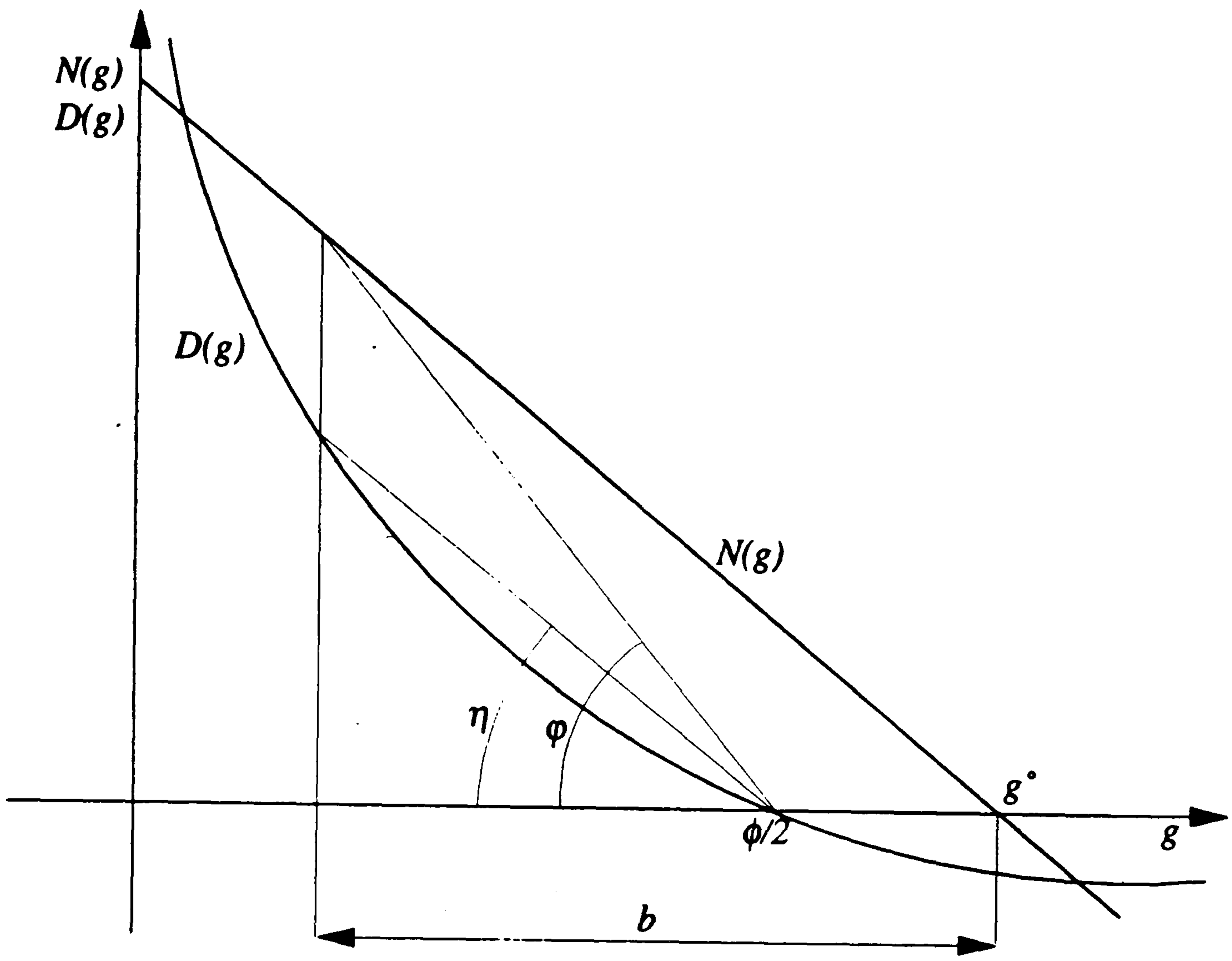

Figure 1. As g approaches $\phi/2$, the ratio of $N(g)$ to $D(g)$ increases.

At $\phi/2$, $B(g)$ has an asymptote because the growth rate approaches the interest rate, causing the explosion of the firm value. The limit for g approaching $\phi/2$ depends on the value of the firm: if it is always positive, as previously required, the limit must approach plus infinity.

As the properties of the derivative of $B(g)$ are the same of those of the derivative of the firm's steady state value, consider:

$$v(g)=\left(\frac{(1-\mu(1-\alpha))\,\phi-(1+\mu\alpha)\,g}{(\phi-g)\,(\phi-2g)}\right)=\frac{N(g)}{D(g)}$$

The numerator of this expression, $N(g)$, is positive if $\mu<\mu^*$, while its denominator, $D(g)$, is always positive, except for the interval $[\phi/2,\ \phi]$. To establish the behaviour of the derivative of $v(g)$ for $g\in[0,\phi/2)$, consider figure 1. $N(g)$ can be expressed as $btg(\varphi)$ and $D(g)$ as $btg(\eta)$. Define g° such that $N(g^{\circ})=0$; if g° is higher than $\phi/2$ when g increases, $tg(\varphi)$ increases, while $tg(\eta)$ decreases; therefore the ratio $D(g)/N(g)$ is always increasing and the derivative of $v(g)$ is positive. Calculations show that g^* is higher than $\phi/2$ if:

$$\mu < 1/(2-\alpha) \qquad (25)$$

One can show that that condition (25) implies both a positive value for the firm and a positive derivative of this value with respect to the growth rate[14].

It is now possible to draw figure 2, which depicts $A(g)$ and $B(g)$ in the interval $[0,\phi/2)$; this is helpful also to recognise that, to have a unique non negative solution for g, a third condition is required. In fact, we need:

$$(1-f)(\phi-\theta) > p(p+\theta)\left(\frac{1-\mu(1-\alpha)}{\phi} + d\right)$$

This condition implies that the growth process can not take off if the probability of death or the asset/income ratio are too high, because the steady state saving is too low. For the same reason, the higher is the ratio between government consumption and national product, the lower will be the growth rate.

5.3 Two non-neutrality results

The effect on the growth rate of an increase of the debt/income ratio, which is considered as a policy instrument, can be seen analysing equation (23). $A(g)$ is clearly unaffected, while $B(g)$ is shifted upwards. Therefore, with a positive probability of death, an increase in d unambiguously reduces the growth rate. In

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[14] Notice that: $(\partial N(\phi/2)/\partial g)|_{\phi/2}=-\phi$ and that $(\partial D(\phi/2)/\partial g)|_{\phi/2}=-(1+\mu\alpha)$; therefore, even if $\mu=1/(2-\alpha)$, the derivative of $v(g)$ is strictly positive.

fact, such a policy action, raising consumption, increases the interest rates; hence it makes research more costly and it lowers the equilibrium capital/output ratio.

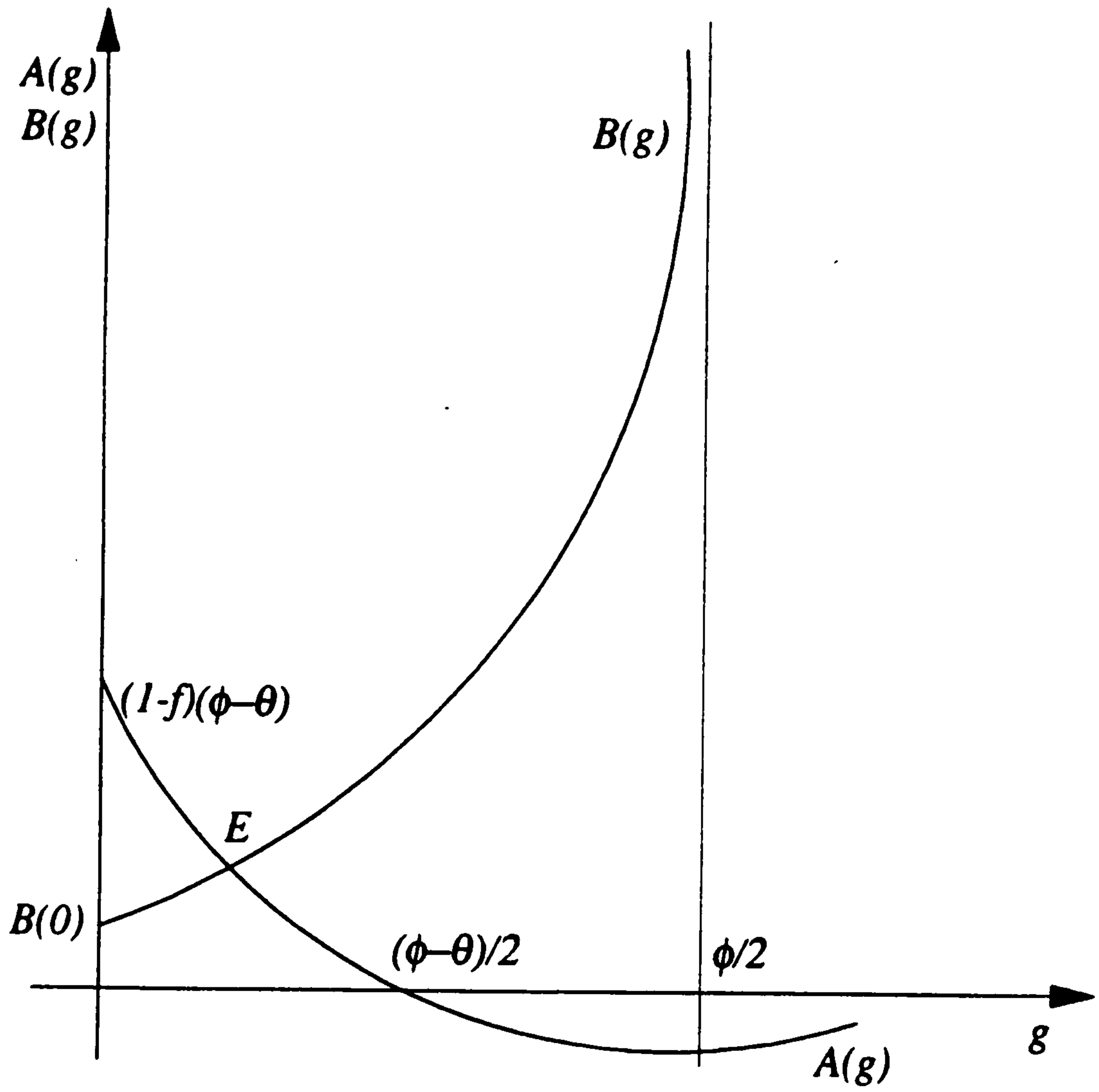

Figure 2. The steady state equilibrium E.

The debt non neutrality has already been pointed out in endogenous growth models where the capital/output ratio is exogenous and the "engine for growth" is provided by externalities only, without any need for research. (Alogoskoufis and Van der Ploeg (1990), Buiter (1991), Saint Paul (1992)) As intuition suggests, a more complex formulation for the production side of the economic system does not affect this result.

In the infinite lives case the growth rate was independent also of the degree of competition. This does not hold true any longer: μ influences both equations (21) and (22). More precisely, an (exogenous) increase in μ shifts downwards $A(g)$ in the interval $[0,(\phi-\theta)/2)$ and $B(g)$ in the interval $[0,\phi/2)$:

$$\frac{\partial A}{\partial \mu} = - \frac{(\phi - \theta - 2g)\,\alpha g}{\phi - g} < 0, \text{ for } g < \frac{\phi - \theta}{2},$$

$$\frac{\partial B}{\partial \mu} = \frac{p(p + \theta)\,(\phi\,(\alpha - 1) - \alpha g)}{(\phi - 2g)\,(\phi - g)} < 0, \text{ for } g < \frac{\phi}{2}$$

(see figure 3)

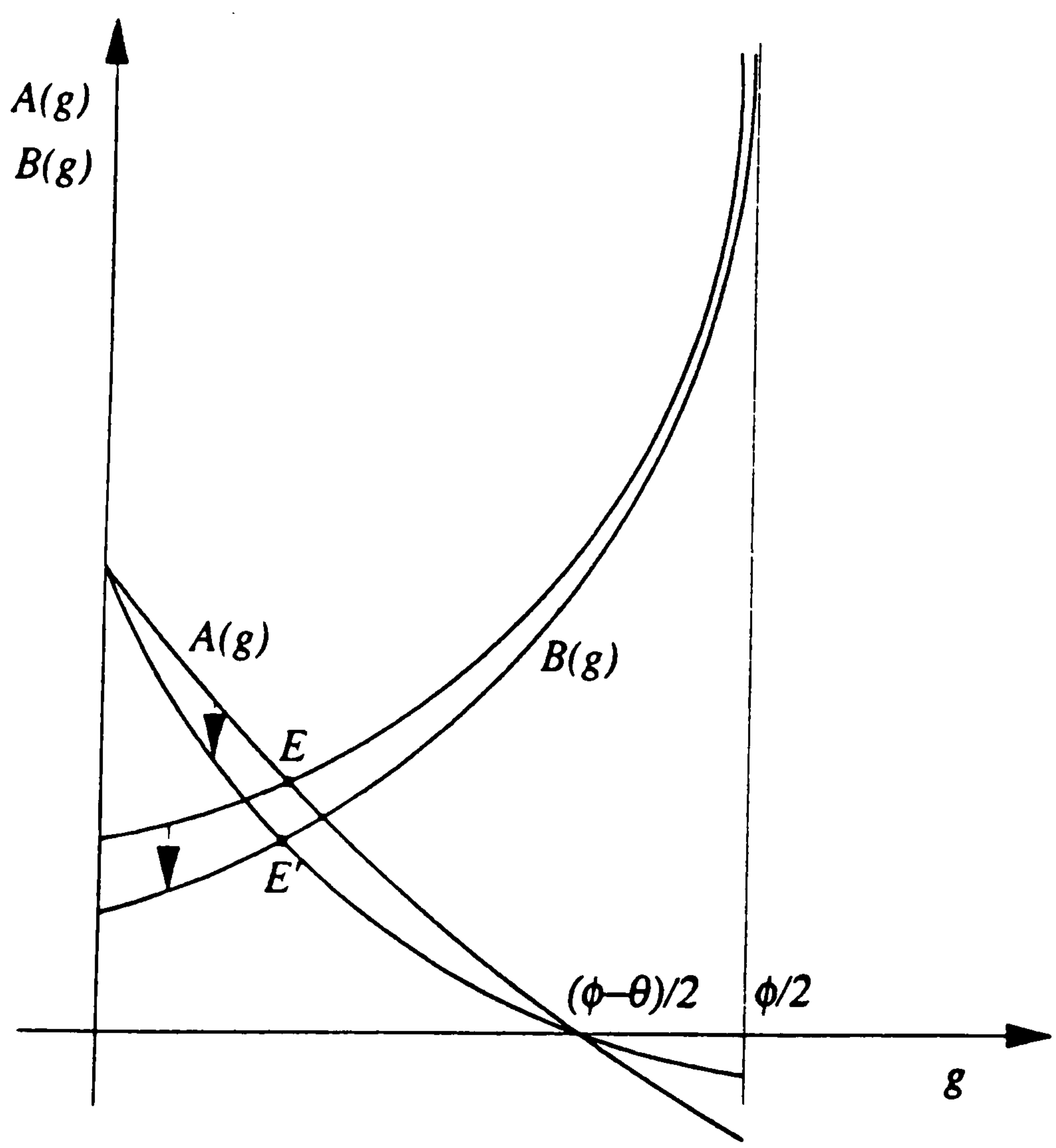

Figure 3. An increase in the degree of competitiveness μ.

The effect on $A(g)$ is due to the fact that an increase in μ raises the equilibrium capital/output ratio, reducing the growth rate for a given volume of savings. $B(g)$ is shifted downwards because the firm value decreases with μ, and this, in a finite lives framework, reduces consumption. We can try to determine the sign of this effect on a comparative static basis, studying the equation:

$$\frac{\partial A}{\partial g}dg + \frac{\partial A}{\partial \mu}d\mu = \frac{\partial B}{\partial g}dg + \frac{\partial B}{\partial \mu}d\mu$$

or:

$$\frac{dg}{d\mu} = \frac{\partial B/\partial\mu - \partial A/\partial\mu}{\partial A/\partial g - \partial B/\partial g}$$

While, if condition (25) is fulfilled, the sign of the denominator is negative, the one of the numerator turns out to be ambiguous. Some algebra shows that, for the interval we are interested in, this sign is positive only if:

$$C(g) = 4\alpha g^3 + \alpha g^2(2\theta - 4\phi) + \alpha g\,[-p(p+\theta)+\phi^2 - \phi\theta] + p(p+\theta)\phi(\alpha - 1) > 0$$

Hence $dg/d\mu<0$ if $C(g)>0$. Unfortunately, the study of such a function does not lead to any brief condition. However, notice that it has a local minimum when: $g = \frac{2\phi-\theta}{6} + \frac{[\phi^2-\phi\theta+\theta^2+3p(p+\theta)]^{1/2}}{6}$ which is always larger than the "Ramsey" solution, $(\phi-\theta)/2$. The value for $C[(\phi-\theta)/2]$ is $[p(p+\theta)(\alpha(\phi+\theta)-2\phi)]/2<0$.

Since $C(g)$ is independent of f and d, we can calculate a combination of these policy parameters such that the growth rate is naught and condition (24) is fulfilled. As $C(0)$ is less than 0, in such a situation an increase in μ would unambiguously increase the steady state growth rate.

Therefore, the possibility that the growth rate, in contrast to what happens in the Grossman and Helpman model, is *positively* related with the competition level can not be rejected. However, the mere presence of this relation contrasts with the infinite lives case and can have some implication for the policy analysis.

6. Policy intervention and static efficiency

Considering the "command" solution for the infinitely-lived case, we argued that a subsidy s to investment can offset the static distortion. The same policy measure is now examined within the perpetual youth framework. Under the hypothesis that the representative firm is affected only by this policy measure, its intertemporal optimisation problem becomes:

$$H: Y^{1-\mu}(K_i^{\alpha}L_{i1}^{1-\alpha}S_i^{\beta})^{\mu} - w - (1-s)I + \lambda_1\phi S(1-L_{i1}) + \lambda_2 I$$

Some algebra and the hypothesis of symmetric equilibrium lead to the following set of first order conditions:

$$H_{L_{i1}} \quad \mu(1-\alpha)\, Y/L_{i1} - \lambda_1\phi\, S_J = 0$$

$$H_K \quad \mu\alpha Y/K_i = (1-s)r$$

$$H_{S_i} \quad \mu\beta Y/S_i = -\overset{o}{\lambda}_1 + r\lambda_1$$

It is possible to show, applying the same procedure carried over in section 3, that the partial solution for the growth rate (equation 17) is unchanged.

By use of the first order conditions, in a steady state equilibrium, the value of the subsidised firm turns out to be:

$$\hat{V}_i(t) = \left(1 - \frac{\mu(1-\alpha)\phi}{\phi - g} - \mu\alpha\right)\frac{\hat{Y}(t)}{r-g} + \frac{\mu\alpha\hat{Y}(t)}{(1-s)r}$$

Correspondingly, the new value/output ratio is:

$$\hat{v} = \left(1 - \frac{\mu(1-\alpha)\phi}{\phi - g} - \mu\alpha\right)\frac{1}{r-g} + \frac{\mu\alpha}{(1-s)r}$$

As $s>0$, $\hat{v}>v$. If we imagine that it is possible to set up a system of lump sum taxes, levied on consumers, such that equation (19) is not affected, the new solution for the finite-lived cases is to be looked for studying the following system:

$$A^*(g) = (\phi - \theta - 2g)\left(1 - f - \frac{\mu}{1-s}\frac{\alpha g}{\phi - g}\right)$$

and

$$B^*(g) = p(p + \theta)\left\{\left[\left(1 - \frac{(1-\alpha)\mu\phi}{\phi - g} - \mu\alpha\right)\frac{1}{r-g} + \frac{\mu\alpha}{(1-s)r}\right] + d\right\}$$

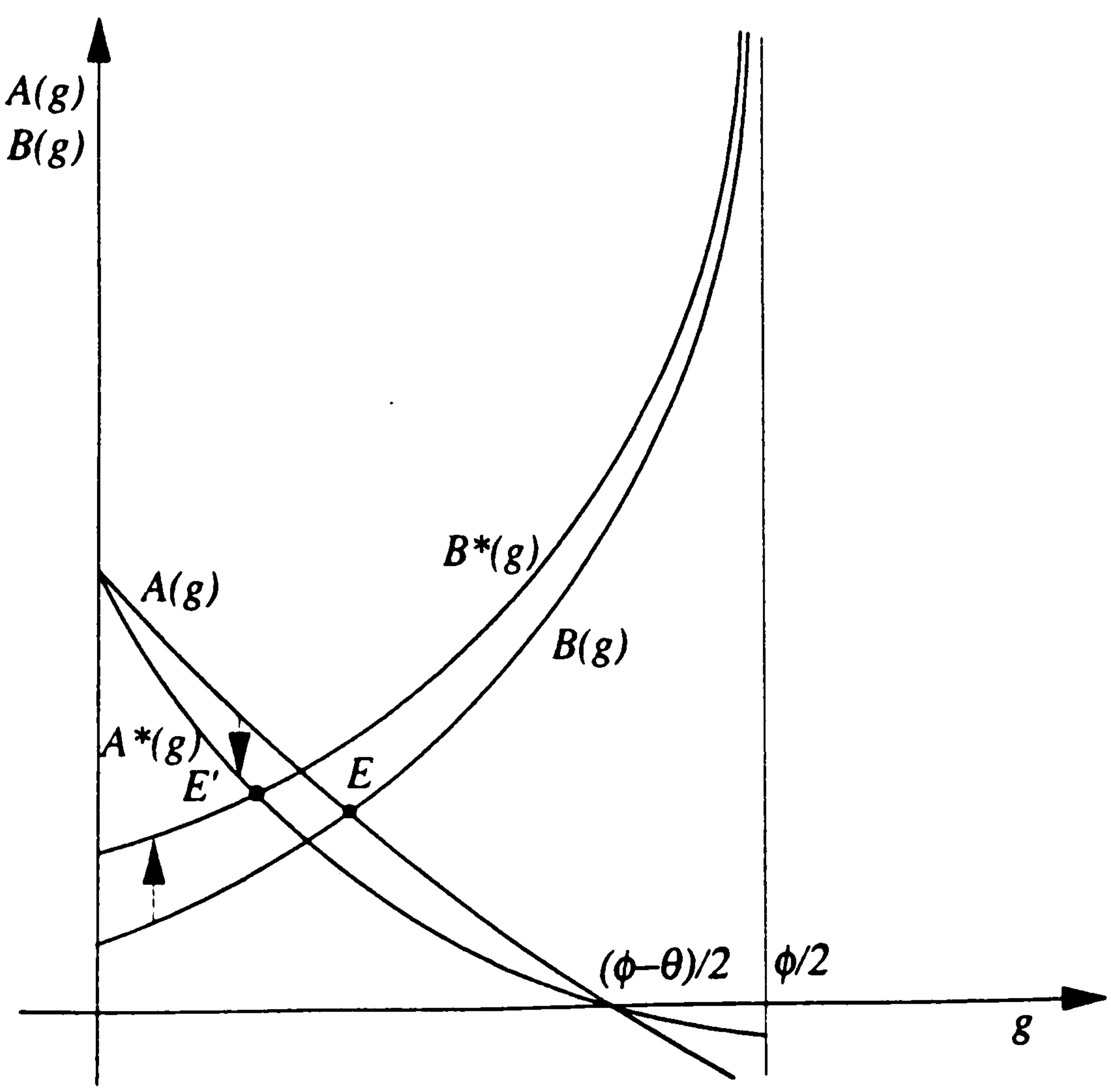

Figure 4. Steady state effects of an investment subsidy.

$A^*(g)$ is shifted downwards, with respect to $A(g)$, because of the increase in the capital/output ratio; $B^*(g)$ is shifted upwards, because of a raised $\hat{v}$. Therefore, as shown in figure 4, the long run equilibrium growth rate unambiguously decreases (from E to E'). Therefore, with finite lives, it is not longer possible to cope with the static distortion problem without affecting the growth rate.

Notice, also, that lump sum taxes levied on firms would have a beneficial effect for growth, because they reduce $\hat{v}$. Clearly, to decide which kind of instrument should be adopted, a social welfare function has to be used. With such a tool, one can address jointly the removal of static and dynamic distortions. It seems that the planner's felicity function proposed by Calvo and Obstfeld (1988) for continuous time overlapping generation models could play an important role for further developments on this point.

7 Concluding remarks

In this chapter, we have developed a model of endogenous growth with imperfect competition and blockaded entry. In our framework, each producer enjoys a partial monopolistic power, obtaining some pure profit, and he determines which share of it to invest in research according to an (intertemporally) optimal plan. The result of this effort is a (continuous) stream of process innovations. A certain minimum degree of monopolistic power has proved to be a necessary condition for growth, because research is funded out of profits; however, provided that this condition is fulfilled, in the infinitely lived case no relation has emerged between the degree of monopolistic power and the rate of growth. This result, which contrasts with the existing literature, has been ascribed to the symmetry among firms and to the presence of a "macroeconomic externality" that characterises the model.

At the policy level, our framework implies that a benevolent social planner cannot simply subsidise research, but he has to correct the distortion entailed by monopolistic competition.

With finite-lives agents, our neutrality result does not hold and policy actions aimed at removing the static inefficiencies caused by imperfect competition affect also the growth rate. Therefore, even for these relatively simple choices, a social welfare function proves to be necessary.

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Chapter III

Growth Models with Money in the Utility Function: a General Treatment of the Transitional Behaviour

1. Introduction

Tobin, in his 1965 seminal contribution, proposed a model where the long run equilibrium level for capital is positively related to the money growth rate. Since then, this result has been obtained in several aggregate frameworks, such as the ones by Sidrauski (1967a), Lucas (1975), Fischer (1979a) and Begg (1980); more recently the role for money has been studied also in non microfounded endogenous growth models (Stadler, (1990)).

In contrast to this stream of literature, one can explicitly derive the macroequations from the optimising behaviour of a representative agent, whose utility function includes real money balances among its arguments. This approach leads to the absence of the "Tobin effect" in the long run (Sidrauski, (1967b)). The potential importance of the connection between capital accumulation and monetary growth for economic policy helps explain the notable number of theoretical attempts which aim at establishing the Tobin effect in microfounded frameworks, either in the long run or during the transition path. Orphanides and Solow (1990) provide a simple survey of this literature.

However, many empirical works do not support this view about the long run effect of monetary policy, rather suggesting a negative relation between capital accumulation and money growth. For example, Fischer (1993) presents a wide cross-sectional regression analysis suggesting a negative link between growth and inflation; De Gregorio (1993), focusing on twelve Latin American countries, provides similar evidence. Smyth (1994) estimates the negative effect of inflation on growth for the US private sector[1]. Accordingly, (not to eliminate money from growth models), different theoretical explanations have been proposed. One can consider money as an argument of the production function, obtaining results that depend on the specification adopted. Alternatively, it is possible to embed the labour-leisure choice into the representative agent optimisation problem: again, the direction of the effect of money growth on capital is not obvious. The transaction role of money is explicitly taken into account in "cash-in-advance" models[2].

This chapter aims at showing that a negative relationship between capital accumulation and money growth is not incompatible with the common practice of including money among the arguments of the utility function. In section 2 we set up the problem, introducing an instantaneous utility function that allows for a

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[1] However, none of those papers is able to disentangle the role of inflation from the one of its variance. Hence, the negative effect on growth could be ascribed to this variable, which is sometimes considered a proxy of macroeconomic uncertainty.

[2] For references, see again, Orphanides and Solow (1990).

constant elasticity of substitution between real money balances and consumption. We then show that in some portions of the parameters space the relation between capital accumulation and money growth is opposite to the one suggested by Tobin. Section 3 is devoted to the discussion of the results attainable by means of analytical techniques, while the subsequent one presents some numerical calculations. In section 5 we study a linear-in-capital growth model underlining its lack of transitional dynamics. We then resort again to numerical techniques to study a framework where the marginal productivity of capital is only asymptotically constant. Again, we are able to highlight situations where the Tobin effect prevails and cases where an increase in monetary growth reduces capital accumulation. Interestingly, we show that, in this endogenous growth model, the stock of capital is permanently altered by variations in the money growth rate. Section 7 concludes.

2. Superneutrality in traditional growth models: a generalisation

The presence of real money balances in the utility function is usually justified on the basis of the "liquidity services" they provide. In other words, it is assumed that money can be introduced into the utility function to stylise the holding of real balances caused by the desire to save on the time and the effort required by the exchange process. Feenstra (1986) demonstrated the equivalence between introducing money in the utility function and entering real balances into an explicit "liquidity costs" technology[3]. Therefore, within Sidrauski's approach, a degree of substitution between consumption and real balances can be ascribed to the

[3] More recently, Croushore, (1993), suggests the equivalence between the money-in-utility approach and shopping-time models. However, this method imply that labour time should be introduced among the arguments of the utility function.

(unknown) characteristics of the transaction technology. For example, the "cash in advance" constraint implies perfect complementarity between consumption and money and hence a Leontief indifference map. The inspection of this situation, due to Asako (1983), is interesting since it establishes the reversal of the Tobin effect for low intertemporal substitution elasticities. However, this extreme case does not gain support from most of the empirical studies on money demand, which usually suggest a negative relationship between real money balances and the nominal interest rates. Therefore, the introduction of a sub-utility function allowing for various degrees of substitution between real money balances and consumption seems a useful generalisation.

The most obvious candidate for such an exercise is a time separable utility function allowing for a constant elasticity of substitution both between its arguments and over time[4]:

$$U = \int_0^\infty \frac{S}{S-1}[ac(t)^{(\sigma-1)/\sigma} + (1-a)m(t)^{(\sigma-1)/\sigma}]^{[\sigma(S-1)]/[S(\sigma-1)]} e^{-\theta t}\, dt \qquad (1)$$

where $c(t)$ and $m(t)$[5] stand for, respectively, consumption and real money balances

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[4] Hartman (1987, p. 475) assumes a utility function with a degree of substitution between consumption and real money balances but which is logarithmic over time. His results about the effects of the money growth rate on capital accumulation crucially depend on the introduction of distortionary income taxation. Also Marini and van der Ploeg (1988, pp. 775-6) and van der Ploeg and Alogoskoufis (1994, section 7) couple an intratemporal CES with a logarithmic intertemporal spefication in continuous time overlapping generations models. Villieu (1992) developed, independently from the author, a model based on the same utility function. However his analysis turned out to be partly incorrect (see below).

[5] From now on, we take as understood the time indexes whenever not confusing.

enjoyed by the representative agent at time t; θ is the intertemporal preference rate, σ is the elasticity of substitution between consumption and real balances and S is the intertemporal substitution elasticity. It is assumed that $\sigma \in [0,\infty)$, $S \in (0,\infty)$[6] and $a \in (0,1)$. When σ is equal to one, this specification specialises to a Cobb Douglas in consumption and money, i.e. to the case analysed by Fischer (1979b) and Cohen (1985).

Assuming constant population[7], the intertemporal budget constraint may be expressed as follows:

$$c + \overset{o}{k} + \frac{\overset{o}{M}}{p} = r\,k + w + \tau \tag{2}$$

where k represents per capita capital and M money in per capita nominal terms; p is the general price level, r is the real interest rate, w the labour income, and τ the lump sum transfer assigned to the representative individual by the Government[8]. The interest rate r, under perfect competition, is equal to the marginal productivity of capital, $f'(k)$, minus δ, the capital depreciation parameter.

The time derivative for any variable x is denoted by $\overset{o}{x}$.

We introduce, as in Blanchard and Fischer, (1989, p. 189), the definition of wealth (h) which, in absence of public debt, is the sum of capital and real money

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[6] With S=0, not only would the utility function degenerate, but also the determinant of the Jacobian of the dynamical system resulting from this model would be zero. (See below.)

[7] We implicitly introduce also the hypothesis of a zero growth rate for the representative family.

[8] Notice that the representative agent should consider τ as independent from his real money holdings, otherwise money would be superneutral even on the transition path. (E.g. Fischer, (1979, p. 1434), and Asako, (1983, p. 1594)).

balances[9], and we recall that $\overset{o}{M}/p = \overset{o}{m} + (\overset{o}{p}/p)\, m$, in order to express the intertemporal budget constraint as follows:

$$\overset{o}{h} = r\,h + w + \tau - c - i\,m$$

where i, the nominal interest rate, is defined as: $r + \overset{o}{p}/p$.

To solve the consumer's problem, it is useful to consider the current values Hamiltonian.

$$H: \frac{S}{S\text{-}1}[ac^{(\sigma-1)/\sigma} + (1\text{-}a)m^{(\sigma-1)/\sigma}]^{[\sigma(S-1)]/[S(\sigma-1)]} + \lambda\,(r\,h + w + \tau - c - i\,m)$$

The first order conditions are:

$$H_c : ac^{-1/\sigma}[ac^{(\sigma-1)/\sigma} + (1\text{-}a)m^{(\sigma-1)/\sigma}]^{(S-\sigma)/[S(\sigma-1)]} = \lambda \tag{3}$$

$$H_m : (1\text{-}a)m^{-1/\sigma}[ac^{(\sigma-1)/\sigma} + (1\text{-}a)m^{(\sigma-1)/\sigma}]^{(S-\sigma)/[S(\sigma-1)]} = i\,\lambda \tag{4}$$

$$H_h : \overset{o}{\lambda} = (\theta - r)\,\lambda \tag{5}$$

Also the following transversality condition, $\lim_{t\to\infty} e^{-\theta t}\lambda(t)h(t) = 0$, must hold.

Dividing (3) by (4) we get:

$$\frac{m}{c} = \left(\frac{a}{1\text{-}a}\,i\right)^{-\sigma} \tag{6}$$

This equation can be interpreted as a money demand function.

It is convenient, as in Villieu (1992, pp. 78 ff.) to frame the dynamical system in terms of consumption, capital and nominal interest rate.

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[9] It is common, in this literature, to rule out public debt. Such a hypothesis is not crucial for the results we are discussing.

Differentiation with respect to time of equation (6) gives an expression which will be useful in what follows:

$$\frac{\overset{o}{c}}{c} = \frac{\overset{o}{m}}{m} + \sigma \frac{\overset{o}{i}}{i} \tag{7}$$

We then recall that $\overset{o}{m}/m = \omega - \overset{o}{p}/p = \omega - i + f'(k) - \delta = i^* - i + f'(k) - \delta - \theta$, where ω is the money growth rate and asterisks denote steady-state values of the corresponding variables, to get:

$$\frac{\overset{o}{c}}{c} = i^* - i + f'(k) - \delta - \theta + \sigma \frac{\overset{o}{i}}{i} \tag{8}$$

Exploiting again equation (6) to substitute out real money balances from equation (3), we obtain, collecting the terms in consumption:

$$a^{\frac{\sigma(S-1)}{S(\sigma-1)}} c^{-1/S} [1 + A^{-1} i^{1-\sigma}]^{\frac{S-\sigma}{S(\sigma-1)}} = \lambda$$

where $A \equiv (a/(1-a))^{\sigma}$; this expression, upon differentiation and by use of (5), can be transformed into:

$$\frac{\overset{o}{i}}{i} = \left(\frac{1+Ai^{\sigma-1}}{\sigma-S}\right)\frac{\overset{o}{c}}{c} + \left(\frac{S(1+Ai^{\sigma-1})}{\sigma-S}\right)[\theta + \delta - f'(k)] \tag{9}$$

Finally, by means of (8), we obtain our first differential equation:

$$\frac{\overset{o}{i}}{i} = \left(\frac{1+Ai^{\sigma-1}}{S+\sigma Ai^{\sigma-1}}\right)(i - i^*) + (S-1)\left(\frac{1+Ai^{\sigma-1}}{S+\sigma Ai^{\sigma-1}}\right)[f'(k) - \theta - \delta] \tag{10}$$

From equations (8) and (10) we immediately get our second law of motion:

$$\frac{\overset{o}{c}}{c} = \left(\frac{S-\sigma}{S+\sigma Ai^{\sigma-1}}\right)(i^* - i + f'(k) - \delta - \theta) + \left(\frac{S\sigma(1+Ai^{\sigma-1})}{S+\sigma Ai^{\sigma-1}}\right)[f'(k) - \theta - \delta] \tag{11}$$

The long run solution of (11) allows us to visualise the superneutrality result: when $\overset{o}{c}=0$ and $i=i^*$ this equation can be satisfied only if $f'(k^*)=\theta+\delta$, which implies that the growth rate of money does not affect the steady state level of output. This is the root of the «friedmanite» prescription of the model. In fact, given the complete ineffectiveness of the monetary policy on output, in the long run it is optimal to «satiate» the individuals with money - when the utility function permits, which here it does not - and hence to reduce M at a rate equal to θ (See Friedman, (1969)).

As for capital, the differential equation is simply:

$$\overset{o}{k} = f(k) - c - \delta k \tag{12}$$

which can be derived from the intertemporal budget constraint (2).

Linearization around the steady state of system (10-11-12) yields the relatively simple jacobian:

$$J = \begin{bmatrix} \left(\dfrac{1+Ai^{*(\sigma-1)}}{S+\sigma A i^{*(\sigma-1)}}\right)i^* & 0 & \left(\dfrac{1+Ai^{*(\sigma-1)}}{S+\sigma A i^{*(\sigma-1)}}\right)(S-1)f''(k^*)i^* \\ \left(\dfrac{\sigma-S}{S+\sigma A i^{*(\sigma-1)}}\right)c^* & 0 & \left[S\sigma\left(\dfrac{1+Ai^{*(\sigma-1)}}{S+\sigma A i^{*(\sigma-1)}}\right)+\dfrac{S-\sigma}{S+\sigma A i^{*(\sigma-1)}}\right]f''(k^*)c^* \\ 0 & -1 & \theta \end{bmatrix}$$

Notice that, in the steady state, consumption is equal to output, net of depreciation, and that the level of capital is such that its marginal productivity matches the intertemporal preference rate plus the capital depreciation parameter. Moreover, the nominal interest rate is equal to the intertemporal preference plus the rate of money growth. Hence, there is a one to one relation between i^* and ω, a fact that will be exploited later.

Two jumping variables are present in this model: consumption and nominal interest rate; being capital the unique predetermined quantity. Hence, to grant saddlepath stability, we need to prove that there is only one negative eigenvalue. This can be done fairly easily, as:

$$Det(J) = \left(\frac{1+Ai^{*(\sigma-1)}}{S+\sigma Ai^{*(\sigma-1)}}\right) S\, i^* c^* f''(k) < 0$$

The sign of the determinant implies that there are either one or three negative eigenvalues. Since

$$Trace(J) = \theta + \left(\frac{1+Ai^{*(\sigma-1)}}{S+\sigma Ai^{*(\sigma-1)}}\right) i^* > 0$$

the negative eigenvalue is unique and the explicit solution for the linearised differential equation for capital is $k(t) = k^* + [k(0)-k^*]e^{\eta^* t}$ where η^* is the stable eigenvalue of J. Therefore, the effect of the money growth rate on the capital level is given by:

$$\frac{\partial k(t)}{\partial \omega} = \frac{\partial \eta^*}{\partial \omega}[k(t)-k^*]t \tag{13}$$

When $k(t)<k^*$ and $t \in (0,\infty)$[10], we are in presence of the Tobin effect on the transition path if $\partial\eta^*/\partial\omega<0$. Therefore, it is necessary to check the sign of the effect on the negative eigenvalue of an increase in the money growth rate. The standard procedure, suggested by Fischer (1979b, p. 1437), exploits the total differential of the characteristic equation, evaluated at η^*:

$$\frac{\partial \mathcal{A}(\eta^*,\omega)}{\partial \omega} d\omega + \frac{\partial \mathcal{A}(\eta^*,\omega)}{\partial \eta} d\eta = 0$$

where $\mathcal{A}(\eta^*,\omega)=$

[10] For t approaching infinity, we have $\lim_{t\to\infty} [k(t)-k^*]t = 0$.

$$= (\eta^{*2}-\theta\eta^*+Sf''(k^*)c^*)\left[\left(\frac{(1+Ai^{*(\sigma-1)})i^*}{S+\sigma Ai^{*(\sigma-1)}}\right)-\eta^*\right]-\eta^*\frac{f''(k^*)c^*(S-1)(\sigma-S)}{S+\sigma Ai^{*(\sigma-1)}}=0 \qquad (14)$$

Total differentiation of $\mathcal{A}(\eta^*,\omega)$ gives:

$$\frac{d\eta^*}{d\omega}=-\left(\frac{\partial\mathcal{A}(\eta^*,\omega)/\partial\omega}{\partial\mathcal{A}(\eta^*,\omega)/\partial\eta}\right) \qquad (15)$$

Since η^* is the unique negative eigenvalue, $\partial\mathcal{A}(\eta^*,\omega)/\partial\eta<0$, and our problem reduces to the evaluation of $\partial\mathcal{A}(\eta^*,\omega)/\partial\omega$. If it is negative, we are in presence of the Tobin effect, otherwise $d\eta^*/d\omega>0$ and an increase in the money growth rate reduces capital accumulation (equation 13).

Differentiating (14) with respect to ω and substituting the resulting expression back into (14) itself we get:

$$\frac{\partial\mathcal{A}(\eta^*,\omega)}{\partial\omega}=\left[\frac{f''(k^*)c^*\eta^*}{(1+Ai^{*(\sigma-1)})i^*-(S+\sigma Ai^{*(\sigma-1)})\eta^*}\right](S-1)(\sigma-S)\,X(S,\sigma,i^*) \qquad (16)$$

where $X(S,\sigma,i^*)=1+\sigma Ai^{*(\sigma-1)}-(\sigma-1)\sigma Ai^{*(\sigma-2)}\,\eta^*(S,\sigma,i^*)$

The sign of the term in the big square bracket in (16) is always positive. Hence, to sign $\partial\mathcal{A}(\eta^*,\omega)/\partial\omega$, we essentially need to study the sign of $X(.)$. The next section surveys the analytical results obtained in the literature under various assumptions about σ and S, while section 4 is devoted to the study of situations where the use of numerical techniques is necessary.

3. Analytical results

When $\sigma=0$ or when $\sigma\geq 1$, it is easy to sign $\partial\mathcal{A}(\eta^*,\omega)/\partial\omega$, since $X(S,\sigma,i^*)>0$.

If consumption and real money balances are strict complements we have that $\text{sign}\left(\frac{\partial\mathcal{A}(\eta^*,\omega)}{\partial\omega}\right)$ = sign(1-S), as shown by Asako (1983, p. 1596). Hence, the Tobin effect is present only if the intertemporal elasticity of substitution exceeds one.

When $\sigma=1$, we are in the situation studied by Fischer and Cohen; $X(.)$ collapses to $1/(1-a)$ and the sign of $\partial\mathcal{A}(\eta^*,\omega)/\partial\omega$ is always negative. Hence, from (15), $d\eta^*/d\omega<0$ and the Tobin effect is estabilished.

Finally, for $\sigma>1$, $\text{sign}\left(\frac{\partial\mathcal{A}(\eta^*,\omega)}{\partial\omega}\right)=\text{sign}((1-S)(S-\sigma))$

Hence, as discussed also by Villieu (1992, pp. 87-89), for $\sigma>S>1$, $d\eta^*/d\omega$ is positive and capital accumulation is reduced by an increase in the money growth rate. In other words, the "anti-Tobin" area is not limited to a zero measure set in the parameters space. As $S>\sigma>1$ or as $\sigma>1$ but $S<1$, however, the traditional positive link between capital accumulation and money growth resists.

So far, we have shown that the same specification for the utility function may generate different behaviours. Therefore, an intuitive explanation clarifying the role of the elasticities of intertemporal and intratemporal substitution seems useful.

It is possible to provide a detailed intuition building on the fact that $\text{sign}\left(\frac{\partial i(t)}{\partial\omega}-1\right)=\text{sign}(1-S)$.

To demonstrate this property of the sequence for the nominal interest rate, we need to start from the explicit solution for the linearised equation that describes the interest rate behaviour. Since there is only one stable eigenvalue, it can be formulated as a linear transformation of the solution of the linearised equation for capital:

$$i(t)-i^*=\beta[k(t)-k^*]$$

The constant β is given by the ratio between the component of the relevant eigenvector associated to the interest rate and the component associated to capital. Calculations show that:

$$i(t)=i^*+(1-S)\left(\frac{f''(k^*)i^*(1+Ai^{*(\sigma-1)})}{(1+Ai^{*(\sigma-1)})i^*-\eta^*(S+\sigma Ai^{*(\sigma-1)})}\right)[k(0)-k^*]e^{\eta^* t} \tag{17}$$

It is immediate to note that sign$[i(t)-i^*]$ = sign$(1-S)$ when $k(0)<k^*$.

Likewise, for consumption we have:

$$c(t)=c^*+(\theta-\eta^*)[k(0)-k^*]e^{\eta^* t} \tag{18}$$

Therefore:

$$\frac{\partial c(t)}{\partial \omega}=-\frac{\partial \eta^*}{\partial \omega}[k(t)-k^*] \tag{19}$$

Time-differentiation of equation (18) gives:

$$\frac{\overset{o}{c}(t)}{c(t)}=\eta^*\left(1-\frac{c^*}{c(t)}\right)$$

Using (19) to substitute out $\partial c(t)/\partial \omega$ from the partial derivative with respect to the money growth rate of the last expression, we obtain an important intermediate result:

$$\frac{\partial[\overset{o}{c}(t)/c(t)]}{\partial \omega}=\frac{\partial \eta^*}{\partial \omega}\left(1-\frac{c^*}{c(t)}-\frac{\eta^* c^*}{c(t)^2}[k(t)-k^*]\right) \tag{20}$$

Notice from equation (19) that the expression inside the big round brackets is negative, since $c(t)<c^*$. Differentiate now the linearised version of equation (11) with respect to the money growth rate, considering as given the sequence for capital[11]. Since $\partial i^*/\partial \omega=1$, we get:

$$\frac{\partial[\overset{o}{c}(t)/c(t)]}{\partial \omega}=\left(\frac{S-\sigma}{S+\sigma Ai^{*\sigma-1}}\right)\left(1-\frac{\partial i(t)}{\partial \omega}\right)+$$

[11] This assumption is common in the literature: see Cohen (1985) and Villieu (1992).

$$+\left(\frac{\sigma(\sigma-1)Ai^{*\sigma-2}(S-\sigma)}{(S+\sigma Ai^{*\sigma-1})^2}\right)\{i-i^*+(S-1)f'(k^*)[k(t)-k^*]\}$$

Exploiting equation (17) to substitute out $(S-1)f'(k^*)[k(t)-k^*]$ from the right hand side of this expression, we obtain:

$$\frac{\partial[\overset{o}{c}(t)/c(t)]}{\partial\omega}=\left(\frac{S-\sigma}{S+\sigma Ai^{*\sigma-1}}\right)\left(1-\frac{\partial i(t)}{\partial\omega}\right)+\left(\frac{\sigma(\sigma-1)Ai^{*\sigma-3}(S-\sigma)\eta^*}{(S+\sigma Ai^{*\sigma-1})(1+Ai^{*(\sigma-1)})}\right)(i-i^*) \tag{21}$$

Equate equations (20) and (21) to get:

$$\left(1-\frac{\partial i(t)}{\partial\omega}\right)=\frac{\partial\eta^*}{\partial\omega}\left(\frac{S+\sigma Ai^{*\sigma-1}}{S-\sigma}\right)\left(1-\frac{c^*}{c(t)}-\frac{\eta^*c^*}{c(t)^2}[k(t)-k^*]\right)-\left(\frac{\sigma(\sigma-1)\eta^*Ai^{*\sigma-3}}{(1+Ai^{*(\sigma-1)})}\right)(i-i^*)$$

Recall that, when σ=0 or σ≥1, $X(S,\sigma,i^*)$ is positive and that sign($\partial\eta^*/\partial\omega$) is equal to sign($(1-S)(S-\sigma)$). Hence, from the last expression, we see that sign$\left(\frac{\partial i(t)}{\partial\omega}-1\right)$ = sign($1-S$).

We are now in the position of being able to provide an intuitive explanation for the results we surveyed. We start from equation (7), which is reproduced for convenience.

$$\frac{\overset{o}{c}}{c}=\frac{\overset{o}{m}}{m}+\sigma\frac{\overset{o}{i}}{i} \tag{7}$$

This expression makes explicit the fact that, for a given interest rate, the representative agent would like to expand consumption and the use of real balances at the same rate, a consequence of the homoteticity of the utility function. However, since the time profile of the nominal interest rate is affected by changes in ω, the higher the substitution possibilities, the more rapid is consumption growth for a given increasing sequence of the interest rate. In fact, with a high σ, a growing interest rate shifts expenditure towards consumption.

With these remarks in mind, we now discuss the three cases briefly surveyed at the beginning of this section.

We see from equation (7) that, in the situation studied by Asako (σ=0)[12]:

$$\frac{\partial[\overset{o}{c}(t)/c(t)]}{\partial\omega}=\frac{\partial[\overset{o}{m}(t)/m(t)]}{\partial\omega}=-\left(\frac{\partial i(t)}{\partial\omega}-1\right)$$

Since sign[$i(t)-i^*$] = sign($1-S$), were $\partial i(t)/\partial\omega$ equal to unity when $S>1$, money would become, in comparison with the initial situation, less convenient in the short run than in the steady state. Hence, real balances would be tilted forward $\left(\frac{\partial[\overset{o}{m}(t)/m(t)]}{\partial\omega}>0\right)$ and so would be consumption. In equilibrium, current money is actually substituted with future money and the reduction in the current demand for real balances entails the undershooting of the interest rate highlighted in figure 1a. Since in the steady state consumption is constant, an increase in its growth rate entails a reduction in its current level, which is the Tobin effect.

When the elasticity of intertemporal substitution is low, the nominal interest "overshoots" its long-run increase (figure 1b), real balances are anticipated, and this causes an increase in consumption.

We now turn our attention to the classic intratemporal Cobb-Douglas case. Differentiate again equation (7) with respect to the money growth rate and substitute out $\frac{\partial[\overset{o}{i}(t)/i(t)]}{\partial\omega}$ using equation (10). One obtains, for a given sequence for capital:

$$\frac{\partial[\overset{o}{c}(t)/c(t)]}{\partial\omega}=-\left(\frac{\partial i(t)}{\partial\omega}-1\right)+\frac{1}{S(1-a)+a}\left(\frac{\partial i(t)}{\partial\omega}-1\right) \qquad (22)$$

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[12] Recall that $\dot{m}/m = i^* - i + f'(k)-\delta-\theta$ and that we are considering capital as given.

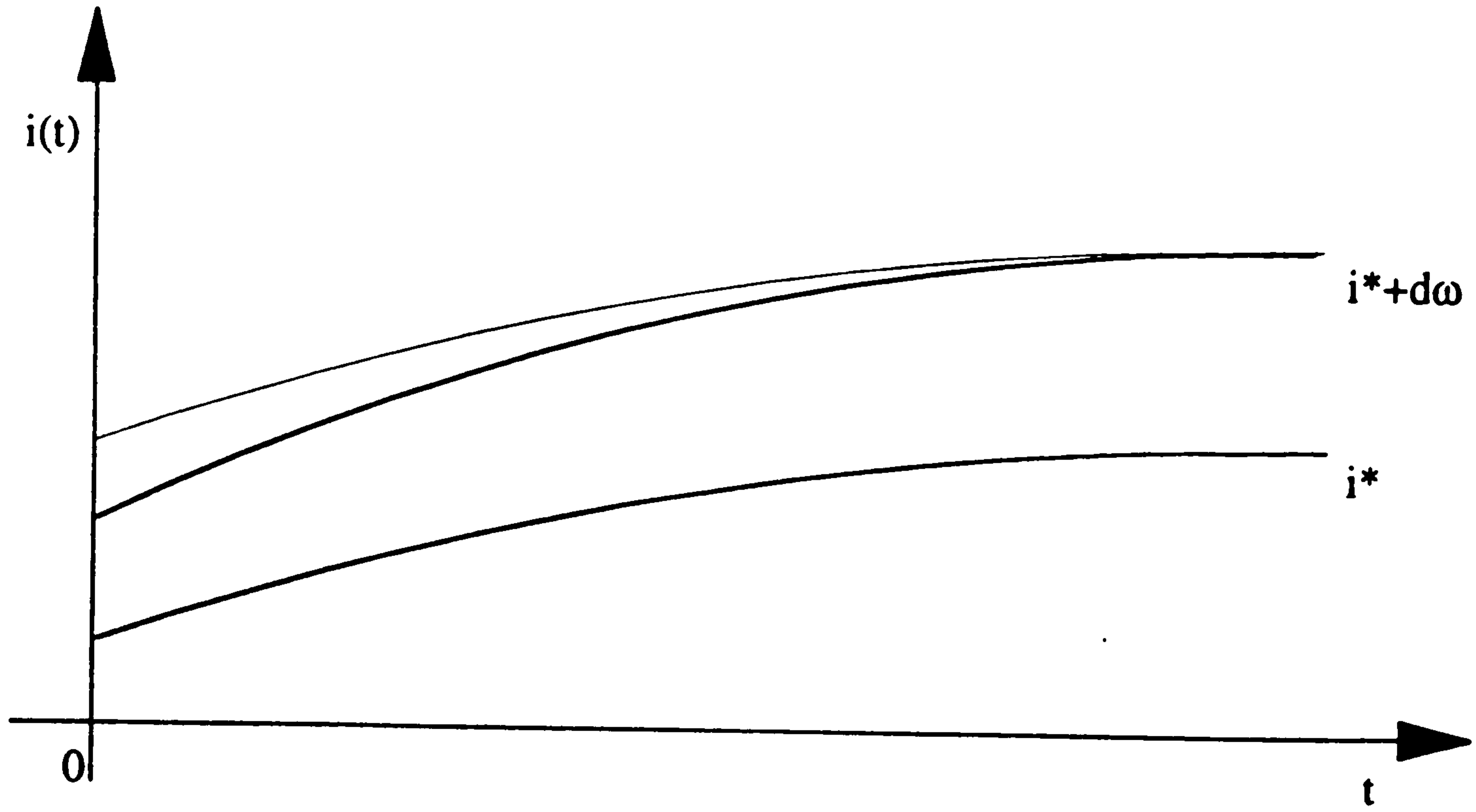

Figure 1a

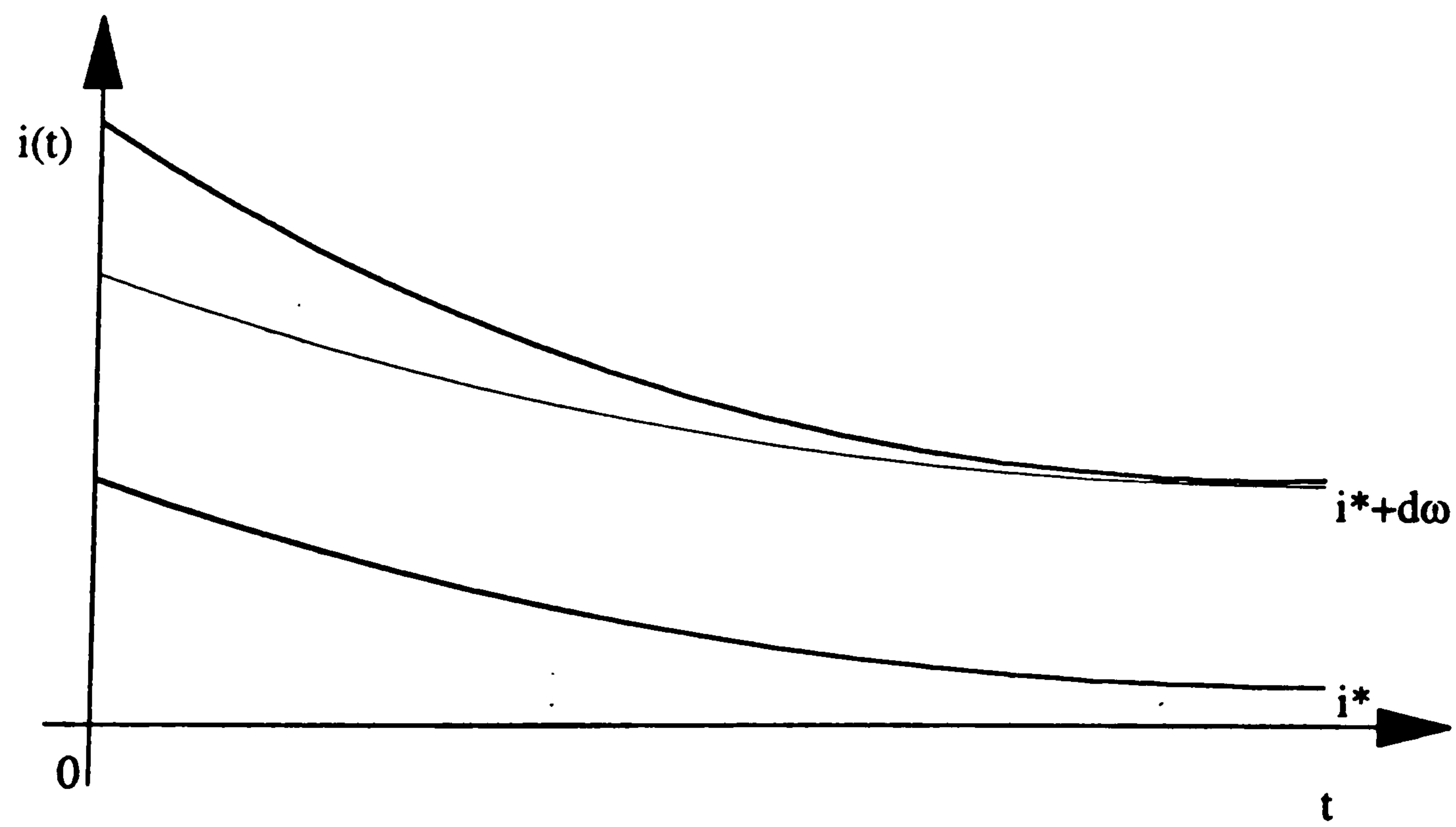

Figure 1b

Figure1. Effects on the time profile for the nominal interest rate when S>1 (figure 1a) and when S<1 (figure 1b).

The two addenda in (22) represent, respectively, the shift in the time profile of real money balances and the variation induced in consumption by the changes in the nominal interest rate sequence. When $S>1$, $\left(\frac{\partial i(t)}{\partial \omega}-1\right)<0$ and $\frac{\partial[\overset{o}{m}(t)/m(t)]}{\partial \omega}$ is positive: money is tilted forward, as in Asako's case, and the effect of the variation in the nominal interest rate, with $\sigma=1$, is not strong enough to reverse the impact of the money growth rate on consumption.

To understand why the effect of a variation in ω on the sequence of nominal interests is low, consider that, if there were no underadjustment in this sequence, the representative agent would be willing to substitute current with future money. Since his elasticity of intertemporal substitution is high, a small reduction in the current nominal interest rate is enough to keep the money market balanced. Therefore $\frac{\partial[\overset{o}{i}(t)/i(t)]}{\partial \omega}$ must be low.

When $S<1$, the second addendum in equation (22), which is now positive, prevails This must happen since, with no overshooting in the sequence for interest rates, money would become more convenient in the short run than in the steady state. Hence, the representative agent becomes willing to substitute future with current money. However, inasmuch as its elasticity of intertemporal substitution is low, a high increase in the current nominal interest rate is required for equilibrium.

Therefore, the Tobin effect is present both with high and with low elasticity of intertemporal substitution, even if for different reasons.

Finally, we examine the more general case analysed by Villieu ($\sigma>1$).

We manipulate equation (7) in the same way we have done in the previous case, although we now use the linearised version of equation (10). We obtain:

$$\frac{\partial[\overset{o}{c}(t)/c(t)]}{\partial \omega}=-\left(\frac{\partial i(t)}{\partial \omega}-1\right)+$$

$$+\sigma\left(\frac{1+Ai^{*\sigma-1}}{S+\sigma Ai^{*\sigma-1}}\right)\left(\frac{\partial i(t)}{\partial\omega}-1\right)+\left(\frac{\sigma(\sigma-1)Ai^{*\sigma-2}(S-\sigma)}{(S+\sigma Ai^{*\sigma-1})^2}\right)\{i-i^*+(S-1)f''(k^*)[k(t)-k^*]\}\quad(23)$$

The first two addenda point out the usual effect on consumption of the variations in the time profile of real money balances and interest rate. However, in comparison with equation (22), a third element emerges, representing the effect on consumption of an increase in the long run interest rate. Notice that such term was naught with Cobb-Douglas preferences, which imply constant expenditure shares: in such case, the consumption profile is affected only by changes in the sequence for the interest rate, but not by variations in i^*. By use of equation (17), we see that the sign of the term in curly brackets in equation (23) is equal to sign(S-1), which implies that the nominal interest rate sequence is increasing if $S>1$[13]. This contributes to tilt forward real money balances and consumption. However, $\sigma>1$ implies a tendency that hinders the previous one, causing an increase in the current level of consumption and hence a reduction in its growth rate: in fact, when substitution between consumption and money is easy, an increase in the interest rate shifts expenditure towards consumption[14]. Which of the two effects prevails depends on sign(σ-S).

As for the two first addenda in equation (23), we notice that they can be written as $\left(\frac{\sigma-S}{1+Ai^{*\sigma-1}}\right)\left(\frac{\partial i(t)}{\partial\omega}-1\right)$: when $S>1$, money is tilted forward and this generates a tendency towards a reduction in consumption. However, when $\sigma>S$, the undershooting of the nominal interest rate is strong enough to reverse the former effect.

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[13] The analysis of the case $S<1$ is omitted, since it is similar to the one for $S>1$, but it is less interesting.

[14] For a similar interpretation, see Villieu, (1995).

Notice that this effect and the former one go in the same direction when $\sigma>1$, but not when $\sigma<1$. Therefore, the analysis of the interval $\sigma\in(0,1)$ is much harder simply because the two effects go in opposite directions and the second one may prevail when $Ai^{*\sigma-2}$ is high, i.e. when i^* is low. This is also the reason why the sign of $X(S,\sigma,i^*)$ can not be easily determined when $\sigma\in(0,1)$[15].

However, this case is relevant: a low but strictly positive elasticity of substitution between consumption and money fits, at the level of stylised facts, with most of the empirical contributions on the demand for money. In such studies, the estimated coefficients on interest rates are often low in absolute value but significant. For an example considering 27 countries, and among them the OECD ones but for Iceland, Luxembourg and Spain, see Fair (1987). At the theoretical level, if $\sigma<1$, the possibility of hyperinflation paths, highlighted by Obstfeld and Rogoff, (1983), is ruled out.

The difficulty for the analytical characterisation of the sign of $X(S,\sigma,i^*)$ arises from the fact that one of the arguments of $X(S,\sigma,i^*)$ is $\eta^*(S,\sigma,i^*)$. However, by inspecting (14) we notice that, if $S=\sigma$, $\eta^*(\sigma)$ becomes independent of i^* and equal to $(\theta-\sqrt{\theta^2-4f''(k^*)c^*\sigma})/2$. In this particular case, we show that $X(\sigma,i^*)$, for $\sigma\in(0,1)$, is negative when i^* approaches 0.

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[15] Villieu (1992, pp. 87-89) investigates the sign of $X(.)$ by inspecting the linearized dynamic behaviour of consumption and of the nominal interest rate. He concludes that Sign$(\partial\mathcal{A}(\eta^*,\omega)/\partial\omega)$ is equal to Sign$((S-1)(\sigma-S))$ also when $\sigma<1$, and hence that $X(.)$ is always positive. However, in his proof, while computing $\partial c(t)/\partial\omega$, $\frac{\partial(\dot{c}(t)/c(t))}{\partial\omega}$ and $\frac{\partial(\dot{r}(t)/r(t))}{\partial\omega}$ he introduces the hypothesis that the sequence for capital is given *at its steady state level* (Villieu, 1992, pp. 87-88). This assumption induces a bias, as shown by Femminis, (1995), since, for a given initial condition, the capital level is affected by η^* and hence by the variations of ω (equation 14).

Reformulate $X(\sigma,i^*)$ as:

$$X(\sigma,i^*)=1+\sigma A\, i^{*(\sigma-1)}\left(1-(\sigma-1)\,\frac{\eta^*(\sigma)}{i^*}\right)$$

Since $\lim\limits_{i^*\to 0^+} i^{*(\sigma-1)}=[+\infty]$ we have:

$$\lim_{i^*\to 0^+} X(\sigma,i^*) = [+\infty]\left\{\lim_{i^*\to 0^+}\left[\sigma A\left(1-(\sigma-1)\,\frac{\eta^*(\sigma)}{i^*}\right)\right]\right\} =$$

$$=[+\infty]\left\{\sigma A\left[1-(\sigma-1)\lim_{i^*\to 0^+}\left(\frac{\eta^*(\sigma)}{i^*}\right)\right]\right\}=[-\infty].$$

This result is limited but relevant, since, by continuity, we argue that $X(S,\sigma,i^*)$ is negative also for some $S\neq\sigma$ and for some $i^*>0$. Since $X(\sigma,i^*)$ is positive for a sufficiently high i^*, also $X(S,\sigma,i^*)$ is greater than zero for some $S\neq\sigma$. Therefore, to asses the quantitative importance of the portion of the parameters space where $X(S,\sigma,i^*)$ is negative we need to resort to numerical techniques.

4. Numerical results

Our computation strategy preliminarily requires the adoption of a specific form for the production function. Normalising per capita labour supply to one, the simplest choice is:

$$y = \phi k^{\gamma} - \delta k \tag{24}$$

where y is per capita output.

Therefore, the model contains eight parameters: S, σ, θ, a, ω, ϕ, γ, δ. Notice, however, that the steady state values of consumption and of the second derivative of the production function always appear in equation (14) as a product. Since $\eta^*(.)$ is the smallest root of that equation, the Cobb-Douglas specification (24) allows to substitute $(\theta+\delta)^2(\gamma-1)/\gamma-\delta(\theta+\delta)/(\gamma-1)$ for $c^*f''(k^*)$ into (14), and $X(.)$ is independent of ϕ.

Moreover, the value of $X(.)$ does not prove to be very sensitive to a; therefore, choosing a = 0.7, and setting the rate of intertemporal preference, θ, to 0.03 and the capital income share, γ, to 0.3, we can focus on S, σ and ω.

Assuming away capital depreciation and choosing a situation close to the friedmanite one, with i^*=0.001, we plot $X(S,\sigma,0.001)$ for $S\in\{0.5, 2\}$, and we obtain figure 2. When S=0.5, $X(.)$ is negative for $\sigma\in[0.000051, 0.931707]$; when S=2, it is negative for $\sigma\in[0.000021, 0.971452]$.

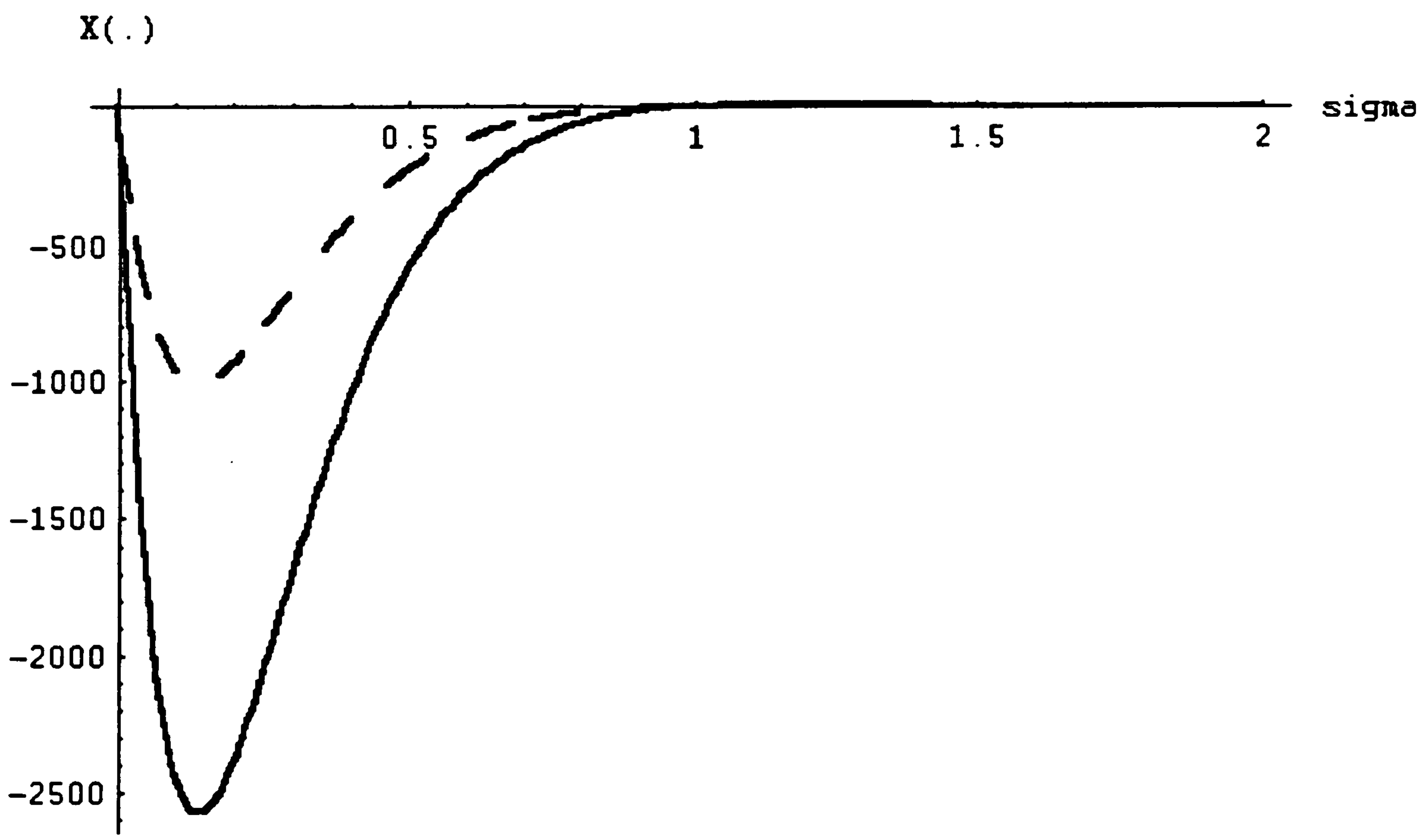

Figure 2. X(S,σ,i*) is plotted as a function of σ, for i*=0.001, when S=0.5 (dashed line) and when S=2 (continuous line).

In general, simulations show that the lower i^*, the larger the interval with negative $X(.)$.

It is notable that, if we accept the policy prescription provided by the model, choosing a nominal interest rate not far from naught, sign($\partial\mathcal{A}(\eta^*,\omega)/\partial\omega$) is opposite to sign($(S-1)(\sigma-S)$). It is natural to investigate situations with more realistic nominal interest rates. We increased i^* by step of 0.001 starting from 0.001 and we first obtained a strictly positive surface, considering $X(.)$ a function of S and σ, for $S\in(0,100)$ and $\sigma\in[0,1]$, with $\hat{i}$=0.066. This proved to be the «critical value» for our parameter set, since a further increase in i^* always produced strictly positive surfaces, i.e. $\hat{i}=\min\{i^*: X(S,\sigma,i^*)>0 \mid S\in(0,100), \sigma\in[0,1]\}$.

Table 1: Critical values of the nominal interest rate.

Parameters values: β=0.03, γ=0.3			
δ=0.0		δ=0.1	
a	$\hat{i}$	a	$\hat{i}$
0.1	0.054	0.1	0.111
0.2	0.056	0.2	0.120
0.3	0.058	0.3	0.126
0.4	0.060	0.4	0.132
0.5	0.062	0.5	0.139
0.6	0.064	0.6	0.145
0.7	0.066	0.7	0.155
0.8	0.070	0.8	0.168
0.9	0.076	0.9	0.192

Note: S varies from 0.1 to 100 by steps of 1; σ varies from 0 to 1 by steps of 0.05.

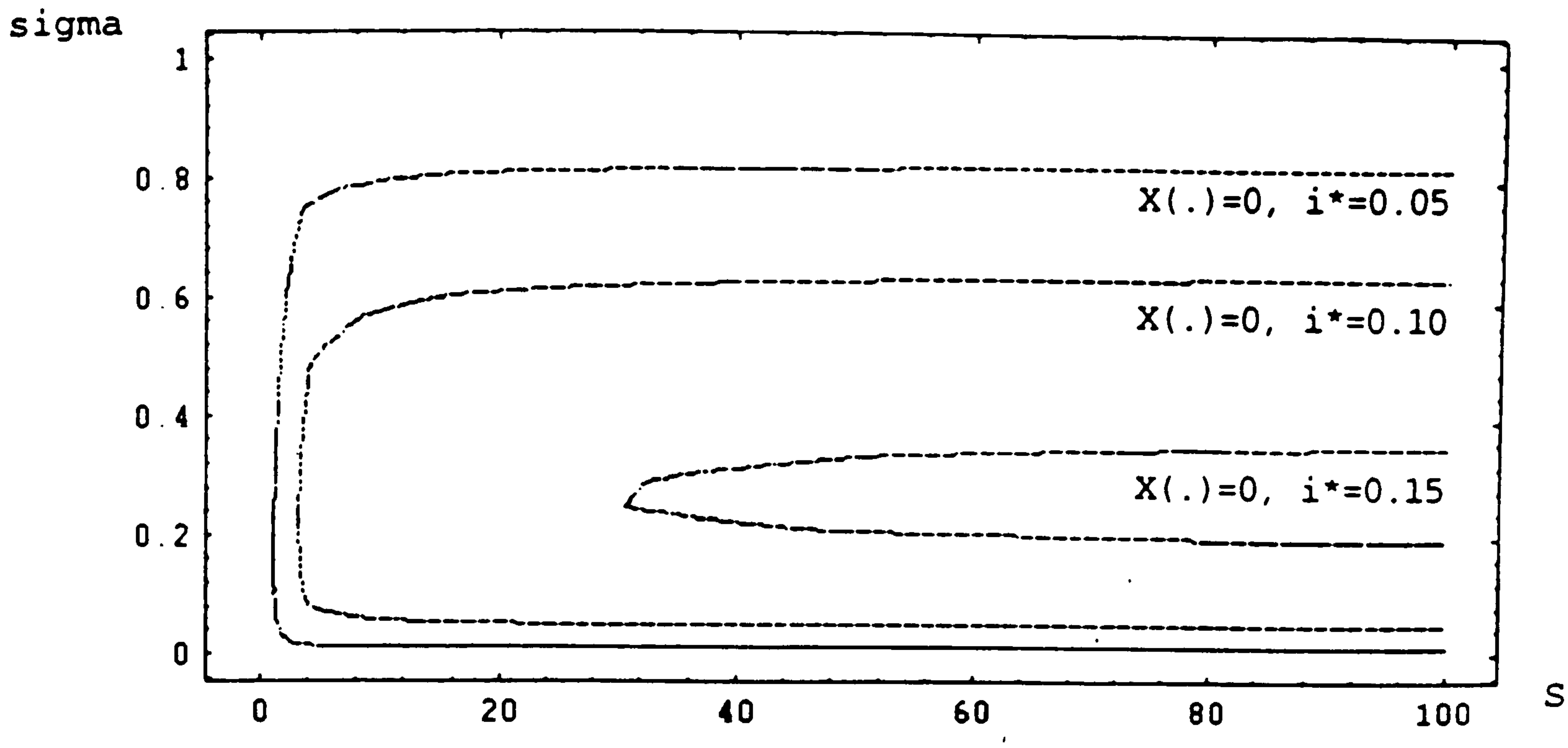

Figure 3. Contours of X(S,σ,i*) at an height equal to zero, for i*∈ {0.05, 0.10, 0.15}.

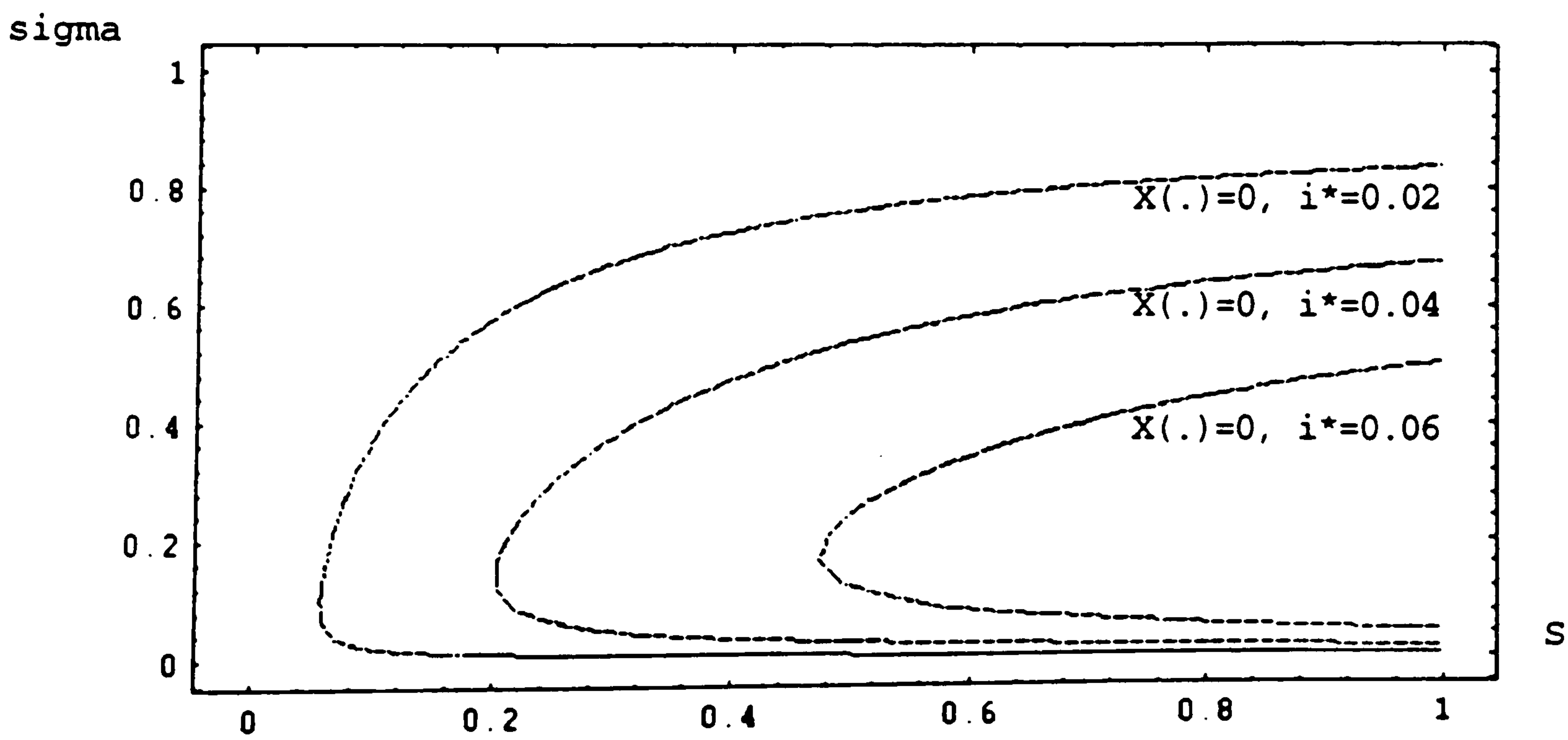

Figure 4. Contours of X(S,σ,i*) at an height equal to zero, for i*∈ {0.02, 0.04, 0.06}.

Therefore, for $i^*>\hat{\imath}$, $X(.)$ is positive and sign$(\partial A(\eta^*,\omega)/\partial\omega)$ is always equal to Sign$((S-1)(\sigma-S))$. Hence, as $\sigma<S<1$, $\partial\eta^*/\partial\omega>0$; however, for $i^*<\hat{\imath}$, the effect on the negative eigenvalue may be reversed. At the level of intuitive description we ascribe this behaviour to the effect highlighted above through equation (23).

Table 1 shows the critical values, $\hat{\imath}$, for parameter a varying from 0.1 to 0.9 by steps of 0.1 and for $\delta\in\{0.0, 0.1\}$.

Figure 3 and 4 provide some evidence about the speed of shrinking of the area where $X(.)$ is negative, when i^* is increased. In figure 3, we draw the contour plots of $X(.)$ at a height of zero for $i^*\in\{0.05, 0.10, 0.15\}$, for a=0.7 and in the case of 10% capital depreciation. Figure 4 draws the contours for $i^*\in\{0.02, 0.04, 0.06\}$, focusing on the area where $S,\sigma\in(0,1]$, which is possibly the most interesting from the perspective of stylised facts. Again, we set a=0.7 and δ= 10%

In conclusion, we have extended the Sidrauski-Fischer model, considering an instantaneous utility function that can encompass various degrees of substitution between consumption and real money balances.

In this version of the model, variations in the rate of money growth change the time profile of the nominal interest rate along the transition path to a steady state. The effects of such alterations are deeper that those highlighted in the existing literature, since they do not only shift expenditure over time, but they also affect its allocation between consumption and real money balances. It turned out that the areas in the parameters space providing an anti-Tobin effect have relevant dimensions. This result is potentially important in helping to reconcile Sidrauski-type models with most of the empirical literature. However, the nominal interest rate becomes a key variable to determine in which portions of the parameter space the anti-Tobin effect occurs. In fact, the econometric estimates of the elasticity of intertemporal substitution suggest a value below unity and $\sigma\in[0,1]$ seems consistent, at least at the level of stylised facts, with the existing empirical literature on the demand for money.

5. Endogenous growth, linear-in-capital technology and super-neutrality

It seems natural, at this stage, to investigate the role of the money growth rate in the light of some recent developments in the theory of growth. Since our specification for the utility function is rather complex, a simple formulation for the production side of the economy is desirable. Hence, we focus on linear-in-capital technologies, that, since Rebelo (1991), have been considered a useful simplification to incorporate into endogenous growth models the finiteness of agents' lives (e.g. Saint-Paul (1992) and van der Ploeg and Alogoskoufis, (1994)), politico-economic equilibria (e.g. Bertola, (1993)) and financial repression or debt monetisation problems (Roubini and Sala-i-Martin (1992a, b)).

We will briefly analyse a model of this type, showing how the concept of "marshallian externality" is crucial to get a linear-in-capital technology and then paying attention to the effects of money growth.

We assume that economic system is perfectly competitive and that every single firm has at its disposal a production function of the type:

$$Y_i = EK_i^{\psi} L_i^{1-\psi}$$

where Y_i is the i-th firm output and K_i and L_i are the productive factors used by it; ψ is a parameter, $\psi \in (0,1)$. It is briefly necessary to pay some attention to E. Each firm regards it as an exogenous parameter, while it is actually affected by the economic system average level of capital (K_m). Therefore, this quantity gives rise to a positive externality on firms.[16]

[16] It is the difference between the private marginal productivity and the social one which

It is usually assumed that $E = \varepsilon K_m^{\varphi}$, with $\varphi = 1-\psi$ in order to allow the economic system the possibility of steady growth.

If we introduce the hypothesis that every firm is equal and if we normalise properly the labour supply, which is assumed to be rigid, it is easy to obtain the aggregate production function:

$$Y = \varepsilon K \tag{25}$$

Notice that the capital rental rate,

$$r = \partial Y_i/\partial K_i = \psi E K_i^{\psi-1} L_i^{1-\psi} = \psi\varepsilon \tag{26}$$

is equal to the share ψ of the social marginal productivity of capital, due to the external effect.

The consumer's intertemporal problem is not substantially affected by the new formulation on the supply side; along the lines established in section 2 we can easily obtain the following equation of motion for aggregate per capita consumption:

$$\frac{\overset{o}{c}}{c} = \left(\frac{\sigma - S}{1+Ai^{\sigma-1}}\right)\frac{\overset{o}{i}}{i} + S(r-\theta) \tag{9'}$$

To close the model we get, differentiating equation (6):

$$\frac{\overset{o}{i}}{i} = -\frac{1}{\sigma}\left(\frac{\overset{o}{m}}{m} - \frac{\overset{o}{c}}{c}\right) = \frac{1}{\sigma}\left(\frac{\overset{o}{c}}{c} - \omega + \pi\right)$$

where π is the inflation rate at time t. Moreover, exploiting the definition of nominal interest, we get:

generates the inefficient growth result typical of this class of models.

$$\frac{\overset{o}{i}}{i} = \frac{\overset{o}{\pi}}{r+\pi}$$

and hence:

$$\frac{\overset{o}{\pi}}{r+\pi} = \frac{1}{\sigma}\left(\frac{\overset{o}{c}}{c} - \omega + \pi\right) \qquad (27)$$

The system composed of equation (9') and (27) can now be reformulated as follows:

$$\frac{\overset{o}{c}}{c} = \left(\frac{\sigma - S}{1+Ai^{\sigma-1}}\right)\frac{\overset{o}{\pi}}{r+\pi} + S(r-\theta) \qquad (28)$$

$$\overset{o}{\pi} = (r+\pi)\left(\frac{1+Ai^{\sigma-1}}{S+\sigma Ai^{\sigma-1}}\right)[S(r-\theta) + \pi - \omega] \qquad (29)$$

Equation (29) can be solved separately and it is unstable around equilibrium. This allows us to conclude that the inflation rate must jump directly to its equilibrium level in response to whatever modification in the parameters affecting it[17]. The level implied by (29) is $\pi = \omega - S(r-\theta)$. Moreover, using this result and substituting (26) into (28), we get:

$$\frac{\overset{o}{c}}{c} = g = S(\psi\varepsilon - \theta) \qquad (30)$$

This equation can be solved together with the dynamic equation for capital, obtained from (25), and expressed in per capita terms:

$$\frac{\overset{o}{k}}{k} = g = \varepsilon - \frac{c}{k} \qquad (31)$$

[17] In Appendix 1 we extend this result to a wider class of time separable utility functions .

Hence, we obtain the growth rate and the equilibrium ratio between consumption and capital. In this framework, like in Sidrauski's, money is superneutral: neither the consumption/capital ratio nor the growth rate are affected by the monetary policy variable. Moreover, it is easy to demonstrate that in system (30-31) there is no transitional dynamics. Therefore, this class of models lacks a source of non neutrality results deeply analysed in the traditional growth framework. Intuitively, this is due to the fact that the nominal interest rate immediately jumps to its steady state value, hence $\mathring{i}(t)/i(t)$, which played an important role in the analysis of sections 2-4, in this model is always naught (but for in the instant of the jump).

Danthine et al. (1987, pp. 488-9 in particular) confirm this strong superneutrality result. These authors, without realising that they were using a structure that will have become, in a few years, a basic one in endogenous growth models, showed that, with intratemporal Cobb-Douglas preferences and a linear technology, money is superneutral even in the presence of stochastic shocks.

6. A specification with a non-linear production function

The result obtained in the previous section suggests that, in order to study transitional dynamics effects in endogenous growth models, it is necessary to consider more general production or preference structures. Therefore, to verify the possibility of the Tobin effect (and of its reversal) we discuss a case where the production function is linear only in the long run. The alternative approach, based on non-homogeneous preferences, would have been feasible.[18]

It has not been possible to obtain an explicit solution for such a model. Hence,

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[18] Recently non homothetic preferences have been used e. g. by Azariadis and Drazen, (1990), and by Rebelo, (1992), to build models which exhibit no-growth traps.

to work out this problem, we use a numerical routine, based on specific functional forms for the utility and the production functions. Maintaining equation (1) as for preferences, we choose to describe the available technology as[19]:

$$\dot{k} = \varepsilon\,(k + \phi k^{\gamma}) - c$$

Hence, with ϕ positive and $\gamma \in (0,1)$, we consider an economic system where the marginal productivity of capital decreases as the accumulation process goes on, approaching the lower bound ε (see Jones and Manuelli, (1990), for the discussion of a similar production function). If such a value is sufficiently high, the growth process keeps on indefinitely. In our exercises, we choose an asymptotic growth rate equal to 2.5% and we adjust the long run marginal productivity of capital according to this value, i.e. $\varepsilon = \theta + 0.025/S$. Moreover, we assign γ=0.3 and we normalise ϕ to unity; as in section 4 we choose θ=0.03 and a=0.7.

Some details about the shooting procedure are supplied in Appendix 2. We based our simulation on the hypothesis that the effects due to the non linearity peter out almost completely during the first seven hundred periods.

The first simulations have been carried out for the case of a high degree of substitution between consumption and real money balances (σ=2). We examined two situations: one where the intertemporal elasticity of substitution is below the logarithmic benchmark (S=0.66) and one where it is above it (S=1.5); we considered three possibilities for the nominal money growth rate: the "friedmanite" rate plus 0.0001, ten and twenty per cent. This corresponds to long run inflation rates of -4.2778%, 5.7221%, 15.7221% (rounded to the fourth decimal) in the case of low intertemporal substitution and of -2.1566%, 7.8433%, 17.8433% when

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19 At the micro level, a production function compatible with the aggregate one used in the main text is: $Y_i = \varepsilon(K_i^{\gamma} L_i^{1-\gamma} K_m^{1-\gamma} + \phi K_i^{\gamma} L_i^{1-\gamma})$.

S=1.5.

Table 2: Percentage variations in the capital level when $\sigma = 2$						
	S=0.66			S=1.5		
	A	B	C	D	E	F
10	3.8799	4.0430	0.1570	-0.3552	-0.3910	-0.0360
20	6.9069	7.1971	0.2715	-0.6292	-0.6918	-0.0630
50	13.1915	13.7468	0.4906	-1.1847	-1.2998	-0.1164
100	18.6826	19.4719	0.6651	-1.6469	-1.8038	-0.1596
200	22.0951	23.0310	0.7665	-1.9204	-2.1015	-0.1847
500	22.8251	23.7924	0.7875	-1.9775	-2.1637	-0.1899
700	22.8288	23.7963	0.7876	-1.9778	-2.1640	-0.1900

Our results concerning the impact of the money growth rate on capital accumulation are summarised in Table 2. Column A shows the percentage increase in the capital level when the nominal money growth rate is increased from the lowest rate to the intermediate one; column B considers an increase in ω from -4.2778% to 15.7221%, while column C provides the effect of a change of the money growth rate from 5.7221% to 15.7221%.

When S=0.66 the Tobin effect is significant: we read in column A a relevant increase in the capital level. A further 10% increase in ω entails a much weaker additional effect.

We checked that the nominal interest rates "overadjusts" with respect to the increase in the money growth rate, as in the basic Fischer-Cohen model analysed in section 3. Hence, the intuitive explanation for such a behaviour is the same we developed above: expenditure is tilted forward and the increase in the nominal interest rate reduces both real money balances and consumption.

It is important to remark that the Tobin effect exhibited by this exercise concerns levels and it does not involves growth rates, as in van der Ploeg and Alogoskoufis (1994, p. 782).

If the intertemporal elasticity of substitution is 1.5, our results are summarised in columns D, E, F[20]. We notice an Anti-Tobin effect, as suggested by our previous analysis. In this case expenditure is tilted forward, but the substitution effect is strong enough to imply an increase in consumption.

As for the real money balances hold by the representative agent, we found that they are sharply reduced by increments in the money growth rate. This seems to imply that the optimal money growth rate cannot be significantly different from the one that allows the representative consumer's satiation.

The second set of simulations has been carried out for the same parameters values, but for the elasticity of substitution between consumption and real money balances, which has been set equal to 0.5. Again, we considered two situations: one where the intertemporal elasticity of substitution is below the logarithmic benchmark (S=0.66) and one where it is above it (S=1.5).

Table 3 shows that, when S=0.66, we find a Tobin effect of very modest moment when ω is increased from -4.2778% to 5.7221%; a further 10% increase in the money growth rate entails an (even weaker) Anti-Tobin effect. For S=1.5, the opposite happens: we notice, first, a reduction in the capital level that is almost completely neutralised when ω increases from 7.8433% to 17.8433%. These

[20] Columns D, E, F correnpond to A, B, C, respectively: column D shows the percentage increase in the capital level when the nominal money growth rate is increased from -2.1566% to 7.8433%; column E considers an increase in ω from -2.1566% to 17.8433%, and column F provides the additional effect of a change from 7.8433% to 17.8433%.

effects are consistent with our analysis of section 4.

Table 3: Percentage variations in the capital level when $\sigma = 0.5$						
	S=0.66			S=1.5		
	A	B	C	D	E	F
10	0.0052	0.0024	-0.0028	-0.0245	-0.0128	0.0117
20	0.0089	0.0040	-0.0049	-0.0421	-0.0217	0.0204
50	0.0157	0.0069	-0.0088	-0.0751	-0.0374	0.0377
100	0.0209	0.0091	-0.0119	-0.1001	-0.0485	0.0517
200	0.0240	0.0103	-0.0137	-0.1141	-0.0544	0.0598
500	0.0246	0.0106	-0.0140	-0.1169	-0.0555	0.0615
700	0.0246	0.0106	-0.0140	-0.1169	-0.0555	0.0615

Therefore, by means of these exercises, we obtain two results. First, we show that, even with infinitely lived representative agents, money is not superneutral in an endogenous growth model where the technology is not linear-in-capital. It is worth to notice that the stock of capital, and therefore production, is permanently altered by variations in the money growth rate. Hence, a temporary change in this policy variable forever affects the production possibilities of the economic system. Therefore, we provide an example of the hysteresis effect that characterises endogenous growth models. Second, we confirm that the sign of the relation between capital accumulation and money growth depends not only on the form of the utility function, but also on the specific values attributed to its parameters and on the level of the money growth rate.

7. Concluding comments

We have generalised the treatment of preferences over consumption and real money balances in the traditional Sidrauski-Fischer-Cohen growth model. Framing it in terms of capital, consumption and interest rates, we have detected non-zero measure sets in the parameters space where the anti-Tobin prevails.

The reduction in the accumulation of capital takes place in two cases.

When the intratemporal elasticity of substitution is higher than the intertemporal elasticity of substitution, and the latter exceeds unity, expenditure is tilted forward, as in the standard Sidrauski-Fischer-Cohen model, but the substitution effect between real money balances and consumption is strong enough to reduce capital accumulation.

When $\sigma<1$ the analysis is more complex and numerical techniques are required. If we consider the case, symmetric to the previous one, when $\sigma<S<1$, we notice that the anti-Tobin effect is present only if the long run nominal interest rate is sufficiently high. However, for low interest rates, the relation between capital accumulation and money growth rate is negative also for other parameters sets, that, again, must be computed numerically. At the level of intuitive description we ascribe this behaviour to the effect exerted by the long-run interest rate on the consumption profile.

We have considered also endogenous growth models. Since the combination of a linear-in-capital technology with homothetic preferences implies the absence of a transition path, we studied numerically a case where the production function is asymptotically linear. Our experiments corroborate the previous results, with an important qualification consisting in the fact that temporary variations in the money growth rate have permanent effects on the income level, in coherence with the hysteresis effects which characterise this class of models.

Hence, our attempt to reconcile the practice of inserting money into the utility function with the negative relation between output and money growth that is suggested by many recent empirical contributions has yielded some fruit.

Appendix 1: Extension of the non-transition result

In section 5 we showed that the endogenous growth model with linear in capital technology and preferences given by (1) lacks transitional dynamics. In this appendix we aim to extend this result to the following more general one:

Proposition: if an economic system displays *i*) linear in capital aggregate production function; *ii*) time separable preferences; *iii*) instantaneous utility homogeneous of degree η with positive but decreasing first derivatives and *iv*) strict concavity for the instantaneous utility, then it lacks transitional dynamics and its growth rate is always equal to $(r-\theta)/(1-\eta)$.

Proof. To prove our proposition, we first show which are the restriction on the degree of homogeneity implied by the hypotheses on preferences and then we maximise the representative consumer's lifetime utility in the face of these qualifications.

Define $f(\tau)=c(\tau)/m(\tau)$ and transform the instantaneous utility as follows: $\mathcal{U}(c,m) = U(f,1)m^{\eta} = U(f)m^{\eta}$ (where we omit the time indexes). Therefore we get $\mathcal{U}_c(c,m)=U_f(f)m^{\eta-1}$ and $\mathcal{U}_m(c,m) = [\eta U(f) - U_f(f)f]m^{\eta-1}$. Hence, the hypothesis according to which the marginal utilities of consumption and real money balances are positive, implies $\eta\neq 0$. Moreover, if $\eta\in(-\infty,0)$, the assumption on the first derivatives requires that $U(f)<0$, and, if $\eta\in(0,\infty)$, that $U(f)>0$. Finally, notice that $\mathcal{U}_{cc}(c,m)=U_{ff}(f)m^{\eta-2}$ and hence that $U_{ff}(f)<0$.

The strict concavity hypothesis implies that the Hessian matrix,

$$\begin{pmatrix} U_{ff}(f) & (\eta-1)U_f(f)-U_{ff}(f)f \\ (\eta-1)U_f(f)-U_{ff}(f)f & \eta(\eta-1)U(f)-2(\eta-1)U_f(f)f+U_{ff}(f)f^2 \end{pmatrix} m^{\eta-2} \tag{A1}$$

has positive determinant and negative trace. Some algebraical derivations show that:

$$det[A1]=m^{2\eta-4}[(\eta-1)[\eta(\eta-1)U(f)U_{ff}(f) - (\eta-1)U_f(f)^2]]> 0$$

only if η<1 and if[21]:

$$(1-\eta)\frac{U_f(f)}{U_{ff}(f)} + \eta\frac{U(f)}{U_f(f)} > 0 \tag{A2}$$

The negativity constraint on the trace of (A1) does not imply any further restriction, as:

$$tr[A1]= m^{\eta-2}[U_{ff}(f) + \eta(\eta-1)U(f)-2(\eta-1)U_f(f)f+U_{ff}(f)f^2] <$$

$$< m^{\eta-2}[U_{ff}(f) +f[(1-\eta)U_f(f)+U_{ff}(f)f]] <$$

$$< m^{\eta-2}[U_{ff}(f) +f[(1-\eta)U_f(f)+U_{ff}(f)\eta U(f)/U_f(f)]] < 0$$

Therefore, the representative consumer's problem can be set out in the following way:

$$max_{\{f(\tau),m(\tau)\}} \int_t^\infty U(f(\tau))m(\tau)^\eta e^{-\theta(\tau-t)}$$

subject to the intertemporal budget constraint:

---

21 Notice that the conditions obtained in the text are necessary and sufficient to grant the concavity for the hessian matrix of the hamiltonian associated with the representative consumer's intertemporal optimization problem. This matrix must be negative semi-definite to avoid violation of the sufficient conditions for optimality. (Beavis and Dobbs, (1990, pp. 334-335 e 348-349)).

$$\overset{o}{a}(t) = ra(t)+w(t)-[f(t)+i(t)]m(t)+x(t)$$

Omitting the time indexes, we express the first order conditions as follows:

$$U_f(f)m^{\eta} = \lambda m \tag{A3}$$

$$\eta U(f)m^{\eta-1} = \lambda(f+i) \tag{A4}$$

$$\overset{o}{\lambda} = (\theta-r)\lambda \tag{A5}$$

The transversality condition is not affected by our formulation of the problem.

Differentiate the ratio between (A4) and (A3),

$$\frac{\eta U(f)}{U_f(f)} = f+i$$

to get a dynamic equation,

$$\eta U_f(f)\,\overset{o}{f} = U_f(f)(\overset{o}{f} + \overset{o}{i}) + U_{ff}(f)\,\overset{o}{f}\,(f+i)$$

which can be made explicit for $\overset{o}{i}$:

$$\overset{o}{i} = \left(-\frac{\eta U(f) U_{ff}(f)}{U_f(f)^2} + (\eta-1)\right)\overset{o}{f} \tag{A6}$$

Consider now the total differential of (A3) and equate it to (A5) to obtain:

$$\frac{U_{ff}(f)}{U_f(f)}\,\overset{o}{f} + (\eta-1)\,\frac{\overset{o}{m}}{m} = \frac{\overset{o}{\lambda}}{\lambda} = \theta - r \tag{A7}$$

Recall the identity $\frac{\overset{o}{m}}{m} = \omega - \pi$ and substitute it into (A7) to get, by use of (A6):

$$\overset{o}{i} = \overset{o}{\pi} = \left((1-\eta)\frac{U_f(f)}{U_{ff}(f)} + \eta\,\frac{U(f)}{U_f(f)} \right)(r-\theta+(1-\eta)(\pi-\omega)) \tag{A8}$$

which corresponds to equation (29) in section 5. It is immediate to remark that, if (A2) is verified, (A8) is unstable around equilibrium. Therefore, the concavity condition grants the absence of transitional dynamics, as the unique acceptable value for inflation is:

$$\pi = \omega - \frac{r-\theta}{1-\eta}$$

Hence, we immediately get, by means of (A7), that also the consumption/money ratio does not vary over time; the system and the real money balances growth rates are equal and given by: $(r-\theta)/(1-\eta)$. •

This proof can obviously be extended to the class of homothetic instantaneous utility functions, which are monotonic transformation of $\mathcal{U}(c(\tau),m(\tau))$.

Hence, we showed that the strong superneutrality result is section 5 does not depend on the specification of preferences over consumption and real money balances but only on the homotheticity of the instantaneous utility function.

Appendix 2: Numerical routines

To generate figure 2 we built the following *Mathematica* routine:

```
Clear[a,b,d,gm,i,F,s,g,f,h,x]
a=0.7;b=0.03;i=.001;d=0;gm=0.3;
F=(b+d-d*gm)*(b+d)*(gm-1)/gm
f[s_,g_] := x/.FindRoot[(x^2-b*x+s*F)*((i+(a/(1-a))^g*i^g)/(s+g*(a/(1-
a))^g*i^(g-1))-
 x)-(x*F*(s-1)*(g-s))/(s+g*(a/(1-a))^g*i^(g-1))==0,{x,-1,-100,0},
MaxIterations->40, WorkingPrecision->20]
h[s_,g_] :=1+g*(a/(1-a))^g*i^(g-1)*(1-((g-1)*f[s,g])/i)
Plot[{h[.5,g],h[2,g]},{g,0,2},AxesLabel->{"sigma","X(.)"},
 PlotStyle->{Dashing[{.04,.04}],{}}]
```

To generate figure 3 and 4 we built *Mathematica* routines similar to the following one:

```
Clear[a,b,d,i,gm,F,s,g,f,h,x]
a=0.7;b=0.03;d=0;i=.02;gm=0.3;
F=(b+d-d*gm)*(b+d)*(gm-1)/gm
f[s_,g_] := x/.FindRoot[(x^2-b*x+s*F)*((i+(a/(1-a))^g*i^g)/(s+g*
(a/(1-a))^g*i^(g-1))- x)-(x*F*(s-1)*(g-s))/(s+g*(a/(1-a))^g*i^(g-1))
==0,{x,-1,-100,0}, MaxIterations->40,
 WorkingPrecision->20]
h[s_,g_] :=1+g*(a/(1-a))^g*i^(g-1)*(1-((g-1)*f[s,g])/i)
i=.05;
ContourPlot[h[s,g],{s,0,100},{g,0,1},AspectRatio->.5,Contours->{0},
 ContourShading->False, ContourSmoothing->True]
i=.10;
ContourPlot[h[s,g],{s,0,100},{g,0,1},AspectRatio->.5,Contours->{0},
 ContourShading->False, ContourSmoothing->True]
i=.15;
ContourPlot[h[s,g],{s,0,100},{g,0,1},AspectRatio->.5,Contours->{0},
 ContourShading->False, ContourSmoothing->True]
Show[%,%%%,%%%%%]
```

where, for convenience, b=β, d=δ, i=i*, gm=γ, s=S, g=σ.

Results in tables 2 and 3 have been obtained using routines of the following type:

```
Clear[a,g,S,te,r,w,A]
Clear[c,i,k]
a=.7;g=2;S=0.66;te=.03;
gr =.025; rr = te+gr/S
b=rr;d=.3;
wmin =-rr(1-S)-S*te
w=wmin+.0001;
A=(a/(1-a))^g;
sol=NDSolve[{c'[t]==c[t]/(S+g*A*i[t]^(g-1))((Sg+g*S(1+A*i[t]^(g1)))
(rr+b*d*k[t]^(d-1)-te)+(S-g)(w+te-i[t])),
i'[t]= =i[t]*(1+A*i[t]^(g-1))/(S+g*A*i[t]^(g-1))((S-1)(rr+b*d*k[t]^(d-1)-te)-
(w+te-i[t])),
k'[t]= = rr*(k[t]+b/rr*k[t]^(d-1))-c[t], k[500]= =1000000,
c[500]= =(rr(1-S)+S*te)*k[500], i[500]= =w+rr(1-S)+S*te},
{c,i,k}, {t,0,700}, AccuracyGoal->10,PrecisionGoal->10, WorkingPrecision->20]
Plot[Evaluate[k[t]/.sol],{t,0,700}]
Plot[Evaluate[c[t]/.sol],{t,0,700}]
Plot[Evaluate[i[t]/.sol],{t,0,700}]
k[0]/.sol
c[0]/.sol
i[0]/.sol
k[700]/.sol
c[700]/.sol
i[700]/.sol
```

where, for convenience, $g=\sigma$; $te=\theta$; gr = growth rate; $rr=\varepsilon, b=\phi\varepsilon, d=\gamma$.

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Chapter IV

On the Optimality of Risk-Sharing in a Stochastic Endogenous Growth Model

1. Introduction

During the last few years, several contributions suggested that the precautionary motive increases savings: Caballero (1990), Weil (1993) and Van der Ploeg (1993) provide various examples based on alternative specifications for the utility functions. In endogenous growth models, built in the spirit of Romer (1986) or Rebelo (1991), precautionary savings have far reaching consequences. Since we observe a "change in the magnitude of the effects", an increase in savings affects not only the level of income but also its growth rate, which may consequently be lowered by a reduction in uncertainty. Therefore, if growth is suboptimal, being based on positive spillover effects, the political stance favourable to financial liberalisation, which, during the post war period, has led to a remarkable lowering of institutional and legal barriers to international capital movements, might prove questionable.

However, insofar we deal with risk averse individuals, a reduction of the growth rate caused by a decrease in uncertainty does not necessarily imply harmful consequences. Thus, a deep analysis of the effects of uncertainty on growth and welfare is necessary in appreciating the role of financial openness.

Devereux and Smith (1994), in an infinite horizon representative agent framework, establish a situation where the lowering of the growth rate due to a reduction of aggregate uncertainty is welfare lessening. However, their findings, which are discussed in the second section, depend on the particular stochastic structure they introduce.

This chapter widens Devereux and Smith's result in a framework where technological uncertainty is modelled by means of geometric brownian motions (as in Obstfeld, (1994)) and the presence of non traded labour income is encompassed by means of contingent claim analysis (section 3). We will consider a single country model; however it is possible, and in principle easy, to show that the international risk pooling, if different countries share the same technology, is equivalent to a reduction in technological uncertainty[1,2]. The effects of an additional distributive shock are considered in section 4, where it is shown that a positive correlation of the capital income share with the technological disturbance reduces the parameters set where the perverse effect occurs. Hence, a distributive

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[1] A recent model that can be very useful to check this statement is Ghosh and Pesenti (1994, section 2 in particular).

[2] If countries face different technological opportunities, international financial integration, channeling savings to projects with high risk and high return, may well promote growth. For references concerning recent developments in this stream of literature, see Devereux and Smith (1994, p. 548).

shock proves important in re-establishing the traditional welfare-augmenting effect of international risk-sharing.

2. A second-best result

Devereux and Smith (1994) consider a discrete time endogenous growth model with C.R.R.A. preferences, where the linearity of the technology in the accumulable factor (capital) is ascribed to a positive marshallian externality, as in Romer (1986).

In a first specification of the model, production is deterministic but each country enjoys a stochastic endowment, i. e. $Y_t=\theta K_t^{\alpha}(M_tL_t)^{1-\alpha} + \varepsilon_t$, where Y_t is output, K_t capital, M_t human capital and L_t labour. Devereux and Smith assume that the stock of human capital, due to the effect of the positive externality, is always equal to the stock of physical capital. Moreover, they notice that the distribution of the random disturbance must grow in proportion to output, otherwise the effect of the shock would become negligible over time. Accordingly, proportionality of the shock to the economy wide capital stock is assumed, i.e. $\varepsilon_t = \gamma_t K_t$, γ_t i.i.d. Notice that the representative consumer does not realise the relation between the distribution of the random shock and total capital.

Secondly, Devereux and Smith consider a model with a production function affected by a multiplicative (Hicks neutral) technology shock: $Y_t=\theta_t K_t^{\alpha}(M_tL_t)^{1-\alpha}$, θ_t i.i.d. In this case we are in presence of a random return on capital.

In the stochastic endowment case, Devereux and Smith show that the growth rate is lessened by a reduction in aggregate uncertainty (by a transition from financial market autarchy to international integration) and that welfare may also be harmed[3]. This result is obtained because we are in presence of three distortions:

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[3] The parameter set for this to be true must be computed numerically. For an example, see

the positive externality implies a suboptimal growth rate, uncertainty is not diversified away as much as possible and the representative agent misperceives the relation between the random disturbance and the accumulable factor. Hence, the possible negative impact on welfare of financial integration is a typical second best result.

However, this outcome is significantly affected by the presence of multiplicative technological shocks. In this case, Devereux and Smith find that an increase in the variance of the shock always damages welfare and that it enhances the growth rate only if the risk aversion index is higher than one (1994, pp. 545-46). The economic intuition for this result is straightforward: with multiplicative shocks the private return on capital is stochastic and it covaries positively with future consumption. Therefore, growth is riskier, in the sense that an increase in the standard deviation of the disturbance enhances the variance of consumption. Hence, Devereux and Smith's results are determined by the fact that, in the case of technological shock, the private return on capital reflects part of the shock so that the externality is less marked than in the case of endowment risk. Therefore, their conclusions are deeply related to the hypothesis concerning the misperception of the link between the random shock and capital.

3. An infinite-horizon representative agent model

3.1 The basic set-up

We now consider a closed economy populated by identical, infinitely lived, individuals who maximise the intertemporal objective:

Devereux and Smith, (1994, pp. 540-44).

$$U_t = E_t\left(\int_t^{\infty} \frac{C_\tau^{1-R}}{1-R} e^{-\rho(\tau-t)} d\tau \right)$$

where E_t is the mathematical expectation conditional on time t information, C_t is time t consumption, ρ is the instantaneous rate of time preference and $R \in [0,\infty)$ represents both the risk aversion index and the reciprocal of the elasticity of intertemporal substitution[4].

The representative firm produces a (gross) output flow dY_i by means of the stochastic Cobb-Douglas production function:

$$dY_i = \beta K_i^{\gamma} L_i^{1-\gamma} \bar{K}^{1-\gamma} (dt + \sigma_z dz) \quad (1)$$

where, respectively, K_i and L_i are the stock of physical capital and the labour managed by the i-th firm; $\bar{K}$ is the average stock of capital, β the inverse of the capital/output ratio and dz a standard Wiener process. Notice that, by assumption, $\gamma \in (0,1)$.

Competition forces firms to have the same capital/labour ratio; hence it is immediate to get the aggregate production function, where the inelastically supplied labour is normalised to one:

$$dY = \beta K (dt + \sigma_z dz)$$

Hence, dz actually is an aggregate shock, affecting all firms, rather than an idiosyncratic one. Despite the simplicity of this formulation, notice that the

[4] We built also a version of the model using the theoretical framework introduced by Epstein and Zin (1989) to distinguish risk aversion from the elasticity of intertemporal substitution. However, our result are not significantly affected by this distinction (see below).

Romer-type production function (1), explicitly providing a productive role for labour, implies a divergence between the gross capital income share determined by the market, and the actual marginal productivity of capital. Therefore the growth rate, from the point of view of a representative agent, is inefficiently low.

In comparison with the standard linear-in-capital technology, the production function (1) has far-reaching consequences, since it implies the existence of an asset, the claim on future stochastic labour income (human wealth), which is assumed to be non traded, due to an obvious moral-hazard problem. The set of available assets, inclusive of the non-traded one, is composed by non-risky bonds (*B*), physical capital (*K*) and human wealth (*H*); the stochastic processes for these assets are:

$$dB = r\,Bdt \tag{2}$$

$$dK = (\gamma\beta - \delta)Kdt + \gamma\beta K\sigma_z dz$$

$$dH = \mu_H Hdt + H\sigma_H dz_H - dS \tag{3}$$

where r and is the safe instantaneous interest rate and δ is the capital depreciation rate.

Notice that not only the implicit discount factor on human wealth, but also the stochastic process dz_H and the standard deviation σ_H are unknowns and must be endogenously determined. On the contrary, the characteristics of the wage process $dS = \mu_S dt + \sigma_S dz_S$ are entirely known; in fact, from (1), $\mu_S = (1-\gamma)\beta K$; $\sigma_S dz_S = (1-\gamma)\beta K\sigma_z dz$ and $\sigma_{SZ} = (1-\gamma)\beta K\sigma_z^2$. (where, in general, $\sigma_{ij}=\sigma_i\sigma_j\rho_{ij}$ and ρ_{ij} is the correlation coefficient between variables i and j). Using the finance jargon, we say that the wage process is spanned by the technological disturbance.

Defining financial (tradable) wealth as the sum of bonds and capital

($F=B+K$), we now assume that the state variables F and H are the unique arguments of the indirect utility function $J(.)$; consequently we set up the optimal portfolio-consumption choice using the following Bellman equation:

$$0 = max_{C,K} \left\{ \frac{C^{1-R}}{1-R} + J_t(F,H) + J_F(F,H)\ [rF + (\gamma\beta - \delta - r)K + \mu_s - C] + \right.$$

$$+J_H(F, H)[\mu_H H - \mu_s] + \frac{J_{FF}(F,H)}{2}[\gamma^2\beta^2\sigma_z^2 K^2 + \sigma_s^2 + 2\gamma\beta\sigma_{sz}K] +$$

$$+ J_{HF}(F,H)\ [\gamma\beta\sigma_{Hz}KH + \sigma_{Hs}H - \gamma\beta\sigma_{sz}K - \sigma_s^2] +$$

$$\left. + \frac{J_{HH}(F,H)}{2}(\sigma_H^2 H^2 + \sigma_s^2 - 2\sigma_{Hs}H) \right\}$$

The associated first order conditions, characterising optimal consumption and portfolio choice, are:

$$C^{-R} = J_F$$

$$(\gamma\beta - \delta - r)\ J_F + J_{FF}(\gamma^2\beta^2\sigma_z^2 K + \gamma\beta\sigma_{sz}) + J_{HF}\ (\gamma\beta\sigma_{Hz}H - \gamma\beta\sigma_{zs}) = 0$$

(where we let as understood the arguments of the indirect utility function)

Hence

$$C^* = J_F^{-1/R} \tag{4a}$$

$$K^* = -\frac{J_F}{J_{FF}}\frac{(\gamma\beta - \delta - r)}{\gamma^2\beta^2\sigma_z^2} - \frac{J_{FH}}{J_{FF}}\frac{\sigma_{Hz}H - \sigma_{sz}}{\gamma\beta\sigma_z^2} - \frac{\sigma_{sz}}{\gamma\beta\sigma_z^2} \tag{4b}$$

The second addendum in (4b) explicits the "state-variable" hedge component of the demand for capital, while the third one represents the "income-hedge" effect.

3.2 Human wealth evaluation

Before guessing a solution for the Bellman equation, it is convenient to "price" human wealth. This evaluation does not entail particular problems, thanks to our assumption according to which the wage process is spanned. We choose to use the contingent claim analysis, since it allows for some intuitive insight concerning the value of the non-traded asset. (For recent examples of this approach, see Merton, (1992), and Bodie, Merton and Samuelson, (1992), sect. 4 in particular)[5].

To price human wealth we notice that the unique productive factor varying over time is capital. Hence, we guess that the value of the non-traded asset depends only on time and on capital itself, i. e.

$$H = H(K,t) \tag{5}$$

Using Ito's lemma we get the total differential for H:

$$dH = H_t dt + H_K dK + \frac{1}{2} H_{KK} (dK)^2$$

However, using (3) we may write

$$\mu_H H - \mu_s = H_K[(\beta - \delta)K - C] + \frac{1}{2} H_{KK} \beta^2 \sigma_z^2 K^2 + H_t \tag{6a}$$

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[5] One could also consider human wealth as if it were freely traded, as suggested by Svensson and Werner, (1993). In this way, one can determine the shadow return at which the claim would be willingly held. In their framework, the shadow return is defined as the return at which an individual would purchase the claim if it would be available on the market, and it might differ through heterogeneous agents.

and

$$H\sigma_H dz_H = \sigma_S dz + H_K\,\beta K\sigma_z dz \tag{6b}$$

An important consequence of this relation is that H, as suggested by intuition, is spanned by the technological shock, i.e. $dz_H = dz$; the standard deviation for human wealth is given by $\sigma_H H = \sigma_S + H_K\,\beta K\sigma_z$.

We now build the "hedge portfolio", i.e. we determine the quantity of capital K^h which minimises the variance of the return of the sum of the non-traded asset and itself.

$$Min_{K^h}\,[Var\,(dH + dK^h)] = min_{K^h}\,[Var\,((\sigma_H H + \gamma\beta\sigma_z K^h)dz)] =$$

$$= min_{K^h}\,[\sigma_H^2 H^2 + \gamma^2\beta^2\sigma_z^2 K^{h2} + 2\gamma\beta\sigma_{Hz}HK^h]$$

Solving for K^h we get:

$$K^h = -\frac{\sigma_{Hz}}{\gamma\beta\sigma_z^2}H \tag{7}$$

With a positive covariance between K and H the optimal hedge is negative, since protection against risk on a non-traded claim on income implies a short position in correlated assets.

It is now important to remark that the spanning of H by the aggregate shock implies that the variance of the return of the portfolio composed of H itself and K^h is zero. Substituting (7) into Var($dH + dK^h$) and using the expressions for $\sigma_H H$ and σ_S obtained above, we get:

$$Var\,(dH + dK^h) = (\sigma_H^2 - \sigma_{Hz}^2/\sigma_z^2)H^2 = 0$$

This result allow us to price human wealth through a simple no-arbitrage assumption. In fact, since the hedge portfolio replicates the characteristics of a tradable one, the return on which has zero variance, its value must be equal to what can be obtained investing the same amount of wealth at the riskless rate, i.e.:

$$H\mu_H + (\gamma\beta - \delta)K^h = r\,(H + K^h) \tag{8}$$

Substituting K^h into the last relation and recalling $\mu_H H$ from (6a) we get:

$$(\gamma\beta - \delta - r)\frac{\sigma_{HZ}}{\gamma\beta\sigma_z^2}H + rH = H_t + H_K\,[(\beta - \delta)K - C] + \frac{1}{2}H_{KK}\,\beta^2\sigma_z^2\,H^2 + \mu_s \tag{9}$$

Finally, specializing (5) into a linear relation between human wealth and capital, i.e. $H = bK$, we get, from (6b):

$$\sigma_{Hz}H = \sigma_{sz} + b\beta\sigma_z^2K$$

Using this last result and the hypothesis of linearity for relation (5), we obtain, from (9):

$$\frac{\gamma\beta - \delta - r}{\gamma\beta}[(1-\gamma)\,\beta + b\beta]\,K + rH = (1-\gamma)\,\beta K + b\left(\beta - \delta - \frac{C}{K}\right)K$$

This equation may be solved to get a (partial) solution for H:

$$H = \frac{(1-\gamma)\,(r+\delta)}{\gamma\beta - \delta - r(1-\gamma) - \gamma g}K \tag{10}$$

where use has been made of the relation $b = H/K$; g is the expected growth rate, i.e. $g = (\beta - \delta - C/K) = E(dK)/K$. Notice that equation (10) corroborates our guess about the linearity of the relation between H and K. However, up to now, H depends also on endogenous variables: the riskless rate and the expected growth rate, whose determination requires the computation of the maximum value function.

3.3 Solution of the model

At this stage, it seems natural to proceed using a tentative solution of the form:

$$J(F,H) = D\frac{(H+F)^{1-R}}{1-R}$$

To verify the correctness of our guess, it is necessary to compute D, substituting the first order conditions (4a) and (4b) into the Bellman equation.

$$0 = D^{(R-1)/R}\frac{(H+F)^{1-R}}{1-R} - \rho D\frac{(H+F)^{1-R}}{1-R} + D(H+F)^{-R}x$$

$$x\left\{rF+(\gamma\beta-\delta-r)\left[\frac{\gamma\beta-\delta-r}{\gamma^2\beta^2\sigma_z^2}\left(\frac{H+F}{R}\right)-\frac{\sigma_{Hz}}{\gamma\beta\sigma_z^2}H\right]+\mu_H H-D^{-1/R}(H+F)\right\}$$

$$-RD\frac{(H+F)^{-(1+R)}}{2}\left\{\gamma^2\beta^2\sigma_z^2\left[\left(\frac{\gamma\beta-\delta-r}{\gamma^2\beta^2\sigma_z^2}\right)^2\left(\frac{H+F}{R}\right)^2+\frac{\sigma_{Hz}^2}{\gamma^2\beta^2\sigma_z^4}H^2+\right.\right.$$

$$\left.\left.-\frac{2(\gamma\beta-\delta-r)\sigma_{Hz}}{\gamma^3\beta^3\sigma_z^4}H\left(\frac{H+F}{R}\right)\right]+2\gamma\beta\left[\frac{\gamma\beta-\delta-r}{\gamma^2\beta^2\sigma_z^2}\left(\frac{H+F}{R}\right)-\frac{\sigma_{Hz}}{\gamma\beta\sigma_z^2}H\right]H\sigma_{Hz}+\sigma_H^2H^2\right\}$$

We notice, from equations (7-8), that:

$$-rH-(\gamma\beta-\delta-r)\frac{\sigma_{Hz}}{\gamma\beta\sigma_z^2}H+\mu_H H = 0$$

Therefore, the first order terms in curly brackets in the Bellman equation become: $\left[r+\frac{(\gamma\beta-\delta-r)^2}{R\gamma^2\beta^2\sigma_z^2}-D^{-1/R}\right](H+F)$, while the second order ones, thanks to the spannng assumption, reduces to $\frac{(\gamma\beta-\delta-r)^2(H+F)^2}{R^2\gamma^2\beta^2\sigma_z^2}$.

Hence:

$$D^{-1/R} = \frac{\rho}{R} + \frac{R-1}{R}\left(r + \frac{(\gamma\beta-\delta-r)^2}{2R\gamma^2\beta^2\sigma_z^2}\right) \tag{11}$$

Therefore our guess is actually correct and consumption is a linear function of the sum of financial and non traded wealth; $D^{-1/R}$ is the marginal and average propensity to consume out of total wealth. K^*, determined by equation (4b), becomes, by means of equation (6b):

$$K^* = \frac{\gamma\beta-\delta-r}{R\gamma^2\beta^2\sigma_z^2}(H+F) - \frac{\sigma_{Hz}}{\gamma\beta\sigma_z^2}H = \frac{\gamma\beta-\delta-r}{R\gamma^2\beta^2\sigma_z^2}(H+F) - \frac{1-\gamma}{\gamma}K^* - \frac{1}{\gamma}H \tag{12}$$

In a representative agent closed economy model, it seems natural to impose that, in equilibrium, the net aggregate quantity of riskless bonds must be zero. Hence $K^* = F$ and we may simplify equation (12), obtaining the "forcing" riskless rate:

$$r = \gamma\beta - \delta - R\gamma\beta^2\sigma_z^2 \tag{13}$$

Using this relation into the definition for $D^{-1/R}$ we get:

$$D^{-1/R} = \frac{\rho}{R} + \frac{(R-1)}{R}[\gamma\beta - \delta - R\,\beta^2\sigma_z^2\,(\gamma - 0.5)] \tag{14}$$

Equation (11) implies that, for a given riskless rate, the precautionary motive is strong enough to cause a reduction in the propensity to consume out of wealth if the risk aversion index is higher than unity (from equation (11), $\partial D^{-1/R}/\partial\sigma_z<0$ if $R>1$)[6]; however, in our general equilibrium framework, an increase in σ_z also induces a "portfolio balance" effect: the representative agent is willing to purchase

[6] This is a standard result: in an early paper Merton neatly interpret this outcome in terms of income and (Hicksian) substitution effect. (See Merton, (1990, ch. 4)).

more of the risk-free asset, whose quantity is held fixed. Hence, the forcing rate diminishes, (equation (13)), so that, being γ lower than 0.5, the propensity to consume, for $R>1$, is actually raised[7].

The determination of the expected growth rate requires more calculation. Starting from the definition for $g = \beta - \delta - D^{-1/R}\left(\frac{K+H}{K}\right)$ and using (10) and (13), we get:

$$(\beta\text{-}\delta\text{-}\ g) = D^{-1/R}\left(1+\frac{(1-\gamma)(\beta-R\beta^2\sigma_z^2)}{\gamma\beta-\delta+R(1-\gamma)\beta^2\sigma_z^2-g}\right)$$

Hence, using (14) and rearranging:

$$\gamma\beta-\delta+(1-\gamma)R\beta^2\sigma_z^2-g = \frac{\rho}{R}+\frac{(R-1)}{R}[\gamma\beta-\delta-R\beta^2\sigma_z^2\ (\gamma-0.5)]$$

and finally:

$$g = \frac{\gamma\beta - \delta - \rho}{R} + \beta^2\sigma_z^2\left(\frac{1+R}{2} - \gamma\right) \qquad (15)$$

We notice that the variance of the technological shock, for realistic parameter values, positively affects the expected growth rate. More precisely, since R must be non negative, we are in presence of such an effect if γ, the capital income share, is lower than 0.5; a sufficient condition which seems to be supported by the data. If we compare this result with the one in Devereux and Smith (1994, p. 545), we

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[7] Notice that $J_W E(dW)$, being equal to $[\gamma\beta-\delta+R\beta^2\sigma_z^2(1-\gamma)]W^{(1-R)}$, increases when σ_z^2 is raised. The term $J_{WW}/2\ E(dW)^2$, representing the loss in welfare due to the impact of technological uncertainty on a concave indirect utility function, boils down to $-R\beta^2\sigma_z^2 W^{(1-R)}/2$. Therefore the effect of technological uncertainty on consumption depends on γ being higher or lower than 0.5.

notice the widening of the parameters set where an increase in technological variability has a positive influence on the expected growth rate.

When $R>1$, equation (15) seems at odds with equation (14); however this apparent contradiction can be explained reformulating the H-valuation expression (10) using equations (13-14):

$$H = \frac{(1-\gamma)\,(\beta - R\beta^2\sigma_z^2)}{\left\{\frac{\rho}{R} + \frac{(R-1)}{R}\left[\gamma\beta - \delta - R\,\beta^2\sigma_z^2\left(\gamma - \frac{1}{2}\right)\right]\right\}} K = \frac{(1-\gamma)\,(\beta - R\beta^2\sigma_z^2)}{D^{-1/R}} K$$

An increase in aggregate uncertainty lowers the numerator of H and it increases the denominator when $R>1$, if $\gamma<0.5$. Therefore, under this parametric restriction, the evaluation of the stochastic income stream is always diminished by an increase in σ_z; this reduction in human wealth more than compensate the increase in the propensity to consume outlined above and explains why the effect of uncertainty on the growth rate is always positive.

3.4 Uncertainty and welfare

To appreciate the effects of an increase in the standard deviation of the technological shock on the maximum value function,

$$J(F,H) = \frac{D(F+H)^{1-R}}{1-R} = \frac{K^{1-R}}{1-R}\frac{\left\{\frac{\rho+(R-\gamma)\,\beta+(1-R)\,\delta}{R} - \beta^2\sigma_z^2\left(\frac{1+R}{2} - \gamma\right)\right\}^{1-R}}{\frac{1}{R}\left\{\rho + (R-1)\left[\gamma\beta - \delta - R\beta^2\sigma_z^2\left(\gamma - \frac{1}{2}\right)\right]\right\}}$$

we define, for convenience,

$$n = \frac{\rho + (R-\gamma)\,\beta + (1-R)\,\delta}{R} - \beta^2\sigma_z^2\left(\frac{1+R}{2} - \gamma\right)$$

$$d = \rho + (R-1)\left[\gamma\beta - \delta - R\,\beta^2\sigma_z^2\left(\gamma - \frac{1}{2}\right)\right]$$

and we compute:

$$\frac{\partial J(F,H)}{\partial \sigma_z} = \frac{K^{1-R}}{1-R}\left\{\frac{R(1-R)\,n^{-R}\,[-\beta^2\sigma_z(1+R-2\gamma)]}{d} - \frac{R(1-R)\,n^{1-R}\,R\beta^2\sigma_z(2\gamma-1)}{d^2}\right\} =$$

$$\left(\frac{K^{1-R}R^2\beta^2\sigma_z n^{-R}}{d^2}\right)\left[-\rho+\beta[(\gamma-1)^2-\gamma(R-\gamma)] + \delta(R-1) - \beta^2R\sigma_z^2\left(\frac{1-2\gamma}{2}\right)(1+R-2\gamma)\right]$$

The first factor is always positive, so the sign of $\partial J(F,H)/\partial\sigma_z$ depends only on the expression contained in the big square brackets[8]. This is a function of five parameters (ρ, β, γ, δ, R) and of the standard deviation of the technological disturbance, and it is therefore hard to interpret. Our strategy consists in setting ρ to 0.03, δ to 0.05 and σ_z to 0.03 (results are not very sensitive to this last figure), and in performing some numerical computations[9].

In an endogenous growth model the choice of an appropriate capital/output ratio (i.e. $1/\beta$) entails some problems. On the one hand, the accumulable factor should represents a "broad measure of capital", possibly including some

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[8] It is immediate to notice that $\partial J(F,H)/\partial\sigma_z^2$ may be positive for some values of the parameters. For example, with logarithmic preferences, we may compute

$$\left.\frac{\partial J(F,H)}{\partial\sigma_z^2}\right|_{\sigma_z^2=0} = \left(\frac{K\,\beta^2\,n^{-1}}{2d^2}\right)[-\rho+\beta(1-\gamma)(1-2\gamma)],$$

which is positive for $1/\beta < (1-\gamma)(1-2\gamma)/\rho$.

[9] Using recursive preferences à la Epstein and Zin (1989) we obtained:

$$\frac{\partial J(F,H)}{\partial\sigma_z} = \left(\frac{K^{1-R}R\varepsilon\beta^2\sigma_z n^{-R}}{d^2}\right)\left[-\rho+\beta[(\gamma-1)^2-\gamma(R-\gamma)]+\delta(R-1)-\beta^2\varepsilon\sigma_z^2\left(\frac{1-2\gamma}{2}\right)(1+R-2\gamma)\right],$$

where ε is the risk aversion index. The parameters set where the sign of this derivative differs from the one of the expression in the main text is negligible.

infrastructure, usually publicly provided, and technological knowledge, at least partly incorporated in "human capital". However, buildings and industrial estates do not seem relevant in generating a positive externality and should be deducted. Figure 1a lets β vary from 0.25 to 0.5 and plots the highest possible value for the risk aversion index which allows for a positive $\partial J(F,H)/\partial\sigma_z$ with a 2% average growth rate. This is held constant by adjusting γ. The highest admissible value for the risk aversion index varies from about 1.25 (when β=0.25 and γ is about 0.385) to about 3 (when β=0.5 and γ is about 0.26). Figure 1b plots the associated gross capital income share[10].

Obviously, the choice of a "realistic" value for R is not straightforward, given the wide set of available estimates. However it is not difficult to find results falling well into the above range. For example, in a recent contribution, Beaudry and van Wincoop (1992), using regional US consumption data, find that one is a rather precise point estimates for $1/R$ (the standard error is 0.3).

Hence, it seems that, within the class of linear in capital aggregate endogenous growth models, the parameter set which entails welfare lowering due to an decrease in uncertainty (due to a transition from financial autarchy to international integration) is not void of empirical relevance.

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[10] Devereux and Smith (1994, p. 544) set β=3.85, γ=0.3, δ=1, R=2 and ρ=0.1. However, since they do not specify the lenght of the time interval in their discrete time model, a precise comparison is not possible. Obstfeld (1994, pp. 1318-19) in his numerical exercises sets R=2; $\sigma_z \in \{0.1;0.02\}$ and δ=0.

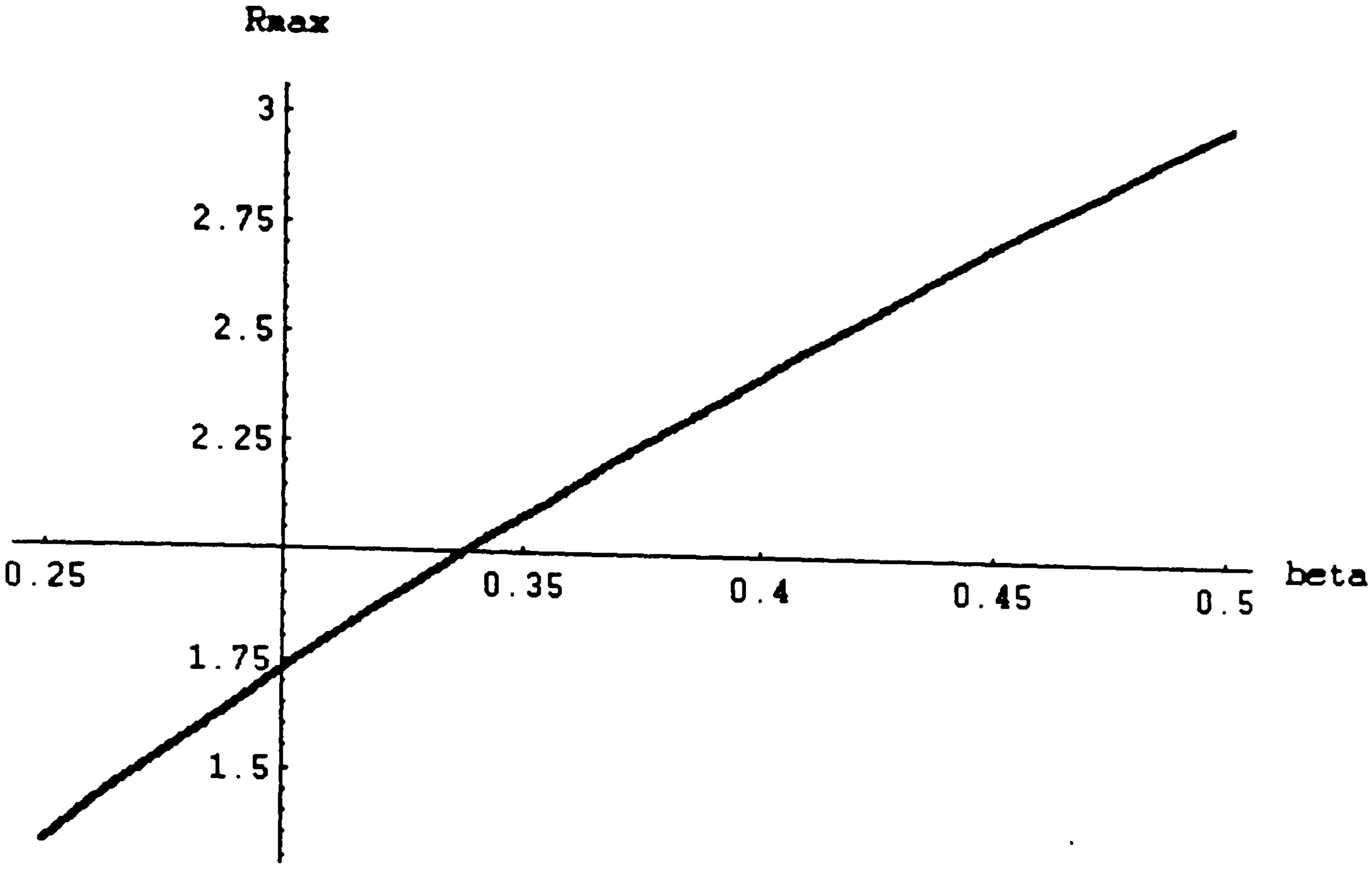

Figure 1.a

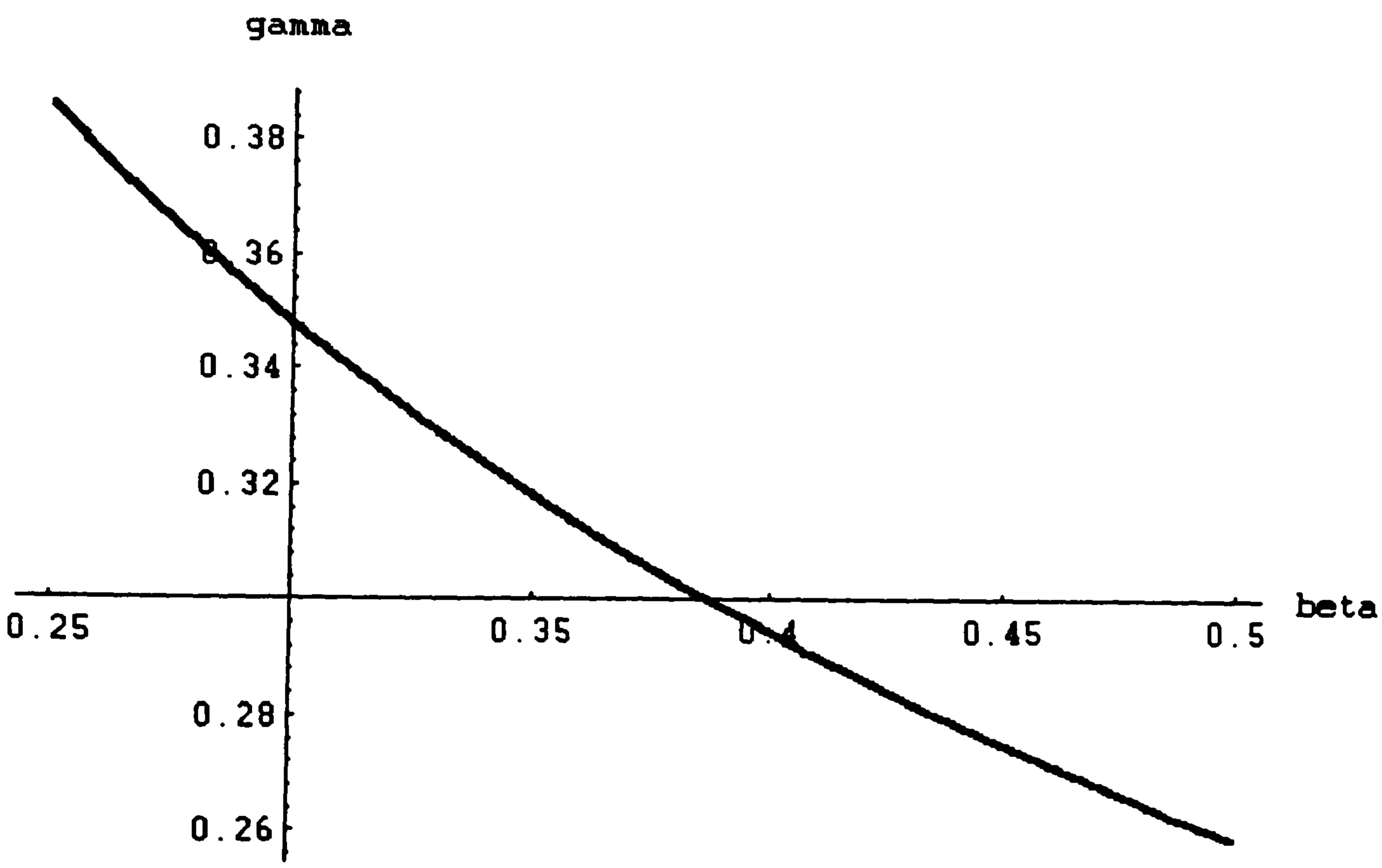

Figure 1.b

Rmax is defined as the highest possible value for the risk aversion index which is compatible with a non negative welfare variation induced by an increase in uncertainty and with a 2% expected growth rate.

Figure 1.a plots *Rmax* against β while figure 1.b plots the gross capital income share associated with *Rmax*.

This model exhibit a striking difference with the one proposed by Devereux and Smith. We showed that the welfare reduction effect of a decrease in uncertainty is possible with Hicks-neutral technology shocks. Our results is to be ascribed to the chosen stochastic structure, which is the continuous time counterpart of a random walk and not of a purely time-independent sequence of shocks. The impact of σ_z on the expected growth rate is much higher in this model, due to the effects of Ito's lemma, which implies first order effects for standard deviations[11]. Regarding our choice, we believe that the use of a geometric brownian motion for dY can be questioned on the ground of the absence of serial correlation, but not because it implies a "cumulative" component.

4. Distributional shocks

Many of the pitfalls of the above framework are attributable to its nature of representative agent equilibrium model and hence cannot be easily removed. However, its irrealism can be loosed in at least one aspect. Up to now, we considered a situation where factors' shares of income are perfectly correlated, while the counterciclical movement of the labour's share is a widely accepted stylised fact. We deal with this phenomenon as if it were due to an exogenous source of randomness[12], however we think that this model should be considered as a "reduced form" for a more complex framework where the distributional shock is endogenously determined by staggered labour contracts, union activity, taxes or government's transfer schemes.

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[11] Taking the limit for the time interval going to zero in Devereux and Smith's discrete model, one may verify that the variance of the random disturbance turns out not to have effects on the expected growth rate.

[12] For a recent exemple of a similar approach, see Ghosh and Pesenti (1994, pp.19-22).

We let the expected utility and the production function be unchanged, but we suggest that the capital yield is given by:

$$dK = (\gamma\beta\text{-}\delta)Kdt + \gamma\beta K\sigma_z dz + K\sigma_D\, dz_D$$

This equation implies that the drift of the wage process is unaffected, but also, more interestingly, that $\sigma_S dz_S = (1\text{-}\gamma)\beta K\sigma_z dz\text{-}K\sigma_D dz_D$ and $\sigma_{SZ} = (1\text{-}\gamma)\beta K\sigma_z{}^2\text{-}\ K\sigma_{Dz.}$ Hence, there is no longer perfect correlation between capital's and labour's shares of income, since they are different linear combinations of dz and dz_D. From the perspective of stylised facts, it seem natural to assume $\sigma_{DZ}>0$. For example, Cardia and Ambler in a recent paper (1993, Table 2) report that, for the US, the ratio between the standard deviation of the wage share of income and the one of output is 0.475, while the ratio between the standard deviation of the profit share and the one of output is 4.

It is important to remark that, in this case, human wealth is not spanned by the available financial assets, hence it is no longer possible to build a "replicating portfolio" with a deterministic return, as we did in section 3.2. To solve the model, we need to proceed, as in Svensson and Werner (1993), as if H were tradable and interpret the resulting μ_H as a "shadow" price. However, to avoid the complete discussion of a three asset model, we will price human wealth through a C.A.P.M. relation.

Also in this case the set of assets is completed by (2) and the Bellman equation for the optimal consumption-portfolio choice is as follows:

$$0 = max_{C,K}\ \frac{C^{1\text{-}R}}{1\text{-}R} + J_F\,(r\,F + (\gamma\beta - \delta - r)\,K + \mu_S - C) + J_H\,(\mu_H\,H - \mu_S) +$$

$$+ \frac{1}{2}\,J_{FF}[(\gamma^2\beta^2\sigma_z{}^2 + \sigma_D{}^2 + 2\gamma\beta\sigma_{Dz})K^2 + \sigma_s{}^2 + 2(\gamma\beta\sigma_{zs} + \sigma_{DS})K] +$$

$$+ J_{HF}[(\gamma\beta\sigma_{Hz}+\sigma_{DH})KH+\sigma_{HS}H-(\gamma\beta\sigma_{zS}+\sigma_{DS})K-\sigma_S^2] +$$

$$+ \frac{1}{2}J_{HH}(\sigma_H^2H^2+\sigma_S^2-2\sigma_{HS}H)$$

The associated first order conditions becomes:

$$C^{-R} = J_F$$

$$(\gamma\beta - \delta - r)\, J_F = - J_{FF}\, [(\gamma^2\beta^2\sigma_z^2 + \sigma_D^2 + 2\gamma\beta\sigma_{Dz})\, K + (\gamma\beta\sigma_{zS} + \sigma_{DS})] +$$

$$+ J_{HF}\, [(\gamma\beta\sigma_{Hz} + \sigma_{DH})\, H - (\gamma\beta\sigma_{zS} + \sigma_{DS})]$$

To price human wealth we specialise again equation (5) into the linear relation $H=bK$, hence $dH = bdK$, or:

$$dH = b[(\beta - \delta)K - C]dt + b\beta K\sigma_z dz \tag{16}$$

Equating this expression with equation (3), we can work out the stochastic characteristics of the human wealth process:

$$\sigma_H H\, dz_H = (1-\gamma)\, \beta K\sigma_z dz - K\sigma_D dz_D + b\beta K\sigma_z dz \tag{17a}$$

and also:

$$\sigma_{Hz}H = (1-\gamma)\, \beta K\sigma_z^2 - K\sigma_{Dz} + \beta H\sigma_z^2 \tag{17b}$$

$$\sigma_{DH}H = (1-\gamma)\, \beta K\sigma_{Dz} - K\sigma_D^2 + \beta H\sigma_{Dz} \tag{17c}$$

Since H is no longer spanned by financial assets, we assume that any portfolio P must be priced by an equilibrium relation similar to equation (8):

$$\mu_P - r = \frac{\sigma_{PW}}{\sigma_W^2}(\mu_W - r) \tag{18}$$

where W is total wealth (in our case $K+H$). Equation (18) is a typical "CAPM" relation. (See Merton, (1990, ch. 11), for a particularly neat exposition of a general equilibrium model of asset market.)

To solve our problem it proved useful to build a portfolio composed of human wealth and of the quantity of capital which solves the problem:

$$Min_{K^p}\,[Var\,(dH + dK)]$$

$$= Min_{K^p}\,[Var\,(\sigma_H H dz_H + \gamma\beta K^p \sigma_z dz + K^p \sigma_D dz_D)]$$

$$= Min_{K^p}\,[(\sigma_H^2 H^2 + \gamma^2\beta^2 K^{p2}\sigma_z^2 + K^{p2}\sigma_D^2 + 2\gamma\beta K^p H\sigma_{Hz} + 2K^p H\sigma_{DH} +$$

$$+ 2\gamma\beta K^{p2}\sigma_{Dz})]$$

i.e.

$$K^p = -\frac{\gamma\beta\sigma_{Hz} + \sigma_{DH}}{\gamma^2\beta^2\sigma_z^2 + \sigma_D^2 + 2\gamma\beta\sigma_{Dz}}H$$

Using equations (17b-17c) we obtain:

$$K^p = -\frac{\gamma\beta[(1-\gamma)\beta K\sigma_z^2 - K\sigma_{Dz} + \beta H\sigma_z^2] + (1-\gamma)\beta K\sigma_{Dz} - K\sigma_D^2 + \beta H\sigma_{Dz}}{\gamma^2\beta^2\sigma_z^2 + \sigma_D^2 + 2\gamma\beta\sigma_{Dz}}H$$

Moreover, it is interesting to notice that:

$$K\text{-}K^p = \frac{\beta(\gamma\beta\sigma_z^2 + \sigma_{Dz})}{\gamma^2\beta^2\sigma_z^2 + \sigma_D^2 + 2\gamma\beta\sigma_{Dz}}(K+H) \tag{19}$$

Some manipulations give

$$\sigma_{PW} = \frac{(1-\gamma)\beta^2\sigma_z^2K - \beta\sigma_{Dz}K + \beta^2\sigma_z^2H + \gamma\beta^2\sigma_z^2K^p + \beta\sigma_{Dz}K^p}{H+K^p} =$$

$$= \frac{\beta^2\sigma_z^2(K+H) + (\gamma\beta^2\sigma_z^2 + \beta\sigma_{Dz})(K^p-K)}{H+K^p}$$

Since $\sigma_W^2 = \beta^2\sigma_z^2$ we get $\frac{\sigma_{PW}}{\sigma_W^2} = \left(\frac{\sigma_z^2\sigma_D^2 - \sigma_{Dz}^2}{\sigma_z^2(\gamma^2\beta^2\sigma_z^2 + \sigma_D^2 + 2\gamma\beta\sigma_{Dz})}\right)\left(\frac{H+K}{H+K^p}\right)$

and we transform equation (18) into:

$$\frac{\mu_H H+(\gamma\beta-\delta)K^p}{H+K^p} - r = \left(\frac{\sigma_z^2\sigma_D^2-\sigma_{Dz}^2}{\sigma_z^2(\gamma^2\beta^2\sigma_z^2+\sigma_D^2+2\gamma\beta\sigma_{Dz})}\right)\left(\frac{H+K}{H+K^p}\right)\left(\frac{\mu_H H+(\gamma\beta-\delta)K}{H+K} - r\right)$$

Multipling both sides by $H+K^p$, grouping terms and rearranging we obtain, by means of (19):

$$\left(\frac{(\gamma\beta\sigma_z^2+\sigma_{Dz})^2}{\sigma_z^2(\gamma^2\beta^2\sigma_z^2+\sigma_D^2+2\gamma\beta\sigma_{Dz})}\right)[\mu_H H+(\gamma\beta-\delta)K - r(H+K)] =$$

$$= (\gamma\beta-\delta-r)\left(\frac{\beta\sigma_z^2(\gamma\beta\sigma_z^2+\sigma_{Dz})}{\sigma_z^2(\gamma^2\beta^2\sigma_z^2+\sigma_D^2+2\gamma\beta\sigma_{Dz})}\right)(K+H) \tag{20}$$

Finally, equating again equations (16) and (3) to substitute out $\mu_H H$ and then simplifying gives:

$$H = \left(\frac{\beta\sigma_z^2(r + \delta)(1-\gamma) + \sigma_{Dz}(\beta-\delta-r)}{\beta\sigma_z^2[\gamma\beta-\delta-r-\gamma(g-r)] - \sigma_{Dz}(g-r)}\right)K \tag{10'}$$

As in section 3.2, we end up with a function which is still relating human wealth to two endogenous variables, r and g, and we need to solve our intertemporal problem. In the Appendix we obtain, respectively, the following expressions for the riskless rate, the expected growth rate, the propensity to consume and for human wealth:

$$r = \gamma\beta - \delta - R\beta(\gamma\beta\sigma_z^2 + \sigma_{Dz})$$

$$g = \frac{\gamma\beta - \delta - \rho}{R} + \beta^2\sigma_z^2\left(\frac{1+R}{2} - \gamma\right) \beta\sigma_{Dz}$$

$$D^{-1/R} = \frac{\rho}{R} + \frac{(R-1)}{R}\{\gamma\beta - \delta - R\beta[\beta\sigma_z^2(\gamma - 0.5) + \sigma_{Dz}]\}$$

$$H = \frac{(1-\gamma)\beta + \beta R[(\gamma-1)\beta\sigma_z^2 + \sigma_{Dz}]}{\gamma\beta - \delta - g - R\beta[(\gamma-1)\beta\sigma_z^2 + \sigma_{Dz}]}K = \frac{(1-\gamma)(\beta - R\beta^2\sigma_z^2) + R\beta\sigma_{Dz}}{D^{-1/R}}K$$

We may now notice several interesting details. An increase in σ_{Dz} reduces the riskless rate, which is consistent with intuition, since capital becomes a riskier asset. Such an increase lowers the propensity to consume, for $R>1$: in this case the "precautionary saving" motive overturns the "portfolio adjustment" one. The value of H, for a given expected growth rate, is positively related to σ_{Dz}: intuition suggests that, since this asset becomes safer, it is more valued. The total effect of the covariance on the growth rate proves to be always negative.

Finally, we analyse again the effects of an increase in the standard deviation of the technological shock on welfare, starting from our maximum value function:

$$J(F,H) = \frac{D(F+H)^{1-R}}{1-R} =$$

$$= \frac{K^{1-R}}{1-R}\frac{\left\{\frac{\rho + (R-\gamma)\beta + (1-R)\delta}{R} + \beta\sigma_{Dz} - \beta^2\sigma_z^2\left(\frac{1+R}{2} - \gamma\right)\right\}^{1-R}}{\frac{1}{R}\left\{\rho + (R-1)\left[\gamma\beta - \delta - R\beta\left(\beta\sigma_z^2\left(\gamma - \frac{1}{2}\right) + \sigma_{Dz}\right)\right]\right\}}$$

As before, we define:

$$n = \frac{\rho + (R-\gamma)\beta + (1-R)\delta}{R} + \beta\sigma_{Dz} - \beta^2\sigma_z^2\left(\frac{1+R}{2} - \gamma\right)$$

$$d = \rho - (1-R)\left[\gamma\beta - \delta - \beta R\left(\beta\sigma_z^2\left(\gamma - \frac{1}{2}\right) + \sigma_{Dz}\right)\right]$$

We now compute the consequence on welfare of an increase in the standard deviation of the distributional process:

$$\frac{\partial J(F,H)}{\partial\sigma_D} = \frac{K^{1-R}}{1-R}\left(\frac{R(1-R)\,n^{-R}\,\beta\sigma_z\rho_{Dz}}{d} + \frac{R(1-R)\,n^{1-R}\,(-\beta R\sigma_z\rho_{Dz})}{d^2}\right)$$

$$= \left(K^{1-R}\,\frac{R\beta n^{-R}\,\sigma_z}{d^2}\right)\rho_{Dz}\,\{R\beta(\gamma-1) + R^2\beta[\sigma_z^2\,\beta(1-\gamma) - \sigma_{Dz}]\}$$

Since the factor in the big round brackets is always positive, for reasonable parameters values we have Sign[$\partial J(F,H)/\partial\sigma_D$] = - Sign[$\rho_{Dz}$], hence, in such case, $\partial J(F,H)/\partial\sigma_D < 0$. Therefore, insofar $\rho_{Dz}>0$, the endogenous part of σ_D should be reduced.

The effect of an increase in σ_z on welfare may be writted as:

$$\frac{\partial J(F,H)}{\partial\sigma_z} =$$

$$\frac{K^{1-R}}{1-R}\left(\frac{R(1-R)\beta n^{-R}[\sigma_D\rho_{Dz} - \beta\sigma_z(1+R-2\gamma)]}{d}\,\frac{R^2(1-R)\beta n^{1-R}[\beta\sigma_z(2\gamma-1) + \sigma_D\rho_{Dz}]}{d^2}\right)$$

$$= \left(\frac{K^{1-R}R\beta n^{-R}}{d^2}\right)\left[\sigma_D\rho_{Dz}\{R\beta(\gamma-1) + R^2\beta[(1-\gamma)\beta\sigma_z^2 - \sigma_{Dz})]\} + \right.$$

$$+ \beta R\sigma_z\,x$$

$$x\left[-\rho+\beta[(\gamma-1)^2-\gamma(R-\gamma)]+\delta(R-1)-R\beta^2\sigma_z^2\left(\frac{1-2\gamma}{2}\right)(1+R-2\gamma)-R\beta\sigma_{Dz}(2\gamma-R)\right]\Bigg\}$$

The meaning of this expression may be grasped more easily if we reformulate it as follows:

$$\frac{\partial J(F,H)}{\partial\sigma_z}=\left(\frac{\sigma_D}{\sigma_z}\right)\left(\frac{\partial J}{\partial\sigma_D}\right)+\left.\frac{\partial J}{\partial\sigma_z}\right|_{\bar{\sigma}_D}+\left(\frac{K^{1-R}R\beta n^{-R}}{d^2}\right)R^2\beta^2\sigma_z\sigma_{Dz}(R-2\gamma)$$

With respect to the case with no distributive shocks we have two further addenda; insofar $\rho_{Dz}>0$, the first is negative for reasonable parameter values while the third is positive if $R>2\gamma$. To appreciate their quantitative effect under this parametric restriction, we run several numerical computation, from which is clear that the higher ρ_{Dz} and σ_D, the more negative the sum of the two terms. As an example consider figure 2, where ρ=0.03, δ=0.05, the capital/output ratio is 3, γ is one third, and σ_z=0.03. For ρ_{Dz}, we choose a rather low value, 0.2. However, from figure 2a it is clear that the region where an increase in σ_z augments welfare, whose first derivative is on the vertical axis, has been sharply reduced: the continuous line is the value of $\partial J(F,H)/\partial\sigma_z$ in absence of distributional shocks, while the dotted line represents the case σ_D=0.02.

Therefore a distributive disturbance, positively correlated with the technological one, may reverse the "perverse" result outlined in section 3.4.

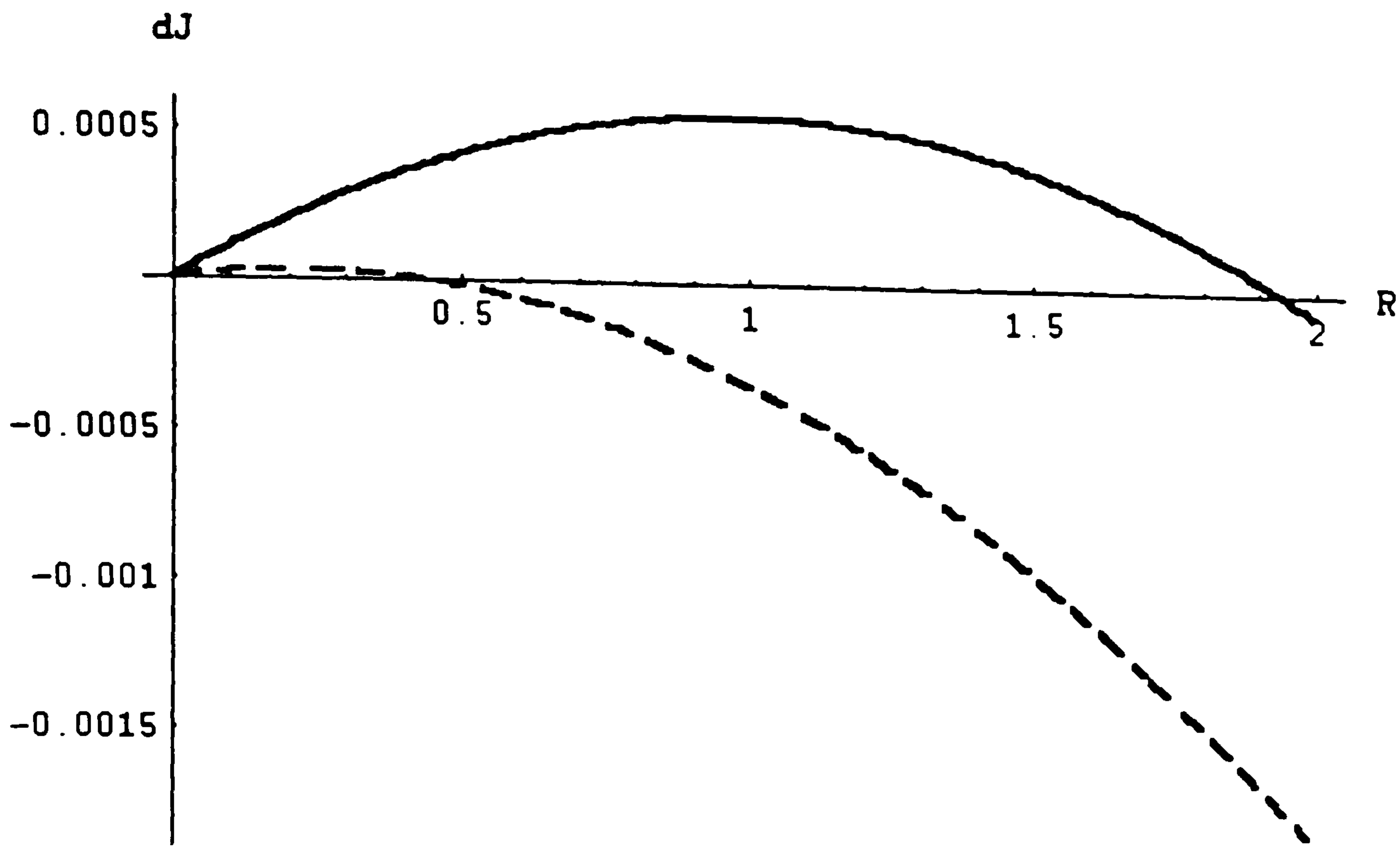

Figure 2.a

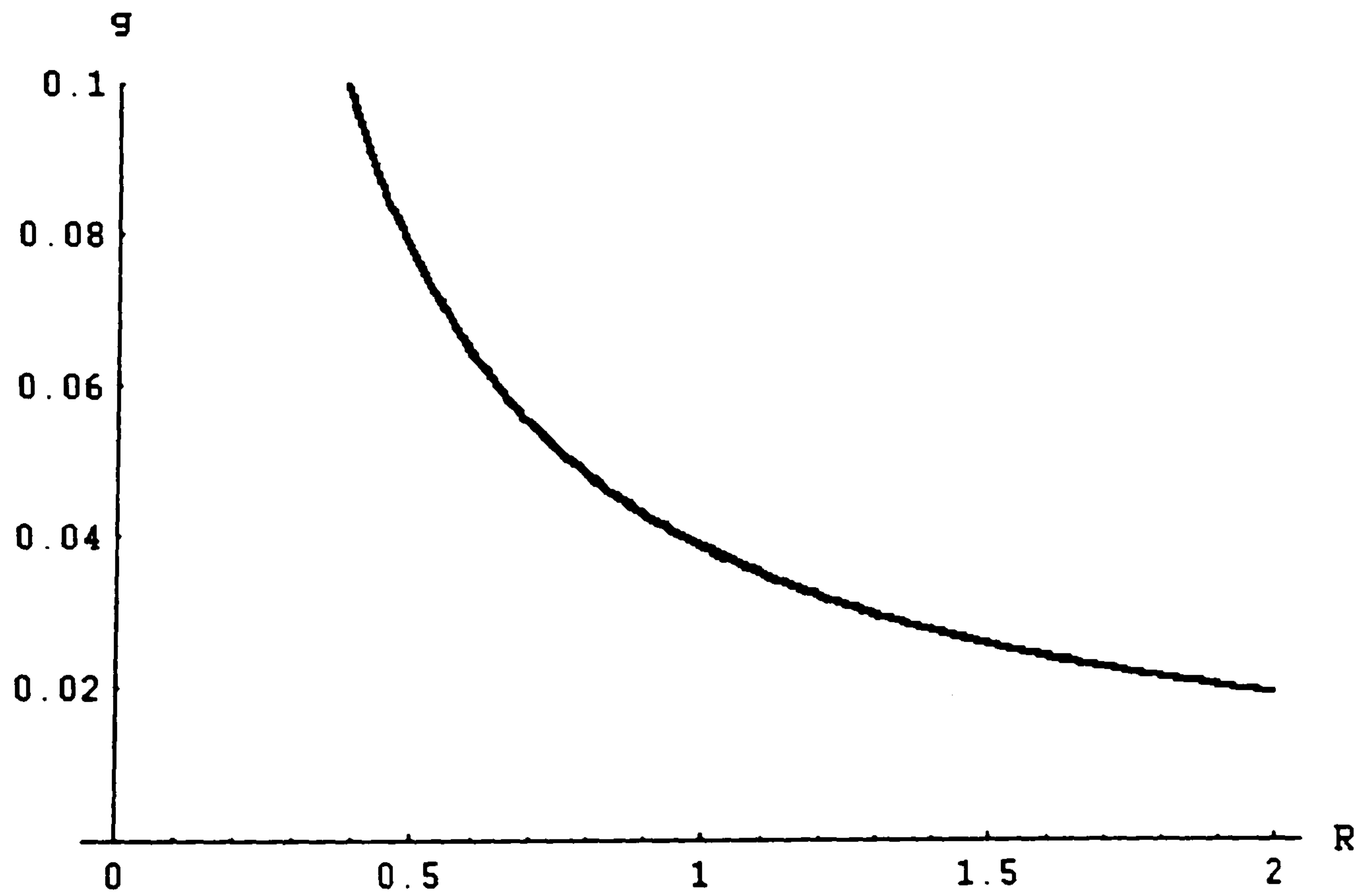

Figure 2.b

Figure 1.a plots the derivative of $J(F,H)$ with respect to σ_z as a function of R when ρ_{zD} is zero (continuous line) and when it is 0.2 (dashed line). Figure 2.b plots the associated expected growth rate.

5. Concluding remarks

We have built a continuous time endogenous growth model where the production function allows for a positive externality in the spirit of Romer (1986) and non accumulable inputs (labour) play a productive role. This assumption entails two main consequences: the divergence of the gross capital income share from the marginal productivity of capital and the presence of an asset, the claim on future labour income, which has been assumed to be non-traded, due to a moral hazard problem.

In this framework, a reduction in technological uncertainty, possibly due to international risk sharing, may be welfare damaging since it lowers savings while the expected growth rate is suboptimal.

The inclusion in the model of non-traded labour income is performed by means of contingent claim analysis and it delivers two important results. If the capital income share is lower than one half, the expected growth rate is always reduced by an increase in technological uncertainty. In the existing literature, this effect is present only if the elasticity of intrtemporal substitution is higher than unity. To grasp the intuition for this result, notice that, in the present model, for $R<1$, an increase in the standard deviation of the technological process induces a reduction in the propensity to consume.

What is more important, we have shown, in contrast to Devereux and Smith (1994), that we may have welfare damaging risk pooling in presence of (Hicks-neutral) technological risk. Moreover, numerical calculations show that the parameters set where risk sharing reduces welfare is relevant. This striking difference in results is to be ascribed to the chosen stochastic structure: the impact of σ_z on the expected growth rate is much higher in our model, due to Ito's lemma, which implies first order effects for standard deviations. The choice of using a geometric brownian motion to model the variability of the output process, in our opinion, can be questioned on the ground of the absence of serial correlation, but not because it implies a "cumulative" component.

To move a step towards "reality", we have considered the possibility of a distributive disturbance. If it increases the correlation between output and the capital income share, the portion of the parameters space where the "perverse" effect on welfare takes place is sharply reduced: in this case, the riskless rate diminishes, since capital becomes a riskier asset, and the propensity to consume is reduced for $R>1$.

Therefore, the more apart an economic environment is to perfect competition (the higher is the distributive disturbance), the less likely is the occurrence of our "perverse" result and the traditional welfare-augmenting effect of international risk-sharing is more likely to hold.

To move further towards "reality" we should consider, for example, the effects of distortionary taxation and of imperfectly competitive markets. Moreover, the role of publicly provided infrastructures should be modelled explicitly and the accumulation of human capital is to be microfounded. Intuition suggest that the first three factors reinforce our result, since they tend to increase the difference between the optimal growth rate and the market one, while the fourth should operate in the opposite direction. Significant extensions of our model, however, prove to be difficult, since the inclusion of a second state variable (either public or human capital) hinders the attainment of an explicit solution, making welfare comparison very difficult even in our representative agent framework.

Appendix: Details of the solution in presence of a distributive shock

As in section 3.3, we use the guess $J(F+H) = D\,\frac{(F+H)^{1-R}}{1-R}$; accordingly the first order conditions become:

$$C = D^{-1/R}(F+H)$$

$$K=\frac{(\gamma\beta-\delta-r)(F+H)\ +\ R(\gamma\beta\sigma_{Hz}+\sigma_{DH})H}{R(\gamma^2\beta^2\sigma_z{}^2+\sigma_D{}^2+2\gamma\beta\sigma_{Dz})} \tag{A1}$$

These expressions and the aggregate constraint B=0 are exploited to compute D from:

$$0 = D^{(R-1)/R}\,\frac{(F+H)^{1-R}}{1-R} - \rho D\,\frac{(F+H)^{1-R}}{1-R} + D\,(F+H)^{-R}\,E(dK+dH) +$$

$$-\frac{R}{2}D\,(F+H)^{-(1+R)}\,E\,(dK+dH)^2 \tag{A2}$$

We first notice that equation (A1), using equations (17b- 17c) yields:

$$R\,(\gamma\beta^2\sigma_z^2 + \beta\sigma_{Dz})(K+H) = (\gamma\beta-\delta-r)\,(F+H)$$

Hence, when B=0, we obtain the forcing rate:

$$r = \gamma\beta-\delta- R\beta(\gamma\beta\sigma_z^2 + \sigma_{Dz}) \tag{A3}$$

Notice that the riskless rate turns out to be reduced by a positive covariance between the technological and the distributive shocks, and that this effect is stronger the higher is the reciprocal of the elasticity of intertemporal substitutionof the individual.

From equation (20), using (A3), we immediately get:

$$\mu_H H + (\gamma\beta-\delta)K - r(H+K) = (\gamma\beta-\delta-r)\left(\frac{\beta\sigma_z^2}{\gamma\beta\sigma_z^2 + \sigma_{Dz}}\right)(K+H) =$$

$$= R\beta^2\sigma_z^2 (K+H)$$

which enables us to obtain, togheter with the equilibrium riskless rate:

$$E(dK+dH) = r(K+H) + R\beta^2\sigma_z^2 (K+H) - C =$$

$$= [\gamma\beta-\delta-\beta R(\gamma\beta\sigma_z^2 + \sigma_{Dz}) + R\beta^2\sigma_z^2](K+H) - D^{-1/R}(K+H)$$

since

$$E(dK+dH)^2 = (\gamma^2\beta^2\sigma_z^2 + \sigma_D^2 + 2\gamma\beta\sigma_{Dz})K^2 + \sigma_H^2 H^2 + 2(\gamma\beta\sigma_{Hz} + \sigma_{DH})\,KH =$$

$= \beta^2\sigma_z^2\,(K+H)^2$ we can compute $D^{-1/R}$ from (A2):

$$D^{-1/R} = \frac{\rho}{R} - \frac{(1-R)}{R}\{\gamma\beta - \delta - R\beta[\beta\sigma_z^2(\gamma - 0.5) + \sigma_{Dz}]\} \qquad \text{(A4)}$$

The explicit expression for the riskless rate gives, from (10'), a formulation for the value of the claim on income where the expected growth rate is the unique endogenous variable, i.e.

$$H = \frac{(1-\gamma)\beta + \beta R[(\gamma-1)\beta\sigma_z^2 + \sigma_{Dz}]}{\gamma\beta - \delta - g - R\beta[(\gamma-1)\beta\sigma_z^2 + \sigma_{Dz}]}K$$

This expression is of great help in computing g. Starting, as in section 3.3, from its definition, we immediately get:

$$(\beta - \delta - g) = D^{-1/R}\left(1 + \frac{(1-\gamma)(\beta - R\beta^2\sigma_z^2) + R\beta\sigma_{Dz}}{\gamma\beta - \delta - g - R\beta[(\gamma-1)\beta\sigma_z^2 + \sigma_{Dz}]}\right)$$

Hence, using (A4) and rearranging:

$$\gamma\beta\text{-}\delta\text{-}g - R\beta[(\gamma - 1)\beta\sigma_z^2 + \sigma_{Dz}]) = \frac{\rho}{R} - \frac{(1-R)}{R}\{\gamma\beta - \delta - R\beta[\beta\sigma_z^2(\gamma - 0.5) + \sigma_{Dz}]\}$$

and finally:

$$g = \frac{\gamma\beta - \delta - \rho}{R} + \beta^2\sigma_z^2\left(\frac{1+R}{2} - \gamma\right) - \beta\sigma_{Dz}$$

The final expression for human wealth becomes:

$$H = \frac{(1-\gamma)(\beta - R\beta^2\sigma_z^2) + R\beta\sigma_{Dz}}{\frac{\rho}{R} - \frac{(1-R)}{R}\{\gamma\beta - \delta - R\beta[\beta\sigma_z^2(\gamma - 0.5) + \sigma_{Dz}]\}} =$$

$$= \frac{(1-\gamma)(\beta - R\beta^2\sigma_z^2) + R\beta\sigma_{Dz}}{D^{-1/R}}$$

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Chapter V

Monetary Uncertainty Relevance and Perpetual Youth in a Stochastic Endogenous Growth Model

1. Introduction

Several contributions consider the role of monetary uncertainty in continuous time Sidrauski-type frameworks. This approach is appealing because, building on Merton's seminal papers, it is possible to consider how volatile money growth affects portfolio choice and investment. Gertler and Grinols (1982) develop a linear-in-capital growth model where the government's revenue is distributed to the representative agent in proportion to wealth. In that paper, they uncover a positive effect of the average money growth rate on investment and a negative impact of money volatility. In Stulz's model, (1986), money growth takes place through lump-sum transfers, which are fully capitalised by agents; in this case investment, and hence the expected growth rate of the economy, are completely independent of the monetary policy parameters. More recently, Grinols and

Turnovsky, (1993), restate Gertler and Grinols' (1982) results in a more widely specified endogenous growth framework where transfers are again distortionary.

None of the existing contributions encompasses the case of finite lives, despite the fact that the results concerning the relations between a positive probability of death and the Tobin effect are well established (e.g. Marini and van der Ploeg, (1988), and van der Ploeg and Alogoskoufis, (1994)). Hence, it seems natural to investigate a Blanchard-type model to disclose the role of an "ongoing" monetary policy volatility in such an overlapping generations framework.

Moreover, the present chapter extends the existing literature on stochastic models, by abandoning, within the intertemporal C.R.R.A. utility function class, the hypothesis of a unitary degree of relative risk aversion.

To solve a dynamic portfolio problem in the presence of overlapping generations, it is necessary to pass through several steps. After a quick set-up of the model (section 2.1), we consider the behaviour of an individual, representative of the cohort born at time s, and we set up her dynamic consumption-portfolio choice. In order to determine the excess return for the risky assets, it is useful to guess at this stage not only a functional form for the individual's maximum value function but also that it is not explicitly related to age. We then proceed (section 2.3) to determine the characteristics of the endogenous stochastic processes by exploiting the constancy of portfolio shares that characterises the model at the aggregate level. This analysis allows us to show that the nominal interest rate is affected by both the risk aversion index and by the covariance between the technological disturbance and the nominal money process. Hence, an "insurance" role for monetary policy emerges. In section 2.4 we move back to the individual behaviour and we determine her portfolio shares, that turn out to depend on the level of her individual wealth. Eventually, we are able to solve the model, verify that the maximum value function is actually age independent and determine consumption and real money demand (section 2.5). At this stage a "second order" Tobin effect emerges, in the sense that both the money supply volatility and the covariance between the output process and the monetary one affect growth.

Section 3 is devoted to the discussion of the policy implications of the model. In particular, we will devote some attention to the possibility of evaluating some alternative central bank operating procedures.

2. The model

2.1 The basic set-up

We consider a closed economy populated by individuals who differ only in their date of birth, $s \in (-\infty, t]$, and who face the same instantaneous probability of death, p, governed by a Poisson process. The representative individual of generation s maximises the intertemporal objective:

$$U_{t,s} = \int_t^{\infty} \left\{ E_t \left[\frac{(c_{\tau,s}{}^{\alpha}\, m_{\tau,s}{}^{1-\alpha})^{1-R}}{1-R}\, e^{-\rho(\tau-t)} \right] e^{-p(\tau-t)} \right\} d\tau$$

where ρ is the instantaneous rate of time preference and $R \in [0, \infty)$ represents both the relative risk aversion index and the reciprocal of the elasticity of intertemporal substitution. E_t is the mathematical expectation conditional on time t information; $c_{\tau,s}$ and $m_{\tau,s}$ are time t consumption and real money balances. Their values depend on the realizations of still unspecified stochastic processes.

Introducing the hypothesis of constancy for the probability of death, we reformulate our objective as follows:

$$U_{t,s} = E_t\left(\int_t^{\infty} \frac{(c_{\tau,s}{}^{\alpha}\, m_{\tau,s}{}^{1-\alpha})^{1-R}}{1-R}\, e^{-(\rho+p)\,(\tau-t)} d\tau \right) \tag{1}$$

The representative firm produces an output flow dY_i by means of a stochastic Cobb-Douglas production function specified in the tradition of Romer (1986):

$$dY_i = \beta K_i{}^{\gamma} L_i{}^{1-\gamma} \bar{K}^{1-\gamma} (dt + \sigma_z dz) \tag{2}$$

where K_i is the stock of physical capital and L_i the labour managed by the i-th firm; $\bar{K}$ is the average stock of capital, β the inverse of the capital/output ratio and dz a standard Wiener process. Hence, a primary source of risk is technological uncertainty. Notice that, from now on, we drop the time indices whenever not confusing.

Competition forces firms to have the same capital/labour ratio; hence it is immediate to get the aggregate production function, where the inelastically supplied labour is normalised to one:

$$dY = \beta K\, (dt + \sigma_z dz)$$

Despite the simplicity of this formulation, the Romer-type production function (2), explicitly providing a productive role for labour, implies the existence of an asset, the claim on future stochastic labour income (human wealth), which is assumed to be non traded, due to an obvious moral-hazard problem.

The dynamic of the nominal money stock, M, is determined by a monetary authority. It is assumed that M follows a lognormal diffusion process:

$$d\,M = \mu_M M\, dt + M\, \sigma_M dz_M \tag{3}$$

where μ_M and σ_M are positive and constant over time. As it is common in this literature, we refer to a situation where the monetary authority is not able or willing to choose a deterministic path for the growth rate of the money stock. However, we will discuss the possibility that the central bank affects either σ_M or the covariance between the monetary policy process and the technological one (or both).

Since we imagine that there is no government expenditure nor debt, the public revenue implied by (3) is distributed to agents. As in Stulz, we accept the hypothesis of fully capitalised lump-sum transfers.

2.2 Individual choice problem

The consumption-portfolio decision is made more complex by the presence of human capital. However, a reward for non-accumulable production factors seems logically necessary on the ground of the considerations developed by Jones and Manuelli (1992): had we abstracted from human wealth, the newly born individual would have no income and could not take part in economic activity[1].

From the production function, we may write the return process for capital, which, at the individual level is:

$$dK_{t,s} = (\gamma\beta + p)\, K_{t,s}dt + \gamma\beta K_{t,s}\, \sigma_z dz \tag{4}$$

Notice that we introduce the hypothesis of a Blanchard-Yaari actuarially fair insurance mechanism, represented by the presence of p, and that we abstract from

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[1] Actually, in this framework a positive value for capitalized transfers is sufficient to allow the youngest cohort of people to "enter" into the economic system. However, the introduction of labour income allows for a comparison with a specification of the model where seignorage is used to finance public expenditure.

capital depreciation.

The human wealth process is[2]:

$$dH_{t,s} = \mu_H H_{t,s}\, dt + \sigma_H H_{t,s}\, dz_H - d\omega \tag{5}$$

where not only the implicit discount factor on human wealth, but also the stochastic process dz_H and the standard deviation σ_H are unknowns and must be endogenously determined. On the other hand, the characteristics of the wage process $d\omega = \mu_\omega dt + \sigma_\omega dz_\omega$ are entirely known; in fact, from (2), $\mu_\omega = (1-\gamma)\beta K$; $\sigma_\omega dz_\omega = (1-\gamma)\beta K\sigma_z dz$ and $\sigma_{\omega z} = (1-\gamma)\beta K\sigma_z^2$. (where, in general, $\sigma_{ij}=\sigma_i\sigma_j\rho_{ij}$ and ρ_{ij} is the correlation coefficient between variables i and j). In the finance jargon, we say that the wage process is spanned by the technological disturbance.

We assume that the stochastic process for the price of money in terms of goods, q_t, is a geometric brownian motion[3], $dq_t = \mu_q q_t dt + \sigma_q q_t dz_q$, and we compute the process for real money balances $m_{t,s} \equiv M_{t,s}\, q_t$[4]:

$$dm_{t,s} = (\mu_q + p)\, m_{t,s}\, dt + \sigma_q\, m_{t,s}\, dz_q \tag{6}$$

Notice that equation (6) implies that real money balances may be insured. As for the capitalised value of future lump-sum transfers ($v_{t,s}$) we postulate the

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[2] More precisely, equation (5) describes the evolution of human wealth of an individual born at s, contingent on that individual still being alive at t. A similar remark applies also for equation (7) that concerns the value of future transfers.

[3] As it is common in this literature, in equilibrium, it will turn out that this hypothesis is verified.

[4] Notice that the terms $qdM + dqdM$, representing the current flow of transfer to the individual, have been omitted. See Stulz, (1986, p. 338), Gertler and Grinols, (1982, p. 242) and Grinols and Turnovsky, (1993, p. 6).

following process:

$$dv_{t,s} = \mu_v \, v_{t,s} \, dt + \sigma_v \, v_{t,s} \, dz_v - d\tau_{t,s} \tag{7}$$

Since transfers are paid to living people, the premium p is absent from equation (7); however, the capitalised value of transfers is insured, although in a way which is different from the one postulated for capital and money. In fact, the individual signs a contract with an insurance company, according to which he will be paid the transfers proportional to the value $v_{t,s}$, irrespective of the fact that the individual might have bought part of these assets from people who are now dead. In the event of her death, the insurance company gets hold of $v_{t,s}$: the Poisson process for death and the law of large numbers ensure that insurance companies balance their budgets. Notice that μ_q, μ_v, σ_q, σ_v and the characteristics of the processes dz_q and dz_v must be endogenously determined.

Finally, it is convenient to introduce a nominal bond (B), which yields a nonstochastic nominal return r (hence, $dB = rBdt$). At the individual level, in real terms, we have:

$$db_{t,s} = (r + \mu_q + p) \, b_{t,s} \, dt + \sigma_q \, b_{t,s} \, dz_q \tag{8}$$

Defining total wealth for the generation-s representative individual as $W_{t,s} = K_{t,s} + m_{t,s} + v_{t,s} + b_{t,s} + H_{t,s}$, we now assume that $W_{t,s}$ is, apart from time, the sole arguments of the indirect utility function $J(.)$. Moreover, we follow Svensson and Werner, (1993), and we consider human wealth as if it were tradable in order to compute the implicit price at which agents would be willing to keep their amount of non traded asset. Consequently, we set up the optimal portfolio-consumption choice using the following Bellman equation:

$$0 = max_{\{c,m,K,H,v\}} \frac{(c_{t,s}{}^{\alpha} m_{t,s}{}^{1-\alpha})^{1-R}}{1-R} - (\rho + p) J + J_W E(dW_{t,s}) + \frac{J_{WW}}{2} E(dW_{t,s}{}^2) \tag{9}$$

where, using (4-8) and the wealth constraint:

$$dW_{t,s} = \{(r + p + \mu_q) W_{t,s} + (\gamma\beta - r - \mu_q) K_{t,s} - r m_{t,s} + (\mu_v - \mu_q - r - p) v_{t,s} +$$

$$+ (\mu_H - r - p - \mu_q) H_{t,s} - c_{t,s}\} dt + \sigma_q W_{t,s} dz_q + (\gamma\beta\sigma_z dz - \sigma_q dz_q) K_{t,s} +$$

$$+ (\sigma_H dz_H - \sigma_q dz_q) H_{t,s} + (\sigma_v dz_v - \sigma_q dz_q) v_{t,s} \tag{10}$$

and:

$$E(dW_{t,s}{}^2) = \sigma_q{}^2 W_{t,s}{}^2 + (\gamma^2\beta^2\sigma_z{}^2 + \sigma_q{}^2 - 2\gamma\beta\sigma_{zq}) K_{t,s}{}^2 + (\sigma_H{}^2 + \sigma_q{}^2 - 2\sigma_{qH}) H_{t,s}{}^2 +$$

$$+ (\sigma_v{}^2 + \sigma_q{}^2 - 2\sigma_{vq}) v_{t,s}{}^2 + 2(\gamma\beta\sigma_{zq} - \sigma_q{}^2) W_{t,s} K_{t,s} + 2(\sigma_{Hq} - \sigma_q{}^2) W_{t,s} H_{t,s} +$$

$$+ 2(\sigma_{vq} - \sigma_q{}^2) W_{t,s} v_{t,s} + 2(\gamma\beta\sigma_{zH} - \gamma\beta\sigma_{zq} - \sigma_{Hq} + \sigma_q{}^2) K_{t,s} H_{t,s} +$$

$$+ 2(\gamma\beta\sigma_{zv} - \gamma\beta\sigma_{zq} - \sigma_{vq} + \sigma_q{}^2) K_{t,s} v_{t,s} + 2(\sigma_{Hv} - \sigma_{Hq} - \sigma_{vq} + \sigma_q{}^2) H_{t,s} v_{t,s} \tag{11}$$

The first order conditions associated to (9) are:

$$(c_{t,s}{}^{\alpha} m_{t,s}{}^{1-\alpha})^{1-R} \frac{\alpha}{c_{t.s}} = - J_W \tag{12a}$$

$$(c_{t,s}{}^{\alpha} m_{t,s}{}^{1-\alpha})^{1-R} \frac{(1-\alpha)}{m_{t.s}} = - r J_W \tag{12b}$$

$$(\gamma\beta - r - \mu_q) J_w = - J_{WW} [(\gamma^2\beta^2\sigma_z{}^2 + \sigma_q{}^2 - 2\gamma\beta\sigma_{zq}) K_{t,s} + (\gamma\beta\sigma_{zq} - \sigma_q{}^2) W_{t,s} +$$

$$+ (\gamma\beta\sigma_{zH} - \gamma\beta\sigma_{zq} - \sigma_{Hq} + \sigma_q^2)\, H_{t,s} + (\gamma\beta\sigma_{zv} - \gamma\beta\sigma_{zq} - \sigma_{vq} + \sigma_q^2) v_{t,s}] \quad (12c)$$

$$(\mu_H - \mu_q - r - p)\, J_w = - J_{WW} [(\sigma_H^2 + \sigma_q^2 - 2\sigma_{Hq})\, H_{t,s} + (\sigma_{Hq} - \sigma_q^2)\, W_{t,s} +$$

$$+ (\gamma\beta\sigma_{zH} - \gamma\beta\sigma_{zq} - \sigma_{Hq} + \sigma_q^2)\, K_{t,s} + (\sigma_{Hv} - \sigma_{Hq} - \sigma_{vq} + \sigma_q^2) v_{t,s}] \quad 12d)$$

$$(\mu_v - \mu_q - r - p)\, J_w = - J_{WW} [(\sigma_v^2 + \sigma_q^2 - 2\sigma_{vq})\, v_{t,s} + (\sigma_{vq} - \sigma_q^2)\, W_{t,s} +$$

$$+ (\gamma\beta\sigma_{zv} - \gamma\beta\sigma_{zq} - \sigma_{vq} + \sigma_q^2)\, K_{t,s} + (\sigma_{Hv} - \sigma_{Hq} - \sigma_{vq} + \sigma_q^2)\, H_{t,s}] \quad (12e)$$

Equation (12a) states that the marginal utility of consumption must be equal to the marginal utility of wealth, while equations (12a-b) imply that the ratio between the marginal utility of money balances and the one of consumption should equal the nominal interest rate, two familiar results. Our simple specification for the instantaneous utility allows us to write the following money demand function:

$$m_{t,s} = \frac{c_{t,s}\,(1-\alpha)}{r\,\alpha} \quad (13)$$

while consumption may be expressed as:

$$c_{t,s} = \left(\frac{J_W}{\alpha}\right)^{-\frac{1}{R}} \left(\frac{1-\alpha}{\alpha\, r}\right)^{\frac{(1-R)(1-\alpha)}{R}} \quad (14)$$

The role of equations (12c-e) is to determine the excess returns of the risky assest K, H and v. Notice, however, that in this set-up the real value of nominal bonds is not risk-free since it is affected by movements in the price of money: it actually has the same risk characteristics that real money balances have.

At this stage it is convenient to guess a functional form for the individuals' maximum value function:

$$J(W_{t,s},t) = \frac{D(W_{t,s})^{1-R}}{1-R}\, e^{-(\rho + p)t} \tag{9'}$$

Notice that we rule out a direct age dependency of the maximum value function. The linearity in wealth which characterises the ratio J_W/J_{WW} when we adopt the functional form (9'), will make tractable the problem of aggregating the individual risk premia equations (12c-e).

2.3 The role of aggregate variables

To determine the distinguishing features of the stochastic processes dz_H, dz_q and dz_v it is necessary to resort to aggregate variables[5]:

$$\frac{dK}{K} = \left(\beta - \frac{C}{K}\right)dt + \beta\sigma_z\, dz \tag{15}$$

$$\frac{dH}{H} = \left(\mu_H - (1-\gamma)\,\beta\frac{K}{H}\right)dt + \sigma_H\, dz_H - (1-\gamma)\,\beta\frac{K}{H}\sigma_z\, dz \tag{16}$$

$$\frac{dm}{m} = (\mu_M + \mu_q + \sigma_{Mq})\, dt + \sigma_M\, dz_M + \sigma_q\, dz_q \tag{17}$$

$$\frac{dv}{v} = \left(\mu_v - (\mu_M + \sigma_{Mq})\,\frac{m}{v}\right)dt + \sigma_v\, dz_v - \sigma_M\, dz_M\, \frac{m}{v} \tag{18}$$

The absence of an explicit relation between time and individual first order conditions (12a-e) and the constancy of the demographic structure allow us to

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[5] Coherence of individual variables with economywide ones can be checked applying Blanchard's aggregation rule. (See Blanchard, (1985, p. 228))

posit that, the portfolio allocation, at the aggregate level, does not change over time. The constancy of portfolio shares implies, in its turn, that all the assets must grow at the same stochastic rate[6]. Hence, we equate the random parts of equations (15) and (16); (15) and (17); (17) and (18), to express the endogenous processes in terms of the exogenous ones, obtaining:

$$\sigma_H dz_H = \left(\beta + (1-\gamma)\, \beta \frac{K}{H}\right) \sigma_z dz$$

$$\sigma_q dz_q = \beta \sigma_z dz - \sigma_M dz_M$$

$$\sigma_v dz_v = \beta \sigma_z dz + \left(\frac{m}{v}\right) \sigma_M dz_M$$

Using the results in Table 1 to substitute out the endogenous stochastic components we get from equation (12c):

$$(\gamma\beta - r - \mu_q) W = R \{(\beta^2 (\gamma-1)^2 \sigma_z^2 + \sigma_M^2 + 2\beta(\gamma-1)\, \sigma_{zM}) K +$$

$$+ (\beta^2 (\gamma-1)\, \sigma_z^2 - \beta(\gamma-1)\, \sigma_{zM} + \beta\, \sigma_{zM} - \sigma_M^2) W +$$

$$+ \left[-\beta^2 (1-\gamma)^2 \frac{K}{H} \sigma_z^2 + \sigma_M^2 + \left(\beta\, (1-\gamma) \frac{K}{H} + \beta(\gamma-1)\right) \sigma_{zM}\right] H +$$

$$+ [\beta\, (\gamma-1)\, \sigma_{zM} + \sigma_M^2] \left(\frac{m}{v} + 1\right) v\}$$

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[6] Grinols and Turnovsky (1993, p. 16) and Turnovsky (1993, p. 962) develop similar argument to propose the same type of solution.

Table 1 - Variances and covariances for the endogenous processes.

$\sigma_H^2 = \beta^2\left(1+(1-\gamma)\frac{K}{H}\right)^2 \sigma_z^2$	$\sigma_{aH} = \beta^2\left(1+(1-\gamma)\frac{K}{H}\right)\sigma_z^2 - \beta\left(1+(1-\gamma)\frac{K}{H}\right)\sigma_{zM}$	$\sigma_{vH} = \beta^2\left(1+(1-\gamma)\frac{K}{H}\right)\sigma_z^2 + \beta\left(1+(1-\gamma)\frac{K}{H}\right)\frac{m}{v}\sigma_{Mz}$	$\sigma_{Hz} = \beta\left(1 + (1-\gamma)\frac{K}{H}\right)\sigma_z^2$
	$\sigma_a^2 = \beta^2\sigma_z^2 + \sigma_M^2 - 2\beta\sigma_{zM}$	$\sigma_{av} = \beta^2\sigma_z^2 + \beta\sigma_{zM}\left(\frac{m}{v} - 1\right) - \left(\frac{m}{v}\right)\sigma_M^2$	$\sigma_{az} = \beta\sigma_z^2 - \sigma_{Mz}$
		$\sigma_v^2 = \beta^2\sigma_z^2 + \left(\frac{m}{v}\right)^2\sigma_M^2 + 2\beta\sigma_{zM}\left(\frac{m}{v}\right)$	$\sigma_{vz} = \beta\sigma_z^2 + \left(\frac{m}{v}\right)\sigma_{zM}$

Similarly, from the excess return equation for human wealth:

$$(\mu_H - \mu_q - r - p)\ W = R\Bigg[\left(\beta^2\ (1-\gamma)^2 \frac{K^2}{H^2}\ \sigma_z^2 + \sigma_M^2 + 2\beta\,(1-\gamma)\ \frac{K}{H}\,\sigma_{zM}\right) H +$$

$$+ \left(\beta^2\ (1-\gamma)\ \frac{K}{H}\,\sigma_z^2 + \beta\sigma_{zM} - \beta\,(1-\gamma)\ \sigma_{zM}\frac{K}{H} - \sigma_M^2\right) W +$$

$$+ \left(-\beta^2\ (1-\gamma)^2\,\frac{K}{H}\,\sigma_z^2 + \beta\,(1-\gamma)\ \frac{K}{H}\,\sigma_{zM} + \sigma_M^2 + \beta\,(\gamma-1)\ \sigma_{zM}\right) K +$$

$$+ \left(\beta\,(1-\gamma)\ \frac{K}{H}\,\sigma_{zM} + \sigma_M^2\right)\left(\frac{m}{v} + 1\right) v\Bigg]$$

Finally, from the individual first order condition (12e):

$$(\mu_v - \mu_q - r - p)\ W = R\ \Bigg[\ \sigma_M^2 \left(\frac{m}{v} + 1\right)^2 v_t + (\beta\ \sigma_{zM} - \sigma_M^2)\left(\frac{m}{v} + 1\right)\ W +$$

$$+ (\beta(\gamma-1)\ \sigma_{zM} + \sigma_M^2)\left(\frac{m}{v} + 1\right) K + \left(\beta(1-\gamma)\ \frac{K}{H}\,\sigma_{zM} + \sigma_M^2\right)\left(\frac{m}{v} + 1\right) H\Bigg]$$

Simplifying where possible and exploiting the aggregate wealth constraint, $W=K+H+m+v$, we obtain, respectively:

$$(\gamma\beta - r - \mu_q)\ = R\ (\beta^2\ (\gamma - 1)\ \sigma_z^2 + \beta\ \sigma_{zM}) \tag{12c'}$$

$$(\mu_H - \mu_q - r - p)\ = R\left(\beta^2\ (1-\gamma)\ \frac{K}{H}\,\sigma_z^2 + \beta\ \sigma_{zM}\right) \tag{12d'}$$

$$(\mu_v - \mu_q - r - p)\ = R\ \left(\beta\ \sigma_{zM}\ \frac{m+v}{v}\right) \tag{12e'}$$

Notice from (12c') that the non stochastic part of the real interest rate, $\gamma\beta\text{-}R[\beta^2(\gamma\text{-}1)\sigma_z^2 + \beta\sigma_{zM}]$, is reduced by increments in the covariance between the monetary policy process and the technological disturbance[7]. This must be so because an increase in σ_{zM} reduces the covariance between the return on capital and that on real bonds. Hence bonds become more useful in hedging against risk on real capital, consequently the real interest rate is reduced.

Equating again equations (17-18) and using the drift components, we get:

$$\mu_q = \mu_v - (\mu_M + \sigma_{Mq})\left(\frac{m+v}{v}\right) = \mu_v - (\mu_M + \beta\sigma_{zM} - \sigma_M^2)\left(\frac{m+v}{v}\right) \tag{19}$$

Substituting (19) into (12c'-e') we reach an important intermediate result: we express the nominal rate, the (implicit) discount factor on human wealth and the return on capitalised transfers as functions of exogenous parameters and of ratios of wealth components, that are still to be determined endogenously.

$$r = \left(\frac{m+v}{v}\right)[\mu_M - \sigma_M^2 + \beta\sigma_{zM}(1-R)] - p \tag{20a}$$

$$\mu_H = \gamma\beta + p + R\left[(1-\gamma)\,\beta^2\,\sigma_z^2\left(\frac{K}{H}+1\right)\right] \tag{20b}$$

$$\mu_v = \gamma\beta + p + R\left((1-\gamma)\,\beta^2\,\sigma_z^2 + \beta\,\sigma_{zM}\,\frac{m}{v}\right) \tag{20c}$$

The covariance between the returns on the discounted flow of lump sum transfers and capital, being equal to $\gamma[\beta^2\sigma_z^2 + \beta\sigma_{zM}(m/v)]$, is increased by the covariance between the monetary policy process and the technological

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[7] If $\sigma_{zM} < \beta\,(1\text{-}\gamma)\,\sigma_z^2$, $r+\mu_q$ might even be higher than the non stochastic component of the return on capital.

disturbance: consequently an increase in σ_{zM} reduces the "hedging" capabilities of v and raises μ_v. As for the nominal interest rate, notice that it is linearly influenced by the term $(\mu_M + \beta\sigma_{zM} - \sigma_M^2)(m+v)/v$. However, the covariance σ_{zM} affects r because it improves the hedging capabilities of bonds against technological risk both directly (the term $R\beta\sigma_{zM}$) and indirectly, through its effect on v (the term $R\beta\sigma_{zM}(m/v)$). When $R>1$, such a risk aversion effect dominates, and the nominal interest rate is reduced by increments in the covariance between the technological disturbance and the monetary policy process, as shown by equation (20a).

2.4 Individual portfolio composition

We now examine the individual level again and we substitute equations (19) and (20a-c) into the first order conditions (12c-e)[8]:

$$[\beta^2 (\gamma-1)\, \sigma_z^2 + \beta\, \sigma_{zM}]\, W_{t,s} = [\beta^2 (1-\gamma)^2\, \sigma_z^2 + \sigma_M^2 - 2\beta(1-\gamma)\, \sigma_{zM}]\, K_{t,s} +$$

$$+ [\beta^2 (\gamma-1)\, \sigma_z^2 + (2\beta - \gamma\beta)\, \sigma_{zM} - \sigma_M^2]\, W_{t,s} +$$

$$+ \left(- \beta^2 (1-\gamma)^2\, \sigma_z^2\, \frac{K_t}{H_t} + \beta(1-\gamma)\, \sigma_{zM}\, \frac{K_t}{H_t} + \sigma_M^2 + \beta(\gamma-1)\, \sigma_{zM}\right) H_{t,s} +$$

$$+ \, (\beta(\gamma-1)\, \sigma_{zM} + \sigma_M^2) \left(\frac{m_t + v_t}{v_t}\right) v_{t,s}$$

$$\left(\beta^2 (1-\gamma)\, \frac{K_t}{H_t}\, \sigma_z^2 + \beta\, \sigma_{zM}\right) W_{t,s} =$$

[8] The subscript t is used to help identify aggregate variables

$$= \left(\beta^2 (1-\gamma)^2 \frac{K_t^2}{H_t^2} \sigma_z^2 + \sigma_M^2 + 2\beta (1-\gamma) \frac{K_t}{H_t} \sigma_{zM}\right) H_{t,s} +$$

$$+ \left(\beta^2 (1-\gamma) \frac{K_t}{H_t} \sigma_z^2 + \beta\sigma_{zM} - \beta(1-\gamma)\, \sigma_{zM} \frac{K_t}{H_t} - \sigma_M^2\right) W_{t,s} +$$

$$+ \left(-\beta^2 (1-\gamma)^2 \frac{K}{H} \sigma_z^2 + \beta(1-\gamma) \frac{K_t}{H_t} \sigma_{zM} + \sigma_M^2 + \beta(\gamma-1)\, \sigma_{zM}\right) K_{t,s} +$$

$$+ \left(\beta (1-\gamma) \frac{K_t}{H_t} \sigma_{zM} + \sigma_M^2\right)\left(\frac{m_t + v_t}{v_t}\right) v_{t,s}$$

$$\beta\, \sigma_{zM} \frac{v_t + m_t}{v_t} W_{t,s} = \sigma_M^2 \left(\frac{m_t + v_t}{v_t}\right)^2 v_{t,s} + (\beta\sigma_{zM} - \sigma_M^2)\left(\frac{m_t + v_t}{v_t}\right) W_{t.s} +$$

$$+ \ (\beta(\gamma-1)\, \sigma_{zM} + \sigma_M^2) \left(\frac{m_t + v_t}{v_t}\right) K_{t,s} + \left(\beta(1-\gamma) \frac{K_t}{H_t} \sigma_{zM} + \sigma_M^2\right)\left(\frac{m_t + v_t}{v_t}\right) H_{t,s}$$

Through some manipulations and exploiting the definitions of aggregate and individual wealth we may reformulate these expressions in terms of deviations between individual and average variables[9]:

$$[\sigma_M^2 - \beta(1-\gamma)\, \sigma_{zM}]\, (W_{t,s} - W_t) =$$

$$= (\sigma_M^2 + \beta(\gamma-1)\, \sigma_{zM}) \left(K_{t.s} - K_t + H_{t.s} - H_t + \frac{v_t + m_t}{v_t} (v_{t.s} - v_t)\right) +$$

$$+ (\beta^2 (1-\gamma)^2 \sigma_z^2 + \beta(\gamma-1)\, \sigma_{zM}) \left(K_{t.s} - \frac{K_t}{H_t} H_{t.s}\right) \qquad (21a)$$

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[9] The demographic structure of Blanchard's model implies that aggregate variables formally coincide with average ones.

$$\left(\sigma_M^2 + \beta(1-\gamma)\frac{K_t}{H_t}\sigma_{zM}\right)(W_{t.s} - W_t) =$$

$$= \left(\beta(1-\gamma)\frac{K_t}{H_t}\sigma_{zM} + \sigma_M^2\right)\left(K_{t.s} - K_t + H_{t.s} - H_t + \frac{v_t + m_t}{v_t}(v_{t.s} - v_t)\right) +$$

$$+ \left(\beta^2(1-\gamma)^2\frac{K_t}{H_t}\sigma_z^2 + \beta(1-\gamma)\sigma_{zM}\right)\left(H_{t.s}\frac{K_t}{H_t} - K_{t.s}\right) \tag{21b}$$

$$\sigma_M^2(W_{t.s} - W_t) = \sigma_M^2\left(K_{t.s} - K_t + H_{t.s} - H_t + \frac{m_t + v_t}{v_t}(v_{t,s} - v_t)\right) +$$

$$+ \beta(\gamma - 1)\sigma_{zM}\left(K_{t.s} - \frac{K_t}{H_t}H_{t.s}\right) \tag{21c}$$

The demand for real money balances is given by equations (13-14), hence the remaining unknowns are $K_{t,s}$, $H_{t,s}$, $v_{t,s}$ and $b_{t,s}$. However, we may notice that the difference between equation (21c) and equation (21a) is equal to the difference between equation (21b) and equation (21c) multiplied by the ratio K_t/H_t. This linear dependence is not surprising: in the model there are two Wiener processes (dz and dz_M), which are able to determine the demand functions for only two assets. Hence, there is a degree of indeterminacy in the optimum portfolio. At this stage it seems natural to assume $H_{t,s}=H_t$, forcing people to hold their entire amount of the non tradable asset.

The system composed by equations (21a)-(21b) reduces to:

$$[\beta^2(1-\gamma)^2\sigma_z^2+\sigma_M^2-2\beta(1-\gamma)\sigma_{zM}](K_{t,s}-K_t) + [\beta(1-\gamma)\sigma_{zM}-\sigma_M^2]\left(\frac{m_t+v_t}{v_t}\right)(v_t-v_{t,s}) =$$

$$= [\beta(1-\gamma)\sigma_{zM}-\sigma_M^2](W_t-W_{t,s}) \tag{22a}$$

$$[\beta(\gamma-1)\,\sigma_{zM} + \sigma_M^2]\,(K_t - K_{t,s}) + \sigma_M^2\left(\frac{m_t + v_t}{v_t}\right)(v_t - v_{t,s}) = \sigma_M^2\,(W_t - W_{t,s}) \quad \text{(22b)}$$

We immediately notice that the vector coefficient for v_t-$v_{t,s}$ in system (22a-b) is a multiple of the vector coefficient for W_t-$W_{t,s}$; hence we get that the desired capital, $K_{t.s}^*$, is equal to K_t and that $v_{t.s}^* = v_t - \frac{v_t}{m_t + v_t}(W_t - W_{t,s})$. Substitution into the individual-s wealth constraint yields also an expression for the real value of bonds holdings: $b_{t.s}^* = m_t - m_{t,s} - \left(\frac{m_t}{v_t + m_t}\right)(W_t - W_{t,s})$. Hence, the optimal investment strategy for an agent is to hold the mean capital stock and to finance this investment by selling his transfers and issuing bonds, when his wealth is less than average.

To grasp some intuition for these portfolio choices, consider the monetary wealth of an individual born at time s, i.e. $m_{t,s} + v_{t.s}^* + b_{t.s}^*$, and compute its variance. Calculations shows that it turns out to be: $[v_t + m_t - (W_t - W_{t,s})]^2 \beta^2 \sigma_z^2$. Hence, the choice of $v_{t.s}^*$ and $b_{t.s}^*$ is effective in insulating the individual from monetary randomness; moreover, if we add such a portfolio to K_t and H_t, we obtain an individual wealth allocation whose variance cannot be reduced further, being $W_{t,s}^2 \beta^2 \sigma_z^2$.

Had we dropped the hypothesis of non-traded human wealth, we could have worked out an alternative optimal portfolio with the same variance.

2.5 Wealth shares and the Tobin effect

To solve the model, we still need to determine the constant of the maximum value function and the wealth shares. We start with the computation of D and we use our guess (9') to simplify the first order conditions (13-14); we substitute them back into the Bellman equation (9), and we get:

$$0 = \left(\frac{D}{\alpha}\right)^{(R-1)/R} \frac{W_{t,s}^{(1-R)}}{1-R} \left(\frac{1-\alpha}{\alpha r}\right)^{(1-\alpha)(1-R)/R} - (p+\rho)\,\frac{D W_{t,s}^{(1-R)}}{1-R} +$$

$$+ DW_{t,s}^{-R}\, E\,(dW_{t,s}) + \frac{1}{2} D\, W_{t,s}^{-(1+R)}\, E(dW_{t,s}^{2})$$

In the Appendix, we compute:

$$E\,(dW_{t,s}) = (r + p + \mu_q)\, W_{t,s} + R\beta\sigma_{zM}\, W_{t,s} - r\, m_{t,s} - c_{t,s}$$

$$E\,(dW_{t,s}^{2}) = \beta^2 \sigma_z^2 W_{t,s}^2$$

Hence, we determine D and therefore, via equations (13-14) the consumption function and the demand for real money balances:

$$D^{-1/R} = \left(\frac{1-\alpha}{\alpha r}\right)^{(1-\alpha)(1-R)/R} \alpha^{(R-1)/R}\, \frac{1}{R} \left\{p + \rho - (1-R)\left[\gamma\beta + p - R\,\beta^2\sigma_z^2\left(\gamma - \frac{1}{2}\right)\right]\right\}$$

$$c_{t,s} = \alpha\, W_{t,s} \left(\frac{p + \rho - (1-R)[\gamma\beta + p - R\,\beta^2\sigma_z^2\,(\gamma - 0{,}5)]}{R}\right) \tag{23a}$$

$$m_{t,s} = \frac{1-\alpha}{r}\, W_{t,s} \left(\frac{p + \rho - (1-R)[\gamma\beta + p - R\,\beta^2\sigma_z^2\,(\gamma - 0.5)]}{R}\right) \tag{23b}$$

Equation (23a) shows that the presence of a monetary policy shocks does not affect the propensity to consume out of wealth; hence monetary policy variability may affect real variables only by altering the evaluation of wealth. This result is ensured by the fact that the computed portfolio allocation implied that the variance of wealth is unaffected by the monetary policy parameters. Equation (23b) is the stochastic version of a familiar result. Notice that D is actually age independent, hence our guess is correct and the aggregate versions of equations (23a-b) can be immediately obtained.

This is very useful, since most of the results obtained so far depend on the values of aggregate wealth components. Hence, exploiting the consumption function and the demand for real balances, we may set up a system of non-linear equations to determine wealth shares and therefore to obtain a complete solution of the model.

Let $\theta_1=K_t/W_t$; $\theta_2=H_t/W_t$; $\theta_3=m_t/W_t$; $\theta_4=v_t/W_t$ and define, for convenience:

$$D_0 = \frac{p + \rho - (1-R)[\gamma\beta + p - R\,\beta^2\sigma_z^2\,(\gamma - 0.5)]}{R}$$

Equating the non stochastic parts of equations (15)-(16), using (20b) to substitute out μ_H, and the aggregate version of (23a) to express consumption, we get the first equation of our system:

$$\beta - \frac{\alpha D_0}{\theta_1} = \gamma\beta + p + R\left[(1-\gamma)\,\beta^2\sigma_z^2\left(\frac{\theta_1+\theta_2}{\theta_2}\right)\right] - (1-\gamma)\beta\frac{\theta_1}{\theta_2} \qquad \text{(24a)}$$

The second equation of the system is obtained from (16) and (18), exploiting our formulation for μ_H, μ_q, μ_v and for σ_{Mq}:

$$R\left((1-\gamma)\,\beta^2\sigma_z^2\frac{\theta_1}{\theta_2} - \beta\sigma_{zM}\frac{\theta_3}{\theta_4}\right) = -\,[\mu_M - \sigma_M^2 + \beta\sigma_{zM}]\,\frac{\theta_3}{\theta_4} + (1-\gamma)\,\beta\,\frac{\theta_1}{\theta_2} \qquad \text{(24b)}$$

From the aggregate version of the demand for real money balances (23b), using the definition of the nominal interest rate, we get:

$$\left([\mu_M - \sigma_M^2 + \beta\sigma_{zM}(1-R)]\,\frac{\theta_3+\theta_4}{\theta_4} - p\right)\theta_3 = (1-\alpha)\,D_0 \qquad \text{(24c)}$$

while the fourth equation is given simply by the aggregate wealth constraint:

$$\Sigma_i\,\theta_i = 1 \qquad \text{(24d)}$$

Collecting terms, we may conveniently reformulate as follows the first two equations of the system:

$$\beta(1-\gamma)\left(1+\frac{\theta_1}{\theta_2}\right)(1-R\,\beta\sigma_z^2) = p + \frac{\alpha D_0}{\theta_1} \tag{24a'}$$

$$\beta(1-\gamma)\frac{\theta_1}{\theta_2}(1-R\,\beta\sigma_z^2) = \frac{\theta_3}{\theta_4}(\mu_M - \sigma_M^2 + \beta\sigma_{zM}(1-R)) \tag{24b'}$$

System (24a'-b'-c-d) is highly nonlinear and some further manipulations are needed. Let:

$$D_1 = \mu_M - \sigma_M^2 + \beta\sigma_{zM}(1-R)$$

$$D_2 = \beta(1-\gamma)(1-R\,\beta\sigma_z^2)$$

Substituting $\theta_1/\theta_2\,D_2$ from the (24b') into (24a') we may express the capital to wealth ratio as a function of parameters and of the ratio beween the value of real money and the one of future transfers:

$$\theta_1 = \frac{\alpha D_0}{D_2 - p + D_1\,\theta_3/\theta_4}$$

(24b') may be reformulated as:

$$\theta_2 = \frac{D_2\,\theta_1}{D_1\,\theta_3/\theta_4}$$

As $D_2>0$, also θ_2 must be positive. Hence, we will restrict our attention to the solution where $D_1\theta_3/\theta_4>0$.

We now substitute

$$\theta_1 + \theta_2 = \frac{\alpha D_0(D_2 + D_1\,\theta_3/\theta_4)}{D_1\,\theta_3/\theta_4\,(D_2 - p + D_1\,\theta_3/\theta_4)}$$

into (24d) to get:

$$1 = \frac{\alpha D_0 (D_2 + D_1\, \theta_3/\theta_4)}{D_1\, \theta_3/\theta_4\, (D_2 - p + D_1\, \theta_3/\theta_4)} + \theta_3 + \theta_4 \tag{25}$$

Notice that equation (24c) and (25) now form a non linear system of two equations and two unknowns.

For convenience, let $\theta_3/\theta_4 = y$. Equation (24c) and (25) become, respectively:

$$\theta_4 = \frac{(1-\alpha)D_0}{(D_1(y+1)-p)\, y}$$

$$1 = \frac{\alpha D_0 (D_2 + D_1 y)}{D_1 y\, (D_2 - p + D_1 y)} + \frac{(y+1)\,(1-\alpha)D_0}{(D_1\,(y+1)-p)\, y} \tag{26}$$

This expression has only one meaningful solution[10]. In fact, yD_1 must be non-negative to ensue positivity of the ratio between human and total wealth; moreover, notice that $D_1\,(y+1) - p$ is the nominal interest rate (equation (20a)). Hence, this expression can not be negative. Let $u = yD_1$ to get, from equation (26):

$$u = D_0 \left(1 + \frac{\alpha p}{D_2 + u - p} + \frac{(1-\alpha)p}{D_1 + u - p} \right) \tag{26'}$$

The left-hand side of (26') is increasing in u, while the right-hand side is decreasing. When D_1-p>0, both of the two asymptotes for the right hand side have negative abscissa and the equilibrium value for u is shown in figure 1a. When D_1-p<0, there is an asymptote with positive abscissa and the unique equilibrium value with u, $r > 0$ lies to its right. (See figure 1b).

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[10] Notice that, if $p=0$, $\frac{\theta_3}{\theta_4} = \frac{D_0}{D_1}$ and $\theta_1 = \frac{\alpha D_0}{D_2+D_0}$; $\theta_2 = \frac{\alpha D_2}{D_2+D_0}$; $\theta_3 = \frac{(1-\alpha)D_0}{D_0 + D_1}$ and $\theta_4 = \frac{(1-\alpha)D_1}{(D_0 + D_1)}$. The ratio of monetary to non monetary wealth is $(1-\alpha)/\alpha$, the result guessed by Stulz, (1986, p. 339).

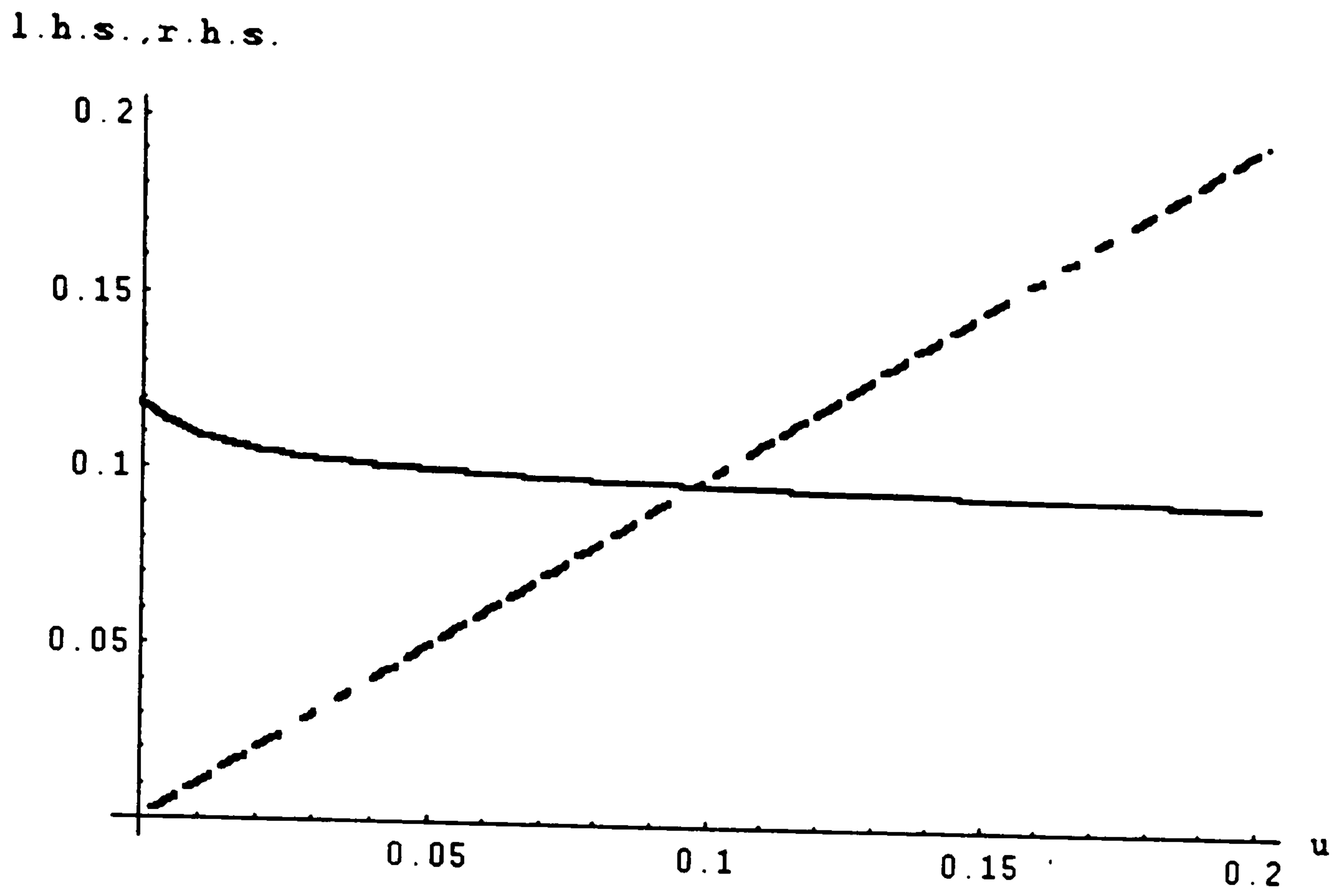

Figure 1.a

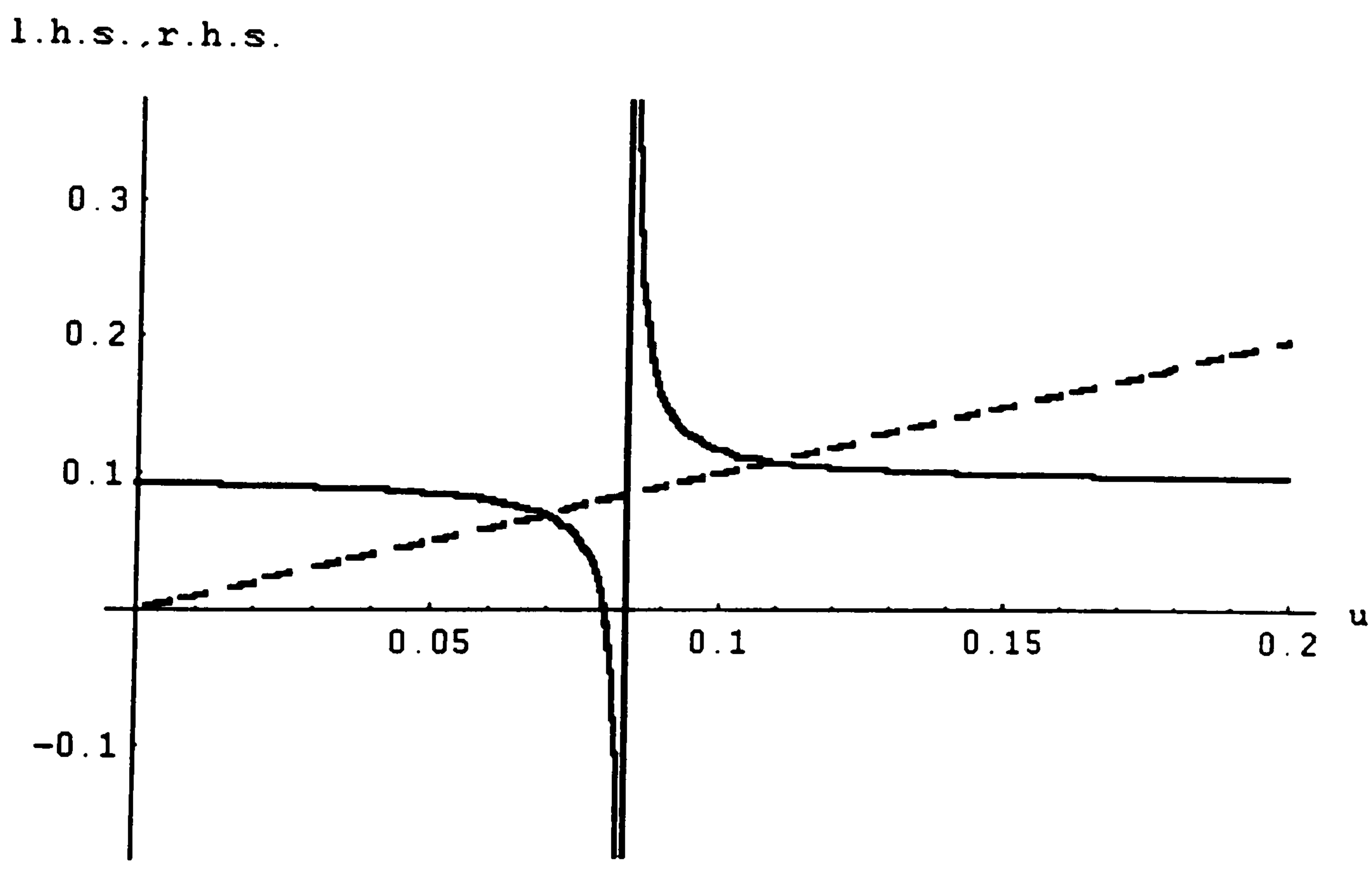

Figure 1.b

Figure 1 plots the left-hand side (dashed line) and the right-hand side (continuous line) of (26'). Figure 1.a considers the case D_1-p>0, when both of the two asymptote for the r.h.s. have negative abscissa; when D_1-p<0, there is an asymptote with positive abscissa and the unique equilibrium value with u, r > 0 lies at its right. (Figure 1b)

The Tobin effect is present if an increase in the nominal interest rate fosters a reduction in consumption for a given level of capital, therefore accelerating growth. In our framework, such a contraction in the aggregate consumption to capital ratio may take place only if θ_1 increases[11]; for this to happen, since θ_1 may be written as $\frac{\alpha D_0}{D_2 - p + u}$, a reduction in u is necessary (and sufficient). Therefore, we need to compute $\partial u/\partial D_1$. Some algebraic manipulations allow us to write:

$$\frac{\partial u}{\partial D_1} = - \frac{(1-\alpha)p(D_2+u-p)^2 D_0}{(D_2+u-p)^2(D_1+u-p)^2 + pD_0[\alpha(D_1+u-p)^2 + (1-\alpha)(D_2+u-p)^2]} < 0$$

If p=0, we are back to Stulz's model and $\partial u/\partial D_1 = 0$, as expected. Notice also that $\partial r/\partial D_1 \equiv \partial u/\partial D_1 + 1 > 0$, since, from the expression above, $\partial u/\partial D_1 > -1$. Hence, an increase in the nominal interest rate always goes together with a reduction in the consumption to capital ratio: the Tobin effect is established. An expansion in the expected nominal money growth rate, μ_M, increasing D_1, depresses consumption. If σ_M^2 increases, on the contrary, the effect on aggregate consumption is positive. The role of the covariance between the monetary policy process and the technological disturbance is less definite. In general, its increase is detrimental for growth if $(1-R)\sigma_{zM}$ is negative. In fact, when $R>1$, the nominal interest rate is reduced by increments in the covariance between the technological disturbance and the monetary policy process. (See equation (20a)).

3. Policy implications and conclusion

Our framework, despite the hypothesis of non distortionary taxes and transfers, delivers a strong monetary uncertainty relevance result. The finiteness of agents'

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[11] From equation (23a), we may formulate aggregate consumption as: $c_t = \alpha D_0 K_t/\theta_1$.

lifetime implies that consumption and growth are affected by the nominal interest rate, which, depends on the stochastic characteristic of the monetary disturbance. In its turn, the impact of monetary uncertainty on the nominal interest rate cannot be ascribed to a pure "convexity" effect unrelated to agents preferences. Even if we relate the definition of "monetary uncertainty" to the inverse of the nominal money stock, as in Rankin (1994, p. 129), monetary randomness, for $R \neq 1$, matters. Let $L=1/M$; we immediately get, through Ito's lemma, that $\mu_L=\mu_M - \sigma_M^2$ and that $\sigma_L dz_L = \sigma_M dz_M$. Hence, the nominal interest rate becomes $r=(m+v)/v[\mu_L+\beta\sigma_{zL}(1-R)] - p$ and the insurance effect entailed by the covariance between the technological disturbance and the monetary policy process still affects the nominal interest rate. Obviously, the existing literature failed to highlight this effect in consequence of the use of logarithmic preferences.

Notice that the covariance effect allows the government to raise the expected seignorage, i.e. $m(\mu_M-\sigma_M^2+\beta\sigma_{zM})$, without altering the nominal interest rate. Since $r = (m+v)/v[\mu_M-\sigma_M^2+\beta\sigma_{zM}(1-R)] - p$, if the correlation between the technological disturbance and the monetary policy process is positive, this goal may be attained increasing σ_M or ρ_{zM} (if possible) in order to let the term $R\beta\sigma_{zM}$ to offset the expansion in $(\mu_M-\sigma_M^2+\beta\sigma_{zM})$. If, on the contrary, ρ_{zM} is negative, the expected seignorage may be increased without augmenting the nominal interest if the correlation coefficient may be set closer to zero, or if σ_M may be reduced. In a better specified model the seignorage coming from our insurance effect might be used to finance public expenditure or the service of government debt instead of being handed back to consumers[12].

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[12] For the "insurance effect" to be present in a model where taxes are levied on wealth (and therefore distortionary, as in Grinols and Turnovsky, (1993, p. 7) and Turnovsky, (1993, p. 957)) it is necessary to include also stochastic government transfers.

In this model, the impact of the interest rate on welfare is twofold: an increase in r reduces the demand for real money balances (equation 23b) and hence utility, but it reduces consumption and speeds up growth. This second effect is welfare enhancing, since the externality characterising the production function (2) entails a sub-optimal growth rate[13].

Hence, if we assume that it is optimal to reduce the nominal interest rate, as it would be with infinitely-lived individuals (see e.g. Fischer, (1979)), we may try to assess the effectiveness of some alternative central banking operating procedures in keeping low the nominal interest rate, for a given μ_M. In our framework, nominal money targeting implies that the monetary authority should try to reduce σ_M as much as possible, while inflation targeting can be translated into the minimisation of the variance of prices.

If we suppose that the central bank may influence only the correlation between the technological process and the monetary one, under nominal money targeting no action is taken, while under inflation targeting ρ_{zM} is maximised and therefore set equal to one. If the risk aversion index is above unity, under the first procedure, r is higher while the expected seignorage is lower.

If we suppose, on the other hand, that ρ_{zM} is exogenous but the standard deviation of nominal money is entirely under control, nominal money targeting leads to an interest rate equal to μ_M. Inflation targeting implies that σ_M should be set to $\beta\sigma_z\rho_{zM}$ if the correlation between the two disturbances is positive and to zero otherwise. In this case, if $\rho_{zM}>0$ and $R>1$, calculations show again that the nominal interest rate is always higher under nominal money targeting while the expected seignorage is the same. If the two stochastic processes are independent or

[13] It is easy to compute, by means of numerical methods, the optimal interest rate, that is positive but very close to zero.

correlated, the two central banking procedures leads to the same results.

If both ρ_{zM} and σ_M may be freely chosen, the nominal interest rate is again equal to μ_M under nominal money targeting, while it becomes μ_M-$R\beta^2\sigma_z^2$ under inflation targeting. Hence, this second procedure is always to be preferred, if the monetary authority aims at reducing r[14].

In conclusion, we showed that monetary policy uncertainty, when preferences are of the C.R.R.A type, affect the nominal interest rate also through an "insurance effect" which depends on the degree of risk-aversion and on the covariance between the technological process and the monetary policy disturbance. Hence, with finite lives, monetary uncertainty, through a Tobin-effect involving the nominal interest rate, affects consumption and growth.

However, even with infinite lives, our "insurance effect" may be important, since it allows for a reduction of the nominal interest rate, given the expected seignorage. In particular, at the level of central banks operating procedures, this framework suggests that inflation targeting provides a performance that is better than the one of nominal money targeting.

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[14] This is the case studied by Canzoneri and Dellas (1995, sec. II in particular). Their framework is a cash-in-advance endowment economy and they show that money targeting is effective, as opposed to nominal interest rate targeting, in reducing r if $R>1$. However, when capital accumulation is taken into account and agents rationally operate portfolio choices, results should be different. Intuition suggests that, while in an endowment economy monetary policy directly affects the marginal utility of consumption, when capital accumulation plays an important role monetary policy influences the marginal utility of future consumption and portfolio effects may dominate.

Appendix: Characterisation of the individual wealth process

To obtain $E(dW_{t,s})$ we start from (10) and we substitute out equations (12c'-e') to get:

$$E(dW_{t,s}) = (r+p+\mu_q)W_{t,s} + R(\beta^2(\gamma-1)\sigma_z^2 + \beta\sigma_{zM})K_{t,s} + R\left(\beta\sigma_{zM}\frac{m_t+v_t}{v_t}\right)v_{t,s}+$$

$$+ R\left(\beta^2(1-\gamma)\frac{K_t}{H_t}\sigma_z^2+\beta\sigma_{zM}\right)H_{t,s} - rm_{t,s} - c_{t,s}dt$$

Hence, substituting the wealth shares computed in section 2.4, we obtain:

$$E(dW_{t,s}) = (r + p + \mu_q)\; W_{t,s} + R\beta\sigma_{zM}\, W_{t,s} - r\, m_{t,s} - c_{t,s}$$

To simplify $E(dW_{t,s}{}^2)$ from (11) we substitute out variances and covariances of the endogenous processes to get:

$$E\,(dW_{t,s})^2 =$$

$$= (\beta^2\,\sigma_z^2 + \sigma_M^2 - 2\beta\sigma_{zM})\; W_{t,s}{}^2 + [\beta^2(\gamma-1)^2\,\sigma_z^2 + \sigma_M^2 + 2\beta(\gamma-1)\sigma_{zM}]K_{t,s}{}^2 +$$

$$+ \left(\beta^2(1-\gamma)^2\,\frac{K^2}{H^2}\,\sigma_z^2 + \sigma_M^2 + 2\beta(1-\gamma)\,\frac{K}{H}\,\sigma_{zM}\right)H_{t,s}{}^2 + \left(\frac{m}{v}+1\right)^2\,\sigma_M^2\,v_{t,s}{}^2 +$$

$$+ 2\,\{[\beta^2\,(\gamma-1)\,\sigma_z^2 - \beta(\gamma-1)\,\sigma_{zM} + \beta\sigma_{zM} - \sigma_M^2]\;W_{t,s}\,K_{t,s}\,+)$$

$$+\left(\beta^2(1-\gamma)\,\frac{K}{H}\,\sigma_z^2 - \beta(1-\gamma)\,\frac{K}{H}\,\sigma_{zM} + \beta\sigma_{zM} - \sigma_M^2\right)W_{t,s}\,H_{t,s} +$$

$$+ (\beta\,\sigma_{zM} - \sigma_M^2)\left(\frac{m}{v}+1\right)W_{t,s}\,v_{t,s} +$$

$$+\left[-\beta^2 (1-\gamma)^2 \frac{K}{H}\sigma_z^2 + \sigma_M^2 + \left(\beta(1-\gamma)\ \frac{K}{H} + \beta(\gamma-1)\right)\sigma_{zM}\right] H_{t,s}K_{t,s} +$$

$$+ [\beta\sigma_{zM}\ (\gamma-1) + \sigma_M^2]\left(\frac{m}{v}+1\right) K_{t,s}\, v_{t,s} + \left(\beta\,(1-\gamma)\ \frac{K}{H}\sigma_{zM} + \sigma_M^2\right)\left(\frac{m}{v}+1\right)\Bigg\}\ H_{t,s}\, v_{t,s}$$

Exploiting again the assumption concerning the impossibility to trade human capital ($H_{t,s} = H_t$) and the fact that $K_{t,s}^* = K_t$, by algebric manipulations, we get:

$$E\,(dW_{t,s})^2 =$$

$$= (\beta^2\,\sigma_z^2 + \sigma_M^2 - 2\beta\sigma_{zM})\ W_{t,s}^2 + \sigma_M^2\,(K+H)^2 + 2(\beta\sigma_{zM} - \sigma_M^2)W_{t,s}(K+H) +$$

$$+ \sigma_M^2\left(\frac{m}{v} + 1\right)^2 v_{t,s}^2 + 2(\beta\sigma_{zM} - \sigma_M^2)\left(\frac{m}{v}+1\right)W_{t,s}\, v_{t,s} + 2\sigma_M^2\left(\frac{m}{v}+1\right)v_{t,s}\,(K+H).$$

Using also the definition of $v_{t,s}^*$, we get the final result:

$$E\,(dW_{t,s})^2 = \beta^2\sigma_z^2 W_{t,s}^2$$

implying that $E\,(dW_{t,s})^2$ is not affected by aggregate variables.

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