Some examples of the spatial evolution of

two-parameter processes with non-adapted initial

conditions.

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Declaration

Unless otherwise stated, the material in this thesis is, to the best of the author's knowledge, original work under the supervision of Dr. Sigurd Assing. It has not previously been submitted for any other degree or diploma.

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Abstract

The central result of this thesis is an enlargement of filtrations result for the filtration $(\mathscr{F}_x; x \geq 0)$, where

$$\mathscr{F}_x = \sigma\{B_{ys} : y \leq x, s \in [0, \infty)\}$$

and $(B_{xt}; x \in \mathbb{R}, t \in [0, \infty))$ is a Brownian sheet on a complete probability space. Although this is a fairly straightforward extension of a result presented in [Yor97] for Brownian filtrations, it is of use to us in a couple of applications. The first is a discussion of 'bridged' Brownian sheets, in which we try to describe the law of a Brownian sheet which is fixed along some curve in the parameter space. The second application is a study of the spatial evolution of solutions to the stochastic heat equation. We fix a starting point in space, and describe the spatial evolution as driven by an $(\mathscr{F}_x; x \geq 0)$-adapted noise. Unfortunately, we find that the initial condition is not in $\mathscr{F}_0$. If we add this initial information to $(\mathscr{F}_x; x \geq 0)$, the driving noise is no longer a martingale, but our enlargement result allows us to write a semimartingale decomposition, in some sense. We are in fact able to write a system of stochastic differential equations which describe the spatial evolution of solutions, such that each equation is driven by a martingale with respect to this larger filtration.

0.1 Introduction

0.1.1 An outline of the thesis.

The principle motivation for this thesis is the study of the stochastic heat equation

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t) + \frac{\partial^2}{\partial x \partial t}B_{xt}, \qquad x \in \mathbb{R},\ t \in [0,\infty)$$
$$u(x,0) = u_0(x) \qquad\qquad \forall\, x \in \mathbb{R} \qquad\qquad (0.1.1)$$

where $(B_{xt}; (x,t) \in \mathbb{R} \times [0,\infty))$ is a Brownian sheet on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. There is much in the literature written about the Markov property of solutions for (0.1.1). Much of this is related to the study of Gaussian random fields, of which [Pit71] and [Roz82] provide a good overview. Recall that a random field $(X(t); t \in \mathbb{R}^n)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is Gaussian if for each finite subset $\{t_1, \ldots, t_k\}$ of $\mathbb{R}^n$ and all $\alpha \in \mathbb{R}^n$, $\sum_{i=1}^{k} \alpha_i X(t_i)$ is a Gaussian random variable. We make the following definitions:

Definition 0.1. *Let $\mathscr{T}$ and $\mathscr{U}$ be two sub-σ-algebras of $\mathscr{F}$. A σ-algebra $\mathscr{S} \subset \mathscr{T}$ is a splitting field for $\mathscr{T}$ and $\mathscr{U}$ if for all bounded $\mathscr{T}$-measurable random variables f and all bounded $\mathscr{U}$-measurable random variables g, $\mathbb{E}(fg|\mathscr{S}) = \mathbb{E}(f|\mathscr{S})\mathbb{E}(g|\mathscr{S})$.*

We remark that this is equivalent to having $\mathbb{E}(g|\mathscr{T}) = \mathbb{E}(g|\mathscr{S})$ for all bounded $\mathscr{U}$-measurable g. Indeed, taking $A \in \mathscr{T}$ and $f = \mathbb{1}_A$, the condition in the defintion becomes $\mathbb{E}[\mathbb{1}_A g|\mathscr{S}] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[g|\mathscr{S}]|\mathscr{S}]$, from which it follows that $\mathbb{E}(g|\mathscr{T}) = \mathbb{E}(g|\mathscr{S})$. For the equivalence, note that if we multiply $\mathbb{E}(g|\mathscr{T}) = \mathbb{E}(g|\mathscr{S})$ through by $\mathbb{1}_A$ and take conditional expectations with respect to $\mathscr{S}$ we obtain $\mathbb{E}(fg|\mathscr{S}) = \mathbb{E}(f|\mathscr{S})\mathbb{E}(g|\mathscr{S})$ for every $f = \mathbb{1}_A$ with $A \in \mathscr{T}$, and the same expression follows by approximation for every bounded, $\mathscr{T}$ measurable f.

For a random variable X on $(\Omega, \mathscr{F}, \mathbb{P})$, we use the notation $\sigma(X)$ to donate the smallest sub-σ-algebra of $\mathscr{F}$ with respect to which makes X measurable, whilst $\sigma(X(t); t \in T)$ denotes the smallest σ-algebra making each $X(t)$ measurable for every t in some indexing set T.

Definition 0.2. *Let $\{X(t) : t \in \mathbb{R}^n\}$ be a stochastic process. If $O \subset \mathbb{R}^n$ is an open set, define*

$$\mathscr{B}^X(O) = \sigma(X(t); \ t \in O).$$

If $D \subset \mathbb{R}^n$ is a closed subset, we set $D_\epsilon = \{t \in \mathbb{R}^n : \inf_{s \in D} |t - s| < \epsilon\}$ and define $\mathscr{B}^X(D) = \cap_{\epsilon > 0} \mathscr{B}^X(D_\epsilon)$. We now say that the random field X is Markov with respect to an open set O if $\mathscr{B}^X(\partial O)$ is a splitting field for $\mathscr{B}^X(\overline{O})$ and $\mathscr{B}^X(O^c)$. (∂O is of course $\overline{O} \backslash O$.)

The Markov property for a random field as we have defined it above is known as Lévy's Markov property. It may seem more natural in definition 0.2 to replace $\mathscr{B}^X(\partial O)$, $\mathscr{B}^X(\overline{O})$ and $\mathscr{B}^X(O^c)$ with $\sigma(X(t) : t \in \partial O)$, $\sigma(X(t); t \in \overline{O})$ and $\sigma(X(t); t \in O^c)$ respectively. This is known as the sharp Markov property, and is naturally a stronger condition.

The benefit of studying Gaussian random fields is that once we understand the covariance structure of the process we can deduce distributional properties such as the Markov property. Indeed the covariance structure is characterised by inner product on the subspace $H = \overline{Sp\{X_t; t \in \mathbb{R}^n\}}$ of $L^2(\Omega, \mathscr{F}, \mathbb{P})$. (Here, $Sp\{X_t; t \in \mathbb{R}^n\}$ denotes the set of linear combinations of elements in $\{X_t; t \in \mathbb{R}^n\}$.) Theorem 5.1 of [Kün79] equates Lévy's Markov property of $(X(t); t \in \mathbb{R}^n)$ for all precompact, open subsets of $\mathbb{R}^n$ to certain properties of the space $\mathcal{H} := \{t \mapsto \mathbb{E}[ZX(t)] : Z \in H\}$, which we couple with the norm $\|f\|_{\mathcal{H}}^2 = \mathbb{E}[Z^2]$. In [NP94], Nualart and Pardoux use this result to demonstrate that the random field $(u(x, t); x \in \mathbb{R}, t \in [0, \infty))$ satisfies Lévy's Markov property for all

precompact, open sets. Suppose we now take as our noise $f(u(x,t))\frac{\partial^2}{\partial x\partial t}B_{xt}$ in place of $\frac{\partial^2}{\partial x\partial t}B_{xt}$. Unless f is constant the approach of [NP94] will no longer work because the solution u is no longer a Gaussian process. The subject of the Markovity of u for general f is a long open problem (see [Par93]). Although we do not approach this problem here, one motivation is to approach some aspects of the Markovity of u when f is constant without relying on the Gaussian structure. We will, in fact, study the process $(u(x,\cdot); x \geq 0)$, that is we allow the process to evolve in the x direction. If we fix t and we define a filtration by $\mathscr{F}_x = \mathscr{B}^u((-\infty, x] \times [0, \infty))$, we may deduce from Nualart and Pardoux's result that $(u(x,\cdot); x \geq 0)$ is Markov with respect to $(\mathscr{F}_x; x \geq 0)$ in the sense of definition 1.1. Naturally, by focusing on a more specific aspect of u we can make a more detailed analysis. For example, can we in fact obtain the sharp Markov property in this case? Is the evolution in the x direction strongly Markovian? Given that the evolution is Markovian, perhaps there may even be a semigroup to describe it.

There is a unique solution to (0.1.1) given by

$$u(x,t) = \int_{\mathbb{R}} u_0(y)g(t,x,y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} g(t-s,x,y)\frac{\partial^2}{\partial s\partial y}B_{sy}\mathrm{d}y\mathrm{d}s, \qquad (0.1.2)$$

where $g(t,x,y)$ is the Green's kernel for the operator Δ, given by

$$g(t,x,y) = \frac{1}{2\sqrt{\pi t}}\exp\left(\frac{-(x-y)^2}{4t}\right).$$

We will define $\int_0^t \int_{\mathbb{R}} g(t-s,x,y)\frac{\partial^2}{\partial s\partial y}B_{sy}\mathrm{d}y\mathrm{d}s$ rigorously as an Itô integral later. Now fix t and define a process $(u(x,t); x \geq 0)$. To describe its Markovian evolution, we might try to describe it through a stochastic differential equation driven by a noise which is adapted to the filtration $(\mathscr{F}_x; x \geq 0)$ where

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$$\mathscr{F}_x = \sigma\{B_{ys} : s \in [0, \infty), y \leq x\}.$$

We immediately observe from (0.1.2) that $u(0, t)$ is not $\mathscr{F}_0$ measurable. To overcome this difficulty, we shall add some extra initial information into our filtration. In fact, we will define a process $((u_x, v_x); x \geq 0)$ which we think of as taking values in a separable Banach space E of the form $X_1^* \times X_2^*$ where X_1 and X_2 can be thought of as test function spaces. Loosely, we take $u_x(h) = \int_0^\infty h(t)u(x, t)dt$ for $h \in X_1$ and we think of v_x as being the derivative in x of u_x. In chapter 3, we will show that for any $X > 0$, $((u_x, v_x); x \in [0, X])$ satisfies an infinite system of stochastic differential equations driven by a noise $(W_x; x \geq 0)$, where

$$W_x(h) = \int_0^x \int_0^\infty h(s)dB_{ys}.$$

It is the initial information (u_0, v_0) that we will add to $(\mathscr{F}_x; x \geq 0)$ to obtain a new filtration $(\tilde{\mathscr{F}}_x; x \geq 0)$.

The problem now is that, whilst $(W_x(h); x \geq 0)$ is an $(\mathscr{F}_x; x \geq 0)$ martingale for any $h \in L^2([0, \infty))$, it is not an $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingale. Our hope instead is that it is an $(\tilde{\mathscr{F}}_x; x \geq 0)$ semimartingale. For this we look to the theory of enlargements of filtrations. For an overview of this subject, see for example [JY85], [Yor97] and [Pro04]. In section 1.2, we will present an enlargement result for $(\mathscr{F}_x; x \geq 0)$ of a similar nature to well a known result for initial enlargements of Brownian filtrations. In particular, we obtain a condition that determines whether or not $(W_x(h); x \geq 0)$ has a semimartingale decomposition for a given h. We use this in the first three sections of chapter 3 to determine an infinite system of stochastic differential equations satisfied by $((u_x, v_x); x \in [0, X])$. The last section in chapter 3 contains for the most part discussion on unresolved issues. In particular, the equations for $((u_x, v_x); x \geq 0)$ give rise

naturally to a martingale problem which one might hope would lead to a strong Markov property. However, there are difficulties defining a suitable space E on which to define this martingale problem, and there is some discussion of this. Furthermore, to obtain a strong Markov property, one requires the uniqueness of one dimensional distributions for solutions to the martingale problem, and unfortunately it has not been possible to answer whether or not this holds for this thesis.

Chapter 2 discusses the outcome of adding initially to $(\mathscr{F}_x; x \geq 0)$ information about the Brownian sheet along some curve. The hope is that the results of section 1.2 allow us to write an equation for W_x driven by a noise which sees this initial information, and that one might read from this a description of Brownian sheet conditioned in some sense along this curve. We provide a general approach using these methods, and although in many cases it quickly becomes too difficult to produce a description of a bridged sheet, we do provide a description of Brownian sheet which is fixed along the minor diagonal.

Let us remark that in order to make use of the results from section 1.2, we require some tools from Malliavin calculus and Gaussian measure theory, which we present in section 1.3 and for which our main references are [Nua06] and [Bog98] respectively. Our use of the Malliavin calculus is similar to that in [Bau02] and [BC07], although these papers deal with finite dimensional processes. [BL06], [FPY93], [GM08], [GM06] and [Sim05] all deal with infinite dimensional bridge processes, albeit using different methods to those presented here, and the author would like to thank the examiners for drawing his attention to them for this final draft.

0.1.2 Notation

In the following, we shall assume that we have an underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ which is rich enough for all stochastic processes that we define. We shall assume that probability space is complete, that is if $A \subset \Omega$ such that there exists $B \in \mathscr{F}$ with $\mathbb{P}(B) = 0$ and $A \subset B$, then $A \in \mathscr{F}$. We will write $\mathcal{N}_{\mathbb{P}}(\mathscr{F}) = \{A \in \mathscr{F} : \mathbb{P}(A) = 0\}$. For an $\mathscr{F}$ measurable random variable f on Ω, $\mathbb{E}[f]$ will denote the expectation of f, $\int_{\Omega} f(\omega)\mathbb{P}(\mathrm{d}\omega)$. For a general probability space $(E, \mathcal{E}, \mu)$, we will often write $\mathbb{E}_{\mu}$ to denote the integral with respect to μ of $\mathcal{E}$ measurable functions from E to $\mathbb{R}$.

If E is a topological vector space and F is some subset of E, we will denote by $Sp(F)$ the linear combinations of elements of F. We denote by $\mathcal{B}(E)$ the Borel σ-algebra on E, that is the σ-algebra generated by the open subsets of E. We will denote by $m(E)$ the space of Borel measurable real valued functions. We define $B(E)$ to be the space of bounded functions from E to $\mathbb{R}$, which we may equip with the supremum norm $\|f\|_{\infty} = \sup_{\{x \in E\}} |f(x)|$ for $f \in B(E)$. We will denote by $C(E)$ the space of continuous functions from E to $\mathbb{R}$, and $C_b(E) = C(E) \cap B(E)$. For $f : E \to \mathbb{R}$, we will denote by $\mathrm{supp} f$ the closure of $\{x \in E : f(x) \neq 0\}$, and we set $C_0(E) = \{f \in C(E) : \mathrm{supp} f \text{ is compact}\}$. If E is a subset of $\mathbb{R}^n$, $n \in \mathbf{N}$, we will denote by $C^k(E)$ the set of $f \in C(E)$ such that $\partial_1^{\alpha_1} \ldots \partial_n^{\alpha} f$ is in $C(E)$ for any multi-index α with $\sum_{i=1}^n \alpha_i \leq k$. Here and in the future, ∂_i refers to the derivative in the ith variable. We take $C^{\infty}(E) = \cap_{k \in \mathbf{N}} C^k(E)$ and $C_0^k(E) = C_0(E) \cap C^k(E)$ for $k \in \mathbf{N} \cup \{\infty\}$.

We shall use parentheses $\langle, \rangle$ to denote the cross variation of two stochastic process $(X(t); t \geq 0)$ and $(Y(t); t \geq 0)$, $\langle X, Y \rangle_t$, with the parameter as a right sub-index. We will also, where it is not confusing, use parentheses to denote the inner product in $L^2(\mathbb{R}^n)$, whilst the norm will be denoted $\|\cdot\|_2$. Otherwise,

we will use $(\cdot, \cdot)_H$ to denote an inner product on a vector space. For a normed vector space V we will denote the topological dual by V^*. For $\phi \in V$ and $l \in V^*$ we will generally write $l(\phi)$ to represent the action of l on ϕ. However, if V is a test function space and l has a representation as a continuous function through

$$l(\phi) = \int l(t)\phi(t)\mathrm{d}t$$

then we may write $l(\phi) = \langle l, \phi \rangle$.

1 Enlargements of Filtrations

1.1 Some preliminary definitions

1.1.1 The Markov property

Suppose that $(X(t); t \in \mathbb{R}^n)$ is a random field on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, taking values in a state space E, which we shall assume for convenience to be a separable Banach space. We have already discussed a couple of possible interpretations of what it means for $(X(t); t \in \mathbb{R}^n)$ to be Markov. When $n = 1$, we have a slightly different intuition: a Markov process X is one such that if we know what the process is doing at time t, we gain no additional information of what the process does after time t from knowing what it did before time t. To turn this into a definition, we need a mathematical description of the our information about the process at any one time, and this is the natural filtration $(\mathscr{F}_t^X; t \geq 0)$ given by

$$\mathscr{F}_t^X = \sigma\{X(s); 0 \leq s \leq t\}.$$

$\mathscr{F}_t^X$ represents the observed information about X up to time t. Our intuition is that for any $t, s \geq 0$, $X(t + s)$ should be independent of $\mathscr{F}_t^X$ conditional upon knowing $X(t)$. We can be slightly more general than this, and allow situations where we have more information at time t than the observed information about X up to time t. If $(\mathscr{F}_t; t \geq 0)$ is a filtration (that is, $\mathscr{F}_t$ is a sub-σ-algebra of $\mathscr{F}$ for all $t \geq 0$ and $\mathscr{F}_s \subset \mathscr{F}_t$ whenever $s < t$), then we say that $(X(t); t \geq 0)$ is adapted to $(\mathscr{F}_t; t \geq 0)$ if $X(t)$ is $\mathscr{F}_t$ measurable for all $t \geq 0$. We say it is $(\mathscr{F}_t; t \geq 0)$ progressively measurable if, for each $t \geq 0$, the map $X : [0, t] \times \Omega \to \mathbb{R}$ is measurable with respect to $\mathcal{B}([0, t]) \otimes \mathscr{F}_t$. This is stronger than adaptedness, however if, for example, $t \mapsto X(t)(\omega)$ is left or right continuous for every ω and $(X(t); t \geq 0)$ is adapted to $(\mathscr{F}_t; t \geq 0)$, then it is progressively measurable with

respect to $(\mathscr{F}_t; t \geq 0)$ (see proposition 1.1.3, [KS98]). Note that if $(X(t); t \geq 0)$ is adapted to $(\mathscr{F}_t; t \geq 0)$, then $\mathscr{F}_t^X \subset \mathscr{F}_t$ for all $t \geq 0$. Here is our definition.

Definition 1.1. *Let* $(X(t); t \geq 0)$ *be a stochastic process on* $(\Omega, \mathscr{F}, \mathbb{P})$ *which is adapted to the filtration* $(\mathscr{F}_t; t \geq 0)$. $(X(t); t \geq 0)$ *is Markov with respect to* $(\mathscr{F}_t; t \geq 0)$ *if*

$$\mathbb{P}(X(t+s) \in \Gamma | \mathscr{F}_t) = \mathbb{P}(X(t+s) \in \Gamma | X(t)) \quad \text{a.s.} \qquad (1.1.1)$$

for all $s, t \geq 0$ *and* $\Gamma \in \mathcal{B}(E)$.

Let $\mathcal{A}$ denote $\{[0, t); t \geq 0\}$, a collection of open subsets of $[0, \infty)$. With regards to the remarks following definition 0.2, we can rephrase definition 1.1 as saying that $(X(t); t \geq 0)$ satisfies the sharp Markov property for every $O \in \mathcal{A}$ (where the parameter space is $[0, \infty)$ instead of $\mathbb{R}^n$).

The Markov property is equivalent to

$$\mathbb{E}[f(X(t+s)) | \mathscr{F}_t] = \mathbb{E}[f(X(t+s)) | X(t)] \quad \text{a.s.}$$

for all $f \in B(\mathbb{R})$. Also, note that if $(X(t); t \geq 0)$ is Markov with respect to $(\mathscr{F}_t; t \geq 0)$ then

$$\mathbb{E}[X(t+s) | \mathscr{F}_t^X] = \mathbb{E}[\mathbb{E}[X(t+s) | \mathscr{F}_t] | \mathscr{F}_t^X] = \mathbb{E}[X(t) | \mathscr{F}_t^X] = X(t).$$

Hence X is also Markov with respect to $(\mathscr{F}_t^X; t \geq 0)$. Suppose instead that we know $X(\tau)$ for a random time τ. Can we still say that the process after time τ is independent of all that went on before τ? We take τ to be an optional time, that is a mapping $\tau : \Omega \mapsto [0, \infty]$ such that $\{\tau < t\} \in \mathscr{F}_t$ for all $t \geq 0$.

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Furthermore, if $\mathscr{F}_{t+} = \cap_{\varepsilon > 0} \mathscr{F}_{t+\varepsilon}$ for all $t \geq 0$, we define $\mathscr{F}_{\tau+}$ by

$$\mathscr{F}_{\tau+} = \sigma\{A \in \mathscr{F} : A \cap \{\tau \leq t\} \in \mathscr{F}_{t+} \ \forall t > 0\}.$$

We make the following definition:

Definition 1.2. *An $(\mathscr{F}_t; t \geq 0)$ progressively measurable stochastic process $(X(t); t \geq 0)$ exhibits the strong Markov property if for every almost surely finite optional time τ and every $s > 0$,*

$$\mathbb{E}[f(X(\tau + s)|\mathscr{F}_{\tau+}] = \mathbb{E}[f(X(\tau + s))|X(\tau)] \quad \text{a.s.}$$

We also have the notion of an $(\mathscr{F}_t; t \geq 0)$ stopping time, that is a random time $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathscr{F}_t$ for all $t \geq 0$. Remark that τ is an $(\mathscr{F}_t; t \geq 0)$ optional time if and only if it is an $(\mathscr{F}_{t+}; t \geq 0)$ stopping time (lemma 2.1.1 of [EK86]). Thus if we can show that $\mathbb{E}[f(X(\tau + s))|\mathscr{F}_\tau] = \mathbb{E}[f(X(\tau + s))|X(\tau)]$ almost surely for all almost surely finite stopping times τ, then $(X(t); t \geq 0)$ is a strong Markov process in the above sense.

1.1.2 Markov Semigroups

In chapter 4 of [EK86], the connection between the theory of semigroups and the theory of Markov processes is discussed. Define $T(t)f(x) = \mathbb{E}[f(X(t))|X(0) = x] =: E_x[f(X(t))]$ for a stochastic process $(X(t); t \geq 0)$ taking values in E and $f \in B(E)$. If $(X(t); t \geq 0)$ is Markov, and furthermore we assume that $\mathbb{E}[f(X(t))|X(0) = x] = \mathbb{E}[f(X(t + s))|X(s) = x]$ (that is, it is time homogeneous) then formally

$$T(t + s)f(x) = \mathbb{E}_x[f(X(t + s))] = \mathbb{E}_x[\mathbb{E}_x[f(X(t + s)|X(s)]] = \mathbb{E}_x[T(t)f(X(s))]$$

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for $x \in E$ and $s, t \geq 0$. (We have used the Markov property here to deduce that $\mathbb{E}_x[f(X(t+s))|X(s)] = \mathbb{E}[f(X(t+s))|X(s)] = T(t)f(X(s))$.) Thus, under certain conditions on X, $(T(t); t \geq 0)$ is a semigroup on $B(E)$, that is $T(t) : B(E) \to B(E)$ is a bounded linear operator for each $t \geq 0$, such that $T(0)$ is the identity and $T(t+s) = T(t)T(s)$ for each $s, t \geq 0$. In general, given a Markov process $(X(t); t \geq 0)$ with respect to $(\mathscr{F}_t; t \geq 0)$ and a semigroup $(T(t); t \geq 0)$, we say that $(X(t); t \geq 0)$ corresponds to $(T(t); t \geq 0)$ if

$$\mathbb{E}[f(X(t+s))|\mathscr{F}_t] = T(s)f(X(t)).$$

For example, if $(X(t); t \geq 0)$ is a Brownian motion, it is a Markov process corresponding to the semigroup $(T(t); t \geq 0)$ given by

$$T(t)f(x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(y) \exp\left(-\frac{(y-x)^2}{2t}\right) \mathrm{d}y.$$

Proposition 4.1.6 of [EK86] demonstrates that the finite dimensional distributions of a Markov process are completely determined by this semigroup and the initial distribution.

Given a semigroup we define an operator $\mathcal{G} : \mathcal{D}(\mathcal{G}) \to B(E)$ by

$$\mathcal{G}f = \lim_{t \to 0} \frac{1}{t}(T(t)f - f),$$

where $\mathcal{D}(\mathcal{G})$ is the subspace of $f \in B(E)$ for which this limit exists. $\mathcal{G}$ is known as the infinitesimal generator of $(T(t); t \geq 0)$. As an example, take a real valued process $(X(t); t \geq 0)$ satisfy an equation of the form $\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}B(t)$, where B is a Brownian motion. One may deduce from Itô's formula that

$$\mathrm{d}f(X(t)) = (b(X(t))f'(X(t)) + \frac{1}{2}\sigma^2(X(t))f''(X(t)))\mathrm{d}t + \sigma(X(t))f'(X(t))\mathrm{d}B(t)$$

for any bounded $C^2(\mathbb{R})$ function. Setting $\mathcal{G}f(x) = \lim_{t\to 0}\frac{1}{t}(\mathbb{E}[f(X(t))|X(0) = x] - f(x))$, we see that $\mathcal{G}f(x) = b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$. One may put conditions on b and σ such that the classical Hille-Yosida implies that there is a (strongly continuous contraction) semigroup with generator $\mathcal{G}$ (see theorem 1.2.6 of [EK86]), and one might hope to show that $(X(t); t \geq 0)$ is a Markov process corresponding to this semigroup. If b or σ are not bounded functions however, $\mathcal{G}f$ will not in general lie in $B(\mathbb{R})$. Our main example will be a situation similar to this. Nevertheless, we may still construct the generator above as a map from $\mathcal{D}(\mathcal{G})$ to $m(E)$. Although we do not construct a semigroup, the above operator is still of use in constructing a martingale problem, as we see in the next section.

1.1.3 The martingale problem.

The idea of a martingale problem is based on the observation that given a Markov process $(X(t); t \geq 0)$ corresponding to a semigroup $(T(t); t \geq 0)$ with generator $\mathcal{G}$, the process $(f(X(t)) - \int_0^t \mathcal{G}f(X(s))\mathrm{d}s; t \geq 0)$ is a martingale for $f \in \mathcal{D}(\mathcal{G}) \subset B(E)$ (see proposition 4.1.7 of [EK86]). We look for some sort of converse: supposing that $f(X(t)) - \int_0^t \mathcal{G}f(X(s))\mathrm{d}s$ is a martingale for all $f \in B(E)$ and some operator $\mathcal{G}$, can we deduce that $(X(t); t \geq 0)$ is Markov? We have to be a little careful here. Suppose that $(X(t); t \geq 0)$ is adapted to the filtration $(\mathscr{F}_t; t \geq 0)$. We ask if the same can be said of $\left(\int_0^t \mathcal{G}f(X(s))\mathrm{d}s; t \geq 0\right)$. Following the remarks of section 4.3 (and also problem 2.2) in [EK86], we can say that given $g \in B(E)$, there is a modification (say $(Y(t); t \geq 0)$) of $\left(\int_0^t g(X(s))\mathrm{d}s; t \geq 0\right)$ which is adapted to $(\mathscr{F}_t; t \geq 0)$. In other words, for any $t \geq 0$, $\mathbb{P}(Y(t) = \int_0^t g(X(s))\mathrm{d}s) = 1$ and $Y(t)$ is $\mathscr{F}_t$ measurable. If the fil-

tration is complete, that is, if $\mathcal{N}_{\mathbb{P}}(\mathscr{F}) \subset \mathscr{F}_t$ for all $t \geq 0$, we can deduce that $\left(\int_0^t g(X(s)) \mathrm{d}s; t \geq 0 \right)$ is $(\mathscr{F}_t; t \geq 0)$ for all $g \in B(E)$. Let us define the complete natural filtration $(\overline{\mathscr{F}}_t^X; t \geq 0)$, given by

$$\overline{\mathscr{F}}_t^X = \sigma\{X(s) : 0 \leq s \leq t\} \vee \mathcal{N}_{\mathbb{P}}(\mathscr{F}).$$

The value of this is that we can always say that $(f(X(t)) - \int_0^t \mathcal{G}f(X(s)) \mathrm{d}s; t \geq 0)$ is $(\overline{\mathscr{F}}_t^X; t \geq 0)$ adapted.

We still need to be a little bit careful because we want to consider cases where $\mathcal{G}f$ may not be bounded. If we take f to be continuous, but possibly not bounded, we may consider the truncated functions $f_n \in B(E)$ for $n \in \mathbf{N}$, where $f_n(x) = f(x)$ if $\|x\|_E \leq n$ and $f_n(x) = 0$ otherwise. If we also define an increasing sequence of stopping times $\tau_n = \inf\{t \geq 0 : \|X(t)\|_E > n\}$, then $f_n(X(\tau_n \wedge t)) = f(X(\tau_n \wedge t)$. Note that for $B \in \mathcal{B}(\mathbb{R})$,

$$\left\{ \int_0^t f(X(s)) \mathrm{d}s \in B \right\} = \bigcup_{n=1}^{\infty} \left(\left\{ \int_0^t f_n(X(\tau_n \wedge s)) \mathrm{d}s \in B \right\} \cap \{\tau_n > t\} \right) \cup A$$

where $A \subset \{\lim_{n \to \infty} \tau_n = \infty\}$. From this it follows that if $(X(t); t \geq 0)$ is adapted to a complete filtration $(\mathscr{F}_t; t \geq 0)$, then $\{\int_0^t f(X(s)) \mathrm{d}s \in B\}$ is the union of countably many elements in $\mathscr{F}_{\tau_n \wedge t}$ and a null set, and is thus in $\mathscr{F}_t$.

In the following, we take A to be some subset of $B(E) \times B(E)$, and let μ be a probability measure on $(E, \mathcal{B}(E))$.

Definition 1.3. $(X(t); t \geq 0)$ *is said to be a solution of the martingale problem for* (A, μ) *with respect to a complete filtration* $(\mathscr{F}_t; t \geq 0)$ *if* $X(0)$ *is distributed according to* μ *and for every* $(f, g) \in A$, $f(X(t)) - \int_0^t g(X(s)) \mathrm{d}s$ *is an* $(\mathscr{F}_t; t \geq 0)$ *martingale. We say it is a solution of the martingale problem for* (A, μ) *if it is a solution to the martingale problem for* (A, μ) *with respect to the filtration*

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$(\overline{\mathscr{F}}_t^X; t \geq 0)$.

Theorem 1.1. *Suppose that any two solutions X and Y of the martingale problem (A, μ) have the same one-dimensional distributions, so that for any $t \geq 0$,*

$$\mathbb{P}(X(t) \in \Gamma) = \mathbb{P}(Y(t) \in \Gamma)$$

for all $\Gamma \in \mathcal{B}(E)$. Then

(a) *any solution of the martingale problem for (A, μ) with respect to $(\mathscr{F}_t; t \geq 0)$ is a Markov with respect to this filtration;*

(b) *if in addition $A \subset C_b(E) \times B(E)$ and X is a solution of the martingale problem for (A, μ) with respect to $(\mathscr{F}_t; t \geq 0)$ whose sample paths are right-continuous with left limits, then X exhibits the strong Markov property with respect to this filtration.*

The proof of this theorem contained in [EK86]. It is not quite sufficient however for our purposes, since we shall wish to choose A so that in general, for $(f, g) \in A$, g is not bounded. Ethier and Kurtz's approach is to take a solution $(X(t); t \geq 0)$ of the martingale problem for (A, μ) with respect to $(\mathscr{F}_t; t \geq 0)$ and study the process $(Y(t); t \geq 0)$ given by $Y(t) = X(t + r)$ for a fixed $r > 0$. They then define the object

$$\eta(Y) = \left[f(Y(t_{n+1})) - f(Y(t_n)) - \int_{t_n}^{t_{n+1}} g(Y(s))\mathrm{d}s \right] \prod_{k=1}^{n} h_k(Y(t_k)) \quad (1.1.2)$$

for $(f, g) \in A$, $h_k \in B(E)$ and arbitrarily chosen $0 \leq t_1 < \ldots < t_n < t_{n+1}$. They define two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ such that $\mathbb{E}_{\mathbb{P}_1}[\eta(Y)] = \mathbb{E}_{\mathbb{P}_2}[\eta(Y)] = 0$. It follows from this that $(f(Y(t)) - \int_0^t g(Y(s))\mathrm{d}s; t \geq 0)$ is both an $(\overline{\mathscr{F}}_t^Y; t \geq 0)$ martingale in $(\Omega, \mathscr{F}, \mathbb{P}_1)$ and an $(\overline{\mathscr{F}}_t^Y; t \geq 0)$ martingale in $(\Omega, \mathscr{F}, \mathbb{P}_2)$. The Markov property then follows from the hypothesis regarding the uniqueness of

one dimensional distributions of solutions and the definitions of $\mathbb{P}_1$ and $\mathbb{P}_2$.

In order to demonstrate that $\mathbb{E}_{\mathbb{P}_1}[\eta(Y)] = \mathbb{E}_{\mathbb{P}_2}[\eta(Y)] = 0$, it is sufficient to show that $\mathbb{E}[\eta(Y)|\mathscr{F}_r] = 0$, which is immediately clear observing that $\mathbb{E}[\eta(Y)|\mathscr{F}_{r+t_n}] = 0$. That $\mathbb{E}_{\mathbb{P}_i}[\eta(Y)] = 0$ implies that $f(Y(t)) - \int_0^t g(Y(s))\mathrm{d}s$ is a $\mathbb{P}_i$ martingale requires the h_k to be bounded, but $f(Y(t_{n+1})) - f(Y(t_n)) - \int_{t_n}^{t_{n+1}} g(Y(s))\mathrm{d}s$ merely needs to be $\mathbb{P}_1$ and $\mathbb{P}_2$ integrable. This follows from the hypothesis since $f(Y(t)) - \int_0^t g(Y(s))\mathrm{d}s$ must be $\mathbb{P}$ integrable for any solution (f, g) of the martingale problem. Thus the condition that f and g are not bounded is not required for part (a).

Part (b) is a little more troublesome. Ethier and Kurtz' approach for part (a) can also be applied for part (b) provided that we can show that for any almost surely finite stopping time τ,

$$\mathbb{E}[\eta(X(\tau + \cdot))|\mathscr{F}_{\tau+t_n}] = 0.$$

To see this, set
$$Z(t) = f(X(t)) - \int_0^t g(X(s))\mathrm{d}s. \tag{1.1.3}$$

We require
$$\mathbb{E}[Z(\tau + t + s) - Z(\tau + t)|\mathscr{F}_{\tau+t}] = 0$$

for all $t, s > 0$. If we take some $T > 0$, the optional sampling theorem tells us that

$$\mathbb{E}[Z((\tau + t + s) \wedge T)|\mathscr{F}_{\tau+t}] = Z((t + \tau) \wedge T)$$

or equivalently $\mathbb{E}[Z((\tau + t + s) \wedge T) - Z((\tau + t) \wedge T)|\mathscr{F}_{\tau+t}] = 0$. This requires that $(Z(t); t \geq 0)$ has a right-continuous modification, hence the requirement that f is continuous. It is then sufficient to show that

$$\mathbb{E}[Z((\tau + t + s) \wedge T) - Z((\tau + t) \wedge T)|\mathscr{F}_{\tau+t}]$$

$$\rightarrow \mathbb{E}[Z(\tau + t + s) - Z(\tau + t)|\mathscr{F}_{\tau+t}] \qquad (1.1.4)$$

almost surely as $T \rightarrow \infty$. This is straightforward when g is bounded by use of the dominated convergence theorem. When g is not bounded however we need to find some other way of showing the above.

Let us make one further remark regarding part (b). In practise we will be dealing with solutions $(X(t); t \geq 0)$ to a martingale problem where $(X(t); t \geq 0)$ is almost surely continuous. In such cases the progressive measurability of $(X(t); t \geq 0)$ is not obvious. Thus we have a set $A \subset \Omega$ with $\mathbb{P}(A) = 1$ on which $t \mapsto X(t)$ is continuous. If we define a process $(Y(t); t \geq 0)$ such that $X_\omega = Y_\omega$ for $\omega \in A$, and $Y_\omega = 0$ otherwise, then assuming that we are working with complete filtrations, $(Y(t); t \geq 0)$ is progressively measurable. Furthermore it is obvious that it satisfies the same martingale problem as $(X(t); t \geq 0)$, so if we may apply theorem 1.1, it follows that $(Y(t); t \geq 0)$ has the strong Markov property, from which it follows that so to does $(X(t); t \geq 0)$.

1.1.4 The Brownian Sheet

In this section we shall define a Brownian sheet and introduce an associated stochastic calculus. In particular, we highlight how the construction of an Itô integral in this specific setting is an application of a more general construction which will be of use later on. The general construction is presented in detail in [Wal86].

We will construct the Brownian sheet, as with Brownian motion, as a centred

Gaussian process with a certain covariance structure. Specifically, we work on $[0, \infty)^2$ and define $C : \mathcal{B}([0, \infty)^2) \times \mathcal{B}([0, \infty)^2) \to \mathbb{R}$ by

$$C(A, B) = |A \cap B|$$

for $A, B \in \mathcal{B}([0, \infty)^2)$, where $|A \cap B|$ is the Lebesgue measure of $A \cap B$. Note that for $A_1, \ldots, A_n \in \mathcal{B}([0, \infty)^2)$ and $a_1, \ldots, a_n \in \mathbb{R}$,

$$\sum_{i,j=1}^{n} a_i a_j C(A_i, A_j) = \int_0^\infty \left(\sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(t) \right)^2 \mathrm{d}t \geq 0.$$

In other words, C is positive definite, and the theory of Gaussian processes implies that there is a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a centred Gaussian process $W : \mathcal{B}([0, \infty)^2) \times \Omega \to \mathbb{R}$ such that $\mathbb{E}[W(A)W(B)] = C(A, B)$ for all $A, B \in \mathcal{B}([0, \infty)^2)$. In particular, for any $|A|, |B| < \infty$,

- $W(A) \sim \mathcal{N}(0, |A|)$, and

- for $A \cap B = \emptyset$, $W(A \cup B) = W(A) + W(B)$ a.s.

which is easily verified by checking that $\mathbb{E}[(W(A \cup B) - W(A) - W(B))^2] = 0$. A centred Gaussian process W with these properties is called a white noise on $[0, \infty)^2$. We now define a Brownian sheet $(B_{xt}; (x, t) \in [0, \infty)^2)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ by

$$B_{xt} = W([0, x] \times [0, t]).$$

The Kolmogorov-Čentsov continuity criterion implies that there is a version of $(B_{xt}; (x, t) \in [0, \infty)^2)$ which is almost surely continuous on $[0, \infty)^2$ (see proposition 1.4 of [Wal86]). Interestingly, the Brownian sheet itself provides a good example of how easily the sharp Markov property can fail. It is readily seen the Brownian sheet satisfies Lévy's Markov property for all bounded open sets (see [Roz82], for example). However, taking D to be the triangle

$\{(x,t) \in [0,\infty)^2 : 0 < t < 1 - x < 1\}$ (which we note is open and bounded), [Wal86] shows that the Brownian sheet fails to be sharp Markov for D. On the other hand, it is sharp Markov for all rectangles. In fact, we may define the sharp Markov property for all Jordan domains, and [DW92] shows that a Brownian sheet exhibits the sharp Markov property with respect to 'almost every' Jordan domain. (Here, 'almost every' is defined using the measure on the Jordan curves induced by a planar Brownian motion forced to reach its starting point at time 1.)

Throughout the thesis we will make use of both this Brownian sheet restricted to $[0,1]^2$ and also a Brownian sheet on $\mathbb{R} \times [0,\infty)$. The latter is simply obtained by taking two independent Brownian sheets B^1 and B^2 and setting $B_{xt} = B^1_{xt}$ for $x \geq 0$ and $B_{xt} = B^2_{-x,t}$ for $x \leq 0$. For now we continue to work with the Brownian sheet on $[0,\infty)^2$. Let us also define a filtration $(\mathscr{F}_x; x \geq 0)$ by $\mathscr{F}_x = \sigma\{B_{ys}; y \in [0,x], s \geq 0\} \vee \mathcal{N}_{\mathbb{P}}(\mathscr{F})$.

Our goal now is to define a stochastic integral $\int_0^x \int_0^\infty f(y,s) \mathrm{d}B_{ys}$ for some class of functions f. We define $f : [0,\infty)^2 \times \Omega \to \mathbb{R}$ by $f(y,s) = \xi \mathbb{1}_{(x_1,x_2]}(y) \mathbb{1}_A(s)$ where $A \in \mathcal{B}([0,\infty))$, $0 \leq x_1 < x_2$ and ξ is a bounded $\mathscr{F}_{x_1}$ measurable random variable. It is then natural to write

$$\int_0^x \int_0^\infty f(y,s) \mathrm{d}B_{ys} = \xi(W(([0,x] \cap (x_1,x_2]) \times A)).$$

In this way we can define $\int_0^x \int_0^\infty f(y,s) \mathrm{d}B_{ys}$ in the space $\mathfrak{E}$ of linear combinations of such f. For all $x \geq 0$ and $A \in \mathcal{B}([0,\infty))$ we define $M_x(A) = W([0,x] \times A)$. Although $M_x(\cdot)$ is merely an additive set function (and not a measure), we can define a σ-finite signed measure Q on $\mathcal{B}([0,\infty)^3)$ by $Q(A,B,[0,x]) =$

$\langle M.(A), M.(B) \rangle_x = x|A \cap B|$. Define a norm $\|\cdot\|_M$ on $\mathfrak{E}$ by

$$\|f\|_M^2 = \mathbb{E}\left[\int_0^\infty \int_0^\infty \int_0^\infty |f(x,s)||f(x,r)||Q|(\mathrm{d}s,\mathrm{d}r,\mathrm{d}x)\right]$$
$$= \int_0^\infty \int_0^\infty \mathbb{E}[|f(x,s)|^2]\mathrm{d}s\mathrm{d}x.$$

and let $\mathscr{P}_M$ be the completion of $\mathfrak{E}$ in this norm. By proposition 2.3 of [Wal86] this is the set of functions $f : [0,\infty)^2 \times \Omega \to \mathbb{R}$ which are measurable with respect to the σ-algebra on $[0,\infty)^2 \times \Omega$ generated by $\mathfrak{E}$ and such that $\|f\|_M < \infty$. For $f \in \mathscr{P}_M$, $(f(x,\cdot); x \geq 0)$ defines a process in $L^2([0,\infty) \times \Omega)$ which is adapted to the filtration $(\mathscr{F}_x; x \geq 0)$. It is not generally true that $(f(\cdot,t); t \geq 0)$ is adapted to $(\sigma\{B_{ys}; y \in [0,\infty), 0 \leq s \leq t\}; t \geq 0)$.

We will now define $\int_0^x \int_0^\infty f(y,s)\mathrm{d}B_{ys}$ for $f \in \mathscr{P}_M$. For this we require that $(M_x(A); x \geq 0, A \in \mathcal{B}([0,\infty)))$ is a martingale measure. In other words, $M_0(A) = 0$ for all $A \in \mathcal{B}([0,\infty))$, $\mathbb{E}[M_x(\cdot)^2]$ is a σ-finite measure on $\mathcal{B}([0,\infty))$ for each $x > 0$, and $(M_x(A); x \geq 0)$ is an $(\mathscr{F}_x; x \geq 0)$ martingale for all $A \in \mathcal{B}([0,\infty))$. Furthermore it is a worthy martingale measure. Walsh provides a precise definition of a worthy martingale measure in [Wal86], although essentially it requires that there is a random σ-finite measure $K : \mathcal{B}([0,\infty))^3 \times \Omega$ which is positive definite in the first two arguments and such that for any $A, B \in \mathcal{B}([0,\infty))$, $|\langle M.(A), M.(B) \rangle_x| \leq K(A, B, [0,x])$. In our case we take $K = |Q|$.

By theorem 2.5 of [Wal86], we may now define an $(\mathscr{F}_x; x \geq 0)$ martingale measure $(f.M_x; x \geq 0)$ for any $f \in \mathscr{P}_M$ where $f.M_x([0,\infty)) = \int_0^x \int_0^\infty f(y,s)\mathrm{d}B_{ys}$ for $f \in \mathfrak{E}$, and furthermore

$$\mathbb{E}[(f.M_x([0,\infty)))^2] = \|f\|_M^2 \quad \forall f \in \mathscr{P}_M$$

We write

$$\int_0^x \int_0^\infty f(y,s)\mathrm{d}B_{ys} := f.M_x([0,\infty)).$$

Note that if $\{f_n; n \in \mathbf{N}\}$ is a sequence in $\mathfrak{E}$ then

$$\mathbb{E}\left[\left(\int_0^x \int_0^\infty (f_n(y,s) - f_m(y,s))\mathrm{d}B_{ys}\right)^2\right] = \|f_n - f_m\|_M^2.$$

Thus, if $f \in \mathscr{P}_M$ and $f_n \to f$ in the $\|\cdot\|_M$ norm, then $\left\{\int_0^x \int_0^\infty f_n(y,s)\mathrm{d}B_{ys}; n \in \mathbf{N}\right\}$ is a Cauchy sequence in $L^2(\Omega)$, and in fact its limit is what we define to be $\int_0^x \int_0^\infty f(y,s)\mathrm{d}B_{ys}$. We also note that

$$\int_0^\infty \int_0^\infty f(y,s)\mathrm{d}B_{ys} = \lim_{x \to \infty} \int_0^x \int_0^\infty f(y,s)\mathrm{d}B_{ys}.$$

where the limit is taken in $L^2(\Omega)$, and that the isometry property implies that this limit exists if and only if $\mathbb{E}[f^2]$ is in $L([0,\infty)^2)$. We make a few remarks regarding the order of integration. It is clear that there is nothing preventing us from going through the same procedure to define an integral in which the roles of the x and t variables are swapped, which we might denote

$$\int_0^t \int_0^\infty f(y,s)\mathrm{d}B_{sy}$$

for some class of f. This is not defined for all $f \in \mathscr{P}_M$, so we have to be careful about the order of integration. However, if $f(x,t)$ is $\sigma\{B_{ys} : 0 \le y \le x,\, 0 \le s \le t\}$ measurable for all (x,t) and furthermore f can be integrated with respect to both $\mathrm{d}B_{xt}$ and $\mathrm{d}B_{tx}$ then it is not difficult to see that

$$\int_0^\infty \int_0^\infty f(y,s)\mathrm{d}B_{ys} = \int_0^\infty \int_0^\infty f(y,s)\mathrm{d}B_{sy}$$

In particular, this is true whenever f is deterministic and belongs to $L^2([0, \infty)^2)$. We will deal often with integrals of deterministic functions, and whilst we shall try to keep notation consistent, the above remarks mean that we can freely interchange the 'order' of integration.

Finally we require a version of Fubini's theorem, which is given as theorem 2.6 in [Wal86].

Theorem 1.2. *Let $h \in L^1([0, \infty))$ and $f : [0, \infty)^3 \times \Omega \to \mathbb{R}$ such that f is measurable with respect to the σ-algebra $\mathcal{B}([0, \infty)) \times \sigma(\mathfrak{E})$ on $[0, \infty)^3 \times \Omega$. Suppose further that*

$$\mathbb{E}\left[\int_0^\infty \left(\int_0^\infty \int_0^\infty |f(y, s, t)|^2 \mathrm{d}y \mathrm{d}s\right) |h(t)| \mathrm{d}t\right] < \infty$$

Then almost surely we have

$$\int_0^\infty \left(\int_0^\infty \int_0^\infty f(y, s, t) \mathrm{d}B_{ys}\right) h(t) \mathrm{d}t = \int_0^\infty \int_0^\infty \left(\int_0^\infty f(y, s, t) h(t) \mathrm{d}t\right) \mathrm{d}B_{ys}.$$

1.1.5 Reformulating equation (0.1.1) as a spatial evolution.

We are now in a position to interpret (0.1.1). The process $(u(x, t); x \in \mathbb{R}, t \in [0, \infty))$ is said to be a solution of (0.1.1) if, for any $h \in C_0^\infty(\mathbb{R} \times [0, \infty))$,

$$\int_0^\infty \int_{\mathbb{R}} \frac{\partial}{\partial t} h(x, t) u(x, t) \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}} h(x, 0) u_0(x) \mathrm{d}x$$
$$+ \int_0^\infty \int_{\mathbb{R}} \Delta h(x, t) u(x, t) \mathrm{d}x \mathrm{d}t + \int_0^\infty \int_{\mathbb{R}} h(x, t) \mathrm{d}B_{tx} = 0.$$

This equation is treated in [Wal86], where it is shown that the unique solution is given by (0.1.2) under certain conditions on u_0, for example $u_0 \in L^1(\mathbb{R})$.

Suppose we try to rewrite equation (0.1.1) as the first order system

$$\frac{\partial}{\partial x}u(x,t) = v(x,t)$$
$$\frac{\partial}{\partial x}v(x,t) = \frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x \partial t}B(x,t).$$

At the moment, this is not well defined since we have no reason to believe that $(v(x,t); x \in \mathbb{R}, t \in [0,\infty))$ is differentiable. One might think instead to test these equations against some functions $h_1, h_2 : [0,\infty) \to \mathbb{R}$, so that we obtain

$$\int_0^\infty u(x,t)h_1(t)\mathrm{d}t = \int_0^\infty u(0,t)h_1(t)\mathrm{d}t + \int_0^x \int_0^\infty h_1(t)v(y,t)\mathrm{d}t\mathrm{d}y$$

and

$$\int_0^\infty v(x,t)h_2(t)\mathrm{d}t = \int_0^\infty v(0,t)h_2(t)\mathrm{d}t - \int_0^x \int_0^\infty h_2'(t)u(y,t)\mathrm{d}t\mathrm{d}y$$
$$- \int_0^x \int_0^\infty h_2(t)\mathrm{d}B_{yt}.$$

This looks like a more well defined system, but of course we still do not have a definition for $v(x,t)$. Formally, if $\int_0^\infty h_2(t)v(x,t)\mathrm{d}t$ did exist we might attempt to use Theorem 1.2 and write

$$\int_0^\infty h_2(t)v(x,t)\mathrm{d}t = \int_\mathbb{R} \int_0^\infty \left(\int_s^\infty \frac{\partial}{\partial x}g(t-s,x,y)h_2(t)\mathrm{d}t \right) \mathrm{d}B_{ys}.$$

As it happens, the term on the right hand side is almost surely finite whenever h_2 is continuous and $\sup_{t\geq 0}|(1+t)^{\frac{3}{4}+\varepsilon}h_2(t)| < \infty$ for some $\varepsilon > 0$. Furthermore, the integral

$$\int_\mathbb{R} \int_0^\infty \left(\int_s^\infty h_1(t)g(t-s,x,y)\mathrm{d}t \right) \mathrm{d}B_{ys}$$

is almost surely finite whenever h_1 is continuous and satisfies $\sup_{t\geq 0}|(1+t)^{\frac{5}{4}+\varepsilon}h_1(t)| < \infty$ for some $\varepsilon > 0$. In chapter 3, we will try to use these tail

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properties to define Banach spaces of test functions X_1 and X_2 and processes $(u_x; x \geq 0)$ and $(v_x; x \geq 0)$ taking values in X_1^* and X_2^* respectively such that

$$\langle h_1, u_x \rangle = \int_{\mathbb{R}} \int_0^\infty \left(\int_s^\infty h_1(t) g(t-s, x, y) \mathrm{d}t \right) \mathrm{d}B_{ys}$$

and

$$\langle h_2, v_x \rangle = \int_{\mathbb{R}} \int_0^\infty \left(\int_s^\infty h_2(t) \frac{\partial}{\partial x} g(t-s, x, y) \mathrm{d}t \right) \mathrm{d}B_{ys}$$

for all $h_1 \in X_1$, $h_2 \in X_2$ and $x \geq 0$. We will further demonstrate that for any $X > 0$, the process $((\langle h_1, u_x \rangle, \langle h_2, v_x \rangle); x \in [0, X])$ satisfies

$$\langle h_1, u_x \rangle = \langle h_1, u_0 \rangle + \int_0^x \langle h_1, v_y \rangle \mathrm{d}y$$

$$\langle h_2, v_x \rangle = \langle h_2, v_0 \rangle - \int_0^x \langle h_2', u_y \rangle \mathrm{d}y - \int_0^x \int_0^\infty h_2(s) \mathrm{d}B_{ys} \qquad (1.1.5)$$

for any $x \geq 0$, where the equalities hold almost surely, provided that $\langle h_2', u_y \rangle$ is defined.

Let us make the following remark about $u(x, t)$. Our goal is to show that $(u(x, t); x \geq 0)$ is Markov with respect to some filtration. We are no longer considering $u(x, t)$, but rather u_x and v_x. Suppose we manage to demonstrate some Markov property for u_x and v_x. What does this say about $u(x, t)$? We should not necessarily think that $(u(x, t); x \geq 0)$ and $(u_x; x \geq 0)$ are the same. Recall theorem 1.2: in order to show that

$$\int_0^\infty h(t) \int_0^t \int_{\mathbb{R}} g(t-s, x, y) \mathrm{d}B_{ys} \mathrm{d}t = \int_0^\infty \int_{\mathbb{R}} \left(\int_s^\infty h(t) g(t-s, x, y) \mathrm{d}t \right) \mathrm{d}B_{ys}$$

we require

$$\int_0^\infty |h(t)| \left(\int_0^t \int_{\mathbb{R}} |g(t-s, x, y)|^2 \mathrm{d}y \mathrm{d}s \right) \mathrm{d}t < \infty.$$

Now,

$$\int_{\mathbb{R}} g(t-s,x,y)^2 \mathrm{d}y = \frac{c}{t-s} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{t-s}\right) \mathrm{d}y = \frac{c}{\sqrt{t-s}}$$

and hence

$$\int_0^\infty |h(t)| \left(\int_0^t \int_{\mathbb{R}} |g(t-s,x,y)|^2 \mathrm{d}y\mathrm{d}s\right) \mathrm{d}t = c \int_0^\infty |h(t)|\sqrt{t}\mathrm{d}t.$$

This integral is not finite for all continuous h such that $\sup_{t\geq 0} |h(t)(1+t)^{\frac{5}{4}+\varepsilon}| < \infty$ unless $\varepsilon > \frac{1}{2}$. Nevertheless, it is finite for all $h \in C_0^\infty([0,\infty))$, so on this class of test functions $u(x,\cdot) = u_x$ in some sense. Our hope is that this is a big enough test function space to carry over properties of u_x to $u(x,t)$. Note that even for $h \in C_0^\infty([0,\infty))$,

$$\int_0^\infty |h(t)| \left(\int_0^t \int_{\mathbb{R}} |\partial_x g(t-s,x,y)|^2 \mathrm{d}y\mathrm{d}s\right) \mathrm{d}t = \infty,$$

so that theorem 1.2 does not allow us define a process $(v(x,t); x \geq 0)$.

We now try to set up a martingale problem which is solved by $((u_x, v_x); x \geq 0)$. Define a filtration $(\mathscr{F}_x; x \geq 0)$ by

$$\mathscr{F}_x = \sigma\{B_{ys}; -\infty \leq y \leq x, \ s \in [0,\infty)]\} \vee \mathcal{N}_{\mathbb{P}}(\mathscr{F}) \qquad (1.1.6)$$

and $F \in B(E)$ by

$$F(u,v) = f(\langle h_1, u\rangle, \ldots, \langle h_n, u\rangle, \langle h_{n+1}, v\rangle, \ldots, \langle h_{2n}, v\rangle)$$

for $f \in C_0^\infty(\mathbb{R}^{n+m})$ and $h_i \in X_1$, $h_{n+i} \in X_2$ for $i = 1, \ldots, n$. The standard approach now is to apply Itô's formula to $F(u_x, v_x)$ to obtain a generator $\mathcal{G}$ such that $((u_x, v_x); x \geq 0)$ solves the martingale problem for some subset of the

graph of $\mathcal{G}$. It is here that our difficulties with the initial conditions begin. Our initial conditions are, we recall,

$$\langle h, u_0 \rangle = \int_{\mathbb{R}} \int_0^\infty \left(\int_s^\infty h(t) g(t-s, 0, y) \mathrm{d}t \right) \mathrm{d}B_{ys}$$
$$\langle l, v_0 \rangle = \int_{\mathbb{R}} \int_0^\infty \left(\int_s^\infty l(t) \partial_2 g(t-s, 0, y) \mathrm{d}t \right) \mathrm{d}B_{ys}$$

In order to determine these we need to know B on the entire parameter space. Thus u_0 and v_0 are not $\mathscr{F}_0$ measurable, and it is meaningless to apply Ito's formula. How might we overcome this problem? One immediate suggestion is to simply take a finer filtration. This brings its own problem, namely that the martingale part in Itô's formula is no longer a martingale with the respect to the larger filtration. However there is hope that it has a semimartingale decomposition, as is discussed in the next section.

1.2 Enlargements of filtrations

1.2.1 Enlargements of Brownian filtrations.

Our hope of obtaining a semimartingale decomposition stems from the following theorem:

Theorem 1.3. *Let B_t, $t \in [0,1]$, be a one-dimensional standard Brownian motion. Fix $L : \Omega \to \mathbb{R}$, which we assume to be $\overline{\mathscr{F}}_1^B$-measurable, and define the stochastic kernel $\dot{\lambda}_s(f)$ by the martingale representation property of the Brownian motion, that is $f(L) = \mathbb{E}f(L) + \int_0^1 \dot{\lambda}_s(f) \, \mathrm{d}B_s$ a.s. for $f : \mathbb{R} \to \mathbb{R}$ which are bounded and measurable. Suppose, for all such f, the stochastic kernel $\dot{\lambda}_s(f)$ admits the factorisation*

$$\dot{\lambda}_s(f) = \mathbb{E}[f(L) \varrho(L, s) | \overline{\mathscr{F}}_s^B] \quad \text{a.s.}$$

for $s \in [0,1]$. *Then the process $(\tilde{B}_t; t \in [0,1])$ given by $\tilde{B}_t := B_t - \int_0^t \varrho(L,s)\,\mathrm{d}s$*
is an $(\overline{\mathscr{F}}_t^B \vee \sigma(L); t \in [0,1])$ standard Brownian motion.

A more precise statement of this theorem, including integrability and measurability conditions on ϱ, is presented in [Yor97]. The proof is essentially to use the martingale representation theorem and the condition placed on $\dot{\lambda}$ to demonstrate that

$$\mathbb{E}\left[f(L)\xi\left(B_{t+r} - \int_0^{t+r} \varrho(L,s)\mathrm{d}s\right)\right] = \mathbb{E}\left[f(L)\xi\left(B_t - \int_0^t \varrho(L,s)\mathrm{d}s\right)\right]$$

for any $t, r > 0$, $f \in B(\mathbb{R})$ and bounded, $\mathscr{F}_t^B$-measurable ξ, from which one may deduce the desired martingale property. We use a similar approach in the proof of the forthcoming theorem 1.5, and hence do not reproduce the proof of theorem 1.3 above, but rather refer to [Yor97], or [MY06] for a more detailed proof.

Suppose $L = B_1$ and f is differentiable. In this case, $\dot{\lambda}_s(f) = \mathbb{E}[f'(B_1)|\overline{\mathscr{F}}_s^B]$, as we shall demonstrate later. We can rewrite this as $\mathbb{E}[f'((B_1 - B_s) + B_s)|\overline{\mathscr{F}}_s^B]$. Observing that $(B_1 - B_s) + B_s$ is equal in law to $N_{1-s} + B_s$, where N_{1-s} is a centred Gaussian random variable with variance $1 - s$ which is independent of $\overline{\mathscr{F}}_s^B$, we note that $\dot{\lambda}_s(f)$ is given by

$$
\begin{aligned}
\mathbb{E}[f'(N_{1-s} + B_s)|\overline{\mathscr{F}}_s^B] &= \int_{\mathbb{R}} \frac{f'(x + B_s)}{\sqrt{2\pi(1-s)}} \exp\left(-\frac{x^2}{2(1-s)}\right) \mathrm{d}x \\
&= \int_{\mathbb{R}} \frac{xf(x + B_s)}{\sqrt{2\pi(1-s)^3}} \exp\left(-\frac{x^2}{2(1-s)}\right) \mathrm{d}x \\
&= \mathbb{E}\left[f(N_{1-s} + B_s)\frac{N_{1-s}}{1-s}\middle| \mathscr{F}_s^B\right] \\
&= \mathbb{E}\left[f(B_1)\frac{B_s}{1-s}\middle| \overline{\mathscr{F}}_s^B\right].
\end{aligned}
$$

Thus $\varrho(L,s) = \frac{B_1 - B_s}{1-s}$. In this case, finding a factorisation for the stochastic

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kernel is a straightforward result of integration by parts. This is something we will wish to bear in mind in infinite dimensions, although the integration by parts is less straightforward and cannot be done in all cases. We would now like to take a Brownian sheet $(B_{xt}; x \in \mathbb{R},\ t \in [0, \infty))$, define a filtration $(\mathscr{F}_x; x \geq 0)$ by (1.1.6), and give a semimartingale decomposition for the term $\int_0^x \int_0^\infty l(s) \mathrm{d}B_{ys}$ upon making some initial enlargement to this filtration. We will write $W_x(l) = \int_0^x \int_0^\infty l(s) \mathrm{d}B_{ys}$. $W_x(l)$ is an $(\mathscr{F}_x; x \geq 0)$ martingale for any $l \in L^2([0, \infty))$, with quadratic variation $\langle W(l) \rangle_x = \int_0^x \int_0^\infty l(s)^2 \mathrm{d}s \mathrm{d}y$.

1.2.2 The Martingale Representation Theorem

The key to theorem 1.3 is being able to write any $\overline{\mathscr{F}}_\infty^B$ measurable random variable as the stochastic integral of a kernel which can by factorised in a certain way. The martingale representation theorem is central to this idea, and we need a similar result for the filtration $(\mathscr{F}_x; x \geq 0)$, as presented below. In the sequel we define $\mathscr{F}_\infty$ by

$$\mathscr{F}_\infty := \sigma\left(\cup_{x \geq 0} \mathscr{F}_x\right).$$

In particular, $\int_0^\infty \int_0^\infty h(y, s) \mathrm{d}B_{ys}$ is $\mathscr{F}_\infty$ measurable for any $h \in L^2([0, \infty)^2)$. We state the theorem for $\mathscr{F}_\infty$ measurable F in $L^2(\Omega)$, although naturally if F is not $\mathscr{F}_\infty$ measurable, the theorem can be restated for $\mathbb{E}[F|\mathscr{F}_\infty]$. We also remark that analogous versions hold for Brownian sheets on restricted parameter spaces, such as $[0, 1]^2$.

Theorem 1.4. *For every $\mathscr{F}_\infty$- measurable random variable in $L^2(\Omega)$ there exists an $(\mathscr{F}_x; x \geq 0)$-adapted measurable process $(\dot{\lambda}_x; x \in [0, \infty))$ in $L^2([0, \infty) \times \Omega; L^2([0, \infty)))$ such that*

$$F = \mathbb{E}F + \int_0^\infty \int_0^\infty \dot{\lambda}_{ys} \,\mathrm{d}B_{ys} \quad a.s.$$

For our purposes we shall use the Clark-Ocone formula, which provides an expression for $\dot{\lambda}$ for a certain class of F. The proof of theorem 1.4 is in [Nua06], except that a small adjustment is needed to translate it into our setting, which we omit in this thesis.

If we return to our definition of the Itô integral, it is straightforward to show for any elementary $f \in \mathscr{P}_M$ that

$$\mathbb{E}\left[\int_0^\infty \int_0^\infty f(y,s)\mathrm{d}B_{ys} \,\middle|\, \mathscr{F}_x \right] = \int_0^x \int_0^\infty f(y,s)\mathrm{d}B_{ys}$$

and it thus follows for any $f \in \mathscr{P}_M$. We thus obtain the form of the martingale representation theorem that we shall make use of: for any $F \in L^2(\Omega)$ there exist an $(\mathscr{F}_x; x \geq 0)$ adapted process $(\dot{\lambda}_{x\cdot}; x \in [0,\infty))$ in $L^2([0,\infty) \times \Omega; L^2([0,\infty)))$ such that

$$\mathbb{E}[F|\mathscr{F}_x] = \mathbb{E}[F] + \int_0^x \int_0^\infty \dot{\lambda}_{ys}\mathrm{d}B_{ys} \quad \text{a.s. } \forall x.$$

1.2.3 Enlargements of $(\mathscr{F}_x; x \geq 0)$.

We now come to the enlargement theorem for $(\mathscr{F}_x; x \geq 0)$. Our aim is to add some initial information to our filtration. The initial information is given by a random variable L. Specifically, we take $(V, \mathcal{E}(V))$ to be a measurable space, and $L: \Omega \to V$ a measurable map. Suppose $F: V \to \mathbb{R}$ is such that $F(L) \in L^2(\Omega)$. With reference to theorem 1.4, there exists some $\dot{\lambda}_{ys}(F)$ such that

$$\mathbb{E}[F(L)|\mathscr{F}_x] = \mathbb{E}[F(L)] + \int_0^x \int_0^\infty \dot{\lambda}_{ys}(F)\mathrm{d}B_{ys}.$$

For our enlargement theorem to work, we need a stochastic factorisation of $\dot{\lambda}_{ys}(F)$ for a large set of F. Let us discuss what we mean by 'large'. Our aim is to demonstrate that for any $l \in L^2([0,1])$, under suitable conditions we may

adjust $(W_x(l); x \geq 0)$ by a drift to obtain a $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingale, where

$$\tilde{F}_x := \mathscr{F}_x \vee \sigma(L)$$

for all $x \geq 0$. In order to show that a process $(\phi_x; x \geq 0)$ is an $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingale, we must show that $\mathbb{E}[\mathbb{1}_A \phi_x] = \mathbb{E}[\mathbb{1}_A \phi_{x'}]$ whenever $x' < x$ and $A \in \tilde{\mathscr{F}}_{x'}$. It is equivalent to show that $\mathbb{E}[\mathbb{1}_A \xi \phi_x] = \mathbb{E}[\mathbb{1}_A \xi \phi_{x'}]$ whenever $x' < x$, ξ is a bounded, $\mathscr{F}_{x'}$ measurable random variable and $A \in \sigma(L)$. If instead we are able to show that for any $A \in \sigma(L)$ there is a sequence of bounded $\sigma(L)$ measurable random variables F_n which converge almost surely to $\mathbb{1}_A$ and such that $\mathbb{E}[F_n \xi \phi_x] = \mathbb{E}[F_n \xi \phi_{x'}]$ then we may deduce the martingale property by taking limits and using, for example, the dominated convergence theorem.

Suppose that V is a normed vector space with topological dual V^*. In such cases we take $\mathcal{E}(V)$ to be the coarsest toplogy on V such that each $h \in V^*$ is measurable. For any subspace $\mathscr{E}$ of V^* we define a space of bounded functions on V by

$$\mathfrak{F}C_b^\infty(\mathscr{E}) := \{F : V \rightarrow \mathbb{R} : F(\phi) = f(h_1(\phi), \dots, h_n(\phi)),$$

$$h_1, \dots, h_n \in \mathscr{E}, f \in Cb^\infty(\mathbb{R}^n), n \in \mathbf{N}\} \qquad (1.2.1)$$

where $C_b^\infty(\mathbb{R}^n)$ are the smooth, bounded $\mathbb{R}$ valued functions on $\mathbb{R}^n$. Our approach is to show that $\mathbb{E}[F(L)\xi \phi_x] = \mathbb{E}[F(L)\xi \phi_{x'}]$ whenever $F \in \mathfrak{F}C_b^\infty(\mathscr{E})$, $x' < x$ and ξ as above. We now ask when is $\mathscr{E}$ sufficiently big that it is possible, for any $A \in \sigma(L)$, to find a sequence of random variables of the form $F(L)$ approximating $\mathbb{1}_A$ a.s. with $F \in \mathfrak{F}C_b^\infty(\mathscr{E})$? We begin by noting that $\sigma(L) = \sigma\{h(L) : h \in V^*\}$. We now have the following

Lemma 1.1. *If $\mathscr{E}$ is dense in V^*, then for any $A \in \sigma(L)$ there exists a sequence*

$(F_n; n \in \mathbf{N}) \subset \mathfrak{F}C_b^\infty(\mathscr{E})$ *such that* $F_n(L) \to \mathbb{1}_A$ *a.s.*

Proof First of all, take $h \in \mathscr{E}$, $B \in \mathcal{B}(\mathbb{R})$ and set $A = \{\omega \in \Omega : h(L)(\omega) \in B\} = h(L)^{-1}(B)$. In this case, $\mathbb{1}_A(\omega) = \mathbb{1}_B(h(L))$. We may now approximate $\mathbb{1}_B$ pointwise by $f_n \in C_0^\infty(\mathbb{R})$, so that putting $F_n(\phi) = f_n(h(\phi))$, $F_n(L)$ converges almost surely to $\mathbb{1}_A$. If we now take $h \in V^*$, we can find a sequence of $h_n \in \mathscr{E}$ such that for each $\omega \in \Omega$, $h_n(L)(\omega) \to h(L)(\omega)$. Thus we may approximate $\mathbb{1}_{h(L)^{-1}(B)}$ almost surely by $\mathbb{1}_{h_n(L)^{-1}(B)}$. It now follows that for any $B \in \mathcal{B}(\mathbb{R})$ and $h \in V^*$, $\mathbb{1}_{h(L)^{-1}(B)}$ can be approximated almost surely by random variables of the form $F(L)$ where $F \in \mathfrak{F}C_b^\infty(\mathscr{E})$.

∎

In what follows we shall take V as above and $\mathscr{E}$ a dense subset of V^*. Recall that for $l \in L^2([0, \infty))$ we define

$$W_x(l) = \int_0^x \int_0^\infty l(s) \mathrm{d}B_{ys}.$$

Theorem 1.5. *Suppose L is $\mathscr{F}_\infty$ measurable and that $l \in L^2([0, \infty))$. Suppose further that for any $F \in \mathfrak{F}C_b^\infty(\mathscr{E})$, $\dot{\lambda}_{ys}(F)$ admits the factorisation*

$$\int_0^\infty \dot{\lambda}_{ys}(F) \, l(s) \, \mathrm{d}s = \mathbb{E}[F(L)\varrho_l(L, y)|\mathscr{F}_y] \quad a.s. \tag{1.2.2}$$

for this l and all $y \in [0, \infty)$, and that $\varrho_l : \Omega \times V \times [0, \infty) \to \mathbb{R}$ is measurable and satisfies

- $\varrho_l(\phi, y)$ *is $\mathscr{F}_y$-measurable for all $\phi \in V$ and $y \in [0, \infty)$*

- $\varrho_l(L, y) \in L^1(\Omega)$ *for all $y \in [0, \infty)$*

- *for any $x > 0$, $y \mapsto \varrho_l(L, y) \in L^1([0, x])$ a.s.*

Define $\tilde{W}_x(l)$ for $x \in [0, \infty)$ by

$$\tilde{W}_x(l) = W_x(l) - \int_0^x \varrho_l(L, y) \, \mathrm{d}y.$$

Under the above assumptions the following holds:

(a) $(\tilde{W}_x(l); x \geq 0)$ is an $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingale;

(b) If for any two $l_1, l_2 \in L^2([0, \infty))$ there exist ϱ_{l_1} and ϱ_{l_2} satisfying (1.2.2) and the above conditions, then

$$\langle \tilde{W}_.(l_1), \tilde{W}_.(l_2) \rangle_x = x \langle l_1, l_2 \rangle \quad \text{a.s.}$$

for all $x \geq 0$.

(c) ϱ_l is unique in the sense that if $\tilde{\varrho}_l : \Omega \times V \times [0, \infty) \to \mathbb{R}$ also satisfies (1.2.2) and the above conditions then for all $\omega \in \Omega$ except on a set of measure 0, $\tilde{\varrho}_l(L, x)(\omega) = \varrho_l(L, x)$ for all $x \geq 0$.

Proof (a) Using lemma 1.1 and the preceding remarks, we need to show that if $F \in \mathfrak{F}C_b^\infty(\mathscr{E})$, $x' < x$ and ξ is a bounded $\mathscr{F}_{x'}$ measurable random variable then

$$\mathbb{E}[F(L)\xi \tilde{W}_x(l)] = \mathbb{E}[F(L)\xi \tilde{W}_{x'}(l)].$$

Now,

$$
\begin{aligned}
\mathbb{E}\left[F(L)\xi\tilde{W}_x(l)\right] &= \mathbb{E}\left[\mathbb{E}[F(L)\xi\tilde{W}_x(l)|\mathscr{F}_x]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[F(L)\xi\left\{W_x(l) - \int_0^x \varrho_l(L, y)\mathrm{d}y\right\} \middle| \mathscr{F}_x\right]\right] \\
&= \mathbb{E}\left[\xi W_x(l)\mathbb{E}\left[F(L)|\mathscr{F}_x\right]\right] - \mathbb{E}\left[\xi\mathbb{E}\left[\int_0^x F(L)\varrho_l(L, y)\mathrm{d}y \middle| \mathscr{F}_x\right]\right].
\end{aligned}
$$

For ease of notation, write Φ_y for $F(L)\varrho_l(L, y)$. Define an $(\mathscr{F}_x; x \geq 0)$ adapted process $(N_x; x \geq 0)$ by

$$N_x = \mathbb{E}\left[\int_0^x \Phi_y \mathrm{d}y \,\middle|\, \mathscr{F}_x\right] - \int_0^x \mathbb{E}\left[\Phi_y \middle| \mathscr{F}_y\right] \mathrm{d}y$$

so that

$$\mathbb{E}[F(L)\xi\tilde{W}_x(l)] = \mathbb{E}\left[\xi W_x(l)\mathbb{E}\left[F(L)\middle|\mathscr{F}_x\right]\right] - \mathbb{E}\left[\xi\left[N_x + \int_0^x \mathbb{E}\left[\Phi_y \middle| \mathscr{F}_y\right]\mathrm{d}y\right]\right].$$

We now demonstrate that $(N_x; x \geq 0)$ is an $(\mathscr{F}_x; x \geq 0)$ martingale. If $z < x$ then

$$
\begin{aligned}
\mathbb{E}[N_x | \mathscr{F}_z] =& \mathbb{E}\left[\mathbb{E}\left[\int_0^x \Phi_y \mathrm{d}y \,\middle|\, \mathscr{F}_x\right]\middle|\mathscr{F}_z\right] - \mathbb{E}\left[\int_0^x \mathbb{E}\left[\Phi_y | \mathscr{F}_y\right]\mathrm{d}y \,\middle|\, \mathscr{F}_z\right]\\
=& \mathbb{E}\left[\int_0^x \Phi_y \mathrm{d}y \,\middle|\, \mathscr{F}_z\right] - \int_0^z \mathbb{E}\left[\Phi_y | \mathscr{F}_y\right]\mathrm{d}y\\
&- \mathbb{E}\left[\int_z^x \mathbb{E}\left[\Phi_y | \mathscr{F}_y\right]\mathrm{d}y \,\middle|\, \mathscr{F}_z\right]\\
=& \mathbb{E}\left[\int_0^z \Phi_y \mathrm{d}y \,\middle|\, \mathscr{F}_z\right] + \mathbb{E}\left[\int_z^x \Phi_y \mathrm{d}y \,\middle|\, \mathscr{F}_z\right]\\
&- \int_0^z \mathbb{E}\left[\Phi_y | \mathscr{F}_y\right]\mathrm{d}y - \mathbb{E}\left[\int_z^x \mathbb{E}\left[\Phi_y | \mathscr{F}_y\right]\mathrm{d}y \,\middle|\, \mathscr{F}_z\right]\\
=& N_z + \mathbb{E}\left[\int_z^x \Phi_y \mathrm{d}y \,\middle|\, \mathscr{F}_z\right] - \mathbb{E}\left[\int_z^x \mathbb{E}\left[\Phi_y | \mathscr{F}_y\right]\mathrm{d}y \,\middle|\, \mathscr{F}_z\right]
\end{aligned}
$$

To show the last two terms cancel, let $A \in \mathscr{F}_z$.

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_A \mathbb{E}\left[\int_z^x \mathbb{E}\left[\Phi_y | \mathscr{F}_y\right]\mathrm{d}y \,\middle|\, \mathscr{F}_z\right]\right]\\
=&\mathbb{E}\left[\int_z^x \mathbb{E}\left[\mathbb{1}_A \Phi_y | \mathscr{F}_y\right]\mathrm{d}y\right]\\
=&\int_z^x \mathbb{E}[\mathbb{E}[\mathbb{1}_A \Phi_y | \mathscr{F}_y]]\mathrm{d}y\\
=&\mathbb{E}\left[\mathbb{1}_A \int_z^x \Phi_y \mathrm{d}y\right]
\end{aligned}
$$

where the final two inequalities require a use of Fubini's theorem. This means that

$$\mathbb{E}\left[\left.\int_z^x \Phi_y \mathrm{d}y\,\right|\,\mathscr{F}_z\right] = \mathbb{E}\left[\left.\int_z^x \mathbb{E}\left[\Phi_y|\,\mathscr{F}_y\right]\mathrm{d}y\,\right|\,\mathscr{F}_z\right]$$

and hence $(N_x; x \geq 0)$ is an $(\mathscr{F}_x; x \geq 0)$ martingale.

We rewrite the term $\int_0^x \mathbb{E}[\Phi_y|\mathscr{F}_y]\,\mathrm{d}y$ using the factorisation of the stochastic kernel in the following way:

$$\begin{aligned}
\int_0^x \mathbb{E}\left[\Phi_y|\,\mathscr{F}_y\right]\mathrm{d}y &= \int_0^x \mathbb{E}\left[F(L)\varrho_l(L,y)|\,\mathscr{F}_y\right]\mathrm{d}y \\
&= \int_0^x \int_0^1 l(s)\dot{\lambda}_{sy}(F(L))\mathrm{d}s\mathrm{d}y \\
&= \left\langle W_\cdot(l), \mathbb{E}[F(L)] + \int_0^\cdot \int_0^1 \dot{\lambda}_{sy}(F(L))\mathrm{d}B_{ys}\right\rangle_x \\
&= \langle W_\cdot(l), \mathbb{E}[F(L)|\mathscr{F}_\cdot]\rangle_x
\end{aligned}$$

where $\langle \cdot, \cdot \rangle_x$ denotes the quadratic co-variation. Thus

$$\begin{aligned}
&\mathbb{E}[F(L)\xi\tilde{W}_x(l)] \\
&= \mathbb{E}[\xi\underbrace{\{W_x(l)\mathbb{E}[F(L)|\mathscr{F}_x] - \langle W_\cdot(l), \mathbb{E}[F(L)|\mathscr{F}_\cdot]\rangle_x\}}_{(\mathscr{F}_x;x\geq 0)\text{ martingale}}] - \mathbb{E}\left[\xi N_x\right] \\
&= \mathbb{E}\left[\xi\{W_{x'}(l)\mathbb{E}[F(L)|\mathscr{F}_{x'}] - \langle W_\cdot(l), \mathbb{E}[F(L)|\mathscr{F}_\cdot]\rangle_{x'}\}\right] - \mathbb{E}\left[\xi N_{x'}\right]
\end{aligned}$$

and by reversing the argument this is $\mathbb{E}[F(L)\xi\tilde{W}_{x'}(l)]$.

Given the conditions in part (b), note that $\tilde{W}_x(l_1)$ and $\tilde{W}_x(l_2)$ differ from $W_x(l_1)$ and $W_x(l_2)$ respectively by processes with bounded variation, so

$$\langle \tilde{W}_\cdot(l_1), \tilde{W}_\cdot(l_2)\rangle_x = \langle W_\cdot(l_1), W_\cdot(l_2)\rangle_x = x\langle l_1, l_2\rangle \quad \text{a.s.}$$

for all $x \geq 0$. Finally, for (c) we remark that if both $(W_x(l) - \int_0^x \varrho_l(L, y)\mathrm{d}y; x \geq 0)$ and $(W_x(l) - \int_0^x \tilde{\varrho}_l(L, y)\mathrm{d}y; x \geq 0)$ are $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingales then $(\int_0^x (\varrho_l(L, y) - \tilde{\varrho}_l(L, y))\mathrm{d}y; x \geq 0)$ is an $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingale. Thus for all $\omega \in \Omega$ except in some null set, $\int_0^x (\varrho_l(L, y)(\omega) - \tilde{\varrho}_l(L, y)(\omega))\mathrm{d}y = 0$ for all $x \geq 0$, and hence $\varrho_l(L, x)(\omega) = \tilde{\varrho}_l(L, x)(\omega)$ for all $x \geq 0$. Finally, if we have the above expression for $\tilde{W}_x(l_1)$ and $\tilde{W}_x(l_2)$, then

$\blacksquare$

Provided that $\tilde{W}_x$ is defined on a large enough space of test functions, it does in fact have a representation as a stochastic integral. Indeed, suppose we can define $\tilde{W}_x(l)$ as above for all l in a dense subset D of $L^2([0, \infty))$, and suppose that $(l_n; n \in \mathbf{N})$ is a sequence of functions in D which converge in $L^2([0, \infty))$ to $\mathbb{1}_{[0,t]}$. Note that

$$\mathbb{E}[(\tilde{W}_x(l_n) - \tilde{W}_x(l_m))^2] = x\|l_n - l_m\|_2^2.$$

Thus, $(\tilde{W}_x(l_n); n \in \mathbf{N})$ is Cauchy in $L^2(\Omega)$, and we can (uniquely) define $\tilde{W}_x(\mathbb{1}_{[0,t]}) \in L^2(\Omega)$ such that $\mathbb{E}[(\tilde{W}_x(l_n) - \tilde{W}_x(\mathbb{1}_{[0,t]})^2] \to 0$ as $n \to \infty$. Define $\tilde{B}_{xt} := \tilde{W}_x(\mathbb{1}_{[0,t]})$, and for a rectangle R with corners (x_1, t_1) and (x_2, t_2) (where $x_1 < x_2$ and $t_1 < t_2$) define

$$\tilde{W}(R) = \tilde{B}_{x_2,t_2} - \tilde{B}_{x_2,t_1} - \tilde{B}_{x_1,t_2} + \tilde{B}_{x_1,t_1}.$$

It is straightforward to show that $\tilde{W}(R)$ is a centred Gaussian random variable with variance $|R|$, and furthermore that if R_1 and R_2 are disjoint rectangles then $\tilde{W}(R_1)$ and $\tilde{W}(R_2)$ are independent. This is sufficient to show that $\tilde{B}$ is a Brownian sheet. Furthermore, if l is a linear combination of indicator functions, then clearly

$$\tilde{W}_x(l) = \int_0^x \int_0^\infty l(s)\mathrm{d}\tilde{B}_{ys}$$

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and this now extends for all $l \in L^2([0, \infty))$.

If $(\tilde{W}_x(l); x \geq 0)$ is defined on $L^2([0, \infty))$ (or indeed densely defined on $L^2([0, \infty))$), it will in particular be defined for all $l \in C_0^\infty([0, \infty))$. We can say more than this: it is in fact a cylindrical Wiener process in the following sense.

Definition 1.4. *Let $\mathcal{D}'([0, \infty))$ denote Schwartz's space of distributions on $[0, \infty)$ (see, for example, chapter 2 of [Hör90]). A $\mathcal{D}'([0, \infty))$ valued process $(W_x; x \geq 0)$ is called a cylindrical Wiener process if*

- *$W_0 = 0$ a.s. and for every $h \in C_0^\infty([0, \infty))$, $W_x(h)$ is a local martingale;*

- *$\mathbb{P}(\langle W.(h), W.(h)\rangle_x = x\|h\|_2^2 \, \forall x \geq 0) = 1$ for all $h \in C_0^\infty([0, \infty))$.*

Our original process $(W_x; x \geq 0)$ is a cylindrical Wiener process. Furthermore, for all $l_1, l_2 \in C_0^\infty([0, \infty))$, $\langle \tilde{W}.(l_1), \tilde{W}.(l_2)\rangle_x = x\langle l_1, l_2\rangle$ since $(W_x(l); l \in C_0^\infty([0, \infty)), x \geq 0)$ and $(\tilde{W}_x(l); l \in C_0^\infty([0, \infty)), x \geq 0)$ share the same co-variation structure. That $(\tilde{W}_x; x \geq 0)$ is a $\mathcal{D}'([0, \infty))$ valued process follows from, for example, lemma 2.2 of [Iwa87].

1.2.4 A brief look forwards.

The challenge is now to find (if possible) a ϱ_l such that

$$\int_0^1 \dot{\lambda}_{ys}(F)\, l(s)\, \mathrm{d}s \;=\; \mathbb{E}[F(L)\varrho_l(L, y)|\mathscr{F}_y] \quad \text{a.s.}$$

for a given L and $l \in L^2([0, \infty))$. In our stochastic heat equation example, we have $L = (u_0, v_0) : \Omega \to E$. This is only one possible L that we might take. Another possibility is to describe a curve

$$\{\gamma(r); r \in [0, 1]\} = \{(x(r), t(r)) \in [0, \infty)^2; r \in [0, 1]\}$$

where we suppose perhaps that x and t are smooth functions. We could then define $L : \Omega \rightarrow C([0,1])$ by

$$L(r) = B_{x(r)t(r)}$$

for $r \in [0,1]$. Thus we add information about the Brownian sheet along some curve into our initial filtration. We shall investigate this further in the second chapter in the hope of producing a stochastic partial differential equation which describes this bridged process.

Before we treat either problem we require a few tools which will help us determine whether or not we can show that

$$\int_0^1 \dot{\lambda}_{ys}(F)\, l(s)\, \mathrm{d}s \;=\; \mathbb{E}[F(L)\varrho_l(L,y)|\mathscr{F}_y] \quad \text{a.s.}$$

It turns out that for certain $\mathscr{F}_\infty$ measurable $F \in L^2(\Omega)$ (and in particular for all $F = F(L)$) there is a closed operator $D : L^2(\Omega) \rightarrow L^2(\Omega; L^2([0,\infty)))$ such that

$$\dot{\lambda}_{ys}(F) = \mathbb{E}[D_{ys}F|\mathscr{F}_y] \quad \text{a.s.} \tag{1.2.3}$$

DF is known as the Malliavin derivative of F, which is a directional derivative in some sense. We will see that

$$\mathbb{E}\left[\int_0^\infty l(s)D_{ys}F(L)\,\bigg|\,\mathscr{F}_y\right] = \mathbb{E}[F(L)\varrho_l(L,y)|\mathscr{F}_y] \quad \text{a.s} \tag{1.2.4}$$

follows through integration by parts with respect to some Gaussian measure (specifically the law of L given $\mathscr{F}_y$).

1.3 A brief review of the Malliavin calculus and Gaussian measures

1.3.1 The derivative and divergence operators

Our interest in Malliavin calculus is solely to find the explicit form of the stochastic kernel $\dot{\lambda}_{ys}(F(L))$ given by the Clark-Ocone formula. In the sequel we work with a Brownian sheet $(B_{xt}; (x, t) \in [0, \infty)^2)$, although the same arguments hold if we restrict the Brownian sheet to $[0, 1]^2$. We define a class of smooth random variables $\mathscr{S}$ by saying that $F \in \mathscr{S}$ if

$$F = f\left(\int_0^\infty \int_0^\infty h_1(y, s)\mathrm{d}B_{ys}, \ldots, \int_0^\infty \int_0^\infty h_n(y, s)\mathrm{d}B_{ys}\right) \qquad (1.3.1)$$

where $f \in C^\infty(\mathbb{R})$ with each derivative having polynomial growth, and $h_1, \ldots, h_n \in L^2([0, \infty)^2)$. In Nualart's notation, we are taking $H = L^2([0, \infty)^2)$ and $W(h) = \int_0^\infty \int_0^\infty h(y, s)\mathrm{d}B_{ys}$. For $A \in \mathcal{B}([0, \infty)^2)$ such that $\mathbb{1}_A \in L^2([0, \infty)^2)$ we will write $W(A) = W(\mathbb{1}_A)$, and in general we write

$$\mathscr{F}_A = \sigma\{B_{ys}; (y, s) \in A\}.$$

Definition 1.5. *The derivative operator* $D : \mathscr{S} \to L^2(\Omega; L^2([0, \infty)^2))$ *is defined for $F \in \mathscr{S}$ by*

$$DF = \sum_{i=1}^n \partial_i f\left(\int_0^\infty \int_0^\infty h_1(y, s)\mathrm{d}B_{ys}, \ldots, \int_0^\infty \int_0^\infty h_n(y, s)\mathrm{d}B_{ys}\right) h_i. \quad (1.3.2)$$

Denote by $\mathbb{D}^{1,2}$ the closure of $\mathscr{S}$ under the norm

$$\|F\|_{1,2}^2 = \mathbb{E}[F^2] + \mathbb{E}[\|DF\|_2^2].$$

The following is proposition 1.2.1 in [Nua06].

Proposition 1.1. $D : \mathbb{D}^{1,2} \to L^2(\Omega; L^2([0, \infty)^2))$ *is a closed operator.*

Remark that since DF is a random variable with values in $L^2(\Omega; L^2([0, \infty)^2))$, we will adopt the notation

$$D_{ys}F = (DF)(y, s).$$

We shall also require the adjoint of D, known as the divergence operator, which we denote by δ. Thus δ is an unbounded operator on $L^2(\Omega; L^2([0, \infty)^2))$ such that for all u in the domain of δ, $\delta(u) \in L^2(\Omega)$ and for any $F \in \mathbb{D}^{1,2}$,

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u \rangle].$$

It is the adjoint of a densely defined unbounded operator, and is therefore closed. We will need the following result:

Lemma 1.2. *Let A be a bounded element of $\mathcal{B}([0, \infty)^2)$ and let $F \in L^2(\Omega)$ be $\mathscr{F}_{A^c}$ measurable. Then $F\mathbb{1}_A \in L^2(\Omega; L^2([0, \infty)^2))$ is in the domain of δ, and furthermore*

$$\delta(F\mathbb{1}_A) = FW(A).$$

This is lemma 1.3.2 in [Nua06]

1.3.2 The Clark-Ocone formula

In this section, we introduce the idea of a Wiener chaos expansion and use it to characterise the derivative and divergence operators. We first introduce the multiple stochastic integral for $f \in L^2(([0, \infty)^2)^m)$. In particular, if $A_1, \ldots, A_m$ are pairwise disjoint, bounded elements of $\mathcal{B}([0, \infty)^2)$ and

$$f(t_1, \ldots, t_m) = \mathbb{1}_{A_1 \times \ldots \times A_m}(t_1, \ldots, t_m)$$

we define

$$I_m(f) = W(A_1) \dots W(A_m)$$

and extend it first to all linear combinations of such f (in a linear way, naturally), and then to $L^2(([0,\infty)^2)^m)$. This is covered in section 1.1.2 of [Nua06], and we also remark that if $\tilde{f}$ represents the symmetrisation of f then $I_m(f) = I_m(\tilde{f})$. We now cite theorem 1.1.2 of [Nua06].

Theorem 1.6. *For any $F \in L^2(\Omega)$ there exist $f_n \in L^2(([0,\infty)^2)^n)$ such that*

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

where the limit is in $L^2(\Omega)$. Here, we define $I_0(f_0) = \mathbb{E}[F]$. Furthermore we may take the f_n to be symmetric, and with this assumption they are unique.

Suppose $F \in \mathbb{D}^{1,2}$ has the above expansion. It now follows that

$$D_{ys}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, (y,s)))$$

(see proposition 1.2.7 of [Nua06] for a proof). The above expansion also leads to a useful description of the divergence operator. Note that if $u \in L^2(\Omega \times [0,\infty)^2)$, since each $u(x,t) \in L^2(\Omega)$ we can write

$$u(x,t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, (x,t)))$$

for some $f_n \in L^2(([0,\infty)^2)^{n+1})$ which are symmetric in the first n variables. Proposition 1.3.7. of [Nua06] tells us that

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

provided that this sum converges in $L^2(\Omega)$, and that this convergence is a suf-

ficient and necessary condition for u to be in the domain of δ.

The Clark-Ocone formula is presented in [Nua06] as proposition 1.3.14 for a one dimensional Brownian motion. Although we are dealing with parameter Brownian sheet, the proof for the equivalent statement is so similar to Nualart's that it is not worth reproducing here.

Proposition 1.2. *Let $F \in \mathbb{D}^{1,2}$ and let $\mathscr{F}_x = \mathscr{F}_{[0,x] \times [0,\infty)}$. Then*

$$F = \mathbb{E}[F] + \delta(u)$$

where we define $u(y, s) = \mathbb{E}[D_{ys} F | \mathscr{F}_y]$.

Where we do need to work a little to adapt the discussion in [Nua06] to our setting is in showing that $\delta(u)$ coincides with the Itô integral $\int_0^\infty \int_0^\infty u(y,s) \mathrm{d}B_{ys}$. For this we need lemma 1.2, which essentially shows that this is true for elementary functions. Specifically, if we take

$$u(x, t) = \sum_{i=1}^{n} F_i \mathbb{1}_{(x_{i-1}, x_i] \times A}(x, t)$$

for $0 \le x_0 < \ldots < x_n < \infty$, $A \in \mathcal{B}([0, \infty))$ bounded and the F_i bounded, $\mathscr{F}_{x_{i-1}}$ measurable random variables for $i = 1, \cdots, n$, then lemma 1.2 gives us

$$\delta(u) = \sum_{i=1}^{n} F_i \int_{x_{i-1}}^{x_i} \int_A \mathrm{d}B_{ys} = \int_0^\infty \int_0^\infty u(y, s) \mathrm{d}B_{ys}.$$

Recall the space $\mathscr{P}_M$ in section 1.1.4 containing limits of such u in the norm $\|u\|_M^2 = \int_0^\infty \int_0^\infty \mathbb{E}[(u(y,s))^2] \mathrm{d}s \mathrm{d}y$, which is of course the norm on $L^2(\Omega; L^2([0,\infty)^2))$. Thus for any $u \in \mathscr{P}_M$ there is a sequence of u_n of the above elementary form such that $\|u_n - u\|_M \to 0$ as $n \to \infty$, and we see immediately that $\delta(u_n)$ converges in $L^2(\Omega)$ to the Itô integral of u. Since δ is a closed operator, the

limit of $\delta(u_n)$ is $\delta(u)$, so that for all $u \in \mathscr{P}_M$, $\delta(u)$ and the Itô integral of u coincide almost surely. $\mathscr{P}_M$ consists of $u \in L^2(\Omega; L^2([0,\infty)^2))$ such that the process $(u(x,\cdot); x \geq 0)$ is $(\mathscr{F}_x; x \geq 0)$ adapted, and in particular contains $(\mathbb{E}[D_{ys}F|\mathscr{F}_y]; (y,s) \in [0,\infty)^2)$ for any $F \in \mathbb{D}^{1,2}$. Thus for all such F we have

$$F = \mathbb{E}[F] + \int_0^\infty \int_0^\infty \mathbb{E}[D_{ys}F|\mathscr{F}_y]\mathrm{d}B_{ys}.$$

1.3.3 Radon Gaussian measures

In section 1.2.4 we made a remark that (1.2.4) can be understood as an integration by parts formula with respect to a Gaussian measure. To understand this we require some tools of Gaussian measure theory. Our setting is a locally convex space X with (topological) dual X^*, and we set $\mathcal{E}(X)$ to be the sigma algebra on X which makes each $l \in X^*$ measurable. Given a measure μ on $(X, \mathcal{E}(X))$ and $l \in X^*$, we may define a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by $\mu \circ l^{-1}(B) = \mu(\{x \in X : l(x) \in B\})$.

Definition 1.6. *A probability measure μ on $(X, \mathcal{E}(X))$ is said to be a centred Gaussian measure if $\mu \circ l^{-1}$ is a centred Gaussian measure on $\mathbb{R}$ for every $l \in X^*$, that is there exists some σ_l^2 such that*

$$\mu \circ l^{-1}(B) = \int_B \frac{1}{\sqrt{2\pi\sigma_l^2}} \exp\left(-\frac{z^2}{2\sigma_l^2}\right) \mathrm{d}z$$

for any $B \in \mathcal{B}(\mathbb{R})$.

Definition 1.7. *A measure μ on $(X, \mathcal{B}(X))$ is Radon if for every $B \in \mathcal{B}(X)$ and $\varepsilon > 0$, there exists a compact K_ε such that $\mu(B \backslash K_\varepsilon) < \varepsilon$.*

Definition 1.8. *A Radon measure μ on $(X, \mathcal{B}(X))$ is a centred Radon Gaussian measure if its restriction to $\mathcal{E}(X)$ is a centred Gaussian measure.*

Let us remark that $\mathcal{E}(X) \subset \mathcal{B}(X)$ is always true, but not necessarily the

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converse. However, if X is, for example, complete, metrisable and separable, then $\mathcal{E}(X) = \mathcal{B}(X)$. Furthermore, all finite measures on $(X, \mathcal{B}(X))$ are Radon (see page 122 of [Sch73] and the appendix of [Bog98]). The examples we shall consider fit into this setting, and therefore to show that they are centred Radon Gaussian measures, we need only check the distributions of $\mu \circ l^{-1}$.

Suppose that L in section 1.2.4 takes values in X and is measurable with respect to $\mathcal{B}(X)$, and define its law on $(X, \mathcal{B}(X))$ by $\mu(A) = \mathbb{P}(L \in A)$. In the examples we shall consider this is a centred Radon Gaussian measure. If $F : X \to \mathbb{R}$ is measurable with respect to $\mathcal{B}(X)$ and $F(L) \in L^1(\Omega)$ then $F \in L^1(X, \mu)$ and

$$\mathbb{E}[F(L)] = \int_X F(x)\mu(\mathrm{d}x).$$

The following lemma is rather weak, but is intended for a specific purpose for which it is strong enough.

Lemma 1.3. *Let $\mathscr{G}$ be a sub σ-algebra of $\mathscr{F}$, and suppose that $L_1, \ldots, L_n$ are real-valued random variables which are independent of $\mathscr{G}$, whilst $L_{n+1}, \ldots, L_{2n}$ are real-valued $\mathscr{G}$-measurable random variables. If $F \in C_b(\mathbb{R}^{2n})$, then*

$$\mathbb{E}[F(L_1, \ldots L_{2n})|\mathscr{G}] = \int_{\mathbb{R}^n} F(x_1, \ldots, x_n, L_{n+1}, \ldots, L_{2n})\mu(\mathrm{d}x) \quad \text{a.s.} \quad (1.3.3)$$

where μ is the law of $(L_1, \ldots, L_n)$ on $\mathbb{R}^n$.

Proof If $F \in B(\mathbb{R}^{2n})$ has the form $F(x) = F_1(x_1) \ldots F_{2n}(x_{2n})$ for bounded F_i, then for any $A \in \mathscr{G}$ we have

$$\mathbb{E}[\mathbb{1}_A F_1(L_1) \ldots F_{2n}(L_{2n})] = \mathbb{E}[\mathbb{1}_A F_{n+1}(L_{n+1}) \ldots F_{2n}(L_{2n})\mathbb{E}[F_1(L_1) \ldots F_n(L_n)]]$$

$$= \mathbb{E}\left[\mathbb{1}_A \int_{\mathbb{R}^n} F_1(x_1) \ldots F_n(x_n)F_{n+1}(L_{n+1}) \ldots F_{2n}(L_{2n})\mu(\mathrm{d}x)\right]$$

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and thus

$$\mathbb{E}[F(L_1, \ldots, L_{2n})|\mathscr{G}] = \mathbb{E}\left[\int_{\mathbb{R}^n} F(x_1, \ldots, x_n, L_{n+1}, \ldots, L_{2n})\mu(\mathrm{d}x) \,\middle|\, \mathscr{G}\right]$$
$$= \int_{\mathbb{R}^n} F(x_1, \ldots, x_n, L_{n+1}, \ldots L_{2n})\mu_1(\mathrm{d}x).$$

If $F \in C_b(\mathbb{R}^{2n})$ it can be approximated pointwise by a sequence $(F_k; k \in \mathbf{N})$ of linear combinations of such functions, so that $F_k(L_1, \ldots, L_{2n}) \to F(L_1, \ldots, L_{2n})$ almost surely. By choosing F_k such that $F_k \leq F$ for all $k \in \mathbf{N}$, it follows that $\mathbb{E}[F_k(L_1, \ldots, F_{2n}|\mathscr{G}] \to \mathbb{E}[F(L_1, \ldots, L_{2n}|\mathscr{G}]$ almost surely as $k \to \infty$. Furthermore, for any $\omega \in \Omega$,

$$\int_{\mathbb{R}^{2n}} F_k(x_1, \ldots, x_n, L_{n+1}(\omega), \ldots, L_{2n}(\omega))\mu(\mathrm{d}x)$$
$$\to \int_{\mathbb{R}^{2n}} F(x_1, \ldots, x_n, L_{n+1}(\omega), \ldots, L_{2n}(\omega))\mu(\mathrm{d}x)$$

as $k \to \infty$, from which (1.3.3) follows.

∎

We will also require the notion of a directional derivative:

Definition 1.9. *We say that $F : X \to \mathbb{R}$ is differentiable in the direction of $h \in X$ at $x \in X$ if*

$$\lim_{t \to 0} \frac{F(x + th) - F(x)}{t}$$

exists, and in such cases we write $\frac{\partial}{\partial h}F(x)$ to denote the limit.

Taking $\mathfrak{F}C_b^\infty(X^*)$ as in (1.2.1) but with X and X^* replacing V and $V*$, we note that for any $h, x \in X$ and $F \in \mathfrak{F}C_b^\infty(X^*)$, F is differentiable in the direction of h at x. In the examples that we shall consider, we will be able to write the left hand side of (1.2.4) as the integral of a directional derivative against a centred Radon Gaussian measure. With this in mind, the following

definition is of use, where we adopt the terminology of [MR92]:

Definition 1.10. *For any Radon measure μ, we say that $h \in X$ is well-μ-admissible if there exists some $\beta^h \in L^1(X, \mu)$ such that*

$$\int_X \frac{\partial}{\partial h} F(x) \mu(\mathrm{d}x) = - \int_X F(x) \beta^h(x) \mu(\mathrm{d}x) \qquad (1.3.4)$$

for all $F \in \mathfrak{F}C_b^\infty(X^*)$.

In the following section we ask which $h \in X$ are well-μ-admissible for a Gaussian measure μ.

1.3.4 Some characterisations of the Cameron-Martin space

Let μ be a centred Radon Gaussian measure on X. For each $l \in X^*$ we have

$$\int_X l(x) \, \mu(\mathrm{d}x) = \int_{\mathbb{R}} (\mu \circ l^{-1})^2(x) \mathrm{d}x < \infty$$

since $\mu \circ l^{-1}$ is an $\mathbb{R}$ valued Gaussian random variable. We define a norm on X by

$$\|h\|_\mu = \sup\{l(h) : l \in X^*, \ \|l\|_{L^2(\mu)} \leq 1\}$$

for $h \in X$, and we set $H_\mu = \{h \in X : \|h\|_\mu < \infty\}$. This is the Cameron-Martin space of μ. We now define a map $\mathcal{C}_\mu : X^* \to (X^*)'$ (where $(X^*)'$ is the algebraic dual of X^*) called the covariance operator by

$$\mathcal{C}_\mu(m)(l) = \int_X m(x) l(x) \mu(\mathrm{d}x)$$

for $m, l \in X^*$. In fact, we may define $\mathcal{C}_\mu : X_\mu^* \to (X^*)'$ in this way, where X_μ^* is the closure of X^* in $L^2(X, \mu)$ under the norm $\| \cdot \|_{L^2(\mu)}$.

Any $h \in X$ can be thought of as an element in the algebraic dual of X^* by

defining

$$h(l) := l(h) \ \forall l \in X^*.$$

We can say more when $h \in H_\mu$. In this case, for $l \in X^*$ we have

$$|h(l)| = |l(h)| = \|l\|_{L^2(\mu)} \left| \left(\frac{l}{\|l\|_{L^2(\mu)}} \right)(h) \right|$$

$$\leq \|l\|_{L^2(\mu)} \|h\|_\mu.$$

We can thus think of h as bounded linear functional on $(X^*, \|\cdot\|_{L^2(\mu)})$, or indeed a bounded linear functional on $(X^*_\mu, \|\cdot\|_{L^2(\mu)})$ by extension. Since X^*_μ is a Hilbert space, the Riesz representation theorem means that there is an $m \in X^*_\mu$ such that, in particular, for any $l \in X^*$,

$$l(h) = h(l) = \langle m, l \rangle_{L^2(\mu)} = \mathcal{C}_\mu(m)(l).$$

We can now identify h with $\mathcal{C}_\mu(m)$. Let us be slightly clearer about what we mean here. For each $m \in X^*_\mu$, $\mathcal{C}_\mu(m)$ is a linear functional on X^*. In our setting, where X is a locally convex space and μ is a Radon measure, it follows from lemma 3.2.1 and theorem A.1.1 of [Bog98] that there is a (unique) element of X (which we shall also denote by $\mathcal{C}_\mu(m)$) such that

$$\mathcal{C}_\mu(m)(l) = l(\mathcal{C}_\mu(m))$$

for all $l \in X^*$. We have shown that if $h \in H_\mu$ then there exists $m \in X^*_\mu$ such that $h = \mathcal{C}_\mu(m)$ in this sense. Furthermore, for any $l \in X^*$ with $\|l\|_{L^2(\mu)} \leq 1$ we see immediately that

$$|l(h)| = |\mathcal{C}_\mu(m)(l)| = |\langle m, l \rangle_{L^2(\mu)}| \leq \|m\|_{L^2(\mu)}$$

and by choosing $l_n \in X^*$ such that $\|l_n - m\|_{L^2(\mu)} \to 0$, we see that $\left\| \left(\frac{l_n}{\|l_n\|_{L^2(\mu)}} \right)(h) \right\|$
$\to \|m\|_{L^2(\mu)}$. We therefore have $\|h\|_\mu = \|m\|_{L^2(\mu)}$. Conversely, if $m \in X^*_\mu$ and
we define $\mathcal{C}_\mu(m) \in X$, the above argument shows that $\|\mathcal{C}_\mu(m)\|_\mu = \|m\|_{L^2(\mu)} <$
∞. We have now shown the following:

Lemma 1.4. $H_\mu = \mathcal{C}_\mu(X^*_\mu)$, and if $h = \mathcal{C}_\mu(m)$, then $\|h\|_\mu = \|m\|_{L^2(\mu)}$. Thus
$\mathcal{C}_\mu$ defines an isometry from X^*_μ onto H_μ, through which H_μ inherits the Hilbert
space structure of X^*_μ.

X^*_μ is known as the reproducing kernel Hilbert space (RKHS for short) of μ.
Note that μ must be Radon for the proof of lemma 3.2.1 [Bog98]. Without this
condition, we cannot define $\mathcal{C}_\mu$ as an isometry from X^*_μ to H_μ. However, if we
denote by Y_μ the closed subspace of X^*_μ which maps into X under $\mathcal{C}_\mu$, then $\mathcal{C}_\mu$
does define an isometry from Y_μ to H_μ.

For any $h \in X$ we define a shift measure μ_h by $\mu_h(A) = \mu(\{x \in X : x + h \in A\})$.
One way to understand the Cameron-Martin space is as the space of shifts h
such that μ_h and μ are equivalent measures. Furthermore, if h does not belong
to H_μ, then μ and μ_h are mutually singular, that is there exist disjoint subsets
A and B of X such that $A \cup B = X$ with $\mu(A) = 1$ and $\mu_h(B) = 1$ (see theorem
2.4.5 of [Bog98]). For $h \in H_\mu$, the Radon-Nikodym density f_h of μ_h with respect
to μ is given by the Cameron-Martin formula

$$f_h(x) = \exp\left(\mathcal{C}_\mu^{-1} h(x) - \frac{1}{2}\|h\|_\mu^2 \right). \qquad (1.3.5)$$

We are now in a position to prove the following, which is proposition 5.1.6 in
[Bog98].

Proposition 1.3. $h \in X$ is well-μ-admissible if and only if $h \in H_\mu$.

Proof This follows from (1.3.5), noting that

$$\int_X \frac{F(x+th) - F(x)}{t} \mu(\mathrm{d}x) = \int_X \left(\frac{\exp\left(t\mathcal{C}_\mu^{-1} h(x) - \frac{t^2}{2}\|h\|_\mu^2 \right) - 1}{t} \right) F(x)\mu(\mathrm{d}x)$$

for $F \in \mathfrak{F}C_b^\infty(X^*)$. The left hand side converges to $\int_X \frac{\partial}{\partial h} F(x)\mu(\mathrm{d}x)$ whilst the right hand side converges to $\int_X \mathcal{C}_\mu^{-1} h(x) F(x)\mu(\mathrm{d}x)$. With reference to (1.3.4), we see that $\beta^h = -\mathcal{C}_\mu^{-1} h$ for $h \in H_\mu$.

Suppose that h is not in H_μ and yet is well-μ-admissible. One may show that well-μ-admissibility implies that

$$\|\mu_{th} - \mu\|_{TV} \le t\|\beta^h \cdot \mu\|_{TV}$$

where $\|\cdot\|_{TV}$ is the total variation norm on signed measures. This is a contradiction since μ and μ_{th} are mutually singular for all $t \in \mathbb{R}$ and hence $\|\mu - \mu_{th}\|_{TV} = 2$ for all t.

2 Bridging the Brownian sheet

2.1 Describing a bridged Brownian sheet

2.1.1 The one dimensional Brownian bridge

In this chapter we will investigate what our enlargement theorem allows us to say about the bridged sheet, that is a sheet which is forced to take some values along a specified curve. To begin with let us consider the one dimensional Brownian bridge. Intuitively this a Brownian motion which we force to be 0, say, at time 1. If B is a Brownian motion, then $B_t - tB_1$ is a Brownian bridge. This is a simple process to understand- we add a drift which pushes B back to 0 when $t = 1$. This ties neatly with the idea that we need some information about the future (in this case, B_1) to describe the bridge. Unfortunately, if we wish to fix a Brownian sheet to be 0 on a curve, it is rather optimistic to think that we might be able to get such a simple expression (except in special cases- see [DPY06] for example). Besides, we still need some reason why the above is the *correct* way to push the Brownian motion towards 0 at 1.

The standard way to define a Brownian bridge B^y forced to hit $y \in \mathbb{R}$ at time 1 is through its finite dimensional distributions:

$$\mathbb{P}(B^y_{t_1} \in \mathrm{d}x_1, \ldots, B^y_{t_n} \in \mathrm{d}x_n) = \mathbb{P}(B_{t_1} \in \mathrm{d}x_1, \ldots, B_{t_n} \in \mathrm{d}x_n | B_1 = y).$$

This we may define using the probability density functions of the B_t, and furthermore, since the distributions are Gaussian, this may be equivalently characterised by the mean $\mathbb{E}[B^y_t] = ty$ and the covariance structure

$$\mathbb{E}[(B^y_s - sy)(B^y_t - ty)] = (s \wedge t)(1 - s \vee t).$$

This equivalence is discussed in [RW87], where it is also demonstrated X is a Brownian bridge from 0 to y if and only if

$$X_t - \int_0^t \frac{y - X_s}{1 - s} \mathrm{d}s$$

is a martingale with quadratic variation t. This ties in with our enlargement result. We have already seen that

$$B_t - \int_0^t \frac{B_1 - B_s}{1 - s} \mathrm{d}s$$

is a martingale with respect to $(\mathscr{F}_t \vee \sigma(B_1); t \in [0, 1])$, from which we may easily calculate the finite dimensional distributions of B conditional on B_1. Suppose now that B^y solves

$$B_t^y - \int_0^t \frac{y - B_s^y}{1 - s} \mathrm{d}s = \tilde{B}_t,$$

where $\{\tilde{B}_t; t \in [0, 1]\}$ is an $(\mathscr{F}_t \vee \sigma(B_1); t \in [0, 1])$ martingale (in fact a Brownian motion, as one soon sees from its quadratic variation). We may define a law μ_y on $C([0, 1])$ by defining it on sets of the form $A = \{\phi \in C([0, 1]) : \phi_{t_1} \in \theta_1, \ldots, \phi_{t_n} \in \theta_n\}$ to be

$$\mu_y(A) = \mathbb{P}(B_{t_1}^y \in \theta_1, \ldots, B_{t_n}^y \in \theta_n).$$

One may show that

$$\mathbb{P}(\{B \in A\} \cap \{B_1 \in D\}) = \int_D \mu_y(A) \mathrm{d}\mathbb{P}_{B_1}(y) \tag{2.1.1}$$

for all $D \in \mathcal{B}(\mathbb{R})$. In this setting we know of the existence of a regular conditional probability with respect to B_1, which we denote by $\mathbb{P}(B \in A | B_1 = y)$, and which satisfies (2.1.1). (The regularity here refers to the measurability of $\mathbb{P}(B \in$

$A|B_1 = y)$ in y.) We can deduce therefore that $\mu_y(A) = \mathbb{P}(B \in A|B_1 = y)$ for $\mathbb{P}_{B_1}$-almost every $y \in \mathbb{R}$. Of course, for $\mathbb{P}_{B_1}$-almost every y in $\mathbb{R}$, we can read Lebesgue-almost every $y \in \mathbb{R}$, which is thus a dense subset of y in $\mathbb{R}$. We do not automatically obtain the above equality for every $y \in \mathbb{R}$ however. This means we cannot say, for example, that $\mathbb{P}(B \in A|B_1 = 0) = \mu_0(A)$. To overcome this, we require some further regularity in y for $\mu_y(A)$- we need that $\mu_y(A)$ is continuous in y. Remark that [RW87] uses the method of Doob's h-transforms to obtain this additional regularity, however this may not be possible for a bridged Brownian sheet.

2.1.2 Enlarging the filtration of a Brownian sheet by the information obtained along a curve

We turn our attention to a Brownian sheet $(B_{xt}; (x, t) \in [0, 1]^2)$ on $(\Omega, \mathscr{F}, \mathbb{P})$, noting that all previous results carry through to the reduced parameter setting. We define a filtration $(\mathscr{F}_x; x \geq 0)$

$$\mathscr{F}_x = \sigma\{B_{ys}; 0 \leq y \leq x, s \in [0, 1]\} \vee \mathcal{N}_{\mathbb{P}}(\mathscr{F}).$$

We would like to take a curve $(x(r), t(r))_{r \in [0,1]}$ with values in $[0, 1]^2$ and define our initial information to be a process $L = (L(r); r \in [0, 1])$ where $L(r) = B_{x(r)t(r)}$ for each $r \in [0, 1]$. Once again we take $\tilde{\mathscr{F}}_x = \mathscr{F}_x \vee \sigma(L)$. Let us make a few remarks about the space in which L takes values. For $h \in L^2([0, 1])$, $\int_0^1 \int_0^1 \int_0^1 |h(r)| \mathbb{1}_{[0,x(r)]}(y) \mathbb{1}_{[0,t(r)]}(s) \mathrm{d}y \mathrm{d}s \mathrm{d}r < \infty$ so by theorem 1.2

$$\int_0^1 h(r) B_{x(r)t(r)} \mathrm{d}r = \int_0^1 \int_0^1 \left(\int_0^1 h(r) \mathbb{1}_{\{r: \, y \leq x(r), \, s \leq t(r)\}} \mathrm{d}r \right) \mathrm{d}B_{ys} \quad \text{a.s.}$$

We will make use of this later, but for now we note that taking $h(t) = 1$ for all $t \in [0, 1]$ gives $B_{x(\cdot)t(\cdot)} \in L^1([0, 1])$ a.s. We thus define a map $L : \Omega \to L^1([0, 1])$.

In fact, we choose $(x(r); r \in [0,1])$ and $(t(r); r \in [0,1])$ to be continuous so that by choosing a continuous version of B, we see that $(L(r); r \in [0,1])$ is almost surely continuous.

Let $F : L^1([0,1]) \to \mathbb{R}$ be bounded with stochastic kernel $\dot{\lambda}_{ys}(F)$. We would like to use the Clark-Ocone formula $\dot{\lambda}_{ys}(F) = \mathbb{E}[D_{ys}F(L)|\mathscr{F}_y]$ (recall section 1.3.2) so we intend that $F(L) \in \mathbb{D}^{1,2}$. This is of course the case if we take $F \in \mathfrak{F}C_b^\infty(\mathscr{E})$, in the notation of lemma 1.1, taking $\mathscr{E}$ to be $C_0^\infty([0,1])$. That is to say, we identify $h \in C_0^\infty([0,1])$ with an element of the dual of $L^1([0,1])$, and we use the notation $\langle h, \cdot \rangle$ to represent the bounded linear functional which maps $\phi \in L^1([0,1])$ to $\int_0^1 h(t)\phi(t)\mathrm{d}t$. Note that $\mathscr{E}$ is dense in the dual of $L^1([0,1])$, so $\mathfrak{F}C_b^\infty(\mathscr{E})$ is sufficiently large to apply theorem 1.5.

For simplicity take $F(\phi) = f(\langle h, \phi \rangle)$, so that $F(L) = f(\langle h, B_{x(\cdot),t(\cdot)} \rangle)$. Theorem 1.2 allows us to write $\langle h, B_{x(\cdot),t(\cdot)} \rangle = \int_0^1 \int_0^1 \left(\int_0^1 \mathbb{1}_{\{z \le x(r)\}} \mathbb{1}_{\{s \le t(r)\}} h(r)\mathrm{d}r \right) \mathrm{d}B_{zs}$, so that

$$F(L) = F\left(\int_0^1 \int_0^1 \langle h, \, \mathbb{1}_{\{z \le x(\cdot)\}} \mathbb{1}_{s \le t(\cdot)}(z,s) \rangle \, \mathrm{d}B_{zs} \right).$$

Recall that 1.2.3 gives us $\dot{\lambda}_{ys}(F(L)) = \mathbb{E}[D_{ys}F(L)|\mathscr{F}_y]$, and we immediately read $D_{ys}F(L)$ in this case to be $\langle h, \mathbb{1}_{\{y \le x(\cdot)\}} \mathbb{1}_{\{s \le t(\cdot)\}} \rangle f'(\langle h, B_{x(\cdot),t(\cdot)} \rangle)$, from which we deduce that for any $l \in L^2([0,1])$

$$\int_0^1 \dot{\lambda}_{ys}(F)\, l(s)\, \mathrm{d}s \;=\; \mathbb{E}\left[f'(\langle h, L \rangle) \left\langle h, \, \mathbb{1}_{\{y \le x(\cdot)\}} \int_0^{t(\cdot)} l(s)\, \mathrm{d}s \right\rangle \, \middle|\, \mathscr{F}_y \right] \quad \text{a.s.}$$

Let us play a bit with the above conditional expectation. For a fixed $y \in (0,1)$ we define processes $(L_y(r); r \in [0,1])$ and $(L^y(r); r \in [0,1])$ by $L_y(r) = B_{y \wedge x(r), t(r)}$ and $L^y(r) = B_{x(r),t(r)} - B_{y \wedge x(r), t(r)}$ for $r \in [0,1]$. Note that $L = L_y + L^y$, that L_y is $\mathscr{F}_y$ measurable, and furthermore that L^y is independent of $\mathscr{F}_y$, which follows from previously discussed properties of the Brownian sheet. Let

$F_y(\phi) = F(L_y + \phi)$ for $\phi \in L^1([0,1])$, and also define a linear operator κ_y : $L^2([0,1]) \to L^1([0,1])$ by

$$\kappa_y l(r) = \mathbb{1}_{\{x_r \geq y\}} \int_0^{t_r} l(s)\, \mathrm{d}s, \quad r \in [0,1].$$

Note that

$$\frac{\partial}{\partial \kappa_y l} F_y(\phi) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (F_y(\phi + \varepsilon \kappa_y l) - F_y(\phi))$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (f(\langle h, L_y \rangle + \langle h, \phi \rangle + \varepsilon \langle h, \kappa_y l \rangle) - f(\langle h, L_y \rangle + \langle h, \phi \rangle))$$

$$= \langle h, \kappa_y l \rangle f'(\langle h, L_y + \phi \rangle)$$

and so

$$f'(\langle h, L \rangle) \left\langle h, \mathbb{1}_{\{x(\cdot) \geq y\}} \int_0^{t(\cdot)} l(s)\, \mathrm{d}s \right\rangle = \left. \frac{\partial}{\partial \kappa_y l} F_y(\phi) \right|_{\phi = L^y}$$

Denoting the law of L^y on $L^1([0,1])$ by μ^y, we may use lemma 1.3 to write

$$\mathbb{E}\left[f'(\langle h, L \rangle) \left\langle h, \mathbb{1}_{\{x(\cdot) \geq y\}} \int_0^{t(\cdot)} l(s)\, \mathrm{d}s \right\rangle \middle| \mathscr{F}_y \right] = \int \frac{\partial}{\partial \kappa_y l} F_y(\phi)\, \mu^y(\mathrm{d}\phi) \quad \text{a.s.}$$

since $L = L_y + L^y$, L^y is independent of $\mathscr{F}_y$ and L_y is $\mathscr{F}_y$ measurable. Our goal is now to find $\varrho_l(\phi, y)$ such that

$$\int \frac{\partial}{\partial \kappa_y l} F_y(\phi) \mu^y(\mathrm{d}\phi) = \int F_y(\phi) \varrho_l(\phi + L_y, y) \mu^y(\mathrm{d}\phi)$$

$$= \mathbb{E}[F(L) \varrho_l(L, y) | \mathscr{F}_y] \tag{2.1.2}$$

in order to be able to apply theorem 1.5.

2.1.3 An equation for $\mathcal{C}_y^{-1}\kappa_y l$

The above integration by parts is rather similar to (1.3.4). So far we have described μ^y as a measure on $L^1([0,1])$, but in fact its support is a smaller space than this. To begin with, $L^y(r)$ is almost surely a continuous function. Furthermore, if $x(r) < y$, or if $t(r) = 0$, then $L^y(r) = 0$. If we set K_y to be the closure of $\{r : x(r) > y\} \cap \{r : t(r) > 0\}$ and $C_{00}(K_y)$ to be the space of continuous functions on K_y which are 0 whenever $x(r) = y$ or $t(r) = y$, then μ^y has its full support on this space. We also define $\mathcal{V}(K_y)$ to be the topological dual of $C_{00}(K_y)$ consisting of signed measures on K_y. We will make the assumption that K_y is the disjoint union of a finite number of closed intervals, $\cup_{i=1}^n \overline{I}_i$. The space of continuous functions on $\overline{I}_i$ which are zero at the end points is separable under the supremum norm (see for example pages 111-112 of [Sch73]). If we lift this restriction at one or both of the end points, the space is still separable since one may show that polynomials with rational coefficients are dense. Since $C_{00}(K_y)$ can now be viewed as a finite product of such spaces, it is a separable Banach space. Thus μ^y is a Radon measure on $C_{00}(K_y)$, and furthermore for any signed measure $\nu \in \mathcal{V}(K_y)$, $\int_{K_y} L^y(r)\nu(\mathrm{d}r)$ is a Gaussian random variable in $\mathbb{R}$ with mean zero. μ^y is therefore a Radon Gaussian measure. We denote by H_y and $\mathcal{C}_y$ the Cameron-Martin space and covariance operator respectively for μ^y.

Proposition 1.3 tells us that if $\kappa_y l \in H_y$, then

$$\int_{C_{00}(K_y)} \frac{\partial}{\partial \kappa_y l} F_y(\phi)\mu^y(\mathrm{d}\phi) = \int_{C_{00}(K_y)} F_y(\phi)\mathcal{C}_y^{-1}\kappa_y l(\phi)\mu^y(\mathrm{d}\phi).$$

If we rewrite this back as an expectation we have

$$\int_0^1 \dot{\lambda}_{ys}(F)\mathrm{d}s = \mathbb{E}[F(L)\mathcal{C}_y^{-1}\kappa_y l(L^y)|\mathscr{F}_y]$$

for all $\mathfrak{F}C_b^\infty(\mathscr{E})$. Thus the main obstacle is to show that $\kappa_y l \in H_y$, which we will generally attempt to do by finding $\mathcal{C}_y^{-1}\kappa_y l$ in $\mathcal{V}(K_y)$ which maps under $\mathcal{C}_y$ to $\kappa_y l$. (Naturally if this fails, it may still be possible to find $\mathcal{C}_y^{-1}\kappa_y l \in H_y'$, but we need not worry about this in our examples.)

Suppose for now that we have $\mathcal{C}_y^{-1}\kappa_y l \in \mathcal{V}(K_y)$. In this case, we define $\varrho_l(\phi, y)$ explicitly by

$$\varrho_l(\phi, y) = \int_{K_y} (\phi(r) - L_y(r)) \mathrm{d}C_y^{-1}\kappa_y l(r) = \int_{K_y} (\phi(r) - B_{y,t(r)}) \mathrm{d}C_y^{-1}\kappa_y l(r).$$

In this case it is clear that $\varrho_l(\phi, y)$ is $\mathscr{F}_y$ measurable for any continuous ϕ and $y \in [0, 1]$. We may use a version of the stochastic Fubini theorem and properties of the Brownian sheet to see that $\varrho_l(L, y)$ is, in fact, in $L^2(\Omega)$, and hence in $L^1(\Omega)$. In order to apply theorem 1.5, it remains to be shown that $y \mapsto \varrho_l(L, y)$ is integrable on $[0, x]$ for any $x \in [0, 1]$. We leave this to be checked in individual cases, however it is clearly the case whenever $y \mapsto \varrho_l(\phi, y)$ is continuous for any $\phi \in C_{00}(K_y)$. This being so, we now have that

$$W_x(l) - \int_0^x \varrho_l(L, y) \mathrm{d}y$$

is an $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingale.

Given $l \in L^2([0, 1])$, our goal is to find some $m \in \mathcal{V}(K_y)$ such that $\kappa_y l = \mathcal{C}_y m$, or rather $m = C_y^{-1}\kappa_y l$. For this it is sufficient to check that $\nu(\kappa_y l) = \nu(\mathcal{C}_y m)$ for every $\nu \in \mathcal{V}(K_y)$, since $\mathcal{V}(K_y)$ separates the points of $C_{00}(K_y)$. $\nu(\mathcal{C}_y m)$ is, by definition, $\langle \nu, m \rangle_{H_y'}$. Thus we search for an m satisfying

$$\nu(\kappa_y l) = \langle \nu, m \rangle_{H_y'}$$

for every $\nu \in \mathcal{V}(K_y)$. The right hand side is given by

$$\mathbb{E}\left[\int_{K_y} L^y(r)\nu(\mathrm{d}r)\int_{K_y} L^y(r')m(\mathrm{d}r')\right]$$

$$= \int_{K_y}\int_{K_y} \mathbb{E}[L^y(r)L^y(r')]m(\mathrm{d}r')\nu(\mathrm{d}r)$$

$$= \int_{K_y}\int_{K_y} (x(r)\wedge x(r') - y)(t(r)\wedge t(r')m(\mathrm{d}r')\nu(\mathrm{d}r)$$

and this must equal $\int_{K_y} \kappa_y l(r)\nu(\mathrm{d}r)$ for all signed measures ν. We will therefore attempt to find a signed measure m such that

$$\kappa_y l(r) = \int_{K_y} (x(r)\wedge x(r') - y)(t(r)\wedge t(r')m(\mathrm{d}r'). \qquad (2.1.3)$$

In fact, our hope is that we can show that $\kappa_y \mathbb{1}_{[0,t]} \in H_y$, so that

$$B_{xt} - \int_0^x\int_0^1 (L(r) - B_{yr})\mathcal{C}_y^{-1}\kappa_y\mathbb{1}_{[0,t]}(\mathrm{d}r)\mathrm{d}y$$

is an $(\tilde{\mathscr{F}}_x; x \in [0,1])$ martingale. If this is the case, we define the process B_{xt}^ϕ by

$$B_{xt}^\phi - \int_0^x\int_0^1 (\phi(r) - B_{yr}^\phi)\mathcal{C}_y^{-1}\kappa_y\mathbb{1}_{[0,t]}(\mathrm{d}r)\mathrm{d}y = M_{xt}$$

where $(M_{xt}; x \in [0,1])$ is the martingale in question. We will usually take ϕ to be either L or some deterministic continuous function. Broadly, we would like to think of it as some continuous function satisfying the boundary conditions of L which is seen by the information $\sigma(L)$. We would like to solve for B^ϕ and determine a law μ_ϕ on $C([0,1]^2)$. Our hope is to be able to treat this in the same way as the law of the Brownian bridge, so that by showing some regularity we might deduce that this law describes a Brownian sheet conditioned to be ϕ

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along $(x(r), t(r))_{r \in [0,1]}$. This would also provide a straightforward approach to determining the covariance structure of the conditioned process. We shall now illustrate this approach with a simple example.

2.1.4 An introductory example

Let $L(r) = B_{1r}$. Our aim is to describe a process which is conditioned to be some continuous function, say, along the line $x = 1$, and our intuition is that we should obtain something which at any time t looks like a bridged Brownian motion. Let us see if this is the case. Our goal is to show that $\kappa_y l$ is in H_y for $l = \mathbb{1}_{[0,t]}$ for any $t \in [0,1]$. It is more convenient however to assume that l is smooth, and then use our results to inform a guess on what $\mathcal{C}_y^{-1} \kappa_y \mathbb{1}_{[0,t]}$ should be. Note that for any $y \in (0,1)$, $K_y = [0,1]$. Thus for a test function $l \in L^2([0,1])$, $\kappa_y l(r) = \int_0^r l(s) \mathrm{d}s$ for all $r \in [0,1]$. We fix $y \in (0,1)$ and we begin by searching for a function $\dot{m}$ such that

$$\int_0^r l(s) \mathrm{d}s = \int_0^1 (1-y)(r \wedge r') \dot{m}(r') \mathrm{d}r' \tag{2.1.4}$$

for all $r \in [0,1]$. Note that we are searching for $m \in \mathcal{V}([0,1])$ with the additional simplification that $m(\mathrm{d}r) = \dot{m}(r)\mathrm{d}r$. Two differentiations of the above equation 2.1.4 lead to

$$\dot{m}(r) = -\frac{\dot{l}(r)}{1-y}.$$

Note that

$$-\int_0^1 (r \wedge r')\dot{l}(r')\mathrm{d}r' = \int_0^r l(s)\mathrm{d}s - l(1)$$

so the above $\dot{m}$ is only a solution if $l(1) = 0$. Now take $l = \mathbb{1}_{[0,t]}$ for $t < 1$. We do indeed have $l(1) = 0$, and it is not difficult to show that

$$\mathcal{C}_y^{-1} \kappa_y \mathbb{1}_{[0,t]}(\mathrm{d}r) = -\frac{\delta_0(\mathrm{d}r) - \delta_t(\mathrm{d}r)}{1-y}.$$

Thus $\mathcal{C}_y^{-1}\kappa_y\mathbb{1}_{[0,t]}(L^y) = \int_0^x \frac{B_{1t}-B_{yt}}{1-y}\mathrm{d}y$ and so

$$B_{xt} - \int_0^x \frac{B_{1t} - B_{yt}}{1-y}\mathrm{d}y$$

is an $(\mathscr{F}_x \vee \sigma(L); x \in [0,1])$ martingale. We denote this martingale by $(M_{xt}; x \in [0,1])$ and remark that its quadratic variation is xt. Referring back to section 1.2.3, there is a modification of M which is a Brownian sheet. We now want to consider putting $L = \phi$ for some continuous function ϕ such that $\phi(0) = 0$. We now define a process B^ϕ on $[0,1] \times [0,1]$ by

$$B_{xt}^\phi = M_{xt} + \int_0^x \frac{\phi(t) - B_{yt}^\phi}{1-y}\mathrm{d}y.$$

We may solve for B^ϕ to get

$$B_{xt}^\phi = x\phi(t) + (1-x)\int_0^x \int_0^t \frac{1}{1-y}\mathrm{d}M_{ys}.$$

For any $t \in [0,1]$, $(M_{xt}; x \in [0,1])$ is a martingale with respect to the enlarged filtration with quadratic variation

$$\int_0^x \int_0^1 \mathbb{1}_{[0,t]}\mathrm{d}r\mathrm{d}y = xt$$

so we find the covariance function of B_{xt}^ϕ, denoted $c((x,t),(x',t'))$, to be

$$
\begin{aligned}
c((x,t),(x',t')) =& (1-x)(1-x')t \wedge t' \int_0^{x \wedge x'} \frac{1}{(1-y)^2}\mathrm{d}y \\
=& (1-x)(1-x')t \wedge t' \left(\frac{1}{1-x \wedge x'} - 1\right) \\
=& \begin{cases} (1-x)x't \wedge t' & x > x' \\ (1-x')xt \wedge t' & x < x' \end{cases}
\end{aligned}
$$

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If we take $t = t'$, this does indeed look like the covariance structure of a scaled one dimensional Brownian bridge.

Let us briefly discuss what happens when $t = 1$. Of course, $\mathbb{1}_{[0,1]}(1) \neq 0$, so we cannot use the same argument as above. On the other hand, if we take $t_n < 1$ such that $t_n \to 1$, then B_{xt_n} and $B_{1t_n} - B_{yt_n}$ converge in $L^2(\Omega)$ to B_{x1} and $B_{11} - B_{y1}$ respectively. Thus if $x' < x$ and $A \in \tilde{\tilde{\mathscr{F}}}_{x'}$ we may deduce that

$$\mathbb{E}\left[\mathbb{1}_A\left(B_{x1} - \int_0^x \frac{B_{11} - B_{y1}}{1-y}\mathrm{d}y\right)\right] = \mathbb{E}\left[\mathbb{1}_A\left(B_{x'1} - \int_0^{x'} \frac{B_{11} - B_{y1}}{1-y}\mathrm{d}y\right)\right],$$

in other words $\left(B_{x1} - \int_0^x \frac{B_{11}-B_{y1}}{1-y}\mathrm{d}y; x \in [0,1]\right)$ is an $(\tilde{\mathscr{F}}_x; x \in [0,1])$ martingale.

Actually we do not need this argument to deal with $\mathbb{1}_{[0,1]}$, or any other test function l with $l(1) \neq 0$. We could instead simply take

$$m(\mathrm{d}r) = -\frac{\dot{l}(r)}{1-y}\mathrm{d}r + \frac{l(1)}{1-y}\delta_1(\mathrm{d}r).$$

The point is we are looking for a semimartingale decomposition for $W_x(l) = \int_0^x \int_0^1 l(s)\mathrm{d}B_{ys}$. If $l(1) \neq 0$, we could simply define $\tilde{l}$ by $\tilde{l}(1) = 0$ and $\tilde{l}(s) = l(s)$ if $s \neq 1$. Now $W_x(l)$ and $W_x(\tilde{l})$ are almost surely the same, and since our original method produces a semimartingale decomposition for $W_x(\tilde{l})$, this is also a semimartingale decomposition for $W_x(l)$ also. In other examples, however, we will see that there really are conditions which we must impose on l so that $\kappa_y l \in H_y$ and which cannot be dealt with in this way, and we discuss this in the next section.

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2.2 An expression for B^ϕ for general curves.

2.2.1 A corollary to theorem 1.5.

Recall that our object is to demonstrate for $l \in L^2([0,1])$ that $\kappa_y l \in H_y$, and for this we aim to find some $m \in \mathcal{V}(K_y)$ satisfying equation (2.1.3). If we can do this, we note that whenever $x(r) = y$ or $t(r) = 0$ then the right hand side of (2.1.3) is 0. Thus we have no hope of finding a solution unless $\kappa_y l \in C_{00}(K_y)$. Of course, if $t(r) = 0$ then $\kappa_y l(r) = 0$. However, if $x(r) = y$ we also require

$$\kappa_y l(r) = \int_0^{t(r)} l(s)\mathrm{d}s = 0.$$

In most cases, there is little hope of finding a large class of l which satisfy this condition for every r that satisfies $x(r) = y$ for some $y \in (0,1)$. Unless $x(r)$ is constant, by varying y we usually obtain at least some interval of values of r on which $x(r) = y$ for some r. The above condition will then imply that l is 0 at least on some interval, unless perhaps $t(r)$ is constant. For example, if $x(r) = t(r) = r$ the only l which satisfies the right conditions for all y is $l = 0$. In many cases it is simply not possible to apply theorem 1.5 for any l except $l = 0$. To get around this, we can rephrase theorem 1.5 as demonstrating that if $\kappa_y l \in H_y$ then

$$\int_0^x \int_0^1 l(s)\mathrm{d}B_{ys} - \int_0^x \mathcal{C}_y^{-1} \kappa_y l(L^y)\mathrm{d}y$$

is an $(\tilde{\mathscr{F}}_x; x \in [0,1])$ martingale. In cases where this is possible, we have $\varrho_l(L, y) = \mathcal{C}_y^{-1} \kappa_y l(L^y)$. Our hope is that if we can find some correction $\sigma_y l$ such that $\kappa_y(l - \sigma_y l) \in H_y$, then we might deduce that $\int_0^x \int_0^1 (l(s) - \sigma_y l(s))\mathrm{d}B_{ys} - \int_0^x \mathcal{C}_y^{-1} \kappa_y(l - \sigma_y l)(L^y)\mathrm{d}y$ is an $(\tilde{\mathscr{F}}_x; x \in [0,1])$ martingale. We note that for those l for which $\kappa_y l \in H_y$, $l \mapsto \varrho_l(L, y)$ defines a linear functional. We will often write $\langle l, \varrho_l(L, y)\rangle$ instead of $\varrho_l(L, y)$ to emphasise this linearity.

Proposition 2.1. *If for every $l \in L^2([0,1])$, $y \in [0,1]$ and $F \in \mathfrak{F}C_b^\infty(C_0^\infty([0,\infty)))$ there is a decomposition of the form*

$$\int_0^1 \dot{\lambda}_{ys}(F)\, l(s)\, \mathrm{d}s \;=\; \mathbb{E}[F(L)\langle l - \sigma_y l, \varrho(L,y)\rangle | \mathscr{F}_y] + \int_0^1 \dot{\lambda}_{ys}(F)\, \sigma_y l(s)\, \mathrm{d}s \quad \text{a.s.}$$

where ϱ satisfies the conditions in theorem 1.5 and $\sigma_y : L^2([0,1]) \to C^\infty[0,1]$, $y \in [0,1]$ is a family of linear operators such that for every $l \in L^2([0,1])$, $y \mapsto \sigma_y l$ is measurable and $\int_0^1 \int_0^1 (\sigma_y l(s))^2 \mathrm{d}s\mathrm{d}y < \infty$, then for each $l \in L^2([0,1])$,

$$M_x(l) \;:=\; W_x(l) - \int_0^x \langle l - \sigma_y l, \varrho(L,y)\rangle\, \mathrm{d}y - \int_0^x \int_0^1 \sigma_y l(s)\, \mathrm{d}B_{ys}$$

is an $(\tilde{\mathscr{F}}_x; x \in [0,1])$ martingale.

Proof We refer to the proof of theorem 1.5. Replace $\tilde{W}_x(l)$ in that case by $W_x(l) - \int_0^x \langle l - \sigma_y l, \varrho(L,\cdot,y)\rangle \mathrm{d}y$ and set $\Phi_y = F(L)\langle l - \sigma_y l, \varrho(L,\cdot,y)\rangle$. Following the same steps we have

$$\int_0^x \mathbb{E}[\Phi_y|\mathcal{F}_y]\mathrm{d}y = \langle \mathbb{E}[F(L)|\mathcal{F}_\cdot], W_\cdot(l)\rangle_x - \left\langle \int_0^\cdot \int_0^1 \sigma_y l(s)\mathrm{d}B_{ys}, \mathbb{E}[F(L)|\mathcal{F}_\cdot]\right\rangle_x.$$

Thus

$\mathbb{E}[F(L)\xi M_x(l)]$

$= \mathbb{E}[\xi(W_x(l)\mathbb{E}[F(L)|\mathcal{F}_x] - \langle W_\cdot(l), \mathbb{E}[F(L)|\mathcal{F}_\cdot]\rangle_x)]$

$\quad - \mathbb{E}\left[\xi\left(\mathbb{E}[F(L)|\mathcal{F}_x]\int_0^x \int_0^1 \sigma_y(s)\mathrm{d}B_{ys} - \left\langle \mathbb{E}[F(L)|\mathcal{F}_\cdot], \int_0^\cdot \int_0^1 \sigma_y l(s)\mathrm{d}B_{ys}\right\rangle_x\right)\right]$

$\quad - \mathbb{E}[\xi N_x]$

and the result follows as before.

$\blacksquare$

We make a couple of points regarding σ_y. We could simply choose $\sigma_y l \in L^2([0,1])$ for each l in an unrelated manner. Once again, the above proposition is on one level a statement about $W_x(l)$ for some individual l. We previously related the different equations through the correlations of the different $W_x(l)$. In this case, we also have the concern that the equations may not be linear. For the purpose of solving these equations later, we will require linearity, and we thus choose σ_y in a linear way. Our decision to take $\sigma_y l \in C^\infty([0,1])$ for all $l \in L^2([0,1])$ is purely for convenience, and as we shall see later, we will never have a problem finding such a σ_y.

2.2.2 Applying our enlargement result for a general curve.

Let us outline a general approach for conditioning the Brownian sheet along a curve $((x(r), t(r)); r \in [0,1])$. We wish to apply proposition 2.1, which we may do if we can find a family of linear operators $\sigma_y : L^2([0,1]) \to C^\infty([0,1])$ for $y \in [0,1]$ and $\varrho : L^2([0,1]) \times C_{00}(K_y) \times [0,1] \times \Omega \to \mathbb{R}$ such that

$$\int_0^1 \mathbb{E}[D_{ys}F(L)|\mathscr{F}_y](l(s) - \sigma_y l(s))\mathrm{d}s = \mathbb{E}[F(L)\langle l - \sigma_y l, \varrho(L, y)\rangle|\mathscr{F}_y] \quad \text{a.s.}$$

for all $y \in [0,1]$ and $F \in \mathfrak{F}C_b^\infty(C_0^\infty([0,\infty)))$ (as well as certain other conditions). We may do this if and only if $\kappa_y(l - \sigma_y l) \in H_y$, and in this case

$$\langle l - \sigma_y l, \varrho(L, y)\rangle = \mathcal{C}_y^{-1}\kappa_y(l - \sigma_y l)(L^y)$$
$$= \int_{K_y} (B_{x(r),t(r)} - B_{y,t(r)})\mathcal{C}_y^{-1}\kappa_y(l - \sigma_y l)(\mathrm{d}r)$$

As we saw with equation (2.1.3), this will be true if we can find a signed measure m such that $\kappa_y(l - \sigma_y l)(r) = \int_{K_y}(x(r) \wedge x(r') - y)(t(r) \wedge t(r'))m(\mathrm{d}r')$ for all $r \in K_y$. If we can find such a solution, then $\mathcal{C}_y^{-1}\kappa_y(l - \sigma_y l) = m$ and it follows

that

$$W_x(l) = M_x(l) + \int_0^x \int_{K_y} (B_{x(r),t(r)} - B_{y,t(r)}) \mathcal{C}_y^{-1} \kappa_y(l - \sigma_y l)(\mathrm{d}r)\mathrm{d}y + \int_0^x \int_0^1 \sigma_y l(s)\mathrm{d}B_{ys}$$
(2.2.1)

for all $l \in L^2([0,1])$. Here, $(M_x(l); x \in [0,1])$ is an $(\tilde{\mathscr{F}}_x; x \in [0,1])$ martingale with quadratic variation $\int_0^x \int_0^1 (l(s) - \sigma_y l(s))^2 \mathrm{d}s\mathrm{d}y$. Suppose we take $l = \mathbb{1}_{[0,t]}$ for some $t \in (0,1)$. We introduce the notation $M_{xt} = M_x(\mathbb{1}_{[0,t]})$ and $\sigma_x(\mathbb{1}_{[0,t]}) = \sigma_{xt}$. We then have

$$B_{xt} = M_{xt} + \int_0^x \int_{K_y} (B_{x(r),t(r)} - B_{y,t(r)}) \mathcal{C}_y^{-1} \kappa_y(l - \sigma_{yt})(\mathrm{d}r)\mathrm{d}y + \int_0^x \int_0^1 \sigma_{yt}(s)\mathrm{d}B_{ys}.$$
(2.2.2)

Our intention in fact is to solve the expression

$$B_{xt}^\phi = M_{xt} + \int_0^x \int_{K_y} (\phi(r) - B_{y,t(r)}^\phi) \mathcal{C}_y^{-1} \kappa_y(\mathbb{1}_{[0,t]} - \sigma_{yt})(\mathrm{d}r)\mathrm{d}y + \int_0^x \int_0^1 \sigma_{yt}(s)\mathrm{d}B_{ys}^\phi$$
(2.2.3)

where $\phi \in C([0,1])$. Intuitively, B^ϕ describes the Brownian sheet conditioned to be ϕ along the curve $((x(r), t(r)); r \in [0,1])$.

We cannot deduce by the same reasoning as the end of section 1.2.3 that there exists a Brownian sheet $\tilde{B}$ such that $M_{xt} = \int_0^x \int_0^1 (\mathbb{1}_{[0,t]}(s) - \sigma_{yt}(s))\mathrm{d}\tilde{B}_{ys}$. Nevertheless, we shall define a stochastic calculus for M using the approach of section 1.1.4. We first define a martingale measure $M_x(A) := M_x(\mathbb{1}_A)$. In this case the co-variation measure is

$$Q(A, B, x) = \int_0^x \int_0^1 (\mathbb{1}_A(s) - \sigma_y \mathbb{1}_A(s))(\mathbb{1}_B - \sigma_y \mathbb{1}_B(s))\mathrm{d}s\mathrm{d}y$$

which again is positive and positive definite. The stochastic integral is defined once again by constructing a martingale measure $f \cdot M$ for a certain class of f and then setting $\int_0^x \int_0^1 f(y,s)\mathrm{d}M_{ys} := f \cdot M_x([0,1])$. For f defined by $f(y,s) =$

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$\xi \mathbb{1}_{(x_1, x_2]}(y) \mathbb{1}_A(s)$, where $A \in \mathcal{B}([0,1]^2)$, $x_1 < x_2$ and ξ is a bounded, $\tilde{\mathcal{F}}_{x_1}$ measurable random variable, then we define $f \cdot M$ by $f \cdot M_x(B) = \xi(M_{x_2 \wedge x}(A \cap B) - M_{x_1 \wedge x}(A \cap B))$. In this case the variance of $\int_0^x \int_0^1 f(y,s) \mathrm{d}M_{ys}$ is given by

$$\mathbb{E}\left[\xi^2 \left(M_{x_2 \wedge x}(A) - M_{x_1 \wedge x}(A)\right)^2\right]$$
$$=\mathbb{E}[\xi^2] \int_{x_1 \wedge x}^{x_2 \wedge x} \int_0^1 (\mathbb{1}_A(s) - \sigma_y \mathbb{1}_A(s))^2 \mathrm{d}s \mathrm{d}y$$
$$=\int_0^x \int_0^1 \mathbb{E}\left[\left(\xi \mathbb{1}_{(x_1,x_2]}(y)\mathbb{1}_A(s) - \xi\mathbb{1}_{(x_1,x_2]}(y)\sigma_y\mathbb{1}_A(s)\right)^2\right]$$

where we have used the martingale property of $M_x(A)$ to deduce that ξ and $M_{x_2 \wedge x}(A) - M_{x_1 \wedge x}(A)$ are independent. We may define $\sigma_y f(y, \cdot)$ naturally by fixing y, and the linearity of σ_y now implies that the above variance is

$$\int_0^x \int_0^1 \mathbb{E}[(f(y,s) - \sigma_y f(y,\cdot)(s))^2]\mathrm{d}s\mathrm{d}y.$$

We take $\|f\|_M^2$ to be the above integral with $x = 1$. It now follows that the natural extension of $\int_0^1 \int_0^1 f(y,s)\mathrm{d}M_{ys}$ for all linear combinations of such f has variance $\|f\|_M^2$, and we may proceed as before to define the integral on the completion of this linear hull under $\|f\|_M$, with the resulting integral an $(\tilde{\mathcal{F}}_x; x \in [0,1])$ martingale satisfying

$$\mathbb{E}\left[\left(\int_0^1 \int_0^1 f(y,s)\mathrm{d}M_{ys}\right)^2\right] = \|f\|_M^2.$$

It is easy to see that, for elementary f as above, that $\int_0^1 \int_0^1 f(y,s)\mathrm{d}M_{ys}$ and L are independent since L is $\tilde{\mathcal{F}}_0$ measurable, and this soon extends to every f for which the integral is defined.

2.2.3 The law of B^ϕ viewed as a regular conditional probability.

For the moment let us think of our original Brownian sheet $B = (B_{xt}; (x,t) \in [0,1]^2)$, which is almost surely continuous on $[0,1]^2$, as a canonical process. We take $(C([0,1]^2), \mathcal{B}(C([0,1]^2)))$ to be our underlying measurable space, on which we define a probability measure μ by $\mu(A) = \mathbb{P}(B \in A)$ for all $A \in \mathcal{B}(C([0,1]^2))$. We now define a measurable map $\mathcal{L} : C([0,1]^2) \to C([0,1])$ by $\mathcal{L}f(r) = f(x(r), t(r))$ for $f \in C([0,1]^2)$ and $r \in [0,1]$. $\mathcal{L}$ corresponds to L in the sense that $\mu(\mathcal{L} \in A) = \mathbb{P}(L \in A)$ for $A \in \mathcal{B}(C([0,1]))$.

Definition 2.1. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space,* $(E, \mathcal{E})$ *a measurable space and* $X : \Omega \to E$ *a measurable function. We say that* $\nu : E \times \mathscr{F} \to [0,1]$ *is a regular conditional probability with respect to* X *if*

- $\nu(x, \cdot)$ *is a probability measure on* $\mathscr{F}$ *for all* $x \in E$;

- $x \mapsto \nu(x, A)$ *is* $\mathcal{E}$ *measurable function for all* $A \in \mathscr{F}$;

- *for all* $A \in \mathscr{F}$ *and* $D \in \mathcal{E}$, $\mathbb{P}(A \cap X^{-1}(D)) = \int_D \nu(x, A) \mathbb{P}_X(\mathrm{d}x)$,

where $\mathbb{P}_X$ *is the law of* X *on* $(E, \mathcal{E})$, *that is* $\mathbb{P}_X(D) = \mathbb{P}(X \in D) = \mathbb{P}(X^{-1}(D))$.

Our setting then is the probability space $(C([0,1]^2), \mathcal{B}(C([0,1]^2)), \mu)$, whilst $(E, \mathcal{E}) = (C([0,1]), \mathcal{B}(C([0,1])))$ and $X = \mathcal{L}$. In this case we know there exists a regular conditional probability with respect to $\mathcal{L}$ (see for example [RW87]), which we denote by $\mathbb{P}(B \in \cdot | \mathcal{L} = \phi)$ for all $\phi \in C([0,1])$. Note that $\mu \mathcal{L}^{-1} = \mathbb{P}_L$.

Suppose now that $\phi \in C([0,1])$ is in $\mathrm{supp}(\mathbb{P}_L)$, the topological support of $\mathbb{P}_L$. By this we mean that for any $N \in \mathcal{B}(C([0,1]))$ such that $\phi \in N$, $\mathbb{P}_L(N) > 0$. One soon sees in examples that we require this condition on ϕ if we are to solve (2.2.3) for B^ϕ. We further suppose that for all such ϕ, the equation (2.2.3) has a solution B^ϕ in $C([0,1]^2)$, and we define a measure μ_ϕ on $(C([0,1]^2), \mathcal{B}(C([0,1]^2)))$ by $\mu_\phi(A) = \mathbb{P}(B^\phi \in A)$ for $A \in \mathcal{B}(C[0,1]^2)$. We would

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like to show that $(\mu_\phi; \phi \in C([0,1]))$ is a regular conditional probability with respect to $\mathcal{L}$. Of course, we have not defined μ_ϕ yet for $\phi \notin \text{supp}(\mathbb{P}_L)$. However, if $D \in \mathcal{B}(C([0,1]))$, then there exists an open set D_1 such that $D_1 \supset D \cap \text{supp}(\mathbb{P}_L)^c$ and $\mu(\mathcal{L}^{-1}(D_1)) = 0$. It follows that for all $A \in \mathcal{B}(C([0,1]^2))$, $\int_D \mu_\phi(A)\mathbb{P}_L(\mathrm{d}\phi)) = \int_{D \cap \text{supp}(\mathbb{P}_L)} \mu_\phi(A)\mathbb{P}_L(\mathrm{d}\phi))$. Thus we may choose μ_ϕ however we wish for $\phi \notin \text{supp}(\mathbb{P}_L)$ and not affect the third property of the regular conditional probability. We shall set $\mu_\phi = 0$ for such ϕ. It is now clear that the first property of the regular conditional probability holds. Furthermore, assuming that $\text{supp}(\mathbb{P}_L)^c$ is in $\mathcal{B}(C([0,1]))$, the second property follows if we can show that the map $\phi \mapsto \mu_\phi(A)$ restricted to $\text{supp}(\mathbb{P}_L)$ is measurable for any $A \in \mathcal{B}(C([0,1]^2))$, for example if we can show this restricted map is continuous. Note that since the underlying probability space is assumed to be complete, μ, μ_ϕ and $\mathbb{P}_L$ have natural extensions to the completions of $\mathcal{B}(C([0,1]^2))$ and $\mathcal{B}(C([0,1]))$. Therefore, since $\text{supp}(\mathbb{P}_L)^c$ is contained in a set of $\mathbb{P}_L$ measure zero, we can drop the above assumption by working with the completions of $\mathcal{B}(C([0,1]^2))$ and $\mathcal{B}(C([0,1]))$ under the measures μ and $\mathbb{P}_L$ respectively. This adjustment does not affect the first and third properties at all.

For the third condition we need to demonstrate for all $D \in \mathcal{B}(C([0,1])) \cap \text{supp}(\mathbb{P}_L)$ and $A \in \mathcal{B}(C([0,1]^2))$ that $\mu(A \cap \mathcal{L}^{-1}(D)) = \int_D \mu_\phi(A)\mathbb{P}_L(\mathrm{d}\phi)$. Noting that $\mu(A \cap \mathcal{L}^{-1}(D)) = \mathbb{P}(\{B \in A\} \cap \{L \in D\})$, the third condition becomes

$$\mathbb{P}(\{B \in A\} \cap \{L \in D\}) = \int_D \mu_\phi(A)\mathbb{P}_L(\mathrm{d}\phi).$$

Suppose that the third property holds for some countable set $\{A_n; n \in \mathbf{N}\} \subset \mathcal{B}(C([0,1]^2))$, and we may as well assume that $\{A_n; n \in \mathbf{N}\}$ is increasing. It is clear that the third property holds for A_1^c, and it also holds for $\bigcup_{n \in \mathbf{N}} A_n$ by the monotone convergence theorem. We now deduce that the third property

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holds if we can show it holds for a collection of sets A which is large enough to generate $\mathcal{B}(C([0,1]^2))$.

If we can show that the third property holds for some $A \in \mathcal{B}(C([0,1]))$, it follows that

$$\int \mathbb{1}_D(\phi)\left(\mathbb{P}(\{B \in A\}|L = \phi) - \mu_\phi(A)\right)\mathbb{P}_L(\mathrm{d}\phi) = 0$$

for all $D \in \mathcal{B}(C[0,1])$. This means that $\mu_\phi(A) = \mathbb{P}(\{B \in A\}|L = \phi)$ for every ϕ except those belong to some $\mathcal{N} \in \mathcal{B}(C([0,1]))$ with $\mathbb{P}_L(\mathcal{N}) = 0$. If we now wish to pick some ϕ and condition on $L = \phi$, we have a problem since we do not know if $\phi \in \mathcal{N}$. To overcome this, we instead look to show that $\phi \mapsto \mu_\phi(A)$ is continuous (where convergence in $C([0,1])$ is in the supremum norm). Recall that this would also confirm the second property, and thus that $(\mu_\phi(A) : A \in \mathcal{B}(C([0,1]^2)), \phi \in C([0,1]))$ is a regular conditional probability. Our aim will be to demonstrate that $\mathcal{N}^c$ is dense in $\mathrm{supp}(\mathbb{P}_L)$. In this case we can deduce that $\mu_\phi(A) = \mathbb{P}(B \in A|L = \phi)$ for all $\phi \in \mathrm{supp}(\mathbb{P}_L)$. If we can show this for enough A to generate $\mathcal{B}(C([0,1]^2))$, we may deduce that $\mu_\phi = \mathbb{P}(B \in \cdot|L = \phi)$ for all such ϕ.

2.2.4 Returning to our introductory example.

Let us return to the example of section (2.1.4). We can read from our previous calculations that the Brownian sheet B is given by

$$B_{xt} = xL(t) + (1-x)\int_0^x \int_0^t \frac{1}{1-y}\mathrm{d}M_{ys}.$$

In particular, if we take $A \in \mathcal{B}(C([0,1]^2))$ of the form

$$A := \{u \in C([0,1]^2) : u(x_1, t_1) \in \theta_1, \ldots, u(x_n, t_n) \in \theta_n, x_i, t_i \in [0,1]\}. \quad (2.2.4)$$

then we may express $\mathbb{P}(B \in A|L)$ as some function

$$\psi(x_1, \ldots, x_n, t_1, \ldots, t_n, L(t_1), \ldots, L(t_n)).$$

At the same time, μ_ϕ, the law on $C([0,1]^2)$ of B^ϕ, is given by

$$\mu_\phi(A) = \psi(x_1, \ldots, x_n, t_1, \ldots, t_n, \phi(t_1), \ldots, \phi(t_n)).$$

by clear analogy for any $\phi \in \mathrm{supp}(\mathbb{P}_L)$. We may now deduce that

$$\mathbb{P}(\{B \in A\} \cap \{L \in D\}) = \mathbb{E}[\mathbb{1}_{\{L \in D\}} \psi(x_1, \ldots, x_n, t_1, \ldots, t_n, L(t_1), \ldots, L(t_n))]$$

$$= \int_D \mu_\phi(A) \mathbb{P}_L(\mathrm{d}\phi). \tag{2.2.5}$$

This shows that for $\mathbb{P}_L$ almost every $\phi \in \mathrm{supp}(\mathbb{P}_L)$, $\mathbb{P}(B \in A|L = \phi) = \mu_\phi$. (It is not hard to see that $\phi \in \mathrm{supp}(\mathbb{P}_L)$ if and only if $\phi \in C([0,1])$ and $\phi(0) = 0$.) It is also straightforward to see that $\phi \mapsto \mu_\phi(A)$ is continuous in ϕ. Thus $(\mu_\phi; \phi \in C([0,1]), \phi(0) = 0)$ defines a regular conditional probability with respect to L. We would also like to say that $\mathbb{P}(B \in A|L = \phi) = \mu_\phi(A)$ for all $\phi \in \mathrm{supp}(\mathbb{P}_L)$. This follows if we can show that any set of $\mathbb{P}_L$ measure 1 is dense under the supremum norm in $\mathrm{supp}(\mathbb{P}_L)$. This is indeed the case, and the proof is very similar to that of the example in the next section.

It is not obvious whether we can find a general method for solving equation (2.2.3). We first need to calculate $\mathcal{C}_y^{-1} \kappa_y(\mathbb{1}_{[0,t]} - \sigma_{yt})$, if indeed we can actually do this in non-trivial cases. Our best bet is to solve equation (2.1.3), but it is not difficult to find examples for which finding an explicit solution is difficult. One would hope to find explicit solutions in order to write an explicit form of (2.2.3), but even when we can do this, it is another matter altogether to solve this expression. In the absence of a general method, we present one example

which is tractable and does not appear to be discussed in the literature.

2.3 Bridging on the minor diagonal

Let us apply the approach of section 2.2.2 when $L(r) = B_{1-r,r}$ for $r \in [0, 1]$. We shall suppose that l is a smooth test function on $[0, 1]$, and for a given $y > 0$ we attempt to find $m \in \mathcal{V}(K_y)$ such that $m = \mathcal{C}_y^{-1}\kappa_y l$. Note that $K_y = [0, 1 - y]$, so that for $\kappa_y l$ to be in $C_{00}(K_y)$ we need $\int_0^{1-y} l(r)\mathrm{d}r = 0$. We assume this for now, and attempt to find a function $\dot{m}$ such that

$$\mathbb{1}_{\{r:1-r\geq y\}}\int_0^r l(s)\mathrm{d}s = \int_0^{1-y}((1-r)\wedge(1-r')-y)(r\wedge r')\dot{m}(r')\mathrm{d}r'$$
$$= \int_0^r (1-r-y)r'\dot{m}(r')\mathrm{d}r' + \int_r^{1-y}(1-r'-y)r\dot{m}(r')\mathrm{d}r'$$

Differentiating once on the region $\{1 - r > y\}$ gives

$$l(r) = -\int_0^r r'\dot{m}(r')\mathrm{d}r' + \int_r^{1-y}(1-r'-y)\dot{m}(r')\mathrm{d}r'$$

and a second time gives

$$\dot{l}(r) = -(1-y)\dot{m}(r).$$

To verify that this $\dot{m}$ is a solution note that

$$-\int_0^r \left(\frac{1-r-y}{1-y}\right)r'\dot{l}(r')\mathrm{d}r' - \int_r^{1-y}\left(\frac{1-r'-y}{1-y}\right)r\dot{l}(r')\mathrm{d}r'$$
$$= \left(\int_0^r l(r')\mathrm{d}r' - \int_0^r \left(\frac{r}{1-y}\right)l(r')\mathrm{d}r'\right) - \int_r^{1-y}\left(\frac{r}{1-y}\right)l(r')\mathrm{d}r'$$
$$= \int_0^r l(r')\mathrm{d}r'$$

since we have already assumed that $\int_0^{1-y} l(r)\mathrm{d}r = 0$. Thus

$$\mathcal{C}_y^{-1}\kappa_y l(\mathrm{d}r) = -\frac{\dot{l}(r)}{1-y}\mathrm{d}r.$$

Assuming we may apply proposition 2.1, for every $l \in L^2([0,1])$ we have (formally)

$$
\begin{aligned}
W_x(l) =& M_x(l) - \int_0^x \int_0^{1-y} \frac{1}{1-y} \frac{\mathrm{d}}{\mathrm{d}r}(l(r) - \sigma_y l(r))(B_{1-r,r} - B_{yr})\mathrm{d}r\mathrm{d}y \\
& + \int_0^x \int_0^1 \sigma_y l(s)\mathrm{d}B_{ys}
\end{aligned}
\tag{2.3.1}
$$

where $(M_x(l); x \in [0,1])$ is an $(\tilde{\mathscr{F}}_x; x \in [0,1])$ martingale.

2.3.1 A (non-unique) equation for the Brownian sheet fixed along the diagonal.

We would like to take $l = \mathbb{1}_{[0,t]}$ for some $t \in [0,1]$ to obtain an expression like (2.2.3). Of course $\mathbb{1}_{[0,t]}$ is not smooth, but intuitively we should have $\dot{l}(r) = \delta_0(r) - \delta_t(r)$. Our goal is to find a signed measure m such that

$$
\kappa_y(\mathbb{1}_{[0,t]} - \sigma_{yt})(r) = \int_0^{1-y} ((1-r) \wedge (1-r') - y)(r \wedge r')m(dr'),
$$

to which end we guess that $\mathcal{C}_y^{-1}\kappa_y(\mathbb{1}_{[0,t]} - \sigma_{yt})$ is given by

$$
\mathcal{C}_y^{-1}\kappa_y(\mathbb{1}_{[0,t]} - \sigma_{yt})(dr') = \frac{\delta_t(dr')}{1-y} + \frac{\dot{\sigma}_{yt}(r')}{1-y}dr'.
\tag{2.3.2}
$$

We check that this is what we want: observe that if $t < 1 - y$

$$
\begin{aligned}
& \int_0^{1-y} ((1-r) \wedge (1-r') - y)(r \wedge r')m(dr') \\
=& \frac{1}{1-y}((1-r) \wedge (1-t) - y)(r \wedge t) + \frac{1}{1-y}\int_0^r (1-r-y)r'\dot{\sigma}_{yt}(r')dr' \\
& + \frac{1}{1-y}\int_r^{1-y}(1-r'-y)r\dot{\sigma}_{yt}(r')dr'
\end{aligned}
$$

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$$=r \wedge t - \frac{rt}{1-y} + \frac{1}{1-y} \int_0^r r\sigma_{yt}(r')\mathrm{d}r' - \int_0^r \sigma_{yt}(r')\mathrm{d}r'$$

$$=r \wedge t - \int_0^r \sigma_{yt}(r')\mathrm{d}r' = \kappa_y(\mathbb{1}_{[0,t]} - \sigma_{yt})(r)$$

where we have used $((1-r)\wedge(1-t)-y)(r \wedge t) = (1-y)(r \wedge t) - rt$. If $t > 1-y$, $\delta_t(\mathrm{d}r')$ is 0 for $r' \in [0, 1-y]$. In this case, the right hand side of (2.3.2) only involves the $\dot{\sigma}_{yt}$ part, which by adjusting the argument above is $r - \int_0^r \sigma_{yt}(r')\mathrm{d}r'$ as required.

Take $\phi \in \text{supp}(\mathbb{P}_L)$, which one soon sees is $C_0([0,1])$, the space of $\phi \in C([0,1])$ with $\phi(0) = \phi(1) = 0$. For such ϕ we define $(B_{xt}^\phi; (x,t) \in [0,1]^2)$ as a process satisfying

$$\begin{aligned}
B_{xt}^\phi &= M_{xt} + \int_0^x \int_0^{1-y} \frac{\phi(r) - B_{yr}^\phi}{1-y} \delta_t(\mathrm{d}r) + \int_0^x \int_0^{1-y} \frac{\phi(r) - B_{yr}^\phi}{1-y} \frac{\mathrm{d}}{\mathrm{d}r}(\sigma_{yt}(r)) \\
&\quad + \int_0^x \int_0^1 \sigma_{yt}(s)\mathrm{d}B_{ys}^\phi \\
&= M_{xt} + \int_0^x \mathbb{1}_{\{t \le 1-y\}} \frac{\phi(t) - B_{yt}^\phi}{1-y} \mathrm{d}y + \int_0^x \int_0^{1-y} \frac{\phi(r) - B_{yr}^\phi}{1-y} \frac{\mathrm{d}}{\mathrm{d}r}(\sigma_{yt}(r))\mathrm{d}r\mathrm{d}y \\
&\quad + \int_0^x \int_0^1 \sigma_{yt}(s)\mathrm{d}B_{ys}^\phi
\end{aligned} \tag{2.3.3}$$

For any $y > 0$, $l - \sigma_y l$ has only one condition to satisfy, so we may take $\sigma_y l$ to be constant. Indeed, if we set $\sigma_y l = c_y$, we see that $\int_0^{1-y} l(s)\mathrm{d}s = (1-y)c_y$ and so

$$\sigma_y l(s) = \frac{1}{1-y} \int_0^{1-y} l(r)\mathrm{d}r \quad \forall s \in [0,1]. \tag{2.3.4}$$

In particular, $\sigma_{yt}(r) = \frac{t \wedge (1-y)}{1-y}$. This simplifies the expression for B^ϕ considerably, since $\frac{\mathrm{d}}{\mathrm{d}r}\sigma_{yt}(r) = 0$.

If $t = 1$ then (2.3.3) reduces to $B_{x1}^\phi = M_{x1} + B_{x1}^\phi$ for all $x \in [0,1]$. There is no contradiction here, since $(M_{x1}; x \in [0,1])$ is an $(\tilde{\mathscr{F}}_x; x \in [0,1])$ martingale with

quadratic variation

$$\int_0^x \int_0^1 (\mathbb{1}_{[0,1]}(s) - \sigma_{y1}(s))^2 \mathrm{d}s\mathrm{d}y.$$

If σ_{y1} is constant, then the constant is in fact 1, so that the above quadratic variation is 0. Thus $(M_{x1}; x \in [0,1])$ is almost surely zero, and (2.3.3) becomes $B^\phi_{x1} = B^\phi_{x1}$ for all $x \in [0,1]$, or rather $(B^\phi_{x1}; x \in [0,1])$ is left undetermined by (2.3.3).

2.3.2 Solving for $t < 1 - x$.

We consider B^ϕ separately on the regions $t < 1 - x$ and when $t > 1 - x$. For the time being we shall consider $t < 1 - x$, in which case (2.3.3) becomes

$$B^\phi_{xt} = M_{xt} + \int_0^x \frac{1}{1-y}(\phi(t) - B^\phi_{yt})\mathrm{d}y + \int_0^x \int_0^1 \frac{t}{1-y}\mathrm{d}B^\phi_{ys}$$

where $\int_0^x \int_0^1 \frac{t}{1-y}\mathrm{d}B^\phi_{ys} = t\int_0^x \frac{1}{1-y}\mathrm{d}B^\phi_{y1}$ comes from the equations on $t > 1 - x$. Our aim is to find a solution for (2.3.3) in $C([0,1]^2)$, so we require that the above equation also holds in the limit where $x = 1 - t$.

When dealing with multi-parameter processes, we shall use d_y to refer to the differential in the parameter y only, and d to refer to the differential in both parameters. In differential form we have (after dividing by $1 - x$)

$$\frac{1}{1-x}\mathrm{d}_x B^\phi_{xt} + \frac{1}{(1-x)^2}B^\phi_{xt}\mathrm{d}x = \frac{1}{1-x}\mathrm{d}_x M_{xt} + \frac{\phi(t)}{(1-x)^2}\mathrm{d}x + \frac{t}{(1-x)^2}\mathrm{d}_x B^\phi_{x1}.$$

The left hand side equals $\mathrm{d}_x\left(\frac{B^\phi_{xt}}{(1-x)}\right)$ so we obtain

$$\frac{B^\phi_{xt}}{1-x} = \int_0^x \frac{1}{1-y}\mathrm{d}_y M_{yt} + \int_0^x \frac{t}{(1-y)^2}\mathrm{d}_y B^\phi_{y1} + \frac{x}{1-x}\phi(t).$$

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In order to satisfy the boundary condition $B^\phi_{x,1-x} = \phi(1-x)$, we require

$$\phi_{1-x} = \int_0^x \frac{1}{1-y} \mathrm{d}_y M_{y,1-x} + \int_0^x \frac{1-x}{(1-y)^2} \mathrm{d}_y B^\phi_{y1}$$

To ensure this is the case, we use the fact that $(B^\phi_{x1}; x \in [0,1])$ is not determined by (2.3.3). We define a process $(U_x,\ x \in [0,1])$ by $U_x = \frac{1}{1-x} \int_0^x \frac{1}{1-y} \mathrm{d}_y M_{y,1-x}$. We now ensure the boundary condition is satisfied without contradicting (2.3.3) by choosing $(B^\phi_{x1}; x \in [0,1])$ to be

$$B^\phi_{x1} = \int_0^x (1-y)^2 \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{\phi(1-y)}{1-y} \right) \mathrm{d}y - \int_0^x (1-y)^2 \mathrm{d}U_y \qquad (2.3.5)$$

which we can tidy up slightly after observing that

$$\int_0^x (1-y)^2 \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{\phi(1-y)}{1-y} \right) \mathrm{d}y = (1-x)\phi(1-x) + 2 \int_0^x \phi(1-y)\mathrm{d}y$$

(using $\phi(1) = 0$).

We can now insert this back into our expression for B^ϕ_{xt} for $t \geq 1 - x$ to obtain

$$
\begin{aligned}
B^\phi_{xt} =& (1-x) \int_0^x \frac{1}{1-y} \mathrm{d}_y M_{yt} - t \int_0^x \frac{1}{1-y} \mathrm{d}_y M_{y,1-x} \\
& + x\phi(t) + t\phi(1-x) \\
=& (1-x) \int_0^x \int_0^t \frac{1}{1-y} \mathrm{d}M_{ys} - t \int_0^x \int_0^{1-x} \frac{1}{1-y} \mathrm{d}M_{ys} \\
& + x\phi(t) + t\phi(1-x). \qquad (2.3.6)
\end{aligned}
$$

Remark that if we put $\phi(t) = B_{1-t,t}$ and go through the same calculations, we see that our original Brownian sheet B satisfies

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$$B_{xt} = \int_0^x \int_0^t \frac{1-x}{1-y} \mathrm{d}M_{ys} - \int_0^x \int_0^{1-x} \frac{t}{1-y} \mathrm{d}M_{ys}$$
$$+ xB_{1-t,t} + tB_{x,1-x}$$

2.3.3   Covariance structure of the bridged sheet on $t < 1 - x$.

We now calculate the conditional covariance of B^ϕ on the region $t < 1 - x$ when σ_y is constant, as given by (2.3.4). In this case

$$\mathbb{E}\left[\left(\int_0^1 \int_0^1 f(y,s)\mathrm{d}M_{ys}\right)^2\right] = \int_0^1 \int_0^1 \left(f(y,s) - \int_0^{1-y} \frac{f(y,t)}{1-y}\mathrm{d}t\right)^2 \mathrm{d}s\mathrm{d}y.$$

Define

$$c((x,t),(x',t')) := \mathbb{E}[(B_{xt}^\phi - \mathbb{E}[B_{xt}^\phi])(B_{x't'}^\phi - \mathbb{E}[B_{x't'}^\phi])].$$

Since $\int_0^x \int_0^t \frac{1}{1-y}\mathrm{d}M_{ys}$ and $\int_0^x \int_0^{1-x} \frac{1}{1-y}\mathrm{d}M_{ys}$ are both centred Gaussian random variables, (2.3.6) implies that $\mathbb{E}[B_{xt}^\phi] = x\phi(t) + t\phi(1-x)$ and hence

$$c((x,t),(x',t')) = \mathbb{E}\left[\left(\int_0^x \int_0^t \frac{1-x}{1-y}\mathrm{d}M_{ys} - \int_0^x \int_0^{1-x} \frac{t}{1-y}\mathrm{d}M_{ys}\right)\right.$$
$$\left. \cdot \left(\int_0^{x'} \int_0^{t'} \frac{1-x'}{1-y}\mathrm{d}M_{ys} - \int_0^{x'} \int_0^{1-x'} \frac{t'}{1-y}\mathrm{d}M_{ys}\right)\right]$$

Writing $f_{xt}(y,s) = \frac{1-x}{1-y}\mathbb{1}_{[0,x]}(y)\mathbb{1}_{[0,t]}(s) - \frac{t}{1-y}\mathbb{1}_{[0,x]}(y)\mathbb{1}_{[0,1-x]}(s)$, we have

$$c((x,t),(x',t')) = \int_0^1 \int_0^1 \left(f_{xt}(y,s) - \int_0^{1-y} \frac{f_{xt}(y,r)}{1-y}\mathrm{d}r\right)$$
$$\left(f_{x't'}(y,s) - \int_0^{1-y} \frac{f_{x't'}(y,r)}{1-y}\mathrm{d}r\right) \mathrm{d}s\mathrm{d}y.$$

This is made considerably more simple by noting that for $t < 1-x$, $\int_0^{1-y} \frac{f_{xt}(y,r)}{1-y}\mathrm{d}r = 0$. What remains is the following:

$$c((x,t),(x',t')) = \int_0^{x \wedge x'} \int_0^{t \wedge t'} \frac{(1-x)(1-x')}{(1-y)^2} \mathrm{d}s \mathrm{d}y - \int_0^{x \wedge x'} \int_0^{t \wedge (1-x')} \frac{(1-x)t'}{(1-y)^2} \mathrm{d}s \mathrm{d}y$$

$$- \int_0^{x \wedge x'} \int_0^{t' \wedge (1-x)} \frac{(1-x')t}{(1-y)^2} \mathrm{d}s \mathrm{d}y + \int_0^{x \wedge x'} \int_0^{(1-x) \wedge (1-x')} \frac{tt'}{(1-y)^2} \mathrm{d}s \mathrm{d}y$$

$$= \left[\frac{1}{1 - x \wedge x'} - 1 \right] \{ (1-x)(1-x')(t \wedge t') - (1-x)t'(t \wedge (1-x'))$$

$$- (1-x')t(t' \wedge (1-x)) + tt'((1-x) \wedge (1-x')) \}$$

Taking $x' < x$, this reduces to

$$c((x,t),(x',t')) = x'\{(1-x)(t \wedge t') - t(t' \wedge (1-x))\}.$$

2.3.4 Regularity of solutions

In this section we define $T = \{(x,t) \in [0,1]^2 : t \leq 1 - x\}$. For $\phi \in C_0([0,1])$, we suppose that there is a modification of $(B_{xt}^\phi; (x,t) \in T)$ taking values in $C(T)$, and we let μ_ϕ denote the law of B^ϕ (with $\mu_\phi = 0$ whenever $\phi \in C([0,1])$ is not in $C_0([0,1])$). Let A be as in (2.2.4), except that $A \subset C(T)$ and each (x_i, t_i) must satisfy $t_i \leq 1 - x_i$. We now have

$$\mu_\phi(A) = \mathbb{P}(B_{x_1 t_1}^\phi \in \mathrm{d}\theta_1, \ldots, B_{x_n t_n}^\phi \in \mathrm{d}\theta_n)$$

$$= \mathbb{P}\left(x_i \phi(t_i) + t_i \phi(1 - x_i) + \int_0^{x_i} \int_0^{t_i} \frac{1 - x_i}{1 - y} \mathrm{d}M_{ys} \right.$$

$$\left. - \int_0^{x_i} \int_0^{1 - x_i} \frac{t_i}{1 - y} \mathrm{d}M_{ys} \in \mathrm{d}\theta_i, \, i = 1, \ldots, n \right)$$

If we denote $\int_0^{x_i} \int_0^{t_i} \frac{1 - x_i}{1 - y} \mathrm{d}M_{ys} - \int_0^{x_i} \int_0^{1 - x_i} \frac{t_i}{1 - y} \mathrm{d}M_{ys}$ by Y_i, $(Y_1, \ldots, Y_n)$ is a centred n dimensional Gaussian random variable with covariance matrix Q, say. If we denote by $\psi_{\phi(1 - x_1), \phi(t_1), \ldots, \phi(1 - x_n), \phi(t_n)}$ the density function of an n dimensional $\mathcal{N}((x_i \phi(t_i) + t_i \phi(1 - x_i))_{i=1}^n, Q)$ normal random variable, then we

have

$$\mu_\phi(A) = \psi_{\phi(1-x_1),\phi(t_1),\ldots,\phi(1-x_n),\phi(t_n)}(\theta_1,\ldots,\theta_n)\mathrm{d}\theta_1\ldots\mathrm{d}\theta_n =: q(A,\phi).$$

The original Brownian sheet satisfies

$$B_{xt} = xB_{1-t,t} + tB_{x,1-x} + \int_0^x \int_0^1 \frac{1-x}{1-y}\mathrm{d}\tilde{B}_{ys} - \int_0^x \int_0^1 \frac{t}{1-y}\mathrm{d}\tilde{B}_{ys}$$

so it is clear that $\mathbb{P}(B \in A|L) = q(A,L)$, and thus

$$\begin{aligned}
\mathbb{P}(\{B \in A\} \cap \{L \in D\}) &= \mathbb{E}[\mathbb{1}_{\{L \in D\}}q(A,L)] \\
&= \int_D q(A,\phi)\mathrm{d}\mathbb{P}_L(\phi) \qquad (2.3.7)
\end{aligned}$$

for any $D \in \mathcal{B}(C_0([0,1]))$.

We now know that $\mu_\phi(A) = \mathbb{P}(B \in A|L = \phi)$ for $\mathbb{P}_L$-almost every ϕ, and we also remark that for any $\phi \in C_0([0,1])$, $\mu_\phi(\{u \in C(T) : u(1 - t_1, t_1) = \phi(t_1),\ldots, u(1 - t_n, t_n) = \phi(t_n)\}) = 1$ since the solution (2.3.6) implies that $B^\phi_{1-t,t} = \phi(t)$. We would like to deduce that $\mu_\phi(A) = \mathbb{P}(B \in A|L = \phi)$ for all $\phi \in C_0([0,1])$. We have some subset $D \subset C_0([0,1])$ such that $\mathbb{P}_L(D) = 1$ and for all $\phi \in D$, $\mu_\phi(A) = \mathbb{P}(B \in A|L = \phi)$. The first step in extending this for all $\phi \in C_0([0,1])$ is to show that D is dense in $C_0([0,1])$.

Proposition 2.2. *If $D \in \mathcal{B}(C_0([0,1]))$ and $\mathbb{P}_L(D) = 1$ then D is dense in $C_0([0,1])$.*

Proof First suppose that $0 \notin \overline{D}$. In this case it is possible to find some $\varepsilon > 0$ such that $\{g \in C_0([0,1]) : \|g\|_\infty < \varepsilon\} \cap D = \emptyset$. This would imply that $\mathbb{P}_L(\{g \in C_0([0,1]) : \|g\|_\infty < \varepsilon\}) = 0$, or rather $\mathbb{P}(\sup_{r \in [0,1]} |L_r| < \varepsilon) = 0$. Thus $0 \in \overline{D}$.

Suppose now that f is any element of $C_0([0,1])$ which is not in $\overline{D}$. There must be an $\varepsilon > 0$ such that $\mathbb{P}_L(\{g \in C_0([0,1]) : \|g - f\|_\infty\} < \varepsilon) = 0$. It is not possible for such an h to be in the Cameron-Martin space of $\mathbb{P}_L$ (which we denote by H_L). Indeed, if $f \in H_L$, then the shift measure $(\mathbb{P}_L)_h$ is equivalent to $\mathbb{P}_L$, however the above statement can be rewritten as $(\mathbb{P}_L)_f(\{g \in C_0([0,1]) : \|g\|_\infty < \varepsilon\}) = 0$, which is not true for any measure which is equivalent to $\mathbb{P}_L$. It now follows that $H_L \subset \overline{D}$.

The proof is complete once we observe that H_L is dense in $C_0([0,1])$ under the infinity norm. For this we note that if $h \in C^1([0,1])$ such that $\int_0^1 h(s)\mathrm{d}s = 0$, then $r \mapsto \int_0^r h(s)\mathrm{d}s$ is in H_L. Indeed, we can show this by finding a signed measure $\mathcal{C}^{-1} \int_0^\cdot h(s)\mathrm{d}s$ such that

$$\int_0^1 \int_0^r h(s)\mathrm{d}s\nu(\mathrm{d}r) = \mathbb{E}\left[\int_0^1 L_r\nu(\mathrm{d}r)\int_0^1 L_s\left(\mathcal{C}^{-1}\int_0^\cdot h(r')\mathrm{d}r'\right)(\mathrm{d}s)\right]$$

for all signed measures ν. Putting $\left(\mathcal{C}^{-1}\int_0^\cdot h(r')\mathrm{d}r'\right)(\mathrm{d}s) = -h'(s)\mathrm{d}s$, we see that

$$-\int_0^1 \mathbb{E}[L_r L_s]h'(s)\mathrm{d}s = -\int_0^r (1-r)sh'(s)\mathrm{d}s - \int_r^1 (1-s)rh'(s)\mathrm{d}s$$
$$= (1-r)\int_0^r h(s)\mathrm{d}s - r\int_r^1 h(s)\mathrm{d}s$$
$$= \int_0^r h(s)\mathrm{d}s - r\int_0^1 h(s)\mathrm{d}s = \int_0^r h(s)\mathrm{d}s.$$

It is clear that such functions are dense in $C_0([0,1])$ (for example if $\phi \in C_0([0,1])$ is smooth then $\phi(r) = \int_0^r \phi'(s)\mathrm{d}s$ and $\int_0^1 \phi'(s)\mathrm{d}s = 0$) and the result follows.

∎

For the sake of simplicity take $A = \{u \in C(T) : u(x_0, t_0) \in \theta\}$ where θ is a bounded open interval in $\mathbb{R}$ and $(x_0, t_0) \in T$. For any $\phi \in C_0([0,1])$ there

exists a sequence of $\phi_k \in C_0([0,1])$ such that $\mu_{\phi_k}(A) = \mathbb{P}(B \in A | L = \phi_k)$ and $\|\mu_{\phi_k} - \mu_\phi\|_\infty \to 0$ using the above proposition. Since $\mu_{\phi_k}(A)$ is a bounded sequence, it has a convergent subsequence, so we may assume without loss of generality that $\mu_{\phi_k}(A)$ converges to some limit, c, say. We aim to show that $\mu_\phi(A) = c$. Observe that $\mu_{\phi_k}(A) = \mathbb{P}(B^{\phi_k}_{x_0,t_0} \in \theta)$ and $\mu_\phi(A) = \mathbb{P}(B^{\phi}_{x_0,t_0} \in \theta)$. Our key observation is that

$$B^{\phi_k}_{x_0,t_0} - B^{\phi}_{x_0,t_0} = x_0(\phi_k(t_0) - \phi(t_0)) + t_0(\phi_k(1 - x_0) - \phi(1 - x_0)).$$

In particular, $|B^{\phi_k}_{x_0,t_0} - B^{\phi}_{x_0,t_0}| \leq 2\|\phi_k - \phi\|_\infty$. Thus there exists an increasing sequence $(K_n; n \in \mathbf{N})$ such that

$$B^{\phi}_{x_0,t_0}(\omega) \in \theta \Rightarrow B^{\phi_{K_n}}_{x_0,t_0}(\omega) \in \theta_{\frac{1}{n}} \tag{2.3.8}$$

and

$$B^{\phi_{K_n}}_{x_0,t_0}(\omega) \in \theta \Rightarrow B^{\phi}_{x_0,t_0}(\omega) \in \theta_{\frac{1}{n}}. \tag{2.3.9}$$

Here, $\theta_{\frac{1}{n}} = \{x \in \mathbb{R} : |x - y| < \frac{1}{n} \text{ for some } y \in \theta\}$. (2.3.9) implies that

$$\mathbb{P}(B^{\phi_{K_n}}_{x_0,t_0} \in \theta) \leq \mathbb{P}(B^{\phi}_{x_0,t_0} \in \theta_{\frac{1}{n}})$$

so that if $n \to \infty$, we see that $c \leq \mathbb{P}(B^{\phi}_{x_0,t_0} \in \theta)$. (Actually, we require here that $\mathbb{P}(B^{\phi}_{x_0,t_0} \in \theta) = \mathbb{P}(B^{\phi}_{x_0,t_0} \in \overline{\theta})$, which follows from the observation that the law of $B^{\phi}_{x_0,t_0}$ is absolutely continuous with respect to Lebesgue measure.) On the other hand, (2.3.8) implies that

$$\mathbb{P}(B^{\phi}_{x_0,t_0} \in \theta) \leq \mathbb{P}(B^{\phi_{K_n}}_{x_0,t_0} \in \theta_{\frac{1}{n}}).$$

If we write $d(x_0, t_0) = x_0(\phi_k(t_0) - \phi(t_0)) + t_0(\phi_k(1 - x_0) - \phi(1 - x_0))$ then

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$$|\mathbb{P}(B_{x_0,t_0}^{\phi K_n} \in \theta_{\frac{1}{n}}) - c| \leq |\mathbb{P}(B_{x_0,t_0}^{\phi K_n} \in \theta_{\frac{1}{n}}) - \mathbb{P}(B_{x_0,t_0}^{\phi K_n} \in \theta)| + |\mathbb{P}(B_{x_0,t_0}^{\phi K_n} \in \theta) - c|$$

$$= |\mathbb{P}(B_{x_0,t_0}^{\phi} \in \theta_{\frac{1}{n}} - d(x_0, t_0)) - \mathbb{P}(B_{x_0,t_0}^{\phi} \in \theta - d(x_0, t_0))|$$

$$+ |\mathbb{P}(B_{x_0,t_0}^{\phi K_n} \in \theta) - c|$$

which converges to zero as $n \to \infty$. It follows that $\mathbb{P}(B_{x_0,t_0}^{\phi} \in \theta) \leq c$. Thus $\mu_{\phi_k}(A) \to \mu_\phi(A)$ as $k \to \infty$. We have shown that for all such A, and for all $\phi \in C_0([0, 1])$, $\mu_\phi(A) = \mathbb{P}(B \in A | L = \phi)$. Since these A are sufficient to generate $\mathcal{B}(C(T))$, it follows that $\mu_\phi = \mathbb{P}(B \in \cdot | L = \phi)$ for all $\phi \in C_0([0, 1])$, and in particular $(\mu_\phi; \phi \in C([0, 1]), A \in \mathcal{B}(C([0, 1])))$ is a regular conditional probability.

2.3.5 Alternative choices of σ_y.

We might expect some changes in our solution if we make a different choice of σ_y. After all, the driving process M changes as we vary σ_y. We continue to focus on the region $t < 1 - x$. In this case equation (2.3.3) for B_{xt}^{ϕ} becomes

$$B_{xt}^{\phi} = M_{xt} + \phi_t \int_0^x \frac{1}{1-y} \mathrm{d}y - \int_0^x \frac{B_{yt}^{\phi}}{1-y} \mathrm{d}y$$
$$+ \int_0^x \int_0^{1-y} \frac{1}{1-y} (\sigma_{yt})'(r)(\phi(r) - B_{yr}^{\phi}) \mathrm{d}r \mathrm{d}y + \int_0^x \int_0^1 \sigma_{yt}(s) \mathrm{d}B_{ys}^{\phi}$$

We are going to assume some form on σ_y, namely that we can write $\sigma_{yt}(s) = \psi(y)\xi'(s)$, where ψ is continuous and ξ is continuously differentiable. As ever this must satisfy

$$\int_0^{1-y} \psi(y)\xi'(s) \mathrm{d}s = \int_0^{1-y} \mathbb{1}_{[0,t]}(s) \mathrm{d}s = t \wedge (1 - y)$$

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and so

$$\psi(y) = \frac{t}{\xi(1-y)}$$

for $y < x$, where ξ is chosen so that $\xi(0) = 0$. For this to be well defined and continuous on any interval $[0, x]$ with $x < 1$ we insist that $\xi(y) \neq 0$ for $y < 1$. (For example, we may take $\xi(y) = y^\alpha$ for any $\alpha \geq 1$.) We therefore wish to solve

$$
\begin{aligned}
B_{xt}^\phi =& M_{xt} + \phi(t) \int_0^x \frac{1}{1-y} \mathrm{d}y - \int_0^x \frac{B_{yt}^\phi}{1-y} \mathrm{d}y \\
&+ t \int_0^x \int_0^{1-y} \frac{\xi''(r)}{(1-y)\xi(1-y)} (\phi(r) - B_{yr}^\phi) \mathrm{d}r \mathrm{d}y \\
&+ t \int_0^x \int_0^1 \frac{\xi'(s)}{(1-y)\xi(1-y)} \mathrm{d}B_{ys}^\phi.
\end{aligned}
\quad (2.3.10)
$$

The last two terms are just polynomials in t, which suggests looking for a solution $u_x(t)$ of the form

$$u_x(t) = t\beta_x + v_x(t).$$

We then obtain

$$
\begin{aligned}
t\beta_x + v_x(t) =& M_{xt} + \phi(t) \int_0^x \frac{1}{1-y} \mathrm{d}y - \int_0^x \frac{v_y(t)}{1-y} \mathrm{d}y \\
&+ t \Bigg\{ \int_0^x \int_0^{1-y} \frac{\xi''(r)}{(1-y)\xi(1-y)} (\phi(r) - u_y(r)) \mathrm{d}r \mathrm{d}y \\
&+ \int_0^x \int_0^1 \frac{\xi'(s)}{(1-y)\xi(1-y)} \mathrm{d}u_y(s) - \int_0^x \frac{\beta_y}{1-y} \mathrm{d}y \Bigg\}
\end{aligned}
$$

We now look to solve

$$v_x(t) = M_{xt} + \phi(t) \int_0^x \frac{1}{1-y} \mathrm{d}y - \int_0^x \frac{v_y(t)}{1-y} \mathrm{d}y$$

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and then with this solution we aim to solve

$$\beta_x = \int_0^x \int_0^{1-y} \frac{\xi''(r)}{(1-y)\xi(1-y)}(\phi(r) - u_y(r))\mathrm{d}r\mathrm{d}y$$
$$+ \int_0^x \int_0^1 \frac{\xi'(s)}{(1-y)\xi(1-y)}\mathrm{d}u_y(s) - \int_0^x \frac{\beta_y}{1-y}\mathrm{d}y$$

for β. We already have the solution for $v_x(t)$:

$$v_x(t) = x\phi(t) + \int_0^x \int_0^t \frac{1-x}{1-y}\mathrm{d}M_{yt}.$$

One might initially think that it is now simply a case of solving for β_x. However, we have once again a consistency condition that needs to be satisfied, namely that

$$u_x(1-x) = \phi(1-x).$$

We now have two conditions on β which need to be satisfied. This is not a hopeless situation: recall that in the case of $\sigma_y(r) \propto 1$ there was no defining equation for B_{x1}, giving a degree of freedom which we then lost for the sake of consistency. B_{x1} was the control which forced the Brownian sheet to take the values ϕ along the diagonal. We hope for something similar here. In the meantime, we obtain an expression for β from the above condition on u_x, that is

$$(1-x)\beta_x + x\phi(1-x) + \int_0^x \int_0^{1-x} \frac{1-x}{1-y}\mathrm{d}M_{ys} = \phi(1-x)$$

and thus

$$u_x(t) = t\phi(1-x) + x\phi(t) + (1-x)\int_0^x \int_0^t \frac{1}{1-y}\mathrm{d}M_{ys} - t\int_0^x \int_0^{1-x} \frac{1}{1-y}\mathrm{d}M_{ys}.$$

Is this the same as our original expression for B_{xt}^ϕ? We will show that, whether or not M changes in some sense through a different choice of σ_y, the solution

we obtain for $(B_{xt}^\phi; (x,t) \in T)$ retains the same covariance structure. Thus all the solutions for $(B_{xt}^\phi : (x,t) \in T)$ for different σ_y are versions of each other. To begin with we show that any two solutions of 2.3.10 for a given choice of σ_y must be the same. Indeed, suppose we have two solutions to this expression, and we denote the difference by Φ. Φ must satisfy

$$\Phi_{xt} = \int_0^x \frac{\Phi_{yt}}{1-y} \mathrm{d}y + \Psi_{xt}$$

where Ψ_{xt} has the form $t\Psi(x)$. Thus, Φ is

$$\frac{1}{1-x}\Phi_{xt} = t \int_0^x \frac{1}{1-y}\mathrm{d}\Psi(y).$$

Since both our solutions must be ϕ along the diagonal, we must have $\Phi_{x,1-x} = 0$. It follows that $\Phi_{xt} = 0$ for all x, t with $t < 1 - x$.

We now remark that our solution u_x above has the same covariance structure on $\{(x,t); t < 1-x\}$ as our original solution for B^ϕ. We can essentially make the same calculation, and we refer to section 2.3.3. Indeed, in this case the covariance $c((x,t),(x',t'))$ is

$$\int_0^1 \int_0^1 \left(f_{xt}(y,s) - \sigma_y f(y,\cdot)(s) \right) \left(f_{x't'}(y,s) - \sigma_y f(y,\cdot)(s) \right) \mathrm{d}s\mathrm{d}y$$

where again, $f_{xt}(y,s) = \frac{1-x}{1-y}\mathbb{1}_{[0,x]}(y)\mathbb{1}_{[0,t]}(s) - \frac{t}{1-y}\mathbb{1}_{[0,x]}(y)\mathbb{1}_{[0,1-x]}(s)$. The same calculation can be performed upon observing that $\sigma_y f_{xt}(y,\cdot)(s) = 0$ when $t < 1 - x$. This is clear since

$$\sigma_y f_{xt}(y,\cdot)(s) = \frac{1-x}{1-y}\mathbb{1}_{[0,x]}(y)\sigma_{yt}(s) - \frac{t}{1-y}\mathbb{1}_{[0,x]}(y)\sigma_{yt}(s)$$

$$= \frac{1-x}{1-y}\mathbb{1}_{[0,x]}(y)\frac{t}{\xi(1-y)}\xi'(s) - \frac{t}{1-y}\mathbb{1}_{[0,x]}(y)\frac{1-x}{\xi(1-y)}\xi'(s) = 0$$

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for $0 < y < x$.

2.3.6 Consistency condition.

We have now defined processes $(\beta_x; x \in [0,1])$ and $(u_x(t); x, t \in [0,1])$ such that $u_{1-t}(t) = \phi(t)\ \forall t \in [0,1]$, however we do not yet know that $u_x(t)$ satisfies (2.3.10). For this we require that

$$\beta_x = \int_0^x \int_0^{1-y} \frac{\xi''(r)}{(1-y)\xi(1-y)}(\phi(r) - B_{yr}^\phi)\mathrm{d}r\mathrm{d}y$$
$$+ \int_0^x \int_0^1 \frac{\xi'(s)}{(1-y)\xi(1-y)}\mathrm{d}B_{ys}^\phi - \int_0^x \frac{\beta_y}{1-y}\mathrm{d}y.$$

This should be satisfied for any appropriate choice of ξ, which at first glance is a cause for concern. However, suppose we consider $\sigma_y \xi'$. We require

$$\int_0^{1-y} \psi(y)\xi'(s)\mathrm{d}s = \int_0^{1-y} \xi'(s)\mathrm{d}s$$

or rather

$$(\psi(y) - 1)\xi(1-y) = 0.$$

Thus $\sigma_y \xi' = \xi'$ for all $y < 1$. If we now put $l = \xi'$ in (2.3.1) we obtain a trivial expression.

Lemma 2.1. $\sigma_y : L^2([0,1]) \to L^2([0,1])$ *defined by*

$$\sigma_y l(s) = \frac{\xi'(s)}{\xi(1-y)} \int_0^{1-y} l(r)\mathrm{d}r$$

is a bounded linear transformation with $\sigma_y \xi' = \xi'$. Define $(B_{xt}^\phi; x, t \in [0,1])$ by (2.3.10) for $t \le 1 - x$ and

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$$B_{xt}^{\phi} = M_{xt} + \int_0^x \int_0^{1-y} \frac{1}{1-y} \sigma_{yt}'(s)(\phi(s) - B_{ys}^{\phi}) \mathrm{d}s \mathrm{d}y$$

$$+ \int_0^x \int_0^1 \sigma_{yt}(s) \mathrm{d}B_{ys}^{\phi} \tag{2.3.11}$$

for $t \geq 1 - x$. This system leaves the term $\int_0^x \int_0^1 \xi'(s) \mathrm{d}B_{ys}^{\phi}$ undetermined. Thus any choice for this term is consistent with the system for B^{ϕ}.

Proof The claims on σ_y follow since

$$\|\sigma_y l\|_2^2 \leq \frac{(1-y)\|\xi\|_2^2}{\xi^2(1-y)} \int_0^{1-y} l^2(r) \mathrm{d}r.$$

Let us now approximate ξ' by linear combinations of characteristic functions, say l_n. Thus $l_n - \sigma_y l_n \to \xi' - \sigma_y \xi' = 0$ in $L^2([0,1])$. Formally, it follows that

$$\int_0^x \int_0^1 l_n(s) \mathrm{d}B_{yr}^{\phi} = \int_0^t \int_0^1 l_n(s) \mathrm{d}M_{ys} + \int_0^x \int_0^1 \sigma_y l_n(s) \mathrm{d}B_{ys}^{\phi}$$

$$- \int_0^x \int_0^{1-y} \frac{1}{1-y} \frac{\mathrm{d}}{\mathrm{d}s} (l_n - \sigma_y l_n)(s)(\phi(s) - B_{ys}^{\phi}) \mathrm{d}s \mathrm{d}y.$$

Both $\int_0^x \int_0^1 (l_n(s) - \sigma_y l_n(s)) \mathrm{d}B_{ys}^{\phi}$ and $\int_0^x \int_0^1 l_n(s) \mathrm{d}M_{ys}$ converge to 0 in $L^2(\Omega)$, the latter owing to the fact that

$$\mathbb{E}\left[\left(\int_0^x \int_0^1 l_n(s) \mathrm{d}M_{ys}\right)^2\right] = \int_0^x \int_0^1 (l_n(s) - \sigma_y l(s))^2 \mathrm{d}s \mathrm{d}y.$$

With a little more work one can show using stochastic integration by parts and Fubini theorems that the last term also converges to 0 in $L^2(\Omega)$.

∎

We have now defined a process B^{ϕ} according to a system of equations which do not determine $\int_0^x \int_0^1 \xi'(s) \mathrm{d}B_{ys}^{\phi}$, which we shall denote by Ξ_x. $(\Xi_x; x \geq 0)$ is

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an $(\mathscr{F}_x; x \geq 0)$ martingale, and we remark that for $x \geq 0$ and any continuously differentiable $f \in C([0, x])$, we may define $\int_0^x f(y)\mathrm{d}\Xi_y$ and furthermore it equals $\int_0^x \int_0^1 f(y)\xi'(s)\mathrm{d}B_{ys}^\phi$. since by stochastic integration by parts and (1.2) we have

$$
\begin{aligned}
\int_0^x f(y)\mathrm{d}\Xi_y &= f(x)\Xi_x - \int_0^x f'(y)\Xi_y\mathrm{d}y \\
&= f(x)\Xi_x - \int_0^x f'(z) \int_0^z \int_0^1 \xi'(s)\mathrm{d}B_{ys}^\phi \mathrm{d}z \\
&= f(x)\Xi_x - \int_0^x \int_0^1 \left(\int_y^x f'(z)\mathrm{d}z \right) \xi'(s)\mathrm{d}B_{ys}^\phi \\
&= \int_0^x \int_0^1 f(y)\xi'(s)\mathrm{d}B_{ys}^\phi
\end{aligned}
$$

We may now define Ξ_x by

$$
\begin{aligned}
\frac{1}{(1-x)\xi(1-x)}\mathrm{d}\Xi_x &= \mathrm{d}\beta_x + \frac{\beta_x}{1-x}\mathrm{d}x \\
&\quad - \int_0^1 \frac{\xi''(r)}{(1-x)\xi(1-x)}(\phi(r) - B_{xr}^\phi)\mathrm{d}r\mathrm{d}x
\end{aligned}
$$

and not contradict the underlying system. This ensures that B_{xt}^ϕ really is the solution we are after. Ξ_x is a control which we must choose carefully in order to ensure that B_{xt}^ϕ is ϕ on the diagonal.

2.3.7 Solving for $t > 1 - x$.

So far we have not discussed the solution on the region $t > 1 - x$. (2.3.11) now becomes

$$
\begin{aligned}
B_{xt}^\phi &= M_{xt} - M_{1-t,t} + \phi(t) + \int_{1-t}^x \int_0^{1-y} \frac{1}{1-y}\sigma'_{yt}(s)(\phi(s) - B_{ys}^\phi)\mathrm{d}s\mathrm{d}y \\
&\quad + \int_{1-t}^x \int_0^1 \sigma_{yt}(s)\mathrm{d}B_{ys}^\phi. \tag{2.3.12}
\end{aligned}
$$

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From this it is also clear that

$$B_{x1}^\phi - B_{1-t,1}^\phi = M_{x1} - M_{1-t,1} + \int_{1-t}^x \int_0^{1-y} \frac{1}{1-y} \sigma'_{y1}(s)(\phi(s) - B_{ys}^\phi)\mathrm{d}s\mathrm{d}y$$
$$+ \int_{1-t}^x \int_0^1 \sigma_{y1}(s)\mathrm{d}B_{ys}^\phi.$$

We now note that if $t > 1 - y$, σ_{yt} is given by

$$\int_0^{1-y} \sigma_{yt}(s)\mathrm{d}s = 1 - y.$$

Thus for $t > 1 - y$, $\sigma_{yt} = \sigma_{y1}$. We now deduce that

$$B_{xt}^\phi = \phi(t) + M_{xt} - M_{1-t,1} - M_{1-t,t} + M_{x1} + B_{x1}^\phi - B_{1-t,1}^\phi.$$

We group the M terms together as $M(R_{xt}^\uparrow)$. ($R_{xt}^\uparrow$ is intended to represent the rectangle with bottom left corner (x, t), so that M gives rise to a random function on such sets.) If we now put $\phi(t) = B_{1-t,t}$ we obtain $M(R_{xt}^\uparrow) = B(R_{xt}^\uparrow)$, from which it follows that $M(R) = B(R)$ for any rectangles in the region $t > 1 - x$. This is not dependent on the choice of σ_y. However, B_{x1}^ϕ and $B_{1-t,1}^\phi$ do change for different σ_y, and consequently so does $B_{xt}^\phi = \phi(t) + M(R_{xt}^\uparrow) + B_{x1}^\phi - B_{1-t,1}^\phi$. For example, if we take σ_y to be constant, then B_{x1}^ϕ and $B_{1-t,1}^\phi$ are chosen by the consistency condition (2.3.5), from which it follows that

$$B_{xt}^\phi = M(R_{xt}^\uparrow) + f(t) + (1-x)\phi(1-x) - t\phi(t) + \int_{1-t}^x \phi(1-y)\mathrm{d}y - \int_{1-t}^x (1-y)^2 \mathrm{d}U_y.$$

Recall that U is given by

$$U_x = \frac{1}{1-x} \int_0^x \int_0^{1-x} \frac{1}{1-y} \mathrm{d}M_{ys}.$$

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An integration by parts gives us

$$\int_{1-t}^x (1-y)^2 \mathrm{d}U_y = (1-x)^2 U_x - t^2 U_{1-t} + 2\int_{1-t}^x (1-y)U_y \mathrm{d}y$$

from which we may deduce that

$$
\begin{aligned}
B_{xt}^\phi =&\, M(R_{xt}^\uparrow) + (1-t)\phi(t) + (1-x)\phi(1-x)\int_{1-t}^x \phi(1-y)\mathrm{d}y \\
&- \int_0^x \int_0^{1-x} \frac{1-x}{1-y}\mathrm{d}M_{ys} + \int_0^{1-t}\int_0^t \frac{t}{1-y}\mathrm{d}M_{ys} \\
&- 2\int_{1-t}^x \int_0^{(1-y)\wedge t} \frac{x\wedge(1-s) - y\vee(1-t)}{1-y}\mathrm{d}M_{ys}
\end{aligned}
$$

where we have used theorem 1.2. From this it is possible to write down explicit expressions for the covariance function.

If we take $\sigma_{yt}(s) = \frac{(t\wedge(1-y))\xi'(s)}{\xi(1-y)}$ we cannot do the same as above since we do not have expressions for B_{x1} and $B_{1-t,1}$. We do however have a different control for which we do have an expression. In order to use this we go back to equation (2.3.11). The last term is of course related to our control, and in the notation of the previous section we have

$$
\begin{aligned}
B_{xt}^\phi =&\, M_{xt} - M_{1-t,t} + \phi(t) + \int_{1-t}^x \frac{(1-y)}{\xi(1-y)}\mathrm{d}\Xi_y \\
=&\, M_{xt} - M_{1-t,t} + \phi(t) + \int_{1-t}^x (1-y)\mathrm{d}\beta_y \\
&+ \int_{1-t}^x \beta_y \mathrm{d}y - \int_{1-t}^x \int_0^1 \frac{\xi''(r)}{\xi(1-y)}(\phi(r) - B_{yr}^\phi)\mathrm{d}r\mathrm{d}y
\end{aligned}
$$

with β as given before. Thus it appears as though a different choice of σ_y corresponds to a different control which forces B^ϕ to behave in the way previously described on the region $\{(x,t) \in [0,1]^2 : 0 \le t \le 1-x \le 1\}$, but which induces a non-unique behaviour outside this region.

3 Spatial evolution of solutions of the heat equation with noise.

3.1 Defining a process $((u_x, v_x); x \geq 0)$.

3.1.1 Tail behaviours for u_x and v_x.

In this section we return to our study of the stochastic heat equation, specifically its Markovian evolution in the spatial direction. Let us recall a little notation. $(B_{xt}; x \in \mathbb{R}, t \geq 0)$ once again represents a Brownian sheet on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We define the filtration $(\mathscr{F}_x; x \geq 0)$ by

$$\mathscr{F}_x = \sigma\{B_{ys}; -\infty \leq y \leq x, \ s \in [0, \infty)\} \vee \mathcal{N}_{\mathbb{P}}(\mathscr{F})$$

and we would like to define processes $(u_x; x \geq 0)$ and $(v_x; x \geq 0)$ by

$$\langle h_1, u_x \rangle = \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} h_1(t) g(t - s, x, y) \mathrm{d}t \right) \mathrm{d}B_{ys}$$

and

$$\langle h_2, v_x \rangle = \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} h_2(t) \partial_2 g(t - s, x, y) \mathrm{d}t \right) \mathrm{d}B_{ys}$$

respectively for some test functions h_1 and h_2. The first thing we will do is identify a class of continuous test functions h_1 and h_2 for which these are defined. To get an idea of what is needed, let us first take h_1 and h_2 to be of the form $(1 + t)^{\alpha}$ and determine values of α for which $E[\langle h_1, u_x \rangle^2]$ and $E[\langle h_2, v_x \rangle^2]$ are defined. We first take $h_1(t) = (1 + t)^{\alpha}$ for all $t \geq 0$. In this case

$$\mathbb{E}[\langle h_1, u_x \rangle^2] = \int_0^{\infty} \int_{-\infty}^{\infty} \left(\int_s^{\infty} (1 + t)^{\alpha} g(t - s, x, y) \mathrm{d}t \right)^2 \mathrm{d}y \mathrm{d}s.$$

This is equal to

$$c \int_0^\infty \int_{-\infty}^\infty \int_s^\infty \int_s^\infty \frac{(1+t)^\alpha (1+t')^\alpha}{\sqrt{(t-s)(t'-s)}} \exp\left(-\frac{(x-y)^2}{4}\left(\frac{1}{t-s}+\frac{1}{t'-s}\right)\right) dt' dt dy ds$$

Since

$$\int_{-\infty}^\infty \exp\left(-\frac{(x-y)^2}{4}\left(\frac{1}{t-s}+\frac{1}{t'-s}\right)\right) dy = c\sqrt{\frac{(t-s)(t'-s)}{(t+t'-2s)}}$$

we obtain

$$
\begin{aligned}
\mathbb{E}[\langle h_1, u_x\rangle^2] =& c \int_0^\infty \int_0^\infty \int_0^{t \wedge t'} \frac{(1+t)^\alpha (1+t')^\alpha}{\sqrt{t+t'-2s}} ds\, dt'\, dt \\
=& c \int_0^\infty \int_0^\infty (1+t)^\alpha (1+t')^\alpha (\sqrt{t+t'} - \sqrt{|t-t'|}) dt'\, dt \quad (3.1.1) \\
=& c \int_0^\infty (1+t)^\alpha t^{\frac{3}{2}} \int_0^\infty (1+ut)^\alpha (\sqrt{1+u} - \sqrt{|1-u|}) du\, dt
\end{aligned}
$$

We need to take $\alpha < 0$, and in this case $(1+ut)^\alpha \leq (ut)^\alpha$. It then follows that

$$\mathbb{E}[\langle h_1, u_x\rangle^2] \leq c \int_0^\infty (1+t)^\alpha t^{\alpha+\frac{3}{2}} dt \cdot \int_0^\infty u^\alpha (\sqrt{1+u} - \sqrt{|1-u|}) du.$$

The t integral is finite provided that $\alpha > -\frac{5}{2}$ and $2\alpha + \frac{3}{2} < -1$, whilst the u integral is finite provided that $\alpha > -2$ and $\alpha < -\frac{1}{2}$ (since $\sqrt{1+u} - \sqrt{|1-u|}$ looks like u for small u and $u^{-\frac{1}{2}}$ for large u). Thus $\mathbb{E}[\langle h_1, u_x\rangle^2] < \infty$ for $-2 < \alpha < -\frac{5}{4}$ (and in fact clearly holds for all $\alpha \leq -2$).

We now do the same for v_x, taking $h_2(t) = (1+t)^\alpha$ for some $\alpha \in \mathbb{R}$. In this case

$$\mathbb{E}[\langle h_2, v_x\rangle^2] = \int_0^\infty \int_{-\infty}^\infty \left(\int_s^\infty (1+t)^\alpha \frac{\partial}{\partial x} g(t-s, x, y) dt\right)^2 dy\, ds$$

This is equal to

$$\int_0^\infty \int_0^\infty \int_0^{t \wedge t'} (1+t)^\alpha (1+t')^\alpha \left(\int_{-\infty}^\infty \frac{\partial}{\partial x} g(t-s,x,y) \frac{\partial}{\partial x} g(t'-s,x,y) \mathrm{d}y \right) \mathrm{d}s\mathrm{d}t'\mathrm{d}t.$$

The integrand in the y integral is proportional to

$$\frac{(x-y)^2}{\sqrt{(t-s)^3(t'-s)^3}} \exp\left(\frac{-(x-y)^2}{4(t-s)} \right) \exp\left(\frac{-(x-y)^2}{4(t'-s)} \right)$$

$$= \frac{(x-y)^2}{\sqrt{(t-s)^3(t'-s)^3}} \exp\left(-(x-y)^2 \left(\frac{t+t'-2s}{4(t-s)(t'-s)} \right) \right).$$

Now,

$$\int_{-\infty}^\infty \frac{(x-y)^2}{\sqrt{\frac{(t-s)(t'-s)}{t+t'-2s}}} \exp\left(-\frac{(x-y)^2}{4} \bigg/ \frac{(t-s)(t'-s)}{t+t'-2s} \right) \mathrm{d}y = c\frac{(t-s)(t'-s)}{t+t'-2s}$$

Thus the y integral is proportional to $\frac{1}{(t+t'-2s)^{\frac{3}{2}}}$ using properties of Gaussian densities, and hence we require that the integral

$$\int_0^\infty \int_0^\infty \int_0^{t \wedge t'} \frac{(1+t)^\alpha(1+t')^\alpha}{(t+t'-2s)^{\frac{3}{2}}} \mathrm{d}s\mathrm{d}t'\mathrm{d}t$$

$$= c \int_0^\infty \int_0^\infty (1+t)^\alpha(1+t')^\alpha \left(\frac{1}{\sqrt{|t-t'|}} - \frac{1}{\sqrt{t+t'}} \right) \mathrm{d}t'\mathrm{d}t$$

$$= c \int_0^\infty (1+t)^\alpha t^{\frac{1}{2}} \int_0^\infty (1+tu)^\alpha \left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}} \right) \mathrm{d}u\mathrm{d}t$$

is finite. Again, we require that $\alpha < 0$, and then the above integral is bounded by

$$c \int_0^\infty (1+t)^\alpha t^{\alpha + \frac{1}{2}} \mathrm{d}t \cdot \int_0^\infty u^\alpha \left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}} \right) \mathrm{d}u.$$

The t integral is finite so long as $\alpha > -\frac{3}{2}$ and $2\alpha + \frac{1}{2} < -1$, whilst the u integral is finite provided that $u > -2$ (noting that $\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}$ behaves again like u for small u, and like $u^{-\frac{3}{2}}$ for large u). Thus for $\alpha < -\frac{3}{4}$, $\mathbb{E}[\langle h_2, v_x \rangle^2]$ is finite.

The above considerations provide conditions on the tails of h_1 and h_2 such that $\langle h_1, u_x \rangle$ and $\langle h_2, v_x \rangle$ are in $L^2(\Omega)$, which we hope might characterise a space E in which $((u_x, v_x); x \geq 0)$ takes values. Ultimately, assuming that E is some separable metric space fitting in the framework of section 1.1.3, we would like to define a martingale problem that is satisfied by $((u_x, v_x); x \geq 0)$. At this point it is not the case that any space will do, but rather the choice of E has a large say in whether we can reap anything of value from the martingale problem. On the other hand, to define a martingale we look to write stochastic differential equations for $\langle h_1, u_x \rangle$ and $\langle h_2, v_x \rangle$, where on the face of it we are dealing with a system of equations for real valued processes, and in fact do not need to define a space for u_x and v_x. However, we have already hinted that we shall require the enlargement theorem 1.5, which requires that the information (u_0, v_0) takes values in a normed vector space.

For $\alpha > 0$ let $C_{0,\alpha}([0, \infty))$ be the space of continuous functions h on $[0, \infty)$ such that $h(t)(1 + t)^\alpha \to 0$ as $t \to \infty$, and let $\|h\|_\alpha = \sup_{t \geq 0} |(1 + t)^\alpha h(t)| < \infty\}$. We observe that $C_{0,\alpha'} \subset C_{0,\alpha} \subset L^2([0, \infty))$ for any $\alpha' > \alpha > \frac{1}{2}$. It now follows that for any $h_1 \in C_{0,\frac{5}{4}+\beta}$ and $h_2 \in C_{0,\frac{3}{4}+\beta}$ (where $\beta > 0$ can be made arbitrarily small), $\langle h_1, u_x \rangle$ and $\langle h_2, v_x \rangle$ are in $L^2(\Omega)$. Our aim eventually is to define a martingale problem on a separable Banach space E which is solved by the processes $(u_x; x \geq 0)$ and $(v_x; x \geq 0)$. The above calculations suggest a certain tail behaviour for u_x and v_x, for each $x \geq 0$. Consequently, we make an assumption that there exist subspaces X_1 and X_2 of $C_{0,\frac{5}{4}+\beta}$ and $C_{0,\frac{3}{4}+\beta}$ respectively such that for all $x \geq 0$, there are modifications of $(u_x; x \geq 0)$ and $(v_x; x \geq 0)$ which belong to X_1^* and X_2^* respectively. We further assume that X_1 and X_2 are complete under norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$ such that $\|\cdot\|_{\frac{5}{4}+\beta} \leq c_1 \|\cdot\|_{X_1}$ and $\|\cdot\|_{\frac{3}{4}+\beta} \leq c_2 \|\cdot\|_{X_2}$. We may immediately take $X_1 = C_{0,\frac{3}{2}+\beta}$. To justify this, we remark that by theorem 1.2, there is a modification of $(u_x; x \geq 0)$ such

that for all $h \in C_{0,\frac{3}{2}+\beta}$, $\langle h, u_x \rangle = \int_0^\infty h(t)u(x,t)\mathrm{d}t$ almost surely, where $u(x,t)$ is almost surely continuous in t and x. We note that for positive $h \in C_{0,\frac{3}{2}+\beta}$,

$$\int_0^\infty h(t)|u(x,t)|\mathrm{d}t = \int_0^\infty h(t)\mathbb{1}_{\{t:u(x,t)\geq 0\}}u(x,t)\mathrm{d}t - \int_0^\infty h(t)\mathbb{1}_{\{t:u(x,t)\leq 0\}}u(x,t)\mathrm{d}t$$

which is almost surely finite. Thus $\left| \int_0^\infty h(t)(1+t)^{\frac{3}{2}+\beta}(1+t)^{-\frac{3}{2}-\beta}u(x,t)\mathrm{d}t \right| \leq$ $\sup_{t\geq 0}|h(t)(1+t)^{\frac{3}{2}+\beta}| \int_0^\infty (1+t)^{-\frac{3}{2}-\beta}|u(x,t)|\mathrm{d}t$, which means u_x is almost surely in $(C_{0,\frac{3}{2}+\beta})^*$. We have relied here on the representation of $(u_x(\cdot); x \geq 0)$ as a continuous function, and the fact that this function is continuous is an application of the Kolmogorov-Čentsov continuity criterion. In general, we think of u and v as processes with parameter spaces $[0,\infty) \times X_1$ and $[0,\infty) \times X_2$, and what we require is a version of the Kolmogorov-Čentsov continuity criterion. We will discuss this in greater detail later on. As a final remark in this section, we observe that the laws of u_x and v_x on X_1^* and X_2^* respectively are Gaussian, and in fact are Radon since X_1^* and X_2^* are separable Banach spaces (see [Bog98]).

3.1.2 A system of SDEs for u_x and v_x.

Proposition 3.1. *If $h_1 \in C_{0,\frac{5}{4}+\beta}$, $h_2 \in C_{0,\frac{3}{4}+\beta}$ and $x \geq 0$ we have*

$$\langle h_1, u_x \rangle = \langle h_1, u_0 \rangle + \int_0^x \langle h_1, v_y \rangle \mathrm{d}y$$

almost surely, and provided that

$$\left| \int_0^\infty \int_0^\infty h_2'(t)h_2'(t')(\sqrt{t+t'} - \sqrt{|t-t'|})\mathrm{d}t'\mathrm{d}t \right| < \infty,$$

then

$$\langle h_2, v_x \rangle = \langle h_2, v_0 \rangle - \int_0^x \langle h_2', u_y \rangle \mathrm{d}y - W_x(h_2)$$

also holds almost surely, where once again, $W_x(h_2) = \int_0^x \int_0^\infty h_2(s)\mathrm{d}B_{ys}$.

Proof We would first like to show that for $h_1 \in C_{0,\frac{5}{4}+\beta}$,

$$\langle h, u_x \rangle = \langle h_1, u_0 \rangle + \int_0^x \langle h_1, v_y \rangle \mathrm{d}y.$$

The integral, by definition of v, is given by

$$\int_{-\infty}^{\infty} \mathbb{1}_{[0,x]}(y) \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} h_1(t) \frac{\partial}{\partial y} g(t-s,y,z) \mathrm{d}t \right) \mathrm{d}B_{zs} \mathrm{d}y.$$

This equals

$$\int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_{-\infty}^{\infty} \int_s^{\infty} \mathbb{1}_{[0,x]}(y) h_1(t) \frac{\partial}{\partial y} g(t-s,y,z) \mathrm{d}t \mathrm{d}y \right) \mathrm{d}B_{zs}$$

provided we may apply theorem 1.2. This we may do since $\int_s^{\infty} h_1(t) \frac{\partial}{\partial y} g(t-s,y,z)\mathrm{d}t$ is square integrable in z and s (since $h_1 \in C_{0,\frac{3}{4}+\beta}$) and the integral of its square is a continuous function in y (in fact a constant), so in turn is integrable on $[0,x]$. The first equation now follows easily.

In order to write an expression for v_x, we are guided by our intuition that $\langle h_2, v_x \rangle = \int_0^{\infty} h_2(t) \frac{\partial}{\partial x} u(x,t)\mathrm{d}t$ for $h_2 \in C_{0,\frac{3}{4}+\beta}$. Formally

$$
\begin{aligned}
\langle h_2, v_x \rangle - \langle h_2, v_0 \rangle &= -\int_{-\infty}^{\infty} (\delta_0(y) - \delta_x(y)) \langle h_2, v_y \rangle \mathrm{d}y \\
&= -\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}y} \mathbb{1}_{[0,x]}(y) \frac{\partial}{\partial y} \int_0^{\infty} h_2(t) u(y,t)\mathrm{d}t\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} h_2(t) \Delta \mathbb{1}_{[0,x]}(y) u(y,t)\mathrm{d}t\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} \int_{-\infty}^{\infty} h_2(t) \Delta \mathbb{1}_{[0,x]}(y) g(t-s,y,z)\mathrm{d}y\mathrm{d}t \right) \mathrm{d}B_{zs}
\end{aligned}
$$

Taking this as our lead, let ϕ_n^x be some sequence in $C_0^{\infty}([0,\infty))$ which converges to $\mathbb{1}_{[0,x]}$ in $L^2(\mathbb{R})$, and consider the integral

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$$\int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} \int_{-\infty}^{\infty} h_2(t) \Delta \phi_n^x(y) g(t-s,y,z) \mathrm{d}y \mathrm{d}t \right) \mathrm{d}B_{zs}.$$

Thanks to well known properties of g we can rewrite the integrand as

$$\int_s^{\infty} \int_{-\infty}^{\infty} h_2(t) \phi_n^x(y) \Delta g(t-s,y,z) \mathrm{d}y \mathrm{d}t$$

$$= \int_s^{\infty} \int_{-\infty}^{\infty} h_2(t) \phi_n^x(y) \frac{\partial}{\partial t} g(t-s,y,z) \mathrm{d}y \mathrm{d}t$$

$$= -h_2(s) \phi_n^x(z) - \int_s^{\infty} \int_{-\infty}^{\infty} h_2'(t) \phi_n^x(y) g(t-s,y,z) \mathrm{d}y \mathrm{d}t$$

Thus

$$\int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} \int_{-\infty}^{\infty} h_2(t) \Delta \phi_n^x(y) g(t-s,y,z) \mathrm{d}y \mathrm{d}t \right) \mathrm{d}B_{zs}$$

$$= -\int_{-\infty}^{\infty} \int_0^{\infty} h_2(s) \phi_n^x(z) \mathrm{d}B_{zs}$$

$$- \int_{-\infty}^{\infty} \int_0^{\infty} \int_s^{\infty} \int_{-\infty}^{\infty} h_2'(t) \phi_n^x(y) g(t-s,y,z) \mathrm{d}y \mathrm{d}t \mathrm{d}B_{zs}.$$

One suspects that this should converge to

$$- \int_0^x \int_0^{\infty} h_2(s) \mathrm{d}B_{zt} - \int_{-\infty}^{\infty} \int_0^{\infty} \int_s^{\infty} \int_0^x h_2'(t) g(t-s,y,z) \mathrm{d}y \mathrm{d}t \mathrm{d}B_{zt}$$

as $n \to \infty$. More precisely, since $h_2 \phi_n^x$ converges to $h_2 \mathbb{1}_{[0,x]}$ in $L^2([0,\infty)^2)$ it is clear that

$$\int_{-\infty}^{\infty} \int_0^{\infty} h_2(s) \phi_n^x(z) \mathrm{d}B_{zs} \to \int_0^x \int_0^{\infty} h_2(s) \mathrm{d}B_{zs},$$

where the convergence is in $L^2(\Omega)$. We would now like to show that

$$\int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} \int_{-\infty}^{\infty} h_2'(t) [\phi_n^x(y) - \mathbb{1}_{[0,x]}(y)] g(t-s,y,z) \mathrm{d}y \mathrm{d}t \right) \mathrm{d}B_{zs}$$

converges to 0 in $L^2(\Omega)$ as $n \to \infty$. This is true if and only if

$$\int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} \int_{-\infty}^{\infty} h_2'(t)[\phi_n^x(y) - \mathbb{1}_{[0,x]}(y)]g(t-s,y,z)\mathrm{d}y\mathrm{d}t \right)^2 \mathrm{d}z\mathrm{d}s$$

converges to 0. We write $\xi_n(y) = \phi_n^x(y) - \mathbb{1}_{[0,x]}(y)$, and we may assume that ξ_n is 0 outside $[0,x]$ for all n. The above integral is then

$$\int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_0^x \xi_n(y) \int_s^{\infty} h_2'(s)g(t-s,y,z)\mathrm{d}t\mathrm{d}y \right)^2 \mathrm{d}s\mathrm{d}z$$

$$\leq \left(\int_0^x \xi_n^2(y)\mathrm{d}y \right) \cdot \left(\int_0^x \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} h_2'(s)g(t-s,y,z)\mathrm{d}t \right)^2 \mathrm{d}s\mathrm{d}z\mathrm{d}y \right)$$

$$\leq cx\|\xi_n\|_2^2 \to 0 \text{ as } n \to \infty.$$

Here we have used the Cauchy-Schwartz inequality, and have observed that $\int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} h_2'(s)g(t-s,y,z)\mathrm{d}t \right)^2 \mathrm{d}s\mathrm{d}z$ is finite and is constant in y.

On the other hand

$$\begin{aligned}
\langle h_2, v_x \rangle - \langle h_2, v_0 \rangle &= \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} h_2(t)\partial_2 g(t-s,x,y)\mathrm{d}t \right) \mathrm{d}B_{ys} \\
&\quad - \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} h_2(t)\partial_2 g(t-s,0,y)\mathrm{d}t \right) \mathrm{d}B_{ys} \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} h_2(t) \int_0^x \Delta_z g(t-s,z,y)\mathrm{d}z\mathrm{d}t \right) \mathrm{d}B_{ys} \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_s^{\infty} \int_{-\infty}^{\infty} h_2(t)\mathbb{1}_{[0,x]}(z)\Delta_z g(t-s,z,y)\mathrm{d}z\mathrm{d}t \right) \mathrm{d}B_{ys}.
\end{aligned}$$

Furthermore,

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \left(\int_{s}^{\infty} \int_{-\infty}^{\infty} h_2(t)\Delta\phi_n^x(y)g(t-s,y,z)\mathrm{d}y\mathrm{d}t \right) \mathrm{d}B_{zs}$$

$$- \int_{-\infty}^{\infty} \int_{0}^{\infty} \left(\int_{s}^{\infty} \int_{-\infty}^{\infty} h_2(t)\mathbb{1}_{[0,x]}(y)\Delta_y g(t-s,y,z)\mathrm{d}y\mathrm{d}t \right) \mathrm{d}B_{zs}$$

$$= \int_{-\infty}^{\infty} \int_{0}^{\infty} \left(\int_{s}^{\infty} \int_{-\infty}^{\infty} h_2(t)\xi_n(y)\Delta_y g(t-s,y,z)\mathrm{d}y\mathrm{d}t \right) \mathrm{d}B_{zs}.$$

This converges to 0 in $L^2(\Omega)$ if and only if $\int_{s}^{\infty} \int_{-\infty}^{\infty} h_2(s)\xi_n(y)\Delta_y g(t-s,y,z)\mathrm{d}y\mathrm{d}t$ converges to 0 in $L^2([0,\infty)^2)$. This integral is equal to

$$-h_2(s)\xi_n(z) - \int_{s}^{\infty} \int_{-\infty}^{\infty} h_2'(t)\xi_n(y)g(t-s,y,z)\mathrm{d}z\mathrm{d}t$$

which converges to 0 in $L^2([0,\infty)^2)$ by previous reasoning. Equating our two limits gives the almost sure identity

$$\langle h_2, v_x \rangle - \langle h_2, v_0 \rangle = - \int_{0}^{x} \int_{0}^{\infty} h_2'(s)\mathrm{d}B_{zt} - \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{s}^{\infty} \int_{0}^{x} h_2'(t)g(t-s,y,z)\mathrm{d}y\mathrm{d}t\mathrm{d}B_{zt}.$$

The second integral is

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \mathbb{1}_{[0,x]}(y) \left(\int_{s}^{\infty} h_2'(t)g(t-s,y,z)\mathrm{d}t \right) \mathrm{d}y\mathrm{d}B_{zs}$$

$$= \int_{0}^{x} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{s}^{\infty} h_2'(s)g(t-s,y,z)\mathrm{d}t\mathrm{d}B_{zs}\mathrm{d}y$$

using theorem 1.2 once again. We therefore have

$$\langle h_2, v_x \rangle = \langle h_2, v_0 \rangle - \int_{0}^{x} \langle h_2', u_y \rangle \mathrm{d}y - W_x(h_2) \quad \text{a.s.}$$

3.1.3 Some remarks on the continuity of $(u_x; x \geq 0)$ and $(v_x; x \geq 0)$.

For each $x \geq 0$, the equations for $\langle h_1, u_x \rangle$ and $\langle h_2, v_x \rangle$ in the system (1.1.5) hold on a set of full measure in $(\Omega, \mathscr{F}, \mathbb{P})$, which we might denote by $\mathcal{A}_x$, say. What

we would really like to say, however, is that for any $X > 0$, there is a set $\mathcal{A}$ of full measure in $(\Omega, \mathscr{F}, \mathbb{P})$ such that (1.1.5) holds on $\mathcal{A}$ for all $x \in [0, X]$. For this we look to the Kolmogorov-Čentsov continuity criterion. This has the additional benefit showing that $(\langle h_1, u_x \rangle; x \in [0, X])$ and $(\langle h_2, v_x \rangle; x \in [0, X])$ are almost surely continuous for $h_1 \in C_{0, \frac{5}{4} + \beta}$ and $h_2 \in C_{0, \frac{3}{4} + \beta}$, which is required if we are to demonstrate that they satisfy a strong Markov property.

Consider first $(\langle h_1, u_x \rangle; x \in [0, X])$ for $h_1 \in C_{0, \frac{5}{4} + \beta}$. For $x, z \in [0, X]$ with $x < z$, $\langle h_1, u_x \rangle - \langle h_1, u_z \rangle$ is a centred Gaussian random variable. We now try to calculate its variance. This is

$$\int_{\mathbb{R}} \int_0^\infty \left(\int_s^\infty h_1(t)(g(t - s, x, y) - g(t - s, z, y)) \mathrm{d}t \right)^2 \mathrm{d}s \mathrm{d}y.$$

Let us look at $g(t - s, x, y)g(t' - s, z, y)$. This is equal to

$$\frac{c}{\sqrt{(t - s)(t' - s)}} \exp\left(\frac{-(x^2 - 2xy + y^2)}{4(t - s)} \right) \exp\left(\frac{-(z^2 - 2zy + y^2)}{4(t' - s)} \right)$$

$$= \frac{c}{\sqrt{(t - s)(t' - s)}} \exp\left(\frac{-(t' + t - 2s)}{4(t - s)(t' - s)} \left(y - \frac{x(t' - s) + z(t - s)}{t' + t - 2s} \right)^2 \right)$$

$$\cdot \exp\left(\frac{(t' + t - 2s)}{4(t - s)(t' - s)} \left(\frac{x^2(t' - s)^2 + 2xz(t - s)(t' - s) + z^2(t - s)^2}{(t' + t - 2s)^2} \right. \right.$$

$$\left. \left. - \frac{x^2}{4(t - s)} - \frac{z^2}{4(t' - s)} \right) \right)$$

$$= \frac{c}{\sqrt{(t - s)(t' - s)}} \exp\left(\frac{-(t' + t - 2s)}{4(t - s)(t' - s)} \left(y - \frac{x(t' - s) + z(t - s)}{t' + t - 2s} \right)^2 \right)$$

$$\cdot \exp\left(\frac{-(x - z)^2}{4(t' + t - 2s)} \right)$$

If we now perform the y integral, we can deduce that

$$\int_{\mathbb{R}} \int_0^\infty \left(\int_s^\infty h_1(t) g(t-s,x,y) \mathrm{d}t \right) \left(\int_s^\infty h_1(t')(g(t'-s,x,y) - g(t'-s,z,y)) \mathrm{d}t' \right) \mathrm{d}s \mathrm{d}y$$

$$= c \int_0^\infty \int_s^\infty \int_s^\infty \frac{h_1(t)h_1(t')}{\sqrt{t'+t-2s}} \left(1 - \exp\left(\frac{-(x-z)^2}{4(t'+t-2s)} \right) \right) \mathrm{d}t' \mathrm{d}t \mathrm{d}s$$

For any $t', t, s \in [0, \infty)$ with $t' + t - 2s > 0$, we may use the mean value theorem to see that there is some $\theta \in (0, (x-z)^2)$ such that

$$1 - \exp\left(-\frac{(x-z)^2}{4(t'+t-2s)} \right) = \frac{(x-z)^2}{4(t'+t-2s)} \exp\left(-\frac{\theta}{4(t'+t-2s)} \right) \leq \frac{(x-z)^2}{4(t'+t-2s)}.$$

We now see that

$$\mathbb{E}[(\langle h_1, u_x \rangle - \langle h_1, u_z \rangle)^2]$$

$$= c \int_0^\infty \int_s^\infty \int_s^\infty \frac{h_1(t)h_1(t')}{\sqrt{t'+t-2s}} \left(1 - \exp\left(\frac{-(x-z)^2}{4(t'+t-2s)} \right) \right) \mathrm{d}t' \mathrm{d}t \mathrm{d}s$$

$$\leq c(x-z)^2 \int_0^\infty \int_s^\infty \int_s^\infty \frac{h_1(t)h_1(t')}{(t'+t-2s)^{\frac{3}{2}}} \mathrm{d}t' \mathrm{d}t \mathrm{d}s \qquad (3.1.2)$$

Since $h_1 \in C_{0, \frac{5}{4}+\beta}$, it is also in $C_{0, \frac{3}{4}+\beta}$, so the above integral is finite. Thus we have a constant C_1 such that for all $0 \leq x < z \leq X$,

$$\mathbb{E}[(\langle h_1, u_x \rangle - \langle h_1, u_z \rangle)^2] \leq C_1 |x - z|^2$$

and more generally, for each $n \in \mathbf{N}$ there is some C_n such that

$$\mathbb{E}[(\langle h_1, u_x \rangle - \langle h_1, u_z \rangle)^{2n}] \leq C_n |x - z|^{2n}.$$

The Kolmogorov-Čentsov continuity criterion now implies that there is a version of $(\langle h_1, u_x \rangle; x \in [0, X])$ which is almost surely continuous. We can also produce a similar argument for $(\langle h_2, v_x \rangle; x \in [0, X])$ for $h_2 \in C_{0, \frac{3}{4}+\beta}$. In this case, for

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$0 \leq x < z \leq X$, we have

$$(\langle h_2, v_z \rangle - \langle h_2, v_x \rangle)^{2n} \leq c \left(\int_x^z \langle h_2', u_y \rangle \mathrm{d}y \right)^{2n} + c(W_z(h_2) - W_x(h_2))^{2n}$$

$$\leq c(z-x)^{2n} \int_x^z \langle h_2', u_y \rangle^{2n} \mathrm{d}y + c(W_z(h_2) - W_x(h_2))^{2n}$$

$$\leq c(z-x)^{2n} \int_x^z (\langle h_2', u_y \rangle - \langle h_2', u_x \rangle)^{2n} \mathrm{d}y$$

$$+ c(z-x)^{2n+1} \langle h_2', u_x \rangle^{2n} + c(W_z(h_2) - W_x(h_2))^{2n}$$

$$\tag{3.1.3}$$

almost surely. We remark that the law of $\langle h_2', u_x \rangle$ does not depend on x, whilst $W_z(h_2) - W_x(h_2)$ is a centred Gaussian random variable with variance $(z-x)\|h_2\|_2$. It now follows that

$$\mathbb{E}[(\langle h_2, v_z \rangle - \langle h_2, v_x \rangle)^{2n}] \leq c(z-x)^{2n} \int_x^z C_n(y-x)^{2n} \mathrm{d}y + c(z-x)^{2n+1} \mathbb{E}[\langle h_2', u_x \rangle^{2n}]$$

$$+ c\mathbb{E}[(W_x(h_2) - W_{x'}(h_2))^{2^n}]$$

$$\leq c_{1,n}(z-x)^{4n+1} + c_{2,n}(z-x)^{2n+1} + c_{3,n}(z-x)^n.$$

Once again, we invoke the Kolmogorov-Čentsov continuity criterion to deduce that $(\langle h_2, v_x \rangle; x \in [0, X])$ has a version which is almost surely continuous.

3.2 A solution to (1.2.2) for certain test functions

3.2.1 The stochastic factorisation as a result of integration by parts.

For any $h_1 \in C_{0, \frac{5}{4}+\beta}$ and $h_2 \in C_{0, \frac{3}{4}+\beta}$, we have equations for $\langle h_1, u_x \rangle$ and $\langle h_2, v_x \rangle$ which are driven by an $(\mathscr{F}_x; x \geq 0)$ martingale, but for which the initial conditions $\langle h_1, u_0 \rangle$ and $\langle h_2, v_0 \rangle$ are not in $\mathscr{F}_0$. We confront this using the enlargement theorem 1.5, taking $L = (u_0, v_0) \in E$ and defining the enlarged filtration $(\tilde{\mathscr{F}}_x; x \geq 0)$ by $\tilde{\mathscr{F}}_x = \mathscr{F}_x \vee \sigma(L)$. Ultimately our goal is to find a drift

$\varrho_{h_2}(L, y)$ such that $\langle h_2, v_x \rangle = \langle h_2, v_0 \rangle - \int_0^x \langle h_2', u_y \rangle \mathrm{d}y - \int_0^x \varrho_{h_2}(L, y) \mathrm{d}y - \tilde{W}_x(h_2)$ almost surely for all $x \geq 0$ and any $h_2 \in C_0^\infty([0, \infty))$, where $(\tilde{W}_x(h_2); x \geq 0)$ is an $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingale. In fact we will show the above for a certain class of $h_2 \in C_{0, \frac{3}{4}+\beta}$, and deduce it for $h_2 \in C_0^\infty$. We therefore need to discuss first whether $(W_x(l); x \geq 0)$ is an $(\tilde{\mathscr{F}}_x; x \geq 0)$ semimartingale for a given $l \in L^2([0, \infty))$. To this end we aim to show that the stochastic factorisation (1.2.2) on page 30 holds for any $F \in \mathfrak{F}C_b^\infty(X_1 \times X_2) \subset B(E)$. Since $X_1 \times X_2$ is dense in E^*, this is a large enough set of F for theorem 1.5 by lemma 1.1.

For the sake of simplicity, we define F by

$$F(u, v) = f(\langle h_1, u \rangle, \langle h_2, v \rangle)$$

with $f \in C_0^\infty(\mathbb{R}^2)$ and $h_i \in X_i$. (The following argument be easily seen to work for all $F \in \mathfrak{F}C_b^\infty(X_1 \times X_2)$.) For F as above,

$$\begin{aligned}
D_{ys}F(L) &= \left(\int_s^\infty h_1(t)g(t-s, 0, y) \mathrm{d}t \right) \partial_1 f(\langle h_1, u_0 \rangle, \langle h_2, v_0 \rangle) \\
&\quad + \left(\int_s^\infty h_2(t)\partial_2 g(t-s, 0, y) \mathrm{d}t \right) \partial_2 f(\langle h_1, u_0 \rangle, \langle h_2, v_0 \rangle)
\end{aligned}$$

and so

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty l(s) D_{ys}F(L) \mathrm{d}s \,\middle|\, \mathscr{F}_y \right] \\
&= \left(\int_0^\infty h_1(t) \int_0^t l(s)g(t-s, 0, y) \mathrm{d}s \mathrm{d}t \right) \mathbb{E}[\partial_1 f(\langle h_1, u_0 \rangle, \langle h_2, v_0 \rangle)|\mathscr{F}_y] \\
&\quad + \left(\int_0^\infty h_2(t) \int_0^t l(s)\partial_2 g(t-s, 0, y) \mathrm{d}s \mathrm{d}t \right) \mathbb{E}[\partial_2 f(\langle h_1, u_0 \rangle, \langle h_2, v_0 \rangle)|\mathscr{F}_y].
\end{aligned}$$

We now fix some $y > 0$ and define

$$\kappa_y l = (l * g(\cdot, 0, y), l * \partial_2 g(\cdot, 0, y))$$

where $l * h$ denotes the convolution

$$l * h(t) = \int_0^t l(s)h(t-s)\mathrm{d}s.$$

For $h_1 \in C_{0,\frac{5}{4}+\beta}$ and $h_2 \in C_{0,\frac{3}{4}+\beta}$, $|\langle h_1, (\kappa_y l)_1 \rangle| < \infty$ and $|\langle h_2, (\kappa_y l)_2 \rangle| < \infty$. For the moment, we shall assume that $\kappa_y l \in E$. We can now write

$$\mathbb{E}\left[\int_0^\infty l(s)D_{ys}(F(L))\mathrm{d}s \,\bigg|\, \mathscr{F}_y\right]$$
$$= \mathbb{E}[\langle h_1, (\kappa_y l)_1 \rangle \partial_1 f(\langle h_1, u_0\rangle, \langle h_2, v_0\rangle)$$
$$+ \langle h_2, (\kappa_y l)_2 \rangle \partial_2 f(\langle h_1, u_0\rangle, \langle h_2, v_0\rangle)|\mathscr{F}_y].$$

We introduce some further notation. For any $h_1 \in X_1$ we write

$$\langle h_1, (u_0)_y \rangle = \int_{-\infty}^y \int_0^\infty \int_s^\infty h_1(t)g(t-s,0,z)\mathrm{d}t\mathrm{d}B_{zs}$$

and set

$$\langle h_1, (u_0)^y \rangle = \langle h_1, u_0 \rangle - \langle h_1, (u_0)_y \rangle.$$

We define $\langle h_2, (v_0)_y \rangle$ and $\langle h_2, (v_0)^y \rangle$ for $h_2 \in X_2$ in a similar way and we assume that we may show that $((u_0)_y, (v_0)_y) \in E$. $(u_0)_y$ and $(v_0)_y$ are $\mathscr{F}_y$ measurable, whilst $(u_0)^y$ and $(v_0)^y$ are independent of $\mathscr{F}_y$. As in section 2.1.2 we define an $\mathscr{F}_y$ measurable function $f_y : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ (which is smooth and has compact support in $\mathbb{R}^2$) by

$$f_y(x_1, x_2) := f(\langle h_1, (u_0)_y \rangle + x_1, \langle h_2, (v_0)_y \rangle + x_2), \ x_1, x_2 \in \mathbb{R}$$

(so that $f_y(\langle h_1, (u_0)^y \rangle, \langle h_2, (v_0)^y \rangle) = f(\langle h_1, u_0\rangle, \langle h_2, v_0\rangle)$) and $F_y : E \to \mathbb{R}$ by

$$F_y(u, v) := f_y(\langle h_1, u\rangle, \langle h_2, v\rangle).$$

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It now follows from lemma 1.3 that

$$\mathbb{E}[\langle h_1, (\kappa_y l)_1 \rangle \partial_1 f(\langle h_1, u_0 \rangle, \langle h_2, v_0 \rangle) + \langle h_2, (\kappa_y l)_2 \rangle \partial_2 f(\langle h_1, u_0 \rangle, \langle h_2, v_0 \rangle) | \mathscr{F}_y]$$

$$= \int_E (\langle h_1, (\kappa_y l)_1 \rangle \partial_1 f_y(\langle h_1, u \rangle, \langle h_2, v \rangle)$$

$$+ \langle h_2, (\kappa_y l)_2 \rangle \partial_2 f_y(\langle h_1, u \rangle, \langle h_2, v \rangle)) \mathrm{d}\mu^y(u, v)$$

$$= \int_E \frac{\partial}{\partial \kappa_y l} F_y(u, v) \mathrm{d}\mu^y(u, v)$$

where μ^y denotes the law of $((u_0)^y, (v_0)^y)$ on E, which is once again a Radon Gaussian measure.

3.2.2 An equation for $\mathcal{C}_y^{-1} \kappa_y l$ II.

As in section (2.1.3), the next step is to determine whether $\kappa_y l$ is in the Cameron-Martin space of μ^y, which we denote as ever by H_y, and as ever we denote the covariance operator of μ^y by $\mathcal{C}_y$. Remark that since $\| \cdot \|_E \leq c \| \cdot \|_{H_y}$, if we can show that $\kappa_y l \in H_y$ then $\kappa_y l \in E$ and the above arguments are justified. In order to show that $\kappa_y l \in H_y$, we intend to show the existence of some ϕ in the reproducing kernel Hilbert space H_y' satisfying $\kappa_y l = \mathcal{C}_y \phi$. In fact, we will restrict ourselves to searching for $m_y(l) \in X_1 \times X_2$ such that $\kappa_y l = \mathcal{C}_y(m_y(l))$. The existence of such an $m_y(l) \in X_1 \times X_2$ is equivalent to the existence of such $m_y(l)$ with $\kappa_y l(h) = \mathcal{C}_y((m_y(l)))(h)$ for every $h \in X_1 \times X_2$. In other words, we wish to find $m_y(l) \in X_1 \times X_2$ such that for all $h \in X_1 \times X_2$, $\kappa_y l(h) = \mathbb{E}[\langle h, ((u_0)^y, (v_0)^y) \rangle \langle m_y(l), ((u_0)^y, (v_0)^y) \rangle]$ is equal to

$$\int_y^\infty \int_0^\infty \left(\int_s^\infty h_1(t)g(t-s,0,z)\mathrm{d}t \right)\left(\int_s^\infty (m_y(l))_1(t)g(t-s,0,z)\mathrm{d}t \right) \mathrm{d}s\mathrm{d}z$$

$$+ \int_y^\infty \int_0^\infty \left(\int_s^\infty h_2(t)\partial_2 g(t-s,0,z)\mathrm{d}t \right)\left(\int_s^\infty (m_y(l))_1(t)g(t-s,0,z)\mathrm{d}t \right) \mathrm{d}s\mathrm{d}z$$

$$+ \int_y^\infty \int_0^\infty \left(\int_s^\infty h_1(t)g(t-s,0,z)\mathrm{d}t \right)\left(\int_s^\infty (m_y(l))_2(t)\partial_2 g(t-s,0,z)\mathrm{d}t \right) \mathrm{d}s\mathrm{d}z$$

$$+ \int_y^\infty \int_0^\infty \left(\int_s^\infty h_2(t)\partial_2 g(t-s,0,z)\mathrm{d}t \right)\left(\int_s^\infty (m_y(l))_2(t)\partial_2 g(t-s,0,z)\mathrm{d}t \right) \mathrm{d}s\mathrm{d}z.$$

For this to hold for all $h \in X_1 \times X_2$, we require $\forall t \in [0,\infty)$ that

$$(l * g(\cdot,0,y))(t) = \int_0^t \int_y^\infty g(t-s,0,z) \left\{ \int_s^\infty \Big((m_y(l))_1(r)g(r-s,0,z) \right.$$

$$\left. + (m_y(l))_2(r)\partial_2 g(r-s,0,z) \Big) \mathrm{d}r \right\} \mathrm{d}z\mathrm{d}s \qquad (3.2.1)$$

and

$$(l * \partial_2 g(\cdot,0,y))(t) = \int_0^t \int_y^\infty \partial_2 g(t-s,0,z) \left\{ \int_s^\infty \Big((m_y(l))_1(r)g(r-s,0,z) \right.$$

$$\left. + (m_y(l))_2(r)\partial_2 g(r-s,0,z) \Big) \mathrm{d}r \right\} \mathrm{d}z\mathrm{d}s. \qquad (3.2.2)$$

3.2.3 Solving for $\mathcal{C}_y^{-1}\kappa_y l$ for certain l.

We consider equation (3.2.1). The natural way to deal with the convolution is to take Laplace transforms. We will denote the Laplace transform by $\mathscr{L}$, and we refer to [Gue91] for a treatment of the subject of Laplace transforms. We refer to the Laplace transform of a locally integrable $f : \mathbb{R} \to \mathbb{R}$ (that is, f such that $\int_a^b |f(t)|\mathrm{d}t < \infty$ for every closed bounded interval $[a,b]$ in $\mathbb{R}$), and we say that f is Laplace transformable whenever $\mathcal{D}(\mathscr{L}f)$ is non-empty. We do not give the general definition of $\mathscr{L}f$ for locally integrable f, but simply observe that whenever $t \mapsto e^{-\lambda t}f(t)$ is an $L^1(\mathbb{R})$ mapping for $\lambda \in \mathcal{D}(\mathscr{L}f)$ and $f(t) = 0$ for

$t < 0$, the definition in [Gue91] becomes $\mathscr{L}f(\lambda) = \int_{\mathbb{R}} e^{-\lambda t} f(t) \mathrm{d}t$. Remark that if $f : [0, \infty) \to \mathbb{R}$ is locally integrable we will simply write $\mathscr{L}f$ for $\mathscr{L}(f \mathbb{1}_{[0,\infty)})$.

We use the following properties, taking f and g to be locally integrable:

(a) if f is differentiable on $(0, \infty)$ with $f(t) = 0$ for $t < 0$, and f' is locally integrable, then $\mathcal{D}(\mathscr{L}f) \subset \mathcal{D}(\mathscr{L}f')$ and

$$\mathscr{L}f'(\lambda) = \lambda \mathscr{L}f(\lambda) - f(0)$$

for $\lambda \in \mathcal{D}(\mathscr{L}f)$;

(b) if f and g are Laplace transformable then so is $f * g$ and

$$\mathscr{L}(f * g)(\lambda) = \mathscr{L}f(\lambda)\mathscr{L}g(\lambda)$$

for $\lambda \in \mathcal{D}(\mathscr{L}f) \cap \mathcal{D}(\mathscr{L}g)$;

(c) if $f, g : [0, \infty) \to \mathbb{R}$ are such that there exists a non-empty open interval with $I \subset \mathcal{D}(\mathscr{L}f) \cap \mathcal{D}(\mathscr{L}g)$ with $\mathscr{L}f(\lambda) = \mathscr{L}g(\lambda)$ for all $\lambda \in I$, then $f(t) = g(t)$ for all t at which both f and g are continuous, and furthermore $f(t) = g(t)$ for Lebesgue-almost every $t \in [0, \infty)$.

In particular set $\mathscr{L}g_y$ to be the Laplace transform of $g_y := g(\cdot, 0, y)$. We recall that

$$g(t, x, y) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(x - y)^2}{4t}\right)$$

and observe that $t \mapsto e^{-\lambda t} g_y(t)$ is integrable for all $\nu > 0$. Note that from exercise 5A.9 of [Gue91],

$$\mathscr{L}g_y(\lambda) = \frac{e^{-|y|\sqrt{\lambda}}}{2\sqrt{\lambda}}.$$

If we assume that l is Laplace transformable, one would now like to rewrite (3.2.1) as

$$\mathscr{L}l(\lambda)\mathscr{L}g_y(\lambda) = \int_y^\infty \mathscr{L}g_z(\lambda)\int_0^\infty e^{-\lambda s}\int_s^\infty \big(g(r-s,0,z)m_y(l)_1(r)$$

$$+ \partial_2 g(r-s,0,z)m_y(l)_2(r)\big)\mathrm{d}r\mathrm{d}s\mathrm{d}z. \quad (3.2.3)$$

Let us assume we can do this, and furthermore, to simplify things we assume that $m_y(l)_2 = 0$. We also observe that much simplification can be made if we suppose that $m_y(l)_1(r) = m_y(l)_1(r-s)m_y(l)_1(s)$. We therefore set $m_y(l)_1(r) = e^{-\nu r}$ for some $\nu > 0$. The above equation now becomes

$$\mathscr{L}l(\lambda)\mathscr{L}g_y(\lambda) = \int_y^\infty \frac{\mathscr{L}g_z(\lambda)\mathscr{L}g_z(\nu)}{\lambda + \nu}\mathrm{d}z.$$

Thus

$$\mathscr{L}l(\lambda)\mathscr{L}g_y(\lambda) = \frac{1}{(\lambda + \nu)(\sqrt{\lambda} + \sqrt{\nu})}\mathscr{L}g_y(\lambda)\mathscr{L}g_y(\nu).$$

Let us now assume that there is function $\psi_\nu : [0,\infty) \to \mathbb{R}$ such that $t \mapsto e^{-\lambda t}\psi_\nu(t)$ is integrable for each $\lambda > 0$ and whose Laplace transform is

$$\mathscr{L}\psi_\nu(\lambda) = \frac{1}{(\lambda + \nu)(\sqrt{\lambda} + \sqrt{\nu})}.$$

What we may now show is that if

$$m_y(\psi_\nu)(r) := \left(\frac{e^{-\nu r}}{\mathscr{L}g_y(\nu)}, 0\right)$$

then for all $\lambda > 0$,

$$\mathcal{L}(\psi_\nu * g_y)(\lambda) = \mathcal{L}\left(\int_0^\cdot \int_y^\infty g(\cdot - s, 0, z)\left\{\int_s^\infty \left((m_y(\psi_\nu))_1(r)g(r - s, 0, z)\right.\right.\right.$$

$$\left.\left.\left. + (m_y(\psi_\nu))_2(r)\partial_2 g(r - s, 0, z)\right)\mathrm{d}r\right\}\mathrm{d}z\mathrm{d}s\right)(\lambda) \qquad (3.2.4)$$

We remark that in order to justify this we require that

$$\mathcal{L}\left(\int_0^\cdot \int_y^\infty g_z(\cdot - s)\int_s^\infty \frac{e^{-\nu r}}{\mathcal{L}g_y(\nu)}g_z(r - s)\mathrm{d}r\mathrm{d}z\mathrm{d}s\right)(\lambda)$$

$$= \int_y^\infty \mathcal{L}\left(\int_0^\cdot g_z(\cdot - s)\int_s^\infty \frac{e^{-\nu r}}{\mathcal{L}g_y(\nu)}g_z(r - s)\mathrm{d}r\mathrm{d}s\right)(\lambda)\mathrm{d}z \qquad (3.2.5)$$

which is a straightforward change in the order of integration, observing that we have already seen that the right hand integral is defined. Equation (3.2.4) follows from this for all $\lambda > 0$. We now use the inverse theorem for the Laplace transforms and the continuity of the functions in question (which in the case of ψ_ν is assumed for now) to deduce equation (3.2.1).

This is all very well, however the form we chose for $m_y(l)$ was by no way unique. For example, suppose we now look for an l such that

$$m_y(l)(r) = (0, e^{-\nu r})$$

is a solution to the two equations 3.2.1 and 3.2.2. Denote by $\mathcal{L}\partial_2 g_y(\lambda)$ the Laplace transform of $\partial_2 g(\cdot, 0, y)$. We remark that

$$\mathcal{L}\partial_2 g_y(\lambda) = -\mathrm{sgn}(y)\sqrt{\lambda}\mathcal{L}g_y(\lambda).$$

We obtain from the first equation the expression

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$$\mathscr{L}l(\lambda)\mathscr{L}g_y(\lambda) = \frac{1}{(\lambda+\nu)(\sqrt{\lambda}+\sqrt{\nu})}\mathscr{L}g_y(\lambda)\mathscr{L}\partial_2 g_y(\nu)$$

and so clearly $m_y(\psi_\nu) = \left(0, \frac{e^{-\nu r}}{\mathscr{L}\partial_2 g_y(\nu)}\right)$ is also a solution for $l = \psi_\nu$. This lack of uniqueness is disturbing- of course $\mathcal{C}_y$ is an isometry between H_y' and H_y, so for any particular l there should be no more than one $m_y(l)$ such that $\kappa_y l = \mathcal{C}_y m_y(l)$. It is also disturbing since the drift is given by

$$\varrho_{\psi_\nu}(L,y) = \int_y^\infty \int_0^\infty \left(\int_s^\infty m_y(\psi_\nu)_1(t)g(t-s,0,z)\right.$$
$$\left. + m_y(\psi_\nu)_2(t)\partial_2 g(t-s,0,z)\mathrm{d}t\right)\mathrm{d}B_{zs} \qquad (3.2.6)$$

and one would hope it is unique. We shall address both issues in one blow. Suppose for now that $m_y(\psi_\nu)(r) = \left(\frac{e^{-\nu r}}{\mathscr{L}g_y(\nu)}, 0\right)$. Then

$$\varrho_{\psi_\nu}(L,y) = \int_y^\infty \int_0^\infty \frac{1}{\mathscr{L}g_y(\nu)}\left(\int_s^\infty g(t-s,0,z)e^{-\nu t}\mathrm{d}t\right)\mathrm{d}B_{zs}.$$

Using $\int_s^\infty g(t-s,0,z)e^{-\nu t}\mathrm{d}t = \frac{e^{-\nu s}e^{-\sqrt{\nu}z}}{2\sqrt{\nu}}$ we have

$$\varrho_{\psi_\nu}(L,y) = \int_y^\infty \int_0^\infty \frac{e^{-\nu s}e^{-\sqrt{\nu}z}}{2\sqrt{\nu}\mathscr{L}g_y(\nu)}\mathrm{d}B_{zs}$$
$$= \int_y^\infty \int_0^\infty e^{-\sqrt{\nu}s}e^{-\sqrt{\nu}z}e^{\nu y}\mathrm{d}B_{zs}$$

We now remark that

$$\sqrt{\nu}\langle e^{-\nu\cdot}, u_y\rangle + \langle e^{-\nu\cdot}, v_y\rangle$$

$$=\sqrt{\nu}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{-\nu s}e^{-\sqrt{\nu}|z-y|}}{2\sqrt{\nu}}\mathrm{d}B_{zs}$$

$$+\int_{-\infty}^{\infty}\int_0^{\infty}\mathrm{sgn}(z-y)\sqrt{\nu}\frac{e^{-\nu s}e^{-\sqrt{\nu}|z-y|}}{2\sqrt{\nu}}\mathrm{d}B_{zs}$$

$$=\int_0^{\infty}\int_y^{\infty}e^{-\nu s}e^{-\sqrt{\nu}(z-y)}\mathrm{d}B_{zs} = \varrho_{\psi_\nu}(L, y)$$

On the other hand, if we take $m_y(\psi_\nu)(t) = \left(0, \frac{e^{-\nu t}}{\mathscr{L}g_y(\nu)}\right)$ then

$$\int_y^{\infty}\int_0^{\infty}\left(\int_s^{\infty}\partial_2 g(t-s,0,z)m_y(\psi_\nu)_2(t)\mathrm{d}t\right)\mathrm{d}B_{zs}$$

is soon seen to be the same, since we get a $\mathscr{L}\partial_2 g_y(\nu)$ term from $\partial_2 g(t-s,0,z)$, which cancels with the $\mathscr{L}\partial_2 g_y(\nu)$ in $m_y(\psi_\nu)_2$. This gives us some hope that $\varrho_{\psi_\nu}(L, y)$ is uniquely defined.

There is of course no problem here. If both $m_y(\psi_\nu)$ and $m'_y(\psi_\nu)$ are elements of $X_1 \times X_2$ with $\kappa_y\psi_\nu = \mathcal{C}_y m_y(\psi_\nu)$ and $\kappa_y\psi_\nu = \mathcal{C}_y m'_y(\psi_\nu)$ then what is important is that $\|m_y(\psi_\nu) - m'_y(\psi_\nu)\|_{H'_y} = 0$ by the isometry property of $\mathcal{C}_y$. Furthermore

$$\varrho_{\psi_\nu}(L, y) = \langle m_y(\psi_\nu), ((u_0)^y, (v_0)^y)\rangle = \langle m'_y(\psi_\nu), ((u_0)^y, (v_0)^y)\rangle \quad \text{a.s.}$$

since

$$\mathbb{E}[(\langle m_y(\psi_\nu), ((u_0)^y, (v_0)^y)\rangle - \langle m'_y(\psi_\nu), ((u_0)^y, (v_0)^y)\rangle)^2]$$

$$=\|m_y(\psi_\nu) - m'_y(\psi_\nu)\|^2_{H'_y} = 0.$$

We have yet to establish whether either of our solutions is in H_y'. Actually, if $m_y(\psi_\nu) \in X_1 \times X_2$ this is immediately obvious since $\langle m_y(\psi_\nu), ((u_0)^y, (v_0)^y) \rangle$ is an $L^2(\Omega)$ random variable from our definition of X_1 and X_2. What we really need to check is that we do have a solution in $X_1 \times X_2$. Suppose we take $m_y(\psi_\nu) = \left(\frac{e^{-\nu r}}{\mathscr{L}g_y(\nu)}, 0 \right)$. We need to show that $\frac{e^{-\nu \cdot}}{\mathscr{L}g_y(\nu)}$ is in X_1, which is obvious since its decays faster than $(1+t)^{\frac{3}{2}+\beta}$. This gives us one form of the solution, although ultimately what is important to us is that we have a solution and an expression for $\langle m_y(\psi_\nu), ((u_0)^y, (v_0)^y) \rangle$.

3.3 Equations for $(\langle h_1, u_x \rangle; x \geq 0)$ and $(\langle h_2, v_x \rangle; x \geq 0)$ for certain test functions h_1 and h_2.

3.3.1 An analysis of the properties ψ_ν.

In the previous section we assumed the existence of a function ψ_ν in $L^2([0, \infty))$ whose Laplace transform is $\frac{1}{(\lambda+\nu)(\sqrt{\lambda}+\sqrt{\nu})}$. In this section we define ψ_ν by

$$\psi_\nu(t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \left(\frac{e^{-\nu s}}{\sqrt{|t-s|}} - \frac{e^{-\nu s}}{\sqrt{t+s}} \right) \mathrm{d}s. \qquad (3.3.1)$$

We have the following

Lemma 3.1. ψ_ν is a continuous function on $[0, \infty)$ which belongs to $L^2([0, \infty))$.

Proof By a change of variable $(s = tu)$, (3.3.1) becomes

$$\psi_\nu(t) = c\sqrt{t} \int_0^\infty e^{-\nu ut} \left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}} \right) \mathrm{d}u.$$

Note that

$$\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}} = \frac{2(1 \wedge u)}{\sqrt{|1-u^2|}(\sqrt{|1-u|} + \sqrt{1+u})}. \qquad (3.3.2)$$

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This blows up like $|1-u|^{-\frac{1}{2}}$ when u is near 1, and converges like $u^{-\frac{3}{2}}$ to 0 as $u \to \infty$. It follows that $u \mapsto \frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}$ is a positive, integrable map on $[0, \infty)$, and since it is an upper bound for $e^{-\nu u t}\left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}\right)$, $\psi_\nu(t)$ is well defined for each t. Furthermore if $t_n \to t$ then $\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}$ dominates the sequence $\left(e^{-\nu u t_n}\left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}\right); n \geq 0\right)$, from which it follows by the dominated convergence theorem that

$$\int_0^\infty e^{-\nu u t_n}\left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}\right) \mathrm{d}u \to \int_0^\infty e^{-\nu u t}\left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}\right) \mathrm{d}u$$

as $n \to \infty$. ψ_ν is thus continuous. To demonstrate that $\psi_\nu \in L^2([0, \infty))$, note that

$$\int_0^\infty \psi_\nu^2(t)\mathrm{d}t$$

$$=c\int_0^\infty t\left(\int_0^\infty e^{-\nu u t}\left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}\right)\mathrm{d}u\right)^2 \mathrm{d}t$$

$$=\int_0^\infty \int_0^\infty \left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}\right)\left(\frac{1}{\sqrt{|1-s|}} - \frac{1}{\sqrt{1+s}}\right)\int_0^\infty t e^{-\nu(u+s)t}\mathrm{d}t\mathrm{d}s\mathrm{d}u$$

$$=c\int_0^\infty \int_0^\infty \frac{1}{\nu^2(s+u)^2}\left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}\right)\left(\frac{1}{\sqrt{|1-s|}} - \frac{1}{\sqrt{1+s}}\right)\mathrm{d}s\mathrm{d}u$$

$$\leq c\left(\int_0^\infty \frac{1}{u}\left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}}\right)\mathrm{d}u\right)^2$$

using $(s+u)^2 \geq 2su$. A glance at (3.3.2) shows that the above integrand converges to 1 as $u \to 0$, hence by the previous comments the above integral is finite.

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Lemma 3.2. *The Laplace transform of ψ_ν is*

$$\mathscr{L}\psi_\nu(\lambda) = \frac{1}{(\nu+\lambda)(\sqrt{\nu}+\sqrt{\lambda})}, \ \lambda > 0.$$

Proof We split the Laplace transform into two parts. The first is

$$\frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-\lambda t}\int_0^\infty \frac{e^{-\nu s}}{\sqrt{|t-s|}}\mathrm{d}s\mathrm{d}t$$

$$=\frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-\lambda t}\left(\int_0^t \frac{e^{-\nu(t-s)}}{\sqrt{s}}\mathrm{d}s + \int_t^\infty \frac{e^{-\nu s}}{\sqrt{s-t}}\right)\mathrm{d}t$$

$$=\frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{e^{\nu s}}{\sqrt{s}}\int_s^\infty e^{-(\nu+\lambda)t}\mathrm{d}t\mathrm{d}s + \frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-(\nu+\lambda)t}\int_0^\infty \frac{e^{-\nu s}}{\sqrt{s}}\mathrm{d}s\mathrm{d}t$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{\nu}}+\frac{1}{\sqrt{\lambda}}\right)\frac{1}{\nu+\lambda}$$

where we have used

$$\int_0^\infty \frac{e^{-\nu s}}{\sqrt{\pi s}}\mathrm{d}s = \frac{1}{\sqrt{\nu}}.$$

The second part is

$$\frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-\lambda t}\int_0^\infty \frac{e^{-\nu s}}{\sqrt{t+s}}\mathrm{d}s\mathrm{d}t$$

$$=\frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-\lambda t}\int_t^\infty \frac{e^{-\nu(s-t)}}{\sqrt{s}}\mathrm{d}s\mathrm{d}t$$

$$=\frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\nu s}}{\sqrt{s}}\int_0^s e^{-\lambda t+\nu t}\mathrm{d}t\mathrm{d}s$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{\lambda}}-\frac{1}{\sqrt{\nu}}\right)\frac{1}{\nu-\lambda}.$$

Combining these gives

$$\mathscr{L}\psi_\nu(\lambda) = \frac{\sqrt{\lambda}\nu - \lambda\sqrt{\nu}}{\sqrt{\lambda}\sqrt{\nu}(\nu^2-\lambda^2)} = \frac{1}{(\sqrt{\lambda}+\sqrt{\nu})(\lambda+\nu)}$$

■

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Our goal is now to show that for any $\nu > 0$ and $X > 0$, $((\langle \psi_\nu, u_x \rangle, \langle \psi_\nu, v_x \rangle); x \in [0, X])$ satisfies the original system (1.1.5). Recall that this follows from proposition 3.1 if we can show that $\psi_\nu \in C_{0, \frac{5}{4}+\beta}$, and further has a continuous derivative such that

$$\int_0^\infty \int_0^\infty \psi_\nu'(t) \psi_\nu'(t')(\sqrt{t+t'} - \sqrt{|t-t'|}) \mathrm{d}t' \mathrm{d}t \qquad (3.3.3)$$

is defined.

Lemma 3.3. *For any* $g \in C^1([0, \infty))$ *such that* $g(t)$ *and* $g'(t)$ *decay faster than* $t^{-\frac{1}{2}}$ *as* $t \to \infty$, *define*

$$\Gamma g(t) = \int_0^\infty g(s) \left(\frac{1}{\sqrt{|t-s|}} - \frac{1}{\sqrt{t+s}} \right) \mathrm{d}s.$$

(This is well defined if we only ask that g *is bounded, and indeed the following holds for such* g. *The above condition is purely for convenience and is sufficient for our application.)* Γg *is differentiable on* $(0, \infty)$ *with*

$$\frac{\mathrm{d}}{\mathrm{d}t} \Gamma g(t) = \frac{1}{2} \mathrm{PV} \int_0^\infty g(s) \left(\frac{\mathrm{sgn}(s-t)}{\sqrt{|t-s|^3}} - \frac{1}{\sqrt{(t+s)^3}} \right) \mathrm{d}s \qquad (3.3.4)$$

$$:= \lim_{\varepsilon \to 0} \frac{1}{2} \int_0^\infty \mathbb{1}_{[0, t-\varepsilon) \cup (t+\varepsilon, \infty)}(s) g(s) \left(\frac{\mathrm{sgn}(s-t)}{\sqrt{|t-s|^3}} - \frac{1}{\sqrt{(t+s)^3}} \right) \mathrm{d}s$$

In particular, ψ_ν *is continuously differentiable on* $(0, \infty)$, *with a derivative which grows like* $t^{-\frac{1}{2}}$ *as* $t \to 0$. *Furthermore,* ψ_ν *is in* $C_{0, \frac{5}{4}+\beta}$ *and* ψ_ν' *satisfies condition (3.3.3) above.*

Proof We show directly that $\lim_{h \to 0} \frac{1}{h}(\Gamma g(t+h) - \Gamma g(t))$ exists. First note that $\int_0^\infty \frac{g(s)}{\sqrt{t+s}} \mathrm{d}s$ is differentiable with derivative $-\frac{1}{2} \int_0^\infty \frac{g(s)}{\sqrt{(t+s)^3}} \mathrm{d}s$ by a standard argument. We now split $\int_0^\infty \frac{g(s)}{\sqrt{|t-s|}} \mathrm{d}s$ into

$$\int_0^{t-\varepsilon} \frac{g(s)}{\sqrt{|t-s|}} \mathrm{d}s + \int_{t-\varepsilon}^{t+\varepsilon} \frac{g(s)}{\sqrt{|t-s|}} \mathrm{d}s + \int_{t+\varepsilon}^\infty \frac{g(s)}{\sqrt{|t-s|}} \mathrm{d}s \qquad (3.3.5)$$

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for some $\varepsilon > 0$. The first and third integrals are differentiable with combined derivative

$$\frac{g(t-\varepsilon)}{\sqrt{\varepsilon}} - \frac{1}{2}\int_0^{t-\varepsilon} \frac{g(s)}{\sqrt{(t-s)^3}}\mathrm{d}s - \frac{g(t+\varepsilon)}{\sqrt{\varepsilon}} + \frac{1}{2}\int_{t+\varepsilon}^{\infty} \frac{g(s)}{\sqrt{(s-t)^3}}\mathrm{d}s.$$

Note that $\frac{g(t-\varepsilon)-g(t+\varepsilon)}{\varepsilon} \cdot \sqrt{\varepsilon} \to 0$ as $\varepsilon \to 0$. We also remark that

$$-\frac{1}{2}\int_0^{t-\varepsilon} \frac{g(s)}{\sqrt{(t-s)^3}}\mathrm{d}s + \frac{1}{2}\int_{t+\varepsilon}^{\infty} \frac{g(s)}{\sqrt{(s-t)^3}}\mathrm{d}s$$

$$= -\left[\frac{g(s)}{\sqrt{t-s}}\right]_{s=0}^{t-\varepsilon} + \int_0^{t-\varepsilon} \frac{g'(s)}{\sqrt{t-s}}\mathrm{d}s - \left[\frac{g(s)}{\sqrt{s-t}}\right]_{s=t+\varepsilon}^{\infty} + \int_{t+\varepsilon}^{\infty} \frac{g'(s)}{\sqrt{s-t}}\mathrm{d}s$$

$$\to \frac{g(0)}{\sqrt{t}} + \int_0^{\infty} \frac{g'(s)}{\sqrt{|t-s|}}\mathrm{d}s$$

as $\varepsilon \to 0$. Thus $-\frac{1}{2}\int_0^{t-\varepsilon} \frac{g(s)}{\sqrt{(t-s)^3}}\mathrm{d}s + \frac{1}{2}\int_{t+\varepsilon}^{\infty} \frac{g(s)}{\sqrt{(s-t)^3}}\mathrm{d}s$ has a limit as $\varepsilon \to 0$, which we define to be $\frac{1}{2}\mathrm{PV}\int_0^{\infty} \frac{g(s)\mathrm{sgn}(s-t)}{\sqrt{|t-s|^3}}\mathrm{d}s$.

The derivative of the middle term in (3.3.5) is the limit as $h \to 0$ of

$$\frac{1}{h}\left(\int_{t+h-\varepsilon}^{t+h+\varepsilon} \frac{g(s)}{\sqrt{|t+h-s|}}\mathrm{d}s - \int_{t-\varepsilon}^{t+\varepsilon} \frac{g(s)}{\sqrt{|t-s|}}\mathrm{d}s\right)$$

$$= \frac{1}{h}\int_{t-\varepsilon}^{t+\varepsilon} \frac{g(s+h)-g(s)}{\sqrt{|t-s|}}\mathrm{d}s$$

which is $\int_{t-\varepsilon}^{t+\varepsilon} \frac{g'(s)}{\sqrt{|t-s|}}\mathrm{d}s$. The absolute value of this is bounded by $\|g'\|_\infty \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\sqrt{|t-s|}}\mathrm{d}s$, which we soon see is $2\|g\|_\infty\sqrt{\varepsilon}$ and converges to 0 as $\varepsilon \to 0$. Therefore, if we calculate $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\infty} \frac{g(s)}{\sqrt{|t-s|}}\mathrm{d}s$ using (3.3.5) and letting $\varepsilon \to 0$ we obtain $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\infty} \frac{g(s)}{\sqrt{|t-s|}}\mathrm{d}s = \frac{1}{2}\mathrm{PV}\int_0^{\infty} \frac{g(s)\mathrm{sgn}(s-t)}{\sqrt{|t-s|^3}}\mathrm{d}s$.

The benefit of this is that $\psi_\nu = c\Gamma e^{-\nu\cdot}$, which implies that ψ_ν is continuously differentiable, and which we may use to check property (3.3.3). However, we first check that $\psi_\nu \in C_{0,\frac{5}{4}+\beta}$ by looking at the behaviour of $\psi_\nu(t)$ for large t,

and we assume that $t > 1$ and $0 < \varepsilon < 1$. Thus $t > t^\varepsilon$ and we have

$$c\psi_\nu(t) = \int_0^{t^\varepsilon} e^{-\nu s} \left(\frac{1}{\sqrt{t-s}} - \frac{1}{\sqrt{t+s}} \right) ds + \int_{t^\varepsilon}^\infty e^{-\nu s} \left(\frac{1}{\sqrt{|t-s|}} - \frac{1}{\sqrt{t+s}} \right) ds.$$

The first integral is bounded by

$$\int_0^{t^\varepsilon} \left(\frac{1}{\sqrt{t-s}} - \frac{1}{\sqrt{t+s}} \right) ds$$

$$= 2(2\sqrt{t} - \sqrt{t-t^\varepsilon} - \sqrt{t+t^\varepsilon})$$

$$= 2 \left(\frac{t^\varepsilon}{\sqrt{t} + \sqrt{t-t^\varepsilon}} - \frac{t^\varepsilon}{\sqrt{t} + \sqrt{t+t^\varepsilon}} \right)$$

$$= \frac{4t^{2\varepsilon}}{(\sqrt{t} + \sqrt{t-t^\varepsilon})(\sqrt{t} + \sqrt{t+t^\varepsilon})(\sqrt{t-t^\varepsilon} + \sqrt{t+t^\varepsilon})}$$

which looks like $t^{-\frac{3}{2}+2\varepsilon}$ for large t. The second integral is bounded by

$$e^{-\nu t^\varepsilon} \int_0^\infty \left(\frac{1}{\sqrt{|t-s|}} + \frac{1}{\sqrt{t+s}} \right) ds$$

which decays exponentially. Thus for large t, $\psi_\nu(t)$ is bounded by $ct^{-\frac{3}{2}+2\varepsilon}$. Choosing ε sufficiently small implies that ψ_ν is in $C_{0,\frac{5}{4}+\beta}$. We now check that property (3.3.3). We first look at the behaviour of $\psi_\nu'(t)$ when $t \to 0$. Note that

$$-\frac{1}{2} \int_0^\infty \frac{e^{-\nu s}}{\sqrt{(t+s)^3}} ds = -\frac{1}{\sqrt{t}} + \nu \int_0^\infty \frac{e^{-\nu s}}{\sqrt{t+s}} ds.$$

Combining this with our expression for $\frac{1}{2}\text{PV} \int_0^\infty \frac{\text{sgn}(s-t)e^{-\nu s}}{\sqrt{|t-s|^3}} ds$ gives $\psi_\nu'(0) = 0$. This leaves us needing to check the behaviour of ψ_ν' at ∞, and again, taking $0 < \varepsilon < 1$ and $t > 1$, we note that

$$\frac{1}{2} \int_0^{t^\varepsilon} e^{-\nu s} \left(\frac{1}{\sqrt{(t-s)^3}} + \frac{1}{\sqrt{(t+s)^3}} \right) ds$$

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is bounded by

$$c\left(\frac{1}{\sqrt{t-t^\varepsilon}} + \frac{1}{\sqrt{t+t^\varepsilon}} - \frac{2}{\sqrt{t}}\right) = \frac{ct^{2\varepsilon}}{\sqrt{t}(\sqrt{t-t^\varepsilon}+\sqrt{t+t^\varepsilon})\sqrt{t-t^\varepsilon}\sqrt{t+t^\varepsilon}}$$

after a manipulation similar to above. This decays like $t^{-2+2\varepsilon}$. We also note that $\text{PV}\frac{1}{2}\int_{t^\varepsilon}^\infty e^{-\nu s}\left(\frac{\text{sgn}(s-t)}{\sqrt{|t-s|^3}} - \frac{1}{\sqrt{(t+s)^3}}\right)\,\mathrm{d}s$ decays exponentially, and so $\psi_\nu'(t)$ decays as fast as $t^{-2+2\varepsilon}$ for small $\varepsilon > 0$.

$\blacksquare$

It will be useful later on to define $(\psi_\nu)_{anti}$, the anti-symmetric extension of ψ_ν, by

$$(\psi_\nu)_{anti}(t) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^\infty \frac{\xi_\nu(s)}{\sqrt{\pi|t-s|}} \tag{3.3.6}$$

for all $t \in \mathbb{R}$, where

$$\xi_\nu(s) = \begin{cases} e^{-\nu s} & s > 0 \\ -e^{-\nu s} & s < 0 \end{cases}$$

3.3.2 A system of equations for $(\langle\psi_\nu, u_x\rangle; x \geq 0)$ and $(\langle\psi_\nu, v_x\rangle; x \geq 0)$.

Let us summarise what we have shown so far. For any $y > 0$ and $\nu > 0$, we have defined $\psi_\nu \in L^2([0,\infty))$ and shown that there exists $m_y\psi_\nu \in X_1 \times X_2$ such that $\mathcal{C}_y m_y \psi_\nu = \kappa_y \psi_\nu$. Taking $\varrho_{\psi_\nu}(\phi, y) = \langle m_y\psi_\nu, \phi - ((u_0)_y, (v_0)_y)\rangle$, we know that $\varrho_{\psi_\nu}(\phi, y)$ is $\mathscr{F}_y$ measurable, $\varrho_{\psi_\nu}(L, y)$ is in $L^1(\Omega)$ for each y and, when considered as a function in y, is in $L^1([0,x])$ for any given $X > 0$. It now follows from theorem (1.5) that

$$\tilde{W}_x(\psi_\nu) = W_x(\psi_\nu) - \int_0^x \varrho_{\psi_\nu}(L, y)\mathrm{d}y$$

is an $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingale. Furthermore, we have shown almost surely that $\varrho_{\psi_\nu}(L, y) = \sqrt{\nu}\langle e^{-\nu\cdot}, u_y\rangle + \langle e^{-\nu\cdot}, v_y\rangle$. One may deduce from the continuity in y of $\langle e^{-\nu\cdot}, u_y\rangle$ and $\langle e^{-\nu\cdot}, v_y\rangle$ that this holds almost surely for all $y \in [0, X]$. Since

we have also shown that $\psi_\nu \in C_{0,\frac{5}{4}+\beta}$ and that $\langle \psi'_\nu, u_x \rangle$ is defined, we have thus shown that for all $\nu > 0$, $\langle \psi_\nu, u_x \rangle$ and $\langle \psi_\nu, v_x \rangle$ almost surely satisfy

$$
\begin{aligned}
\langle \psi_\nu, u_x \rangle =& \langle \psi_\nu, u_0 \rangle + \int_0^x \langle \psi_\nu, v_y \rangle \mathrm{d}y \\
\langle \psi_\nu, v_x \rangle =& \langle \psi_\nu, v_0 \rangle - \int_0^x \left(\langle \psi'_\nu, u_y \rangle + \sqrt{\nu} \langle e^{-\nu \cdot}, u_y \rangle + \langle e^{-\nu \cdot}, v_y \rangle \right) \mathrm{d}y \\
& - \tilde{W}_x(\psi_\nu)
\end{aligned}
\tag{3.3.7}
$$

for all $x \in [0, X]$.

Let $Y = Sp\langle \{ \psi_\nu; \nu > 0 \} \rangle$. We can naturally define equations for u_x and v_x tested against anything in Y. This is easier to write if we can find linear operators A_1 and A_2 such that $A_1(\psi_\nu) = \sqrt{\nu} e^{-\nu \cdot}$ and $A_2(\psi_\nu) = e^{-\nu \cdot}$. This form allows us to guess equations for $\langle h_1, u_x \rangle$ and $\langle h_2, v_x \rangle$ for a more general class of h_1 and h_2.

A_1 is straightforward to deal with. Define

$$
A_1 h_2(t) = - \int_t^\infty \frac{h'_2(s)}{\sqrt{\pi(s-t)}} \mathrm{d}s.
$$

If $h_2(s) = e^{-\nu s}$ then

$$
\begin{aligned}
A_1 h_2(t) =& \nu \int_t^\infty \frac{e^{-\nu s}}{\sqrt{\pi(s-t)}} \mathrm{d}s \\
=& \nu \int_0^\infty \frac{e^{-\nu(t+s)}}{\sqrt{\pi s}} \mathrm{d}s \\
=& \sqrt{\nu} e^{-\nu t}
\end{aligned}
$$

In order to determine A_2, we apply Fourier transforms, which we will denote by F (taking $F\psi(z) = \int_{\mathbb{R}} \psi(t) e^{-itz} \mathrm{d}t$), to (3.3.6).

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Thus

$$F((\psi_\nu)_{anti})(z) = \frac{1}{2}F\left(\frac{1}{\sqrt{\pi|\cdot|}}\right)(z)F(\xi_\nu)(z).$$

Note that

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\sqrt{t}}\cos(zt)\mathrm{d}t = \frac{1}{\sqrt{z}} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\sqrt{t}}\sin(zt)\mathrm{d}t$$

for $z > 0$. We therefore get

$$\int_0^\infty \frac{1}{\sqrt{\pi t}}e^{-izt}\mathrm{d}t = \frac{1-i}{\sqrt{2z}}$$

and

$$\int_{-\infty}^0 \frac{1}{\sqrt{\pi|t|}}e^{-izt}\mathrm{d}t = \int_0^\infty \frac{1}{\sqrt{\pi t}}e^{izt}\mathrm{d}t = \frac{1+i}{\sqrt{2z}}.$$

So for $z > 0$ we get $F\left(\frac{1}{\sqrt{\pi|\cdot|}}\right)(z) = \sqrt{\frac{2}{z}}$. Furthermore, since $\frac{1}{\sqrt{\pi|\cdot|}}$ is symmetric, so its Fourier transform is also, and hence

$$F\left(\frac{1}{\sqrt{\pi|\cdot|}}\right)(z) = \sqrt{\frac{2}{|z|}}.$$

This implies that

$$\begin{aligned}
F(\xi_\nu)(z) &= \sqrt{2|z|}F((\psi_\nu)_{anti})(z) \\
&= \frac{\sqrt{2|z|}}{iz}izF((\psi_\nu)_{anti})(z) \\
&= \sqrt{2}\frac{\mathrm{sgn}(z)}{i\sqrt{|z|}}F((\psi_\nu)'_{anti})(z)
\end{aligned}$$

where $(\psi_\nu)'_{anti}$ refers to the derivative of $(\psi_\nu)_{anti}$. Our previous observations regarding the Fourier sine and cosine transformations of $1/\sqrt{|t|}$ show that for

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$z > 0$, $F\left(\frac{\text{sgn}(\cdot)}{\sqrt{|\cdot|}}\right)(z) = \sqrt{\frac{\pi}{2}} \cdot \frac{-2i}{\sqrt{z}} = \frac{\sqrt{2\pi}}{i\sqrt{z}}$. The antisymmetry of $\frac{\text{sgn}(t)}{\sqrt{|t|}}$ gives

$$F\left(\frac{\text{sgn}(\cdot)}{\sqrt{|\cdot|}}\right)(z) = \sqrt{2\pi}\frac{\text{sgn}(z)}{i\sqrt{|z|}}$$

for all $z \in \mathbb{R}$. This implies that

$$F(\xi_\nu)(z) = F\left((\psi_\nu)'_{anti} * \frac{\text{sgn}(\cdot)}{\sqrt{\pi|\cdot|}}\right)(z)$$

for all $z \in \mathbb{R}$ and hence

$$e^{-\nu t} = \int_{-\infty}^{\infty} \frac{\text{sgn}(t-s)}{\sqrt{\pi|t-s|}}(\psi_\nu)'_{anti}(s)\mathrm{d}s =: A_2\psi_\nu(t)$$

for all $t > 0$. Note that $(\psi_\nu)'_{anti}$ is well defined since $\psi_\nu(0) = 0$.

3.3.3   Extending these equations for test functions $h_1, h_2 \in C_0^\infty([0, \infty))$.

We can now write that for any $h_1, h_2 \in Y$ and $X \geq 0$, $\langle h_1, u_x\rangle$ and $\langle h_2, v_x\rangle$ almost surely satisfy

$$
\begin{aligned}
\langle h_1, u_x\rangle =& \langle h_1, u_0\rangle + \int_0^x \langle h_1, v_y\rangle\mathrm{d}y \\
\langle h_2, v_x\rangle =& \langle h_2, v_0\rangle - \int_0^x (\langle h_2', u_y\rangle + \langle A_1 A_2 h_2, u_y\rangle + \langle A_2 h_2, v_y\rangle)\mathrm{d}y - \tilde{W}_x(h_2)
\end{aligned}
$$

$$(3.3.8)$$

for all $x \in [0, X]$. As a shorthand, we shall say that $(\langle h_1, u_x\rangle, \langle h_2, v_x\rangle)$ satisfies system (3.3.8) when the above is satisfied almost surely for a given x, even though we should also include the terms $\langle h_2', u_x\rangle$, $\langle A_1 A_2 h_2, u_x\rangle$ and $\langle A_2 h_2, v_x\rangle$. To what extent can we extend this for $h_1 \in C_{0,\frac{5}{4}+\beta}$ and $h_2 \in C_{0,\frac{3}{4}+\beta}$? It seems unlikely to extend for all $h_1 \in C_{0,\frac{5}{4}+\beta}$ and $h_2 \in C_{0,\frac{3}{4}+\beta}$ since for this we would require for all such h_2 that $A_1 A_2 h_2 \in C_{0,\frac{5}{4}+\beta}$ and $A_2 h_2 \in C_{0,\frac{3}{4}+\beta}$. This seems

implausible, however we do have the following:

Proposition 3.2. *For all* $h_2 \in C_0^\infty([0,\infty))$, $A_1 A_2 h_2 \in C_{0,\frac{5}{4}+\beta}$ *and* $A_2 h_2 \in C_{0,\frac{3}{4}+\beta}$.

Proof Suppose that the support of h_2 lies within $[0,T]$ and take $t \gg T$. Then

$$
\begin{aligned}
A_2 h_2(t) &= c \int_0^T \left(\frac{1}{\sqrt{t-s}} + \frac{1}{\sqrt{t+s}} \right) h_2'(s) \mathrm{d}s \\
&= c \int_0^T \left(\frac{1}{(t-s)^{\frac{3}{2}}} - \frac{1}{(t+s)^{\frac{3}{2}}} \right) h_2(s) \mathrm{d}s
\end{aligned}
$$

Thus

$$
\begin{aligned}
|A_2 h_2(t)| &\leq c \left| \int_0^T \left(\frac{1}{(t-s)^{\frac{3}{2}}} - \frac{1}{(t+s)^{\frac{3}{2}}} \right) \mathrm{d}s \right| \\
&= c \left| \left[(t-s)^{-\frac{1}{2}} + (t+s)^{-\frac{1}{2}} \right] \Big|_{s=0}^{T} \right| \\
&\leq c \left(\frac{1}{\sqrt{t-T}} - \frac{1}{\sqrt{t}} \right) - \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t+T}} \right).
\end{aligned}
$$

Now

$$
\frac{1}{\sqrt{t-T}} - \frac{1}{\sqrt{t}} = \frac{\sqrt{t} - \sqrt{t-T}}{\sqrt{t}\sqrt{t-T}} = \frac{T}{\sqrt{t}\sqrt{t-T}(\sqrt{t} + \sqrt{t-T})}
$$

which for large t looks like $t^{-\frac{3}{2}}$. A similar expression for $\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t+T}}$ allows us to at least deduce that $A_2 h_2(t) \leq c t^{-\frac{3}{2}}$ provided that $t \geq T$, and hence $A_2 h_2$ is in $C_{0,\frac{3}{4}+\beta}$, in fact in $C_{0,\frac{5}{4}+\beta}$ even.

For $t > T$, the kernel in the expression for $A_2 h_2(t)$ has no singularities, so we may differentiate with respect to t, taking derivatives inside the integral to obtain

$$(A_2 h_2)'(t) = c \int_0^T \left(\frac{1}{(t-s)^{\frac{3}{2}}} + \frac{1}{(t+s)^{\frac{3}{2}}} \right) h_2'(s) \mathrm{d}s$$

$$= c \int_0^T \left(\frac{1}{(t-s)^{\frac{5}{2}}} - \frac{1}{(t+s)^{\frac{5}{2}}} \right) h_2(s) \mathrm{d}s.$$

This is bounded by

$$c \left(((t-T)^{-\frac{3}{2}} - t^{-\frac{3}{2}}) - (t^{-\frac{3}{2}} - (t+T)^{-\frac{3}{2}}) \right).$$

The first term here is

$$\frac{t^3 - (t-T)^3}{(t-T)^{\frac{3}{2}} t^{\frac{3}{2}} (t^{\frac{3}{2}} + (t-T)^{\frac{3}{2}})}.$$

For large t this looks like $t^{-\frac{5}{2}}$. Therefore, provided that $s > T$, $|(A_2 h_2)'(t+s)| \leq c(t+s)^{-\frac{5}{2}}$, so that

$$|A_1 A_2 h_2(s)| \leq c \int_0^\infty \frac{(t+s)^{-\frac{5}{2}}}{t^{\frac{1}{2}}} \mathrm{d}t$$

$$= c \int_0^\infty \frac{(su+s)^{-\frac{5}{2}}}{s^{\frac{1}{2}} u^{\frac{1}{2}}} s \mathrm{d}u$$

$$= cs^{-2} \int_0^\infty \frac{(u+1)^{-\frac{5}{2}}}{u^{\frac{1}{2}}} \mathrm{d}u.$$

This is finite, and so $A_1 A_2 h_2(s)$ goes to 0 at least as fast as s^{-2}. Thus $A_1 A_2 h_2 \in C_{0, \frac{5}{4} + \beta}$.

$\blacksquare$

We now know that the system (3.3.8) is well defined for h_1, h_2 in $C_0^\infty([0, \infty))$. What we do not know is whether $\langle h_1, u_x \rangle$ and $\langle h_2, v_x \rangle$ satisfy such a system for any $x \geq 0$. We have already shown that $\langle h_1, u_x \rangle$ and $\langle h_1, v_x \rangle$ satisfy the first equation in (3.3.8) for any $h_1 \in C_{0, \frac{5}{4} + \beta}$. It is the second equation which is more

problematic. Given $f_n \in Y$ we know that

$$\langle f_n, v_x \rangle = \langle f_n, v_0 \rangle - \int_0^x (\langle f_n', u_y \rangle + \langle A_1 A_2 f_n, u_y \rangle + \langle A_2 f_n, v_y \rangle) \mathrm{d}y - \tilde{W}_x(f_n).$$
(3.3.9)

Our aim, given $h_2 \in C_0^\infty([0, \infty))$, is to choose f_n in such a way that each term in (3.3.9) converges in $L^2(\Omega)$ to the equivalent term with h_2 replacing f_n. It is not immediately obvious that we may choose f_n to even converge pointwise to h_2. However we deduce from the form of ψ_ν given by (3.3.1) that

$$(f_n - h_2)(t) = c \int_0^\infty \left(\frac{1}{\sqrt{|t-s|}} - \frac{1}{\sqrt{t+s}} \right) A_2(f_n - h_2)(s) \mathrm{d}s. \qquad (3.3.10)$$

The reason this is of use is because for each n, $A_2 f_n$ is a linear combination of exponential functions. We may therefore look to approximate $A_2 h_2$ by such functions and look to deduce some information regarding the convergence of the f_n from (3.3.10). The following is an application of corollary 3.7 of [dP71], which is an extension of the Stone-Weierstrass theorem for weighted topologies.

Proposition 3.3. *Let A be the subspace of linear combinations of exponential functions. Then A is dense in $C_{0,\alpha}([0, \infty))$ for all $\alpha > 0$.*

We have previously seen that for any $h_2 \in C_0^\infty([0, \infty))$ and $0 < \alpha < \frac{3}{2}$, $A_2 h_2 \in C_{0,\alpha}([0, \infty))$. We may thus choose $f_n \in Y$ such that $\|A_2(f_n - h_2)\|_\alpha \to 0$ as $n \to \infty$. As an aside, we remark that if $\alpha' < \alpha$ then $\|A_2(f_n - l)\|_{\alpha'} \to 0$. Thus in the following arguments, we may allow α to represent different values as required, so long as we may eventually choose an $\alpha \in (0, \frac{3}{2})$ that is big enough for all of our arguments to work.

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In particular, we observe that $|f_n(t) - h_2(t)|$ is bounded by

$$\int_0^\infty \left| \frac{1}{\sqrt{|t-s|}} - \frac{1}{\sqrt{t+s}} \right| |A_2(f_n - h_2)(s)| \mathrm{d}s$$

$$\leq \|A_2(f_n - h_2)\|_\alpha \int_0^\infty \left(\frac{1}{\sqrt{|t-s|}} - \frac{1}{\sqrt{t+s}} \right) (1+s)^{-\alpha} \mathrm{d}s$$

$$= \|A_2(f_n - h_2)\|_\alpha \int_0^\infty \left(\frac{1}{\sqrt{|t+1-(s+1)|}} - \frac{1}{\sqrt{(t+1)+(s+1)-2}} \right) (s+1)^{-\alpha} \mathrm{d}s$$

$$= \|A_2(f_n - h_2)\|_\alpha (1+t)^{-\alpha+\frac{1}{2}} \int_{\frac{1}{1+t}}^\infty \left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u-\frac{2}{1+t}}} \right) u^{-\alpha} \mathrm{d}u$$

$$\leq \|A_2(f_n - h_2)\|_\alpha (1+t)^{-\alpha+\frac{1}{2}} \int_0^\infty \left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u-\frac{2}{1+t}}} \right) u^{-\alpha} \mathrm{d}u$$

where we have used the substitution $(1+s) = u(1+t)$. Since $-1/\sqrt{1+u-\frac{2}{1+t}} < -1/\sqrt{1+u}$ for all $t > 0$, it follows that

$$|f_n(t) - h_2(t)| \leq \|A_2(f_n - h_2)\|_\alpha (1+t)^{-\alpha+\frac{1}{2}} \int_0^\infty \left(\frac{1}{\sqrt{|1-u|}} - \frac{1}{\sqrt{1+u}} \right) u^{-\alpha} \mathrm{d}y$$

and we have already noted that the integral is finite for any $\alpha < 2$. Thus, $\|f_n - h_2\|_\beta \leq c\|A_2(f_n - h_2)\|_\alpha \sup_{t \geq 0}(1+t)^{\beta-\alpha+\frac{1}{2}}$ and thus $\|f_n - h_2\|_{\alpha-\frac{1}{2}} \to 0$ as $n \to \infty$.

We now investigate the convergence of each of the terms in (3.3.9). To begin with, note that

$$\int_0^\infty (f_n(t) - h_2(t))^2 \mathrm{d}t \leq \|f_n - h_2\|_{\alpha-\frac{1}{2}}^2 \int_0^\infty (1+t)^{-2\alpha+1} \mathrm{d}t$$

which converges to 0 for any $\alpha > 1$. For such α, it follows that $\tilde{W}_x(f_n)$ converges to $\tilde{W}_x(h_2)$ in $L^2(\Omega)$. We now look at $\langle f_n - h_2, v_x \rangle$. The norm of this in $L^2(\Omega)$

is

$$c \int_0^\infty \int_0^t (f_n - h_2)(t)(f_n - h_2)(t') \left(\frac{1}{\sqrt{t-t'}} - \frac{1}{\sqrt{t+t'}} \right) \mathrm{d}t' \mathrm{d}t.$$

This is bounded by

$$c\|f_n - h_2\|^2_{\alpha - \frac{1}{2}} \int_0^\infty \int_0^t (1+t)^{-\alpha + \frac{1}{2}}(1+t')^{-\alpha + \frac{1}{2}} \left(\frac{1}{\sqrt{t-t'}} - \frac{1}{\sqrt{t+t'}} \right) \mathrm{d}t' \mathrm{d}t$$

which converges to 0 provided that the integral is finite. We have already seen that this is the case provided that $\alpha - \frac{1}{2} > \frac{3}{4}$, which is an acceptable choice of α. This naturally deals with $\langle f_n, v_0 \rangle$ as well.

We may follow the above argument to show that $\langle A_2(f_n - h_2), v_y \rangle$ converges to 0 in $L^2(\Omega)$ provided that $\frac{3}{4} < \alpha < \frac{3}{2}$. For the $\langle A_1 A_2(f_n - h_2), u_x \rangle$ term, we recall that

$$A_1 A_2(f_n - h_2)(t) = -\int_t^\infty \frac{(A_2(f_n - h_2))'(s)}{\sqrt{\pi(s-t)}} \mathrm{d}s$$

(provided of course that this is well defined, which we have seen is certainly the case for $f_n - h_2$ above). We thus have

$$\mathbb{E}[\langle A_1 A_2 h, u_x \rangle^2]$$
$$= c \int_0^\infty \int_0^\infty \left(\int_t^\infty \frac{(A_2 h)'(s)}{\sqrt{s-t}} \mathrm{d}s \right) \left(\int_{t'}^\infty \frac{(A_2 h)'(s')}{\sqrt{s'-t'}} \mathrm{d}s' \right) (\sqrt{t+t'} - \sqrt{|t-t'|}) \mathrm{d}t' \mathrm{d}t$$
$$= c \int_0^\infty \int_0^\infty (A_2 h)'(s)(A_2 h)'(s') \int_0^s \int_0^{s'} \frac{\sqrt{t+t'} - \sqrt{|t-t'|}}{\sqrt{s-t}\sqrt{s'-t'}} \mathrm{d}t' \mathrm{d}t \mathrm{d}s' \mathrm{d}s.$$

We deal with the t, t' part of the integral first, noting that $t + t' \geq 2\sqrt{tt'}$ and thus

$$\sqrt{t+t'} - \sqrt{|t-t'|} = \frac{t' \wedge t}{\sqrt{t+t'} + \sqrt{|t-t'|}} \leq \frac{t' \wedge t}{\sqrt{2}(tt')^{\frac{1}{4}}}. \tag{3.3.11}$$

This now implies, along with the change of variables $t = su$ and $t' = s'u'$, that

$$\int_0^s \int_0^{s'} \frac{\sqrt{t+t'} - \sqrt{|t-t'|}}{\sqrt{s-t}\sqrt{s'-t'}} \mathrm{d}t' \mathrm{d}t$$

$$\leq c \int_0^s \int_0^{s'} \frac{t' \wedge t}{\sqrt{s-t}\sqrt{s'-t'}(tt')^{\frac{1}{4}}} \mathrm{d}t' \mathrm{d}t$$

$$\leq c(s' \wedge s)(ss')^{\frac{1}{4}} \int_0^1 \int_0^1 \frac{u' \vee u}{\sqrt{1-u}\sqrt{1-u'}(uu')^{\frac{1}{4}}} \mathrm{d}u' \mathrm{d}u$$

where we have also used $(s'u') \wedge (su) \leq (s' \wedge s)(u' \vee u)$. It now follows that

$$\begin{aligned}
\mathbb{E}[\langle A_1 A_2 h, u_x \rangle^2] &\leq c \int_0^\infty \int_0^\infty (A_2 h)'(s)(A_2 h)'(s')(s' \wedge s)(ss')^{\frac{1}{4}} \mathrm{d}s' \mathrm{d}s \\
&= c \int_0^\infty \int_0^s (A_2 h)'(s)(A_2 h)'(s')(s')^{\frac{5}{4}} s^{\frac{1}{4}} \mathrm{d}s' \mathrm{d}s \\
&= c \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}((A_2 h(s))^2) s^{\frac{3}{2}} \mathrm{d}s + c \int_0^\infty \int_{s'}^\infty (A_2 h)'(s) s^{\frac{1}{4}} A_2 h(s')(s')^{\frac{1}{4}} \mathrm{d}s \mathrm{d}s' \\
&= c \int_0^\infty (A_2 h(s))^2 s^{\frac{1}{2}} \mathrm{d}s + c \int_0^\infty A_2 h(s')(s')^{\frac{1}{4}} [A_2 h(s) s^{\frac{1}{4}}]_{s=s'}^\infty \mathrm{d}s' \\
&\quad + c \int_0^\infty \int_{s'}^\infty A_2 h(s) s^{-\frac{3}{4}} A_2 h(s')(s')^{\frac{1}{4}} \mathrm{d}s \mathrm{d}s' \\
&\leq c \|A_2 h\|_\alpha^2 \int_0^\infty (1+s)^{-2\alpha} s^{\frac{1}{2}} \mathrm{d}s \\
&\quad + c \|A_2 h\|_\alpha^2 \int_0^\infty \int_0^s s^{-\frac{3}{4}} (s')^{\frac{1}{4}} (1+s)^{-\alpha} (1+s')^{-\alpha} \mathrm{d}s' \mathrm{d}s.
\end{aligned}$$

Of course, for this argument to work and for the final integrals to be finite, there are restrictions on α, and one soon sees that it is sufficient that $\alpha > \frac{5}{4}$. Replacing h by $f_n - h_2$, we have $\mathbb{E}[\langle A_1 A_2 (f_n - h_2), u_x \rangle^2] \leq c \|A_2 (f_n - h_2)\|_\alpha \to 0$ as $n \to \infty$.

The remaining term is $\langle (f_n - h_2)', u_y \rangle$. This is a little tricky since we do not have a grip on the convergence of $(f_n - h_2)'$. However, (3.3.4) allows us to

deduce that

$$(f_n - h_2)'(t) = c\text{PV} \int_0^\infty A_2(f_n - h_2)(s) \left(\frac{\text{sgn}(s-t)}{\sqrt{|t-s|^3}} - \frac{1}{\sqrt{(t+s)^3}} \right) \mathrm{d}s.$$

From this it follows that

$$\mathbb{E}[\langle (f_n - h_2)', u_x \rangle^2]$$
$$= c \int_0^\infty \int_0^t \left(\text{PV} \int_0^\infty \left(\frac{\text{sgn}(t-s)}{\sqrt{|t-s|^3}} - \frac{1}{\sqrt{(t+s)^3}} \right) A_2(f_n - h_2)(s)\mathrm{d}s \right)$$
$$\cdot \left(\text{PV} \int_0^\infty \left(\frac{\text{sgn}(t'-s')}{\sqrt{|t'-s'|^3}} - \frac{1}{\sqrt{(t'+s')^3}} \right) A_2(f_n - h_2)(s')\mathrm{d}s' \right) (\sqrt{t+t'} - \sqrt{t-t'})\mathrm{d}t'\mathrm{d}t$$
$$= c\text{PV} \int_0^\infty \text{PV} \int_0^\infty \left(\frac{\text{sgn}(1-s)}{\sqrt{|1-s|^3}} - \frac{1}{\sqrt{(1+s)^3}} \right) \left(\frac{\text{sgn}(1-s')}{\sqrt{|1-s'|^3}} - \frac{1}{\sqrt{(1+s')^3}} \right)$$
$$\cdot \int_0^\infty \int_0^t \frac{A_2(f_n - h_2)(st)A_2(f_n - h_2)(s't')}{\sqrt{tt'}}(\sqrt{t+t'} - \sqrt{t-t'})\mathrm{d}t'\mathrm{d}t\mathrm{d}s'\mathrm{d}s$$

The change of order of integration is justified by first defining the s and s' integrals on $[0, t - \varepsilon) \cup (t + \varepsilon, \infty)$ and $[0, t' - \varepsilon) \cup (t' + \varepsilon, \infty)$ respectively (for $\varepsilon > 0$) and applying Fubini's theorem in this case. We may then allow $\varepsilon \to 0$ to obtain the above equality, provided of course that the final integral is finite. If we now use (3.3.11) we see that the t, t' integral is bounded above by

$$c\|A_2(f_n - h_2)\|_\alpha^2 \int_0^\infty \int_0^t (1 + st)^{-\alpha}(1 + s't')^{-\alpha} \frac{t'}{(tt')^{\frac{3}{4}}} \mathrm{d}t'\mathrm{d}t$$
$$= c \int_0^\infty \int_0^{\frac{s'}{s}} (1 + r)^{-\alpha}(1 + vr)^{-\alpha} \frac{r^{\frac{1}{2}}v^{\frac{1}{4}}}{s^{\frac{1}{4}}(s')^{\frac{5}{4}}} \mathrm{d}v\mathrm{d}r$$

following a change of variables ($t' = tu$ followed by $r = st$ and $v = \frac{s'u}{s}$). Now

$$\int_0^\infty \int_0^\infty (1+r)^{-\alpha}(1+vr)^{-\alpha}r^{\frac{1}{2}}v^{\frac{1}{4}}\mathrm{d}v\mathrm{d}r = \int_0^\infty (1+r)^{-\alpha}r^{\frac{1}{2}} \int_0^\infty (1+u)^{-\alpha}\frac{u^{\frac{1}{4}}}{r^{\frac{5}{4}}}\mathrm{d}u\mathrm{d}r < \infty$$

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provided $\alpha > \frac{5}{4}$. In this case we have

$$\mathbb{E}[\langle (f_n - h_2)', u_y \rangle^2] \leq c \|A_2(f_n - h_2)\|_\alpha^2 \mathrm{PV} \int_0^\infty \mathrm{PV} \int_0^\infty \left(\frac{\mathrm{sgn}(1-s)}{\sqrt{|1-s|^3}} - \frac{1}{\sqrt{(1+s)^3}} \right)$$
$$\left(\frac{\mathrm{sgn}(1-s')}{\sqrt{|1-s'|^3}} - \frac{1}{\sqrt{(1+s')^3}} \right) \frac{1}{s^{\frac{1}{4}}(s')^{\frac{5}{4}}} \mathrm{d}s' \mathrm{d}s.$$

Any fears that the s' integrand might have too great a singularity at $s' = 0$ are allayed by the fact that $\frac{1}{\sqrt{(1-s')^3}} - \frac{1}{\sqrt{(1+s')^3}}$ looks like s' for small s'. We may now fix $\alpha \in (\frac{5}{4}, \frac{3}{2})$, and given $h_2 \in C_0^\infty([0, \infty))$ we may choose $f_n \in Y$ such that $\|A_2(f_n - h_2)\|_\alpha \to 0$ as $n \to \infty$. What we have shown is that we may take $L^2(\Omega)$ limits in (3.3.9) to obtain

Proposition 3.4. *For any* $h_1, h_2 \in C_0^\infty([0, \infty))$ *and* $x \geq 0$, $(\langle h_1, u_x \rangle, \langle h_2, v_x \rangle)$ *satisfies system (3.3.8).*

Suppose we now take $X > 0$. We are now able to find a set $\mathcal{A}_1 \in \mathscr{F}$ such that $\mathbb{P}(\mathcal{A}_1) = 1$ on which $(\langle h_1, u_x \rangle, \langle h_2, v_x \rangle)$ satisfies (3.3.8) for all $x \in [0, X] \cap \mathbb{Q}$. We can also find $\mathcal{A}_2 \in \mathscr{F}$ with $\mathbb{P}(\mathcal{A}_2) = 1$ on which all the terms in (3.3.8) are continuous on $[0, X]$. Thus, $\mathcal{A}_1 \cap \mathcal{A}_2$ is a set of measure 1 on which $(\langle h, u_x \rangle, \langle h, v_x \rangle)$ satisfies (3.3.8) for all $x \in [0, X]$.

3.4 The Martingale Problem and other unresolved issues.

3.4.1 Defining a generator.

Let $f \in C_0^\infty(\mathbb{R}^{n+m})$ and suppose that $h_1, \cdots, h_{n+m} \in C_0^\infty(\mathbb{R})$. We define a process $(F(u_x, v_x); x \in [0, X])$ by

$$F(u_x, v_x) = f(\langle h_1, u_x \rangle, \ldots, \langle h_n, u_x \rangle, \langle h_{n+1}, v_x \rangle, \ldots, \langle h_{n+m}, v_x \rangle).$$

Itô's formula now implies that

$$F(u_x, v_x) - F(u_0, v_0) - \sum_{i=1}^{n} \int_0^x \partial_i f(\langle h_1, u_y \rangle, \ldots, \langle h_{n+m}, v_y \rangle) \langle h_i, v_y \rangle \mathrm{d}y$$

$$- \sum_{j=1}^{m} \int_0^x \partial_{n+j} f(\langle h_1, u_y \rangle, \ldots, \langle h_{n+m}, v_y \rangle)(\langle h'_{n+j}, u_y \rangle + \langle A_1 A_2 h_{n+j}, u_y \rangle + \langle A_2 h_{n+j}, v_y \rangle) \mathrm{d}y$$

$$- \sum_{j=1}^{m} \sum_{k=1}^{m} \int_0^x \langle h_{n+j}, h_{n+k} \rangle \partial_{j+n} \partial_{k+n} f(\langle h_1, u_y \rangle, \ldots, \langle h_{n+m}, v_y \rangle) \mathrm{d}y$$

is an $(\tilde{\mathscr{F}}_x; x \in [0, X])$ martingale. Of course, we may view F as an element of $\mathfrak{F}C_b^\infty(C_0^\infty([0, \infty)) \times C_0^\infty([0, \infty))) \subset B(E)$, and it is now tempting to define a generator $\mathcal{G}$ on this subset of $B(E)$ by

$$\mathcal{G}F(u, v) = \sum_{i=1}^{n} \langle h_i, v \rangle \partial_i f(\langle h_1, u \rangle, \ldots, \langle h_{n+m}, v \rangle)$$

$$- \sum_{j=1}^{m} (\langle h'_{n+j}, u \rangle + \langle A_1 A_2 h_{n+j}, u \rangle + \langle A_2 h_{n+j}, v \rangle) \partial_{j+n} f(\langle h_1, u \rangle, \ldots, \langle h_{n+j}, v \rangle)$$

$$- \sum_{j=1}^{m} \sum_{k=1}^{m} \langle h_{n+j}, h_{n+k} \rangle \partial_{j+n} \partial_{k+n} f(\langle h_1, u \rangle, \ldots, \langle h_{n+m}, v \rangle). \quad (3.4.1)$$

There is no hope that $\mathcal{G}F \in B(E)$, but more pressingly, at the moment we do not even know if it defines a function on E. We have demonstrated that $\mathcal{G}F(u_x, v_x)$ is defined for all $x \geq 0$, but for $\mathcal{G}F(u, v)$ to be well defined for all $(u, v) \in E$, we need to know that for all $h \in C_0^\infty([0, \infty))$, that $h, h', A_1 A_2 h \in X_1$ and that $h, A_2 h \in X_2$. If this is the case, then we have some hope of setting up a martingale problem to which $((u_x, v_x); x \in [0, X])$ is a solution. As a result, the next section discusses in greater detail our assumption that we have a suitable space E in which $((u_x, v_x); x \geq 0)$ takes values.

3.4.2 A separable Banach space setting for $(v_x; x \geq 0)$.

Let $\mathcal{C} : \mathcal{D}(\mathcal{C}) \rightarrow L^2([0,\infty))$ be a positive, densely defined closed operator on $L^2([0,\infty))$ defined by

$$\mathcal{C}h(t) = \int_0^\infty C(t,s)h(s)\mathrm{d}s$$

for some positive kernel C. If C is symmetric, then $\mathcal{C}$ is a symmetric operator. We can extend this as a symmetric operator on $L^2([0,\infty); \mathbb{C})$ by defining $\mathcal{C}(h_1 + ih_2) = \mathcal{C}h_1 + i\mathcal{C}h_2$ for $h_1, h_2 \in \mathcal{D}(\mathcal{C})$, and we remark that

$$\langle \mathcal{C}(h_1 + ih_2), h_1 + ih_2 \rangle = \langle \mathcal{C}h_1, h_1 \rangle + \langle \mathcal{C}h_2, h_2 \rangle + i\langle \mathcal{C}h_2, h_1 \rangle - i\langle \mathcal{C}h_1, h_2 \rangle$$

from which it follows that the extended $\mathcal{C}$ is also positive. We now look at $\mathscr{R}(\mathcal{C}-iI)^\perp$ and $\mathscr{R}(\mathcal{C}+iI)^\perp$, where $\mathscr{R}$ denotes the range. For $h_1, h_2, g_1, g_2 \in \mathcal{D}(\mathcal{C})$, one soon sees that $\langle (\mathcal{C} - iI)(h_1 + ih_2), g_1 + ig_2 \rangle = 0$ for all $h_1, h_2 \in \mathcal{D}(\mathcal{C})$ if and only if $\mathcal{C}g_1 = g_2$ and $\mathcal{C}g_2 = -g_1$, and that $\langle (\mathcal{C} - iI)(h_1 + ih_2), g_1 + ig_2 \rangle = 0$ for all $h_1, h_2 \in \mathcal{D}(\mathcal{C})$ if and only if $\mathcal{C}g_1 = -g_2$ and $\mathcal{C}g_2 = g_1$. Clearly then

$$\mathscr{R}(\mathcal{C} - iI)^\perp \cap (\mathcal{D}(\mathcal{C}) + i\mathcal{D}(\mathcal{C})) = \mathscr{R}(\mathcal{C} + iI)^\perp \cap (\mathcal{D}(\mathcal{C}) + i\mathcal{D}(\mathcal{C})).$$

Thus the dimensions of $\mathscr{R}(\mathcal{C} + iI)^\perp$ and $\mathscr{R}(\mathcal{C} - iI)^\perp$ are the same. Referring to section 13.20 of [Rud91], $\mathcal{C}$ thus has a self-adjoint extension, which we also denote by $\mathcal{C}$, and which furthermore has a self adjoint square root, that is $\mathcal{C}^{\frac{1}{2}} : \mathcal{D}(\mathcal{C}^{\frac{1}{2}}) \rightarrow L^2([0,\infty))$ such that $\mathcal{C}^{\frac{1}{2}}\mathcal{C}^{\frac{1}{2}} = \mathcal{C}$ (see theorem 13.31 of [Rud91]).

Suppose now that $L^2([0,\infty))$ is embedded in a Hilbert space H and that $\mathcal{C}^{\frac{1}{2}} : L^2([0,\infty)) \rightarrow H$. If $\mathcal{C}^{\frac{1}{2}}$ is Hilbert-Schmidt, then for a cylindrical Wiener measure μ_0 on $L^2([0,\infty))$, $\mathcal{C}^{\frac{1}{2}}\mu_0$ is a Radon Gaussian measure on H by Sazonov's theorem (see [Sch73]). Define an operator $K : \mathcal{D}(K) \rightarrow L^2([0,\infty))$ on $L^2([0,\infty))$ by $Kh(t) = \int_0^\infty k(t,s)h(s)\mathrm{d}s$ for some kernel k, and take H to be the closure

of $\mathcal{D}(K)$ under the norm $\|Kh\|_2$. We require that K is injective so that $\|\cdot\|_H$ really is a norm. We shall assume that $k(t,s)$ is of the form $k(t,s) = k(t-s)$, where $k : \mathbb{R} \to \mathbb{R}$ is symmetric. The covariance of $\mathcal{C}^{\frac{1}{2}}$ is given by $\int_H (h,\phi)_H (g,\phi)_H \mathcal{C}^{\frac{1}{2}} \mu_0(\mathrm{d}\phi)$. Assume that we may find a basis $\{e_k\} \subset \mathcal{D}(\mathcal{C}^{\frac{1}{2}})$ for $L^2([0,\infty))$. We look to approximate the covariance by

$$
\int_{L^2} \left(h, \mathcal{C}^{\frac{1}{2}} \left(\sum_{k=0}^n \langle \phi, e_k \rangle e_k \right) \right)_H \left(g, \mathcal{C}^{\frac{1}{2}} \left(\sum_{k=0}^n \langle \phi, e_k \rangle e_k \right) \right)_H \mu_0(\mathrm{d}\phi)
$$
$$
= \sum_{i=0}^n \sum_{j=0}^n (h, \mathcal{C}^{\frac{1}{2}} e_i)_H (g, \mathcal{C}^{\frac{1}{2}} e_j)_H \int_{L^2} \langle \phi, e_i \rangle \langle \phi, e_j \rangle \mu_0(\mathrm{d}\phi)
$$
$$
= \sum_{i=0}^n (h, \mathcal{C}^{\frac{1}{2}} e_i)_H (g, \mathcal{C}^{\frac{1}{2}} e_i)_H
$$

Suppose that h, g are such that that $Kh, Kg \in \mathcal{D}(K)$, and furthermore that $K^2 h, K^2 g \in \mathcal{D}(\mathcal{C}^{\frac{1}{2}})$. It then follows by the symmetry of both K and $\mathcal{C}^{\frac{1}{2}}$ that the above expression is

$$
\sum_{i=0}^n \langle \mathcal{C}^{\frac{1}{2}} K^2 h, e_i \rangle \langle \mathcal{C}^{\frac{1}{2}} K^2 g, e_i \rangle.
$$

This converges as $n \to \infty$ to $\langle \mathcal{C}^{\frac{1}{2}} K^2 h, \mathcal{C}^{\frac{1}{2}} K^2 g \rangle = \langle \mathcal{C} K^2 h, K^2 g \rangle$. Provided that $\mathcal{C}^{\frac{1}{2}} : L^2([0,\infty)) \to H$ is continuous, this is the covariance of $\mathcal{C}^{\frac{1}{2}} \mu_0$ for h and g in the domain of K^2. In fact, as mentioned above, our aim is to choose K so that $\mathcal{C}^{\frac{1}{2}}$ is Hilbert-Schmidt. Suppose that we choose k so that $k(t,\cdot) \in \mathcal{D}(\mathcal{C}^{\frac{1}{2}})$ for all $t \in [0,\infty)$. Note that if $k(t,\cdot) \in \mathcal{D}(\mathcal{C})$, then $\|\mathcal{C}^{\frac{1}{2}} k(t,\cdot)\|_2^2 = \langle \mathcal{C} k(t,\cdot), k(t,\cdot) \rangle$, so we may deduce that $k(t,\cdot) \in \mathcal{D}(\mathcal{C}^{\frac{1}{2}})$. We then note that

$$
\sum_{i=0}^\infty \|\mathcal{C}^{\frac{1}{2}} e_i\|_H^2 = \sum_{i=0}^\infty \int_0^\infty \left(\int_0^\infty k(t,s) \mathcal{C}^{\frac{1}{2}} e_i(s) \mathrm{d}s \right)^2 \mathrm{d}t
$$

$$= \sum_{i=0}^{\infty} \int_0^{\infty} \left(\int_0^{\infty} \mathcal{C}^{\frac{1}{2}} k(t,\cdot)(s) e_i(s) \mathrm{d}s \right)^2 \mathrm{d}t$$

$$= \int_0^{\infty} \int_0^{\infty} (\mathcal{C}^{\frac{1}{2}} k(t,\cdot)(s))^2 \mathrm{d}s \mathrm{d}t$$

$$= \int_0^{\infty} \int_0^{\infty} \mathcal{C} k(t,\cdot)(s) k(t,s) \mathrm{d}s \mathrm{d}t$$

$$= \int_0^{\infty} \int_0^{\infty} C(s,r) \left(\int_0^{\infty} k(t,r) k(t,s) \mathrm{d}t \right) \mathrm{d}s \mathrm{d}r$$

which provides a condition which k must satisfy in order that $\mathcal{C}^{\frac{1}{2}}$ is a Hilbert-Schmidt operator.

Let us put this in the context of the process $(v_x; x \geq 0)$. We have already seen, for $h_1, h_2 \in C_{0, \frac{3}{4} + \beta}$, that

$$\mathbb{E}[\langle h_1, v_x \rangle \langle h_2, v_x \rangle] = c \int_0^{\infty} \int_0^{\infty} h_1(t) h_2(s) \left(\frac{1}{\sqrt{|t-s|}} - \frac{1}{\sqrt{t+s}} \right) \mathrm{d}s \mathrm{d}t.$$

Set $C_v(t,s) = \frac{1}{\sqrt{|t-s|}} - \frac{1}{\sqrt{t+s}}$ and $\mathcal{C}_v h(t) = \int_0^{\infty} C_v(t,s) h(s) \mathrm{d}s$. This fits into the above setting. We need to find some $k(t,s)$ such that $k(t,\cdot) \in \mathcal{C}_v^{\frac{1}{2}}$ and such that

$$\left| \int_0^{\infty} \int_0^{\infty} C_v(t,s) \int_0^{\infty} k(t,r) k(r,s) \mathrm{d}r \mathrm{d}s \mathrm{d}t \right| < \infty.$$

Suppose we take $k(t,s) = (1 + |t-s|)^{\alpha}$ for some $\alpha < -\frac{1}{2}$. Remark that for $r > 0$,

$$(1 + |t-r|)^{\alpha} \leq (1+r)^{\alpha} \left(1 + \left| 1 - \left(\frac{1+t}{1+r} \right) \right| \right)^{\alpha}.$$

Thus a little manipulation implies that

$$\int_0^\infty \int_0^\infty C_v(t,s) \int_0^\infty k(t,r)k(r,s)\mathrm{d}r\mathrm{d}s\mathrm{d}t$$

$$\leq \int_0^\infty \int_{\frac{1}{1+r}}^\infty \int_{\frac{1}{1+r}}^\infty \left(\frac{1}{\sqrt{|t-s|}} - \frac{1}{\sqrt{t+s-\frac{2}{1+r}}} \right)(1+r)^{\frac{3}{2}+2\alpha}$$

$$\cdot (1+|1-t|)^\alpha (1+|1-s|)^\alpha \mathrm{d}s\mathrm{d}t\mathrm{d}r$$

$$\leq \int_0^\infty (1+r)^{\frac{3}{2}+2\alpha}\mathrm{d}r \cdot \int_0^\infty \int_0^\infty C_v(t,s)(1+|1-t|)^\alpha (1+|1-s|)^\alpha \mathrm{d}s\mathrm{d}t.$$

We have already seen that the s,t integral is finite if $\alpha < -\frac{3}{4}$, whilst the r integral is finite if $\alpha < -\frac{5}{4}$. Remark that $\mathcal{C}_v k(t,\cdot) \in L^2([0,\infty))$ and $|\langle \mathcal{C}_v k(t,\cdot), k(t,\cdot)\rangle| < \infty$ for such α, so we may assume that $k(t,\cdot) \in \mathcal{D}(\mathcal{C}_v^{\frac{1}{2}})$ for all $t > 0$. We also remark that $\mathcal{D}(\mathcal{C}_v^{\frac{1}{2}})$ includes $C_0^\infty([0,\infty))$ so it is possible to find an orthonormal basis for $L^2([0,\infty))$ in $\mathcal{D}(\mathcal{C}_v^{\frac{1}{2}})$.

We thus take H to be the closure of $\{h \in L^2([0,\infty)) : K^2h \in L^2([0,\infty))\}$ under the norm $\|h\|_H = \|Kh\|_2$. In this setting, $\mathcal{C}_v^{\frac{1}{2}}\mu_0$ is a Radon Gaussian measure on H. Suppose that $g \in H$, and $h \in \mathcal{D}(K^2)$. We can define a linear functional on $\mathscr{R}(K^2)$ by $g(K^2h) = (h,g)_H$. Note that $|g(K^2h)| \leq \|h\|_H \|g\|_H$. Suppose further that $K^2h \in C_{0,\frac{3}{4}+\beta}$. Then

$$\|h\|_H^2 = \int_0^\infty (1+t)^{\frac{3}{4}+\beta}K^2h(t) \cdot (1+t)^{-\frac{3}{4}-\beta}h(t)\mathrm{d}t \leq c\|K^2h\|_{\frac{3}{4}+\beta}\|h\|_2.$$

Of course, $\|K^2h\|_{\frac{3}{4}+\beta}\|h\|_H \leq c(\|K^2h\|_{\frac{3}{4}+\beta} + \|h\|_2)^2$, so we define $\|K^2h\|_{X_2} := \|K^2h\|_{\frac{3}{4}+\beta} + \|h\|_2$ and set X_2 to be the closure under this norm of $\{K^2h \in C_{0,\frac{3}{4}+\beta} : h \in \mathcal{D}(K^2)\}$. Thus any $g \in H$ defines an element of X_2^*, so we may think of $\mathcal{C}_v^{\frac{1}{2}}\mu_0$ as a Radon Gaussian measure on X_2^*. (More precisely, we may embed H in X_2^*, and use lemma 2.2.2 of [Bog98] to deduce that the push-forward of $\mathcal{C}_v^{\frac{1}{2}}\mu_0$ is a Gaussian measure on X_2^*. The fact that the push-forward is Radon

follows from the continuity of the embedding.)

Intuitively now, the law of v_x should be $\mathcal{C}_v^{\frac{1}{2}}$. The problem with this is that all we can say about v_x is that it has a modification, which we also denote by v_x, which takes values in $\mathbb{R}^{X_2}$. To spell this out, the law of v_x is defined on the cylinder sets $\{\phi \in \mathbb{R}^{X_2} : \phi(h_1) \in B_1, \ldots, \phi(h_n) \in B_n\}$ where $n \in \mathbf{N}$, $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ and $h_1, \ldots, h_n \in X_2$, whilst $\mathcal{C}_v^{\frac{1}{2}} \mu_0$ is defined on the cylinder sets $\{\phi \in X_2^* : h_1(\phi) \in B_1, \ldots, h_n(\phi) \in B_n\}$, where again $n \in \mathbf{N}$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$, and $h_1, \ldots, h_n \in (X_2^*)^*$. We denote the σ-algebras generated by these cylinder sets $\mathcal{E}(\mathbb{R}^{X_2})$ and $\mathcal{E}(X_2^*)$ respectively. (This is consistent with Bogachev's notation in [Bog98], where $\mathcal{E}(X)$ is defined to be the smallest σ-algebra making all elements of X^* measurable. In our case, the topology on $\mathbb{R}^{X_2}$ is the coarsest topology which makes the maps $\phi \mapsto \phi(h)$ continuous for all $h \in X_2$.) Define $j : X_2^* \to \mathbb{R}^{X_2}$ by $j(\phi) = \phi$. One clearly sees that the preimages under j of the cylinder sets in $\mathbb{R}^{X_2}$ are cylinder sets in X_2^*, since each $h \in X_2$ defines an element in $(X_2^*)^*$. We may thus use lemma 2.2.2 of [Bog98] once again to deduce that $\mathcal{C}_v^{\frac{1}{2}} \mu_0$ defines a Gaussian measure on $\mathcal{E}(\mathbb{R}^{X_2})$. Furthermore, if $\phi_n \to \phi$ in X_2^* then $\phi_n(h) \to \phi(h)$ for all $h \in X_2$. Thus j is continuous and it follows that $\mathcal{C}_v^{\frac{1}{2}} \mu_0$ is Radon on $\mathcal{E}(\mathbb{R}^{X_2})$. We have constructed $\mathcal{C}_v^{\frac{1}{2}} \mu_0$ to have the same covariance as the law of v_x, so it is easily seen, for example, that $\mathcal{C}_v^{\frac{1}{2}} \mu_0$ coincides with the law of v_x on the cylinder sets of $\mathbb{R}^{X_2}$, and hence on $\mathcal{E}(\mathbb{R}^{X_2})$. Thus for each $x \geq 0$ the law of v_x has support in X_2^*.

3.4.3 A martingale problem associated with $\mathcal{G}$.

We now take $\Psi \subset C_0^\infty([0, \infty)) \times C_0^\infty([0, \infty))$ such that, for $(h_1, h_2) \in \Psi$, $h_1, h_2', A_2 h_1 \in X_1$, and $h_1, h_2, A_1 A_2 h_2 \in X_2$. Define

$$\mathfrak{F}\Psi = \{F \in B(E) : F(u, v) = f(u(h_1), \ldots, u(h_n), v(h_{n+1}), \ldots, v(h_{2n})),$$
$$f \in C_0^\infty(\mathbb{R}^{n+m}), (h_i, h_{n+i}) \in \Psi\},$$

and define $\mathcal{G}F$ as before for $F \in \mathfrak{F}\Psi$. We thus define a martingale problem for (A_Ψ, μ), where $A_\Psi \in B(E) \times m(E)$ is defined by

$$A_\Psi = \{(F, G) : F \in \mathfrak{F}\Psi, G = \mathcal{G}F\}.$$

and μ is the law of (u_0, v_0) on E. We now have a modification of $((u_x, v_x); x \in [0, X])$ which takes values in E. Suppose that this modification, which we also denote by $((u_x, v_x); x \in [0, X])$, is a solution to the martingale problem for (A_Ψ, μ), and specifically that $F(u_x, v_x) - \int_0^x \mathcal{G}F(u_y, v_y) \mathrm{d}y$ is an $(\tilde{\mathscr{F}}_x; x \in [0, X])$ martingale for all $F \in \mathfrak{F}\Psi$. (In particular, we need to prove either directly or using the form of (3.3.8) that $((u_x, v_x); x \in [0, X])$ really is adapted to $(\tilde{\mathscr{F}}_x; x \geq 0)$.) Note that although we do not know that $\mathcal{G}F$ is continuous in E for $F \in \mathfrak{F}\Psi$, it may be viewed as a continuous function on a Euclidean space, so that in reference to the remarks in section 1.1.3 which relate to the adaptedness of $(F(u_x, v_x) - \int_0^x \mathcal{G}F(u_y, v_y) \mathrm{d}y; x \in [0, X])$ to $(\tilde{\mathscr{F}}_x; x \geq 0)$, this is easily seen to follow from a suitable choice of τ_n.

If we can show that for any x the law of (ξ_x, η_x) is the same for any solution $((\xi_x, \eta_x); x \in [0, X])$ of the martingale problem for (A_Ψ, μ) with respect to $(\tilde{\mathscr{F}}_x; x \in [0, X])$ then the Markov property follows from the first part of theorem 1.1. The strong Markov property does not immediately follow however because

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it is not true that for $F \in \mathfrak{F}\Psi$, $\mathcal{G}F \in B(E)$. In order to overcome this, and also to better understand the one dimensional distributions for solutions of the martingale problem, we shall first show that any solution of the martingale has the same form as $((u_x, v_x); x \in [0, X])$ in some sense.

Proposition 3.5. *Let $((\xi_x, \eta_x); x \in [0, X])$ be a solution of the martingale problem for (A_Ψ, μ) with respect to some complete filtration $(\mathscr{U}_x; x \in [0, X])$. Then for every $(h_1, h_2) \in \Psi$ there exists a $(\mathscr{U}_x; x \in [0, X])$ martingale $\hat{W}_x(h_2)$ such that for each $x \geq 0$,*

$$\langle h_1, \xi_x \rangle = \langle h_1, u_0 \rangle + \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y$$

$$\langle h_2, \eta_x \rangle = \langle h_2, v_0 \rangle - \int_0^x (\langle h_2', \xi_y \rangle + \langle A_1 A_2 h_2, \xi_y \rangle + \langle A_2 h_2, \eta_y \rangle) \mathrm{d}y$$

$$- \hat{W}_x(h_2) \qquad\qquad (3.4.2)$$

almost surely. Furthermore, for any two (h_1, h_2) and (h_3, h_4) in Ψ, the quadratic variation of $\hat{W}_x(h_2)$ and $\hat{W}_x(h_4)$ is given by

$$\langle \hat{W}_x(h_2), \hat{W}_x(h_4) \rangle = \langle h_2, h_4 \rangle_2.$$

Proof For $(h_1, h_2) \in \Psi$ we define $(\mathscr{U}_x; x \geq 0)$ local martingales $(M_x^{h_1}; x \geq 0)$ and $(\hat{W}_x(h_2); x \in [0, X])$ by

$$M_x^{h_1} = \langle h_1, \xi_x \rangle - \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y$$

$$\hat{W}_x(h_2) = \langle h_2, \eta_x \rangle - \int_0^x \langle h_2, \zeta_y \rangle \mathrm{d}y$$

Here we have written

$$\langle h_2, \zeta_y \rangle = -\langle h_2', u_y \rangle - \langle A_1 A_2 h_2, u_y \rangle - \langle A_2 h_2, v_y \rangle$$

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for convenience. The first equation in (3.4.2) is $M_x^{h_1} = M_0^{h_1}$ almost surely, which we obtain by showing that $\mathbb{E}[(M_x^{h_1} - M_0^{h_1})^2] = 0$. To this end, we note that

$$\langle h_1, \xi_x \rangle^2 - 2 \int_0^x \langle h_1, \xi_y \rangle \langle h_1, \eta_y \rangle \mathrm{d}y$$

is a $(\mathscr{U}_x; x \in [0, X])$ local martingale. As a result

$$\left(\langle h_1, \xi_x \rangle - \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y \right)^2$$

$$= 2 \int_0^x \langle h_1, \xi_y \rangle \langle h_1, \eta_y \rangle \mathrm{d}y - 2 \langle h_1, \xi_x \rangle \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y + \left(\int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y \right)^2 + loc.\ mart$$

$$= 2 \int_0^x \left(M_y^{h_1} + \int_0^y \langle h_1, \eta_z \rangle \mathrm{d}z \right) \langle h_1, \eta_y \rangle \mathrm{d}y - 2 \left(\int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y \right)^2 - 2 M_x^{h_1} \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y$$

$$+ \left(\int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y \right)^2 + loc.\ mart$$

Remark that since the function $\langle h_1, \eta_y \rangle \langle h_1, \eta_z \rangle$ is symmetric in y and z,

$$2 \int_0^x \int_0^y \langle h_1, \eta_y \rangle \langle h, \eta_z \rangle \mathrm{d}z \mathrm{d}y = \int_0^x \int_0^x \langle h_1, \eta_y \rangle \langle h_1, \eta_z \rangle \mathrm{d}z \mathrm{d}y = \left(\int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y \right)^2$$

and furthermore, by stochastic integration by parts,

$$2 \int_0^x M_y^{h_1} \langle h, \eta_y \rangle \mathrm{d} = 2 M_x^{h_1} \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y - 2 \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}M_y^{h_1}$$

where the last term is a $(\mathscr{U}_x; x \in [0, X])$ local martingale. Thus $\left(\langle h_1, \xi_x \rangle - \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y \right)^2$ is a $(\mathscr{U}_x; x \in [0, X])$ local martingale. Consequently

$$\mathbb{E} \left[\left(\langle h_1, \xi_x \rangle - \langle h_1, u_0 \rangle - \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y \right)^2 \right] = 0$$

and we obtain

$$\langle h_1, \xi_x \rangle = \langle h_1, u_0 \rangle + \int_0^x \langle h_1, \eta_y \rangle \mathrm{d}y \quad \text{a.s.}$$

We now need to calculate the quadratic variation $\langle \hat{W}(h_2), \hat{W}(h_4) \rangle_x$ for $(h_1, h_2), (h_3, h_4) \in \Psi$, so we look at $\hat{W}_x(h_2)\hat{W}_x(h_4)$. This is given by

$$\left(\langle h_2, \eta_x \rangle - \langle h_2, v_0 \rangle - \int_0^x \langle h_2, \zeta_y \rangle \mathrm{d}y \right) \left(\langle h_4, \eta_x \rangle - \langle h_4, v_0 \rangle - \int_0^x \langle h_4, \zeta_y \rangle \mathrm{d}y \right)$$

$$= - \hat{W}_x(h_2)\langle h_4, v_0 \rangle - \hat{W}_x(h_4)\langle h_2, v_0 \rangle - \langle h_2, v_0 \rangle \langle h_4, v_0 \rangle + \langle h_2, \eta_x \rangle \langle h_4, \eta_x \rangle$$

$$- \langle h_2, \eta_x \rangle \int_0^x \langle h_4, \zeta_y \rangle \mathrm{d}y - \langle h_4, \eta_x \rangle \int_0^x \langle h_2, \zeta_y \rangle \mathrm{d}y + \int_0^x \langle h_2, \zeta_y \rangle \mathrm{d}y \int_0^x \langle h_4, \zeta_y \rangle \mathrm{d}y$$

Furthermore

$$\langle h_2, \eta_x \rangle \langle h_4, \eta_x \rangle - \langle h_2, v_0 \rangle \langle h_4, v_0 \rangle - \int_0^x \langle h_2, \zeta_y \rangle \langle h_4, \eta_y \rangle \mathrm{d}y$$

$$- \int_0^x \langle h_4, \zeta_y \rangle \langle h_2, \eta_y \rangle \mathrm{d}y - \int_0^x \int_0^\infty h_2(s) h_4(s) \mathrm{d}s \mathrm{d}y$$

is an $(\tilde{\mathscr{F}}_x; x \geq 0)$ martingale, and so $\hat{W}_x(h_2)\hat{W}_x(h_4)$ is

$$\int_0^x \int_0^\infty h_2(s) h_4(s) \mathrm{d}s \mathrm{d}y + \int_0^x \langle h_4, \zeta_y \rangle \langle h_2, \eta_y \rangle \mathrm{d}y - \langle h_2, \eta_x \rangle \int_0^x \langle h_4, \zeta_y \rangle \mathrm{d}y$$

$$+ \int_0^x \langle h_2, \zeta_y \rangle \langle h_4, \eta_y \rangle \mathrm{d}y - \langle h_4, \eta_x \rangle \int_0^x \langle h_2, \zeta_y \rangle \mathrm{d}y$$

$$+ \int_0^x \langle h_2, \zeta_y \rangle \mathrm{d}y \int_0^x \langle h_4, \zeta_y \rangle \mathrm{d}y + loc.\ mart.$$

As before, a stochastic integration by parts gives us

$$\hat{W}_x(h_{2i}) \int_0^x \langle h_{2(3-i)}, \zeta_y \rangle \mathrm{d}y = \int_0^x \int_0^y \langle h_{2(3-i)}, \zeta_z \rangle \mathrm{d}z \mathrm{d}\hat{W}_y(h_{2i})$$

$$+ \int_0^x \hat{W}_y(h_{2i}) \langle h_{2(3-i)}, \zeta_y \rangle \mathrm{d}y$$

for $i = 1, 2$, from which it follows that

$$\int_0^x \langle h_{2i}, \eta_y \rangle \langle h_{2(3-i)}, \zeta_y \rangle \mathrm{d}y - \langle h_{2i}, \eta_x \rangle \int_0^x \langle h_{2(3-i)}, \zeta_y \rangle \mathrm{d}y$$
$$= -\left(\int_0^x \langle h_{2i}, \zeta_y \rangle \mathrm{d}y \right) \left(\int_0^x \langle h_{2(3-i)}, \zeta_y \rangle \mathrm{d}y \right) + \int_0^x \int_0^y \langle h_{2i}, \zeta_y \rangle \langle h_{2(3-i)}, \zeta_z \rangle \mathrm{d}z \mathrm{d}y$$
$$+ \textit{loc. mart.}$$

Since

$$\int_0^x \int_0^y \langle h_2, \zeta_y \rangle \langle h_4, \zeta_z \rangle \mathrm{d}z \mathrm{d}y + \int_0^x \int_0^y \langle h_4, \zeta_y \rangle \langle h_2, \zeta_z \rangle \mathrm{d}z \mathrm{d}y$$
$$= \left(\int_0^x \langle h_2, \zeta_y \rangle \mathrm{d}y \right) \left(\int_0^x \langle h_4, \zeta_y \rangle \mathrm{d}y \right)$$

it soon follows that

$$\hat{W}_x(h_2) \hat{W}_x(h_4) = \int_0^x \int_0^\infty h_2(s) h_4(s) \mathrm{d}s \mathrm{d}y + \textit{loc. mart.}$$

Thus

$$\langle W(h_2), W(h_4) \rangle_x = \int_0^x \int_0^\infty h_2(s) h_4(s) \mathrm{d}s \mathrm{d}y.$$

$\blacksquare$

3.4.4 The Markov property for solutions to the martingale problem for (A_Ψ, μ).

For the moment, we suppose that we have spaces X_1, X_2 and Ψ such that all solutions to the martingale problem for (A_Ψ, μ) have the same finite dimensional distributions, so that $((u_x, v_x); x \in [0, \infty))$ is Markov with repsect to $(\tilde{\mathscr{F}}_x; x \geq 0)$. We thus have

$$\mathbb{E}[F(u_{x+z}, v_{x+z}) | \tilde{\mathscr{F}}_x] = \mathbb{E}[F(u_{x+z}, v_{x+z}) | u_x, v_x] \quad \text{a.s.} \qquad (3.4.3)$$

We would now like to translate this into something meaningful about $(u(x,t); x \in \mathbb{R}, t \in [0,\infty))$, the original solution of (0.1.1). If we let $f \in C_0^\infty(\mathbb{R}^n)$ and $h_i \in C_0^\infty([0,\infty))$ for $i = 1, \ldots, n$, then one possible form of $F(u_x, v_x)$ is

$$f\left(\int_0^\infty h_1(t)u(x,t)\mathrm{d}t, \ldots, \int_0^\infty h_n(t)u(x,t)\mathrm{d}t\right).$$

(3.4.3) holds for such functions, and by approximating $n\mathbb{1}_{(t_i - \frac{1}{n}, t_i]}$ by functions of compact support and using the dominated convergence theorem, we can replace each h_i by $n\mathbb{1}_{(t_i - \frac{1}{n}, t_i]}$. A second use of the dominated convergence theorem as $n \to \infty$ gives (3.4.3) for $F(u_x, v_x) = f(u(x, t_1), \ldots, u(x, t_n))$ for any $t_1, \ldots, t_n \in (0, \infty)$. It soon follows by approximating $\bigotimes_{i=1}^n \mathbb{1}_{A_i}$ (where $A_i \in \mathcal{B}(\mathbb{R})$) in $L^1(\mathbb{R})$ by functions with compact support that

$$\mathbb{P}(u(x+z, t_1) \in A_1, \ldots, u(x+z, t_n) \in A_n | \tilde{\mathscr{F}}_x)$$
$$= \mathbb{P}(u(x+z, t_1) \in A_1, \ldots, u(x+z, t_n) \in A_n | u_x, v_x),$$

or rather by a monotone class argument

$$\mathbb{P}(u(x+z, \cdot) \in A | \tilde{\mathscr{F}}_x) = \mathbb{P}(u(x+z, \cdot) \in A | u_x, v_x)$$

for any $A \in \mathcal{B}(C([0, \infty)))$. Strictly speaking, for $(u(x, \cdot); x \geq 0)$ to be Markov with respect to $(\tilde{\mathscr{F}}_x; x \geq 0)$ we require $\mathbb{P}(u(x+z, \cdot) \in A | \tilde{\mathscr{F}}_x) = \mathbb{P}(u(x+z, \cdot) \in A | u(x, \cdot))$. However, we can still think of this as a Markov property in terms of splitting fields. We can set $\mathscr{H}([0, x] \times [0, \infty)) = \sigma(u(y, \cdot); 0 \leq y \leq x) \vee \sigma(u_x, v_x)$ and $\mathscr{H}([x, \infty) \times [0, \infty)) = \sigma(u(y, \cdot); x \leq y < \infty) \vee \sigma(u_x, v_x)$, and since $\sigma(u_x, v_x) \subset \mathscr{H}([0, x] \times [0, \infty)) \subset \tilde{\mathscr{F}}_x$ it follows that $\sigma(u_x, v_x)$ is a splitting field for $\mathscr{H}([0, x] \times [0, \infty))$ and $\mathscr{H}([x, \infty) \times [0, \infty))$.

We cannot compare directly with the result of [NP94], which demonstrates that

$\mathscr{B}^u(\{x\} \times [0, \infty))$ is a splitting field for $\mathscr{B}^u((-\infty, x])$ and $\mathscr{B}^u([x, \infty))$, and is in fact the minimal splitting field. (Here we have recalled the notation of definition 0.2.) In our case, $\sigma(u_x, v_x) \subset \mathscr{B}^u(\{x\} \times [0, \infty))$, but also $\mathscr{H}([0, x] \times [0, \infty)) \subset \mathscr{B}^u([0, x] \times [0, \infty))$ and $\mathscr{H}([x, \infty) \times [0, \infty)) \subset \mathscr{B}^u([x, \infty) \times [0, \infty))$. In order to compare with [NP94] we need to be able to say more about $\sigma(u_x, v_x)$. In any case, one may hope that further analysis of $(u_x; \geq 0)$ and $(v_x; x \geq 0)$ may give a clearer description of the splitting field $\sigma(u_x, v_x)$. For example, if we can show that $\sigma(u_x, v_x) = \mathcal{B}(\{x\} \times [0, \infty))$ then we obtain the result of [NP94] for these specific sets, and in particular we see that the additional information required at the boundary is exactly that given by v_x. On the other hand, it could be the case that $\sigma(u_x, v_x)$ does not contain any more information than $\sigma(u(x, \cdot))$, and in this case we really have the sharp Markov property.

Let us discuss briefly the strong Markov property, and in particular whether we can apply the second part of theorem 1.1 for progressively measurable solutions $((\xi_x, \eta_x); x \in [0, \infty))$ to the martingale problem for (A_Ψ, μ) with respect to a filtration $(\mathscr{U}_x; x \in [0, \infty))$. For $F \in \mathfrak{F}\Psi$ defined by $F(u, v) = f(\langle h_1, u \rangle, \langle h_2, v \rangle)$,

$$F(\xi_x, \eta_x) - \int_0^x \mathscr{G}F(\xi_z, \eta_z) \mathrm{d}z = F(u_0, v_0) + \int_0^x \partial_2 f(\langle h, \xi_y \rangle, \langle l, \eta_y \rangle) \mathrm{d}\hat{W}_y(h_2)$$

almost surely, where $(\hat{W}_x(h_2); x \in [0, \infty))$ is a $(\mathscr{U}_x; x \in [0, \infty))$ martingale. With reference to (1.1.3) in the proof of theorem 1.1, we denote the left hand side by $Z(x)$. Note that, regardless of whether $(\langle h_1, \xi_x \rangle; x \in [0, \infty))$ and $(\langle h_2, \eta_x \rangle; x \in [0, \infty))$ are continuous, $(Z(x); x \in [0, \infty))$ has a continuous modification using the Kolmogorov-Čentsov continuity criterion. In order to make the proof of theorem 1.1 work, we need to show that for any almost surely finite $(\mathscr{U}_x; x \in [0, \infty))$ stopping time θ and $x, y \geq 0$,

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$$\mathbb{E}[Z(x+y+\theta)|\mathcal{U}_{x+\theta}] = Z(x+\theta).$$

The optional sampling theorem tells us that for any $X \geq 0$

$$\mathbb{E}[Z((x+y+\theta) \wedge X)|\mathcal{U}_{x+\theta}] = Z((x+\theta) \wedge X).$$

Take $A \in \mathcal{U}_{x+\theta}$. Suppose we can show that

$$\mathbb{E}[(Z((x+y+\theta) \wedge X) - Z((x+\theta) \wedge X))\mathbb{1}_A] \to \mathbb{E}[(Z(x+y+\theta) - Z(x+\theta))\mathbb{1}_A]$$

as $X \to \infty$. Since the left hand side is 0 for all X, this would give the desired result. To show this, it is enough to show that

$$Z((x+y+\theta) \wedge X) - Z((x+\theta) \wedge X) \to Z(x+y+\theta) - Z(x+\theta)$$

in $L^2(\Omega)$ as $X \to \infty$. For $a < b$, $Z(b) - Z(a) = \int_a^b \partial_2 f(\langle h_1, \xi_z \rangle, \langle h_2, \eta_z \rangle) \mathrm{d}\hat{W}_z(h_2)$ and so

$$(Z((x+y+\theta) \wedge X) - Z((x+\theta) \wedge X))) - (Z(x+y+\theta) - Z(x+\theta))$$
$$= \int_0^\infty \partial_2 f(\langle h_1, \xi_z \rangle, \langle h_2, \eta_z \rangle) \left(\mathbb{1}_{((x+\theta) \wedge X, (x+y+\theta) \wedge X]}(z) - \mathbb{1}_{(x+\theta, x+y+\theta]}(z) \right) \mathrm{d}\hat{W}_z(h_2).$$

This converges to 0 in $L^2(\Omega)$ if and only if

$$\int_0^\infty \mathbb{E}\left[\left(\partial_2 f(\langle h_1, \xi_z \rangle, \langle h_2, \eta_z \rangle) \left(\mathbb{1}_{((x+\theta) \wedge X, (x+y+\theta) \wedge X]}(z) - \mathbb{1}_{(x+\theta, x+y+\theta]}(z) \right) \right)^2 \right] \mathrm{d}z \mathrm{d}s$$

does. This is bounded by

$$\|\partial_2 f\|_\infty \int_0^\infty \mathbb{E}\left[\left(\mathbb{1}_{((x+\theta) \wedge X, (x+y+\theta) \wedge X]}(z) - \mathbb{1}_{(x+\theta, x+y+\theta]}(z) \right)^2 \right] \mathrm{d}z$$

which is equal to

$$c \int_0^\infty \int_\Omega \mathbb{1}_{\{\omega:x+\theta(\omega)<X\leq x+y+\theta(\omega)\}} \mathbb{1}_{(X,x+y+\theta(\omega)]}(z)$$

$$+ \mathbb{1}_{\{\omega:X<x+\theta(\omega)\}} \mathbb{1}_{(x+\theta(\omega),x+y+\theta(\omega)]}(z)\mathbb{P}(\mathrm{d}\omega)\mathrm{d}z$$

$$=c \int_\Omega (x+y+\theta(\omega)-X)\mathbb{1}_{\{\omega:x+\theta(\omega)<X\leq x+y+\theta(\omega)\}} + y\mathbb{1}_{\{\omega:X<x+\theta(\omega)\}}\mathbb{P}(\mathrm{d}\omega)$$

$$\leq cy\mathbb{P}(x+y+\theta>X).$$

This converges to 0 as $X \to \infty$ since θ is almost surely finite.

Assuming that all solutions of the martingale problem for (A_Ψ, μ) have the same one dimensional distributions, and that $((u_x, v_x); x \in [0, \infty))$ is progressively measurable with respect to $(\tilde{\mathscr{F}}_x; x \geq 0)$, it follows that

$$\mathbb{E}[F(u_{x+\theta}, v_{x+\theta})|\tilde{F}_\theta] = \mathbb{E}[F(u_{x+\theta}, v_{x+\theta})|u_x, v_x]$$

for all bounded $F : E \to \mathbb{R}$ and all almost surely finite stopping times θ and $x \geq 0$. Let us comment on the progressive measurability of $((u_x, v_x); x \geq 0)$, assuming the process is adapted to $(\tilde{\mathscr{F}}_x; x \geq 0)$. For $X > 0$ define u^X and v^X to be the processes u and v respectively restricted to $[0, X]$. Define a set $B = \{(u, v) \in E : \langle h_1, u \rangle \in B_1, \ldots, \langle h_n, u \rangle \in B_n, \langle h_{n+1}, v \rangle \in B_{n+1}, \ldots, \langle h_{2n}, v \rangle \in B_{2n}\}$ where $h_i \in X_1$ for $i = 1, \ldots, n$, $h_i \in X_2$ for $i = n+1, \ldots, 2n$ and $B_i \in \mathcal{B}(\mathbb{R})$ for $i = 1, \ldots, 2n$. We wish to show that $(u^X, v^X)^{-1}(B) \in \mathcal{B}([0, X]) \times \tilde{F}_X$. The remarks of section 3.1.3 imply that each $(\langle h_i, u_x \rangle; x \in [0, X])$ and $(\langle h_{n+i}, v_x \rangle; x \in [0, X])$ has a continuous modification. Since $(\tilde{\mathscr{F}}_x; x \in [0, X])$ is a complete filtration, it follows that $(u^X, v^X)^{-1} \in \mathcal{B}([0, X]) \times \tilde{\mathscr{F}}_X$ provided that

$$\{(x, \omega) : \langle h_i, u_x \rangle(\omega) \in B_i, \langle h_{n+i}, v_x \rangle(\omega) \in B_{n+i}, \, i = 1 \ldots, n\}$$

$$\cap \left([0, X] \times \{\omega : \langle h_i, u^X \rangle(\omega), \langle h_{n+i}, v^X \rangle(\omega) \text{ continuous for } i = 1, \ldots, n\}\right)$$

is, which follows from the remarks in [EK86], section 2.1.

3.4.5 Uniqueness of solutions to the martingale problem.

There are clearly a number of issues here yet to be resolved. Perhaps the biggest open question is whether it is possible to show that the solutions of the martingale problem for (A_Ψ, μ) have unique one-dimensional distributions. Unfortunately we are as yet unable to answer this, although we make the following remarks about it. If $((\xi_x, \eta_x); x \in [0, X])$ is a solution of the martingale problem for (A_Ψ, μ), then there is a specific noise $(\hat{W}_x; x \in [0, X])$ such that $((\xi_x, \eta_x); x \in [0, X])$ satisfies the system (3.4.2). We would like to think of $((\xi_x, \eta_x); x \in [0, X])$ as being a strong solution for the noise $\hat{W}$ and the initial condition $(\xi_0, \eta_0) = (u_0, v_0)$. One way we might make this more precise is by thinking of $((\langle h_1, u_x \rangle, \langle h_2, v_x \rangle); x \in [0, X])$ as a strong solution to (3.4.2) for $(h_1, h_2) \in \Psi$. The problem with this is that our system is not closed- it depends also on the term $\langle h'_2, \xi_x \rangle + \langle A_1 A_2 h_2, \xi_x \rangle + \langle A_2 h_2, \eta_x \rangle$. In any case, our hope is that for the given initial condition (u_0, v_0) and a noise $\hat{W}$ there is one, and only one, solution to (3.4.2). We denote this by $(\xi_x, \eta_x) = \Phi(x, (u_0, v_0), \hat{W})$. We would thus like to show that any solution of the martingale problem for (A_Ψ, μ) has the form $(\Phi(x, (u_0, v_0), \hat{W})$ for some noise $\hat{W}$ such that $\langle \hat{W}(h_2), \hat{W}(h_4) \rangle = \langle h_2, h_4 \rangle$ for all $(h_1, h_2), (h_3, h_4) \in \Psi$, and hence all solutions of the martingale (A_Ψ, μ) have the same one-dimensional distributions (in fact, the same law, so this is clearly more than sufficient).

The success of this approach depends largely on whether we can show that the solutions to (3.4.2) are unique given a noise $\hat{W}$. Although we are attempting to show more than we need here, this approach has the benefit that uniqueness reduces to the uniqueness of a deterministic system of equations. Indeed, if we denote the difference between two solutions of (3.4.2) by (Ξ_x, Θ_x), then for all $(h_1, h_2) \in \Psi$ we have

$$\langle h_1, \Xi_x \rangle = \int_0^x \langle h_1, \Theta_y \rangle \mathrm{d}y$$

$$\langle h_2, \Theta_x \rangle = -\int_0^x (\langle h_2', \Xi_y \rangle + \langle A_1 A_2 h_2, \Xi_y \rangle + \langle A_2 h_2, \Theta_y \rangle) \mathrm{d}y \qquad (3.4.4)$$

for all $x \in [0, X]$, and $\Xi_0 = \Theta_0 = 0$. For any $h \in C_0^\infty([0, \infty))$ such that $(h, h) \in \Psi$ we can write this as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \langle h, \Xi_x \rangle + \langle h', \Xi_x \rangle + \langle A_1 A_2 h, \Xi_x \rangle + \frac{\mathrm{d}}{\mathrm{d}x} \langle A_2 h, \Xi_x \rangle = 0.$$

Intuitively then we would like to be able to show that the equation

$$\frac{\partial^2}{\partial x^2} \phi(x, t) - \frac{\partial}{\partial t} \phi(x, t) + A_2^* A_1^* \phi(x, t) + A_2^* \frac{\partial \phi}{\partial x}(x, t) = 0$$

has only the zero solution when supplied with boundary conditions $\phi(0, t) = \frac{\partial}{\partial x} \phi(x, t)|_{x=0} = 0$ for all $t > 0$. In fact, there is very little hope for this without also knowing that $\phi(x, 0) = 0$ for $x \in [0, X]$. However, we have assumed this condition for our original process $u(x, t)$ satisfying equation (0.1.1), so we may include this condition as some further property of the space E.

Since this uniqueness problem is unresolved, we do not attempt to discuss it here. However, we point out that if the uniqueness of the above partial differential equation is to be enough to provide uniqueness for solutions to the weak form of our equation, we need to know that the weak form is defined for 'sufficiently'

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many test functions h, whatever is meant by that. This involves choosing Ψ as large as possible such that $((u_x, v_x); x \geq 0)$ solves the martingale problem for (A_Ψ, μ). Naturally, the smaller the set Ψ, the less hope we have of finding uniqueness to the above weak equation. Let us offer some heuristic why it seems plausible that we can take $\Psi = C_0^\infty([0, \infty)) \times C_0^\infty([0, \infty))$. To begin with, it is clear that for any $h \in C_0^\infty([0, \infty))$, h and h' are in X_1. In fact, we can easily see that $A_2 h \in X_1$ also. Indeed, when we investigated the tail of $A_2 h$ we saw that $|A_2 h(t)| \leq c \left(\left(\frac{1}{\sqrt{t-T}} - \frac{1}{\sqrt{t}} \right) - \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t+T}} \right) \right)$ for $t \gg T$, where the support of h is in $[0, T]$. Our previous estimate was rather crude, and it is in fact possible to show this bound is

$$\frac{cT^3}{\sqrt{t-T}\sqrt{t+T}(\sqrt{t-T} + \sqrt{t+T})(\sqrt{t} + \sqrt{t-T})^2(\sqrt{t} + \sqrt{t+T})^2}$$

which looks like $t^{-\frac{7}{2}}$.

For X_2, we still look to use the ideas of section 3.4.2, but a more fruitful route could be to take

$$H = \{h : (I - \Delta)^{-\frac{1}{2}} m (I - \Delta)^{-\frac{1}{2}} h \in L^2([0, \infty))\}$$

where Δ is the Laplacian with Dirichlet boundary conditions at zero, and m is a positive weight function. If one assumes that $\mathcal{C}_v^{\frac{1}{2}}$ and $(I - \Delta)^{-\frac{1}{2}}$ have representations as symmetric kernel operators, where the kernels have certain favourable properties, and furthermore that they commute (and we observe that Δ and $\mathcal{C}_v$ do commute), then it is possible to show that

$$\sum_i \|\mathcal{C}_v^{\frac{1}{2}} e_i\|_H^2 = c \int_0^\infty m(t) \int_0^\infty \left(\frac{1}{\sqrt{|t - t'|}} - \frac{1}{\sqrt{t + t'}} \right) k(t, t') \mathrm{d}t' \mathrm{d}t$$

where $k(t, t')$ is the kernel of $(I - \Delta)^{-1}$, which looks like $e^{-|t-t'|}$. Formally we have

$$\langle h, v_x \rangle = ((I - \Delta)^{\frac{1}{2}} m^{-1} (I - \Delta)^{\frac{1}{2}} h, v_x)_H.$$

Thus we might think of v_x as a bounded linear functional on a subspace of $C_{0, \frac{3}{4}+\beta}$ containing h satisfying $m^{-\frac{1}{2}} (I - \Delta)^{\frac{1}{2}} h \in L^2([0, \infty))$. The benefit of this approach then is that we reduce to checking tail properties of $(I - \Delta)^{\frac{1}{2}} h$. It is tempting to compare this to the tail properties of h'. In particular, for $h \in C_0^\infty$, h' can be integrated against any continuous weight m, and we already have an idea of how quickly $(A_2 h)'$ decays. However, at the time of writing more work is needed to turn these ideas into a precise argument.

References

[Bau02]   Fabrice Baudoin. Conditiones stochastic differential equations: theory, examples and applications to finance. *Stochastic Processes and their Applications*, 100:109–145, 2002.

[BC07]   Fabrice Baudoin and Laure Coutin. Volterra bridges and applications. *Markov Processes and Related Fields*, 13(3):587–596, 2007.

[BL06]   Zdzisław Brzeźniak and Remi Léandre. Fellerian pants. *C. R. Math. Acad. Sci. Paris*, 343(5):333–338, 2006.

[Bog98]   Vladimir I. Bogachev. *Gaussian Measures*, volume 62 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1998.

[dP71]   João de Prolla. Bishop's generalized stone-weierstrass theorem for weighted spaces. *Mathematische Annalen*, 191(4):283–289, 1971.

[DPY06]   Paul Deheuvels, Giovanni Peccati, and Marc Yor. On quadratic functionals of the brownian sheet and related processes. *Stochastic Processes and their Applications*, 116(3):493–538, 2006.

[DW92]   Robert C. Dalang and John N. Walsh. The sharp markov property of the brownian sheet and related processes. *Acta Mathematica*, 168(3-4):153–218, 1992.

[EK86]   Stewart N. Ethier and Thomas G. Kurtz. *Markov Processes: Charactirezation and Convergence*. Wiley Series in Probability and Statistics. John Wiley and Sons, Inc, 1986.

[FPY93]   Patrick J. Fitzsimmons, Jim Pitman, and Marc Yor. Markovian bridges: construction, palm interpretation, and splicing. *Progress in Probability*, 33:101–134, 1993.

[GM06]   Ben Goldys and Bohdan Maslowski. Lower estimates of transition densities and bounds on exponential ergodicity for stochastic pde's. *The Annals of Probability*, 34(4):1451–1496, 2006.

[GM08]   Ben Goldys and Bohdan Maslowski. The ornstein-uhlenbeck bridge and applications to markov semigroups. *Stochastic Processes and their Applications*, (10):1738–1767, 2008.

[Gue91]   P. B. Guest. *Laplace Transforms and an Introduction to Distributions*. Ellis Horwood, 1991.

[Hör90]   Lars Hörmander. *The analysis of linear partial differential operators*, volume 1. Springer-Verlag, second edition, 1990.

[Iwa87]   Koichiro Iwata. An infinite-dimensional stochastic differential equation with state space c($\mathbb{R}$). *Probability and Related Fields*, 74(1):141–159, 1987.

[JY85]   Thierry Jeulin and Marc Yor. *Grossissements de filtrations*. Number 1118 in Lecture Notes in Mathematics. Springer, Berlin, 1985.

[KS98]   Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, second edition, 1998.

[Kün79]   Hans Künsch. Gaussian markov random fields. *Journal of the Faculty of Science University of Tokyo Section IA Mathematics*, 26(1):53–73, 1979.

[MR92]   Zhi Ming Ma and Michael Röckner. *Introduction to the theory of (non-symmetric) Dirichlet forms*. Universitext. Springer-Verlag, Berlin, 1992.

[MY06]   Roger Mansuy and Marc Yor. *Random times and enlargements of filtrations in a Brownian setting*. Number 1873 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2006.

[NP94]   David Nualart and Etienne Pardoux. Markov field properties of solutions of white noise driven quasi-linear parabolic pdes. *Stochastics and Stochastics Reports*, 48(1-2):17–44, 1994.

[Nua06]   David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[Par93]   Etienne Pardoux. Stochastic partial differential equations, a review. *Bulletin des Sciences Mathmatiques*, 117(1):29–47, 1993.

[Pit71]   Loren D. Pitt. A markov property for gaussian processes with a multi-dimensional parameter. *Archive for Rational Mechanics and Analysis*, 43:367–391, 1971.

[Pro04]   Philip E. Protter. *Stochastic integration and differential equations*. Number 21 in Applications of Mathematics (New York). Springer-Verlag, Berlin, second edition, 2004.

[Roz82]   Yu. A. Rozanov. *Markov Random Fields*. Applications of Mathematics. Springer-Verlag, New York/Berlin, 1982.

[Rud91]   Walter Rudin. *Functional Analysis*. MacGraw Hill, Inc., New York, second edition, 1991.

[RW87]   L.C.G. Rogers and David Williams. *Diffusions, Markov processes, and martingales*. Wiley Series in Probability and Mathematical Statistics. John Wiley and Sons, Inc, 1987.

[Sch73] Laurent Schwartz. *Radon measures on arbitrary topological spaces and cylindrical measures.* Number 6 in Tata Institute of Fundamental Research Studies in Mathematics. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973.

[Sim05] Isabel Simão. A continuous kernel for the transition semigroup associated with a diffusion process in a hilbert space. *Semigroup Forum*, 71(1):49–72, 2005.

[Wal86] John B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilits de Saint-Flour, XIV—1984*, number 1180 in Lecture Notes in Mathematics, pages 265–439. Springer, Berlin, 1986.

[Yor97] Marc Yor. *Some aspects of Brownian motion, Part II: Some recent martingale problems.* Lectures in Mathematics ETH Zrich. Birkhäuser Verlag, Basel, 1997.