

**Original citation:**

Berndt, Bruce C. and Hart, William B.. (2007) An identity for the Dedekind eta-function involving two independent complex variables. Bulletin of the London Mathematical Society, Volume 39 (Number 2). pp. 345-347. ISSN 0024-6093

**Permanent WRAP url:**

<http://wrap.warwick.ac.uk/31797/>

**Copyright and reuse:**

The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions. Copyright © and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable the material made available in WRAP has been checked for eligibility before being made available.

Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

**A note on versions:**

The version presented here may differ from the published version or, version of record, if you wish to cite this item you are advised to consult the publisher's version. Please see the 'permanent WRAP url' above for details on accessing the published version and note that access may require a subscription.

For more information, please contact the WRAP Team at: [wrap@warwick.ac.uk](mailto:wrap@warwick.ac.uk)

warwick**publications**wrap

highlight your research

<http://wrap.warwick.ac.uk>

# AN IDENTITY FOR THE DEDEKIND ETA-FUNCTION INVOLVING TWO INDEPENDENT COMPLEX VARIABLES

BRUCE C. BERNDT<sup>1</sup> and WILLIAM B. HART

## 1. INTRODUCTION

Recall that the Dedekind eta-function  $\eta(\tau)$  is defined for  $q = e^{2\pi i\tau}$  and  $\tau \in \mathcal{H} = \{\tau : \text{Im } \tau > 0\}$  by

$$\eta(\tau) = q^{1/24}(q; q)_\infty,$$

where

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

The purpose of this paper is to prove the following striking identity for the eta-function, of which we know no other examples of a similar type.

**Theorem 1.1.** *For  $w, z \in \mathcal{H}$ ,*

$$\begin{aligned} 27\eta^3(3w)\eta^3(3z) &= \eta^3\left(\frac{w}{3}\right)\eta^3\left(\frac{z}{3}\right) \\ &+ i\eta^3\left(\frac{w+1}{3}\right)\eta^3\left(\frac{z+1}{3}\right) - i\eta^3\left(\frac{w+2}{3}\right)\eta^3\left(\frac{z+2}{3}\right). \end{aligned} \quad (1.1)$$

We describe now the genesis of (1.1). In preparing his doctoral thesis [2], the second author searched for modular equations involving

$$u_1(\tau) := \frac{\eta(\tau/m)}{\eta(\tau)} \quad \text{and} \quad v_1(\tau) := u_1(n\tau), \quad (1.2)$$

(and various modular transforms thereof). His goal was to generalize the modular equations of ‘irrational kind’ for the Weber functions

$$f(\tau) := e^{-\pi i/24} \frac{\eta\left(\frac{x+1}{2}\right)}{\eta(\tau)}, \quad f_1(\tau) := \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}, \quad f_2(\tau) := \sqrt{2} \frac{\eta(2\tau)}{\eta(\tau)},$$

discussed in §75 of Weber’s book [3], i.e., the case  $m = 2$  in (1.2). For example, if  $n = 3$ , letting

$$u(\tau) := f(\tau), \quad u_1(\tau) := f_1(\tau), \quad u_2(\tau) := f_2(\tau)$$

and

$$v(\tau) = f(3\tau), \quad v_1(\tau) := f_1(3\tau), \quad v_2(\tau) := f_2(3\tau),$$

---

<sup>1</sup>Research partially supported by grant MDA904-00-1-0015 from the National Security Agency.

one can prove the identity

$$u^2v^2 = u_1^2v_1^2 + u_2^2v_2^2, \quad \tau \in \mathcal{H}.$$

Generally, Weber's modular equations depend on  $n$  and increase in complexity as  $n$  increases.

In attempting to generalize these modular equations, the second author began with an appropriately normalized set of transforms (under modular substitutions) of  $u_3(\tau) := \eta(\tau/3)/\eta(\tau)$ . However, he eventually realized that the modular equations obtained for these 'generalized Weber functions' did not appear to vary as  $n$  increased. Moreover, the single identity that he found was completely general in that the second parameter  $n\tau$  was not related to  $\tau$  in any way, i.e., the equation held for two completely independent complex variables. Simplification then gave the identity for the eta-function given in Theorem 1.1 above. The identity was then verified in many cases to tens of thousands of decimal places.

## 2. PROOF OF THEOREM 1.1

Let  $q = e^{2\pi iw}$ ,  $Q = e^{2\pi iz}$ , and  $\rho = e^{2\pi i/3}$ . Then (1.1) is equivalent to the identity

$$\begin{aligned} 27q^{3/8}Q^{3/8}(q^3; q^3)_\infty^3(Q^3; Q^3)_\infty^3 &= q^{1/24}Q^{1/24}(q^{1/3}; q^{1/3})_\infty^3(Q^{1/3}; Q^{1/3})_\infty^3 \\ &\quad + i\rho^{1/4}q^{1/24}Q^{1/24}(\rho q^{1/3}; \rho q^{1/3})_\infty^3(\rho Q^{1/3}; \rho Q^{1/3})_\infty^3 \\ &\quad - i\rho^{-1/4}q^{1/24}Q^{1/24}(\rho^{-1}q^{1/3}; \rho^{-1}q^{1/3})_\infty^3(\rho^{-1}Q^{1/3}; \rho^{-1}Q^{1/3})_\infty^3, \end{aligned}$$

or

$$\begin{aligned} 27q^{1/3}Q^{1/3}(q^3; q^3)_\infty^3(Q^3; Q^3)_\infty^3 &= (q^{1/3}; q^{1/3})_\infty^3(Q^{1/3}; Q^{1/3})_\infty^3 \\ &\quad + i\rho^{1/4}(\rho q^{1/3}; \rho q^{1/3})_\infty^3(\rho Q^{1/3}; \rho Q^{1/3})_\infty^3 \\ &\quad - i\rho^{-1/4}(\rho^{-1}q^{1/3}; \rho^{-1}q^{1/3})_\infty^3(\rho^{-1}Q^{1/3}; \rho^{-1}Q^{1/3})_\infty^3. \end{aligned} \tag{2.1}$$

To prove (2.1), we use Jacobi's identity [1, p. 285]

$$(q; q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}. \tag{2.2}$$

Observe that

$$\rho^{n(n+1)/2} = \begin{cases} 1, & \text{if } n \equiv 0, 2 \pmod{3}, \\ \rho, & \text{if } n \equiv 1 \pmod{3}, \end{cases}$$

Hence, by (2.2),

$$\begin{aligned} (\rho q^{1/3}; \rho q^{1/3})_\infty^3 &= \sum_{n=0}^{\infty} (-1)^n (2n+1) \rho^{n(n+1)/2} q^{n(n+1)/6} \\ &= \sum_{\substack{n=0 \\ n \equiv 0, 2 \pmod{3}}}^{\infty} (-1)^n (2n+1) q^{n(n+1)/6} + \rho \sum_{\substack{n=0 \\ n \equiv 1 \pmod{3}}}^{\infty} (-1)^n (2n+1) q^{n(n+1)/6} \end{aligned}$$

$$\begin{aligned}
&= (q^{1/3}; q^{1/3})_{\infty}^3 + (\rho - 1) \sum_{n=0}^{\infty} (-1)^{3n+1} (6n+3) q^{(3n+1)(3n+2)/6} \\
&= (q^{1/3}; q^{1/3})_{\infty}^3 - 3(\rho - 1) q^{1/3} \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{3n(n+1)/2} \\
&= (q^{1/3}; q^{1/3})_{\infty}^3 - 3(\rho - 1) q^{1/3} (q^3; q^3)_{\infty}^3, \tag{2.3}
\end{aligned}$$

where we used (2.2) twice again. For brevity, set

$$A := (q^{1/3}; q^{1/3})_{\infty}^3, \quad B := (Q^{1/3}; Q^{1/3})_{\infty}^3, \quad C := q^{1/3} (q^3; q^3)_{\infty}^3, \quad D := Q^{1/3} (Q^3; Q^3)_{\infty}^3.$$

Using the notation above, (2.3), its analogue with  $\rho$  replaced by  $\rho^{-1}$ , and their analogues, with  $q$  replaced by  $Q$ , in (2.1), we find that it suffices to prove that

$$\begin{aligned}
27CD &= AB + i\rho^{1/4} (A - 3(\rho - 1)C) (B - 3(\rho - 1)D) \\
&\quad - i\rho^{-1/4} (A - 3(\rho^{-1} - 1)C) (B - 3(\rho^{-1} - 1)D). \tag{2.4}
\end{aligned}$$

Observe that  $\rho^{1/4} = (\sqrt{3} + i)/2$ . Thus, the coefficient of  $AB$  on the right-hand side of (2.4) is equal to

$$1 + i\rho^{1/4} - i\rho^{-1/4} = 1 + i \left( \frac{\sqrt{3}}{2} + \frac{i}{2} \right) - i \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right) = 0. \tag{2.5}$$

Next, the coefficients of  $AD$  and  $BC$  on the right-hand side of (2.4) are each equal to

$$\begin{aligned}
&-3i\rho^{1/4}(\rho - 1) + 3i\rho^{-1/4}(\rho^{-1} - 1) \\
&= -3i \left( \frac{\sqrt{3}}{2} + \frac{i}{2} \right) \left( \frac{-3 + i\sqrt{3}}{2} \right) + 3i \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right) \left( \frac{-3 - i\sqrt{3}}{2} \right) \\
&= -\frac{3i}{4}(-4\sqrt{3}) + \frac{3i}{4}(-4\sqrt{3}) = 0. \tag{2.6}
\end{aligned}$$

The coefficient of  $CD$  on the right-hand side of (2.4) is equal to

$$\begin{aligned}
&9i\rho^{1/4}(\rho - 1)^2 - 9i\rho^{-1/4}(\rho^{-1} - 1)^2 \\
&= 9i \left( \frac{\sqrt{3}}{2} + \frac{i}{2} \right) \left( \frac{-3 + i\sqrt{3}}{2} \right)^2 - 9i \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right) \left( \frac{-3 - i\sqrt{3}}{2} \right)^2 \\
&= 9i \left( \frac{6\sqrt{3} - 6i}{4} - \frac{6\sqrt{3} + 6i}{4} \right) \\
&= 9i(-3i) = 27. \tag{2.7}
\end{aligned}$$

Hence, using the calculations (2.5)–(2.7) in (2.4), we see that (2.4) indeed has been shown, and so this completes the proof.

## REFERENCES

- [1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Clarendon Press, Oxford, 1960.
- [2] W. B. Hart, *Evaluation of the Dedekind Eta Function*, Ph.D. Thesis, Macquarie University, Sydney, 2004.
- [3] H. Weber, *Lehrbuch der Algebra*, Dritter Band, Friedrich Vieweg und Sohn, Braunschweig, 1908; reprinted by Chelsea, New York, 1961.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, IL 61801, USA

*E-mail address:* `berndt@math.uiuc.edu`

*E-mail address:* `wbhart@math.uiuc.edu`