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AN IDENTITY FOR THE DEDEKIND ETA-FUNCTION INVOLVING TWO INDEPENDENT COMPLEX VARIABLES

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1. Introduction

Recall that the Dedekind eta-function $\eta(\tau)$ is defined for $q = e^{2\pi i \tau}$ and $\tau \in \mathcal{H} = \{\tau : \text{Im } \tau > 0\}$ by

$$\eta(\tau) = q^{1/24}(q; q)_{\infty},$$

where

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).$$

The purpose of this paper is to prove the following striking identity for the eta-function, of which we know no other examples of a similar type.

**Theorem 1.1.** For $w, z \in \mathcal{H}$,

$$27\eta^3(3w)\eta^3(3z) = \eta^3 \left( \frac{w}{3} \right) \eta^3 \left( \frac{z}{3} \right)$$

$$+ i\eta^3 \left( \frac{w+1}{3} \right) \eta^3 \left( \frac{z+1}{3} \right) - i\eta^3 \left( \frac{w+2}{3} \right) \eta^3 \left( \frac{z+2}{3} \right).$$

(1.1)

We describe now the genesis of (1.1). In preparing his doctoral thesis [2], the second author searched for modular equations involving

$$u_1(\tau) := \eta(\tau/m) \eta(\tau) \quad \text{and} \quad v_1(\tau) := u_1(n\tau),$$

(1.2)

(and various modular transforms thereof). His goal was to generalize the modular equations of ‘irrational kind’ for the Weber functions

$$f(\tau) := e^{-\pi i/24} \eta \left( \frac{\tau + 1}{2} \right), \quad f_1(\tau) := \frac{\eta(\tau)}{\eta(\tau)}, \quad f_2(\tau) := \sqrt{2} \frac{\eta(2\tau)}{\eta(\tau)},$$

discussed in §75 of Weber’s book [3], i.e., the case $m = 2$ in (1.2). For example, if $n = 3$, letting

$$u(\tau) := f(\tau), \quad u_1(\tau) := f_1(\tau), \quad u_2(\tau) := f_2(\tau)$$

and

$$v(\tau) := f(3\tau), \quad v_1(\tau) := f_1(3\tau), \quad v_2(\tau) := f_2(3\tau),$$

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one can prove the identity
\[ u^2 v^2 = u_1^2 v_1^2 + u_2^2 v_2^2, \quad \tau \in \mathcal{H}. \]

Generally, Weber’s modular equations depend on \( n \) and increase in complexity as \( n \) increases.

In attempting to generalize these modular equations, the second author began with an appropriately normalized set of transforms (under modular substitutions) of \( u_3(\tau) := \eta(\tau/3)/\eta(\tau) \). However, he eventually realized that the modular equations obtained for these ‘generalized Weber functions’ did not appear to vary as \( n \) increased. Moreover, the single identity that he found was completely general in that the second parameter \( n\tau \) was not related to \( \tau \) in any way, i.e., the equation held for two completely independent complex variables. Simplification then gave the identity for the \( \eta \)-function given in Theorem 1.1 above. The identity was then verified in many cases to tens of thousands of decimal places.

2. Proof of Theorem 1.1

Let \( q = e^{2\pi i\omega} \), \( Q = e^{2\pi i z} \), and \( \rho = e^{2\pi i/3} \). Then (1.1) is equivalent to the identity
\[
27q^{3/8} Q^{3/8} (q^3; q^3)_\infty (Q^3; Q^3)_\infty = q^{1/24} Q^{1/24} (q^{1/3}; q^{1/3})_\infty (Q^{1/3}; Q^{1/3})_\infty \\
+ i\rho^{1/4} q^{1/24} Q^{1/24} (\rho q^{1/3}; \rho q^{1/3})_\infty (\rho Q^{1/3}; \rho Q^{1/3})_\infty \\
- i\rho^{-1/4} q^{1/24} Q^{1/24} (\rho^{-1} q^{1/3}; \rho^{-1} q^{1/3})_\infty (\rho^{-1} Q^{1/3}; \rho^{-1} Q^{1/3})_\infty,
\]
or
\[
27q^{1/3} Q^{1/3} (q^3; q^3)_\infty (Q^3; Q^3)_\infty = (q^{1/3}; q^{1/3})_\infty (Q^{1/3}; Q^{1/3})_\infty \\
+ i\rho^{1/4} (\rho q^{1/3}; \rho q^{1/3})_\infty (\rho Q^{1/3}; \rho Q^{1/3})_\infty \\
- i\rho^{-1/4} (\rho^{-1} q^{1/3}; \rho^{-1} q^{1/3})_\infty (\rho^{-1} Q^{1/3}; \rho^{-1} Q^{1/3})_\infty.
\]

(2.1)

To prove (2.1), we use Jacobi’s identity [1, p. 285]
\[
(q; q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/2}.
\]

(2.2)

Observe that
\[
\rho^{n(n+1)/2} = \begin{cases} 
1, & \text{if } n \equiv 0, 2 \pmod{3}, \\
\rho, & \text{if } n \equiv 1 \pmod{3}, 
\end{cases}
\]

Hence, by (2.2),
\[
(\rho q^{1/3}; \rho q^{1/3})_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) \rho^{n(n+1)/2} q^{n(n+1)/6} \\
= \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/6} + \rho \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/6}
\]

\[
\sum_{n=1}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/6}.
\]
\[
= (q^{1/3}; q^{1/3})_\infty^3 + (\rho - 1) \sum_{n=0}^{\infty} (-1)^{3n+1} (6n + 3) q^{(3n+1)(3n+2)/6}
\]
\[
= (q^{1/3}; q^{1/3})_\infty^3 - 3(\rho - 1)q^{1/3} \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{3n(n+1)/2}
\]
\[
= (q^{1/3}; q^{1/3})_\infty^3 - 3(\rho - 1) q^{1/3} (q^3; q^3)_\infty^3,
\]
where we used (2.2) twice again. For brevity, set
\[
A := (q^{1/3}; q^{1/3})_\infty^3, \quad B := (Q^{1/3}; Q^{1/3})_\infty^3, \quad C := q^{1/3}(q^3; q^3)_\infty^3, \quad D := Q^{1/3}(Q^3; Q^3)_\infty^3.
\]
Using the notation above, (2.3), its analogue with \(\rho\) replaced by \(\rho^{-1}\), and their analogues, with \(q\) replaced by \(Q\), in (2.1), we find that it suffices to prove that
\[
27CD = AB + i\rho^{1/4} (A - 3(\rho - 1)C) (B - 3(\rho - 1)D)
\]
\[
- i\rho^{-1/4} (A - 3(\rho^{-1} - 1)C) (B - 3(\rho^{-1} - 1)D).
\]
Observe that \(\rho^{1/4} = (\sqrt{3} + i)/2\). Thus, the coefficient of \(AB\) on the right-hand side of (2.4) is equal to
\[
1 + i\rho^{1/4} - i\rho^{-1/4} = 1 + i \left( \frac{\sqrt{3}}{2} + \frac{i}{2} \right) - i \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right) = 0.
\]
Next, the coefficients of \(AD\) and \(BC\) on the right-hand side of (2.4) are each equal to
\[
-3i\rho^{1/4}(\rho - 1) + 3i\rho^{-1/4}(\rho^{-1} - 1)
\]
\[
= -3i \left( \frac{\sqrt{3}}{2} + \frac{i}{2} \right) \left( -3 + i\sqrt{3} \right) + 3i \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right) \left( -3 - i\sqrt{3} \right)
\]
\[
= -\frac{3i}{4} (-4\sqrt{3}) + \frac{3i}{4} (-4\sqrt{3}) = 0.
\]
The coefficient of \(CD\) on the right-hand side of (2.4) is equal to
\[
9i\rho^{1/4}(\rho - 1)^2 - 9i\rho^{-1/4}(\rho^{-1} - 1)^2
\]
\[
= 9i \left( \frac{\sqrt{3}}{2} + \frac{i}{2} \right) \left( -3 + i\sqrt{3} \right)^2 - 9i \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right) \left( -3 - i\sqrt{3} \right)^2
\]
\[
= 9i \left( 6\sqrt{3} - 6i \right) - 9i \left( 6\sqrt{3} + 6i \right)
\]
\[
= 9i(-3i) = 27.
\]
Hence, using the calculations (2.5)–(2.7) in (2.4), we see that (2.4) indeed has been shown, and so this completes the proof.
REFERENCES


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