Constant flux relation for driven dissipative systems

Colm Connaughton,[1,*] R. Rajesh,[2,†] and Oleg Zaboronski[3,‡]

[1] *Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*
[2] *Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai-600013, India*
[3] *Mathematics Institute, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, UK*
(Dated: February 6, 2008)

Conservation laws constrain the stationary state statistics of driven dissipative systems because the average flux of a conserved quantity between driving and dissipation scales should be constant. This requirement leads to a universal scaling law for flux-measuring correlation functions, which generalizes the 4/5-th law of Navier-Stokes turbulence. We demonstrate the utility of this simple idea by deriving new exact scaling relations for models of aggregating particle systems in the fluctuation-dominated regime and for energy and wave action cascades in models of strong wave turbulence.

PACS numbers: 05.20.-y, 47.27.-i, 47.35.Bb, 61.43.Hv

A major challenge of theoretical physics is to develop a general formalism for understanding properties of systems far from equilibrium. In the absence of such a formalism, exact results play an important role. They act as precursors for more general theories and as checkpoints for phenomenological theories and numerical methods. In this paper we derive an exact relation satisfied by a particular correlation function for an important class of *turbulent* non-equilibrium systems.

By turbulent systems, we understand not just fluid turbulence, but a more general class of driven dissipative systems characterised by the presence of an inertial range. This is a wide range of scales separating the driving and dissipation. In the inertial range, dynamics is expected to be universal, i. e. independent of details of driving and dissipation. In the steady state, the inertial range dynamics is characterised by constant fluxes of conserved quantities from driving to dissipation scale. A well known example is Navier-Stokes (NS) turbulence [1]. Energy injected into the system at a large length scale L is dissipated at a length scale $l \sim \nu^{3/4}$, where ν is the kinematic viscosity. In the inertial range $l \ll r \ll L$, there is a constant energy flux directed to small scales.

If the inertial range dynamics is scale invariant, the constant flux of a conserved quantity always fixes the scaling of a certain correlation function. An example is Kolmogorov's 4/5 law of 3 dimensional NS turbulence [2] which states that, in the inertial range,

$$\langle \left[v_l(\vec{r},t) - v_l(\vec{0},t)\right]^3 \rangle = -\frac{4}{5}\epsilon r, \quad l \ll r \ll L, \tag{1}$$

where $v_l(\vec{r},t)$ is the longitudinal component of the velocity at $\vec{r}$ at time t and ϵ is the average energy flux. Pedagogical derivations of this result are in [1, 3]. Other examples are the Monin-Yaglom relation for passive advection (see [4] and references therein) and Burgers turbulence where integrals of motion can be used to prove extreme anomalous scaling of all structure functions [5].

A general law relating scaling properties of correlation functions to inertial range conservation laws is still missing. As a result, no counterpart of the 4/5-law is known for such important turbulent systems such as 3− and 4-wave turbulence and cluster-cluster aggregation. In this paper we derive a general constant flux relation (CFR) which should be obeyed by all flux-determined correlations of turbulent systems. We start with a heuristic argument, followed by an analytic application of this argument to the following specific models:

- Mass model (MM): Consider particles of positive mass on a d-dimensional lattice. They evolve in time by diffusion, at rate D, aggregation of particles at the same site, at rate $\lambda(m_1, m_2)$, and injection of particles of mass m_0 at rate J/m_0. For more detail, see [6].

- Charge model (CM): The model is similar to MM except for the following modifications. The masses (now charges) can be positive or negative. Particles of charge $\pm m_0$ are input at the same rate J/m_0^2.

- Wave turbulence (WT): An ensemble of interacting dispersive waves maintained far from equilibrium by interaction with external sources and sinks. For details and a review see [7, 8].

Together, these models relate to a broad range of disciplines, far beyond the common applications of fluid turbulence. Applications of the MM or CM include submonolayer epitaxial thin film growth [9], river networks [10, 11], force fluctuations in granular bead packs [12] and non-equilibrium phase transitions [13, 14]. The constant kernel MM maps [15] onto the directed abelian sandpile, one of the first models generating power laws from simple dynamical rules [16]. WT is realized in such diverse situations as capillary and gravity waves in liquids, electromagnetic waves in non-linear optics and Alfven waves in plasmas (see [7, 17] for more applications).

We present a heuristic argument for a general turbulent system. It is necessarily abstract, hiding much of the physics of specific systems. We believe that the general approach only acquires clarity after application to concrete examples. The idea is very simple. First identify the conserved quantity expected to cascade, we shall call it I, and the space in which it flows. Second, use the equation of motion to write down a Boltzmann-like continuity equation for the average density of I in this space. This equation identifies a flux-carrying correlation function, Π, and a nonlinear coupling, T, controlling the redistribution of I. Third, use dimensional analysis to find the scaling of Π corresponding to constant flux. Suppose the flux of I occurs in a d-dimensional space, spanned by $\vec{p}$. The degrees of freedom are at least partially indexed by $\vec{p}$. We denote them by $u_{\vec{p}}$. For example, in three dimensional NS turbulence, energy transfer occurs between all modes in Fourier space, so $d = 3$. Also $\vec{p} = \vec{k}$, the wave number, and $u_{\vec{p}}$ is $\vec{v}(\vec{k})$, the velocity field. For surface waves, motion occurs in three dimensions but energy is transferred between wave trains indexed by a two dimensional wave-vector, so $d = 2$. For particle aggregation problems, mass is transferred in one dimensional mass space so $d = 1$ even though the particles may be diffusing on a lattice of arbitrary spatial dimension.

The system must have an inertial range in $\vec{p}$-space, within which the dynamics is entirely dominated by the I-conserving interactions. Denote the flux by $\vec{J}(\vec{p})$. Assume for simplicity, that the inertial range dynamics is isotropic, so we can average over angles in $\vec{p}$-space. Let ρ_I be the density, and $J(p)$ the flux, of I in p-space, where $p = |\vec{p}|$. In the inertial range the flux of I through a sphere of radius p does not depend on p. Therefore, $\vec{J}(\vec{p}) = C\ p^{-d}\vec{p}$. On the other hand, from the equation of motion, one can always derive a continuity equation for I in p-space taking the general form:

$$\partial_p J(p) = -\dot{\rho}_I = \int \prod_{m=1}^{n-1} d^d p_m T(p, p_1, p_2, \ldots, p_{n-1})\Pi. \tag{2}$$

T is a non-linear coupling and Π is a correlation function of the $u_{\vec{p}}$, whose precise forms must be calculated for each specific example. In Eq. (2) it is assumed that the non-linearity is of degree $n-1$ in the $u_{\vec{p}}$. We take T to be homogeneous of degree β: $T(\Lambda p) = \Lambda^\beta T(p)$. Equating scaling exponents, we find

$$\Pi(p) \sim p^{-nd-\beta}. \tag{3}$$

How does this relate to Kolmogorov's 4/5 law? Eq. (1) is usually derived in real space although the flux is in $\vec{k}$-space, so the correspondence is not clear. A $\vec{k}$-space derivation of the 4/5 law, along exactly the lines outlined here is indeed possible. It can be found in [18]. Even earlier, a similar approach was taken by Kraichnan in his seminal work on 2D turbulence [19]. To summarise, for NS, $n = 3$ (nonlinearity is quadratic), $\beta = 1$ (nonlinearity contains one derivative) and $\Pi \sim \langle v_l(\vec{k}_1)v_l(\vec{k}_2)v_l(\vec{k}_3)\rangle$. By Eq. (3), $\Pi \sim k^{-1-3d}$. Thus $\langle(v_l(r) - v_l(0))^3\rangle \sim r^{1+3d-3d} = r$, consistent with Eq. (1).

The remainder of this letter is dedicated to the illustration and analytical verification Eq. (3) for the specific models listed above. We illustrate the practical meaning of the steps outlined above by explicitly identifying Π and T and, by exact computations, show that Eq. (3) is more than dimensional analysis. The examples extend the applicability of the 4/5-law beyond the usual cases. MM is a pedagogical example of flux conservation outside of hydrodynamics. CM illustrates that the idea is also relevant to statistical conservation laws. WT demonstrates the approach for non-quadratic invariants.

First, consider the MM. The conserved quantity is mass. The mass flux is in mass space. Let the reaction rate λ_{m_1,m_2} be a homogeneous function of degree β. Let $N(m, \vec{x}, t)$ be the local mass distribution. The continuity equation for the average mass density, $m\langle N(m, \vec{x}, t)\rangle$ takes the following form in the limit $m_0 \to 0,\ t \to \infty$:

$$m \int_0^\infty dm_1 dm_2 \left[I_{0;1,2} - I_{2;1,0} - I_{1;0,2}\right] = 0, \tag{4}$$

where $I_{0;1,2} = \lambda_{m_1,m_2} C(m_1,m_2)\delta(m - m_1 - m_2)$ and $C(m_1, m_2, t) = \langle N(m_1, \vec{x}, t)N(m_2, \vec{x}, t) - \frac{1}{\Delta x^d}\delta(m_1 - m_2)N(m_1, \vec{x}, t)\rangle$ is the average number of pairs of particles of masses m_1 and m_2 per lattice site. Δx is the lattice spacing. Eq. (4) is not supposed to be obvious. A detailed derivation for the case $\beta = 0$ can be found in [6]. Let us look for homogeneous solutions to Eq. (4) of degree h, *i.e.*, $C(\Lambda m_1, \Lambda m_2) = \Lambda^h C(m_1, m_2)$. The value of h is determined by the balance between the 'plus' and 'minus' terms in the left hand side of Eq. (4). This balance can be made explicit by applying the following Zakharov Transformations [7, 20] to the 'minus' terms:

$$\text{Second integral:} \quad (m_1, m_2) \to \left(\frac{mm_1}{m_2}, \frac{m^2}{m_2}\right), \tag{5}$$

$$\text{Third integral:} \quad (m_1, m_2) \to \left(\frac{m^2}{m_1}, \frac{mm_2}{m_1}\right), \tag{6}$$

after which we obtain

$$0 = \int_0^\infty \int_0^\infty dm_1 dm_2 I_{m;m_1,m_2}\left[m^y - m_1^y - m_2^y\right] \tag{7}$$

where $y = -h - \beta - 3$. Eq. (7) is satisfied only if $y = 1$: the function $f(m_1, m_2) = (m_1 + m_2)^y - m_1^y - m_2^y$ is sign definite for any $y \neq 1$ whereas I_{m,m_1,m_2} is non-negative. Therefore the integral in the right hand side of Eq. (7) is zero only for $y = 1$. Consequently, $h = -3 - \beta$, is the unique scaling solution, under an extra assumption of convergence of collision integrals, which will be discussed elsewhere. For $\beta = 0$, the scaling $h = -3 - \beta$ is already established. It is trivially true for $d > 2$ where mean field holds, in $d = 2$ due to cancellation of logarithmic corrections [6] and in $d < 2$ by an exact solution [21].

The scaling of the flux correlation function $\Pi = mC\delta(m) \sim m^{-3-\beta}$ agrees with general CFR, Eq. (3), for $d = 1$, $n = 3$. Note that the result, remarkably, does not depend on the transport properties of the particles such as mass dependence of the diffusion coefficient.

Next, consider CM. There are two conserved quantities: average charge and average squared charge. Charge is conserved in elementary collisions, but squared charge is conserved only on average. To see this, consider the coagulations between the following pairs of charges: (m_1, m_2), $(-m_1, m_2)$, $(m_1, -m_2)$ and $(-m_1, -m_2)$. They occur with equal probability and the total squared charge in the initial and final states are equal. Therefore, squared charge is conserved *on average.* Total flux of charge is zero by the symmetry of input. Therefore, for the CM, CFR is associated with a flux of squared charge. The corresponding continuity equation is

$$m^2 \int_0^\infty dm_1 dm_2 \left[I'_{0;1,2} - I'_{2;1,0} - I'_{1;0,2}\right] = 0, \qquad (8)$$

where $I'_{0;1,2} = K_{m_1,m_2} C(m_1,m_2)\delta[m \pm m_1 \pm m_2]$. Here $C(m_1, m_2)$ has the same meaning as in the MM. Due to symmetry $C(-m_1, m_2) = C(m_1, m_2)$. We assume K_{m_1,m_2} to be homogeneous of degree β. Also, $\delta(x \pm y \pm z) = \sum_{m,n=0}^{1} \delta(x - (-1)^m y - (-1)^n z)$. The following ZT is applied to Eq. (8):

$$\text{First integral:} \quad (m_1, m_2) \to \left(\frac{m^2}{m_1}, \frac{m m_2}{m_1}\right), \qquad (9)$$

$$\text{Second integral:} \quad (m_1, m_2) \to \left(\frac{m^2}{m_2}, \frac{m m_1}{m_2}\right), \qquad (10)$$

and integration over m_1 is performed to obtain

$$\begin{aligned} 0 &= \int_0^\infty dm_2 K(m, m_2) C(m, m_2) \left[m^2 \left(\frac{m}{|m - m_2|}\right)^{-y} \right. \\ &+ \left. m^2 \left(\frac{m}{|m + m_2|}\right)^{-y} - 2m^2 - 2m^2 \left(\frac{m}{m_2}\right)^{-y} \right], \qquad (11) \end{aligned}$$

where $y = -h - \beta - 2$ and h is the homogeneity exponent of $C(m_1, m_2)$ to be determined. If $y = 2$, then Eq. (11) is satisfied as the integrand vanishes identically due to square charge conservation. Generalizing the argument made in case of MM, it is possible to check that $y = 2$ is the unique scaling solution, given locality. Thus, $h = -4 - \beta$. The scaling of the flux measuring correlation function $\Pi = m^2 C(m) \delta(m) \sim m^{-3-\beta}$ is consistent with Eq. (3) for $d = 1$ and $n = 3$.

Finally we consider wave turbulence. It is a Hamiltonian system with canonical variables $\{a_{\vec{k}}, \bar{a}_{\vec{k}}\}_{\vec{k} \in \mathbf{R}^d}$:

$$H = \int d\vec{k} [\ \omega(k) \bar{a}_{\vec{k}} a_{\vec{k}} + u(\vec{k})\ ]. \qquad (12)$$

ω is a homogeneous function of $\vec{k}$ and $u(\vec{k})$ is the nonlinear part of the energy density. When $u = 0$, the Hamiltonian describes free waves with dispersion law $\omega = \omega(k)$, $\vec{k}$ being the wave vector. The equations of motion are

$$\dot{a}_{\vec{k}} = i \left[\omega(k) a_{\vec{k}} + \frac{\delta U}{\delta \bar{a}_{\vec{k}}}\right], \qquad (13)$$

where $U = \int d\vec{k} u(\vec{k})$. The equation of motion conserves H. There may be additional conservation laws depending on the form of $u(\vec{k})$. The system is maintained far from equilibrium by adding appropriate forcing and dissipation terms to Eq. (13). Most commonly, $u(\vec{k})$ has degree 3 or 4 in $a_{\vec{k}}$.

We first study the energy cascade which conserves energy in the inertial range. As a result, the average flux of energy is constant, which leads to the continuity equation

$$\left\langle \dot{u}(\vec{k}) - \dot{\bar{a}}_{\vec{k}} \frac{\delta U}{\delta \bar{a}_{\vec{k}}} - \dot{a}_{\vec{k}} \frac{\delta U}{\delta a_{\vec{k}}} \right\rangle = 0, \qquad (14)$$

where U is the interaction part of the Hamiltonian. We now consider two cases: 3-wave turbulence (n=3) where

$$u = \int \prod_{i=1}^{2} d\vec{k}_i \delta(\vec{k} - \vec{k}_1 - \vec{k}_2) T_{k;k_1,k_2} [\bar{a}_{\vec{k}} a_{\vec{k}_1} a_{\vec{k}_2} + c.c], \qquad (15)$$

and 4-wave turbulence (n=4) where

$$\begin{aligned} u &= \int \prod_{i=1}^{3} d\vec{k}_i \delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3) T_{k,k_1;k_2,k_3} \\ &\quad \times [\bar{a}_{\vec{k}} \bar{a}_{\vec{k}_1} a_{\vec{k}_2} a_{\vec{k}_3} + c.c.]. \qquad (16) \end{aligned}$$

In both cases, we assume T to be homogeneous of degree β. For 3-wave turbulence Eq. (14), reduces to

$$\int \prod_{i=1}^{2} (dk_i k_i^{d-1}) \left[T_{k;k_1,k_2} \Pi_{0;1,2} - T_{k_1;k,k_2} \Pi_{1;0,2}\right] = 0, \qquad (17)$$

where $\Pi_{0;1,2} = \int \prod_{i=0}^{2} d\Omega_i \langle Re(a_{\vec{k}} \dot{\bar{a}}_{\vec{k}_1} \bar{a}_{\vec{k}_2}) \rangle$ is the flux correlation function. Note an essential difference between the flux correlation function for total energy and that for quadratic energy. Total energy flux depends on correlations between fields and time derivatives of fields whereas quadratic energy flux would depend on correlations between fields only. Transforming $(k_1, k_2) \to (k^2/k_1, k k_2/k_1)$ in the second integral, we obtain

$$0 = \int \prod_{i=1}^{2} (dk_i k_i^{d-1}) T_{k;k_1,k_2} \Pi_{0;1,2} \left[k^y - k_1^y\right], \qquad (18)$$

where $y = -3d - h - \beta$ and h is the degree of homogeneity of Π. Therefore, $y = 0$, or $h = -3d - \beta$ solves Eq.(18). The result $\Pi \sim k^{-3d-\beta}$ agrees with Eq. (3), as the order of nonlinearity is $n = 3$, dimensionality of flux space is d and homogeneity index of interaction kernel T is β. It is also consistent with Kolmogorov-Zakharov (KZ) theory describing turbulence of weakly non-linear waves [7]: in this limit the wave action density corresponding to an energy cascade is $n(k) \sim k^{-(\beta+d)}$. Within KZ theory, $\Pi \sim \omega Im\langle a\bar{a}^2\rangle \sim \omega T\delta(\omega)\delta^d(k)n(k)^2 \sim k^{-3d-\beta}$. We stress however that, unlike KZ theory, this result is equally valid for strong wave turbulence.

A similar analysis for the 4 wave case yields $h = -4d - \beta$ for the homogeneity degree of the correlation function $\Pi_{0,1;2,3} = \int \prod_{i=0}^{3} d\Omega_i \langle Re(\dot{a}_{\vec{k}} a_{\vec{k}_1} \bar{a}_{\vec{k}_2} \bar{a}_{\vec{k}_3})\rangle$. This is also consistent with KZ theory for weak non-linearity and with Eq. (3) for the order of non-linearity $n = 4$.

In addition to energy, 4-wave turbulence also conserves wave action $N = \int d\vec{k} n(\vec{k})$, where $\langle \bar{a}_{\vec{k}_1} a_{\vec{k}_2}\rangle = n(\vec{k}_1)\delta(\vec{k}_1 - \vec{k}_2)$. There is a CFR associated with the cascade of wave action. The flux correlation function is $\Pi_{0,1;2,3} = \int \prod_{i=0}^{3} d\Omega_i Im\langle \bar{a}_{\vec{k}} \bar{a}_{\vec{k}_1} a_{\vec{k}_2} a_{\vec{k}_3})\rangle$. Assume that Π is a scaling function of degree h. Applying ZT to the continuity equation we find This is consistent with Eq. (3), for order of non-linearity $n = 4$, dimensionality of flux space d and homogeneity degree of interaction kernel β. In the limit of small non-linearity, the scaling $h = -4d - \beta$ can be also derived from KZ theory: for weakly non-linear waves, wave action density in the wave action cascade is $n(k) \sim k^{-(2\beta-\alpha+3d)/3}$, where α is dispersion exponent. On the other hand, within KZ theory, $\Pi \sim T\delta^d(k)\delta(\omega)n(k)^3 \sim k^{-4d-\beta}$.

This ends our illustration of specific examples where the CFR idea, expressed heuristically in Eq. (3), is used to derive new scaling laws respected even by strongly interacting systems. Probably there are other examples waiting to be analysed using the ideas presented in this letter. In closing we should emphasise that, although the results for specific systems analysed here are exact, the CFR is generally not a theorem. Application of the ZT implicitly assumes convergence of the collision integral on the scaling solution for the flux carrying correlation function (in the wave turbulence literature, this is referred to as the condition of locality). This assumption is usually not readily verifiable, even a posteriori, and will require considerable additional effort both theoretically and numerically. Nevertheless, there are specific cases where these issues are simplified which we would like to mention. For the MM, the constant kernel case ($\beta = 0$), can be reduced to a differential equation which can be explicitly solved. For wave turbulence with certain types of ultra-local interaction coefficients, CFR can be derived as a theorem by solving some simple recursion relations. It is not our intention to present more detailed analysis of these particulars here but rather to give an assurance that the assumption of locality has a basis in reality.

To summarize, we demonstrated, using a number of examples, that the scaling of a particular correlation function can be determined when there is a constant flux of a physical quantity in scale-invariant turbulent dynamics. This provides an extension of Kolmogorov's 1941 work on NS turbulence and Yaglom's work on passive advection to a diverse class of systems.

Acknowlegements
We would like to thank G. Falkovich and I. Kolokolov for interesting discussions and useful suggestions. This work was partially carried out under the auspices of the National Nuclear Security Administration of the U.S. Department of Energy at Los Alamos National Laboratory under Contract No. DE-AC52-06NA25396.

* Electronic address: connaughtonc@gmail.com
† Electronic address: rrajesh@imsc.res.in
‡ Electronic address: olegz@maths.warwick.ac.uk

[1] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[2] A. N. Kolmogorov, Dokl. Akad. Nauk. SSSR **32**, 16 (1941), reprinted in Proc. Roy. Soc. Lond. A, 434, 15-17, (1991).
[3] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, London, 1987), 2nd ed.
[4] G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Modern Phys. **73**, 913 (2001).
[5] G. Falkovich, in *Encyclopedia of Nonlinear Science*, edited by A. Scott (Routledge, New York and London, 2004).
[6] C. Connaughton, R. Rajesh, and O. Zaboronski, Physica D **222**, 97 (2006).
[7] V. Zakharov, V. Lvov, and G. Falkovich, *Kolmogorov Spectra of Turbulence* (Springer-Verlag, Berlin, 1992).
[8] A. Newell, S. Nazarenko, and L. Biven, Physica D **152-153**, 520 (2001).
[9] P. L. Krapivsky, J. F. F. Mendes, and S. Redner, Phys. Rev. B **59**, 15950 (1999).
[10] A. E. Scheidegger, Bull. I. A. S. H. **12**, 15 (1967).
[11] P. S. Dodds and D. H. Rothman, Phys. Rev. E **59**, 4865 (1999).
[12] S. N. Coppersmith, C. Liu, S. Majumdar, O. Narayan, and T. A. Witten, Phys. Rev. E **53**, 4673 (1996).
[13] S. N. Majumdar, S. Krishnamurthy, and M. Barma, Phys. Rev. E **61**, 6337 (2000).
[14] R. Rajesh, Phys. Rev. E **69**, 036128 (2004).
[15] D. Dhar, preprint arXiv:cond-mat/9909009 (1999).
[16] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).
[17] V. S. Zakharov, F. Dias, and A. Pushkarev, Phys. Rep. **398**, 1 (2004).
[18] E. Kuznetsov and V. Lvov, Physica D **2**, 203 (1981).
[19] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[20] C. Connaughton, R. Rajesh, and O. Zaboronski, Phys. Rev. E **69**, 061114 (2004).
[21] R. Rajesh and S. N. Majumdar, Phys. Rev. E **62**, 3186 (2000).