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No-Cycle Algebras and Representation Theory

by

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DEDICATION

To the memory of my grandmother Phyllis Jean Simmonds.

Contents

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Declarations

I declare that, to the best of my knowledge, all the work presented here is original (except for those parts of which the contrary is clearly stated). No part of this thesis has been submitted for any other degree.

Abstract

In the first half of this dissertation we study certain quotient algebras of preprojective algebras called no-cycle algebras N. These are studied via one-cycle algebras, which are introduced here. Results include detailed combinatorial information on N, and in certain special cases a presentation for N as a quiver with relations.

In the second half we consider deformations of coordinate algebras of Kleinian singularities. Results include an explicit presentation for the deformations of a type D singularity.

These two themes are tied together at the end by some mainly speculative comments about the role the various objects studied have to play in representation theory.

Chapter 1

Preliminary Facts

1.1 Basic Notation and Terminology.

(Number Systems.)

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_q$ will denote, respectively, the natural numbers, the integers, the rationals, the reals, the complex numbers, and the field of q elements. 0 *is* a natural number. If $n \in \mathbb{Z}$ and p is prime then $\overline{n} \in \mathbb{F}_p$ will denote the reduction of n modulo p. (Sets, Maps and Categories.)

All maps, morphisms and functors will be written on the *right*. Thus the composition of $\alpha : S \to T$ and $\beta : T \to U$ will be written $\alpha\beta$.

(Linear Algebra.)

k will denote an algebraically closed field of characteristic $p > 0$. K will denote an arbitrary field. (Actually K barely appears, but when it does it is clear that it could just as easily be a commutative ring with certain mild restrictions.) Given a vector space V, V^* will denote its dual space, $T(V)$ its tensor algebra and $S(V)$ its symmetric algebra.

(Rings, Algebras and Modules.)

The word non-commutative will be used to mean not necessarily commutative while the admittedly awful word 'not-commutative' should be self-explanatory. All rings and algebras are assumed to have a 1 and all modules are assumed to be unital. All modules are *right* modules unless otherwise stated. If M is an R-module and $S \subseteq M$ then SR denotes the R-submodule of M generated by S. If $m_1, \cdots, m_n \in M$ then the R-submodule of M generated by the m_i will be denoted $(m_1, \cdots, m_n)R$. Similar notation will be used for ideals (left, right and two-sided) of rings and algebras. The centre of a ring or algebra R will be denoted by $Z(R)$.

The socle of an algebra (defined as the sum of all irreducible submodules) is used at one point. See [3] for further details.

Suppose that A is an algebra, and M, S are A-modules with M of finite length and S irreducible. Then $|M : S|$ will denote the multiplicity of S as a composition factor of M. If $S_1, \cdots, S_n$ are a complete set of pairwise nonisomorphic irreducible modules for the finite-dimensional algebra A, and P_i denotes the projective cover of S_i, then the *Cartan matrix* of A is the matrix $|P_i : S_j|$.

(Lie Algebras and Algebraic Groups)

Algebraic groups will be denoted using the letters $G, H, \cdots$. A letter such as $\mathfrak{g}, \mathfrak{h}, \cdots$ will denote the Lie algebra of the corresponding group. $\mathfrak{g}$ will also be used for an abstract Lie algebra. Given a Lie algebra $\mathfrak{g}$ we write ad for the representation $\mathfrak{g} \to \mathrm{End}_k(\mathfrak{g})$ given by

$$x(y\mathrm{ad}) = [xy].$$

We will also sometimes regard associative algebras as being Lie algebras (with commutators as the Lie bracket in the usual way) without explicit explanation. In particular we will sometimes use the notation ad in associative algebras.

The *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ will be denoted $U(\mathfrak{g})$.

(Gradings and Filtrations.)

Given an algebra A, a *grading* for A is a decomposition

$$A = \bigoplus_{n \geq 0} A_n$$

where $1 \in A_0$ and $A_nA_m \subseteq A_{n+m}$. If each space A_n is finite-dimensional we define the Poincaré series of A to be

$$\sum_{n\geq 0}(\text{dim}A_n)q^n$$

If A itself is finite dimensional then we may refer to its Poincaré polynomial. We will also use the terms Poincaré series and Poincaré polynomial in more general situations, but the intended meaning should always be clear.

A *filtration* for an algebra A is a chain $F_0 \subseteq F_1 \subseteq \cdots$ of subspaces satisfying $A = \bigcup F_i$, $K \subseteq F_0$, $F_nF_m \subseteq F_{n+m}$. An algebra possessing a filtration is called a *filtered algebra*. The *associated graded algebra* of a filtered algebra A is defined to be the algebra gr$A$ which, as a vector space is

$$F_0 \oplus \bigoplus_{n\geq 1}(F_n/F_{n-1}),$$

and which has multiplication determined by $(\alpha+F_{n-1})(\beta+F_{m-1}) = \alpha\beta+F_{n+m-1}$. gr$A$ has an obvious grading. If B is a graded algebra we call any filtered algebra whose associated graded algebra is isomorphic to B a *deformation* of B.

(Groups and Group Actions.)

If G is a group and M is a kG-module, then M^G will denote the space of invariants. Suppose A is a graded commutative algebra A, G acts on A by automorphisms of A (as a graded algebra) and $A_0 \subseteq A^G$.Then A_G will denote the *coinvariant algebra* $A \otimes_{A^G} A_0$, where A_0 is made into a left A^G-module by identifying it with $A^G/\bigoplus_{n\geq 1}(A^G)_n$. In other words we simply kill all homogeneous invariants of positive degree.

If G is finite and $|G|$ is invertible in k then $(\cdot,\cdot)_G$ will denote the form on $(kG)^*$ given by

$$(\chi,\psi)_G = \frac{1}{|G|}\sum_{g\in G}(g\chi)(g^{-1}\psi).$$

If M, N are kG-modules then $(M,N)_G$ is interpreted by substituting characters for modules.

(Graphs, Quivers and Path Algebras)
A *graph* Γ consists of *finite* sets Γ_0 and Γ_1 whose elements are called vertices and edges. Each edge has associated to it an unordered pair of verices called its end points. We allow *loops* (edges whose end points coincide) and *multiple edges* (ie two edges may have the same pair of endpoints).

A *quiver* is simply a quadruple $Q = (Q_0, Q_1, t, h)$ where Q_0 and Q_1 are *finite* sets and t, h are maps $Q_1 \to Q_0$. We can represent quivers diagrammatically by drawing a small circle for each vertex (element of Q_0) and an arrow from αt to αh for each $\alpha \in Q_1$.

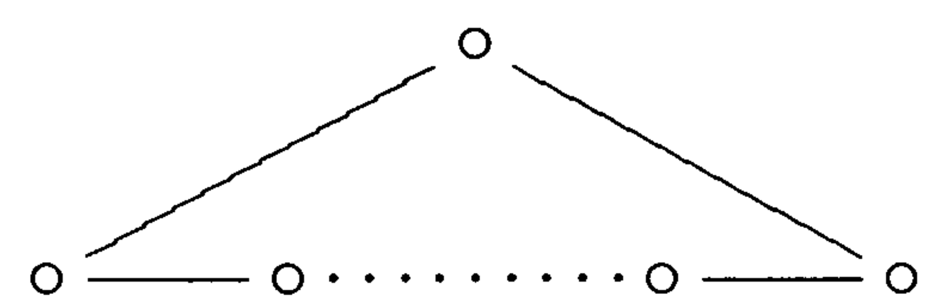
Given a quiver Q we generalize the notation Q_i by defining Q_n $(n \geq 2)$ to be the set of all n-tuples $(\alpha_1, \cdots, \alpha_n)$ of arrows such that $\alpha_i h = \alpha_{i+1} t$ for each i. Elements of the set $P = \bigcup_{n \geq 0} Q_n$ will be called paths. We extend the maps t, h to all of P in the obvious way. Given paths p, q such that $ph = qt$ we define the path pq also in the obvious way. Define the *path algebra* KQ to be the K-algebra whose underlying vector space has basis P and where the product of paths p, q is as above for $ph = qt$ and zero otherwise. Given $i \in Q_0$ we usually write e_i for i regarded as an element of KQ. Obviously the e_i are idempotents whose sum is the 1 of KQ. Obviously KQ is a graded algebra with $(KQ)_n$ equal to the span of Q_n.

(Miscellaneous terminology) There is a great deal of basic terminology which we shall use but which it would not be appropriate to define in detail here. Examples include *PBW basis, Levi subalgebra, Basic algebra, Gabriel quiver, Global Dimension, Symmetric algebras and Frobenius algebras* (in the sense of Nakayama). All of these and other terms we shall use are defined in the union of [7], [3], [52], [15].

1.2 Affine Coxeter Graphs.

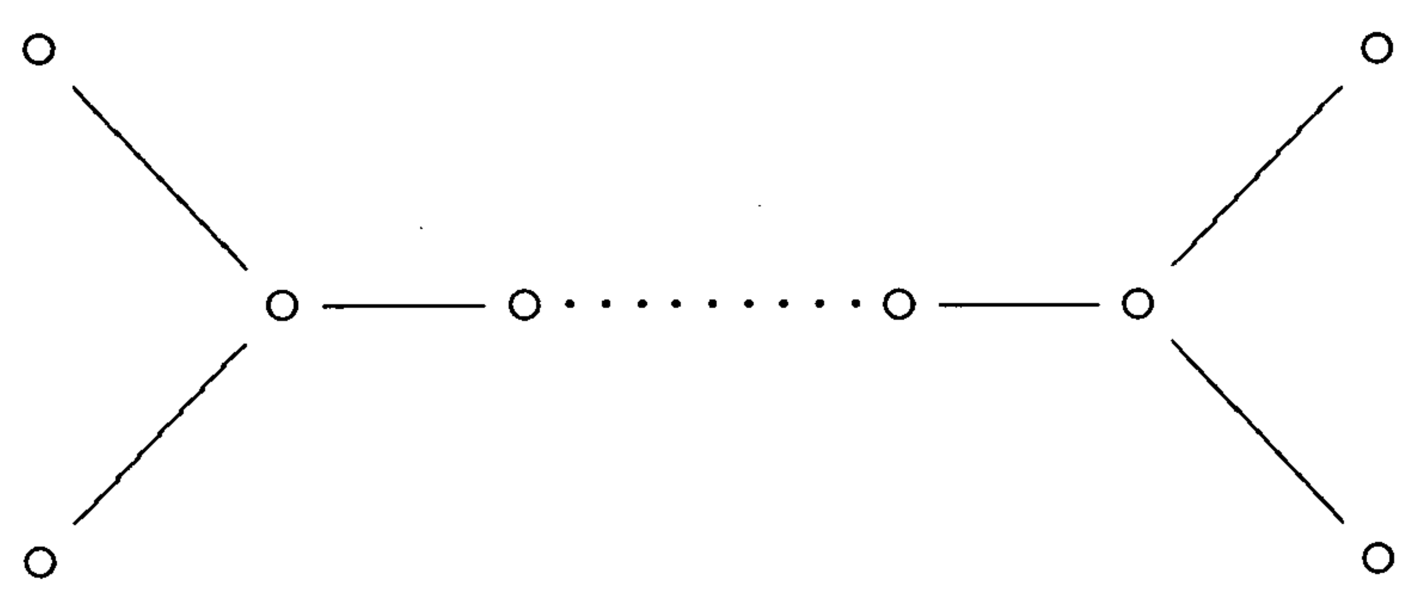
The starting point for the present dissertation is the famous ADE classification. At its simplest level this is merely a set of graphs which classify many different types of mathematical objects. Indeed, each of the sections 1.2 - 1.7 is based on an ADE classification relevent to the present thesis, while other ADE classifications are described in [26]. The graphs are as follows.

Type A_{n-1} $(n \geq 2)$:



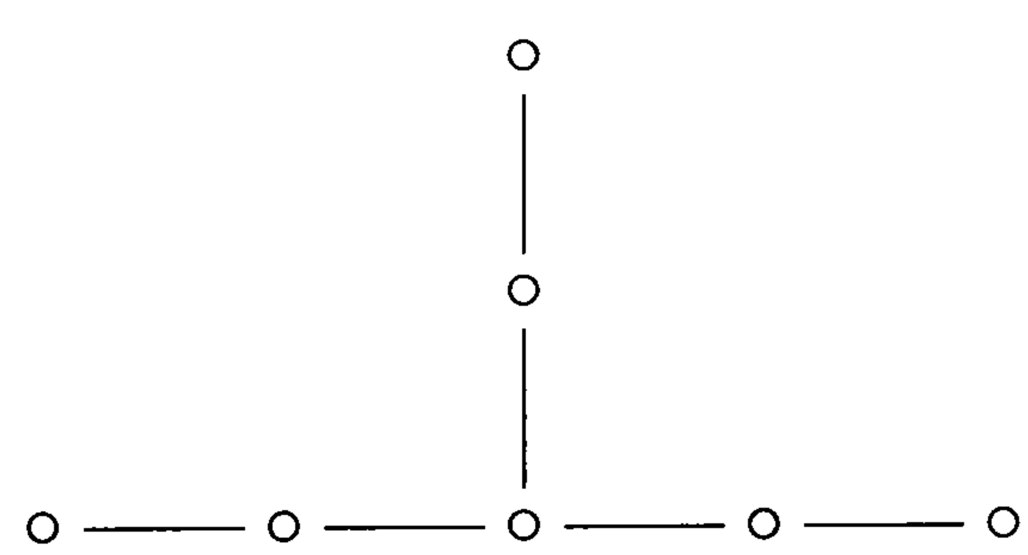
(n vertices)

Type D_n $(n \geq 4)$:

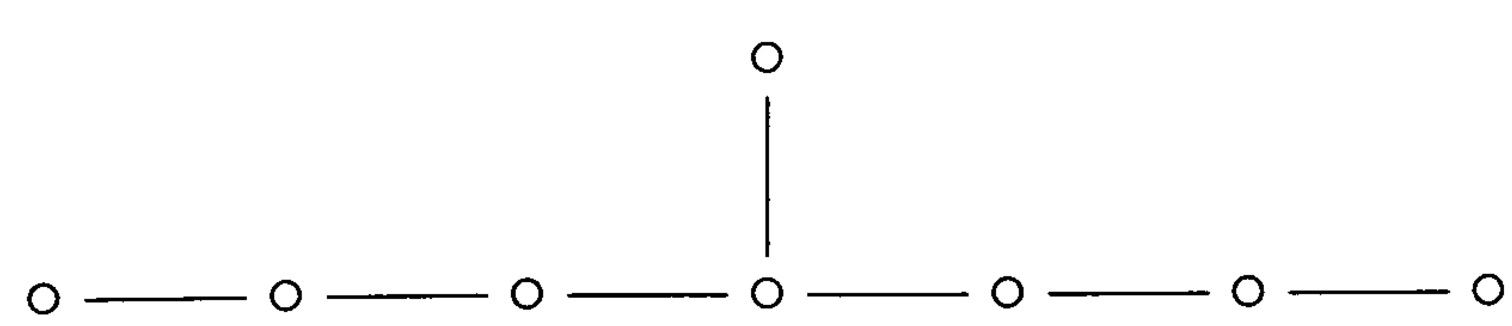


($n+1$ vertices)

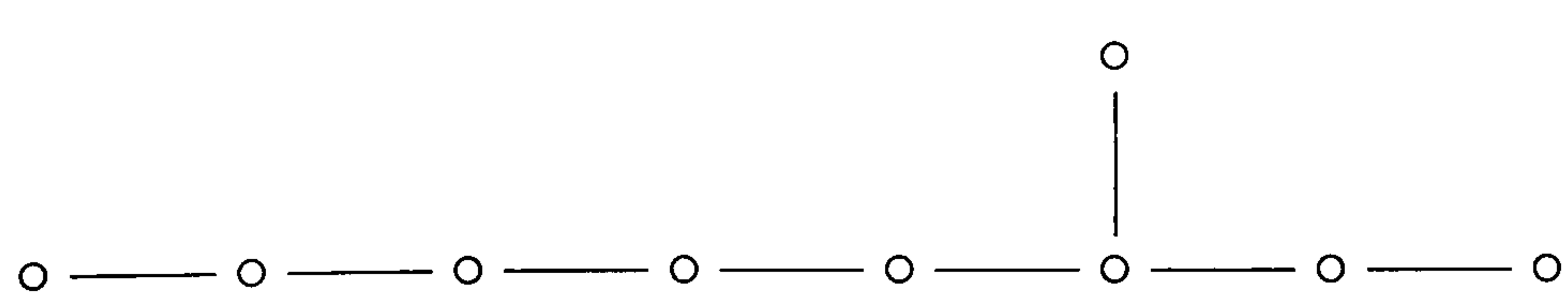
Type E_6:



Type E_7:



Type E_8:



The simplest classification problem whose solution is given by these graphs is that of finding all connected finite graphs Γ (with no loops) together with a map $\Gamma_0 \to \mathbb{N}$, written $i \mapsto d_i$ such that at least one vertex has $d_i = 1$ and such that for all vetices i we have $2d_i = \sum_{j-i} d_j$. The only solutions to this problem are the graph A_{n-1} with $d_i = 1$ for all i, and the graphs D_n, E_6, E_7, E_8 with the following vertex labels:

$$\begin{array}{cccccc} 1 & & & & & 1 \\ & 2 & 2 & \cdots & 2 \;\; 2 & \\ 1 & & & & & 1 \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{array}$$

$$\begin{array}{ccccccc} & & & 2 & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{array} \qquad \begin{array}{cccccccc} & & & & & 3 & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{array}$$

(See [40] for a discussion of related problems.) Any vertex i with $d_i = 1$ is called an affine vertex, and a graph obtained by removing one affine vertex is called a finite or non-affine Coxeter graph. This process gives a bijective correspondence between the two types of Coxeter graphs. In fact when most authors talks of the ADE classication they usually use the non-affine Coxeter graphs and thus employ the messier notation $\tilde{A}_{n-1}$, $\tilde{D}_n$, $\tilde{E}_n$ for the affine ones. We shall always use affine graphs and therefore drop the tildes.

In the next few sections we shall see numerous other types of objects classified by Coxeter graphs. But these are not all - for a more comprehensive survey the best reference is [26].

1.3 Representation type of quivers.

A good reference for the material of this subsection is [38]. A *representation* of a quiver Q is a vector space V_i for eace vertex i and a linear map $V_{\alpha t} \to V_{\alpha h}$ for each arrow α. This is easily seen to be the same thing as a module for the path algebra kQ, and we shall always deal with the latter. A fundamental problem is to understand the indecomposable modules for a path algebra, and a first step is to find the *representation type* of any given quiver - or rather of its path algebra. (The *representation type* of an algebra A is said to be *finite* if A has only finitely many indecomposable modules, *tame* if A has infinitely many indecomposable modules but these are in some suitable sense classifyable (see eg. [3]) for a precise definition) and *wild* otherwise.) The crucial theorem is as follows. (See [38])

Theorem 1.3.1 *[17] Let Q be a quiver whose underlying graph is connected. Then Q is of finite representation type if and only if its underlying graph is a non-affine Coxeter graph. Q is of tame representation type if and only if its underlying graph is an affine Coxeter graph (or a graph with one vertex and one loop).*

This result can also be understood in terms of preprojective algebras and preprojective modules. We shall not need this result so we refer the reader to [40]. Preprojective algebras, on the other hand, will be central to the present dissertation.

1.4 Preprojective Algebras.

Let Q be a finite quiver with vertex set I. For the moment Q may be arbitrary. Let $\overline{Q}$ be the double quiver of Q, that is the quiver obtained by adjoining an opposite arrow α^* for each arrow $\alpha \in Q_1$. Define $\gamma \in k\overline{Q}$ by

$$\gamma = \sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha).$$

Definition 1.4.1 *The preprojective algebra $\Pi = \Pi(\Gamma)$ ([1],[16],[18],[43]) is defined to be the quotient of $k\overline{Q}$ by the ideal generated by γ.*

The isomorphism class of Π does not depend on the orientation of Q, so only depends on the underlying graph Γ of Q. Sometimes we will talk of the preprojective algebra of a graph. $k\overline{Q}$ is a graded algebra (graded by the length of paths) and γ is homogeneous (of degree 2). Thus Π is also graded.

We have the following theorem. See [40], p 553.

Theorem 1.4.2 *Let Π denote the preprojective algebra of a connected graph Γ. Then Π is finite dimensional if and only if Γ is a non-affine Coxeter graph. It is infinite-dimensional but Noetherian if and only if Γ is an affine Coxeter graph.*

1.5 Finite Subgroups of $\mathrm{SL}_2(\mathbb{C})$.

Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, and let $M_0, \cdots, M_r$ be a complete set of pairwise nonisomorphic irreducible right $\mathbb{C}G$ modules with M_0 trivial. Set $I := \{0, \cdots, r\}$. Let V be the natural two-dimensional module. Define a matrix $A = (a_{ij})$ by

$$a_{ij} = (M_i \otimes V, M_j)_G.$$

McKay's observation [35] is that A is a symmetric matrix which is the adjacency matrix of an *affine Coxeter graph* Γ (with vertex set I), and that this sets up a bijective correspondence between conjugacy classes of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ and isomorphism types of affine Coxeter graphs. It follows that the numbers d_i of section 1.2 now have a concrete interpretation as the dimensions of the modules M_i. We shall fix once and for all a specific group for each type. In types E_6, E_7, E_8 the matrix representations we use will *not* be those appearing in [26]. Our choice is more convenient for many purposes.

Type A_{n-1}. Let ε be a primitive nth root of unity. Take G to be the cyclic group of order n generated by

$$\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}.$$

Type D_n. Let ε be a primitive $(2n-4)$th root of unity. Take G to be the group generated by

$$\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We have $|G| = 4n - 8$.

Type E_6. Here take G to be the set of all matrices of one of the three forms

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \begin{bmatrix} 0 & \varepsilon \\ -\varepsilon^{-1} & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} \varepsilon & \alpha \\ -\alpha^{-1} & \varepsilon^{-1} \end{bmatrix},$$

where α can be any 4th root of unity and ε any *primitive* 8th root of unity. There are 4 matrices of each of the first two types and 16 of the third type. Thus $|G| = 24$.

Type E_7. Here we take G to be the group of order 48 generated by the group of type E_6 and the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Type E_8. Fix a primitive 20th root of unity ε. Here we set G to be the group generated by

$$\begin{bmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon^{-2} \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} \varepsilon^2 - \varepsilon^{-2} & \varepsilon + \varepsilon^{-1} \\ -\varepsilon - \varepsilon^{-1} & -\varepsilon^2 + \varepsilon^{-2} \end{bmatrix}.$$

We have $|G| = 120$.

Define numbers a, b, h by the following table.

Type	a	b	h
A_{n-1}	2	n	n
D_n	4	$2n-4$	$2n-2$
E_6	6	8	12
E_7	8	12	18
E_8	12	20	30

We have the following relations: $r = |I| - 1$; $\sum_{i \in I} d_i = h$; $\sum_{i \in I} d_i^2 = \frac{ab}{2} = |G|$; $a + b = h + 2$; $(r-1)ab = 2h^2 - 4h$; $a = 2\max d_i$.

It is well-known (see for example [32]) that $S(V)^G$ is generated by three homogeneous polynomials A, B, H of degrees a, b, h respectively and that these are subject to a single relation of the form

$$H^2 = f(A, B),$$

where $f(A, B)$ is homogeneous of degree $2h$. Set $x = (1, 0), y = (0, 1)$ thus identifying $S(V)$ with $k[x, y]$. We shall now make explicit choices for A,B,H.

Type A_{n-1}. Set

$$\begin{aligned} A &= xy, \\ B &= x^n + y^n, \\ H &= x^n - y^n. \end{aligned}$$

These satisfy

$$H^2 = B^2 - 4A^n.$$

Type D_n. Set

$$\begin{aligned} A &= x^2y^2, \\ B &= x^{2n-4} + y^{2n-4}, \\ H &= xy(x^{2n-4} - y^{2n-4}). \end{aligned}$$

These satisfy

$$H^2 = AB^2 - 4A^{n-1}.$$

Type E_6. Set

$$\begin{aligned} A &= xy(x^4 + y^4), \\ B &= x^8 - 14x^4y^4 + y^8, \\ H &= (x^4 - y^4)(x^8 + 34x^4y^4 + y^8). \end{aligned}$$

These satisfy

$$H^2 = 108A^4 + B^3.$$

Type E_7. Set

$$\begin{aligned} A &= x^8 - 14x^4y^4 + y^8, \\ B &= x^2y^2(x^4+y^4)^2, \\ H &= xy(x^8-y^8)(x^8+34x^4y^4+y^8). \end{aligned}$$

These satisfy

$$H^2 = A^3B + 108B^3.$$

Type E_8. Let $i := \varepsilon^5$, a primitive 4th root of unity. Set

$$\begin{aligned} A &= xy(x^{10} - 11ix^5y^5 + y^{10}), \\ B &= x^{20} + 228ix^{15}y^5 - 494x^{10}y^{10} + 228ix^5y^{15} + y^{20}, \\ H &= (x^{10}-y^{10})(x^{20} - 522ix^{15}y^5 + 10006x^{10}y^{10} - 522ix^5y^{15} + y^{20}). \end{aligned}$$

These satisfy

$$H^2 = B^3 - 1728iA^5.$$

In type E_8, there is some beautiful geometry associated to these polynomials. Each complex root $\lambda x + \mu y$ of one of the polynomials gives a point $\lambda : \mu$ on the complex projective line - or Riemann sphere - $P^1(\mathbb{C})$. Then the roots of A,B,H give respectively the vertices of, the centres of the faces of, and the midpoints of the edges of a regular icosahedron drawn on the sphere (see [32]).

We shall now explain a connection between preprojective algebras and the so-called *skew group algebra* associated to G. G acts on V, so we can form the skew group algebra $G * S(V)$. As a vector space we have

$$G * S(V) = \mathbb{C}G \otimes S(V),$$

while multiplication is determined by

$$(x \otimes \alpha)(y \otimes \beta) = xy \otimes \alpha^y \beta.$$

$G * S(V)$ is a graded algebra (give $S(V)$ its usual grading and the group elements degree 0). For each irreducible right $\mathbb{C}G$-module M_i pick an idempotent f_i in $\mathbb{C}G$ such that $f_i(\mathbb{C}G) \cong M_i$. Set $f = \sum f_i$.

Theorem 1.5.1 *We have the following isomorphism of graded algebras*

$$f[G * S(V)]f \cong \Pi.$$

In particular Π *is graded Morita equivalent to* $G * S(V)$.

The proof of this result can be found in [14] (corollary 3.6), although it first appeared in [41] and is proved implicitly during the proof of Proposition 2.13. The basic explanation is as follows. The degree 1 part of $f[G * S(V)]f$ can clearly be identified with

$$\mathrm{Hom}_{\mathbb{C}G}(M, M \otimes V),$$

where $M = \oplus M_i$. By the McKay correspondence this latter space can be identified with the degree one part of $\mathbb{C}\overline{Q}$ (the space spanned by the arrows in the McKay quiver). Choices have to be made when making these identifications. In this way we see that

$$f[G * T(V)]f \cong \mathbb{C}\overline{Q}.$$

Here $T(V)$ is the tensor algebra. Now, $S(V)$ is obtained from $T(V)$ by imposing relations in degree 2, and it is possible, by making the correct choices, to make these relations correspond to the preprojective relations.

This implies that the centre of Π is (graded) isomorphic to the centre of the skew group algebra, which is easily seen to be isomorphic to the algebra of G-invariant polynomials $\mathbb{C}[A, B, H]$.

1.6 Kleinian Singularities.

A Kleinian singularity is a variety in $\mathbb{C}^3$ defined by one of the following equations.

Type A_{n-1}	$u^n = vw$	$(n \geq 2)$
Type D_n	$u^{n-1} + uv^2 + w^2 = 0$	$(n \geq 4)$
Type E_6	$u^4 + v^3 + w^2 = 0$	
Type E_7	$u^3v + v^3 + w^2 = 0$	
Type E_8	$u^5 + v^3 + w^2 = 0.$	

These equations are the usual choices but we note that in type A_{n-1} a different choice of generators would give the defining equation $u^n + v^n + w^2 = 0$. This alternative form is often more convenient. In particular it means that *any* Kleinian singularity is defined by an equation of the form $f(u, v) + w^2 = 0$, a fact which we shall use to give uniform descriptions of certain objects.

By comparing these equations with those of section 1.5 we see that a variety is a Kleinian singularity if and only if it is a quotient of $\mathbb{C}^2$ by a nontrivial finite subgroup of $SL_2(\mathbb{C})$. There are a large number of other characterizations of Kleinian singularities available (see [26]).

1.7 Simple Lie Algebras of Simply-Laced Type over $\mathbb{C}$.

Another important instance of the ADE classification, is the classification of simply-laced simple Lie algebras over $\mathbb{C}$, which we now describe. Basic referenses for this material are [7], [23], [27], [44], [45].

Suppose $\mathfrak{g}_{\mathbb{C}}$ is a simple Lie algebra over $\mathbb{C}$, and that $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra (often abbreviated to the awful 'CSA') - that is, a nilpotent subalgebra that coincides with its own normalizer. Now, for any $\alpha \in \mathfrak{h}_{\mathbb{C}}^*$ one defines

$$(\mathfrak{g}_{\mathbb{C}})_\alpha := \{x \in \mathfrak{g}_{\mathbb{C}} : [x, y] = y\alpha x \text{ for all } y \in \mathfrak{h}_{\mathbb{C}}\}$$

One knows that $(\mathfrak{g}_{\mathbb{C}})_0 = \mathfrak{h}_{\mathbb{C}}$ and that there is a set $\Delta \subseteq (\mathfrak{h}_{\mathbb{C}}^* - \{0\})$, whose elements are called *roots*, such that $(\mathfrak{g}_{\mathbb{C}})_\alpha \neq 0$ precisely when $\alpha \in \Delta \cup \{0\}$. Moreover for $\alpha \in \Delta$, $(\mathfrak{g}_{\mathbb{C}})_\alpha$ is one-dimensional, and there is a decomosition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta}(\mathfrak{g}_{\mathbb{C}})_\alpha$. Let $\Psi \subseteq \Delta$ denote a set of simple roots (so that $|\Psi| = r$), and let $\Delta = \Delta^+ \cup \Delta^-$ be the decomposition of Δ into the corresponding sets of positive and negative roots. Recall that the Killing form on $\mathfrak{g}_{\mathbb{C}}$ (given by $(x, y) := \mathrm{tr}(\mathrm{ad}x \circ \mathrm{ad}y)$) restricts to a nonegenerate symmetric biliear form on $\mathfrak{h}_{\mathbb{C}}$, and so we are able to identify $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}^*$, and in particular consider the Killing form applied to a pair of elements of $\mathfrak{h}_{\mathbb{C}}^*$. It is helpful to consider the space E defined as the $\mathbb{R}$-span of Δ. The Killing form restricts to a positive-definite symmetric $\mathbb{R}$-bilinear form on E, so, in a nutshell, E is a Euclidean space. Let $P \subseteq E$ denote the weight lattice. $\mathfrak{g}_{\mathbb{C}}$ is said to be of *simply laced type* if all roots have the same length. *FROM NOW ON* assume that $\mathfrak{g}_{\mathbb{C}}$ is of simply laced type. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean form on E suitably scaled so that $\langle \alpha, \alpha \rangle = 2$ for all roots α. Now for any distinct simple roots α, β we find that $\langle \alpha, \beta \rangle$ equals 0 or -1. We can form a graph whose vertices correspond to the simple roots and with an edge between a pair of simple roots α, β precisely when $\langle \alpha, \beta \rangle = -1$. In this way one gets a bijective correspondence between isomorphism classes of simply-laced simple Lie algebras and *non-affine* Coxeter graphs. The choice of the set of simple roots determines a so called *highest root* α_0, and forming the same graph for the set $\Psi \cup \{-\alpha_0\}$ one gets the corresponding affine Coxeter graph (except that in type A_1 one must join the two vertices by *two* edges because in this case we get the value -2 for the inner product).

Having seen that one can identify simply laced simple Lie algebras $\mathfrak{g}_{\mathbb{C}}$ with affine Coxeter graphs, one can think of everything considered in sections 1.2 - 1.5 (I, r, d_i) as being associated to $\mathfrak{g}_{\mathbb{C}}$. In fact there are the following interpretations of these things. I is identified with the set $\Psi \cup \{-\alpha_0\}$, r is just the *rank* of $\mathfrak{g}_{\mathbb{C}}$ (ie the dimension of $\mathfrak{h}$), while the numbers d_i for $i \neq 0$ give the coeffiecients of

α_0 relative to the simple roots.

The Lie algebra associated to the graph A_{n-1} is, of course, $\mathfrak{sl}_n$, while the Lie Algebra associated to the graph D_n is $\mathfrak{so}_{2n}$. In contrast the algebras associated to the graphs E_6, E_7, E_8, usually denoted $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, have no straightforward description.

1.8 Modular Representation Theory.

Part 2) of the following is proved in [27], while part 3) appears in [42] - 1) is of course completely trivial and is only included for completeness.

Lemma 1.8.1 *Let x, y denote commuting indeterminates. We have*
1) $(\lambda x)^p = \lambda^p x^p$ for all $\lambda \in k$.
2) $x(\mathrm{ad} y)^p = [x, y^p]$.
3) Let L denote the Lie subalgebra of $k\langle x, y\rangle$ generated by x and y. Then L is a free Lie algebra on x, y. Moreover the noncommutative polynomial $C_p(x, y)$ defined by

$$C_p(x, y) := (x + y)^p - x^p - y^p,$$

lies in L. □

Part 3) implies that $C_p(x, y)$ can be expressed in terms of iterated commutators in x and y, and the freeness of L means that it actually makes sense to talk of $C_p(x, y)$ if x, y are any elements of any Lie algebra over k. We define a *restricted* Lie alegebra to be a Lie algebra $\mathfrak{g}$ over k equipped with a map $\mathfrak{g} \to \mathfrak{g}$, written $x \mapsto x^{[p]}$, satisfying
1) $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ for all $\lambda \in k$ and $x \in \mathfrak{g}$,
2) $x(\mathrm{ad} y)^p = \left[x, y^{[p]}\right]$ for all $x, y \in \mathfrak{g}$,
3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + C_p(x, y)$ for all $x, y \in \mathfrak{g}$.

This is my preferred way of defining restricted Lie algebras, as it cunningly avoids having to give an explicit formula for $C_p(x, y)$ in terms of commutators (see [27] for such a formula). However, there is another such avoidance trick involving the map $\theta : \mathfrak{g} \to U := U(\mathfrak{g})$ given by $x\theta = x^p - x^{[p]}$; one can easily check that conditions 1), 3) are equivalent to the statement that θ is *p-linear* (ie $(\lambda x)\theta = \lambda^p x\theta$, and $(x+y)\theta = x\theta + y\theta$), while 2) is equivalent to the statement that $\mathfrak{g}\theta \subseteq Z(U)$.

Now suppose that $\mathfrak{g}$ is a finite-dimensional restricted Lie algebra. Any irreducible $\mathfrak{g}$-module M has a unique *p-character* χ. χ is an element of $\mathfrak{g}^*$ with the property that for all $x \in \mathfrak{g}$, $x^p - x^{[p]} - (x\chi)^p$ annihilates M. This is equivalent to saying that M is a quotient of the *reduced enveloping algebra* U_χ defined as the quotient of U obtained by killing all $x^p - x^{[p]} - (x\chi)^p$.

We will now return to the situation of the previous section. $\mathfrak{g}_{\mathbb{C}}$ has a so-called *Chevalley* basis e_α, h_β where $\alpha \in \Delta$, $\beta \in \Psi$. The structure constants of $\mathfrak{g}_{\mathbb{C}}$ with respect to this basis are integers, so one can define a Lie algebra over $\mathbb{Z}$, and tensor over the field k to obtain a Lie algebra over k. We shall denote the resulting Lie algebra by $\mathfrak{g}$. There exists a simply-connected semisimple algebraic group G whose Lie algebra is $\mathfrak{g}$. Now, $\mathfrak{g}$ has a natural p-structure given by the formulae $e_\alpha^{[p]} = 0$, $h_\beta^{[p]} = h_\beta$.

We will assume that p is greater than the Coxeter number h, although one should always consult the references to see the best possible restrictions on p required for the results we mention to hold.

The centre of $U(\mathfrak{g})$ has two distinguished subalgebras. The G-invariants U^G (familiar from characteristic zero) and the *p-centre* Z_p generated by all elements of the form $x^p - x^{[p]}$, where $x \in \mathfrak{g}$. It is a fact that $Z(U)$ is a free Z_p-module (of rank p^r) with free basis $\tilde{f}_1^{a_1} \cdots \tilde{f}_r^{a_r}$ where $a_i < p$ and where $\tilde{f}_n$ generate U^G (familiar from characteristic zero). This result was first proved by Veldkamp in [51] and certain improvements were made in [37].

The representation theory of U_χ reduces, by a result of Kac and Weisfeiler, to the case where χ is *nilpotent*, meaning that it kills some Borel subalgebra. (In fact, under our hypotheses on p, there is a G-isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, and under this isomorphism this corresponds to the usual definition of nilpotence for an element of a Lie algebra). Now the representation theory of U_χ clearly depends on the G-orbit of χ. Under our hypothesis on p there is a classification of nilpotent orbits similar to the one in characteristic zero (see [24]). We are interested in the case where χ lies in the so-called *subregular* nilpotent orbit, characterized by the condition that its dimension is $2(N-1)$, where $N = \frac{1}{2}|\Delta|$. Fix a subregular nilpotent character χ.

Now by [11] we have a block decomposition of U_χ

$$U_\chi = \bigoplus_\lambda B_{\chi,\lambda},$$

where λ runs through a set of representatives of the $W\cdot$ orbits (see [7]) on P/pP.

1.9 Gröbner bases.

This section contains a rather crude introduction to Grobner bases tailored to our needs. For a full treatment, see for example [5].

Let $R = k\langle x_1, \cdots, x_l\rangle$ be a free associative algebra. Let $\mathcal{M}$ denote the set of monomials in the x_i, that is the smallest set containing 1 and closed under right multiplication by the x_i. An *admissible order* on $\mathcal{M}$ is a total order $<$ on $\mathcal{M}$ satisfying for all M, N, $P \in \mathcal{M}$:

· $1 \leq M$.

· If $M < N$ then $MP < NP$ and $PM < PN$.

· There are only finitely many Q with $Q < M$.

From now on in this section assume that $\mathcal{M}$ is equipped with an admissible order.

By a *rewrite rule* I shall mean an ordered pair (M, a) where $M \in \mathcal{M}$ and a is an element of R only involving monomials strictly less than M with respect to $<$. Usually we shall say $M \rightsquigarrow a$ is a rewrite rule.

Now suppose we have a fixed set of rewrite rules. We want to extend the notation $\rightsquigarrow$ to say when an element of R can be rewritten to another one using a sequence of rewrite rules. So suppose $b \in R$ contains a monomial P with (nonzero) coefficient λ and that P has a *submonomial* M for which there is a rewrite rule $M \rightsquigarrow a$. Write $P = NMQ$ where $N, Q \in \mathcal{M}$ (this is what I mean by a submonomial). Then we say that

$$b \rightsquigarrow b - \lambda P + \lambda NaQ.$$

This looks formidable, but all we have done is replace M by a. However, one does need to be slightly careful since it is essential to understand that λ must be the coefficient of P in b. We cannot, for example, write

$$0 = 3M - 3M \rightsquigarrow 3a - 3M.$$

We finally extend $\rightsquigarrow$ by reflexivity and transitivity.

Now we come to the point of all this. Suppose that $S \subseteq R - \{0\}$. We want to understand the quotient algebra R/RSR. Without loss of generality we assume that for all $a \in S$, the highest monomial $\overline{a}$ appears with coefficient 1. Now, each $a \in S$ gives 'obvious' linear relations among the images of monomials in R/RSR, obtained from the rewrite rules $\overline{a} \rightsquigarrow \overline{a} - a$. In an ideal situation, this would be 'it'- in the sense that the elements of $\mathcal{M}$ not containing any $\overline{a}$ as a submonomial would give a basis for R/RSR. If this is the case we say that S (or the corresponding set of rewrite rules) gives a *Gröbner basis* for R/RSR. The following technical result gives a sufficient condition for this to happen.

Theorem 1.9.1 *Suppose R, $\mathcal{M}$, $<$, S are as in the above discussion. Consider the relation $\rightsquigarrow$ obtained from the rewrite rules $\overline{a} \rightsquigarrow \overline{a} - a$ for $a \in S$. Suppose further that*

· *Whenever $a, b \in S$ are such that there exist monomials M, N, O with $O \neq 1$ for which $\overline{a}M = N\overline{b} = NOM$, then we have $aM - Nb \rightsquigarrow 0$.*

· *Whenever a, b are distinct elements of S then $\overline{a}$ does not occur as a submonomial of $\overline{b}$.*

Then S gives a Gröbner basis for R/RSR. $\square$

We shall denote the element $aM - Nb$ in the above theorem by $(\overline{a}, \overline{b})_O$, O being the (nonempty) 'overlap'.

Chapter 2

Introduction

Assuming that the reader is familiar with the material of the previous chapter we now summarise the main results of the present dissertation, and the ideas behind them.

The starting point for the present investigation was the problem of finding the relationship between deformations of Kleinian singularities and subregular reprentations of Lie algebras of the corresponding type. This problem was investigated by Premet [39] in characteristic zero while in the modular case it was considered, in type A only, by Gordon and Rumynin in [21]. The main problem was to attempt to generalise the results of [21] to types D and E.

Let us describe those results which we are attempting to generalise. Fix $n \geq 2$ and suppose that p, the characteristic of k, is greater than n. We shall be interested in the subregular representations of the Lie algebra $\mathfrak{sl}_n(k)$. So let χ be a subregular nilpotent functional and let U_χ denote the associated reduced enveloping algebra. Fix a regular weight λ. Gordon and Rumynin consider the algebra

$$U_{\chi,\lambda} := \frac{B_{\chi,\lambda}}{JB_{\chi,\lambda}},$$

where J is the unique maximal ideal of $Z(B_{\chi,\lambda})$. The Gabriel quiver of $B_{\chi,\lambda}$ was shown by Jantzen in [30] to be the double of an affine Coxeter graph of type

A_{n-1}, and since $U_{\chi,\lambda}$ is a quotient of $B_{\chi,\lambda}$ by a nilpotent ideal, these algebras must have the same Gabriel quivers. Gordon and Rumynin show that in fact the basic algebra of $U_{\chi,\lambda}$, is the so called *no-cycle algebra* $N(n)$ obtained from the path algebra of this quiver by killing all nontrivial paths beginning and ending at the same vertex. The algebra $N(n)$ can be described alternatively as the quotient of the corresponding preprojective algebra obtained by killing the ideal generated by all homogeneous central elements of positive degree. Gordon and Rumynin show further that $U_{\chi,\lambda}$ is actually a matrix algebra over a certain algebra $t(v)$ constructed as a quotient of a deformation $T(v)$ of the coordinate algebra of a Kleinian singularity of type A_{n-1}. The deformations $T(v)$, depending on a polynomial v, and given explicitly by generators and relations, had been constructed by Smith and Hodges in [49], [22].

The main motivation for the present work was the following suggestions. 1) That for *all* types the algebra $U_{\chi,\lambda}$ should be Morita equivalent to the no-cycle algebra N, here *defined* to be the quotient of the preprojective algebra Π of an affine Coxeter graph of corresponding type obtained by killing all homogeneous central elemnts of positive degree. (Note that they certainly have the same Gabriel quivers by [29]). 2) That, moreover, $U_{\chi,\lambda}$ should be isomorphic to a matrix algebra over some central quotient of certain deformations of the corresponding Kleinian singularity.

There are several reasons why the problem is significantly more difficult in types D and E. Firstly no adequate description of the no-cycle algebras N was available at the time - what *are* the central elemants of Π? Similarly there was no construction of deformations of Kleinian singularities, analogous to that of Smith and Hodges, available in types D and E. I have been able to find satisfactory solutions to both of these problems in type D, and given some interesting properties, but the various objects in type D are substantially more complicated than in type A and it seems very difficult (but, I would say, *not* completely hopeless) to prove the conjectures on subregular representations by direct algebraic methods. As a result of this, the present thesis is almost entirely about no-cycle algebras and deformations of Kleinian singularites - the representation theory described above serves mainly to motivate the work.

Chapter 3 begins with a discussion of preprojective algebras. We describe a simple combinatorial procedure for finding the Poincaré series of a preprojective algebra. This is not a new observation (at least if there are no loops) - it follows trivially from a description of preprojective algebras in terms of preprojective *modules.* It is, however, instructive to give a direct argument. (This combinatorial information is later used to give nice properties of no-cycle algebras).

We then specialise to the case we are really interested in; preprojective algebras Π of affine Coxeter graphs. Because of a Morita equivalence of Π with a skew group algebra, it is known that $Z(\Pi)$ is isomorphic to $\mathbb{C}[x,y]^G$ (the coordinate algebra of a Kleinian singularity of the corresponding type), so is generated by three homogeneous elements A, B, H (whose degrees I consider to be in ascending order). By considering a certain group $\overline{G}$ containing G, and systematically applying old results about *pseudo-reflection groups*, we are able to deduce several consequences for Π. Firstly it is a free module over the central subalgebra $\mathbb{C}[A,B]$. Next, defining the *one-cycle algebra* Ω to be $\Pi \otimes_{\mathbb{C}[A,B]} \mathbb{C}$ we get an algebra of dimension $2h^2$ (h the corresponding Coxeter number) which is only slightly bigger than the no-cycle algebra N. More precisely, Ω has no graded component of degree any greater than h, while Ω_h is the kernel of the epimorphism $\Omega \to N$ and is a direct sum of one-dimensional spaces e_iHe_i. It is shown that there is a great deal of interesting combinatorics associated to one-cycle algebras. Finally (for the consequences of the theory of pseudo-reflection groups) Ω is also shown shown to be a symmetric algebra in the sense of [8] satisfying a kind of *Poincaré duality.*

We then consider one-cycle algebras and the no-cycle algebras in the individual types. In type A they can be described extremely easily (and the description will also explain the names). We are also able to show that in this case Ω, and hence N, has tame representation type give a classification of its simple modules. It should be emphasized that this is essentially done by Gordon and Rumynin in [21]. In type D we show that the algebras have wild representation type, thus excuding the possibility of using a classification of modules to deduce some consequences about modular representation theory as done for type A. On the positive side, we are able to write down explicit presentations for Ω and N in type D which are about as neat as one could hope for. It is to be hoped that in the future it will be possible to use these presentations to prove the conjectures about subregular representations. In type E we are able to do little except make some speculations.

In chapter 4 we consider deformations of coordinate algebras of Kleinian singularities. For type A_{n-1}, such a family of deformations was first introduced by Smith and Hodges in [50], [22]. Their algebras $T(v)$ are given explicitly by (three) generators and relations, and depend on a polynomial v of degree n. It had been observed in [21] that in the modular case under suitable restrictions on p and f, the centre of $T(v)$ is isomorphic to a the coordinate algebra of a type A_{n-1} Kleinian singularity. We show that under these circumstances to one-cycle algebra occurs as a central reduction of $T(v)$. (The corresponding result for the no-cycle algebra had previously proved in [21])

The next problem I considered was to find a family of deformations in type D_n analogous to that of Smith and Hodges. This problem is solved in section 4.4.6 where I describe a family of deformations $T(\xi, f)$ depending on a number ξ and a polynomial f. In fact we are able to prove that *any* deformation of the coordinate algebra of a Kleinian singularity of type D_n is isomorphic to some $T(\xi, f)$. The next problem was to prove that in the modular case, and under suitable conditions on ξ, f, the centre of $T(\xi, f)$ is isomorphic to the coordinate algebra of a type D Kleinian singularity, and that the corresponding no-cycle algebra one-cycle algebra occur as certain central reductions of $T(\xi, f)$. Unfortunately, the centre of $T(\xi, f)$

is harder to exhibit than the centre of $T(v)$. As an indication of this we have that the generator of $Z(T(v))$ of lowest degree is $h^p - h$, where h is the generator of $t(v)$ of lowest degree. The corresponding element for $T(\xi, f)$ is

$$u^p + u - \tfrac{1}{2} + \tfrac{1}{2}(1-4u)^{\frac{p+1}{2}}.$$

In fact, except for various special cases, I have not succeeded in describing the centre of $T(\xi, f)$. This should not be too difficult, and I intend to return to the subject in the future.

Also in chapter 4 we describe the 'most natural' family $\mathcal{O}^\lambda$ of deformations of a Kleinian singularity, introdeced by Crawley-Boevey and Holland in [14]. Here λ can be thought of as being an element of the $\mathbb{C}$-span of the set I. We make some remarks concerning the relationship between the $\mathcal{O}^\lambda$ and the deformations $T(v)$, $T(\xi, f)$. In type A it is the case that the family $T(v)$ is 'essentially the same' as the family $T(v)$, in a sense which we make precise. I expect a similar result to hold in type D although no such theorem appears in the present dissertation. In fact this is a problem which has been holding me back (in the modular type A case, the conditions on v which give the required results about the centre of $T(v)$ correspond to straightforward conditions on λ. In type D I do not know how one should identify the set $\mathbb{C}I$ with the set of pairs (ξ, f)).

Finally we make some speculations concerning the role that the various objects considered might have to play in representation theory.

Chapter 3

One-cycle Algebras and No-cycle Algebras

3.1 Introduction.

In the first half of this chapter we shall define one-cycle algebras and no-cycle algebras as certain quotients of the preprojective algebras Π of quivers whose underlying graph is an affine Coxeter graph. We shall then expoit the Morita equivalence of Π with a certain skew group algebra to read off many nice properties of no-cycle algebras and, especially one-cycle algebras. For example the one-cycle algebra has dimension $2h^2$ where h is the corresponding Coxeter number, and we give very precise information on the Loewy structure of the indecomposable. We are also able to prove that one-cycle algebras satisfy a kind of Poincaré duality.

In the second half of the chapter we consider one-cycle algebras and no-cycle algebras in the individual types (with varying success for the different types). In type A we can easily describe the algebras as well as give a complete classification of the indecomposable modules. In type D we are able, with a lot of tedious computation, to describe the algebras as a quivers with relations. We are also able to prove that in this case no classification of indecomposable modules is possible because the algebras are of *wild representation type*. In type E we do little except make some speculations.

3.2 Poincaré Series of Preprojective Algebras.

Lemma 6.1 of [1] applies to give the following:

Theorem 3.2.1 *If Γ is an affine Coxeter graph, then Π is a prime ring.*

It follows easily that if Γ is as in the theorem and $i \in I$ then the canonical homomorphism $Z(\Pi) \to e_i\Pi e_i$ is injective. Indeed if $\zeta \in Z(\Pi)$ satisfies $e_i\zeta e_i = 0$, then

$$\zeta\Pi e_i = \zeta\Pi e_i^2 = \Pi e_i\zeta e_i = 0.$$

Thus $\zeta = 0$ since Π is prime. Note that this result is false for the *non-affine* Coxeter graph of type A_3; a path of length 2 beginning and ending at the middle vertex is central, but has no component at the outer two vertices.

Let $A = (a_{ij})$ denote the adjacency matrix of Γ, so that if $i \neq j$ then a_{ij} is the number of edges connecting i, j, while a_{ii} is *twice* the number of loops at i. Define the *quantum Cartan matrix* C by

$$C = I - qA + q^2 I,$$

where q is an indeterminate. Define further a matrix $P = (P_{ij}(q))$ by

$$P_{ij}(q) = \sum_{n \geq 0} \dim(e_i\Pi_n e_j)q^n.$$

Thus the (i, j) entry of P is a generating function giving dimensions of spaces of 'paths' in Π from i to j.

The following result is well known (at least in the case where there are no loops), but a direct argument is worthwhile.

Theorem 3.2.2 *If Γ is an affine Coxeter graph then $P = C^{-1}$.*

Proof. Let $I = (k\overline{Q})\gamma(k\overline{Q})$ be the ideal defining Π. Subscripts will be used to denote graded components. For any $n \geq 0$ and $i, j \in I$ write

$$e_i(k\overline{Q})_n e_j = e_i I_n e_j \oplus W(n,i,j).$$

Here $W(n,i,j)$ is an arbitrary vector space complement. Set

$$W_n = \bigoplus_{i,j \in I} W(n,i,j).$$

The key observation of the proof is that if $n \geq 2$ then

$$I_n = I_{n-1}(k\overline{Q})_1 \oplus W_{n-2}\gamma.$$

First we show that the sum is direct. Suppose $\omega \in W_{n-2}$ is such that $\omega\gamma \in I_{n-1}(k\overline{Q})_1$. Write

$$\omega\gamma = \sum_{\alpha \in Q_1} (\iota_\alpha \alpha + \iota_{\alpha^*}\alpha^*),$$

where $\iota_\alpha, \iota_{\alpha^*} \in I_{n-1}$. Since $\gamma = \sum_{\alpha \in Q_1}(\alpha\alpha^* - \alpha^*\alpha)$ it follows that $\omega\alpha = \iota_{\alpha^*}$ and $\omega\alpha^* = -\iota_\alpha$ for all $\alpha \in Q_1$. Thus for all arrows β in Q we have $\omega\beta \in I_{n-1}$. Thus there is some nonzero θ in Π such that $(\omega + I)\Pi\theta = 0$ (we may take θ to be an arrow). Since Π is a prime ring it follows that $\omega \in I$. Since ω is also in W_{n-2} we must have $\omega = 0$. Hence $\omega\gamma = 0$ as required.

Thus to complete the proof of the above observation we need to show that

$$I = \bigoplus_{n \geq 2} \left[I_{n-1}(k\overline{Q})_1 \oplus W_{n-2}\gamma \right].$$

The right hand side is certainly a right ideal of $k\overline{Q}$ which is contained in I and which contains γ. Thus we only need to see that it is a left ideal, which amounts to proving that if $n \geq 2$ then

$$(k\overline{Q})_1 W_{n-2}\gamma \subseteq I_n(k\overline{Q})_1 \oplus W_{n-1}\gamma.$$

Well, if β is an arrow in $\overline{Q}$ and $\omega \in W_{n-2}$ then certainly $\beta\omega \in (k\overline{Q})_{n-1}$. Write $\beta\omega = \iota + \omega'$, where $\iota \in I_{n-1}, \omega' \in W_{n-1}$. Then $\beta\omega\gamma = \iota\gamma + \omega'\gamma \in I_n(k\overline{Q})_1 +$

$W_{n-1}\gamma$. This completes the proof of the key observation. We now show how to complete the proof of the theorem. The statement is easily seen to be equivalent to saying that if $n \geq 2$ and $i, k \in I$ then

$$\dim(e_i \Pi_n e_k) = \big[\sum_{j \in I} \dim(e_i \Pi_{n-1} e_j) a_{jk}\big] - \dim(e_i \Pi_{n-2} e_k).$$

Obviously

$$\dim(e_i \Pi_n e_k) = \dim(e_i (k\overline{Q})_n e_k) - \dim(e_i I_n e_k).$$

But,

$$\dim(e_i (k\overline{Q})_n e_k) = \sum_{j \in I} \dim(e_i (k\overline{Q})_{n-1} e_j) a_{jk},$$

and

$$\dim(e_i I_n e_k) = \big[\sum_{j \in I} \dim(e_i I_{n-1} e_j) a_{jk}\big] + \dim(e_i W_{n-2} e_k),$$

by the key observation. Clearly the difference of these two expressions is exactly what we want. □

3.3 Some Combinatorics.

Let V, $M_0, \cdots, M_r$, I, Γ, $A = (a_{ij})$ all be as in chapter 1. Let $\chi_0, \cdots, \chi_r$ be the characters of $M_0, \cdots, M_r$. Let $\Pi = \Pi(\Gamma)$ and let $P = (P_{ij}(q))$ be the matrix of section 3.2. Several different characterizations of this matrix exist.

Theorem 3.3.1 *Let $n \geq 0$ and let $i, j \in I$. Then the following numbers are all equal.*

1. $\dim(e_i \Pi_n e_j)$,
2. *The coefficient of q^n in the (i,j)-entry of $(I - qA + q^2 I)^{-1}$,*
3. $(M_i \otimes S^n(V), M_j)_G$,
4. *The coefficient of q^n in the series*

$$\frac{1}{|G|} \sum_{g \in G} \frac{\chi_i(g)\chi_j(g^{-1})}{\det(I - qg)}.$$

The equality of 1. and 2. follows from Theorem 2. The equality of 3. and 4. follows from a straightforward generalization of Molien's theorem (modify the proofs of [2], Proposition 2.5.1 and Theorem 2.5.3). The equality of 2. and 3. can be proved in a way reminiscent of the proof of Theorem 2, and so I omit the rather tedious details. Alternatively the equality of 1. and 3. is a direct consequence of an isomorphism of Π with $f[G * T(V)]f$, outlined in the introductory chapter. □

Recall the numbers a, b, h of the introductory chapter. It turns out that we can always write

$$P_{ij}(q) = \frac{S_{ij}(q)}{(1-q^a)(1-q^b)},$$

where the numerators $S_{ij}(q)$ are *polynomials* (This fact will become clear later, but a case-by-case argument can be given by finding the matrix P for each type. We shall do this at the end of the section). The polynomials $N_i(q) = S_{i0}(q)$ appear in [36]. In fact, later results of this thesis provide an affirmative answer to a suggestion made there that the symmetry of these polynomials should be explained by some kind of Poincaré duality. The (uniquely determined) polynomials $B_i(q)$ of degree $\leq \frac{h}{2}$ satisfying $B_i(q) + q^h B_i(q^{-1}) = N_i(q)$ appear in [33] (p. 1199).

We shall now see that the map $(i,j) \mapsto S_{ij}(q)$ provides an appropriate q-analogue of the map $i \mapsto d_i$, or more correctly of the map $(i,j) \mapsto 2d_i d_j$. If n is a nonnegative integer define the *quantum integer* $[n]$ by

$$[n] = \frac{1-q^n}{1-q}.$$

If n is a positive half integer define $[n]$ by the same formula. All expressions I will write involving this notation will give polynomials for all types. For $i, j \in I$ let $d(i,j)$ denote the length of the shortest path in $\overline{Q}$ from i to j, and $n(i,j)$ the number of such shortest paths. $n(i,j)$ is 1 unless i, j are opposite vertices in type A_{2m-1}, in which case it is 2.

Proposition 3.3.2 *The following hold for all* $i, j, k \in I$.

1. *$S_{ij}(q) = q^h S_{ij}(q^{-1})$.*

2. *$\deg S_{ij}(q) = h - d(i,j)$.*

3. *The coefficient of $q^{d(i,j)}$ in $S_{ij}(q)$ is $n(i,j)$.*

4. *$S_{ij}(q) = S_{ji}(q)$.*

5. *$(1+q^2)S_{ik}(q) - q\sum_{j\in I} a_{ij}S_{jk}(q) = \delta_{ik}(1-q^a)(1-q^b)$.*

6. *$S_{00}(q) = 1 + q^h$.*

7. *$S_{ij}(1) = 2d_i d_j$.*

8. *$\sum_{j\in I} d_j S_{ij}(q) = d_i[a][b]$.*

9. *$\sum_{i\in I} S_{0i}(q) = \frac{[a][b][\frac{h}{2}]}{[\frac{a}{2}][\frac{b}{2}]} = \frac{(1+q^{\frac{a}{2}})(1+q^{\frac{b}{2}})(1-q^{\frac{h}{2}})}{1-q}$.*

10. *$\sum_{i,j\in I} S_{ij}(q) = (r-1)[a][b] + 2\frac{[a][b][\frac{h}{2}]}{[\frac{a}{2}][\frac{b}{2}]}$.*

11. *$S_{ij}(q)$ can be written in the form*

$$n(i,j)q^{d(i,j)}\prod_s \frac{[n_s]}{[d_s]},$$

where $d_s | n_s$ for all s.

Proof. We clearly have

$$(1-qA+q^2I)^{-1}(1-q^a)(1-q^b) = q^h(1-q^{-1}A+q^{-2}I)^{-1}(1-q^{-a})(1-q^{-b}),$$

since $a+b=h+2$. Thus 1 follows from the second characterisation of the $P_{ij}(q)$ in theorem 3.3.1. Next observe that we obviously have

$$S_{ij}(q) = n(i,j)q^{d(i,j)} + O(q^{d(i,j)+1}).$$

Hence 2 and 3 both follow from 1. 4, 5 follow from the second characterization of the $P_{ij}(q)$ of Theorem 3. 6 follows from the well known fact $e_0\Pi e_0$ is isomorphic to a Kleinian singularity. 7 follows trivially from 5, 6. 8 follows from the third characterization of the $P_{ij}(q)$, since when left hand size is divided through by $(1-q^a)(1-q^b)$ the coefficient of q^n counts the dimension of $M_i \otimes S^n(V)$. This dimension is $d_i(n+1)$, and $n+1$ is the coefficient of q^n in the series $(1-q)^{-2}$. 9,10 and 11 can be checked on a case-by-case basis after the polynomials $S_{ij}(q)$ have been given. However, I do not know of a good reason why these hold. □

We will now explicitly give the matrix $S = (S_{ij}(q))$ for each type. In each case it is a simple matter to check that the matrix S satisfies the required equation.

Type A_{n-1}. In this case we simply have

$$S_{ij}(q) = q^{d(i,j)} + q^{n-d(i,j)}.$$

Type D_n. Label the vertices of Γ as follows:

$$\begin{array}{ccccccccc} 0 & & & & & & & & y \\ & 1 & 2 & 3 & \cdots & n-4 & & n-3 & \\ x & & & & & & & & z \end{array}$$

Taking graph automorphisms and the fact that S is symmetric into account, S can be determined from the following entries:

$$\begin{array}{rcll} S_{00}(q) & = & 1+q^{2n-2} & \\ S_{0x}(q) & = & q^2(1+q^{2n-6}) & \\ S_{0y}(q) & = & q^{n-2}(1+q^2) & \\ S_{0i}(q) & = & q^i(1+q^2)(1+q^{2n-2i-4}) & (1 \leq i \leq n-3) \\ S_{ij}(q) & = & q^{j-i}(1+q^2)(1+q^{2i})(1+q^{2n-2j-4}) & (1 \leq i \leq j \leq n-3). \end{array}$$

Type E_6. Label (and order) the vertices as follows:

$$\begin{array}{ccccc} & & 6 & & \\ & & 5 & & \\ 0 & 1 & 2 & 3 & 4 \end{array}$$

With respect to this ordering, the matrix S is as follows:

$$\begin{bmatrix} \frac{[24]}{[12]} & q\frac{[8][12]}{[4][6]} & q^2\frac{[6][8]}{[2][4]} & q^3\frac{[8]}{[2]} & q^4\frac{[8]}{[4]} & * & * \\ & & & & & & \\ * & \frac{[8][12]}{[2][6]} & q\frac{[6][8]}{[2][2]} & q^2\frac{[4][8]}{[2][2]} & * & * & * \\ & & & & & & \\ * & * & \frac{[6][6][8]}{[2][2][4]} & * & * & * & * \\ & & & & & & \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{bmatrix}$$

All entries marked $*$ can be determined from the nine given entries using graph automorphisms and the fact that S is symmetric.

Type E_7. Label the vertices as follows:

$$\begin{array}{ccccccc} & & & 7 & & & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

Then S is:

$$\begin{bmatrix}
\frac{[36]}{[18]} & q\frac{[20][12]}{[10][6]} & q^2\frac{[12][12]}{[4][6]} & q^3\frac{[8][12]}{[2][6]} & q^4\frac{[12]}{[2]} & q^5\frac{[4][12]}{[2][6]} & q^6\frac{[12]}{[6]} & q^4\frac{[8][12]}{[4][6]} \\
* & \frac{[4][12][20]}{[2][6][10]} & q\frac{[12][12]}{[2][6]} & q^2\frac{[4][8][12]}{[2][2][6]} & q^3\frac{[4][12]}{[2][2]} & q^4\frac{[4][4][12]}{[2][2][6]} & * & q^3\frac{[8][12]}{[2][6]} \\
* & * & \frac{[12][12]}{[2][4]} & q\frac{[8][12]}{[2][2]} & q^2\frac{[6][12]}{[2][2]} & * & * & q^2\frac{[8][12]}{[2][4]} \\
* & * & * & \frac{[8][8][12]}{[2][2][6]} & * & * & * & q\frac{[8][8][12]}{[2][4][6]} \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & \frac{[12][16]}{[6][4]}
\end{bmatrix}$$

Type E_8. Label the vertices as follows:

$$\begin{array}{cccccccc}
 & & & & & 8 & & \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}$$

In this case the matrix S looks rather formidable so we give a simple description (similar descriptions exist for other types). To any vertex i associate a rational function $S_i^L(q)$ as follows:

$$\begin{array}{cccccccc}
 & & & & & q^{-4}\frac{[12]}{[4]} & & \\
1 & q^{-1}\frac{[4]}{[2]} & q^{-2}\frac{[6]}{[2]} & q^{-3}\frac{[8]}{[2]} & q^{-4}\frac{[10]}{[2]} & q^{-5}\frac{[12]}{[2]} & q^{-6}\frac{[16]}{[4]} & q^{-7}\frac{[28]}{[14]}.
\end{array}$$

Similarly define polynomials $S_i^R(q)$ by:

$$\begin{array}{cccccccc} & & & & & q^6\frac{[12][20]}{[4][10]} & & \\ \frac{[60]}{[30]} & q\frac{[20][36]}{[10][18]} & q^2\frac{[24][20]}{[8][10]} & q^3\frac{[12][16][20]}{[6][8][10]} & q^4\frac{[12][20]}{[6][4]} & q^5\frac{[12][20]}{[2][10]} & q^6\frac{[4][12][20]}{[2][6][10]} & q^7\frac{[12][20]}{[6][10]}. \end{array}$$

Now if $i, j \in I$ with $(i, j) \neq (8, 8)$, and with i not lying further right than j, we have

$$S_{ij}(q) = S_{ji}(q) = S_i^L(q)S_j^R(q).$$

This determines all entries of S except $S_{88}(q) = \frac{[12][18][20]}{[4][6][10]}$.

3.4 Invariants, One-cycle Algebras and No-cycle Algebras.

Recall the G-invariant polynomials A, B, H introduced in the introductory chapter. Define $\Lambda := \mathbb{C}[A, B]$. Note that A,B are algebraically independent so Λ is a polynomial algebra. Define a group $\tilde{G}$ by

$$\tilde{G} = \{g \in GL(V) : g \text{ fixes } \Lambda \text{ elementwise } \}.$$

Lemma 3.4.1

1. $|\tilde{G} : G| = 2$.
2. *$\tilde{G}$ is generated by G and* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
3. $\Lambda = S(V)^{\tilde{G}}$.

Proof. The first part is well known. The other parts follow immediately from the form of the polynomials A, B, H. $\square$

Since Λ is a polynomial algebra it follows that $\tilde{G}$ is a *pseudo-reflection group* (see [49]). Many of the results which follow are obtained by systematically exploiting this fact.

Recall from section 1.1.5 that there is an isomorphism between Π and $f[G * S(V)]f$. We shall assume from now on that such an isomorphism has been fixed and, in particular, regard Λ as being a subalgebra of $Z(\Pi)$. (The shrewd reader will be aware that there are graded automorphisms of the algebra $\mathbb{C}[A, B, H]/(H^2 - B^2 + 4A^n)$ under which the algebra $\mathbb{C}[A, B]$ is *not* invariant (for example $(A, B, H) \mapsto (A, iH, iB)$), so in type A the subalgebra Λ of Π is not specified by simply saying that an isomorphism between Π and $f[G * S(V)]f$ exists - one has to specify central elements A, B of Π (we shall do this in section 3.6). In the meantime the following arguments hold for any choice of Λ, so this is not a major problem. This problem does not arise in types D, E because in these cases we have $a, b < h$ so Λ is merely the subalgebra of Π generated by all central elements of degrees a and b.)

Theorem 3.4.2 *Each of the spaces $e_i\Pi e_j$ is a graded free Λ-module.*

Proof. Since $\tilde{G}$ is a pseudo-reflection group, the Shepherd-Todd thoerem (see [49]) implies that $S(V)$ is a graded free Λ-module. The same is therefore true of the skew group algebra since each space $g \otimes S(V)$ is clearly isomorphic as a graded Λ-module to $S(V)$. Thus Π is a graded projective Λ-module. (To see this note that the condition that Π is a graded projective Λ-module is equivalent to the condition that any progenerator of the category of Π modules is a projective Λ module. A similar statement holds for the skew group algebra. Then use the fact that the two categories are equivalent as graded Λ-categories.) Hence each summand $e_i\Pi e_j$ is a projective Λ-module and hence a graded free Λ-module since Λ is a polynomial algebra. $\square$

We can now give the crucial definition.

Definition 3.4.3 *Define the* one-cycle algebra Ω *by*

$$\Omega = \Pi \otimes_\Lambda \mathbb{C} = \Pi/\Lambda^+\Pi.$$

Here $\Lambda^+ = \oplus_{n>0}\Lambda_n$*, and we are identifying* $\mathbb{C}$ *with* Λ/Λ^+ *and regarding it as a* Λ*-module.*

Note that we have

$$\Omega = f[G * S(V)_{\tilde{G}}]f.$$

Here $S(V)_{\tilde{G}}$ denotes the *coinvariant* algebra $S(V) \otimes_\Lambda \mathbb{C}$.

Proposition 3.4.4

1. $\dim \Omega = 2h^2$
2. *The Poincaré polynomial of each space* $e_i\Omega e_j$ *is* $S_{ij}(q)$.
3. *The Poincaré polynomial of* Ω *is*

$$(r-1)\frac{(1-q^a)(1-q^b)}{(1-q)^2} + 2\frac{(1+q^{\frac{a}{2}})(1+q^{\frac{b}{2}})(1-q^{\frac{h}{2}})}{1-q}.$$

Proof. By Proposition 3.3.2 it suffices to prove 2, but this follows trivially from Theorems 2 and 4, the definitions of Λ and the $S_{ij}(q)$, and the fact that Λ has Poincaré series $\frac{1}{(1-q^a)(1-q^b)}$. □

It is known that the image of H spans the degree h part of $S(V)_{\tilde{G}}$. Therefore we have

$$S(V)_G = S(V)_{\tilde{G}}/[S(V)_{\tilde{G}}]_h.$$

Definition 3.4.5 *Define the* no-cycle algebra *to be* Ω/Ω_h.

It follows from what has been said that we have the following interpretations of the no-cycle algebra:

$$\Omega/\Omega_h = \Pi \otimes_{Z(\Pi)} \mathbb{C} = \Pi/Z(\Pi)^+\Pi = f[G * S(V)_G]f.$$

By Proposition 3.3.2 it is clear that the dimension of the no-cycle algebra is $2h^2 - r - 1$.

We can combine proposition 3.3.2 and the combinatorics of section 3.3 to find the Loewy structure of the indecomposable projective Ω-modules $e_i\Omega$. The radical series of any right Ω-module M is given by

$$\mathsf{rad}^s M = M\mathsf{rad}^s\Omega.$$

But clearly

$$\mathsf{rad}^s\Omega = \bigoplus_{t \geq s} \Omega_s.$$

Thus

$$\begin{aligned} \frac{\mathsf{rad}^s(e_i\Omega)}{\mathsf{rad}^{s+1}(e_i\Omega)} &\cong \bigoplus_j \mathsf{dim}(e_i\Omega_s e_j)k_j \\ &= \bigoplus_j (\text{coeff. of } q^s \text{ in } S_{ij}(q))k_j. \end{aligned}$$

For example in type D_4 our graph has five vertices which we call 0, x, y, z, 1. 1 is the central vertex. Then

$$S_{11} = 1 + 3q^2 + 3q^4 + q^6.$$

$$S_{10} = S_{1x} = S_{1y} = S_{1z} = q + 2q^3 + q^5.$$

Thus the Loewy structure of $e_1\Omega$ is (in obvious notation)

$$\begin{array}{c} 1 \\ 0\ x\ y\ z \\ 1\ 1\ 1 \\ 0\ 0\ x\ x\ y\ y\ z\ z \\ 1\ 1\ 1 \end{array}$$

$$0 \; x \; y \; z$$

$$1$$

We see easily that

$$|e_i\Omega : k_j| = S_{ij}(1).$$

But $S_{ij}(1) = 2d_id_j$ by proposition 3.3.2. Thus the Cartan matrix of Ω is $(2d_id_j)$, a singular matrix.

On the other hand the Cartan matrix of the no-cycle algebra is obviously $(2d_id_j - \delta_{ij})$. A simple calculation shows that

$$\det(2d_id_j - \delta_{ij}) = (-1)^r(1 + 2\sum_{i \neq 0} d_i^2).$$

Thus using $\sum d_i^2 = |G| = ab/2$ we see that the determinant of the Cartan matrix of the no-cycle algebra algebra is simply

$$(-1)^r(ab - 1).$$

We end this section with a nice property of one-cycle algebras.

Proposition 3.4.6 *Let A be any finite-dimensional algebra which is Morita equivalent to Ω. Then $\dim A = 2M^2$, where $M = \sum_{i \in I} d_i \dim S_i$, S_i being the irreducible module corresponding to the vertex i.*

Proof. Let S_i $(i \in I)$ be the irreducible A-modules, and let P_i be the projective cover of S_i. We get

$$\begin{aligned} \dim A &= \sum_i \dim S_i \dim P_i \\ &= \sum_i \dim S_i \sum_j |P_i : S_j| \dim S_j \\ &= \sum_i \dim S_i \sum_j 2d_id_j \dim S_j \\ &= 2M^2. \end{aligned}$$

The first equality is a general property of finite-dimensional algebras. (see [3]). □

3.5 Poincaré Duality.

Lemma 3.5.1 *Suppose $A = \bigoplus_{t\geq 0} A_t$ is a graded algebra.*

1) Any central idempotent of A necessarily lies in A_0.

2) Suppose that the primitive central orthogonal idempotents of A_0 are $e_1, \cdots, e_s$. Define a graph Δ as follows:

$$\Delta_0 = \{1, \cdots, s\}.$$

$$\Delta_1 = \{(i,j) : i \neq j \textit{ and } e_iAe_j + e_jAe_i \neq 0.\}.$$

For each connected component $C \subseteq \Delta_0$ of Δ set

$$e_C = \sum_{i \in C} e_i.$$

Then the e_C are precisely the primitive central orthogonal idempotents of A.

Proof. 1) Let $e = \sum_{t\geq 0} e_t$ be a central idempotent. It is clear that e_0 is central. Now from $e^2 = e$ we deduce

$$e_0^2 = e_0,\ \ 2e_0e_1 = e_1,\ \ 2e_0e_2 + e_1^2 = e_2, \text{ etc.}$$

Now assume inductively that $t \geq 1$ and that $e_1 = \cdots = e_{t-1} = 0$. Then we clearly get $2e_0e_t = e_t$. Multiplying both sides by e_0 we deduce $e_0e_t = 0$. Hence $e_t = 2e_0e_t = 0$.

2) Obviously the e_C are orthogonal idempotents of A. I claim that each e_C is central. Indeed, let $a \in e_iAe_j$. Then

$$e_Ca = \delta_{i\in C}a = \delta_{j\in C}a = ae_C.$$

The central equality holds because if i, j lie in different connected components then $a = 0$.

Now, to prove that the e_C are primitive it suffices to see that any central idempotent e is a sum of some of the e_C. Well such an e must lie in A_0 by the first part, so must be of the form

$$e = \sum_{i \in T} e_i,$$

where $T \subseteq \Delta_0$. Now suppose $i \in T$ and $(i, j) \in \Delta_1$. The proof will be complete if we can show $j \in T$. Without loss we of generality we may assume $e_i A e_j \neq 0$, so let $a \in e_i A e_j - 0$. Then

$$\delta_{j \in T} a = ae = ea = a \neq 0$$

Thus $j \in T$ as required. □

Definition 3.5.2 *Let $A = \bigoplus_{t \geq 0} A_t$ be a finite-dimensional graded algebra. A is said to be a* generalized Poincaré duality algebra of depth n *if the following conditions hold.*

1) A_0 is isomorphic to a direct sum of copies of k.

2) $n = max\{t : A_t \neq 0\}$.

3) There exists a linear map $\eta : A_n \mapsto k$ such that for any $t = 0, \cdots, n$ the bilinear form

$$(\cdot, \cdot)_t : A_t \times A_{n-t} \mapsto k$$

(multiplication followed by η) is nondegenerate.

4) $(\alpha, \beta)_t = (\beta, \alpha)_{n-t}$ whenever the two sides make sense.

If, in addition, A is commutative and we have $A_0 = k$, then we simply call A a Poincaré duality algebra.

Proposition 3.5.3 *Let (A, η) be a generalized Poincare duality algebra of depth n, and let Δ be the graph of lemma 3.5.1. Then*

1) A is a symmetric algebra.

2) $dim(e_i A_n e_j) = \delta_{ij}$ for all i, j.

3) If A is indecomposable and η' is another map giving A the structure of a generalized Poincaré duality algebra, then η' is a nonzero scalar multiple of η.

Proof.

1) The bilinear form $A \times A \mapsto k$ given by

$$(\alpha, \beta) = \sum_{t=0}^{n} (\alpha_t, \beta_{n-t})_t$$

is obviously nondegenerate, symmetric and associative.

2) Fix $i \in \Delta_0$. Now if $j \neq i$ then obviously $(e_i, e_j A_n)_0 = 0$ since $e_i e_j = 0$. Also $(e_i, A_n e_j)_0 = (A_n e_j, e_i)_n = 0$. Thus, since $(e_i, A_n)_0 \neq 0$, $e_i A_n e_i$ must be nonzero. But since $\dim A_n = \dim A_0$ (by the nondegeneracy of $(.,.)_0$), the result follows.

3) Fix $i \in \Delta_0$. Now, by the argument of 2) we know $e_i A_n e_i$ is one dimensional and $\eta|_{e_i A_n e_i}, \eta'|_{e_i A_n e_i}$ are nonzero. Without loss of generality we may assume that

$$\eta|_{e_i A_n e_i} = \eta'|_{e_i A_n e_i}.$$

Since A is indecomposable it follows from part 2) of the above lemma that Δ is connected. Thus it suffices to prove that if j is such that $(i, j) \in \Delta_1$, then

$$\eta|_{e_j A_n e_j} = \eta'|_{e_j A_n e_j}.$$

Assume without loss that $e_i A e_j \neq 0$. Let $\alpha \in e_i A e_j - 0$. We may as well assume α is homogeneous. By nondegeneracy there exists a homogeneous $\beta \in e_j A e_i$ such that $(\beta, \alpha) \neq 0$. Thus $\beta\alpha$ must span $e_j A_n e_j$. But now

$$(\beta\alpha)\eta = (\alpha\beta)\eta = (\alpha\beta)\eta' = (\beta\alpha)\eta'.$$

The result follows. □

Theorem 3.5.4 *The one-cycle algebra Ω is a generalized Poincaré duality algebra of depth h.*

Proof. Recall that

$$\Omega = f[G * S(V)_{\tilde{G}}]f,$$

so we will regard Ω as being a subspace of the skew coinvariant algebra. Now, under the hypotheses of the Shephard-Todd theorem, the corresponding coinvariant algebra is a Poincaré duality algebra (see [49]). Thus $S(V)_{\tilde{G}}$ is a Poincaré duality algebra of depth h. Let

$$\eta_0 : [S(V)_{\tilde{G}}]_h \to k$$

be any vectorspace isomorphism. Notice that η_0 is invariant under G since $[S(V)_{\tilde{G}}]_h$ is the image of kH and H is G-invariant. Define a linear map

$$\eta_1 : [G * S(V)_{\tilde{G}}]_h \to k$$

by

$$(g\alpha)\eta_1 = \delta_{ge}(\alpha\eta_0) \text{ for all } g \in G, \alpha \in [S(V)_{\tilde{G}}]_h.$$

Finally define $\eta : \Omega_h \to k$ by restricting η_1 to Ω_h.

We now check the conditions for η to make Ω into a generalized Poincaré duality algebra. To prove that $(\alpha\beta)\eta = (\beta\alpha)\eta$ for all $\alpha \in \Omega_t, \beta \in \Omega_{h-t}$ it suffices to prove the corresponding property for η_1. So let $x, y \in G$ and $\alpha \in [S(V)_{\tilde{G}}]_t, \beta \in [S(V)_{\tilde{G}}]_{h-t}$. We have to see that

$$[(x\alpha)(y\beta)]\eta_1 = [(y\beta)(x\alpha)]\eta_1.$$

Well,

$$\begin{aligned} [(x\alpha)(y\beta)]\eta_1 &= (xy\alpha^y\beta)\eta_1 \\ &= \delta_{xy=e}(\alpha^y\beta)\eta_0 \end{aligned}$$

Similarly $[(y\beta)(x\alpha)]\eta_1 = \delta_{yx=e}(\beta^x\alpha)\eta_0$. Thus we may as well assume $y = x^{-1}$. Then

$$\begin{aligned} (\alpha^y\beta)\eta_0 &= (\alpha^{x^{-1}}\beta)\eta_0 \\ &= ((\alpha\beta^x)^{x^{-1}})\eta_0 \\ &= (\alpha\beta^x)\eta_0 \\ &= (\beta^x\alpha)\eta_0 \end{aligned}$$

The result follows. Finally we need to prove the nondegeneracy part. So let $\sum_{g\in G} g\alpha_g$ be a nonzero element of Ω_t. Let $x \in G$ be such that $\alpha_x \neq 0$. By the nondegeneracy property for η_0 we may pick $\beta \in [S(V)_{\tilde{G}}]_{h-t}$ such that $(\alpha_x\beta)\eta_0 \neq 0$. Then

$$\begin{aligned}
[(\sum g\alpha_g)fx^{-1}\beta^{x^{-1}}f]\eta &= [f(\sum g\alpha_g)fx^{-1}\beta^{x^{-1}}]\eta_1 \\
&= [\sum g\alpha_g x^{-1}\beta^{x^{-1}}]\eta_1 \\
&= (\alpha_x^{x^{-1}}\beta^{x^{-1}})\eta_0 \\
&= (\alpha_x\beta)\eta_0 \\
&\neq 0
\end{aligned}$$

The result follows, and so the proof is complete. □

It follows that Ω is a self-injective algebra, which means that Ω is injective as a right module over itself. The following easy result is therefore applicable in our case.

Proposition 3.5.5 *Suppose A is a self-injective Artinian algebra. Then every non-projective indecomposable right A-module M is annihilated by soc(A).*

Proof Write

$$A = P_1 \oplus \cdots \oplus P_t,$$

a direct sum of indecomposable right A-modules. Each P_i is injective, so has an irreducible socle. Fix $i \in 1, \cdots, t$ and $m \in M$. The map $P_i \to M$ given by $a \mapsto ma$ (for all $a \in P_i$) is a homomorphism of right A-modules. Hence the kernel is either zero or contains soc(P_i). The former implies that P_i is a submodule of M, and hence a summand of M, being an injective module. But this is impossible since M is indecomposable and non-projective. It follows that m is annihilated by soc(P_i). Hence result. □

Since we obviously have soc(Ω) $= \Omega_h$, we get the following.

Theorem 3.5.6 *Let M be an indecomposable right Ω-module. Then either $M \cong e_i\Omega$ for some i, or M is a module for the no-cycle algebra Ω/Ω_h.* $\square$

3.6 One-cycle Algebras over Other Fields

In the sequel we shall sometimes make use of one-cycle algebras defined over k, and we need to make a few remarks about how to interpret this. Everything that has been said applies as long as the matrix group G makes sense (ie the roots of unity required exist), and p does not divide the its order. In types D and E is defined as the quotient of the preprojective algebra obtained by killing all homogeneous central elements of degrees a, b, while in type A our explicit presentation can be taken as the definition.

3.7 One-cycle algebras in Type A

The previous few subsections have been very theoretical in nature, and in this subsection we take a detailed look at the one-cycle algebras in type A. So fix $n \geq 2$. We take Q to be the quiver with n vertices labelled by the elements of $\mathbb{Z}/n\mathbb{Z}$ and n arrows $\alpha_i : i \to i+1$. The double quiver $\overline{Q}$ then has n additional arrows $\beta_i : i+1 \to i$. The preprojective relations then take the form

$$\alpha_i\beta_i = \beta_{i-1}\alpha_{i-1}.$$

Now it is easy to write down elements which generate the centre of Π. Set:

$$A := \sum_i \alpha_i\beta_i,$$

$$B := \sum_i (\alpha_i\alpha_{i+1}\cdots\alpha_{i-1} + \beta_{i-1}\beta_{i-2}\cdots\beta_i),$$

$$H := \sum_i (\alpha_i\alpha_{i+1}\cdots\alpha_{i-1} - \beta_{i-1}\beta_{i-2}\cdots\beta_i).$$

Notice that these elements do indeed satisfy the required relation, namely $H^2 = B^2 - 4A^n$.

Recall the problems (unique to type A) mentioned in section 3.4. *From now on when we consider* Λ *as a subalgebra of* Π *we mean the algebra generated by the* A*,* B *above. The one-cycle algebra is interpreted as the quotient of* Π *obtained by killing these elements.*

We therefore have a concrete description of the one-cycle algebra Ω - it is the quotient of $\mathbb{C}\overline{Q}$ by the relations

$$\alpha_i\beta_i = \beta_i\alpha_i = 0,$$

$$\alpha_i\alpha_{i+1}\cdots\alpha_{i-1} + \beta_{i-1}\beta_{i-2}\cdots\beta_i = 0.$$

An even simpler description exists for the no-cycle algebra - it is merely the quotient of $\mathbb{C}\overline{Q}$ obtained by killing all nontrivial paths which begin and end at the same vertex.

Recall the discussion of section 3.4 on the Loewy structure of the indecomposable projective Ω-modules $e_i\Omega$. In the present case the Loewy structure is, in obvious notation

$$\begin{array}{c} i \\ i-1 \quad i+1 \\ i-2 \quad i+2 \\ \vdots \\ \vdots \\ i+1 \quad i-1 \\ i \end{array}$$

(It should be mentioned that in type A_{n-1}, Ω is a *Brauer graph algebra* for the graph with two vertices, each of multiplicity one, connected by n edges. A good reference on Brauer graph algebras is *B*, and there is no value in giving further details here as this observation is just another way of expressing the description just given of the indecomposable projective modules.)

Theorem 3.5.6 implies that Ω and $\Omega/\mathrm{soc}(\Omega)$ have the same representation type. In type A it is shown in [21] that the latter has tame representation type. It is, however, instructive to give a direct argument for the one-cycle algebra (especially as this approach is independent of the theory of polynomial invariants of finite groups). We now present such an approach.

We now introduce an intermediate algebra. Define

$$W := \frac{k\overline{Q}}{k\overline{Q}(\alpha_i\beta_i, \beta_i\alpha_i, \alpha_i\alpha_{i+1}\cdots\alpha_i, \beta_i\beta_{i-1}\cdots\beta_i)k\overline{Q}}.$$

W is a *string algebra* [12] so there is a simple combinatorial procedure for determining its indecomposable modules. I shall use a different language from [12], but it is clear that the result is the same.

A *string* consists of the following:

· A nonempty, finite set N of *nodes*.

· A total ordering on N. I will display a string diagrammatically with the nodes in a line increasing from left to right.

· A labelling on each node by an element of $\mathbb{Z}/n\mathbb{Z}$.

· An arrow, either $\rightarrow$ or $\leftarrow$, between each pair of adjacent nodes.

This should all satisfy:

$S1)$ The labels on the nodes increase by one each time as one reads from left to right.

$S2)$ No $n+1$ consecutive arrows point in the same direction.

Thus, for example if $n=3$ then

$$2 \leftarrow 0 \rightarrow 1 \leftarrow 2 \leftarrow 0 \leftarrow 1$$

is a string. The *length* of a string is the number of arrows it contains. Note that there is a string of length 0 correponding to each element of $\mathbb{Z}/n\mathbb{Z}$.

Now suppose that S is a string of length l with labels $a_0, \cdots, a_l$ (from left to right). I shall now describe a *string module* $M = M(S)$. Firstly, as a vector space, M has basis consisting of the symbols $m_0, \cdots m_l$. Next, m_i is concentrated at the vertex e_i for all i. Now suppose $0 \leq i < l$. If the arrow between a_i and a_{i+1} is $\rightarrow$ then set $m_i \alpha_{a_i} = m_{i+1}$. Otherwise set $m_{i+1} \beta_{a_i} = m_i$. Finally if $0 \leq i < l$, $\gamma \in \{\alpha_j, \beta_j\}$ and $m_i \gamma$ has not been defined above then set it to be zero. It is clear that this does indeed define a W-module

Having considered strings and string modules we now consider *bands* and *band modules*. A band consists of the following:

· A nonempty finite set N of nodes,

· A circular ordering on N. This is a map $N \rightarrow N$ which determines for each node ω the 'next' node ω'. One can represent a band diagrammatically with the nodes in a circle with ω' being the next node around from ω reading clockwise.

· A labelling of each node by an element of $\mathbb{Z}/n\mathbb{Z}$.

· An arrow between each node and the next, pointing in one of the two possible directions.

This should all satisfy:

$B1)$ If a node ω is labelled i, then ω' is labelled $i+1$.

$B2)$ No $n+1$ consecutive arrows point in the same direction.

The *length* of a band is defined to be the number of arrows it contains. By $B1)$ the length of a band is a multiple of n. By $B2)$ the number of bands of length n is $2^n - 2$.

We now consider *powers* of bands and *reduced* bands. If B is a band of length l and $r \geq 1$ then define the rth power of B as follows. Choose a node in B, and break B at that node to form a string of length l whose first and last nodes are labelled with the same number. Then join r copies of this string together to form a band of length lr. This construction clearly does not depend on the node chosen. Call a band reduced if it is not a power of some band of smaller length.

Finally we can define band modules. Suppose B is a reduced band of length l, a_ω is the label on a vertex ω, V is a vector space and θ is an indecomposable automorphism of V. We shall construct a W-module $M = M(B, [\theta])$ which depends up to isomorphism on B and $[\theta]$, the similarity class of θ. Firstly, as a vector space we have

$$M = \bigoplus_{\omega \in N} V_\omega,$$

where each V_ω is just a copy of V. Now fix a node δ and chosse automorphisms θ_ω of V, one for each node ω such that

$$\theta_\delta \theta_{\delta'} \cdots \theta_{\delta^{(l-1)}} = \theta.$$

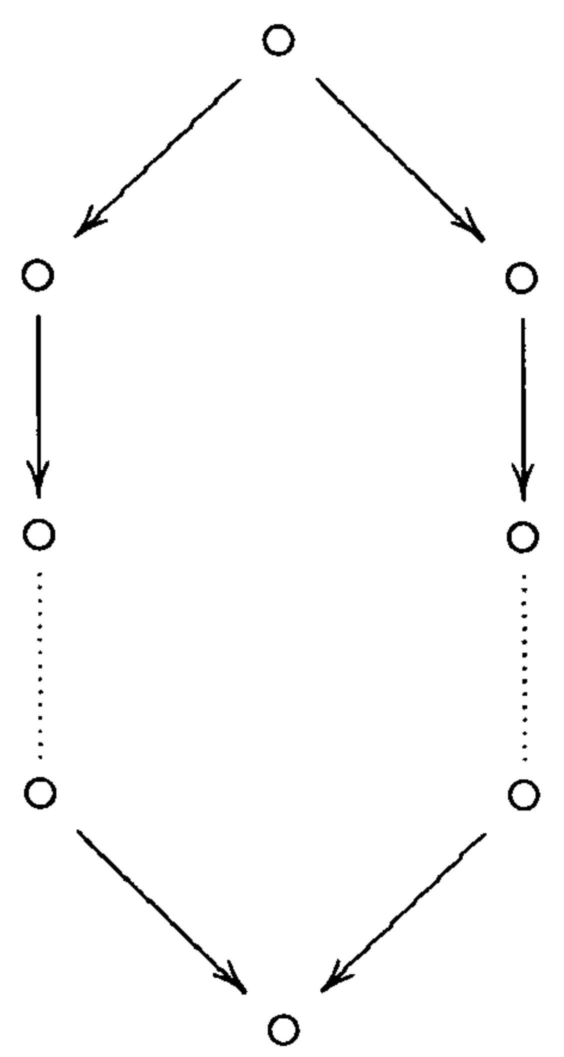
Next, V_ω is concentrated at a_ω. Now let ω be a node. If the arrow between ω and ω' points from ω to ω' then define the action of α_{a_ω} from V_ω to $V_{\omega'}$ to be θ_ω. If it points in the other direction, then define the action of β_{a_ω} from $V_{\omega'}$ to V_ω to be $\theta_{\omega^{-1}}$. Finally, if ω is a node, $\gamma \in \{\alpha_i, \beta_i\}$ and the action of γ on V_ω has not been defined then set it to be zero. It is clear that this does define a W-module which is independent of the choices made. We can now state the classification theorem of [12] for the special case of our algebra W.

Theorem 3.7.1 *The modules $M(S)$ and $M(B, [\theta])$ provide a complete set of pairwise nonisomorphic indecomposable right W-modules.* $\square$

Of course it is now an easy matter to find the classification of all indecomposable Ω-modules; one merely needs to find which of the modules of the above classification are killed by the elements

$$\alpha_i \alpha_{i+1} \cdots \alpha_{i+n-1} + \beta_{i-1} \beta_{i-2} \cdots \beta_{i-n}.$$

This clearly includes all modules corresponding to strings/bands with no $n+1$ arrows pointing in the same direction. It is equally clear that the only other possibilities are band modules corresponding to bands of the form



(n arrows on each side)

and with $\theta = -1 : k \to k$. These are easily seen to be the indecomposable projective Ω-modules. In summary we have:

Theorem 3.7.2 *The modules $M(S)$ and $M(B, [\theta])$ corresponding to strings/bands with no $n+1$ consecutive arrows pointing in the same direction, together with the indecomposable projective Ω-modules $e_i\Omega$ provide a complete set of pairwise indecomposable right Ω-modules.* $\square$.

3.8 Proof of Wildness in Type D

In this section and the next we will consider in some detail the preprojective algebras and one-cycle algebras in type D_n. So fix $n \geq 4$, let $m := n - 4$ and let Q be the quiver

$$\begin{tikzcd}
a \arrow[dr] & & & & & c \\
& 0 \arrow[r] & 1 \arrow[r, dotted, dash] & m-1 \arrow[r] & m \arrow[ur] \arrow[dr] & \\
b \arrow[ur] & & & & & d
\end{tikzcd}$$

Notice that this is NOT the labelling used in earlier sections and that 0 is not an affine vertex, contrary to our usual notation.

We shall see that Ω and Ω/Ω_h are of wild representation type, in contrast to the type A situation. The crucial argument is as follows.

Proposition 3.8.1 *Let J be the two-sided ideal of Π generated by all paths of length ≥ 4 beggining and ending at the same vertex. Then Π/J is of wild representation type.*

Proof. Let C denote the category of representations for the quiver with one vertex and two loops. The statement that Π/J is wild is equivalent to saying that there is a full subcategory D of mod$-(\Pi/J)$ and functors

$$F : C \to D$$

$$G : D \to C$$

such that FG is naturally isomorphic to the identity functor on C. Take D to be the full subcategory of mod-Π/J consisting of those representations

$$\begin{tikzcd}
M_a \arrow[dr, shift left, "\alpha_a"] & & & & & & M_c \arrow[dl, shift left, "\alpha_c^*"] \\
& M_0 \arrow[ul, shift left, "\alpha_a^*"] \arrow[dl, shift left, "\alpha_b^*"] \arrow[r, shift left, "\alpha_0"] & M_1 \arrow[l, shift left, "\alpha_0^*"] \arrow[r, phantom, "\cdots\cdots\cdots"] & M_{m-1} \arrow[r, shift left, "\alpha_{m-1}"] & M_m \arrow[l, shift left, "\alpha_{m-1}^*"] \arrow[urr, shift left, "\alpha_c"] \arrow[drr, shift left, "\alpha_d"] & & \\
M_b \arrow[ur, shift left, "\alpha_b"] & & & & & & M_d \arrow[ull, shift left, "\alpha_d^*"]
\end{tikzcd}$$

satisfying

- α_a is injective and α_a^* is surjective.
- imα_a = ker$\alpha_a^* =: U$.
- im$\alpha_b \subseteq U \subseteq$ kerα_b^*.
- $M_0 = \cdots = M_m =: M$ and $\alpha_0 = \cdots = \alpha_{m-1} = 1$.
- im$\alpha_c^* \subseteq U \subseteq$ kerα_c.

Our functor F takes the representation

$$\begin{tikzcd}
V \arrow[loop right, "\alpha"] \arrow[loop left, "\beta"]
\end{tikzcd}$$

to the representation

$$\begin{tikzcd}
V \arrow[dr, shift left, "(0\ 1)"] & & & & & & V \arrow[dl, shift left, "(0\ \beta)"] \\
& V \oplus V \arrow[ul, shift left, "\binom{1}{0}"] \arrow[dl, shift left, "\binom{1}{0}"] \arrow[r, shift left, "1"] & V \oplus V \arrow[l, shift left, "\begin{pmatrix} 0 & \alpha+1 \\ 0 & 0 \end{pmatrix}"] \arrow[r, phantom, "\cdots\cdots\cdots"] & V \oplus V \arrow[r, shift left, "1"] & V \oplus V \arrow[l, shift left, "\begin{pmatrix} 0 & \alpha+1 \\ 0 & 0 \end{pmatrix}"] \arrow[urr, shift left, "\binom{1}{0}"] \arrow[drr, shift left, "\binom{1}{0}"] & & \\
V \arrow[ur, shift left, "(0\ \alpha)"] & & & & & & V \arrow[ull, shift left, "(0\ 1+\alpha-\beta)"]
\end{tikzcd}$$

of $k\overline{Q}$

It is easy to see that the preprojective relations are satisfied. Next suppose that P is a path in $k\overline{Q}$ of length ≥ 4 beginning and ending at the same vertex. Clearly P must contain at least two leftward arrows (that is arrows of the form α^* for α an arrow in the original quiver Q. But it is a simple matter to check that any path of the form

$$\alpha_{i_1}^* \alpha_{i_2} \alpha_{i_3} \cdots \alpha_{i_{t-1}} \alpha_{i_t}^*$$

for $t \geq 2$ lies in the annihilator of our representation. Thus we do indeed have a representation of Π/J, which obviously lies in D. It is obvious how to deal with morphisms and easy to check that we do indeed have a functor $C \to D$.

Now we show how to construct a suitable functor G. Suppose $((M_i), (\alpha_i), (\alpha_i^*))$ is an object in D, and let U,M be as before. For $i \in \{a, b, c\}$ we have maps

$$M/U \to M_i \to U$$

induced by α_i and α_i^*. Let ξ_i denote the composite map $M/U \to U$. Note that ξ_a is an isomorphism. Now define the image of our object under G to be the representation

$$\begin{tikzcd} U \arrow[loop right, "\xi_a^{-1}\xi_b"] \arrow[loop left, "\xi_a^{-1}\xi_c"] \end{tikzcd} .$$

Given a morphism

$$\theta : ((M_i), (\alpha_i), (\alpha_i^*)) \to ((N_i), (\beta_i), (\beta_i^*))$$

in D it is clear that

$$(\mathrm{ker}\alpha_a^*)\theta \subseteq \mathrm{ker}\beta_a^*,$$

making it clear how to define θG. It is a simple matter to check functoriality.

Finally it is clear that $FG \cong 1_C$.□

Theorem 3.8.2 *The one-cycle algebra Ω and the no-cycle algebra Ω/Ω_h of type D are of wild representation type.*

Proof. Ω is obtained from Π by killing certain central homogeneous elements A,B of degrees $4, 2n-4 \geq 4$. Since A,B are central they are linear combinations of paths beginning and ending at the same vertex. Thus Ω has Π/J as a quotient algebra, so is wild. Similarly for the no-cycle algebra.□

We have been able to deduce the above theorem with limited information about the defining relations, which we consider next.

3.9 Explicit Relations in Type D

This section is devoted to exhibiting the subalgebra Λ of $Z(\Pi)$ explicitly, thus describing Ω and Ω/Ω_h as quivers with relations. We require $n \geq 5$ for the arguments. We deal with the case D_4 at the end.

Let $\Pi := \Pi(Q)$, and let Γ be the underlying graph of Q. If $e_1, \cdots e_t$ is a sequence of vertices of Γ with e_i connected to e_{i-1} for all $i \geq 1$ then let $p(e_1, \cdots, e_n)$ denote the path

$$e_1 \rightarrow e_2 \rightarrow \cdots \rightarrow e_t.$$

Also given *any* sequence of vertices $e_1, \cdots, e_t$ let $S(e_1, \cdots, e_t)$ denote the shortest path beginning at e_1, ending at e_n and visting the other vertices in the correct order.

Of course we could have taken Q to be any quiver with underlying graph Γ. Our choice gives us the following preprojective relations:

$$p(a,0,a) = p(b,0,b) = p(c,m,c) = p(d,m,d) = 0,$$

$$p(0,1,0) = p(0,a,0) + p(0,b,0),$$

$$p(m,m-1,m) = p(m,c,m) + p(m,d,m),$$

$$p(i,i-1,i) = p(i,i+1,i) \text{ for } 0 < i < m.$$

Our choice of Q was motivated by the fact that for any other choice we would have to keep track of sign changes in the arguments that follow. In particular we get a particularly neat form of the following lemma, which we use repeatedly.

Lemma 3.9.1 *(Alternating vertex principle) Let P be any path in $\overline{Q}$ which begins at the vertex a, which ends at the vertex $i \geq 0$ and which visits none of the vertices a, b, c, d. Then in Π we have*

$$P = p(a,0,b,0,a,0,b,\cdots,0,1,\cdots,i).$$

This result follows trivially from the preprojective relations. There are clearly similar results about paths beginning at b, c, d or paths ending at a, b, c, d. There are also simple obvious corollories. For example any path P of length divisible by 4 beginning at a and ending at b and not visiting c, d must be zero in Π. We will refer to all such results as the *alternating vertex principle.*

We will write G for the group of graph automorphisms of Γ. Since $n \geq 5$ this is a dihedral group of order 8. An element of G is determined by its effect on $\{a, b, c, d\}$ so we will regard G as being a subgroup of the symmetric group of the set $\{a, b, c, d\}$. In particular we will make use of the sign representation written $g \mapsto (-1)^g$. G acts on $k\overline{Q}$ (and on Π by algebra automorphisms via

$$p(e_1, \cdots, e_t)^g = p(e_1^g, \cdots, e_t^g).$$

We will also make use of the antiautomorphism $*$ of Π given by

$$p(e_1, \cdots, e_t)^* = p(e_t, \cdots, e_1).$$

We know that the subalgebra

$$\Lambda := \{\alpha \in Z(\Pi) : \alpha^* = \alpha\}$$

is a polynomial algebra in two homogeneous generators of degrees 4, $2n-4$. The aim of this section is to give explicit generators for Λ, thus exhibiting the one-cyle algebra

$$\Omega := \Pi \otimes_\Lambda k$$

as a quiver with relations.

We will frequently need to explain why two explicitly given elements of Π are different. One approach to this would be to find a Gröbner basis for Π. Unfortunately I cannot find a Gröbner basis using only finitely many rewrite rules. However, using the results of sections 3.2 and 3.3 we know the dimension of each space $e_i \Pi_t e_j$, and it is usually easy - using the alternating vertex principle - to write down a spanning set of each such space containing the right number of vectors. This method will be used repeatedly.

A generator of degree 4 is easy to describe.

Proposition 3.9.2 *The element A of Π_4 equal to*

$$\begin{aligned} S(a,b,a) + S(b,a,b) + S(c,d,c) + S(d,c,d) \\ + \sum_{i=0}^{m-1} p(i,i+1,i,i+1,i) \\ + p(m,m-1,m,m-1,m) \end{aligned}$$

is a nonzero element of Λ.

Proof A obviously satisfies $A^* = A$. Next observe that if $0 < i < m$ then $p(i,i+1,i,i+1,i) = p(i,i-1,i,i-1,i)$. It is therefore obvious that $A \in \Pi^G$.

Also A is nonzero since $S(a, b, a)$ is nonzero, by the argument indicated above. It therefore suffices to prove that A commutes with the arrows $p(a, 0)$ and $p(i, i+1)$ for $0 \leq i < m/2$. Firstly,

$$\begin{aligned} Ap(a,0) &= p(a,0,b,0,a,0) \\ &= p(a,0,b,0,1,0) \\ &= p(a,0,1,0,1,0) \\ &= p(a,0)A. \end{aligned}$$

Also,

$$\begin{aligned} Ap(i,i+1) &= p(i,i+1,i,i+1,i,i+1) \\ &= p(i,i+1,i+2,i+1,i+2,i+1) \\ &= p(i,i+1)A. \end{aligned}$$

This completes the proof. $\square$

To describe a generator of degree $2n-4$ is more difficult and requires dealing with the cases n is odd and n is even separately. The result is as follows.

Proposition 3.9.3 *Let $B \in \Pi_{2n-4}$ be defined as follows.*

· *If n is odd then let B be equal to*

$$\begin{aligned} &S(a,c,a) + S(b,d,b) + S(c,a,c) + S(d,b,d) \\ &+ \sum_{i \text{ even}} [S(i,a,c,i) - S(i,d,a,i)] \\ &+ \sum_{i \text{ odd}} [S(i,a,c,i) - S(i,c,b,i)]. \end{aligned}$$

· *If n is even then let B be equal to*

$$\begin{aligned} &S(a,c,a) + S(b,d,b) + S(c,a,c) + S(d,b,d) \\ &+ \sum_{i \text{ even}} [S(i,a,c,i) + S(i,c,a,i)] \end{aligned}$$

$$+\sum_{i\ odd}[S(i,a,c,i)+S(i,d,b,i)].$$

Then B is an element of Λ outside $k[A]$.

Proof. Throughout the proof we apply the altenating vertex principle repeatedly, without explicit mention.

Set

$$B_0 := (e_a+e_b+e_c+e_d)B,$$

$$B_1 := \sum_{i\text{ even}} e_iB,$$

$$B_{-1} := \sum_{i\text{ odd}} e_iB.$$

Now suppose that n is odd.

We first establist the claim that $B^g=(-1)^gB$ for all $g\in G$. B_0 is certainly fixed by $(ac)(bd)$. Also,

$$\begin{aligned}
B_0^{(ab)} &= S(b,c,b)+S(a,d,a)+S(c,b,c)+S(d,a,d)\\
&= -S(b,d,b)-S(a,c,a)-S(c,a,c)-S(d,b,d)\\
&= -B_0
\end{aligned}$$

Here we have applied the alternating vertex principle four times. For example by a preprojective relation we have

$$S(b,c,b)=S(b,m,m-1,m,b)-S(b,d,b),$$

but the term $S(b,m,m-1,m,b)$ is zero in Π. Note that this would be false if n were even. Since $(ac)(bd),(ab)$ generate G we get $B_0^g=(-1)^gB_0$.

Next,

$$\begin{aligned}
B_1^{(ab)} &= \sum_{i\text{ even}}[S(i,b,c,i)-S(i,d,b,i)]\\
&= \sum_{i\text{ even}}[S(i,0,1,0,c,i)-S(i,a,c,i)-S(i,d,0,1,0,i)+S(i,d,a,i)]
\end{aligned}$$

Notice that

$$S(i,0,1,0,c,i) = S(i,d,c,d,\cdots,c,i) = S(i,d,0,1,0,i),$$

by two applications of the alternating vertex principle. Hence $B_1^{(ab)} = -B_1$. Similarly one can check that $B_1^{(ac)(bd)} = B_{-1}$, $B_{-1}^{(ac)(bd)} = B_1$ and $B_{-1}^{(ab)} = -B_{-1}$. Thus it follows that $B^g = (-1)^g B$ for all g, thus establishing the claim.

Next we prove that $B^* = B$. Obviously $B_0^* = B_0$. Also

$$B_1^* = -B_1^{(cd)} = -(-1)^{(cd)} B_1 = B_1.$$

Similarly for B_{-1}. The result follows.

Next we prove that B is central. It suffices to prove that B commutes with $p(a,0)$ and with $p(i,i+1)$ for $0 \leq i < m/2$. Well,

$$\begin{aligned}
p(a,0)B &= S(a,0,a,c,0) - S(a,d,a,0) \\
&= -S(a,d,a,0) \\
&= S(a,c,a,0) - S(a,m,m-1,a,0) \\
&= S(a,c,a,0) \\
&= Bp(a,0)
\end{aligned}$$

Next, suppose $0 \leq i < m/2$ is *even*. Then

$$\begin{aligned}
p(i,i+1)B &= S(i,i+1,a,c,i+1) - S(i,c,b,i+1) \\
&= S(i,b,a,c,i+1) - S(i,c,b,i+1) \\
&= S(i,b,d,c,i+1) - S(i,c,b,i+1) \\
&= S(i,b,d,i,i+1) - S(i,c,b,i+1) \\
&= B^{(ab)(cd)} p(i,i+1) \\
&= Bp(i,i+1)
\end{aligned}$$

Essentially the same argument deals with the case i is odd.

Finally we want to see that $B \notin k[A]$. Since 4 does not divide $2n-4$ is suffices to prove that B is nonzero. This follows easily since $S(a,c,a)$ is nonzero, by the argument involving ite Poincaré series. This completes the proof in the case n is odd.

Now suppose that n is even. The proof has to be longer in this case since it is *not* true here that $B^g = (-1)^g B_1$ for $g \in G$.

We first prove that $B^* = B$. Certainly $B_0^* = B_0$ and $B_1^* = B_1$. Also

$$\begin{aligned}
B_{-1}^* &= \sum_{i \text{ odd}} [S(i,a,c,i) + S(i,d,b,i)] \\
&= \sum_{i \text{ odd}} [S(i,a,m,m-1,m,i) - S(i,a,d,i) + S(i,d,0,1,0,i) - S(i,d,a,i)] \\
&= \sum_{i \text{ odd}} [S(i,a,b,a,\cdots,a,i) - S(i,a,d,i) + S(i,d,c,d,\cdots,d,i) - S(i,d,a,i)],
\end{aligned}$$

which is obviously invariant under $*$. Thus we do indeed have $B^* = B$. Notice that $B^{(ac)(bd)} = B^* = B$. Thus to prove that B is central it suffices to prove that it commutes with each of $p(a,0)$, $p(b,0)$, $p(i,i+1)$ $(0 \le i < m,\ i$ even). Firstly

$$\begin{aligned}
p(a,0)B &= S(a,0,a,c,0) + S(a,c,a,0) \\
&= S(a,c,a,0) \\
&= Bp(a,0).
\end{aligned}$$

Next,

$$\begin{aligned}
p(b,0)B &= S(b,a,c,0) + S(b,c,a,0) \\
&= S(b,d,c,0) + S(b,m,m-1,m,a,0) - S(b,d,a,0) \\
&= S(b,d,0,1,0) - S(b,d,a,0) \\
&= S(b,d,b,0) \\
&= Bp(b,0).
\end{aligned}$$

Finally, suppose $0 \leq i < m$ is even. Then

$$\begin{aligned}
p(i,i+1)B &= S(i,i+1,a,c,i+1) + S(i,d,b,i+1) \\
&= S(i,i+1,0,1,0,c,i+1) - S(i,i+1,b,c,i+1) + S(i,d,b,i+1) \\
&= S(i,c,d,c,\cdots,c,i+1) - S(a,b,c,i+1) + S(i,d,b,i+1) \\
&= S(i,c,0,1,0,i+1) - S(i,a,d,c,i+1) + S(i,d,b,i+1) \\
&= S(i,c,a,i+1) + S(i,c,b,i+1) - S(i,a,d,c,i+1) + S(i,d,b,i+1) \\
&= S(i,c,a,i+1) + S(i,m,m-1,m,b,i+1) - S(i,a,d,c,i+1) \\
&= S(i,c,a,i+1) + S(i,a,b,a,\cdots,b,i+1) - S(i,a,d,i,i+1) \\
&= S(i,c,a,i+1) + S(i,a,m,m-1,m,i,i+1) - S(i,a,d,i,i+1) \\
&= S(i,c,a,i+1) + S(i,a,c,i,i+1) \\
&= Bp(i,i+1).
\end{aligned}$$

Hence $B \in \Lambda_{2n-4}$. To prove that $B \notin k[A]$ it suffices to prove that $S(a,c,a) \notin k[S(a,b,a,\cdots,a)]$. However, it is clear by the alternating vertex principle that any paths of length $2n-4$ beginning and ending at a and not visiting c or d is equal to $S(a,b,a,\cdots,a)$. It follows that $e_a\Pi_{2n-4}e_a$ is spanned by $S(a,c,a)$ and $S(a,b,a,\cdots a)$. Since it is 2-dimensional, the result follows. $\square$

This enables us to give a uniform description of the one-cycle algebra in types D_n for all $n \geq 5$ even though a description of Λ required two separate cases. The result is as follows.

Theorem 3.9.4 *The one-cycle algebra Λ is a quotient of the preprojective algebra Π given by the following relations*

$$p(a,0,b,0,a) = p(b,0,a,0,b) = p(c,m,d,m,c) = p(d,m,c,m,d) = 0,$$

$$p(i,i+1,i,i+1,i) = 0 \text{ for } 0 \leq i < m,$$

$$p(i,i-1,i,i-1,i) = 0 \text{ for } 0 < i \leq m,$$

$$P + P^* = 0 \text{ for any cyclic path of length } 2n-4.$$

The last relations do not come as any great surprise. Any cyclic path P of length 2 obviously satisfies $P^* = P$, so the symmetric property of Ω implies that these relations must hold.

We now briefly deal with the D_4 case. The proof is rather easy and is ommitted.

Proposition 3.9.5 *Let Π be the preprojective algebra of the quiver with five vertices $0, a, b, c, d$ and four arrows $i \to 0$ for $i \in \{a, b, c, d\}$. Then the corresponding one-cycle algebra is given by the following relations ($i, j \in \{a, b, c, d\}$).*

$$p(i, 0, j, 0, i) = 0,$$

$$p(0, i, 0, j, 0) + p(0, j, 0, i, 0) = 0. \square$$

3.10 One-cycle Algebras in Types E_6, E_7 and E_8.

One-cycle algebras in the exceptional types seem rather complicated and it is certainly completely impractical to attempt to find their presentations by hand, as I did for type D. However it should be mentioned that there is a simple algorithm which will find the central elements A, B, H of Π and which can be implemented fairly easily on a computer. Simply choose some total ordering on the set of vertices of Q and then order the paths in $k\overline{Q}$ first by length and then lexicographically (after identifying paths with certain sequences of vertices). One can then obtain a Gröbner basis for Π using the usual algorithm (whenever one encounters a composition which cannot be written to zero, one simply adjoins a new rewrite rule to deal with this). Of course in practise the process will never end, but all we actually need is a basis for Π_n for $n \leq h+1$, which we will find in finite time. Once these bases are found it is clearly possible, in principle, to find A, B, H by solving linear equations. The downside to all this is that the answer will be expressed in terms of our unnatural choice of basis for $k\overline{Q}$, while experience in type D suggests that there will always be a nicer description expressing these elements as linear combinations of very few paths. In type E_8 where A, B, H are of degrees 12, 20, 30 one cannot expect these efficient descriptions to be easy to find. I have, however, been able to find a neat description of A in type E_6 which also reflects the symmetry of the affine Coxeter graph in this case. Labelling the vertices of Q by

$$\begin{array}{ccccc} & & 6 & & \\ & & 5 & & \\ 0 & 1 & 2 & 3 & 4 \end{array}$$

and with all arrows of Q directed towards the vertex 2, one can take A to be (in the notation we used for type D)

$$\begin{aligned} S(0,3,0) + S(6,1,6) + S(4,5,4) \\ +S(1,6,1) - S(1,4,1) \\ +S(3,0,3) - S(3,6,3) \\ +S(5,4,5) - S(5,0,5). \end{aligned}$$

This is about the best one could hope for.

It is likely that the one-cycle algebras in the exceptional cases are of wild representation type. It seems a fruitless task to adapt the argument employed for type D to these cases since the result is almost certainly true and is, in any case, of a rather negative nature.

Chapter 4

Deformations

4.1 Introduction.

Cooordinate algebras of Kleinian singularities are given gradings by putting u, v, w in degrees a, b, h respectively. (Recall the description of Kleinian singularities in the introductory chapter). We shall be interested in filtered deformations of these algebras.

The history of such deformations is as follows. In type A_{n-1} a family of non-commutative deformations $T(v)$ was introduced by Smith and Hodges in [50], [22]. These algebras were given explicitly by (three) generators and relations and depend on a polynomial v of degree n. Hodges noted: "It seems unlikely that there will be a practical way of generalizing the above work to include these other fixed rings". Crawley-Boevey and Holland subsequently showed [14] that there is a uniform, and surprisingly simple, way to write down a family $\mathcal{O}^\lambda$ of deformations for any type. Their algebras depend on a vector $\lambda \in \mathbb{C}^I$, where I is the set of vertices of the corresponding affine Coxeter graph. $\mathcal{O}^\lambda$ is commutative precisely when λ lies in the hyperplane defined by $\sum \lambda_i d_i = 0$, and in type A_{n-1} the varity of noncommutative $\mathcal{O}^\lambda$'s is essentially the same family as $T(v)$ (although this is not competely trivial to see). Crawley-Boevey and Holland's work certainly settled the question of existence of a family of deformations for all types, and gave numerous ring-theoretic and homological properties of the algebras, but left open some questions. Is *every* deformation of the coordinate algebra of a Kleinian singularity isomorphic, an a filtered algrebra, to some $\mathcal{O}^\lambda$? Can one give neat and useable presentations for these algebras in types other than A? Was Hodges assessment correct?

4.2 The Deformations $\mathcal{O}^\lambda$ of Crawley-Boevey and Holland.

We now describe the construction of Crawley-Boevey and Holland ([14]). Let G be a finite, nontrivial subgroup of $\mathsf{SL}_2(\mathbb{C})$, and pick $\lambda \in Z(\mathbb{C}G)$. G acts naturally on the algebra $\mathbb{C}\langle x, y\rangle$ (noncommuting polynomials) so we may form the skew group algebra $G * \mathbb{C}\langle x, y\rangle$. Then define

$$\mathcal{O}^\lambda := e_0 \frac{G * \mathbb{C}\langle x, y\rangle}{[x, y] = \lambda} e_0,$$

where e_0 is the sum of the group elements. With certain care, one can also consider these algebras over an algebraically closed field k of prime characteristic p. The following points should be borne in mind. 1) One should avoid primes for which Maschke's theorem fails. 2) One should not allow G to be an arbitrary nontrivial subgroup of $\mathsf{SL}_2(k)$ - instead one should consider only the specific matrix groups described in Chapter 1 (which makes pefect sense so long as the required roots of unity exist).

Let us return to the complex case. The set Ψ of simple roots is identified with the set $I - \{0\}$. Since any root (ie an element of Δ) is an integer linear combination of simple roots we may consider them as being elements of the set $\mathbb{Z}^I$. Here we want to consider the *affine root system* $\Delta^{\mathsf{aff}} \subseteq \mathbb{Z}^I$. This consists of *imaginary roots* (nonzero integer linear combinations of δ, which is defined by

$\delta_i = d_i$, and *real roots* which are images of the coordinate vectors under sequences of *reflections* which we now describe. For $i \in I$ the reflection $\sigma_i : \mathbb{Z}^I \to \mathbb{Z}^I$ is given by

$$e_j\sigma_i = \begin{cases} -e_i & i = j, \\ e_j + \delta_{i-j}e_i & i \neq j. \end{cases}$$

The following appears in [14]

Theorem 4.2.1 $\mathcal{O}^\lambda$ *has the following properties.*
1) It is commutative if and only if $\lambda \cdot \delta = 0$.
2) It is a simple ring if and only if $\lambda \cdot \alpha \neq 0$ *for all* $\alpha \in \Delta^{\mathit{aff}} - \Delta$.
3) It has global dimension given by

$$\textit{gl.dim}\mathcal{O}^\lambda = \begin{cases} 1 & \textit{if } \lambda \cdot \alpha \neq 0 \textit{ for all } \alpha \in \Delta^{\mathit{aff}}, \\ 2 & \textit{if } \lambda \cdot \alpha = 0 \textit{ for some } \alpha \in \Delta^{\mathit{aff}}, \textit{ but} \\ & \lambda \cdot \alpha \neq 0 \textit{ for all } \alpha \in \Delta, \\ \infty & \textit{if } \lambda \cdot \alpha = 0 \textit{ for some } \alpha \in \Delta. \ \square \end{cases}$$

We mention this result mainly for the following reason. We shall describe later a family of deformations $T(\xi, f)$ of type D Kleinian singularities. This family is presumably the same as $\mathcal{O}^\lambda$ and I want to know how the pair (ξ, f) corresponds to λ. One way to tackle this problem might be to find the global dimension of $T(\xi, f)$ using methods similar to those of Hodges, and then compare with part 3) of the above.

4.3 The Deformations $U(u)$ and $T(v)$ of Smith and Hodges.

We shall now consider the algebras of Smith and Hodges. Initially we shall work over the complex algebras (as did Smith and Hodges), but we shall principally be interested in the same algebras defined over k. We shall not distinguish these cases in our notation for these algebras. Other abuses of notation which occur in the next few sections include the use of letters A, B, H, a, b, h, u, v, w, Ω for things other than what they mean in the rest of the dissertation. This was in an attempt to be consistent with Smith and Hodges, and should not cause confusion.

Fix $n \geq 2$ and let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $n-1$. Following Smith [50] we define an algebra with generators A, B, H and relations

$$[H, A] = A, \quad [H, B] = -B, \quad [A, B] = f(H).$$

This algebra has basis $A^i B^j H^k$ where $i, j, k \in \mathbb{N}$. Its centre is generated by the element $\Omega := AB + BA + \frac{1}{2}(u(H+1) + u(H))$, where $u(x)$ is chosen so that $f(x) = \frac{1}{2}(u(x+1) - u(x)) = \frac{1}{2}\Delta u$. We shall denote this algebra by $U(u)$. (Here we have *not* used the notation of Smith - he displays f. We shall see later that in the modular case displaying f is certainly *not* the 'correct' thing to do.) Notice that in the algebra $U(u)/U(u)\Omega U(u)$ we have

$$AB = -u(H), \quad BA = -u(H+1).$$

Thus writing a, b, h for the images of A, B, H respectively and setting $v(x) := -u(x+1)$ we get the following presentation for the quotient

$$[h, a] = a, \quad [h, b] = -b, \quad ba = v(h), \quad ab = v(h-1).$$

These are exactly the presentations considered by Hodges in [22], and we shall follow his notation and call this algebra $T(v)$. $T(v)$ is known to be a deformation of a Kleinian singularity of type A_{n-1}. Recall that the filtrations we use are by giving a, b, h (or A, B, H) degrees n, n, 2 respectively. This is the filtration used by all authors on the subject.

The following easy result is not stated explicitly in the literature yet it almost certainly well-known.

Proposition 4.3.1 *1) $T(v)$ and $T(w)$ are isomorphic as filtered algebras if and only if there exists $\theta \in \mathbb{C}^\times$, $\varphi \in \mathbb{C}$ and $\varepsilon \in \{\pm 1\}$ such that $w(x) = \theta v(\varepsilon x + \varphi)$.*
2) Any not-commutative deformation of $\mathbb{C}[a, b, h]/(ab = h^n)$ is isomorphic, as a filtered algebra, to some $T(v)$.

Proof. 1) Denote the filtration of $T(v)$ by $F_0 \subseteq F_1 \subseteq \cdots$. We want to prove that the only way to pick generators h', a', b' for $T(v)$ (lying in F_2, F_n, F_n respectively) such that we have

$$[h', a'] = a', \quad [h', b'] = -b', \quad b'a' = v(h'), \quad a'b' = v(h' - 1),$$

is if w is of the above form. Since h' lies in F_2 it has to be of the form $\varepsilon h - \varphi$ with $\varepsilon \neq 0$. Then the eigenvalues of $[h', ?]$ on the space F_n are ε, $-\varepsilon$, 0 (0 with multiplicity $n+1$). It follows that we must take $\varepsilon = \pm 1$. Suppose we take $\varepsilon = 1$. Then a', b' must lie in $\mathbb{C}^\times a$, $\mathbb{C}^\times b$ respectively. Then $b'a' \in \mathbb{C}^\times v(h' + \varphi)$ as required. The argument for $\varepsilon = -1$ is similar.

The converse implication is of course completely trivial.
2) We shall not include the details for the second part since we shall subsequently prove the corresponding result for type D, and the present argument is just a (much) simpler version of that. □

4.4 $U(u)$ and $T(v)$ in the Modular case.

We shall now make a few remarks about the algebras analogous to $U(u)$ - which we shall still call $U(u)$ - which are defined over k. The first point is that the map $\Delta : k[x] \to k[x]$ given by $(f\Delta)(x) = f(x+1) - f(x)$ is *not* surjective in this case, so the appropriate analogues of Smith's algebras in characterisic p are the algebras $U(u)$ given by presentations

$$[H, A] = A, \quad [H, B] = -B, \quad [A, B] = \tfrac{1}{2}(u(H+1) - u(H)),$$

for some $u(x) \in k[x]$ of degree n. We still get the basis $A^iB^jH^k$, but it is not true here that the centre of $U(u)$ is generated by Ω (defined as before). There is a second distinguished subalgebra (which we may call the p-centre) generated by A^p, B^p, $H^p - H$.

From now on in this section, suppose that $n \geq 2$, that the characteristic p of k is $> n$, and that the polynomial v is monic polynomial of degree n and has distinct roots , including 0, which all lie in $\mathbb{F}_p$. Let the roots be $\overline{a_0}, \cdots, \overline{a_{n-1}}$ where

$$0 = a_0 < a_1 < \cdots < a_{n-1} < p.$$

Let $T(v)$ be defined as the algebra over k with the same presentation as before. These conditions correspond to the condition that λ is a regular integral weight. It is shown in [21] that, under our restrictions on p, that the centre of $T(v)$ is generated by a^p, b^p, $h^p - h$ and is again isomorphic to the coordinate algebra of a Kleinian singularity of type A_{n-1}. It therefore makes sense to consider a quotient of $T(v)$ analogous to the construction of the one-cycle algebra.

Definition 4.4.1 *Define*

$$\omega(v) := \frac{T(v)}{T(v)(a^p + b^p, h^p - h)T(v)}.$$

We will show that $\omega(v)$ has dimension $2p^2$ and is Morita equivalent to the one-cycle algebra of corresponding type.

Let $\mathcal{M}$ be the set of monomials in the free associative algebra $k < a, b, h >$. We will consider two collections $(*)$, $(**)$ of rewrite rules. $(*)$ will consist of:

$$\begin{aligned} ha &\rightsquigarrow a(h+1), \\ hb &\rightsquigarrow b(h-1), \\ ba &\rightsquigarrow v(h), \\ ab &\rightsquigarrow v(h-1), \end{aligned}$$

while $(**)$ will consist of:

$$\begin{aligned} b^p &\rightsquigarrow -a^p, \\ h^p &\rightsquigarrow h, \\ a^{p+1} &\rightsquigarrow -b^{p-1}v(h). \end{aligned}$$

Proposition 4.4.2 *1) There exists an admissible order on $\mathcal{M}$ such that each rewrite rule in $(*) \cup (**)$ expresses a monomial as a linear combination of strictly lower monomials.*

2) The rewrite rules $(*)$ give a Gröbner basis for $T(v)$.

3) The rewrite rules $(*) \cup (**)$ give a Gröbner basis for $\omega(v)$.

Proof. 1) Let $\delta : \mathcal{M} \to \mathbb{N}$ be the ordinary degree map, so that $a\delta = b\delta = h\delta = 1$. Let $\delta' : \mathcal{M} \to \mathbb{N}$ be the *alternative* degree map determined by $a\delta' = b\delta' = 1$, $h\delta' = 0$, $(MN)\delta' = M\delta' + N\delta'$. Now define a total order on $\mathcal{M}$ as follows. i) If $M\delta' < N\delta'$ then $M < N$. ii) If $M\delta' = N\delta'$ and $M\delta < N\delta$ then $M < N$. iii) Monomials having the same ordinary degree and the same alternative degree are ordered by a lexicographically with a coming before b and b coming before h. It is easy to check that this works.

2) This follows from the known basis for $T(v)$.

3) Note first that

$$\begin{aligned} (b^p, ba)_b &= a^{p+1} + b^{p-1}v(h) \\ &\rightsquigarrow 0. \end{aligned}$$

This shows that the third rewrite rule of $(**)$ does actually represent a relation in $\omega(v)$ and also deals with a composition. We next deal with the remaining compositions involving one rewrite rule from each of $(*)$, $(**)$:

$$\begin{aligned} (hb, b^p)_b &= -b(h-1)b^{p-1} - ha^p \\ &\rightsquigarrow -b^p h - a^p h \\ &\rightsquigarrow 0. \end{aligned}$$

$$\begin{aligned}
(ab, b^p)_b &= -v(h-1)b^{p-1} - a^{p+1} \\
&\rightsquigarrow -b^{p-1}v(h) - a^{p+1} \\
&\rightsquigarrow 0.
\end{aligned}$$

$$\begin{aligned}
(h^p, ha)_h &= -ha + h^{p-1}a(h+1) \\
&\rightsquigarrow -a(h+1) + a(h+1)^p \\
&= -ah + ah^p \\
&\rightsquigarrow 0.
\end{aligned}$$

$$\begin{aligned}
(h^p, hb)_h &= -hb + h^{p-1}b(h-1) \\
&\rightsquigarrow -b(h-1) + b(h-1)^p \\
&= -bh + bh^p \\
&\rightsquigarrow 0.
\end{aligned}$$

$$\begin{aligned}
(ha, a^{p+1})_a &= -a(h+1)a^p - hb^{p-1}v(h) \\
&\rightsquigarrow -a^{p+1}(h+1) - b^{p-1}(h+1)v(h) \\
&\rightsquigarrow 0.
\end{aligned}$$

$$\begin{aligned}
(ba, a^{p+1})_a &= -v(h)a^p - b^p v(h) \\
&\rightsquigarrow -a^p v(h) - b^p v(h) \\
&\rightsquigarrow 0.
\end{aligned}$$

$$\begin{aligned}
(a^{p+1}, ab)_a &= b^{p-1}v(h)b + a^p v(h-1) \\
&\rightsquigarrow b^p v(h-1) + a^p v(h-1) \\
&\rightsquigarrow 0.
\end{aligned}$$

Finally we have to deal with compostions involving some rewrite rule in $(**)$ with itself. We first note that it is easy to see by induction that for all $r \geq 0$ we have

$$\begin{aligned} a^r b^r &\rightsquigarrow \prod_{i=1}^{r} v(h-i), \\ b^r a^r &\rightsquigarrow \prod_{i=o}^{r-1} v(h+i). \end{aligned}$$

Now if $0 < t < p$ we have

$$\begin{aligned} (b^p, b^p)_{b^{p-t}} &= a^p b^t - b^t a^p \\ &\rightsquigarrow a^{p-t}\big[\prod_{i=1}^{t} v(h-i)\big] - \big[\prod_{i=0}^{t-1} v(h+i)\big] a^{p-t} \\ &\rightsquigarrow a^{p-t}\big[\prod_{i=1}^{t} v(h-i)\big] - a^{p-t}\big[\prod_{i=0}^{t-1} v(h-t+i)\big] \\ &= 0. \end{aligned}$$

Similarly if $0 < t < p$ we have

$$\begin{aligned} (a^{p+1}, a^{p+1})_{a^{p+1-t}} &= b^{p-1} v(h) a^t - a^t b^{p-1} v(h) \\ &\rightsquigarrow b^{p-1} a^t v(h+t) - \big[\prod_{i=1}^{r} v(h-i)\big] b^{p-t-1} v(h) \\ &\rightsquigarrow b^{p-t-1}\big[\prod_{i=0}^{r} v(h+i)\big] - b^{p-t-1}\big[\prod_{i=1}^{r} v(h+r+1-i)\big] v(h) \\ &= 0. \end{aligned}$$

Finally,

$$\begin{aligned} (a^{p+1}, a^{p+1})_a &= b^{p-1} v(h) a^p - a^p b^{p-1} v(h) \\ &\rightsquigarrow b^{p-1} a^p v(h) - a\big[\prod_{i=1}^{p-1} v(h-i)\big] v(h) \\ &\rightsquigarrow \big[\prod_{i=0}^{p-2} v(h+i)\big] a v(h) - a \prod_{i=0}^{p-1} v(h-i) \\ &\rightsquigarrow a\big[\prod_{i=0}^{p-2} v(h+i+1)\big] v(h) - a \prod_{i=0}^{p-1} v(h-i) \\ &= 0. \end{aligned}$$

□

Corollary 4.4.3 *$\omega(v)$ has basis consisting of all monomials of one of the two forms $a^r h^s$, $b^{r-1}h^s$ where $1 \leq r \leq p$ and $0 \leq s \leq p-1$. In particular $\omega(v)$ has dimension $2p^2$.*□

Theorem 4.4.4 *$\omega(v)$ is Morita equaivalent to the one-cycle algebra of type A_{n-1}. In particular it is self-injective.*

Proof. Let $u(v)$ denote the subalgebra of $\omega(v)$ generated by a and h. Let M be any finite-dimensional irreducible $\omega(v)$-module. I claim that $D(a) := \{m \in M : ma = 0\}$ is a $u(v)$-submodule of M. Certainly it is stable under a but also if $m \in D(a)$ then

$$(mh)a = ma(h+1) = 0,$$

hence it is also stable under h. Next observe that in $\omega(v)$ we have

$$\begin{aligned} a^{2p} &= a^{p-1}a^{p+1} \\ &= -a^{p-1}b^{p-1}v(h) \\ &= -\prod_{i=0}^{p-1} v(h-i) \\ &= -(h^p - h)^n \\ &= 0. \end{aligned}$$

(The third equality follows from the proof of the previous proposition). It follows that $D(a) \neq 0$. But since $h^p = h$ we know that h acts diagonalizably on $D(a)$. Thus M has a one-dimensional $u(v)$-submodule. It follows that M is a quotient of some $\omega(v)$-module of the form

$$V(\lambda) := k_\lambda \otimes_{u(v)} \omega(v),$$

where $\lambda \in \mathbb{Z}$ and k_λ denotes k made into a $u(v)$-module via $1a = 0$, $1h = \overline{\lambda}$. By [GR], $V(\lambda) = 0$ if $\overline{\lambda}$ is not a root of v while if $\lambda = a_i$ then $V(\lambda)$ has an irreducible head S_i of dimension r_i. [21] tells us that $\omega(v)/(a^p)\omega(v)$ is Morita equivalent to the no-cycle algebra. Thus the Gabriel quiver (see [3]) of $\omega(v)$ must have n vertices indexed by the elements of $\mathbb{Z}/n\mathbb{Z}$ and $2n$ arrows $i \to i \pm 1$.

Let B be the basic algebra which is Morita equivalent to $\omega(v)$. We think of B as being a quotient of the path algebra of the above quiver. The argument of [21] shows that the relations

$$\alpha_i\beta_i = 0 = \beta_i\alpha_i$$

are satisfied. However since $a^p + b^p = 0$ we also get the relations

$$\alpha_i\alpha_{i+1}\cdots\alpha_{i+n-1} + \beta_{i+n-1}\beta_{i+n-2}\cdots\beta_i = 0.$$

Thus B is a quotient of Ω. Thus the dimension of an indecomposable projective $\omega(v)$-module is at most $2\sum r_i = 2p$. Using the general result from [3] we know

$$\dim\omega(v) = \sum \dim S_i \dim P_i,$$

where the S_i are the irreducible $\omega(v)$-modules and P_i denotes the projective cover of S_i. Since $\dim\omega(v) = 2p^2$, this forces $\dim P_i = 2p$ for all i. Hence $B = \Omega$. $\Box$.

Note that we have also proved

Proposition 4.4.5 *The socle of $\omega(v)$ is generated by a^p.* $\Box$

4.5 A Certain Map.

In the next section we shall give the construction of a family of deformations of the coordinate algebra of a Kleinian singularity of type D. The construction is analogous to that of Smith and Hodges for type A. In order to even define the algebras we will need to consider certain polynomials and a certain map of polynomials.

For $k \geq 0$ define

$$L_k(x) := \prod_{i=0}^{k-1}(x + i(i-1)).$$

Note that $L_k(x)$ has degree k so this gives a $\mathbb{Z}$-basis for the polynomial ring $\mathbb{Z}[x]$. Define a $\mathbb{Z}$-linear map $* : \mathbb{Z}[x] \to \mathbb{Z}[x]$ by setting

$$L_k^*[x] := \frac{k}{x} L_k(x).$$

We shall also use the map $*$ over more general commutative rings K, with the same definition.

It is natural to ask what $*$ does to the polynomial x^k, so define $f_k(x) := (x^k)^*$. The first few are

$$\begin{aligned}
f_0(x) &= 0 \\
f_1(x) &= 1 \\
f_2(x) &= 2x \\
f_3(x) &= 3x^2 + 2x \\
f_4(x) &= 4x^3 + 8x^2 + 8x \\
f_5(x) &= 5x^4 + 20x^3 + 56x^2 + 56x \\
f_6(x) &= 6x^5 + 40x^4 + 216x^3 + 608x^2 + 608x.
\end{aligned}$$

(According to [47], The numbers $2, 8, 56, 608, \cdots$ are called *Gennochi medians,* and all characterizations of this sequence given there are at least as complicated as ours. On the other hand, other diagonals of the triangle of coefficients do not appear as sequences in [47], so there is unlikely to be a simple formula for the $f_n(x)$)

Proposition 4.5.1 *1)* $f_k(x) \in \mathbb{N}[x]$ *for all* k*.*

2) Let R denote the algebra $\mathbb{Q}[a,b]/(a^2+2a+b^2+2b-2ab)$. For all k we have

$$f_k(a)+f_k(b) = 2(a^k-b^k)/(a-b),$$

where the right-hand side is shorthand for $2\sum_{i=0}^{k-1} a^i b^{k-1-i}$.

3) The conclusion of 2) remains valid if we replace $\mathbb{Q}$ by any commutative ring.

Proof. We shall prove the first two parts together. The first claim is that for all $k \geq 1$ there are polynomials $p_k(x)$, $q_k(x) \in \mathbb{Q}[x]$ satisfying

i) $p_k(x)$ and $q_k(x)$ have degree at most k,

ii) $p_k(x)$ and $q_k(x)$ have no constant term,

iii) All coefficients of $p_k(x)$ and $q_k(x)$ are nonnegative,

iv) We have

$$a^k b = \tfrac{k}{k+1}a^{k+1} + p_k(a) + \tfrac{1}{k+1}b^{k+1} + q_k(b),$$

$$ab^k = \tfrac{k}{k+1}b^{k+1} + p_k(b) + \tfrac{1}{k+1}a^{k+1} + q_k(a).$$

This is clearly true for $k=1$ since we can take $p_1(x)=q_1(x)=x$. Now suppose that $k \geq 1$ and that the result is true for $1,\cdots k$. Then we can write

$$a^{k+1}b - \tfrac{1}{k+1}ab^{k+1} = \tfrac{k}{k+1}a^{k+2} + ap_k(a) + aq_k(b),$$

$$ab^{k+1} - \tfrac{1}{k+1}a^{k+1}b = \tfrac{k}{k+1}b^{k+2} + bp_k(b) + bq_k(a).$$

Multiplying the first of these by $\frac{(k+1)^2}{k(k+2)}$, the second by $\frac{k+1}{k(k+2)}$ and adding gives

$$\begin{aligned} a^{k+1}b &= \tfrac{k+1}{k+2}a^{k+2} + \tfrac{(k+1)^2}{k(k+2)}ap_k(a) + \tfrac{(k+1)^2}{k(k+2)}aq_k(b) \\ &\quad + \tfrac{1}{k+2}b^{k+2} + \tfrac{k+1}{k(k+2)}bp_k(b) + \tfrac{k+1}{k(k+2)}bq_k(a). \end{aligned}$$

Now by induction $aq_k(b)$ and $bq_k(a)$ are each of the form $s(a)+t(b)$ where $s(x)$, $t(x)$ are rational polynomials of degree at most $k+1$, with no constant term, and with nonnegative coefficients. The same is clearly true of $ap_k(a)$ and $bp_k(b)$. It follows that we can write $a^{k+1}b$ in the required form, and by symmetry in a and b

the corresponding formula will also hold for ab^{k+1}. The claim has therefore been established.

The next claim is that for all $1 \leq l \leq k$ there is a rational polynomial $p_{k,l}(x)$ satisfying
i) $p_{k,l}(x)$ is monic and of degree $l+k$,
ii) $p_{k,l}(x)$ has no constant term,
iii) All coefficients of $p_{k,l}(x)$ are nonnegative,
iv) $a^l b^k + a^k b^l = p_{k,l}(a) + p_{k,l}(b)$.
We do induction on l. If $l = 1$ the claim has already been established. (The monic property holds because $\frac{k}{k+1} + \frac{1}{k+1} = 1$). Therefore suppose $2 \leq l \leq k$. Then we have

$$\begin{aligned} a^l b^k + a^k b^l &= ab(a^{l-1}b^{k-1} + a^{k-1}b^{l-1}) \\ &= ab(p_{k-1,l-1}(a) + p_{k-1,l-1}(b)). \end{aligned}$$

Since any monomial occurring in the latter expression (when expanded) is of the form $a^t b$ or ab^t the claim follows from the case $l = 1$.

It follows that for all $k \geq 2$ we can write $2(a^k - b^k)/(a-b) = g_k(a) + g_k(b)$, where $g_k(x)$ is a rational polynomial with nonnegative coefficients, no constant term and leading term kx^{k-1}. Also set $g_0(x) = 0$, $g_1(x) = 1$. Define $\circ$ to be the $\mathbb{Q}$-linear map $\mathbb{Q}[x] \to \mathbb{Q}[x]$ satisfying $(x^k)^\circ = g_k(x)$ for all k. Suppose $k \geq 2$. Since x^2 divides $L_k(x)$ it is clear that $L_k^\circ(0) = 0$. Next observe that if (α, β) is a pair of complex numbers satisfying $\alpha^2 + 2\alpha + \beta^2 + 2\beta = 2\alpha\beta$ then we have

$$2(L_k(\alpha) - L_k(\beta))/(\alpha - \beta) = L_k^\circ(\alpha) + L_k^\circ(\beta).$$

It is trivial to check that for all $i \in \mathbb{N}$, $(-i(i-1), -i(i+1))$ is such a pair. Since

$$L_k(0) = L_k(-2) = L_k(-6) = \cdots = L_k(-(k-1)(k-2)) = 0$$

it follows that $L_k^\circ(x)$ is a scalar multiple of $\frac{1}{x}L_k(x)$. By considering the highest coefficient we get that this scalar is k. Hence $\circ = *$, $f_k(x) = g_k(x)$ for all k, and the proof of the first two parts is complete.

Part 3) is a simple consequence of part 2) as follows. It suffices to prove the result for the ring $\mathbb{Z}$, and this holds since $\mathbb{Q}[a, b]$ is a unique factorization domain, and since the element $a^2 + 2a + b^2 + 2b - 2ab$ is an irreducible element of that ring. □

4.6 Type D deformations: $U(\xi, f)$ and $T(\xi, f)$.

Fix $n \geq 4$. Suppose that the field K has characteristic different from 2. Let $f(x) \in K[x]$ have degree $n - 1$, and let $\xi \in K$. Define $U(\xi, f)$ to be the K-algebra with generators u, v, w and relations

$$\begin{aligned} [u, v] &= 2w + 2v \\ [u, w] &= -2uv - \xi \\ [v, w] &= f^*(u) + v^2. \end{aligned}$$

Define also $T(\xi, f)$ to be the quotient of $U(\xi, f)$ given by the additional relation

$$f(u) + uv^2 + w^2 + \xi v = 0.$$

These algebras are given filtrations by putting K in degree 0 and u, v, w in degrees 4, $2n - 4$, $2n - 2$ respectively.

The main result of this chapter is the following

Theorem 4.6.1 *1) gr($T(\xi, f)$) $= K[u, v, w]/(u^{n-1} + uv^2 + w^2)$.*
2) In the case $K = \mathbb{C}$, any not-commutative deformation R of $\mathbb{C}[u, v, w]/(u^{n-1} + uv^2 + w^2)$ is isomorphic, as a filtered algebra, to some $T(\xi, f)$.
3) Suppose $n \geq 5$. Then $T(\xi, f) \cong T(\xi', f')$ (as filtered algebras) if and only if there exists a nonzero scalar μ such that $\xi' = \mu\xi$, $f' = \mu^2 f$. Moreover in this case the isomorphism is unique.

The proof will occupy the rest of this section. It is probably longer than required, but the result is certainly nontrivial as any proof has to explain the appearance of the map $*$.

We begin with a result about $U(\xi, f)$.

Proposition 4.6.2 *gr*$(U(\xi, f)) = K[u, v, w]$.

Proof. We will apply Shirshov's composition lemma. The total order $\leq$ we shall place on the set of monomials in $K\langle u, v, w\rangle$ is as follows. If the degree of M is less than the degree of N (giving u, v, w the degrees indicated above) then we put $M < N$. If M and N have the same degree but M comes before N lexicographically (with u before v before w) then put $M < N$. It is trivial to see that this does define an admissible order. Our relations now become rewrite rules

$$\begin{aligned} vu &\rightsquigarrow uv - 2v - 2w, \\ wu &\rightsquigarrow uw + 2uv + \xi, \\ wv &\rightsquigarrow vw - v^2 - f^*(u). \end{aligned}$$

There is only one composition to check, namely

$$C := (wv - vw + f^*(u) + v^2)u - w(vu - uv + 2v + 2w).$$

We have

$$\begin{aligned} C &= -vwu + f^*(u)u + v^2u + wuv - 2wv - 2w^2 \\ &\rightsquigarrow -vuw - 2vuv - \xi v + f^*(u)u + vuv - 2v^2 - 2vw + uwv + 2uv^2 \\ &\quad +\xi v - 2wv - 2w^2 \\ &= -vuw - vuv + f^*(u)u - 2v^2 - 2vw + uwv + 2uv^2 - 2wv - 2w^2 \\ &\rightsquigarrow -uvw + 2vw + 2w^2 - uv^2 + 2v^2 + 2wv + f^*(u)u - 2v^2 - 2vw + uvw \\ &\quad -uv^2 - uf^*(u) + 2uv^2 - 2wv - 2w^2 \\ &= 0. \quad \square \end{aligned}$$

Proposition 4.6.3 *The element* $z := f(u) + uv^2 + w^2 + \xi v$ *lies in the centre of* $U(\xi, f)$.

Proof. We have

$$\begin{aligned}
[u,z] &= u^2v^2 - uv^2u + uw^2 - w^2u + \xi uv - \xi vu \\
&= u^2v^2 - uvuv + 2uv^2 + 2uvw + uw^2 - wuw - 2wuv - \xi w + \xi uv \\
&\quad -\xi uv + 2\xi v + 2\xi w \\
&= u^2v^2 - u^2v^2 + 2uv^2 + 2uwv + 2uv^2 + 2uvw + uw^2 - uw^2 - 2uvw \\
&\quad -2uwv - 4uv^2 - 2\xi v - \xi w + 2\xi v + 2\xi w \\
&= 0.
\end{aligned}$$

We will now show that

$$[z,v] = f(u)v - vf(u) - wf^*(u) - f^*(u)v - f^*(u)w - vf^*(u),$$

$$[z,w] = f(u)w - wf(u) + \xi f^*(u) + uf^*(u)v + uvf^*(u).$$

For brevity we shall underline any term in the anticipated answer when it appears and omit it from subsequent lines, writing $+\cdots$ to represent the missing terms.

$$\begin{aligned}
[z,v] &= \underline{f(u)v} - \underline{vf(u)} + uv^3 - vuv^2 + w^2v - vw^2 \\
&= wv^2 + 2v^3 + wvw - \underline{wf^*(u)} - vw^2 + \cdots \\
&= vwv - \underline{f^*(u)v} + v^3 - \underline{f^*(u)w} - v^2w + \cdots \\
&= -\underline{vf^*(u)} + \cdots .
\end{aligned}$$

$$\begin{aligned}
[z,w] &= \underline{f(u)w} - \underline{wf(u)} + uv^2w - wuv^2 + \xi vw - \xi wv \\
&= uv^2w - uwv^2 - 2uv^3 - \xi v^2 + \xi vw - \xi vw + \xi v^2 + \underline{\xi f^*(u)} + \cdots \\
&= uv^2w - uvwv + uv^3 + \underline{uf^*(u)v} - 2uv^3 + \cdots \\
&= uv^2w - uv^2w + uv^3 + \underline{uvf^*(u)} + uv^3 - 2uv^3 + \cdots \\
&= 0 + \cdots .
\end{aligned}$$

Define M to be the matrix

$$\begin{bmatrix} u-2 & 2u & 0 \\ -2 & u & 0 \\ 0 & \xi & u \end{bmatrix}.$$

It is immediate from the relations for $[u, v]$ and $[u, w]$ that for any polynomial $g(x) \in K[x]$ we have

$$vg(u) = g(M)_{11}v + g(M)_{21}w + g(M)_{31},$$

$$wg(u) = g(M)_{12}v + g(M)_{22}w + g(M)_{32}.$$

Thus the condition $[z, v] = [z, w] = 0$ translates into the matrix equation

$$(f(M) - f(u)I) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = f^*(M) \begin{bmatrix} -1 & u \\ -1 & 0 \\ 0 & 0 \end{bmatrix} + f^*(u) \begin{bmatrix} -1 & u \\ -1 & 0 \\ 0 & \xi \end{bmatrix}.$$

This really is an if and only if condition since $U(\xi, f)$ is known to have $K[u]$-basis consisting of all monomials of the form $v^j w^k$, while all entries in any of our matrices are polynomials in u alone. By the Cayley-Hamilton theorem applied to the matrix

$$N := \begin{bmatrix} u-2 & 2u \\ -2 & u \end{bmatrix}$$

we get that $2uN = N^2 + 2N + u^2 I + 2uI$. It therefore follows from proposition 4.5.1 that we have

$$f(N) - f(u)I = (f^*(N) + f^*(u)I)(\tfrac{1}{2}(N - uI)).$$

Since

$$\tfrac{1}{2}(N - uI) = \begin{bmatrix} -1 & u \\ -1 & 0 \end{bmatrix}$$

this establishes the result for the top two rows (here we use the fact that the $(1, 3)$ and $(2, 3)$ entries of M are zero). We now establish the result for the bottom row. Using the fact that

$$L_k(M) = L_{k-1}(M)(M - (k-1)(k-2)I)$$

it is trivial to prove by induction that for all $k \geq 1$ the bottom row of $L_k(M)$ is

$$\begin{bmatrix} \frac{L_{k-1}u}{u}(-\xi k(k-1)) & \frac{L_{k-1}(u)}{u}\xi ku & L_k(u) \end{bmatrix}.$$

Similarly the bottom row of $\frac{1}{M}L_k(M)$ is

$$\left[\begin{array}{ccc} \frac{L_{k-1}(u)}{u^2}(-\xi(k-1)(k-2)) & \frac{L_{k-1}(u)}{u^2}\xi(k-1)(u+k-2) & \frac{L_k(u)}{u} \end{array}\right].$$

Since $L_k^*(x) = \frac{k}{x}L_k(x)$ this enables us to explicitly write down the bottom rows of the two sides of our matrix equation (in terms of the $L_l(u)$) for $f(x) = L_k(x)$, and hence verify that, in this case, it works (we omit the details). Since the $L_k(x)$ form a basis for the polynomial algebra this gives us the general case. The proof is complete. □

Part 1) of theorem 4.6.1 follows easily from the last two results. Indeed by an obvious filtration argument, and taking into account the basis for $U(\xi, f)$, left multiplication by z gives an injective map $U(\xi, f) \to zU(\xi, f)$. Thus the Poincaré series of the $U(\xi, f)$ module $U(\xi, f)/zU(\xi, f)$ is obtained from the Poincaré series of $U(\xi, f)$ by multiplying it by $1-q^{4n-4}$. However since z is central this right ideal is actually a two-sided ideal, so this quotient module coincides with $T(\xi, f)$. Now the basis for $U(\xi, f)$ tells us that its Poincaré series is $1/(1-q^4)(1-q^{2n-4})(1-q^{2n-2})$. Thus $T(\xi, f)$ has Poincaré series $(1+q^{2n-2})/(1-q^4)(1-q^{2n-4})$. The same argument shows that this is also the Poincaré series for the Kleinian singularity. The result follows. □

Proof of part 2) of Theorem 4.6.1. Denote the filtration of R by $F_0 \subseteq F_1 \subseteq \cdots$, and for negative k set $F_k = 0$. Pick u, v, $w \in R$ such that the corresponding graded elements correspond to the u, v and w of $A := \mathbb{C}[u, v, w]/(u^{n-1} + uv^2 + w^2)$. This double usage of letters should not cause confusion. When we talk about coeeficients of elements of R we shall mean relative to the basis consisting of the monomials $u^iv^jw^k$ where i, j, $k \in \mathbb{N}$, $k \leq 1$. Clearly $[u, v] \in F_{2n-2}$, $[u, w] \in F_{2n}$, $[v, w] \in F_{4n-8}$. Since R is not commutative, it makes sense to define k to be the minimum nonnegative integer such that at least one of the following statements holds:

$$[u, v] \in F_{2n-2-2k} - F_{2n-4-2k},$$

$$[u, w] \in F_{2n-2k} - F_{2n-2k-2},$$

$$[v, w] \in F_{4n-8-2k} - F_{4n-10-2k}.$$

Notice that $[u, v] \in F_{2n-2-2k}$, $[u, w] \in F_{2n-2k}$, $[v, w] \in F_{4n-8-2k}$. Assume first that $k > 0$. Write

$$[u, v] + F_{2n-4-2k} = \alpha u^* + \beta v \in A,$$

where $\alpha, \beta \in \mathbb{C}$ and $*$ denotes 'some power' In fact $*$ has to be $(2n - 2 - 2k)/4$ since the right hand side is homogeneous of degree $2n - 2 - 2k$, but to keep the presentation neat we shall leave out such expressions since they are not critical to what follows. α and β are well-defined complex numbers because everything occuring on the right has degree $2n - 2 - 2k$. In particular if $k \neq 1$ then β is zero, while if $k = 1$ then β is the coefficient of v in $[u, v]$. α is the coefficient of $u^{(2n-2-2k)/4}$ in $[u, v]$ (the fact that this exponent need not be an integer is irrelevant). The main point is that we have taken into account all possible monomials that could occur in $[u, v]$.

Similarly write

$$[u, w] + F_{2n-2k-2} = \gamma w + \delta v + \varepsilon u^* \in A,$$

$$[v, w] + F_{4n-10-2k} = \zeta u^* + \eta u^* v + \theta u^* w \in A.$$

Write

$$u^{n-1} + uv^2 + w^2 = \Psi$$

A few points are needed to make sense of the calculations which follow. When we replace, say, wu by $uw - \gamma w - \delta v - \varepsilon u^* + \cdots$, the terms involving greek letters represent a drop in degree by $2(k + 1)$, and $+\cdots$ means that we ignore any terms which involve a larger drop. Ignoring all drops larger than $2(k + 1)$ essentially means that we don't have to bother with terms involving a product of two lower case greek letters. Also when we replace w^2 by $-u^{n-1} - uv^2 + \Psi$, everything occuring in Ψ has degree lower than that of w^2, so we can ignore any term involving Ψ and a lower case greek letter. Another simple consequence of all this is that as soon as a term contains any greek letter (including Ψ) we are automatically allowed to bring the greek letter to the front and put all the u's, v's and w's in that term into the correct order.

We have

$$
\begin{aligned}
0 &= (w^2)u - w(wu) \\
&= -uv^2u - u^n + \Psi u - wuw + \gamma w^2 + \delta vw + \varepsilon u^* w + \cdots \\
&= -uvuv + \alpha u^* v + \beta uv^2 - u^n + \Psi u - uw^2 + 2\gamma w^2 + 2\delta vw + 2\varepsilon u^* w + \cdots \\
&= 2\alpha u^* v + 2\beta uv^2 - 2\gamma uv^2 - 2\gamma u^n + 2\delta vw + 2\varepsilon u^* w + \cdots
\end{aligned}
$$

By considering the coefficients of distinct monomials it follows that $\alpha = \beta = \gamma = \delta = \varepsilon = 0$. Similarly we get:

$$
\begin{aligned}
0 &= (w^2)v - w(wv) \\
&= -uv^3 - u^{n-1}v + \Psi v - wvw + \zeta u^* w + \eta u^* vw + \theta u^* w^2 + \cdots \\
&= -uv^3 - u^{n-1}v + \Psi v - vw^2 + 2\zeta u^* w + 2\eta u^* vw + 2\theta u^* w^2 + \cdots \\
&= 2\zeta u^* w + 2\eta u^* vw - 2\theta u^* v^2 - 2\theta u^* + \cdots
\end{aligned}
$$

Hence $\zeta = \eta = \theta = 0$. It follows that $[u,v] \in F_{2n-4-2k}$, $[u,w] \in F_{2n-2k-2}$, $[v,w] \in F_{4n-10-2k}$, contradicting the definition of k. It follows that $k = 0$.

We have no further use for the lower case greek letters of the proof so far, so we shall now redefine them using similar ideas. Write

$$[u,v] + F_{2n-4} = \alpha w + \beta u^* \in A,$$

$$[u,w] + F_{2n-2} = -\gamma uv - \delta u^* \in A,$$

$$[v,w] + F_{4n-8} = \varepsilon u^* + \zeta v^2 + \eta u^* v + \theta u^* w \in A.$$

Then,

$$\begin{aligned}
0 &= (w^2)u - w(wu) \\
&= -uv^2u - u^n + \Psi u - wuw - \gamma uvw - \delta u^* + \cdots \\
&= -uvuv + \alpha uvw + \beta u^*v - u^n + \Psi u - uw^2 - 2\gamma uvw - 2\delta u^*w + \cdots \\
&= (\alpha - 2\gamma)uvw + 2\beta u^*v - 2\delta u^*w + \cdots
\end{aligned}$$

It follows that $\alpha = 2\gamma$, $\beta = \delta = 0$. Similarly

$$\begin{aligned}
0 &= (w^2)v - w(wv) \\
&= -uv^3 - u^{n-1}v + \Psi v - wvw + \varepsilon u^*w + \zeta v^2w + \eta u^*vw + \theta u^*w^2 + \cdots \\
&= -uv^3 - u^{n-1}v + \Psi v - vw^2 + 2\varepsilon u^*w + 2\zeta v^2w + 2\eta u^*vw + 2\theta u^*w^2 + \cdots \\
&= -uv^3 - u^{n-1}v + vuv^2 + vu^{n-1} + 2\varepsilon u^*w + 2\zeta v^2w + 2\eta u^*vw + 2\theta u^*w^2 + \cdots \\
&= -2\gamma v^2w - 2(n-1)\gamma u^*w + 2\varepsilon u^*w + 2\zeta v^2w + 2\eta u^*vw - 2\theta u^* - 2\theta u^*v + \cdots
\end{aligned}$$

Hence $\varepsilon = (n-1)\gamma$, $\zeta = \gamma$, $\eta = \theta = 0$.

The upshot of all this is that there exists a *nonzero* scalar μ such that the relations take the form

$$\begin{aligned}
[u, v] &= 2\mu w + \cdots \\
[u, w] &= -2\mu uv + \cdots \\
[v, w] &= \mu(n-1)u^{n-2} + \mu v^2 + \cdots
\end{aligned}$$

where $+\cdots$ means that all missing terms have degree less than what precedes. (It should be mentioned that this statement follows trivially from an observation referred to in [39] (pages 1 and 2) that " [every not-commutative deformation of a Kleinian singularity] has the same Poisson bracket up to a scalar multiple". I have been able to find no proof of this statement in the literature). By replacing u, v, w by $\mu^{-2}u$, $\mu^{2-n}v$, $\mu^{1-n}w$ respectively we can take $\mu = 1$. It follows that there exist α, β, $\gamma \in \mathbb{C}$ (again completely redefining these letters) and polynomials $p(x)$,

$q(x)$, $r(x)$, $s(x)$, $t(x) \in \mathbb{C}[x]$ such that the first three relations take the form

$$\begin{aligned}
[u, v] &= 2w + \alpha v + s(u) \\
[u, w] &= -2uv + \beta v + \gamma w - t(u) \\
[v, w] &= r(u) + v^2 + v^2 + p(u)v + q(u)w.
\end{aligned}$$

In fact $r(x)$ must begin $(n-1)x^{n-2}$ and there are obvious restrictions on the degrees of the other polynomials. Next defining w' by the equation $2w + \alpha v + s(u) = 2w' + 2v$, and replacing w by w', we can actually assume that $\alpha = 2$, $s(x) = 0$. Next, replacing u by $u - \frac{1}{2}\beta$ we can assume $\beta = 0$. Also observe that by replacing v, w by $v + c$, $w - c$ for a number c we can make the coeffiecient of x in $t(x)$ anything we like. We choose to make it zero. Notice that none of these reductions messes up the good work done by the previous ones.

This gives us the following relations in R, which we shall use as if they were rewrite rules.

$$\begin{aligned}
vu &= uv - 2v - 2w \\
wu &= uw + 2uv + \gamma w + t(u) \\
wv &= vw - r(u) - v^2 - p(u)v - q(u)w
\end{aligned}$$

We have

$$
\begin{aligned}
0 &= (wv - vw + r(u) + v^2 + p(u)v + q(u)w)u - w(vu - uv + 2v + 2w) \\
&= -vwu + r(u)u + v^2u + p(u)vu + q(u)wu + wuv - 2wv - 2w^2 \\
&= -vuw - 2vuv + \gamma vw + vt(u) + r(u)u + vuv - 2v^2 - 2vw + p(u)uv \\
&\quad -2p(u)v - 2p(u)w + q(u)uw + 2q(u)uv - \gamma q(u)w - q(u)t(u) + uwv \\
&\quad +2uv^2 - \gamma wv - t(u)v - 2wv - 2w^2 \\
&= -uvw + 2vw + 2w^2 - uv^2 + 2v^2 + 2wv + \gamma vw + vt(u) + r(u)u - 2v^2 \\
&\quad -2vw + p(u)uv - 2p(u)v - 2p(u)w + q(u)uw + 2q(u)uv - \gamma q(u)w \\
&\quad -q(u)t(u) + uvw - ur(u) - uv^2 - up(u)v - uq(u)w + 2uv^2 - \gamma wv \\
&\quad -t(u)v - 2wv - 2w^2 \\
&= \gamma vw - 2p(u)v - 2p(u)w + 2q(u)uv - \gamma q(u)w - \gamma wv - q(u)t(u) + vt(u) \\
&\quad -t(u)v \\
&= -2p(u)v - 2p(u)w + 2q(u)uv - \gamma q(u)w + \gamma r(u) + \gamma v^2 + \gamma p(u)v \\
&\quad +\gamma q(u)w - q(u)t(u) + vt(u) - t(u)v \\
&= -2p(u)v - 2p(u)w + 2q(u)uv - \gamma q(u)w + \gamma r(u) + \gamma v^2 + \gamma p(u)v \\
&\quad +\gamma q(u)w - q(u)t(u) - t(u)v + t(M)_{11}v + t(M)_{21}w + t(M)_{31},
\end{aligned}
$$

where as before M is the matrix

$$
\begin{bmatrix} u-2 & 2u & 0 \\ -2 & u & 0 \\ 0 & \lambda & u \end{bmatrix}.
$$

Now since a basis of R is known, we can equate coefficients to zero. From the coefficient of v^2 we see that $\gamma = 0$. Since $t(x)$ has no term in x it is either a constant or of degree ≥ 2. We shall now show that the latter leads to a contradiction. So suppose the leading term in $t(x)$ is αx^d where $\alpha \neq 0$, $d \geq 2$. By a trivial induction argument, for all $k \geq 1$, the left hand column of M^k has the form

$$\begin{bmatrix} u^k - 2k^2u^{k-1} + \cdots \\ -2ku^{k-1} + \cdots \\ 2(k-1)\alpha u^{d+k-2} + \cdots \end{bmatrix},$$

where $+\cdots$ indicates that all other terms involve a strictly lower power of u. By comparing coefficients of 1, w, v in our expression for the composition we get

$$\begin{aligned} q(u)t(u) &= t(M)_{31} && (1) \\ 2p(u) &= t(M)_{21} && (2) \\ 2p(u) - 2q(u)u &= t(M)_{11} - t(u) && (3). \end{aligned}$$

Now, the $(3,1)$ entry of $t(M)$ has leading term $-d(d-1)\alpha^2u^{2d-2}$, so, by (1), $q(u)$ has leading term $-d(d-1)\alpha^2u^{d-2}$. Similarly by using (2) we can determine the leading term of $p(u)$, and we get that the leading term of $2p(u) - 2q(u)u$ is $(2d^2 - 4d)\alpha u^{d-1}$. However the leading term of $t(M)_{11} - t(u)$ is easily seen to be $-2\alpha d^2u^{d-1}$. Hence from (3) we get that $4d^2 - 4d = 0$, ie $d = 0$ or 1, a contradiction.

We deduce that $t(u)$ is a constant ξ, say. It then follows from (2), (3) that $p(u) = q(u) = 0$.

It is now only necessary to consider the final relation, which we choose to write in the form $z = 0$, where

$$z = f(u) + uv^2 + w^2 + k(u)v + l(u)w + \delta v^2 + \varepsilon vw.$$

Here $f(u)$ is a monic polynomial of degree $n-1$, δ, ε are complex numbers, and $k(u)$, $l(u)$ are polynomials with the appropriate restrictions on the degrees. We want to show that $r(u) = f^*(u)$, $k(u) = \xi$, $l(u) = \delta = \varepsilon = 0$. We shall use the following relations:

$$\begin{aligned}
vu &= uv - 2v - 2w \\
wu &= uw + 2uv + \xi \\
wv &= vw - r(u) - v^2 \\
w^2 &= -f(u) - uv^2 - k(u)v - l(u)w - \delta v^2 - \varepsilon vw
\end{aligned}$$

In R we have

$$\begin{aligned}
0 &= w(wu) - (w^2)u \\
&= wuw + 2wuv + \xi w + f(u)u + uv^2u + k(u)vu + l(u)wu + \delta v^2 u + \varepsilon vwu \\
&= uw^2 + 2uvw + \xi w + 2uwv + 4uv^2 + 2\xi v + \xi w + f(u)u + uvuv - 2uv^2 \\
&\quad -2uvw + k(u)uv - 2k(u)v - 2k(u)w + l(u)uw + 2l(u)uv + \xi l(u) \\
&\quad +\delta vuv - 2\delta v^2 - 2\delta vw + \varepsilon vuw + 2\varepsilon vuv + \varepsilon\xi v \\
&= -uf(u) - u^2v^2 - uk(u)v - ul(u)w - \delta uv^2 - \varepsilon uvw + 2uvw + \xi w \\
&\quad +2uwv + 4uv^2 + 2\xi v + \xi w + f(u)u + u^2v^2 - 2uv^2 \\
&\quad -2uwv - 2uv^2 - 2uvw + k(u)uv - 2k(u)v - 2k(u)w + l(u)uw \\
&\quad +2l(u)uv + \xi l(u) + (\delta + 2\varepsilon)uv^2 + 2(\delta + 2\varepsilon)v^2 + 2(\delta + 2\varepsilon)wv - 2\delta v^2 \\
&\quad -2\delta vw + \varepsilon uvw - 2\varepsilon v^2 - 2\varepsilon w^2 + \varepsilon\xi v \\
&= 2\xi w + 2\xi v - 2k(u)v - 2k(u)w + 2l(u)uv + \xi l(u) + 2\varepsilon uv^2 + 2\varepsilon v^2 \\
&\quad +2(\delta + 2\varepsilon)(vw - r(u) - v^2) - 2\delta vw - 2\varepsilon v^2 + 2\varepsilon f(u) + 2\varepsilon uv^2 \\
&\quad +2\varepsilon k(u)v + 2\varepsilon l(u)w + 2\delta\varepsilon v^2 + 2\varepsilon^2 vw + \varepsilon\xi v.
\end{aligned}$$

The coefficient of uv^2 tells us that $\varepsilon = 0$. Then looking at the coefficient of w (a polynomial in u) we see $k(u) = \xi$. Then the coefficients of vw tells us $\delta = 0$, and the coefficient of v (a polynomial in u again) tells us that $l(u) = 0$. It remains to prove that $r(u) = f^*(u)$.

Our relations now take the following simpler form

$$\begin{aligned} vu &= uv - 2v - 2w \\ wu &= uw + 2uv + \xi \\ wv &= vw - r(u) - v^2 \\ w^2 &= -f(u) - uv^2 - \xi v. \end{aligned}$$

We will now show that in R the following hold

$$0 = -wr(u) + f(u)v - r(u)w - vf(u) - r(u)v - vr(u),$$

$$0 = -wf(u) + f(u)w + \xi r(u) + ur(u)v + uvr(u).$$

Notice that these are the same as we had in the proof of part 1) but with $f^*(u)$ replaced by $r(u)$. In the following calculations we shall underline any term in the anticipated answer when it appears and omit it from subsequent lines, writing $+\cdots$ to represent the missing terms.

$$\begin{aligned} 0 &= w(wv) - (w^2)v \\ &= wvw - \underline{wr(u)} - wv^2 + \underline{f(u)v} + uv^3 + \xi v^2 \\ &= vw^2 - \underline{r(u)w} - v^2w - wv^2 + uv^3 + \xi v^2 + \cdots \\ &= -\underline{vf(u)} - vuv^2 - v^2w - wv^2 + uv^3 + \cdots \\ &= 2v^3 + wv^2 - v^2w + \cdots \\ &= v^3 + vwv - \underline{r(u)v} - v^2w + \cdots \\ &= -\underline{vr(u)} + \cdots \end{aligned}$$

This establishes the first equation. Next,

$$\begin{aligned} 0 &= w(w^2) - (w^2)w \\ &= -\underline{wf(u)} - wuv^2 - \xi wv + \underline{f(u)w} + uv^2w + \xi vw \\ &= -uwv^2 - 2uv^3 + \underline{\xi r(u)} + uv^2w + \cdots \\ &= -uvwv + \underline{ur(u)v} - uv^3 + uv^2w + \cdots \\ &= \underline{uvr(u)} + \cdots \end{aligned}$$

We now have to prove a converse of part of the proof of the first part, namely that that $f^*(x)$ is the *unique* polynomial satisfying the matrix equation

$$(f(M) - f(u)I)\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = f^*(M)\begin{bmatrix} -1 & u \\ -1 & 0 \\ 0 & 0 \end{bmatrix} + f^*(u)\begin{bmatrix} -1 & u \\ -1 & 0 \\ 0 & \xi \end{bmatrix},$$

where M is as before. But this is easy - we saw in the proof of part 1 that the $(3,1)$, $(3,2)$ entries of $L_k(M)$ are polynomials in u of degrees $k-2$, $k-1$ repectively. Hence the $(3,1)$ entry of

$$L_k(M)\begin{bmatrix} -1 & u \\ -1 & 0 \\ 0 & 0 \end{bmatrix} + L_k(u)\begin{bmatrix} -1 & u \\ -1 & 0 \\ 0 & \xi \end{bmatrix},$$

is a polynomial in u of degree $k-2$. The result follows immediately. □

Proof of part 3) of Theorem 4.6.1.

The 'if' part is obvious since in $T(\xi, f)$ the elements u, μv, μw clearly satisfy the relations for $T(\mu\xi, \mu^2 f)$. For the converse suppose that there are elements u', v', w' in $T(\xi, f)$ which satisfy the relations of $T(\xi', f')$, lie in the appropriate filtered component and for which $T(\xi, f)$ has basis consisting of monomials $(u')^i(v')^j(w')^k$ with $k \leq 1$.

Write

$$\begin{aligned} u' &= \lambda u + \alpha \\ v' &= \mu v + p(u) \\ w' &= \nu w + \beta v + q(u). \end{aligned}$$

(the proof breaks down here for $n = 4$ since u' may have a coefficient of v). Now,

$$\begin{aligned} [w', u'] &= 2u'v' + \xi' \\ &= 2(\lambda u + \alpha)(\mu v + p(u)) + \xi' \\ &= 2\lambda\mu uv + 2\alpha\mu v + 2\lambda u p(u) + 2\alpha p(u) + \xi'. \end{aligned}$$

Thus, in particular, $[w', u']$ has coefficient of w equal to zero. On the other hand

$$\begin{aligned} [w', u'] &= [\nu w + \beta v + q(u), \lambda u + \alpha] \\ &= \lambda\nu[w, u] - \beta\lambda[u, v] \\ &= 2\lambda\nu uv + \lambda\nu\xi - 2\beta\lambda w - 2\beta\lambda v. \end{aligned}$$

It follows that $\beta\lambda = 0$. But clearly $\lambda \neq 0$ since u', 1 and u, 1 must have the same spans. Thus $\beta = 0$, so we get

$$[w', u'] = \lambda\nu(2uv + \xi).$$

looking back at our first expression for $[w', u']$ we deduce $\alpha = 0 = p(u)$, $\nu = \mu$ and $\xi' = \lambda\mu$.

Next we get

$$\begin{aligned} 2w' + 2v' &= [u', v'] \\ &= [\lambda u, \mu v] \\ &= \lambda\mu(2w + 2v) \\ &= 2\lambda(w' - q(u'/\lambda)) + 2\lambda v'. \end{aligned}$$

By comparing coefficients we get $\lambda = 1$, $q(u) = 0$. We have thus shown $u' = u$, $v' = \mu v$, $w' = \mu w$, $\xi' = \mu\xi$. Finally we get

$$f'(u) = -(w')^2 - u'(v')^2 - \xi' v' = \mu^2 f(u).$$

So $f' = \mu^2 f$. The result follows. $\square$

4.7 Identities in the Algebras $U(\xi, f)$.

In this section we simply record some identities. We do not give proofs as they are all trivial to prove by induction. I suspect that there are further simple identities which I am yet to discover, making computation within these algebras easier than one might at first imagine. We have

$$\left[\frac{1}{u}L_{k+1}(u), v\right] = 2k^2\frac{1}{u}L_k(u)v + 2k\frac{1}{u}L_k(u)w + \xi k(k-1)\frac{1}{u^2}L_k(u).$$

$$\left[\frac{1}{u}L_{k+1}(u), w\right] = -2kL_k(u)v + 2k(k-1)\frac{1}{u}L_k(u)w - \xi k(u+k-1)\frac{1}{u^2}L_k(u),$$

$$\left[v^k, w\right] = kv^{k+1} + \sum_{i=0}^{k-1} v^i f^*(u) v^{k-1-i},$$

$$\left[v^k, u\right] = k(k-3)v^k - 2kv^{k-1}w + 2\sum_{i=0}^{k-2}(i+1)v^i f^*(u) v^{k-2-i}.$$

Notice that if the characteristic of K is $p > 0$ then we get

$$[L_p(u), v] = [L_p(u), w] = 0,$$

$$[v^p, w] = f^*(u)(\mathrm{ad}v)^{p-1},$$

$$[v^p, u] = f^*(u)(\mathrm{ad}v)^{p-2}.$$

4.8 Central Elements of the Algebras $U(\xi, f)$ in the Modular case.

In this section we work over k rather than K. For suitable primes p I had hoped to be able to write down three central elements of $U(\xi, f)$ beginning $u^p + \cdots$, $v^p + \cdots$, $w^p + \cdots$ (which should surely exist) analogous to the situation in type A. Disappointingly I have only been able to describe the first of these, but I feel that the 'missing' identities mentioned in the previous section should make the other two equally obvious. The first one, however, is given by

Lemma 4.8.1 *If p is an odd prime, then $L_p(u)$ lies in the centre of $U(\xi, f)$.*

Proof. Noting that $L_p(u) = \frac{1}{u}L_{p+1}(u)$, the result follows immediately from an identity recorded in the previous section. $\square$

I am also able to describe a central element of the form $v^p + \cdots$ in the case $p = 3$. I will not provide a proof here since the result will hopefully be superceded by a more satisfactory one.

Proposition 4.8.2 *Consider the linear map $k[x] \to k[x]$ given by*

$$L_k(x) \mapsto \begin{cases} 0 & k \equiv 0, 1 (\textit{mod}\, 3) \\ L_{k-2}(x) & k \equiv 2 (\textit{mod}\, 3) \end{cases}$$

Let $g(x)$ be the image of $f(x)$ under this map. Then $v^3 - g(u)v$ is a central element of $U(\xi, f)$. $\square$

4.9 Some Miscellaneous Observations.

In this section we make some simple observations regarding the various deformations.

1) Firstly, I would like to bring out certain similarities between the presentations for the algebras $T(v)$ and $T(\xi, f)$. Notice that the map $\Delta : \mathbb{C}[x] \to \mathbb{C}[x]$ given by $(f\Delta)(x) = f(x+1) - f(x)$, which was used in the description of Smith's and Hodges's algebras, is characterized by the property that it takes $\prod_{i=0}^{k-1}(x+i)$ to $\frac{k}{x}$ times itself. The map $*$ was defined in exactly the same way, but using the polynomials $\prod_{i=0}^{k-1}(x + i(i-1))$ instead. Also, the presentations in type D are much simpler than one may have suspected, and are strikingly similar to those of $U(u)$ (at least if one replaces A, B by $A + B$, $A - B$). All this suggests, to me at least, that maybe there is a completely uniform presentation of deformations depending on two polynomials in u (the generator of lowest degree), perhaps expressed in terms of the corresponding root system (like Serre's relations for simple Lie algebras). (In type A_{n-1} the 'missing' polynomial is zero, while in type D it's

the constant ξ. Only in the exceptional types do I expect both polynomials to be at least linear).

2) The family $T(v)$ for v of degree 4 should be 'the same as' the family $T(\xi, f)$ for f quadratic (since $A_3 = D_3$), yet this is not at all immediate from the presentations. In fact it is so. To verify it (and we shall not spoil the reader's fun by giving the details) notice that $h \in T(v)$ should correspond to some nonzero scalar multiple h' of $v \in T(\xi, f)$, the scalar chosen so that the adh' has ± 1 as eigenvalues. Then let a', b' span these eigenspaces and check that $b'a'$ is a quartic in h' as required.

3) Next we consider the algebras $\mathcal{O}^\lambda$ in types D and E. Since [14] tells us that $\mathcal{O}^\lambda$ is a deformation of the corresponding Klienian singularity, we know that there are three generators u, v, w lying in the filtered components F_a, F_b, F_h and (by Gröbner basis arguments) that we get a presentation if we can express $[u, v]$, $[u, w]$, $[v, w]$, w^2 in terms of lower degree monomials. Define $\mathcal{P}^\lambda$ to be the algebra defined by the relations for the commutators only. We found in type D that $\mathcal{P}^\lambda$ is a deformation of affine 3-space and that the algebra $\mathcal{O}^\lambda$ is obtained by killing a central element. It therefore seems natural to ask: 'If G is a finite subgroup of $\mathsf{SL}_2(\mathbb{C})$ and $\lambda \in Z(\mathbb{C}G)$ is there a natural and uniform way to define an algebra $\mathcal{P}^\lambda$ with these properties?'.

4.10 Deformations in Representation Theory I.

In this section we shall make some remarks concerning the connections between the objects considered in this thesis and representation theory over $\mathbb{C}$. We shall freely use the notation from sections 1.1.7, as well as Chapter 2, except that we shall drop the clumsy subscripts $\mathbb{C}$

Most of the notation follows [39]. Let E, F, H denote an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Define $\chi \in \mathfrak{g}^*$ correspond to E under an isomorphism of G-modules $\mathfrak{g} \cong \mathfrak{g}^*$. We assume that E is chosen so that χ lies in the subregular orbit of the G-action on $\mathfrak{g}$. There is a one-dimensional torus λ in G such that $E \in \mathfrak{g}(2)$, $F \in \mathfrak{g}(-2)$, ker ad$E \subseteq \oplus_{i \geq 0}\mathfrak{g}(i)$, where $\mathfrak{g} = \oplus_{\mathbb{Z}}\mathfrak{g}(i)$ is the grading corresponding to the action of λ. Following [39] we define $\mathfrak{m}_\chi := \oplus_{i \leq -2}\mathfrak{g}(i)$, a nilpotent Lie subalgebra of $\mathfrak{g}$ of dimension $N-1$. Let $\mathbb{C}_\chi$ denote $\mathbb{C}$ considered as a left $U(\mathfrak{m}_\chi)$-module where $x \in \mathfrak{m}_\chi$ acts as $x\chi$. Set

$$\tilde{Q}_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m}_\chi)} \mathbb{C}_\chi,$$

a $\mathfrak{g}$-module, and

$$\tilde{H}_\chi = \mathsf{End}_{\mathfrak{g}}(\tilde{Q}_\chi)^{\mathsf{opp}}.$$

Then for any algebra homomorphism $\eta : Z(\mathfrak{g})$ define

$$\tilde{H}_{\chi,\eta} := \tilde{H}_\chi \otimes_{Z(\mathfrak{g})} \mathbb{C}_\eta.$$

Premet proves in [39] that in type A the algebra $\tilde{H}_{\chi,\eta}$ is isomorphic to Hodges algebra $T(v)$ for some choice of v. This is essentially true because (for any type) this algebra is known to be a deformation of the corresponding Kleinian singularity. Premet observes that similar arguments allow one to describe all subregular algebras $\tilde{H}_{\chi,\eta}$. In fact the computations required for type D are essentially the proof of our Theorem 4.6.1 part 2). We state this as a theorem.

Theorem 4.10.1 *If $\mathfrak{g} = \mathfrak{so}_{2n}$, then $\tilde{H}_{\chi,\eta} \cong T(\xi, f)$ for some suitable choice of ξ, f.* $\square$.

4.11 Deformations in Representation Theory II.

In this final section we make some tantalising but enormously speculative comments about the modular case. We shall employ the notation and the restrictions on p from section 1.1.8 and chapter 2 (in particular λ is a regular weight). As indicated in Chapter 2, Janzten has proved in [29] that the block $B_{\chi,\lambda}$ has $r+1$

irreducible modules in bijective correspondence with the set I, and under this bijection, the Ext^1 groups between nonisomorphic irreducible modules is equal to a_{ij}, the number of edges connecting i, j in the affine Coxeter graph (which is usually at most 1). In fact Jantzen also gives the dimension of the irreducible module S_i.

A suggestion made by my supervisor Dmitriy Rumynin was that the central quotient algebra $U_{\chi,\lambda}$ should in all cases be Morita equivalent to the no-cycle algebra. One anticipated method of proof should combine the derived version of the McKay correspondence (see [31], [9]) with results of [4] on localization techniques. An alternative approach might be to find the connection between $U(\mathfrak{g})$ and the algebras $\mathcal{P}^\lambda$ mentioned in section 4.9. It seems that somehow the p-centre of $U(\mathfrak{g})$ should correspond to a central subalgebra of $\mathcal{P}^\lambda$ isomorphic to a Kleinian singularity, while the Harish-Chandra centre should correpond to central subalgebra of $\mathcal{P}^\lambda$ generated by a single element.

It is interesting to speculate on the possible role of the one-cycle algebras. Let us suppose that it is the case that the algebra $U_{\chi,\lambda}$ is Morita equivalent to the no-cycle algebra. One can ask if there is an algebra $U^*_{\chi,\lambda}$ and surjective homomorphisms

$$U \to U^*_{\chi,\lambda} \to U_{\chi,\lambda},$$

such that $U^*_{\chi,\lambda}$ is Morita equivalent to the corresponding one-cycle algebra. If this is the case then the algebra $U^*_{\chi,\lambda}$ is certainly a very interesting object. Firstly it is a Frobenius algebra (by Morita equivalence). Secondly, by combining Jantzen's dimension formulae for the S_i with Proposition 3.4.6, we find that the dimension of $U^*_{\chi,\lambda}$ is $2p^{2N}$. (In fact, by Propostion 2.1 of [39], $U^*_{\chi,\lambda}$ is necessary a matrix algebra over an algebra of dimension $2p^2$ which is Morita equivalent to the one-cycle algebra. One can speculate (and it is true in type A, by [21]) that this latter algebra is a quotient of some deformation of the corresponding Kleinian singularity). The dimension of $U^*_{\chi,\lambda}$ suggests that it should have some nice PBW-type basis.

Appendix A

Some Information...

This Appendix contains certain interesting facts (all loosely related to this thesis) which to me seem mysterious. Since the observations are of a very elementary nature and concern classical objects, they are surely not new. On the other hand, except in the one case where the contrary is clearly indicated, I have not seen them stated explicitly in the literature.

A.1 ...On Finite-dimensional Preprojective algebras.

Although we did not state it in this way, it is clear that theorem 3.3.2 is equivalent to the following recurrence

$$\mathsf{dim}(e_i\Pi_n e_k) = \left[\sum_{j-k} \mathsf{dim}(e_i\Pi_{n-1}e_j)\right] - \mathsf{dim}(e_i\Pi_{n-2}e_k).$$

One may wonder what happens if one attempts to use this recurrence to work out the Poincaré series for the Preprojective algebra of a *non-affine* graph. Let's try this for type D_5:

$$\begin{array}{cccc} 1 & & & \\ & 1 & 1 & 1 \\ 1 & & & \end{array} \rightsquigarrow \begin{array}{cccc} 1 & & & \\ & 3 & 2 & 1 \\ 1 & & & \end{array} \rightsquigarrow \begin{array}{cccc} 2 & & & \\ & 3 & 3 & 1 \\ 2 & & & \end{array} \rightsquigarrow \begin{array}{cccc} 2 & & & \\ & 4 & 2 & 2 \\ 2 & & & \end{array}$$

$$\rightsquigarrow \begin{array}{cccc} 2 & & & \\ & 3 & 3 & 1 \\ 2 & & & \end{array} \rightsquigarrow \begin{array}{cccc} 1 & & & \\ & 3 & 2 & 1 \\ 1 & & & \end{array} \rightsquigarrow \begin{array}{cccc} 1 & & & \\ & 1 & 1 & 1 \\ 1 & & & \end{array} \rightsquigarrow \begin{array}{cccc} 0 & & & \\ & 0 & 0 & 0, \\ 0 & & & \end{array}$$

and after that we get negative numbers. A similar thing happens for all types. In fact for all types this procedure gives the correct answer for all degrees up to and including $h - 2$, while the preprojective algebra has no nonzero components of degree greater than $h - 2$.

Using this method one finds, on a case-by-case basis, that for any type the Poincaré polynomial of the preprojective algebra is given by the formula

$$\frac{r - 2(\sum q^{m_i}) + rq^h}{(1 - q)^2},$$

where r is the rank, h the Coxeter number, and $m_1, \cdots, m_r$ the exponents of the corresponding Coxeter group. I cannot see any obvious reason why the recurrence described above should give this formula.

A.2 ...On Certain Poincaré Series.

It is a rather bizarre fact that all complex roots of the Poincaré series of a one-cycle algebra have absolute value 1. The same is true of finite-dimensional preprojective algebras. There must, surely, be some sensible reason behind this.

A.3 ...On Finite Subgroups of $\mathsf{SL}_2(\mathbb{C})$.

There are certain properties of finite subgroups G of $\mathsf{SL}_2(\mathbb{C})$ that I cannot explain. The first is that if g is a nonidentity element of G then $\sum \chi(g) \in \{0, 2\}$, where the sum is over all irreducible characters of G. This is equivalent to the statement that if M is a sum of all irreducible $\mathbb{C}G$-modules, then $M \otimes M \cong 2M \oplus \text{free}$.

The second (related) fact is that if V denotes the natural two-dimensional module of G then

$$\sum_{n \geq 0} \text{length}_{\mathbb{C}G}(S^n(V))q^n = \frac{1 - q^{\frac{h}{2}}}{(1 - q^{\frac{a}{2}})(1 - q^{\frac{b}{2}})(1 - q)},$$

where length denotes composition length (in this case the number of irreducible summands) and a, b, h are as usual.

Finally we simply quote a question from [26]: "What are the Coxeter exponents and the Coxeter number for a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, and why? (It is known that the Coxeter number equals the largest degree of the three homogeneous generators of the G-invariant polynomial ring. But why?)"

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