PROBLEMS IN THE REPRESENTATION THEORY OF
ALGEBRAIC GROUPS

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General Remarks

This thesis is in two parts. Part A is concerned with a problem in the representation theory of semisimple algebraic groups. In part B we are interested in the endomorphism rings of comodules, in particular in the endomorphism ring of a certain induced comodule. The parts are more or less self contained each having its own introduction and references. There is however an underlying theme of induction and so the exposition of 2, (A) is relevant to both parts. Other points of overlap are the second theorem of 4 in part A and the digression of 2 in part B.

We assemble below definitions and remarks, mostly of a general nature, in order to clarify notation and acquaint the reader with the various facts which will be used without reference later on.

A coalgebra over a field \( k \) is a triple \((C, \mu, \varepsilon)\) where \( C \) is a vector space over \( k \) and \( \mu : C \to C \otimes_k C \) and \( \varepsilon : C \to k \) are \( k \)-maps satisfying

\[
(1 \otimes \mu) \mu = (\mu \otimes 1) \mu \quad \text{and} \quad (\varepsilon \otimes 1) \mu = (1 \otimes \varepsilon) \mu = 1_C.
\]

We often refer to a coalgebra \((C, \mu, \varepsilon)\) simply as \( C \). The most important coalgebras occurring in this thesis are Hopf algebras.
A Hopf algebra \((C, \mu, \varepsilon)\) over \(k\) is a coalgebra over \(k\) such that \(C\) is a \(k\) algebra, \(\mu\) and \(\varepsilon\) are \(k\) algebra maps and there is a \(k\) map \(S : C \to C\) which has the property that
\[
(1) \text{ for any } c \in C, \sum_i S(c_i) c'_i = \sum_i c_i S(c'_i) = \varepsilon(c) 1
\]
where \(\mu(c) = \sum_i c_i \otimes c'_i\). \(S\) is called the antipode of \((C, \mu, \varepsilon)\) and is the unique \(k\) map satisfying (1). A Hopf algebra \((C, \mu, \varepsilon)\) is called commutative if \(C\) is a commutative \(k\) algebra.

Hopf algebras occurring in this thesis are commutative except where an indication to the contrary is given. The most interesting Hopf algebras from our point of view are those arising from algebraic groups. Algebraic groups should be understood to be affine throughout.

Let \(K\) be an algebraically closed field and let \((G, K[G])\) be an algebraic group over \(K\). We regard \(K[G]\) as a set of functions on \(G\) and \(K[G] \otimes K[G]\) as a set of functions on \(G \times G\). \(K[G]\) is naturally a Hopf algebra with structure maps \(\mu, \varepsilon\) given by \(\mu(a)(x,y) = a(xy)\) for any \(a \in K[G], x, y \in G\) and \(\varepsilon(b) = b(1)\) for any \(b \in K[G]\).

Let \((C, \mu, \varepsilon)\) be a Hopf algebra over \(k\), a Hopf ideal is an ideal \(I\) of \(C\) such that \(\mu(I) \leq I \otimes C + C \otimes I\), \(\varepsilon(I) = 0\) and \(S(I) \leq I\). It is clear how \(0/I\) may be given the structure of a Hopf algebra. A Hopf ideal \(I\) is normal if
\[
m_{13}(1 \otimes I \otimes S)(\mu \otimes 1)\mu(I) \leq C \otimes I\]
where \(m_{13} : C \otimes C \otimes C \to C \otimes C\) is the \(k\) map such that \(m_{13}(a \otimes b \otimes c) = ac \otimes b\) for any \(a, b, c \in C\).
If \((G, K[G])\) is an algebraic group and \(H\) is a closed subgroup then \(I = \{a \in K[G] \mid a(h) = 0 \text{ for all } h \in H\}\) is a Hopf ideal. If \(H\) is in addition normal then \(I\) is a normal Hopf ideal. However there are other Hopf ideals of \(K[G]\) which do not arise in this way, in general.

Let \((C, \mu, \varepsilon)\) be a coalgebra over \(k\). A left \(C\) comodule is a pair \((V, \tau)\) where \(V\) is a \(k\) space and \(\tau : V \to V \otimes_k C\) is a \(k\) map satisfying

\[
(\tau \otimes 1)\tau = (1 \otimes \mu)\tau \quad \text{and} \quad (1 \otimes \varepsilon)\tau = 1_V.
\]

We often denote a \(C\) comodule \((V, \tau)\) simply by \(V\). \(\mathcal{M}(C)\) denotes the category of left comodules (and comodule maps).

We may similarly define right comodules.

Let \(V = (V, \tau)\) be a left \(C\) comodule and \(X\) a \(k\) space, we use

\((X) \otimes V\) to denote the left comodule \((X \otimes V, 1 \otimes \tau)\). Let \(\{v_i\}_{i \in I}\) be a basis of \(V\), the elements \(t_{ij}\) determined by

\[
\tau(v_i) = \sum_{j \in I} v_j \otimes t_{ji}
\]

are called coefficient functions and we put \(\operatorname{cf}(V) = k \text{span}\{t_{ij}\}_{i, j \in I}\). Now suppose \(V\) is finite dimensional. Define elements \(f_i\) of \(V^* = \operatorname{Hom}_k(V, k)\) by

\[
f_i(v_j) = \delta_{ij} \text{ for } i, j \in I.
\]

We may give \(V^*\) the structure of a right comodule \((V^*, \tau_R^*)\) by defining

\[
\tau_R^*(f_j) = \sum_{i \in I} t_{ji} \otimes f_i.
\]

If \((C, \mu, \varepsilon)\) is a Hopf algebra having antipode \(S\) then we may give \(V\) the structure of the dual left comodule \((V, \tau_L^*)\) where

\[
\tau_L^*(v_i) = \sum_{j \in I} v_j \otimes S(\tau_{ij})
\]

for any \(i \in I\).
From a coalgebra \((C, \mu, \varepsilon)\) over \(k\) we obtain an algebra \(C^* = \text{Hom}_k(C, k)\) where multiplication is defined by

\[
Y_1 Y_2(c) = \sum_i Y_1(c_i) Y_2(c_i') \quad \text{for} \quad Y_1, Y_2 \in C^* \text{ and } c \in C
\]

with \(\mu(c) = \sum_i c_i \otimes c_i'\). A left \(C\) comodule \((V, \gamma)\) may be regarded as a left \(C^*\) module by defining

\[
\gamma v = \sum_i v_1 Y(c_i) \quad \text{for} \quad \gamma \in C^*, v \in V \text{ and } \gamma(v) = \sum_i v_1 \otimes c_i.
\]

We may refer to a \(C^*\) module arising in this way as a rational \(C^*\) module.

Similarly we may regard a right \(C\) comodule as a right \(C^*\) module, in particular \(C\) becomes a \(C^*\) bimodule in this way.

Let \((V, \gamma)\) and \((V', \gamma')\) be left \(C\) comodules. A \(C\) morphism, or comodule map, \(\theta : V \rightarrow V'\) is a \(k\) map such that

\[
\gamma' \circ \theta = (\theta \otimes 1) \gamma. \quad \text{We write } \text{Hom}_C(V, V') \text{ for the } k \text{ space of all } C \text{ morphisms. } \text{Hom}_C(V, V') \text{ may be naturally identified with } \text{Hom}_{C^*}(V, V').
\]

If \(C = K[G]\) for an algebraic group \(G\) over \(k\), we may identify \(G\) with a subset of \(C^*\) by \(g \rightarrow \varepsilon_g\) where

\[
\varepsilon_g(a) = a(g) \quad \text{for any } g \in G, a \in K[G]. \quad \text{By a rational } KG \text{ module we mean a rational } K[G]^* \text{ module. Thus } K[G] \text{ is a rational } KG \text{ bimodule. If } V \text{ and } V' \text{ are rational } G \text{ modules we may, and frequently do, identify } \text{Hom}_{KG}(V, V') \text{ and } \text{Hom}_{K[G]}(V, V').
\]

Further details of much of the above material may be found in [6] and [12] of the references for part A.
F Complements and injective comodules

Introduction

In this part we investigate U.L. Hopf algebras (see 1.3). Our aim is to understand, in good cases, the injective indecomposable comodules of a Hopf algebra in terms of a finite dimensional subcomodule, an F complement.

In section 1 we give examples of U.L. algebras and establish some general properties. In section 2 we follow a well worn path to reach a theory of restriction and induction for Hopf algebras. This theory is used in section 3 where we concern ourselves with U.L. algebras arising from algebraic Chevalley groups. We give a proof of a conjecture of Humphreys on the nature of the injective comodules of a simply connected algebraic Chevalley group in characteristic $p > 0$. We also calculate fairly explicitly some related F complements. In section 4 we make some general remarks on the nature of the blocks of a simply connected algebraic Chevalley group in characteristic $p > 0$. 
Complements and U.L. algebras

Notation

Throughout part $A, K$ denotes a perfect field of non zero characteristic $p$. In this section we simply write $\otimes$ for $\otimes_K$ and $\dim$ for $\dim_K$.

Let $A$ be a commutative algebra over $K$, $I$ an ideal of $A$, $V, W$ subspaces of $A$ and $n \in \mathbb{N}$.

Define

$$I[p^n] = \sum_{a \in I} A p^n, \quad V(p^n) = \{v p^n | v \in V\}$$

(a subspace) and

$$V \cdot W = K \text{ span } \{vw | v \in V, w \in W\}.$$

Put $V_1 = V, V_n = V_n(V_{n-1}(p)$ so that $V_n$ is spanned by the elements of the form $v_1 v_2^p \ldots v_n^p$, all $v_i \in V$.

By an augmentation algebra we mean a pair $(A, \mathcal{E})$, where $A$ is a commutative algebra over a field $k$, say, and $\mathcal{E}: A \to k$ is a $k$ algebra map. We put $\mathcal{M} = \ker \mathcal{E}$.

Definition A uniformly layered (U.L.) algebra is a Noetherian augmentation algebra $(A, \mathcal{E})$ over $K$ such that

$$\dim A/\mathcal{M}[p^n] = p^{nl} \text{ for each } n \in \mathbb{N} \text{ where } l = \dim \mathcal{M}/\mathcal{M}^2.$$
Remarks

1. For any Noetherian augmentation algebra $A$ over $K$, we have $(1) \dim A/\mathcal{M}^{[p^n]} < p^n$, where $l = \dim \mathcal{M}/\mathcal{M}^2$ (see proof of the lemma of (2) in [11]). Further if 
\[ \{ a_i + \mathcal{M}^2 \}_{i=1}^{l} \text{ is a basis of } \mathcal{M}/\mathcal{M}^2, \text{ we have equality in (1) if and only if } a_1 p^{n-1} a_2 p^{n-1} \ldots a_l p^{n-1} \notin \mathcal{M}^{[p^n]} \, . \]

2. If $(A, \xi)$ and $(A', \xi')$ are U.L. algebras finitely generated over $K$ then $(A \otimes A', \xi \otimes \xi')$ is U.L.

3. Let $(A, \xi)$ be an augmentation algebra over $K$ such that $A$ is an integral domain, we may form an augmentation algebra $(A, \xi)$ where $\xi(a/b) = \xi(a)/\xi(b)$ for $a, b \in A$, $b \notin N$. If $A$ is Noetherian then $(A, \xi)$ is U.L. if and only if $(A, \xi)$ is U.L.

Examples (a) $A = K[X_1, X_2, \ldots, X_l]$, $\xi: A \to K$ any $K$ algebra map.

By a change of coordinates, if necessary, we may assume that $\xi(X_i) = 0$ for all $1 < i < l$. So
\[ \mathcal{M} = \sum_{i=1}^{l} AX_i, \mathcal{M}^{[p^n]} = \sum_{i=1}^{l} AX_i p^n \text{ and } A/\mathcal{M}^{[p^n]} \]

may be identified with the polynomials in $\{X_1, \ldots, X_l\}$ of degree less than $p^n$ in each variable.

In particular if $K$ is algebraically closed and $U$ is a connected unipotent algebraic group over $K$, then $K[U]$ is a free polynomial ring by $\mathbb{G}, (3.3)]$. By the above, if $g \in U$
and \( \xi_g : K[U] \to K \) is evaluation at \( g \) then \((K[U], \xi_g)\) is a U.L. algebra.

(b) Let \( H \) be a torus over \( K \), algebraically closed,
\[ \xi : K[H] \to K \text{ evaluation at 1}. \]
\[ K[H] = K\left[ T_1, \ldots, T_n, \frac{1}{T_1}, \frac{1}{T_2}, \ldots, \frac{1}{T_n} \right] \]
for some \( n \), \( \xi(T_i) = 1 \) for all \( i \), so by example (a) and remark 3, \((K[H],\xi)\) is U.L.

(c) Let \( K \) be algebraically closed, \( B \) any soluble connected algebraic group over \( K \) and \( \xi : K[B] \to K \) evaluation at 1.
Then \( B = H.U \), the semidirect product of a maximal torus \( H \) and \( U \) the subgroup of \( B \) consisting of the unipotent elements.
\[ K[B] = K[H] \otimes K[U] \text{ so } (K[B],\xi) \text{ is U.L. by (a), (b) and remark 2}. \]

(d) Let \( G \) be a connected semisimple algebraic group over \( K \) algebraically closed and \( \xi \) evaluation at 1. \((K[G],\xi)\) is U.L. This is proved in \([1], \text{ Proposition of } 2\) . It also follows from Chevalley's Theorem \([4, \text{ Proposition 1}\) , (a), (b) together with remarks 2 and 3.

1.2

**Definition** A complement of a U.L. algebra \((A,\xi)\) is a subspace \( V \) of \( A \) such that \( 1 \in V \) and \( A = V \oplus M[D] \).

The importance of complements lies in the following.
Theorem. If $V$ is a complement for a U.L. algebra $(A, \mathcal{E})$ then

$$A = V_n + \mathcal{M}[p^{n}]$$

for any $n \in \mathbb{N}$. 

Proof. True for $n=1$ by definition of complement. Assume that $n > 1$ and the theorem holds for $n-1$.

Put $X = V_n + \mathcal{M}[p^{n}]$ and $T_m = (\mathcal{M} \cap V)(p^{n-1}) \cdots (\mathcal{M} \cap V)(p^{n-1})$

$m$ times.

We will prove by induction that, for all $m$, $A = X + AT_m$ (A).

Since $A/\mathcal{M}[p^{n}]$ is local, for some $m'$, $T_{m'} \subseteq \mathcal{M}[p^{n}]$ so it will follow from (A) that $A = X$ as required.

$$\mathcal{M} = \mathcal{M} \cap V + \mathcal{M}[p],$$

by the modular law, so

$$\mathcal{M}[p^{n-1}] = A(\mathcal{M} \cap V)(p^{n-1}) + \mathcal{M}[p^{n+1}] \quad (1).$$

$$A = V_{n-1} + \mathcal{M}[p^{n-1}] = V_{n-1} + A(\mathcal{M} \cap V)(p^{n-1}) + \mathcal{M}[p^{n}]$$

$$= X + A(\mathcal{M} \cap V)(p^{n-1})$$

$$\quad (1 \in V \text{ implies } V_1 \subseteq V_2 \subseteq \ldots \ldots \ldots).$$

Hence (A) is true for $m=1$.

Note $A = V + \mathcal{M}[p]$ gives $A(p^{n-1}) \subseteq V(p^{n-1}) + \mathcal{M}[p^{n}]$

and so $A(p^{n-1})V_{n-1} \subseteq X$ (2).

Suppose $A = X + AT_m$ then

$$A = X + V_{n-1} + \mathcal{M}[p^{n-1}]T_m$$

$$= X + A(\mathcal{M} \cap V)(p^{n-1})T_m = X + AT_{m+1}$$

using (1), (2) and the hypothesis $A = V_{n-1} + \mathcal{M}[p^{n-1}]$. 

Hence (A) is true by induction so

\[ A = X = V_n + M[p^n] \quad (3). \]

Since \( A \) is U.L., \( \dim A/M[p^n] = p \ ndim M/M^2 \)

\[ = (\dim A/M[p^n])^n = (\dim V)^n. \]

There is a K epimorphism

\[ V \otimes V(p) \otimes \ldots \otimes V(p^{n-1}) \rightarrow VV(p) \ldots V(p^{n-1}) = V_n \quad (4) \]

given by \( v_1 \otimes v_2 \otimes \ldots \otimes v_n \rightarrow v_1 v_2 \ldots v_n \).

Hence \( \dim V_n \leq (\dim V)^n \). This shows that the sum (3) is
direct and proves the theorem.

\[ \]

\[ \]

\[ \]

\[ \]

Remark \ As a corollary to the above argument we obtain

\[ \dim V_n = (\dim V)^n, \dim V(p^n) = \dim V \text{ for any } n \geq 0 \]

and the map of (4) is an isomorphism.

\[ \]

Definition \ For \( W \leq A, A \) a commutative K algebra put

\[ W_\infty = \bigcup_{n>0} W_n. \] Say a complement \( V \) of a U.L. algebra \( (A, \mathcal{E}) \)

is strong if \( V_\infty = A \).

Examples \ (a) \( A = K[x], \mathcal{E}(x) = 0. \) Put \( V = K + Kx + \ldots + Kx^{p-1}. \)

\[ M[p] = AX^p \text{ so } V \text{ is a complement. } V_\infty = A. \]

(b) \( A = K[y, \frac{1}{y}], \mathcal{E}(y) = 1. \)
(i) \( V = K + KY + \ldots + K^{p-1} \) is a complement, but \( V_\infty = K[Y] \not= A \), \( V \) is not strong.

(ii) For \( p > 2 \), \( V = K \) span \( \left\{ \frac{1}{y^{p-1}}, \frac{1}{y^{p-2}}, \ldots, \frac{1}{y^{1}}, y \right\} \)

is a strong complement.

We conclude 1.2 with a proposition and its limiting version.

Let \((A, \mathcal{E})\) be a U.L. algebra over \( K \), \( V \) a subspace of \( A \) such that \( V \cap \mathcal{M}^\mathcal{E} = 0 \). Let \( V = V(0) \oplus V(1) \oplus \ldots \oplus V(n) \) be any \( K \) space decomposition of \( V \). Put \( T_m = \) the set of functions \( \{0, 1, \ldots, m\} \to \{0, 1, \ldots, m\} \). For \( f \in T_m \) define

\[ V(f) = V(f(0)) \oplus V(f(1)) \oplus \ldots \oplus V(f(m)) \]

**Proposition (a)** With the above hypotheses

\[ V_{m+1} = \sum_{f \in T_m} V(f). \]

**Proof** \( V_{m+1} \) is spanned by elements of the form

\[ v_{i_0} v_{i_1} \ldots v_{i_m} \]

where \( v_{i_r} \in V(i_r), 0 \leq i_r \leq n. \)

But \( v_{i_0} v_{i_1} \ldots v_{i_m} \in V(f), \) where \( f(r) = i_r \) for \( 0 \leq r \leq m. \)

Hence \( V_{m+1} = \sum_{f \in T_m} V(f) \) \hspace{1cm} (1).

Let \( W \) be a complement for \( A \) containing \( V \). By the remark following the theorem we have \( \dim W(\mathcal{E}) = \dim W \) for \( r > 0 \)
and the natural map \( W \otimes W(\mathcal{E}) \otimes \ldots \otimes W(\mathcal{E}) \to W_{m+1} \) is an
isomorphism of K spaces. Hence $V \leq W$ gives $\dim V^{(P^r)} = \dim V$ and $\dim V(s)^{(P^r)} = \dim V(s)$ for $r \geq 0$, $0 \leq s \leq n$. Also the natural maps $V \otimes V^{(p)} \otimes \ldots \otimes V^{(p^m)} \to V_{m+1}$ and

$$V(f(0)) \otimes V(f(1))^{(p)} \otimes \ldots \otimes V(f(m))^{(p^m)} \to V(f)$$

are isomorphisms of K spaces for any $f \in T_m$.

Thus $\dim V_{m+1} = (\dim V)^{m+1}$ (2) and

$$\dim V(f) = \dim V(f(0)) \cdot \dim V(f(1)) \cdot \ldots \cdot \dim V(f(m))$$

(3).

Using (2) and (3) one can check that $\sum_{f \in T_m} \dim V(f) = \dim V_{m+1}$.

It follows that the sum in (1) is direct as required.

Assume further that $1 \in V(0)$. Put $S = \text{the set of functions } \mathbb{Z}^+ \to \{0, 1, \ldots, n\}$ of finite support. For $f \in S$ put

$$\alpha(f) = \max \{r \mid f(r) \neq 0\}.$$ For $m \in \mathbb{Z}^+$, $f \in S$ define

$$V(f, m) = V(f(0)) \cdot V(f(1))^{(p)} \cdot \ldots \cdot V(f(m))^{(p^m)}$$

and

$$V(f) = \bigcup_{m \geq \alpha(f)} V(f, m).$$

Proposition (b) With the above hypotheses

$$V_\infty = \sum_{f \in S} \bigoplus V(f).$$

Proof. Put $S_m = \{f \in S \mid \alpha(f) \leq m\}$. We see that

$$V_{m+1} \subseteq \sum_{f \in S_m} V(f).$$

Since $V_\infty = \bigcup_{m \geq 1} V_m$ this shows that

$$V_\infty \subseteq \sum_{f \in S} V(f).$$
\( V(f) \subseteq V_\infty \) for any \( f \in S \) is clear so \( V_\infty = \bigcup_{f \in S} V(f) \).

Now suppose \( \sum_{i=1}^{\infty} v_{f_i} = 0 \), each \( v_{f_i} \in V(f_i) \) and \( f_1, \ldots, f_\ell \) distinct elements of \( S \). Pick \( m \in \mathbb{Z}^+ \) such that \( m \geq (f_i) \) for all \( i \) and \( v_{f_i} \in V(f_i, m) \) for all \( i \).

Then \( f_1, \ldots, f_\ell \) are distinct elements of \( S_m \) and the directness of the sum in proposition (a) gives each \( v_{f_i} = 0 \).

---

1.3 We shall be most interested in the augmentation algebra \( (R, \varepsilon) \) arising from a Hopf algebra \((R, \mu, \varepsilon)\) over \( K \).

**Definition** Let \((R, \mu, \varepsilon)\) be a U.L. Hopf algebra over \( K \) (i.e. \((R, \varepsilon)\) is U.L.). An \( F \) complement is a complement for \((R, \varepsilon)\) which is also a left subcomodule of \( R \). That is, an \( F \) complement is a subspace \( V \) of \( R \) such that

\[
1 \in V, \quad \mu(V) \subseteq V \otimes R \quad \text{and} \quad V \oplus [R] = R.
\]

**Remark** One can see that with respect to the appropriate Hopf structures the complements of 1.2(a), 1.2(b) (ii) and 1.2(b) (ii) are \( F \) complements.

**A Criterion for an \( F \) complement**

What follows is largely a reworking of [2] in a Hopf algebra context. Let \((R, \mu, \varepsilon)\) be a Hopf algebra over \( K \).

Put \( T(R) = \{ Y \in R^* \mid Y(rs) = \varepsilon(r)Y(s) + Y(r)\varepsilon(s) \text{ for all } r, s \in R \} \). \( T(R) \) is a Lie algebra with Lie product
\[
[\mathbf{y}_1, \mathbf{y}_2] = \mathbf{y}_1 \mathbf{y}_2 - \mathbf{y}_2 \mathbf{y}_1 \quad \text{for } \mathbf{y}_1, \mathbf{y}_2 \in T(R), \text{ multiplication given by formula 2 of General Remarks. The Lie algebra } T(R) \text{ is restricted with } p \text{ operation } \mathbf{y}^{[p]} = \mathbf{y}^p \text{ for } \mathbf{y} \text{ in } T(R).
\]

Now suppose \(\mathcal{M}^{[p]} = 0\) and \(R\) is finite dimensional.

Put \(L(N) = K \text{ span } \{ \mathbf{y}_1 \mathbf{y}_2 \ldots \mathbf{y}_t \mid y_1 \in T(R) \text{ and } t < N \}\)

\(= K + T(R) + T(R)^2 + \ldots + T(R)^N\).

**Lemma 1** If \(y \in L(N), 1 > N\) then \(ym_{1-1}^N \subseteq \mathcal{M}_{1-N}^N\).

**Proof.** Clearly it is enough to prove the result for \(N = 1\).

If \(y \in T(R), m_1, \ldots, m_1 \in \mathcal{M}, r \in R\) then

\[
y_0(m_1 \ldots m_1) = (y_0(m_1 m_2 \ldots m_1 m_r))m_1
\]

\(= (m_1 \ldots m_1 m_r^0) y_0 m_1^0\).

It is easy to see that the above equation may be used to give a proof of the lemma by induction on \(l\).

**Corollary** If \(y \in L(N), y(\mathcal{M}^N) = 0\).

Let \(\{x_1, \ldots, x_n\}\) be a basis for \(T(R)\), by 13.2.3 of [14] \(T(R)\) generates the restricted enveloping algebra of \(T(R)\), it follows that \(\dim R = p^n\) and that

\[
\mathcal{B} = \{ x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mid 0 \leq \alpha_1 \leq p-1 \} \text{ is a basis for } R^*.
\]

We refer to an element of \(\mathcal{B}\) as a monomial. Let

\[
x_0 = x_1^{p-1} \ldots x_n^{p-1} \text{ and define } \phi: R^* \to K \text{ by }
\]

\[
\phi(D) = \text{coefficient of } x_0 \text{ in an expression of } D \text{ as a sum of monomials, for } D \in R^*.
\]
Define degree of

\[ \lambda_{\alpha} = \sum_{0 \leq \alpha_i \leq p-1} \lambda_{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \]

\[ = \max \{ \sum i \alpha_i \mid \lambda_{\alpha} \neq 0 \} \]

For \( D \in R^* \) write \( \partial(D) = \) degree of \( D \). We define \( \partial(0) = -1 \).

The following facts may easily be verified: if

\[ D_1, D_2 \in R^*, \ D_1 \neq 0 \neq D_2 \] then

\[ \partial(D_1 D_2) < \partial(D_1) + \partial(D_2) \] and \( \partial(D_1 + D_2) < \partial(D_1) + \partial(D_2) \) \( (1) \)

where \( [D_1, D_2] = D_1 D_2 - D_2 D_1 \).

**Lemma 2** If \( \phi(x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_1^{\beta_1} \cdots x_n^{\beta_n}) \neq 0 \) and

\[ \sum_i \alpha_i + \sum_i \beta_i < n(p-1) \] then \( \alpha_i + \beta_i = p-1 \)

for \( 1 \leq i \leq n \).

**Proof** Clearly by \( (1) \) we must have \( \sum i \alpha_i + \sum i \beta_i = n(p-1) \).

If \( (\alpha_1, \ldots, \alpha_n) = (0,0,\ldots,0) \) then it is obvious. Now we prove the lemma by induction on the number of nonzero \( \alpha_i \)'s.

Suppose \( \alpha_t \neq 0 \), then

\[ x_1^{\alpha_1} \cdots x_r^{\alpha_r} \cdots x_n^{\alpha_n} x_1^{\beta_1} \cdots x_n^{\beta_n} = \]

\[ x_1^{\alpha_1} \cdots x_{r-1}^{\alpha_r-1} x_{r+1}^{\alpha_{r+1}} \cdots x_n^{\alpha_n} x_1^{\beta_1} \cdots x_r^{\beta_r} \cdots x_n^{\beta_n} \]

\[ + x_1^{\alpha_1} \cdots x_{r-1}^{\alpha_r-1} x_{r+1}^{\alpha_{r+1}} \cdots x_n^{\alpha_n} x_1^{\beta_1} \cdots x_r^{\beta_r} \cdots x_n^{\beta_n} \]

\[ + \text{terms of smaller degree, by (1).} \]
Hence \( \varphi(x_1^r \cdots x_{n-1}^r \cdot x_{n+1}^{r+1} \cdots x_n^r x_n^{r+1} \cdots x_n^r) \neq 0 \) and so by induction we get \( \alpha_i^r + \beta_i = p^{-1} \), \( i \neq r \) and \( 0 + (\alpha_r^r + \beta_r) = p^{-1} \).

Thus for all \( i, \alpha_i^r + \beta_i = p^{-1} \).

**Proposition** If \( \{a_i + M^2\}_{i=1}^n \) is a basis for \( M/M^2 \) then \( R \) is a free cyclic left \( R^* \) module generated by \( a_1, a_2, \ldots, a_n, p^{-1} \).

**Proof** It is easy to check that \( \{a_1, \ldots, a_n\} \) generate \( R \) as a \( K \) algebra. \( \mathfrak{m}^2 = 0 \) and so each \( a_i^p = 0 \). Now since \( \dim R = p^n \) we get that \( \{a_1^1, \ldots, a_n^1\} \) is a basis for \( R \). By lemma 1 for \( 0 < \alpha_i < p-1 \)

\begin{enumerate}
    \item \( (x_1^{\alpha_1} \cdots x_n^{\alpha_n})(a_0) = 0 \) unless \( \alpha_1 = \alpha_2 = \ldots = \alpha_n = p-1 \)
    \item where \( a_0 = a_1^{p-1} \cdots a_n^{p-1} \).
\end{enumerate}

Hence since \( \{x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n} | 0 < \alpha_i < p-1\} \) is a basis for \( R^* \),
\[
x_1^{p-1} \cdots x_n^{p-1}(a_0) \neq 0.
\]

Thus
\[
(x_1^{p-1} \cdots x_n^{p-1} a_0) \notin \mathfrak{m}.
\]

Now suppose \( D \cdot a_0 = 0 \) for some \( 0 \neq D \in R^* \),
\[
D = \sum \lambda_x x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \text{ say. Suppose that}
\]
\[
\partial(D) = \sum \lambda_x \beta_i \text{ for } \lambda_x \neq 0. \text{ Then}
\]
\[
(x_1^{p-1} - \beta_1 \cdots x_n^{p-1} - \beta_n) a_0 = 0. \text{ Now by the}
\]
above lemma the coefficient of $x_1^{p-1} \ldots x_n^{p-1}$ in an expression of $x_1^{p-1} - \beta_1 \ldots x_n^{p-1} - \beta_n \ D$ as a linear combination of monomials is $\lambda_{\beta}$. And so by (2) we have

$$\lambda_{\beta} (x_1^{p-1} \ldots x_n^{p-1}) (a_0) = 0.$$ But $\lambda_{\beta} \neq 0$ so we have a contradiction to (3). Thus $D = 0$ and so the left annihilator of $a_0 = a_1^{p-1} \ldots a_n^{p-1}$ is 0. Hence

$$\dim R \cdot a_0 = p \dim T(R) = \dim R$$

and so $R \cdot a_0 = R$ and we are done.

**Corollary** $R^*$ is a Frobenius algebra.

**Proof** The natural map $R \rightarrow (R^*)^*$ is a left $R^*$ module isomorphism since $\dim R < \infty$. But the proposition shows $R \cong (R^*)^*$ as left $R^*$ modules thus $\displaystyle \bigoplus_{R^*} R \cong (R^*)^*$, i.e. $R^*$ is Frobenius.

We no longer insist that $M_{[p]} = 0$.

**Lemma 3** Suppose $R$ is a finitely generated Hopf algebra over $K$ and $\{a_i + M^2\}_{i=1, \ldots, n}$ a $K$ basis for $M/M^2$.

$$a_0 = a_1^{p-1} \ldots a_n^{p-1} + M_{[p]}$$ and $\overline{\pi}: R \rightarrow R/M_{[p]}$ the natural map. If $\overline{\pi}(r) = a_0$ then $L_R(r) + M_{[p]} = R$ where $L_R(r)$ is the left subcomodule of $R$ generated by $r$.

**Proof** Work mod $M_{[p]}$ and apply the proposition.
Criterion In the situation of the above lemma, if $R$ is U.L., $\dim L_R(r) \leq p \dim M/M^2$ and $1 \in L_R(r)$ then $L_R(r)$ is an $F$ complement for $R$.

Examples (a) Winter's $F$ complement

$R = K[SL_2(K)] = K[A,B,C,D \mid AD - BC = 1]$ where $K$ is algebraically closed and the functions $A,B,C,D$ are defined by

$$g = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}$$ for any $g \in G = SL_2(K)$.

Put $V = K \text{ span } \{ A^\alpha B^\beta C^\gamma D^\delta \mid 0 \leq \alpha + \beta \leq p-1 \text{ and } 0 \leq \gamma + \delta \leq p-1 \}.$

One can easily check that $V$ is a left rational $G$ module.

$M = R(A - 1) + RB + RC + R(D - 1)$ and

$\{ A - 1 + M^2, B + M^2, C + M^2 \}$ is a basis of $M/M^2$.

Let $\equiv$ denote congruence mod $M/M^2$. Note $A^p - 1 \equiv (A - 1)^p$. Then

$$(A - 1)^{p-1}B^{p-1}C^{p-1} = (A - 1)^{p-1}(AD - 1)^{p-1}$$

for

$$\sum_{r,s=0}^{p-1} A^r B^s C^\gamma D^\delta \equiv \sum_{r,s=0}^{p-1} A^r D^s \in V (1).$$

Clearly $1 = AD - BC \in V$ and using this relation

$V = K \text{ span } \{ A^\alpha B^\beta C^\gamma D^\delta \mid 0 \leq \alpha + \beta, \gamma + \delta \leq p-1 \text{ and } \alpha + \beta = p-1 \text{ or } \gamma + \delta = p-1 \}.$

Hence $\dim_K V \leq p(1 + 2 + \ldots + p) + p(1 + 2 + \ldots + p-1)$

$$= \frac{1}{2}p^2(p + 1) + \frac{1}{2}p^2(p-1) = p^3$$ (2).

Thus by (1), (2) and our criterion $V$ is an $F$ complement.
(b) \( R = K[G] \), \( G = U(3, K) \) the group of upper triangular unipotent 3 by 3 matrices.

\[ R = K[x, y, z] \] where \( x, y, z \) are defined by

\[
\begin{pmatrix}
1 & x(g) & y(g) \\
0 & 1 & z(g) \\
0 & 0 & 1
\end{pmatrix}
\]

for any \( g \in G \).

Define \( A \) and \( B \) by \( A(g) = Y(g^{-1}) \) and \( B(g) = Z(g^{-1}) \) for \( g \in G \). So \( A = XZ - Y, B = -Z \).

Put \( V = K \text{ span } \{ x^a y^b z^c \mid 0 \leq a + b, c \leq p-1 \} \).

Check \( V \) is a left \( G \) submodule of \( R \). The relation

\[
BX + A + Y = 0
\]
gives

\[
V = K \text{ span } \{ x^a y^b z^c \mid 0 \leq a + b, c \leq p-1 \text{ and } \min\{a, c\} = 0 \}.
\]

Hence \( \dim V \leq p^3 \) as in example (a) (1).

\( \tilde{M} = RX + RY + RZ \) and \( \{ X + \gamma^2, Y + \gamma^2, Z + \gamma^2 \} \) is a basis for \( M/\gamma M^2 \). As before \( \equiv \) denotes congruence mod \( \gamma M^2 \).

We have \( x^{p-1}y^{p-1}z^{p-1} = y^{p-1}(A + Y)^{p-1} \equiv A^{p-1}Y^{p-1} \in V \).

\( \uparrow \in V \) so by (1) and the criterion \( V \) is an \( F \) complement.

Remark \( \uparrow \) The \( V \) of example (b) is a special case of the \( F \) complement described in 3.3.

2 It is not difficult to do examples (a) and (b) without appeal to the criterion, see \([5]\).
2 Restriction and Induction

2.1

In this section we develop a theory of restriction and induction for coalgebras and Hopf algebras. This closely parallels, and may be regarded as a generalisation of, the set up in [10]. The theory will be used in the next section to show that certain comodules are injective.

Notation

Throughout this section comodule means left comodule.

Let \((C, \mu, \xi)\) be a coalgebra over a field \(k\). Let \((V, \gamma)\) be a comodule, if \(X\) is any \(k\) space \((X) \otimes V\) denotes the \(C\) comodule \((X \otimes V, 1_X \otimes \gamma)\) as in \([5, 1.2f]\). If \((C, \mu, \xi)\) is a Hopf algebra we denote by \(k\) the trivial one dimensional comodule.

\(\varphi_0\) and \(\varphi^0\)

Let \((A, \rho_A, \xi_A)\) and \((B, \rho_B, \xi_B)\) be coalgebras over \(k\) and let \(\varphi: A \rightarrow B\) be a coalgebra map (see \([12, 1.3f]\)). \(\varphi\) determines functors \(\varphi_0: \mathcal{M}(A) \rightarrow \mathcal{M}(B)\) the \(\varphi\) restriction functor and \(\varphi^0: \mathcal{M}(B) \rightarrow \mathcal{M}(A)\) the \(\varphi\) induction functor.

If \((V, \gamma)\) is an \(A\) comodule then \(\varphi_0(V)\) is defined to be the \(B\) comodule \((V, (1 \otimes \varphi) \gamma)\). If \((V', \gamma')\) is also an \(A\) comodule and \(f \in \text{Hom}_A(V, V')\) then write \(\varphi_0(f)\) for the \(k\) map \(f\) regarded as an element of \(\text{Hom}_B(\varphi_0(V), \varphi_0(V'))\). It is easy to see that \(\varphi_0\) is a covariant functor.
Let \((W, \sigma)\) be a \(B\) comodule, \(\varphi^0(W)\) is the \(A\) subcomodule of \((W) \otimes A\) consisting of the elements \(\sum_i w_i \otimes a_i, w_i \in W, a_i \in A\) such that

\[(T \otimes 1)(\sum_i w_i \otimes a_i) = (1 \otimes (\varphi \otimes 1)\mu_A)(\sum_i w_i \otimes a_i).
\]

This is a subcomodule since \(T \otimes 1\) and \(1 \otimes (\varphi \otimes 1)\mu_A\) may be regarded as \(A\) morphisms : \((W) \otimes A \to (W \otimes B) \otimes A\) and \(\varphi^0(W)\) is the kernel of \(T \otimes 1 - (1 \otimes (\varphi \otimes 1)\mu_A)\).

Suppose \((W', \sigma')\) is also a \(B\) comodule and \(f \in \text{Hom}_B(W, W')\), we define \(\varphi^0(f) : \varphi^0(W) \to \varphi^0(W')\) by

\[\varphi^0(f)(\sum_i w_i \otimes a_i) = \sum_i f(w_i) \otimes a_i.\]

One can check that \(\varphi^0(f)\) is well defined, an element of \(\text{Hom}_A(\varphi^0(W), \varphi^0(W'))\) and that \(\varphi^0\) is a covariant functor.

Define the \(k\) map \(e : \varphi^0(W) \to W\) by

\[e(\sum_i w_i \otimes a_i) = \sum_i E_A(a_i)w_i \text{ for } \sum_i w_i \otimes a_i \in \varphi^0(W).
\]

Using that \(1 \otimes (\varphi \otimes 1)\mu_A - T \otimes 1\) is zero on \(\varphi^0(W)\), one may check that the diagram

\[
\begin{array}{ccc}
\varphi_o(\varphi^0(W)) & \xrightarrow{e} & W \\
(1 \otimes 1 \otimes \varphi)(1 \otimes \mu_A) \downarrow & & \downarrow \sigma \\
\varphi_o(\varphi^0(W)) \otimes B & \xrightarrow{e \otimes 1} & W \otimes B
\end{array}
\]

commutes, i.e. \(e\) is a \(B\) morphism.
Universal Mapping Property

Let \((X, \tau)\) be any \(A\) comodule, then if \(\alpha \in \text{Hom}_B(\varphi^o(X), W)\) there is a unique \(\hat{\alpha} \in \text{Hom}_A(X, \varphi^o(W))\) such that \(e \hat{\alpha} = \alpha\), i.e. the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\hat{\alpha}} & \varphi^o(W) \\
\downarrow{\alpha} & & \downarrow{e} \\
W & & \\
\end{array}
\]
commutes.

**Proof** Define maps \(\Lambda_1 : \text{Hom}_B(\varphi^o(X), W) \rightarrow \text{Hom}_A(X, \varphi^o(W))\) and \(\Lambda_2 : \text{Hom}_A(X, \varphi^o(W)) \rightarrow \text{Hom}_B(\varphi^o(X), W)\) by
\[
\Lambda_1(\alpha) = (\alpha \otimes 1)_\tau \quad \text{and} \quad \Lambda_2(\theta) = \theta \quad \text{for} \quad \alpha \in \text{Hom}_B(\varphi^o(X), W)
\]
and \(\theta \in \text{Hom}_A(X, \varphi^o(W))\). Check \(\Lambda_1, \Lambda_2\) are well defined and are inverse isomorphisms.

**Remark** If \(M\) is an \(A\) comodule, \(f \in \text{Hom}_B(\varphi^o(M), W)\) and \((M, f)\) has the above universal mapping property possessed by \((\varphi^o(W), e)\) it follows that there exist unique \(A\) isomorphisms \(u\) and \(v\) such that
\[
\begin{array}{ccc}
M & \xleftarrow{u} & \varphi^o(W) \\
\downarrow{f} & & \downarrow{e} \\
& & \\
\end{array}
\]
commutes.

**2.2** Restriction and induction have the following properties:

a) \(\varphi^o\) and \(\varphi\) commute with taking a direct sum of comodules;
b) \(\varphi^o\) is left exact, \(\varphi\) is exact;
c) For any $A$ comodule $X$ and any $B$ comodule $W$

\[ \text{Hom}_A(X, \varphi^o(W)) \cong \text{Hom}_B(\varphi^o(X), W) \text{ as k spaces (reciprocity)} \]

d) Let $(C, \mu_C, \xi_C)$ be a coalgebra over $k$ and $\psi: B \to C$ a coalgebra map then (i) $(\psi \varphi)_o$ is naturally isomorphic to $\varphi \psi \varphi$

and (ii) $(\psi \varphi)^o$ is naturally isomorphic to $\varphi^o \psi^o$ (transitivity of induction);

e) Regard $B$ as a left $B$ comodule, then $\varphi^o(B) \cong A$ as left comodules;

f) If $W$ is an injective in $\mathcal{M}(B)$ then $\varphi^o(W)$ is an injective in $\mathcal{M}(A)$.

Suppose further that $A$ and $B$ are (commutative) Hopf algebras and that $\varphi$ is a Hopf algebra map. Recall that the tensor product of two comodules for a Hopf algebra has a natural comodule structure. We have the following additional properties:

g) $\varphi^o$ commutes with taking a tensor product of comodules;

h) If $(V, \gamma)$ is an $A$ comodule and $(W, \Gamma)$ is a $B$ comodule then

\[ \varphi^o(\varphi^o(V) \otimes W) \cong V \otimes \varphi^o(W) \] as $A$ comodules (the adjoint law).

We shall sketch proofs of d) (ii), e) and h).

**Proposition** d) (ii) $(\psi \varphi)^o$ is naturally isomorphic to $\varphi^o \psi^o$.

**Proof** Suppose $(W, S)$ is a $C$ comodule. Let $f_1 : \gamma^o(W) \to W$ be the $C$ morphism defined $f_1(\bigotimes_i w_i \otimes b_i) = \bigotimes_i w_i \xi_C(b_i)$, similarly define a $B$ morphism $f_2 : \varphi^o(\gamma^o(W)) \to \gamma^o(W)$.
Put $f = f_1 f_2$.

Now if $X$ is any $A$ comodule and $\alpha \in \text{Hom}_A((\psi \varphi)_0(X), W)$, then by the universal mapping property for $(\psi^0(W), f_1)$ there exists a unique $\hat{\alpha} \in \text{Hom}_B(\varphi^0(X), \psi^0(W))$ such that $f_1 \hat{\alpha} = \alpha$. By the universal mapping property for $(\varphi^0(\psi^0(W)), f_2)$ there is a unique $\hat{\alpha} \in \text{Hom}_A(X, \varphi^0(\psi^0(W)))$ such that $f_2 \hat{\alpha} = \alpha$. Hence $f \hat{\alpha} = f_1 f_2 \hat{\alpha} = \alpha$. By reversing the steps we get that $\alpha$ is the unique element of $\text{Hom}_A(X, \varphi^0(\psi^0(W)))$ with this property. Thus $(\varphi^0(\psi^0(W)), f)$ has the universal mapping property of 2.1. Hence there exist unique $A$ isomorphisms $f_W, e_W$ such that

$$
\begin{array}{ccc}
\varphi^0(\psi^0(W)) & \xrightarrow{f} & (\psi \varphi) = \psi^0(W) \\
& f_W & \\
& e_W & \xrightarrow{e} \\
\end{array}
$$

Naturality may be checked by constructing the isomorphisms $f_W$ and $e_W$ explicitly.

**Corollary** e) If $B$ is regarded as a left comodule then 

$\varphi^0(B) \cong A$ as left $A$ comodules.

**Proof** Let $k$ denote the one dimensional $k$ coalgebra $(k, \mu, \varepsilon)$ defined by $\mu(1) = 1 \otimes 1$, $\varepsilon(1) = 1$. If $(C, \Delta_C, \varepsilon_C)$ is any $k$ coalgebra then $\varepsilon_C : C \rightarrow k$ is a $k$ coalgebra map and we claim $\varepsilon_C^0(k) \cong C$, as left $C$ comodules.
Now \( \mathcal{E}_C^\circ(k) = \{1 \otimes c | (\mu \otimes 1)(1 \otimes c) = 1 \otimes (\mathcal{E}_C \otimes 1)\mu_0(1 \otimes c)\} \)
\[ = \{1 \otimes c | c \in C\} \cong C \text{ as left } C \text{ comodules.} \]

Now by d) (ii) \( \varphi^\circ(B) \cong \varphi^\circ(\mathcal{E}_B^\circ(k)) \cong (\mathcal{E}_B \varphi)^\circ(k) \)
\[ \cong \mathcal{E}_A^\circ(k) \cong A \text{ as } A \text{ comodules.} \]

**Proposition h)** Let \( (A, \mu_A, \mathcal{E}_A^\circ), (B, \mu_B, \mathcal{E}_B^\circ) \) be Hopf algebras and \( \varphi : A \to B \) a Hopf algebra map. Let \( (V, \tau) \) be a left \( A \) comodule and \( (W, \tau) \) a left \( B \) comodule. Then
\[ \varphi^\circ(V \otimes \varphi^\circ(W)) \cong V \otimes \varphi^\circ(W) \text{ as } A \text{ comodules.} \]

**Proof** Let \( \lambda_1 : \varphi^\circ(V \otimes \varphi^\circ(W)) \to \varphi^\circ(V) \otimes W \) be the \( B \) map given by \( \lambda_1(v \otimes \sum_i w_i \otimes a_i) = v \otimes \sum_i \mathcal{E}_A(a_i)w_i \) for \( v \in V, \sum_i w_i \otimes a_i \in \varphi^\circ(W) \). Let \( \Lambda_1 \) be the \( A \) morphism
\[ V \otimes \varphi^\circ(W) \to \varphi^\circ(V \otimes \varphi^\circ(W)) \] corresponding to \( \lambda_1 \), given by the universal mapping property. \( \Lambda_1 \) satisfies
\[ \Lambda_1(v \otimes \sum_i w_i \otimes a_i) = \sum_{i, l} v_l \otimes w_i \otimes f_i a_i \text{ for } v \in V, \]
\[ \sum_i w_i \otimes a_i \in \varphi^\circ(W) \text{ and } \tau(v) = \sum_i v_i \otimes f_i. \]

Let \( S \) be the antipode of \( A \). Define
\[ \Lambda_2 : \varphi^\circ(V \otimes W) \to V \otimes \varphi^\circ(W) \] to be the \( k \) map such that
\[ \Lambda_2(\sum_{i, j} v_i \otimes w_j \otimes a_{ij}) = \sum_{i, j, l} v_{il} \otimes w_j \otimes S(f_{il})a_{ij} \]
where \( v_i \in V, w_j \in W, a_{ij} \in A \) and \( \tau(v_i) = \sum_l v_{il} \otimes f_{il}. \)
Check that $A_2$ is well defined and that $A_1$ and $A_2$ are inverse isomorphisms of $A$ comodules.

2.3

Theorem Let $(\hat{A}_r, \hat{\mu}_A^r, \epsilon_A)$ and $(B_r, \hat{\mu}_B^r, \epsilon_B)$ be Hopf algebras over $k$ and $\varphi : A \to B$ a Hopf algebra map such that $J = \ker \varphi$ is a normal Hopf ideal of $A$ (see [13]). The identification $\varphi^0(B) \otimes 1 : \varphi^0(B) \to A$ takes $\varphi^0(k)$ to a sub Hopf algebra of $A$.

Proof Put $V = (\hat{\epsilon}_B \otimes 1) \varphi^0(k) = \{a \in A | (\varphi \otimes 1)\mu_A(a) = 1 \otimes a \}$

$\{a \in A | \mu_A(a) \in (J + k) \otimes A \}.$

Clearly $V$ is a sub $k$ algebra of $A$. Let $\{v_i\}_{i \in I}$ be a basis of $V$ and let $S$ be the antipode of $A$.

$\mu(v_i) = \sum_j v_j \otimes f_{ij}$, say. We have

$(\mu \otimes 1)\mu(v_i) = \sum_{j, l} v_l \otimes f_{lj} \otimes f_{ij},$ so $J$ normal gives

$\sum_{j, l} v_l S(f_{ij}) \otimes f_{lj} \in \hat{A} \otimes (J + k).$ Hence for any $m$

$\sum_{j, l, i} v_l S(f_{ij}) f_{im} \otimes f_{lj} \in \hat{A} \otimes (J + k)$ and so

$\sum_{j, l, i} v_l \delta_{j m} \otimes f_{lj} = \sum_{i} v_l \otimes f_{im} \in \hat{A} \otimes (J + k).$ By this and a similar argument it follows that

$\{a \in A | \mu_A(a) \in (J + k) \otimes A \} = \{a \in A | \mu_A(a) \in \hat{A} \otimes (J + k) \}$ (1).

From this it follows that $V$ is a right and left subcomodule of $A$ and it remains to show that $V$ is $S$ invariant.
Let $T: A \otimes A \to A \otimes A$ be the $k$ map defined by
$$T(\sum a_i \otimes a_i') = \sum a_i \otimes a_i'$$
for $a_i, a_i' \in A$.

Since $\mu_A S = T(S \otimes S)\mu_A$ (Corollary 1), we have, for $a \in V$,
$$\mu_A(S(a)) \in T(S \otimes S)((J + k) \otimes A) = A \otimes (J + k).$$
Hence by (1)
$$S(a) \in V.$$

**Corollary 1** If $(A, \mu, \varepsilon)$ is a Hopf algebra over $k$, $H$ a sub
Hopf algebra and $J = A(H \cap \ker \varepsilon)$ then if $\varphi: A \to A/J$ is the
natural map and $\overline{\varepsilon}: A/J \to k$ is defined by $\overline{\varepsilon}(a + J) = \varepsilon(a)$
for any $a \in A$, we have $(\overline{\varepsilon} \otimes 1)\varphi^0(k) = H$.

**Proof** Put $H_1 = (\overline{\varepsilon} \otimes 1)\varphi^0(k)$. We may easily check that
$H \leq H_1$ and $H_1 \cap \ker \varepsilon = J$. Hence $A(H_1 \cap \ker \varepsilon) = J$ and so
by Takeuchi's theorem [13, 4.3], $H_1 = H$.

**Corollary 2** Let $(A, \mu, \varepsilon)$ be a Hopf algebra over $k$, a perfect
field of characteristic $p > 0$. Let $J = \mathcal{M}_{p^n}$, $n \in \mathbb{N}$ and
$\varphi: A \to A/J$ be the natural map. Then $(\overline{\varepsilon} \otimes 1)\varphi^0(k) = A(p^n)$.

**Definition** Let $(A, \mu, \varepsilon)$ be a Hopf algebra over $K$ and let $k$
be a subfield of $K$. A Hopf $k$ form of $A$ is a $k$ subalgebra $A_k$
having the following properties: 1) $A_k$ contains a $k$ basis
which is also a $K$ basis of $A$, 2) $\mu(A_k) \subseteq A_k \otimes A_k$, 3) $\varepsilon(A_k) \subseteq k$,
4) $S(A_k) = A_k$ where $S$ is the antipode of $A$. 
Say $A$ is defined over $k$ if $A$ has a Hopf $k$ form.

Suppose $(A, \mu, \varepsilon)$ is defined over $k$, the field of $q$ elements.
Let $A_k$ be a Hopf $k$ form, define $\varphi_k(a) = a^q$ for each $a \in A_k$.
Let $\varphi : A \rightarrow A$ be the unique $K$ linear extension of $\varphi_k$. One may easily check that $\varphi$ is a Hopf algebra map.

**Proposition** Let $(A, \mu, \varepsilon)$ be as above and $A$ a reduced ring.
Let $\pi : A \rightarrow A/\mathfrak{m}[q]$ be the natural map. If $X$ is an $A$ comodule such that $\pi_0(X)$ is injective and $Y$ an injective $A$ comodule then $X \otimes \varphi_0(Y)$ is an injective $A$ comodule.

**Proof** $Y$ is injective and hence is a direct summand of a direct sum of copies of the left $A$ comodule $A$. Thus by 2.2a it is enough to take $Y = A$.

One may check that $\varphi : \varphi_0(A) \rightarrow A^{(q)}$ is an isomorphism of $A$ comodules. $A^{(q)} \cong \pi^0(K)$ by the second corollary to theorem 2.3. Now $X \otimes \pi^0(K) \cong \pi^0(\pi_0(X))$ by 2.2h, hence by 2.2f this is an injective $A$ comodule.

**Remark** This proposition will be extremely useful in 3 where $A = K[Q]$, the coordinate ring of a simply connected Chevalley group over $K$, an algebraically closed field of characteristic $p > 0$. We will be most interested in the case $p = q$, where $\varphi_0$ corresponds to the Frobenius operation on rational $G$ modules.
2.4. As an illustration of the restriction, induction machinery developed so far we give a proof of part of Steinberg's twisted tensor product theorem.

Our proof makes use of the following general result.

**Lemma.** Let \((R, \mu, \xi)\) be a Hopf algebra over \(K\), algebraically closed, \(J\) a normal Hopf ideal, \(R_J\) the corresponding sub Hopf algebra (see [13]) and \(\varphi: R \rightarrow R/J\) the natural map. If \(V\) is an \(R\) comodule such that \(\varphi_0(V)\) is simple and \(W\) is a simple \(R\) comodule such that \(\text{cf}(W) \leq R_J\) then \(V \otimes W\) is simple.

**Proof.** (Based on an idea of H. Blau).

Let \(0 \neq U \leq V \otimes W\), \(U\) an \(R\) subcomodule of \(V \otimes W\).

Since \(\text{cf}(W) \leq R_J\), \(\varphi_0(V \otimes W) \cong \varphi_0(V) \otimes (W)\) and so \(\varphi_0(U)\) is a direct sum of copies of \(\varphi_0(V)\).

Thus by the simplicity of \(\varphi_0(V)\) and the algebraic closure of \(K\), \(\dim \text{Hom}_{R/J}(\varphi_0(U), \varphi_0(V)) = \dim U / \dim V\) (1).

But \(\text{Hom}_{R/J}(\varphi_0(U), \varphi_0(V)) \cong \text{Hom}_R(U, \varphi^0(\varphi_0(V)))\)

\[\cong \text{Hom}_R(U, V \otimes \varphi^0(K)) \cong \text{Hom}_R(U, V \otimes R_J)\] (2)

by 2.2c, 2.2h, and corollary 1 of the proposition of 2.3.

Now \(R_J\) may be regarded as a Hopf algebra in its own right, thus there is a decomposition

\[R_J = \bigoplus_{\beta \in B, i=1, \ldots, \dim \mathbb{F}} \mathbb{F} \beta_i\]
by \([6, 1.5.1]\), where \(\{W_\beta\}_{\beta \in B}\) is a full set of simple \(R_J\) comodules and \(I_{\beta_i} = I_\beta\) for \(i = 1, \ldots, \dim W_\beta\) \(\beta \in B\) and \(I_\beta\) is an \(R_J\) injective envelope of \(W_\beta\). The \(I_{\beta_i}\) and \(W_\beta\) may be regarded as \(R\) comodules.

Now \(W \cong W_\gamma\) for some \(\gamma \in B\) and so

\[
\dim \text{Hom}_R(U, V \otimes R_J) \geq \sum_{\beta \in B} \dim \text{Hom}_R(U, V \otimes I_{\beta_i}) \implies \dim \text{Hom}_R(U, V \otimes R_J) \geq \dim W \dim \text{Hom}_R(U, V \otimes W).
\]

Now \(\text{soc}_R I_\gamma = W_\gamma \cong W\) so

\[
\dim \text{Hom}_R(U, V \otimes R_J) \geq \dim W \dim \text{Hom}_R(U, V \otimes W).
\]

But \(U \leq V \otimes W\) so by (t) and (2) we get

\[
\dim U \leq \dim W. \quad \text{However } U \leq V \otimes W \text{ gives } \\
\dim U \leq (\dim V)(\dim W) \text{ thus } \dim U = (\dim V)(\dim W)
\]

and \(U = V \otimes W\).

**Proposition** Let \((A, \mu, \Delta)\) be a Hopf algebra over \(K\), algebraically closed of characteristic \(p > 0\), suppose \(A\) is a reduced ring.

Suppose \(A\) is defined over \(F_p\), the field of \(p\) elements and let \(A_p\) be a Hopf \(F_p\) form. For \(n > 0\) let \(\phi(n)\) be the unique \(K\) linear extension of the map \(A_p \to A_p\) taking \(a\) to \(a_p^n\) for each \(a \in A_p\). Let \(\Pi(n) : A \to A_n = A/\mathfrak{m}^n\) be the natural map. Let \(V, W\) be \(A\) comodules such that \(\Pi(n)_o(V)\) is simple and \(\Pi(1)_o(W)\) is simple. Then \(\Pi(n+1)_o(V \otimes \phi(n)_o(W))\) is a simple \(A_{n+1}\) comodule.
Proof. \( \Pi(n+1) \circ (V \otimes \varphi(n) \circ (W)) \)

\[ \cong \left( \Pi(n+1) \circ (V) \otimes (\Pi(n+1) \circ \varphi(n)) \right) \circ (W) \]

by 2.2g and 2.2(d)(i).

We now apply the above lemma with \( R = A_{n+1} \) and \( J = \mathcal{J}_1 \mathcal{J}_1 \).

Let \( \Pi(n,n+1) : A_{n+1} \rightarrow A_n \) be the map taking

\[ a + \mathcal{M}[\mathcal{P}^{n+1}] \rightarrow a + \mathcal{M}[\mathcal{P}^{n}] \text{ for any } a \in A. \]

\[ \Pi(n,n+1) \circ (\Pi(n+1) \circ (V)) \cong (\Pi(n,n+1) \circ \Pi(n+1)) \circ (V) \]

by 2.2d(ii) \( \cong \Pi(n) \circ (V) \) since \( \Pi(n,n+1) \circ \Pi(n+1) = \Pi(n). \)

However \( \Pi(n) \circ (V) \) is simple by hypothesis. It remains to show that \( (\Pi(n+1) \circ \varphi(n)) \circ (W) \) is simple and that its coefficient space lies in \( A_{n+1}(\mathcal{P}^n) \).

\[ \text{cf}((\Pi(n+1) \circ \varphi(n)) \circ (W)) \leq \Pi(n+1) \circ \varphi(n)(\text{cf}(W)) \]

\[ \leq \Pi(n+1) \circ \varphi(n)(A) = \Pi(n+1)(A(\mathcal{P}^n)) = A_{n+1}(\mathcal{P}^n). \]

\( \Pi(n+1) \circ \varphi(n) : A \rightarrow A_{n+1} \) has \( \mathcal{M}[\mathcal{P}] \) in its kernel, thus gives rise to a map \( \hat{\xi} : A_1 \rightarrow A_{n+1}(\mathcal{P}^n) \). By the argument of \( \hat{\xi} \) second lemma of 2, it follows that \( \hat{\xi} \) is an isomorphism of Hopf algebras. However

\[ (\Pi(n+1) \circ \varphi(n)) \circ (W) \cong (\hat{\xi} \Pi(1)) \circ (W) \]

\[ \cong \hat{\xi} \circ \Pi(1) \circ (W), \quad \Pi(1) \circ (W) \text{ is simple by the hypotheses hence } \]

\( \hat{\xi} \) is an isomorphism implies

\[ (\Pi(n+1) \circ \varphi(n)) \circ (W) \text{ is simple. This completes the proof of the proposition.} \]
**Theorem**  Let \((A, \mu, \xi), \varpi(n), \varphi(n)\) be as in the above proposition. Let \(V_0, \ldots, V_n\) be \(A\) comodules such that each \(\varpi(1)_o(V_i)\) is simple, then
\[
\varpi(n+1)_o(V_0 \otimes \varphi(1)_o(V_1) \otimes \ldots \otimes \varphi(n)_o(V_n))
\]
is a simple \(A_{n+1}\) comodule.

**Proof** By induction on \(n\). For \(n = 0\), \(\varpi(1)_o(V_0)\) is simple by hypothesis. Assume true for \(n-1\). This gives
\[
\varpi(n)_o(V_0 \otimes \varphi(1)_o(V_1) \otimes \ldots \otimes \varphi(n-1)_o(V_{n-1}))\]
is simple.

Now in the above proposition we put
\[
V_i = V_0 \otimes \varphi(1)_o(V_1) \otimes \ldots \otimes \varphi(n-1)_o(V_{n-1}), \quad W = V_{n-1}
\]
to obtain that
\[
\varpi(n+1)_o(V_0 \otimes \varphi(1)_o(V_1) \otimes \ldots \otimes \varphi(n-1)_o(V_{n-1}) \otimes \varphi(n)_o(V_n))
\]
is simple as required.

Let \(A = K[G]\) for \(G\) a simply connected algebraic Chevalley group over \(K\), algebraically closed of characteristic \(p > 0\). Let \(\{M(\lambda)\}_{\lambda \in X_p^+}\) be a full set of restricted simple rational modules for \(G\), \(M(\lambda)\) having unique highest weight \(\lambda \in X_p^+\). Here \(X^+\) is the set of dominant weights and \(X_p^+\) the dominant weights which when expressed as a linear combination of fundamental dominant weights have all coefficients between 0 and \(p-1\).

By theorem 6.4 of \([3]\) \(\varpi(1)_o(M(\lambda))\) is simple for \(\lambda \in X_p^+\) and so the above theorem gives
\[(\Pi(n+1))_0 (M(\lambda_0) \otimes M(\lambda_1)^{Fr} \otimes \ldots \otimes M(\lambda_n)^{Fr^n})\]
is simple where \(Fr\) denotes the Frobenius operation on modules and each \(\lambda_i \in \chi_p^+\). This, of course, is stronger than saying that \(M(\lambda_0) \otimes M(\lambda_1)^{Fr} \otimes \ldots \otimes M(\lambda_n)^{Fr^n}\) is simple as an \(R\) comodule which is part of the Steinberg twisted tensor product theorem.
3 F Complements and Chevalley Groups

3.1

Let \((R, \mu, \xi)\) be a coalgebra over \(k\), a field. Let \((V, \gamma)\) be a finite dimensional right (resp. left) \(R\) comodule.

\(V^* = \text{Hom}_k(V, k)\) may be given a right (resp. left) \(R\) comodule structure (see \([6, 1.2]\)) . As in \([6, 1.2]\) we define a \(k\) linear map \(c : V^* \otimes V \to cf(V)\) by

\[c(\alpha \otimes v) = (\alpha \otimes 1)^{-1}(v),\]

where \(\alpha \in V^*\) and \(v \in V\).

Note \(c\) is onto and if we regard \(V^* \otimes V\) as the left comodule \((V^*) \otimes V\) and as the right comodule \(V^* \otimes (V)\) then \(c\) is an \(R^*\) bimodule map. We denote \(c(\alpha \otimes v)\) by \(c_{\alpha, v}\) \((\alpha \in V^*, v \in V)\) and observe \(a c_{\alpha, v} b = c_{a \alpha b, av}\) for \(a, b \in R^*\). We may easily show that if \(c_{\alpha, v} = 0\) for all \(\alpha \in V^*\) then \(v = 0\).

We now put \(R = K[G]\), the coordinate ring of a simply connected Chevalley group \(G\) over \(k\), an algebraically closed field of characteristic \(p > 0\). Put \(R_n = R/\gamma_1^p \vdash P^n\) for \(n \in \mathbb{N}\). We identify \(R_n^*\) with the subspace of \(R^*\) consisting of those elements which vanish on \(\gamma_1^p \vdash P^n\) and further with the \(u_n\) of \([9]\). The identification \(\varphi : u_n \to R_n^*\) is given on the generators \(X_{\alpha, i}\) \((\text{see}[9])\), \(0 \leq i \leq p^n - 1\), \(\alpha\) a root by

\[\sum_{i=0}^{p^n-1} t^i \varphi(X_{\alpha, i})(r) = r(x_{\alpha}(t))\]

for \(r \in R\) and elements \(x_{\alpha}(t)\) of the root subgroup \(X_{\alpha}\).
Roots and root subgroups are computed with respect to the standard maximal torus of diagonal matrices $H$ arising from the Chevalley construction. $U$ (resp. $U'$) denotes the maximal unipotent subgroup which consists of upper (resp. lower) triangular matrices arising from the Chevalley construction.

We identify $G$ with a subset of $\mathbb{R}^*$ by $g \to \mathcal{E}_g$ where $\mathcal{E}_g(r) = r(g)$ for $g \in G$, $r \in \mathbb{A}$.

**Definition** $N^+_0(n)$ (resp. $N^-_0(n)$) = the $K$ span of products of the elements $X_{\alpha, i}$ for $0 \leq i \leq p-1, \alpha > 0$ (resp. $\alpha < 0$).

$N^+(n) = N^+_0(n) + K \mathcal{E}$, $N^-(n) = N^-_0(n) + K \mathcal{E}$. When $n$ is clear we will omit it from the notation just given.

**Lemma 1 (J.A. Green)** Let $L, M$ be left $R_n$ comodules such that

(i) $L = \Delta_n c$ where $N^+_0 c = 0$.

and (ii) $M = \eta_0 m$ where $N^-_0 m = 0$.

Then $L \otimes M = \Delta_n c (l \otimes m)$.

**Proof** We first find an expression for the action of $X_{\alpha, i}$ on a tensor product of $R_n$ comodules. Let $(u_n, \Delta, \epsilon)$ be the noncommutative Hopf algebra dual to $R_n$. If $V, W$ are $R_n$ comodules and $y \in u_n$ then $\gamma_v (v \otimes w) = \sum_{i} \gamma_{i \otimes v} \otimes \gamma_{i \otimes w}$ for $v \in V$, $w \in W$ and $\Delta (y) = \sum_{i} \gamma_{i \otimes i}$.

Now for $r, s \in R$ $(rs)(x_{\alpha}(t)) = r(x_{\alpha}(t))s(x_{\alpha}(t))$

$= \left( \sum_{u} t^u X_{\alpha, u}(r) \right) \left( \sum_{v} t^v X_{\alpha, v}(s) \right)$. Coefficient of $t^i$ is
\[ \sum_{j} X_{\alpha, \beta}^{(r)} X_{\alpha, \beta-1}^{(s)}. \] Hence \[ \Delta(X_{\alpha, \beta}) = \sum_{j<\beta} X_{\alpha, \beta} \times X_{\alpha, \beta-1-j}. \]

Put \( M_1 = \{ y \in M \mid 1 \otimes y \in \mathcal{Z} \} \) where \( Z = u_n(1 \otimes \mathbb{F}). \)

If \( y \in M_1 \), \( 1 \otimes y \in \mathcal{Z} \), so for \( \alpha > 0 \), \( X_{\alpha, 1} \circ (1 \otimes \mathbb{F}) \)

\[ = 1 \otimes X_{\alpha, 1} \circ y \in \mathcal{Z}. \] Hence \( X_{\alpha, 1} \circ y \in M_1 \) and so \( m \in M_1 \) gives \( N^t m = M \leq M_1 \) by \( (ii) \) and so \( M_1 = \mathcal{M} \). Put \( L_1 = \{ x \in L \mid x \otimes M \leq \mathcal{Z} \} \), by a similar argument we get \( L_1 = L \) so \( \mathcal{Z} = L \otimes \mathcal{M} \).

Remark Clearly the right handed version of this lemma works just as well.

Definition If \( X \) is a left comodule for \( R_n \) we define \( X^+ \) (resp. \( X^- \)) to be \( \{ r \in cf(X) \mid r \circ N_0^- = 0 \) (resp. \( r \circ N_0^+ = 0 \} \). \]

Note if \( X \) is simple \( cf(X) \) is a direct sum of copies of \( X \) by \( [6, 1.3a(ii)] \). Every simple \( u_n \) module is the restriction of a simple \( R \) comodule (\([6,2]\)) and it follows from the structure of such comodules that \( X \cong X^+ \cong X^- \), as left \( R_n \) comodules.

\( \{ \xi \in X^* \mid \xi \circ X_{\alpha, \beta} = 0, \text{ all } \alpha < 0, \alpha \leq 0 < 1 \leq p^{n-1} \} = K_l \) for some \( l \neq 0 \). Further \( c : X^* \otimes X \to cf(X) \) is an isomorphism (\([6,1.3a.l]\)) and so \( X^+ = \{ clx \mid x \in X \} \). Similarly for \( Y \) a simple left \( R_n \) comodule we have \( Y^* = \{ c'm, y \mid y \in Y \} \) where \( 0 \neq m \in Y^* \) is such that \( m \circ X_{\alpha, 1} = 0 \) for all \( \alpha > 0 \), \( 0 < j < p^{n-1} \) and \( c' : Y^* \otimes Y \to cf(Y) \) is the natural map.
Lemma 2 (J.A. Green) Let $X, Y$ be simple $R_n$ comodules as above, define 

$$
\psi : X \otimes Y \rightarrow X^+ \otimes Y^- \rightarrow X^+ Y^-
$$

where $x \otimes y \rightarrow c_1, x \otimes c'_m y \rightarrow c_1, x c'_m y$. 

$\psi$ is an isomorphism of $R_n$ comodules.

Proof $\text{cf}(X \otimes Y) = \text{cf}(X) \text{cf}(Y)$ so we have a map $d : (X \otimes Y)^* \otimes (X \otimes Y) \rightarrow \text{cf}(X) \text{cf}(Y)$ and we note that $d_\lambda ^\gamma, x \otimes y = c_\xi ^\lambda, x c'_\mu ^\gamma, y$ for $\xi \in X^*, \lambda \in Y^*, x \in X$, $y \in Y$. So if $z \in X \otimes Y$, $\psi(z) = d_1 \otimes m, z^*$.

Now if $\psi(z) = 0$, $d_1 \otimes m, z^* b = 0$ for all $b \in u_n$. Hence $d(1 \otimes m) c, z = 0$ for all $b \in u_n$. By the right handed version of lemma 1 we have $d_\gamma ^\gamma, z = 0$ for all $\gamma \in (X \otimes Y)^*$. So by a previous remark $z = 0$. Thus $\psi$ is an injective map. By dimensions it is an isomorphism.

Definition For $V$ a left $R = K[G]$ comodule we define $V^+$ (resp. $V^-$) to be $\{r \in \text{cf}(V) \mid \text{reg} = r \text{ for all } g \in U^- \text{ (resp. } U^+)\}$

Definition $R(0) = \{r \in R \mid \text{rch} = r \text{ for all } h \in R\}$. $R(0)$ may be identified with $K[G/H]$ and with $\varphi^0(K)$ where

$\varphi : R \rightarrow K[H]$ is the restriction map.

Theorem $R(0)$ is a U.L. augmentation algebra for which $St^+ St^-$ is an $F$ complement.

Remarks $St_n$ denotes the simple rational $G$ module (i.e., $R$ comodule) having unique highest weight $(\frac{n}{2}) \xi$ where $\xi$ is the half sum of the positive roots. We write simply $St$ for $St_1$. 

R(O) is an augmentation algebra on which G acts as a group of K algebra automorphisms. \( St^+St^- \) is a G submodule of R(O). In view of this structure we refer to \( St^+St^- \) as an \( F \) complement rather than simply a complement.

**Proof of theorem** Let \( \rho_n : R \rightarrow R_n \) be the natural map. We first show \( 1 \in St^+_nSt^-_n \). By arguments similar to those of the above lemmas \( St^+_nSt^-_n \cong St_n \otimes St_n \). Thus since \( St_n \) is self dual we have \( \text{Hom}_G(K, St_n \otimes St_n) \neq 0 \) and so \( \exists 1 \in St^+_nSt^-_n \).

Now \( \rho_n(1) \neq 0 \) gives \( \rho_n(St^+_n) \neq 0 \). \( St_n \) is simple as an \( R_n \) comodule (or \( R_n^* = \mathfrak{u}_n \) module) by [9, Prop 2.3] and \( \rho_n : R \rightarrow R_n \) is an \( R_n \) comodule map. It follows that

\[
\rho_n(St^+_n) = \left[ (\rho_n)_{o(St^+_n)} \right]^+. \text{Hence by the preceding lemma } \\
\rho_n(St^+_nSt^-_n) = ((\rho_n)_{o(St^-_n)})(\rho_n)_{o(St^+_n)} \text{ has dimension } (\dim St_n)^2 = 2^n \dim \mathfrak{u}. \text{Since } \rho_n \text{ restricted to } St^+_nSt^-_n \text{ is injective we have } St^+_nSt^-_n \cap \mathfrak{m}[p^n] = 0, \text{where } \mathfrak{m} \text{ is the augmentation ideal of } R. \text{ Hence}

\[
(1) \quad \dim R(0)/(R(0) \cap \mathfrak{m})[p^n] \geq 2^n \dim \mathfrak{u}.
\]

But \( R(0) = K[G/H] \) so since \( \dim G/H = 2 \dim \mathfrak{u} \) we get\[
\dim R(0) \cap \mathfrak{m} / (R(0) \cap \mathfrak{m})^2 = 2 \dim \mathfrak{u} \text{ (the dimension of the tangent space at 1 = the dimension of G/H since G/H is a smooth affine variety).}
\]

Now by remark 1 of 1.1 we have the reverse inequality to (1). Hence (1) is an equality and \( R(0) \) is a U.L.
augmentation algebra. (augmentation map is the restriction of $\xi$).

Now by dimensions and the fact that $St^+St^- \cap (R(0) \cap M)_{\mathbb{F}} = 0$ we have $St^+St^-$ is an $F$ complement of $R(0)$.

**Lemma 3** Let $W$ be a finite dimensional rational $G$ module (i.e. an $R$ comodule). Let $W = \bigoplus \mathbb{C} W_{\mu}$ be a weight space decomposition of $W$, $W_{\mu}$ being the space of vectors of weight $\mu$.

For all $n$ sufficiently large

$$\dim \text{Hom}_G(St_n, W \otimes St_n) = \dim W_0.$$

**Proof** Put $m(\mu) = \dim W_{\mu}$ for a weight $\mu$ and let $X(\mathbb{H})$ denote the lattice of all weights. Pick $n$ large enough so that

(a) $\mu + (p^n-1)\delta$ is dominant for each weight $\mu$ of $W$ and

(b) $m(\mu) \neq 0$ and $\mu \in p^nX(\mathbb{H})$ implies $\mu = 0$.

Let $\overline{g}$ be the complex Lie algebra from which $G$ has been obtained via the Chevalley construction. For a dominant weight let $V(\lambda)$ be a finite dimensional simple $\overline{g}$ module of highest weight $\lambda$. Let $V(\lambda)$ be a reduction mod $p$ of $V(\lambda)$, reduced via a minimal admissible lattice. For a rational finite dimensional $G$ module $M$, $\text{ch}(M)$ denotes the character of $M$ (computed relative to $H$).

It follows from the proposition of 7.2 of [7] that

(1) $\text{ch}(W)\text{ch}(St_n) = \bigoplus_{\mu} m(\mu)\text{ch}(\mu + (p^n-1)\delta)$

where $\text{ch}(V) = \text{ch} \cdot V(\nu)$ for a dominant weight $\nu$. 
Now as a $\mathfrak{u}_n$ module $\text{St}_n$ is projective and simple. It is therefore the unique simple $\mathfrak{u}_n$ module in its block, say $B(e)$ where $e$ is the corresponding central idempotent of $\mathfrak{u}_n$.

$G$ acts rationally on $\mathfrak{u}_n$ by $\theta^g = \varepsilon_{g^{-1}} \theta \varepsilon_g$ for $\theta \in \mathfrak{u}_n$ and $g \in G$. It acts as a group of $K$ algebra automorphisms and hence permutes the central idempotents. Since $G$ is connected and $\mathfrak{u}_n$ has only finitely many central idempotents $G$ fixes each central idempotent. It follows that any $\mathfrak{u}_n$ block component of a rational $G$ module is a $G$ component.

Let $Y$ be the $B(e)$ block component of $W \otimes \text{St}_n$. From (1) we deduce that $\text{St}_n$ occurs at least $m(0)$ times as a $G$ composition factor of $W \otimes \text{St}_n$.

Hence (2) \[ \dim Y / \dim \text{St}_n \geq m(0). \]

Now suppose $M$ is a $G$ composition factor of $Y$. Since $Y$ as a $\mathfrak{u}_n$ module is a direct sum of copies of $\text{St}_n$ we must have

\[ M \cong \text{St}_n \otimes M(\lambda)^{Fr_n}, \]

for some dominant weight $\lambda$ where $Fr$ denotes the Frobenius operator on modules. Here $M(\lambda)$ denotes the simple rational $G$ module having highest weight $\lambda$. (3) follows from the Steinberg twisted tensor product theorem and the proposition in 2.3 of [9].

Since $V(\overline{\lambda'})$ is an indecomposable $G$ module for any dominant $\lambda'$, a $G$ composition factor of $V(\overline{\lambda'})$ is in $B(e)$ if and only if $V(\overline{\lambda'})$ is in $B(e)$ if and only if $M(\overline{\lambda'})$ is in $B(e)$. 
If \( M((p^n - 1) \delta + \mu) \) is in \( B(e) \) for a weight \( \mu \) of \( W \) then
\[ (p^n - 1) \delta + \mu = (p^n - 1) + p^n \lambda \] for some dominant weight \( \lambda \) by (3). Hence \( \mu \in p^nX(H) \) and so \( \mu = 0 \) by (b). Now (1) gives
\[ \dim Y / \dim St_n = m(0). \] From (3) and (4) we obtain that all \( G \) composition factors of \( Y \) are isomorphic to \( St_n \). Thus by the lemma of 4.1 of \([9]\) \( Y \) is completely reducible and it follows that
\[ \dim Hom_G(St_n, W \otimes St_n) = m(0). \]

Remark The argument used in the above lemma is similar to, and modelled on, the argument used by Humphreys in 5.1 of \([9]\).

**Proposition** The \( F \) complement \( St^n + St^- \) for \( R(0) \) is strong.

**Proof** Put \( B = (St^n + St^-)_{\infty} \). Let \( V \) be any finite dimensional rational \( G \) module, then
\[ \text{Hom}_G(V, (St^n + St^-)_n) \cong \text{Hom}_G(V, St_n \otimes St_n) \cong \text{Hom}_G(St_n^*, V^* \otimes St_n) \cong \text{Hom}_G(St_n, V^* \otimes St_n) \] (1) since \( St_n \) is self dual.

For a finite dimensional rational left \( G \) module \( W \) we are using \( W^* \) to denote the left dual module.

By (1) and lemma (3) we have
\[ \dim Hom_G(V, (St^n + St^-)_n) = m(0) \text{ for all } n \text{ large enough,} \]
where \( m(0) \) is the dimension of the zero weight space of \( V \).

Hence \( \dim Hom_G(V, B) = m(0). \)

Note also that if \( \varphi : R \rightarrow K[H] \) is the restriction map
\[ \dim \text{Hom}_G(V, R(0)) = \dim \text{Hom}_G(V, \varphi^0(K)) \]
\[ = \dim \text{Hom}_H(\varphi_o(V), K) = m(0). \]

Hence the natural map \( \text{Hom}_G(V, B) \to \text{Hom}_G(V, R(0)) \) is an identification. Now suppose that \( B \neq R(0) \). Pick \( v \in R(0) \setminus B \) and put \( V = KG_v \). Consider the embedding \( i : V \to R(0) \) given by \( i(v) = v \) for all \( v \in V \). Since \( V \notin B \)
\[ i \in \text{Hom}_G(V, R(0)) \setminus \text{Hom}_G(V, B) \] which is a contradiction. Thus \( B = R(0) \) and \( St^+St^- \) is a strong F complement.

3.2
Let \( X(H)_p \) denote the set of restricted dominant weights. For \( \mu \in X(H)_p \) we define \( \hat{\mu} \) to be \( (p-1)\xi + \omega_0(\mu) \) where \( \omega_0 \) is the unique element of the Weyl group of \( G \) which takes all positive roots to negative roots.
\[ M(\mu) \] occurs exactly once as a \( G \) and \( \sim \mu \) submodule of \( M(\hat{\mu}) \otimes \text{St} \) \([7, 8.2]\). Let \( \text{K}(\hat{\mu}) \otimes \text{St} = Y(\mu) \oplus Z \) be a G decomposition such that \( Y(\mu) \) is indecomposable and \( K(\mu) \) occurs as a \( G \) submodule of \( Y(\mu) \). We call \( Y(\mu) \) the infinitesimally injective cover of \( K(\mu) \). It is determined up to \( G \) isomorphism by the Krull-Schmidt theorem.

Theorem \([1]\) Let \( h \) be the Coxeter number of \( G \). If \( p > 2h - 2 \) then, for each \( \mu \in X(H)_p \), \( Y(\mu) \) regarded as a \( \sim \mu \) module is the \( \sim \mu \) injective cover of \( K(\mu) \).
We will say that hypothesis \( H \) holds if, for all \( \mu \in \mathcal{X}(H)_p \), \( Y(\mu) \) regarded as a \( \mathcal{A}_1 \) module is the \( \mathcal{A}_1 \) injective cover of \( K(\mu) \).

Note (H) is true in all known cases and conjectured to be true in general. We assume for the rest of the section that \( G \) is a group for which (H) is true.

**Theorem** Let \( Y(0) \) be realised inside \( \text{St}^+ \text{St}^- \). \( I = Y(0)_\infty \) is the rational \( G \) injective cover of \( K(0) \). The rational \( G \) injective cover of \( K(\lambda_0) \otimes K(\lambda_1)^{\text{Fr}} \otimes \cdots \otimes K(\lambda_n)^{\text{Fr}^n} \), where all \( \lambda_i \in \mathcal{X}(H)_p \) and \( \lambda_n \neq 0 \) is

\[
Y(\lambda_0) \otimes Y(\lambda_1)^{\text{Fr}} \otimes \cdots \otimes Y(\lambda_n)^{\text{Fr}^n} \otimes I^{\text{Fr}^{n+1}}.
\]

**Proof** \( Y(0) \mid \text{St}^+ \text{St}^- \) hence by proposition (b) of 1.2

\( I = Y(0)_\infty \mid (\text{St}^+ \text{St}^-)_\infty \). By the proposition of 3.1 \( (\text{St}^+ \text{St}^-)_\infty = R(0) \) which is injective (since it is a \( G \) summand of \( R \)).

Now by repeated application of the proposition of 2.3 we get \( Y(\lambda_0) \otimes Y(\lambda_1)^{\text{Fr}} \otimes \cdots \otimes Y(\lambda_n)^{\text{Fr}^n} \otimes I^{\text{Fr}^{n+1}} \) is injective. Put \( \lambda = \sum_{i=0}^{n} p_i \lambda_i \) and

\[
I(\lambda) = Y(\lambda_0) \otimes Y(\lambda_1)^{\text{Fr}} \otimes \cdots \otimes Y(\lambda_n)^{\text{Fr}^n} \otimes I^{\text{Fr}^{n+1}}.
\]

It remains to prove that the \( G \) socles of \( I \) and \( I(\lambda) \) are simple.

By the argument of proposition (a) of 1.2

\( Y(0)_m \cong Y(0) \otimes Y(0)^{\text{Fr}} \otimes \cdots \otimes Y(0)^{\text{Fr}^{m-1}} \). Hence by the theorem of [8, 1.1] \( \text{soc}_m Y(0)_m \cong K(0) \) as \( \mathcal{A}_m \) modules for any \( m > 0 \).
Now by a Clifford's theorem argument

$$\soc_G(Y(0)_m) \leq \soc_{\mathfrak{o}_m}(Y(0)_m)$$
and so

$$\soc_G(Y(0)_m) \equiv M(0).$$

Finally since $I = \bigcup_{m>0} Y(0)_m$, $\soc_G(I) \equiv M(0)$.

A similar argument shows that

$$\soc_G(I(\lambda)) \cong M(\lambda_0) \otimes M(\lambda_1)^{Fr} \otimes \ldots \otimes M(\lambda_n)^{Fr^n}.$$ 

Notes 1. We did not use the full strength of hypothesis (H) in proving the theorem. It is enough to assume that for each $\mu \in X(H)_F$, $Y(\mu)$ is a $G$ module which when viewed as a $u_\lambda$ module is isomorphic to the $u_{\lambda}$ injective cover of $M(\mu)$ and that $Y(0)$ is a $G$ summand of $St \otimes St$.

2. This theorem was first proved for $SL_2(K)$ by P. H. Winter [14]. It was conjectured to be true generally by Humphreys. A version of it has been proved by J. W. Ballard [14].

3.3

We now apply the proposition of 3.3 to produce a strong $F$ complement for $K[U]$.

Lemma Let $\overline{\pi} : R \rightarrow K[U]$ be the restriction map.

$$\overline{\pi}(R(0)) = K[U].$$

Proof (a) $G = SL_n(K)$.

$U =$ the unipotent upper triangular matrices in $SL_n(K)$

$H =$ the diagonal matrices in $SL_n(K)$.

Let $X_{ij}$ be the coordinate functions for $G$, $1 \leq i, j \leq n$. 

Let \( Y_{ij}, 1 \leq i < j \leq n \) be the coordinate functions for \( U \).

It is easy to see that

\[
Z_{ij} = X_{i1} \cdots X_{i-1, i-1} X_{ij} X_{i+1, i+1} \cdots X_{mn} \in R(O).
\]

Moreover for \( i < j \), \( \Pi(Z_{ij}) = Y_{ij} \). \( K[U] \) is generated by the \( Y_{ij} \) and \( \Pi \) is a \( K \) algebra map, hence \( \Pi(R(O)) = K[U] \).

(b) Let \( G \) be an arbitrary simply connected Chevalley group over \( K \). \( G \leq \text{SL}_n(K) \) and

\( U \leq U_0 = \) the upper triangular unipotent matrices

\( H \leq H_0 = \) the diagonal matrices in \( \text{SL}_n(K) \).

Put \( G_o = \text{SL}_n(K), K[G_o](H_0) = \) the elements of \( K[G_o] \) fixed by the right \( H_0 \) action. The diagram

\[
\begin{array}{ccc}
K[G_o](H_0) & \xrightarrow{\alpha} & K[U] \\
\downarrow & & \downarrow \\
R(O) & \xrightarrow{\gamma} & K[U]
\end{array}
\]

commutes where all maps are restrictions. Moreover \( \alpha \) and \( \beta \) are onto, hence \( \gamma \) is onto and the result is established.

**Theorem** \( \Pi(St^+) \) is a strong \( F \) complement for \( K[U] \).

**Proof** First observe that if \( f \in St^- \) then for \( u \in U \),

\[
f(u) = f(\text{eu}(1)) = f(1) \text{ so } \Pi(St^-) < K.
\]

Now \( 1 \in St^+St^- \) and \( \Pi(1) = 1 \) so \( \Pi(St^+St^-) \neq 0 \). It follows that \( \Pi(St^-) = K \) and \( \Pi(St^+) = \Pi(St^+St^-) \).
Let \( \mathcal{M}_0 \) = the augmentation ideal of \( R(0) \)
\[ \mathcal{M}_U = \text{the augmentation ideal of } K[U] \]
\( \text{St} + \text{St}^- + \mathcal{M}_0 \mathcal{[p]} = R(0) \) (theorem of 3.1) and so
\[ \mathcal{W}(\text{St} + \text{St}^-) + (\mathcal{M}_0 \mathcal{[p]}) = \mathcal{W}(R(0)) = K[U] \] by the above lemma.

But \( \mathcal{W}(\mathcal{M}_0 \mathcal{[p]}) \leq \mathcal{W}_U \mathcal{[p]} \) hence
\[ (1) \quad \mathcal{W}(\text{St} + \text{St}^-) + \mathcal{W}_U \mathcal{[p]} = K[U] \]

Now \( \dim \mathcal{W}(\text{St} + \text{St}^-) = \dim \mathcal{W} \text{St}^+ < p \dim U \) and
\[ \dim K[U] / \mathcal{W}_U \mathcal{[p]} = p \dim U. \]

Hence the sum of (1) is direct and \( \mathcal{W}(\text{St}^+) = \mathcal{W}(\text{St} + \text{St}^-) \)
is an \( F \) complement. It is strong since
\[ (\mathcal{W}(\text{St} + \text{St}^-))_\infty = \mathcal{W}(\text{St} + \text{St}^-)_\infty = \mathcal{W}(R(0)) = K[U]. \]

3.4

Let \( V \) be an \( F \) complement for a U.L. Hopf algebra \( (A, \mu, \lambda) \) over \( K \).

Let \( \rho : A \to A_1 = A/\mathcal{M}_0 \mathcal{[p]} \) be the natural map.

**Definition** Say \( V \) is a full \( F \) complement for \( A \) if there is an
\( A \) comodule decomposition
\[ V = V(0) \oplus V(1) \oplus \ldots \oplus V(n) \]
such that each \( \rho_0(V(r)) \) is indecomposable (as an \( A_1 \) comodule).

**Theorem** \( R = K[G] \) has a strong and full \( F \) complement.

**Proof** Let \( R = \bigoplus_{\lambda \in \mathcal{X}(H)^+} \oplus (I(\lambda, 1) \oplus \ldots \oplus I(\lambda, n_\lambda)) \)
be a decomposition of \( R \) into injective indecomposables, where
$\text{soc}_R(I(\lambda, i)) \cong M(\lambda)$, $m_\lambda = \dim M(\lambda)$ and $X(H)^+$ denotes the set of dominant weights. For $\lambda \in X(H)_p$, $1 \leq i \leq n$ let $Y(\lambda, i)$ be a copy of the infinitesimally injective comodule $Y(\lambda)$ realised inside $I(\lambda, i)$. The existence of such a $Y(\lambda, i)$ is ensured by the injectivity of $I(\lambda, i)$ and the fact that $\text{soc}_R(Y(\lambda)) \cong \text{soc}_R(I(\lambda, i))$.

Put $V = \bigoplus_{\lambda \in X(H)_p} \bigoplus(Y(\lambda, 1) \oplus \cdots \oplus Y(\lambda, m))$ (1).

Claim $V$ is a strong end full $F$ complement for $R$. $V$ contains $1$ since it contains a copy of $N(0)$.

$$\dim V = \sum_{\lambda \in X(H)_p} \dim M(\lambda) \dim Y(\lambda) = \dim \mu_1 = p \dim G.$$ This is true since $\{M(\lambda)\}_{\lambda \in X(H)_p}$ is a full set of simple $R/\mathfrak{m}[p]$ comodules [3, 6.6] and $Y(\lambda)$ as an $R/\mathfrak{m}[p]$ comodule is an $R/\mathfrak{m}[p]$ injective cover of $M(\lambda)$ for each $\lambda \in X(H)_p$.

To complete the proof that $V$ is an $F$ complement we must show that $V \cap \mathfrak{m}[p] = 0$.

Now $V$ and $\mathfrak{m}[p]$ are $\mu_1$ modules so it is enough to prove that $\text{soc}_{\mu_1}(V) \cap \mathfrak{m}[p] = 0$.

$$\text{soc}_{\mu_1}(V) = \bigoplus_{\lambda \in X(H)_p} \text{soc}_{\mu_1}(Y(\lambda, i)) = \bigoplus_{\lambda \in X(H)_p} \text{soc}(M(\lambda)).$$ Since $\{M(\lambda)\}_{\lambda \in X(H)_p}$ remain simple on restriction to $\mu_1$ it follows that the natural map $\bigoplus_{\lambda \in X(H)_p} \text{soc}(M(\lambda)) \rightarrow R/\mathfrak{m}[p]$ is an injection. Hence $\text{soc}_{\mu_1}(V) \cap \mathfrak{m}[p] = 0$. 


Thus $V$ is an $F$ complement moreover it is full by construction. Using the decomposition (1) and proposition (b) of 1.2 we may decompose $V_\infty$ as the direct sum of comodules, each of which is injective by the theorem of 3.2. Hence $V_\infty$ is injective.

Put $W = \text{soc}_R(V) = \bigoplus_{\lambda \in \mathcal{X}(H)} \text{soc}(M(\lambda))$.

We may deduce from Steinberg's twisted tensor product theorem that $\text{soc}_R(R) = W_\infty$. Hence $\text{soc}_R(R) \leq V_\infty$. Now it follows that an $R$ comodule complement to $V_\infty$ in $R$ must have zero socle, hence $V_\infty = R$.

Remarks 1 We have shown that the decomposition (1) gives rise to, via proposition (b) of 1.2, a decomposition of $R$ into injective indecomposables.

2 This method of producing an $F$ complement is, in practice, non constructive. We would very much like to be able to write down, in terms of polynomials in the coordinate functions, $F$ complements for simply connected Chevalley groups other than $\text{SL}_2$. 
4. Remarks on Blocks

As usual $G$ is a simply connected algebraic Chevalley group over $K$, an algebraically closed field of characteristic $p > 0$. Let $\{I(\lambda)\}_{\lambda \in X(H)^+}$ be a full set of injective indecomposable $K[G]$-comodules such that $\text{soc}_{G}(I(\lambda)) \cong \Pi(\lambda)$ for each $\lambda \in X(H)^+$. For $\lambda, \lambda' \in X(H)^+$, $c_{\lambda', \lambda} = \dim \text{Hom}_{G}(I(\lambda), I(\lambda'))$ (see [6, 2.5]). It follows from [6, 2.5.5] that $c_{\lambda', \lambda} \neq 0$ if and only if $c_{\lambda, \lambda'} \neq 0$ (in fact as Winter suggests [5, p53] $c_{\lambda, \lambda'}$ is either $0$ or $\infty$). We say $\lambda$ is adjacent to $\lambda'$ if $c_{\lambda, \lambda'} \neq 0$ and let $\sim$ be the equivalence relation generated by adjacency. We say $M(\lambda')$ and $M(\lambda'')$ are in the same block or $\lambda'$ and $\lambda''$ are in the same block if $\lambda' \sim \lambda''$. We think of a block as an equivalence class of $X(H)^+$ under $\sim$.

Let $B$ be a block, we define $B^C = \{(p-1)\delta + p\lambda \mid \lambda \in B\}$.

**Proposition** If $B$ is a block then $B^C$ is a block.

**Proof** To prove this we use the following fact (*) which will be proved later.

(*) If $\lambda \in X(H)^+$ then $I((p-1)\delta + p\lambda) \cong \text{St} \otimes I(\lambda)^{Fr}$.

Suppose that $\lambda'$ and $\lambda''$ are adjacent, i.e. $M(\lambda'')$ is a composition factor of $I(\lambda')$. Using (*) and the Steinberg twisted
tensor product theorem we see that $M((p-1)\delta + p\lambda''')$ is a composition factor of $I((p-1)\delta + p\lambda') \cong St \otimes I(\lambda)^{Fr}$. Hence $(p-1)\delta + p\lambda'''$ and $(p-1)\delta + p\lambda'$ are adjacent and so since $\sim$ is the equivalence relation generated by adjacency any two elements of $B^\theta$ are equivalent.

We must show that if $\lambda' \in B^\theta$ and $\lambda'' \sim \lambda'$ then $\lambda'' \in B^\theta$.

Clearly we may assume that $\lambda''$ is adjacent to $\lambda'$.

$\lambda' \in B^\theta$ implies that $\lambda' = (p-1)\delta + p\lambda$ for some $\lambda \in B$ and so $I(\lambda') \cong St \otimes I(\lambda)^{Fr}$. But now $M(\lambda''')$ is a composition factor of $I(\lambda')$ implies that $\lambda''' = (p-1)\delta + p\mu$ for some $\mu$ such that $M(\mu)$ is a composition factor of $I(\lambda)$, by Steinberg's twisted tensor product theorem. Thus $\lambda''' \in B^\theta$.

Definition (see[7,3.1,3.2]) If $\lambda, \lambda' \in X(\mathbb{H})$ we say that $\lambda$ and $\lambda'$ are linked if there is a $w \in W$, the Weyl group, such that $w(\lambda + \delta) = \lambda' + \delta$ mod $px(\mathbb{H})$. Clearly linkage defines an equivalence relation.

We have the following linkage principle ([7,3.1,3.2]) ; if $M(\lambda)$ and $M(\mu)$ are composition factors of an indecomposable rational $G$ module then $\lambda$ and $\mu$ are linked in $X$.

Let $L$ be any linkage class of $X(\mathbb{H})$ not containing $(p-1)\delta$, put $B_{i,L} = \{ \lambda \in X(\mathbb{H})^* | \lambda = \lambda_0 + p\lambda_1 + \ldots + p^{i-1}\lambda_{i-1} + p^i\lambda_i + \ldots + p^r\lambda_r \}$ where $\lambda_0 = \lambda_1 = \ldots = \lambda_{i-1} = (p-1)\delta$, $\lambda_i \in L$ and all $\lambda_j \in I(H)_{p^j}$. 

\[(A)\]
The following theorem has been proved by Humphreys in [8] using the theorem of 3.2; we now give a proof which does not involve theorem 3.2 and hence proves the theorem for all primes.

**Theorem** Weights belonging to a given block of $K[G]$ all lie in one of the sets $B_{i,L}$.

**Proof** By induction on $i$. For $i = 0$ this is just the linkage principle. Suppose true for $i \geqslant 0$, then

$$B_{i,L} = \bigcup_{\alpha \in J_{i,L}} B_{\alpha}$$

lying in $B_{i,L}$. Now apply the operation $\Theta$ to both sides to get

$$B_{i+1,L} = \bigcup_{\alpha \in J_{i,L}} B_{\alpha}^\Theta.$$ 

Thus $B_{i+1}$ is a union of blocks by the above proposition and we are done.

**Notes**

1. Humphreys suggests that after further partitioning the $B_{i,L}$ corresponding to cosets of the root lattice in $X(H)$ we should obtain precisely the blocks of $K[G]$.

2. By the proposition of 2.3 $\text{St} \otimes I(\lambda)^{Fr}$ is injective for any $\lambda \in X(H)^+$. We may show it is isomorphic to $I((p-1)\delta + p\lambda)$ either by showing it has a simple socle or that it is indecomposable. We may show it has a simple socle by a generalisation of the lemma of 2.4. However for the sake of variety we adopt the second approach and proceed by the following general result.
Theorem  Let $G$ be an algebraic group over $K$, an algebraically closed field, $R = K[G]$, $J$ a normal Hopf ideal, $R/J$ the corresponding sub Hopf algebra and $\varphi : R \to R/J$ the natural map. Suppose $V$ and $W$ are $R$ comodules such that $\varphi_0(V)$ is an indecomposable $R/J$ comodule of finite width and $W$ is an indecomposable $R$ comodule of finite width such that $\varphi(W) \leq R_J$.

Then $V \otimes W$ is an indecomposable $R$ comodule of finite width.

Proof  We first show that $V \otimes W$ has finite width (see (3) definition in 1.1). Let $M$ be a finite dimensional $R$ comodule. Since $\varphi_0(V)$ has finite width there is a finite dimensional subspace $X \leq \varphi_0(V)$ such that

$$\Theta(M) \leq X \text{ for all } \Theta \in \text{Hom}_{R/J}(\varphi_0(M), \varphi_0(V)).$$

It follows that if $V_1$ is a finite dimensional submodule of $V$ containing $X$ $\text{Hom}_R(K_*, V_1 \otimes W)$ may be identified with

$$\text{Hom}_R(M, V_1 \otimes W).$$

But

$$\text{Hom}_R(K_*, V_1 \otimes W) \cong \text{Hom}_R(M \otimes V_1^*, W)$$

which is finite dimensional by the proposition of 1.1 of (B), using the fact that $W$ is an $R$ comodule of finite width. Hence for any finite dimensional $R$ comodule $W$, $\text{Hom}_R(K, V \otimes W)$ is finite dimensional and so $V \otimes W$ has finite width.

Put $E = \text{End}_R(V \otimes W)$, $D = \text{End}_{R/J}(\varphi_0(V))$ and $F = \text{End}_R(W)$. 


G acts on D by conjugation, i.e. for \( g \in G \), \( \alpha \in D \), \( \alpha^g \) is defined by \( \alpha^g(v) = g^{-1} \alpha (gv) \) for \( v \in V \). Now \( \varphi_\alpha(v) \) is indecomposable of finite width and so by (B) \( \mathcal{F} \) proposition 1 \( D/J(D) \cong K \), where \( J(D) \) is the Jacobson radical of \( D \).

Let \( \lambda : D \to K \) be the natural map. \( J(D) \) is fixed under conjugation by \( g \) since \( g \) acts as an algebra automorphism of \( D \), thus \( \lambda(\alpha^g) = \lambda(\alpha) \) \( (1) \) for any \( \alpha \in D \), \( g \in G \).

Let \( \{ x_i \}_{i \in I} \) be a fixed basis of \( W \), then if \( \Theta \in E \) we define \( K \) endomorphisms \( \Theta_{ij} \) for \( (i,j) \in I \times I \) by

\[
\Theta(v \otimes x_i) = \sum_{j \in I} \Theta_{ji}(v) \otimes x_j.
\]

Now \( \text{cf}(W) \leq R_j \) implies \( \varphi_\alpha(v \otimes v) \cong \varphi_\alpha(v) \otimes (v) \) and so each \( \Theta_{ij} \) is an \( R/J \) map.

Define \( \Lambda : E \to F \) by

\[
\Lambda(\Theta)(x_i) = \sum_{j \in I} \lambda(\Theta_{ji})x_j \quad \text{for} \quad \Theta \in E.
\]

There are two things to check: (a) that the sum is finite; and (b) that \( \Lambda(\Theta) \) is an \( R \) map. (a) follows from an argument similar to one used in the theorem of 3.4 of (B). We now prove (b).

For any \( g \in G \),

\[
\Lambda(\Theta)(gx_i) = \Lambda(\Theta)(\sum_{j \in I} r_{ji}(g)x_j)
\]

\[
= \sum_{j, k \in I} \lambda(\Theta_{kj})r_{ji}(g)x_k \quad \text{where} \quad r_{ji} \text{ are defined by}
\]

\[
gx_i = \sum_{j \in I} r_{ji}(g)x_j.
\]
\[ g \Delta(\Theta)(x_i) = g \sum_{j \in I} \lambda(\Theta_{ji})x_j = \sum_{k,j \in I} \lambda(\Theta_{ji})r_{kj}(g)x_k. \]

Hence we must prove
\[
\sum_{j \in I} \lambda(\Theta_{kj})r_{ji}(g) = \sum_{j \in I} \lambda(\Theta_{ji})r_{kj}(g) \quad \text{for all } i,k,g.
\]

Now \( \Theta \) is a \( G \) map, hence
\[ g \Theta(v \otimes x_i) = \Theta(gv \otimes gx_i). \]

Thus,
\[
g \sum_{j \in I} \Theta_{ji}(v) \otimes x_j = \sum_{j,k \in I} g \Theta_{ji}(v)r_{kj}(g) \otimes x_k
\]

\[
= \Theta \left( \sum_{j \in I} r_{ji}(g) gv \otimes x_j \right) = \sum_{j,k \in I} r_{ji}(g) \Theta_{kj}(gv) \otimes x_k.
\]

Hence for all \( v \in \mathcal{V}, \)
\[
\sum_{j \in I} g \Theta_{ji}(v)r_{kj}(g) = \sum_{j \in I} r_{ji}(g)\Theta_{kj}(gv),
\]
i.e.
\[
\sum_{j \in I} \Theta_{ji}r_{kj}(g) = \sum_{j \in I} r_{ji}(g)\Theta_{kj}g.
\]

Applying \( \lambda \) to the above and using (1) we get (2).

Now \( \Delta \) is a \( K \) algebra map and
\[
\ker \Delta = \{ \Theta \mid \Theta_{ij} \in \mathcal{J}(D) \text{ for all } i,j \in I \}.
\]

Clearly \( \ker \Delta \) is locally nilpotent (see (3) definition of 1.2 for locally nilpotent) and so since \( \mathcal{F} \) is local, \( \mathcal{E} \) is local.

Thus \( \mathcal{E} \) only has the idempotents 0 and 1, i.e. \( V \otimes W \) is indecomposable.

**Note** When \( G \) is finite the above theorem says that if \( H \) is a normal subgroup of \( G \), \( V,W \) finite dimensional indecomposable
G modules such that $V|_H$ is indecomposable and $H$ acts trivially on $W$ then $V \otimes W$ is indecomposable.

It now follows that $St \otimes I(\lambda)^{Fr}$ is indecomposable by taking $G$ to be the simply connected Chevalley group over $K$, an algebraically closed field of characteristic $p > 0$, $J = \mathfrak{m}^{[p]}$ (so that $R_J = R^{(p)}$), $V = St$ and $W = I(\lambda)^{Fr}$. 
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Endomorphism Rings and Induction over a Unipotent Quotient

Introduction

Let $K$ be an algebraically closed field of characteristic $p > 0$, $G$ a finite group and $N$ a normal subgroup such that $G/N$ is a $p$ group. It is a well known theorem of Green [5] that if $M$ is a finite dimensional indecomposable $KN$ module then $\text{Ind}_N^G(M)$ is indecomposable.

The work in this part was motivated by a desire to find a generalisation of this in the context of algebraic groups and their rational representations. The main theorem (3.1) is such a generalisation; it is not, however, as good as we can conjecture.

In order to prove the main theorem we develop, in section 1, results on the endomorphism ring of a comodule of finite width which may be of independent interest.
1 The endomorphism ring of a comodule of finite width

1.1

Let $(R, \mu, \varepsilon)$ be a coalgebra over $K$, an algebraically closed field.

By a comodule we shall mean a left comodule.

**Definition** An $R$ comodule $V$ is uniformly bounded if for some $n \in \mathbb{N}$, there is a short exact sequence

$$0 \to V \to \bigoplus_{i=1}^{n} R \to R \text{ of } R \text{ comodules.}$$

**Lemma** Let $V$ be an $R$ comodule, $\{S_\tau\}_{\tau \in \mathcal{B}}$ a full set of simple $R$ comodules and

$V_\tau = \text{the sum of all simple subcomodules of } V \text{ isomorphic to } S_\tau$, for each $\tau \in \mathcal{B}$. The following are equivalent:

(a) $V$ is uniformly bounded;

(b) $\text{soc}_R(V)$ is uniformly bounded;

(c) the injective envelope of $V$ is uniformly bounded;

(d) there exists an $N \in \mathbb{N}$ such that

$$\dim V / (\dim S_\tau)^2 \leq N \text{ for all } \tau \in \mathcal{B}.$$ 

**Proof** (a) $\Rightarrow$ (b) clear.

(b) $\Rightarrow$ (a). Using the injectivity of $\bigoplus_{i=1}^{n} R$ we may complete

\[ 0 \to \text{soc}_R(V) \to V \]

\[ \Downarrow \alpha \]

\[ \bigoplus_{i=1}^{n} R \]

\[ \uparrow \beta \]

\[ R \]

to a commutative diagram via an $R$ morphism $\alpha$. Here $\alpha, n$ have
been chosen so that

\[ 0 \to \text{soc}_R(V) \xrightarrow{\cong} \bigoplus_{i=1}^{n} R \] is a short exact sequence of \( R \) comodules.

Now \( \ker \hat{\alpha} \cap \text{soc}_R(V) = \ker \alpha = 0 \). Hence \( \hat{\alpha} \) is a monomorphism and \( V \) is uniformly bounded.

Let \( I(V) \) be an injective envelope of \( V \), then \((b) \iff (c)\) is clear since \( \text{soc}_R(I(V)) \cong \text{soc}_R(V) \).

\((b) \Rightarrow (d)\)

There exists a short exact sequence

\[ 0 \to \text{soc}_R(V) \to \bigoplus_{\tau \in \mathbb{B}} \bigoplus_{\rho \in \mathbb{B}} V_\rho \to \bigoplus_{i=1}^{n} \text{soc}_R(R) \] for some \( n \), hence there is a short exact sequence

\[ 0 \to \bigoplus_{\tau \in \mathbb{B}} V_\tau \to \bigoplus_{i=1}^{n} \text{soc}_R(R) . \]

Now by the algebraic closure of \( K \) and the structure theorem

\[ \text{soc}_R(R) \cong \bigoplus_{\tau \in \mathbb{B}} \bigoplus_{j=1}^{\dim S_\tau} S_\tau . \]

It follows that for each \( \tau \in \mathbb{B} \) we have the exact sequence

\[ 0 \to V_\tau \to \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{\dim S_\tau} S_\tau \]

and so \( \dim V_\tau \leq n (\dim S_\tau)^2 \) for all \( \tau \).

\((d) \Rightarrow (b) \) Reverse the steps of \((b) \Rightarrow (d)\).

**Definition** (see \([6, 2.3]\)) An \( R \) comodule \( V \) has finite width if for each simple \( R \) comodule \( S_\tau \), \( S_\tau \) occurs only finitely many times in \( \text{soc}_R(V) \).
Definition. If \( \{S_\gamma\} \subset B \) is a full set of simple \( R \) comodules, \( W \) an \( R \) comodule and \( \mathfrak{X} \) a subset of \( B \) then
\[
O_{\mathfrak{X}}(W) = \text{the unique maximal } R \text{ subcomodule of } W \text{ each of whose composition factors is isomorphic to } S_\gamma \text{ for some } \gamma \in \mathfrak{X}.
\]
For a comodule \( M \) we put \( \mathfrak{X}(M) = \{ \gamma \in B \mid S_\gamma \text{ is a composition factor of } M \} \).

Remark. It is clear from the definition that an \( R \) comodule \( V \) has finite width if and only if \( \dim \text{Hom}_R(C_\gamma, V) < \infty \) for all \( \gamma \in B \). We now show that this is equivalent to an apparently stronger condition.

Proposition. \( V \), an \( R \) comodule, has finite width if and only if for any finite dimensional \( R \) comodule \( M \), \( \dim \text{Hom}_R(M, V) < \infty \).

Proof. \( \Longleftarrow \) is evident.

\( \Longrightarrow \) Let \( M \) be a finite dimensional \( R \) comodule. Clearly \( \text{Hom}_R(M, V) \) may be identified with \( \text{Hom}_R(M, O_{\mathfrak{X}}(V)) \).

\( \text{Soc}_R(O_{\mathfrak{X}}(V)) \) involves only finitely many simples each with a finite multiplicity. Hence \( O_{\mathfrak{X}}(V) \) is uniformly bounded by the above lemma. Thus there exists an exact sequence
\[
0 \to O_{\mathfrak{X}}(V) \to \bigoplus_{i=1}^{n} R \text{ for some } n \in \mathbb{N}.
\]
Hence \( \text{Hom}_R(M, V) \) may be identified with a subspace of
\[
\text{Hom}_R(M, \bigoplus_{i=1}^{n} R) \cong \bigoplus_{i=1}^{n} \text{Hom}_R(M, R).
\]
Thus it is enough to prove that \( \text{Hom}_R(M, R) \) is finite dimensional.
Regarding $\mathcal{E} : R \to K$ as a coalgebra map we get $R \cong \mathcal{E}^0(K)$ by 2.2e of part A. Hence by 2.2c of part A

$\text{Hom}_R(M,R) \cong \text{Hom}_R(\mathcal{E}^0(M),K)$ as $K$ spaces. The latter is finite dimensional and hence so is the former. This completes the proof of the proposition.

**Definition** Let $V$ be an $R$ comodule, $V_1$ an $R$ subcomodule of $V$. We say $V_1$ is fully invariant in $V$ if for each $\varnothing \in \text{End}_R(V)$

$$\varnothing(V_1) \subseteq V_1.$$ 

**Definition** Let $V$ be an $R$ comodule, say $V$ is locally fully invariant if it is the union of its finite dimensional fully invariant subcomodules.

**Corollary 1** If $V$ is an $R$ comodule of finite width then $V$ is locally fully invariant.

**Proof** Let $M \subset V$, $M$ a finite dimensional subcomodule. Put $E = \text{End}_R(V)$, $H = \text{Hom}_R(M,V)$. $H$ is finite dimensional by the proposition, hence $M_0 = \bigoplus_{x \in H} x(M)$ is finite dimensional.

$E.M \subset M_0$ hence $E.M$ is a finite dimensional fully invariant subcomodule containing $M$. It follows that $V$ is locally fully invariant.

**Corollary 2** Let $(S, \mu, \varepsilon)$ be a coalgebra over $K$ and let

$\varphi : R \to S$ be a morphism of coalgebras. If $W$ is an $S$ comodule of finite width then $\varphi^0(W)$ is an $R$ comodule of finite width.
Proof Let $X$ be any finite dimensional $R$ comodule, then

$$\text{Hom}_R(X, \varphi^0(\mathcal{V})) \cong \text{Hom}_S(\varphi_0(X), \mathcal{W})$$
by 2.2c of part A. By the proposition, the latter and hence the former is finite dimensional. Thus $\varphi^0(\mathcal{W})$ has finite width.

Remarks 1 If $R$ is any coalgebra, $V = \sum_{i \in \mathbb{N}} \otimes R$ does not have finite width and is not locally fully invariant.

To see that $V$ is not locally fully invariant observe that $E = \text{End}_R(V)$ contains a copy of the restricted symmetric group on $\mathbb{N}$, $\text{Resym}(\mathbb{N})$. $\varphi \in \text{Resym}(\mathbb{N})$ acts on $V$ by

$$\varphi(r_1, r_2, \ldots, r_i, \ldots) = (r_{\varphi(1)}, r_{\varphi(2)}, \ldots, r_{\varphi(i)}, \ldots)$$
where each $r_j \in R$ and almost all $r_j$ are zero. It is easy to see that if $0 \neq v \in V$ then the $\text{Resym}(\mathbb{N})$ orbit of $v$ lies in no finite dimensional subspace of $V$. Hence $V$ has no nonzero fully invariant, finite dimensional subspaces.

2 The class of $R$ comodules of finite width is closed under taking submodules, but not quotients. Let $K$ be an algebraically closed field of characteristic $p > 0$. Put $R = \text{the Hopf algebra } K[X]$, with structure maps $\mu, \varepsilon$ determined by

$$\mu(X) = 1 \otimes X + X \otimes 1$$
and $\varepsilon(X) = 0$.

$R$ viewed as a left $R$ comodule has finite width (in fact $\text{soc}_R(R) = K$), but $\text{soc}_R(R/K) = V/K$ where $V = K + \sum_{i \geq 0} KX^i$. Every simple $R$ comodule is isomorphic to $K$ hence $V/K$ does not have finite width.
1.2

In 1.2 and 1.3 $(R, \mu, \epsilon)$ is a coalgebra over $K$ algebraically closed, $V$ is an $R$ comodule of finite width, and

$\mathcal{A}$ = the set of fully invariant finite dimensional subcomodules of $V$. For $U \in \mathcal{A}$ we define the map $p_U : \text{End}_R(V) \to \text{End}_R(U)$ to be restriction and put $E = \text{End}_R(V)$, $E_U = p_U(E)$.

**Proposition 1 (Local Fitting Lemma)** For any $\alpha \in E$,

$$V = \bigoplus_{n=1}^{\infty} \ker \alpha^n \oplus \bigcup_{U \in \mathcal{A}} \bigcap_{n=1}^{\infty} \alpha^n(U).$$

**Proof** Put

$$X = \bigcup_{n=1}^{\infty} \ker \alpha^n, \quad Y = \bigcup_{U \in \mathcal{A}} \bigcap_{n=1}^{\infty} \alpha^n(U).$$

Let $v \in X \cap Y$. Then $v \in \bigcap_{n \geq 1} \alpha^n(U)$ for some $U \in \mathcal{A}$.

Now $U$ is finite dimensional so there exists an $N > 0$ such that $\alpha^N + s(U) = \alpha^N(U)$ for all $s \geq 0$, clearly since $v$ is in $X$ we may pick $N$ so that $v \in \ker \alpha^N$. Thus $v = \alpha^N(u)$ for some $u \in U$ and $\alpha^N(v) = 0$.

But $\beta = \alpha^N |_{\alpha^N(U)} : \alpha^N(U) \to \alpha^N(U)$ is an isomorphism since $\beta$ is an epimorphism and $\alpha^N(U)$ is finite dimensional. Hence $\alpha^N(v) = 0$ implies $v = 0$. Thus $X \cap Y = 0$.

Now if $v \in V$, $v \in U_1$ for some $U_1 \in \mathcal{A}$ since $V$ is locally fully invariant. There exists an $M > 0$ such that $\alpha^M(U_1) = \alpha^M(U_1)$ for all $s \geq 0$.

$$\alpha^2M(v_1) = \alpha^{2M}(v_1)$$ for some $v_1 \in U_1$ and hence
v - \alpha^H(v_1) \in \ker \alpha^H \subseteq X. Also \alpha^H(v_1) \in \bigcap_{n\geq 1} \alpha^P(U_1) \subseteq Y.

v = v - \alpha^H(v_1) + \alpha^H(v_1) \in X \oplus Y \text{ so } V = X \oplus Y.

**Definition** Say an ideal I of E is locally nilpotent if for each \( U \in \mathcal{A} \) there is an integer \( n_U \) such that \( I^{n_U}(U) = 0 \).

**Lemma** If I and J are locally nilpotent ideals of E, then 

I + J is locally nilpotent.

**Proof** For \( U \in \mathcal{A} \) there exist \( n_U \) and \( m_U \) such that 

\[ I^{n_U}(U) = J^{m_U}(U) = 0 \]

and we get 

\[ I^{n_U} + m_U - 1(U) = 0. \]

**Definition** \( N_L(E) \) = the sum of all locally nilpotent ideals of E.

**Proposition 2** \( N_L(E) \) = the Jacobson radical of E and is locally nilpotent.

**Proof** Note from the Local Fitting Lemma we have \( \alpha \in E \) is a unit if and only if \( \ker \alpha = 0 \), since if \( \ker \alpha = 0 \),

\[ V = \bigcup_{U \in \mathcal{A}} \cap \alpha^n(U) \] showing \( \alpha \) is onto.

Let \( \beta \in N_L(E) \) and \( v \in \ker(1 + \beta) \), \( v \in U \) say for some \( U \in \mathcal{A} \). \( \beta|_U \) is nilpotent and so \( (1 + \beta)|_U \) is a unit, hence \( v = 0 \). Thus \( \ker(1 + \beta) = 0 \), \( 1 + \beta \) is a unit and \( \beta \) is quasiregular. Thus \( N_L(E) \) is a quasiregular ideal and so 

\( N_L(E) \subseteq J(E) \), the Jacobson radical.

Let \( W \in \mathcal{A} \). \( W \) is finite dimensional so \( J(E)^{m_w} = J(E)^{m+1}W \) for some \( m \in \mathbb{N} \). Hence by Nakayama's lemma \( J(E)^mW = 0 \) and so
\( J(E) \) is locally nilpotent. Hence \( N_U(E) = J(E) \) and is locally nilpotent as required.

**Corollary** \( \bigcap_{n=1}^{\infty} J(E)^n = 0. \)

**Proof** Clearly \( \rho_U(J(E)) \subseteq J(E_U) \) for any \( U \in \mathcal{A} \) and so

\[
\rho_U \left( \bigcap_{n=1}^{\infty} J(E)^n \right) \subseteq \bigcap_{n=1}^{\infty} J(E_U)^n = 0 \quad \text{since} \quad J(E_U) \quad \text{is a nilpotent ideal of the finite dimensional algebra} \quad E_U.
\]

But \( V \cup U \in \mathcal{A} \) and so \( \bigcap_{n=1}^{\infty} J(E)^n \) acts trivially on \( V \), hence \( \bigcap_{n=1}^{\infty} J(E)^n = 0. \)

**Proposition 3** \( J(E) = \bigcap_{U \in \mathcal{A}} \rho_U^{-1}(J(E_U)). \)

**Proof** Similar to the proof of the corollary.

**Example** \( R = K[X] \) the Hopf algebra over \( K \), an algebraically closed field of characteristic zero with structure maps

\( \mu, \xi \) determined by \( \mu(X) = 1 \otimes X + X \otimes 1, \xi(X) = 0. \)

Put \( V = R \) (considered as a left comodule) and

\[ N = \{ \theta \in E | \theta(1) = 0 \}. \]

Since \( \text{soc}_R(V) = K \) we get that for any finite dimensional subcomodule \( W \) of \( V \) and \( \theta \in N \), \( \dim \sigma(W) < \dim W \) (since \( K \leq \ker \theta \cap W \)). It follows that if \( \dim W = n \) and \( \theta_1, \ldots, \theta_n \in N \) then \( \theta_1 \theta_2 \cdots \theta_n(W) = 0 \) and so \( N \) is locally nilpotent.

Clearly \( N \) has codimension 1 so \( N = J(E) \).
Let \( D : K[X] \to K[X] \) be the differentiation map. One may easily see that \( D \in \mathbb{N} \) and \( D^n \neq 0 \) for all \( n \in \mathbb{N} \). Hence \( J(\mathbb{E})^n \neq 0 \) for all \( n \in \mathbb{N} \) in fact \( J(\mathbb{E}) \) is not nil.

1.3

**Lemma 1** If \( a + J(\mathbb{E}) \) is an idempotent in \( \mathbb{E}/J(\mathbb{E}) \) then there exists an \( e = e^2 \in \mathbb{E} \) such that \( a + J(\mathbb{E}) = e + J(\mathbb{E}) \).

**Proof** Put \( a_1 = a, \quad x_1 = a_1^2 - a_1 \) and for \( n > 1 \) define \( a_n, x_n \) inductively by \( a_n = a_{n-1} + x_{n-1} - 2x_{n-1}a_{n-1}; \quad x_n = a_{n-1}^2 - a_{n-1}^2 \).

It is easily verified that \( x_n \in J(\mathbb{E})^{2^n} \) and so if \( U \in \mathbb{U} \), \( x_m(U) = 0 \) for all \( m \geq N_U \) say. We define \( e = a_\infty \) at \( U \in \mathbb{U} \) by

\[
a_\infty(u) = a_{N_U}^\infty(u) \quad \text{for} \quad u \in U.
\]

Clearly this defines an \( R \) endomorphism of \( V \) and \( a_\infty^2 = a_\infty \). It remains to prove that \( a_\infty - a \in J(\mathbb{E}) \).

For \( U \in \mathbb{U} \), \( \rho_U(a_\infty - a) = \rho_U(a_{N_U}^\infty - a) \). But \( a_{N_U}^\infty - a \in J(\mathbb{E}) \) hence \( \rho_U(a_\infty - a) \in J(E_U) \). It follows from proposition 3 of 1.2 that \( a_\infty - a \in J(\mathbb{E}) \) as required.

**Remark** The above procedure for producing idempotents is, of course, a limiting version of the well known Brauer idempotent lifting process.

**Lemma 2** (cf [4, Theorem 44.3 (2)]) If \( 1 = f_1 + \ldots + f_n \) is an orthogonal decomposition of \( 1 \) in \( E/J(\mathbb{E}) \) there is an orthogonal decomposition \( 1 = e_1 + \ldots + e_n \) of \( 1 \) in \( E \) such that \( e_i + J(\mathbb{E}) = f_i \) for all \( 1 \leq i \leq n \).
Proof By induction on $n$, $n = 2$ being the preceding lemma.

Suppose $n > 2$ and the lemma is true for $n - 1$. We may choose a lift \{e, e_2, ..., e_n\} of \{f_1 + f_2, f_3, ..., f_n\}. Let $b + J(E) = f_1$; put $a = ebe$ then

$$a + J(E) = ebe + J(E) = (f_1 + f_2)f_1(f_1 + f_2) = f_1.$$ 

It follows that if $a_\infty$ is defined as in the proof of lemma 1

$$a_\infty = e_\infty = a_\infty \quad (\text{since } a_\infty \text{ is defined at a point by a polynomial in } a).$$

Put $e_1 = a_\infty$ then

\{e_1, e - e_1, e_2, ..., e_n\} is a lift of \{f_1, ..., f_n\}.

**Proposition 1** $V$ is indecomposable if and only if $E/J(E) \cong k$.

**Proof** ($\Rightarrow$) We first show that $E_U$ is local for all $U \in \mathcal{A}$. Suppose $\alpha \in E$ and $\rho_U(\alpha)$ is an idempotent in $E_U$. By the Local Fitting Lemma $\bigcup_{n>1} \ker \alpha^n = 0$ or $V$. Hence $\alpha$ is locally nilpotent or a unit which implies $\rho_U(\alpha)$ is nilpotent or a unit. Hence $\rho_U(\alpha) = 0$ or 1. Hence $E_U/J(E_U) \cong k$ for all $U \in \mathcal{A}$.

Now if $\theta \in E$, $U \in \mathcal{A}$ there exists a unique $\lambda_U \in K$ such that $\theta - \lambda_U$ is nilpotent on $U$. If $W$ is also an element of $\mathcal{A}$ there exists a unique $\lambda_W \in K$ such that $\theta - \lambda_W$ is nilpotent on $W$ and a unique $\lambda_{U + W} \in K$ such that $\theta - \lambda_{U + W}$ is nilpotent on $U + W$. Now since $\theta - \lambda_{U + W}$ is nilpotent on $U$, $\lambda_{U + W} = \lambda_U$, similarly $\lambda_{U + W} = \lambda_W$ hence $\lambda_U = \lambda_W$. Thus there is a unique $\lambda \in K$ such that $\rho_W(\theta - \lambda) \in J(E_W)$ for all $W \in \mathcal{A}$. Hence by
Proposition 3 of 1.2, $\emptyset - \lambda \in J(B)$, showing that $J(B)$ has codimension 1, i.e., $B/J(B) \cong K$.

$(\Leftarrow)$ Let $\pi : B \to B/J(B)$ denote the natural map. If $e = e^2 \in B$, then $B/J(B) \cong K$ implies $\pi(e) = J(B)$ or $1 + J(B)$. Hence $e$ or $1 - e \in J(B)$, but $e$ and $1 - e$ are idempotents so $e$ or $1 - e \in \bigcap_{n \geq 1} J(B)^n = 0$ (corollary to proposition 2 of 1.2).

Thus $e$ or $1 - e = 0$, $B$ has only one nonzero idempotent and $V$ is indecomposable.

**Proposition 2** $B/J(B)$ is finite dimensional if and only if there exists an $R$ comodule decomposition

$V = V_1 \oplus \ldots \oplus V_n$ for some $n \in \mathbb{N}$, such that each $V_i$ is indecomposable.

**Proof** $(\Rightarrow)$ Let $1 = f_1 + \ldots + f_n$ be an orthogonal decomposition of 1 in $B/J(B)$ such that each $f_i$ is primitive. Lift to an orthogonal decomposition $1 = e_1 + \ldots + e_n$ of 1 in $B$ by lemma 2 of 1.3.

$V = e_1V \oplus \ldots \oplus e_nV$ and each $e_iV$ is indecomposable.

$(\Leftarrow)$ Let $V = V_1 \oplus \ldots \oplus V_n$ be an $R$ comodule decomposition of $V$ such that each $V_i$ is indecomposable.

Put $I = \{1, \ldots, n\}$. We identify $\nabla \in \text{End}_R(V) = E$ with a matrix $(\nabla_{ij})_{i,j \in I}$ where $\nabla_{ij} \in \text{Hom}_R(V_j, V_i)$. Note that each $\text{End}_R(V_i)$ is local. Now by Fitting's theorem
\( J(\text{End}_R(V)) = \{ \alpha \mid \text{each } \alpha_{ij} \text{ is not an isomorphism} \} \).

Choose a decomposition \( I = \bigcup_{t=1}^{r} I_t \) of \( I \) such that

\( i \) and \( j \) are in the same \( I_t \) if and only if \( V_i \cong V_j \). Renumber the \( V_i \)'s, if necessary, so that \( 1, 2, \ldots, n_1 \in I_1, n_1 + 1, \ldots, n_2 \in I_2 \) etcetera. Put \( X_i = V_{n_i} \) for \( i = 1, 2, \ldots, r \) and put

\[
D_i = \text{End } X_i / J(\text{End } X_i) \cong K \text{ by the above proposition.}
\]

Say an element \( \beta \in \text{Hom}_R(V_j, V_i) \) is singular if \( \beta \) is not an isomorphism. We define a map \( \overline{\beta} : \text{End}_R(V) \to \mathcal{Y} \)

by \( \overline{\beta}(\alpha_{ij}) = (\alpha_{ij} + \text{the set of singular elements of } \text{Hom}_R(V_j, V_i)) \)

where \( (\alpha_{ij}) \) is the matrix representing an \( R \) map \( \alpha \) and \( V \) is a matrix ring identifiable with \( \bigoplus_{t=1}^{r} (D_t) \mid I_t \mid \). The kernel of \( \overline{\beta} \) is \( J(E) \) hence \( E/J(\mathcal{Z}) \) is finite dimensional.

\[\textbf{Note}\] This proposition together with lemma 2 of 1.3 shows that

if \( V \) has a decomposition \( V = V_1 \oplus \ldots \oplus V_n \) such that each \( V_i \) is indecomposable then \( E \) is semiperfect in the sense of \([1]\).
2 Algebra automorphisms of a finite dimensional simple algebra

Let $E$ be a finite dimensional simple algebra over $K$, an algebraically closed field. Let $E_0$ denote the set of units in $E$. $E_0$ is an algebraic subgroup of $GL(E)$, $E_0$ is isomorphic to $GL_n(K)$ where $E \cong M_n(K)$.

We put $E_1 = \{\text{the elements of } E_0 \text{ having determinant } 1\}$.

Now there is a map $\Theta : E_1 \to GL(E)$ given by $\Theta(x)a = xax^{-1}$ for $x \in E_1$, $a \in E$. It is easy to check that is a morphism of algebraic groups. This gives rise naturally to a map $d\Theta : \mathfrak{L}(E_1) \to \mathfrak{L}(GL(E))$ where $\mathfrak{L}(E_1)$ and $\mathfrak{L}(GL(E))$ denote the Lie algebras of $E_1$ and $GL(E)$ respectively. It is our first aim to find $\ker d\Theta$.

We identify $E$ with $\text{End}_K(V)$ for some $K$ space $V$. Let $\{v_1, \ldots, v_n\}$ be a basis of $V$ and for $1 \leq i, j \leq n$ let $e_{ij}$ be the element of $E$ defined by $e_{ij}(v_1) = \sum_{1 \leq l \leq n} e_{ij}^l v_1$ for $1 \leq l \leq n$. We define functions $Z_{\alpha, \beta, ij} : GL(E) \to K$ for $1 \leq \alpha, \beta, i, j \leq n$ by $g e_{ij}^l = \sum_{1 \leq \alpha, \beta, i, j \leq n} Z_{\alpha, \beta, ij}(g) e_{\alpha, \beta}^l$. For any $g \in GL(E)$.

Define functions $X_{ij} : E_1 \to K$ by $x = \sum_{1 \leq i, j \leq n} X_{ij}(x) e_{ij}$ for $x \in E_1$. 

Let $\Delta : \text{GL}(E) \to K$ be the inverse determinant function, then

$$K[\text{GL}(E)] = K \left[ Z_{\alpha \beta, i j} \Delta \mid 1 \leq \alpha, \beta, i, j \leq n \right]$$ and

$$E_k = K \left[ x_{i j} \mid 1 \leq i, j \leq n \right].$$

$\Theta$ gives rise to a comorphism $\Theta^* : K[\text{GL}(E)] \to K[ E_k ]$,

$$\Theta^*(Z_{\alpha \beta, i j})(x) = Z_{\alpha \beta, i j}(\Theta(x)), \text{ for any } g \in E_k.$$

Now $\Theta(x).e_{i j} = x e_{i j} x^{-1} = \sum_{\alpha, \beta} x_{\alpha \beta}(x)e_{\alpha \beta} e_{i j} x^{-1}$

$$= \sum_{\alpha, \beta} x_{\alpha i}(x)e_{\alpha j} x^{-1} \sum_{x, r, s} x_{\alpha i}(x)e_{\alpha j} y_{rs}(x)e_{rs}$$

where $y_{rs} : E_k \to K$ is defined by $x^{-1} = \sum_{1 \leq r, s \leq n} y_{rs}(x)e_{rs}$.

Hence $\Theta(x).e_{i j} = \sum_{\alpha, \beta} x_{\alpha i}(x)e_{\alpha j} y_{\beta r}(x)$ for any $x \in E_k$ and so

$$\Theta^*(Z_{\alpha \beta, i j}) = x_{\alpha i} y_{\beta j}.$$

Suppose $\gamma \in \ker d\Theta$, then

$$\gamma(x_{\alpha i} y_{\beta j}) = 0 \text{ for all } 1 \leq i, j, \alpha, \beta \leq n.$$

$\gamma$ is a derivation so $0 = \gamma(x_{\alpha i} y_{\beta j})$

$$= E(x_{\alpha i}) \gamma(y_{\beta j}) + \gamma(x_{\alpha i}) y_{\beta j} = E(x_{\alpha i}) \gamma(y_{\beta j}) + \gamma(x_{\alpha i}).$$

Thus for $\alpha \neq i$, $j = \beta$ we get $\gamma(x_{\alpha i}) = 0$.

For $\alpha = i$, $j = \beta$ we get $\gamma(y_{\alpha \beta}) + \gamma(x_{\alpha i}) = 0$.

Hence $\gamma(x_{11}) = \ldots = \gamma(x_{nn})$. Since $\gamma$ is a derivation and

$$\det( (x_{i j}) ) = 1 \text{ we get } \gamma \left( \det( (x_{i j}) ) \right) = 0.$$

Expanding this relation we obtain

$$\gamma(x_{11}) + \ldots + \gamma(x_{nn}) = 0.$$

We can now describe the kernel of $d\Theta$. 

(a) If char $K = p > 0$ and $p \nmid n$ or char $K = 0$ then ker $d\mathcal{O} = 0$.

(b) If char $K = p > 0$ and $p \mid n$ then ker $d\mathcal{O}$ is the $K$ span of the derivation $D$ defined on the generators by

$$D(X_{ij}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$  

Since $D$ acts as $\mathcal{E}$ on the generators $X_{ij}$, we have $D^g = \mathcal{E}_g D \mathcal{E}_{g^{-1}} = D$ for all $g \in B_1$.

Let $U_1 = \{ x \in B_1 \mid X_{ij}(x) = \begin{cases} 1 & i = j \\ 0 & i > j \end{cases} \}$.

$\mathcal{L}(U_1)$ may be identified with $\{ \gamma \in \mathcal{L}(B_1) \mid \gamma(X_{ij}) = 0 \text{ for } i \geq j \}$.

Now if $U$ is any unipotent subgroup of $B_1$, $U$ may be put into upper triangular form, i.e., there is a $g \in B_1$ such that $U^g \subseteq U_1$. Now $(KD \cap \mathcal{L}(U))^g = KD^g \cap \mathcal{L}(U^g)$.

$KD \cap \mathcal{L}(U_1) = 0$. Hence for any unipotent subgroup $U$ of $B_1$, $d\Theta | \mathcal{L}(U) : \mathcal{L}(U) \to \mathcal{L}(GL_K(E))$ is injective.

It can be checked that the set of $K$ algebra automorphisms of $E$ form a closed subgroup of $GL(E)$. We put $G = \text{Aut}_K \text{alg}(E)$. Clearly $\text{Im} \Theta \subseteq G$ and by Theorem 2.35 of [7] it is onto. Let $H$ be a closed connected unipotent subgroup of $G$, put $L = \Theta^{-1}(H)$. $\Theta(L^0)$ has finite index in $H$ hence $\Theta(L^0) = H$. 
Now as abstract groups \( L/\ker\Theta \cong \text{Im}\Theta = H \).
\[
\ker\Theta = \left\{ \left( \lambda^{\alpha}, \lambda^{\beta} \right) \mid \lambda^n = 1 \right\} = \text{centre of } B_1 ,
\]
hence \( L \) is soluble.

Thus \( L^0 \) is a soluble connected algebraic group hence
\( L^0 = T.L_u \), the semi direct product of a maximal torus \( T \) and
the unipotent radical of \( L, L_u \). Now \( \Theta(T) \) is a torus in \( K \)
which is unipotent, hence \( \Theta(T) = 1 \). But \( \ker\Theta \) is finite and
\( T \) is connected so \( T = 1 \). Thus \( L^0 \) is unipotent.

Clearly \( \Theta : L^0 \to H \) is an isomorphism of abstract groups,
a morphism of varieties and separable since
\[
d\Theta \mid L^0 : L^0 \to \mathfrak{f}(H) \text{ is injective and } \dim L^0 = \dim H .
\]

Thus \( \Theta \mid L^0 : L^0 \to H \) is an isomorphism of algebraic groups.

Let \( \gamma : H \to L^0 \) be the inverse. The discussion so far shows the following.

**Proposition** Let \( E \) be a finite dimensional simple algebra over
\( K \), an algebraically closed field. Let \( E^o \) be the group of units in
\( E \) considered as an algebraic group. Let \( H \) be a unipotent algebraic

group over \( K \) acting rationally as a group of \( K \) algebra automorphisms
of \( E \) giving rise to the map \( \xi : H \to \text{Aut}_K\text{alg}(E) \). There exists a

morphism of algebraic groups \( \gamma : H \to E_1 \), the elements of \( E \) of
determinant 1, such that \( \gamma(h)a\gamma(h)^{-1} = \xi(h)a \) for all \( h \in H, a \in E \).

**Remark** Clearly \( \gamma \) is uniquely determined.
A Digression Let $G$ be a simply connected algebraic Chevalley group over $K$, algebraically closed of characteristic $p > 0$, and let $\mathfrak{u}'$ be the restricted enveloping algebra of the Lie algebra of $G$, realised inside $k[G]^*$. If $V$ is a simple $\mathfrak{u}'$ module we may use the above procedure to give $V$ a rational $G$ module structure.

Put $I$ the annihilator of $V$ in $\mathfrak{u}'$, an ideal, $J(\mathfrak{u}')$ the Jacobson radical of $\mathfrak{u}'$ and $B = \mathfrak{u}'/J(\mathfrak{u}')$. Let

$$1 = e_1 + \ldots + e_n$$

be an orthogonal decomposition of $1$ in $B$, into centrally primitive idempotents. $G$ acts rationally on $\mathfrak{u}'$ by conjugation, i.e. $a^g = \xi_g a \xi_g^{-1}$ and hence on $B$. $G$ permutes the finite set of idempotents $\{e_1, \ldots, e_n\}$ and since $G$ is connected it must fix each $e_1$.

Now $A/I \cong e_i B$ for some $i$. Put $e = e_1$. $B = eB$ is a simple algebra isomorphic to $\text{End}_K(V)$ on which $G$ acts as a group of $K$ algebra automorphisms.

Let $U$ and $U'$ be the standard maximal unipotent subgroups of $G$, arising from the Chevalley construction. Let

$$\Pi : eB \to \text{End}_K(V)$$

be the natural map.

Now $U$ acts on $eB$ by $K$ algebra automorphisms so by the proposition there is a map $\gamma : U \to eB$ such that

$$(eb)^X = \gamma(x) \Pi (eb) \gamma(x^{-1})$$

for all $x \in U$, $b \in B$.

Similarly we obtain a map $\gamma' : U' \to eB$ and $\gamma$ and $\gamma'$ are uniquely determined. We also know that $V$ has the structure of
a simple rational $G$ module $([S_j, G])$ via $\rho$ say, in such a way that $\rho$ on restriction gives rise to the natural map

$$U \rightarrow \text{End}_K(V).$$

Now for $x \in U$ we have $\rho(x) \equiv (eb) \rho(x^{-1}) = (eb)^x$ and so $\rho|_U = \tilde{\psi}$, similarly $\rho|_{U'} = \tilde{\psi}'$. Since $G$ is generated by $U$ and $U'$ $\rho$ is completely determined on $G$ by $\tilde{\psi}$ and $\tilde{\psi}'$. 
3 Induction over a unipotent quotient

3.1

We make the following conjecture.

Conjecture Let $G$ be an algebraic group over $K$, an algebraically closed field. Let $H$ be a closed normal subgroup and suppose $G/H$ is unipotent. Let $\varphi : K[G] \to K[H]$ be the restriction map. If $V$ is an indecomposable rational $KH$ module of finite width then $\varphi^0(V)$ is an indecomposable $KG$ module of finite width.

Remark By corollary 2 of the proposition of 1.1 $\varphi^0(V)$ has finite width. We cannot prove the conjecture but we prove a restricted version in this section.

Definition Let $G$ be an algebraic group over $K$ and let $H$ be a closed subgroup. We say a rational $KH$ module $V$ is a finite $G$ component if there is a rational $G$ module $W$ such that

$$\varphi_0(W) \cong Y_1 \oplus \ldots \oplus Y_n$$

as $KH$ modules, for some $n \in \mathbb{N}$, where each $Y_i$ is indecomposable of finite width $Y_i \cong V$ and $\varphi : K[G] \to K[H]$ is the restriction map.

Theorem Let $G$ be an algebraic group over $K$, algebraically closed, $H$ a closed normal subgroup such that $G/H$ is unipotent. Let $\varphi : K[G] \to K[H]$ be the restriction map. If $V$ is an indecomposable rational $KH$ module and a finite $G$ component then $\varphi^0(V)$ is a rational indecomposable $G$ module of finite width.
Proof Claim it is enough to prove

(*) $G/H$ connected implies the theorem is true.

Put $N = H\mathbb{G}^0$, $\psi_1 : K[G] \to K[N]$, $\psi_2 : K[H] \to K[H]$, restriction maps. $\varphi = \psi_2 \psi_1$ so $\varphi^0(\mathcal{V}) \cong \psi_1^0(\psi_2^0(\mathcal{V}))$.

Now $H\mathbb{G}^0/H$ is connected so (*) implies $\psi_2^0(\mathcal{V})$ is indecomposable of finite width. $|G:H\mathbb{G}^0| < \infty$ and $G/H\mathbb{G}^0$ is unipotent. Thus if $\text{char } K = 0$ then $G = H\mathbb{G}^0$ and $\varphi^0(\mathcal{V}) = \psi_2^0(\mathcal{V})$ is indecomposable of finite width. If $\text{char } K = p > 0$ then $G/H\mathbb{G}^0$ is a finite $p$ group. It follows from the proof of [5, theorem 8] that $\varphi^0(\mathcal{V}) \cong \psi_1^0(\psi_2^0(\mathcal{V}))$ is indecomposable.

We now assume that $G/H$ is connected.

Let $W$ be a rational $G$ module such that

$$\varphi^0(W) = Y_1 \oplus \cdots \oplus Y_n$$

where each $Y_i$ is indecomposable of finite width (as an $H$ module) and $Y_1 \cong \mathcal{V}$.

Let $D = \text{End}_{K[G]}(W)$ and $\mathcal{B} = \text{Ind}_{K[G]}(W \otimes K[G])$ where $K[G] \otimes K[G] = \{f \in K[G] \mid f_{\otimes h} = f \text{ for all } h \in H\}$.

We will show that the length of any orthogonal decomposition of $1$ in $\mathcal{B}$ is at most $n$.

Let $\{x_i\}_{i \in I}$ be a basis for $K[G]_{(H)}$. If $\Theta \in \mathcal{B}$ we may define $\Theta_{ij} : W \to W$ by $\Theta(w \otimes x_i) = \sum_{j \in I} \Theta_{ji}(w) \otimes x_j$.

One can easily check that the $\Theta_{ij}$ so defined are elements of $D$.

We wish to show that for a fixed $i$

$$(1) \quad \Theta_{ji} \in J(D) \text{ for all but a finite number of } j \text{ s.}$$

Here $J(D)$ denotes the Jacobson radical of $D$. 

By proposition 2 of 1.3 \( D/J(D) \) is finite dimensional. Let\( C \) be a \( K \) space complement to \( J(D) \) in \( D \). \( C \) is finite dimensional so there is a finite dimensional fully invariant \( K \) submodule \( U \) of \( W \) such that the restriction map

\[
\rho : D \to \text{End}_{K[H]}(U) \text{ is injective on } C.
\]

It follows that if \( D_U = \rho(D) \), \( \rho \) induces an isomorphism

\[
\overline{\rho} : D/J(D) \to D_U/J(D_U), \quad \text{(2)}.
\]

Now since \( W \otimes K[G]_H \cong \varphi^0(U) \) has finite width it is locally fully invariant. Hence for fixed \( i \), \( \mathcal{O}(U \otimes x_i) \leq U \), a finite dimensional \( K \) space, for all \( \theta \in \mathcal{O} \).

Thus \( \mathcal{O}(U \otimes x_i) \leq \sum_{j \in I_1} U \otimes x_j \) for some finite subset \( I_1 \) of \( I \).

Hence \( \mathcal{O}_{j_1}(U) = 0 \) for \( j \notin I_1 \) and so \( \mathcal{O}_{j_1} \in J(D) \) for all \( j \notin I_1 \) by (2), proving (1).

\( G \) acts on \( D \) as a group of \( K \) algebra automorphisms \( \alpha^g \) being defined for \( \alpha \in D \), \( g \in G \) by

\[
\alpha^g(z) = g^{-1} \alpha(gz) \quad \text{for each } z \in U.
\]

This gives rise to an action of \( G \) on \( D/J(D) \). We shall show that this action is rational.

Put \( U_1 = KGU \), for \( \alpha \in D \), \( g \in G \)

\[
\alpha(gU) = g \cdot \alpha^g(U) \subseteq gU,
\]

since \( U \) is fully invariant under \( D \).

Since \( U \leq U_1 \) it follows that if \( \rho_1 : D \to \text{End}_{K[H]}(U_1) \) is the restriction map \( \rho_1 \) gives rise to an isomorphism
\[ \tilde{\rho}_1 : D/J(D) \to D_{U_1}/J(D_{U_1}) \] where \( D_{U_1} = \rho_1(D) \).

Now \( G \) acts naturally on \( \text{End}_K(U_1) \) by conjugation and with this action \( \text{End}_K(U_1) \cong U_1 \otimes U_1^* \) hence \( \text{End}_K(U_1) \) is a rational \( G \) module. \( D_{U_1} \) is a sub \( G \) module of \( \text{End}_K(U_1) \) hence rational, \( D_{U_1}/J(D_{U_1}) \) is a quotient of a rational module hence rational, finally \( \tilde{\rho}_1 \) is an isomorphism of \( G \) modules hence \( D/J(D) \) is a rational \( G \) module.

Now by the Wedderburn theorem
\[ D/J(D) \cong R_1 \oplus \ldots \oplus R_{\ell} \] say, a direct sum of rings, where
\[ R_i = \text{End}_K(V_i) \] for finite dimensional \( K \) spaces \( V_i \).

\( H \) acts trivially by conjugation on \( D \), hence on \( D/J(D) \). Thus \( D/J(D) \) is a rational \( G/H \) module. Clearly \( G/H \) permutes the central idempotents of \( D/J(D) \), since they form a finite set and \( G/H \) is connected \( G \) fixes each central idempotent.

Thus \( G/H \) acts rationally on each \( R_i \) as a group of \( K \) algebra automorphisms. \( G/H \) is unipotent so we may give each \( V_i \) the structure of a rational \( G/H \) module by the proposition of 2.

Note that the length of an orthogonal decomposition of 1 in \( D/J(D) \) into a sum of primitive idempotents is \( \sum_{i=1}^{\ell} \dim V_i \).

Let \( \lambda_s : D \to R_s \) be the natural map for \( 1 \leq s \leq n \), \( F_s = \text{End}_{KG}(V_s \otimes K[G](H)) \) and \( F = F_1 \oplus \ldots \oplus F_{\ell} \), the direct sum of rings. Let \( V_s \) afford the representation of \( G/H \) \( \tau_s \).
For $1 < s < 1$ define $\Delta_s : E \rightarrow F$ by

$$\Delta_s(\Theta)(v \otimes x_i) = \sum_{j \in I} \lambda_s(\Theta_{ji})(v) \otimes x_j$$

for $v \in V_s$, $i \in I$, $\Theta \in B$. By (1) this is a well defined $K$ space map, we must check that $\Delta_s(\Theta) \in F_s$ for any $\Theta \in B$.

$$\Delta_s(\Theta)(g(v \otimes x_i)) = \Delta_s(\Theta)(\sigma_s(g)v \otimes g x_i)$$

$$= \Delta_s(\Theta)\left(\sum_{j \in I} r_{ji}(g) \sigma_s(g)v \otimes x_j\right)$$

where the $r_{ji}$ are defined by $g x_i = \sum_{j \in I} r_{ji}(g)x_j$.

So $\Delta_s(\Theta)(g(v \otimes x_i)) = \sum_{j, k \in I} r_{kj}(g) \lambda_s(\Theta_{kj})(\sigma_s(g)v) \otimes x_k$.

$$g \Delta_s(\Theta)(v \otimes x_i) = g(\sum_{j \in I} \lambda_s(\Theta_{ji})(v) \otimes x_j)$$

$$= \sum_{j, k \in I} \sigma_s(g) \lambda_s(\Theta_{ji})(v) \otimes g x_j$$

$$= \sum_{j, k \in I} r_{kj}(g) \sigma_s(g) \lambda_s(\Theta_{ji})(v) \otimes x_k.$$ 

So we must show that

$$\sum_{j \in I} r_{ji}(g) \lambda_s(\Theta_{kj}) \sigma_s(g) = \sum_{j \in I} r_{kj}(g) \sigma_s(g) \lambda_s(\Theta_{ji}),$$

i.e. using the definition of the $G$ action on $V_s$, that

$$(3) \sum_{j \in I} r_{ji}(g) \lambda_s(\Theta_{kj} g) = \sum_{j \in I} r_{kj}(g) \lambda_s(\Theta_{ji}).$$

Now $\Theta$ is a $G$ map so $\Theta(g(v \otimes x_i)) = g \Theta(v \otimes x_i)$ for all $w \in V$, $i \in I$. 
\[
\Theta(g(w \otimes x_i)) = \left( \sum_{j \in I} r_{ji}(g) g^w \otimes x_j \right)
\]
\[
= \sum_{j, k \in I} r_{ji}(g) \Theta_{kj}(g^w) \otimes x_k.
\]
\[
g \Theta(v \otimes x_i) = g \sum_{j \in I} \Theta_{ji}(v) \otimes x_j = \sum_{j \in I} g \Theta_{ji}(v) \otimes g x_j
\]
\[
= \sum_{j, k \in I} r_{kj}(g) g \Theta_{ji}(v) \otimes x_k.
\]

Hence
\[
\sum_{j \in I} r_{ji}(g) \Theta_{kj}^g = \sum_{j \in I} r_{kj}(g) \Theta_{ji}; \text{ applying } \lambda_s
\]
to this equation we get (3).

One may easily check that \( A_s \) is an algebra homomorphism.

Define \( A : E \rightarrow F \) to be \( A_1 \oplus \cdots \oplus A_{1'} \).

Now if \( \Theta \in \ker A \) then for all \( i, j \) \( \Theta_{ij} \in \ker \lambda_s \) for all \( s \),
i.e. \( \Theta_{ij} \in J(D) \). It is easily seen that
\[
\{ \Theta \in E | \Theta_{ij} \in J(D) \text{ for all } i, j \}
\]
is a locally nilpotent ideal of \( E \). Thus \( \ker A \) is locally nilpotent. Now
\[
F_s = \text{End}_{KG}(V_s \otimes K[G](H)) \cong \text{End}_{KG}(\varphi^0(V_s)) \cong \text{End}_{KG}(V_s \otimes K[G](H))
\]
\[
\cong M_{n_s} (\text{End}_{KG}(K[G](H))) \text{ where } n_s = \dim V_s.
\]

Now \( \text{End}_{KG}(K[G](H)) \) is a local ring since \( G/H \) is

unipotent and so

length of an orthogonal decomposition of \( 1 \) in \( F_s \) into a sum

of primitive idempotents = \( \dim V_s \).

Since \( A \) has a locally nilpotent kernel we have

number of idempotents in an orthogonal decomposition of \( 1 \) in

\( E \) into a sum of primitive idempotents \( < \sum_s \dim V_s \) = number of
idempotents in an orthogonal decomposition of $1$ in $D$ into a
sum of primitive idempotents $= n$.

Now $\varphi^0(W) \cong \bigoplus_{i=1}^{n} \varphi^0(Y_i)$ and $H$ normal in $G$, $Y_i \neq 0$
implies $\varphi^0(Y_i) \neq 0 \quad (4)$. 

So if $\mathbf{e}_i : \varphi^0(W) \rightarrow \varphi^0(W)$ are projection maps such that
$
\mathbf{e}_i$ acts as the identity on $\varphi^0(Y_i)$ and zero on $\varphi^0(Y_j)$ for $j \neq i$,
then $1 = \mathbf{e}_1 + \cdots + \mathbf{e}_n$ is an orthogonal decomposition of $1$ of
length $n$. Thus $\mathbf{e}_i$ must be primitive and so $\varphi^0(Y_i) \cong \varphi^0(Y)$
is indecomposable.

Proof of (4) : Let $0 \neq X$ be a rational module for $H$, suppose
$\varphi^0(X) = 0$. Without loss of generality $X$ is simple since $\varphi^0$ is
left exact. By [2, theorem 4] $X$ embeds in a rational $G$ module
$Z$ which we may also take to be simple. Since $G/H$ is connected and
$G/H$ permutes the finite set of $H$ homogeneous components of $Z$,
$\varphi^0(Z) \cong \bigoplus X \quad (as H modules)$.

Thus $0 \neq \varphi^0(Z) \cong Z \otimes X G_{(H)} = \bigoplus \varphi^0(X)$ and so
$\varphi^0(X) \neq 0$.

This completes the proof of the theorem.

Example We give an example of a finite dimensional $KH$ module
$V$ which is not a finite $G$ component but $\varphi^0(V)$ is indecomposable.
In order to smooth our way we give a definition and a general
lemma.
Definition  Let $G$ be an algebraic group over $K$ and $H$ a closed subgroup. Let $\varphi : K[G] \to K[H]$ be the restriction map. We say a rational $H$ module $V$ is a $G$ component if

$$V \mid \varphi_0(V)$$

for some rational $G$ module $V$.

Remark  If $G$ is finite then any $H$ module $V$ is a $G$ component since $V \mid \varphi_0(\varphi^0(V))$. In general we have the following.

Lemma  Let $G$ be an algebraic group over $K$, $H$ a closed subgroup and $\varphi : K[G] \to K[H]$ the restriction map. A rational $H$ module $H$ is a $G$ component if and only if the sequence

$$0 \to \ker e \to \varphi^0(V) \xrightarrow{e} V \to 0$$

is the natural map, is exact and split.

Proof  ($\Leftarrow$) Obvious.

($\Rightarrow$) Let $V \mid \varphi_0(W), W$ a rational $G$ module. Let

$$\alpha : V \to \varphi_0(W) \text{ and } \beta : \varphi_0(W) \to V$$

be $H$ maps such that $\beta \alpha = 1_V$. By the universal mapping property there is a unique $G$ map

$$\tilde{\beta} : W \to \varphi^0(V)$$

such that $\beta = e \tilde{\beta}$. Now

$$e(\tilde{\beta} \alpha) = e \tilde{\beta} \alpha = \beta \alpha = 1_V.$$ 

Which implies that $e$ is onto and

$$0 \to \ker e \to \varphi_0(\varphi^0(V)) \xrightarrow{e} V \to 0$$

is split.

Remark  If $V$ is a finite dimensional rational $KH$ module then $V$ is observable (see [2]) if and only if $e : \varphi^0(V) \to V$ is surjective.
Now let $G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in K \right\}$.

$H = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & t \end{pmatrix} \mid y, z \in K \right\}$

$K$ algebraically closed.

Let $\varphi : K[G] \to K[H]$ be the restriction map.

$K[G] = K[X, Y, Z]$ where $X, Y, Z$ are as in example 1.3 of part A.

Put $V = K + K \varphi(Y)$ a rational left $KH$ module.

$\varphi^0(V) = \{ 1 \otimes a + \varphi(Y) \otimes b \mid 1 \otimes a + h_{y,z} \varphi(Y) \otimes b \}

= 1 \otimes ah_{y,z} + \varphi(Y) \otimes bh_{y,z}$ for all $y, z \in K$

$\leq (V) \otimes K[G]$.

Here $h_{y,z} = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$

Now $h_{y,z} \varphi(Y) = y + \varphi(Y)$ so we require

$1 \otimes (a + yb) + \varphi(Y) \otimes b = 1 \otimes ah_{y,z} + \varphi(Y) \otimes bh_{y,z}$

So $b \in K[G](H)$ and $ah_{y,z} = a + yb$ for all $y, z \in K$.

$K[G](H) = K[X]$ so if $a = a(X, Y, Z)$, $a(X, Y, Z)h_{y,z} = a(X, Y, Z) + yb(X)$

for all $y, z \in K$. Using the equations

$Xh_{y,z} = x$, $Yh_{y,z} = y + Y$, and $Zh_{y,z} = z + Z$ we eventually find $a = a_0(X) + Ya_1(X)$ and $b(X) = a_1(X)$ for some polynomials $a_0$ and $a_1$. Thus we may identify $\varphi^0(V)$ with

$K[X] + YK[X] \leq K[G]$.

$G$ is unipotent hence $\text{soc}_G(K[G]) = K$ and so $\varphi^0(V)$ is indecomposable as a $G$ module.
We show further that the KH module \( \varphi_0(\varphi^0(V)) \) is indecomposable!

Let \( \Theta \) be a KH endomorphism of \( \varphi^0(V) \), \( \Theta \) determines \( K \) space endomorphisms of \( K[x] \), \( \Theta_{ij} \) for \( 1 \leq i, j \leq 2 \) defined by

\[
\Theta(a(x)) = \Theta_{11}(a(x)) + \Theta_{12}(a(x)) \\
\Theta(b(x)) = \Theta_{21}(b(x)) + \Theta_{22}(b(x)) \text{ for } a(x) \text{ and } b(x) \text{ in } K[x].
\]

Using the fact that \( \Theta \) is a KH map we get

\[
\Theta_{21} = 0, \text{ and } \Theta_{11} = \Theta_{22} = \alpha \text{ say where } \alpha(b(x)) = x\alpha(b(x)) \text{ for all } b(x) \in K[x].
\]

Hence \( \lambda \) is multiplication by \( f(x) \) for some \( f(x) \in K[x] \). We may represent \( \Theta \) as

\[
\begin{pmatrix}
\alpha_f(x) & \beta \\
0 & \alpha_f(x)
\end{pmatrix}
\]

where \( \alpha_f(x) \) and \( \beta \) are \( K \) endomorphisms of \( K[x] \), \( \alpha_f(x) \) being multiplication by \( f(x) \).

It follows that if \( \Theta \) is an idempotent then \( \Theta = 0 \) or \( 1 \).

Hence the KH module \( \varphi_0(\varphi^0(V)) \) is indecomposable.

By the above lemma \( V \) cannot be a \( G \) component.

3.2

Let \( K \) be a field which is the algebraic closure of its prime subfield. We now show that if \( G \) is an algebraic group over \( K \), \( H \) a normal closed subgroup such that \( G/H \) is unipotent and \( \dim G = 1 \) then every KH module of finite width is a finite \( G \) component.
Lemma If \( G = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in K \} \), \( K \) algebraically closed of characteristic \( p > 0 \) and

\[
H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in K, \; x^q = x \right\}
\]

where \( q \) is some power of \( p \), then every \( KH \) module extends to a rational \( KG \) module.

Proof Let \( \varphi : K[G] \to K[H] \) be the restriction map.

\( K[G] = K[X] \) where \( g = \begin{pmatrix} 1 & X(g) \\ 0 & 1 \end{pmatrix} \) for any \( g \in G \).

Put \( Y = \varphi(X) \). \( K[H] = K \text{ span}\{1, Y, \ldots, Y^{q-1}\} \).

Define \( \psi : K[H] \to K[G] \) to be the \( K \) map given by

\[
\psi(Y^i) = x^i \text{ for } 1 \leq i \leq q-1.
\]

Note \( \psi \) is not a Hopf algebra map since \( \psi(Y^q) = \psi(Y) = x \neq x^q = \psi(Y)^q \).

It is however a coalgebra map.

Thus if \( V \) is a \( KH \)-comodule \( \psi_0(V) \) is a \( K[G] \)-comodule and \( \varphi_0(\psi_0(V)) \cong (\varphi \psi)_0(V) \cong V \).

Proposition Let \( K \) be a field which is the algebraic closure of its prime subfield and let \( G \) be an algebraic group over \( K \).

Suppose \( G^0 \cong K^+ \), the one dimensional unipotent group. If \( H \) is a finite subgroup of \( G \) then any finite dimensional \( KH \) module is a finite \( G \) component.
Proof Let $V$ be a finite dimensional $KH$ module.

Step 1 Without loss of generality (WLOG) $G^0H = G$.


Suppose $V \mid \psi'(\omega')$ for some finite dimensional rational $\mathbb{H}$ module $W$. Now $|G:L| < \infty$ implies $W \mid \psi'(\omega')$ and so $V \mid \psi'(\omega') = \varphi'(\psi'(\omega'))$, $\psi'(\omega')$ is finite dimensional and so $V$ is a finite $G$ component.

Step 2 WLOG $\text{char } K = p > 0$.

Proof Suppose $\text{char } K = 0$, $H \cap G^0$ is a finite torsion free group so $H \cap G^0 = 1$. $K[H]$ is semisimple so if $\varepsilon' : K[H] \rightarrow K$ is evaluation at 1

$$V \mid \varepsilon'(\varepsilon'(\omega)) = (V) \otimes K[H] \cong (V) \otimes \varphi'(K[G]_{(G^0)})$$

the last isomorphism holds since multiplication $H \times G^0 \rightarrow G$ is an isomorphism of varieties. Thus $V$ is a finite $G$ component.

We now assume $\text{char } K = p > 0$.

Step 3 WLOG $H$ is a $p$ group.

Proof If $P$ is a Sylow $p$ subgroup of $H$, $V \mid \Theta'(\Theta'(\omega))$ where $\Theta : K[H] \rightarrow K[P]$ is restriction; this follows from D.G.Higman's theorem (see lemma 51.2 of [4]). Suppose

$\Theta'(\omega) \mid \xi'(\omega)$ where $\xi : K[G] \rightarrow K[P]$ is the restriction map and $W$ is a finite dimensional rational $K\mathbb{H}$ module.
Then

\[(1) \forall \mid G^0(\mathcal{O}_o(v)) \mid G^0(\mathcal{O}_o(\mathcal{W})) = G^0(\mathcal{O}_o(\mathcal{W}))\]

where \(\mathcal{N} : K[G] \rightarrow K[G^0]\) is the restriction map. The above isomorphism follows from the fact that any cross-section for \(P\) in \(H\) is also a cross-section for \(PG^0\) in \(G\). This in turn is true since

\[|G: PG^0| = |HG^0: PG^0| = \frac{|H|}{|H/G^0|} = \frac{|H|}{|P|/|P^0|} \]

because \(H \cap G^0 = H \cap G^0\) as \(H \cap G^0\) is a normal \(p\) sub-group of \(H\).

Thus by (1), \(V\) is a finite \(G\) component.

**Step 4** \(G^0 < Z(G)\), the centre of \(G\).

**Proof** \(H\) is a \(p\) group hence \(G = G^0H\) is unipotent hence nilpotent.

Put \(H_r = H, H_r = N_G(H_{r-1})\) the normaliser of \(H_{r-1}\) for \(r > 1\).

By nilpotence there is an \(n \in \mathbb{N}\) such that \(H_n\) is finite and \(H_{n+1}\) is infinite. Since \(H_n / C_G(H_n)\) is finite, \(C_G(H_n)\) is infinite. Hence \(C_G(H_n)^0 = G^0\) as \(G\) is one dimensional. Hence \(G^0\) centralises \(H < H_n\). \(G = HG^0\) and \(G^0\) is abelian so \(G^0 < Z(G)\).

**Step 5** WLOG \(H \cap G^0\) may be identified with a finite subfield of \(K\).

**Proof** We identify \(G^0\) with \(K^+\) and denote by \(K(q)\) the set of elements of \(G^0\) which correspond to the subfield of \(K\) having \(q\) elements, where \(q\) is some power of \(p\).
$\mathbb{H} \cdot G^0$ is a finite subgroup of $G^0$, thus $\mathbb{H} \cdot G^0 \leq K(q)$ for some $q$.

Put $\mathcal{V} = \mathcal{H} \cdot G^0$ where $\mathcal{H} = K(q)$; $\mathcal{H} \cap G^0 = K$; moreover

$\mathcal{V} \mid \alpha^0(\mathcal{V})$, where $\alpha : K[\mathcal{H}] \rightarrow K[\mathcal{H}]$ is the restriction map,

since $\mathcal{H}$ is finite. Hence if $\alpha^0(\mathcal{V})$ is a finite $G$ component,

so is $\mathcal{V}$.

**Step 6** $\mathcal{V}$ is a finite $G$ component.

**Proof** $\mathcal{V}$ affords representation $\rho : \mathcal{H} \rightarrow \text{End}_K(\mathcal{V})$ say. Since

$\mathcal{H} \cap G^0 = K(q)$ we get by the above lemma

$$\rho|_{\mathcal{H} \cap G^0} : \mathcal{H} \cap G^0 \rightarrow \text{End}_K(\mathcal{V}) \text{ extends to}$$

$$\rho_1 : G^0 \rightarrow \text{End}_K(\mathcal{V}).$$

There is an epimorphism of algebraic groups

$$\pi : \mathcal{H} \times G^0 \rightarrow G, \quad \pi(x,y) = xy \text{ for } (x,y) \in \mathcal{H} \times G^0.$$  

Ker $\pi = \{(x,x^{-1}) \mid x \in \mathcal{H} \cap G^0\}$. Define a representation

$$\mathcal{G}_1 : \mathcal{H} \times G^0 \rightarrow \text{End}_K(\mathcal{V}) \text{ by } \mathcal{G}_1(x,y) = \rho(x) \rho_1(y) \text{ for } (x,y) \in \mathcal{H} \times G^0.$$  

Since $G^0 \leq Z(G)$ this defines a rational representation of $\mathcal{H} \times G^0$. Now $\mathcal{G}_1(\text{ker} \pi) = 1$ and so $\mathcal{G}_1$ determines a rational representation $\mathcal{G} : G \rightarrow \text{End}_K(\mathcal{V})$ which extends $\mathcal{V}$. Hence if $\mathcal{G}$ affords the $G$ module $\mathcal{Y}$, $\mathcal{G}_0(\mathcal{Y}) \cong \mathcal{V}$ and so $\mathcal{V}$ is a finite $G$ component.

This proposition clearly proves the claim made at the beginning of the section.
Remarks 1  We wonder whether it is true that if $H$ is a finite subgroup of an algebraic group $G$ over $K$ then every finite dimensional $KH$ module is a finite $G$ component.

2  By analogy with Green's theorem [5, theorem 3] and the theory of relative projectivity for the finite dimensional modules of a finite group we had hoped that a proof of the conjecture of 3.1 would shed light on the theory of relative injectivity for algebraic groups, in the sense of [3]. We no longer believe this would be the case.
References

1 Bass, H : Finistic dimension and a homological generalization of semi-primary rings, Trans. A.M.S. 95 (1950).


