NONLINEAR CONTROLLABILITY AND OBSERVABILITY WITH

APPLICATIONS TO GRADIENT SYSTEMS

José Agostinho Basto Gonçalves

Thesis submitted for the degree of Ph.D. at the University of Warwick.

May 1981.
# TABLE OF CONTENTS

Acknowledgements  
Summary  
Introduction  

## CHAPTER I - NONLINEAR ACCESSIBILITY  
1. Control systems  
2. Distributions  
3. Reachability and accessibility  
4. Strong accessibility  

## CHAPTER II - NONLINEAR OBSERVABILITY  
1. Systems: basic definitions  
2. Construction of strongly equivalent weakly observable system  
3. Observability and local observability  
4. Duality  
5. Examples  
6. Minimal realizations  

## CHAPTER III - GRADIENT SYSTEMS  
1. Gradient systems  
2. Nonlinear equivalence  
3. Equivalence of "bilinear" gradient systems  

## CHAPTER IV - HAMILTONIAN SYSTEMS  
1. Symplectic actions  
2. Realizations of Hamiltonian systems  
3. Example  

REFERENCES
ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr. P. Crouch for his encouragement and advice during the completion of this work.

I would also like to thank P. Barros for the very helpful discussions we had and all my friends, whose contribution was not so explicit but not less valuable.

Thanks are due, also, to "Fundação Calouste Gulbenkian" for their financial support, and to Peta McAllister for the typing of this thesis.
SUMMARY

We extend the theory of nonlinear observability due to Hermann-Krener [5] to the non-regular case, in which the observability codistribution is not constant dimensional, and we obtain results in some sense dual of the ones already known for accessibility.

We discuss a conjecture of P. Varaya [15], namely that the isomorphism of two locally controllable gradient systems is an isometry for the underlying pseudo Riemannian manifolds, proving it to be false without further, or different, assumptions; we also prove some positive results, and the analogue of the above for Hamiltonian systems, with weaker conditions: an isomorphism of reachable Hamiltonian systems is a symplectomorphism.

Finally we prove that a Hamiltonian system with finite-dimensional Lie algebra, satisfying standard conditions, has an accessible Hamiltonian realization, constructed in a canonical way.
INTRODUCTION

The basic definitions, results and notation follow, as much as possible, those adopted in [5] and [11-13], when concerning control theory, and in [1] where relative to differential geometry.

The construction of a locally weakly observable system, made in chapter II, is based on a result (proposition II.2.3) that can be formulated without reference to control systems: If $\mathcal{F}$ is a codimension $q$ foliation on $M$, and $D$ is a family of complete vector fields on $M$ leaving $\mathcal{F}$ invariant and spanning the tangent space at every point, the leaf space $M_1 = M/\mathcal{F}$ can be given the structure of a paracompact Hausdorff manifolds of dimension $q$ such that the canonical projection $\pi_1: M \to M_1$ is a submersion, and moreover a fibre map (a fibration in the $C^\infty$ case).

In chapter IV, the construction of an accessible realization of a Hamiltonian system is inspired by the use made in Mechanics of the "method of orbits" introduced in [7] to study Hamiltonian systems with symmetry and obtain a reduced phase space.
In this chapter we present the known results on nonlinear accessibility that are needed later, essentially based on the work by H. Sussmann [11, 13].
1 - Control Systems

A $C^K (K = \infty, \omega)$ control system is a 4-tuple $\Sigma = (M, \Omega, f, \mathcal{U})$ verifying:

i) The state space $M$ is a $n$-dimensional $C^K$ differentiable manifold, which we assume to be Hausdorff, paracompact and connected.

ii) The control space $\Omega$ is a metric space, usually a $m$-dimensional Euclidian space or discrete.

iii) The dynamics $f: M \times \Omega \to TM$ is such that, for any $u \in \Omega$, $x^u = f(\cdot, u): M \to TM$ is a $C^K$ vector field.

iv) The class of admissable controls $\mathcal{U}$ is a family of functions $\mathbb{R} \to \Omega$ defined on intervals of the form $[0, T]$ with $T \in \mathbb{R}^+$, and closed under concatenation, i.e., if $u : [0, T_1] \to \Omega$ and $v : [0, T_2] \to \Omega$ are in $\mathcal{U}$, the map $u*v : [0, T_1 + T_2] \to \Omega$ defined by $u*v|_{[0, T_1]} = u$, $u*v|_{[T_1, T_1 + T_2]} = v(\cdot - T_1)$ is also in $\mathcal{U}$.

Let $\mathcal{U}_m$ be the set of functions $\mathbb{R} \to \Omega$ with domains on above, such that every $u \in \mathcal{U}_m$ is the limit almost everywhere of a sequence of piecewise constant $K$-valued functions, where $K \subset \Omega$ is compact, and let $\mathcal{U}_{pc}$ be the set of piecewise constant $\Omega$-valued functions with the same type of domain. We assume $\mathcal{U}_{pc} \subset \mathcal{U} \subset \mathcal{U}_m$.

In order to ensure the existence and uniqueness of solution of $\dot{x} = f(x, u)$, $x(0) = x_0$ for every $x_0 \in M$ and $u \in \mathcal{U}$, we assume that
the family of associated vector fields \( D = \{ X^u \}_{u \in \Omega} \) satisfies a local Lipschitz condition uniformly in \( u \) over any compact subset \( K \) of \( \Omega \), i.e., given \( x_0 \in M \) and a compact \( K \subseteq \Omega \) there is a chart \( U \) around \( x_0 \) and a constant \( C \in \mathbb{R}^+ \) such that every component of the local representatives in \( U \) of \( \{ X^u \}_{u \in K} \) is Lipschitz with constant \( C \).

We denote by \( \pi(x_0, u; t) \) the solution at time \( t \) of the equation \( \dot{x} = f(x, u) \), \( x(0) = x_0 \).

The following approximation lemma [12] allows us, from now on, to consider \( \mathcal{U} = \mathcal{U}_p \) in every case:

**Lemma I.1.1.**

Let \( \Sigma \) be a control system and \( \{ u_i \} \) a sequence of admissible controls defined on \( [0, T] \) with values in the compact set \( K \subseteq \Omega \). If \( \{ u_i \} \rightarrow u \) almost everywhere and \( \pi(x_0, u; t) \) is defined on \( [0, T] \) then, for sufficiently large \( i \), \( \pi(x_0, u_i; t) \) is defined on \( [0, T] \) and
\[
\lim_{i \to \infty} \pi(x_0, u_i; t) = \pi(x_0, u; t).
\]

2 - **Distributions**

A distribution (codistribution) \( \Delta \) on a manifold \( M \) is the assignment to each \( x \in M \) of a subspace \( \Delta(x) \) of \( T_x M (T^*_x M) \).

\( \Delta \) is a \( C^k \) (co)distribution if it is spanned by a family \( D \) of \( C^k \) vector fields (1-forms), i.e., if \( \Delta(x) \) is the linear subspace of
Let $\mathcal{X} = \{X(x)\}$ with $X \in D$ for every $x \in M$. A $C^k$ distribution $\Delta$ is involutive if whenever the vector fields $X^1, X^2$ belong to $\Delta$ (i.e. $X^i(x) \in \Delta(x)$ for $i = 1, 2$ and $x \in M$) so does $[X^1, X^2]$. We say $\Delta$ is $r$-dimensional if $\Delta(x)$ has dimension $r$ for each $x \in M$.

Let $D$ be a family of vector fields; the pseudo group $G_D$ (pseudo semigroup $S_D$) of $D$ is the set of all finite products of elements belonging to the pseudo-groups of flows (pseudo-semigroups of flows for non-negative time) of vector fields in $D$.

We say that $\Delta$ is $D$-invariant if for every $g \in G_D$ and $x \in \text{dom}(g)$ we have $g_*(\Delta(x)) \subseteq \Delta(gx)$, or in the case of codistributions $g^* \Delta(gx) \subseteq \Delta(x)$.

We denote by $\Delta_D$ the distribution spanned by $D$, by $\overline{\Delta}(D)$ the smallest involutive distribution containing $\Delta_D$, and by $P_D$ the smallest $D$-invariant distribution containing $\Delta_D$.

A submanifold $S$ of $M$ is said to be an integral submanifold of $\Delta$ if $T_x S = \Delta(x)$ for $x \in S$, and a maximal integral submanifold of $\Delta$ is a connected integral submanifold of $\Delta$ maximal for the relation of inclusion.

$\Delta$ is integrable if through every point of $M$ there passes a maximal
integral submanifold of $\Delta$.

The orbit of $D$ containing $x \in M$ is the set $G_Dx$.

**Theorem 1.2.1[11]:**

$P_D$ is an integrable distribution, and the maximal integral submanifolds are the orbits of $D$.

This in particular implies $\mathcal{J}(D) \subseteq P_D$.

**Theorem 1.2.2[11]:**

Let $\Delta$ be a $C^K$ distribution spanned by a family $D$ of vector fields. $\Delta$ is integrable iff $\Delta = P_D$.

As corollaries we have

**Nagano Theorem 1.2.3 [11,8]:**

An analytic involutive distribution is integrable.

**Frobenius Theorem 1.2.4:**

A $C^\infty$ constant dimensional involutive distribution is integrable.
3 - Reachability and accessibility

Let \( \Sigma \) be a control system, and \( D \) the family of associated vector fields.

\( \Sigma \) is said to have the reachability (controllability) property if \( G_Dx = M \) (\( S_Dx = M \)) for any \( x \in M \).

\( \Sigma \) is accessible at \( x \in M \) if \( S_Dx \) has non-empty interior in \( M \), and locally accessible at \( x \) if for any neighbourhood \( U \) of \( x \) the set of points attainable from \( x \), i.e. of the form \( y = gx \) with \( g \in S_D \), with trajectories not leaving \( U \) has non-empty interior in \( M \). Equivalently, we can say \( \Sigma \) is locally accessible at \( x \in M \) if its restriction to any neighbourhood of \( x \) is accessible at \( x \).

Accessibility and local accessibility are defined in the natural way; and clearly local accessibility implies accessibility.

\textbf{Chow Theorem [11] 1.3.1:}

\( \Sigma \) is reachable iff \( P_D \) has dimension \( n \).

We say that \( \Sigma \) satisfies the controllability rank condition at \( x \in M \) if \( \dim \mathcal{J}(D)(x) = n \).

\textbf{Theorem I.3.2 [5]:}

If \( \Sigma \) satisfies the controllability rank condition at \( x \in M \), then it is locally accessible at \( x \).

We have a partial converse for the above theorem:
Theorem 1.3.3 [5]:

If $\Sigma$ is locally accessible, the controllability rank condition is satisfied on an open dense set in $M$.

In the analytic case $P_D = \mathcal{T}(D)$ (Nagano theorem) and the same is true if $\mathcal{T}(D)$ is constant dimensional, from Frobenius theorem.

Thus we get

Theorem 1.3.4:

If $\Sigma$ is an analytic system, or $\mathcal{T}(D)$ has constant dimension, reachability, accessibility and local accessibility are equivalent.

It is quite easy to construct counter examples to the above result for the $C^\infty$ case, where $\mathcal{T}(D)$ has not constant dimension.

Example 1.3.5:

Let $M = \mathbb{R}^2$ and $D$ be $\{ \frac{3}{\partial x}, \phi(x) \frac{3}{\partial y} \}$ where $\phi(x) = 0$ if $x \geq 0$ and $\phi(x) = \exp(-1/x)$ if $x < 0$.

Then the corresponding control system has the reachability property, but $\Sigma$ is not accessible at a point $x = (x_1, x_2)$ if $x_1 \geq 0$ (and it is not locally accessible at the same points).

Example 1.3.6:

Consider now the above example changing $\phi$ into $\psi$, defined by $\psi(x) = \phi(-x)$.
The corresponding system is then accessible but it is not locally accessible at points \( x = (x_1, x_2) \) if \( x_1 < 0 \).

4 - Strong accessibility

A system is said to be strongly accessible at \( x \in M \) if, for some \( t \in \mathbb{R}^+ \), the set of points attainable from \( x \) in time \( t \) has non-empty interior. Strong accessibility is defined in the natural way.

Let \( \mathcal{T}_0(D) \) be the distribution spanned by all sums \( Z_1 + Z_2 \), where \( Z_1 \) is a linear combination of vectors in \( D \) such that the sum of the coefficients is zero, and \( Z_2 \) is an element in the derived Lie algebra of \( \mathcal{T}(D) \), i.e. \( Z_2 \in [\mathcal{T}(D), \mathcal{T}(D)] \) if we interpret \( \mathcal{T}(D) \) as a Lie algebra of vector fields.

Lemma I.4.1 [13]:

\( \mathcal{T}_0(D) \) is an ideal of \( \mathcal{T}(D) \) of codimension one or zero, as Lie algebras, and at each \( x \in M \), \( \mathcal{T}(D) (x) \) and \( \mathcal{T}_0(D) (x) \) are subspaces of \( T_xM \) such that the codimension of \( \mathcal{T}_0(D) (x) \) in \( \mathcal{T}(D) (x) \) is zero or one.

Define \( P_0^D \) as the distribution spanned by the differences of vector fields belong to \( P_D \).

Lemma I.4.2. [4]:

\( P_0^D \) is a \( D \)-invariant distribution, and its codimension in \( P_D \) is zero or one. If the system is analytic \( P_0^D \) is the distribution spanned by
Theorem I.4.3 [13]:

An analytic system is strongly accessible iff  $\mathcal{T}_0(D)$ is n-dimensional.

Theorem I.4.4 [4]:

If $\Sigma$ is strongly accessible, $\mathcal{P}_D^0$ is n-dimensional.

A particular case is of interest: suppose we are given a $C^k$ system, in which $f(x,u) = X^0(x) + \sum_{i=1}^{\ell} u_i X_i(x)$, where $X_i$, $i = 0,1,...,n$ are complete vector fields in $M$, and $\mathcal{T}(D)$ is a finite dimensional Lie algebra.

Theorem I.4.5 [6]:

i) $\mathcal{T}(D)$ is the Lie algebra generated by $\{X^0,X^1,...,X^\ell\}$, and every element of $\mathcal{T}(D)$ is a complete vector field.

ii) $\mathcal{T}_0(D)$ is the ideal of $\mathcal{T}(D)$ generated by $\{X^1,...,X^\ell\}$.

iii) If we define $\text{ad}^K_{X^0}$ by $\text{ad}^K_{X^0} X = X^0 [X^0, \text{ad}^{K-1}_{X^0} X]$ for $K > 0$, $\mathcal{T}_0(D)$ is the Lie algebra generated by $\{\text{ad}^r_{X^0} X^i, i = 1,2,...,K: r = 0,1,...\}$.

From this theorem, it follows that control systems as above are strongly accessible iff they are accessible and $X^0(x) \in \mathcal{T}_0(D)(x)$ at every $x \in M$. 
CHAPTER II - NONLINEAR OBSERVABILITY

Our aim is to extend the results of Hermann-Krener [5] to the non-regular case, obtaining some kind of duality of the known results about accessibility, presented in the previous chapter.

After the basic definitions in the first section, section two consists of the construction of strongly equivalent complete locally weakly observable and weakly observable systems from a given complete reachable system.

The construction is used in section three to give criteria for observability, and section four presents the duality with corresponding results for accessibility.

In section five we give some examples of the application of these results, and section six contains the basic notions about minimal realizations that are used later.
1 - Systems : basic definitions

A $C^K$ system is a $\sigma$-tuple $\Sigma = (M, \Omega, f, \mathcal{U}, N, h)$ where

i) $(M, \Omega, f, \mathcal{U})$ is a $C^K$ control system

ii) $N$ is a $\lambda$-dimensional Euclidean space

iii) $h : M \to N$ is a $C^K$ map.

$N$ is called the output space, and $h$ the output map. Properties of the associated control system, also denoted by $\Sigma$, will be attributed to the system $\Sigma$.

We say two points $x_0, x_1$ in $M$ are (weakly) indistinguishable if for every $\gamma \in S_D (\gamma \in G_D)$, such that $\gamma x_0$ and $\gamma x_1$ are defined, we have $h \circ \gamma x_0 = h \circ \gamma x_1$. We denote by $S_D^* h (G_D^* h)$ the set of functions of the form $h \circ \gamma, \gamma \in S_D (\gamma \in G_D)$.

$\Sigma$ is (weakly) observable if no two points are (weakly) indistinguishable.

$\Sigma$ is strictly locally (weakly) observable at $x \in M$ if there exists an open nhd $U$ of $x$ such that the restriction of $\Sigma$ to any nhd of $x V \subset U$ is (weakly) observable.

$\Sigma$ is locally (weakly) observable at $x$ if there exists a nhd $U$ of $x$ such that there are no two points in $U$ (weakly) indistinguishable.

Clearly the weak definitions are equivalent to the usual ones if $\Sigma$ is
symmetric, i.e. \( X \in D \Rightarrow -X \in D \), or analytic.

If the system is complete (every \( X \in D \) is a complete vector field) the domain of any \( \gamma \in G_D \) is the whole state space; in that case, or if the system is analytic, indistinguishability and weak indistinguishability are equivalence relations \([12]\).

Example II.I.1.

Let \( M \) be the bounded open set enclosed by the following polygon:

\[
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\]

Let \( h: M \rightarrow \mathbb{R} \) be defined by \( h|_{M_1} = 0 \) and \( h(x,y) = \exp(-1/(x-1)) \) if \( (x,y) \in M_2 \). Let \( D = \{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y} \} \).

Then, if \( x_0 = (1/2, 1/2) \), \( x_1 = (1/2, 3/2) \) and \( x_2 = (1/2, 5/2) \) we have \( x_0, x_1 \) and \( x_1, x_2 \) indistinguishable, but \( x_0 \) and \( x_2 \) are not indistinguishable.

Given two systems \( \Sigma_1 \) and \( \Sigma_2 \), with the same control class and output space, we say \( x_1 \in M_1 \) and \( x_2 \in M_2 \) are (weakly) indistinguishable if \( h_1 \circ g_1 \circ x_1 = h_2 \circ g_2 \circ x_2 \), where \( g_1 \in S_0 \) (\( g_1 \in G_0 \)) and \( g_2 \) is obtained
using the same piecewise constant control as the one involved in $g_1$.

If $\Sigma_1$ and $\Sigma_2$ are not complete, the above equality has to be satisfied when $x_1 \in \text{dom } g_1$ and $x_2 \in \text{dom } g_2$.

$\Sigma_1$ and $\Sigma_2$ are strongly equivalent if every state in $\Sigma_1$ is weakly indistinguishable from some state in $\Sigma_2$ and vice-versa.

2 - Construction of strongly equivalent weakly observable system

We assume $\Sigma$ is a complete, reachable system with output map $h : M \to \mathbb{R}^\ell$, $h = (h_1, ..., h_\ell)$. We construct the two following codistributions:

i) $\mathcal{H}(D)$ is the smallest codistribution containing $dh_i$, $i = 1, 2, ..., \ell$ and closed for Lie differentiation by elements of $D$.

ii) $\mathcal{P}(D)$ is the smallest $D$-invariant codistribution containing $dh_i$, $i = 1, 2, ..., \ell$.

From the definition of Lie derivative, it is clear that $\mathcal{H}(D) \subset \mathcal{P}(D)$.

Since $\Sigma$ is reachable, i.e. $M$ is an orbit of $D$, $\mathcal{P}(D)$ is constant dimensional. We denote by $q$ its dimension.

Let $F_D$ be the smallest subspace (over $\mathbb{R}$) of $C^\infty(M)$ containing $h_i$, $i = 1, ..., \ell$ and closed under Lie differentiation by elements of $D$.

Since the exterior differentiation commutes with Lie derivative on functions, $\mathcal{H}(D) = dF_D$ in the sense that $\mathcal{H}(D)$ is the codistribution spanned by the differentials of the functions in $F_D$. 
Let $E_D$ be the smallest subspace of $C^k(M)$ containing $h_i$, $i = 1, \ldots, l$ and closed for the action of $G_D$ on $C^k(M)$ defined by $(g,f) \mapsto g^* f$. Then $\mathcal{D}(D) = dE_D$, as above, since $dg^* f = g^* df$.

Let $\Lambda$ be the $C^k$ distribution defined by $\Lambda = \text{Ker} \mathcal{D}(D)$, i.e. $\Lambda$ is spanned by the vector fields $X \in \mathcal{V}^k(M)$ such that, for any differential form $\alpha$ belonging to $\mathcal{D}(D)$, $\alpha \cdot X = 0$. Equivalently, we can characterize $X$ by $X\psi = 0$ for every $\psi \in E_D$.

**Lemma II.2.1:**

$\Lambda$ is an integrable distribution.

**Proof:**

Since $\mathcal{D}(D)$ is constant dimensional, so is $\Lambda$.

Let $X,Y$ be two vector fields belonging to $\Lambda$, and $\psi \in E_D$. Then:

$$[X,Y]\psi = X(Y\psi) - Y(X\psi) = X.0 - Y.0 = 0$$

and thus $[X,Y]$ is in $\Lambda$, therefore $\Lambda$ is involutive. By Frobenius theorem, it is integrable.

**Lemma II.2.2:**

The maximal integral submanifolds of $\Lambda$ are closed, and any two of them are diffeomorphic.
Proof:

It follows from the definition of $\Lambda$ that its maximal integral submanifold containing some point $x \in M$ is the connected component of the set $\{x' \in M : \psi(x) = \psi(x') \psi \in E_D\}$ containing $x$. As the connected component of a closed set is closed, the first part of the lemma is proved.

Now let $L_i, L'_{i'}$ be two maximal integral submanifolds of $\Lambda$ passing through $x, x'$ respectively. As the system is reachable, there exists $g \in G$ such that $gx = x'$.

For every $g \in G$, if $X$ is a vector field in $\Lambda$, $g_*X$ is also in $\Lambda$ (i.e. $\Lambda$ is $D$-invariant):

$$g_*X.\psi = d\psi.g_*X = (g^*d\psi)X = X.(g^*\psi) = 0$$

since $\psi \in E_D \Rightarrow g^*\psi \in E_D$.

Thus $g$ takes maximal integral submanifolds into maximal integral submanifolds, and as $gx = x'$ we have $gL = L'$.

Proposition II.2.3:

There exists a Hausdorff paracompact $C^k$ manifold $M_\parallel$ of dimension $q$, and a map $\pi_1 : M \to M_\parallel$ such that:

i) $\pi_1$ is a submersion.
ii) The fibres of $\pi_1$ are the maximal integral submanifolds of $\Lambda$.

iii) $\pi_1$ is a fibre map, and in the $C^\infty$ case it is a fibration.

Given two manifolds $E, M$ and a map $p: E \to M$ we say $p$ is a fibre map if $M$ has an open cover of sets $\{U_i\}$ such that, for every $i$ there exists a diffeomorphism $\phi_i : p^{-1}(U_i) \to U_i \times F$ making the following diagram commutative

$$
\begin{array}{ccc}
p^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times F \\
p \downarrow & & \downarrow p_1 \\
U_i & & 
\end{array}
$$

where $p_1$ is the projection on the first factor, $F$ is a manifold called the fibre.

Proof:

Consider the equivalence relation $R$ defined on $M$ by $x \sim y \iff x$ and $y$ belong to the same maximal integral submanifold of $\Lambda$. Let $M_1 = M/R$ and $\pi$ be the natural projection.

Take $[x] \in M_1$, $[x] = \pi_1(x)$. We can find $\phi_1, \ldots, \phi_q \in E_D$ such that $d\phi_1(x), \ldots, d\phi_q(x)$ span $\mathcal{G}(D)(x)$. Then this is also true for an open nbhd $U'$ of $x$. 
Let $U_1 = \pi_1(U')$ and $U = \pi_1^{-1}(U_1)$. We want to show that $d\phi_1, \ldots, d\phi_q$ span $\mathcal{R}(D)$ on $U$; it is enough to show that if $d\phi(y) \neq 0$ for some $\phi \in E_D$ and $y \in M$, then $d\phi$ is non-zero in every point of the maximal integral submanifold $L_y$ passing though $y$.

Let $y' \in L_y$ and suppose $d\phi(y') = 0$. Then for every vector field $X$ in $D' = G_xD$ (i.e. $x \in D'$ iff there exists $g \in G$ and $Y \in D$ such that $X = g_xY$) we have $L_X\phi(y') = d\phi(y').X(y') = 0$.

We remark that if $\Lambda$ is $D$-invariant it is also $D'$-invariant, and as the system is reachable $P(D)$ and therefore $D'$ span the tangent space at all points. Clearly if $\Lambda$ is $D'$-invariant so is $\mathcal{R}(D)$.

Now $L_X\phi$ is constant on the maximal integral submanifolds of $\Lambda$, since $d(L_X\phi)$ is contained in $\mathcal{R}(D)$

$$d(L_X\phi) = L_Xd\phi$$

and from $D'$-invariance we have $L_Xd\phi \in \mathcal{R}(D)$. Thus $L_X\phi(y) = 0$ for every $X \in D'$, or equivalently $d\phi(y) \cdot X(y) = 0$; as $D'(y)$ spans $T_yM$ we conclude $d\phi(y) = 0$, and the contradiction proves our claim.

If we denote by $\phi : M \to \mathbb{R}^q$ the map $\phi(y) = (\phi_1(y), \ldots, \phi_q(y))$ we see that $\phi$ is a submersion at every point of $U$. Moreover the equivalence relation $R$ considered on $U$ is the same as $x = y \iff \phi(x) = (\phi(y))$. 
Then, by \([3, \text{ pg. 91}]\) we have the following commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\phi} & \mathbb{R}^q \\
\pi_1 & \downarrow & \\
U_1 & \xrightarrow{\phi_1} & \\
\end{array}
\]

where \(\phi(U) = \phi_1(U_1)\) is an open set, since \(\phi\) is an open map and \(\phi(U) = \phi(U')\), \(\phi_1\) is a diffeomorphism onto its image and \(\pi_1\) is a submersion on \(U\).

This way, when we vary \([x]\) in \(M_1\) we construct an atlas on \(M_1\), with charts \((U_1, \phi_1)\), and if \(M_1\) is given the \(C^K\) structure defined by this atlas \(\pi_1\) is a submersion, with the maximal integral submanifolds of \(\Lambda\) as fibres.

As the induced topology on \(M_1\) is the quotient topology, \(M_1\) is paracompact \([3]\).

Let \([x] \in M_1\), and consider a chart \((U_1, \phi_1)\) around \([x]\) constructed as above. Take \(x \in U\).

We have vector fields \(X_1, \ldots, X^q \in D_x\) such that the linear subspace \(S(x)\) spanned by \(X_1(x), \ldots, X^q(x)\) verifies \(S(x) \oplus \Lambda(x) = T_x M\). Define the map \(\phi : \mathbb{R}^q \times L_x \to M\) by \(\phi(t_1, \ldots, t_q, y) = X^1_{t_1} \circ \cdots \circ X^q_{t_q}\); clearly this map has rank \(n\) at \((0, x)\), so it is a diffeomorphism of some open set \(A \subset \mathbb{R}^q \times L_x\) containing \((0, x)\), onto an open neighbourhood \(N\) of \(x\).
We can take \( N \subset U \), and \( A \) with the form \( C \times W \), when \( C \) is an open cubic neighbourhood of the origin in \( \mathbb{R}^q \) and \( W \) is an open neighbourhood of \( x \) in \( L_x \).

Then \( \sigma = \pi_1 \circ \phi_A \) is a submersion, and by \( D^i \)-invariance the fibres of that map have the form \( \{s\} \times W \) with \( s \in C \). So we have: \( C \) is diffeomorphic to \( A/\sigma \), then to \( \sigma(A) = V_1 \), i.e. we have the commutative diagram:

\[
\begin{array}{ccc}
A = C \times W & \xrightarrow{\phi_A} & M \\
\downarrow P_1 & & \downarrow \pi_1 \\
C & \xrightarrow{\sigma} & M_1 \\
\end{array}
\]

where \( \sigma_1 \) is a diffeomorphism onto its image.

Define the map \( \psi: V_1 \times L_x \rightarrow M \) by taking \( \psi = \phi \circ (\sigma_1^{-1}, \text{id}) \); it is a \( C^K \) map, and its image is contained in \( V = \pi_1^{-1}(V_1) \).

Define \( \theta: V \rightarrow V_1 \times L_x \) by \( \theta(y) = (\pi_1(y), x_{-s_q}^q \circ \ldots \circ x_{-s_1}^1 y) \)
where \( (s_1, \ldots, s_q) = \sigma_1^{-1} \circ \pi_1(y) \). Clearly \( \theta \) is a \( C^K \) map.

It is trivial to check that \( \psi \circ \theta = \text{id}_V \) and \( \theta \circ \psi = \text{id}_{V_1 \times L_x} \); therefore \( \theta \) is a diffeomorphism and if \( p_1: V_1 \times L_x \rightarrow V_1 \) denotes the projection on the first factor, we have the commutative diagram:
Since we can obtain a cover of $M_1$ by this process, we have proved that $\pi_1$ is a fibre map.

Now let $[x] = \pi_1(x)$, $[y] = \pi_1(y)$ be two distinct points in $M_1$.

We can find an open nhd $U'$ of $x$ such that $U' \cap L_y = \emptyset$. Let $V_1$ be an open nhd of $[x]$ over which we have as before

$$V = \pi_1^{-1}(V_1) \rightarrow V_1 \times L \rightarrow (L_1^{-1}, \text{id}) \rightarrow C \times L$$

We can assume $U' \subset V$; if $U = \pi_1^{-1}(\pi_1(U'))$ we can see, using the above diffeomorphism onto $C \times L$, that $U$ is the union of the maximal integral submanifolds of $L$ passing through $U'$, and hence $U \cap L_y = \emptyset$.

Then we can find an open nhd $W'$ of $y$ such that $W' \cap U = \emptyset$; as before, if we take $W = \pi_1^{-1}(\pi_1(W'))$, we can assume $W \cap U = \emptyset$. As $U, W$ are open sets, $\pi_1(U)$ and $\pi_1(W)$ are open nhds of $[x]$ and $[y]$ respectively, and moreover they are disjoint. Therefore $M_1$ is Hausdorff.
In the $C^\infty$ case, if $\pi:M \to M_1$ is a fibre map and $M_1$ is a paracompact Hausdorff manifold, then $\pi_1$ is a fibration \[10\].

**Corollary II.2.4:**

The maximal integral submanifolds of $\Lambda$ are regular submanifolds of $M$, of dimension $n-q$.

**Proof:**

Since $\pi_1$ is a submersion, and the maximal integral submanifolds of $\Lambda$ are the inverse image of (then regular) points in $M_1$, the result follows.

$\Lambda$ is then called a regular submanifold, from the regularity properties of its maximal integral submanifolds.

**Theorem II.2.5:**

Let $\Sigma$ be a complete system. Then we can construct a locally weakly observable system $\Sigma_1$, complete, such that $\Sigma$ and $\Sigma_1$ are strongly equivalent.

**Proof:**

Consider, as in the previous proposition, the fibre map $\pi_1:M \to M_1$. Define a family of vector fields $D_1$ by taking $X_1 \in D_1$ if $X_1 = \pi_1^* X$; this is well defined, since as $\Lambda$ is $D$-invariant the projection $X_1(\bar{x})$ is independent of the point taken on the fibre above $\bar{x}$. Moreover if $X = f(\cdot,u)$ with $u \in \mathcal{U}$, then $X_1 = f_1(\cdot,u)$ for the same $u \in \mathcal{U}$ with
\( f_1 = \pi_1 \ast f \), which is similarly well defined.

Since the output map is constant along maximal integral submanifolds of \( A \) we have

\[
\begin{array}{ccc}
M & \xrightarrow{h} & N \\
\downarrow{\pi_1} & & \downarrow{h_1} \\
M_1 & & \\
\end{array}
\]

Let \( \Sigma_1 = (M_1, \Omega, f_1, \mathcal{U}, N, h_1) \); it is a complete system, from the construction of the associated vector fields.

Since \( X_t h_i \) with \( X \in \mathcal{D} \) is constant on the fibres of \( \pi_1 \) we have

\[
\begin{array}{ccc}
M & \xrightarrow{X_t h_i} & R \\
\downarrow{\pi_1} & & \downarrow{\eta} \\
M_1 & & \\
\end{array}
\]

Clearly \( \eta = X_t^* h_i \), and hence the way we construct the charts \( (U_1, \phi_1) \) around each point \([x]\) in \( M_1 \) shows that \( \Sigma_1 \) is locally weakly observable (the points in \( U_1 \) can be distinguished).

From the construction we see that \( x \) and \( \pi_1(x) \) are indistinguishable, concluding the proof.

The proof that \( \Sigma_1 \) is defined on a Hausdorff manifold does not assume local accessibility, or better, the controllability rank condition.
satisfied at every point, for the system $\Sigma$.

In the proof that $\Sigma_1$ is Hausdorff is involved a family of vector fields for which the distribution $\Lambda$ is invariant; the family $D$ is not rich enough, in general, but as Hermann-Krener used it in [5] they had to assume the controllability rank condition at every point to do the job.

In our proof, instead of using $D$ we used $D'$ which trivially satisfies the controllability rank condition, since in fact $D'$ spans the tangent space at every point.

Related to this, we can remark that example 3.10 in [5], which is meant as a counter-example for the non-assumption of the controllability rank condition, is misleading: the important fact is that the original system is not reachable in that example, not that it is not locally accessible.

The equivalence relation used for the construction of $M_1$ has as equivalence classes the connected components of the sets on which all $\phi \in E_D$ are constant; the resulting system $\Sigma_1$ is only locally weakly observable.

If we take as equivalence classes the sets (and not their connected components) on which $E_D$ is constant, we will be able to obtain a new system $\Sigma_2$, now weakly observable. Clearly the new manifold $M_2$ thus defined will be the same as the one obtained from $M_1$ with the equivalence
relation $[x] \sim [y] \iff E_{D_1}([x]) = E_{D_1}([y])$, i.e. we have the following commutative diagram:

\[ \begin{array}{ccc}
M & \xrightarrow{\pi_1} & M_1 \\
\downarrow \Pi & & \downarrow \pi_2 \\
M_2 & \end{array} \]

**Proposition II.2.6:**

There exists a Hausdorff paracompact $C^K$ manifold $M_2$ of dimension $q$, and a map $\pi_2 : M_1 \rightarrow M_2$ such that:

i) $\pi_2$ is a covering projection.

ii) the fibres of $\pi_2$ are the equivalence classes for the weak indistinguishability relation $\sim$ on $M_1$.

iii) $\pi_2 \circ \pi_1$ is a fibre map, and its fibres are the equivalence classes for the equivalence relation $R$ on $M$ $x R_y \iff \phi E_D \phi(x) = \phi(y)$, i.e. $x R_y \Rightarrow x, y$ are weakly indistinguishable.

**Proof:**

iii) is a consequence of i) and ii) and the previous remarks.

Let $M_2 = M_1/\sim$ and $\pi_2$ the natural projection.
Take $[[x]] \in M_2$ and $[x] \in M_1$ such that $\pi_2([x]) = [[x]]$. Construct around $[x]$ a chart $(U_1, \phi^1)$ as in proposition 4.

Then any two points in $U_1$ are weakly indistinguishable, and therefore $\pi_2|U_1$ is a bijection onto its image.

Let $U_2 = \pi_2(U_1)$. Then $(U_2, \phi^1 \circ (\pi_2|U_1)^{-1})$ is a chart for $M_2$; it is clear that in this way we can construct a $C^K$ structure in $M_2$, for which $\pi_2$ is a local diffeomorphism; since the induced topology is the quotient topology, $M_2$ is paracompact.

If we prove $\pi_2$ is a covering projection, then it results, as in proposition 4, that $M_2$ is Hausdorff.

Now, noting that the flows of the vector fields in $D_1$ take equivalence classes into equivalence classes, i.e. fibres of $\pi_2$, we can use the same reasoning as in the proof of proposition 4 to show that every point $[[x]]$ has an open nhbd $V_2$ such that

$$\pi_2^{-1}(V_2) \xrightarrow{0_1} V_2 \times \pi_2^{-1}([[x]])$$

$\pi_2^{-1}([[x]])$ is a discrete space, i.e., its tangent space is $\{0\}$ at every point.
This shows that $\pi_2$ is a covering projection, and the proof is complete.

**Theorem II.2.7:**

Let $\Sigma$ be a complete system. Then we can construct a weakly observable complete system $\Sigma_2$, such that $\Sigma$ and $\Sigma_2$ are strongly equivalent.

**Proof:**

The proof is in any way the analogue of the one of theorem 5.

3 - **Observability and local observability**

We assume $\Sigma$ is a complete $C^K$ system, as previously.

$\Sigma$ is said to satisfy the observability rank condition at $x \in M$ if $\dim \mathcal{H}(D)(x) = n$.

**Theorem II.3.1:**

$\Sigma$ is locally weakly observable iff $\mathcal{P}(D)$ has dimension $n$.

**Proof:**

If $\mathcal{P}(D)$ has dimension less than $n$ we can construct, as in the previous chapter, a strongly equivalent system of lower dimension. Then any two points in the same fibre are weakly indistinguishable, and so
the original system is not locally weakly observable (note that the fibre is at least one dimensional).

On the other hand, if $\dim \mathcal{O}(D)(x) = n$ we can take $\phi_1, \ldots, \phi_n \in E_D$ such that $\{d\phi_i\}_{i=1}^n$ span $\mathcal{O}(D)(x)$ and $T_x^* M$.

By the inverse function theorem, $\phi: M \to \mathbb{R}^n$ given by $\phi = (\phi_1, \ldots, \phi_n)$ is a diffeomorphism on some neighbourhood $U$ of $x$; then if $x_1, x_2 \in U$ we can find $\phi_i$ such that $\phi_i(x_1) \neq \phi_i(x_2)$.

Since in our definition of weak indistinguishability we can substitute $E_D$ for $G^*_D$, it follows that no two points in $U$ are weakly indistinguishable. As the construction can be made for any $x \in M$, $\Sigma$ is locally weakly observable.

**Theorem II.3.2 [5]:**

If $\Sigma$ satisfies the observability rank condition at $x \in M$, then $\Sigma$ is strictly locally observable at $x$.

**Theorem II.3.3 [5]:**

If $\Sigma$ is strictly locally observable, the observability rank condition is satisfied on an open dense subset of $M$.

As when considering accessibility, we have the following:

**Theorem II.3.4:**

If $\mathcal{H}(D)$ has constant dimension, in particular when $\Sigma$ is analytic,
local observability, local weak observability and strict local observability are equivalent.

Proof:

Let $\Delta = \ker \mathcal{H}(D)$; if $\mathcal{H}(D)$ is constant dimensional so is $\Delta$. Now, following [9], $\Delta$ is $D$-invariant if and only if, for every locally defined vector field $Y$ contained in $\Delta$ and every $X \in D$ we have $[X, Y]$ contained in $\Delta$ (note that constant dimensionality is essential).

Take $\phi \in F_D$; then $d\phi [X, Y] = X.(d\phi Y) - Y.(d\phi X) - dd\phi(X, Y) = 0$, and so $[X, Y]$ is in $\Delta$.

If $\Delta$ is $D$-invariant, so is $\mathcal{H}(D)$ and then $\mathcal{P}(D) = \mathcal{H}(D)$.

This proves the equivalence stated above. The proof of theorem 3.12 in [5] shows that if the system is analytic $\mathcal{H}(D)$ is constant dimensional.

4 - Duality

Comparing the results of the previous section with those in I.3 we can see we have some kind of duality, in the sense that theorems II.3.1, II.3.2, II.3.3, II.3.4 can be obtained from the ones in the first chapter (and vice-versa) using a table of correspondences as follows:
controllability rank condition

(I.3.2) \downarrow \uparrow (I.3.3)

local accessibility

\downarrow \uparrow (I.3.4)

accessibility

\downarrow \uparrow (I.3.4)

reachability

dim P_D = n

observability rank condition

(II.3.2) \downarrow \uparrow (II.3.3)

strict local observability

\downarrow \uparrow (II.3.4)

local observability

\downarrow \uparrow (II.3.4)

local weak observability

\uparrow (II.3.1)

dim \mathcal{A}(D) = n

Note that \mathcal{P}(D) and \mathcal{H}(D), apart from having dual roles of P_D and \mathcal{F}(D) as shown above, have dual constructions, in the sense that:

\mathcal{F}(D) (\mathcal{H}(D)) is the (co)distribution spanned by all the Lie derivatives with respect to vector fields in D of vector fields in D (of differentials of the output map components).

P_D(\mathcal{P}(D)) is the smallest D-invariant (co)distribution containing D (dh).

Thus, in a certain sense, the vector fields in D are "dual" of the differentials of the components of the output map.

As another case of duality we have the following result:
Proposition II.4.1:

Whenever $S_D = G_D$, controllability and reachability are equivalent, and so are weak observability and observability. In particular, that is true for systems defined on compact Lie groups.

5 - Examples

Example II.5.1: Consider the system defined on $\mathbb{R}^2$ the family of vector fields $D = \{\frac{\partial}{\partial x}, \phi(x)\frac{\partial}{\partial y}\}$, where $\phi(x) = e^{-1/x^2}$ for $x \leq 0$, and $\phi(x) = 0$ for $x > 0$, and $h: \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) = \sin x$.

This system is reachable, but not accessible, and it is not weakly observable, since any two points in $\mathbb{R}^2$ with the same first coordinate are weakly indistinguishable.

If we construct $\mathcal{H}(D)$ we see it is spanned by $\cos x \, dx$, i.e., it is not constant dimensional, since it has dimension 1 on $\mathbb{R}^2 - A$, where $A = \{(x,y) \in \mathbb{R}^2, x = \pi/2 + k \text{ with } k \in \mathbb{Z}\}$, and dimension zero on $A$. On the other hand $\mathcal{P}(D)$ is spanned by $dx$, and has constant dimension, equal to 1.

The construction in proposition 2.4 gives rise to the system $\Sigma_1$, define on $\mathbb{R}$, with $D_1 = \{\frac{\partial}{\partial x}\}$ and $h_1: \mathbb{R} \to \mathbb{R}$, $h_1(x) = \sin x$.

This new system is accessible, and locally observable: we can distinguish any two points $x_1, x_2$ in positive turn, if $x_1 - x_2 \neq 2K\pi$ with $K \in \mathbb{Z}$.

The construction in proposition 2.6 gives the system $\Sigma_2$, defined on $M_2 = S^1$, with $D_2$ consisting of the unit tangent vector, and
\( h_2 : S^1 \to \mathbb{R} \) given by the sinus. This system is accessible and observable.

The maps \( \pi_1 \) and \( \pi_2 \) are the usual projections \( \mathbb{R}^2 \to \mathbb{R} \) and \( \mathbb{R} \to S^1 \), respectively.

**Example 11.5.2:** In general, the final system \( \Sigma_2 \) we obtain is not accessible or observable.

Let \( \Sigma \) be defined on \( \mathbb{R}^3 \) by \( D = \left\{ -\frac{3}{\partial x}, \phi(-x)\frac{\partial}{\partial y}, \frac{3}{\partial z} \right\} \) with \( \phi \) as above, and \( h : \mathbb{R}^3 \to \mathbb{R}^2 \), \( h(x,y,z) = (\phi(-x),y) \). Then \( \Sigma_1 = \Sigma_2 \) is defined on \( \mathbb{R}^2 \) by \( D_1 = D_2 = \left\{ -\frac{3}{\partial x}, \phi(-x)\frac{\partial}{\partial y} \right\} \) and \( h_1 = h_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) the map \( (x,y) \to (\phi(-x),y) \).

\( \Sigma_1 \) is not accessible, because the set of points attainable from \( (x,y) \) with \( x < 0 \) is \( \{(x',y') \mid x' \leq x \text{ and } y' = y\} \), which has empty interior in any nhd of \( (x,y) \). But it is reachable.

Also, it is not observable, since the points \( (x_1,y) \) and \( (x_2,y) \) with \( x_1, x_2 < 0 \) are indistinguishable, but it is weakly observable because that type of pair can be distinguished if we allow the negative trajectories of \( -\frac{3}{\partial x} \).

**6 - Minimal Realizations**

Given an initialized system \( (\Sigma,x) \), i.e., a pair where \( \Sigma \) is a system and \( x \) some point in \( M \), its input-output map \( u(\Sigma,x) \) is the map from \( \mathcal{U} \) into the set of curves in \( N \) defined as follows if \( u \in \mathcal{U} \), with domain \( [0,1] \), \( u(\Sigma,x)(u) \) is the curve \( t \to h \circ \pi(x,u;t) \), which has a domain \( J \) such that \( 0 \in J \subseteq [0,1] \).
An input output map, with inputs \( \mathcal{U} \) and outputs in \( \mathcal{N} \), is a map \( \mu \) such that, if \( u \in \mathcal{U} \) is as above, \( \mu(u) \) is some curve in \( \mathcal{N} \), with domain \( J = [0,1] \) and \( 0 \in J \).

We say two input-output maps are equivalent, \( \mu = \nu \), if for every \( u \in \mathcal{U} \), \( \mu(u) \) and \( \nu(u) \) coincide in the common domain of definition.

An initialized system \((\Sigma, x)\) is a realization of the input-output map \( \mu \) if \( \mu(\Sigma_1 x) = \mu \). A realization is called minimal if \( \Sigma \) is reachable and weakly observable, and quasi-minimal if \( \Sigma \) is reachable and locally weakly observable.

Given two systems \( \Sigma_1, \Sigma_2 \) we say \( F: M_1 \rightarrow M_2 \) is an isomorphism of systems if

i) \( F \) is a \( C^k \) diffeomorphism

ii) \( f_2 = F_* f_1 \)

iii) \( h_1 = F_* h_2 \)

\( F \) is an isomorphism of initialized systems \((\Sigma_1, x_1), (\Sigma_2, x_2)\) if it is an isomorphism of the systems \( \Sigma_1 \) and \( \Sigma_2 \), and moreover \( F(x_1) = x_2 \).

Theorem II[.6.1]:

Given a complete initialized system \((\Sigma, x)\) there exist a minimal
and a quasiminimal equivalent complete realization. Moreover the complete minimal realizations are unique up to isomorphism.

Proof:

To prove existence, we first obtain a reachable realization of \((\Sigma, x)\): if \(S\) is the orbit of \(x\) under \(D\), the system \((\Sigma_S, x)\), obtained by restriction to \(S\), is an equivalent reachable complete realization.

Now if \(\Sigma_1\) and \(\Sigma_2\) are the locally weakly observable, respectively weakly observable, systems obtained from \(\Sigma\) by using the constructions of proposition 2.4 and 2.5, \((\Sigma_1, \pi_1(x))\) and \((\Sigma_2, \pi_2(x))\) are the required realizations.

The uniqueness of the complete minimal realizations has been proved by Sussmann [12]. Of course we need a slight adaptation, in view of the differences in the definitions.
CHAPTER III - GRADIENT SYSTEMS

We discuss a conjecture presented by P. Varaya [15], and then proved for the linear case: an isomorphism of two locally controllable gradient systems in an isometry of the underlying pseudo-Riemannian manifolds.

In the first section we present the basic definitions and results, extending the class of systems to allow for the non-symmetric case.

In section two we prove:

i) the conjecture is false (example 2.4), but some positive results can be obtained, with different kinds of conditions.

ii) the above conjecture is true for Hamiltonian systems, assuming only reachability instead of local controllability (theorem 2.3).

Section three is the statement and proof of a positive result for "bilinear" gradient systems.
Following [15], we consider a nonlinear RLC electrical network, with \( k_1 \) capacitors, \( k_2 \) inductors, \( s_1 \) current sources and \( s_2 \) voltage sources.

Let \( q: \mathbb{R} \to \mathbb{R}^{k_1} \) be the change on the capacitors as function of time, and \( \phi: \mathbb{R} \to \mathbb{R}^{k_2} \) be the fluxes through the inductors. We denote by \( i: \mathbb{R} \to \mathbb{R}^{k_1 \times k_2 \times s_1 \times s_2} \) and \( v: \mathbb{R} \to \mathbb{R}^{k_1 \times k_2 \times s_1 \times s_2} \) the current and voltage.

Assuming the capacitors are voltage controlled and the inductors current-controlled, we have

\[
q_{\alpha}(t) = \bar{q}_{\alpha}(v_{\alpha}(t)) \quad \alpha = 1, \ldots, k_1
\]

\[
\phi_{\beta}(t) = \bar{\phi}_{\beta}(i_{\beta+k_1}(t)) \quad \beta = 1, \ldots, k_2
\]

for some \( \bar{q}: \mathbb{R}^{k_1} \to \mathbb{R}^{k_1} \) and \( \bar{\phi}: \mathbb{R}^{k_2} \to \mathbb{R}^{k_2} \).

By Maxwell's equations

\[
i_{\alpha}(t) = \frac{d}{dt} q_{\alpha}(t) = \frac{d}{dv_{\alpha}} \bar{q}_{\alpha}(v_{\alpha}(t)) \, \dot{v}_{\alpha}(t)
\]

\[
\dot{v}_{k_1+\beta}(t) = \frac{d}{dt} \phi_{\beta}(t) = \frac{d}{di_{\beta+k_1}} \bar{\phi}_{\beta}(i_{\beta+k_1}(t)) \, i_{\beta+k_1}(t)
\]

or denoting \( \frac{d}{dv_{\alpha}} \bar{q}_{\alpha} \) by \( C_{\alpha} \), and \( \frac{d}{di_{\beta+k_1}} \bar{\phi}_{\beta} \) by \( L_{\beta} \).
\[ i_\alpha(t) = C_\alpha(v_\alpha(t)) \dot{v}_\alpha(t), \quad \alpha = 1, \ldots, k_1 \]

\[ v_{k_1+\beta}(t) = L_\beta(i_{\beta+k_1}(t)) \dot{i}_{k_1+\beta}(t), \quad \beta = 1, \ldots, k_2 \]

We assume \( C_\alpha \) and \( L_\beta \) are positive. We can think of the network as converting

\[ z = (v_1, \ldots, v_{k_1}, -i_{k_1+1}, \ldots, -i_{k_1+k_2}, i_{k_1+k_2+1}, \ldots, i_{k_1+k_2+s_1}, v_{k_1+k_2+s_1+1}, \ldots, v_{k_1+k_2+s_1+s_2}) \]

into

\[ w = (-i_1, \ldots, -i_{k_1}, v_{k_1+1}, \ldots, v_{k_1+k_1}, v_{k_1+k_2+1}, \ldots, v_{k_1+k_2+s_1}, i_{k_1+k_2+s_1+1}, \ldots, i_{k_1+k_2+s_1+s_2}) \]

and we assume there is a map \( f: \mathbb{R}^{k_1 \times k_2 \times s_1 \times s_2} \to \mathbb{R}^{k_1 \times k_2 \times s_1 \times s_2} \) making the conversion.

If the network is reciprocal, i.e. \( \frac{\partial f}{\partial z} \) is a symmetric matrix, there exists a map \( P: \mathbb{R}^{k_1 \times k_2 \times s_1 \times s_2} \to \mathbb{R}^{k_1 \times k_2 \times s_1 \times s_2} \) such that \( f = \frac{\partial}{\partial z} P \). We can then write

\[ -i_\alpha = -C_\alpha(v_\alpha(t)) \dot{v}_\alpha(t) = \frac{\partial}{\partial z} P, \quad \alpha = 1, \ldots, k_1 \]
\[ v_{k_1 + \beta} = L_{\beta}(i_{k_1 + \beta}(t)) \quad i_{k_1 + \beta}(t) = \frac{\partial}{\partial z_{k_1 + \beta}} \quad \beta = 1, \ldots, k_2 \]

\[ v_{k_1 + k_2 + \gamma} = \frac{\partial}{\partial z_{k_1 + k_2 + \gamma}} \quad \gamma = 1, \ldots, s_1 \]

\[ i_{k_1 + k_2 + s_1 + \delta} = \frac{\partial}{\partial z_{k_1 + k_2 + s_1 + \delta}} \quad \delta = 1, \ldots, s_2 \]

If we take \( x \in \mathbb{R}^{k_1 + k_2} \) as \( x = (v_1, \ldots, v_{k_1}, i_{k_1 + 1}, \ldots, i_{k_1 + k_2}) \),
\( u \in \mathbb{R}^{s_1 + s_2} \) as \( u = (i_{k_1 + k_2 + 1}, \ldots, \ldots, i_{k_1 + k_2 + s_1}, v_{k_1 + k_2 + s_1 + 1}, \ldots, v_{k_1 + k_2 + s_1}) \),
\( y \in \mathbb{R}^{s_1 + s_2} \) as \( y = (v_{k_1 + k_2 + 1}, \ldots, v_{k_1 + k_2 + s_1}) \),
\( i_{k_1 + k_2 + s_1 + 1}, \ldots, i_{k_1 + k_2 + s_1 + s_2} \) the above equations can be written:

\[ A(x) = \frac{\partial}{\partial x} p(x, u) \]

\[ y = \frac{\partial}{\partial u} p(x, u) \]

where \( A = \text{diag} (-C_1, \ldots, -C_{k_1}, L_1, \ldots, L_{k_2}) \).

We can generalize the above model assuming it is only a local description, and allowing for mutual inductance we substitute a diagonal matrix by a non-singular symmetric matrix.

Then we can take the state space as \( M \), a \( k_1 + k_2 \) dimensional
matrix, \( \xi \) as a nondegenerate symmetric bilinear tensor field on \( M \), and assuming \( P \) is globally defined we have

\[
\begin{align*}
\dot{x} &= f(x,u) \\
\{ & \\
y &= h(x,u)
\end{align*}
\]

where

\[
\begin{array}{ccc}
M \times \mathbb{R}^\ell & \xrightarrow{f} & TM \\
\downarrow P_1 & & \downarrow \\
M & \xrightarrow{p} & M
\end{array}
\]

is a commutative diagram, \( f(\cdot,u)\xi = dP(\cdot,u) \) and \( h(x,u) = \frac{\partial}{\partial u} P(x,u) \).

Systems of this form, with \( \xi \) non-degenerate but not necessarily symmetric, will be called gradient systems, and if \( \xi \) is a symplectic form, i.e. skew symmetric and closed, the system will be said to be Hamiltonian.
1 - Gradient systems

Gradient systems are defined on $C^k$ manifolds having additional structure, namely a non-singular bilinear tensor field $\xi$. This means that at every $x \in M$, $\xi(x)$ is a non-singular bilinear map $\pi^x : T_xM \times T_xM \rightarrow \mathbb{R}$.

If $z : M \rightarrow \mathbb{R}$ is a $C^k$ map, we can define $\text{grad}_\xi z$ as the vector field $X$ such that $\xi(x) \cdot (X(x), v) = dz(x) \cdot v$, for every $x \in M$ and $v \in T_xM$, or equivalently $X \cdot \xi = dz$.

The systems we will consider are defined as follows. Let $P : M \times \mathbb{R}^m \rightarrow \mathbb{R}$ be a $C^k$ map, and denote by $P_u$ the map $x \mapsto P(x, u)$. Define $f(x, u) = \text{grad}_\xi P_u(x)$.

The control system will be $\Sigma = (M, \mathbb{R}^m, f, \Gamma)$. Some difficulty arises with the definition of the output space and output map. We want the output to be $y(x, u) = \frac{\partial}{\partial u} P(x, u)$; if we consider it as being a family of maps $\{y_u\}_{u \in \mathbb{R}^m}$ with $y_u : M \rightarrow \mathbb{R}^m, y_u(x) = y(x, u)$ we see that the output space is not finite dimensional.

The results of the previous chapter can be carried through, if we adopt the following definition of indistinguishability: two points $x_1, x_2$ in $M$ are (weakly) indistinguishable if for every $\gamma \in S_D$ ($\gamma \in G_D$) such that $\gamma x_1$ and $\gamma x_2$ are defined, we have $y_u \circ \gamma x_1 = y_u \circ \gamma x_2$ for every $u \in \mathbb{R}^m$.

It is clear that the results in chapter II do not depend on the system.
having just one output map or having a family of output maps. Thus our
system will be \( \Sigma = (M, \mathbb{R}^m, f, \mathcal{U}_{pc}, \mathbb{R}^m, \{y_u\}) \).

Thus give an intuitive meaning to the above definition of
indistinguishability we can prove:

**Lemma III.1.1:** \( x_1 \) and \( x_2 \) are indistinguishable iff, for any
\( u \in \mathcal{U}_{pc} \), \( y(\pi(x_1, u; t), u(t)) = y(\pi(x_2, u; t), u(t)) \), whenever \( t \) is in
the common domain of definition of \( \pi(x_1, u; \cdot) \) and \( \pi(x_2, u; \cdot) \).

**Proof:**

It is clear that indistinguishability implies the above result: for
simplicity assume \( u \) assumes two values, \( u_1 \) in \([0, t_1]\) and \( u_2 \) in
\([t_1, t_2]\). Then for \( t \in [0, t_1] \), \( y(\pi(x_1, u; t), u(t)) \neq y_{u_1}(\pi(x_1, u; t)) \) and thus
the property is verified on \([0, t_1]\). Similarly, if \( t \in [t_1, t_2]\)
\( y(\pi(x_1, u; t), u(t)) = y_{u_2}(\pi(x_1, u; t)) \) and our claim follows. Clearly
the reasoning is independent of the number of values \( u \) assumes.

Consider \( \gamma \in S_D \) and let \( v \in \mathcal{U}_{pc} \) be the corresponding control,
defined on \([0, T]\). Consider the control \( v' \) such that \( v' = v \) on
\([0, T]\) and \( v' = u \) on \([T, T']\).

By assumption \( y(\pi(x_1; v', t), v'(t)) = y(\pi(x_2, v', t), v'(t)) \) for every
\( t \in [0, T'] \). Then on \([T, T']\) we have \( y_u(\pi(x_1, v'; t)) = y_u(\pi(x_2, v'; t)) \) and
by continuity, if we let \( t \to T \) we get \( y_u(\gamma x_1) = y_u(\gamma x_2) \).

Since \( u \) can be chosen arbitrarily, this concludes the proof.

The two main cases we are interested in are: i) \( \xi \) is symmetric; ii) \( \xi \) is skew-symmetric, thus a 2-form.

In case ii) if we assume \( \xi \) is closed, we obtain what we will call a Hamiltonian system. We list now some properties and definitions about Hamiltonian vector fields, that are used in the sequel.

Let \((M, w)\) be a symplectic manifold, i.e., \( w \) is a closed non-degenerate 2-form on \( M \). A vector field \( X \) on \( M \) is said to be Hamiltonian if \( X \perp w = df \) for some function \( f \) on \( M \), which we call the Hamiltonian of \( X \).

Since \( w \) is non-degenerate, given any function \( f \) we can find a vector field \( X_f \) such that \( X_f \perp w = df \).

**Proposition III.1.2:**

i) \( X \) is Hamiltonian if \( L_X w = 0 \).

ii) The set of Hamiltonian vector fields is a subalgebra of the Lie algebra of vector fields.

Denote by \( \{f, g\} = w(X_f, X_g) \). Then we have

**Proposition III.1.3:**

i) \( X_{\{f, g\}} = [X_g, X_f] \)
A map \( \phi: M_1 \to M_2 \) between two symplectic manifolds is a symplectomorphism iff \( \phi^* w_2 = w_1 \). The diffeomorphisms associated to a Hamiltonian vector field are symplectomorphisms.

2 - Nonlinear equivalence

Assume we are given two equivalent gradient systems \( \Sigma_1 \) and \( \Sigma_2 \), i.e. there exists an isomorphism \( \phi \) between them. We are interested in knowing how \( \phi \) acts on \( \xi_1 \), more precisely, we want to study conditions under which \( \xi_1 = \phi^* \xi_2 \).

There are two specially interesting cases:

i) \( \xi_1 \) and \( \xi_2 \) are pseudo-Riemannian metrics, and we want \( \phi \) to be an isometry;

ii) \( \xi_1 \) and \( \xi_2 \) are symplectic forms, i.e. are skew-symmetric, thus differential 2-forms, and closed, \( d\xi_1 = d\xi_2 = 0 \), and then we want \( \phi \) to be a symplectomorphism.

Proposition III.2.1 \([15]\):

Let \( \Sigma_1, \Sigma_2 \) be two equivalent systems, and \( \phi \) an isomorphism \( \phi: M_1 \to M_2 \) between them. Then if \( \xi' = \phi^* \xi_2 - \xi_1 \), there exists a map \( S: M_1 \to \mathbb{R} \) such that \( f_1(\cdot, u) - \xi' = dS \), for every \( u \in \mathbb{R}^m \).

**Proof:**

If \( \Sigma_1 \) and \( \Sigma_2 \) are equivalent, we have \( \frac{\partial}{\partial u} P_1(x, u) = \frac{\partial}{\partial u} P_2(\phi(x), u) \).
and so there exists $S : M \to \mathbb{R}$ such that $P_2(\phi(x), u) = P_1(x, u) + S(x)$.

Differentiating with respect to $x$, we obtain

$$\phi^* dP_2u = dP_1u + dS = f_1(x, u) \perp \xi_1 + dS.$$  

On the other hand $\phi^* dP_2u = \phi^*(f_2(x, u) \perp \xi_2)$ and as $f_2(x, u) = \phi_* f_1(x, u)$ we get

$$\phi^*(\phi_* f_1(x, u) \perp \xi_2) = f_1(x, u) \perp \xi_1 + dS.$$  

The first member is $f_1(x, u) \perp \phi^* \xi_2$ and so

$$f_1(x, u) \perp \phi^* \xi_2 = f_1(x, u) \perp \xi_1 + dS$$  

or

$$f_1(x, u) \perp \xi^* = dS.$$  

A system $\Sigma$ is said to be locally controllable at $x \in M$ if, given any neighbourhood $U$ of $x$, we can find a neighbourhood $V$ of $x$ such that any point in $V$ can be attained from $x$, following a trajectory not leaving $U$.

We remark that local controllability implies, but is not implied by, controllability.
Lemma III.2.2:

If one of the following conditions is verified, then \( dS = 0 \), i.e. \( S \) is constant:

i) \( \xi_1, \xi_2 \) are skew-symmetric, and \( \Sigma_1 \) is reachable.

ii) \( \xi_1, \xi_2 \) are symmetric and \( \Sigma_1 \) is locally controllable.

Proof:

Assume i) is verified. Then if \( X \in D_1 \) we have \( L_X S = \xi'(X,X) = 0 \), since \( \xi' \) is skew-symmetric when both \( \xi_1 \) and \( \xi_2 \) are.

This means \( S \) is constant along trajectories of vector fields in \( D_1 \), and as \( \Sigma_1 \) is reachable, \( S \) is constant.

Now suppose ii) is verified. Then \( L_X S \) is independent of the vector field \( X \) in \( D_1 \); let \( X, Y \in D_1 \) then \( L_Y S = dS.Y = \xi'(X,Y) = \xi'(Y,X) = dS.X = L_X S \) since the symmetry of \( \xi_1 \) and \( \xi_2 \) implies the symmetry of \( \xi' \).

If \( L_X S(x) > 0 \), let \( U \) be an open neighbourhood of \( x \) on which \( L_X S \) is still positive, we can find \( V \) as in the definition of local controllability.

A point \( x_1 \in V \) is attainable from \( x_1 \) and therefore \( S(x_1) > S(x) \). This is a contradiction, if we choose \( x_1 \) in the non-empty set \( \{ x' \mid x' \in V, S(x') = S(x) \} \).
Similarly we prove that \( L_x S \) cannot be negative, therefore \( L_x S = 0 \) for every \( x \in D_1 \). As local controllability clearly implies local accessibility and therefore reachability, we can conclude, as above, that \( S \) is constant.

Theorem III.2.3:

Let \( \Sigma_1 \) and \( \Sigma_2 \) be equivalent Hamiltonian systems; then if \( \Sigma_1 \) is reachable, \( \phi \) is a symplectomorphism.

Proof:

Let \( \gamma_1 \in G_{D_1} \); since all elements of \( G_{D_1} \) are symplectomorphisms, \( \gamma_1^* w_1 = w_1 \).

As \( \Sigma_1 \) and \( \Sigma_2 \) are equivalent, there exists \( \gamma_2 \) in \( G_{D_2} \) such that \( \phi \gamma_1 = \gamma_2 \circ \phi \) : Thus we have \( \gamma_1^* \phi^* w_2 = (\phi \gamma_1)^* w_2 = (\gamma_2 \circ \phi)^* w_2 = \phi^* \gamma_2^* w_2 = \phi^* w_2 \) and therefore \( \gamma_1^* w' = w' \).

From lemma 2, we have \( X \perp w' = 0 \) for every \( x \in D_1 \). If \( \gamma_1 \in G_{D_1} \), \( ((\gamma_1^* X) \perp w')(x).v = w'(x) (\gamma_1^* X(\gamma_1^{-1}(x)), v) = w'(\gamma_1^{-1}(x)) (X(\gamma_1^{-1}(x)), \gamma_1^{-1} v) = 0 \), and thus \( \gamma_1^* X \perp w' = 0 \).

Since the system \( \Sigma_1 \) is reachable, the set of vector fields of the form \( \gamma_1^* X \) as above spans the tangent space of \( M_1 \) at every point, therefore \( w' \equiv 0 \), and the result is proved.

We would like to have a result as above for the symmetric case, with the condition ii) in lemma 2, as it was conjectured in [15], but
the following example shows that is impossible:

Example III.2.4:

Let \( M_1 = \mathbb{R}^4 \), and define

\[
P_1(x_1, x_2, x_3, x_4, u_1, u_2) = u_1 x_1 + u_2 (x_2 + x_3 + x_4) .
\]

Let \( \xi_1(x) = \text{diag}(1, e^{-x_4}, e^{-x_1}, e^{-x_3}) \).

An easy computation shows that the vector fields in \( D_1 \) are linear combinations of the vector fields

\[
X_1 = \frac{\partial}{\partial x_1} \quad (u_1 = 1, u_2 = 0) \quad \text{and} \quad X_2 = \frac{\partial}{\partial x_2} + e_1 \frac{\partial}{\partial x_3} + e_3 \frac{\partial}{\partial x_4} \quad (u_1 = 0, u_2 = 1) .
\]

Then

\[
X_3 = [X_1, X_2] = e_1 \frac{\partial}{\partial x_3} \quad \text{and} \quad X_4 = [X_3, X_2] = e_1 x_3 \frac{\partial}{\partial x_4} .
\]

This shows that \( \mathcal{T}(D_1) \) spans the tangent space of \( M_1 \) at every point, so the system is locally accessible, and as it is symmetric, it is locally controllable.

The output functions are \( y_1 = \frac{\partial p}{\partial u_1} = x_1 \) and \( y_2 = \frac{\partial p}{\partial u_2} = x_2 + x_3 + x_4 \).

Take \( X_5 = e_2 x_4, X_2 = e_1 (x_1 + x_3) \).

Then

\[
\text{det}(dy_1) = (1, 0, 0, 0), \quad \text{det}(dy_2) = (0, 1, 1, 1) \quad \text{in the basis} \quad (dx_1, dx_2, dx_3, dx_4) , \quad \text{and so we have} \quad d(dy_1) = (e_1, 0, e_3, 0) \quad \text{and} \quad d(dy_2) = (e_1 + e_3(e_4 - 2e_1), 0, (x_4 - x_1)e_3(x_4 - x_1), x_1 e_1 x_4) .
\]

Taking \( x_1 = x_3 = x_4 = 1 \), the corresponding covectors

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
e & 0 & e & 0 \\
(-e^3) & 0 & 0 & e
\end{pmatrix}
\]
and linearly independent. As the system is analytic and accessible, that means the observability rank condition is verified everywhere, and $\Sigma_1$ is observable.

Now consider $M_2 = M_1$, $P_2 = P_1$ and let $\xi_2$ be defined by

$$
\xi_2(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-x_4} & 0 & 0 \\ 0 & 0 & e^{-x_1-x_3} & -e^{x_1} \\ 0 & 0 & e^{x_1} & -e^{-x_3} & 2x_1 \end{bmatrix}
$$

Computation shows us that this gives rise to the same system, more precisely the identity is an isomorphism between $\Sigma_1$ and $\Sigma_2$, but $\xi' = \xi_2 - \xi_1 \neq 0$.

$\Sigma_1$ is analytic complete, locally controllable, observable, and therefore we cannot expect to prove any positive result without further assumptions.

We remark that in the skew-symmetric case we had to assume the 2-form to be closed, in theorem 3, and then $L_X\omega = 0$ for every $X \in D$.

If $\xi$ is a (pseudo) Riemannian metric on $M$, and $L_X\xi = 0$, $X$ is called a Killing vector field, and we can prove:

**Theorem III.2.5:**

Let $\Sigma_1$ and $\Sigma_2$ be equivalent gradient systems, $\xi_1$ and $\xi_2$
(pseudo) Riemannian metrics, and assume the elements of $D_i$ are Killing vector fields for $\xi_i$. Then, if $\Sigma_1$ is locally controllable, $\phi$ is an isometry.

Proof:

If $L_X \xi = 0$ and $\gamma$ is a diffeomorphism associated to $X$, then $\gamma^* \xi = \xi$, i.e. $\gamma$ is an isometry. This was the condition involved in the proof of theorem 3 and since local controllability implies reachability, we have just to reproduce the argument of that proof to get our result.

In his Ph.D. thesis [15], Verma states the following:

Theorem III.2.6:

Assume $M_1 = M_2 = \mathbb{R}^n$, and $\xi_1 = \xi_2$ is the usual Riemannian metric. If $\Sigma_1$ is locally controllable and

i) these systems have the form

$$\begin{align*}
\dot{x} &= Fg(F^T x) + FBu \\
y &= (FB)^T x
\end{align*}$$

where $F$ and $B$ are matrices, and $g: \mathbb{R}^n \to \mathbb{R}^n$ is such that $g_i$ depends only of $x_i$, i.e. $\frac{\partial g_i}{\partial x_j} = 0$ if $i \neq j$.

ii) $\mathcal{V}(D_i)$ can be generated as a vector space by the associated vector fields and Lie brackets of them in which at least one of the vectors is constant.
Then \( \phi \) is an isometry.

When trying to generalize this result, we found that its proof is (at least) uncomplete, since it fails to consider the case in which the constant vector field is not an input vector field. The proof uses the fact that \( \phi \) takes a constant vector field into another constant vector field, which is true for input vector fields, but is not proved in general.

If instead of condition ii) we assume a stronger condition avoiding that situation, we can obtain:

**Theorem III.2.7:**

\( E_1 \) is locally controllable, and \( E_1, E_2 \) are such that:

i) \( f_i(.,u) = \sum_{j=1}^{e} u_j X_i^j \), where \( X_i^r \) are \( C^k \) vector fields, and

\[
\forall i \quad X_i^j = 0, \quad j = 1,2,\ldots, e
\]

where \( \forall i \) is the Levi-Civita connection associated to \( \xi_i \).

ii) \( \mathcal{F}(D_i) \) can be generated as a vector space by the associated vector fields and by successive Lie bracketting with input vector fields.

Then \( \phi \) is an isometry.

**Proof:**

If \( (M, \xi) \) is a manifold, and \( \nu \) its Levi-Civita connection, we have
\[ 2\xi(v, Y, Z) = X.\xi(Y, Z) + Y.\xi(X, Z) - Z.\xi(X, Y) + \\
\quad + \xi([Z, X], Y) + \xi([Z, Y], X) + \xi([X, Y], Z). \]

Also \( X \perp \xi' = 0 \) for any vector field \( X \in D_1 \), we shall prove by induction that the same is true for the vector fields in \( \mathcal{P}(D_1) \) generating it, as in ii).

The induction is on the order of the brackets: it is already verified for order zero. Now assume it has been proved up to order \( p \).

Let \( X \in \mathcal{P}(D_1) \) be a vector field of the considered type of order \( p+1 \), i.e. \( X = [Y, Z] \) where \( Y \) is an input vector field and \( Z \) is the order \( p \).

We have
\[ Y \perp \xi' = 0 \quad \text{(local controllability)} \]
\[ Z \perp \xi' = 0 \quad \text{(induction)} \]
and also \( \nabla_{Z}^{1} Y = 0 \) since \( \nabla^{1} Y = 0 \).

Then, by the above formula:
\[
\xi_1(X, A) = Z.\xi_1(Y, A) + Y.\xi_1(Z, A) - A.\xi_1(Y, Z) + \\
\quad + \xi_1([A, Y], Z) + \xi_1([A, Z], Y) \\
= \phi^{*Z}.\xi_2(\phi^{*Y}, \phi^{*A}) + \phi^{*Y}.\xi_2(\phi^{*Z}, \phi^{*A}) \\
\quad - \phi^{*A}.\xi_2(\phi^{*Y}, \phi^{*Z}) + \xi_2([\phi^{*A}, \phi^{*Y}], \phi^{*Z}) \\
\quad + \xi_2([\phi^{*A}, \phi^{*Z}], \phi^{*Y}).
\]
As \( \nabla^2_{\phi_Y} \phi_Y = 0 \) since \( \nabla^2 \phi_Y = 0 \) (note \( \phi_Y \) is an input vector field for \( \Sigma_1 \)) we have

\[
\xi_1(X, A) = \phi^* \xi_2(\phi_Y, \phi_Z, \phi_A) = \\
= \phi^* \xi_2(\phi_Y, \phi_A). 
\]

As \( A \) is an arbitrary vector field, this means \( X \perp \xi' = 0 \), and the proof is complete.

Instead of assuming certain properties of the family of associated vector fields, or relations of them with the pseudo Riemannian metric, we can assume convenient properties of the diffeomorphism \( \phi \) in order to obtain positive results.

**Theorem III.2.8:**

If \( \phi \) is an affine diffeomorphism, i.e. if \( \phi_Y^2 = \phi^2_X \phi_Y \) and \( \Sigma_1 \) is strictly locally observable, \( \phi \) is an isometry.

**Proof:**

If \( \Sigma_1 \) is strictly locally observable, the observability rank condition is satisfied on an open dense subset of \( M \). As will be seen from the proof, we can assume it is satisfied everywhere.

Let \( \mathcal{D}_1 \) be the family of vector fields which are gradient of functions in \( F_{D_1} \); then, since we are assuming \( \mathcal{H}(D_1) \) spans the cotangent space at every point, \( \mathcal{D}_1 \) spans the tangent space at every point.
From proposition III.2.1 and taking derivatives with respect to $u$, we have $X \lhd \xi' = 0$ for every $X$ which is the gradient in $(M_1, \xi_1)$ of a component of the output function.

The other vector fields in $\mathcal{D}_1$ are gradient of Lie derivatives of components of output maps with respect to vector fields in $D_1$. We can prove that $Y \lhd \xi' = 0$ for every $Y \in \mathcal{D}_1$ by induction on the order of the Lie derivatives.

The above remark proves the case of 0-th order derivatives. Assume we have proved our claim for vector fields gradient of Lie derivatives of order $p$.

Let $Y = \text{grad}_{\xi_1} L_{\chi^p}$, where $\phi$ is a Lie derivative of order $p$, and denote by $Z$ the gradient of $\phi$.

Claim, $Y = v_X^1 Z + v_Z^1 X$.

Denote the second member by $Y'$; then we have, from the formula stated in the beginning of the proof of the previous theorem:

$$2 \xi_1 (Y' A) = 2 (X \xi_1 (Z, A) + Z \xi_1 (X, A) - A \xi_1 (X, Z) +$$
$$+ \xi_1 ([A, X], Z) + \xi_1 ([A, Z], X))$$

by just substitution, for any vector field $A$.

Now

$$\xi_1 ([A, X], Z) = d\phi [A, X] = A (d\phi X) - X (d\phi A)$$
and if \( X = \text{grad} \xi_1 \eta \)

\[
\xi_1([A,Z],X) = d\eta |A,Z| = A.(d\eta.Z) - Z.(d\eta.A)
\]

By substitution

\[
\xi_1(Y',A) = -A \xi_1(Z,X) + A.\xi_1(Z,X) + A \xi_1(X,Z) = A.\xi_1(X,Z)
\]

since \( X.(d\phi A) = X \xi_1(Z,A) \) and \( Z(d\eta.A) = Z.\xi_1(X,A) \).

Thus we have proved:

\[
Y' \perp \xi_1 = d\xi_1(X,Z)
\]

and since \( \xi_1(X,Z) = d\phi.X = L_X\phi \), we have

\[
Y' \perp \xi_1 = dL_X\phi
\]

and \( Y' \) is the gradient of \( L_X\phi \), therefore \( Y = Y' \).

From the equivalence of \( \Sigma_1 \) and \( \Sigma_1 \) we have \( \phi L_{\phi X}(\phi^{-1}) = L_X\phi \).

Consider the vector field \( \phi_*Y \).

Since \( \phi \) preserves the covariant derivative, we have

\[
\phi_*Y = \nabla_{\phi_*X}^2 \phi_*Z + \nabla_{\phi_*Z}^2 \phi_*X
\]

and therefore

\[
\phi_*Y \perp \xi_2 = d\xi_2(\phi_*X,\phi_*Z)
\]
As $Z \perp \xi' = 0$ by the induction hypothesis, we have

$$\phi_* Y \perp \xi_2 = d(\phi^{-1})^* \xi_1(X,Z) = (\phi^{-1})^* d\xi_1(X,Z)$$
$$= (\phi^{-1})^* (Y \perp \xi_1)$$

or equivalently

$$\phi^* (\phi_* Y \perp \xi_2) = Y \perp \xi_1$$
$$Y \perp \phi^* \xi_2 = Y \perp \xi_1$$

Thus $Y \perp \xi' = 0$, and this will be true for every $Y$ in $\mathcal{D}_1$. As they span all the tangent space, $\xi' = 0$ i.e. $\phi$ is an isometry.

We can look for other conditions by making assumptions on the metrics and type of diffeomorphisms allowed: consider gradient systems of the form

$$f_i(x_i) \dot{x}_i = \frac{\partial P}{\partial x_i}(x,u)$$
$$y_j(x,u) = \frac{\partial P}{\partial u_j}(x,u)$$

defined on $M = \mathbb{R}^n$, and restrict the diffeomorphisms $\phi: \mathbb{R}^n \to \mathbb{R}^n$ to have the form

$$\phi(x_1, \ldots, x_n) = (\phi_1(x_1), \ldots, \phi_n(x_n))$$

Working inside this class of systems we obtain:
Proposition III.2.9:

Assume $\Sigma_1, \Sigma_2, \phi$ are as above, and $\Sigma_1$ is locally controllable. Then $\phi$ is an isometry.

Proof:

We have $X \perp \xi' = 0$, for every $X$ in $D_1$, or in coordinates

$$
\xi'_{ij}(x) X_j(x) \equiv 0 .
$$

Now, if $Z = [X, Y]$, where $X, Y \in D_1$, we have

$$
Z \perp \xi' = [X, Y] \perp \xi',
$$

which we can write in coordinates as

$$
\xi'_{ij} \frac{\partial}{\partial x_k} (Y_j) X_k - \xi'_{ij} \frac{\partial}{\partial x_k} (X_j) Y_k
$$

for the $i$-th component, summing over $K$, and $j$.

From the form of $\xi_1, \xi_2$ and $\phi$, it is clear that

$$i \neq j \Rightarrow (\xi'_{ij} \equiv 0 , \text{ and } \frac{\partial}{\partial x_j} \xi'_{ii} \equiv 0 ) .$$

Therefore, we have
(Z ∩ ξ')\_i = \xi'\_i - \frac{d}{dx\_i}(\xi'\_i)X\_i \equiv 0

Thus, we have X ∩ ξ' = 0, Y ∩ ξ' = 0 \Rightarrow [X, Y] ∩ ξ' = 0

and by induction on the order of the brackets, we get

X ∈ T(D) \Rightarrow X ∩ ξ' = 0

and therefore ξ' = 0, i.e. ϕ is an isometry.

3 - Equivalence of "bilinear" gradient systems

We shall consider a special class of systems, by taking

M = \mathbb{R}^n, \xi(v_1, v_2) = <v_1, Av_2> when A is a symmetric non-singular matrix (<,> is the usual inner product in \mathbb{R}^n) and P: \mathbb{R}^n × \mathbb{R}^k → \mathbb{R}

is given by P(x,u) = <x,Fx> + \sum\_{k=1}^{k} u\_k<x,N\_kx> + <x,Gu> where F, N\_k and
G are matrices, F and \( N_K \) symmetric.

Then \( \Sigma \) has the form

\[
\dot{x} = \begin{bmatrix} 2F & 0 & \cdots & 0 \\ \Sigma_{Kx} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{Kx} & 0 & \cdots & 0 \\ \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \end{bmatrix} + g_u
\]

\[
y_K = \langle G^K, x \rangle + \langle x, N_K x \rangle \quad K = 1, 2, \ldots, \ell
\]

when \( G^K \) is a vector with components the elements of the \( k \)-th column of \( G \).

\( \Sigma \) is not bilinear, since the output is not linear in \( x \), and the system obtained by eliminating those terms in the output is called the associated bilinear system.

We say \( \Sigma \) is irreducible if the associated system is a minimal bilinear realization.

**Theorem III.3.1:**

A linear isomorphism \( \phi \) is an equivalence of two irreducible gradient systems \( \Sigma_1 \) and \( \Sigma_2 \) iff it is an isomorphism of the associated bilinear systems and then it is an isometry.

**Proof:**

Let \( Z = \phi X \) be a linear equivalence of \( \Sigma_1 \) and \( \Sigma_2 \). It is clear that \( G_2^{\top} \phi = G_1^{\top} \), from the condition on the outputs of equivalent systems.
Since $\phi$ takes the dynamics of the first system into the dynamics of the second, it does the same to the associated bilinear systems.

Together with $G_2^T = G_1^T$ means that $\phi$ is an isomorphism of the associated bilinear systems.

Now suppose $\phi$ is an isomorphism of the a.b.s., then $\phi$ is the unique solutions of the equations:

i) $\phi^{-1}A_2^{-1}F_2\phi = A_1^{-1}F_1$

ii) $\phi^{-1}A_2^{-1}N_2K\phi = A_1^{-1}N_1K$

iii) $\phi^{-1}A_2^{-1}G_2 = A_1^{-1}G_1$

iv) $G_2^T\phi = G_1^T$.

An easy computation shows that $A_2^{-1}(\phi^{-1})^TA_1$ satisfies the same set of equations:

i) $= F_1A_1^{-1} = \phi^TF_2A_2^{-1}(\phi^{-1})^T$ =>

$$A_1^{-1}F = A_1^{-1}\phi^TA_2^{-1}A_2^{-1}F_2 A_2^{-1}(\phi^{-1})^TA_1$$

$$A_1^{-1}F = (A_2^{-1}(\phi^{-1})^TA_1)^{-1}A_2^{-1}F_2 (A_2^{-1}(\phi^{-1})^TA_1)^{-1}$$.

Similarly, from ii) we can obtain the equation

$$A_1^{-1}N_1K = (A_2^{-1}(\phi^{-1})^TA_1)^{-1}A_2^{-1}N_2K (A_2^{-1}(\phi^{-1})^TA_1)^{-1}$$.
Now \( \text{iii}) \Rightarrow A_1 \phi^{-1} A_2^{-1} G_2 = G_1 \)

and we obtain

\[
G_1^T = G_2^T \left( A_2^{-1} (\phi^{-1})^T A_1 \right).
\]

From \( \text{iv}) \) we get \( G_1 = \phi^T G_2 \), and then \( A_1^{-1} G_1 = A_1^{-1} \phi^T \),

thus

\[
A_1^{-1} G_1 = \left( A_2^{-1} (\phi^{-1})^T A_1 \right)^{-1} A_2^{-1} G_2.
\]

Since \( \phi \) is the unique solution of these equations
\( \phi = A_2^{-1} (\phi^{-1})^T A_1 \) or equivalently \( A_1 = \phi^T A_2 \phi \).

Now \( \phi^T N_{2K} \phi = A_1 \phi^{-1} A_2^{-1} N_{2K} = A_1 A_1^{-1} N_{1K} \) and therefore
\( \phi^T N_{2K} \phi = N_{1K} \). This together with the other equations shows that \( \phi \) is an equivalence of the gradient systems, and as \( A_1 = \phi^T A_2 \phi \), \( \phi \) is an isometry.
CHAPTER IV - HAMILTONIAN SYSTEMS

Section one presents the background material, and in section two we prove that an initialized complete analytic Hamiltonian system, with finite dimensional Lie algebra and satisfying quite standard assumptions, has an equivalent accessible Hamiltonian realization defined on an orbit in the dual of the Lie algebra of the coadjoint action; moreover, if that realization is strongly accessible it is quasi-minimal.

Section three consists of an example for the above results.
1 - Symplectic actions

Let $G$ be a connected Lie group acting on a symplectic manifold $M$ by symplectormorphisms. Then, to each vector $v$ in $\mathfrak{g}$, the Lie algebra of $G$, there is a vector field $X_v$ in $M$, given by $X_v(x) = (\phi_x)_* v$, where $\phi: G \times M \to M$ is the action and $\phi_x(y) = \phi(y, x)$; the derivative is taken at the identity in $G$.

$X_v \wedge w$ is a closed form, since $G$ acts by symplectormorphisms.

The action $\phi$ is said to be Poisson if the two following conditions are verified:

i) $X_v \wedge w$ is exact, i.e. there exists a function $f_v$ such that $X_v \wedge w = df_v$.

ii) It is possible to define a map $\beta: \mathfrak{g} \to C^0(M)$ which is a homomorphism of Lie algebras and such that $X_v \wedge w = d\beta(v)$.

If $\phi$ is a Poisson action, we can define a map $I: M \to \mathfrak{g}^*$, the dual of $\mathfrak{g}$, by $I(x).v = \beta(v)(x)$. $I$ is called the moment map.

We denote by $Ad$, respectively $Ad^*$, the adjoint, respectively coadjoint, action of $G$ on $\mathfrak{g}$, respectively $\mathfrak{g}^*$.

Then we have:

Theorem IV.1.1 [2].

The following diagram is commutative:
Theorem IV.1.2 [7]:

Let \( S \) be an orbit of the coadjoint action on \( \mathfrak{g}^* \). Then \( S \) has a canonical symplectic form \( \omega_S \) defined as follows: let \( \xi_{\upsilon_1} \) and \( \xi_{\upsilon_2} \) be the two vector fields in \( S \) corresponding to \( \upsilon_1, \upsilon_2 \in \mathfrak{g} \); then

\[ \omega_S (\xi_{\upsilon_1}, \xi_{\upsilon_2}) (\alpha) = \alpha([\upsilon_1, \upsilon_2]) , \alpha \in S \] . Moreover, \( \xi_\upsilon \) is Hamiltonian, corresponding to the restriction to \( S \) of the function \( \alpha \to \alpha(\upsilon) \).

We remark that if \( \upsilon \in \mathfrak{g} \), the vector field \( X_\upsilon \) is tangent to the orbits of \( G \) on \( M \), and thus, by theorem 1, induces a vector field \( \xi_\upsilon \) on \( \mathfrak{g}^* \) tangent to the orbits of the coadjoint action; it is the restriction of this vector field to \( S \) that we call the vector field in \( S \) corresponding to the vector \( \upsilon \) in \( \mathfrak{g} \).

Corollary IV.1.3:

The Hamiltonian of \( \xi_\upsilon \) is the restriction of \( \tilde{\beta}(\upsilon) \) to \( S \), where \( \tilde{\beta}(\upsilon) \) satisfies \( \beta(\upsilon) = \tilde{\beta}(\upsilon) \circ J \).

Proof:

It is enough to show that \( \alpha \to \alpha(\upsilon) \) is the same as \( \tilde{\beta}(\upsilon) \).
Now, if we denote the above map by $\overline{T}_v$, we have

$$\overline{T}_v (\mathcal{I}(x)) = \mathcal{I}(x)(v) = \beta(v)(x)$$

i.e. $\overline{T}_v$ satisfies the relation $\overline{T}_v \circ \mathcal{J} = \beta(v)$ and thus $\overline{T}_v = \overline{\beta}(v)$.

2 - Realizations of Hamiltonian systems

Let $(M,\omega)$ be a symplectic manifold; we assume we have defined on $M$ a Hamiltonian system $\Sigma$, as in the previous chapter, such that

i) $\Sigma$ is analytic and complete.

ii) $\mathcal{H}(\mathcal{D})$ is a finite dimensional Lie algebra.

From [6] we know that $G$ is a Lie group, the Lie algebra $\mathfrak{g}$ of $G$ is isomorphic to $\mathcal{H}(\mathcal{D})$, and all the vector fields in $\mathcal{H}(\mathcal{D})$ are complete.

We shall assume the action of $G$ on $M$ is Poisson, with moment $\mathcal{J}$.

Lemma IV.2.1:

If $\mathcal{J}(x_1) = \mathcal{J}(x_2)$ then $y_u(x_1) = y_u(x_2)$ for every $u \in \mathbb{R}^\mathbb{R}$.

Proof:

$\mathcal{J}(x_1) = \mathcal{J}(x_2)$ in particular means that $\mathcal{P}_u(x_1) = \mathcal{P}_u(x_2)$ for every $u \in \mathbb{R}^\mathbb{R}$; let $v_u$ be the vector in $\mathfrak{g}$ corresponding to the associated vector field $\chi^u$; then $\mathcal{J}(x_1).v_u = \mathcal{J}(x_2).v_u$ or equivalently
\[ P_u(x_1) = P_u(x_2) \] since \( P_u \) is the Hamiltonian of \( X^u \).

So the two maps \( u \rightarrow P(x_1,u) \) and \( u \rightarrow P(x_2,u) \) are equal at every point, so \( \frac{\partial}{\partial u} P(x_1,u) = \frac{\partial}{\partial u} P(x_2,u) \) or \( y_u(x_1) = y_u(x_2) \) for every \( u \in \mathbb{R}^\ell \).

From this lemma and theorem IV.1.1 we see that we can define a strongly equivalent system \( \Sigma' \) on \( \mathcal{G}^* \), since both the dynamics and the output commute with \( \mathcal{I} \).

Suppose we have an initialized system \( (\Sigma, x) \); and let \( S \) be the orbit in \( \mathcal{G}^* \) of \( \mathcal{I}(x) \) by the coadjoint action of \( G \). Let \( \Sigma_S \) be the restriction of \( \Sigma' \) to \( S \).

**Theorem IV.2.2.**

\( (\Sigma_S, \mathcal{I}(x)) \) is strongly equivalent to \( (\Sigma, x) \), and moreover it is an accessible Hamiltonian system.

**Proof:**

It is clear that \( \Sigma_S \) is accessible, since \( S \) is an orbit, and strongly equivalent (as an initialized system) to \( \Sigma \).

As remarked on the previous lemma, we can define \( \mathcal{P}: \mathcal{G}^* \times \mathbb{R}^\ell \rightarrow \mathbb{R} \) such that \( \mathcal{P}(x,u) = \mathcal{P}(\mathcal{I}(x),u) \) for every \( (x,u) \in \mathbb{M} \times \mathbb{R}^\ell \). Let \( \mathcal{P}_S \) be its restriction to \( S \times \mathbb{R}^\ell \).

By theorem IV.1.3 we know that the associated vector field \( \xi^u \)
corresponding to the control $u \in \mathbb{R}^k$ (and to $X^u$) has Hamiltonian given by $P_{S_u}: S \rightarrow \mathbb{R}^k$, $P_{S_u}(\alpha) = P_S(\alpha, u)$.

The output $y_u$ is given by the restriction to $S$ of the map $\overline{y}_u$ which satisfies $y_u(x) = \overline{y}(f(x))$ for every $x \in M$ (the existence of $\overline{y}_u$ is proved by the previous lemma). Therefore, to finish the proof it is enough to show that

$$y_u^S(\alpha) = \frac{\partial}{\partial u} P_S(\alpha, u) .$$

From $P(f(x), u) = P(x, u)$ for every $(x, u) \in M \times \mathbb{R}^k$ we get

$$\frac{\partial}{\partial u} P(f(x), u) = \frac{\partial}{\partial u} P(x, u) , \text{ i.e. } \overline{y}_u(\alpha) \text{ is given by } \frac{\partial}{\partial u} P(\alpha, u) .$$

Now if $\alpha \in S$ $\frac{\partial}{\partial u} P_S(\alpha, u) = \frac{\partial}{\partial u} P(\alpha, u) = \overline{y}_u(\alpha) = y_u^S(\alpha)$ and the proof is complete.

We remark that $\Sigma_S$ is accessible, but not necessarily observable or locally observable.

From the results in [14] we can derive:

Theorem IV.2.3:

If $\Sigma_S$ is strongly accessible it is locally observable, thus a quasiminimal realization.
Proof:

Since strong accessibility and local observability are local conditions, we can assume one system to be defined $\mathbb{R}^n$:

$$\begin{align*}
\dot{x} &= w^{-1}(x) \frac{\partial P(x,u)}{\partial x} \\
y &= \frac{\partial}{\partial u} P(x,u)
\end{align*}$$

where $P : \mathbb{R}^n \times \mathbb{R}^\ell \to M$ and $w(x)$ is an invertible skew-symmetric matrix.

As in [14] we define the associated extended system as being

$$\begin{align*}
\dot{x} &= w^{-1}(x) \frac{\partial}{\partial x} P(x,y) \\
\dot{v} &= u \\
y(x,v) &= \frac{\partial}{\partial v} P(x,v)
\end{align*}$$

on the phase space $\mathbb{R}^n \times \mathbb{R}^\ell$.

In [14] it has been proved:

1) strong accessibility of the extended system is equivalent to strong accessibility of the original system.

2) The strong accessibility distribution at $(x,v)$ for the extended system is given by $\{0\} \times \mathbb{R}^\ell + V \times \{0\}$, where $V$ is a distribution on $\mathbb{R}^n$ such that $V \perp w$ is the observability codistribution for the original system at $x$, for a fixed $u = v$.

Therefore, from i) and ii), we see that strong accessibility implies local observability.
3 - Example

Let $M = \mathbb{R}^6$, with $w$ the usual symplectic form and

$$P(x,u) = g_0(x) + u_1g_1(x) + u_2g_2(x), \text{ thus } P:M \times \mathbb{R}^2 \rightarrow \mathbb{R}, \text{ where}$$

$$g_0(x) = x_3x_5 - x_2x_6, \quad g_1(x) = x_1x_5 - x_2x_4, \quad g_2(x) = x_1x_6 - x_3x_4.$$  

In this case $w = \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix}$

An easy computation shows that, if we define $X^i$ by $X^i \perp w = dg_i$, we have

$$X^0 = \begin{bmatrix} 0 \\ x_3 \\ -x_2 \\ 0 \\ x_6 \\ -x_5 \end{bmatrix}, \quad X^1 = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ -x_5 \\ x_4 \\ 0 \end{bmatrix}, \quad X^2 = \begin{bmatrix} -x_3 \\ 0 \\ x_1 \\ -x_6 \\ 0 \\ x_4 \end{bmatrix}$$

and also $[X^1 , X^2] = X^0$, $[X^2 , X^0] = X^1$, $[X^0 , X^1] = X^2$.

Therefore $\mathcal{F}(D) = \mathcal{F}_0(D)$ and finite dimensional; from the above relations we see that $\mathcal{F}(D)$ is isomorphic to $\text{so}(3)$, the Lie algebra of the skew symmetric $3 \times 3$ matrices, with the isomorphism given by

$$X^0 \rightarrow A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$
Moreover, we can identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ by means of

\[
\begin{align*}
(X_1, X_2, X_3) &\rightarrow \begin{bmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{bmatrix}
\end{align*}
\]

Then the bracket in $\mathfrak{so}(3)$ corresponds to the usual vector product in $\mathbb{R}^3$.

If we define an action of $\text{SO}(3)$ on $\mathbb{R}^6$ by means of

\[
\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}x = A_1 \cdot x
\]

and take $Y_i(x) = \frac{d}{dt}|_{t=0} \exp(tA_i \cdot x)$ we see that $Y_i \equiv X^i$, $i=0,1,2$. 

Thus we can assume we have been given an action $\text{SO}(3) \times \mathbb{R}^6 \rightarrow \mathbb{R}^6$; the maps $x \rightarrow Ax$ are symplectormorphisms for every $A \in \text{SO}(3)$, since they are the same as $x \rightarrow \alpha x$ for some $\alpha \in G_D$. ($\alpha$ depends on $A$), which follows from [6, pg 52-53].

If we identify $\mathbb{R}^6$ with $T^*\mathbb{R}^3$, we see the above action on $\mathbb{R}^6$ is the one induced by the usual action of $\text{SO}(3)$ on $\mathbb{R}^3$, and thus it is a Poisson action [2, pg 377].

Having identified $\text{so}(3)$ with $\mathbb{R}^3$, we can identify $\text{so}(3)^*$ with $\mathbb{R}^3$ as well, using the Euclidean structure.

It is easy to see that the adjoint action of $\text{SO}(3)$ on $\mathbb{R}^3$ (identified with $\text{so}(3)$) is equivalent, but not the same, to the usual action of $\text{SO}(3)$ on $\mathbb{R}^3$, for instance $\text{Ad}_{\exp tA_1}x = (\exp tA_0)x$, and so the orbits are as follows the orbit of $x \in \mathbb{R}^3$ is the set 
\[ \{y \in \mathbb{R}^3 | \|y\| = \|x\|\} \]

From the definition of coadjoint action, its orbits are exactly the same as the ones above.

As we know that the moment map does exist, we can compute it as follows:

\[ \mathcal{I}(x_1,x_2,x_3,x_4,x_5,x_6) \quad (1,0,0) = g_1(x) = x_1x_5 - x_2x_4 \]
\[ \mathcal{I}(x_1,x_2,x_3,x_4,x_5,x_6) \quad (0,1,0) = g_2(x) = x_1x_6 - x_3x_4 \]
\[ \mathcal{I}(x_1,x_2,x_3,x_4,x_5,x_6) \quad (0,0,1) = g_0(x) = x_2x_6 - x_3x_5 \]
and therefore

\[ \mathcal{I} : \mathbb{R}^6 \to \mathbb{R}^3, \mathcal{I}(x) = (x_1 \cdot x_5 - x_2 \cdot x_4, x_1 \cdot x_6 - x_3 \cdot x_4, x_2 \cdot x_6 - x_3 \cdot x_5) \]

If we denote by \( p_i \) the projection on the \( i \)-th factor of \( \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) \((i=1,2,3)\) we have

\[
\begin{align*}
g_1(x) &= p_1 \circ \mathcal{I}(x) \\
g_2(x) &= p_2 \circ \mathcal{I}(x) \\
g_0(x) &= -p_3 \circ \mathcal{I}(x)
\end{align*}
\]

and from the original system

\[
\begin{align*}
\dot{x} &= x^0 + u_1 x^1 + u_2 x^2 \\
y_1 &= g_1(x) \\
y_2 &= g_2(x)
\end{align*}
\]

we obtain a strongly equivalent system in \( \mathbb{R}^3 \), in which the output-maps are the projections on the first and second factor of \( \mathbb{R}^3 \).

To obtain an accessible realization, in this case a minimal realization and strongly accessible since \( [x^1, x^2] = x^0 \), we have to restrict it to an orbit of the coadjoint action, i.e. if we initialize the system at \( x \in \mathbb{R}^3 \) the state space of the minimal realization will be

\[ M_1 = \{ y \in \mathbb{R}^3 \mid ||y|| = ||x|| \} \], the output functions will be the
restriction to $M_1$ of $p_1$ and $p_2$, and the associated vector fields will be obtained from $p_1, p_2$ and $-p_3$ using the canonical symplectic form on the sphere.

Therefore, beginning with $x \in \mathbb{R}^6$, we obtain a new Hamiltonian system on $M_1$, where $P_1 : M_1 \times \mathbb{R}^2 \to \mathbb{R}$ is given by $P_1(x_1, x_2, x_3, u_1, u_2) = -x_3 + u_1 x_1 + u_2 x_2$. The symplectic form is defined as follows: if $v_1, v_2 \in \mathbb{R}^3$ are tangent to $M_1$, $\omega(v_1, v_2)(x) = (x, v_1 \cdot v_2)$ where $(\ , \ )$ denotes the usual inner product in $\mathbb{R}^3$. The associated vector fields are tangent to the sets $\{ y \in M_1, p_1(y) = \text{const} \}$, so the corresponding flows are rotations around the axes.

Of course, if $J(x) = 0$, then the orbit is just the origin, and the analysis above is not needed, nor valid.


   Dunod, 1968.


    To appear.
