Geometric ergodicity of a bead-spring pair with stochastic Stokes forcing

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Abstract

We consider a simple model for the fluctuating hydrodynamics of a flexible polymer in dilute solution, demonstrating geometric ergodicity for a pair of particles that interact with each other through a nonlinear spring potential while being advected by a stochastic Stokes fluid velocity field. This is a generalization of previous models which have used linear spring forces as well as white-in-time fluid velocity fields.

We follow previous work combining control theoretic arguments, Lyapunov functions, and hypo-elliptic diffusion theory to prove exponential convergence via a Harris chain argument. To this, we add the possibility of excluding certain "bad" sets in phase space in which the assumptions are violated but from which the systems leaves with a controllable probability. This allows for the treatment of singular drifts, such as those derived from the Lennard-Jones potential, which is an novel feature of this work.

1 Introduction

The study of polymer stretching in random fluids has been identified as a first step in the much larger project of modeling and understanding drag reduction in polymer solutions [Che00] and theoretical focus has been brought on the dynamics of simple dumbbell models [LMV02], [CMV05], [AV05]. Of particular interest is the experimentally observed phenomenon called the coiled state / stretched state phase transition [GCS05]. Mathematically this transition has been characterized by seeking models can admit solutions that are ergodic for certain regions of parameter space, while being null recurrent for other parameters [CMV05]. In this paper we address the topic of how to prove ergodicity for a wide range of models that generalize preceding work.

Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ denote the respective positions in $\mathbb{R}^2$ of two polymer "beads" connected by a "spring" at time t. As noted in [DE86, Ö96] these beads are not intended to

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model individual monomers, nor is the spring intended to capture the mechanics of an actual molecule. Rather the hope is to study qualitative features of generic chains interacting in a randomly fluctuating incompressible fluid environment.

Having made this caveat, the canonical Langevin model for two spherical particles in a passive polymer system is given by

$$m\ddot{\mathbf{x}}_i = -\nabla\Phi(|\mathbf{x}_i - \mathbf{x}_j|) + \zeta(\mathbf{u}(\mathbf{x}_i(t), t) - \dot{\mathbf{x}}_i(t)) + \kappa\dot{\mathbf{W}}(t) \ . \tag{1}$$

The mass m is considered to be vanishingly small and so the inertial term, $m\ddot{\mathbf{x}}_i$, will be ignored. On the right hand side, the first term is the restorative force exerted on the beads due to the potential energy of the polymer's current configuration. The second term is an expression for the drag force exerted by a time-dependent fluid velocity field $\mathbf{u}$ with friction coefficient $\zeta := 6\pi a\eta$ following from Stokes drag law for a spherical particle of radius a in a fluid with viscosity η. The final term is the force due to thermal fluctuations in the fluid. The diffusive constant κ is often taken to be $\kappa = \sqrt{2k_B T\zeta}$, where k_B is the Boltzmann constant and T is the temperature of the system in Kelvin, in accordance with the fluctuation-dissipation theorem [CMV05].

The goal of the present work is to achieve rigorous results about the ergodicity of the *connector* process

$$\mathbf{r}(t) := \frac{1}{2}(\mathbf{x}_1(t) - \mathbf{x}_2(t))$$

in both the $\kappa = 0$ and $\kappa \neq 0$ regimes with nonlinear spring interaction in the presence of a spatially and temporally correlated incompressible fluid velocity field.

In the simplest possible setting one ignores the fluid and assumes a Hookean (quadratic) spring potential Φ. In this case (1) is a simplification of the classical Rouse model [DE86]. We define

$$\Phi(\mathbf{x}, \mathbf{y}) := \frac{\gamma}{2}|\mathbf{x} - \mathbf{y}|^2$$

and the particle dynamics satisfy the system of SDE

$$d\mathbf{x}_1(t) = \gamma\left[\mathbf{x}_2(t) - \mathbf{x}_1(t)\right]dt + \kappa\, d\mathbf{W}_1(t)$$
$$d\mathbf{x}_2(t) = \gamma\left[\mathbf{x}_1(t) - \mathbf{x}_2(t)\right]dt + \kappa\, d\mathbf{W}_2(t)$$

where $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent standard Brownian motions. The dynamics of the connector $\mathbf{r}(t)$ are given by

$$d\mathbf{r}(t) = -\gamma\mathbf{r}(t) + \kappa\sqrt{2}d\mathbf{W}(t).$$

where $\mathbf{W} = \frac{1}{\sqrt{2}}(\mathbf{W}_1 - \mathbf{W}_2)$ is a standard Brownian motion. We see that each of the connector components is an Ornstein-Uhlenbeck process which theqrefore has the unique invariant measure

$$r_i(t) \sim N\left(0, \frac{\kappa^2}{\gamma}\right).$$

This exactly solvable model does not yield physical results, so one must adopt nonlinear models for either or both of the spring potential and fluid forces.

Significant theoretical advances exist for the dynamics of a single tracer particle convected by a wide variety of fluid models [MK99]. One popular fluid model for non-interacting two-point motions [BCH07] [MWD$^+$05] as well as for Hookean bead-spring systems [Che00, LMV02, CMV05] is a time-dependent random field satisfying the statistics of the Kraichnan-Batchelor ensemble [Bat59] [Kra68]. Such a fluid is still white in time, but is coloured in space.

In the case where $\kappa = 0$ with non-interacting beads, the spatial correlations in the convecting fluid velocity field allow for concentration and aggregation phenomena [SS02b] [MWD$^+$05] [BCH07]. This happens because when the two beads are very close together, the fluid forces on the respective beads are so strongly correlated there is no force encouraging separation.

The presence of a diffusive term with $\kappa \neq 0$ prevents such aggregation and the long term behavior of the connector depends on so-called Weissenberg number $Wi = \zeta/2\gamma$ [CMV05]. It is shown that when $Wi < 1$ the connector $\mathbf{r}$ will have a non-trivial stationary distribution, dubbed the "coiled" state. For $Wi > 1$, the connector does not have a stationary distribution and is called "stretched." The authors express interest in the case where the fluid is not assumed to be white-in-time.

In this work we use the incompressible stochastic Stokes equations to generate a fluid that is coloured in space and time (see Section 1.2). In the Hookean spring case with $\kappa = 0$, this model leads to degenerate dynamics (Proposition 1.1). However, in a more general setting with a non-linear spring potential, we show that dynamics are nondegenerate, although the coiled / stretched state dichotomy discussed in [CMV05] is not present. We find that $\mathbf{r}(t)$ is ergodic regardless of the physical parameters (Theorem 2.1).

The method used here to establish ergodicity builds on the Harris Chain theory developed in [Har56, Has80, Num84]. It is particularly indebted to the uniform ergodic results in weighted norms developed in [MT93a, MT93b]. The argument follows the path outlined in [MS02] [MSH02] for unique ergodicity of degenerate diffusions, but requires some nontrivial extensions to deal with the multiplicative nature of the noise and to permit the type of singular vector fields that arise as natural choices for the spring potential Φ. We build a framework around a general ergodic result from [HM08] and then develop the needed analysis to apply this framework.

Mathematically, as in [MSH02, MS02], this paper combines control theory with techniques from the theory of hypoelliptic diffusions to invoke results in the spirit of [MT93a, MT93b], where ergodicity is obtained by proving a minorization condition on a class of "small sets" (see [MT93a, MT93b]) while establishing a matching Lyapunov function. However, the current setting presents a number of difficulties which prevent the application of the results [MSH02] directly. In particular the spring potential, and hence the drift term, has a singularity (Assumption 2). Theqrefore the natural candidates for "small sets" are not compact. This difficulty is overcome by splitting the small sets into "good" and "bad" sets. On the compact "good" set, defined in Eq. (17), we demonstrate uniform controllability as in [MSH02, MS02]. On the bad set, one cannot obtain uniform control; however, the deterministic dynamics move the system into the good set in finite time so that geometric ergodicity still holds (Section 2.2). Allowing the spring potential to be singular extends the applicability of the theory to many interesting, physically important potentials such as the Lennard-Jones potential. Related ideas have been also recently been used to prove related, but different, ergodic and homogenization results in different settings (see [Bub09, HP07]).

1.1 Structure of Paper and overview of results

For the remainder of Section 1, we propose the model and explore its dynamics when the distance between the two beads, $\mathbf{r}$, is close to zero. Proposition 1.1 shows that when the spring is Hookean with zero rest length (a quadratic potential), $\mathbf{r}(t) \to \mathbf{0}$ as $t \to \infty$ almost surely if the spring constant γ is sufficiently strong relative to a quantity that depends on the typical spatial gradients in the random forcing. Lemma 1.2 shows that, in the case of a nonlinear spring with non-zero rest length, $\mathbf{r}(t)$ never equals zero nor converges to it as $t \to \infty$.

In Section 2, we quote an abstract ergodic result from the literature through which the results in this paper proceed. The quoted result requires proving a minorization condition and the existence of a Lyapunov function. Section 2.1 contains a general prescription for how to deduce the minorization condition from the existence of a continuous transition density and a weak form of topological irreducibility for the Markov process. In Section 2.2 the needed topological irreducibility is proven via a control theoretic argument. In Section 2.3 we invoke Hörmander's "sum of squares" theorem to prove that the associated hypoelliptic diffusion has a smooth transition density. Section 2.4 contains the calculations establishing the existence of a Lyapunov function and Section 2.5 contains a number of generalizations and implications of the preceding results. The appendix contains the derivation of the model used.

1.2 Definition of the model

In the overdamped, highly viscous regime, it is reasonable to neglect the nonlinear term in Navier-Stokes equations. Following [OR89], [MS02], [MSH02] and [SS02a] we consider the bead-spring system advected by a random field $\mathbf{u}$ satisfying the incompressible time-dependent stochastic Stokes equations. Following [Wal86], [DZ92], [Dal99] and [McK06] we have the stochastic PDE

$$\begin{aligned} \partial_t \mathbf{u}(\mathbf{x}, t) - \nu \Delta \mathbf{u}(\mathbf{x}, t) + \nabla p(\mathbf{x}, t) &= \mathbf{F}(d\mathbf{x}, dt) \\ \nabla \cdot \mathbf{u}(\mathbf{x}, t) &= 0 \end{aligned} \tag{2}$$

with periodic boundary conditions on the rectangle $R := [0, L] \times [0, L]$ where L is presumed to be very large. We assume that the space-time forcing is a Gaussian process with covariance

$$\mathbb{E}[\mathbf{F}(\mathbf{x}, t)] = \mathbf{0}, \qquad \mathbb{E}\big[F^i(\mathbf{x}, t) F^j(\mathbf{y}, s)\big] = (t \wedge s) 2 k_B T \nu \delta_{ij} \Gamma(\mathbf{x} - \mathbf{y}) \tag{3}$$

where $t \wedge s$ is the minimum of t and s; the components $i, j \in \{x_1, x_2\}$ and δ_{ij} is a Kronecker delta function; and the remaining constants each have physical meaning: k_B is Boltzmann's constant, T is the temperature of the system and ν is the viscosity. As is shown in the Appendix, it follows that

$$\mathbf{F}(\mathbf{x}, t) = \frac{\sqrt{2k_BT\nu}}{L} \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \mathbf{0}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}/L} \sigma_{\mathbf{k}} \mathbf{B}_{\mathbf{k}}(t) \ . \tag{4}$$

where the $\mathbf{B}_{\mathbf{k}}$ are independent standard 2-d Brownian motions and the coefficients $\sigma_{\mathbf{k}}$ are related to the spatial correlation function Γ through the Fourier relation

$$\Gamma(\mathbf{x}) = \frac{2k_BT\nu}{L^2} \sum_{k \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}/L} \sigma_{\mathbf{k}}^2 \ .$$

This relation is possible because Γ is a covariance function, and theqrefore positive definite. By Bochner's Theorem, Γ is realizable as the the Fourier inverse transform of a positive real "spectral" measure. In the periodic setting, this is the counting measure

$$\hat{\Gamma}(\mathbf{k}) = \sqrt{2k_BT\nu}\sigma_{\mathbf{k}}^2.$$

Often, one defines the correlation structure on the Fourier side directly. We choose a radial, summable shape structure for the $\{\sigma_{\mathbf{k}}\}$.

Assumption 1. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}_+$ be given with $\phi(\mathbf{x}) = \phi(|\mathbf{x}|)$ and let the set active mode directions $\mathcal{K} \subset \mathbb{Z}^2$ be given. We define $\sigma_{\mathbf{k}}^2 := \phi^2(\mathbf{k})$ and require that the norm*

$$\|\sigma\|_s^2 := \sum_{\mathbf{k} \in \mathcal{K}} \sigma_{\mathbf{k}}^2 |\mathbf{k}|^{-2s}$$

is finite for all $s \geq 1$.

Our interest will be rigorous analysis of the long-term behavior of the connector process $\mathbf{r}$ whose dynamics, after slight simplification, are given by the following system (see Appendix for details). We define the Markov process $\mathbf{X}(t) = \binom{\mathbf{r}(t)}{\mathbf{z}(t)}$ to be the unique solution to the system of SDE

$$\frac{d}{dt}\mathbf{r}(t) = -\nabla\Phi(\mathbf{r}(t)) + \sum_{\mathbf{k} \in \mathcal{K}} \sin(\lambda\, \mathbf{k} \cdot \mathbf{r}(t)) \frac{\mathbf{k}^\perp}{|\mathbf{k}|} z_{\mathbf{k}}(t) \tag{5}$$

where the $z_{\mathbf{k}}$ satisfy

$$dz_{\mathbf{k}}(t) = -\lambda^2\nu|\mathbf{k}|^2 z_{\mathbf{k}}(t)dt + \sqrt{2\beta\nu}\sigma_{\mathbf{k}} dW_{\mathbf{k}}(t) \tag{6}$$

where $\lambda = 2\pi/L$ and $\beta = k_BT/4\pi^2$. We define the norm on this family of processes, $\|\mathbf{z}\|^2 := \sum_{\mathbf{k}\in\mathcal{K}} |z_{\mathbf{k}}|^2$.

We note that the eigenmodes $\{z_{\mathbf{k}}\}$ are a mutually independent collection of Ornstein-Uhlenbeck processes which have the stationary distribution

$$z_{\mathbf{k}} \sim N\left(0, \beta\frac{\sigma_{\mathbf{k}}^2}{|\mathbf{k}|^2}\right)$$

and autocorrelation function

$$\mathbb{E}[z_{\mathbf{k}}(t)z_{\mathbf{k}}(s)] = \beta\frac{\sigma_{\mathbf{k}}^2}{|\mathbf{k}|^2}\, e^{-\lambda^2|t-s|}\ .$$

For a position $\mathbf{r} \in \mathbb{R}^2$ and the family $\{z_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{K}}$ chosen from their respective stationary distributions, we note that the state-dependent fluid forcing

$$\mathbf{U}(\mathbf{r}, \mathbf{z}) := \sum_{\mathbf{k}\in\mathcal{K}} \sin(\lambda\mathbf{k}\cdot\mathbf{r})\, \frac{\mathbf{k}^{\perp}}{|\mathbf{k}|} z_{\mathbf{k}} \tag{7}$$

is a mean zero random variable whose variance is bounded by a constant times the H_1 norm of the spectral measure

$$\mathbb{E}[\mathbf{U}(\mathbf{r}, \mathbf{z})] = \mathbf{0}, \qquad \mathbb{E}\left[|\mathbf{U}(\mathbf{r}, \mathbf{z})|^2\right] \le \beta\|\sigma\|_1^2 < \infty.$$

Furthermore, we will use the pathwise estimate

$$|\mathbf{U}(\mathbf{r}, \mathbf{z})|^2 \le \|\mathbf{z}\|^2\ . \tag{8}$$

As mentioned earlier, the choice of quadratic potential Φ corresponds to a Hookean spring model, but this yields degenerate dynamics (Proposition 1.1). In the sequel, we place the following assumptions on the nonlinear spring potential.

Assumption 2. *Let $\Phi(\mathbf{r}) = \Phi(|\mathbf{r}|)$ be a continuously differentiable function satisfying the following*

(i) For every $R \ge 0$, the set $\{r \in \mathbb{R}_+ \text{ s.t. } \Phi(r) \le R\}$ is compact.

(ii) There exists an R_0 such that for all $r \ge R_0$

$$\nabla\Phi(\mathbf{r})\cdot\mathbf{r} \ge \gamma|\mathbf{r}|^2 \tag{9}$$

and there exists $c > 0$ and $\epsilon_0 > 0$ such that for all $\mathbf{r}$ satisfying $|\mathbf{r}| \in (0, \epsilon_0]$

$$-\nabla\Phi(\mathbf{r})\cdot\mathbf{r} \ge c \tag{10}$$

Remark 1. In is in the context of this assumption that we choose the length of the periodicity of the forcing fluid. Henceforth we take $L \gg 4R_0$.

The above description includes as an example the linear spring potential with non-zero rest length. In this case,

$$\Phi(\mathbf{r}) = \frac{\gamma}{2}\left(|\mathbf{r}| - R\right)^2$$

for some constant R denoting the rest length. The theory presented also allows for potentials with singularities, as seen in the family of functions

$$\Phi(\mathbf{r}) = \frac{1}{2q}|\mathbf{r}|^{2q} + \frac{1}{\alpha|\mathbf{r}|^{\alpha}} \tag{11}$$

where $q \geq 1$ and $\alpha > 0$. The choice of $q = 1$ and $\alpha = 12$ corresponds to a Lennard-Jones singularity at zero and a classical linear spring at infinity.

A perhaps more common choice is the *finite extensible nonlinear elastic* FENE model

$$\Phi(\mathbf{r}) = \ln\left(1 - \frac{|\mathbf{r}|^2}{R^2}\right).$$

While this choice is acceptable away from zero, this logarithmic term alone does not satisfy the near-zero condition and some repulsive force must be added at the origin.

1.3 Near-zero dynamics

Behavior near the origin is critical to determining whether our bead-spring model supports non-trivial dynamics. The following quick calculation demonstrates that without a repulsive force at the origin, under mild conditions on the spring constant and spectral measure of the fluid, the two beads become trapped near each other in the long run almost surely.

Proposition 1.1 (Degeneracy of the Hookean spring case)**.** *Let* $\mathbf{r}$ *and the family* $\{z_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{K}}$ *satisfy the system of differential equations* (5) *and* (6)*, with the spectral measure* $\{\sigma_{\mathbf{k}}\}$ *satisfying Assumption 1. Let* $\Phi(\mathbf{r}) = \frac{\gamma}{2}|\mathbf{r}|^2$ *and suppose that* $\|\sigma\|_0 < \infty$*. Then there exists a* $\gamma_0 = \gamma_0(\|\sigma\|_0)$ *so that if* $\gamma > \gamma_0$ *then*

$$\lim_{t\to\infty} \mathbf{r}(t) = \mathbf{0}$$

almost surely.

Proof. Note that the process $|\mathbf{r}(t)|^2$ satisfies the following pathwise ODE, where ω denotes the sample space variable.

$$\begin{aligned}
\frac{d}{dt}|\mathbf{r}(t;\omega)|^2 &= -2\gamma|\mathbf{r}(t;\omega)|^2 + 2\sum_{\mathbf{k}\in\mathcal{K}} \sin(\lambda\mathbf{k}\cdot\mathbf{r}(t;\omega))\,\frac{\mathbf{k}^{\perp}\cdot\mathbf{r}(t;\omega)}{|\mathbf{k}|} z_{\mathbf{k}}(t;\omega) \\
&\leq -2\gamma|\mathbf{r}(t;\omega)|^2 + 2\lambda\sum_{\mathbf{k}\in\mathcal{K}} |\mathbf{k}\cdot\mathbf{r}(t;\omega)||\mathbf{k}^{\perp}\cdot\mathbf{r}(t;\omega)|\frac{|z_{\mathbf{k}}(t;\omega)|}{|\mathbf{k}|} \\
&\leq -2\gamma|\mathbf{r}(t;\omega)|^2 + 2\lambda|\mathbf{r}(t;\omega)|^2\sum_{\mathbf{k}\in\mathcal{K}} |\mathbf{k}||z_{\mathbf{k}}(t;\omega)|
\end{aligned}$$

This differential inequality implies (and suppressing the dependence on ω)

$$|\mathbf{r}(t)|^2 \le \exp\left[-2\gamma t + 2\lambda \int_0^t \sum_{\mathbf{k}\in\mathcal{K}} |\mathbf{k}||z_{\mathbf{k}}(s)|ds\right] .$$

By the Law of Large Numbers

$$\frac{1}{t}\int_0^t \sum_{\mathbf{k}\in\mathcal{K}} |\mathbf{k}||z_{\mathbf{k}}(s)|ds \to \sqrt{\beta}\|\sigma\|_0$$

almost surely as $t\to\infty$, and so we see that for sufficiently large γ one has that $|\mathbf{r}(t)|^2 \to 0$ almost surely as $t\to\infty$.

□

The remainder of the paper is devoted to the nonlinear spring case. We first establish that the origin is an unattainable point for the process $\mathbf{r}(t)$ and that after a certain interval of time, there is a positive probability of escape from a neighborhood of the origin. Due to the exponential decay of memory in the system, we have in fact shown connector process $\mathbf{r}$ will escape a near-origin neighborhood with probability 1.

We begin by noting the following growth inequality. Since the strength of the fluid forcing is limited in a neighborhood of the origin, the deterministic spring force dominates and sends the two beads apart with positive probability.

Lemma 1.2. *Suppose the spectral measure $\{\sigma_{\mathbf{k}}\}$ satisfies Assumption 1 and the spring potential Φ satisfies Assumption 2. Then the origin is unattainable for the connector vector process, i.e. $|\mathbf{r}(t)| > 0$ for all t almost surely. Furthermore, $\mathbf{r}(t)$ leaves any sufficiently small open neighborhood of the origin in finite time.*

Proof. Fixing any $M > 0$, we define $t_0 = 0$ and $t_n = \inf\{t \ge 1 + t_{n-1} : \|\mathbf{z}(t)\| \le M\}$. Standard properties of the $z_{\mathbf{k}}$ ensure that $t_n < \infty$ with probability one. Furthermore, for any $\tilde{M} > M$ there exists an $\alpha > 0$ so that $\mathbb{P}\{\Omega_n\} \ge \alpha$ where $\Omega_n = \{\sup_{s\in[t_n,t_n+1]} \|\mathbf{z}(s)\| \le \tilde{M}\}$. Let ϵ_0 and c be the constants from Assumption 2. Then for any $\vartheta \in (0,1)$ and $\epsilon \in (0,\epsilon_0]$ if one sets $\tau = \inf\{s \ge 0 : |\mathbf{r}(s)| > \epsilon\}$ then for $t \in [0,\tau]$,

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}|\mathbf{r}(t)|^2 &= -\nabla\Phi(\mathbf{r}(t))\cdot\mathbf{r}(t) + \mathbf{U}(\mathbf{r}(t),\mathbf{z}(t))\cdot\mathbf{r}(t)\\ &\ge c - \frac{1}{4\vartheta}|\mathbf{r}(t)|^2 - \vartheta|\mathbf{U}(\mathbf{r}(t),\mathbf{z}(t))|^2\\ &\ge c - \frac{1}{4\vartheta}|\mathbf{r}(t)|^2 - \vartheta\|\mathbf{z}(t)\| .\end{aligned}$$

where we recall the ω-by-ω estimate (8). Further restricting to any $\omega \in \Omega_n$ and fixing $\vartheta = c/(2\tilde{M})$, one has

$$\frac{d}{dt}\frac{1}{2}|\mathbf{r}(t)|^2 \ge \frac{c}{2} - \frac{\tilde{M}}{2c}|\mathbf{r}(t)|^2 .$$

Assuming that $\tau > t_n$, integrating the preceding estimate produces

$$|\mathbf{r}(t \wedge \tau)|^2 \geq e^{-(t\wedge\tau - t_n)\tilde{M}/c}|\mathbf{r}(t_n)|^2 + c\int_{t_n}^{t\wedge\tau} e^{-(s-t_n)\tilde{M}/c}\,ds$$
$$\geq \frac{1}{\tilde{M}}\left(1 - e^{-(1\wedge(\tau - t_n))\tilde{M}/c}\right).$$

Fixing $\epsilon = \epsilon_0 \wedge (1 - e^{-\tilde{M}/c})/\tilde{M}$, we see that on Ω_n it is impossible to have $\tau > t_n + 1$. Hence we see that $\mathbb{P}\{\tau \in [t_n, t_n + 1] | \tau > t_n\} \geq \alpha$. Using the strong Markov property of the family $\mathbf{z}$ and the fact that the t_n are stopping times, we have $\mathbb{P}\{\tau > t_n + 1\} \leq (1 - \alpha)^n$ which concludes the proof. □

We have in fact proven the following which will be used later.

Corollary 1.3. *Given $\tilde{M} > M > 0$, there exists an ϵ and an $\alpha \in (0, 1)$ such that*

$$\tau(\mathbf{r}_0, \mathbf{z}_0) := \inf\{t \geq 0 : |\mathbf{r}(t)| \geq \epsilon \text{ and } \|\mathbf{z}(t)\| < \tilde{M}\}$$

satisfies

$$\inf_{\{\mathbf{z}_0 :\, \|\mathbf{z}_0\| < M\}} \inf_{\{\mathbf{r}_0 :\, 0 < |\mathbf{r}_0| \leq \epsilon_0\}} \mathbb{P}\{\tau(\mathbf{r}_0, Z_0) \leq 1\} \geq \alpha$$

2 Ergodicity

We begin by setting some notation. Let $\mathbf{X}(t) = \binom{\mathbf{r}(t)}{\mathbf{z}(t)}$ satisfy the system (5)-(6). For technical reasons we henceforth restrict ourselves to the case where the set of active modes $\mathcal{K}$ (from Assumption 1) is finite. It follows from Proposition 1.2 that the Markov process $\mathbf{X}(t)$ is well-defined on the state space

$$\mathbb{X} := \left\{(\mathbf{r}, \mathbf{z}) \in (\mathbb{R}^2 \setminus \mathbf{0}) \times \mathbb{R}^N\right\}.$$

For a bounded, measurable function $\varphi : \mathbb{X} \to \mathbb{R}$ we define the action of the Markov semigroup $\mathcal{P}_t$ by

$$(\mathcal{P}_t\varphi)(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[\varphi(\mathbf{X}(t))].$$

To measure convergence to equilibrium we introduce the following weighted norm on such functions φ relative to a given Lyapunov function $V : \mathbb{X} \to [0, \infty)$,

$$\|\varphi\|_1 := \sup_{\mathbf{x}\in\mathbb{X}} \frac{|\varphi(\mathbf{x})|}{1 + V(\mathbf{x})};.$$

In this paper, we take

$$V(\mathbf{x}) := |\mathbf{r}|^2 \vee R_0^2 + \eta|\mathbf{z}|^2$$

where R_0 is the value given in equation (9) of the spring potential Assumption 2 and η is a constant to be chosen later in Lemma 2.7. We note that the Markov semigroup $\mathcal{P}_t$ can be extended to act on all functions φ bounded pointwise above by V.

The main result of this article is the following statement about the geometric ergodicity of the full Markov process $\mathbf{X}$, and by corollary, the marginal process $\mathbf{r}$ converges to a unique non-trivial stationary distribution in exponential time.

Theorem 2.1. *Suppose that the set of active modes $\mathcal{K}$ is finite, but contains at least three pairwise linearly independent vectors, and let the spring potential Φ satisfy Assumption 2. Then there exists a unique non-trivial invariant measure π and constants $C > 0$ and $\lambda > 0$ so that*

$$\|\mathcal{P}_t\varphi - \pi\varphi\| \leq Ce^{-\lambda t}\|\varphi\|$$

where $\pi\varphi = \int \varphi d\pi$.

We begin by recording in a more abstract setting the norm in which the dynamics converge to equilibrium. It is a weighted version of the total variation distance between measures. We generalize the preceding weighted L^∞-norm with a family of equivalent norms depending on a scale parameter $\beta > 0$. For any $\varphi\colon \mathbb{X} \to \mathbb{R}$ define

$$\|\varphi\|_\beta = \sup_{\mathbf{x}} \frac{|\varphi(\mathbf{x})|}{1 + \beta V(\mathbf{x})}$$

We use this to define an associated norm on probability measures.

$$\rho_\beta(\mu_1, \mu_2) = \sup_{\varphi : \|\varphi\|_\beta \leq 1} \int \varphi(\mathbf{x})\mu_1(d\mathbf{x}) - \int \varphi(\mathbf{x})\mu_2(d\mathbf{x})$$

The conclusion of Theorem 2.1 follows from the main theorem in Ref [HM08] adapted to our setting:

Theorem 2.2. *Suppose that for some $t > 0$, $c_1 > 0$ and $c_0 \in (0, 1)$, a function $V\colon \mathbb{X} \to [0, \infty)$ having compact level sets with $\lim_{\|\mathbf{x}\|\to\infty} V(\mathbf{x}) = \infty$ satisfies*

$$(\mathcal{P}_t V)(\mathbf{x}) \leq c_0 V(\mathbf{x}) + c_1 \tag{12}$$

for all $\mathbf{x} \in \mathbb{X}$. Furthermore suppose there exists a probability measure ν and constant $\alpha \in (0, 1)$ such that

$$\inf_{\mathbf{x}\in\mathcal{C}} \mathcal{P}_t(\mathbf{x}, \cdot\,) \geq \alpha\nu(\,\cdot\,) \tag{13}$$

with $\mathcal{C} := \{\mathbf{x} \in \mathbb{X} : V(\mathbf{x}) \leq R\}$ for some $R > 2c_1/(1 - c_0)$.

Then there exists an $\alpha_0 \in (0, 1)$ and $\beta > 0$ so that

$$\rho_\beta(\mathcal{P}_t^*\mu_1, \mathcal{P}_t^*\mu_2) \leq \alpha_0\rho_\beta(\mu_1, \mu_2)$$

for any probability measure μ_1 and μ_2 on $\mathbb{X}$.

The first condition (12) states that V is a Lyapunov function for the dynamics, established in Lemma 2.7. In Lemma 2.6 we construct a minorizing measure, as required by condition (13). This lemma follows from the combination of a form of topological irreducibility (Proposition 2.4) and local smoothing (Proposition 2.6). The local smoothing follows from hypoellipticity of the generator of the Markov process $\mathbf{X}$ and a version of Hörmander's sum of squares theorem (cf. [Hör85, Str08]).

2.1 Conditions for measure-theoretic irreducibility

We now show how to use a very weak topological irreducibility to prove the measure-theoretic minorization/irreducibility property given in (13). We begin by fixing the set $\mathcal{C}$ which should be thought of as the "center" of the state space:

$$\mathcal{C} := \{\mathbf{x} \in \mathbb{X} : V(\mathbf{x}) \leq 2R_0^2\}. \tag{14}$$

Proposition 2.3. *If the following two conditions hold, then there exists a constant $\alpha \in (0,1)$ and probability measure ν so that (13) holds.*

(i) Uniformly Accessible Neighborhood Condition: *There exists a $\mathbf{x}_* \in \mathcal{C}$ such that for any $\delta > 0$ there exists a $t_1 > 0$ and a function $\alpha_1 = \alpha_1(t,\delta) > 0$, continuous in t, such that*

$$\inf_{\mathbf{x}\in\mathcal{C}} \mathcal{P}_t(\mathbf{x}, B_\delta(\mathbf{x}_*)) \geq \alpha_1 \tag{15}$$

for all $t > t_1$.

(ii) Continuous Density Condition: *There exists $s > 0$ and an open set $\mathcal{O} \subset \mathcal{C}$, with $\mathbf{x}_* \in \mathcal{O}$, such that for $\mathbf{x} \in \mathcal{O}$ and measurable $A \subset \mathcal{O}$ one has*

$$\mathcal{P}_s(\mathbf{x}, A) = \int_A p_s(\mathbf{x},\mathbf{y})d\mathbf{y}$$

with $p_s(\mathbf{x},\mathbf{y})$ jointly continuous in $(\mathbf{x},\mathbf{y})$ for $\mathbf{x},\mathbf{y} \in \mathcal{O}$ and $p_s(\mathbf{x}_*,\mathbf{y}_*) > 0$ for some $\mathbf{y}_* \in \mathcal{O}$.

Proof. By the continuity assumption on p_s there exists $\delta > 0$ so that $B_\delta(\mathbf{x}_*), B_\delta(\mathbf{y}_*) \subset \mathcal{O}$ and

$$\inf_{\mathbf{x}\in\mathcal{B}_\delta(\mathbf{x}_*)} \inf_{\mathbf{y}\in\mathcal{B}_\delta(\mathbf{y}_*)} p_s(\mathbf{x},\mathbf{y}) \geq \frac{1}{2} p_s(\mathbf{x}_*,\mathbf{y}_*) > 0 \,.$$

We define the minorizing probability measure ν by

$$\nu(A) = \frac{\lambda(A \cap B_\delta(\mathbf{y}_*))}{\lambda(B_\delta(\mathbf{y}_*))}$$

where λ is Lebesgue measure and A is any measurable set.

For any $t \geq t_1 + s$, we define $\alpha(t) = \frac{1}{2}p_s(\mathbf{x}_*,\mathbf{y}_*)\alpha_1(t-s)$, where α_1 is the function from the Uniformly Accessible Neighborhood Condition (i). Given any measurable set A and $\mathbf{x}_0 \in \mathcal{C}$ we have

$$\begin{aligned}\mathcal{P}_t(\mathbf{x}_0, A) &= \int_A \int_{\mathbb{R}^{2+N}} \mathcal{P}_{t_1}(\mathbf{x}_0, d\mathbf{x})\mathcal{P}_s(\mathbf{x}, d\mathbf{y}) \\ &\geq \int_{A\cap B_\delta(\mathbf{y}_*)} \int_{B_\delta(\mathbf{x}_*)} \mathcal{P}_{t_1}(\mathbf{x}_0, d\mathbf{x}) p_s(\mathbf{x},\mathbf{y})d\mathbf{y} \geq \alpha\nu(A) \,,\end{aligned}$$

which proves the claim. □

2.2 Topological irreducibility via controllability

Recall the "center" $\mathcal{C}$ of the state space $\mathbb{X}$ from (14). In order to demonstrate the Uniformly Accessible Neighborhood Condition (i) given in Lemma 2.3, we wish to use the $\mathbf{z}$ process to drive the $\mathbf{r}$ process to some reference point $\mathbf{r}_*$. However, due to the possible singularity at the origin (see Assumption 2) the differential equation (5) for $\mathbf{r}$ may have unbounded coefficients. We theqrefore will designate a region of bad control within the center $\mathcal{C}$, as well as a compact region of good control.

To this end, let ϵ_1 be the constant derived from applying Corollary 1.3 with $M = R_0$ and $\tilde{M} = \sqrt{2}R_0/\eta$. We define the set of "bad" points in $\mathcal{C}$ by

$$\mathcal{B} = \left\{(\mathbf{r}, \mathbf{z}) \in \mathcal{C} : |\mathbf{r}| < \epsilon_1\right\} . \tag{16}$$

We define the set of "good" points $\mathcal{G}$ to be the remaining portion of $\mathcal{C}$ plus an additional region for technical reasons. Precisely,

$$\mathcal{G} = \mathcal{G}_{\mathbf{r}} \times \mathcal{G}_{\mathbf{z}} := \left\{(\mathbf{r}, \mathbf{z}) \in \mathbb{X} : |\mathbf{r}| \in \left[\epsilon_1, \sqrt{2}R_0\right], \|\mathbf{z}\| \leq M\right\} . \tag{17}$$

where the constant $M := \max\{\frac{2R_0}{\sqrt{\eta}}, |\mathcal{K}|^3(1+\max_{\mathbf{r}\in\mathcal{G}_{\mathbf{r}}} |\nabla\Phi(\mathbf{r})|)\}$ with η defined later in Lemma 2.7.

We now use a controllability argument to establish the weak form of uniform topological irreducibility on $\mathcal{G}$ given in Eq. (15) (for the set $\mathcal{C}$).

Lemma 2.4 (Topological irreducibility on the "good" set $\mathcal{G}$)**.** *There exists a reference point $\mathbf{x}_* \in \mathcal{C}$ such for any $\delta > 0$, there exists a $t_1 > 0$ so that for any closed interval $I \subset [t_1, \infty)$ there is an $\alpha_1 > 0$ with*

$$\inf_{t\in I} \inf_{\mathbf{x}\in\mathcal{G}} \mathcal{P}_t(\mathbf{x}, B_\delta(\mathbf{x}_*)) \geq \alpha_1 \tag{18}$$

Sketch of Proof: We begin by finding a suitable reference point $\mathbf{x}_* \in \mathcal{G}$. By Assumption 2, the spring potential Φ has a (possibly non-unique) global minimum $\mathbf{r}_*$. Since the global minimum of the norm $\|\cdot\|$ is achieved at the origin, $\mathbf{z} = \mathbf{0}$, we set $\mathbf{x}_* := \binom{\mathbf{r}_*}{\mathbf{0}}$.

Now, recall that the $\mathbf{r}$-dynamics have the form

$$\frac{d}{dt}\mathbf{r} = -\nabla\Phi(\mathbf{r}) + S(\mathbf{r})\mathbf{z}$$

where S is the $2 \times N$ Stokes forcing matrix where the j-th column is given by the vector $\sin(\mathbf{k}_j \cdot \mathbf{r})\, \mathbf{k}_j^\perp/|\mathbf{k}_j|$ and $\{\mathbf{k}_1, \ldots, \mathbf{k}_N\}$ is an enumeration of the active mode set $\mathcal{K}$.

Since there is a positive probability that an Ornstein-Uhlenbeck process will remain in a small tube about any given continuous curve over some compact interval of time, as long as S remains non-degenerate (has rank 2), we may use the $\{z_{\mathbf{k}}\}$ to drive the connector $\mathbf{r}$ to a neighborhood of $\mathbf{r}_*$. We shall later fix a $\delta_r < \delta$ and denote

$$\tau_1 := \inf\{t > 0 : |\mathbf{r}(t) - \mathbf{r}_*| < \delta_r\}.$$

It may be that $\|\mathbf{z}(\tau_1)\|$ is rather large. We must then bring the control to $\mathbf{0}$ sufficiently fast so that at time

$$\tau_2 := \inf\{t > \tau_1 : \|\mathbf{z}(t)\| < \delta_z\}$$

we have $|\mathbf{r}(\tau_2) - \mathbf{r}_*| < 2\delta_r$. After this it must be shown that the process $\mathbf{X}$ can be held in place near the reference point $\mathbf{x}_*$ through to the end of the designated interval I. The value for δ_z will be chosen to accomplish this last requirement.

This argument may be extended to include general initial starting position due to the restriction to the compact region $\mathcal{G}$, where the coefficients of $\mathbf{r}$ are bounded uniformly.

We now supply the details.

Proof of Lemma 2.4: *Construction of the norm-minimizing control:* First, we discuss the non-degeneracy of the Stokes matrix S. Note that some of the columns of S may become zero for certain $\mathbf{r}$; however, by hypothesis, $\mathcal{K}$ contains at least three pairwise linearly independent vectors, so $S(\mathbf{r})$ will still have rank 2 for all $\mathbf{r} \in \mathcal{G}_{\mathbf{r}}$.

Let $\Gamma(t)$ be a piecewise smooth curve in the interior of the annulus $\mathcal{G}_{\mathbf{r}}$ with $\Gamma(0) = \mathbf{r}_0$ and $\Gamma(T) = \mathbf{r}_*$. Without loss of generality, we will suppose that this path is a single linear segment since any two points in an annulus can be connected by two line segments. Furthermore, we suppose that $\Gamma(t)$ is an arclength parametrization so that $\left|\frac{d}{dt}\Gamma(t)\right| = 1$. We will write $\Gamma(t) = \mathbf{v}t + \mathbf{r}_0$ where $\mathbf{v}$ is the unit vector $\frac{1}{T}(\mathbf{r}_* - \mathbf{r}_0)$.

For each fixed $t \in [0, T]$, the linear system

$$S(\Gamma(t))\mathbf{z}(t) = \frac{d}{dt}\Gamma(t) + \nabla\Phi(\Gamma(t))$$

has a unique minimal norm solution, namely

$$\mathbf{z}(t) = S^{\dagger}(\mathbf{v}t + \mathbf{r}_0)\left(\mathbf{v} + \nabla\Phi(\mathbf{v}t + \mathbf{r}_0)\right) \tag{19}$$

where

$$S^{\dagger} = S^*(SS^*)^{-1}$$

is the Moore-Penrose pseudoinverse [BIG80] and S^* is the transpose of S. Note that SS^* is a symmetric 2×2 matrix and so its inverse can be written explicitly. Let S_i denote the ith row of S. Then

$$(SS^*)^{-1} = \begin{pmatrix} |S_2|^2 & -S_1 \cdot S_2 \\ -S_1 \cdot S_2 & |S_1|^2 \end{pmatrix}$$

The key observation is that each component is a sum of the form $\sum c_j \sin^2(\mathbf{k}_j \cdot (\mathbf{v}t + \mathbf{r}_0))$ where the coefficients $c_j \in [-1, 1]$ do not depend on the path Γ. Theqrefore each component is a continuous function of t. This is true of S^* as well and so $\mathbf{z}$ is in fact continuous.

It remains to show uniform bounds on $\mathbf{z}$, but this is immediate since $\nabla\Phi$ and S are continuous functions over the compact domain $\mathcal{G}_{\mathbf{r}}$. Altogether we see that if $N = |\mathcal{K}|$, it follows from (19) that

$$\begin{aligned} \|\mathbf{z}\| &\leq \sup_{\mathbf{r} \in \mathcal{G}_{\mathbf{r}}} |S^{\dagger}(\mathbf{r})||1 + \nabla\Phi(\mathbf{r})| \\ &\leq |\mathcal{K}|^3(1 + \max_{\mathbf{r} \in \mathcal{G}_{\mathbf{r}}} |\nabla\Phi(\mathbf{r})|). \end{aligned}$$

This is the constant seen in the definition of the good region, Eq. (17).
Error bounds for fuzzy control: Let the $\mathbf{r}$-path Γ be specified and let $\mathbf{z}$ be the associated norm-minimizing control. Let $\tilde{\mathbf{z}}(t)$ be a solution to the fluid velocity SDE (6) and define Ω_ϵ to be the event where the sample path $\tilde{\mathbf{z}}(t;\omega)$ satisfies

$$\sup_{t\in[0,\tau_1]} |\mathbf{z}(t) - \tilde{\mathbf{z}}(t;\omega)| < \epsilon\,.$$

We note that $\mathbb{P}\{\Omega_\epsilon\} =: \alpha_\epsilon > 0$. Now, let $\tilde{\mathbf{r}}(t;\omega)$ be the solution to the ω-by-ω ODE,

$$\frac{d}{dt}\tilde{\mathbf{r}} = -\nabla\Phi(\tilde{\mathbf{r}}) + S(\tilde{\mathbf{r}})\tilde{\mathbf{z}}\,.$$

The control error $\mathbf{h}(t;\omega) := \Gamma(t) - \tilde{\mathbf{r}}(t;\omega)$ satisfies the ODE

$$\frac{d}{dt}|\mathbf{h}|^2 = -2\langle\nabla\Phi(\Gamma) - \nabla\Phi(\tilde{\mathbf{r}})), \mathbf{h}\rangle + 2\langle S(\Gamma)\mathbf{z} - S(\tilde{\mathbf{r}})\tilde{\mathbf{z}}, \mathbf{h}\rangle\,. \tag{20}$$

Since the spring potential Φ is continuously differentiable, it is Lipschitz in the region of good $\mathbf{r}$-control $\mathcal{G}_\mathbf{r}$ with constant λ_Φ. The first term of (20) is theqrefore bounded for every t by

$$-2\langle\nabla\Phi(\Gamma) - \nabla\Phi(\tilde{\mathbf{r}})), \mathbf{h}\rangle \leq 2|\nabla\Phi(\Gamma) - \nabla\Phi(\tilde{\mathbf{r}}))||\mathbf{h}| \leq 2\lambda_\Phi|\mathbf{h}|^2.$$

The Stokes matrix is similarly Lipschitz with constant λ_S and together we have for all $\omega \in \Omega_\epsilon$

$$\begin{aligned}\frac{d}{dt}|\mathbf{h}|^2 &\leq 2\lambda_\Phi|\mathbf{h}|^2 + 2\langle S(\Gamma)\,(\mathbf{z} - \tilde{\mathbf{z}}), \mathbf{h}\rangle + 2\langle(S(\Gamma) - S(\tilde{\mathbf{r}}))\tilde{\mathbf{z}}, \mathbf{h}\rangle \\ &\leq 2|\mathbf{h}|^2(\lambda_\Phi + \lambda_S|\tilde{\mathbf{z}}|) + 2\epsilon|S(\Gamma)||\mathbf{h}| \\ &\leq |\mathbf{h}|^2(2\lambda_\Phi + 2\lambda_S|\tilde{\mathbf{z}}| + \epsilon|S(\Gamma)|) + \epsilon|S(\Gamma)|\end{aligned}$$

where in the last line we have used the polarization inequality $|\mathbf{h}| \leq \frac{1}{2}(1+|\mathbf{h}|^2)$. By Duhamel's principle we have

$$|\mathbf{h}(t)|^2 \leq \epsilon \int_0^t e^{\int_s^t(2\lambda_\Phi + 2\lambda_S|\tilde{\mathbf{z}}(s')| + \epsilon|S(\Gamma(s'))|)ds'}|S(\Gamma(s))|ds$$

We conclude that, given Γ, τ_1, and any $\delta_r > 0$, we may choose ϵ_r sufficiently small so that

$$\mathbb{P}\left\{\sup_{t\in[0,\tau_1]} |\Gamma(t) - \tilde{\mathbf{r}}(t)| < \delta_r\right\} \geq \alpha_{\epsilon_r} > 0. \tag{21}$$

Settling the noise and holding $\mathbf{r}$ *in place:* Although $\tilde{\mathbf{r}}(\tau_1)$ is within a ball of arbitrarily small size about the target $\mathbf{r}_*$, it is likely that the magnitude of the forcing terms $\mathbf{z}$ will be nontrivial. We must show that it is possible to decrease the forcing terms rapidly enough that $\tilde{\mathbf{r}}$ does not leave a prescribed ball around $\mathbf{r}_*$, in this case radius $2\delta_r$. Subsequently, we must hold $\mathbf{r}$ in place through to the end of and prescribed interval I.

For all $t \in I$ with $t > \tau_1$, we estimate as follows:

$$|\tilde{\mathbf{r}}(t) - \mathbf{r}_*| \leq |\tilde{\mathbf{r}}(t) - \tilde{\mathbf{r}}(\tau_2)| + |\tilde{\mathbf{r}}(\tau_2) - \tilde{\mathbf{r}}(\tau_1)| + |\tilde{\mathbf{r}}(\tau_1) - \mathbf{r}_*|\,.$$

where we recall that τ_1 is the arrival time of $\mathbf{r}$ in a neighborhood $\mathbf{r}_*$ and τ_2 is the arrival of the full Markov process near $\mathbf{x}_*$. The estimates follow from the integral representation of the $\tilde{\mathbf{r}}$ process. Over the interval $[\tau_2, t]$ we have

$$\begin{aligned} |\tilde{\mathbf{r}}(t) - \tilde{\mathbf{r}}(\tau_2)| &= \left| \int_{\tau_2}^{t} -\nabla\Phi(\tilde{\mathbf{r}}(s)) + S(\tilde{\mathbf{r}}(s))\tilde{\mathbf{z}}(s) ds \right| \\ &\leq (t - \tau_2) \sup_{\mathbf{r} \in B_{\delta_r}(\mathbf{r}_*)} \left\{ |\nabla\Phi(\mathbf{r})| + |S(\mathbf{r})| \sup_{s \in [\tau_2, t]} \{\|\tilde{\mathbf{z}}(s)\|\} \right\} \end{aligned}$$

Since the Stokes matrix satisfies the bound $|S(\mathbf{r})| < N$, we require $\|\mathbf{z}(s)\| < \delta/6Nt_2$ for all $s \in [\tau_2, t]$. Next note that since the spring potential Φ is continuously differentiable and since $\mathbf{r}_*$ is located at a local minimum, we may choose $\delta_r \in (0, \delta/3)$ small enough so that $\sup\{|\nabla\Phi(\mathbf{r})| : \mathbf{r} \in B_{2\delta_r}(\mathbf{r}_*)\} \leq \frac{\delta}{6t_2}$.

In the interval $[\tau_1, \tau_2]$ we prescribe the ideal path for the fluid vector $\mathbf{z}$ to be the linear interpolation between the initial condition $\mathbf{z}(\tau_1)$ and $\mathbf{z}(\tau_2) = \mathbf{0}$. We demand that the sample paths $\tilde{\mathbf{z}}(t)$ satisfy

$$\|\mathbf{z}(t) - \tilde{\mathbf{z}}(t)\| \leq \epsilon_r - (\epsilon_r - \delta_r)\,\frac{t - \tau_1}{\tau_2 - \tau_1}$$

for all $t \in (\tau_1, \tau_2)$ where ϵ_r is the constant chosen in the inequality (21) and τ_2 will be chosen in a moment. We have the estimate

$$\begin{aligned} |\tilde{\mathbf{r}}(\tau_2) - \tilde{\mathbf{r}}(\tau_1)| &\leq (\tau_2 - \tau_1) \sup_{\mathbf{r} \in \mathcal{G}_{\mathbf{r}}} \Big[|\nabla\Phi(\mathbf{r})| + S(\mathbf{r}) \sup_{s \in [\tau_1, \tau_2]} \{\|\tilde{\mathbf{z}}(s)\|\} \Big] \\ &\leq (\tau_2 - \tau_1) \big[\sup_{\mathbf{r} \in \mathcal{G}_{\mathbf{r}}} |\nabla\Phi(\mathbf{r})| + N(M + \epsilon) \big] \end{aligned}$$

where M is the constant in the definition (17) of the good set $\mathcal{G}_{\mathbf{r}}$, and the ϵ allows for the tube surrounding the ideal path $\mathbf{z}$. We may theqrefore choose a τ_2 sufficiently close to τ_1 so that $|\tilde{\mathbf{r}}(\tau_2) - \tilde{\mathbf{r}}(\tau_1)| < \delta/6$.

Altogether, we have for any $\tau_2 < t < t_2$, that $|\mathbf{r}(t) - \mathbf{r}_*| < \frac{5}{6}\delta$ and $\|\mathbf{z}(t)\| \leq \frac{1}{6}\delta$ with positive probability, so the claim (18) is satisfied.

□

We now use the properties of neighborhoods of the $\mathbf{r}$ singularity (and hence the "bad" set $\mathcal{B}$) established in Corollary 1.3 to extend the previous proposition to all of $\mathcal{C}$.

Lemma 2.5 (Topological irreducibility on $\mathcal{C}$). *Given a $\delta > 0$, there exists a $t_1' > 0$ so that for any $t \geq t_1'$ there is an $\alpha_1' > 0$ with*

$$\inf_{\mathbf{x} \in \mathcal{C}} \mathcal{P}_t(\mathbf{x}, B_\delta(\mathbf{x}_*)) \geq \alpha_1'$$

Proof. Set $t'_1 = t_1 + 1$ where t_1 is the constant from Lemma 2.4. Now for any $t \geq t'_1$

$$\inf_{\mathbf{x}\in\mathcal{B}} P_t(\mathbf{x}, B_\delta(x_*)) \geq \Big(\inf_{\mathbf{x}\in\mathcal{B}} \mathbb{P}_{\mathbf{x}}\{\tau \leq 1\}\Big)\Big(\inf_{\mathbf{x}\in\mathcal{G}} \inf_{s\in[t-1,t]} \mathcal{P}_s(\mathbf{x}, B_\delta(\mathbf{x}_*))\Big)$$
$$\geq \alpha\alpha_1 > 0$$

where α comes from Corollary 1.3 and α_1 from Lemma 2.4. Setting $\alpha'_1 = \alpha\alpha_1$ completes the proof. □

2.3 Measure Irreducibility via Hörmander's Condition

Lemma 2.6 (Absolute continuity of the transition density)**.** *For any $t > 0$, the Markov process $\{\mathbf{X}(t) = \binom{\mathbf{r}(t)}{\mathbf{z}(t)}\}_{t\geq 0}$ with transition kernel $\mathcal{P}_t(\mathbf{x}, U)$, possesses a transition density $p_t(\mathbf{x}, \mathbf{y})$, i.e.,*

$$\mathcal{P}_t(\mathbf{x}, U) = \int_U p_t(\mathbf{x}, \mathbf{y}) d\mathbf{y}$$

for every $U \in \mathscr{B}(\mathcal{C})$, where $p_t(\mathbf{x}, \mathbf{y})$ is jointly continuous in $(\mathbf{x}, \mathbf{y}) \in \mathcal{C} \times \mathcal{C}$.

Remark 2. In fact, the system has a density for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{X} \times \mathbb{X}$. However, due to the periodicity of our forcing, proving this would require an additional small argument. Since we do not need this fact, we refrain.

Proof. The claim follows from a now classical theorem of Hörmander which states that if a diffusion on an open manifold satisfies a certain algebraic condition then $L_1 = \partial_t - \mathcal{L}$ and $L_2 = \partial_t - \mathcal{L}^*$ are both hypoelliptic in $\mathcal{C}$ where $\mathcal{L}$ is the generator of the diffusion $\mathbf{X}(t)$ and $\mathcal{L}^*$ is its adjoint. A combination of Itô's formula and the fact that we have shown that the singularities of the potential are unattainable demonstrates that $L_1 u = 0$ and $L_2 u = 0$ have distribution-valued solutions. Hypoellipticity of the operators ensures first that these distribution-valued solutions are in fact smooth. Furthermore, hypoellipticity implies the existence of fundamental solution, which in turn yields continuity in the second variable throughout the center of the space $\mathcal{C}$.

The fact that the density is jointly continuous follows after a little more work. The argument is laid out in its entirety for $\mathbf{R}^N$ valued diffusions in Section 7.4 of [Str08]. In particular, see Theorem 7.4.3 and Theorem 7.4.20. Essentially, the same proofs follow in our setting since we have shown the system is a well defined diffusion on the manifold $\mathbb{X}$ with distribution-valued solution. Hypoellipticity and the properties which follow are local statements, and theqrefore still apply. The needed results in the general setting, as opposed to $\mathbb{R}^N$, can be found in Chapter 22 of [Hör85], noting in particular Theorem 22.2.1. However, the presentation in [Str08] is closer to the exact statements we need.

We now turn to the explicit calculations needed to show that Hörmander's condition is satisfied. We recast the system of equations (5) and (6) as

$$d\mathbf{X}(t) = A(\mathbf{X}) + Bd\mathbf{W}(t)$$

where $A(\mathbf{x}) \in \mathbb{R}^{2+N}$ and $B \in \mathbb{R}^{(2+N)\times(2+N)}$ with

$$A(\mathbf{x}) = \begin{pmatrix} -\nabla\Phi(\mathbf{r}) + \sum_{\mathbf{k}} \sin(\lambda\mathbf{k}\cdot\mathbf{r})\frac{\mathbf{k}^\perp}{|\mathbf{k}|} z_{\mathbf{k}} \\ -\lambda^2\nu\mathbf{z} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Here I is the identity in $\mathbb{R}^N$. In this notation, the generator $\mathcal{L}$ of the diffusion is given in terms of a test function φ by

$$(\mathcal{L}\varphi)(\mathbf{x}) = (A\cdot\nabla)\varphi(\mathbf{x}) + \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{K}}(B_{\mathbf{k}}\cdot\nabla)^2\varphi(\mathbf{x})$$

where $B_{\mathbf{k}}$ is the column of B associated with the mode direction $\mathbf{k} \in \mathcal{K}$. Hörmander's condition at the point $\mathbf{x}$ requires that

$$\operatorname{span}\Big\{B_{\mathbf{k}}, [A(\mathbf{k}), B_{\mathbf{k}}] : \mathbf{k}\in\mathcal{K}\Big\} = \mathbb{R}^{2+N}$$

where $[A(\mathbf{x}), B_{\mathbf{k}}]$ is the commutator or Lie bracket between the two vector fields $A(\mathbf{x})$ and $B_{\mathbf{k}}$. In our simplified setting where $B_{\mathbf{k}}$ is a constant vector-field one has

$$[A(\mathbf{x}), B_{\mathbf{k}}] = \frac{\partial}{\partial z_{\mathbf{k}}}A(\mathbf{x}) = \begin{pmatrix} \sin(\lambda\mathbf{k}\cdot\mathbf{r})\frac{\mathbf{k}^\perp}{|\mathbf{k}|} \\ -\lambda^2\nu|\mathbf{k}|^2\,\mathbf{e}_{\mathbf{k}} \end{pmatrix}$$

where $\mathbf{e}_{\mathbf{k}}$ is the is the unit basis vector in $\mathbb{R}^N = \mathbb{R}^{|\mathcal{K}|}$ associated to the mode direction $\mathbf{k}\in\mathcal{K}$.

The set $\{[A(\mathbf{x}), B_{\mathbf{k}}]\}_{\mathbf{k}\in\mathcal{K}}$ will span $\mathbb{R}^{2+N}$ if and only if the set $\{\sin(\mathbf{k}\cdot\mathbf{r})\mathbf{k}^\perp\}_{\mathbf{k}\in\mathcal{K}}$ spans $\mathbb{R}^2$ since the $\{\mathbf{e}_{\mathbf{k}} : \mathbf{k}\in\mathcal{K}\}$ spans $\mathbb{R}^N$. We recall that by assumption $\mathcal{K}$ contains at least three pairwise independent vectors which we label $\mathbf{k}_1, \mathbf{k}_2$, and $\mathbf{k}_3$. One may note that due to the periodicity of the forcing, $\sin(\lambda\mathbf{k}\cdot\mathbf{r}) = 0$ for all $\mathbf{r}\in\mathbb{Z}^2/\lambda^2$. However, by construction, all of these points lie outside of $\mathcal{C}$. Thus restricting to $\mathbf{x}\in\mathcal{C}$ at least two of $\mathbf{r}\cdot\mathbf{k}_i$ are nonzero. □

2.4 The Lyapunov function

We now provide the Lyapunov function required by Theorem 2.2 to match the minorization established in Lemma 2.5. We only need the function to control the return to the center of space. As such we will truncate the function inside of a radius R_0.

Recall the definition of the Lyapunov function

$$V(\mathbf{x}) := |\mathbf{r}|^2 \vee R_0^2 + \eta\|\mathbf{z}\|^2.$$

where η is specified below.

Lemma 2.7 (Lyapunov function)**.** *There exists an interval $I_1 \subset (0,\infty)$, a function $V : \mathbb{X} \to [0,\infty]$ with compact level sets, at least one point $\mathbf{x}$ such that $V(\mathbf{x}) < \infty$, and a constant $c_1 \geq 0$ such that for any $t \in I_1$ there exists a $c_0 \in (0,1)$ with*

$$(\mathcal{P}_t V)(\mathbf{x}) \leq c_0 V(\mathbf{x}) + c_1.$$

Proof. Again, let $\mathcal{L}$ be the generator for the Markov process $\mathbf{X}(t) := \binom{\mathbf{r}(t)}{\mathbf{z}(t)}$, which we write explicitly as

$$\mathcal{L} := (-\nabla\Phi(\mathbf{r}) + \mathbf{U}(\mathbf{r}, \mathbf{z})) \cdot \nabla_{\mathbf{r}} + \eta\nu\lambda^2 \left(\sum_{\mathbf{k}\in\mathcal{K}} -2|\mathbf{k}|^2 z_{\mathbf{k}} \frac{\partial}{\partial z_{\mathbf{k}}} + \beta\sigma_{\mathbf{k}}^2 \frac{\partial^2}{\partial z_{\mathbf{k}}^2} \right)$$

Because the definition of V and estimates for the spring potential vary depending on the value of $|\mathbf{r}|$ we divide the proof into three cases. First we address the case, $|\mathbf{r}| > R_0$, where $\nabla\Phi(\mathbf{r}) \cdot \mathbf{r} \geq \gamma|\mathbf{r}|^2$. Denoting $k := \min_{\mathbf{k}\in\mathcal{K}}\{|\mathbf{k}|\}$, we have

$$\begin{aligned}
\mathcal{L}V(\mathbf{x}) &= -2\nabla\Phi(\mathbf{r}) \cdot \mathbf{r} + 2\mathbf{U}(\mathbf{r}, \mathbf{z}) \cdot \mathbf{r} + \eta\nu\lambda^2 \sum_{\mathbf{k}\in\mathcal{K}} \left(-2|\mathbf{k}|^2 z_{\mathbf{k}}^2 + \beta\sigma_{\mathbf{k}}^2\right) \\
&\leq -2\gamma|\mathbf{r}|^2 + 2\left(\vartheta|\mathbf{U}(\mathbf{r}, \mathbf{z})|^2 + \frac{1}{\vartheta}|\mathbf{r}|^2\right) + \eta\nu\lambda^2 \left(-2k^2\|\mathbf{z}\|^2 + \beta\|\sigma\|_0^2\right) \\
&\leq -2\left(\gamma - \frac{1}{\vartheta}\right)|\mathbf{r}|^2 + 2\left(\eta\nu\lambda^2 k^2 - \vartheta\right)\|\mathbf{z}\|^2 + \eta\nu\lambda^2\beta\|\sigma\|_0^2
\end{aligned}$$

For $\delta \in (0, 1)$, letting $\vartheta^{-1} = (1 - \delta)\gamma$ and defining $C_1 := \eta\beta\nu\lambda^2\|\sigma\|_0^2$ yields

$$\mathcal{L}V(\mathbf{x}) = -2\delta\gamma|\mathbf{r}|^2 - \left(\eta\nu\lambda^2 k^2 - [(1 - \delta)\gamma]^{-1}\right)\|\mathbf{z}\|^2 + C_1 \ .$$

Further demanding that $\delta < \min(1, \nu k^2\lambda^2/\gamma)$ allows us to pick $\eta \geq [\gamma(1 - \delta)(\lambda^2\nu k^2 - \delta\gamma)]^{-1}$ so that

$$\mathcal{L}V(\mathbf{x}) \leq -\delta\gamma V(\mathbf{x}) + C_1$$

when $|\mathbf{r}| > R_0$.

When $|\mathbf{r}| < R_0$, the $\mathbf{r}$-dependent portion of the Lyapunov function is constant. Theqrefore

$$\begin{aligned}
\mathcal{L}V(\mathbf{x}) &\leq -2\eta\nu\lambda^2 k^2\|\mathbf{z}\|^2 + C_1 \\
&= -2\nu\lambda^2 k^2(R_0 + \eta\|\mathbf{z}\|^2) + 2\lambda^2\nu k^2 R_0 + C_1 \\
&= -2\lambda^2\nu k^2 V(\mathbf{x}) + C_2
\end{aligned}$$

Let $a = \delta\gamma \wedge 2\lambda^2\nu k^2$ then, taken together, the above two estimates yield for $\mathbf{X}(t_0) = \mathbf{x}_0$ with $|\mathbf{r}| \neq R_0$,

$$\frac{d}{dt}\mathbb{E}_{\mathbf{x}_0}\left[V(\mathbf{X}(t))\right] \leq -aV(\mathbf{X}_t) + C_2$$

so that

$$\mathbb{E}_{\mathbf{x}_0}[V(\mathbf{X}(t))] \leq e^{-at}V(\mathbf{x}_0) + \frac{C_2}{a}(1 - e^{-at})$$

Given t, we may set $c_0 = e^{-at}$ and $c_1 = C_2/a$ to give (12).

It remains to include the case where $\mathbf{X}(t_0) = \binom{\mathbf{r}_0}{\mathbf{z}_0}$ where $|\mathbf{r}_0| = R_0$. We first observe that the measure irreducibility Lemma 2.6 guarantees that for any small $h > 0$, $\mathbb{P}\{|\mathbf{X}(t_0 + h)| = R_0\} = 0$. Furthermore, continuity of the dynamics and of the Lyapunov function ensures that there exists an $\epsilon > 0$ such that if we define $\tilde{\mathbf{X}}(t) = (\tilde{\mathbf{r}}(t), \tilde{\mathbf{z}}(t))$ with $\tilde{\mathbf{r}}(t_0) = (R_0 + \epsilon)\mathbf{X}(t_0)/R_0$ and $\tilde{\mathbf{z}}(t_0) = \mathbf{z}(t_0)$, then we have

$$V(\mathbf{X}(t_0 + s)) \leq V(\tilde{\mathbf{X}}(t_0 + s))$$

for all $s < h$.

It follows that

$$\mathbb{E}_{\mathbf{x}_0}[V(\mathbf{X}(t))] \leq e^{-at}V(\tilde{\mathbf{x}}_0) + \frac{C_2}{a}(1 - e^{-at}).$$

and taking limits as h and ϵ go to zero, we recover (12). □

2.5 Ergodicity of generalizations

In the derivation of the model equations (5) and(6) we imposed the simplifying assumption that the center of mass $\mathbf{m}(t) := \frac{1}{2}(\mathbf{x}_1(t) + \mathbf{x}_2(t))$ is held at zero (see Appendix). This greatly simplified the presentation, and did not affect the conclusion that the bead-spring system has an ergodic connector process $\mathbf{r}(t)$. Indeed the fluid velocity term with nonzero $\mathbf{m}(t)$ is given by Eq. (27):

$$\begin{aligned} &\frac{1}{2}[\mathbf{u}(\mathbf{x}_1(t), t) - \mathbf{u}(\mathbf{x}_2(t), t)] \\ &\qquad = \sum_{\mathbf{k} \in \mathcal{K}} [\cos(\lambda \mathbf{k} \cdot \mathbf{m}) z_{\mathbf{k}} - \sin(\lambda \mathbf{k} \cdot \mathbf{m}) y_{\mathbf{k}}] \sin(\lambda \mathbf{k} \cdot \mathbf{r}) \frac{\mathbf{k}^\perp}{|\mathbf{k}|} \end{aligned}$$

where the $\{y_{\mathbf{k}}\}$ are a second set of OU-processes defined exactly as the $\{z_{\mathbf{k}}\}$.

Because the $\mathbf{m}$ terms appear inside of cosines and sines, there is no new significant contribution to the Lyapunov function calculation. For the Hörmander condition, the additional terms in the coefficients of the noise introduces more "dead spots" in the forcing, but still one needs only *four* pairwise linearly independent vectors $\mathbf{k}_i$ in the mode set $\mathcal{K}$ to ensure that at least two of the vectors

$$\left\{ [\cos(\lambda \mathbf{k}_i \cdot \mathbf{m}) - \sin(\lambda \mathbf{k}_i \cdot \mathbf{m})] \sin(\lambda \mathbf{k}_i \cdot \mathbf{r}) \mathbf{k}_i^\perp \right\}$$

are nonzero. This guarantees the existence of a continuous transition density and it remains to show the δ-ball controllability as in Lemma 2.4. While the calculation is more involved, the principle of identifying the region of good control $\mathcal{G}$, where the coefficients of the $\mathbf{r}$-differential equation are uniform, still applies. Furthermore, since the differential equation for $\mathbf{r}$ is linear in the $\{y_{\mathbf{k}}\}$ and $\{z_{\mathbf{k}}\}$, we may still solve for stochastic control explicitly in terms of the desired path Γ as long as the new Stokes matrix is non-degenerate. Again, this is guaranteed by the hypothesis that $\mathcal{K}$ contains at least four pairwise linearly independent vectors.

We take a moment to consider the model closest to that of Celani, et. al. [CMV05], where in the canonical Langevin Equation (1), the spring potential is quadratic, the mass m is still

0, but the coefficient of the Brownian motion is nonzero: $\kappa = \sqrt{2k_B T\zeta}$. Our generalization is the replacement of the Kraichnan-ensemble with the stochastic Stokes equations.

Again, for small values of $\mathbf{r}$ the force on the connector $\mathbf{r}$ due to the fluid velocity becomes negligible, however, the remaining terms an Ornstein-Uhlenbeck and by standard ergodic properties of such processes, $\mathbf{r}$ quickly leaves any neighborhood of the origin with probability 1. For large values of $|\mathbf{r}|$, the quadratic spring potential dominates and the Lyapunov function calculation still holds. Since the diffusion is elliptic, trivially implying the existence of a continuous transition density, and all arguments in the derivation of the stochastic δ-ball controllability still apply, we see that the ergodic theorem holds for $\mathbf{r}(t)$.

This stands in contrast to the results in [CMV05] where it was argued that there exists a range of parameters where no stationary distribution exists. It is not immediately clear to us how to construct a model with coloured-in-space-and-time fluid velocity field that supports the "stretched" and "coiled" regimes cited in the physics literature.

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A Derivation of the model

In the overdamped, highly viscous regime, it is reasonable to neglect the nonlinear term in Navier-Stokes equations [OR89]. Following [Wal86], [DZ92], [Dal99] and [McK06] we have the stochastic PDE given in Section 1, Eq. 2,

$$\partial_t \mathbf{u}(\mathbf{x},t) - \nu\Delta\mathbf{u}(\mathbf{x},t) + \nabla p(\mathbf{x},t) = \mathbf{F}(\mathbf{x},dt)$$
$$\nabla\cdot\mathbf{u}(\mathbf{x},dt) = 0$$

with periodic boundary conditions on the rectangle $R := [0,L]\times[0,L]$ where L is presumed to be very large. We assume that the space-time forcing is a Gaussian process with covariance

$$\mathbb{E}\big[F^\alpha(\mathbf{x},t)F^\beta(\mathbf{y},s)\big] = (t\wedge s)2k_B T\nu\delta_{\alpha\beta}\Gamma(\mathbf{x}-\mathbf{y})$$

where $\alpha,\beta\in\{x_1,x_2\}$ and $\delta_{\alpha\beta}$ is a Kronecker delta function. The constants each have physical meaning: k_B is Boltzmann's constant, T is the temperature of the system and ν is the viscosity. It follows that

$$\mathbf{F}(\mathbf{x},t) = \frac{\sqrt{2k_B T\nu}}{L}\sum_{\mathbf{k}\in\mathbb{Z}^2\setminus\mathbf{0}} e^{\lambda i\mathbf{k}\cdot\mathbf{x}}\sigma_{\mathbf{k}}\mathbf{B}_{\mathbf{k}}(t)\ .$$

where the coefficients $\sigma_{\mathbf{k}}$ are related to the spatial correlation function Γ through the Fourier relation

$$\Gamma(\mathbf{x}) = \frac{1}{L^2} \sum_{k \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} e^{\lambda i \mathbf{k}\cdot\mathbf{x}} 2k_B T \nu \sigma_{\mathbf{k}}^2 \,.$$

Indeed, from the definition (4) we have

$$\begin{aligned}
\mathbb{E}\big[F^\alpha(\mathbf{x},t)F^\beta(\mathbf{y},s)\big] &= \frac{2k_B T\nu}{L^2} \sum_{\mathbf{k}} \sum_{\mathbf{j}} e^{\lambda i (\mathbf{k}\cdot\mathbf{x} - \mathbf{j}\cdot\mathbf{y})} \sigma_{\mathbf{k}} \sigma_{\mathbf{j}} \, \mathbb{E}\Big[B_{\mathbf{k}}^\alpha(t) B_{\mathbf{j}}^\beta(s)\Big] \\
&= (t \wedge s) 2k_B T \nu \delta_{\alpha\beta} \frac{1}{L^2} \sum_{\mathbf{k}} e^{\lambda i \mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \sigma_{\mathbf{k}}^2
\end{aligned}$$

demonstrating (3).

We turn our attention to the Fourier transform of the SPDE, noting the transform of the noise is given by

$$\begin{aligned}
\iint_R e^{-\lambda i \mathbf{k}\cdot\mathbf{x}} \mathbf{F}(\mathbf{x},t) d\mathbf{x} &= \iint_R e^{-\lambda i \mathbf{k}\cdot\mathbf{x}} \frac{\sqrt{2k_B T\nu}}{L} \sum_{\mathbf{j} \in \mathbb{Z}^2 \setminus \mathbf{0}} e^{\lambda i \mathbf{j}\cdot\mathbf{x}} \sigma_{\mathbf{j}} \mathbf{B}_{\mathbf{j}}(t) d\mathbf{x} \\
&= \frac{\sqrt{2k_B T\nu}}{L} \sum_{\mathbf{j} \in \mathbb{Z}^2 \setminus \mathbf{0}} \sigma_{\mathbf{j}} \mathbf{B}_{\mathbf{j}}(t) \iint_R e^{-\lambda i (\mathbf{k}-\mathbf{j})\cdot\mathbf{x}} d\mathbf{x} \\
&= \sqrt{2k_B T\nu} L \sum_{\mathbf{j} \in \mathbb{Z}^2 \setminus \mathbf{0}} \sigma_{\mathbf{j}} \mathbf{B}_{\mathbf{j}}(t) \delta_{\mathbf{kj}} \\
&= \sqrt{2k_B T\nu} L \sigma_{\mathbf{k}} \mathbf{B}_{\mathbf{k}}(t)
\end{aligned}$$

The SPDE transforms into the infinite dimensional system

$$d\hat{\mathbf{u}}_{\mathbf{k}}(t) + \lambda^2 \nu |\mathbf{k}|^2 \hat{\mathbf{u}}_{\mathbf{k}}(t) + \lambda i \mathbf{k} \hat{p}_{\mathbf{k}}(t) = \sqrt{2k_B T\nu} L \sigma_{\mathbf{k}} d\mathbf{B}_{\mathbf{k}}(t) \tag{22}$$

$$\lambda i \mathbf{k} \cdot \hat{\mathbf{u}}(t) = \mathbf{0} \tag{23}$$

For the sake of completing the formal argument, suppose for the moment that the forcing term is smooth with derivative $\mathbf{f}$. By taking the dot product of $\mathbf{k}$ with the terms of equation (22), the first two terms vanish – via incompressibility condition (23) – leaving the identity

$$\lambda i |\mathbf{k}|^2 \hat{p}_{\mathbf{k}}(t) = \sqrt{2k_B T\nu} L \sigma_{\mathbf{k}} \mathbf{k} \cdot \mathbf{f}(t) \tag{24}$$

Substituting back into (22) and gathering $\mathbf{f}(t)$ terms on the right-hand side yields

$$d\hat{\mathbf{u}}_{\mathbf{k}}(t) + \lambda^2 \nu |\mathbf{k}|^2 \hat{\mathbf{u}}_{\mathbf{k}}(t) = \sqrt{2k_B T\nu} L \sigma_{\mathbf{k}} \left(\mathbf{f}(t) - \frac{\mathbf{k}\cdot\mathbf{f}(t)}{|\mathbf{k}|^2} \mathbf{k} \right) \tag{25}$$

The projection on the right hand side has two standard representations.

$$\mathbf{f} - \frac{\mathbf{k}\cdot\mathbf{f}}{|\mathbf{f}|^2} \mathbf{k} = \left(\mathbf{I} - \frac{\mathbf{f} \otimes \mathbf{k}}{|\mathbf{k}|^2} \right) \mathbf{f} = \frac{\mathbf{f}\cdot\mathbf{k}^\perp}{|\mathbf{k}|^2} \mathbf{k}^\perp$$

where $\mathbf{k}^\perp := \binom{-k_2}{k_1}$. Applying Duhamel's principle and assuming vanishing initial conditions, we have the following representation for solutions to the fluid mode equations

$$\begin{aligned}\hat{\mathbf{u}}_{\mathbf{k}}(t) &= \sqrt{2k_BT\nu}\sigma_{\mathbf{k}}L\int_0^t e^{-\lambda^2\nu|\mathbf{k}|^2(t-s)}\left(\mathbf{I}-\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)d\mathbf{B}_{\mathbf{k}}(t)\\ &= \left(\mathbf{I}-\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\mathbf{z}_{\mathbf{k}}\end{aligned}$$

where we define $\mathbf{z}_{\mathbf{k}}$ to be the appropriate Ornstein-Uhlenbeck process,

$$d\mathbf{z}_{\mathbf{k}}(t) = -\lambda^2\nu|\mathbf{k}|^2\mathbf{z}_{\mathbf{k}}(t)dt + \sqrt{2k_BT\nu}L\sigma_{\mathbf{k}}d\mathbf{B}_{\mathbf{k}}(t)\ .$$

We theqrefore have the solution for the fluid velocity field,

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) &= \frac{1}{L^2}\sum_{\mathbf{k}\in\mathbb{Z}^2\backslash\mathbf{0}} e^{\lambda i\mathbf{k}\cdot\mathbf{x}}\left(\mathbf{I}-\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\mathbf{z}_{\mathbf{k}}\\ &= \frac{1}{L^2}\sum_{\mathbf{k}\in\mathbb{Z}^2\backslash\mathbf{0}} e^{\lambda i\mathbf{k}\cdot\mathbf{x}}\frac{\mathbf{z}_{\mathbf{k}}\cdot\mathbf{k}^\perp}{|\mathbf{k}|^2}\mathbf{k}^\perp\end{aligned}$$

After defining $z_{\mathbf{k}} := \frac{1}{L^2}\frac{\mathbf{z}_{\mathbf{k}}\cdot\mathbf{k}^\perp}{|\mathbf{k}|}$, we have the 1-$d$ OU-processes that drive the dynamics

$$dz_{\mathbf{k}}(t) = -\frac{4\pi^2\nu|\mathbf{k}|^2}{L^2}z_{\mathbf{k}}(t)dt + \frac{\sqrt{2k_BT\nu}\sigma_{\mathbf{k}}}{L}dB_{\mathbf{k}}(t)$$

Imposing the condition that we require real-valued solutions, after Fourier inversion we have the following trigonometric expansion for 2-d stochastic Stokes.

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}\in\mathbb{Z}^2\backslash\mathbf{0}}\left(\cos(\lambda\,\mathbf{k}\cdot\mathbf{x})y_{\mathbf{k}} + \sin(\lambda\,\mathbf{k}\cdot\mathbf{x})z_{\mathbf{k}}\right)\frac{\mathbf{k}^\perp}{|\mathbf{k}|} \tag{26}$$

where the $y_{\mathbf{k}}$ and $z_{\mathbf{k}}$ satisfy

$$\begin{aligned}dy_{\mathbf{k}}(t) &= -\frac{4\pi^2\nu|\mathbf{k}|^2}{L^2}y_{\mathbf{k}}(t)dt + \frac{\sqrt{2k_BT\nu}\sigma_{\mathbf{k}}}{L}dW^y_{\mathbf{k}}(t)\\ dz_{\mathbf{k}}(t) &= -\frac{4\pi^2\nu|\mathbf{k}|^2}{L^2}z_{\mathbf{k}}(t)dt + \frac{\sqrt{2k_BT\nu}\sigma_{\mathbf{k}}}{L}dW^z_{\mathbf{k}}(t)\end{aligned}$$

and $\{W^y_{\mathbf{k}}\}$ and $\{W^z_{\mathbf{k}}\}$ are i.i.d. sequences of standard 1-d Brownian motions. In the bulk of the paper we will express the above SDEs in terms of the constants $\lambda = 2\pi/L$ and $\beta = k_BT/2\pi$.

In the case where $\sigma_{\mathbf{k}} = 1$ for all $\mathbf{k}$, we have spatially white noise forcing. However, in dimensions two and higher [Dal99] [Wal86], there do not exist function-valued solutions for stochastic Stokes. It is sufficient to assume $\sum_{\mathbf{k}}\frac{\sigma^2_{\mathbf{k}}}{|\mathbf{k}|^2} < \infty$, the condition specified in

Assumption 1. (Note that the sum does not converge with $\sigma_{kv} \equiv 1$ because $\sum_{\mathbf{k}\in\mathbb{Z}^2\setminus\mathbf{0}}$ is the equivalent of a double integral.)

In this paper, we study the dynamics of the two beads in normal coordinates: $\mathbf{m}(t) = \frac{1}{2}(\mathbf{x}_1(t) + \mathbf{x}_2(t))$ and $\mathbf{m}(t) = \frac{1}{2}(\mathbf{x}_1(t) - \mathbf{x}_2(t))$,

$$\begin{aligned}\frac{d}{dt}\mathbf{m}(t) &= \frac{1}{2}[\mathbf{u}(\mathbf{x}_1(t), t) + \mathbf{u}(\mathbf{x}_2(t), t)]\\ \frac{d}{dt}\mathbf{r}(t) &= -\nabla\Phi(|\mathbf{r}(t)|) + \frac{1}{2}[\mathbf{u}(\mathbf{x}_1(t), t) - \mathbf{u}(\mathbf{x}_2(t), t)].\end{aligned}$$

In light of equation (2), we may write the radial process and the noise together as a Markovian system of SDE with two degenerate directions. In order to write the system in this form, we first record the identity

$$\begin{aligned}&\frac{1}{2}[\mathbf{u}(\mathbf{x}_1(t), t) - \mathbf{u}(\mathbf{x}_2(t), t)] \\ &\quad = \sum_{\mathbf{k}\in\mathcal{K}} [\cos(\lambda\mathbf{k}\cdot\mathbf{m})z_{\mathbf{k}} - \sin(\lambda\mathbf{k}\cdot\mathbf{m})y_{\mathbf{k}}]\sin(\lambda\mathbf{k}\cdot\mathbf{r})\frac{\mathbf{k}^\perp}{|\mathbf{k}|}\end{aligned} \tag{27}$$

For the majority of the paper, we used the simplification $\mathbf{m}(t) = 0$ for all t. This does not have any effect on the ergodic results as is discussed in Section 2.5, but it does significantly streamline the presentation. Altogether we have the definition of the dynamics given in Section 1, Eq. 5.

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