EQUALITY OF CRITICAL POINTS FOR POLYMER DEPINNING TRANSITIONS WITH LOOP EXPONENT ONE

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Abstract. We consider a polymer with configuration modeled by the trajectory of a Markov chain, interacting with a potential of form $u + V_n$ when it visits a particular state 0 at time n, with $\{V_n\}$ representing i.i.d. quenched disorder. There is a critical value of u above which the polymer is pinned by the potential. A particular case not covered in a number of previous studies is that of loop exponent one, in which the probability of an excursion of length n takes the form $\varphi(n)/n$ for some slowly varying φ; this includes simple random walk in two dimensions. We show that in this case, at all temperatures, the critical values of u in the quenched and annealed models are equal, in contrast to all other loop exponents, for which these critical values are known to differ at least at low temperatures.

1. Introduction

A polymer pinning model is described by a Markov chain $(X_n)_{n\geq 0}$ on a state space containing a special point 0 where the polymer interacts with a potential. The space-time trajectory of the Markov chain represents the physical configuration of the polymer, with the nth monomer of the polymer chain located at (n, X_n) (or just at X_n, for an undirected model.) When the chain visits 0 at some time n, it encounters a potential of form $u+V_n$, The *i.i.d.* random variables $(V_n)_{n\geq 1}$ typically model variation in monomer species. We study the phase transition in which the polymer depins from the potential when u goes below a critical value. We denote the distribution of the Markov chain (started from 0) in the absence of the potential by P^X and we assume that it is recurrent. This recurrence assumption is merely a convenience and does not change the essential mathematics; see [Al08], [GT05]. Of greatest interest is the case when the excursion length distribution decays as a power law:

excursion_law

$$P^X(\mathcal{E} = n) = n^{-c}\varphi(n), \qquad n \geq 1. \tag{1.1}$$

Here the *loop exponent* is $c \geq 1$, $\mathcal{E}$ denotes the length of an excursion from 0 and φ is a slowly varying function, that is, a function satisfying $\varphi(\kappa n)/\varphi(n) \to 1$ as n tends to

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infinity, for all $\kappa > 0$. Without loss of generality we will assume that $\varphi(n)$ converges to 0, as n converges to infinity.

A large part of the existing rigorous literature on such models omits the case $c = 1$, because it is often technically different and not covered by the methods that apply to $c > 1$; see e.g. [Al08], [GT07], [To08c], [To08a]. That omission is partially remedied in this paper, and we will see that the behavior for $c = 1$ can be quite different from $c > 1$. The case $c = 1$ includes symmetric simple random walk in two dimensions, for which $\varphi(n) \sim \pi/(\log n)^2$ [JP72]. The essential feature of $c = 1$ is that $P^X(\mathcal{E} > n)$ is a slowly varying function of n, so that for example the longest of the first m excursions typically has length greater than any power of m. This in effect enables the polymer to (at low cost) bypass stretches of disorder in which the values V_n are insufficiently favorable, and make returns to 0 in more-favorable stretches.

The quenched version of the pinning model is described by the Gibbs measure

olymer_measure
$$d\mu_N^{\beta,u,\mathbf{V}}(\mathbf{x}) = \frac{1}{Z_N} e^{\beta H_N^u(\mathbf{x},\mathbf{V})}\, dP^X(\mathbf{x}) \tag{1.2}$$

where $\mathbf{x} = (x_n)_{n\geq 0}$ is a path, $\mathbf{V} = (V_n)_{n\geq 0}$ is a realization of the disorder, and

Hdef
$$H_N^u(\mathbf{x},\mathbf{V}) = \sum_{n=1}^{N} (u + V_n)\delta_0(x_n). \tag{1.3}$$

The normalization

$$Z_N = Z_N(\beta, u, \mathbf{V}) = E^X\left[e^{\beta H_N^u(\mathbf{x},\mathbf{V})}\right]$$

is the partition function. The disorder $\mathbf{V}$ is a sequence of i.i.d. random variables with mean zero, variance one. We denote the distribution of this sequence by P^V. We assume that V_1 has exponential moments of all orders and we denote by $M_V(\beta)$ the moment generating function of P^V.

Let

$$L_N^X = L_N^X(\mathbf{x}) := \sum_{n=1}^{N} \delta_0(x_n)$$

denote the local time at 0, and define the *quenched free energy*

$$f_q(\beta, u) := \frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta, u, \mathbf{V}).$$

The fact that this limit exists and does not depend on $\mathbf{V}$ (except on a null set) is standard; see [Gi07]. In fact, one has by an easy subadditivity argument, that *a.s.* the following equality holds

$$f_q(\beta, u) := \frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} E^V \log Z_N(\beta, u, \mathbf{V}).$$

The parameter $u \in \mathbb{R}$ can be thought of as the mean value of the potential, while the parameter $\beta > 0$ is the inverse temperature. It is known that the phase space in

(β, u) is divided by a critical line $u = u_c^q(\beta)$ into two regions: the *localized* and the *delocalized* one. In the delocalized region $u < u_c^q(\beta)$ we have $f_q(\beta, u) = 0$ P^V-a.s., while in the localized region $u > u_c^q(\beta)$ we have $f_q(\beta, u) > 0$ P^V-a.s. It is proved in [GT06] that $f_q(\beta, \cdot)$ is infinitely differentiable for all $u > u_c(\beta)$. An alternate, more phenomenological, characterization of the two regions is as follows. From convexity we have for fixed β that

qcontact

$$\left\langle \frac{L_N}{N} \right\rangle_{[0,N]}^{\beta,u,\{V_i\}} = \frac{1}{\beta}\frac{\partial}{\partial u}\left(\frac{1}{N}\log Z_N(\beta, u, \mathbf{V})\right) \to \frac{\partial}{\partial u} f_q(\beta, u) \text{ for all } u, \tag{1.4}$$

P^V-a.s. This limiting value is called the *contact fraction*, denoted $C_q(\beta, u)$, and it is positive in the localized region and zero in the delocalized region. When the contact fraction is positive we say the polymer is *pinned*.

The effect of the quenched disorder on the phase transition is quantified by comparing the quenched model to the corresponding *annealed model*, which is obtained by averaging the Gibbs weight over the disorder to give

$$E^V\left(e^{\beta H_N^u(\mathbf{x},\mathbf{V})}\right) = e^{\beta \Delta L_N(\mathbf{x})},$$

where $\Delta = u + \beta^{-1}\log M_V(\beta)$, so the corresponding annealed partition function is

$$Z_N^a = Z_N^a(\beta, u) := E^X\left(e^{\beta\Delta L_N}\right)$$

and the Gibbs measure is

anngibbs

$$d\mu_N^{\beta,u}(\mathbf{x}) = \frac{1}{Z_N^a} e^{\beta\Delta L_N(\mathbf{x})}\, dP^X(\mathbf{x}). \tag{1.5}$$

The corresponding annealed free energy is denoted $f_a(\beta, u)$. The annealed critical point is readily shown to be $u_c^a(\beta) = -\beta^{-1}\log M_V(\beta)$ for all $\beta > 0$ (see [AS06]), so $\Delta = u - u_c^a(\beta)$. It is a standard consequence of Jensen's Inequality that $f_a(\beta, u) \geq f_q(\beta, u)$, so $u_c^a(\beta) \leq u_c^q(\beta)$. The effect, or lack of effect, of the disorder on the depinning transition may be seen in whether these two critical points actually differ, and whether the specific heat exponent (describing the behavior of the free energy as u decreases to the critical point) is different in the quenched case.

Though most mathematically rigorous work is relatively recent, there is an extensive physics literature on polymer pinning models—see the recent book [Gi07] and surveys [Gi08], [To08b] and the references therein. In [Al08] (see also [To08a] for a slightly weaker statement with simpler proof) it was proved that for $1 < c < 3/2$, and for $c = 3/2$ with $\sum_{n=1}^\infty 1/n(\varphi(n))^2 < \infty$, for sufficiently small β, one has $u_c^q(\beta) = u_c^a(\beta)$ and the specific heat exponents are the same. Both works considered Gaussian disorder, though the method in [Al08] can be extended to accommodate more general disorder having a finite exponential moment.

By contrast, it follows straightforwardly from the sufficient condition ([To08c], equation (3.6)) that for $c > 1$, if V_1 is unbounded, or if V_1 is bounded and its essential supremum v satisfies $P^V(V_1 = v) = 0$, then for sufficiently large β one has

$u_c^q(\beta) > u_c^a(\beta)$; the method is based on fractional moment estimates. These results together with [Al08] suggest that for $1 < c < 3/2$ there should be a transition from weak to strong disorder, i.e. there should exist a value $\beta_0 > 0$ below which the annealed and quenched critical curves coincide, i.e. $u_c^q(\beta) = u_c^a(\beta)$, for $\beta < \beta_0$, while for $\beta > \beta_0$, one has $u_c^a(\beta) < u_c^q(\beta)$, but this has not been proved.

For $1 < c < 3/2$ and certain choices of bounded V_1 (necessarily with $P(V_1 = v) > 0$), it is known that the quenched and annealed critical points are equal for all $\beta > 0$ [DGLT08]. But in these examples $\mathrm{Var}(e^{\beta V_1})/[E(e^{\beta V_1})]^2$ stays bounded as $\beta \to \infty$, so there is no true "strong disorder" regime.

For $c > 3/2$ it follows from [GT05] that the quenched and annealed specific heat exponents are different, and it was proved in [DGLT07] that the critical points are strictly different for all $\beta > 0$, that is, $\beta_0 = 0$. In [AZy08] the distinctness of critical points at high temperature was extended to include $c = 3/2$ with $\varphi(n) \to 0$ as $n \to \infty$, and the asymptotic order of the gap $u_c^q(\beta) - u_c^a(\beta)$ was given. Recently in [GLT08] the critical points were shown to be distinct for all $\beta > 0$ for the case of $c = 3/2$ and $\varphi(n)$ asymptotically a positive constant, a case about which physicists had long disagreed ([DHV92], [FLNO88].)

Here we show that even with true strong disorder, the critical points remain the same in the case $c = 1$.

theorem **Theorem 1.1.** *Consider the quenched model* (1.2) *and suppose* $E(e^{tV_1}) < \infty$ *for all* $t \in \mathbb{R}$ *and* (1.1) *holds with* $c = 1$. *For all* $\beta > 0$ *and all* $u > u_c^a(\beta)$, *the quenched free energy* $f_q(\beta, u) > 0$, *and thus* $u_c^q(\beta) = u_c^a(\beta)$ *for all* $\beta > 0$.

2. Notation and Idea of the Proof.

Denote the local time at zero over a time interval I by

local_time
$$L_I^X = \sum_{n \in I} \delta_0(x_n), \tag{2.1}$$

so $L_N^X = L_{[0,N]}^X$. The overlap between two paths X, X' in an interval I is defined as

overlap
$$B_I^{X,X'} = \sum_{n \in I} \delta_0(x_n)\delta_0(x_n'). \tag{2.2}$$

We denote by $P^{X,X'}$ the measure corresponding to two independent copies X, X' of the Markov chain. The "energy gained over an interval I" is

energy
$$H_I^u(\mathbf{x}, V) = \sum_{n \in I} (u + V_n)\delta_0(x_n). \tag{2.3}$$

The annealed correlation length is defined to be $M = M(\beta, u) := 1/\big(\beta f_a(\beta, u)\big)$. From (1.5), both $\beta f_a(\beta, u)$ and M are functions of only the product $\beta\Delta$. Moreover, the annealed contact fraction can be defined analogously to (1.4), for the annealed system. Using Laplace aymptotics and the large deviations for the local time L_N, one can deduce the asymptotics of M and $C_a(\beta,u)$, for $\beta\Delta\sim 0$. Specifically, letting

$$\Psi(t)=\int_t^\infty \varphi(e^s)\ ds,$$

we obtain

$$\beta\Delta\sim\Psi(\log M)\quad\text{and}\quad C_a(\beta,u)\sim\frac{1}{M\varphi(M)}\quad\text{as }\beta\Delta\to 0.$$

For example, if $\varphi(n)\sim K(\log n)^{-\alpha}$ for some $\alpha>1$ then

$$\log M=\log\frac{1}{\beta f_a(\beta,u)}\sim\left(\frac{\alpha-1}{K}\beta\Delta\right)^{-1/(\alpha-1)}\qquad\text{as }\beta\Delta\to 0,$$

so $f_a(\beta,\cdot)$ is C^∞ even at $u=u_c^a(\beta)$. The details are similar to the case $c>1$ carried out in [Al08], but we do not include them here as they are not required for our analysis.

We use length scales $K_1(\beta,M),K_2(\beta,M)$ related as follows, for $\beta,M>0$. Let $\Lambda_V(\beta):=\log M_V(2\beta)-2\log M_V(\beta)$. By our convention, see below (1.1), we have that $\varphi(x)\to 0$ as $x\to\infty$. For a slowly varying function $\varphi(x)\to 0$, we have

$$\frac{\log x}{\log\frac{1}{\varphi(x)}}\to\infty\quad\text{as }x\to\infty.$$

Therefore we can choose K_1,K_2 satisfying

K_2 (2.4)
$$32K_2<e^{\Lambda_V(\beta)K_2}$$

and

K_1 (2.5)
$$4(M\vee 1)\log\frac{1}{\varphi(K_1)}<K_2<\frac{1}{2\Lambda_V(\beta)}\log\frac{K_1}{2}.$$

For fixed β, as $\Delta\to 0$ (i.e. $M\to\infty$) we then have $M\ll K_2\ll K_1$. We assume henceforth that K_1,K_2 are even integers.

Define intervals

$$I_i=[iK_1,(i+1)K_1]\cap\mathbb{Z},\qquad I_i^\gamma=[iK_1,(i+\gamma)K_1]\cap\mathbb{Z},$$

for $0<\gamma<1$. For an interval I, let $\tau_I=\inf\{n\in I\colon x_n=0\}$ and $\sigma_I=\sup\{n\in I\colon x_n=0\}$. We set $\tau_I=\sigma_I=\infty$ if the path does not visit the interval I. We denote by Ξ_{NK_1} the set of all paths of length NK_1 which have the following property : if $\tau_{I_i}<\infty$ for some $i\le N$, then $\tau_{I_i}\in I_i^{1/2}$ and $\sigma_{I_i}-\tau_{I_i}<K_2$.

Idea of the proof. We will look at a scale NK_1 and restrict the partition function $Z_{NK_1}(u,\beta,\mathbf{V})$ to paths that belong to the set Ξ_{NK_1}. Further we will restrict attention to paths within Ξ_{NK_1} which bypass *bad* blocks of length K_1. Roughly speaking a bad block is defined to be a block for which the quenched partition function of a path starting at a random , uniformly distributed, point in the block, and not spending more than K_2 units of time in this block, is less than half the corresponding annealed partition function. In Lemma 3.2 we control the probability of having a bad block. What is left is to make an energy-entropy balancing of the paths that belong into Ξ_{NK_1} and bypass bad blocks, and show that for $\beta > 0$ and $\Delta = u + \beta^{-1} \log M_V(\beta) > 0$ this balance is uniformly (in N) bounded away from zero. For this we will use the fact that in a *good* block the free energy gained is of the order K_2/M (this is essentially Lemma 3.1), and the fact that because $P^X(\mathcal{E} > k)$ is a slowly varying function of k the cost of bypassing bad blocks is small.

3. Proof of the Theorem

e_correlation **Lemma 3.1.** *Let $\beta > 0, u \in \mathbb{R}, \Delta = u + \beta^{-1} \log M_V(\beta)$ and $M = M(\beta, u)$. Then for all $N > 2\beta\Delta M$,*

correlations
$$\log E^X \left[e^{\beta \Delta L_N} \right] \geq \frac{1}{2} \frac{N}{M}. \tag{3.1}$$

Proof. It is observed in [Al08] that $a_N := \beta\Delta + \log E^X \left[e^{\beta\Delta L_N} \right]$ is subadditive in N. Since $a_N / N \to \beta f_a(\beta, u)$, it follows that

$$\beta\Delta + \log E^X \left[e^{\beta\Delta L_N} \right] \geq N\beta f_a(\beta, \Delta) = \frac{N}{M},$$

and the result is immediate. □

The block I_i is called *good* if it satisfies

$$\begin{aligned} \sum_{b \in I_i^{1/2}} E^X \left[e^{\beta H^u_{[b, b+K_2]}(\mathbf{x}, \mathbf{V})} \big| x_b = 0 \right] \ &> \ \frac{1}{2} \sum_{b \in I_i^{1/2}} E^V E^X \left[e^{\beta H^u_{[b, b+K_2]}(\mathbf{x}, \mathbf{V})} \big| x_b = 0 \right] \\ &= \ \frac{|I_i^{1/2}|}{2} E^X \left[e^{\beta\Delta L_{K_2}} \right], \end{aligned}$$

and *bad* otherwise. Let $p^V_{good} := P^V\, (I_i \text{ is good})$ and $p^V_{bad} := P^V\, (I_i \text{ is bad})$.

good **Proposition 3.2.** *For K_1, K_2 satisfying* (2.4), (2.5) *we have $p^V_{good} > 1/2$.*

Proof. By Chebyshev's inequality,

$$\begin{aligned} p^V_{bad} \ &\leq \ 4 \frac{\mathrm{Var}^V \left(\sum_{b \in I_i^{1/2}} E^X \left[e^{\beta H^u_{[b, b+K_2]}(\mathbf{x}, \mathbf{V})} \big| x_b = 0 \right] \right)}{\left(\sum_{b \in I_i^{1/2}} E^V E^X \left[e^{\beta H^u_{[b, b+K_2]}(\mathbf{x}, \mathbf{V})} \big| x_b = 0 \right] \right)^2} \\ &< \ 4 \frac{\sum_{b, b' \in I_i^{1/2}} 1_{|b - b'| \leq K_2} E^V E^{X, X'} \left[e^{\beta H^u_{[b, b+K_2]}(\mathbf{x}, \mathbf{V}) + \beta H^u_{[b', b'+K_2]}(\mathbf{x}', \mathbf{V})} \,\big|\, x_b = x'_{b'} = 0 \right]}{\sum_{b, b' \in I_i^{1/2}} E^{V, V'} E^{X, X'} \left[e^{\beta H^u_{[b, b+K_2]}(\mathbf{x}, \mathbf{V})(X) + \beta H^u_{[b', b'+K_2]}(\mathbf{x}', \mathbf{V}')} \,\big|\, x_b = x'_{b'} = 0 \right]}. \end{aligned}$$

Here we used the fact that whenever the two independent paths $\mathbf{x}, \mathbf{x}'$ visit zero at points b, b' such that $|b-b'| > K_2$, then the energies $H^u_{[b,b+K_2]}(\mathbf{x}, \mathbf{V})$ and $H^u_{[b',b'+K_2]}(\mathbf{x}', \mathbf{V})$ are independent.

An easy calculation shows that the above is equal to

$$
\begin{aligned}
&4\,\frac{\sum_{b,b'\in I_i^{1/2}} 1_{|b-b'|\le K_2} E^{X,X'}\left[e^{\beta\Delta(L^X_{[b,b+K_2]}+L^{X'}_{[b',b'+K_2]})}e^{\Lambda_V(\beta)B^{X,X'}_{[b,b+K_2]\cap[b',b'+K_2]}}\,\Big|\, x_b = x'_{b'} = 0\right]}{\sum_{b,b'\in I_i^{1/2}} E^{X,X'}\left[e^{\beta\Delta(L^X_{[b,b+K_2]}+L^{X'}_{[b',b'+K_2]})}\,\Big|\, x_b = x'_{b'} = 0\right]}\\
&\le 4\,\frac{\sum_{b,b'\in I_i^{1/2}} 1_{|b-b'|\le K_2}\; e^{\Lambda_V(\beta)K_2}\; E^{X,X'}\left[e^{\beta\Delta(L^X_{[b,b+K_2]}+L^{X'}_{[b',b'+K_2]})}\,\Big|\, x_b = x'_{b'} = 0\right]}{\sum_{b,b'\in I_i^{1/2}} E^{X,X'}\left[e^{\beta\Delta(L^X_{[b,b+K_2]}+L^{X'}_{[b',b'+K_2]})}\,\Big|\, x_b = x'_{b'} = 0\right]}\\
&= \frac{4}{|I_i^{1/2}|^2}\sum_{b,b'\in I_i^{1/2}} 1_{|b-b'|\le K_2}\; e^{\Lambda_V(\beta)K_2}\\
&< \frac{32K_2}{K_1}e^{\Lambda_V(\beta)K_2}\\
&< \frac{1}{K_1}e^{2\Lambda_V(\beta)K_2}\\
&< \frac{1}{2},
\end{aligned}
$$

for K_1, K_2 satsifying (2.4), (2.5). In the third line we have used the fact that the expectations in the second line do not depend on b and b'. $\square$

We now return to the proof of Theorem 1.1. Let

$$J_N := \{i \le N\colon I_i \text{ is good}\} \cup \{0\} = \{i_1 < \cdots < i_{|J_N|}\}.$$

By convention we let $I_{|J_N|+1} := \{N\}$. Under P^V, the sequence $(i_j - i_{j-1})_{j\ge 2}$ is an i.i.d. sequence of geometric random variables with parameter p^V_{good}.

We denote by $\Xi^{J_N}_{NK_1} = \Xi^{J_N}_{NK_1}(\mathbf{V})$ the set of paths that belong to Ξ_{NK_1} and make no returns to 0 in bad blocks after the first block. In the following computation a_j and b_j are the starting and ending points, respectively, of the excursion from I_{i_j} to $I_{i_{j+1}}$. For notational convenience we denote by $p_n = P^X(\mathcal{E} = n)$, as this is introduced in (1.1). By convention we set $b_0 := 0$ and $b_{|J_N|} := N$. Also, $Z_{NK_1}\left(\Xi^{J_N}_{NK_1}\right)$ denotes the partition function restricted on the set of paths $\Xi_{NK_1}^{J_N}$. We then have

$$Z_{NK_1}\left(\Xi_{NK_1}^{J_N}\right) = \sum_{a_1\le K_2}\sum_{b_1\in I_{i_2}^{1/2}}\sum_{a_2-b_1\le K_2}\cdots\sum_{b_{|J_N|-1}\in I_{i_{|J_N|}}^{1/2}}\sum_{a_{|J_N|}-b_{|J_N|-1}\le K_2}$$

$$\prod_{j=1}^{|J_N|} E^X\left[e^{\beta H^u_{[b_{j-1},a_j]}(\mathbf{x},\mathbf{V})}\delta_0(x_{a_j})\,\middle|\, x_{b_{j-1}}=0\right] p_{b_j-a_j}$$

$$\ge \sum_{a_1\le K_2}\sum_{b_1\in I_{i_2}^{1/2}}\sum_{a_2-b_1\le K_2}\cdots\sum_{b_{|J_N|-1}\in I_{i_{|J_N|}}^{1/2}}\sum_{a_{|J_N|}-b_{|J_N|-1}\le K_2}$$

$$\prod_{j=1}^{|J_N|} E^X\left[e^{\beta H^u_{[b_{j-1},a_j]}(\mathbf{x},\mathbf{V})};\ \sigma_{[b_{j-1},b_{j-1}+K_2]}=a_j\,\middle|\, x_{b_{j-1}}=0\right] p_{b_j-a_j}.$$

Moreover, on the set $\{\sigma_{[b_{j-1},b_{j-1}+K_2]}=a_j\}$ we have that $H^u_{[b_{j-1},a_j]}(\mathbf{x},\mathbf{V}) = H^u_{[b_{j-1},b_{j-1}+K_2]}(\mathbf{x},\mathbf{V})$, and therefore for some C the above is bounded below by

$$\sum_{a_1\le K_2}\sum_{b_1\in I_{i_2}^{1/2}}\sum_{a_2-b_1\le K_2}\cdots\sum_{b_{|J_N|-1}\in I_{|J_N|}^{1/2}}\sum_{a_{|J_N|}-b_{|J_N|-1}\le K_2}$$

$$\prod_{j=1}^{|J_N|} E^X\left[e^{\beta H^u_{[b_{j-1},b_{j-1}+K_2]}(\mathbf{x},\mathbf{V})};\ \sigma_{[b_{j-1},b_{j-1}+K_2]}=a_j\,\middle|\, x_{b_{j-1}}=0\right] \min_{a\in I_{i_j}^{3/4},\, b\in I_{i_{j+1}}^{1/2}} p_{b-a}$$

$$= E^X\left[e^{\beta H^u_{[b_0,b_0+K_2]}(\mathbf{x},\mathbf{V})}\right] \min_{a_1\in I_{i_1}^{3/4},\, b_1\in I_{i_2}^{1/2}} p_{b_j-a_j}$$

$$\cdot\prod_{j=2}^{|J_N|}\sum_{b_{j-1}\in I_{i_j}^{1/2}} E^X\left[e^{\beta H^u_{[b_{j-1},b_{j-1}+K_2]}(\mathbf{x},\mathbf{V})}\,\middle|\, x_{b_{j-1}}=0\right] \min_{a\in I_{i_j}^{3/4},\, b\in I_{i_{j+1}}^{1/2}} p_{b-a}$$

$$\ge E^X\left[e^{\beta H^u_{[b_0,b_0+K_2]}(\mathbf{x},\mathbf{V})}\right]\left(\min_{a\in I_{i_1}^{3/4},\, b\in I_{i_2}^{1/2}} p_{b-a}\right)\prod_{j=2}^{|J_N|}\frac{|I_{i_j}^{1/2}|}{2}E^X\left[e^{\beta\Delta L_{K_2}}\right]\min_{a\in I_{i_j}^{3/4},\, b\in I_{i_{j+1}}^{1/2}} p_{b-a}$$

$$\ge E^X\left[e^{\beta H^u_{[b_0,b_0+K_2]}(\mathbf{x},\mathbf{V})}\right]\ C\frac{\varphi\big((i_2-i_1+1)K_1\big)}{(i_2-i_1+1)K_1}$$

$$\cdot\prod_{j=2}^{|J_N|}\left(C\frac{\varphi\big((i_{j+1}-i_j+1)K_1\big)}{(i_{j+1}-i_j+1)K_1}\,\frac{|I_{i_j}^{1/2}|}{2}\ \ E^X\left[e^{\beta\Delta L_{K_2}}\right]\right)$$

$$= \frac{1}{K_1}E^X\left[e^{\beta H^u_{[b_0,b_0+K_2]}(\mathbf{x},\mathbf{V})}\right]\left[\prod_{j=1}^{|J_N|} C\frac{\varphi\big((i_{j+1}-i_j+1)K_1\big)}{4(i_{j+1}-i_j+1)}\right]\ \left(E^X\left[e^{\beta\Delta L_{K_2}}\right]\right)^{|J_N|-1}.$$

In the second inequality we used the fact that the interval I_{i_j} is good, while the last equality makes essential use of $c=1$ in the cancellation of factors K_1. We then have that

$$\begin{aligned}
\frac{1}{NK_1}\log Z_{NK_1} &\geq \frac{1}{NK_1}\log Z_{NK_1}\left(\Xi_{NK_1}^{J_N}\right)\\
&\geq \frac{1}{NK_1}\log\left(\frac{1}{K_1}E^X\left[e^{\beta H^u_{[b_0,b_0+K_2]}(\mathbf{x},\mathbf{V})}\right]\right)+\frac{|J_N|-1}{NK_1}\log E^X\left[e^{\beta\Delta L_{K_2}}\right]\\
&\quad+\frac{1}{NK_1}\sum_{j=1}^{|J_N|}\log\frac{C\varphi\big((i_{j+1}-i_j+1)K_1\big)}{4(i_{j+1}-i_j+1)}
\end{aligned}$$

Letting $N\to\infty$ we get that the left hand side converges to the quenched free energy $f_q(\beta,u)$, while the right hand side converges to

$$\frac{1}{K_1}p^V_{good}\log E^X\left[e^{\beta\Delta L_{K_2}}\right]+\frac{1}{K_1}p^V_{good}E^V\log\frac{C\varphi\big(i_2K_1\big)}{i_2},$$

where C is a constant different from what appears above. Recall that i_2-1 is a geometric random variable under P^V with parameter p^V_{good}. For K sufficiently large we have

$$C_\varphi:=\inf\left\{\frac{x\varphi(kx)}{\varphi(k)}:x\geq 1,k\geq K\right\}>0,$$

and we may assume $K_1\geq K$. We then have

$$\begin{aligned}
f_q(\beta,u) &\geq \frac{1}{K_1}p^V_{good}\left(\log E^X\left[e^{\beta\Delta L_{K_2}}\right]+E^V\log\frac{CC_\varphi\varphi(K_1)}{i_2^2}\right)\\
&= \frac{1}{K_1}p^V_{good}\left(\log E^X\left[e^{\beta\Delta L_{K_2}}\right]+\log\Big(CC_\varphi\varphi(K_1)\Big)-2E^V[\log i_2]\right)\\
&\geq \frac{1}{K_1}p^V_{good}\left(\log E^X\left[e^{\beta\Delta L_{K_2}}\right]+\log\Big(CC_\varphi\varphi(K_1)\Big)-2\log\left(\frac{1}{p^V_{good}}+1\right)\right)
\end{aligned}$$

and by Lemma 3.1 and Proposition 3.2 this is bounded below by

scale2.1

$$\frac{1}{2K_1}\left(\frac{K_2}{2M}+\log\Big(CC_\varphi\varphi(K_1)\Big)-2\log 3\right). \tag{3.2}$$

Then using (2.4), (2.5) we get that provided M is sufficiently large, i.e. Δ is small,

$$f_q(\beta,u)>\frac{1}{2K_1}\left(\frac{K_2}{4M}+\log\frac{CC_\varphi}{9}\right)>0.$$

□

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