THE STRONG WEAK CONVERGENCE OF THE QUASI-EA

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ABSTRACT. In this paper we investigate the convergence of a novel simulation scheme to the target diffusion process. This scheme, the Quasi-EA, is closely related to the Exact Algorithm (EA) for diffusion processes, as it is obtained by neglecting the rejection step in EA. We prove the existence of a myopic coupling between the Quasi-EA and the diffusion. Moreover, an upper bound for the coupling probability is given. Consequently we establish the convergence of the Quasi-EA to the diffusion with respect to the total variation distance.

1. INTRODUCTION

In this paper, we shall present two convergence results about a novel simulation scheme to the target diffusion process. This scheme is closely related to the Exact Algorithm (EA) for the simulation of diffusion process which was introduced in Beskos and Roberts [2005]. In particular, the simulation scheme we consider is essentially EA without the acceptance-rejection correction.

This scheme (which we call the Quasi-EA) is studied for two reasons. Firstly we are interested in the properties of Quasi-EA as a simulation scheme in its own right. Secondly a thorough understanding of Quasi-EA contributes to a fuller understanding of EA scheme itself.

Its main appeal is that it allows for the exact simulation (i.e. free from any time discretisation error) of any skeleton of the diffusion sample path. Moreover, it is possible to simulate exactly from some classes of path-dependent functionals of the diffusion process. Because EA plays a central rule in our work, we briefly introduce the main ideas behind EA. For a more exhaustive exposition of EA we refer to Beskos et al. [2006a] and Beskos et al. [2006b].

We consider the diffusion process Y a unique strong solution of the Stochastic Differential Equation (SDE)

$$\begin{aligned} dY_t &= b\left(Y_t\right)dt + \sigma\left(Y_t\right)dB_t \qquad\qquad 0 \le t \le T \\ Y_0 &= y \end{aligned} \tag{1}$$

B is the scalar Brownian Motion (BM) on the bounded time interval $[0, T]$ and y is the initial condition. The drift coefficient b and the diffusion coefficient σ are implicitly assumed to satisfy the proper conditions that imply the existence and uniqueness of such diffusion Y.

The mild assumption that σ is continuously differentiable and strictly positive guarantees the existence and uniqueness of a bjective function η such that the

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transformed diffusion process $X_t := \eta(Y_t)$ satisfies the SDE

$$\begin{aligned} dX_t &= \alpha(X_t)\,dt + dB_t \qquad 0 \le t \le T \\ X_0 &= x := \eta(y) \end{aligned} \tag{2}$$

The drift coefficient α depends on the functional form of b and σ and the diffusion coefficient is the unitary constant function. The SDE (2) is assumed to admit a unique strong and non-explosive solution and we denote with $\mathbb{X}$ the state space of X. From now on this SDE will be our starting point.

Let $\mathbb{Q}_T^x$ and $\mathbb{W}_T^x$ denote the law of the diffusion X and the law of a BM respectively on $[0,T]$ both started at x. From now on the following hypotheses are assumed to hold:

- $\forall x \in \mathbb{X}$ $\mathbb{Q}_T^x \ll \mathbb{W}_T^x$ and the Radon-Nikodym derivative is given by the Girsanov's formula:

$$\frac{d\mathbb{Q}_T^x}{d\mathbb{W}_T^x}(\omega) = \exp\left\{\int_0^T \alpha(\omega_s)\,d\omega_s - \frac{1}{2}\int_0^T \alpha^2(\omega_s)\,ds\right\} \tag{3}$$

- α is continuously differentiable on $\mathbb{X}$;
- $\alpha^2 + \alpha'$ is bounded below on $\mathbb{X}$.

We introduce the Biased Brownian Motion (BBM) Z and its law $\mathbb{Z}_T^x$. This process is defined as a BM on $[0,T]$ started at x and conditioned on having its terminal value Z_T distributed according to the density

$$h_{x,T}(u) := \eta_{x,T} \times \exp\left\{A(u) - \frac{(u-x)^2}{2T}\right\} \tag{4}$$

Here $A(u) := \int_c^u \alpha(r)\,dr$ for some $c \in \mathbb{X}$ and the normalising constant $\eta_{x,T}$ is assumed to be finite. Hence, conditionally on the value of Z_T the process Z is distributed as a Brownian Bridge (BB). Given the hypothesis it is possible to prove that

$$\frac{d\mathbb{Q}_T^x}{d\mathbb{Z}_T^x}(\omega) = \eta_{x,T}\exp\{-A(x)\}\exp\left\{-\int_0^T \frac{\alpha^2+\alpha'}{2}(\omega_s)\,ds\right\} \tag{5}$$

$$\propto \exp\left\{-\int_0^T \phi(\omega_s)\,ds\right\} \le 1 \tag{6}$$

where $\phi(u) := \left(\alpha^2(u) + \alpha'(u)\right)/2 - \inf_{r\in\mathbb{X}}\left(\alpha^2(r) + \alpha'(r)\right)/2$. Equation (6) suggests the use of a rejection sampling algorithm to generate realisations from $\mathbb{Q}_T^x$. However it is not possible to generate a sample from Z, being Z an infinite dimensional variate, and moreover it is not possible to compute analytically the value of the integral in (6). For ease of exposition we only consider the case of EA1, where $\alpha^2 + \alpha'$ is assumed to be bounded. It should be noted that this hypothesis can be weakened or even removed, leading to EA2 (Beskos et al. [2006a]) and to EA3 (Beskos et al. [2006b]) respectively.

We denote by m the finite supremum of ϕ. Let $\Phi = (\mathcal{X}, \Psi)$ be a unit rate Poisson Point Process (PPP) on $[0,T]\times[0,m]$ and let N be the number of points of Φ below $\phi(\omega_s)$ conditionally on a path ω. If the event Γ is defined as $\Gamma(\omega,\Phi) := \{N = 0\}$

it follows that

$$\mathcal{P}r\left[\Gamma|\omega\right] = \exp\left\{-\int_0^T \phi\left(\omega_s\right) ds\right\} \tag{7}$$

This equivalence is an immediate consequence of the properties of PPPs. The lhs of (7) is the probability that no points of Φ fall in the region of the plane bounded by 0 and $\phi(\omega_s)$, a region whose area is given by the integral on the rhs of (7). Thus there is no need to calculate the integral analytically. We have implicitly assumed that it was possible to generate the infinite dimensional variate $\omega \sim \mathbb{Z}_T^x$. However to compute the the acceptance event Γ we only need to know the value of Z on a finite set of random times only, this set corresponding to the time coordinates of the realisations of Φ. It is then possible to exchange the order in which Z and Φ are generated, obtaining Algorithm 1.

Algorithm 1 Exact Algorithm 1 (EA1)

1. SAMPLE $Z_T \sim h_{x,T}$
2. SAMPLE $\Phi = (\mathcal{X}, \Psi)$
3. SAMPLE $\left\{Z_{\chi_j}; 1 \leq j \leq |\mathcal{X}|\right\} \sim \mathbb{Z}_T^x \mid Z_T$
4. IF Γ RETURN $\mathcal{S} := \left\{Z_{\chi_j}; 1 \leq j \leq |\mathcal{X}|\right\}, Z_T$
   ELSE GOTO 1

As already observed, step 3 of Algorithm 1 results in the generation of independent BBs. EA1 returns a skeleton $\mathcal{S}$ distributed according to the true law of the diffusion X. Furthermore, from Beskos et al. [2006a] it is known that the conditional law $\mathbb{Z}_T^x \mid \mathcal{S}$ can be expressed as the product measure of independent BBs. Using this result it is possible to complete $\mathcal{S}$ on any finite arbitrary number of times if required and even to generate (exactly) certain functionals of the path of X.

The remaining part of this paper is organised as follows. In Section 2 the Quasi-EA is motivated and introduced and the connection with EA is examined. In Section 3 the two main theorems are proved. We start by establishing a local convergence result that is further extended by means of the maximal coupling inequality. We show that the Quasi-EA is an accurate continuous time approximation of the law of the diffusion process. In fact we prove the existence of a myopic (sequential) coupling between the diffusion and the simulation scheme. The existence of such a coupling implies the convergence with respect to the total variation distance, a strong form of weak convergence according to Jacka and Roberts [1997]. Section 4 concludes the paper with some remarks about possible future research on the topic and practical considerations about our scheme.

The Euler scheme does not generally converge with respect to the total variation distance, see Peter Glynn and Goodman [2006]. However, under mild technical conditions, the Euler scheme does converge with respect to total variation distance if the diffusion process has a constant diffusion term (see Jacod and Shiryaev [1987]). Genon-Catalot [2007] extended this result to prove that the rate of convergence is of order $\Delta^{1/2}$, where Δ is the length of the (equally spaced) discretisation interval.

Here we focus on the existence of a sequential (or myopic) coupling between the diffusion and the simulation scheme. A sequential coupling is a coupling in which, for each time increment in turn, we try to maximise the probability that the two processes stay together. It is therefore a natural and practical coupling.

Whilst total variation convergence is equivalent to the existence of a coupling, and thus, the existence of a sequential coupling implies total variation convergence, the converse is false, as we shall see for the Euler scheme. To illustrate this, consider

$$dY_t = \alpha dt + dB_t \tag{8}$$

$$dX_t = dB_t \tag{9}$$

where α is a constant and B_t is a scalar BM on $[0,T]$. Both X_t and Y_t start at the same value. Let $[0,T]$ be partitioned in intervals of length Δ. Then by Taylor expansion arguments it is possible to show that the total variation distance between Y_Δ and X_Δ is of order $\mathcal{O}\left(\Delta^{1/2}\right)$, hence the probability of the coupling succeeds is $\left(1-\Delta^{1/2}\right)^{1/\Delta}$ and this last quantity tends to 0 and $\Delta \downarrow 0$.

Moreover, the bound given by the sequential coupling on the total variation distance is not very strict. A trivial example is the Ornstein-Uhlenbeck process. Simple computations show that its Euler discretisation converges with respect to the total variation distance on the single interval $[0,\Delta]$ with rate Δ. Similarly Quasi-EA can be shown to converge with order $\Delta^{3/2}$ on the same interval (implying that the bound (20) in Theorem 1 is quite sharp). We already stated that the Euler discretisation converges on the whole interval $[0,T]$ with rate $\Delta^{1/2}$. However, the coupling inequality of Theorem (2) implies that Quasi-EA converges with the same order $\Delta^{1/2}$.

Finally, it should be observed that the hypothesis used in the derivation of the two main convergence theorems include the conditions that permits the simulation of X using EA3. Because of this we gain more insight into the role of the proposal measure $\mathbb{Z}_T^x$ in the context of EA.

2. The quasi-EA

The main idea behind the construction of Quasi-EA is very simple. In step 2 of Algorithm 1 we sample a PPP with unit rate on $[0,T]\times[0,m]$. The acceptance rate in EA increases as $T \downarrow 0$, as the likelihood that no points from Φ are sampled increases too. Clearly in this eventuality the proposed path is accepted. Similar considerations can be made in the case of EA3 too. As the acceptance rate can be interpreted as a measure of the quality of the proposal measure, we develop a scheme that always accept the proposal variate distributed as $\mathbb{Z}_T^x$. Given the previous considerations, this approximation is accurate only if T is quite small. Hence the time interval $[0,T]$ is partitioned into smaller intervals on which the scheme is applied sequentially.

We now describe the scheme more precisely. The time interval $[0,T]$ is divided into n smaller intervals having the same length $\Delta = T/n$. The continuous time scheme Y is defined by the following equations

$$Y_0 = x \tag{10}$$

$$Y_{i\Delta} \sim h_{Y_{i\Delta},\Delta} \qquad (i = 1,\cdots,n) \tag{11}$$

$$Y_s \sim \mathbb{BB}\left(Y_{i\Delta}, Y_{(i+1)\Delta}, \Delta\right) \qquad (i\Delta < s < (i+1)\Delta) \tag{12}$$

where $\mathbb{BB}(x, dy, t)$ is the measure of a BB starting at $(0,x)$ and ending at (t,dy). It can be easily seen the process Y thus defined consists of n sequential BBMs. The simulation of the Quasi-EA involves the sampling of the sequence of random variables in (11) only. However we are going to prove a stronger result than the convergence of the discretized process $\{Y_{i\Delta}; i = 0, \cdots, n\}$ to the diffusion process. We shall prove that the law of the continuous time process Y defined by equations (10)-(12) converges with respect to the total variation distance to the law of the diffusion process X as $n \uparrow \infty$. This result suggests that the BBs are good process to *fill-in the gaps* between the simulated values of the discretized process.

We denote by $\mathbb{Y}_{0,T}^{x,n}$ the probability measure induced by the Quasi-EA scheme Y started at x and consisting of n steps on $[0,T]$. Consistently with the previous notation $\mathbb{Y}_{0,\Delta}^{y}$ denotes the probability measure induced by this scheme on the single step $[0,\Delta]$ when $Y_0 = y$. To sum up

$$\{Y_s; 0 \le s \le T \mid Y_0 = x\} \sim \mathbb{Y}_{0,T}^{x,n} \tag{13}$$

$$\{Y_s; 0 \le s \le \Delta \mid Y_0 = y\} \sim \mathbb{Y}_{0,\Delta}^{y} = \mathbb{Z}_{\Delta}^{y} \tag{14}$$

3. Two convergence results

The two convergence theorems require the two following lemmas.

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, such that for sufficiently small $\Delta > 0$*

$$\frac{1}{\sqrt{2\pi\Delta}} \int_{\mathbb{R}} |f(y)| \, e^{-\frac{(y-x)^2}{2\Delta}} dy < \infty \tag{15}$$

then (for any fixed $x \in \mathbb{R}$)

$$\lim_{\Delta \downarrow 0} \frac{1}{\sqrt{2\pi\Delta}} \int_{\mathbb{R}} f(y) \, e^{-\frac{(y-x)^2}{2\Delta}} dy = f(x) \tag{16}$$

Proof. Let $\varepsilon > 0$, $f(y) = f(y) 1_{\{y \in \overline{B}_\varepsilon(x)\}} + f(y) 1_{\{y \in \overline{B}^c_\varepsilon(x)\}}$ where $\overline{B}_\varepsilon(x)$ is the closed ball centred in x with radius ε and $\overline{B}^c_\varepsilon(x)$ is the complementary set of $\overline{B}_\varepsilon(x)$. Remember that $\mathcal{N}(x, \Delta)$ converges weakly to the Dirac delta function δ_x as it is easily proved via characteristic function arguments. Additionally, $f(y) 1_{\{y \in \overline{B}_\varepsilon(x)\}}$ is bounded and the measure with respect to δ_x of the set of its points of discontinuity is zero. Using the Skorohod representation theorem and the bounded convergence theorem, it follows that

$$\lim_{\Delta \downarrow 0} \frac{1}{\sqrt{2\pi\Delta}} \int_{\mathbb{R}} f(y) 1_{\{y \in \overline{B}_\varepsilon(x)\}} e^{-\frac{(y-x)^2}{2\Delta}} dy = f(x)$$

As $\frac{|f(y)|}{\sqrt{2\pi\Delta}} e^{-\frac{(y-x)^2}{2\Delta}} 1_{\{y \in \overline{B}^c_\varepsilon(x)\}}$ is decreasing in Δ (and integrable for sufficiently small Δ), it is possible to apply the dominated convergence theorem obtaining

$$\begin{aligned} &\lim_{\Delta \downarrow 0} \left| \frac{1}{\sqrt{2\pi\Delta}} \int_{\mathbb{R}} f(y) 1_{\{y \in \overline{B}^c_\varepsilon(x)\}} e^{-\frac{(y-x)^2}{2\Delta}} dy \right| \\ \le &\lim_{\Delta \downarrow 0} \frac{1}{\sqrt{2\pi\Delta}} \int_{\mathbb{R}} |f(y)| 1_{\{y \in \overline{B}^c_\varepsilon(x)\}} e^{-\frac{(y-x)^2}{2\Delta}} dy = 0 \end{aligned}$$

The linearity of the integral ends the proof. □

For ease of exposition we assume that the state space of X is $\mathbb{R}$. This assumption does not have any impact on our results. The following conditions, to be used selectively in the following results, are now introduced

- $(E1)$ $\exists k \in (0,1), \exists c \in \mathbb{R}_+ : \alpha(u) \le c + \frac{k}{T} u \; (u \ge 0) \; and \; \alpha(u) \ge c + \frac{k}{T} u \; (u < 0)$

- $(E2)$ $\exists k \in (0,1), \exists c \in \mathbb{R}_+ : |\alpha(u)| \leq c + \frac{k}{T}|u|, u \in \mathbb{R}$
- $(S1)$ α is twice continuously differentiable on $\mathbb{R}$

Lemma 2. *If condition* $(E1)$ *holds, then* $\forall r > 0$

$$\sup_{0 \leq s \leq T} \mathbb{E}_{X \sim h_{x,s}}[e^{r|X|}] < \infty \tag{17}$$

Proof. From the definition of A we get

$$A(u) \leq r|u| + \frac{k}{2T}u^2 \ (u \in \mathbb{R})$$

and so

$$h_{x,s}(u) \leq \exp\left\{-\frac{(u-x)^2}{2s} + r|u| + \frac{k}{2T}u^2\right\} (u \in \mathbb{R}, 0 \leq s \leq T)$$

If $u \geq 0$

$$h_{x,s}(u) \leq \exp\left\{-\frac{(u-\mu_+)^2}{2\sigma_+^2}\right\} r_+$$

where $\mu_+ = \left(\frac{x+2sr}{1-\frac{sk}{T}}\right), \sigma_+ = \frac{s}{\left(1-\frac{sk}{T}\right)}, r_+ = \exp\left\{-\frac{x^2+x+2sr}{1-\frac{sk}{T}}\right\}$. Similarly if $u < 0$

$$h_{x,s}(u) \leq \exp\left\{-\frac{(u-\mu_-)^2}{2\sigma_-^2}\right\} r_-$$

where $\mu_- = \left(\frac{x-2sr}{1-\frac{sk}{T}}\right), \sigma_+ = \sigma_-, r_- = \exp\left\{-\frac{x^2+x-2sr}{1-\frac{sk}{T}}\right\}$.
As $k \in (0,1)$ it follows that $r_+, r_-, \mu_+, \mu_-, \sigma_+, \sigma_-$ are bounded for $s \in (0,T]$ and the result follows. □

For any two probability measures $\mathbb{M}, \mathbb{N}$ on a measurable space $(E, \mathcal{E})$, let $||\mathbb{M} - \mathbb{N}||$ be their total variation metric, that is

$$||\mathbb{M} - \mathbb{N}|| := \sup_{A \in \mathcal{E}} |\mathbb{M}(A) - \mathbb{N}(A)| \tag{18}$$

We are now ready to state the following localised result:

Theorem 1. *If condition* $(S1)$ *hold, then for any fixed* x *the law of the BBM* $\mathbb{Z}_\Delta^x$ *converges towards the law of the diffusion process* $\mathbb{Q}_\Delta^x$ *with respect to_the total variation metric_as* $\Delta \downarrow 0$ *:*

$$\lim_{\Delta \downarrow 0} ||\mathbb{Z}_\Delta^x - \mathbb{Q}_\Delta^x|| = 0 \tag{19}$$

and for Δ *sufficiently small,* $\varepsilon > 0$

$$||\mathbb{Z}_\Delta^x - \mathbb{Q}_\Delta^x|| \leq k_x \Delta^{3/2-\varepsilon} \tag{20}$$

where the leading order constant k_x *is a continuous function of* x *(i.e. the rate of convergence is at least* $\mathcal{O}_x\left(\Delta^{3/2-\varepsilon}\right)$ *for any* $\varepsilon > 0$*).*

Proof. To ease the notation let $\mathbb{W} = \mathbb{W}_\Delta^x$ and $\mathbb{Q} = \mathbb{Q}_\Delta^x$ in the scope of this proof. We introduce a new probability measure $\mathbb{Q}^{Tr}$ via the following Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{Tr}}{d\mathbb{Q}} = 1_{Tr}/q_{x,\Delta}$$
$$q_{x,\Delta} = \mathbb{Q}[Tr]$$

where the measurable event Tr is defined as $Tr = \left\{\sup_{0\le s\le\Delta}|\omega_s - \omega_0| \le \Delta^{1/2-\varepsilon}\right\}$. Consequently,

$$\begin{aligned}\frac{d\mathbb{Q}^{Tr}}{d\mathbb{Z}} &= 1_{Tr}e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}/d_{\alpha,\Delta}\times q_{x,\Delta}\\ &= 1_{Tr}e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}/dq_{\alpha,\Delta}\\ dq_{x,\Delta} &= \mathbb{E}_{\mathbb{Z}}\left[1_{Tr}e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}\right]\end{aligned}$$

The triangular inequality gives

$$\sup_{A\in\mathcal{C}}|\mathbb{Z}(A)-\mathbb{Q}(A)| \le \sup_{A\in\mathcal{C}}\left|\mathbb{Z}(A)-\mathbb{Q}^{Tr}(A)\right| + \sup_{A\in\mathcal{C}}\left|\mathbb{Q}^{Tr}(A)-\mathbb{Q}(A)\right|$$

We now proceed by establishing the convergence of the two terms on the rhs of the inequality. Concerning the first term we have that

$$\begin{aligned}\mathbb{Z}(A) &= \int_A 1_{Tr} + 1_{Tr^c}d\mathbb{Z}(\omega)\\ \mathbb{Q}^{Tr}(A) &= \frac{1}{dq_{x,\Delta}}\int_A 1_{Tr}e^{-\int_0^t\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}d\mathbb{Z}(\omega)\end{aligned}$$

and trivial computations gives

$$\begin{aligned}&\sup_{A\in\mathcal{C}}\left|\mathbb{Z}(A)-\mathbb{Q}^{Tr}(A)\right|\\ \le&\mathbb{Z}[Tr^c] + \frac{1}{dq_{x,\Delta}}\left\{2\int 1_{Tr}\left|1-e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}\right|d\mathbb{Z}(\omega)+\mathbb{Z}[Tr^c]\right\}\end{aligned}$$

Regarding the second term it suffice to observe that

$$\begin{aligned}\mathbb{Q}(A) &= \frac{1}{d_{x,\Delta}}\int_A e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}d\mathbb{Z}(\omega)\\ \mathbb{Q}^{Tr}(A) &= \frac{1}{d_{x,\Delta}q_{x,\Delta}}\int_A 1_{Tr}e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}d\mathbb{Z}(\omega)\end{aligned}$$

and therefore similar computations leads to

$$\begin{aligned}&\sup_{A\in\mathcal{C}}\left|\mathbb{Q}^{Tr}(A)-\mathbb{Q}(A)\right|\\ \le&\frac{1}{dq_{x,\Delta}}\left\{\mathbb{Z}[Tr^c]+\frac{1}{d_{x,\Delta}}\mathbb{E}_{\mathbb{Z}}\left[1_{Tr^c}e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}\right]\right\}\end{aligned}$$

Combining the two results yields

$$\begin{aligned}&\sup_{A\in\mathcal{C}}|\mathbb{Q}(A)-\mathbb{Z}(A)|\\ \le&\frac{1}{dq_{x,\Delta}}\left\{(2+dq_{x,\Delta})\,\mathbb{Z}[Tr^c]+\left(2+\frac{1}{d_{x,\Delta}}\right)\mathbb{E}_{\mathbb{Z}}\left[1_{Tr^c}e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}\right]\right\}\end{aligned}$$

Moreover $dq_{x,\Delta}\in(0,1)$ and the functions $1_{Tr}e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}$ and 1_{Tr} are both positive and increasing as $\Delta\downarrow 0$. By two applications of the monotone convergence theorem it follows that

$$\lim_{\Delta\downarrow 0}\mathbb{E}_{\mathbb{Z}}\left[1_{Tr}e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}\right]$$
$$=\lim_{\Delta\downarrow 0}\mathbb{Z}\left[Tr\right]$$

As we will shortly show $\mathbb{Z}\left[Tr^c\right]\downarrow 0$ as $\Delta\downarrow 0$, so $dq_{x,\Delta}\uparrow 1$. Similarly an application of the monotone convergence theorem shows that $d_{x,\Delta}\uparrow 1$ as $\Delta\downarrow 0$. As a consequence we can find two positive constants c_1 , c_2 (independent of x) such at that for Δ sufficiently small

$$\sup_{\mathcal{A}\in\mathcal{C}}\left|\mathbb{Q}\left(A\right)-\mathbb{Z}\left(A\right)\right|$$
$$\leq c_1\mathbb{Z}\left[Tr^c\right]+c_2\mathbb{E}_{\mathbb{Z}}\left[1_{Tr^c}e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}\right]$$

All that remains to be checked is the rate of convergence of the two terms of the inequality's rhs. The first one is

$$\mathbb{Z}\left[Tr^c\right]=\frac{1}{\mathbb{E}_{\mathbb{W}}\left[e^{A(\omega_t)}\right]}\mathbb{E}_{\mathbb{W}}\left[Tr^c e^{A(\omega_t)}\right]$$

and an application of the Cauchy-Schwartz inequality to $\mathbb{E}_{\mathbb{W}}\left[Tr^c e^{A(\omega_t)}\right]$ shows that

$$\mathbb{E}_{\mathbb{W}}\left[Tr^c\right]\leq\left(\mathbb{W}\left[Tr^c\right]\right)^{\frac{1}{2}}\frac{\left(\mathbb{E}_{\mathbb{W}}\left[e^{2A(\omega_\Delta)}\right]\right)^{\frac{1}{2}}}{\mathbb{E}_{\mathbb{W}}\left[e^{A(\omega_\Delta)}\right]}$$

From Lemma 1 $\mathbb{E}_{\mathbb{W}}\left[e^{A(\omega_\Delta)}\right]\to e^{A(x)}$ as $\Delta\downarrow 0$ and $\left(\mathbb{E}_{\mathbb{W}}\left[e^{2A(\omega_\Delta)}\right]\right)^{\frac{1}{2}}\to\left(e^{2A(x)}\right)^{\frac{1}{2}}=e^{A(x)}$. Hence we can find a positive constant c_3 (independent of x) so that for Δ sufficiently small

$$\mathbb{E}_{\mathbb{W}}\left[Tr^c\right]\leq c_3\left(\mathbb{W}\left[Tr^c\right]\right)^{\frac{1}{2}}$$

By writing $\mathbb{W}_0$ for the law of the BM started at zero on $[0,\Delta]$, by knowing that if B is a BM , $-B$ is a BM too, and by using the reflection principle, we obtain

$$\mathbb{W}\left[\sup_{0\leq s\leq\Delta}\left|\omega_s-\omega_0\right|>\Delta^{\frac{1}{2}-\varepsilon}\right]$$
$$\leq 4\mathbb{W}_0\left[\omega_\Delta>\Delta^{\frac{1}{2}-\varepsilon}\right]$$

By changing the variable and by applying the bound

$$\int_x^\infty e^{-\frac{y^2}{2}}dy\leq\frac{1}{x^2}e^{-\frac{x^2}{2}}$$

it follows that

$$\left(\mathbb{W}\left[Tr^c\right]\right)^{\frac{1}{2}}\leq c_4\Delta^\varepsilon e^{-\frac{1}{c_4\Delta^{2\varepsilon}}}$$

that decreases exponentially as $\Delta\downarrow 0$.

We now consider $\mathbb{E}_{\mathbb{Z}}\left[1_{Tr}\left(1-e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s)-\inf_\omega\frac{\alpha^2+\alpha'}{2}\right)ds}\right)\right]$.

Let

$$r=\arg\inf_{\omega\in Tr}\inf_{s\in[0,\Delta]}\frac{\alpha^2+\alpha'}{2}\left(\omega_s\right)>-\infty$$

For Δ sufficiently small

$$k_x := \sup_{|y|\le|x|+1} \frac{\left|\frac{\alpha^2+\alpha'}{2}(y) - \frac{\alpha^2+\alpha'}{2}(x)\right|}{|y-x|} \ge \sup_{|x-y|\le\Delta^{1/2-\varepsilon}} \frac{\left|\frac{\alpha^2+\alpha'}{2}(y) - \frac{\alpha^2+\alpha'}{2}(x)\right|}{|y-x|}$$

and as $(S1) \Rightarrow \frac{\alpha^2+\alpha'}{2}$ is locally Lipschitz it follows that k_x is continuous in x and for Δ sufficiently small

$$\left|\frac{\alpha^2+\alpha'}{2}(\omega_s) - \frac{\alpha^2+\alpha'}{2}(r)\right| \le k_x |\omega_s - r| \le 2k_x\Delta^{1/2-\varepsilon}$$

by_the definition of Tr . Finally

$$\mathbb{E}_{\mathbb{Z}}\left[1_{Tr}\left(1 - e^{-\int_0^\Delta\left(\frac{\alpha^2+\alpha'}{2}(\omega_s) - \inf_\omega \frac{\alpha^2+\alpha'}{2}\right)ds}\right)\right] \le 1 - e^{2k_x\Delta^{3/2-\varepsilon}} \le 2k_x\Delta^{3/2-\varepsilon}$$

for sufficiently small Δ. □

The extension of this localised result to the global case relies on the maximal coupling inequality. The coupling method (see Thorisson [2000]) is already prevalent in the field of SDEs, mainly in the multi-dimensional case. Our approach is very similar to that of Odasso [2005]. Given the relevance of the coupling method it is sensible to briefly introduce its basic elements. We recall:

Definition 1. Let $(E,\mathcal{E})$ be a Polish measurable space, and $\mathbb{M},\mathbb{N}$ be two probability measures on $(E,\mathcal{E})$. We state that a probability measure $\hat{\mathbb{P}}$ on $\left(E^2,\mathcal{E}^2\right)$ is a coupling of $(\mathbb{M},\mathbb{N})$ if its marginals are $\mathbb{M}$ and $\mathbb{N}$. We also say that a random object $(\Omega',\mathcal{F}',\mathbb{P}',(X',Y'))$, where $(\Omega',\mathcal{F}',\mathbb{P}')$ is a probability space and (X',Y') is a $\mathcal{F}'/\mathcal{E}^2$-measurable function, is a coupling of $(\mathbb{M},\mathbb{N})$ if the image measure $\mathbb{P}'(X',Y')^{-1}$ is a coupling of $(\mathbb{M},\mathbb{N})$.

The power of the coupling argument comes from the following Lemma

Lemma 3. *Let* $||\mathbb{M}-\mathbb{N}||$ *be the total variation metric, that is*

$$||\mathbb{M}-\mathbb{N}|| := \sup_{A\in\mathcal{E}} |\mathbb{M}(A) - \mathbb{N}(A)| \tag{21}$$

(Coupling Inequality) *For any coupling* $(\Omega',\mathcal{F}',\mathbb{P}',(X',Y'))$ *of* $(\mathbb{M},\mathbb{N})$

$$||\mathbb{M}-\mathbb{N}|| \le \mathbb{P}'[X' \ne Y'] \tag{22}$$

(Maximal Coupling Equality) *There is a coupling* $\left(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}},\left(\hat{X},\hat{Y}\right)\right)$ *of* $(\mathbb{M},\mathbb{N})$ *s.t.*

$$||\mathbb{M}-\mathbb{N}|| = \hat{\mathbb{P}}\left[\hat{X} \ne \hat{Y}\right] \tag{23}$$

This coupling is called the maximal coupling of $(\mathbb{M},\mathbb{N})$.

Theorem 2. *(**Main Convergence Theorem**) If in addition to condition $(S1)$ conditions $(E2)$ and $(S2)$ also hold, where*

- *$(S2)$ α' is sub-quadratic , that is*

$$|\alpha'(u)| \le c\left(1+u^2\right) \qquad (u \in \mathbb{R}) \tag{24}$$

or condition $(E1)$ and $(S3)$ also hold, where

- *$(S3)$ α and α' are sup-exponential , which are:*

$$|\alpha(u)|, |\alpha'(u)| \le c\left(1+e^{c|u|}\right) \qquad (u \in \mathbb{R}) \tag{25}$$

then there exist a (myopic) coupling $\left(\widehat{\mathbb{P}}, \hat{X}, \hat{Y}\right)$ of $\left(\mathbb{Q}^x_{0,T}, \mathbb{Y}^{x,n}_{0,T}\right)$ such that

$$\lim_{n\to\infty} \hat{\mathbb{P}}\left[\hat{X} \neq \hat{Y}\right] = 0 \tag{26}$$

and the rate of convergence is at least $\mathcal{O}\left(\Delta^{1/2-\varepsilon}\right)$ for any $\varepsilon > 0$. As a consequence of the coupling inequality, $\mathbb{Y}^{x,n}_{0,T}$ converges towards $\mathbb{Q}^x_{0,T}$ with respect to the total variation_metric with the same rate of convergence.

Remark 1. If $(E2)$ holds then condition $(S2)$ is a weak assumption. As $\alpha^2+\alpha'$ is bounded below, this additional condition means that the drift coefficient cannot oscillate too quickly as $|u| \to \infty$. Moreover in most diffusion models condition $(E1)$ is satisfied, as otherwise the diffusion would exhibit explosive behaviour.

Proof. We build a probability space $\left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}\right)$ and two measurable functions

$$\begin{aligned}\left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}\right) &\to \tilde{X}(s,x,y) & s \in (0,\Delta] \\ &\to \tilde{Y}(s,x,y) & s \in (0,\Delta]\end{aligned}$$

that define the maximal coupling of $\left(\mathbb{Q}^x_{0,\Delta}, \mathbb{Y}^y_{0,\Delta}\right)$. A coupling of $\left(\mathbb{Q}^x_{0,T}, \mathbb{Y}^{x,n}_{0,T}\right)$ that starts from this maximal coupling is constructed on the single time interval $(0,\Delta]$. The initial step is defined by setting $\hat{X}_0 = \hat{Y}_0 = x$ and $\hat{X}_s = \tilde{X}(s,x,x), \hat{Y}_s = \tilde{Y}(s,x,x)$ for $s \in (0,\Delta]$. Now suppose_that $\left(\hat{X}, \hat{Y}\right)$ is a coupling of $\left(\mathbb{Q}^x_{0,i\Delta}, \mathbb{Y}^{x,n}_{0,i\Delta}\right)$. Let

$$\begin{aligned}\hat{X}_{i\Delta+s} &:= \tilde{X}\left(s, \hat{X}_{i\Delta}, \hat{Y}_{i\Delta}\right) & s \in (0,\Delta] \\ \hat{Y}_{i\Delta+s} &:= \tilde{Y}\left(s, \hat{X}_{i\Delta}, \hat{Y}_{i\Delta}\right) & s \in (0,\Delta]\end{aligned}$$

independent of $\left(\hat{X}, \hat{Y}\right)$ on $[0, i\Delta]$. From the time homogeneity and Markov property of the processes X, Y (limited to the set of times $\{i\Delta; i = 1, \cdots, n-1\}$) it follows that $\left(\hat{X}, \hat{Y}\right)_{s\in(i\Delta,(i+1)\Delta]}$ is a coupling of $\left(\mathbb{Q}^{\hat{X}_{i\Delta}}_{i\Delta,(i+1)\Delta}, \mathbb{Y}^{\hat{Y}_{i\Delta}}_{i\Delta,(i+1)\Delta}\right)$ and that the extended process $\left(\hat{X}, \hat{Y}\right)_{s\in[0,(i+1)\Delta]}$ is a coupling of $\left(\mathbb{Q}^x_{0,(i+1)\Delta}, \mathbb{Y}^{x,n}_{0,(i+1)\Delta}\right)$. The induction step is thus satisfied. No measurability problem arise in the definition of $\tilde{X}, \tilde{Y}$ and they can be chosen to be jointly measurable in (x,y), see Odasso [2005] for the technical details. The coupling inequality yields

$$\begin{aligned}
&\left|\left|\mathbb{Y}_{0,T}^{x_0,n} - \mathbb{Q}_{0,T}^{x_0}\right|\right| \\
\leq &\hat{\mathbb{P}}\left[\hat{X} \neq \hat{Y}\right] = \hat{\mathbb{P}}\left[\exists s \in [0,T] : \hat{X}_s \neq \hat{Y}_s\right] \\
\leq &\hat{\mathbb{P}}\left[\hat{X}_s \neq \hat{Y}_s \, on \, [0,\Delta]\right] \\
&\sum_{i=1}^{n-2} \hat{\mathbb{P}}\left[\hat{X} \neq \hat{Y} \, on \, ((i+1)\Delta, i\Delta] \mid \hat{X}_s = \hat{Y}_s \, on [0,i\Delta]\right] \\
= &\hat{\mathbb{P}}\left[\hat{X}_s \neq \hat{Y}_s \, on \, [0,\Delta]\right] \\
&\sum_{i=0}^{n-2} \hat{\mathbb{P}}\left[\hat{X} \neq \hat{Y} \, on \, ((i+1)\Delta, i\Delta] \mid \hat{X}_{i\Delta} = \hat{Y}_{i\Delta}\right]
\end{aligned}$$

from the structure of the coupling $\left(\hat{X}, \hat{Y}\right)$, and the generic term of this last quantity is exactly the maximal coupling of $\left(\mathbb{Q}_{i\Delta,(i+1)\Delta}^{\hat{X}_{i\Delta}}, \mathbb{Y}_{i\Delta,(i+1)\Delta}^{\hat{Y}_{i\Delta}}\right)$. Hence, it is possible to use the maximal coupling equality. By defining a family of sub-probability measures on $\mathbb{R}$ by $\mathbb{S}_i(\mathcal{A}) := \hat{\mathbb{P}}\left[\hat{X}_{i\Delta} = \hat{Y}_{i\Delta} \in \mathcal{A}\right]$ and by using this last consideration we obtain by the disintegration of the conditional probability that

$$\begin{aligned}
&\left|\left|\mathbb{Y}_{0,T}^{x_0,n} - \mathbb{Q}_{0,T}^{x_0}\right|\right| \\
\leq &\left|\left|\mathbb{Y}_{0,\Delta}^{x_0} - \mathbb{Q}_{0,\Delta}^{x_0}\right|\right| + \sum_{i=0}^{n-2} \int \left|\left|\mathbb{Y}_{i\Delta,(i+1)\Delta}^{s} - \mathbb{Q}_{i\Delta,(i+1)\Delta}^{s}\right|\right| d\mathbb{S}_i(s) \\
= &\left|\left|\mathbb{Y}_{0,\Delta}^{x_0} - \mathbb{Q}_{0,\Delta}^{x_0}\right|\right| + \sum_{i=0}^{n-2} \int \left|\left|\mathbb{Y}_{0,\Delta}^{s} - \mathbb{Q}_{0,\Delta}^{s}\right|\right| d\mathbb{S}_i(s) \\
\leq &k_{x_0}\Delta^{3/2-\varepsilon} + \Delta^{3/2-\varepsilon} \sum_{i=0}^{n-2} \int k_s d\mathbb{S}_i(s)
\end{aligned}$$

As in the proof of Theorem 1

$$k_x := \sup_{|y| \leq |x|+1} \frac{\left|\frac{\alpha^2+\alpha'}{2}(y) - \frac{\alpha^2+\alpha'}{2}(x)\right|}{|y-x|}$$

We see that the behaviour in the tails as $|x| \to \infty$ is determined by

$$\overline{\lim}_{|x|\to\infty} \tilde{k}_x := \overline{\lim}_{|x|\to\infty} \frac{\left|\frac{\alpha^2+\alpha'}{2}(x)\right|}{|x|}$$

If condition $(S2)$ holds the positive function $\tilde{k}_x$ can diverge_at most linearly. So $k_x \leq k(1+|x|)$ and

$$\int k_s d\mathbb{S}_i(s) \leq k + k\int |s| \, d\mathbb{S}_i(s) \leq k + k\int |s| \, d\mathbb{Q}_{i\Delta}^{x_0}(s)$$

where the last integral is the absolute momentum of $X_{i\Delta}$: $\mathbb{E}\left[|X_{i\Delta}| \mid X_0 = x_0\right]$. It is a known result that $(E2)$ implies

$$\mathbb{E}\left[|X_{i\Delta}| \mid X_0 = x_0\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq T} |X_s|^2 \mid X_0 = x_0\right] < \infty$$

$\forall i \in \mathbb{N}$. Therefore

$$\begin{aligned}&\left|\left|\mathbb{Y}^{x_0,n}_{0,T} - \mathbb{Q}^{x_0}_{0,T}\right|\right| \\ \leq & k_{x_0}\Delta^{3/2-\varepsilon} + \Delta^{3/2-\varepsilon}\sum_{i=0}^{n-2} k\left(1 + \mathbb{E}\left[\sup_{0\leq s\leq T}|X_s|^2 \mid X_0 = x_0\right]\right) \\ \leq & d\Delta^{3/2-\varepsilon}n = \frac{d}{T}\Delta^{1/2-\varepsilon}\end{aligned}$$

for Δ sufficiently small.

If instead condition $(S3)$ holds, by looking at $\overline{\lim}_{|x|\to\infty}\tilde{k}_x$, we obtain the sub-exponential growth condition on k_x, that is

$$k_x \leq k\left(1 + e^{k|x|}\right) \; (x \in \mathbb{R})$$

As a consequence

$$\int k_s d\mathbb{S}_i(s) \leq k + k\int e^{k|s|}d\mathbb{S}_i(s) \leq k + k\int e^{k|s|}d\mathbb{Z}^{x_0}_{i\Delta}(s)$$

Again, the last integral is $\mathbb{E}\left[e^{k|Y_{i\Delta}|} \mid Y_0 = x_0\right]$. As condition $(E1)$ holds, Lemma 2 implies that

$$\sup_{i=1,\cdots,n-1}\mathbb{E}\left[e^{k|Y_{i\Delta}|} \mid Y_0 = x_0\right] < \infty$$

Finally

$$\begin{aligned}&\left|\left|\mathbb{Y}^{x_0,n}_{0,T} - \mathbb{Q}^{x_0}_{0,T}\right|\right| \\ \leq & k_{x_0}\Delta^{3/2-\varepsilon} + \Delta^{3/2-\varepsilon}\sum_{i=0}^{n-2} k\left(1 + \sup_{i=1,\cdots,n-1}\mathbb{E}\left[e^{k|Y_{i\Delta}|} \mid Y_0 = x_0\right]\right) \\ \leq & d\Delta^{3/2-\varepsilon}n = \frac{d}{T}\Delta^{1/2-\varepsilon}\end{aligned}$$

□

4. Conclusion

In this paper we proved two convergence results about the Quasi-EA simulation scheme.

We shown the convergence of the law of the BBM Z to the law of the diffusion process X when both are started at the same value and the time interval $[0, \Delta]$ shrinks to zero. The convergence is obtained with respect to the total variation distance and an upper bound for the rate of the convergence is shown to be $\mathcal{O}_x\left(\Delta^{3/2-\varepsilon}\right)$ $\forall\varepsilon > 0$. The notation underlines that this speed of convergence is not necessarily uniform in x.

We also extend this convergence to the global case of a fixed time interval $[0, T]$. In this case $[0, T]$ is uniformly partitioned in n intervals and the convergence is obtained as $n \uparrow \infty$. The main difficulty that the starting point for X and Y on each single interval is not the same anymore is overcome using the coupling method.

We are thus able to construct a successful myopic coupling of the Quasi-EA and the diffusion. Consequently we obtain the convergence with respect to the total variation distance and an upper bound for the rate of convergence $\mathcal{O}\left(\Delta^{1/2-\varepsilon}\right)$ $\forall\varepsilon >$ 0.

Very efficient algorithms to sample from the parametric family of densities $\{h_{x,T}\}_{x\in\mathbb{X}}$ are introduced in Peluchetti [2007]. However, while the quasi-EA is more efficient than the Euler scheme, a brief simulation study suggests that Predictor-Corrector schemes result in a more accurate simulation.

As already noted, the hypothesis used in the derivation of these results include the conditions that permits the simulation of X using EA3. It is possible to weaken the condition used in our work and prove the convergence of the Quasi-EA even in models where EA3 can not be applied, and this will be the focus of future research. This could be worked out in future research. The main contribute in this paper is to obtain an insight into the role of the BBM Z in the context of EA.

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