COHOMOLOGY AND THE
SUBGROUP STRUCTURE OF A
FINITE SOLUBLE GROUP.

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Acknowledgements.

Declaration.

The work described in this thesis is original except where the contrary is clearly stated.

SUMMARY

The main topic of this thesis is the discovery and study of a cohomological property of the subgroups called $\mathcal{F}$-normalizers in finite soluble groups; namely, the property that with certain coefficient modules the restriction map in cohomology from a soluble group to its $\mathcal{F}$-normalizers vanishes in non-zero degrees. Chapter 3 is devoted to a proof of this fact. It turns out that in some classes of soluble groups the $\mathcal{F}$-normalizers are characterized by this property, and the study of these classes occupies Chapters 4 and 5. Various connections with cohomology and group theory are found; the approach seems to offer some unification of disparate results from the theory of soluble groups.

The relation between $\mathcal{F}$-normalizers and cohomology was discovered through study of the work of Jacques Thévenaz on the action of a soluble group on its lattice of subgroups. Chapter 1 is a summary of this work and its background, and is included to provide motivation. A link with the rest of the thesis arises through a new result, in which certain subgroups crucial to Thévenaz's analysis of soluble groups are shown to coincide with their system normalizers. A proof of this is given in Chapter 2, which also contains some miscellaneous results on soluble groups from the class considered by Thévenaz, comprising those groups whose lattices of subgroups are complemented.

The problem of characterizing $\mathcal{F}$-normalizers in soluble groups by the results of Chapter 3 is proposed in Chapter 4, and in Chapters 4 and 5 two essentially different approaches to this problem are taken, which lead to partial solutions in different sets of circumstances. In Chapter 4, the first cohomology groups of soluble groups are considered, and an application is given to a proof of a recent theorem of Volkmar Welker described in Chapter 1 on the homotopy type of the partially ordered set of conjugacy classes of subgroups of a soluble group. Another application is to the study of local conjugacy of subgroups of soluble groups, and these are combined in a result which shows that the set of conjugacy classes considered by Welker is homotopy equivalent to an analogous set obtained from local conjugacy classes.

In Chapter 5 some known results on the local conjugacy of $\mathcal{F}$-normalizers are exhibited, as evidence for a cohomological characterization of these subgroups. The results are used to study groups of p-length one by a 'local' analysis, whereby the problem of characterizing $\mathcal{F}$-normalizers is translated into a question concerning the action of automorphisms on the cohomology rings of p-groups. In the study of this question a natural place to start is the case of abelian groups, whose cohomology rings are known; calculations in this case lead to results on the $\mathcal{F}$-normalizers of A-groups. The question is then considered for other p-groups, revealing an elegant relationship between the cohomology of p-groups, the theory of varieties, and some well-known results on automorphisms of p-groups.

CHAPTER 1.

The questions which are considered in this thesis arose from the work of Jacques Thévenaz and others on topological aspects of the structure of the lattice of subgroups of a finite soluble group. In order to provide a context for the original part of the thesis, we begin in this expository chapter by briefly describing the results of Thévenaz and their background.

1.1 The order complex of a partially ordered set.

Let $\Lambda^*(G)$ denote the set of subgroups of the group G. The relation of inclusion between subgroups of G is clearly a partial ordering of $\Lambda^*(G)$, which is in fact a lattice with this partial ordering, where the meet and join of two subgroups are respectively their intersection and the subgroup they generate. We reserve the simpler notation $\Lambda(G)$ for the subset of $\Lambda^*(G)$ which consists of the *proper, nontrivial* subgroups of G; in other words

$$\Lambda(G) = \Lambda^*(G) - \left\{ 1, G \right\}.$$

Unlike $\Lambda^*(G)$, the partially ordered set $\Lambda(G)$ is not a lattice in general, because two proper, nontrivial subgroups of G may have their intersection equal to 1, or their join equal to G, or both. (In fact it is easy to see that $\Lambda(G)$ is a lattice if and only if G is cyclic of prime-power order.)

The study of partially ordered sets is a branch of combinatorics, and many concepts are available which can be applied, in particular, to the lattices $\Lambda^*(G)$ of subsets of groups. An early result of this kind is the well – known theorem of Iwasawa [1], which asserts that a finite group G is supersoluble if and only if all the *maximal chains* in $\Lambda^*(G)$ are *of the same length*. (A chain in a partially ordered set is just a totally ordered subset.) The monograph of Suzuki [1] contains

many interesting results of this kind.

However, a slightly different approach to the study of the lattice $\Lambda^*(G)$ has proved particularly fruitful. This is to use a construction first given by Folkman [1] whereby a simplicial complex called the *order complex* may be associated in a natural way to any partially ordered set:

Definition 1.1.1. (The order complex.) Let $\mathcal{P}$ be a partially ordered set. Then the order complex $|\mathcal{P}|$ is the simplicial complex whose n-simplices, for each $n \geq 0$, are the chains in $\mathcal{P}$ of length $n + 1$, and whose face relations are the relations of inclusion between these chains.

Through this construction, topological ideas may be used to study partially ordered sets. In particular, one can say that two partially ordered sets are homotopy equivalent, meaning that there is a homotopy equivalence between their order complexes. An order-preserving or reversing map between two partially ordered sets induces a simplicial map between their order complexes; given two such maps, one can therefore ask if they are homotopic. An elementary but useful criterion for such a homotopy to exist is the following lemma, due to Quillen:

Lemma 1.1.2 (Hawkes, Isaacs and Ozaydin [1, Lemma 9.3]). *If $\alpha, \beta: \mathcal{P} \to \mathcal{Q}$ are order-preserving maps such that $\alpha(x) \leq \beta(x)$ for all $x \in \mathcal{P}$, then α is homotopic to β.*

The simplicial homology of the order complex carries combinatorial information about the original partially ordered set; most importantly, if the set is finite then the order complex has an Euler characteristic $\chi(|\mathcal{P}|)$, defined as the alternating sum of the dimensions of the rational simplicial homology groups of $|\mathcal{P}|$. This definition makes it clear that the Euler characteristic depends only on

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the homotopy type of $|\mathcal{P}|$, but on the other hand $\chi(|\mathcal{P}|)$ has a well-known combinatorial interpretation in terms of the Möbius number of a partially ordered set associated to $\mathcal{P}$. (For the definition of the Möbius number, and a summary of the history of this concept, see Hawkes, Isaacs and Ozaydin [1, Section 1] and the references given there, especially Rota [1].)

Proposition 1.1.3 (See Hawkes, Isaacs and Ozaydin [1 Section 9]). *Let $\mathcal{P}^*$ be a finite partially ordered set with a smallest element a and a largest element b. Let $\mathcal{P}$ be the interior of $\mathcal{P}^*$, i.e $\mathcal{P} = \mathcal{P}^* - \{a, b\}$. Then*

$$\mu(\mathcal{P}^*) = \tilde{\chi}(|\mathcal{P}|)$$

where $\tilde{\chi}(|\mathcal{P}|) = \chi(|\mathcal{P}|) - 1$ is the reduced Euler characteristic of $|\mathcal{P}|$.

Recall that a *Galois connection* between partially ordered sets $\mathcal{P}$ and $\mathcal{Q}$ is a pair of order-reversing maps α: $\mathcal{P} \to \mathcal{Q}$ and β: $\mathcal{Q} \to \mathcal{P}$ such that $\beta \circ \alpha$: $\mathcal{P} \to \mathcal{P}$ and $\alpha \circ \beta$: $\mathcal{Q} \to \mathcal{Q}$ are both increasing maps. Rota ([1, Theorem 4.1]) shows that the Möbius numbers of two partially ordered sets with a Galois connection are equal, but the order complex allows more information to be expressed:

Theorem 1.1.4 (Hawkes, Isaacs and Ozaydin [1, Proposition 9.5]). *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite partially ordered sets. If α and β constitute a Galois connection between $\mathcal{P}$ and $\mathcal{Q}$ then the maps they induce between $|\mathcal{P}|$ and $|\mathcal{Q}|$ are homotopy equivalences.*

Proof. $\beta \circ \alpha$ and $\alpha \circ \beta$ are homotopic to the identity maps on $\mathcal{P}$ and $\mathcal{Q}$ respectively, by Lemma 1.1.2. $\square$

Corollary 1.1.5. *Let $\mathcal{P}$ be a partially ordered set containing either a smallest or a largest element. Then $|\mathcal{P}|$ is contractible.*

Proof. If $b \in \mathcal{P}$ is a largest element then the unique map $\mathcal{P} \to \{b\}$ and the inclusion map $\{b\} \to \mathcal{P}$ constitute a Galois connection. The order complex $|\mathcal{P}|$ is therefore homotopy equivalent to the point $|b|$. If $\mathcal{P}$ has a smallest element a, then a is the largest element in the opposite partially ordered set $\mathcal{P}^{opp}$, whose order complex $|\mathcal{P}^{opp}|$ is clearly identical with $|\mathcal{P}|$. $\square$

In the next section we discuss the application of these ideas to the specific case of the order complex $|\Lambda(G)|$, where G is a finite group. The reduced Euler characteristic of this space is, by Proposition 1.1.3, equal to the Möbius number of the lattice $\Lambda^*(G)$, which is usually referred to as the Möbius number of G, written $\mu(G)$. Notice that we study $|\Lambda(G)|$ and not $|\Lambda^*(G)|$ – the latter space is contractible, by Corollary 1.1.5.

1.2 The homotopy type of $|\Lambda(G)|$ for soluble G.

In a paper [1] which predates the introduction of the order complex, Gaschütz gave implicitly a remarkable formula for the Möbius number $\mu(G)$ of any finite *soluble* group G. This formula is stated and proved explicitly by Kratzer and Thévenaz [1, Théoremè 2.6] and by Hawkes, Isaacs and Ozaydin [1, Corollary 3.4]. We select a chief series for G,

$$1 = N_0 < N_1 < \ldots < N_n = G,$$

and for each i with $1 \leq i \leq n$, we let c_i be the number of subgroups of G which are (relative) complements of the chief factor N_i/N_{i-1}. The formula of Gaschütz is given in the following theorem.

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Theorem 1.2.1 (Gaschütz). *Let G be a finite soluble group, and let numbers c_i be defined as above. Then the Möbius number $\mu(G)$ is given by the following formula:*

$$\mu(G) = (-1)^n c_1 c_2 \ldots c_n.$$

Note that in particular the product on the right must be independent of the chief series chosen, although this is not in itself obvious. Clearly the Möbius number $\mu(G)$ is non-zero if and only if each of the numbers c_i on the right is non-zero; in other words, for $\mu(G)$ to be non-zero it is necessary and sufficient that each factor of the chief series used in the formula above should have at least one complement, and since this series was arbitrary we may simply say that all the chief factors of G must be complemented.

Gaschütz's formula gives the reduced Euler characteristic $\tilde{\chi}\big(|\Lambda(G)|\big) = \mu(G)$ in terms of the internal structure of G, when G is a soluble group. It is natural to ask what other topological information about $|\Lambda(G)|$ can be obtained from the structure of G, and this question was addressed and in a sense completely answered by Kratzer and Thévenaz, who determined the homotopy type of $|\Lambda(G)|$ for any finite soluble group G. The answer turns out to be extremely simple:

Theorem 1.2.2 (Kratzer and Thévenaz [2, Corollaire 4.10]). *Let G be a finite soluble group. Let n be the chief length of G and let m be the product $c_1 \ldots c_n$ which appeared in Theorem 1.2.1. Then the homotopy type of the space $|\Lambda(G)|$ is determined as follows:*

(i) If $m = 0$, then $|\Lambda(G)|$ is contractible;

(ii) If m > 0, then |Λ(G)| is homotopy equivalent to a bouquet of spheres of dimension n − 2. The number of spheres in the bouquet is precisely m.

It is easy to calculate the reduced Euler characteristic of a bouquet of spheres using, for example, the Mayer–Vietoris theorem; if the space X is a bouquet of m spheres of dimension n−2, as in case (ii) of the above theorem, then one finds that $\tilde{\chi}(X)$ is equal to $(-1)^n m$. Since the Euler characteristic of a space is an invariant of its homotopy type, Theorem 1.2.1 is a corollary of Theorem 1.2.2. Notice in particular that m = 0, that is μ(G) = 0, if and only if |Λ(G)| is contractible. From Theorem 1.2.2 we can deduce not just the Euler characteristic of |Λ(G)|, but the whole integral homology of that space:

Corollary 1.2.3 (Thévenaz [1, Introduction]). *The integral homology of the space |Λ(G)| is as follows:*

$$H_0\left(|\Lambda(G)|\right) = \mathbb{Z};$$

$$H_{n-2}\left(|\Lambda(G)|\right) = \text{free abelian of rank } m;$$

$$H_r\left(|\Lambda(G)|\right) = 0 \text{ for } r \text{ unequal to } 0 \text{ or } n.$$

Thus the space |Λ(G)| has only one non-zero integral homology group in dimensions greater than zero, and this is a free abelian group whose rank is equal to the absolute value of the Möbius number μ(G).

1.3 The action of G on |Λ(G)|.

The group G acts on $\Lambda(G)$ by conjugation, and since this action is order-preserving there is an induced simplicial action on the order complex $|\Lambda(G)|$, and therefore on the homology groups $H_*\big(|\Lambda(G)|\big)$. If G is soluble, then these homology groups are given by Corollary 1.2.3, but their G-module structure is not. The G-structure of $H_*\big(|\Lambda(G)|\big)$ was determined by Thévenaz [1]; in this section we describe the results of that paper.

We restrict our attention to the non-trivial case, i.e. where $|\Lambda(G)|$ is not contractible. By Corollary 1.2.3 there is only one homology group to consider, that of dimension n-2 (the action of G on $H_0\big(|\Lambda(G)|\big)$ is trivial). Again the answer is very elegant and satisfactory, for $H_{n-2}\big(|\Lambda(G)|\big)$ turns out to be a multiple of a permutation module of the form $\mathbb{Z}[G/T]$, where T is a certain subgroup which complements the derived subgroup of G. Thévenaz's description of the stabilizer T depends on his concept of an *upper - infiltrated complement* to a normal subgroup of a soluble group:

Definition 1.3.1 (Thévenaz [1, Section 2]). G be a soluble group, and let N be a normal subgroup of G. Suppose that N has a complement C in G, and let

$$1 = N_0 < . \; . \; . < N = N_k < . \; . \; . < N_n = G$$

be a chief series for G which passes through N. Then C is said to be *upper infiltrated* provided there exists a chain of subgroups

$$C = C_k < C_{k-1} < . \; . \; . < C_0 = G$$

such that C_i complements the subgroup N_i in G, for $1 \le i \le k$.

Thévenaz shows ([1 Section 2]) that this definition does not depend on the

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choice of the chief series which passes through N. Furthermore, in a soluble group whose Möbius number is non-zero, every normal subgroup has at least one upper-infiltrated complement. Thévenaz's key result on upper-infiltrated complements in soluble groups is the following:

Theorem 1.3.2 (Thévenaz [1, Theorem 2.2]). *Let G be a finite soluble group. Then all upper-infiltrated complements of the derived subgroup of G are conjugate (when they exist).*

Since the G-set $[G/T]$ only depends on the conjugacy class of the subgroup T, the set $[G/T]$ for T an upper-infiltrated complement to the derived subgroup of G, is uniquely defined. By showing that $H_{n-2}(|\Lambda(G)|)$ is a sum of permutation modules whose stabilizers are upper-infiltrated complements of the derived subgroup of G, and then applying Theorem 1.3.2, Thévenaz proves the following result:

Theorem 1.3.3 ([1, Theorem 3.2]). *Let G be a finite soluble group, and suppose that $|\Lambda(G)|$ is not contractible. Then the derived subgroup G' of G has an upper-infiltrated complement T, and as G-modules*

$$H_{n-2}(|\Lambda(G)|) \cong \mathbb{Z}[G/T] \oplus \mathbb{Z}[G/T] \oplus \ldots \oplus \mathbb{Z}[G/T].$$

The discovery which gave rise to most of the work below was that the upper-infiltrated complements of Theorem 1.3.3 coincide with the *system normalizers* of the group G. We prove this in Theorem 2.2.1 below. It is well-known that the system normalizers of a soluble group lie in a single conjugacy class, so we obtain Theorem 1.3.2 as a corollary, but ironically Thévenaz's proof of Theorem 1.3.2 is the starting point of the work in Chapters 3, 4 and 5 of this thesis.

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1.4 The partially ordered set of conjugacy classes of subgroups.

Besides $\Lambda(G)$ and the associated Euler characteristic $\mu(G)$, another partially ordered set associated with finite groups has received attention in recent years. This is the partially ordered set of *conjugacy classes* of subgroups of G, whose order relation is defined by 'subconjugacy', that is, by the inclusion of a representative for the smaller conjugacy class in a representative for the larger. For a finite group G, we denote this partially ordered set $\Delta^*(G)$, and, just as for the set of subgroups, we write $\Delta(G)$ for the subset of $\Delta^*(G)$ consisting of conjugacy classes of proper, nontrivial subgroups of G. If G is soluble, then the Möbius number $\lambda_G(G)$ of $\Delta^*(G)$ is directly related to that of $\Lambda^*(G)$:

Theorem 1.4.1 (Hawkes, Isaacs and Ozaydin [1, Theorem 7.2]). *Let $\mu(G)$ and $\lambda_G(G)$ be the Möbius numbers of the sets $\Lambda^*(G)$ and $\Delta^*(G)$ respectively, where G is a finite soluble group. Then the following equation holds:*

$$\mu(G) = \lambda_G(G)/|G|.$$

It can be shown that $\lambda_G(G)$ is also equal to the product $(-1)^n d_1 \ldots d_n$, where the notation is as in Theorem 1.2.1, except that here we let d_i be the number of *conjugacy classes* of complements to the factor N_i/N_{i-1} of the chief series of 1.2.1. As is well known, the numbers d_i can also be written as the orders of certain 1-dimensional cohomology groups associated with G, and in fact this is a sign of the significance of cohomology in the study of the set $\Delta(G)$, which we exploit in Chapter 4.

The homotopy type of the complex $|\Delta(G)|$ has recently been determined by Volkmar Welker. It turns out that this complex, like $|\Lambda(G)|$, is homotopy equivalent to a bouquet of spheres of dimension n-2, where n is the chief length of G. (It should not be surprising that the common dimension of the spheres in

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this bouquet is the same as for $|\Lambda(G)|$, since $|\Delta(G)|$ is a sort of quotient of $|\Lambda(G)|$ by the action of the finite group G.)

Theorem 1.4.2 (Volkmar Welker [1, Satz 2.9]). *Let G be a finite soluble group, and let n be the chief length of G. Then the complex $|\Delta(G)|$ has the homotopy type of a bouquet of spheres of dimension n–2.*

Welker's result is actually more general than this, and gives a determination of the homotopy types of all 'intervals' in the partially ordered set of conjugacy classes of G, as well as a formula for the number of spheres in the bouquet. Of course, from the formula for the reduced Euler characteristic of any bouquet of spheres we know that the number of spheres in the bouquet of Theorem 1.4.2 is just the absolute value of the Möbius number $\lambda_G(G)$, so in effect Welker's formula is another expression for this number. Theorem 1.4.2 also follows from the work in Chapter 4 of this thesis.

1.5 Insoluble groups.

No results on the order complex $|\Lambda(G)|$ have been mentioned which are supposed to be valid for insoluble groups. This is a reflection of a true dichotomy which arises in the study of the complex $|\Lambda(G)|$; none of the results above are true for insoluble groups, and in fact very little seems to be known about insoluble groups in this context. For example, no formula is known which expresses the Möbius number of an insoluble group in terms of the internal structure of the group, as Theorem 1.2.1 does in the case of soluble groups. Indeed the existence of a formula resembling that of Theorem 1.2.1 seems to be precluded by an example of Hawkes, Isaacs and Ozaydin [1, Corollary 8.7] of a family of insoluble groups whose Möbius numbers are divisible by arbitrary primes which do not divide the orders of the groups. Furthermore, the complex $|\Lambda(G)|$ does not have

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the homotopy type of a bouquet of spheres in general. An example is when $G =$ $PSL_2(\mathbb{Z}_7)$ (see Kratzer and Thévenaz [2, Remarques]) – in fact this group has $\mu(G)$ $= 0$, while $|\Lambda(G)|$ is not contractible, so that $|\Lambda(G)|$ cannot be homotopic to a bouquet of spheres. (The situation seems to be different for *congruences* involving Möbius numbers, most of which are valid for all groups regardless of solubility. See Hawkes, Isaacs and Ozaydin [1] for a good collection of these.)

CHAPTER 2

2.1. The class of nC-groups.

From Theorem 1.2.1 we deduce that a soluble group has non-vanishing Möbius number if and only if it has a chief series all of whose factors are complemented. Kratzer and Thévenaz [2, Proposition 4.13] give a number of conditions on a soluble group G equivalent to this one, among which is the condition that each normal subgroup possess a complement in G. They call soluble groups satisfying these conditions *complemented*, but to avoid confusion with the stronger condition introduced by Philip Hall [1], who considered groups *all* of whose subgroups have complements, we use instead a term introduced by Christiensen [1, 2] who studied the class of groups with complemented normal subgroups from a group-theoretic point of view.

Definition 2.1.1. An nC-group is a group in which every normal subgroup has at least one complement.

Definition 2.1.1 does not require that G be soluble – for example all simple groups are nC-groups – but like Christiensen [1, 2] and H.Bechtell [1] we consider only soluble nC-groups in the sequel. For convenience, we state the connection between this definition and the non-vanishing of the Möbius number for soluble groups:

Theorem 2.1.2 (Kratzer and Thévenaz [2, Proposition 4.13]). *Let G be a finite soluble group. Any two of the following conditions on G are equivalent:*

(i) G is an nC-group;

(ii) The Möbius number $\mu(G)$ is non-zero;

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(iii)   The order complex $|\Lambda(G)|$ is non-contractible;

(iv)   G has a chief series

$$1{=}N_0 < N_1 < . \; . \; . < N_n = G,$$

all of whose factors N_i/N_{i-1} are complemented.

The equivalence of (ii) and (iv) is an immediate consequence of Gaschütz's formula (Theorem 1.2.1), and as remarked there, the formula also shows that if G has non-vanishing Möbius number then *all* chief series for G have all their factors complemented as in (iv). □

Corollary 2.1.3.   *The property of being a soluble nC-group passes to quotients.*

Proof. Let N be a normal subgroup of the soluble nC-group G.   We may choose a chief series passing through N, all of whose factors are complemented.   The part of this series which lies above N is also a chief series for G/N whose factors are complemented, so G/N is an nC-group by Theorem 2.1.2, (iv).

The independence of condition (iv) above on the choice of a chief series of G can also be deduced from the following 'generalized Jordan – Hölder theorem', which is valid for arbitrary groups.   Note that in a soluble group a non-Frattini chief factor is the same thing as a complemented one.

Theorem 2.1.4 (Doerk and Hawkes [1, Chapter I, Theorem 9.13]).   *Suppose that a finite group G has chief series*

$$1 = N_0 < N_1 < . \; . \; . < N_n < G$$

and

$$1 = M_0 < M_1 < \ldots < M_n < G.$$

Then there exists a bijective correspondence between the sets of factors of these two series, such that a pair of corresponding factors have the same isomorphism class and are either both Frattini or both non-Frattini. In particular, the total number of Frattini factors is the same for both series.

Finally we note the criterion, due to Philip Hall, for a p-group to be nC:

Theorem 2.1.5 (See Hawkes, Isaacs and Ozaydin [1, Corollary 3.5]). *A p-group P is nC if and only if it is elementary abelian.*

Proof If P is nC then $\Phi(P) = 1$, so that P is elementary abelian. On the other hand, any subgroup of an elementary abelian p-group has a complement. □

In the next section we summarise the parts of the theory of formations which are used in the sequel. This theory is the setting for a generalization, due to Carter and Hawkes [1], of the concept of the system normalizers of a soluble group, and it turns out that all of the theory below works naturally in this general context.

2.2 Saturated and local formations.

A *formation* is a class $\mathcal{F}$ of groups with the following two 'closure properties':

(i) If $G \in \mathcal{F}$ and $N \lhd G$ then $G/N \in \mathcal{F}$

(ii) If $N_1, N_2 \lhd G$ with $N_1 \cap N_2 = 1$ and if G/N_1 and $G/N_2 \in \mathcal{F}$, then $G \in \mathcal{F}$.

The class of nC-groups is an example of a formation (Bechtell [1, Theorem

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1.3]). For an account of the theory of formations of groups, see Doerk and Hawkes [1], Chapter IV. We use the concepts of formation theory throughout the sequel, but we are concerned mostly with *saturated* formations, that is classes of groups which satisfy (i), (ii) and the additional condition;

(iii) (Saturation.) If $G/\Phi(G) \in \mathcal{F}$ then $G \in \mathcal{F}$, where $\Phi(G)$ is the Frattini subgroup of G.

It is clear that the formation of soluble nC-groups is not saturated, since an nC-group can have no Frattini subgroup. The central example of a saturated formation is the class of finite nilpotent groups, which we write $\mathcal{N}$. There is a rich theory of saturated formations, of which we need only a part in the sequel, namely the concept, due to Gaschütz [2], of a *local formation*, and the construction, due to Carter and Hawkes [1], of the so-called $\mathcal{F}$-normalizers of a finite soluble group corresponding to the local formation $\mathcal{F}$. A full and recent account is given in Doerk and Hawkes [1, Chapter IV, Section 3 and Chapter V, Section 2], from which we extract the definition of a local formation given below.

Definitions2.2.1.

(i) A *formation function* f is a set of (possibly empty) formations of groups $f(p)$, one for each rational prime p. The *support* of f is the set π of primes p such that $f(p)$ is non-empty.

(ii) If f is a formation function, then a chief factor H/K of a group G is *f-central* if $A_G(H/K) \in f(p)$ for all p dividing $|H/K|$, where $A_G(H/K)$ is the group of automorphisms of H/K induced by the conjugation action of G. A chief factor which is not f-central is called *f-eccentric*. Where $f(p)$ is empty, the condition is interpreted as saying that a chief factor H/K is f-central if and only if p does not divide $|H/K|$. Note that, if G is soluble, then H/K is an irreducible module for G over $\mathbb{Z}_p$, where p is the unique prime divisor of $|H/K|$. The same

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definition applies equally well to any such irreducible module, whether or not it occurs as a chief factor of G.

(iii) If f is a formation function, then *the local formation defined by f*, written LF(f), is the class of groups all of whose chief factors are f-central. □

A key result, due to Gaschütz, is that *the class LF(f) is a saturated formation.* (Gaschütz [2], Doerk and Hawkes [1, Chapter IV, Theorem 3.3].) For example, if for each prime p we set f(p) = {1}, the formation consisting of the trivial group only, then LF(f) is the formation $\mathcal{N}$ of finite nilpotent groups, because finite nilpotent groups are characterised by the fact that all their chief factors are central. In general, a formation $\mathcal{F}$ which can be written as LF(f) for some formation function f is called a *local formation.* The function f is then called a *local definition* of $\mathcal{F}$. By Gaschütz's theorem a local formation is necessarily saturated, but in fact a celebrated theorem, proved for formations of soluble groups by Lubeseder and later for arbitrary formations by Schmid, shows that the converse is also true – every saturated formation is of the form LF(f) for some formation function f. (See Doerk and Hawkes [1, Chapter IV, Section 4].)

In general a saturated formation $\mathcal{F}$ has many local definitions f. It is always possible to choose one which satisfies the relation f(p) $\leq \mathcal{F}$ for all primes p, for the function f' defined by f'(p) = f(p)$\cap\mathcal{F}$, also locally defines the formation $\mathcal{F}$. A local definition which satisfies this condition is said to be *integrated.* Amongst the integrated local definitions of a given formation there is a measure of uniqueness (Doerk and Hawkes [1, Chapter IV, 3.7]); in particular the definition (2.2.1, (iii) above) of f-central and eccentric modules does not depend on the choice of f amongst integrated local definitions of f. (Doerk and Hawkes [1, Chapter V, (3.1)].) In the sequel we always use integrated local definitions, and speak of $\mathcal{F}$-central and eccentric modules, meaning those which are f-central or eccentric for any integrated local definition of $\mathcal{F}$.

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Note. By the theorem of Lubeseder and Schmid referred to above, the concepts of local and saturated formations actually refer to the same objects. However, in order to keep the treatment below logically independent of this difficult theorem, we refer throughout to 'local formations' where the existence of a formation function is necessary for the theory. In any specific case, as in the example of nilpotent groups, a formation function can be found without appeal to the general existence theorem.

$\mathcal{F}$-normalizers.

The $\mathcal{F}$-normalizers of a soluble group were first defined by Carter and Hawkes [1]. They generalize the notion of a system normalizer, the system normalizers corresponding to the case where $\mathcal{F}$ is the formation of nilpotent groups. The original definition requires that $\mathcal{F}$ contain the formation of nilpotent groups, but this is generalized to arbitrary local formations in Doerk and Hawkes [1, Chapter 5, Definition 3.1]. We summarize the construction of these subgroups below:

Suppose that $\mathcal{F}$ is a saturated formation defined locally by the integrated formation function f. Let π be the support of f. If G is a soluble group, then for each prime $p \in \pi$, we write $G^{f(p)}$ for the $f(p)$ -residual of G, that is the smallest normal subgroup N of G with $G/N \in f(p)$. (Note that such an N exists by the axioms for a formation.) Let Σ be a Hall system of G, and for each prime p let G^p be the Hall p-complement belonging to Σ. Let G_π be the Sylow π-subgroup in Σ.

Definition 2.2.2 (Doerk and Hawkes [1, Chapter V, 2.1 and 2.2]). Let the subgroup $D = D_{\mathcal{F}}(\Sigma)$ of G be defined by

$$D = G_\pi \cap \left(\bigcap_{p \in \pi} N_G(G^p \cap G^{f(p)}) \right).$$

Then D is the $\mathcal{F}$-*normalizer of G* associated to Σ. □

The $\mathcal{F}$-normalizers corresponding to different Hall systems form a conjugacy class of G, because of the conjugacy of the Sylow systems. It follows from the definition that the $\mathcal{F}$-normalizer D is a Sylow π-subgroup of the subgroup $\bigcap N_G(G^p \cap G^{f(p)})$ (for the index of $\bigcap N_G(G^p \cap G^{f(p)})$ is a product of primes in π). $\mathcal{F}$-normalizers have many group-theoretic properties, which are discussed in detail in Doerk and Hawkes [1, Chapter V]. We need only the following:

Theorem 2.2.3 (Doerk and Hawkes [1, Chapter V, Theorem 3.2]). *The $\mathcal{F}$-normalizers of a finite soluble group G have the following properties:*

(i) (Cover – avoidance.) They cover all $\mathcal{F}$-central chief factors of G and avoid all $\mathcal{F}$-eccentric ones.

(ii) (Epimorphism – invariance.) If $N \triangleleft G$ then the $\mathcal{F}$-normalizers of G/N are precisely the subgroups DN/N, where D is an $\mathcal{F}$-normalizer of G;

(iii) The $\mathcal{F}$-normalizers of G belong to $\mathcal{F}$, and G is equal to its $\mathcal{F}$-normalizers if and only if $G \in \mathcal{F}$.

(iv) Let S be the set of subgroups D of G which can be joined to G by a chain of subgroups of the form

$$D = G_r < G_{r-1} < \ldots < G_0 = G,$$

where G_i is an $\mathcal{F}$-abnormal maximal subgroup of G_{i+1} for $1 \leq i \leq r$. Then the minimal elements of the set S are precisely the $\mathcal{F}$-normalizers of G. (An $\mathcal{F}$-abnormal maximal subgroup of a soluble group H is one which complements an $\mathcal{F}$-eccentric chief factor of H.)

2.3 $\mathcal{F}$-normalizers and upper-infiltrated complements.

In this section we establish the claim of Chapter 1 that upper-infiltrated complements to the derived subgroup of a finite soluble group G (Chapter 1, Definition 1.3.1) coincide, when they exist, with the system normalizers of the group. This will provide the start for our investigation in the succeeding chapters of the behaviour of the restriction map in group cohomology.

Upper-infiltrated complements *do* always exist in nC-groups (Thévenaz [1, Section 2]), but the existence of such complements to the derived subgroup is not a sufficient condition for a soluble group to be nC. For example, let G be the semidirect product of a cyclic group Q of order q with a cyclic group P of order p^2 (where p divides q-1 and $\Phi(P) = C_P(Q)$). Then clearly the derived subgroup Q of G has P as an upper-infiltrated complement, but G is not an nC-group since $\Phi(G) = \Phi(P) \neq 1$. Thévenaz's main interest was in the case where G is an nC-group, but his results are valid under the weaker hypothesis that the relevant upper-infiltrated complements exist, and we work with the same weaker hypothesis.

Let $\mathcal{F}$ be a local formation, defined by the integrated formation function f. The following is the main theorem of this section.

Theorem 2.3.1. *Suppose that in the soluble group G the $\mathcal{F}$-residual $G^{\mathcal{F}}$ has an upper-infiltrated complement C. Then C is an $\mathcal{F}$-normalizer of G.*

Corollary 2.3.2. (See Theorem 1.3.2.) *The set of upper-infiltrated complements of $G^{\mathcal{F}}$ is either empty or consists of a single conjugacy class of G.*

The derived subgroup G' of G is not in general an $\mathcal{F}$-residual, since the formation of abelian groups is not saturated, but the following lemma shows that in the case we are considering the derived subgroup is the identical with the residual which corresponds to the formation $\mathcal{N}$ of nilpotent groups:

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Lemma 2.3.3. *Suppose that the derived subgroup G' of the soluble group G has an upper-infiltrated complement. Then $G' = G^{\mathcal{N}}$, the nilpotent residual of G.*

Thus by taking $\mathcal{F}$ to be the formation of nilpotent groups in Theorem 2.3.1 we obtain the promised result that the upper-infiltrated complements to the derived subgroup of G, when they exist, coincide with the system normalizers of G.

We begin by proving Lemma 2.3.3. We certainly have $G' \geq G^{\mathcal{N}}$, since abelian groups are nilpotent. If the containment were strict we could choose a normal subgroup N of G lying between the two subgroups, and with $N/G^{\mathcal{N}}$ a chief factor of G. This factor would lie in the derived subgroup of the nilpotent group $G/G^{\mathcal{N}}$ and therefore in its Frattini subgroup; it would therefore be a Frattini chief factor of G. On the other hand, if C is a complement of N in G whose existence is guaranteed by the hypothesis that G' has an *upper-infiltrated* complement, then the subgroup $CG^{\mathcal{N}}$ complements the factor $N/G^{\mathcal{N}}$, a contradiction. Therefore $G'=G^{\mathcal{N}}$, as required. $\square$

To prove Theorem 2.3.1 we need the following lemma.

Lemma 2.3.4. *Let $\mathcal{F}$ be a local formation. Suppose that G is a finite soluble group having normal subgroups H and K, with*

$$H < K \leq G^{\mathcal{F}} \leq G$$

and K/H a chief factor of G. If K/H is $\mathcal{F}$-central then it is a Frattini factor of G.

Proof. Since H is contained in the $\mathcal{F}$-residual $G^{\mathcal{F}}$, we have

$$G^{\mathcal{F}}/H = (G/H)^{\mathcal{F}},$$

and we may therefore assume that H=1 in the remainder of the proof. Then K is a minimal normal subgroup of G, and

$$A_G(K) \cong G/C_G(K) \in \mathcal{f}(p) \subseteq \mathcal{F},$$

where p is the prime dividing the order of K. (Recall that $A_G(K)$ is the group of automorphisms which G induces on K.)

Suppose that K has a complement D in G. Let M be the core of D, that is the largest normal subgroup of G contained in D. Then it is easy to see that

$$M = D \cap C_G(K)$$

and

$$D/M \cong G/C_G(K) \in \mathcal{f}(p).$$

The quotient group G/M is isomorphic with the primitive group $[K](D/M)$ having self–centralizing minimal normal subgroup K and core-free complement D/M. Since $D/M \in \mathcal{f}(p) \subseteq \mathcal{F}$ and K is an $\mathcal{F}$-central chief factor of $[K](D/M)$, it follows that G/M belongs to $\mathcal{F}$. Therefore

$$K \leq G^{\mathcal{F}} \leq M = D \cap C_G(K) \leq D,$$

a contradiction, since D was supposed to be a complement of K in G. □

Proof of Theorem 2.3.1.

Fix a chief series of G which passes through the $\mathcal{F}$-residual $G^{\mathcal{F}}$, say

$$1 = N_0 < N_1 < \ldots < N_r = G^{\mathcal{F}} < \ldots < N_n = G.$$

The hypothesis that the complement C is upper-infiltrated means we can find subgroups $C_0, \ldots, C_r$ such that C_i complements N_i for $1 \le i \le r$, and the C_i lie in a chain

$$G = C_0 > C_1 > \ldots > C_r = C.$$

Write D_i for the subgroup $C_i N_{i-1}$, for $1 \le i \le r$; then D_i is a complement of the factor N_i/N_{i-1} of the chief series, and so by Lemma 2.3.4 the factors N_i/N_{i-1}, are $\mathcal{F}$-eccentric for $1 \le i \le r$. (The remaining factors are $\mathcal{F}$-central.)

The intersection of the chief series with a subgroup C_i is the series

$$1 = N_i \cap C_i < N_{i+1} \cap C_i < \ldots < C_i = G \cap C_i$$

of C_i; the factors $N_{j+1} \cap C_i/N_j \cap C_i$ are isomorphic as C_i-modules with the factors N_{j+1}/N_j of the original series, and since for $j \ge i$ we have $C_i N_j = G$, we see that C_i induces the same automorphisms as G on these factors. The above is therefore a chief series of C_i.

In particular, each factor $N_{j+1} \cap C_i/N_j \cap C_i$ for $i \le j \le r$, is $\mathcal{F}$-eccentric, so C_{i+1}, which complements the factor $N_{i+1} \cap C_i/N_i \cap C_i$ in C_i, is an $\mathcal{F}$-abnormal maximal subgroup of C_i for $1 \le i \le r$. It follows from Theorem 2.2.3 (iv) that C contains an $\mathcal{F}$-normalizer of G. On the other hand, the order of C is equal to the product of the orders of the $\mathcal{F}$-central factors in our chief series, because as we remark above these are the factors between $G^{\mathcal{F}}$ and G. By Theorem 2.2.3 (i) the same is true of the $\mathcal{F}$-normalizers of G. Therefore C is an $\mathcal{F}$-normalizer of G, as required. $\square$

2.4 Some properties of nC-groups.

In this section we study soluble nC-groups from a group - theoretic point of view. Many results in this area have been obtained by Christiensen [1, 2], who showed in particular that the class of soluble nC-groups is closed under taking normal subgroups:

Theorem 2.4.1 (Christiensen [2, Theorem 3.5]). *Let G be a soluble nC-group, and let N be a normal subgroup of G. Then N is an nC-group.*

Christiensen proves his result by showing that a soluble group is nC if and only if all its *characteristic* subgroups are complemented. (This is reproved by Kratzer and Thévenaz [2, Proposition 4.13].) Christiensen remarks that his proof does not work for insoluble groups - I do not know whether the truth or falsehood of Theorem 2.4.1 for insoluble groups has been established.

Gaschütz has shown [3] that in every soluble group G there is a characteristic conjugacy class of subgroups, the *prefrattini subgroups*, which respectively cover and avoid the Frattini and complemented factors in any chief series for G. These subgroups are epimorphism invariant and intersect in the Frattini subgroup of G. In fact their relationship to the Frattini subgroup is similar to the relationship between the system normalizers and the hypercentre of G - this has been formalised in the notion of *precursive subgroups*; see Doerk and Hawkes [1, Chapter V, Section 5]. It follows that a soluble group is an nC-group if and only if its prefrattini subgroups are trivial (Gaschütz [3, Satz 6.6]); more generally, a result of Kurzweil and Hauck [1] shows that the prefrattini subgroups of a soluble group G are the minimal subgroups U of G for which the order complex $|\Lambda(U, G)|$ is non -contractible (The notation $\Lambda(H, G)$ for a subgroup H of G, means the partially ordered set of subgroups lying strictly between H and G.) We use the prefrattini subgroups of soluble groups in Chapter 4: Here we describe an

invariant θ(G) which is a measure of the failure of a soluble group G to be nC and which is 'dual' to the measure provided by the size of the prefrattini subgroups of G in the sense that it concerns the sizes of nC-subgroups of G, that is, subgroups H for which |Λ(1, H)| is non − contractible.

Definition 2.4.2. Let G be a finite soluble group. Define the set Θ(G) as follows;

$$\Theta(G) = \{H: \ H \le G \ and \ H \ is \ an \ nC\text{-}group\},$$

and define θ(G) by

$$\theta(G) = h.c.f. \ \{|G:H|; \ H \in \Theta(G)\}.$$

Thus θ(G) is the highest common factor of the indices of the nC-subgroups of G. Clearly if G is itself an nC-group then θ(G) = 1, but the converse is also true. To prove this, we need a well-known result of Gaschütz:

Theorem 2.4.3 (Gaschütz [4]). *Let G be a finite group and suppose that G has a normal subgroup V which is an abelian p-group for some prime p. Let P be a Sylow p-subgroup of G (thus V ≤ P). Then V has a complement in G if and only if V has a complement in P.*

Corollary 2.4.4. *If in the above situation H is any subgroup of G of p′-index, then V will have a complement in G if and only if V has a complement in H.*

(Theorem 2.4.3 is now thought of as a form of the 'stable element theorem' of group cohomology, Theorem 3.2.2.) We also need the fact that the invariant θ(G) behaves well with respect to normal subgroups and quotient groups:

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Lemma 2.4.5. *Let N be a normal subgroup of the finite soluble group G. Then $\theta(G)$ is divisible by $\theta(N)\theta(G/N)$.*

Proof. For any subgroup H of G we have $|G{:}H| = |G/N{:}HN/N| \times |N{:}H\cap N|$. On the other hand, if H is an nC-group then so are HN/N and $H\cap N$, by Corollary 2.1.3 and Theorem 2.4.1 respectively. Therefore the index of an nC-subgroup of G is of the form rs, where r and s are the indices of nC-subgroups of G/N and N respectively. The result follows. □

Theorem 2.4.6. *Let G be a finite soluble group. Then G is an nC-group if and only if $\theta(G) = 1$.*

Proof. If G is an nC-group then $\theta(G) = 1$, as remarked above. Conversely if $\theta(G)=1$, then by Lemma 2.4.5 we also have $\theta(G/N)=1$ for each quotient G/N of G, so that by Theorem 2.1.2 (ii) and induction on $|G|$, it suffices to show that G has a complemented minimal normal subgroup. Let V be any minimal normal subgroup of G, and let p be the prime dividing the order of V. The hypothesis that $\theta(G)=1$ implies that G has an nC-subgroup H of p'-index. By definition V has a complement in H, and it follows from Corollary 2.4.4 that V has a complement in G, as required. □

The proof of Theorem 2.4.6 shows that if a soluble group G has $\theta(G)$ prime to p, then all p-chief factors in any chief series for G must be complemented. (The property that all p-factors be complemented is independent of the choice of chief series, by Theorem 2.1.4, so as with nC-groups it is sufficient to check the factors of any one such series.) However, the converse is not true; for any prime p there exist soluble groups all of whose p-chief factors have complements but which do not have an nC-subgroup of p'-index, as the following example shows:

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Example 2.4.7. Let p and q be primes such that q divides p-1, let P be a cyclic group of order p, and let N be a nontrivial irreducible module for P over the field $\mathbb{Z}_q$ of q elements. The dual module $\text{Hom}_{\mathbb{Z}_q}(N, \mathbb{Z}_q)$ is also irreducible and nontrivial for P, so if

$$\varphi: N \to \mathbb{Z}_q$$

is nonzero, then the conjugates φ^x, for $x \in P$, are pairwise distinct. We fix a choice of (nonzero) φ. Since q divides p-1 we can choose an embedding of $\mathbb{Z}_q$ in the multiplicative group $\mathbb{Z}_p^x$. By composing this embedding with φ we obtain an action of the abelian group N on $\mathbb{Z}_p$. We write X for $\mathbb{Z}_p$ regarded as a module for N via this action. The conjugates of X under the action of P on N are pairwise distinct.

Let M be the homocyclic q-group of exponent q^2 whose Frattini quotient $M/\Phi(M)$ is isomorphic to N. It is easy to find an action of P on M such that $M/\Phi(M)$ is isomorphic to N as $\mathbb{Z}_q P$-module. The map $x \mapsto x^p$ induces an isomorphism from $N \cong M/\Phi(M)$ to $\Phi(M)$, so the latter is an irreducible $\mathbb{Z}_q P$-module isomorphic to N. Let H be the semidirect product of M with P for this action; thus H has a chief series

$$1 < N < M < MP = H$$

with the bottom two factors isomorphic to N (as H-modules by inflation) and at the top a central factor of order p.

Next let W be the (H/N)-module induced from the M/N-module X constructed above. We assert that W is an *irreducible* (H/N)-module. Indeed, by Nakayama reciprocity (Doerk and Hawkes [1, Chapter B, Theorem 6.5]) we have:

$$\text{Hom}_{\mathbb{Z}_p(H/N)}(W, W) \cong \text{Hom}_{\mathbb{Z}_p(M/N)}(X, W_{(M/N)}),$$

while by Mackey's theorem, $W_{M/N}$ is the sum of the p conjugates of X by the action of P. Since these conjugates are distinct,

$$\text{Hom}_{\mathbb{Z}_p(M/N)}(X, W_{(M/N)}) = \text{Hom}_{\mathbb{Z}_p(M/N)}(X, X) \cong \mathbb{Z}_p,$$

so that W is (absolutely) irreducible, by Schur's lemma. To complete the construction we inflate the module W to H and form the semidirect product,

$$G = [W]H.$$

It is clear that each p-factor in the chief series

$$1 < W < WN < WM < WH = G$$

has a complement; WN/W is the only Frattini factor. However, suppose that K is an nC-subgroup of G of p'-index. Certainly K contains W, so that (by the modular law) K is a semidirect product $[W]E$, where $E = K \cap H$. Replacing K by a conjugate if necessary we may assume that E contains P, and so (again by the modular law) E is of the form SP for some subgroup S of M. If K is to be an nC-group then E, being a quotient of K, must be an nC-subgroup of H, and therefore S, being normal in E, must also be nC. The Frattini subgroup of S is therefore trivial, so that S has exponent p, and lies in $\Phi(M)$. The group G was constructed so that $\Phi(M)$ acts trivially on W, so the subgroup K must have the form $[(W \times S)]P$. Thus S is a normal subgroup of K, and the quotient K/S is the p-group $[W]P \cong C_p$ wr C_p. This last group is not an nC-group, in contradiction to the hypothesis that K is nC. So G has no nC-subgroup of p'-index. Therefore p divides $\theta(G)$ in spite of the fact that every p-chief factor of G has a complement. □

It would be nice if Theorem 2.4.6 remained true, in terms of Möbius numbers, for insoluble groups; unfortunately this is not the case. For example the simple group $PSL_2(\mathbb{Z}_7)$, is the product of the symmetric group of order 24 and a cyclic group of order 7, which have Möbius numbers -12 and -1 respectively, but the

Möbius number $\mu(PSL_2(\mathbb{Z}_7))$ is zero. (See Section 1.5.)

In soluble groups G with abelian Sylow subgroups, or A-groups, as they are known, the set $\Theta(G)$ of nC-subgroups can be described completely. Note that the term 'A-group' applies only to *soluble* groups.

Theorem 2.4.8. *Let G be an A-group. Then G has a subgroup H with the property that the nC-subgroups of G are precisely the subgroups of H and their conjugates in G. Any other subgroup of G with the same property is a conjugate of H.*

In other words, for an A-group G, there exists $H \leq G$ such that $\Theta(G) = \{K \leq G: K \leq_G H\}$. The uniqueness up to conjugacy part is trivial; if H_1 and H_2 are two such subgroups then by hypothesis H_1 is conjugate to a subgroup of H_2 and vice versa, so that H_1 and H_2 are conjugate. To prove the existence of the subgroup H we need an easy case of the Hall – Higman theorem [1, Theorem A], namely that A-groups have p-length one for every prime p. This special case has a well – known direct proof, which we give next:

Proposition 2.4.9. *Let P be a Sylow p-subgroup of the soluble group G. Let Z(P) be the centre of P. Then $Z(P) \leq O_{p',p}(G)$.*

Proof. It is easy to see that Z(P) is centralizes every p-chief factor of G. On the other hand, the intersection of the centralizers of the p-chief factors of G is the p-Fitting subgroup $O_{p',p}(G)$ (Huppert [1, Chapter VI, Satz 5.4]). □

Corollary 2.4.10. *If G is soluble and has an abelian Sylow p-subgroup, then G has p-length one. In particular, an A-group has p-length one for all primes p.*

Recall that a *Sylow basis* B of a finite soluble group G is a set consisting of the identity subgroup together with one Sylow p-subgroup for each prime p dividing |G|, such that the groups in B permute pairwise. (Doerk and Hawkes [1, Chapter I, Definition 4.7].) The following result on the Sylow bases of groups of p-length one is due to Huppert:

Theorem 2.4.11 (Huppert [1, Satz 6.11]). *Let G be a finite soluble group which has p-length one for all primes p. Let $B = \left\{1, P_1, . . . ,P_k\right\}$ be a Sylow basis of G. Then for any i and j, $1 \le i, j \le k$, any characteristic subgroup of P_i permutes with any characteristic subgroup of P_j.*

Proof of 2.4.8. We first establish that an A-group G is nC if and only if its Sylow subgroups are elementary abelian. (An equivalent form of this result has been obtained by Bechtell [2, Theorem 2.3].) On the one hand, G has p-length one for all primes p by Corollary 2.4.10, so each Sylow p-subgroup P of G is isomorphic to a normal subgroup of $G/O_{p'}(G)$. Therefore if G is an nC-group then so is P by Corollary 2.1.4 and Theorem 2.4.1. It follows that P is elementary abelian, by Theorem 2.1.5. Conversely if the Sylow subgroups of G are elementary abelian, or in other words nC-groups by Theorem 2.1.5, then $\theta(G) = 1$, so that G is an nC-group by Theorem 2.4.6.

We can now prove Theorem 2.4.8. Note that among A-groups the nC-property is inherited by arbitrary subgroups. Let G be an A-group as in the statement, let $B = \left\{1, P_1, . . . ,P_r\right\}$ be a Sylow basis for G, and for each i let Q_i be the characteristic subgroup ΩP_i of P_i generated by the elements of prime order. From Theorems 2.4.10 and 2.4.11 it follows that the product

$$H = Q_1 Q_2 . . . Q_r$$

is a subgroup of G; this subgroup has the required properties, as we show next.

The Sylow subgroups of H are elementary abelian; they are $Q_1, \ldots, Q_r$ and their conjugates, so H and all its subgroups are nC-subgroups of G. On the other hand, suppose $K \leq G$ is an nC-group. By replacing K with a conjugate if necessary we may assume that the Sylow basis B reduces into K, i.e. that for each i the intersection of P_i with K is a Sylow subgroup of K. (Here we are using the fact that the Sylow bases of G are conjugate.) Since the Sylow subgroups of K are elementary abelian, we must have $P_i \cap K \leq Q_i$, for each i, and it follows that

$$K = (P_1 \cap K) \ldots (P_r \cap K) \leq Q_1 \ldots Q_r = H,$$

as required. □

The Sylow subgroups of nC-groups.

In the proof of Theorem 2.4.8 we encountered a class of p-groups that cannot be the Sylow subgroup of a soluble nC-group, namely those which are abelian but not elementary. In fact Sylow subgroups of nC-groups are not common amongst p-groups, for as Henn and Priddy [1] show, in a (rather technical) sense 'most' p-groups have the 'opposite' property that they are *p-nilpotent forcing*, that is, any group containing them as a Sylow p-subgroup automatically has a normal p-complement. (A classical case is Burnside's theorem that cyclic 2-groups are 2-nilpotent forcing.) Except for the cyclic group of order 2, a p-nilpotent forcing group cannot be the Sylow p-subgroup of an nC-group. (It is easy to see that, with this single exception, elementary abelian groups are not p-nilpotent forcing.) If 'most' p-groups cannot be the Sylow subgroup of an nC-group, then one might expect to obtain strong structural restrictions on those p-groups that *can* appear in this way. A few such restrictions are given below, but first we point out a known result in the opposite direction:

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Theorem 2.4.12 (Hawkes [1, Theorem 1]). *Any finite soluble group is isomorphic to a subgroup of a finite soluble group which has a unique chief series, all of whose factors are complemented.*

The following two results are generalizations of the fact that an abelian Sylow subgroup of a soluble nC–group is elementary:

Theorem 2.4.13. *Let G be a soluble nC–group, and let P be a Sylow p–subgroup of G. Then the centre Z(P) is elementary abelian.*

Proof. By Corollary 2.1.3 we may assume that $O_{p'}(G) = 1$. Therefore $O_{p',p}(G) = O_p(G)$ is elementary abelian, by Theorem 2.4.1. But Z(P) is contained in $O_{p',p}(G)$, by Proposition 2.4.9. □

Theorem 2.4.14. *Suppose that the p–group P is a Sylow p–subgroup of the nC–group G. Then the factors in the derived series of P are elementary abelian.*

Proof. As before we may suppose that $O_{p'}(G) = 1$, and so we may assume that G has a minimal normal p–subgroup N. Let C be a complement for N in G and let $Q = C \cap P$; then by the modular law P is isomorphic to the semidirect product

$$P \cong [N]Q.$$

The group Q is isomorphic with a Sylow subgroup of the nC–group G/N, so by induction the derived factors of Q are elementary abelian. This is enough to ensure that the derived factors of P have the same property, as the following lemma shows:

Lemma 2.4.15. *Suppose that P is a semidirect product $[N]Q$, where N is abelian. For $r \geq 1$ let $P^{(r)}$ and $Q^{(r)}$ be the rth derived subgroups of P and Q, respectively.*

Then

$$P^{(r)} = [N, Q, Q^{(1)}, \ldots, Q^{(r-1)}] Q^{(r)}$$

and there are surjective homomorphisms

$$N \times Q/Q^{(1)} \to P/P^{(1)}$$

and

$$[N, Q, \ldots, Q^{(r-1)}] / [N, Q, \ldots, Q^{(r)}] \times Q^{(r)}/Q^{(r+1)} \to P^{(r)}/P^{(r+1)}.$$

In particular, if N and $Q^{(r)}/Q^{(r+1)}$ are elementary abelian then so is $P^{(r)}/P^{(r+1)}$.

Proof of Lemma 2.4.15. The subgroups $N, [N, Q], \ldots, [N, Q, \ldots, Q^{(r-1)}]$ are all normal in P, for clearly $NQ = P, NQ^{(1)}, \ldots, NQ^{(r-1)}$ are normal subgroups of P, and from the identity $[n, n'q] = [n,q][n,n']^q = [n,q]$, which holds for any n, n' in N and any q in Q (N being abelian) we deduce that $[N, Q] = [N, P]$ is normal in P, and inductively,

$$[N, Q, Q^{(1)}, \ldots, Q^{(r)}] = [[N, \ldots, Q^{(r-1)}], NQ^{(r)}] \lhd P$$

(using each time that the commutator of two normal subgroups of P is itself normal). Inductively, from

$$P^{(r)} = [N, Q, Q^{(1)}, \ldots, Q^{(r-1)}] Q^{(r)}$$

we deduce that

$$P^{(r+1)} = \langle [[N, Q, Q^{(1)}, \ldots, Q^{(r-1)}], Q^{(r)}], Q^{(r+1)} \rangle,$$

and finally that $P^{(r+1)} = [N, Q, Q^{(1)}, \ldots, Q^{(r)}].Q^{(r+1)}$, because the first factor is normal in P. The assertion about $P^{(r)}$ follows by induction on r.

The homomorphisms are just given by multiplication; they are clearly surjective because of the form of $P^{(r)}$, and the last remark follows from this and the fact that $[N, Q, \ldots, Q^{(r)}]$ is a subgroup of N for each r. This completes the proof of the lemma and of Proposition 2.4.14. □

The same proof shows that the derived factors of any *Hall* subgroup of a soluble nC-group have *square-free exponent.*

We can deduce from Theorems 2.4.13 and 2.4.14 that the factors of the upper and lower central series of a Sylow subgroup of a soluble nC-group are elementary abelian, because of the following result.

Proposition 2.4.16 (Huppert [1, Chapter III, Satz 2.13]). *Let P be a p-group with upper and lower central series*

$$G = \gamma_1(G) > \gamma_2(G) > \ldots > \gamma_{c+1}(G) = 1,$$

and

$$1 = Z_0(G) < Z_1(G) < Z_2(G) < \ldots < Z_c(G) = G.$$

Then for each i, $1 \le i \le c$, the exponent of $Z_{i+1}(G)/Z_i(G)$ divides the exponent of $Z_i(G)/Z_{i-1}(G)$ and the exponent of $\gamma_i(G)/\gamma_{i+1}(G)$ that of $\gamma_{i-1}(G)/\gamma_i(G)$.

Corollary 2.4.17. *Let P be a Sylow subgroup of the soluble nC-group G. Then the factors of the upper and lower central series of P are elementary abelian.*

Proof Apply Proposition 2.4.16 to 2.4.13 and 2.4.14 respectively. □

The subgroup P itself can have any exponent-this is obvious from Theorem

2.4.12. In fact, one of the theorems of Hall and Higman [1, Theorem 3.3.1] shows that where the Sylow p-subgroup of a soluble group has exponent p, the p-length of the group is usually 1; exceptions are only possible when p=2, or when p is a Fermat prime and the Sylow 2-subgroups of G are non-abelian. An nC-group G of p-length one has elementary abelian Sylow p-subgroups (see the proof of 2.4.8). On the other hand, if P is the Sylow p-subgroup of a soluble nC-group G and if P contains an element of order p^2, then P has a subgroup isomorphic to the regular wreath product C_p wr C_p, as we show next:

Lemma 2.4.18. *Let Q be a cyclic group of order p^n and let V be a module for Q over the field of p elements. Then the semidirect product $[V]Q$ has exponent p^n unless V has the regular $\mathbb{Z}_p Q$-module as a direct summand, in which case the exponent of $[V]Q$ is p^{n+1}.*

Proof. Let $Q = \langle y \rangle$. Since the characteristic polynomial of y is $(X-1)^{p^n}$, which splits completely over $\mathbb{Z}_p$, we may choose a $\mathbb{Z}_p$-basis for V with respect to which the operator y is in Jordan normal form. There is no loss in assuming that the module V is *indecomposable*, or in other words that y is represented by a single Jordan block,

$$
\begin{bmatrix}
1 & 1 & . & . & . & . \\
. & 1 & 1 & . & . & . \\
. & . & 1 & 1 & . & . \\
. & . & . & 1 & 1 & . \\
. & . & . & . & 1 & 1 \\
. & . & . & . & . & 1
\end{bmatrix}
$$

of size $r \times r$, where r is the dimension of V.

Thus as an endomorphism of V we have $y = 1+\nu$, where ν is a nilpotent linear map with $\nu^r = 0$ but $\nu^{r-1} \neq 0$. To find the exponent of the semidirect product we

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calculate directly the orders of elements; in this calculation there is no loss in considering only elements of the form (v, y), for v an element of V, for y was an arbitrary generator of Q, and elements whose Q-component is not a generator have order at most p^n by induction. For $s \in \mathbb{Z}$ we have

$$(v, y)^s = (v^{1+y+\cdots+y^{s-1}}, y^s)$$

and for $s = p^t$ we may use the polynomial identity

$$1+(1+X)+(1+X)^2+\cdots+(1+X)^{s-1} = X^{s-1},$$

valid over $\mathbb{Z}_p$, to rewrite this as

$$(v, y)^s = (v^{s-1}(v), y^s).$$

Thus, since y has order p^n, this will be the exponent of the semidirect product unless there exists $v \in V$ such that $v^{p^n-1}(v) \neq 0$, in other words unless the Jordan block above has dimension p^n, in which case the formula shows that the exponent will be just p^{n+1}. Finally, if the dimension of V is p^n then V must be the regular module, for example because from the existence of the Jordan form for y we know that there is only one indecomposable module for Q of each dimension. □

Theorem 2.4.19. *Suppose that P is a Sylow p-subgroup of the nC-group G. where p is an odd prime. If P is not elementary abelian then at least one of the following holds:*

(i) p is a Fermat prime, and the Sylow 2- subgroups of G are non-abelian.

(ii) P has a subgroup isomorphic to the regular wreath product Q wr Q, where Q is cyclic of order p; therefore P is irregular and of nilpotent class at least p.

Proof. If P has exponent p then (i) must hold, by Hall and Higman [1, Theorem 3.3.1], because the p-length of P must be greater than 1. Suppose therefore that P contains an element of order p^2. As usual, we may assume without loss of generality that G has no p'-normal subgroup, and choose a minimal normal

subgroup V of G, whose order is say p^n, and a complement C for V in G. Let Q be the subgroup $C \cap P$, a complement for V in P. If Q contains an element of order p^2 then Q, and hence P, satisfy (ii) by induction, so we are left with the case where the exponent of Q is p. If $x \in P$ has order p^2 and we write $x = v.g$, where $v \in V$ and $g \in Q$, then $x \in V\langle g \rangle$, and it follows from Lemma 2.4.18 that V has a $\mathbb{Z}_p\langle g \rangle$-direct summand W which is regular. It is easy to see that the subgroup $[W]\langle g \rangle$ of P is isomorphic to the wreath product of $\langle g \rangle$ with itself. The other assertions of (ii) follow from this. □

The construction in Example 2.4.7 yields a group whose p–chief factors are complemented and whose Sylow p–subgroup is isomorphic to C_p wr C_p. We can use the same construction, but with an elementary abelian q–group in place of the homocyclic q–group of exponent q^2 used there, to give an nC–group with the same Sylow p–subgroup C_p wr C_p.

It seems interesting to consider the difference in strength between the nC–condition on a soluble group and its p–local analogue, that is the condition that a soluble group have all its p–chief factors complemented. In the results about Sylow subgroups proved above we have used only the weaker condition that the p–group in question be the Sylow p–subgroup of a soluble group with complemented *p–chief factors*, and in fact there do exist p–groups which are the Sylow p–subgroup of such a group, but which still cannot be embedded as a Sylow subgroup of a soluble nC–group. For example, if P is the non–abelian group of order 27 and exponent 3, then P is a Sylow 3–subgroup of the semidirect extension $[V]G$ of $G = SL_2(\mathbb{Z}_3)$ by its natural module $V = (\mathbb{Z}_3)^2$. On the other hand, it is easy to see that no soluble nC–group has a Sylow 3–subgroup isomorphic to P. However, this counterexample depends on the presence of the exceptional case of the Hall–Higman theorem–here we have the Fermat prime 3,

and the Sylow 2-subgroups of $[V]G$ are non-abelian. In all the exceptional cases of Hall–Higman [1, Theorem 3.3.1] the order of the group is even; we end this chapter with the following conjecture:

Conjecture 2.4.20. *If G is a finite soluble group of odd order, all of whose p-chief factors are complemented, then there is an nC-group H whose Sylow p-subgroups are isomorphic with those of G.*

CHAPTER 3

3.1 Introduction.

We deduced Thévenaz's Theorem 1.3.2 on the conjugacy of upper-infiltrated complements from Theorem 2.3.1, in which it is shown that where upper infiltrated complements to the derived subgroup of a soluble group exist, they coincide with the system normalizers of the group. Thévenaz, being unaware of Theorem 2.3.1, gave a proof of his result based on a curious cohomological property of the upper-infiltrated complements:

Lemma 3.1.1 (Thévenaz [1, Lemma 2.2]). *Suppose that T is an upper-infiltrated complement of the derived subgroup of a finite soluble group G. Let k be a field and S a simple kG-module. If S is not the trivial module. then the restriction map*

$$\text{res: } H^1(G, S) \to H^1(T, S)$$

is the zero map.

We may regard Lemma 3.1 as a property of system normalizers, temporarily with the proviso that upper-infiltrated complements exist, but this proviso is unnecessary; the main result of this chapter is a direct proof that the $\mathcal{F}$-normalizers of all soluble groups have an analogous property for any local formation $\mathcal{F}$. In Chapters 4 and 5 we consider the extent to which the cohomological properties of $\mathcal{F}$-normalizers proved in this chapter characterize these subgroups. It turns out that there are interesting connections with other questions in the theory of groups and their cohomology.

3.2 Cohomology of groups.

This section is a review of results from the cohomology theory of groups which are used below. Everything that we use is covered by either Brown [1] or Evens [1]; most is standard theory and may be found in any text on homological algebra.

Throughout the sequel, 'group cohomology' means the ordinary cohomology theory; for any group G and G-module V, we have a collection of abelian groups $H^r(G, V)$, one in each degree or dimension $r \geq 0$. We use the standard notation

$$H^*(G, V)$$

to denote the direct sum of the cohomology groups $H^r(G, V)$, for $r \geq 0$. When $V = k$ is a ring on which G acts trivially (in our case k is invariably a field of characteristic p) the additive group $H^*(G, V)$ has a natural ring structure, given by the cup product;

$$\cup: H^r(G, k) \otimes H^s(G, k) \to H^{r+s}(G, k),$$

which is associative and commutative in the graded sense. (This means that for $\xi \in H^r(G, k)$ and $\eta \in H^s(G, k)$, we have $\xi \cup \eta = (-1)^{rs} \eta \cup \xi$. In particular, note that the subalgebra of $H^*(G, k)$ which consists of the direct sum of the cohomology groups of *even* degree, is a commutative algebra over k.) We use the ring structure of $H^*(G, k)$ in Chapter 5.

Functorality.

The cohomology ring $H^*(G, V)$ is functorial in the pair (G, V) in the following sense (for a fuller discussion, see Brown [1, Chapter III, Section 8] or Evens [1, page 3]). Given another group H and an H-module W, a *compatible pair* (ρ, π) is a pair of group homomorphisms $\rho: H \to G$ and $\pi: V \to W$ (note the direction of

the map ρ) such that the following equation holds for all $v \in V$ and $h \in H$:

$$\pi \left(\rho(h)v \right) = h \; \pi(v).$$

This equation simply says that if V is regarded as an H-module via ρ, then the map π: $V \to W$ is a homomorphism of H-modules. A compatible pair (ρ, π) induces a map $(\rho, \pi)^* = \pi^* \circ \rho^*$: $H^*(G, V) \to H^*(H, W)$. The following are the most important cases:

(i) *Restriction and Inflation.* If ρ: $H \to G$ is a homomorphism, and V is a G-module then V may be regarded as a $\rho(H)$-module by restriction and then as an H-module by inflation. The pair consisting of the inclusion map $H \to G$ and the identity map $V \to V$ is clearly compatible, and so there is an induced map

$$\rho^*: \; H^*(G, V) \to H^*(H, V).$$

In the extreme case where the map ρ is the inclusion of a subgroup H of G, the map ρ^* is traditionally called *restriction* from G to H, and at the opposite extreme, where ρ is an epimorphism, ρ^* is the *inflation map*.

(ii) If V and W are modules for the same group G and π: $V \to W$ is a G-module homomorphism then the pair $(1, \pi)$ is compatible, where 1 is the identity map on G. Thus there is an induced morphism π_*: $H^*(G, V) \to H^*(G, W)$. Such maps are collectively known as *coefficient morphisms.*

(iii) *Conjugation.* If $H \le G$ is a subgroup and V is a G-module then it is easy to check that the maps

$$c_g: \; H \to g^{-1}Hg$$

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and

$$m_g: V \to V$$

given respectively by $c_g(h) = g^{-1}hg$ and $m_g(v) = gv$, are a compatible pair. The induced map

$$(c_g, m_g)^*: H^*(H, V) \to H^*(g^{-1}Hg, V)$$

is also called conjugation by G. If $\xi \in H^*(H, V)$ then we write ξ^g for the image of ξ under this map; unfortunately then $\xi^{gh} = (\xi^h)^g$, but if this seems suspicious the reader can substitute the notation $g^*(\xi)$, which indicates composition correctly.

In terms of the standard resolutions $F_*(G)$ and $F_*(H)$ for G and H (see Brown [1, Chapter I, Section 5]), the map of cohomology groups induced by a compatible pair (ρ, π) corresponds to the chain map

$$\text{Hom}(\rho_*, \pi): \text{Hom}_G(F_*(G), V) \to \text{Hom}_H(F_*(H), W)$$

$$\eta \mapsto \pi \circ \eta \circ \rho_*.$$

where $\rho_*: F_*(H) \to F_*(G)$ is the extension by linearity of $\rho: H \to G$. For computations with arbitrary projective resolutions for H and G, the map ρ_* is replaced by any chain map 'compatible with ρ' in the sense that $\rho_*(hx) = \rho(h)\rho_*(x)$ for all $h \in H$ and $x \in F_*(H)$. If $\rho: G \to G$ is conjugation by an element $g \in G$ then the map $F_*(G) \to F_*(G)$ given by $x \mapsto g^{-1}x$ is compatible with ρ (this is not the same as the extension of ρ by linearity when $F_*(G)$ is the standard resolution, although it is easy to write down a chain homotopy between the two maps). With this choice of compatible map the formula above shows that $\xi^g \in \text{Hom}(F_*(G), V)$ is the map $g \circ \xi \circ g^{-1}$. In particular *the conjugation action of G on its own cohomology (with any coefficients) is trivial.* In the sequel we assume implicitly the following corollary of this fact: For any G-module V, the kernel of the

restriction map $H^*(G, V) \to H^*(H, V)$ depends only on the conjugacy class of H in G.

If G acts trivially on V, then given any map $\rho\colon H \to G$ we may regard V as a trivial module for H as well, and then the pair $(\rho, 1)$ is compatible. In particular the group of automorphisms of G acts on the cohomology ring $H^*(G, k)$ for any ring k.

Relation with Sylow subgroups.

The following results are proved using the transfer map in cohomology (Brown [1, Chapter III, Section 9], Evens [1, Section 4.2]). They relate the cohomology of a group to that of its Sylow subgroups.

Proposition 3.2.1 (Brown [1, Chapter III, Corollary 10.2], Evens [1, Corollary 4.2.3]). *Let G be a finite group, and let M be a G-module. Then multiplication by $|G|$ annihilates $H^r(G, M)$ for all $r \geq 1$. In particular, if multiplication by $|G|$ is an isomorphism from M to M, then $H^r(G, M) = 0$ for all $r \geq 1$.*

If M is finitely generated, Proposition 3.2.1 implies that the groups $H^r(G, M)$ are finitely generated torsion abelian groups, hence finite. Their primary decomposition is given by the following theorem:

Theorem 3.2.2 (Brown [1, Chapter III, Theorem 10.3]). *Let G be a finite group and let M be a G-module. Let p be a prime and let P be the Sylow p-subgroup of G. Then the restriction map*

$$\text{res}\colon\ H^r(G, M) \to H^r(P, M)$$

is a monomorphism on the p-primary component of $H^r(G, M)$, and zero on the other components. □

The image of the restriction map in Theorem 3.2.2 is actually the subgroup of $H^r(P, M)$ consisting of G-stable elements, but we do not need this.

The inflation-restriction exact sequence.

This exact sequence of cohomology groups gives a relationship between the cohomology of a group and that of a normal subgroup. A more sophisticated relationship is the Lyndon-Hochschild-Serre spectral sequence, from which the restriction-inflation sequence can be deduced.

Proposition 3.2.3. (See Evens [1, Corollary 7.2.3].) *Let N be a normal subgroup of the group G. Let M be a G-module and write M^N for the module of N-fixed points of M, regarded as a module for G/N by deflation. Then there is a map τ:*

$$H^1(N, M) \to H^2(G/N, M^N) \text{ (the transgression map) and an exact sequence:}$$

$$0 \to H^1(G/N, M^N) \to H^1(G, M) \to H^1(N, M) \xrightarrow{\tau} H^2(G/N, M^N) \to H^2(G, M).$$

If $r > 1$ is such that $H^i(N, M) = 0$ for $1 \le i < r$, then the sequence

$$0 \to H^r(G/N, M^N) \to H^r(G, M) \to H^r(N, M)$$

is also exact.

Corollary 3.2.4. *If multiplication by $|N|$ is an isomorphism from M to M, then $H^r(G/N, M^N) \cong H^r(G, M)$, for $r \ge 1$.*

Proof The groups $H^r(N, M)$ are zero for $r \ge 1$, by Proposition 3.2.1. Therefore the restriction-inflation sequence degenerates into the claimed isomorphism. □

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3.3 $\mathcal{F}$-normalizers and cohomology.

In all our results the 'module of coefficients' V will be an irreducible module for the group algebra kG, where k is a field. We will generally restrict our attention to the case $k = \mathbb{Z}_p$, since the concept of $\mathcal{F}$-eccentricity applies to these modules. For modules over fields of characteristic zero (or prime to the order of the group) all cohomology groups in dimension ≥ 1 are zero, by Lemma 3.3.2, while on the other hand it is easy to see that any irreducible G-module is either an irreducible $\mathbb{Z}_pG$-module for some prime p, or an irreducible $\mathbb{Q}G$-module; since the latter case is not interesting cohomologically, we take the liberty of using the term 'irreducible G-module' to mean an irreducible module over $\mathbb{Z}_pG$, for some prime p.

The following is the main result of this chapter. As usual, $\mathcal{F}$ denotes a local formation (see Chapter 2, Section 2).

Theorem 3.3.1. *Let D be an $\mathcal{F}$-normalizer of the finite soluble group G. Then the restriction map*

$$\text{res: } H^r(G, V) \to H^r(D, V)$$

is the zero map for all $r \geq 1$, whenever the module of coefficients V is an $\mathcal{F}$-eccentric irreducible module for G.

We are not claiming that the cohomology groups themselves are zero; in fact it is quite possible for both $H^r(G, V)$ and $H^r(D, V)$ to be non-zero for all $r \geq 1$, as the following example shows:

Example 3.3.2. Let G be the product $Z \times S_4$ of a group Z of order 2 and the symmetric group of degree 4. The system normalizers of G are of the form $Z \times T$,

where T is a subgroup of S_4 generated by a transposition. Let V be the vierergruppe in S_4, regarded by inflation as a module for G over the field $\mathbb{Z}_2$ of two elements. Then V is an irreducible, eccentric module for G. However, the cohomology groups $H^r(G, V)$ and $H^r(Z \times T, V)$ are non-zero for all $r \geq 1$. For $H^r(Z \times T, V)$, this is easy to show using the long exact sequence in cohomology associated with the sequence $0 \to \mathbb{Z}_2 \to V \to \mathbb{Z}_2 \to 0$ of $Z \times T$-modules. From the fact that $H^r(Z \times T, \mathbb{Z}_2)$ has dimension r+1 over $\mathbb{Z}_2$ (see Proposition 5.4.1, or Evens [1, Section 3.5]), and the sections of long exact sequence,

$$H^{r-1}(Z \times T, \mathbb{Z}_2) \to H^r(Z \times T, \mathbb{Z}_2) \to H^r(Z \times T, V),$$

one concludes immediately that the dimension of $H^r(Z \times T, V)$ over $\mathbb{Z}_2$ is at least one-in fact these groups have dimension precisely one for all r.

The calculation of $H^*(G, V)$ is more involved, and can be avoided for the first cohomology group $H^1(G, V)$ by an appeal to Gaschütz's theorem that all of the complemented chief factors of a soluble group have non-zero 1-cohomology. (See page 58 below.) However, we sketch the calculation of all the cohomology groups $H^*(G, V)$, if only for amusement's sake.

Since V is a module over a field, the Künneth formula shows that

$H^*(G, V) = H^*(Z, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(S_4, V)$.(See Evens [1, page 17-18].) The ring $H^*(Z, \mathbb{Z}_2)$ is a polynomial ring in one variable over $\mathbb{Z}_2$, generated by the non-zero element $\xi \in H^1(Z, \mathbb{Z}_2)$. (Evens [1, Section 3.5].) It follows that

$$H^n(G, V) \cong \bigoplus_{0 \leq r \leq n} H^r(S_4, V).$$

To determine the groups $H^*(S_4, V)$, we think of V as the Vierergruppe, with S_4 acting by conjugation. Since V acts trivially on itself and is a module over a field, we have $H^*(V, V) = H^*(V, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} V$. (Evens [1, page 30].) Furthermore, V is *projective* as an S_4/V-module by deflation, so $H^*(V, V) = H^*(V, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} V$ is projective as an S_4/V-module also. Thus in the Hochschild-Serre spectral

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sequence, $E_2^{p,q} = H^p(S_4/V, H^q(V, V)) \Rightarrow H^{p+q}(S_4, V)$, the E_2 page has no nonzero entries for $p \geq 1$. The spectral sequence therefore has no non-zero differentials, and so restriction gives an isomorphism

$$H^*(S_4, V) \cong \left(H^*(V, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} V\right)^{S_3}.$$

The ring $H^*(V, \mathbb{Z}_2)$ is a polynomial ring over $\mathbb{Z}_2$ in two variables of degree 1 (Evens [1, Section 3.5]), so the right-hand side of the above, in degree n, is the S_3-fixed point subspace $\left(\mathbb{Z}_2[X, Y]_n \otimes_{\mathbb{Z}_2} V\right)^{S_3}$, where the $\mathbb{Z}_2$-span of X and Y is an S_3-module isomorphic to V, and the subscript n indicates the subspace of $\mathbb{Z}_2[X, Y]$ consisting of homogeneous polynomials of degree n.

Let C be the subgroup of S_3 of order 3, and let d(n) be the dimension of the subspace of C-fixed points of $\mathbb{Z}_2[X, Y]_n$. Then $\mathbb{Z}_2[X, Y]_n$ is the direct sum of the C-fixed point subspace, and $\left(n+1-d(n)\right)/2$ copies of V (for V is the only irreducible $\mathbb{Z}_2S_3$-module on which C acts nontrivially, and V is projective).

After tensoring with V, using the fact that the trivial S_3-module $\mathbb{Z}_2$ occurs once as a submodule of $V \otimes_k V$, and does not occur as a submodule of $V \otimes_k U$ where U is centralized by C, we obtain

$$\dim_{\mathbb{Z}_2} H^*(S_4, V) = \left(n+1-d(n)\right)/2.$$

Finally, we calculate d(n) by extending scalars to the field of four elements, where the action of C is diagonalizable: If $\pi \in C$ is a 3-cycle, then we may suppose that $\pi(X) = \lambda X$, $\pi(Y) = \lambda^{-1}Y$, where $\lambda \in \mathbb{F}_4$ is a cube root of unity. The Brauer character of the space generated by monomials $X^n, X^{n-1}Y, \ldots, Y^n$ is then given by $\chi(1) = n+1$ and $\chi(\pi) = \chi(\pi^{-1}) = \lambda^n + \lambda^{n-2} + \ldots + \lambda^{-n}$. The multiplicity d(n) of the trivial C- module in $\mathbb{Z}_2[X, Y]_n$ is $\left(\chi(1)+2\chi(\pi)\right)/3$, and, substituting this in the equation for $\dim_{\mathbb{Z}_2} H^*(S_4, V)$, we arrive at last at the answers:

$$\dim_{\mathbb{Z}_2} H^n(S_4, V) = \left[(n+1)/3\right], \text{ and } \dim_{\mathbb{Z}_2} H^n(G, V) = \sum_{0 \leq r \leq n} \left[(r+1)/3\right]. \quad \square$$

For the proof of Theorem 3.3.1 we require the following lemma:

Lemma 3.3.3. *Let p be a rational prime, and let G be a group with a Sylow p-complement H. If k is a field of characteristic p and V is a nontrivial, irreducible kG-module then no non-zero element of V is fixed by H.*

Proof. The assertion is equivalent to the statement that $\mathrm{Hom}_{kH}(k_H, V) = 0$, where k_H denotes the field k regarded as the trivial kH-module. By Nakayama reciprocity (Doerk and Hawkes [1, Chapter B, Theorem 6.5]),

$$\mathrm{Hom}_{kH}(k_H, V) = \mathrm{Hom}_{kG}(\mathrm{Ind}^G(k_H), V).$$

As is well known, the induced module $\mathrm{Ind}^G(k_H)$ is the projective cover of the trivial kG-module k_G. (Doerk and Hawkes [1, Chapter 1]). Therefore the head of $\mathrm{Ind}^G(k_H)$ is just the trivial module k_G. In other words there can be no non-zero map from $\mathrm{Ind}^G(k_H)$ to any simple G module other than k_G, and since by hypothesis V is not isomorphic with k_G, the groups of homomorphisms above must indeed be zero, as required. □

Proof of theorem 3.3.1.

Suppose that $\mathcal{F}$ is defined by the integrated formation function f whose support is π. The construction of the $\mathcal{F}$-normalizers of G is given in Section 2.2; if Σ is a Sylow system of G we write $T^p = G^{f(p)} \cap G^p$ for each prime $p \in \pi$ which divides the order of G, where G^p is the Sylow p-complement belonging to Σ. Then the $\mathcal{F}$-normalizer $D = D(\Sigma)$ is defined to be the intersection over $p \in \pi$ of the normalizers $N_G(T^p)$ and the Sylow π-complement belonging to Σ. In particular the $\mathcal{F}$-normalizers of G are π-groups so by Proposition 3.2.1 we only have to consider $\mathcal{F}$-eccentric simple G-modules whose characteristics lie in π. If

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V is such a module, of characteristic say p, we can factorize the restriction map in cohomology according to the inclusion,

$$D \subseteq N_G(T^p) \subseteq G.$$

That is, the restriction map from $H^*(G, V)$ to $H^*(D, V)$ is the composite of two restrictions

$$H^*(G, V) \to H^*(N_G(T^p), V) \to H^*(D, V).$$

The middle cohomology group is zero in dimensions ≥ 1, as we show next. The group T^p is a normal p'-subgroup of $N_G(T^p)$, so by Corollary 3.2.4 we have the following isomorphism:

$$H^r(N_G(T^p), V) \cong H^r((N_G(T^p)/T^p), V^{T^p}).$$

Now T^p is a Sylow p-complement of $G^{f(p)}$, which in turn is a normal subgroup of G. By Clifford's theorem the restriction of the irreducible module V to $G^{f(p)}$ is a semisimple $G^{f(p)}$-module whose irreducible summands are conjugate in G; furthermore, the hypothesis that V is $\mathcal{F}$-eccentric implies that one, and therefore all, of these summands is nontrivial. By Lemma 3.3.3 applied to each summand, the fixed-point submodule V^{T^p} is zero. Therefore $H^r(N_G(T^p), V) \cong H^r((N_G(T^p)/T^p), V^{T^p})$. is zero, as claimed. The theorem follows immediately, since a map which factors through the zero group must be zero. $\square$

Remark. Theorem 3.3.1 bears a relationship with a theorem of Barnes, Schmid and Stammbach [1] on the cohomological characterisation of saturated formations of finite groups. They prove [1, Theorem A] that for a saturated formation $\mathcal{F}$, a

finite group G belongs to $\mathcal{F}$ if and only if $H^1(G, V) = 0$ for all irreducible $\mathcal{F}$-eccentric G-modules V, and that moreover if $G \in \mathcal{F}$, then $H^r(G, V) = 0$ for all such modules and all $r \geq 1$. (The authors give a definition of $\mathcal{F}$-eccentricity which does not mention formation functions; their paper (just) predates the theorem of Schmid that any saturated formation is local. Theorem A is also stated for local formations, with our definition of $\mathcal{F}$-eccentricity [1, Theorem B].) A soluble group belongs to $\mathcal{F}$ if and only if it is equal to its $\mathcal{F}$-normalizers, by Theorem 2.2.3(iii), so Theorem 3.3.1 represents a generalization of part of Theorems A and B in the case of soluble groups. Where a converse to 3.3.1 can be proved, we have a strict generalization of these theorems, as for example in the case of soluble nC-groups (see Corollary 4.4.5). It is interesting to note that in Barnes, Schmid and Stammbach's results the first cohomology group 'governs' the behaviour of the others; this is also the case in Corollary 4.4.5. In Chapter 5, we find cases where differences appear between the behaviour of $H^1(G, V)$ and that of the higher cohomology groups $H^r(G, V)$, $r \geq 2$.

In the special case of the second cohomology group we give another proof of Theorem 3.3.1 which uses the description of this group in terms of extensions of G by its module V. (See Brown [1, Chapter IV, Section 3].) For any group G and G-module V, the elements $H^2(G, V)$ correspond to equivalence classes of short exact sequences of groups and homomorphisms

$$0 \to V \to E \to G \to 1$$

in which the conjugation action of E on V (which can be deflated to G, since V is abelian) agrees with the given action of G. The zero element of $H^2(G, V)$ corresponds to the split extension, for which E is the semidirect product $[V]G$, and for a subgroup of G, the restriction map $H^2(G, V) \to H^2(D, V)$ is just the map

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which takes the representative class of an extension,

$$0 \longrightarrow V \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1,$$

to the representative of the extension of D given by

$$0 \longrightarrow V \longrightarrow \pi^{-1}(D) \xrightarrow{\pi} D \longrightarrow 1.$$

Thus Theorem 3.3.1 (for $H^2(G, V)$) is equivalent to the following statement:

Suppose that D is an $\mathcal{F}$-normalizer of the finite soluble group G and that V is an $\mathcal{F}$-eccentric irreducible G-module. Then for any extension

$$0 \to V \to E \to G \to 1,$$

the corresponding extension of D;

$$0 \to V \to \pi^{-1}(D) \to D \to 1,$$

is split.

This is a simple consequence of the covering and avoidance properties of $\mathcal{F}$-normalizers and the fact that they are preserved under epimorphisms. (Theorem 2.2.3.) Thus if K is an $\mathcal{F}$-normalizer of the extension group E, the image $\pi(K)$ of K in G is conjugate to D. Therefore $\pi^{-1}(D)$ is the product of V with K, and moreover $V \cap K = 1$ because V is by hypothesis an $\mathcal{F}$-eccentric chief factor of the group E. That is to say, K splits the extension of D by V. □

3.4 The hypercentre.

The intersection of the system normalizers of any soluble group G is the hypercentre $Z_\infty(G)$. (Doerk and Hawkes [1, Chapter I, Theorem 5.9].) Therefore the restriction map

$$\text{res: } H^r(G, V) \to H^r(Z_\infty(G), V),$$

is zero for a soluble group G and all eccentric irreducible G-modules V, by Theorem 3.3.1. The hypothesis that G be soluble is unnecessary here-certainly the definition of the hypercentre, unlike that of the system normalizers, does not require a soluble group G, and we can give a simple direct proof that restriction to the hypercentre is zero for eccentric coefficient modules without using the solubility of G, as follows:

Theorem 3.4.1. *Let G be a finite group and let $Z_\infty(G)$ be the hypercentre of G. If V is an eccentric, irreducible module for G then the restriction map*

$$\text{res: } H^r(G, V) \to H^r(Z_\infty(G), V)$$

is zero for all $r \geq 1$.

Proof. Suppose that the characteristic of V is the prime p, and let Q be the (unique) Sylow p-subgroup of $Z_\infty(G)$. Since the map

$$\text{res: } H^r(Z_\infty(G), V) \to H^r(Q, V)$$

is a monomorphism by Corollary 3.2.3, it suffices to prove that

$$\text{res: } H^r(G, V) \to H^r(Q, V)$$

is the zero map.

The image of this map is contained in the subgroup of $H^r(Q, V)$ consisting of G-stable elements. Such an element is *a fortiori* stable under the action of $O^p(G)$, the normal subgroup of G generated by the elements of p'-order. This latter subgroup centralizes Q, and so acts trivially on the cohomology groups $H^r(Q, \mathbb{Z}_p)$. On the other hand, Q is contained in $O_p(G)$, and therefore acts

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trivially on the simple module V, so that we have a factorization (Evens [1, page 30]);

$$H^r(Q, V) \cong H^r(Q, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} V$$

which is an isomorphism of G-modules if G acts diagonally on the tensor product. It follows that as a module for $O^p(G)$, $H^*(Q, V)$ is just a sum of copies of V. By Clifford's theorem V is a semisimple $O^p(G)$-module whose summands are conjugate by G; since $O^p(G)$ cannot centralize V, they are all non-trivial. Therefore there are no non-zero $O^p(G)$-stable elements, and the result follows. □

CHAPTER 4

4.1 Introduction.

In this chapter and the next we try to find a converse to Theorem 3.3.1 in the following form (where as usual $\mathcal{F}$ is a local formation):

4.1.1 Ideal converse. *Let K be a subgroup of the finite soluble group G, and suppose that for all $\mathcal{F}$-eccentric irreducible G-modules V and all $r \geq 1$, the restriction map*

$$\text{res:} \ H^r(G, V) \to H^r(K, V)$$

is zero. Then K is contained in an $\mathcal{F}$-normalizer of G.

It must be stressed that this statement is *false in general.* However there are important classes of soluble groups for which it or something similar can be proved. My own opinion is that this is evidence of some more complex but universal cohomological property of $\mathcal{F}$-normalizers which I have been unable to discover, and a small amount of evidence for this point of view is given in Chapter 5. In this chapter the approach is to consider an easy case; we impose the condition of 4.1.1 on the first cohomology groups $H^1(G, V)$ only. This is obviously less likely to ensure that K is contained in an $\mathcal{F}$-normalizer than the full condition, but because of the tractability of the degree-1 cohomology of soluble groups, we can identify a class of groups including the nC-groups of Chapter 2, in which 4.1.1 is true. We begin with a brief discussion of the properties of degree 1 cohomology of (p-)soluble groups and the related concept, due to Gaschütz [3, Section 4] of crowns in soluble groups.

4.2 The 1-cohomology of p-soluble groups.

The following two results are well known; they are essentially due to Gaschütz. (See also Stammbach [1, Theorem A].)

Lemma 4.2.1. *Let G be a p-soluble group, and let V be a faithful, simple module for G over a field of characteristic p. Then $H^r(G, V) = 0$ for all $r \geq 1$.*

Proof. The subgroup $O_p(G)$ of G centralizes V (because by Clifford's theorem V is a semisimple $O_p(G)$-module, while on the other hand $O_p(G)$ is a p-group). Since V is supposed to be faithful, $O_p(G) = 1$, and so, since G is p-soluble, $O_{p'}(G) > 1$. Therefore $O_{p'}(G)$ acts nontrivially on V, so by Clifford's theorem, $V^{O_{p'}(G)}$ is zero. By Corollary 3.2.4, $H^r(G, V) \cong H^r(G/O_{p'}(G), V^{O_{p'}(G)})$ for all $r \geq 1$, and the result follows.

Theorem 4.2.2. *Let G be a p-soluble group. Let W be a homogeneous semisimple module for G over a field of characteristic p. If $C_G(W)$ is the centralizer of W in G then the restriction map*

$$\text{res: } H^1(G, W) \to H^1(C_G(W), W)$$

gives rise to an isomorphism

$$H^1(G, W) \cong \text{Hom}_{\mathbb{Z}G}(C_G(W)^{ab}, W).$$

Proof. Let V be a simple direct summand of W. Then $C_G(V) = C_G(W)$ because W is homogeneous. Write N for $C_G(V)$. Then every simple summand of $W = W^N$ is a faithful, simple G/N-module, and so, by Lemma 4.2.1, $H^1(G/N, W^N) = H^2(G/N, W^N) = 0$. Consider the five-term exact sequence associated with G, N

and V (Proposition 3.2.3); in this case, the terms are as follows:

$$0 \to 0 \xrightarrow{\text{inf}} H^1(G, W) \xrightarrow{\text{res}} H^1(N, W)^G \xrightarrow{\tau} 0 \longrightarrow H^2(G, W).$$

Therefore the restriction map $H^1(G, W) \to H^1(N, W)^G$ is an isomorphism. Since N acts trivially on W, we have $H^1(N, W) = \text{Hom}_{\mathbb{Z}}(N^{ab}, W)^G$, where N^{ab} is the abelianization of N. It is easy to check that

$$\text{Hom}_{\mathbb{Z}}(N^{ab}, W)^G = \text{Hom}_{\mathbb{Z}G}(N^{ab}, W),$$

and the result follows. □

The relationship between chief factors and 1-cohomology for a p-soluble group G is best expressed in terms of the *crowns* of G, as follows.

Crowns of p-soluble groups.

Suppose that the p-soluble group G has a simple module V over the field $\mathbb{Z}_p$ of p elements. It is easy to see that the centralizer $C_G(V)$ has a well defined smallest normal subgroup $R_G(V)$ with the property that the factor $X = C_G(V)/R_G(V)$ is an elementary abelian p-group which, as a G-module, is a sum of modules isomorphic to V. (Gaschütz [1, Section 4].) This factor is called the *crown* of G associated with V. (Gaschütz uses the word 'Kopf'.)

We write $\kappa(V)$ for the crown of G corresponding to V. Thus by Theorem 4.2.2 we have

$$H^1(G, W) \cong \text{Hom}_{\mathbb{Z}G}(\kappa(V), W)$$

for any homogeneous module W all of whose summands are isomorphic with V.

As a factor of G, any crown $\kappa(V)$ has a unique conjugacy class of complements (Gaschütz [1, Satz 5.1].) – this is immediate from Theorem 4.2.1, because $\kappa(V)$ is a faithful homogeneous semisimple module for $G/C_G(V)$. Gaschütz [1, Satz 5.3] proves various properties of complements of crowns in a soluble group. We need only a few of these properties, and since Gaschütz assumes that G is soluble, rather than just p–soluble, we give direct proofs here.

Lemma 4.2.3 (Gaschütz [1]). *Let V be a simple module over $\mathbb{Z}_p$ for the p–soluble group G, and suppose that $H < G$ complements a factor of G which is isomorphic to V. Then $R_G(V) \le H$.*

Proof. Suppose that the factor is a minimal normal subgroup of G, so that G is isomorphic to the semidirect product $[V]H$. Then clearly $C_G(V) = V \times C_H(V)$, so $C_G(V)/C_H(V)$ is a factor of G isomorphic to V. Therefore $R_G(V) \le C_H(V) \le H$. The general case follows similarly. □

Corollary 4.2.4. *In any chief series for G, the number of complemented factors which are isomorphic to V is equal to the number of summands in $\kappa(V)$.*

Proof. This number is independent of the chief series, by the generalized Jordan–Hölder theorem (Theorem 2.1.4), while for a series chosen to pass through $R_G(V)$ and $C_G(V)$, no factor isomorphic to V can lie above $C_G(V)$, and by Theorem 4.2.3 no such complemented factor can lie below $R_G(V)$. □

Proposition 4.2.5 (Gaschütz [3]). *Let V be a simple module over $\mathbb{Z}_p$ for the p–soluble group G, and let $1 < N_0 < N_1 < \ldots < N_n = G$ be a chief series for G. For each factor N_i/N_{i-1} of this series which is complemented and isomorphic to V, choose any complement C_i. Then the intersection D of the subgroups C_i is a complement for the crown $\kappa(V)$ of G.*

Proof. Let $i_1 < \ldots < i_r$ be the indices of the factors N_i/N_{i-1} above, so that by definition $D = C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_r}$. Clearly N_{i_1} is contained in $C_{i_2}, \ldots, C_{i_r}$, so by the modular law $N_{i_1}(C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_r}) = (N_{i_1}C_{i_1}) \cap (C_{i_2} \cap \ldots \cap C_{i_r}) = C_{i_2} \cap \ldots \cap C_{i_r}$. By the same reasoning, $N_{i_2}N_{i_1}(C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_r}) = N_{i_2}(C_{i_2} \cap \ldots \cap C_{i_r}) = C_{i_3} \cap \ldots \cap C_{i_r}$, and we deduce finally that $N_{i_r}(C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_r}) = G$. Clearly $N_{i_r} \leq C_G(V)$, since N_{i_r}/N_{i_r-1} is isomorphic to V, so we also have $DC_G(V) = G$. The formula $N_{i_1}(C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_r}) = C_{i_2} \cap \ldots \cap C_{i_r}$ above shows that $C_{i_1}(C_{i_2} \cap \ldots \cap C_{i_r}) = G$, and of course $C_{i_2}(C_{i_3} \cap \ldots \cap C_{i_r})$ etc. $= G$ similarly. Therefore the index in G of $D = C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_r}$ is the product of those of the C_i, or in other words of the orders of the chief factors complemented by the C_i. By Corollary 4.2.4 this is just $|C_G(V)/R_G(V)|$, and now we are done since D contains $R_G(V)$ by Lemma 4.2.3. $\square$

Corollary 4.2.6. *Suppose that G is π-soluble, and that $V_1, \ldots, V_r$ are nonisomorphic simple modules for G over (possibly different) prime fields $\mathbb{Z}_p$, where each prime $p \in \pi$. Let $D_1, \ldots, D_r$ be complements of their respective crowns. Then the index in G of $D = D_1 \cap \ldots \cap D_r$ is the product $|G:D_1| \ldots |G:D_r|$. Furthermore, any two subgroups of the form $D_1 \cap \ldots \cap D_r$ are conjugate in G.*

Proof. By Theorem 4.2.5, we may choose a chief series for G and express each D_i as an intersection of subgroups which complement distinct factors of this series. Since the D_i are complements of different crowns, the subgroups in the expressions for different D_i complement disjoint sets of factors of the series, so the intersection of all of them has the stated index, as in the penultimate line of the proof of Theorem 4.2.5. The conjugacy also follows from this line, since in general if M and N are subgroups of a group G and $MN = G$, then all intersections $M^g \cap N^h$ are conjugate in G. $\square$

The above is closely related to a theorem of Gaschütz (Huppert and Blackburn [1, Remark 15.6]) on the second Loewy layer of the projective covers of trivial modules for a soluble group. To see the relationship, regard $\mathbb{Z}_p$ as a trivial module for G and consider the first few modules in the minimal projective resolution of $\mathbb{Z}_p$ over $\mathbb{Z}_pG$;

$$\cdots \to P_1 \to P_0 \to \mathbb{Z}_p \to 0.$$

Thus P_0 is the projective cover of $\mathbb{Z}_p$, and because the resolution is minimal, the head P_1/JP_1 is isomorphic to the second Loewy layer JP_0/J^2P_0 of this projective cover. Again because of the minimality of the resolution, we have for any simple $\mathbb{Z}_pG$-module V;

$$H^1(G, V) = \text{Hom}_G(P_1/JP_1, V) = \text{Hom}_G(JP_0/J^2P_0, V).$$

By Theorem 4.2.2, the first of these is equal to $\text{Hom}_G(\kappa(V), V)$. Therefore each simple module V over $\mathbb{Z}_p$ occurs in the modules JP_0/J^2P_0 and $\kappa(V)$ with the same multiplicity, and it follows that JP_0/J^2P_0 is isomorphic to the the direct sum of the crowns $\kappa(V)$, one for each simple $\mathbb{Z}_pG$-module V. After Corollary 4.2.4, this is the same as the sum of all complemented p-factors of G which occur in a given chief series.

Derivations.

Recall the interpretation of the first cohomology group $H^1(G, M)$ in terms of derivations. A derivation or 1-cocycle is a map

$$\delta\colon G \to M$$

from G to a (left) G-module M, which satisfies the condition

$$\delta(gh) = \delta(g) + g.\delta(h)$$

for all g, h ∈ G. A derivation is said to be *inner* if there exists an element m ∈ M such that

$$\delta(g) = m - g.m$$

holds for all g∈ G. The derivations from G to M form a group Der(G, M) under pointwise addition: the inner derivations constitute a subgroup Inn(G, M). The first cohomology group H^1(G, M) can be realized as the quotient group Der(G, M)/Inn(G, M). We write [δ] for the class of a derivation δ; that is its coset in the first cohomology group H^1(G, M).

If a group G is the semidirect product [V]H of a subgroup H with an abelian group V on which G acts, then as is well known the map δ given by δ(vh) = v is a derivation from G to V, and defines a class [δ] ∈ H^1(G, V). (See Brown [1, Chapter 4, Section 2].) The same formula works if V is a chief factor of G complemented by H, and we may use this to give an explicit description of H^1(G, W) in terms of derivations from G to W, when G is a p-soluble group and W is a sum of crowns of G. To do this we use the natural action of Hom$_G$(M, M) on H^1(G, M) for any module M; this is given by associating to φ ∈ Hom$_G$(M, M) the coefficient morphism φ$_*$: H^1(G, M) → H^1(G, M), and in terms of derivations it is just the action on H^1(G, M) induced by composition of derivations G → M with endomorphisms of M. Notice that with this action, H^1(G, M) becomes a *right* module over Hom$_G$(M, M).

Proposition 4.2.7. *Let π be a set of primes and suppose the group G is p-soluble for each prime p ∈ π. Let V_1, . . . ,V_r be nonisomorphic simple G-modules over fields $\mathbb{Z}_p$, where each prime p ∈ π, and let W be the sum*

$$\kappa(V_1) \oplus \kappa(V_2) \oplus \ldots \oplus \kappa(V_r)$$

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of the corresponding crowns of G. Then there is a surjective derivation

$$\delta\colon G \longrightarrow W$$

such that the cohomology class [δ] generates $H^1(G, W)$ as a regular (right) module over $\mathrm{Hom}_G(W, W)$.

Proof. Let D_i be a complement of the crown $\kappa(V_i)$, and define a map δ_i from G to $\kappa(V_i)$ by writing

$$\delta_i(c.d) = c.R_G(V_i)$$

for $c \in C_G(V_i)$ and $d \in D_i$. Thus δ_i is a derivation from G to $\kappa(V_i)$, as explained above. Notice that δ_i restricts to the identity map on $\kappa(V_i)$. By Theorem 4.2.2, restriction gives an isomorphism between $H^1(G, \kappa(V_i))$ and $\mathrm{Hom}_G(\kappa(V_i), \kappa(V_i))$, which clearly commutes with the right action of $\mathrm{Hom}_G(W, W)$, and it follows that $H^1(G, \kappa(V_i)) = [\delta_i]\mathrm{Hom}_G(W, W)$. Let $\delta\colon G \to W$ be the diagonal sum of the δ_i, so that [δ] generates $H^1(G, W)$ regularly over $\mathrm{Hom}_G(W, W)$. The kernel of δ_i is just D_i, and so the kernel of δ is the intersection of the D_i. We have;

$$|\delta(G)| = |G\colon \cap D_i| = \prod|G\colon D_i| = \prod|\kappa(V_i)| = |W|,$$

where the second equation is Corollary 4.2.6. Therefore δ is surjective. $\square$

It is the surjectivity of δ that is crucial; it is easy to see by considering minimal resolutions that all the other properties of the first cohomology group can be achieved for an arbitrary finite group G (where instead of 'sum of crowns' one writes 'sum of homogeneous components of Loewy layers JP_0/J^2P_0, for various primes p'. See the discussion after the proof of Corollary 4.2.6.)

If $K \leq G$ then res: $H^1(G, W) \to H^1(K, W)$ is just the quotient of the obvious map from Der(G, W) to Der (K, W), and is clearly a morphism of (right) $Hom_G(W, W)$-modules for the action of $Hom_G(W, W)$ described above. Therefore the kernel of the restriction map is a $Hom_G(W, W)$-submodule of $H^1(G, W)$, so that we have an order-reversing map from the set Δ^* of conjugacy classes of subgroups of G defined in Section 1.4 to the set of $Hom_G(M, M)$-submodules of $H^1(G, M)$. The next result, which is our main theorem in this section, shows that, when M is a sum of crowns of G, there is a natural map in the other direction. We use this in Section 3 to construct Galois connections between subsets of $\Delta^*(G)$ and sets of submodules of $H^1(G, M)$.

Theorem 4.2.8. *Let G be a finite π-soluble group, and let W be a G-module which is a sum of nonisomorphic crowns of G, whose characteristics lie in π. Suppose that $U \leq H^1(G, W)$ is a $Hom_G(W, W)$-submodule, and consider the set $\Gamma(U)$ of subgroups of G given by*

$$\Gamma(U) = \left\{ K \leq G: \text{The kernel of} \left(\text{res: } H^1(G, W) \to H^1(K, W) \right) \text{contains } U \right\}.$$

Then G has a conjugacy class of subgroups whose members are precisely the maximal elements of the set $\Gamma(U)$.

Proof. By Proposition 4.2.7, the first cohomology group $H^1(G, W)$ is a regular module of the form $[\delta].Hom_G(W, W)$, where δ is a derivation of G *onto* W. The submodules of this module are just the sets $[\delta].I$, where I is a right ideal of the ring $Hom_G(W, W)$. Since W is semisimple, $Hom_G(W, W)$ is a semisimple ring; all its right ideals are of the form $e.Hom_G(W, W)$ for some idempotent element $e \in Hom_G(W, W)$. (See Curtis and Reiner [1, Proposition 3.18].) Thus we may choose $e = e^2 \in Hom_G(W, W)$ such that $U = [\delta].e.Hom_G(W, W)$, or in other

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words U consists precisely of the cohomology classes of those derivations which factor through the derivation $e \circ \delta$: $G \longrightarrow W$.

Suppose that K and L are two subgroups of G contained in $\Gamma(U)$. The cohomology class $[e \circ \delta]$ vanishes on restriction to K, or in other words there exists an inner derivation η: $K \longrightarrow W$ such that

$$e \circ \delta(k) = \eta(k)$$

holds for all $k \in K$. Since $e^2 = e$ we also have

$$e \circ \delta(k) = e \circ \eta(k)$$

for all $k \in K$. Since η is inner, there exists $w \in W$ such that $\eta(k) = w - k.w$ for all $k \in K$, and since δ is surjective we can find an element h of G such that $w = \delta(h)$. We calculate as follows:

$$
\begin{aligned}
h.\delta(h^{-1}kh) &= h.\Big(\delta(h^{-1}k) + h^{-1}k.\delta(h)\Big) \\
&= h\Big(\delta(h^{-1}) + h^{-1}\delta(k) + h^{-1}k.\delta(h)\Big) \\
&= -\delta(h) + \delta(k) + k.\delta(h),
\end{aligned}
$$

where in the last line we have used the fact, valid for any derivation δ, that $0 = \delta(hh^{-1}) = \delta(h) + h.\delta(h^{-1})$. Since $w = \delta(h)$ we find that

$$e \circ \delta(h^{-1}kh) = e \circ h^{-1}\Big(\delta(k) - \eta(k)\Big) = h^{-1} \circ e\Big(\delta(k) - \eta(k)\Big) = 0,$$

for all $k \in K$. Thus the derivation $e \circ \delta$ vanishes on the subgroup $K^h = h^{-1}Kh$ of G.

The same argument gives an element t of G such that $e \circ \delta$ vanishes on L^t, and so, since the kernel of a derivation is a subgroup of G, $e \circ \delta$ vanishes on the subgroup $\langle K^h, L^t \rangle$ generated by these two subgroups. It follows that $[e \circ \delta]$, and

with it every cohomology class in the submodule U, vanishes upon restriction to $\langle K^h, L^t \rangle$.

Finally, suppose that K is *maximal* in $\Gamma(U)$. Then, since $\Gamma(U)$ is a union of conjugacy classes, K^h is also maximal. Since $K^h \leq \langle K^h, L^t \rangle$, which we have just shown also to belong to $\Gamma(U)$, we must have $L^t \leq K^h$; that is, L is contained in a conjugate of K. □

4.3 The partially ordered set of conjugacy classes of subgroups.

For any finite G-module W, we write $\Xi^*(G)$ for the lattice of $\text{Hom}_G(W, W)$-submodules of $H^1(G, W)$. As usual, $\Xi(G)$ denotes the proper part of $\Xi^*(G)$; i.e

$$\Xi(G) = \Xi^*(G) - \left\{ \partial_j H^1(G, W) \right\}.$$

As in Section 1.5 we write $\Delta^*(G)$ for the partially-ordered set of conjugacy classes of subgroups of G and $\Delta(G)$ for the proper part of $\Delta^*(G)$. If $U \in \Xi^*(G)$, write $\Xi(G, U)$ for the subset of $\Xi(G)$ consisting of those submodules V of $H^1(G, W)$ with $U < V < H^1(G, W)$. Similarly, if $(K) \in \Delta^*(G)$, then $\Delta(G, K)$ means the partially ordered set of conjugacy classes of subgroups H with $K <_G H <_G G$.

Corollary 4.3.1. *Suppose, as in Theorem 4.2.8, that G is π-soluble and that W is a direct sum of nonisomorphic crowns of G, of characteristics belonging to π. Then there are order-reversing maps*

$$\gamma: \Xi^*(G) \longrightarrow \Delta^*(G)$$
$$\sigma: \Delta^*(G) \longrightarrow \Xi^*(G)$$

such that $\gamma \circ \sigma: \Delta^*(G) \longrightarrow \Delta^*(G)$ and $\sigma \circ \gamma: \Xi^*(G) \longrightarrow \Delta^*(G)$ are increasing maps.

Proof. The map σ is defined by letting σ(K) be the kernel of the restriction map

$$\text{res: } H^1(G, W) \rightarrow H^1(K, W).$$

(This kernel depends only on the conjugacy class of K - see Section 3.2.) The existence of the other map, γ, follows from Theorem ... .8; for a submodule U of $H^1(G, W)$ we let γ(U) be the maximal elements of the set

$$\left\{ K \leq G; \ U \leq \ker \left(\text{res: } H^1(G, W) \rightarrow H^1(K, W)\right)\right\}.$$

Theorem 4.2.8 shows that γ(U) consists of a single conjugacy class of subgroups of G. The assertions about γ ∘ σ and σ ∘ γ are purely set-theoretic and easy to verify. □

Note that γ(U) is just the maximal subset of the set of subgroups K of G having σ(K) ≥ U.

The maps γ and σ define a Galois connection between the partially ordered sets Ξ*(G) and Δ*(G), which by Theorem 1.1.4 induces a homotopy equivalence between their order complexes. However, we are interested not in Δ*(G), whose order complex is contractible (Corollary 1.1.4), but in its proper part Δ(G). Fortunately the maps we have constructed also give a Galois connection between the proper part Ξ(G) and an appropriate interval Δ(G, K) in the proper part Δ(G), where K is a subgroup of G which depends on the module W.

Theorem 4.3.2. *Suppose, as above, that G is a π-soluble group, and let W be the direct sum of all the crowns of G whose characteristics lie in π. Let the maps*

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$$\gamma: \Xi^*(G) \longrightarrow \Delta^*(G)$$

$$\sigma: \Delta^*(G) \longrightarrow \Xi^*(G)$$

be defined as in Corollary 4.3.1. Further, define $T \leq_G G$ by

$$T = \gamma(H^1(G, W)).$$

Then the restrictions of γ and σ define a Galois connection between $\Xi(G)$ and $\Delta(G, T)$.

Proof. We have to show that $\gamma(\Xi(G)) \subseteq \Delta(G, T)$ and $\sigma(\Delta(G, T)) \subseteq \Xi(G)$. This amounts to proving the following four things:

(i)   If $(K) < G$, then $\sigma\big((K)\big)$ is nonzero;

(ii) If $(K) > (T)$ then $\sigma\big((K)\big)$ is a proper submodule;

(iii) If $U < H^1(G, W)$ then $\gamma(U) > (T)$;

(iv) If $U > 0$ then $\gamma(U) < G$.

Of these, (ii) follows from the definition of T and (iv) is just obvious. To prove (iii) we find a formula for the order of $\gamma(U)$ in terms of that of U. Recall that since $H^1(G, W)$ is a regular module over the semisimple ring $\mathrm{Hom}_G(W, W)$, each submodule U of $H^1(G, W)$ is of the form $[\delta] I$, where δ is the derivation we constructed in Proposition 4.2.7 and I is a right ideal of $\mathrm{Hom}_G(W, W)$ generated by some idempotent element e.

Lemma 4.3.3. *Suppose that $U \in \Xi^*(G)$ is the submodule $[\delta].I$, where I is the right ideal generated by an idempotent $e \in \mathrm{Hom}_G(W, W)$. Then the index of a subgroup K of G belonging to the conjugacy class $\gamma(U)$ is given by*

$$|G:K| = |e(W)|.$$

In particular, the common index of the subgroups T in the conjugacy class $\gamma(H^1(G, W))$ is equal to the order of W.

Proof. From Theorem 4.2.8 it is clear that $\gamma(U)$ is the conjugacy class in G of the kernel of the derivation $e \circ \delta$: $G \longrightarrow W$. In other words, K is the pre-image under δ of the G-submodule ker e of W. For any surjective derivation δ_1 from a group G onto a G-module M, one has $|G:\ker \delta_1| = |\delta_1(G)|$; the desired result is obtained by applying this to the derivation $\delta_1 = \pi \circ \delta$ above. $\square$

The proof of (iii) is now straightforward: If $U < H^1(G, W)$ then the projection e of Lemma 4.3.3 is not surjective, and we deduce from the lemma that the order of a subgroup of G in $\gamma(U)$ is greater that the order of any of the subgroups T in $\gamma(H^1(G, W))$.

We now deal with (i), which says that for any proper subgroup of G containing T, the map

$$\text{res: } H^1(G, W) \to H^1(K, W)$$

is not injective.

By Lemma 4.3.3 the index of T in G is a number all of whose prime factors belong to π, and so the same is true of K. We may clearly assume that K is a maximal subgroup of G, which therefore complements a chief factor V in any chief series for G. If V is any such factor, then $C_G(V)/C_K(V)$ is a factor of G isomorphic to V, and clearly complemented by K. (See the proof of Lemma 4.2.3.) In particular, $C_G(V)/C_K(V)$ is a summand of W, because the characteristic of V lies in π. The formula $\delta(xk) = xC_K(V)$, for $x \in C_G(V)$ and $k \in K$, defines a derivation from G to $C_G(V)/C_K(V)$, and therefore from G to W,

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which cannot be inner since it does not vanish on $C_G(V)$, but which vanishes on K. Thus the kernel of the restriction map $H^1(G, W) \to H^1(K, W)$ contains a non-zero class, namely $[\delta]$, which is what we wanted to prove.

We have verified (i), (ii), (iii) and (iv), and the proof of Theorem 4.3.2 is complete. □

We next identify the conjugacy class of subgroups T defined in Theorem 4.3.2 in terms of the structure of the group G. We return to the slightly more general situation where W may be any sum of distinct crowns of G, rather than just a sum of all the crowns whose characteristics lie in a given set.

Lemma 4.3.4. *Let G be π-soluble, let $V_1, \ldots, V_r$ be nonisomorphic simple modules for G over prime fields $\mathbb{Z}_p$, where each p belongs to π, and for $1 \leq i \leq r$ let W_i be the crown $\kappa(V_i)$ of G. Let W be the direct sum of the W_i, and let $T = \gamma(H^1(G, W))$, as in Theorem 4.3.2. Then the subgroups in the conjugacy class of T are the intersections of the form*

$$D_1 \cap \ldots \cap D_r$$

where D_i is a complement of W_i, for $1 \leq i \leq r$.

Proof. The D_i give rise to derivations δ_i, as in Theorem 4.2.7 (we did not need to make any choice among the different complements to each crown), and the subgroup T may be taken to be the kernel of the derivation δ of that theorem, the other subgroups in $\gamma(H^1(G, W))$ being conjugates of T by Theorem 2.4.8. The kernel of δ_i is clearly D_i, and the result follows. □

If W is the sum of all the crowns whose characteristics lie in π, then we

remarked in the proof of Theorem 4.3.2 that the index of T in G is a π-number.

Recall that since G is π-soluble it contains a unique conjugacy class of Sylow π-complements (Gorenstein [1, Chapter 6, Section 3]):

Corollary 4.3.5. *Let W be the sum of all crowns of G whose characteristics are primes in π. Then the subgroups $T = \gamma(H^1(G, W))$ are equal to the Sylow π-complements of G if and only if in any chief series for G, each factor whose characteristic belongs to π is complemented in G.*

Proof. By Lemma 4.3.3 the common index of the subgroups T is equal to the product of the orders of the crowns in G, or, by Corollary 4.2.4, to the product of the orders of those complemented factors in any chief series for G whose characteristics belong to π. On the other hand the index of a Sylow π-complement is the product of the orders of all the π-factors in any chief series. $\square$

In the final result of this section we demonstrate how the Galois connection we have defined can be used to determine homotopy types. The next result was first proved, essentially, by Volkmar Welker, although he works with soluble groups.

Corollary 4.3.6 (See Welker [1, Satz 2.9]). *Let G be a π-soluble group, and suppose that every π-chief factor of G is complemented. Let T be the conjugacy class of Sylow π-complements in G. Then the order complex of the partially ordered set $\Delta(G, T)$, is homotopy equivalent to a bouquet of spheres.*

Proof. By Theorem 4.3.2 and Corollary 4.3.5, the order complexes of $\Delta(G, T)$ and $\Xi(G)$ are homotopy equivalent. However, $\Xi^*(G)$ is the lattice of submodules of a module over a ring, and is therefore a modular lattice. It follows immediately that $|\Xi(G)|$ has the homotopy type of a bouquet of spheres, by a theorem of

Kratzer and Thévenaz [2, Théorème 3.4]. (The result about modular lattices can also be deduced from a theorem of Folkman; see Hawkes, Isaacs and Ozaydin [1, page 1030] or Quillen [1, page 118] for a discussion.) □

4.4 $\mathcal{F}$-prefrattini subgroups and $\mathcal{F}$-normalizers.

In this section we apply the theory of Section 4.3 to our problem of estabishing a converse to Theorem 3.3.1. Once again we let $\mathcal{F}$ denote a local formation, and here we write $W(\mathcal{F})$ for the sum of the crowns of a soluble group G whose summands are $\mathcal{F}$-eccentric. Then in the notation of Section 4.2, the maximal subgroups T of G for which the restriction map

$$\text{res: } H^1(G, W(\mathcal{F})) \to H^1(T, W(\mathcal{F}))$$

is zero, are the subgroups $\gamma(H^1(G, W(\mathcal{F})))$ of G. They form a single conjugacy class by Theorem 4.2.8, and by Lemma 4.3.4 we know that the subgroups in this conjugacy class are just the various intersections of one complement from each of the crowns of G that are summands of $W(\mathcal{F})$. From this description we can identify the subgroups T immediately; they are the $\mathcal{F}$-prefrattini subgroups of G which were introduced by Hawkes [2]. The description in terms of crown complements can be taken as the definition of the $\mathcal{F}$-prefrattini subgroups, but for convenience we give the original definition:

Definition 4.4.1 (See Hawkes [2, page 149]). Let Σ be a Sylow system of G, and for each prime p, let G^p be the Sylow p-complement in Σ. Let

$$1 = N_0 < N_1 < \ldots < N_n = G,$$

be any chief series for G, and for each complemented $\mathcal{F}$-eccentric p-factor in this series choose a complement which contains G^p. Then the intersection of these complements is the $\mathcal{F}$-prefrattini subgroup of G corresponding to Σ.

It is shown in [2] that this definition is independent of the choices of chief series and of the complements. The definition makes it look as if an arbitrary intersection of complements of distinct factors might not be an $\mathcal{F}$-prefrattini subgroup, but in fact all such intersections are conjugate; once we know that an $\mathcal{F}$-prefrattini subgroup can be expressed as an intersection of crown complements, this follows from Proposition 4.2.5.

Lemma 4.4.2. *If $V_1, \ldots, V_r$ are the nonisomorphic $\mathcal{F}$-eccentric irreducible modules for G, and $\kappa(V_1), \ldots, \kappa(V_r)$ are their crowns, then each intersection $D_1 \cap \ldots \cap D_r$, where D_i is a complement of $\kappa(V_i)$, is an $\mathcal{F}$-prefrattini subgroup of G, and vice-versa.*

Proof. Immediate from Proposition 4.2.5. and Corollary 4.2.6 □

Corollary 4.4.3. *Let G be a finite soluble group and let K be a subgroup of G. Then the following are equivalent:*

(i) The restriction map

$$\text{res:} \ H^1(G, V) \to H^1(K, V)$$

is zero for all $\mathcal{F}$-eccentric, irreducible G-modules V.

(ii) K is contained in an $\mathcal{F}$-prefrattini subgroup of G.

Proof. The $\mathcal{F}$-prefrattini subgroups of G are the subgroups in $\gamma(H^1(G, W(\mathcal{F})))$, by Lemmas 4.3.4 and 4.4.2. □

Fortunately there is a close relationship between the $\mathcal{F}$-prefrattini subgroups and the $\mathcal{F}$-normalizers of a soluble group G; in particular these classes of

subgroups coincide when G is an nC-group. When $\mathcal{F}$ is the formation consisting of the trivial group only, the $\mathcal{F}$-prefrattini subgroups of G are the prefrattini subgroups defined by Gaschütz [3], which are trivial if and only if all chief factors of the group are complemented, that is if and only if G is an nC-group. (See page 25.) In general there is the following theorem of Hawkes:

Theorem 4.4.4 (Hawkes [2, Theorem 4.1]). *Let $\mathcal{F}$ be a local formation and let G be a finite soluble group. Each $\mathcal{F}$-prefrattini subgroup of G is the permutable product of a prefrattini subgroup and an $\mathcal{F}$-normalizer. In particular, if G is an nC-group, then the $\mathcal{F}$-prefrattini subgroups coincide with the $\mathcal{F}$-normalizers.*

From Theorem 4.4.4 and Corollary 4.4.3, we obtain

Corollary 4.4.5. *Statement 4.1.1 is true for nC-groups.*

4.5 Local conjugacy.

We can use the splitting of cohomology according to the Sylow structure of a group (Proposition 3.2.1) to obtain from Theorem 4.3.2 and Corollary 4.4.3 some new results about local conjugacy in finite soluble groups.

Definition 4.5.1. Suppose that H and K are subgroups of the finite group G. Then H and K are *locally conjugate* if and only if each Sylow subgroup of H is conjugate in G to a Sylow subgroup of K.

More generally, if every Sylow subgroup of H is conjugate to a *subgroup* of a Sylow p-subgroup of K, then we will say that H is *locally subconjugate to K.* This is weaker in general than saying that H is locally conjugate to a subgroup of K, as the following example shows:

Example 4.5.2. Let G be the wreath product of the alternating group of degree 4 with a cyclic group of order 2, so that G is the semi-direct product $\left[\left(A_4 \times A_4\right)\right] C_2$. Let x and t be elements of A_4 with orders 3 and 2 respectively, and let H be the subgroup of G generated by the element (x, t) of the base group $A_4 \times A_4$. Then H is cyclic of order 6. If K is the subgroup $1 \times A_4$ of G, then clearly both the Sylow 2-and 3-subgroups of H are conjugate to a subgroup of K, so that H is locally subconjugate to K, but on the other hand H cannot be locally conjugate to a subgroup of K because K has no subgroup of order 6. □

Local conjugacy is a weaker relation on subgroups of a finite group than true conjugacy. Losey and Stonehewer [1] give conditions under which two locally conjugate subgroups of a finite soluble group must be truly conjugate, as well as examples where they are not. Here we exploit the fact that where, as in Theorem 4.2.8, a conjugacy class of subgroups T of a soluble group is known to be characterized by the behaviour of the restriction map $H^1(G, W) \to H^1(T, W)$ for some module W, the splitting of cohomology allows us to deduce that the conjugacy class of T is also a local conjugacy class:

Lemma 4.5.3. *Let G be a finite group and let K be a subgroup of G. Let $Q_1, \ldots, Q_r$ be a set of Sylow subgroups of K, one for each prime which divides |K|. Then for any G-module W, the kernel of the map*

$$res\colon H^1(G, W) \to H^1(K, W)$$

is equal to the intersection over $1 \le i \le r$ of the kernels of the maps

$$res\colon H^1(G, W) \to H^1(Q_i, W).$$

Proof. By Theorem 3.2.2, the product res: $H^1(G, W) \to \prod H^1(Q_i, W)$ is a

monomorphism, and the result follows immediately. □

Corollary 4.5.4. *In the notation of Theorem 4.2.8, let $T = \gamma(U)$ for some submodule U of $H^1(G, W)$, and let K be a subgroup of G which is locally subconjugate to T. Then K is conjugate to a subgroup of T.*

Proof. By hypothesis, for every Sylow subgroup Q of K, the kernel of the map res: $H^1(G, W) \to H^1(Q, W)$ contains U. By Lemma 4.5.3, it follows that the kernel of res: $H^1(G, W) \to H^1(K, W)$ contains U. Therefore $K \leq_G T$ by definition of T. □

For example, from our characterisation of $\mathcal{F}$-prefrattini subgroups, Corollary 4.4.3, we deduce the following theorem.

Theorem 4.5.5. *Let T be an $\mathcal{F}$-prefrattini subgroup of the finite soluble group G, and suppose that the subgroup K of G is locally subconjugate to T. Then K is conjugate to a subgroup of T.*

Corollary 4.5.6. *The $\mathcal{F}$-prefrattini subgroups of a finite soluble group form a conjugacy class which is closed under local conjugacy.*

Another application of Corollary 4.5.4 is to the Galois connection between $\Delta(G, T)$ and $\Xi(G)$ established in Theorem 4.3.2. We consider the simplest case, where G is a soluble nC-group and W is the sum of all the crowns of G, so that $T = 1$. (This is Corollary 4.3.5 in with π = all primes.) Theorem 4.3.2 says that in those circumstances σ and γ define a Galois connection between $\Delta(G)$ and $\Xi(G)$.

We introduce a new partially ordered set, written $\Delta^{loc}(G)$, whose elements are the *local* conjugacy classes of subgroups of G, with the partial ordering given by

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local subconjugacy. Then there is a natural projection map π: $\Delta(G) \longrightarrow \Delta^{loc}(G)$, which associates to each conjugacy class of subgroups of G the local conjugacy class containing it, and it is clear that π is order-preserving. The next result shows that in fact π is a homotopy equivalence between the order complexes of the two partially ordered sets.

Corollary 4.5.7. *Suppose that G is a finite soluble nC-group. Then π is a homotopy equivalence between the order complexes of $\Delta(G)$ and of $\Delta^{loc}(G)$.*

Proof. By Lemma 4.5.3, the map σ: $\Delta(G) \to \Xi(G)$ factors through π. If we write $\sigma = \bar{\sigma} \circ \pi$, where $\bar{\sigma}$: $\Delta^{loc}(G) \to \Xi(G)$ is the quotient of σ, then clearly π is order preserving, $\bar{\sigma}$ is order reversing, and $\tau = \gamma \circ \bar{\sigma}$: $\Xi(G) \to \Delta(G)$ is order preserving. It is easy to check (using the surjectivity of π) that $\tau \circ \bar{\sigma}(H) \geq H$ for all $H \in \Delta^{loc}(G)$, while for all $U \in \Xi(G)$, $\bar{\sigma} \circ \tau(U) \geq U$, so the simplicial maps induced by $\bar{\sigma}$ and τ between the order complexes of $\Pi^{loc}(G)$ and $\Xi(G)$ are homotopy equivalences. (Note that $\bar{\sigma}$ and τ do not constitute a Galois connection between $\Pi^{loc}(G)$ and $\Xi(G)$, because they are order-*preserving* rather than order-reversing. We therefore cannot use Theorem 1.1.3 itself, but its proof from Lemma 1.1.2 works equally well in this case.) □

Remark Corollary 4.5.6 can also be deduced in a straightforward way from a special case of a theorem of Losey and Stonehewer ([1]). We sketch the argument in the case where $\mathcal{F}$ is the formation consisting of the trivial group only, that is for the prefrattini subgroups of G. Since these are characterized (Gaschütz [3, Satz 6.2]) as the largest subgroups of G which are contained in some conjugate of every maximal subgroup of G, it is enough to show that a conjugacy class of *maximal* subgroups of a soluble group is also a local conjugacy class. Suppose, therefore that H and K are two subgroups of G which are locally conjugate, with

H maximal. Let V be a minimal normal subgroup of G. If V is contained in H, then K must also contain V, and by induction H/V and K/V are conjugate in G/V, which shows that H and K are conjugate in G. Otherwise V is a common normal complement for H and K, and by Losey and Stonehewer ([1], Theorem B), H and K are indeed conjugate, as required.

4.6 The Frattini subgroup.

The Frattini subgroup of a soluble group is the largest normal subgroup contained in its prefrattini subgroups (Gaschütz [3, Satz 6.5], or Doerk and Hawkes [1, page 422]). Therefore Corollary 4.4.3 implies that the Frattini subgroup of a soluble group G is the unique maximal element of the set of normal subgroups N of G for which the restriction map

$$\text{res: } H^1(G, V) \to H^1(N, V)$$

is zero for *all* simple coefficient modules V. (When $\mathcal{F}$ consists of the trivial group only, all simple G-modules are $\mathcal{F}$-eccentric.) The same characterization holds for insoluble groups:

Theorem 4.6.1. *Let N be a normal subgroup of the finite group G. Then the following are equivalent:*

(i) The restriction map
$$\text{res: } H^1(G, V) \to H^1(N, V)$$

is zero, for all irreducible coefficient modules V.

(ii) N is contained in the Frattini subgroup of G.

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Proof Suppose first that N satisfies (i). If V is a simple G-module with nonvanishing 1-cohomology, then the inflation-restriction sequence

$$0 \to H^1(G/N, V^N) \xrightarrow{\text{inf}} H^1(G, V) \xrightarrow{\text{res}} H^1(N, V)$$

shows that V^N must be non-zero, which since V is irreducible implies that N must be in the kernel of V. The intersection of the kernels of all the simple G-modules that have non-vanishing 1-cohomology is the Fitting subgroup of G (Griess and Schmid [1, Theorem 1]), so at least $N \le \text{Fit}(G)$. To show that in fact N $\le \text{Frat}(G)$, let

$$1 = N_0 < N_1 < \ldots < N_r = N$$

be a piece of chief series of G from 1 to N. Since N is nilpotent, each factor is abelian and is either Frattini or complemented by a maximal subgroup of G. Suppose, for a contradiction, that there is a complemented factor $V = N_i/N_{i-1}$, and let C be a complement for this factor in G. The map $\delta: G \to V$ defined by writing $\delta(nc) = nN_{i-1}$ for $n \in N_i$ and $c \in C$, is easily checked to be a derivation (see page 59 above), and if $[\delta]$ is the class of δ in $H^1(G, V)$, the restriction of $[\delta]$ to $H^1(N, V)$ non-zero because δ is not zero on N, while on the other hand any inner deruivation for N to V is certainly zero, since N centralizes V. Therefore all the factors N_i/N_{i-1} are Frattini, so $N \le \text{Frat}(G)$, as required.

On the other hand, suppose that N satisfies (ii). Then certainly $N \le \text{Fit}(G)$ so by the result on Fit(G) above, N centralizes any simple module with non-vanishing 1-cohomology. Therefore, if V is any such module, $H^1(N, V) = \text{Hom}(N, V)$ and the image under restriction to N of $H^1(G, V)$ is contained in the G-invariant subspace $\text{Hom}_G(N, V)$. Any non-zero element of this image must be a *surjective* homomorphism, because V is G-irreducible, so if $\delta: G \to V$ is a derivation such that $[\delta]$ does not vanish on restriction to N, we have $\delta(N) = V$, and we may

calculate that $|G:\ker \delta| = |\delta(G)| = |\delta(N)| = |N:N \cap \ker \delta|$, or in other words $N.\ker \delta = G$. But $\ker \delta$ is certainly a proper subgroup of G, so the last equation contradicts the hypothesis that N is contained in the Frattini subgroup of G. This completes the proof. □

In Chapter 3 we carried out a similar analysis for the hypercentre of any group; it is amusing to note the following (well known) result as a corollary:

Corollary 4.6.2. *Let G be any finite group, and let G', $Z_{\infty}(G)$ and $\Phi(G)$ denote the derived subgroup, hypercentre, and Frattini subgroup of G respectively. Then $G' \cap Z_{\infty}(G) \leq \Phi(G)$.*

Proof. If V is a *central* simple module for G, then an element of $H^1(G, V)$ is a homomorphism from G to V, which therefore vanishes on G'. If V is eccentric then such an element vanishes on restriction to $Z_{\infty}(G)$, by Theorem 3.4.1. The result now follows from Theorem 4.6.1. □

Finally, we show that Theorem 2.4.1 follows from Corollary 4.4.3. Recall that a soluble group is an nC-group if and only if its prefrattini subgroups are trivial.

Theorem 4.6.3. *Let N be a normal subgroup of the soluble group G and let D be a prefrattini subgroup of N. Then D is contained in a prefrattini subgroup of G.*

Proof. Let V be an irreducible module for G. Then by Clifford's theorem the restriction of V to N is a semisimple module for N. Therefore, by Corollary 4.4.3 the map res: $H^1(N, V) \to H^1(D, V)$ is zero. The restriction map from $H^1(G, V)$ to $H^1(D, V)$ factors through $H^1(N, V)$, and is therefore also zero, so that D must lie in a prefrattini subgroup of G, again by Corollary 4.4.3. □

CHAPTER 5

5.1 Introduction.

Corollary 4.4.3 shows that the 'ideal converse' 4.1.1 is true for soluble nC-groups, but it also shows that in groups where the $\mathcal{F}$-prefrattini subgroups and $\mathcal{F}$-normalizers do not happen to coincide, there is no chance of obtaining 4.1.1 by considering only first cohomology groups. To try to distinguish between $\mathcal{F}$-prefrattini subgroups and $\mathcal{F}$-normalizers in general we are forced to look at the behaviour of the restriction map in higher-dimensional cohomology, which is what we do in this chapter.

A consequence of our analysis in Chapter 4 is the fact that the $\mathcal{F}$-prefrattini subgroups of any soluble group form a local conjugacy class (Theorem 4.5.5). If the ideal converse 4.1.1 were true then we could deduce the same thing about the conjugacy class of $\mathcal{F}$-normalizers. The truth or otherwise of this in general does not seem to be known, but in fact special cases have been proved (without using cohomology). For example, compare Theorem 4.5.5 with the following result, which is due to Alperin:

Theorem 5.1.1 (Alperin [1, Theorem 9]). *Let D be a system normalizer of the finite soluble group G. If K is a subgroup of G, each of whose Sylow subgroups is conjugate to a subgroup of D, then K is conjugate to a subgroup of D.*

In Section 2 we discuss similar results, due to Chambers [1], on the $\mathcal{F}$-normalizers of soluble groups with abelian Sylow subgroups, and we show that the analogue of Theorem 5.1.1 is true for such groups. Our approach in this chapter makes it necessary to use these results, a reversal of the situation in Chapter 4 where Theorem 4.5.5 follows from the analysis there.

To deal in this chapter with local formations $\mathcal{F}$ whose support π is not the whole set of primes, we will sometimes use the following result of Evens:

Theorem 5.1.2 (Evens [1, Corollary 6.1.2]). *Let H be a subgroup of the finite group G, and suppose that p is a prime which divides the order of H. Then the map*

$$\text{res: } H^r(G, \mathbb{Z}_p) \to H^r(H, \mathbb{Z}_p)$$

is non-zero for infinitely many even values of r. □

A similar result for integral cohomology was proved earlier by Swan [1]. We derive the following easy corollary:

Corollary 5.1.3. *Let G be a soluble group, and let π be a set of primes. If H is any subgroup of G, then the following are equivalent:*

(i) The map

$$\text{res: } H^r(G, V) \to H^r(H, V)$$

is zero for all sufficiently large even r, for all irreducible modules V over fields whose characteristic does not belong to π

(ii) H is contained in a Hall π-subgroup of G.

Proof. (i) $\Rightarrow$ (ii) is immediate from Theorem 5.1.2. Conversely, (ii) $\Rightarrow$ (i) follows from Proposition 3.2.1. □

Notice how similar Corollary 5.1.3 looks to Statement 4.1.1. In fact, although we

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do not use this, the corollary can be construed as a special case of Statement 4.1.1, if one is prepared to work with non-integrated formation functions f. This is because the Hall π-subgroups of a group are the 'f-normalizers' corresponding to the non-integrated formation function f given by $f(p) = \{$all soluble groups$\}$ for $p \in \pi$, and $f(p)$ empty for $p \notin \pi$. (A normalizer corresponding to a non-integrated formation function is defined the formula of Definition 2.2.2, as for the integrated case, but the subgroups so defined share only some of the properties of normalizers defined by integrated functions.)

Evens' proof of Theorem 5.1.2 is a simple application of the Evens norm map, which we also use in Section 5.6. We need a result like Theorem 5.1.2 in this chapter to ensure that subgroups which satisfy the condition of Statement 4.1.1 must be π-subgroups, where π is the support of the relevant local formation, and in my opinion the results of this chapter are best seen as 'equivariant' versions of Theorem 5.1.2. Until Section 6, where we need it anyway, we avoid the use of the norm map by essentially proving special cases of Corollary 5.1.3 as we go along, but in Section 6 we need to use the norm map anyway, and so appeal direct to Theorem 5.1.2.

5.2 $\mathcal{F}$-normalizers of A-groups.

The $\mathcal{F}$-normalizers of A-groups were shown by Chambers [1] to have special properties from which we can deduce all that we need concerning their behaviour. In the case of soluble groups with *elementary* abelian Sylow subgroups, which are also nC-groups, we could use Theorem 4.5.5, but the results of this section are equally applicable in the general case. As usual, $\mathcal{F}$ is a local formation.

Definition 5.2.1 (Chambers [1, Section 2]). A subgroup H of a soluble group G is said to be *p-normally embedded* if each Sylow p-subgroup of H is also a Sylow subgroup of a *normal* subgroup of G. A subgroup H which is p-normally embedded for every prime p is said to be *normally embedded*.

Theorem 5.2.2 (Chambers [1, Corollary 3.4]). *Let G be a finite A-group. Then the $\mathcal{F}$-normalizers of G are normally embedded in G.*

(Chambers states his theorem under the assumption that $\mathcal{F}$ contains the formation of nilpotent groups, because the original definition of $\mathcal{F}$-normalizers (Carter and Hawkes [1, Section 4]) required this condition. His proof applies equally to the general case.)

Recall Definition 4.5.1; for K, H $\leq$ G we say that K is locally subconjugate to H if every Sylow subgroup of K is conjugate to a subgroup of H. The following theorem is inspired by Chambers [1, Theorem 2.6].

Theorem 5.2.3. *Let H be a normally embedded subgroup of a finite soluble group G, and suppose that the subgroup K of G is locally subconjugate to H. Then K is conjugate to a subgroup of H.*

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Proof. We proceed by induction. Let N be a minimal normal subgroup of G, and let p be the prime dividing the order of N. By Chambers [1, Proposition 2.2], HN/N is a normally embedded subgroup of G/N, and the relationship between K and H passes to quotients, so by induction we may assume, replacing K with a conjugate subgroup if necessary, that $KN \leq HN$. Since N is a p-group, H contains a Sylow p-complement of HN, so by replacing KN, if necessary, with a conjugate by an element of HN, we may suppose at the same time that there is a Sylow p-complement K^p of K such that $K^p \leq H$. We choose any Sylow p-subgroup K_p of K, and a Sylow p-subgroup H_p of H such that

$$K_p \leq (HN)_p = H_pN.$$

By hypothesis G has a normal subgroup T which contains H_p as a Sylow subgroup. The subgroup T must also contain K_p, since K_p is supposed to be conjugate in G to a subgroup of H_p. Therefore the join $\langle K_p, H_p \rangle$, being a subgroup of T, contains H_p as a Sylow subgroup, and it follows that there exists an element $x \in \langle K_p, H_p \rangle$ for which $K_p \leq (H_p)^x$. Both K_p and H_p are contained in H_pN, and we deduce that x is contained in H_pN also. But $H_p = T \cap (H_pN)$ is *normal* in H_pN, so really $K_p \leq H_p$. Thus both K_p and K^p are contained in H, and we are done. □

Corollary 5.2.4. *Let D be an $\mathcal{F}$-normalizer of the finite A-group G. If $K \leq G$ is locally subconjugate to D, then K is conjugate to a subgroup of D.*

Proof. Immediate from 5.2.2 and 5.2.3. □

If we wish to establish that a p-subgroup Q of a soluble group G belongs to an $\mathcal{F}$-normalizer of G, we may consider instead of G the quotient G/N of G by any

normal subgroup of G of p'-order; epimorphic maps preserve $\mathcal{F}$-normalizers (Theorem 2.2.3(ii)),and it follows that Q is contained in an $\mathcal{F}$-normalizer of G if and only if QN/N is contained in an $\mathcal{F}$-normalizer of G/N. The same applies to $\mathcal{F}$-prefrattini subgroups, because these are also epimorphism-invariant (Hawkes [2, Corollary 3.5]). For convenience we record this remark in a lemma:

Lemma 5.2.5. *Let Q be a p-subgroup of the soluble group G. Then Q is contained in an $\mathcal{F}$-normalizer of G if and only if $QO_{p'}(G)/O_{p'}(G)$ is contained in an $\mathcal{F}$-normalizer of $G/O_{p'}(G)$. The same holds for $\mathcal{F}$-prefrattini subgroups.*

If G has p-length one then $G/O_{p'}(G)$ *has a normal Sylow p-subgroup.* A criterion for a p-subgroup of such a group to be contained in an $\mathcal{F}$-normalizer is given by the next lemma, whose content is well known.

Lemma 5.2.6. *Suppose that the local formation $\mathcal{F}$ is defined by the integrated formation function f. Let G be a group with a normal Sylow p-subgroup P, and let $Q \leq P$ be a p-subgroup of G, where p belongs to the support of $\mathcal{F}$. Let K be a p-complement of the $f(p)$-residual $G^{f(p)}$. Then the following are equivalent:*

(i) Q *is contained in an $\mathcal{F}$-normalizer of G;*

(ii) K *centralizes a P-conjugate of Q.*

Proof. By Definition 2.2.2 and the remark immediately following it, the $\mathcal{F}$-normalizers of G are the Sylow π-subgroups of the subgroups of G of the form

$$\bigcap_{s \in \pi} N_G(G^s \cap G^{f(s)}),$$

where $\{G^s : s \in \pi\}$ is any set of s-complements, $s \in \pi$. Since Q is a p-group,

where $p \in \pi$, we may choose the G^s to contain Q, except for s = p; then $Q \nleqslant$ $N_G(G^s \cap G^{f(s)})$ for all $s \in \pi$ except p. Therefore Q is contained in an $\mathcal{F}$-normalizer of G if and only if Q normalizes $G^P \cap G^{f(p)}$ for some G^P; in other words, if and only if Q normalizes a p-complement K of $G^{f(p)}$. The set of p-complements of $G^{f(p)}$ is also the set of p-complements of the normal subgroup $PK = PG^{f(p)}$ of G, so they are all conjugate by elements of P; furthermore, one has

$$P \bigcap N_G(K) = C_P(K),$$

because P is normal in G. The result follows. □

If P is *abelian* then condition (ii) of the above lemma simply says that K centralizes Q.

5.3 Reduction to p-group cohomology.

In this section we show that for a group G of p-length one, the result we seek, 4.1.1, is equivalent to a statement about the action of automorphisms on the cohomology ring with $\mathbb{Z}_p$-coefficients of a Sylow p-subgroup of G. Our first reduction of the problem is to groups with $O_{p'}(G) = 1$:

Lemma 5.3.1. *Suppose that N is a normal subgroup of p'-order of the finite group G. Then for any G-module M of characteristic p, the inflation map*

$$\text{inf: } H^r(G/N, M^N) \longrightarrow H^r(G, M)$$

is an isomorphism for all $r \geq 1$. If K is a subgroup of G then the diagram

$$
\begin{array}{ccc}
H^*(G/N, M^N) & \xrightarrow{\quad\text{inf}\quad} & H^*(G, M) \\
\Big\downarrow{\text{res}} & & \Big\downarrow{\text{res}} \\
H^*(KN/N, M^N) & \xrightarrow{\quad\text{inf}\quad} & H^*(K, M)
\end{array}
$$

is commutative.

Proof. The first part is Corollary 3.2.4. The commutativity of the diagram follows directly from the definitions of inflation and restriction (Section 3.2). □

Corollary 5.3.2. *Suppose that $V_1, \ldots, V_m$ are irreducible G-modules (of any characteristics). Then for a p-subgroup Q of G, the following are equivalent:*

(i) The map

$$\text{res: } H^r(G, V_i) \to H^r(Q, V_i)$$

is zero, for each i, $1 \le i \le m$.

(ii) The map

$$\text{res: } H^r(G/N, V_i) \to H^r(QN/N, V_i)$$

is zero whenever V_i is of characteristic p and $N \le \ker_G(V_i)$ (V_i is then regarded as a G/N-module by deflation).

Proof. Since the V_i are irreducible and N is a normal subgroup of G, the N-fixed point subgroup V_i^N is either zero of the whole of V_i. In the former case the condition (i) is vacuous, by the first part of Lemma 5.3.1, and the same is true if the characteristic of V_i is different from p, by Proposition 3.2.1. Thus we need only consider (i) for those V_i which deflate to G/N, and for such a module (i) and (ii) state respectively that the right and left hand verticals of the diagram of Lemma 5.3.1 are zero. These statements are clearly equivalent, because the diagram is commutative and its horizontal maps are isomorphisms. $\square$

The next theorem is the main result of this section. The study of 4.1.1 for groups of p-length one is reduced to a problem concerning the action of automorphisms on the cohomology ring of a p-group. The extra structure of the cup product in this ring will be useful in the investigation in Sections 4, 5 and 6.

Theorem 5.3.3. *Let G be a finite group with a normal Sylow p-subgroup P. Let Q be a subgroup of P and let N be a normal subgroup of G which contains P. Let K be a complement of P in N. Then for any $r \geq 1$, the following are equivalent:*

(i) The restriction map

$$\text{res: } H^r(G, V) \to H^r(Q, V)$$

is zero for all simple G-modules V with $N \not\leq \ker_G(V)$;

(ii) For all $k \in K$ and $\omega \in H^r(P, \mathbb{Z}_p)$, the following condition holds:

$$\text{res}_Q(\omega) = \text{res}_Q(\omega^k)$$

where the notation ω^k refers to the natural action of G on $H^*(P, \mathbb{Z}_p)$.

The proof is based on the Eckmann–Shapiro lemma, which is most natural to state in terms of the notion of a *coinduced* module (Brown [1, Chapter III, Section 5]). If H is a subgroup of G and M is a (left) module for H over $\mathbb{Z}_p$ then the coinduced module $\text{Coind}^G(M)$ is the abelian group $\text{Hom}_{\mathbb{Z}_p H}(\mathbb{Z}_p G, M)$, made into a left $\mathbb{Z}_p G$-module by writing

$$g(\varphi)(z) = \varphi(zg)$$

for $\varphi \in \text{Hom}_{\mathbb{Z}_p H}(\mathbb{Z}_p G, M)$ and all $g \in G$ and $z \in \mathbb{Z}G$. Note that for finite groups, induced and coinduced modules are naturally isomorphic (Brown [1, Chapter III, Proposition 5.9]); our use here of coinduction rather than induction is a matter of convenience.

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Lemma 5.3.4 (Eckmann-Shapiro). *Let G be a group with a subgroup H, and let M be a module for H. Then for any G-module X there is a natural isomorphism*

$$Hom_H(X, M) \longrightarrow Hom_G(X, Coind^G(M))$$

given by $f \longmapsto F$, where

$$F(x)(g) = f(gx), \text{ for } x \in X, g \in G.$$

If $X \longrightarrow \mathbb{Z}_p$ is a projective resolution over G (and therefore also over H) then the above map is an isomorphism of chain complexes

$$Hom_H^*(X, M) \longrightarrow Hom_G^*(X, Coind^G(M)).$$

Therefore the cohomology groups $H^*(H, M)$ and $H^*(\mathcal{G}, Coind^G(M))$ are naturally isomorphic.

Proof. One checks that the map $f \longmapsto F$ is an isomorphism of abelian groups and commutes with the chain maps of the complexes. The inverse of the map $f \longmapsto F$ is given by $f(x) = F(x)(1_G)$. See Evens [1, Proposition 4.1.3] for the details. □

We need the following well-known description of the G-module (co)induced from the trivial $\mathbb{Z}_p P$-module, where P is a normal subgroup of G:

Lemma 5.3.5. *Suppose that P is a normal Sylow p-subgroup of the group G. Then the module $Coind_P^G(\mathbb{Z}_p)$ is isomorphic to the inflation to G of the regular G/P-module. It is therefore a direct sum of irreducible $\mathbb{Z}_p G$-modules, in which each such module occurs with multiplicity at least one.*

Proof. There is a natural isomorphism of G–modules

$$\text{Hom}_{\mathbb{Z}_p P}(\mathbb{Z}_p G, \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p(G/P),$$

given by

$$\varphi \longmapsto \sum_{s \in G/P} \varphi(s^{-1})\, s .$$

The group algebra $\mathbb{Z}_p(G/P)$ is semisimple, because the order of G/P is prime

to p; it is therefore a sum, with non–zero multiplicities, of all the simple modules

for the group G/P over $\mathbb{Z}_p$. On the other hand, any simple module for $\mathbb{Z}_p G$ is

centralized by $P = O_p(G)$, and so occurs as the inflation to G of some simple

G/P–module. $\square$

Proof of 5.3.3.

Write W for the module $\text{Coind}_P^G(\mathbb{Z}_p) = \text{Hom}_{\mathbb{Z}_p P}(\mathbb{Z}_p G, \mathbb{Z}_p)$ in Lemma 5.3.5.

Let W^N be the subgroup of W consisting of N–fixed points. Then W^N is a $\mathbb{Z}_p G$–

submodule of W, because N is normal in G. By Lemma 5.3.5, W is a semisimple

$\mathbb{Z}_p G$–module, so there exists a $\mathbb{Z}_p G$–submodule U of W such that

$$W = W^N \oplus U.$$

A simple submodule V of W is contained in W^N if and only if $N \leq \ker_G(V)$. It

follows that in a decomposition of W into homogeneous components (see Doerk

and Hawkes [1, Chapter B, Definition 3.4]), the submodule W^N is the direct sum

of the homogeneous components of the simple submodules V of W that have $N \leq$

$\ker_G(V)$, and U is the sum of the other homogeneous components of W. In

particular, U is *unique*.

The submodule W^N consists of those maps $\varphi \in \operatorname{Hom}_{\mathbb{Z}_p P}(\mathbb{Z}_p G, \mathbb{Z}_p)$ which satisfy the equation $\varphi(zn) = \varphi(z)$ for $n \in N$ and $z \in \mathbb{Z}_p G$. Thus the formula

$$\varphi \longmapsto \sum_{s \in G/N} \varphi(s^{-1})\, s$$

defines an isomorphism of G-modules: $W^N \longrightarrow \mathbb{Z}_p(G/N)$. This is a similar situation to Lemma 5.3.5.

The projection $\pi \colon W \longrightarrow W^N$ of the decomposition $W = W^N \oplus U$ is given by

$$\pi(\varphi) = \left(1/|N{:}P|\right) \sum_{t \in N/P} t(\varphi).$$

(This makes sense because p does not divide the index of P in N.) It is easy to check that π is a map of G-modules such that $\pi(\varphi) = \varphi$ for $\varphi \in W^N$.

The Sylow subgroup P of G acts trivially on W; we can define an isomorphism of $\mathbb{Z}_p P$-modules,

$$W \longrightarrow \bigoplus_{s \in G/P} \mathbb{Z}_p \qquad\qquad (*)$$

by evaluation at a transversal of G/P:

$$\varphi \longmapsto \left(\varphi(s^{-1})\right)_{s \in G/P}.$$

In view of the description of the simple summands of the modules W^N and U above, assertion (i) in the statement of Theorem 5.3.3 says equivalently that the restriction map res: $H^r(G, U) \to H^r(Q, U)$ is zero, or in other words:

$$\operatorname{res}_Q(\omega) = \operatorname{res}_Q(\pi_*(\omega))$$

for all $\omega \in H^r(G, W)$. Since restriction is transitive and π_* commutes with restriction, this condition may be rewritten as

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$$\mathrm{res}_Q\big(\mathrm{res}_P(\omega)\big) = \mathrm{res}_Q\big(\pi_*(\mathrm{res}_P(\omega))\big), \text{ for all } \omega \in H^r(G, W).$$

We now use the Eckmann–Shapiro isomorphism $H^r(P, \mathbb{Z}_p) \xrightarrow{\sim} H^r(G, W)$ (Lemma 5.3.4) to rewrite this condition in terms of elements of $H^r(P, \mathbb{Z}_p)$. Thus if $\eta \in H^r(P, \mathbb{Z}_p)$ is represented by a cocycle $f \in \mathrm{Hom}_{\mathbb{Z}_p P}(X_r, \mathbb{Z}_p)$ (where as in Lemma 5.3.4, we have chosen a $\mathbb{Z}_p G$-projective resolution X of $\mathbb{Z}_p$) then the image ω of η under the Eckmann–Shapiro isomorphism is the cohomology class of the map $F \in \mathrm{Hom}_{\mathbb{Z}_p G}(X_r, W)$, where $F(x)(g) = f(gx)$ for all $x \in X_r$. Using the evaluation map $(*)$ to identify W with $\oplus \, \mathbb{Z}_p$ as P-modules, we see that $\mathrm{res}_P(\omega)$ is represented by the cocycle $F: X_r \longrightarrow \oplus \, \mathbb{Z}_p$ which takes x to $\big(f(sx)\big)_{s \in G/P}$. In other words, $\mathrm{res}_P(\omega) \in H^r(P, \oplus \, \mathbb{Z}_p)$ is just the direct sum of the conjugates of η by the elements of G/P. (See Section 3.2.) On the other hand, $\pi_*(\mathrm{res}_P(\omega))$ is represented by $\pi \circ F: X_r \longrightarrow \oplus \, \mathbb{Z}_p$. The component $(\pi \circ F)_s: X_r \longrightarrow \mathbb{Z}_p$ of $\pi \circ F$ is the map $x \longmapsto (\pi \circ F)(x)(s^{-1})$, and using the formula for π, we obtain

$$(\pi \circ F)(x)(s^{-1}) = \big(1/|N{:}P|\big) \sum_{t \in N/P} \big(f(s^{-1}tx)\big).$$

For each $s \in G/P$, we therefore have;

$$\big(\mathrm{res}_P(\omega)\big)_s = \eta^s;$$
$$\big(\pi_*(\mathrm{res}_P(\omega))\big)_s = \big(1/|N{:}P|\big) \sum_{t \in N/P} (\eta^{t^{-1}})^s.$$

Condition (i) holds if and only if the two expressions are equal upon restriction to Q, for all $s \in G/P$. This happens if and only if $\mathrm{res}_Q(\eta) = \mathrm{res}_Q(\eta^t)$ for all $t \in N/P$ and all $\eta \in H^r(P, \mathbb{Z}_p)$. (We have written t instead of t^{-1}, which makes no difference since $\{t^{-1}\}$ is another transversal of N/P.) This is exactly condition (ii), since we may take K for the transversal to N/P. $\square$

The following is an immediate corollary of the above proof:

Corollary 5.3.6. *In the situation of Theorem 5.3.3, the following two conditions are equivalent:*

(i) The restriction map

$$\text{res: } H^r(G, V) \to H^r(Q, V)$$

is zero for all simple G-modules V.

(ii) For all $\omega \in H^r(P, \mathbb{Z}_p), \text{res}_Q(\omega) = 0.$

Proof. As in the proof of Theorem 5.3.3, we let η be an element of $H^r(P, \mathbb{Z}_p)$ and let ω be its image under the Eckmann – Shapiro isomorphism $H^r(P, \mathbb{Z}_p) \to H^r(G, W)$, where W is the coinduced module $\text{Coind}_P^G(\mathbb{Z}_p)$. In the notation of the proof of 5.3.3,

$$\left(\text{res}_P(\omega)\right)_s = \eta^s \in H^r(P, \mathbb{Z}_p).$$

Thus (i) holds if and only if for all $s \in G/P$ and all $\eta \in H^r(P, \mathbb{Z}_p)$, the element η^s vanishes on restriction to $H^r(Q, \mathbb{Z}_p)$. In other words, the restriction map from $H^r(P, \mathbb{Z}_p)$ to $H^r(Q, \mathbb{Z}_p)$ must be zero, which is (ii). $\square$

In the applications we can use Corollary 5.3.6 to deduce that the subgroup Q must be trivial. Really we could just use Theorem 5.1.2 for this, but by essentially proving Theorem 5.1.2 in the special cases where we require it, we avoid the Evens norm map until Section 6.

5.4 The cohomology rings of abelian p-groups.

Our aim is to use Theorem 5.3.3 in conjunction with Lemma 5.2.5, which is our criterion for a p-subgroup of a group of p-length one to be contained in an $\mathcal{F}$-normalizer. In order to apply 5.2.6, we need to be able to deduce from condition (ii) of Theorem 5.3.3 that the subgroup Q of that theorem, or some conjugate of Q by an element of P, is *fixed pointwise by K*. This fails in general, but when P is abelian we can control the situation by calculating with the known cohomology rings of abelian groups. These calculations are the subject of this section, but we begin with two general remarks.

Remark (i). If K does fix pointwise a subgroup of P which is conjugate to Q in P, then condition (ii) of Theorem 5.3.3 will be certainly be satisfied for Q. To see this, first note that we may assume that Q itself is fixed by K, since the kernel of the restriction map in question depends only on the conjugacy class of Q in P. Then the restriction map from $H^*(P, \mathbb{Z}_p)$ to $H^*(Q, \mathbb{Z}_p)$ commutes with the action of K, but the action of K on $H^*(Q, \mathbb{Z}_p)$ is trivial. (See the definitions of these maps in Section 3.2.) Therefore, for all $\omega \in H^*(P, \mathbb{Z}_p)$, we have $\mathrm{res}_Q\big(\omega - \alpha^*(\omega)\big)$ $= \mathrm{res}_Q(\omega) - \alpha^*\big(\mathrm{res}_Q(\omega)\big) = 0$, as required.

Remark (ii). Condition (ii) of Theorem 5.3.3 still holds in the apparently weaker circumstance that each *element* k of K fixes *some* P-conjugate of Q, where perhaps different conjugates occur. Actually this still implies that one of the P-conjugates of Q is fixed by the whole of K. (Write N for the product PK, as in Theorem 5.3.3. If every element of K centralizes Q^p for some $p \in P$, we find that K is contained in the product $PC_N(Q)$. Since K is a p'-group, the Sylow p-complements of $PC_N(Q)$ are K and its P-conjugates. However, among the Sylow p-complements of $PC_N(Q)$ are all the Sylow p-complements of $C_N(Q)$.) Of course, in this section P is abelian, so the question of different P-conjugates does not arise.

We now proceed to examine the $\mathbb{Z}_p$-cohomology rings of abelian p-groups. The structure of these rings is well known; they are the tensor product of a polynomial algebra over $\mathbb{Z}_p$ with an exterior algebra in the same number of variables. The following proposition is a detailed statement of the structure of $H^*(P, \mathbb{Z}_p)$, for any abelian p-group P:

Proposition 5.4.1. (Cohomology over $\mathbb{Z}_p$ of an abelian p-group.)

Let $P = Q_1 \times \ldots \times Q_n$, where Q_i is cyclic of order p^{e_i}, for $1 \leq i \leq n$. Then the cohomology ring $H^*(P, \mathbb{Z}_p)$ is the graded-commutative $\mathbb{Z}_p$-algebra,

$$\mathbb{Z}_p[\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_n]$$

where $\eta_i \in H^1(P, \mathbb{Z}_p)$ and $\xi_i \in H^2(P, \mathbb{Z}_p)$, subject to the relations

$$\eta_i^2 = 0 \text{ (for p odd or } e_i > 1);$$

$$\eta_i^2 = \xi_i \text{ (if p = 2 and } e_i = 1).$$

The $\mathbb{Z}_p$-span of the degree 2 generators ξ_i is the subgroup of $H^2(P, \mathbb{Z}_p)$ which consists of the classes representing abelian extensions of P by $\mathbb{Z}_p$. The ξ_i may be chosen so that for each i from 1 to n, the class ξ_i represents an extension

$$0 \to \mathbb{Z}_p \to Q_1 \times \ldots \times E_i \times \ldots \times Q_n \to Q_1 \times \ldots \times Q_i \times \ldots \times Q_n \to 1,$$

obtained in the obvious way from a non-split extension E_i of the cyclic group Q_i by $\mathbb{Z}_p$.

Proof. Apart from the description of the generators ξ_i in terms of extensions, this

is given in Evens [1, Section 3.5]. The method is to calculate $H^*(Q, \mathbb{Z}_p)$ directly when Q is cyclic, using an explicit minimal resolution X for $\mathbb{Z}_p$ over $\mathbb{Z}_pQ$. (Evens [1, Section 3.2]. This resolution is simple to write down, but unfortunately the formulae for a 'diagonal approximation' $X \to X \otimes X$ to calculate the ring structure are not particularly straightforward even in this simplest case.) The cohomology of a direct product of cyclic groups is then obtained using the Künneth formula, which for cohomology with coefficients in a field simply says that there is an isomorphism of graded rings, given by the 'outer product', $H^*(G, k) \otimes_k H^*(H, k) \to H^*(G \times H, k)$. (For fields k, we may see this directly by choosing minimal resolutions X and Y for H and K respectively; their tensor product $X \otimes_k Y$ is then a resolution for $G \times H$, and it is easy to write down an isomorphism of chain complexes, $\mathrm{Hom}_G(X, k) \otimes_k \mathrm{Hom}_H(Y, k) \xrightarrow{\sim}$ $\mathrm{Hom}_{G \times H}(X \otimes_k Y, k)$. The differentials in the left-hand complex $\mathrm{Hom}_G(X, k) \otimes_k \mathrm{Hom}_H(Y, k)$ are all zero, because of the minimality of X and Y; therefore the same is true of the complex $\mathrm{Hom}_{G \times H}(X \otimes_k Y, k)$, and so the above isomorphism is at the same time an isomorphism in cohomology, which coincides with the cross product of the Künneth theorem.)

We must also verify the statement about the generators ξ_i. In the case of a *cyclic* p-group Q, the second cohomology group $H^2(Q, \mathbb{Z}_p)$ is just $\mathbb{Z}_p$, the p-1 non-zero classes representing the p-1 inequivalent non-split central extensions of Q by $\mathbb{Z}_p$,

$$0 \to \mathbb{Z}_p \to E \xrightarrow{\pi} Q \to 1,$$

where $E = \langle x \rangle$ is cyclic and $\pi(x)$ may be any generator of Q, except that for t = 1 (mod p), the choices $\pi(x) = y$ and $\pi(x) = y^t$ give equivalent extensions. If we choose any such non-zero class, say $\xi^{(i)}$, in $H^2(Q_i, \mathbb{Z}_p)$, and write

$$\xi_i = 1 \otimes . . \otimes \xi^{(i)} \otimes . . \otimes 1, \in H^2(P, k),$$

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then these are the generators ξ_i of the statement (any other choice of ξ_i is just a scalar multiple of this one). From this description of ξ_i, it follows that the extension of P by $\mathbb{Z}_p$ represented by ξ_i is just the sum of an extension of Q_i which $\xi^{(i)}$ represents, and the trivial extension $0 \to Q_j \to Q_j \to 0$ of the other summands-a complete proof of this unsurprising fact is given in Lemma 5.4.2. In particular, each ξ_i represents an abelian extension of P.

On the other hand, the subgroup of $H^2(P, \mathbb{Z}_p)$ which consists of elements representing abelian extensions of P by $\mathbb{Z}_p$ is just $\text{Ext}(P, \mathbb{Z}_p)$, whose dimension over $\mathbb{Z}_p$ is just n in this case, because P has n cyclic summands. (To calculate the Ext group we need a resolution of P by free $\mathbb{Z}$-modules; an obvious candidate is the direct sum of the resolutions

$$0 \longrightarrow \mathbb{Z} \xrightarrow{q_i} \mathbb{Z} \longrightarrow 0,$$

where q_i is multiplication by the order of Q_i. We find that $\text{Ext}(P, \mathbb{Z}_p)$ is the direct sum of n copies of $\mathbb{Z}_p$.) The dimension of the subspace of $H^2(P, \mathbb{Z}_p)$ generated by the ξ_i is also n, which shows that the ξ_i span $\text{Ext}(P, \mathbb{Z}_p)$ as $\mathbb{Z}_p$-module. This completes the proof, except for Lemma 5.4.2. $\square$

Lemma 5.4.2. *Suppose that G and H are finite groups, and that V is a kG–module for some commutative ring k. Suppose that $\xi \in H^2(G, V)$ represents the equivalence class of the extension*

$$0 \longrightarrow V \longrightarrow E \xrightarrow{\pi} G \longrightarrow 0.$$

If k is regarded as a trivial H–module, so that $H^0(H, k) = k$, then the image of ξ ⊗ 1 under the external product map $H^*(G, V) \otimes H^*(H, k) \longrightarrow H^*(G \times H, V)$ is the element of $H^2(G \times H, V)$ which represents the extension

$$0 \longrightarrow V \longrightarrow E \times H \xrightarrow{\;(\pi,1)\;} G \times H \longrightarrow 0.$$

Proof. Let $F_*(G)$, $F_*(H)$ and $F_*(G \times H)$ be the standard resolutions for G and H and for $G \times H$, respectively. The Alexander-Whitney formula gives an augmentation preserving chain homotopy equivalence φ from $F_*(G \times H)$ to $F_*(G)$ $\otimes$ $F_*(H)$; in terms of the bar notation (Evens [1, Section 2.3]) the formula is

$$\varphi\big([(g_1,h_1)|.|.|.|(g_n,h_n)]\big) = \sum [g_1|\ldots|g_p] \otimes h_1 h_2 \cdot . \cdot h_p[h_{p+1}|\ldots|h_n],$$

where the summation runs from p = 0 to n. (Recall that, in the bar notation, H^0 is spanned by the 'empty bracket' [ ], where the augmentation map of the bar resolution takes [ ] to 1_k.) We represent the cohomology class ξ by a cocycle f: $F_2(G) \longrightarrow V$; the values of f on the kG-basis elements $[x|y]$ of $F_2(G)$ determine a function from $G \times G$ to V which is an ordinary 2-cocycle associated with the given extension of G by V. The cocycle whose class is the image of $\xi \otimes 1$, say t, is the composition of the map $f \otimes 1 : F_2(G) \otimes F_0(H) \longrightarrow V \otimes k = V$ with the map φ of the Alexander-Whitney formula. Thus

$$t\big([(g_1,h_1)|(g_2,h_2)]\big) = f \otimes 1 \big([g_1|g_2] \otimes h_1 h_2[\,]\big) = f\big([g_1|g_2]\big) \otimes 1.$$

Therefore, if the cocycle f corresponds to the section s: $G \longrightarrow E$ then t corresponds to the section $s \times 1$: $G \times H \longrightarrow E \times H$ in the extension of $G \times H$ by V, as required. □

In our calculations with abelian groups, we work with the subalgebra of $H^*(P, \mathbb{Z}_p)$ generated by the degree-2 generators ξ_i of Proposition 5.4.1. For convenience we make the following definition:

Definition 5.4.3. If P is an abelian p-group, then in the notation of Proposition 5.4.1, let $R(P)$ be the subalgebra of $H^*(P, \mathbb{Z}_p)$ generated by the ξ_i, for $1 \leq i \leq n$.

The next proposition lists the relevant properties of the subalgebra R(P). Recall that if P is a p-group, then ΩP is the subgroup of P generated by the elements of prime order.

Proposition 5.4.4. *Let P be an abelian p-group. Then R(P) has the following properties:*

(i) *R(P) is a polynomial ring of the form $\mathbb{Z}_p[\xi_1, \ldots, \xi_n]$, where the degree of each generator ξ_i is 2;*

(ii) *$H^*(P, \mathbb{Z}_p)$ is generated by R(P) together with the elements of degree 1;*

(iii) *R(P) is invariant under any automorphism α^* of $H^*(P, \mathbb{Z}_p)$ induced by an automorphism α of P;*

(iv) *The restriction map $H^*(P, \mathbb{Z}_p) \longrightarrow H^*(\Omega P, \mathbb{Z}_p)$ induces an isomorphism of R(P) with $R(\Omega P)$;*

(v) *If Q is a nontrivial subgroup of P, then the restriction map $R(P) \longrightarrow R(Q)$ is non-zero;*

(vi) *If Q is a subgroup of P, and α is an automorphism of P of p'-order, then the following are equivalent;*

(I) *$res_Q(\xi - \alpha^*(\xi)) = 0$ for all $\xi \in R(P)$;*

(II) *The action of α on P fixes $\Omega Q = Q \cap \Omega P$ pointwise.*

Proof. As in Proposition 5.4.1, let P be the product of cyclic groups Q_i, for $1 \leq i \leq$ n. Then (i) and (ii) follow immediately from Proposition 5.4.1. To establish (iii), it suffices to note that the linear span of the ξ_i in $H^2(P, \mathbb{Z}_p)$ is invariant under automorphisms of P, or in other words that the abelian extensions of P by $\mathbb{Z}_p$ are permuted amongst themselves by Aut P, which is clear. (The image under an automorphism α of P of the central extension $0 \to \mathbb{Z}p \to E \xrightarrow{\pi} P \to 0$ is just the extension $0 \to \mathbb{Z}p \to E \xrightarrow{\alpha^{-1} \circ \pi} P \to 0$: This may be seen by noting that α acts on cocycles by composition; or see Brown [1, Chapter IV, Exercise 3.1]).

To verify (iv), note that the map res: $\text{Ext}(P, \mathbb{Z}_p) \to \text{Ext}(\Omega P, \mathbb{Z}_p)$ is the direct sum from $i = 1$ to n of the restriction maps

$$\text{res: } \text{Ext}(Q_i, \mathbb{Z}_p) \to \text{Ext}(\Omega Q_i, \mathbb{Z}_p).$$

Each of these maps is injective, because a non-split extension of Q_i by $\mathbb{Z}_p$ remains non-split on restriction to any subgroup of Q_i. On the other hand, both $\text{Ext}(Q_i, \mathbb{Z}_p)$ and $\text{Ext}(\Omega Q_i, \mathbb{Z}_p)$ have order p. Therefore the restriction map is an isomorphism from $\text{Ext}(P, \mathbb{Z}_p)$ to $\text{Ext}(\Omega P, \mathbb{Z}_p)$, and so a ring isomorphism from $R(P)$ to $R(\Omega P)$, since both are polynomial rings.

It remains to verify (v) and (vi). We devote our attention to the proof of (vi), and remark where (v) follows incidentally. To see that (II) implies (I), consider that the restriction map from $R(P)$ to $R(\Omega Q)$ can be factored into two maps

$$R(P) \longrightarrow R(Q) \xrightarrow{\sim} R(\Omega Q)$$

of which the second is an isomorphism by (iv). If an automorphism α of P fixes ΩQ pointwise, then it clearly acts trivially on $R(Q)$, and since restriction commutes with α^*, we have

$$\text{res}_{\Omega Q}(\xi - \alpha^*(\xi)) = 0 \text{ for all } \xi \in R(P),$$

that is, the composite map is zero. The first map must therefore also be zero, as

required.

Next, suppose that the subgroup Q of P satisfies (I). Then ΩQ also satisfies (I), so since (II) relates only to ΩQ, we may replace Q by ΩQ and thus assume that Q is elementary abelian. Then $Q \leq \Omega P$, and by (iv) we may replace P with ΩP and assume that P is also elementary abelian. We may reduce further to the case where Q is cyclic of prime order, since then in general every cyclic subgroup of Q, sharing property (I), must be fixed pointwise by α.

Thus we have an elementary abelian p-group P, with a cyclic subgroup $Q = \langle x \rangle$, and an automorphism α which acts on P such that (I) holds. We can construct an abelian extension of P by $\mathbb{Z}_p$ by choosing any complement U for Q in P, and defining E to be $T \oplus U$, where $T = \langle y \rangle$ is cyclic of order p^2 with a projection $\pi: T \longrightarrow Q$ given by $y \longmapsto x$; the sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow T \oplus U = E \xrightarrow{(\pi,1)} Q \oplus U = P \longrightarrow 0$$

is an abelian extension of P, which clearly remains non-split on restriction to Q. (The existence of this extension proves (v).) If this extension is represented by, say, ξ, where $\xi \in R(P)$, then as remarked above, $\alpha^*(\xi)$ represents the extension

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow T \oplus U = E \xrightarrow{\alpha^{-1}\circ(\pi,1)} Q \oplus U = P \longrightarrow 0$$

whose restriction to Q is

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \pi^{-1}\circ\alpha(Q) \longrightarrow Q \longrightarrow 0. \tag{$*$}$$

If $\alpha(Q) \neq Q$ then we may choose U to contain $\alpha(Q)$. For this choice of U the extension ($*$) splits, so that $\text{res}_Q(\alpha^*(\xi)) = 0$, contradicting (I). Therefore α must stabilize Q, at least as a group. But then we may think of α as an automorphism of Q (using the fact that the action of α commutes with the restriction map). The

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group $\mathbb{Z}_p^{\times}$ of p'-automorphisms of Q acts regularly on $\text{Ext}(Q, \mathbb{Z}_p)$, while our hypothesis is now that α fixes the non-zero element $\text{res}_Q(\xi)$ of $\text{Ext}(Q, \mathbb{Z}_p)$. Therefore α centralizes Q, as required. The proof of 5.4.4 is complete. $\square$

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5.5 $\mathcal{F}$-normalizers and cohomology.

In this section we use our knowledge of the cohomology of abelian groups to study A-groups. There is an overlap with the results of Chapter 4; in particular we find that we have a different proof of Corollary 4.4.5 in the case of groups whose Sylow subgroups are elementary abelian, for these groups are both A-groups and nC-groups (see Section 2.4). As usual, $\mathcal{F}$ denotes a locally defined formation in each of the statements that follow, and $\pi = \pi(\mathcal{F})$ is the support of $\mathcal{F}$.

Theorem 5.5.1. *Suppose that G is a finite soluble group, each of whose Sylow subgroups is either cyclic, or elementary abelian. Let D be a subgroup of G. Then any two of the following are equivalent:*

(i) The map

$$\text{res: } H^2(G, V) \to H^2(D, V)$$

is zero for all irreducible, $\mathcal{F}$-eccentric G-modules V;

(ii) The map

$$\text{res: } H^n(G, V) \to H^n(D, V)$$

is zero for all sufficiently large n, for the same coefficients as (i);

(iii) D is contained in an $\mathcal{F}$-normalizer of G.

Of course, (iii) implies that the restriction map vanishes in *all* dimensions, by Theorem 3.3.1. Notice also that the group G need not be an nC-group, since some of its Sylow subgroups may be cyclic of non-prime order. Even for nC-groups with abelian Sylow subgroups, we obtain the new result that the $\mathcal{F}$-

normalizers are characterized by the vanishing of the restriction map in cohomology of degree 2; in Chapter 4 we looked only at degree-1 cohomology. It may be that this characterization in terms of degree-2 cohomology holds true for all nC-groups; after all, the short proof of Theorem 3.3.1 for the second cohomology group seems to show that 2 is the most natural degree to consider for normalizers.

Proof of Theorem 5.5.1. We divide the proof into three steps for the sake of lucidity.

Step 1. We may assume that D is a p-group, for some prime p.

Proof. From Theorem 3.2.2 we deduce that condition (i) or (ii) has the property that it holds for a general subgroup D of G if and only if it is satisfied by a Sylow p-subgroup of D, for each prime p. On the other hand, Corollary 5.2.4 says exactly that condition (iii) also has this property. Thus we may pass to the general case from the case where D is a p-group.

Step 2. Having assumed that D is a p-group, we may assume that $O_{p'}(G) = 1$.

Proof. We demonstrate that the conditions (i), (ii) and (iii) of the statement are respectively equivalent to the same conditions on the subgroup $DO_{p'}(G)/O_{p'}(G)$ of $G/O_{p'}(G)$. For conditions (i) and (ii), this is Corollary 5.3.2,(the irreducible $\mathcal{F}$-eccentric modules for $G/O_{p'}(G)$ are just the irreducible $\mathcal{F}$-eccentric modules for G whose kernels contain $O_{p'}(G)$), while for condition (iii) it is Lemma 5.2.5.

Recall that G has p-length one (Corollary 2.4.10); we may therefore assume from now on that the Sylow p-subgroup of G is normal:

Step 3. G has a normal Sylow p-subgroup P (containing D).

First suppose that p belongs to the support π of $\mathcal{F}$. Let K be a p-complement of the $f(p)$-residual $G^{f(p)}$ of G. Lemma 5.2.6 shows that D belongs to an $\mathcal{F}$-normalizer of G if and only if D is centralized by K. Let N be the subgroup of G generated by P and K; thus $N = PK = PG^{f(p)}$ is a normal subgroup of G containing P, and the irreducible $\mathcal{F}$-eccentric G-modules in characteristic p are precisely those with $N \not\leq \ker_G(V)$ (for P, being normal, must centralize any irreducible module of characteristic p). This is the situation of Theorem 5.3.3, and on applying that result we find that (i) and (ii) are respectively equivalent to (i)' and (ii)' below:

(i)' For all $k \in K$ and $\omega \in H^2(P, \mathbb{Z}_p)$, $\operatorname{res}_D(\omega - \omega^k) = 0$;

(ii)' If n is sufficiently large then for all $k \in K$ and $\omega \in H^n(P, \mathbb{Z}_p)$, $\operatorname{res}_D(\omega - \omega^k) = 0$.

Both (i)' and (ii)' imply that $\operatorname{res}_D(\xi - \xi^k) = 0$ for all $\xi \in R(P)$ and $k \in K$, as we show next. For (i)' this is obvious, since R(P) is generated by elements of degree 2. From (ii)' we deduce that, in particular,

$$\operatorname{res}_D\left((\xi)^{p^m} - (\xi^k)^{p^m}\right) = 0$$

for any $\xi \in H^*(P, \mathbb{Z}_p)$, and m sufficiently large (because the degree of ξ^{p^m} is p^m times the degree of ξ.) The map which takes p^{th} powers is an endomorphism of R(P) (the Frobenius endomorphism), so since restriction is a ring homomorphism, this is equivalent to

$$\left(\mathrm{res}_D(\xi - \xi^k)\right)^{p^m} = 0$$

from which we deduce the same conclusion as for (i)', because R(D) is a polynomial ring and has no nilpotent elements.

Next, we apply Proposition 5.4.4 to the automorphisms of P induced by the action of K. From 5.4.4(vi), (I) $\Rightarrow$ (II), we deduce that

K fixes ΩD.

If the Sylow p–subgroup P of G is elementary abelian, then D = ΩD; otherwise P is cyclic, in which case we deduce that K fixes D anyway (the group of p'–automorphisms of a cyclic p–group acts regularly.) Thus D satisfies the criterion of Lemma 5.2.6, and we may deduce that D belongs to an $\mathcal{F}$–normalizer of G, as required.

Finally, if p $\notin$ π, then we must deduce that D is trivial. We use Corollary 5.3.6 in place of Theorem 5.3.3, to deduce that (i) and (ii) are equivalent respectively to (i)″ and (ii)″:

(i)″ For all $\omega \in H^2(P, \mathbb{Z}_p)$, $\mathrm{res}_D\omega = 0$;

(ii)″ If n is sufficiently large then for all $\omega \in H^n(P, \mathbb{Z}_p)$, $\mathrm{res}_D\omega = 0$.

The triviality of D follows from (i)″ by Theorem 5.4.4(v) (which takes the place of (vi) in the argument for the case p $\in$ π), and (ii)″ implies (i)″ just as in the above argument. $\square$

The proof also yields the implication (iii) $\Rightarrow$ (i); if we know in the situation of Step 3, that D is fixed by K, we may use Theorem 5.4.2(vi), (II) $\Rightarrow$ (I) to deduce (i)′ and therefore (i), to which (i)′ is equivalent.

The only place where the restrictions on Sylow p-subgroups of G in Theorem 5.5.1 are used in the proof is in Step 3, where we deduce that K fixes D, knowing only that ΩD is fixed by K. When P is abelian, this deduction is also valid as long as we have the additional information that D is stabilized *as a group* by the action of K, a fact which we exploit in Section 6. Unfortunately it is not true in general that an automorphism from K which acts on P so as to fix ΩD must fix the whole of D, and this is entirely responsible for the fact that Theorem 5.5.1 fails to be valid for A-groups in general. The next result shows what does happen for these groups; not surprisingly, in view of the comments we have just made, the statement involves the maximal nC-subgroups ΩK of A-groups which we constructed in Chapter 2 (Theorem 2.4.8). (Recall that the subgroup ΩK of an A-group K is really defined only up to conjugacy in K; it will be apparent in the statement below that this ambiguity is unimportant here.)

Theorem 5.5.2. *Let G be a soluble group with abelian Sylow subgroups. Let D be a subgroup of G. Then any two of the following are equivalent:*

(i) For n = 1 and 2, the map

$$\text{res: } H^n(G, V) \to H^n(D, V)$$

is zero for all irreducible $\mathcal{F}$-eccentric coefficient modules V;

(ii) The map of (i) is zero for all n $\geq$ 1;

(iii) D is contained in an $\mathcal{F}$-prefrattini subgroup of G, and ΩD is contained in an $\mathcal{F}$-normalizer of G.

Proof. This is very similar to the proof of Theorem 5.5.1, except for a few extra

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technicalities. We cannot appeal to Theorem 3.3.1 this time because the implication (iii) ⇒ (i) of the present theorem is stronger than that result. Again there are three steps:

Step 1. We reduce to the case where D is a p-subgroup.

Proof. Again a general subgroup D satisfies (i) or (ii) if and only if each of its Sylow subgroups does, by Theorem 3.2.2. The same holds for the first half of (iii) (here we are using our Theorem 4.5.5 and not Corollary 5.2.4.) Next, it follows from the construction of ΩD in the proof of Theorem 2.4.8 that a Sylow subgroup of ΩD is equal to ΩQ for some Sylow subgroup Q of G. If for each Sylow subgroup Q of D, we have ΩQ contained in an $\mathcal{F}$-normalizer of G, then given a fixed $\mathcal{F}$-normalizer S, we know that ΩD is locally subconjugate to S (Definition 4.5.1), and so conjugate to a subgroup of S by Corollary 5.2.4. Thus D satisfies (i), (ii) or (iii) if and only if the same is true of each of its Sylow subgroups, as in the proof of Theorem 5.5.1.

Step 2. We reduce to the case that $O_{p'}(G) = 1$.

Proof. Conditions (i) and (ii) are equivalent to the same condition on $DO_{p'}(G)/O_{p'}(G)$, by Corollary 5.3.2. To see that the same is true of (iii), we use Lemma 5.2.5, noting that $\Omega(DO_{p'}(G)/O_{p'}(G)) = (\Omega D)O_{p'}(G)/O_{p'}(G)$.

Step 3. Since G has p-length one, we may now assume that G has a normal Sylow p-subgroup P, which contains D.

First suppose that p belongs to the support π of $\mathcal{F}$. Let K be a p-complement of the $f(p)$-residual $G^{f(p)}$. By Lemma 5.2.6, ΩD is contained in an $\mathcal{F}$-

normalizer of G if and only if ΩD is centralized by K. On the other hand, as in the proof of Theorem 5.5.1 we use Theorem 5.3.3 to deduce that (i) and (ii) are respectively equivalent to the conditions:

(i)': $\text{res}_D(\omega - \omega^k) = 0$ for all $\omega \in H^r(P, \mathbb{Z}_p)$, for $r = 1$ or 2;

(ii)': $\text{res}_D(\omega - \omega^k) = 0$ for all $\omega \in H^r(P, \mathbb{Z}_p)$, for all $r \geq 1$.

By Proposition 5.4.4, the ring $H^*(P, \mathbb{Z}_p)$ is generated over $\mathbb{Z}_p$ by elements of degrees 1 and 2, of which the degree 2 elements on their own generate R(P). Therefore (i)' and (ii)' are each equivalent to:

(∗) For all $k \in K$, $\text{res}_D(\omega - \omega^k) = 0$ for all $\omega \in R(P)$, and all $\omega \in H^1(P, \mathbb{Z}_p)$.

By Proposition 5.4.4 (vi), $\text{res}_D(\omega - \omega^k) = 0$ for all $k \in K$ and all $\omega \in R(P)$ if and only if ΩD is centralized by K, that is, if and only if ΩD is contained in an $\mathcal{F}$-normalizer of G. By Theorem 5.3.3, $\text{res}_D(\omega - \omega^k) = 0$ for all $\omega \in H^1(P, \mathbb{Z}_p)$ if and only if the map

$$\text{res: } H^1(G, V) \to H^1(D, V),$$

is zero for all $\mathcal{F}$-eccentric irreducible G-modules V, that is (by Corollary 4.4.3) if and only if D is contained in an $\mathcal{F}$-prefrattini subgroup of G. Thus (∗) is satisfied if and only if D is contained in an $\mathcal{F}$-prefrattini subgroup, and ΩD in an $\mathcal{F}$-normalizer of G, as required.

Finally, suppose that the prime p does not belong to the support π of $\mathcal{F}$. Then condition (iii) implies that ΩD, and therefore D, is trivial, so we need only show that (i) and (ii) also imply that D is trivial. This is an application of Corollary 5.3.6, precisely as in the corresponding part of the proof of Theorem 5.5.1. □

We conclude this section with an example to show that condition (iii) of Theorem 5.5.2 does not generally imply that a subgroup is contained in an $\mathcal{F}$-normalizer.

Example 5.5.3. Let p be an odd prime, and let A and B be the cyclic groups C_{p^2} = $\langle x \rangle$ and C_{p^3} = $\langle y \rangle$ respectively. Let P be the direct product A × B, generated by (x, 1) and (1, y). Choose a nontrivial automorphism $x \mapsto x^n$ of A which has prime order q dividing p–1 (the condition is that $n^q = 1 \pmod{p^2}$, and n is not congruent to 1 mod p), and define $\nu: P \to P$ by the equations $\nu(x, 1) = (x^n, 1)$ and $\nu(1, y) = (1, y)$. Let G be the semidirect product $[P]\langle\nu\rangle$ and let Q be the cyclic subgroup of P generated by the element (x^p, y^p). Then $\Omega Q = \langle(1, y^{p^2})\rangle$ is fixed pointwise by ν, and therefore lies in the centre of G, and Q itself is contained in the Frattini subgroup $\Phi G = \Phi P$ of G. However Q does not lie in any system normalizer of G, because its generator is not fixed by ν. □

5.6 Normal subgroups.

If G is a group of p–length 1, then a p–subgroup Q of G is normally embedded in G (Definition 5.2.1) if and only if $QO_{p'}(G)/O_{p'}(G)$ is a normal subgroup of $G/O_{p'}(G)$. If the test subgroup D in the statement of Theorem 5.5.1 happens to be normally embedded in G (so that each of its Sylow subgroups is also normally embedded), then in Step 3 of the proof of that theorem we have the additional information that D (a different D, because of the reductions of the first two steps) is *normal* subgroup of G. When we have this extra knowledge, we do not need the restrictive hypotheses about the Sylow subgroups of G that appear in that theorem, because we can use the following well-known result:

Theorem 5.6.1. *Let D be an abelian p-group, and suppose that α is an automorphism of D whose order is prime to p. If α fixes ΩD pointwise, then α acts trivially on D.*

Proof. See Gorenstein [1, Chapter 5, Theorem 2.4]. □

Thus the proof of Theorem 5.5.1, with Theorem 5.6.1 used in Step 3 for the automorphisms induced by the action of K on the normal subgroup D, immediately yields:

Theorem 5.6.2. *Let G be a finite A-group, and let D be a normally embedded subgroup of G. Then any two of the following are equivalent:*

(i) The map

$$\text{res: } H^2(G, V) \to H^2(D, V)$$

is zero for all irreducible, $\mathcal{F}$-eccentric G-modules V.

(ii) The map

$$\text{res: } H^n(G, V) \to H^n(D, V)$$

is the zero map for the same coefficients as (i), for all sufficiently large n.

(iii) D is contained in an $\mathcal{F}$-normalizer of G.

Since the $\mathcal{F}$-hypercentre of G is the largest normal subgroup of G contained in any $\mathcal{F}$-normalizer of G (Doerk and Hawkes [1, Chapter V, Theorem 2.4]), we obtain the following as a special case of 5.6.2:

Corollary 5.6.3. *If D is a normal subgroup of G, then D satisfies (i) or (ii) of Theorem 5.6.2, if and only if D is contained in the $\mathcal{F}$-hypercentre of G.*

After another, more interesting, study of the cohomology rings of p-groups we will show that Corollary 5.6.3 holds true for any group which has p-length one for each prime p, except for possible difficulties associated with the prime p = 2. This perhaps marginal improvement requires considerable further work, but the reduction to p-groups which we employed in Section 5.5 is still valid, so that the extra difficulty comes in dealing with the p-groups themselves. The improved version of Corollary 5.6.3 is as follows:

Theorem 5.6.4. *Let G be a soluble group and let N be a normal subgroup of G. Suppose that G has p-length one for every prime p which divides the order of N, and that the Sylow 2-subgroup of N is abelian. Suppose that for every sufficiently large even number n the map*

$$\text{res: } H^n(G, V) \to H^n(N, V)$$

is zero for all $\mathcal{F}$-eccentric irreducible modules V. Then N is contained in the $\mathcal{F}$-hypercentre of G.

Cohomology of p-groups again.

For any group G and field k, let $H^{ev}(G, k)$ be the sum of the even - degree cohomology groups $H^{2n}(G, k)$, $0 \le n < \infty$. This is a subalgebra of the whole cohomology algebra $H^*(G, k)$, and the graded commutativity of the big ring shows that $H^{ev}(G, k)$ is commutative, a fact which we use implicitly below. We use the letter J to stand for the *nilradical* of $H^{ev}(G, k)$, although it suggests the Jacobson radical - in fact the two radicals coincide for any finite group G,

because the cohomology ring of a finite group over a field is a finitely-generated algebra. However, we do not need to use this, except for the reinterpretation of our results in terms of the theory of varieties in Section 5.6.

The remainder of this section is devoted to the proof of the following result, and the discussion of some interesting consequences.

Theorem 5.6.5. *Let P be a finite p-group, and let k be a field of characteristic p. Suppose that α is an automorphism of P of p'-order, and that Q is a normal subgroup of P for which $\alpha(Q) = Q$. Suppose that at least one of the following is satisfied:*

(i) Q is abelian;

(ii) p is odd.

Then the following condition is (necessary and) sufficient for Q to be fixed pointwise by α:

$$\text{For all } \xi \in H^{ev}(P, k), \ res_Q(\xi - \alpha^*(\xi)) \in J\left(H^{ev}(Q, k)\right). \tag{C}$$

We first show how Theorem 5.6.4 follows from Theorem 5.6.5 - the line of argument is familiar from the proofs of Theorems 5.5.1, 5.5.2 and 5.6.2.

Proof of Theorem 5.6.4.

This time we appeal to Evens' theorem, 5.1.2, which tells us immediately that N is a π-group, where π is the support of $\mathcal{F}$. (Recall that for $p \notin \pi$, the trivial G-module $\mathbb{Z}_p$ is $\mathcal{F}$-eccentric.) It is therefore sufficient to show that for each prime $p \in \pi$, a Sylow p-subgroup Q of N is contained in the $\mathcal{F}$-hypercentre. By Corollary 5.3.2, we may suppose that $O_{p'}(G) = 1$, and therefore that G has a normal Sylow p-subgroup P. Therefore $Q = P \cap N$ is normal in G.

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As usual, let $G^{\mathcal{F}(p)}$ be the $\mathcal{F}(p)$-residual of G, and let K be a p-complement of $G^{\mathcal{F}(p)}$. By Lemma 5.2.6, the subgroup Q is contained in the $\mathcal{F}$-hypercentre if and only if Q is fixed pointwise by the action of K on P. By Theorem 5.3.3, the condition on Q in the statement is equivalent to the following:

For all $k \in K$ and $\xi \in H^{ev}(P, \mathbb{Z}_p)$ of sufficiently large degree, $res_Q(\xi - \xi^k) = 0$.

Suppose this condition is satisfied, and let $\xi \in H^{ev}(P, k)$. We may choose m to make the degree of $\tau = \xi^{p^m}$ sufficiently large; then by hypothesis $res_Q(\tau - \tau^k) = 0$. But $res_Q(\tau - \tau^k) = res_Q(\xi - \xi^k)^{p^m}$, so that $res_Q(\xi - \xi^k) \in J\left(H^{ev}(Q, k)\right)$. Thus condition (C) of Theorem 5.6.5 is satisfied. By hypothesis, either Q is abelian or p is odd, so K fixes Q pointwise, as required. □

We turn to the proof of Theorem 5.6.5. We need several preliminary results, of which the one which fails in general when p = 2 is the following powerful generalisation of Proposition 5.6.1, due to Thompson:

Theorem 5.6.6. (Gorenstein [1, Chapter 5, Theorem 3.13]). *Let Q be a p-group, where p is odd. Then Q has a characteristic subgroup T of exponent p, such that every nontrivial automorphism of Q of p'-order acts nontrivially on T.*

Notice that if Q is abelian, then this is the same as Theorem 5.6.1. Theorem 5.6.6 is a consequence of a result of Thompson from the 'odd-order paper' (Feit and Thompson [1, Lemma 8.2]) which asserts that a p-group Q has a characteristic subgroup T of nilpotent class 1 or 2, on which any p'-automorphism of Q acts nontrivially. When p > 2 the subgroup T must be regular, so that ΩT has exponent p (this fails, of course, for p = 2, as the dihedral group of order 8 shows); Theorem 5.6.6 then follows from an earlier, similar result of Huppert ([2,

Hilfssatz 1.5]) which asserts that for p odd, a nontrivial automorphism of p'-order of the p-group Q, acts nontrivially on ΩQ. (Huppert's result is also false for p = 2; for example, the quaternionic group Q_8 has an automorphism of order 3 which clearly fixes ΩQ_8 since the latter has order 2.) See Gorenstein [1, Section 5.3] for a full discussion.

These results are dual to the following more elementary result, which is due originally to Burnside, and which is equally valid for p = 2.

Theorem 5.6.7 (Gorenstein [1, Chapter 5, Theorem 1.4]). *If α is a nontrivial automorphism of the p-group P, and the order of α is prime to p, then α acts nontrivially on $P/\Phi(P)$.*

(The relation between this result and Theorem 5.6.6 can be expressed exactly when P is abelian, using the duality between an abelian group P and its character group $P\hat{} = \text{Hom}(P, \mathbb{Q}/\mathbb{Z})$. The nature of the duality is more obscure in general.)

Corollary 5.6.8. *Let k be a field of characteristic p > 0. A nontrivial p'-automorphism α of P acts nontrivially on $H^1(P, k)$.*

Proof. Since P acts trivially on k, we have $H^1(P, k) = \text{Hom}(P/\Phi P, k)$, which is the vector space dual to $k \otimes_{\mathbb{Z}_p} P/\Phi(P)$. If α acts trivially on the dual, then its action on $k \otimes P/\Phi(P)$, and therefore on $P/\Phi(P)$ since the tensor product is taken over a field, is trivial. Therefore α acts trivially on P itself, by Theorem 5.6.7. $\square$

Lemma 5.6.9. *Let P be a p-group and let Q be a nontrivial normal subgroup of P. If X is a subgroup of P which is properly contained in Q, then the union of the subgroups of P conjugate to X is a proper subset of Q.*

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Proof. First, the union of conjugates is indeed a subset of Q, because Q is normal in P. We may assume that X is a maximal subgroup of Q (we may have X = 1). Suppose for a contradiction that every element of Q is contained in a conjugate of X. Since P is a p-group, $Q \cap Z(P)$ is nontrivial. Let x be a nontrivial element of order p in this intersection, and let S = $\langle x \rangle$. Then S , being normal in P, is contained in all the conjugates of X; therefore Q/S is the union of the conjugates in P/S of its proper subgroup X/S. We arrive inductively at the case where Q is of prime order, where the hypothesis is absurd. This completes the proof. □

Remark. If Q is elementary abelian, it is easy to see that Q is the union of p+1 of its maximal subgroups, but not of any smaller number. In particular, when Q has rank 2 all of the maximal subgroups are needed.

In order to prove the next proposition we need two constructions in the cohomology of groups. The first is the (mod p) *Bockstein homomorphism*,

$$\beta \colon H^r(P, k) \to H^{r+1}(P, k), r \geq 0.$$

When $k = \mathbb{Z}_p$, the Bockstein is defined as the connecting homomorphism in the long exact sequence of cohomology groups which arises from the short exact sequence of coefficient modules

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

For general fields k, one may define β by extending scalars, using the natural algebra isomorphism (Evens [1, page 30]):

$$H^*(P, k) = H^*(P, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} k.$$

(There is an alternative approach to defining β, where the short exact sequence above is replaced by a 'k-version' constructed using Witt vectors. See Evens [1, Sections 3.3 and 3.4] for the details of this constuction, and the properties of the Bockstein map in general, but note that Evens uses δ to denote the mod p Bockstein which we have called β; his β is the *integral* Bockstein map from $H^*(G, \mathbb{Z}/p\mathbb{Z})$ to $H^*(G, \mathbb{Z})$.)

We collect some standard properties of the Bockstein homomorphism:

Lemma 5.6.10. *The Bockstein homomorphism β has the following properties:*

(i) It is functorial with respect to maps of groups. That is, if $\rho\colon Q \to P$ is a homomorphism of groups then the diagram

$$
\begin{array}{ccc}
H^*(P, k) & \xrightarrow{\ \rho^*\ } & H^*(Q, k) \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\
H^*(P, k) & \xrightarrow{\ \rho^*\ } & H^*(Q, k)
\end{array}
$$

is commutative. In particular, β commutes with restriction and inflation and with the map $\alpha^*\colon H^*(P, k) \to H^*(P, k)$, where α^* is an automorphism of P.

(ii) If P is cyclic of order p, then $\beta\colon H^1(P, k) \longrightarrow H^2(P, k)$ is an isomorphism. In particular, if $y_i \in H^1(P, k)$ are any non-zero elements then the product of the elements βy_i is non-nilpotent in $H^{ev}(P, k)$.

Proof. (i) is proved by straightforward 'diagram chasing', using the definition of the connecting homomorphism arising from a short exact sequence of cochain

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complexes. To verify (ii) it is sufficient to prove that in this case β is a monomorphism, since both cohomology groups are 1-dimensional over k by Proposition 5.4.1. In case $k = \mathbb{Z}_p$, the long exact sequence gives us

$$0 \longrightarrow \mathrm{Hom}(P, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{Hom}(P, \mathbb{Z}/p^2\mathbb{Z}) \longrightarrow \mathrm{Hom}(P, \mathbb{Z}/p\mathbb{Z}) \overset{\beta}{\longrightarrow} H^2(P, \mathbb{Z}/p\mathbb{Z}).$$

The map directly before β is zero, because any map from P to $\mathbb{Z}/p^2\mathbb{Z}$ goes into the kernel of the projection $\mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$, so β is a monomorphism by exactness. (The same argument holds good for any group P of exponent p.) For general k note that tensoring over a field preserves monomorphisms. The second part of (ii) follows, because the elements of $H^2(P, k)$ generate a polynomial subalgebra of $H^*(P, k)$, by Proposition 5.4.1 and the isomorphism $k \otimes \mathbb{Z}_p[\xi] = k[\xi]$, valid for arbitrary commutative rings k. $\square$

We need the Evens norm map or multiplicative transfer (Evens [2]). The construction and properties of this map are described in Evens [1, Chapter 6]. If G is any finite group with a subgroup K, and k is a field, then the norm is a map

$$N_{K \to G}\colon\ H^{\mathrm{ev}}(K, k) \to H^{\mathrm{ev}}(G, k)$$

which is multiplicative with respect to the cup-products (it is not a ring homomorphism in general). If $\xi \in H^{\mathrm{ev}}(K, k)$ is homogeneous of degree n, then $N_{K \to G}(\xi)$ is a homogeneous element of $H^{\mathrm{ev}}(G, k)$ of degree $n|G{:}K|$. The norm map satisfies a multiplicative version of the 'Mackey formula' for the additive transfer or corestriction map; we only need this in the case where K is a normal subgroup of G, and we record it seperately for clarity.

Lemma 5.6.11 (Evens [1, Theorem 6.1.1, (N4)]). *If $N_{K \to G}$ is the norm map,*

then

$$\mathrm{res}_K \, N_{K \to G}(\xi) = \prod_{t \in T} \xi^t,$$

where the product is taken over a transversal T to G/K (and is independent of the choice of transversal).

Corollary 5.6.12. *Let α be an automorphism of G such that $\alpha(K) = K$. Then for any $\xi \in H^{ev}(K, k)$,*

$$\mathrm{res}_K \, N_{K \to G}(\alpha^*(\xi)) = \alpha^* \, \mathrm{res}_K \, N_{K \to G}(\xi).$$

Proof. By the lemma, $\mathrm{res}_K \, N_{K \to G}(\alpha^*(\xi)) = \prod (\alpha^*(\xi))^t$, where t runs over any transveral to K in G. The action of G (by conjugation) and of α on K combine to give an action of the semidirect product $[G] \rtimes \langle \alpha \rangle$ on K, so that

$$(\alpha^*(\xi))^t = \alpha^*(\xi^{\alpha(t)}).$$

The result follows, because $\{ \alpha(t) : t \in T \}$ is a transversal to $\alpha(K) = K$ in G, and the products are taken in the commutative ring $H^{ev}(K, k)$. (Evens shows that the norm map itself commutes with α^* (Evens [1, Theorem 6.1.1, (N5)]), but we do not need this.) $\square$

The proposition below, with Theorem 5.6.6, is the key to Theorem 5.6.5:

Proposition 5.6.13. *Let P be a p-group and let Q be a normal subgroup of P. Suppose that Q has exponent p. Let k be a field of characteristic p. If y is an element of $H^1(Q, k)$, define $\chi(y) \in H^{2|P:Q|}(Q, k)$ by the formula*

$$\chi_{K,G}(y) = \mathrm{res}_K \, N_{K \to G}(\beta(y)).$$

If y is nonzero, then $\chi(y)$ is a non-nilpotent element of $H^{ev}(Q, k)$.

Proof. Since $H^1(Q, k) = \text{Hom}(Q, k)$, we are saying that y is a nonzero homomorphism from Q to the additive group of k. Let X be the kernel of y, so that X is a proper subgroup of Q. We observe that for $t \in P$, the homomorphism y^t: $Q \to k$ is just y composed with conjugation by t, so that in particular ker (y^t) is the subgroup X^t of Q. By Lemma 5.6.8, there is an element $g \in Q$ such that $y^t(g)$ is nonzero for all $t \in P$. Let S be the subgroup of Q that g generates. S is cyclic of prime order, because by hypothesis Q has exponent p. We have

$$\text{res}_S\left(\chi_{Q,P}(\beta y) \right) = \text{res}_S\left(\prod_{t \in T}(\beta y)^t \right)$$

$$= \prod_{t \in T} \beta\left(\text{res}_S (y^t) \right),$$

(where the multiplication is over a transversal T to P/Q), because β commutes with restriction and with the action of t and restriction is multiplicative (note that we have not said that β is multiplicative; in fact β is a derivation of $H^*(Q, k)$).

By the choice of S, each term $\text{res}_S (y^t)$ is a nonzero element of $H^1(Q, k)$, and so by Lemma 5.6.9 the Bocksteins of these terms are nonzero elements of degree 2 whose product cannot be nilpotent in $H^*(S, k)$. If $\chi_{Q,P}(\beta y)$ were nilpotent, then its restriction to S would certainly be nilpotent as well, a contradiction. Therefore $\chi_{Q,P}(\beta y)$ cannot be nilpotent, which is what we wished to prove. □

Proof of theorem 5.6.5. We first show that we may assume that k is *algebraically closed.* Let K be the algebraic closure of k. Then $H^*(P, K)$ is naturally isomorphic to $H^*(P, k) \otimes_k K$ (Evens [1, page 30]), and the maps res: $H^*(P, K) \to H^*(Q, K)$, and α^*: $H^*(P, K) \to H^*(P, K)$, are obtained from the corresponding maps over k by extension of scalars. If condition (C) is satisfied

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over k, then for all $\xi \in H^{ev}(P, k)$ there exists m (depending on ξ) such that $\left(res_Q(\xi - \alpha^*(\xi))\right)^m = 0$. If $\xi' \in H^{ev}(P, K) = H^{ev}(P, k) \otimes_k K$ is of the form $\xi \otimes 1$, we have $res_Q(\xi' - \alpha^*(\xi')) = res_Q(\xi - \alpha^*(\xi)) \otimes 1$, which is clearly nilpotent. A general element of $H^{ev}(P, K)$ is a sum of these terms, so the same condition (C) holds over K. (We are saying, in effect, that for any commutative algebra R over k, the nilradical of $K \otimes_k R$ contains the K - span of the nilradical of R.)

Next, Thompson's Theorem 5.6.6 (if p is odd), or Theorem 5.6.1 (if Q is abelian) shows that Q has a characteristic subgroup T of exponent p on which any nontrivial automorphism of Q of p'-order acts nontrivially. Clearly condition (C) on Q is inherited by T, and so it is sufficient to prove that α acts trivially on T. In effect we may assume that *Q has exponent p*. This is really the crucial step in the proof, for it enables us to employ Proposition 5.6.13 which is definitely false for some groups of exponent greater than p.

Since we may take k to be algebraically closed, we may assume that the eigenvalues of the action of α^* on the k-vector space $H^1(Q, k)$ lie in k. Since the order of α is prime to p this means that α is diagonalizable over k, and so we may choose a basis $y_1, \ldots, y_l$ of $H^1(Q, k)$ (where $|Q/\Phi(Q)| = p^l$) such that for each y in this basis there is a p'-root of unity λ in k with $\alpha^*(y) = \lambda y$. Since α^* commutes with the Bockstein homomorphism β, we also have $\alpha^*(\beta y) = \lambda.\beta y$, and so if $\chi = \chi_{Q,P}$ is defined as in Proposition 5.6.13, we find that

$$\alpha^*\left(\chi(y)\right) = \lambda^{|P:Q|} \chi(y).$$

By its definition $\chi(y)$ is the restriction to Q of an element of $H^{ev}(P, k)$, so since α^* commutes with restriction, condition (C) says that $\chi(y) - \alpha^* \chi(y)$ is in the nilradical of $H^{ev}(Q, k)$. However,

$$\chi(y) - \alpha^* \chi(y) = \left(1 - \lambda^{|P:Q|}\right)\chi(y),$$

which by Proposition 5.6.13 is non-nilpotent unless the coefficient $1 - \lambda^{|P:Q|} = 0$, or in other words unless $\lambda = 1$, since $|P:Q|$ is a power of p while λ is a p'-root of unity. This must be the case for each y_i, $1 \leq i \leq l$. Therefore, if condition (C) holds, α must act trivially on $H^1(Q, k)$. Finally, Corollary 5.6.8 shows that α acts trivially on Q, and we are done. $\square$

We can deduce some results of independent interest from Theorem 5.6.5. In particular, taking Q = P in Theorem 5.6.5 we obtain the following corollary:

Theorem 5.6.14. *If P is a group of order a power of p, where p is an odd prime, and k is a field of characteristic p, then a nontrivial automorphism of P of p'- order acts nontrivially on the ring $H^{ev}(P, k)/J\big(H^{ev}(P, k)\big)$.*

It would be interesting to have a purely cohomological proof of this result (that is, one which does not make essential use of Thompson's Theorem 5.6.6) because a substantial part of the strength of Theorem 5.6.6, namely Huppert's result that a p'-automorphism which fixes ΩP must be trivial, can be recovered from Theorem 5.6.14 by the use of Quillen's famous characterization of the radical of $H^{ev}(P, k)$:

Theorem 5.6.15 (Quillen and Venkov [1], Evens [1, Corollary 8.3.4]). *An element ξ of $H^{ev}(P, k)$ is nilpotent if and only if the same is true of its restriction to every elementary abelian subgroup of P.*

Corollary 5.6.16. *The element ξ is nilpotent if and only if $res_{\Omega P}(\xi)$ is nilpotent.*

Proof The two groups P and ΩP have the same sets of elementary abelian subgroups. $\square$

The promised deduction of Huppert's theorem now follows:

Corollary 5.6.17. *If p is odd and the p'-automorphism α of P fixes ΩP, then α is the identity map.*

Proof. If α fixes ΩP, then $\mathrm{res}_{\Omega P}(\xi - \alpha^*(\xi)) = \mathrm{res}_{\Omega P}(\xi) - \alpha^* \mathrm{res}_{\Omega P}(\xi)$ is zero for all $\xi \in H^{\mathrm{ev}}(P, k)$. The action of α on $H^{\mathrm{ev}}(P, k)$ therefore fixes that ring modulo its radical, by Corollary 5.6.16, and it follows from Theorem 5.6.14 that α acts trivially on P. □

We remarked earlier that Corollary 5.6.17 is dual to Burnside's Theorem 5.6.7. This duality is visible in Theorem 5.6.14: For p odd, a non-trivial automorphism of P of p'-order acts non-trivially on the radical $J\big(H^*(P, k) \big)$ of the cohomology ring of P. This follows immediately from Corollary 5.6.7, since for odd p the graded-commutativity of $H^*(P, k)$ shows that elements of odd degree in that ring have their squares equal to zero – in particular the radical $J\big(H^*(P, k) \big)$ contains $H^1(P, k)$.

The action of p'-automorphisms on the cohomology ring $H^*(P, k)$ itself has been studied by Diethelm [1]). He shows that the representation of p'-automorphisms of P on this ring is effective in a very strong sense – namely, if G is a group of p' – automorphisms of P then ($H^*(P, k)$ being degreewise a finite-dimensional, and therefore semisimple kG-module) every irreducible representation of G over k is contained with infinite multiplicity in $H^*(P, k)$. His results are equally valid for p = 2. It is natural to ask if some combination of these results is possible, to find representations of G on $H^*(P, k)/J\big(H^*(P, k)\big)$, but I have not pursued this question.

5.7 Varieties.

The results above can be phrased alternatively in terms of the theory of varieties in the cohomology theory of groups. We give the definition of varieties for finite groups in a special case, and refer to Evens [1, Chapter 8] and the references given there, for a full account of the theory.

Definition 5.7.1 (Evens [1, Chapter 8]). Let G be a finite group, and let k be a field of characteristic p > 0. The *variety of G over k*, written $X_G(k)$, is the prime ideal spectrum of the commutative ring H(G, k), where H(G, k) is defined to be $H^{ev}(G, k)$ if p is odd, or $H^*(G, k)$ if p = 2.

The theory of varieties depends inevitably upon the deep fact that the cohomology ring of a finite group over a field is a finitely-generated algebra. This is the Venkov-Evens theorem (Evens [1, Corollary 7.4.6]). The proof above of Theorem 5.6.5 does not use the Venkov-Evens theorem, or anything so deep, but we now assume this result implicitly in order to translate Theorem 5.6.5 into the language of varieties.
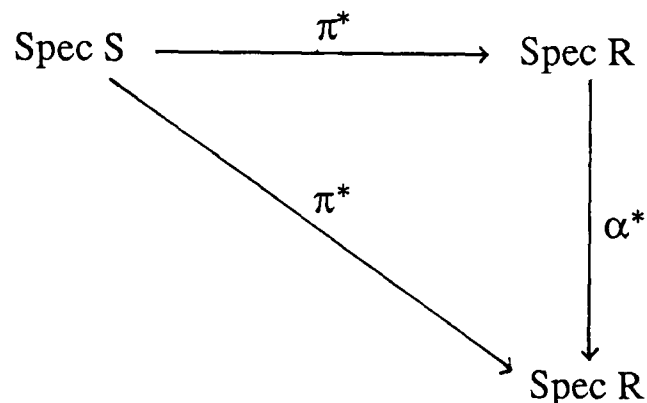
If $\eta\colon K \to G$ is a group homomorphism then $\eta^*\colon H^{ev}(G, k) \to H^{ev}(K, k)$ induces a covariant map $\eta_*\colon X_K(k) \to X_G(k)$. In particular, the automorphism group of G acts naturally on the variety $X_G(k)$. The coordinate ring of $X_P(k)$ is just $H^{ev}(P, k)/J\!\left(H^{ev}(P, k)\right)$ (for p odd), and we may translate Theorem 5.6.5 into a result about the action of p'-automorphisms of a p-group P on the variety $X_P(k)$, using the following lemma, whose content is presumably well-known.

Lemma 5.7.2. *Let k be an algebraically closed field (of any characteristic) and let R and S be finitely generated algebras over k with nilradicals J(R) and J(S) respectively. Let π: R → S be a morphism of k-algebras, and let α be a k-*

algebra automorphism of R. Then the following are equivalent:

(i) $\pi(x - \alpha(x)) \in J(S)$ *for all* $x \in R$.

(ii) The diagram

$$\begin{array}{ccc} \text{Spec S} & \xrightarrow{\;\;\pi^*\;\;} & \text{Spec R} \\ & \searrow{\scriptstyle\pi^*} & \downarrow{\scriptstyle\alpha^*} \\ & & \text{Spec R} \end{array}$$

is commutative.

Proof. It is easy to check that (i) $\Rightarrow$ (ii). To prove that (i) holds it is sufficient to show that $\pi(x-\alpha(x)) \in \omega$ for any $x \in R$ and maximal ideal ω of S, because in a finitely generated algebra over a field the nilradical is the intersection of the maximal ideals (Atiyah and MacDonald [1, Chapter 5, Exercise 24]). Thus let ω be a maximal ideal of S. By Hilbert's Nullstellensatz (Atiyah and MacDonald[1, Chapter 5]), $S/\omega \cong k$, so as k-spaces $S = k \oplus \omega$, and for any $x \in R$ we may write $\pi(x) = \lambda + t$ where $t \in \omega$. Since α is k-linear, $\pi(x-\lambda) \in \omega$, or $x-\lambda \in \pi^*(\omega)$, which by hypothesis implies that $\alpha(x) - \lambda \in \pi^*(\omega)$ also. Therefore $\pi(x-\alpha(x)) \in \omega$ for all maximal ideals ω, as required. $\square$

To spare ourselves notational difficulties we record the varieties version of Theorem 5.6.5 only the case where p is odd:

Theorem 5.7.3. *Suppose that* α *is a* p'-*automorphism of the* p-*group P, where p is odd. Then the following are equivalent:*

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(i) α fixes pointwise the image of $X_Q(k)$ in $X_P(k)$.

(ii) α fixes Q pointwise.

Corollary 5.7.4. *If P is a p-group, where p is odd, then a nontrivial p'-automorphism of P acts nontrivially on $X_P(k)$.*

NOTE The author is grateful to Dr D. Benson for pointing out the following short proof of Theorem 5.7.3.

Lemma. Let α be a p'-automorphism of the p-group P. Suppose the semidirect product $P \rtimes \langle \alpha \rangle$ acts on an F_p-vector space V. If α acts nontrivially on V then α acts nontrivially on the set of P-orbits of V.

Proof. Let $0 < V_1 < \ldots < V_{n-1} < V$ be a chain of subspaces defined by
$$V_j / V_{j-1} = \{ x \in V/V_{j-1} : x \text{ is fixed by } P \}$$
Clearly each V_j is $\langle \alpha \rangle$-invariant, so α acts nontrivially on some V_j / V_{j-1}. But each coset of V_{j-1} in V_j is a union of P-orbits. □

Proof of 5.7.4. By the Thompson Theorem (5.6.6 above), P has a characteristic subgroup D of exponent p (recall p is odd) on which α acts nontrivially. Thus $(P/D) \rtimes \langle \alpha \rangle$ acts on $V = D/D'$ with α acting nontrivially (see 5.6.8) so by the lemma, α acts nontrivially on the set of P-orbits on D/D'.

Consider the sequence of maps of varieties
$$(D/D' \otimes_{F_p} K) = V_{D/D'} \to V_D \to V_Q \to V_P . \quad \binom{\text{notation as}}{\text{in } 5.7.1}$$
Since D has exponent p, a line in $V_{D/D'}$ defined over F_p injects into V_P. The image in V_P of $V_{D/D'}$ is just $(V_{D/D'})/(P/D)$, so α acts nontrivially on the image, as required. □

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