On the Theory of Fitting Classes of Finite Soluble Groups

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Abstract

We continue the study of Fitting classes begun by Fischer in 1966 and carried on by (notably) Gaschütz and Hartley. Disappointingly the theory has, as yet, failed to display the richness of its predecessor, the theory of Formations. Here we present our contributions, embedded in a survey of the progress so far made in this tantalizing part of finite soluble group theory.

Chapter 0 indicates the group theoretic notation we use, while Chapter 1 contains the basic results and terminology of Fitting class theory. Broadly speaking this theory comprises a study of the classes themselves, a study of the embedding of the $\mathfrak{F}$-subgroups (subgroups which belong to $\mathfrak{F}$) of an arbitrary group G and a study of the relation between $\mathfrak{F}$ and the $\mathfrak{F}$-subgroups of G . As in Formation theory we focus attention on canonical sets of $\mathfrak{F}$-subgroups , namely the $\mathfrak{F}$-injectors, the Fischer $\mathfrak{F}$-subgroups and the maximal $\mathfrak{F}$-subgroups containing the radical.

Chapter 2 begins with analyses of several examples of Fitting classes, establishing the coincidence of these three sets of $\mathfrak{F}$-subgroups (in all groups) in many cases, a property not enjoyed by all Fitting classes. Here too we examine some known 'new classes from old' procedures, and introduce a new one $\mathfrak{F}^{\pi}$ (defined for any $\mathfrak{F}$ and set of primes π), showing how this concept may be used to characterize the injectors for the product $\mathfrak{F}_1\mathfrak{F}_2$ of two Fitting classes. The chapter ends with some remarks on the thorny problem of generating Fitting classes from given groups and we present an imitation of work of Dark, the only person to achieve progress in this direction. Finally we show how one of

the classes so constructed settles a question posed by Gaschütz.

Chapter 3 develops the theory of pronormal subgroups, based on key theorems of Mann-Alperin and Fischer, showing in particular that a permutable product of pronormal subgroups is again pronormal. This approach yields a more compact version of work on subgroups of a group which are p-normally embedded for all primes p (we use the term strongly pronormal), published by Chambers. The injectors for a Fischer class (in particular a subgroup closed Fitting class) have this property.

Chapter 4 attacks the problem of determining the injectors for the class $\mathfrak{F}^{\pi}$, and shows, in the light of chapter 3, that the natural guess (a product of an $\mathfrak{F}$-injector and a Hall π'-subgroup) holds good when, for instance, $\mathfrak{F}$ is a Fischer class. However, modification of the example of Dark denies that this is in general the case. So arises the concept of permutability of a Fitting class and, after giving a new proof of a related lemma of Fischer, we establish conditions on a Fitting class equivalent to its being permutable, involving system normalizers.

Chapter 5 takes a preliminary look at the analogue of Cline's partial ordering of strong containment ($\ll$) for Fitting classes, and we show that a Fitting class maximal in this sense and having strongly pronormal injectors in all groups, is necessarily a normal Fitting class. In our final section we examine the radical of a direct power of a group G and show that, for a normal class, the radical is never the corresponding direct power of the radical of G (unless of course G lies in the class). This investigation puts the set of normal Fitting classes in a new setting, and we demonstrate that to each Fitting class $\mathfrak{F}$ there corresponds a well defined class $\mathfrak{F}^*$ with properties close to those of $\mathfrak{F}$.

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To the best of my knowledge results presented here, and not attributed to others or described as well-known, are original.

Contents

Chapter 0 Notation and assumed results

. $H \leqslant G$, $H < G$, $H \lhd G$ mean that H is a subgroup, proper subgroup, normal subgroup of G respectively; while $X \subseteq Y$, $X \subset Y$ indicate that X is a subset, proper subset of Y respectively. Apart from this, our notation and terminology is that of Gorenstein's book, supplemented by the following, we hope well accepted, notation.

$\mathfrak{X}$ is called a class of groups if $\mathfrak{X}$ contains each group of order 1 , and any group isomorphic to one of its members. An $\mathfrak{X}$-group is then a member of $\mathfrak{X}$. Unless otherwise stated, each group considered in this dissertation will be finite and belong to $\mathfrak{S}$, the class of soluble groups. $\mathfrak{A}$ denotes the class of Abelian groups and $\mathfrak{N}$ the class of nilpotent groups. $\mathfrak{S}_\pi$ is the class of soluble π-groups (where π is a set of primes), and for any class $\mathfrak{X}$, $\mathfrak{X}_\pi$ is $\mathfrak{X} \cap \mathfrak{S}_\pi$. Frequent use will be made of the now familiar closure operation notation of Hall. A closure operation is an expanding, idempotent and monotonic function which maps classes of groups to classes of groups. For example $S\mathfrak{X}$ is the class of all subgroups of $\mathfrak{X}$-groups, and $Q\mathfrak{X}$ the class of all quotients of $\mathfrak{X}$-groups.

We assume knowledge of Hall's fundamental theorems on the existence and conjugacy of Hall subgroups in finite soluble groups, and also his work on Sylow systems and system normalizers. G_p will always denote a member of $\mathrm{Syl}_p(G)$, the set of Sylow p-subgroups of G , and G_π a member of $\mathrm{Hall}_\pi(G)$, the set of Hall π-subgroups of G . We say G_π reduces into the subgroup H of G , written $G_\pi \searrow H$, if $G_\pi \cap H \in \mathrm{Hall}_\pi(H)$. A Sylow system Σ of G reduces into H , written $\Sigma \searrow H$, if $G_\pi \searrow H$ for all $G_\pi \in \Sigma$. $|G|_\pi$ denotes the greatest π-divisor of $|G|$, so $|G|_\pi = |G_\pi|$.

$O_\pi(G)$ is the unique largest normal π-subgroup of G and $G/O^\pi(G)$ its unique largest π-quotient.

We also assume the reader is familiar with the basic properties of Carter subgroups and the elements of Formation theory. For these and the other results we need, Huppert's book is the natural reference.

G (soluble) is called primitive iff G has a unique minimal normal subgroup which is complemented iff G has a self-centralizing minimal normal subgroup.

If $A, B \leqslant G$; $A \perp B$ will mean that A and B permute, that is $AB = BA$ or equivalently that AB is a subgroup of G . If $A \lhd B \leqslant G$ and $H \leqslant G$, we say H covers A/B if $B = A(H \cap B)$ and avoids A/B if $H \cap B \leqslant A$. $\prod_{i\in I} G_i$ will denote the direct product of the groups $\{ G_i \}_{i\in I}$. C_t is our notation for the cyclic group of order t , Σ_n the symmetric group of degree n , A_n its alternating subgroup, D_{2n} the dihedral group of order $2n$ and Q_8 the quaternion group . $A \wr B$, the wreath product of A by B , will be constructed from the regular permutation representation of B unless another representation is specified. A conjugacy class H^G of subgroups of G will be called a characteristic conjugacy class if each automorphism of G permutes the members of H^G . H is a subnormal subgroup of G , written H sn G , if there exists a series $H \lhd H_1 \lhd H_2 \lhd \ldots\ldots \lhd H_{n-1} \lhd G$. H is a pronormal subgroup of G , written H pro G , if H and H^g are conjugate in their join $< H,H^g >$ for all $g \in G$. Soc G denotes the socle of G , the product of all the minimal normal subgroups of G ; and π-soc G , the π-socle of G , the product of all the minimal normal π-subgroups of G .

If * is a function which assigns to each group G a subgroup (respectively set of subgroups) $*(G)$, then if $N \lhd G$, $*(G \bmod N)$ will denote the inverse image of (respectively inverse images of the members of) $*(G/N)$ in G .

Groups and sets of operators (including matrices) will always act on the right.

We write $G = [H]K$ whenever K complements the normal subgroup H of G , that is, G is a split extension of H by K . The following key corollary of the Schur-Zassenhaus theorem will be well used.

If $G = [H]K$ where $|H|$ and $|K|$ are coprime then

a) $H = [H , K]C_H(K)$ and the product is direct when $H \in \mathfrak{A}$

b) $[H , K] = [H , K , K]$, so $[H , K , K , \ldots , K] = 1$ eventually (that is with sufficient terms K) $\Rightarrow [H , K] = 1$.

c) If A/B is a K-admissable section of H (that is $\overset{B}{\cancel{A}} \lhd \overset{A}{\cancel{B}} \leq H$ and K normalizes A and B) then $C_{A/B}(K) = C_A(K)B/B$.

The proof requires the solubility of either H or K , which holds in this dissertation by the assumed solubility of each G , but is true in any case by the theorem of Feit and Thompson.

Frequent appeals will be made to the fact that the Fitting subgroup $F(G)$ of a finite soluble group contains its centralizer in G ; and also to the well-known theorems of Maschke and Clifford.

Chapter 1 Introduction and preliminary results

1.1 Fitting Classes and $\mathfrak{F}$-injectors

The concept of a Fitting class was introduced by Fischer in [1] (the main results of which appear in [2]), to dualize Gaschütz's elegant theory of Formations. The name is a memorial to Hans Fitting, whose celebrated theorem [3] shows that the class of nilpotent groups is a Fitting class.

A Fitting class $\mathfrak{F}$ is a class of groups closed under the operations of taking normal subgroups and forming normal products, that is

$$N \lhd G \in \mathfrak{F} \Rightarrow N \in \mathfrak{F} \qquad (\mathfrak{F} = S_n\mathfrak{F})$$

$$\mathfrak{F} \ni N_1, N_2 \lhd G = N_1N_2 \Rightarrow G \in \mathfrak{F} \qquad (\mathfrak{F} = N_0\mathfrak{F})$$

Broadly speaking, the theory of Fitting classes comprises a study of the classes themselves, a study of the embedding of those subgroups of an arbitrary group G which lie in a particular Fitting class $\mathfrak{F}$ (the $\mathfrak{F}$-subgroups of G), and finally a study of the relation between $\mathfrak{F}$ and the $\mathfrak{F}$-subgroups of G. As in Formation theory we focus our attention on canonical sets of $\mathfrak{F}$-subgroups of G.

If $\mathfrak{X}$ is any N_0-closed class, then in each group G the set of normal $\mathfrak{X}$-subgroups has a unique maximal member called the $\mathfrak{X}$-radical of G, clearly characteristic in G and denoted by $G_{\mathfrak{X}}$. For example $G_{\mathfrak{N}}$ is the Fitting subgroup of G. Suppose N sn G, specifically $N = N_0 \lhd N_1 \lhd N_2 \lhd \dots \lhd N_r = G$. Then for $i = 1,\dots,r$ $(N_{i-1})_{\mathfrak{X}}$ char $N_{i-1} \lhd N_i$, and so $N_{\mathfrak{X}} = (N_0)_{\mathfrak{X}} \lhd (N_1)_{\mathfrak{X}} \lhd (N_2)_{\mathfrak{X}} \lhd \dots \lhd (N_r)_{\mathfrak{X}} = G_{\mathfrak{X}}$. Letting $N \in \mathfrak{X}$ we see that $G_{\mathfrak{X}}$ contains every subnormal $\mathfrak{X}$-subgroup of G. If now $\mathfrak{F}$ is a Fitting class and N sn G then $N_{\mathfrak{F}} = N \cap G_{\mathfrak{F}}$. For, we have just seen that $N_{\mathfrak{F}} \leq G_{\mathfrak{F}}$, but certainly $N \cap G_{\mathfrak{F}} \lhd N$ and $N \cap G_{\mathfrak{F}}$ sn $G_{\mathfrak{F}}$, so $N \cap G_{\mathfrak{F}} \in S_n\mathfrak{F} = \mathfrak{F}$.

The <u>characteristic of a Fitting class</u> $\mathfrak{F}$ (denoted by char $\mathfrak{F}$), is defined to be the set π of those primes p which divide the order of some $\mathfrak{F}$-group. So of course $\mathfrak{F} \subseteq \mathcal{S}_\pi$.

<u>1.1.1 Proposition</u> (Hartley [4])

If $\mathfrak{F}$ is a Fitting class and char $\mathfrak{F} = \pi$ then $\mathcal{N}_\pi \subseteq \mathfrak{F} \subseteq \mathcal{S}_\pi$.

<u>Proof</u>

We need only prove $\mathcal{N}_\pi \subseteq \mathfrak{F}$. To do this it suffices to show that each cyclic p-group lies in $\mathfrak{F}$ for all $p \in \pi$, since a nilpotent π-group is generated by such subgroups and they are of course subnormal. So let $p \in \pi$, then by definition there exists a group $G \in \mathfrak{F}$ such that $p \mid |G|$. Therefore there exists a normal subgroup H of G such that H has a normal subgroup K of index p in H. Then $H \times H/K$ is the product of normal subgroups $\{(h,1) : h \in H\}$ and $\{(h,Kh) : h \in H\}$, each isomorphic to H. Now $H \lhd G \in \mathfrak{F}$, and so since $\mathfrak{F}$ is a Fitting class, we have $H \times H/K \in \mathfrak{F}$. It follows that $\mathfrak{F}$ contains $H/K \cong C_p$. An easy induction argument shows that C_{p^n} is a subgroup of the repeated wreath product $(\ldots(C_p \wr C_p) \wr C_p \ldots) \wr C_p$ with n terms, which is clearly generated by its (necessarily subnormal) subgroups of order p. Again by subnormality we deduce $C_{p^n} \in \mathfrak{F}$ and the proof is complete. □

We now define the central object of our study. Let $\mathfrak{X}$ be an arbitrary class of groups. A subgroup V of G is called an <u>$\mathfrak{X}$-injector</u> of G if $V \cap N$ is a maximal $\mathfrak{X}$-subgroup of N for each subnormal subgroup N of G. We say $V \cap N$ is <u>$\mathfrak{X}$-maximal</u> in N in this case. It is clear that if G has an $\mathfrak{X}$-injector V, then the image V^θ of V under an automorphism θ of G is an $\mathfrak{X}$-injector of G also, and $V \cap N$ is an $\mathfrak{X}$-injector of N for each subnormal subgroup N of G.

The following theorem sets the theory of Fitting classes in motion and was proved in 1967. For completeness and future reference we reproduce its elegant proof here.

1.1.2 THEOREM (Fischer, Gaschütz and Hartley [5] Satz 1)
If $\mathfrak{F}$ is a Fitting class, then each group G has precisely one conjugacy class of $\mathfrak{F}$-injectors.

PROOF

The argu¢ment depends on the following key lemma.

1.1.3 Lemma (Hartley [5] Lemma)
Let $\mathfrak{F}$ be a Fitting class and G a group with $N \lhd G$ and $G/N \in \mathfrak{N}$. If V_1 and V_2 are $\mathfrak{F}$-maximal subgroups of G and $W = V_1 \cap N = V_2 \cap N$ is $\mathfrak{F}$-maximal in N, then V_1 and V_2 are conjugate in G.

Proof

Clearly $V_1, V_2 \leq N_G(W) = G^*$ say, and $G^*/N \cap G^* \in \mathfrak{N}$, hence we may assume $W \lhd G$. For $i = 1,2$ let $T_i = N_G(V_i)$, then $[V_i, T_i, T_i, \ldots, T_i] \leq N$ eventually since $G/N \in \mathfrak{N}$. Of course $[V_i, T_i, T_i, \ldots, T_i] \leq V_i$, so $[V_i, T_i, T_i, \ldots, T_i] \leq N \cap V_i = W$ eventually, which means V_i/W is a hypercentral normal subgroup of T_i/W. So V_i/W lies in a system normalizer of T_i/W, which in turn lies in a Carter subgroup of T_i/W, which we may write as WC_i/W for some Carter subgroup C_i of T_i. Now $WC_i/W \in \mathfrak{N}$, so V_i is an $\mathfrak{F}$-maximal subnormal subgroup of WC_i and therefore $V_i = (WC_i)_{\mathfrak{F}}$. Suppose $x \in G$ normalizes C_i, then x normalizes WC_i and hence also its $\mathfrak{F}$-radical V_i. Therefore $x \in T_i$, but C_i is self-normalizing in T_i, so in fact $x \in C_i$. This shows C_i is a Carter subgroup of G and so C_1 and C_2 are conjugate in G, $C_1 = C_2{}^g$ say, so finally $V_1 = (WC_1)_{\mathfrak{F}} = (WC_2{}^g)_{\mathfrak{F}} = (WC_2)_{\mathfrak{F}}{}^g = V_2{}^g$. $\square$

The theorem clearly holds when $|G| = 1$, so we use induction, supposing the result true for all groups of order less than $|G|$. So let M be a proper normal subgroup of G with $G/M \in \mathfrak{N}$ and R an $\mathfrak{F}$-injector of M. We show that if V is $\mathfrak{F}$-maximal in G with $R \leq V$, then V is an $\mathfrak{F}$-injector of G. It will clearly suffice to show that $V \cap G^*$ is an $\mathfrak{F}$-injector of G^* whenever G^* is a maximal normal subgroup of G. Let V^* be an $\mathfrak{F}$-injector of G^*, this exists by induction. Let $N = G^* \cap M$, then $G/N \in \mathfrak{N}$, and furthermore $V \cap N$ and $V^* \cap N$ are $\mathfrak{F}$-injectors of N. By induction they are conjugate, so (replacing V^* by a conjugate if necessary) we assume $V \cap N = V^* \cap N = W$ say. Now if $\overline{V}$ is an $\mathfrak{F}$-maximal subgroup of G containing V^*, then Lemma 1.1.3. implies that $\overline{V}$ and V are conjugate in G. Hence $\overline{V} \cap G^* = V^*$ and $V \cap G^*$ are too, which implies $V \cap G^*$ is an $\mathfrak{F}$-injector of G^* as required.

Now suppose V_1 and V_2 are $\mathfrak{F}$-injectors of G, and again that G^* is a maximal normal subgroup of G. As injectors of G^*, $V_1 \cap G^*$ and $V_2 \cap G^*$ are conjugate in G^* by induction, and so as before we may actually assume they are equal. It is then an immediate consequence of Lemma 1.1.3. that V_1 and V_2 are conjugate in G, and the proof is complete. □

Fundamental use will be made of the following corollaries of 1.1.2 and its proof.

<u>1.1.4 THEOREM</u> (Fischer, Gaschütz and Hartley [5] Korollar and Satz 2)

Let $\mathfrak{F}$ be a Fitting class and G a group. Then

a) If $N \lhd G$ with $G/N \in \mathfrak{N}$, and V is an $\mathfrak{F}$-maximal subgroup of G such that $V \cap N$ is an $\mathfrak{F}$-injector of N, then V is an $\mathfrak{F}$-injector of G.

b) If $1 = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_r = G$ and $G_i/G_{i-1} \in \mathfrak{N}$ for $i = 1,..,r$ then V is an $\mathfrak{F}$-injector of G if and only if $V \cap G_i$ is $\mathfrak{F}$-maximal in G_i for $i = 1,..,r$.

c) If V is an $\mathfrak{F}$-injector of G and $V \leqslant H \leqslant G$ then V is an $\mathfrak{F}$-injector of H.

PROOF

a) follows from the proof of 1.1.2 and b) by induction.

To prove c) suppose $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_r = G$ is a series with nilpotent factors. Then so also is $1 = H_0 \lhd H_1 \lhd \ldots \lhd H_r = H$, where $H_i = H \cap G_i$ for $i = 1,..,r$. Now $H_i \cap V = G_i \cap H \cap V = G_i \cap V$ is $\mathfrak{F}$-maximal in $G_i \cap H = H_i$, so by b) V is an $\mathfrak{F}$-injector of H. $\square$

We shall denote the set of $\mathfrak{F}$-injectors of a group G by $I_{\mathfrak{F}}(G)$.

The proof of the following converse to 1.1.2 is clear.

1.1.5 Proposition

If $\mathfrak{X}$ is a class of groups such that each group G has an $\mathfrak{X}$-injector, then $\mathfrak{X}$ is a Fitting class.

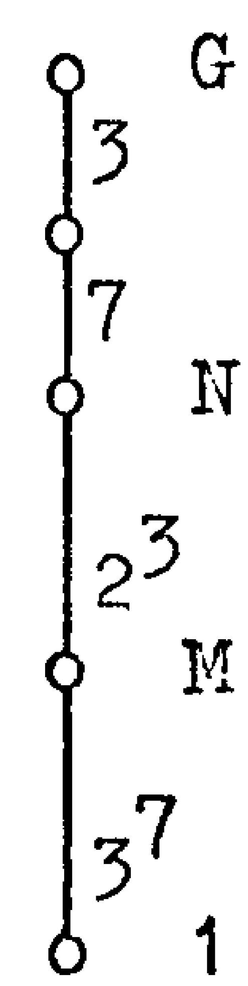
Considered together, 1.1.2 and 1.1.5 show that Fitting classes are precisely the classes for which injectors exist in all groups. This demonstrates that, despite what the closure operations suggest, a Fitting class is best regarded as the dual of a Schunck class [6]. Such classes are precisely those for which projectors exist in all groups. This point of view is strongly endorsed by Gaschütz in [7] and [8].

The fact that $\mathfrak{H}$-projectors (where $\mathfrak{H} = Q\mathfrak{H}$) are conjugate whenever they exist [9], prompted the offene Frage 1 of [5]: Are the $\mathfrak{X}$-injectors of a group G (where $\mathfrak{X} = S_n\mathfrak{X}$) necessarily conjugate when they exist?

The question was closed in the negative by Hawkes, and we sketch his example here. It is worth noting that in [10] Chambers shows the answer is yes if G satisfies the strong condition of having p-length 1 for all primes p.

1.1.6 Example

There exists a group G with the unique chief series indicated.



Let H be the non-Abelian group of order 21 and $\mathfrak{X}$ the S_n-closed class it generates, so $\mathfrak{X} = \{H, C_7, 1\}$.

N has complements H_1 and H_2 not conjugate in G, both isomorphic to H, each of which is an $\mathfrak{X}$-injector of G.

1.1.2 and 1.1.4 have the following easy but important consequences.

1.1.7 Proposition

Let $\mathfrak{F}$ be a Fitting class, V an $\mathfrak{F}$-injector of G and $N \lhd G$.

a) $V \cap N$ is pronormal in G and in particular V pro G.

b) $G = NN_G(V \cap N)$.

c) V covers or avoids each chief factor of G.

d) If $\mathfrak{F}$ has characteristic π, then an $\mathfrak{F}$-injector V_o of a Hall π-subgroup G_π of G is an $\mathfrak{F}$-injector of G.

Proof

a) Suppose $x \in G$, then $V \cap N$ and $(V \cap N)^x$ are $\mathfrak{F}$-injectors of N. By 1.1.4 c) they are $\mathfrak{F}$-injectors of their join $J \leqslant N$, and therefore conjugate in J by 1.1.2. This establishes $V \cap N$ pro G.

b) This follows from the Frattini argument (which may be applied whenever we have H pro G and $H \leq N \lhd G$, to yield $G = NN_G(H)$.), and a).

c) If N/K is a chief factor of G, then b) implies that C normalizes $K(V \cap N)$, so $K(V \cap N) = N$ or $V \cap N \leq K$.

d) By the definition of characteristic, V is a π-group and so $V^x \leq G_\pi$ for some $x \in G$. Then V^x is an $\mathfrak{F}$-injector of G_π by 1.1.4. c), therefore is conjugate to V_0 by 1.1.2. Thus V_0 is an $\mathfrak{F}$-injector of G as required. $\square$

1.2 Fischer classes and the sets $II_{\mathfrak{F}}(G)$ and $III_{\mathfrak{F}}(G)$.

If $\mathfrak{F}$ is a Fitting class, an $\mathfrak{F}$-subgroup T of G is called a Fischer $\mathfrak{F}$-subgroup of G if T contains each $\mathfrak{F}$-subgroup of G which it normalizes. This is clearly equivalent to the condition that T contains each normal $\mathfrak{F}$-subgroup of any $H \leq G$ with $T \leq H$. Thus the Fischer $\mathfrak{F}$-subgroups are dual to the $\mathfrak{X}$-covering subgroups of Gaschütz. (T is an $\mathfrak{X}$-covering subgroup of G if $T \in \mathfrak{X}$ and covers each $\mathfrak{X}$-quotient of any $H \leq G$ with $T \leq H$.)

It is an immediate consequence of 1.1.4 c) that an $\mathfrak{F}$-injector of a group G is also a Fischer $\mathfrak{F}$-subgroup of G. Thus denoting the set of Fischer $\mathfrak{F}$-subgroups of G by $II_{\mathfrak{F}}(G)$, we have $I_{\mathfrak{F}}(G) \subseteq II_{\mathfrak{F}}(G)$. It is one of the major disappointments of the theory that there exist $\mathfrak{F}$ and G where this inclusion is proper, that is the Fischer $\mathfrak{F}$-subgroups of G do not form a single conjugacy class, a disparity with the dual theory. (It is of course clear that in general $II_{\mathfrak{F}}(G)$ is a union of conjugacy classes of G.) The only known example of this is due to Dark [11], and we shall exhibit some Corollaries of this complicated work later. (See 2.7 and 4.1.3 c).)

However, if $\mathfrak{F} = S\mathfrak{F}$ then $I_{\mathfrak{F}}(G) = II_{\mathfrak{F}}(G)$ for all G. In fact we have the following usefully stronger result, due to Fischer, a proof of which appears in [4]. A Fitting class $\mathfrak{F}$ is called a Fischer class if

$N \lhd G \in \mathfrak{F}$, $N \leqslant H \leqslant G$ and H/N has prime power order imply $H \in \mathfrak{F}$.

So certainly an S-closed Fitting class is a Fischer class.

<u>1.2.1 THEOREM</u> (Fischer)

$\mathfrak{F}$ is a Fischer class implies $I_{\mathfrak{F}}(G) = II_{\mathfrak{F}}(G)$ for all G.

A further canonical set of $\mathfrak{F}$-subgroups of a group G are the $\mathfrak{F}$-maximal subgroups of G which contain the $\mathfrak{F}$-radical of G. We denote this set by <u>$III_{\mathfrak{F}}(G)$</u>, and observe that each Fischer $\mathfrak{F}$-subgroup of G belongs to $III_{\mathfrak{F}}(G)$. Thus for arbitrary $\mathfrak{F}$ and G we have

$$I_{\mathfrak{F}}(G) \subseteq III_{\mathfrak{F}}(G) \subseteq III_{\mathfrak{F}}(G).$$

It is surprisingly frequently true of a Fitting class $\mathfrak{F}$ that these three sets coincide for each G, that is, $\mathfrak{F}$-injectors are characterised as those $\mathfrak{F}$-maximal subgroups containing the $\mathfrak{F}$-radical. For example when $\mathfrak{F} = \mathfrak{N}$ (see 2.1.2), and of course when $\mathfrak{F} = \mathfrak{S}_{\pi}$, for any set of primes π. (The Hall π-subgroups are the $\mathfrak{S}_{\pi}$-injectors of G) However, interesting examples of this phenomenon only occur when $\mathfrak{F}$ has ' full characteristic ', as we prove in the next result.

<u>1.2.2 Proposition</u>

$I_{\mathfrak{F}}(G) = III_{\mathfrak{F}}(G)$ for all G implies $\mathfrak{N} \subseteq \mathfrak{F}$ or $\mathfrak{F} = \mathfrak{S}_{\pi}$ for some π.

<u>Proof</u>

Suppose $\mathfrak{N} \nsubseteq \mathfrak{F}$, that is there exists a prime $p \notin \pi = \operatorname{char} \mathfrak{F}$. Let G be an arbitrary π-group. Each normal subgroup of $W = C_p \wr G$ has order divisible by p, and so $W_{\mathfrak{F}} = 1$. Now if $1 \neq G_q$ is a Sylow q-subgroup of G, then $G_q \in \mathfrak{F}$ since $q \in \pi = \operatorname{char} \mathfrak{F}$.

Therefore G_q lies in some $\mathfrak{F}$-maximal subgroup H of W which, by the triviality of $W_{\mathfrak{F}}$ and the hypothesis of the proposition, is an $\mathfrak{F}$-injector of W. This holds for all q | $|G|$ and therefore $|H| \geq |G|$. But of course $p \nmid |H|$ since $p \notin \pi$ so $|H| = |G|$ and $G \in \mathfrak{F}$. We deduce $\mathfrak{S}_\pi \subseteq \mathfrak{F}$, and equality follows since $\pi = \operatorname{char} \mathfrak{F}$. □

This shows that if $\mathfrak{F}$ is a Fitting class of characteristic π and $\mathfrak{F} \neq \mathfrak{S}_\pi$, then the question of the validity of $I_{\mathfrak{F}}(G) = III_{\mathfrak{F}}(G)$ is best asked for $G \in \mathfrak{S}_\pi$, the class of π-groups. Of course, for such $\mathfrak{F}$, the $\mathfrak{F}$-subgroups of an arbitrary group G are just the $\mathfrak{F}$-subgroups of the Hall π-subgroups of G. Thus the problem of the embedding of the $\mathfrak{F}$-subgroups of G divides into two, namely the embedding of the $\mathfrak{F}$-subgroups of a Hall π-subgroup G_π in G_π and the embedding of G_π in G. Only the first problem is dependent on the Fitting class $\mathfrak{F}$, so when $\mathfrak{F}$ has characteristic π it is natural to work within the category of π-groups.

In order to produce an example which shows that the converse of 1.2.2 is false, we mention a well-known and useful technique for constructing groups with a unique chief series.

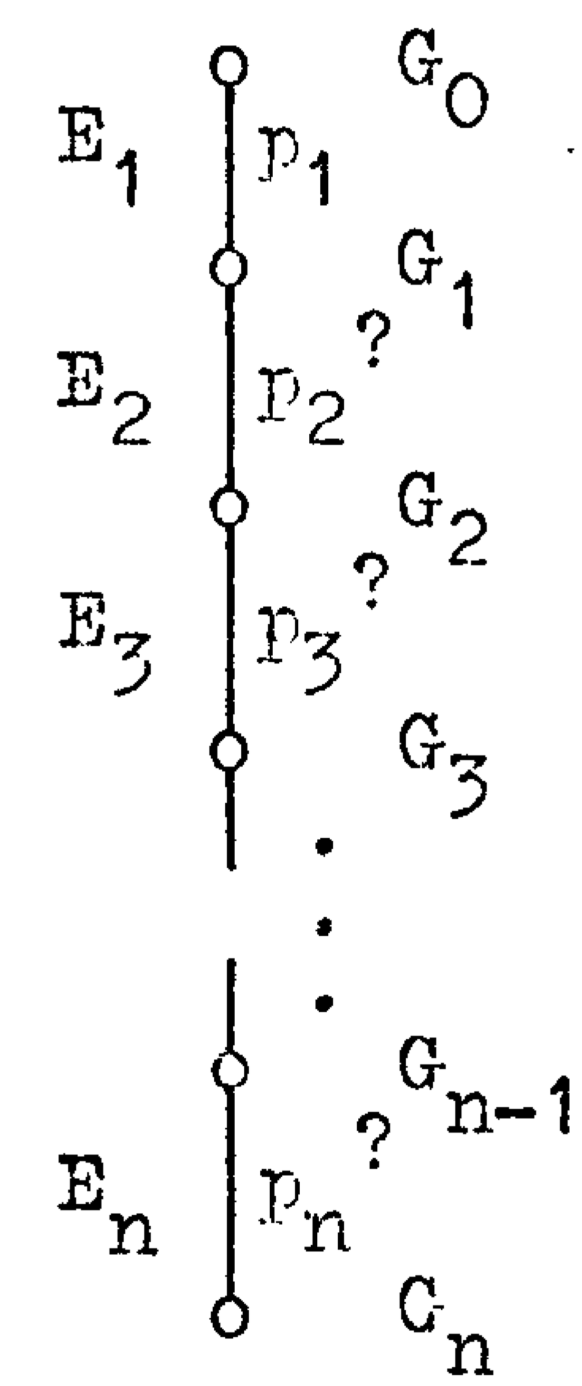
1.2.3 Lemma

Let H be a group with a unique minimal normal subgroup M where M is a q-group. If p is a prime distinct from q then H has a faithful irreducible representation over GF(p).

Proof

Let V be the regular GF(p)H-module. Then H acts faithfully on some composition factor of V. For, if not, M (as the unique minimal normal subgroup of H) lies in the kernel of the representation of H on each composition factor of V and therefore, by a well-known result, must act trivially on V since $q \nmid p$. This contradicts the fact that V is the regular $GF(p)H$-module, so our assertion holds and we have a faithful irreducible $GF(p)H$-module. □

1.2.4 Corollary



Let $p_n, \ldots, p_2, p_1$ be a sequence of primes with no two consecutive terms equal. Then there exists a group G with a unique chief series $1 = G_n \lhd G_{n-1} \lhd \ldots \lhd G_2 \lhd G_1 \lhd G_0 = G$, and elementary Abelian subgroups $E_n, \ldots, E_2, E_1$ such that E_i is a p_i-group which covers the chief factor G_{i-1}/G_i and avoids the rest, and E_i normalizes E_j for all $1 \leqslant i \leqslant j \leqslant n$, so $G = E_n E_{n-1} \ldots E_2 E_1$.

Proof

We use induction on n. If $n = 1$ put $G = E_1 \cong C_{p_1}$.
By induction suppose $H = E_{n-1} \ldots E_2 E_1$ has the required properties. Then H has a unique minimal normal subgroup E_{n-1} which is a p_{n-1}-group. By Lemma 1.2.3 there exists a faithful irreducible $GF(p_n)H$-module and we let E_n be its additive group. Clearly the split extension $G = E_n H$ satisfies the statement of the Corollary, and for each $i = 1, \ldots, n$, G/G_i is a primitive soluble group with unique minimal normal subgroup G_{i-1}/G_i which is self-centralizing in G/G_i. Knowledge of familiar small groups and their representations usually indicates the orders of the first few terms in the series $E_1, E_2, \ldots$. □

To specify the Fitting class in our example, we need the concept of the product of two Fitting classes, introduced by Gaschütz in [7].

If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are Fitting classes, Gaschütz defines $\underline{\mathfrak{F}_1\mathfrak{F}_2} = \{G : G/G_{\mathfrak{F}_1} \in \mathfrak{F}_2\}$. This is a subclass of $\{G :$ there exists $N \lhd G$ such that $N \in \mathfrak{F}_1,\ G/N \in \mathfrak{F}_2\}$ and the inclusion can be strict, but it is easily seen that the classes coincide when $\mathfrak{F}_2 = Q\mathfrak{F}_2$. Certainly $\mathfrak{F}_1 \subseteq \mathfrak{F}_1\mathfrak{F}_2$, but $\mathfrak{F}_2 \nsubseteq \mathfrak{F}_1\mathfrak{F}_2$ can occur. We now perform the routine verification of some elementary facts concerning $\mathfrak{F}_1\mathfrak{F}_2$, returning to a fuller discussion in 2.6 .

<u>1.2.5 Proposition</u>

If $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ are Fitting classes then

a) $\mathfrak{F}_1\mathfrak{F}_2$ is a Fitting class of characteristic char $\mathfrak{F}_1 \cup$ char $\mathfrak{F}_2$

b) $G_{\mathfrak{F}_1\mathfrak{F}_2}/G_{\mathfrak{F}_1} = (G/G_{\mathfrak{F}_1})_{\mathfrak{F}_2}$ for all G

c) $(\mathfrak{F}_1\mathfrak{F}_2)\mathfrak{F}_3 = \mathfrak{F}_1(\mathfrak{F}_2\mathfrak{F}_3)$.

<u>Proof</u>

a) If $N \lhd G \in \mathfrak{F}_1\mathfrak{F}_2$, then $N/N_{\mathfrak{F}_1} = N/N \cap G_{\mathfrak{F}_1} \cong NG_{\mathfrak{F}_1}/G_{\mathfrak{F}_1}$ and $NG_{\mathfrak{F}_1}/G_{\mathfrak{F}_1} \lhd G/G_{\mathfrak{F}_1} \in \mathfrak{F}_2$, hence $N \in \mathfrak{F}_1\mathfrak{F}_2$.

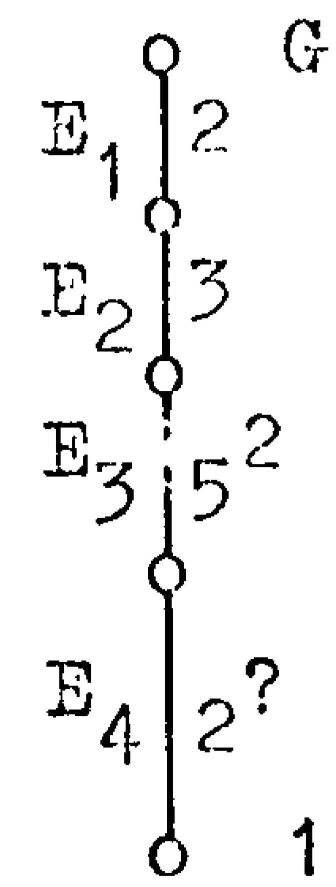
If $\mathfrak{F}_1\mathfrak{F}_2 \ni N_1, N_2 \lhd N_1N_2 = G$ then $G/G_{\mathfrak{F}_1} = N_1N_2/G_{\mathfrak{F}_1} = N_1G_{\mathfrak{F}_1}/G_{\mathfrak{F}_1} . N_2G_{\mathfrak{F}_1}/G_{\mathfrak{F}_1}$ and $N_iG_{\mathfrak{F}_1}/G_{\mathfrak{F}_1} \cong N_i/N_i \cap G_{\mathfrak{F}_1} = N_i/(N_i)_{\mathfrak{F}_1} \in \mathfrak{F}_2$, so $G \in \mathfrak{F}_1\mathfrak{F}_2$. $C_p \in \mathfrak{F}_1\mathfrak{F}_2$ if and only if $C_p \in \mathfrak{F}_1 \cup \mathfrak{F}_2$, gives the rest of the statement.

b) As we have remarked above, $\mathfrak{F}_1 \subseteq \mathfrak{F}_1\mathfrak{F}_2$, so $G_{\mathfrak{F}_1} \leq G_{\mathfrak{F}_1\mathfrak{F}_2}$. But certainly $(G_{\mathfrak{F}_1\mathfrak{F}_2})_{\mathfrak{F}_1} \lhd G$, therefore we must have $G_{\mathfrak{F}_1} = (G_{\mathfrak{F}_1\mathfrak{F}_2})_{\mathfrak{F}_1}$. So $G_{\mathfrak{F}_1\mathfrak{F}_2}/G_{\mathfrak{F}_1}$ is a normal $\mathfrak{F}_2$-subgroup of $G/G_{\mathfrak{F}_1}$, that is $G_{\mathfrak{F}_1\mathfrak{F}_2}/G_{\mathfrak{F}_1} \leq (G/G_{\mathfrak{F}_1})_{\mathfrak{F}_2} = N/G_{\mathfrak{F}_1}$ say. Again $N_{\mathfrak{F}_1} \lhd G$ so $N_{\mathfrak{F}_1} = G_{\mathfrak{F}_1}$ and $N \in \mathfrak{F}_1\mathfrak{F}_2$, which yields the result.

c) $G \in (\mathfrak{F}_1\mathfrak{F}_2)\mathfrak{F}_3 \Leftrightarrow G/G_{\mathfrak{F}_1\mathfrak{F}_2} \in \mathfrak{F}_3 \Leftrightarrow (G/G_{\mathfrak{F}_1})/(G/G_{\mathfrak{F}_1})_{\mathfrak{F}_2} \in \mathfrak{F}_3$ (by b)) $\Leftrightarrow G/G_{\mathfrak{F}_1} \in \mathfrak{F}_2\mathfrak{F}_3 \Leftrightarrow G \in \mathfrak{F}_1(\mathfrak{F}_2\mathfrak{F}_3)$. $\square$

1.2.6 Example

Let $\mathfrak{F} = \mathfrak{N}\mathfrak{S}_3$ and G a group constructed as in 1.2.4 from the sequence 2, 5, 3, 2 .
Then $E_4 = G_{\mathfrak{N}\mathfrak{S}_3}$ and using 1.1.4 b) we see that $E_4E_2 \in I_{\mathfrak{F}}(G)$.
However $E_4E_1 \in III_{\mathfrak{F}}(G) - I_{\mathfrak{F}}(G)$.
Thus $\mathfrak{N} \subseteq \mathfrak{F}$, but $I_{\mathfrak{F}}(G) \nsubseteq III_{\mathfrak{F}}(G)$.



Using similar techniques it is easy to construct a group G for which $III_{\mathfrak{F}}(G)$ comprises arbitrarily many conjugacy classes.

We end this introductory chapter with an elementary lemma and some cautionary observations.

1.2.7 Lemma

Suppose $H, K \leq G$, $H \perp K$ and that the Hall π-subgroup G_π of G reduces into H and K. Then $G_\pi \searrow HK$, $H \cap K$ and $(H \cap G_\pi)(K \cap G_\pi) = HK \cap G_\pi$.

Proof

Clearly $(H \cap G_\pi)(K \cap G_\pi) \subseteq HK \cap G_\pi$ (1)

and $|(H \cap G_\pi)(K \cap G_\pi)| = |H \cap G_\pi||K \cap G_\pi|/|H \cap K \cap G_\pi|$ (2)

Now $|HK| = |H||K|/|H \cap K|$ and G reduces into H and K, so taking π-divisors we have

$|HK|_\pi = |H \cap G_\pi||K \cap G_\pi|/|H \cap K|_\pi$ (3)

By (1) $|(H \cap G_\pi)(K \cap G_\pi)| \leq |HK \cap G_\pi| \leq |HK|_\pi$ (4)

and of course $|H \cap K \cap G_\pi| \leq |H \cap K|_\pi$ (5)

Then (2) and (3) show that we must have equality in (4) and (5) which implies the statement of the lemma. $\square$

1.2.8. Proposition

Suppose $\mathfrak{F}$ is a Fitting class for which $N_1, N_2 \lhd G = N_1N_2$ and $V = \mathfrak{F}$-injector of G imply $V = (N_1 \cap V)(N_2 \cap V)$.
Then $\mathfrak{F} = \mathfrak{S}_\pi$ for some set of primes π .

Proof

Let $\pi = \operatorname{char} \mathfrak{F}$ and suppose $\mathfrak{F} \neq \mathfrak{S}_\pi$. Let N be a group of minimal order in $\mathfrak{S}_\pi - \mathfrak{F}$. So $N_{\mathfrak{F}} = F$ say, is the unique maximal normal subgroup of N, of index p say. By assumption $N \notin \mathfrak{F}$, so F is $\mathfrak{F}$-maximal in N and therefore its unique $\mathfrak{F}$-injector. N is a π-group, so $p \in \operatorname{char} \mathfrak{F}$ and therefore $N/F \cong C_p \in \mathfrak{F}$. Now $G = N \times N/F$ is the product of normal subgroups $\{(n,1) : n \in N\}$ and $\{(n,Fn) : n \in N\}$, N_1 and N_2 say, each isomorphic to N and each having $\{(f,1) : f \in F\} \cong F$ as its unique $\mathfrak{F}$-injector. Thus if V is an $\mathfrak{F}$-injector of G, by hypothesis $V = (N_1 \cap V)(N_2 \cap V)$, so $V = \{(f,1) : f \in F\}$. However $F \times C_p \in \mathfrak{F}$, so V is not $\mathfrak{F}$-maximal in G, a contradiction giving $\mathfrak{F} = \mathfrak{S}_\pi$. □

This proposition shows why $\{G : \mathfrak{F}_1\text{-injector of } G \in \mathfrak{F}_2\}$ (where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are Fitting classes), can fail to be a Fitting class when $\mathfrak{F}_1 \neq \mathfrak{S}_\pi$ for some π. For example $G = \Sigma_3 \times C_2$ is the product of two normal subgroups isomorphic to Σ_3, each of which has an $\mathfrak{N}$-injector which is a 3-group. However G itself does not enjoy this property.

1.2.9 Proposition

Suppose $\mathfrak{F}$ is a non-trivial Fitting class for which $N_1, N_2 \lhd G = N_1N_2$ implies $G_{\mathfrak{F}} = (N_1)_{\mathfrak{F}}(N_2)_{\mathfrak{F}}$. Then $\mathfrak{F} = \mathfrak{S}$.

Proof

We may repeat the proof of the previous proposition to show that $\mathfrak{F}$ necessarily has the form $\mathfrak{S}_\pi$, for some set of primes π. Now suppose p and q are primes such that $p \in \pi$ but $q \notin \pi$. Let N be a group with unique chief series constructed in the manner of 1.2.4 from the sequence p, q, p. Then $N_{\mathfrak{F}} = M$ say, the unique minimal normal subgroup of N. $G = N \times C_p$ is the product of two normal copies of N (in the usual way), each with $\mathfrak{F}$-radical $M \times 1$. However it is apparent that $G_{\mathfrak{F}} = M \times C_p$, which contradicts our hypothesis. Thus our supposed choice of primes is not possible and we deduce that $\mathfrak{F}$ is trivial or equal to $\mathcal{S}$. $\square$

We shall even see in Chapter 5 that there exist Fitting classes $\mathfrak{F}$ for which $(A \times B)_{\mathfrak{F}} = A_{\mathfrak{F}} \times B_{\mathfrak{F}}$ can be false.

These observations are in sharp contrast to the situation when $\mathfrak{F}$ is a saturated formation. For it is easily seen that if G is a subdirect product of groups A and B, then an $\mathfrak{F}$-projector (respectively $\mathfrak{F}$-normalizer) of G is a subdirect product of the $\mathfrak{F}$-projectors (respectively $\mathfrak{F}$-normalizers) of A and B .

Chapter 2 Examples of Fitting classes

In this chapter we shall try to indicate most of the known Fitting classes, a promise which in itself reveals that our supply of examples is very limited. Contributing to this sparseness, many of the techniques for deriving new formations and Schunck classes from old ones fail to dualize. In particular no satisfactory 'local definition' for a Fitting class has yet emerged. Here we examine some established 'new from old' procedures, including the product $\mathfrak{F}_1 \mathfrak{F}_2$, and introduce a new one, paying special attention to discovering the properties which the new class inherits from the old. However we emphasise that fundamental examples of Fitting classes, the building blocks for these procedures, are far from plentiful.

2.1 The nilpotent injectors

We begin with a mild generalization of a construction, due we understand to Dade but also published by Mann, which characterizes the $\mathfrak{N}$-injectors of a group G. A statement of the characterization also appears on page 705 of Huppert's book.

Let $\{\pi_i\}_{i \in I}$ be a family of pairwise disjoint sets of primes (so certainly I is countable and possibly finite), and put $\pi = \bigcup_{i \in I} \pi_i$ and $\pi_i' = \pi - \pi_i$ for each $i \in I$. Let $\mathfrak{X}$ be the class of those π-groups which have a normal Hall π_i-subgroup for each $i \in I$, so every $\mathfrak{X}$-group is a direct product of π_i-groups for various i. $\mathfrak{X}$ is clearly a Fitting class, in fact $\mathfrak{X} = \bigcap_{i \in I} \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_i'}$.

2.1.1 THEOREM

If $G \in \mathfrak{S}_\pi$ then $I_{\mathfrak{X}}(G) = III_{\mathfrak{X}}(G) = \left\{ \prod_{i \in I} V_{\pi_i} : V_{\pi_i} \in \mathrm{Hall}_{\pi_i}(C_G(G_{\mathfrak{N}_{\pi_i'}})) \right\}$

PROOF

Clearly $G_{\mathfrak{X}} = \prod_{i \in I} G_{\mathcal{S}_{\pi_i}}$. Suppose $G_{\mathfrak{X}} \leq T \leq G$ and $T \in \mathfrak{X}$.

So T_{π_i}, the unique Hall π_i-subgroup of T, centralizes $T_{\pi_i'}$ and $T_{\pi_i'} \geq \prod_{j \neq i} G_{\mathcal{S}_{\pi_j}}$. Now $\prod_{j \neq i} G_{\mathcal{S}_{\pi_j}} \geq G_{\mathfrak{N}_{\pi_i'}}$, and so T_{π_i} centralizes $G_{\mathfrak{N}_{\pi_i'}}$. Let V_{π_i} be a Hall π_i-subgroup of $C_G(G_{\mathfrak{N}_{\pi_i'}})$ containing T_{π_i} . Notice $V_{\pi_i} \geq G_{\mathfrak{N}_{\pi_i}}$.

Suppose this selection has been made for each $i \in I$.

If $i, j \in I$ with $i \neq j$, then $[V_{\pi_i}, V_{\pi_j}] \leq C_G(G_{\mathfrak{N}_{\pi_i'}}) \cap C_G(G_{\mathfrak{N}_{\pi_j'}})$ which lies in $C_G(G_{\mathfrak{N}})$ since $\pi_i' \cup \pi_j' = \pi$.

As is well-known $C_G(G_{\mathfrak{N}}) \leq G_{\mathfrak{N}}$, so $[V_{\pi_i}, V_{\pi_j}] \leq G_{\mathfrak{N}}$.

In particular V_{π_i} normalizes $G_{\mathfrak{N}} V_{\pi_j}$. Now by a previous remark and the construction of V_{π_j} it follows that $G_{\mathfrak{N}} V_{\pi_j} = G_{\mathfrak{N}_{\pi_j'}} V_{\pi_j}$, of which V_{π_j} is the $\mathcal{S}_{\pi_j}$-radical, a characteristic subgroup. Thus V_{π_i} actually normalizes V_{π_j} , so by symmetry $[V_{\pi_i}, V_{\pi_j}] \leq V_{\pi_i} \cap V_{\pi_j} = 1$. We have shown $T \leq \prod_{i \in I} V_{\pi_i} = V$ say.

Clearly $V \in \mathfrak{X}$, so it remains to prove that all subgroups constructed in the manner of V are conjugate in G .

G is a finite group, so there are a finite number n say, of non-trivial subgroups in the set $\{G_{\mathcal{S}_{\pi_i}} : i \in I\}$. We may suppose the labelling makes $G_{\mathcal{S}_{\pi_1}}, \ldots, G_{\mathcal{S}_{\pi_n}} \neq 1$. Obviously $G_{\mathcal{S}_{\pi_i}} = 1$ iff $G_{\mathfrak{N}_{\pi_i}} = 1$ and furthermore $V_{\pi_i} = 1$ iff $G_{\mathfrak{N}_{\pi_i}} = 1$.

For if $G_{\mathfrak{N}_{\pi_i}} = 1$ then $G_{\mathfrak{N}_{\pi_i'}} = G_{\mathfrak{N}}$ which contains its centralizer in G, so $V_{\pi_i} = 1$. The converse is clear from $V_{\pi_i} \geq G_{\mathfrak{N}_{\pi_i}}$.

Let $V = V_{\pi_1} \times V_{\pi_2} \times \ldots \times V_{\pi_n}$ and $V^* = V^*_{\pi_1} \times V^*_{\pi_2} \times \ldots \times V^*_{\pi_n}$ be subgroups of G with V_{π_i}, $V^*_{\pi_i} \in \mathrm{Hall}_{\pi_i}(C_G(G_{\mathcal{N}_{\pi_i'}}))$ for $i = 1, \ldots n$. Clearly there exists $x_1 \in G$ such that $V_{\pi_1}{}^{x_1} = V^*_{\pi_1}$. As an induction hypothesis, suppose there exists $x_r \in G$ such that $V_{\pi_i}{}^{x_r} = V^*_{\pi_i}$ for $i = 1, \ldots, r < n$. Now $V_{\pi_{r+1}}$ and $V^*_{\pi_{r+1}}$ are Hall π_{r+1}-subgroups of $C_G(G_{\mathcal{N}_{\pi'_{r+1}}}) \cap C_G(V^*_{\pi_1} \times \ldots \times V^*_{\pi_r})$, so there exists $y \in C_G(V^*_{\pi_1} \times \ldots V^*_{\pi_r})$ such that $V_{\pi_{r+1}}{}^{x_r y} = V^*_{\pi_{r+1}}$. Then $V_{\pi_i}{}^{x_r y} = (V^*_{\pi_i})^y = V^*_{\pi_i}$ for $i = 1, \ldots, r$, and putting $x_r y = x_{r+1}$, we have $V_{\pi_i}{}^{x_{r+1}} = V^*_{\pi_i}$ for $i = 1, \ldots, r+1$. The induction step is complete and we deduce $V^{x_n} = V^*$. □

When $\pi_i = \{ p_i \}$ the set consisting of the ith prime for each positive integer i, $\mathcal{X} = \mathcal{N}$. So we have the immediate special case ~~corollary~~ :

2.1.2 THEOREM (Fischer-Dade)

$I_{\mathcal{N}}(G) = III_{\mathcal{N}}(G) = \{ \prod_{p \mid |G|} V_p : V_p \in \mathrm{Syl}_p(C_G(G_{\mathcal{N}_{p'}})) \}$ for all G.

Fischer, in [1], was the first to prove (though by a far more complicated argument) that all the $\mathcal{N}$-maximal subgroups of a group G which contain the Fitting subgroup are conjugate, and Gaschütz has designated these duals of the Carter subgroups the 'Fischer subgroups' of G. To avoid confusion with the previously defined Fischer $\mathfrak{F}$-subgroups, we shall adhere to the term $\mathcal{N}$-injector.

Now suppose that, for each $i \in I$, $\mathfrak{F}_i$ is a Fitting class of characteristic π_i, where as before $\{\pi_i\}_{i \in I}$ is a family of pairwise disjoint sets of primes. Again let $\pi_i' = \bigcup_{i \in I} \pi_i - \pi_i$. Clearly $\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i \mathcal{S}_{\pi_i'}$ is a Fitting class comprising those

π-groups which have a normal Hall π_i-subgroup lying in $\mathfrak{F}_i$, for all $i \in I$. As a second corollary of 2.1.1 we have :

2.1.3 THEOREM

If $G \in \mathcal{S}_\pi$ then $I_{\mathfrak{F}}(G) = \left\{ \prod_{i\in I} H_{\pi_i} : H_{\pi_i} \text{ is an } \mathfrak{F}_i\text{-injector of } V_{\pi_i} \text{ where } V_{\pi_i} \text{ is a Hall } \pi_i\text{-subgroup of } C_G(G_{\mathfrak{N}_{\pi_i'}}) \right\}$.

PROOF

Let $H = \prod_{i\in I} H_{\pi_i} \leq \prod_{i\in I} V_{\pi_i} = V$ say, be subgroups of G constructed as in the statement. Let T be an $\mathfrak{F}$-injector of G, $T = \prod_{i\in I} T_{\pi_i}$ say. Certainly $\mathfrak{N}_\pi \subseteq \mathfrak{F}$, so $G_{\mathfrak{N}} \leq T$ and therefore T_{π_i} centralizes $G_{\mathfrak{N}_{\pi_i'}}$. Let W_{π_i} be a Hall π_i-subgroup of $C_G(G_{\mathfrak{N}_{\pi_i'}})$ containing T_{π_i}. Then $T \leq \prod_{i\in I} W_{\pi_i} = W$ say. By 1.1.4 c) T is an $\mathfrak{F}$-injector of W and by 2.1.1 $V = W^x$ for some $x \in G$. It follows that T^x is an $\mathfrak{F}$-injector of V. Now $V_{\pi_i} \lhd V$ and the $\mathfrak{F}$-subgroups of V_{π_i} are precisely its $\mathfrak{F}_i$-subgroups, so $V_{\pi_i} \cap T^x = T_{\pi_i}{}^x$ is actually an $\mathfrak{F}_i$-injector of V_{π_i}. Thus for each $i \in I$ there exists $y_i \in V_{\pi_i}$ such that $H_{\pi_i}{}^{y_i} = T_{\pi_i}{}^x$. Notice that y_j centralizes V_{π_i} whenever $i \neq j$, so putting $y = \prod_{i\in I} y_i$ we have $H_{\pi_i}{}^y = T_{\pi_i}{}^x$, and hence $H^y = T^x$. We have shown that a fixed injector is conjugate to any subgroup constructed in the manner of H which completes the proof. □

2.2 A class due to Gaschütz.

For the duration of this section let $\mathcal{H}$ be a fixed but arbitrarily chosen Fitting class of characteristic π say. If p is a prime, Gaschütz defines $e_p(\mathcal{H}) = \{ G : \text{all } p\text{-chief factors of } G \text{ between } 1 \text{ and } G_{\mathcal{H}} \text{ are central} \}$. Given two chief series of G through $G_{\mathcal{H}}$, the correspondence of Zassenhaus's lemma makes factors between 1 and $G_{\mathcal{H}}$ correspond, so $G \in e_p(\mathcal{H})$ if all the p-chief factors between 1 and $G_{\mathcal{H}}$ in a particular chief series of G are central. If $p \notin \pi$ then the centrality condition is vacuous and $e_p(\mathcal{H}) = \mathcal{S}$, so we shall always assume that $p \in \pi$.

2.2.1 Proposition

$e_p(\mathcal{H})$ is a Fitting class and actually a Fischer class.

Proof

For simplicity write $\mathfrak{F} = e_p(\mathcal{H})$.

Suppose $N \triangleleft G \in \mathfrak{F}$. Consider a chief series of G through $N_{\mathcal{H}} = N \cap G_{\mathcal{H}}$ and $G_{\mathcal{H}}$. Then by hypothesis all p-chief factors of G below $N_{\mathcal{H}}$ are central and it follows immediately that the p-chief factors of N below $N_{\mathcal{H}}$ in a refinement of this series are central in N, so $N \in \mathfrak{F}$.

Now suppose $\mathfrak{F} \ni N_1, N_2 \triangleleft G = N_1N_2$. An easy induction argument shows that, to prove $G \in \mathfrak{F}$, we may assume that N_1 and N_2 are maximal normal subgroups of G. Let $N = N_1 \cap N_2$ so G/N is Abelian. Consider a chief series of G through $N_{\mathcal{H}}$ and $G_{\mathcal{H}}$. The chief factors between $N_{\mathcal{H}}$ and $G_{\mathcal{H}}$ are G-isomorphic to factors between N and G, so are central. A p-chief factor of G below $N_{\mathcal{H}}$ is completely reducible as N_i-module by Clifford's theorem, and hence by the assumption $N_i \in \mathfrak{F}$ is central in N_i. It is therefore central in G and we have $G \in \mathfrak{F}$ as required.

To prove $\mathfrak{F}$ is a Fischer class we suppose $N \triangleleft G \in \mathfrak{F}$ and $N \leq H \leq G$ with H/N a q-group for some prime q (possibly $p = q$) and we must show $H \in \mathfrak{F}$. Whatever q is, the chief factors of H between $N_{\mathcal{H}} = N \cap H_{\mathcal{H}}$ and $H_{\mathcal{H}}$ are H-isomorphic to factors between N and H, so are central as $H/N \in \mathcal{S}_q$. Since $G \in \mathfrak{F}$, the p-chief factors in a chief series of G below $N_{\mathcal{H}}$ are central, so the p-chief factors of H in a refinement of this series below $H_{\mathcal{H}}$ are central in H. $\square$

Now suppose σ is a subset of π and define $e_\sigma(\mathcal{H}) = \bigcap_{p\in\sigma} e_p(\mathcal{H})$. Again set $\mathcal{F} = e_\sigma(\mathcal{H})$ for simplicity. It follows immediately from the definition that the intersection of a collection of Fischer classes is also a Fischer class, so by Fischer's theorem 1.2.1, the previous proposition yields $I_{\mathcal{F}}(G) = II_{\mathcal{F}}(G)$ for all G. However still more is true as we prove in the following result.

2.2.2 THEOREM

$I_{\mathcal{F}}(G) = III_{\mathcal{F}}(G)$ for all G.

PROOF

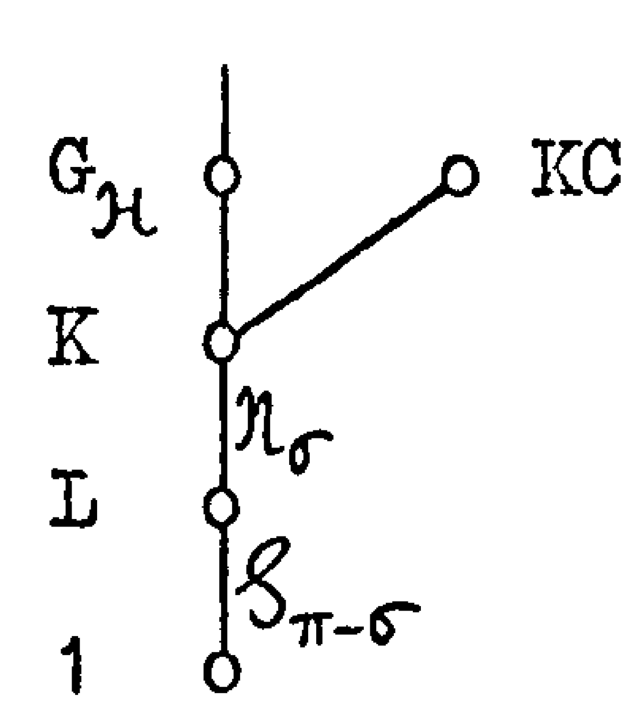
As in 2.1.1 we identify the members of $III_{\mathcal{F}}(G)$ and exhibit their conjugacy, so we must first look for $G_{\mathcal{F}}$.

(1) $(G_{\mathcal{H}})_{\mathcal{F}} = G_{\mathcal{H}} \cap G_{\mathcal{F}} = (G_{\mathcal{H}})_{\mathcal{S}_{\pi-\sigma}\mathcal{N}_\sigma} = K$ say.

Obviously $(G_{\mathcal{F}})_{\mathcal{H}} = (G_{\mathcal{H}})_{\mathcal{F}} = G_{\mathcal{H}} \cap G_{\mathcal{F}}$ holds for arbitrary $\mathcal{F}$ and $\mathcal{H}$. In our case a normal $\mathcal{F}$-subgroup of $G_{\mathcal{H}}$ has all its σ-chief factors central, so is p-nilpotent for all $p \in \sigma$, and hence lies in K. On the other hand $K \in \mathcal{F}$ is clear.

Put $L = (G_{\mathcal{H}})_{\mathcal{S}_{\pi-\sigma}}$ and $C = C_G(K/L) \lhd G$

(2) $K = G_{\mathcal{H}} \cap KC$ and $G \rhd KC \in \mathcal{F}$.



For, $K/L = (G_{\mathcal{H}}/L)_{\mathcal{N}}$ by (1) and 1.2.5 b), so K/L contains its centralizer in $G_{\mathcal{H}}/L$. Therefore $G_{\mathcal{H}} \cap C \leq K$ and so $K = K(G_{\mathcal{H}} \cap C) = G_{\mathcal{H}} \cap KC$ by the Dedekind law. Thus $(KC)_{\mathcal{H}} = K$ and $KC \in \mathcal{F}$ follows immediately from the definition of $\mathcal{F}$.

(3) If $KC \leq T \in \mathcal{F}$ then T_p centralizes $O_{p'}(K/L)$ and T/KC is a σ-group.

Here p is any prime and, as always, T_p denotes a Sylow p-subgroup of T. $T_{\mathcal{H}}$ certainly contains K, so all chief factors of T between L and K are central. In particular T_p centralizes all the chief factors of T between L and $O_{p'}(K \bmod L)$ and we deduce as usual that T_p centralizes $O_{p'}(K/L)$. So if $p \notin \sigma$, then T_p centralizes K/L and therefore lies in C. Hence T/KC is a σ-group.

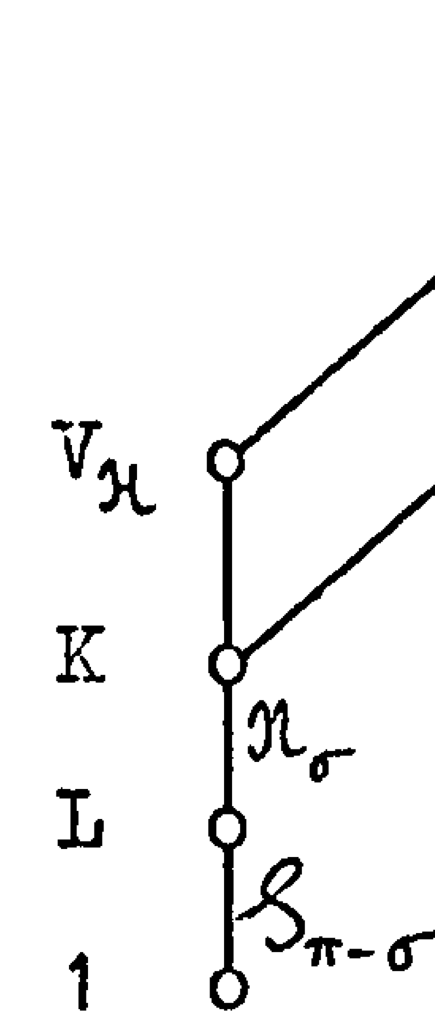
(4) $III_{\mathfrak{F}}(G) = \left\{ \prod_{p\in\sigma} KCV_p = V : V_p \in Syl_p(C_G(O_{p'}(K/L))) \right\}$.

Suppose $KC \leq T \in \mathfrak{F}$. By (3), T/KC is a σ-group and for each $p \in \sigma$ $T_p \leq V_p$ for some $V_p \in Syl_p(C_G(O_{p'}(K/L)))$. If $p, q \in \sigma$ and $p \neq q$ then $[V_p, V_q] \leq C_G(O_{p'}(K/L)) \cap C_G(O_{q'}(K/L)) = C$. Letting $\bar{G} = G/KC$, this means $\bar{V}_p$ and $\bar{V}_q$ centralize each other

product not direct

so $V = \prod_{p\in\sigma} KCV_p$ is indeed a group ~~which~~ and it contains T, and furthermore $V/KC \in \mathfrak{N}$. Since $KC \leq G_{\mathfrak{F}}$, to prove (4) it will suffice to show that $V \in \mathfrak{F}$.

$V_{\mathcal{H}} \cap KC = (KC)_{\mathcal{H}} = K$ by (2). So chief factors of V between K and $V_{\mathcal{H}}$ are V-isomorphic to factors between KC and V, which are central since $V/KC \in \mathfrak{N}$. If $p \in \sigma$, $O_p(K/L)$ is centralized by C and by each V_q with $q \neq p$, so V induces a p-group of automorphisms on each p-chief factor of V between L and K, hence such a chief factor must actually be central in V. L is a σ'-group so we have $V \in \mathfrak{F}$.



(5) $I_{\mathfrak{F}}(G) = III_{\mathfrak{F}}(G)$

In view of (4) it is sufficient to show that all the subgroups V are conjugate. Now $\bar{V}_p$ is a Sylow p-subgroup of the normal subgroup $\overline{C_G(O_{p'}(K/L))}$ of $\bar{G}$ and we have seen $[\bar{V}_p, \bar{V}_q] = 1$.

It follows as in the proof of 2.1.1 that all the subgroups $\overline{V}$ are conjugate in $\overline{G}$ and therefore since $KC \leq V$, all the V are conjugate in G. (We shall see in Chapter 3 that conjugacy in such situations follows from a more general result.)

From the proof ~~so far~~ it is apparent that :

(6) $G_{\mathfrak{F}} = \prod_{p \in \sigma} O_p(KC_G(O_{p'}(K/L)) \bmod KC)$ $\square$

2.2.3 Example

As an example of a class of this type , consider $e_p(\mathfrak{S}_p)$. It is not difficult to show that this class is precisely $\{ G : O_p(G)$ is centralized by all the p'-elements of $G \}$. Furthermore $e_3(\mathfrak{S}_3)$ contains Σ_4 but not its subgroup Σ_3 , so $e_3(\mathfrak{S}_3)$ is a Fischer class (by 2.2.1) which is not S-closed.

Notice that for the classes in both 2.1.1 and 2.2.2 , it happens that an injector of an arbitrary group G has Sylow subgroups which are also Sylow subgroups of certain normal subgroups of G. This property is in fact a consequence of the classes being Fischer classes , a result due to Fischer himself , and we give a proof in Chapter 3. However, example 1.2.6 shows that $I_{\mathfrak{F}}(G) = III_{\mathfrak{F}}(G)$ does not follow when $\mathfrak{F}$ is a Fischer class or even an S-closed class.

2.3 Groups with central socle

We now turn our attention to a non-Fischer class. If p is a prime , let $\mathfrak{F}_p$ denote $\{ G :$ each minimal normal p-subgroup of G is central $\}$, the class of groups with central p-socle. Clearly $\mathfrak{N} \subseteq \mathfrak{F}_p$. We suspect that this class has its origins in Kiel too.

2.3.1 Proposition

$\mathfrak{F}_p$ is a Fitting class.

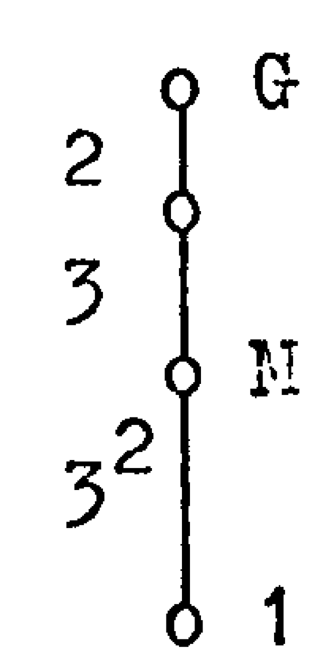
Proof

Let M be a minimal normal p-sugroup of N , where $N \lhd G \in \mathfrak{F}_p$. If $g \in G$ then M^g is also a minimal normal subgroup of N , so $< M^G > = \prod_{g \in G} M^g = M^{g_1} \times M^{g_2} \times \ldots \times M^{g_r}$ for suitable $g_i \in G$, and is a p-group. By hypothesis , a minimal normal subgroup of G contained in $< M^G >$ is central in G , so there exists $x = m_1 m_2 \ldots m_r \in Z(G)$ such that $m_i \in M^{g_i}$ and $m_j \neq 1$ for some j. Now if $n \in N$ then $x^n = m_1{}^n m_2{}^n \ldots m_r{}^n$ and $x = x^n$ since $x \in Z(G)$, so $m_i{}^n = m_i$ for all $n \in N$ and for all i because the product of the M^{g_i} is direct. Therefore $1 \neq m_j \in M^{g_j} \cap Z(N)$, but $Z(N) \lhd G$, so $M \cap Z(N) \neq 1$ and by minimality $M \leqslant Z(N)$. So we have $N \in \mathfrak{F}_p$.

Now let M be a minimal normal subgroup of G , where $G = N_1 N_2 \rhd N_1, N_2 \in \mathfrak{F}_p$. If $M \cap N_1 \neq 1$ then $M \cap N_1$ contains a minimal normal subgroup of N_1 which by hypothesis is central in N_1. Thus $M \cap Z(N_1) \neq 1$, but certainly $Z(N_1) \lhd G$, so by minimality $M \leqslant Z(N_1)$. If $M \cap N_1 = 1$ then $[M, N_1] = 1$. In either case N_1 (and similarly N_2) centralizes M , so $M \leqslant Z(G)$. Therefore $G \in \mathfrak{F}_p$, completing the argu~~m~~ment. $\square$

2.3.2 Example

$\mathfrak{F}_3$ is not a Fischer class. For let $M \cong C_3 \times C_3$ be the natural module (under right action) for the subgroup $A = < \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} , \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} >$ $(\cong \Sigma_3)$ of $GL(2,3)$, and put $G = MA$.



M has a unique minimal A-admissable subgroup , which is central , so $G \in \mathfrak{F}_3$. However , MA_2 , an extension of M by a group of order 2, clearly does not lie in $\mathfrak{F}_3$.

2.3.3 Proposition

For each G , $G_{\mathfrak{F}_p} = C_G(p\text{-soc } G)$.

Proof

Put $C = C_G(\text{p-soc } G)$ and let M be a minimal normal p-subgroup of C. $C \lhd G$, so $<M^G> = M^{g_1} \times M^{g_2} \times \ldots \times M^{g_r}$ for suitable $g_i \in G$, and M^{g_i} is minimal normal in C. Now a minimal normal subgroup of G contained in $<M^G>$ is central in C, and it follows as in the proof of 2.3.1 that M is central in C. Thus $C \in \mathfrak{Z}_p$ and so $C \leqslant G_{\mathfrak{Z}_p}$.

On the other hand, each minimal normal p-subgroup N of G lies in $G_{\mathfrak{Z}_p}$, and therefore a minimal normal subgroup of $G_{\mathfrak{Z}_p}$ contained in N lies in $Z(G_{\mathfrak{Z}_p})$. Thus $N \cap Z(G_{\mathfrak{Z}_p}) \neq 1$ and by minimality it follows that $G_{\mathfrak{Z}_p}$ centralizes N. Hence $G_{\mathfrak{Z}_p} \leqslant C$. $\square$

If π is a set of primes let $\mathfrak{Z}_\pi$ denote $\bigcap_{p \in \pi} \mathfrak{Z}_p$, the class of groups with central π-socle. (We hope that the context will prevent confusion arising from our basic convention that $\mathfrak{Z}_\pi = \mathfrak{Z} \cap \mathfrak{S}_\pi$.) Clearly:

$$G_{\mathfrak{Z}_\pi} = \bigcap_{p \in \pi} G_{\mathfrak{Z}_p} = \bigcap_{p \in \pi} C_G(\text{p-soc } G) \text{ (by 2.3.3)} = C_G(\pi\text{-soc } G).$$

In particular if Ω denotes the set of all primes, $G_{\mathfrak{Z}_\Omega} = C_G(\text{soc } G)$

We gather from Professor Gaschütz that the following result has also been obtained by one of his students.

2.3.4 THEOREM

For each G $I_{\mathfrak{Z}_p}(G) = III_{\mathfrak{Z}_p}(G) = \left\{ C_G(C_P(G_p)) : P = \text{p-soc } G_{\mathfrak{Z}_p} \text{ and } G_p \in \mathrm{Syl}_p(G) \right\}$.

PROOF

As in 2.3.3 let $C = C_G(\text{p-soc } G) = G_{\mathfrak{Z}_p}$. So $P = \text{p-soc } C$.

(1) $C = C_G(P)$

$C \in \mathfrak{Z}_p$, so certainly C centralizes P, that is $C \leqslant C_G(P)$. Each minimal normal p-subgroup of G is central in C, so must lie in p-soc C. Thus p-soc G $\leqslant$ P and hence C $\geq C_G(P)$, giving equality.

(2) If C $\leqslant$ H $\leqslant$ G then p-soc H $\leqslant$ P. If also H $\in \mathfrak{F}_p$ then p-soc H $= C_p(H)$.

For let M be a minimal normal p-subgroup of H , then either M $\leqslant$ P or M $\cap$ P = 1. In the second case M centralizes P and therefore lies in C by (1). Then Clifford's theorem tells us that M is completely reducible under the action of C , so M $\leqslant$ P after all. If also H $\in \mathfrak{F}_p$, then p-soc H $\leqslant C_p(H)$ follows from what we have just proved. The reverse inclusion holds because P is elementary Abelian.

(3) If C $\leqslant$ H $\in \mathfrak{F}_p$ then p-soc H = $C_p(H) = C_p(H_p)$ where $H_p \in Syl_p(H)$.

In view of (2) it suffices to prove $C_p(H) = C_p(H_p)$, which we do by induction on $|H/C|$, the statement being obvious when H = C. Let N be a maximal normal subgroup of H containing C. Let $N_p = H_p \cap N$. Then using our induction hypothesis we have :

$$C_p(N) = C_p(N_p) \geq C_p(H_p) \geq C_p(H) \quad . \qquad *$$

a) If $|H : N| \neq p$, then $C_p(N)$ is normalized by H , and by Maschke's theorem is completely reducible under its action. So since H $\in \mathfrak{F}_p$ we have $C_p(N) \leqslant C_p(H)$, and hence equality holds throughout * .

b) If $|H : N| = p$, then (from *) $C_p(H_p)$ is centralized by both N and H_p , and therefore also by their product H. So again $C_p(H_p) \leqslant C_p(H)$ and equality follows immediately.

(4) If C $\leqslant$ H $\in \mathfrak{F}_p$ then C_G(p-soc H) $\in \mathfrak{F}_p$.

Certainly H $\leqslant C_G$(p-soc H) = T say. Now let M be a minimal normal p-subgroup of T. Then M $\leqslant$ P by the argument (2) , so M contains a minimal normal subgroup of H which , by definition , is central in T.

Then by minimality $M \leq Z(T)$ and so $T \in \mathfrak{F}_p$.

(5) $III_{\mathfrak{F}_p}(G) = \{ C_G(C_P(G_p)) : G_p \in Syl_p(G) \}$

Suppose $G_{\mathfrak{F}_p} = C \leq H \in \mathfrak{F}_p$ and $Syl_p(H) \ni H_p \leq G_p \in Syl_p(G)$.

Then using (3) $p\text{-soc } H = C_P(H_p) \geq C_P(G_p)$ $\quad$ **

Now by (1) CG_p induces a p-group of automorphisms of P so from (2) it is clear that $CG_p \in \mathfrak{F}_p$, and so $p\text{-soc}(CG_p) = C_P(G_p)$.

Then by (4) we have $C_G(C_P(G_p)) \in \mathfrak{F}_p$. Thus from ** we deduce $H \leq C_G(p\text{-soc } H) \leq C_G(C_P(G_p)) \in \mathfrak{F}_p$, which establishes (5).

Conjugacy of the Sylow subgroups of G now implies $III_{\mathfrak{F}_p}(G)$ is a conjugacy class, giving the statement of the theorem. $\square$

We have been unable to prove an analogous result for the class $\mathfrak{F}_\pi$, but we suspect that $I_{\mathfrak{F}_\pi}(G) = III_{\mathfrak{F}_\pi}(G)$ for all G can be established using similar techniques. In particular (1) - (4) of the proof of 2.3.4 hold with p replaced by π and P by $S = \pi\text{-soc } G_{\mathfrak{F}_\pi}$, so the problem is to determine a suitable replacement for G_p in (5). At least we have :

2.3.5 THEOREM

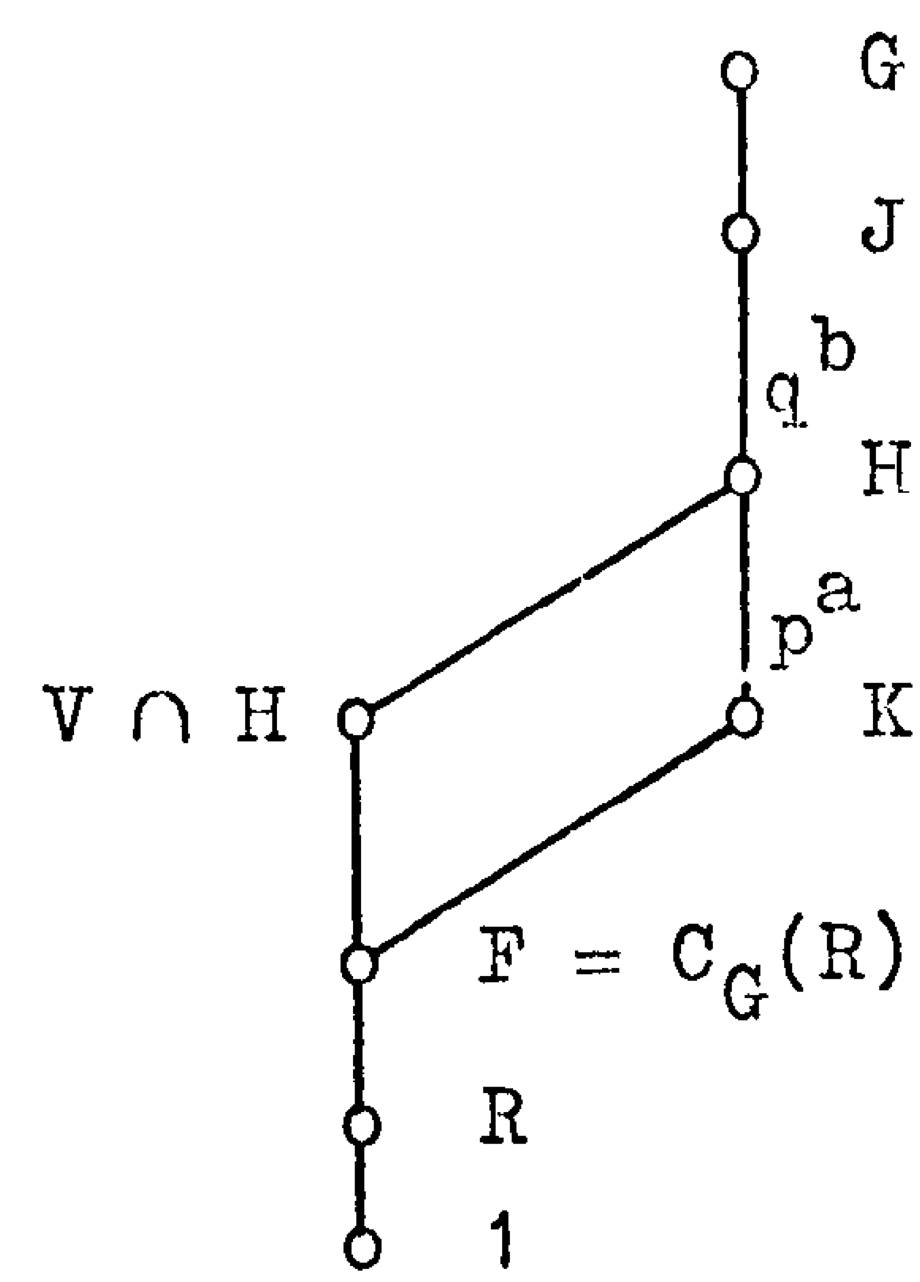
For each G $I_{\mathfrak{F}_\pi}(G) = II_{\mathfrak{F}_\pi}(G)$.

PROOF

Example 2.3.2 shows $\mathfrak{F}_\pi$ need not be a Fischer class, so we are denied an immediate proof by Fischer's theorem 1.2.1. However, we can argue from the structure of a group G of minimal order which contains a non-injective Fischer $\mathfrak{F}_\pi$-subgroup E, a structure which Fischer uses in the proof of his theorem. Proofs of the details we require ((1) - (3) below) appear in section 2.3 of [4].

Consider such a group G and subgroup E. Let $F = G_{\mathfrak{F}_\pi}$, then $F = C_G(R)$ where $R = \pi\text{-soc } G$, by the remark following 2.3.3 .

F is not an $\mathfrak{F}_\pi$-injector of G , for otherwise F would be $\mathfrak{F}_\pi$-maximal in G and so F = E would be an $\mathfrak{F}_\pi$-injector of G contrary to assumption. Let V be an $\mathfrak{F}_\pi$-injector of G and choose a normal subgroup H of G containing F and minimal subject to $V \cap H > F$. Let K be maximal among the normal subgroups of G lying between F and H. Then by minimality of H , F is $\mathfrak{F}_\pi$-maximal in K , so $V \cap K = F$. By 1.1.7 a) V covers or avoids the chief factors of G , so it covers H/K since $V \cap H > F$.



Furthermore :

(1) With H chosen , K is unique.

(2) $HE = G$ and $H \cap E = F$.

(3) H/K is the unique minimal normal subgroup of G/K.

Suppose H/K is a p-chief factor of G . Let J/H be a chief factor of G. From (2) and (3) it follows that G/K is a primitive group , and so J/H is a q-group for some prime $q \neq p$.

Let $V_0 = V \cap H$ and $E_0 = E \cap J$, so E_0 covers J/H by (2).

F centralizes R , so by Maschke's theorem $O_{p'}(R)$ is completely reducible under the action of V_0. Since $V_0 \in \mathfrak{F}_\pi$ it follows that V_0 centralizes $O_{p'}(R)$. (4)

Similarly E_0 centralizes $O_{q'}(R)$. Therefore $N = C_G(O_{q'}(R))$ is a normal subgroup of G containing F , which covers J/H. By (3) it follows that $H \leq NK$, and the Dedekind law gives

$H = K(N \cap H)$. (5)

If $N \cap H \neq H$, then $N \cap H \leq K$ by (1) , which contradicts (5). Thus $H \leq N$ which implies that V_0 centralizes $O_{q'}(R)$. Then (4) and the fact that $p \neq q$ show that V_0 centralizes R , which is a final contradiction, showing that no such G exists and the theorem follows. □

2.4. Normal Fitting classes

A Fitting class $\mathfrak{X}$ is called <u>normal</u> if each group G has a normal (and therefore by conjugacy unique) $\mathfrak{X}$-injector, equivalently , if $G_{\mathfrak{X}}$ is $\mathfrak{X}$-maximal in G for all G. Such classes are studied by Blessenohl and Gaschütz in [8] , where some examples are presented. As the authors remark , the bizarre nature of these examples seems to indicate that an easy classification of normal Fitting classes is improbable.

However , we may add to our arsenal :

<u>2.4.1 THEOREM</u> (Blessenohl and Gaschütz [8] Satz 3.2)
$\{$ G : each element of G induces (by conjugation) an even permutation of set of elements of $O_{2'}(G)$ $\}$ is a normal Fitting class.

<u>2.4.2 THEOREM</u> (Blessenohl and Gaschütz [8] from Satz 3.3)
Let p be a prime and G a group. Let $M_1, \ldots, M_r$ be the p-chief factors in a chief series of G , and $D_i : G \to GL(M_i)$ the representation afforded by M_i. By the Jordan- Holder theorem the map $d_G : g \to \prod_{i=1}^{r} \det(D_i(g)) \in GF(p)$, is well-defined.
Then $\{ G : d_G(g) = 1$ for all $g \in G \}$ is a normal Fitting class.

We shall require the following main result of [8] in Chapter 5.

<u>2.4.3 THEOREM</u> (Blessenohl and Gaschütz [8] Satz 5.3)
$\mathfrak{X}$ is a non-trivial normal Fitting class iff $G' \leq G_{\mathfrak{X}}$ for all G.

<u>2.4.4 Remarks</u>

a) the substance of 2.4.3 is the implication $\Rightarrow$, the converse being clear.

b) An easy arguement of Cossey shows that each non-trivial normal Fitting class contains $\mathfrak{N}$, reinforcing the indication of 2.4.3 that normal Fitting classes are big.

c) 2.4.3 shows that the intersection of an arbitrary collection of normal Fitting classes is again normal , for it tells us that the intersection contains $\{ G' : G \in \mathcal{S} \} = \mathfrak{D}$ say. In particular there is a unique smallest normal Fitting class. Nobody has as yet solved the tantalizing problem of identifying this class , though it is not difficult to verify that it is the Fitting class generated by $\mathfrak{D}$. Hawkes has pointed out that it does not actually coincide with $\mathfrak{D}$, since $\mathfrak{D}$ fails to contain D_8. For if $D_8 \cong G'$ for some G , then the fact that $Aut(D_8)$ is a 2-group implies that we may assume G is a 2-group. But this contradicts an old result of Burnside which asserts that a non-Abelian group of order p^3 is not the derived group of a p-group.

d) 1.2.5 b) and 2.4.3 show immediately that , for any Fitting class $\mathfrak{F}$ and normal Fitting class $\mathfrak{X}$, both $\mathfrak{F}\mathfrak{X}$ and $\mathfrak{X}\mathfrak{F}$ are normal. Another consequence worth remembering is that if the Fitting class $\mathfrak{F}$ contains a normal Fitting class $\mathfrak{X}$, then $\mathfrak{F}$ too is normal.

2.5 Fitting formations

Many of the easy examples of Fitting classes are formations too. Naturally such classes are called Fitting formations. $\mathfrak{N}, \mathcal{S}_\pi, \mathcal{S}_{p'}, \mathcal{S}_p\mathcal{S}_{p'} = \{$ groups of p-length 1 $\}$ are examples. Notable among those formations which are not Fitting classes are the class $\mathfrak{A}$ and the class of supersoluble groups. This is exhibited by the groups Q_8 and a primitive split extension of $C_5 \times C_5$ by Q_8 , respectively. It is well-known (proved for example in [2]) that a saturated formation locally defined by Fitting formations , is again a Fitting formation.

That Fitting formations need not be saturated was demonstrated by Hawkes and his example appears in [12]. We have had little success in analysing , in the manner of previous results, the behaviour of the class he defines , though we feel a successful investigation would be illuminating.

This is a convenient place to set out a well-known result which we shall need later.

2.5.1 Proposition

If $\mathfrak{X}$ is a Fitting formation and N_1, $N_2 \lhd G = N_1N_2$ then $G^{\mathfrak{X}} = N_1^{\mathfrak{X}}N_2^{\mathfrak{X}}$. (Here $G^{\mathfrak{X}}$ denotes the $\mathfrak{X}$-residual of G , that is the unique minimal member of the set of $N \lhd G$ with $G/N \in \mathfrak{X}$)

Proof

$N_1/(N_1 \cap G^{\mathfrak{X}}) \cong N_1G^{\mathfrak{X}}/G^{\mathfrak{X}} \in S_n\{G/G^{\mathfrak{X}}\}$ so $N_1^{\mathfrak{X}} \leq G^{\mathfrak{X}} \cap N_1$ since $\mathfrak{X} = S_n\mathfrak{X}$.
$G/N_1^{\mathfrak{X}}N_2^{\mathfrak{X}} = N_1N_2/N_1^{\mathfrak{X}}N_2^{\mathfrak{X}} = N_1N_2^{\mathfrak{X}}/N_1^{\mathfrak{X}}N_2^{\mathfrak{X}}.N_1^{\mathfrak{X}}N_2/N_1^{\mathfrak{X}}N_2^{\mathfrak{X}}$ and
$N_1N_2^{\mathfrak{X}}/N_1^{\mathfrak{X}}N_2^{\mathfrak{X}} \cong N_1/(N_1 \cap N_1^{\mathfrak{X}}N_2^{\mathfrak{X}}) = N_1/N_1^{\mathfrak{X}}(N_1 \cap N_2^{\mathfrak{X}}) \in Q\{N_1/N_1^{\mathfrak{X}}\}$,
hence $G/N_1^{\mathfrak{X}}N_2^{\mathfrak{X}}$ is a normal product of $\mathfrak{X}$-groups since $\mathfrak{X} = Q\mathfrak{X}$.
Therefore $G^{\mathfrak{X}} \leq N_1^{\mathfrak{X}}N_2^{\mathfrak{X}}$ since $\mathfrak{X} = N_0\mathfrak{X}$. The two inequalities together yield the result. $\square$

2.6 The classes $\mathfrak{F}^{\pi}$, $\mathfrak{F}_1\mathfrak{F}_2$ and $\mathfrak{F}_1 \cap \mathfrak{F}_2$

If $\mathfrak{F}$ is a Fitting class and π is a set of primes we define $\mathfrak{F}^{\pi}$ to be the class of groups G in which an $\mathfrak{F}$-injector of G contains a Hall π-subgroup of G. This is clearly equivalent to the requirement that an $\mathfrak{F}$-injector of G covers all the π-chief factors of G. We make the obvious convention that $\mathfrak{F}^{\emptyset} = \mathfrak{S}$ for all $\mathfrak{F}$. Our chosen notation is perhaps poor , but has the useful advantage of brevity.

2.6.1 Proposition

For each $\mathfrak{F}$ and π : a) $\mathfrak{F}^{\pi}$ is a Fitting class

b) $\mathfrak{F}, \mathcal{S}_{\pi'} \subseteq \mathfrak{F}\mathcal{S}_{\pi'} \subseteq \mathfrak{F}^{\pi} = \mathfrak{F}^{\pi}\mathcal{S}_{\pi'} = (\mathfrak{F}^{\pi})^{\pi}$

c) The following are equivalent :

i) $\mathfrak{F} = \mathfrak{F}\mathcal{S}_{\pi'}$

ii) For each G , an $\mathfrak{F}$-injector of G has π-index in G.

iii) $\mathfrak{F}^{\pi'} = \mathcal{S}$

iv) $\mathfrak{F} = \mathfrak{F}^{\pi}$

d) If $\mathfrak{F}$ is a Fischer class , so also is $\mathfrak{F}^{\pi}$.

Proof

a) Suppose $N \lhd G \in \mathfrak{F}^{\pi}$. So let V be an $\mathfrak{F}$-injector of G containing the Hall π-subgroup G_{π} of G. Then $V \cap N$ is an $\mathfrak{F}$-injector of N containing the Hall π-subgroup $G_{\pi} \cap N$ of N. Therefore $N \in \mathfrak{F}^{\pi}$.

Now suppose $\mathfrak{F}^{\pi} \ni N_1, N_2 \lhd G = N_1N_2$. Let V be an $\mathfrak{F}$-injector of G with a Hall π-subgroup V_{π}. Then for i = 1, 2 , $V_{\pi} \cap (V \cap N_i) = V_{\pi} \cap N_i$ is a Hall π-subgroup of the $\mathfrak{F}$-injector $V \cap N_i$ of N_i. Now $T \in \mathfrak{F}^{\pi}$ clearly implies that each Hall π-subgroup of any $\mathfrak{F}$-injector of T is a Hall π-subgroup of T. Thus V_{π} contains Hall π-subgroups of N_1 and N_2 so it follows from 1.2.7 that V_{π} is a Hall π-subgroup of G and therefore $G \in \mathfrak{F}^{\pi}$.

b) The inclusions $\mathfrak{F}, \mathcal{S}_{\pi'} \subseteq \mathfrak{F}\mathcal{S}_{\pi'} \subseteq \mathfrak{F}^{\pi} \subseteq \mathfrak{F}^{\pi}\mathcal{S}_{\pi'}$ are clear. If $G \in \mathfrak{F}^{\pi}\mathcal{S}_{\pi'}$ then an $\mathfrak{F}$-injector of $G_{\mathfrak{F}^{\pi}}$ contains a Hall π-subgroup of $G_{\mathfrak{F}^{\pi}}$ which is of course a Hall π-subgroup of G, and it follows that $G \in \mathfrak{F}^{\pi}$.

$\mathfrak{F}^{\pi} \subseteq (\mathfrak{F}^{\pi})^{\pi}$ is clear , so it remains to prove the reverse inclusion Suppose $G \in (\mathfrak{F}^{\pi})^{\pi}$ and inductively that $(\mathfrak{F}^{\pi})^{\pi}$-groups of order less than $|G|$ lie in $\mathfrak{F}^{\pi}$. So a maximal normal subgroup N of G lies in $\mathfrak{F}^{\pi}$. If $|G : N| = p \notin \pi$ then $G \in \mathfrak{F}^{\pi}\mathcal{S}_{\pi'} = \mathfrak{F}^{\pi}$ as above and we are done. If $p \in \pi$ then since $G \in (\mathfrak{F}^{\pi})^{\pi}$, G/N is covered by an $\mathfrak{F}^{\pi}$-injector of G , which already contains N so $G \in \mathfrak{F}^{\pi}$ in this case too. Certainly $1 \in \mathfrak{F}^{\pi}$ and the induction is complete so we have $(\mathfrak{F}^{\pi})^{\pi} \subseteq \mathfrak{F}^{\pi}$ as required.

c) i) $\Rightarrow$ ii) Suppose $\mathfrak{F} = \mathfrak{F}\mathfrak{S}_{\pi'}$, G is an arbitrary group, and inductively that an $\mathfrak{F}$-injector of a group G of order less than $|G|$ has π-index. Let V be an $\mathfrak{F}$-injector of G and N a maximal normal subgroup of G of index p say. By the induction hypothesis $V \cap N$ has π-index in N. Thus if $p \subset \pi$ it follows that V has π-index in G as required. Assume $p \in \pi'$ then by 1.1.7 b) (a Frattini argument) it follows that $(V \cap N)$ is normalized by a Hall π'-subgroup $G_{\pi'}$ say, of G, and $V \cap N < (V \cap N)G_{\pi'} = H$ say. By supposition $\mathfrak{F} = \mathfrak{F}\mathfrak{S}_{\pi'}$ so $H \in \mathfrak{F}$. Now H must be $\mathfrak{F}$-maximal in G, for otherwise $V \cap N < H \cap N \in \mathfrak{F}$, contrary to the fact that $V \cap N$ is an $\mathfrak{F}$-injector of N. So by 1.1.3 a), H is an $\mathfrak{F}$-injector of G. H certainly has π-index in G and so the induction step holds. The statement is trivially true when $G \cong 1$ so the proof is complete.

ii) $\Leftrightarrow$ iii) is clear

ii) $\Rightarrow$ iv). Certainly $\mathfrak{F} \subseteq \mathfrak{F}^{\pi}$. If $G \in \mathfrak{F}^{\pi}$ then an $\mathfrak{F}$-injector of G has π'-index, which with ii) implies $G \in \mathfrak{F}$.

iv) $\Rightarrow$ ii) follows from b) immediately.

d) Suppose $N \lhd G \in \mathfrak{F}^{\pi}$, $N \leqslant H \leqslant G$ where H/N is a p-group. We must show $H \in \mathfrak{F}^{\pi}$. First assume $p \notin \pi$ and let T be an $\mathfrak{F}$-injector of H. $N \in \mathfrak{F}^{\pi}$, so $T \cap N$ contains a Hall π-subgroup of N, which by our assumption is a Hall π-subgroup of H. Therefore $H \in \mathfrak{F}^{\pi}$. Now suppose $p \in \pi$, and let H_{π} be a Hall π-subgroup of H. Since $G \in \mathfrak{F}^{\pi}$ we may choose an $\mathfrak{F}$-injector V of G which contains H. Then $V \cap N \lhd V \in \mathfrak{F}$, $V \cap N \leqslant V \cap H$ and $V \cap H/V \cap N$ is a p-group so $V \cap H \in \mathfrak{F}$ since $\mathfrak{F}$ is a Fischer class. Now $V \cap H$ contains H_{π} which covers $H/N \in \mathfrak{S}_p$, and $V \cap N$ is an $\mathfrak{F}$-injector of N so it follows that $V \cap H$ is an $\mathfrak{F}$-maximal subgroup of H. Then by 1.1.4 a) $V \cap H$ is an $\mathfrak{F}$-injector of H, and hence $H \in \mathfrak{F}^{\pi}$.

$\square$

2.6.2 Remarks

a) The inclusion $\mathfrak{F}\mathcal{S}_{\pi'} \subseteq \mathfrak{F}^{\pi}$ can be strict, for example $\Sigma_4 \in \mathfrak{N}^{\{2\}} - \mathfrak{N}\mathcal{S}_{2'}$.

b) The operation $\mathfrak{F} \to \mathfrak{F}^{\pi}$ is no respecter of closure operations. For instance Σ_3 is both a quotient and a subgroup of $\Sigma_4 \in \mathfrak{N}^{\{2\}}$, but $\Sigma_3 \notin \mathfrak{N}^{\{2\}}$ so this class does not inherit either the Q- or the S-closed property of $\mathfrak{N}$.

c) The equivalence of i) and ii) in 2.6.1 c) (first pointed out by Gaschütz), together with 2.6.1 b), shows that an $\mathfrak{F}^{\pi}$-injector H of an arbitrary group G has π-index in G, that is H contains a Hall π'-subgroup $G_{\pi'}$ say, of G. By definition of $\mathfrak{F}^{\pi}$, an $\mathfrak{F}$-injector T of H contains a Hall π-subgroup of H, so it follows by order that $H = TG_{\pi'}$. Thus the problem of determining the $\mathfrak{F}^{\pi}$-injectors of G reduces to the identification of T. (Clearly all such subgroups T are conjugate.) A natural guess is that T is actually an $\mathfrak{F}$-injector of G and this is indeed the case when $\mathfrak{F}$ is a Fischer class, as we shall see in 4.1. Unfortunately this is not always true and in the same section we shall present a group G with an $\mathfrak{F}$-injector which permutes with no Hall π'-subgroup of G. In preparation we conclude this part of our discussion of $\mathfrak{F}^{\pi}$ with:

2.6.3 Proposition

If $\mathfrak{F}$ is a Fitting class and π a set of primes, and an $\mathfrak{F}$-injector V of G permutes with a Hall π'-subgroup $G_{\pi'}$ of G, then $H = VG_{\pi'}$ is an $\mathfrak{F}^{\pi}$-injector of G.

Proof

The proposition is trivially true if $|G| = 1$, so we use induction, supposing the result true for groups of order less than $|G|$. Let N be a maximal normal subgroup of G. By order, a Hall π-subgroup V_{π} of V is a Hall π-subgroup of H, so $H = V_{\pi}G_{\pi'}$. It follows that $(V_{\pi} \cap N)$ and $(G_{\pi'} \cap N)$ are Hall subgroups of $H \cap N \lhd H$, and therefore $H \cap N = (V_\pi \cap N)(G_{\pi'} \cap N)$. But $(V_\pi \cap N)(G_{\pi'} \cap N) \leq (V \cap N)(G_{\pi'} \cap N) \leq H \cap N$, and so we have equality throughout. $V \cap N$ and $G_{\pi'} \cap N$ are an $\mathfrak{F}$-injector and a Hall π'-subgroup of N respectively , so by induction $H \cap N$ is an $\mathfrak{F}^\pi$-injector of N. We now show H is $\mathfrak{F}^\pi$-maximal in G , then by 1.1.4 a) it follows that H is an $\mathfrak{F}^\pi$-injector of G completing the induction argument. So suppose $H \leq K$ where K is $\mathfrak{F}^\pi$-maximal in G. K contains V and $G_{\pi'}$ and by 1.1.4 c) V is an $\mathfrak{F}$-injector of K. Therefore V contains a Hall π-subgroup of K and hence $K = VG_{\pi'}$ by order , which yields $K = H$. □

The following characterization of the injectors for the Fitting class $\mathfrak{F}_1\mathfrak{F}_2$ is an application of our $\mathfrak{F}^\pi$ concept.

2.6.4 THEOREM

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be Fitting classes and G a group. Put $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2$ and $\pi_2 = \text{char}\ \mathfrak{F}_2$. Let T be an $\mathfrak{F}_1$-injector of $G_{\mathfrak{F}_1^{\pi_2}}$. By definition of $\mathfrak{F}_1^{\pi_2}$ and the Frattini argument , T is normalized by some Hall π_2-subgroup G_{π_2} of G. Let V/T be an $\mathfrak{F}_2$-injector of TG_{π_2}/T , then V is an $\mathfrak{F}$-injector of G.

PROOF

We first establish the result in two special cases.

(1) If $G \in \mathfrak{F}_1^{\pi_2}$, then an $\mathfrak{F}_1$-injector of G is an $\mathfrak{F}$-injector of G .

This is trivial if $G = 1$, so we argue by induction , supposing (1) holds for groups of order less than $|G|$. Let N be a maximal normal subgroup of G of index p say , and H an $\mathfrak{F}_1$-injector of G. Certainly $N \in \mathfrak{F}_1^{\pi_2}$. If H covers G/N , (as is of course the case when $p \in \pi_2$ since $G \in \mathfrak{F}_1^{\pi_2}$ by assumption) then $H \in \mathfrak{F}$ and H properly contains $H \cap N$, which by induction is an

$\mathfrak{F}$-injector of N. Hence N is $\mathfrak{F}$-maximal in G and so by 1.1.4 a) is an $\mathfrak{F}$-injector of G.

Thus we may assume $H < N$ and $p \notin \pi_2$, so H is an $\mathfrak{F}_1$-injector of N and therefore by induction also an $\mathfrak{F}$-injector of N. But then if $H < J \in \mathfrak{F}$, it follows that $H = J \cap N$ and so $H = J_{\mathfrak{F}_1}$ because H is $\mathfrak{F}_1$-maximal in G. Then $J/J \cap N \cong G/N \cong C_p \notin \mathfrak{F}_2$ which contradicts $J \subset \mathfrak{F}$. We deduce that H is $\mathfrak{F}$-maximal in G and by 1.1.4 a) again, H is an $\mathfrak{F}$-injector of G.

(2) If $G \in \mathfrak{F}_1{}^{\pi_2}\mathfrak{N}$, then TG_{π_2} is an $\mathfrak{F}$-injector of G.

Since $\mathfrak{F}_1{}^{\pi_2} = \mathfrak{F}_1{}^{\pi_2}\mathfrak{S}_{\pi_2'}$ by 2.6.1 b), $G \in \mathfrak{F}_1{}^{\pi_2}\mathfrak{N}$ implies $G/G_{\mathfrak{F}_1{}^{\pi_2}} \in \mathfrak{N}_{\pi_2}$, and $\mathfrak{N}_{\pi_2} \subseteq \mathfrak{F}_2$ by definition of π_2.

Therefore TG_{π_2} lies in $\mathfrak{F}$ and covers $G/G_{\mathfrak{F}_1{}^{\pi_2}}$. (2) then follows from (1) and 1.1.4 b).

Now let $K = F(G \bmod G_{\mathfrak{F}_1{}^{\pi_2}}) = G_{\mathfrak{F}_1{}^{\pi_2}\mathfrak{N}}$. Put $K_{\pi_2} = K \cap G_{\pi_2}$, a Hall π_2-subgroup of K. By (2) TK_{π_2} is an $\mathfrak{F}$-injector of K.

(3) T is $\mathfrak{F}_1$-maximal in K.

For if not suppose $T < T^* \leq K$ with $T^* \in \mathfrak{F}_1$. Then by 1.1.4 a) T^* is an $\mathfrak{F}_1$-injector of $G_{\mathfrak{F}_1{}^{\pi_2}}T^* = S$ say. It follows that $S \in \mathfrak{F}_1{}^{\pi_2}$, but S sn $K \lhd G$, so that $S \leq G_{\mathfrak{F}_1{}^{\pi_2}}$, which implies $T^* \leq G_{\mathfrak{F}_1{}^{\pi_2}}$, a contradiction to the fact that T is an $\mathfrak{F}_1$-injector of $G_{\mathfrak{F}_1{}^{\pi_2}}$.

(4) Suppose either a) $TK_{\pi_2} \leq H \in \mathfrak{F}$
or b) $TK_{\pi_2} \leq H \leq TG_{\pi_2}$, then $H_{\mathfrak{F}_1} = T$.

We have $T = H \cap G_{\mathfrak{F}_1{}^{\pi_2}} \lhd H$ and $TK_{\pi_2} = H \cap K \lhd H$, because in case a) T and TK_{π_2} are $\mathfrak{F}$-maximal in $G_{\mathfrak{F}_1\pi_2}$ and K respectively by (1) and (2), and in case b) T and TK_{π_2} have π'-index in $G_{\mathfrak{F}_1\pi_2}$ and K respectively. So $[TK_{\pi_2}, H_{\mathfrak{F}_1}] \leq TK_{\pi_2} \cap H_{\mathfrak{F}_1} =$ $K \cap H_{\mathfrak{F}_1} \in \mathfrak{F}_1$. Certainly $T \leq K \cap H_{\mathfrak{F}_1}$, since T is a normal $\mathfrak{F}_1$-subgroup of H, lying in K. Then by (3) $T = K \cap H_{\mathfrak{F}_1}$, so we have $[K_{\pi_2}, H_{\mathfrak{F}_1}] \leq T$ and therefore $[G_{\mathfrak{F}_1\pi_2}, H_{\mathfrak{F}_1}] \leq G_{\mathfrak{F}_1\pi_2}$ which says $H_{\mathfrak{F}_1}$ centralizes $F(G/G_{\mathfrak{F}_1\pi_2})$. Now a standard result implies $H_{\mathfrak{F}_1} \leq K$ and so $T = H_{\mathfrak{F}_1}$ as required.

(5) The canonical map induces a 1-1 correspondence between

$\{\mathfrak{F}$-subgroups of TG_{π_2} containing $TK_{\pi_2}\}$ and

$\{\mathfrak{F}_2$-subgroups of TG_{π_2}/T containing $TK_{\pi_2}/T\}$.

For if $TK_{\pi_2} \leq H \leq TG_{\pi_2}$, then by (4) $H_{\mathfrak{F}_1} = T$ so that $H \in \mathfrak{F}$ iff $H/T \in \mathfrak{F}_2$.

Now let H be an $\mathfrak{F}$-injector of G. Passing to a conjugate if necessary, we may assume that H contains TK_{π_2}, since by (2) TK_{π_2} is an $\mathfrak{F}$-injector of the normal subgroup K of G. Then by (4) $H_{\mathfrak{F}_1} = T$, so H/T is a π_2-subgroup of $N_G(T)/T$, of which TG_{π_2}/T is a Hall π_2-subgroup. Thus we may assume further that $H \leq TG_{\pi_2}$.

Now let $T \lhd TK_{\pi_2} \lhd \ldots \lhd TG_{\pi_2}$ be a series with nilpotent factors. Then using 1.1.4 b) and (5) we see that H/T is an $\mathfrak{F}_2$-injector of TG_{π_2}/T and hence conjugate to the subgroup V/T constructed in the statement of the theorem. So V is an $\mathfrak{F}$-injector of G as asserted. $\square$

2.6.5 Corollary

Suppose $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ are Fitting classes and put

$\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2 \ldots \mathfrak{F}_n$. For $i = 2,\ldots,n$, let $\pi_i = \mathrm{char}(\mathfrak{F}_i \ldots \mathfrak{F}_n)$ and put $\pi_{n+1} = \emptyset$. Let $G_0 = 1$ and for $i = 1,\ldots,n$ let $G_i = G_{\mathfrak{F}_1^{\pi_2}\mathfrak{F}_2^{\pi_3}\ldots\mathfrak{F}_i^{\pi_{i+1}}}$ so by 1.2.5 b) $G_i/G_{i-1} = (G/G_{i-1})_{\mathfrak{F}_i^{\pi_{i+1}}}$

Notice $\mathfrak{F}_n^{\pi_{n+1}} = \mathfrak{S}$ so $G_n = G$.

$$\begin{array}{ccl}
G_n & \circ & G \\
 & | & \\
G_{n-1} & \circ & G_{\mathfrak{F}_1^{\pi_2}\ldots\mathfrak{F}_{n-1}^{\pi_n}} \\
 & \vdots & \\
G_2 & \circ & G_{\mathfrak{F}_1^{\pi_2}\mathfrak{F}_2^{\pi_3}} \\
 & | & \\
G_1 & \circ & G_{\mathfrak{F}_1^{\pi_2}} \\
 & | & \\
G_0 & \circ & 1
\end{array}$$

Suppose V is an $\mathfrak{F}$-injector of G , then $(V \cap G_i)G_{i-1}/G_{i-1}$ is an $\mathfrak{F}_i$-injector of G_i/G_{i-1} . Loosely V covers precisely an $\mathfrak{F}_i$-injector of the ' ith ' layer of G.

<u>Proof</u>

The result is obvious when $n = 1$.

Suppose $n = 2$. By 2.6.4 and the conjugacy of injectors , V/T is an $\mathfrak{F}_2$-injector of TG_{π_2}/T for some $\mathfrak{F}_1$-injector T of G_1 and Hall π_2-subgroup G_{π_2} of G . Now for $x \in TG_{\pi_2}$, the map $Tx \to G_1x$ is an isomorphism $TG_{\pi_2}/T \cong G_1G_{\pi_2}/G_1$ since $T = TG_{\pi_2} \cap G_1$. Thus VG_1/G_1 , the image of V/T under this map , is an $\mathfrak{F}_2$-injector of $G_1G_{\pi_2}/G_1$ which in turn is a Hall π_2-subgroup of G/G_1 . Since char $\mathfrak{F}_2 = \pi_2$, it follows that VG_1/G_1 is an $\mathfrak{F}_2$-injector of G/G_1 and we already know $V \cap G_1 = T$ is an $\mathfrak{F}_1$-injector of G_1 , so the result holds in this case.

Now suppose $n > 2$ and inductively that the result holds for the product of at most n-1 classes. Setting $\mathfrak{H} = \mathfrak{F}_2\mathfrak{F}_3\cdots\mathfrak{F}_n$, the induction hypothesis implies that VG_1/G_1 is an $\mathfrak{H}$-injector of G/G_1 and $V \cap G_1$ is an $\mathfrak{F}_1$-injector of G_1. Since $G_i/G_1 = (G/G_1)_{\mathfrak{F}_2^{\pi_3}\ldots\mathfrak{F}_i^{\pi_{i+1}}}$ for $i = 2,\ldots,n$, a second application of induction (with a suitable isomorphism theorem) gives the statement for n classes and the proof is complete. $\square$

2.6.5' Corollary

Suppose $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are Fitting classes with $\mathfrak{N} \subseteq \mathfrak{F}_2$. Then for each G a) $I_{\mathfrak{F}_1\mathfrak{F}_2}(G) = I_{\mathfrak{F}_2}(G \bmod G_{\mathfrak{F}_1})$

and b) $III_{\mathfrak{F}_1\mathfrak{F}_2}(G) = III_{\mathfrak{F}_2}(G \bmod G_{\mathfrak{F}_1})$.

In particular, if also $I_{\mathfrak{F}_2}(G) = III_{\mathfrak{F}_2}(G)$ for all G then $I_{\mathfrak{F}_1\mathfrak{F}_2}(G) = III_{\mathfrak{F}_1\mathfrak{F}_2}(G)$ for all G.

Proof

a) If $\mathfrak{N} \subseteq \mathfrak{F}_2$ then char $\mathfrak{F}_2$ is the set of all primes. Clearly $\mathfrak{F}_1^{\{\text{all primes}\}} = \mathfrak{F}_1$ so in this case 2.6.4 states that the inverse image of an $\mathfrak{F}_2$-injector of $G/G_{\mathfrak{F}_1}$ in G, is an $\mathfrak{F}_1\mathfrak{F}_2$-injector of G and by their conjugacy all $\mathfrak{F}_1\mathfrak{F}_2$-injectors of G have this form. This is statement a)

b) Suppose $G_{\mathfrak{F}_1\mathfrak{F}_2} \leqslant H \leqslant G$. Since $G_{\mathfrak{F}_1\mathfrak{F}_2} \lhd H$ we have $G_{\mathfrak{F}_1\mathfrak{F}_2} \cap H_{\mathfrak{F}_1} = (G_{\mathfrak{F}_1\mathfrak{F}_2})_{\mathfrak{F}_1} = G_{\mathfrak{F}_1}$ as usual, so $[G_{\mathfrak{F}_1\mathfrak{F}_2}, H_{\mathfrak{F}_1}] \leqslant G_{\mathfrak{F}_1}$. We deduce that $H_{\mathfrak{F}_1}$ centralizes $G_{\mathfrak{F}_1\mathfrak{F}_2}/G_{\mathfrak{F}_1}$, which by the supposition $\mathfrak{N} \subseteq \mathfrak{F}_2$ contains the Fitting subgroup of $G/G_{\mathfrak{F}_1}$. So by the often quoted result, $H_{\mathfrak{F}_1} \leqslant G_{\mathfrak{F}_1\mathfrak{F}_2}$ and hence $H_{\mathfrak{F}_1} = G_{\mathfrak{F}_1}$. (The crux of the proof of 2.6.4 is a disguised form of this argument) We have $H \in \mathfrak{F}_1\mathfrak{F}_2$ iff $H/G_{\mathfrak{F}_1} \in \mathfrak{F}_2$ and b) is now clear.

$\square$

Despite $I_{\mathfrak{S}_\pi}(G) = III_{\mathfrak{S}_\pi}(G)$ for all G, example 1.2.6 shows that the final sentence of the statement of 2.6.5 does not follow without the assumption $\mathfrak{N} \subseteq \mathfrak{F}_2$. (Recall that by 1.2.2, $\mathfrak{N} \nsubseteq \mathfrak{F}_2$ and $I_{\mathfrak{F}_2}(G) = III_{\mathfrak{F}_2}(G)$ for all G implies $\mathfrak{F}_2 = \mathfrak{S}_\pi$ for some π.) However $I_{\mathfrak{S}_{\pi'}\mathfrak{S}_\pi}(G) = III_{\mathfrak{S}_{\pi'}\mathfrak{S}_\pi}(G) = \mathrm{Hall}_\pi(G \bmod O_{\pi'}(G))$ holds for each G by a similar proof since $O_{\pi',\pi}(G)/O_{\pi'}(G)$ contains its centralizer in $G/O_{\pi'}(G)$ for all G, as is well known.

2.6.6 THEOREM

If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are Fischer classes then so also is $\mathfrak{F}_1\mathfrak{F}_2$.

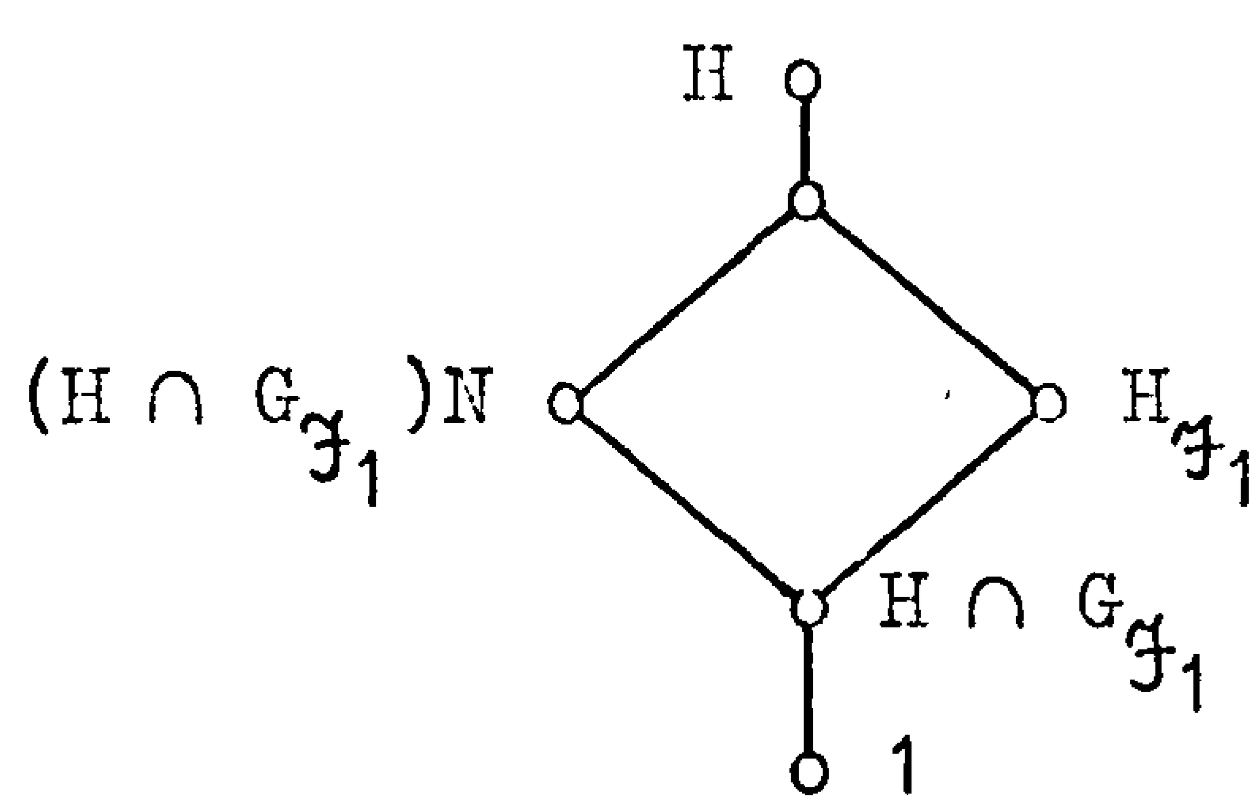
PROOF

Suppose $H \lhd G \in \mathfrak{F}_1\mathfrak{F}_2$ and $N \leq H \leq G$ where H/N is a p-group. We require $H \in \mathfrak{F}_1\mathfrak{F}_2$.

$H/(H \cap G_{\mathfrak{F}_1}) \cong HG_{\mathfrak{F}_1}/G_{\mathfrak{F}_1} \geq NG_{\mathfrak{F}_1}/G_{\mathfrak{F}_1} \lhd G/G_{\mathfrak{F}_1} \in \mathfrak{F}_2$ and $(HG_{\mathfrak{F}_1}/G_{\mathfrak{F}_1})/(NG_{\mathfrak{F}_1}/G_{\mathfrak{F}_1})$ is a p-group, so $H/(H \cap G_{\mathfrak{F}_1}) \in \mathfrak{F}_2$ since $\mathfrak{F}_2$ is a Fischer class by assumption.

$H \rhd H \cap G_{\mathfrak{F}_1} \geq N \cap G_{\mathfrak{F}_1} \lhd G_{\mathfrak{F}_1} \in \mathfrak{F}_1$ and $H \cap G_{\mathfrak{F}_1}/N \cap G_{\mathfrak{F}_1}$ is a p-group, so $H \cap G_{\mathfrak{F}_1} \in \mathfrak{F}_1$ since $\mathfrak{F}_1$ is a Fischer class by assumption. Therefore $H \cap G_{\mathfrak{F}_1} \leq H_{\mathfrak{F}_1}$, and so

$H_{\mathfrak{F}_1} \cap (H \cap G_{\mathfrak{F}_1})N = (H \cap G_{\mathfrak{F}_1})(H_{\mathfrak{F}_1} \cap N) = (H \cap G_{\mathfrak{F}_1})N_{\mathfrak{F}_1} = H \cap G_{\mathfrak{F}_1}$ with help from the modular law. Thus we have :



$H/(H \cap G_{\mathfrak{F}_1}) \in \mathfrak{F}_2$ and

$H/(H \cap G_{\mathfrak{F}_1})N$ is a p-group

so it too lies in $\mathfrak{F}_2$.

Applying part b) of the following lemma, we deduce $H/H_{\mathfrak{F}_1} \in \mathfrak{F}_2$.

2.6.7 Lemma

Let $\mathfrak{F}$ be a Fitting class and suppose $A, B \lhd G$ with $A \cap B = 1$ and $G/AB \in \mathfrak{N}$. Then a) G/A, $G/B \in \mathfrak{F}$ implies $G \in \mathfrak{F}$,

and b) G/A, $G \in \mathfrak{F}$ implies $G/B \in \mathfrak{F}$.

Proof

We need only show that G is isomorphic to a subnormal subgroup G_0 of $X = G/A \times G/B$ and furthermore that X is generated by G_0 and $G/A \times 1$. The conclusions then follow immediately.

Let $N = AB$ and put $G_0 = \{ (Ag, Bg) : g \in G \}$, then $G \cong G_0$ since $A \cap B = 1$. Also $G_0 \geq N/A \times N/B$ and so G_0 sn X because $G/N \in \mathfrak{N}$. $(A, Bg) = (Ag, Bg)(Ag, B)^{-1}$ so $X = G_0(G/A \times 1)$, and the proof is complete. □

2.6.8 Remarks

a) In section 3.2 of [4] Hartley gives the short argument which shows that for an arbitrary Fitting class $\mathfrak{F}$ and set of primes π, $\mathfrak{F}\mathcal{S}_\pi\mathcal{S}_{\pi'}$ is actually a Fischer class, so a non-Fischer class cannot be expressed in the form $\bigcap_p \mathfrak{F}(p)\mathcal{S}_p\mathcal{S}_{p'}$ for some choice of Fitting classes $\{\mathfrak{F}(p)\}$. Notice that $\mathfrak{F}\mathfrak{N} = \bigcap_{\text{all } p} \mathfrak{F}\mathcal{S}_p\mathcal{S}_{p'}$ so this too is a Fischer class for any $\mathfrak{F}$. It is not difficult to show that the Fischer class $e_q(\mathcal{S}_q)$, mentioned in 2.2.3 cannot be put in this form either, so there is no hope that this direct analogue of local definition for saturated formations, yields even all the Fischer classes. We would point out that the class $\mathfrak{F} = \bigcap_{\text{all } p} \mathfrak{F}(p)\mathcal{S}_{p'}\mathcal{S}_p$, considered by Hartley in section 3.3 of his paper, actually coincides with $\overline{\mathfrak{F}}\mathfrak{N}$ where $\overline{\mathfrak{F}} = \bigcap_{\text{all } p} \mathfrak{F}(p)\mathcal{S}_{p'}$, so by 2.6.5 $I_{\mathfrak{F}}(G) = III_{\mathfrak{F}}(G) = I_{\mathfrak{N}}(G \bmod G_{\mathfrak{F}})$ in this case.

b) We have been unable to settle the problem :

* $\qquad S\mathfrak{F}_i = \mathfrak{F}_i$ for $i = 1,2$ implies $S\mathfrak{F}_1\mathfrak{F}_2 = \mathfrak{F}_1\mathfrak{F}_2$?

It can easily be seen that an example which exhibits the falsehood of * would necessarily have $\mathfrak{F}_2 \neq Q\mathfrak{F}_2$, and conversely it is very likely that such a class $\mathfrak{F}_2$ could be incorporated into an example to deny * . Nobody seems to know a Fitting class $\mathfrak{F}$ with $S\mathfrak{F} = \mathfrak{F} \neq Q\mathfrak{F}$, though we expect one exists. Since $\{S,N_0\}$-closed implies $\{S,D_0\}$-closed implies R_0-closed, this means that all known S-closed Fitting classes are actually S-closed Fitting formations. Hawkes has conjectured that these latter classes are the so-called primitive saturated formations, ~~that is, those with a local definition $\{ f(p) \}$ of the form $f(p) = \mathcal{S}_{\pi(p)}$~~.

c) As in 2.3 let $\mathfrak{F}_p$ be the class of groups with central p-socle.

The following example shows that the $\mathfrak{F} \to \mathfrak{F}^{\pi}$ operation behaves poorly with respect to products , specifically $(\mathfrak{F}_2\mathfrak{F}_3)^{\{2\}} \neq \mathfrak{F}_2{}^{\{2\}}\mathfrak{F}_3{}^{\{2\}}$. Let $H \cong SL(2,3) \sim [Q_8]C_3$ and let $G = H \wr \Sigma_3$ constructed with the natural permutation representation of Σ_3 , so the base group B of G is isomorphic to $H \times H \times H$. H has a central unique minimal normal subgroup Z say, of order 2, and hence $B \in \mathfrak{F}_2$. $BC_3 \notin \mathfrak{F}_2$ (where C_3 is the Sylow 3-subgroup of Σ_3) since C_3 acts in a non-trivial and completely reducible manner on $Z \times Z \times Z$. It follows that BC_3 is an $\mathfrak{F}_2\mathfrak{F}_3$-injector of G , so $G \notin (\mathfrak{F}_2\mathfrak{F}_3)^{\{2\}}$. However BC_2 (where C_2 is a Sylow 2-subgroup of Σ_3) is an $\mathfrak{F}_2$-injector of G and covers all its 2-chief factors , so $G \in \mathfrak{F}_2{}^{\{2\}}$ and hence $G \in \mathfrak{F}_2{}^{\{2\}}\mathfrak{F}_3{}^{\{2\}}$.

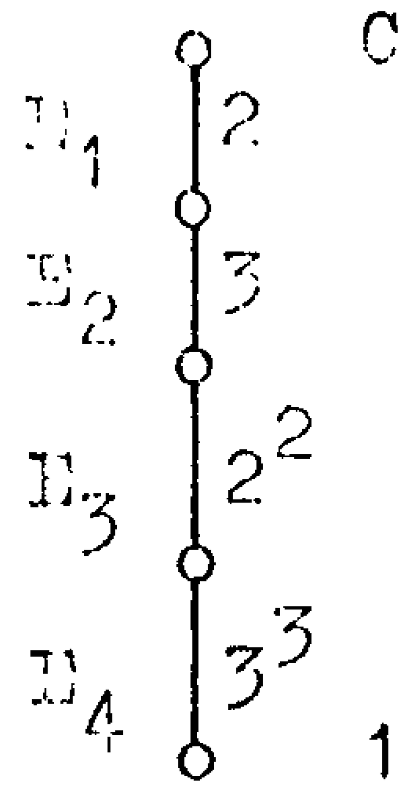
We end our ' new from old ' section with some observations on the intersection $\mathfrak{F}_1 \cap \mathfrak{F}_2$ of two Fitting classes $\mathfrak{F}_1$ and $\mathfrak{F}_2$. It seems rarely possible to relate the injectors of a group G for the class $\mathfrak{F}_1 \cap \mathfrak{F}_2$ to those for the classes $\mathfrak{F}_1$ and $\mathfrak{F}_2$. In particular $(\mathfrak{F}_1 \cap \mathfrak{F}_2)$-injector = $\mathfrak{F}_1$-injector $\cap$ $\mathfrak{F}_2$-injector in each group , is nearly always false and we give two simple examples.

<u>2.6.9 Examples</u>

a) Again let $\mathfrak{F}_p$ be the class of groups with central p-socle. Put $\mathfrak{F} = \mathfrak{F}_2 \cap \mathfrak{F}_3$. Let $M_3 \cong C_3 \times C_3$, $M_2 \cong C_2 \times C_2$ and $M = M_3 \times M_2$. Let $A \cong \Sigma_3$ act on M_3 as in example 2.3.2 and on M_2 so that $M_2A \cong \Sigma_4$. Put $G = MA$. Then $G \in \mathfrak{F}_3$ and MA_2 is an $\mathfrak{F}_2$-injector of G , but M is the unique $\mathfrak{F}$-injector of G .

b) Now put $\mathfrak{F}(p) = \mathcal{S}_{p'}\mathcal{S}_p\mathcal{S}_{p'}$ the classof groups of p-length 1. Let $\mathfrak{F} = \mathfrak{F}(2) \cap \mathfrak{F}(3)$ and let G be a group with unique chief series constructed in the manner of 1.2.4 from the sequence 3, 2, 3, 2 .

$E_2E_3E_4$ is the unique $\mathfrak{F}(2)$-injector of G and $E_1E_3E_4$ is an $\mathfrak{F}(3)$-injector of G. Their intersection is E_3E_4, whereas $E_1E_3E_4$ is also an $\mathfrak{F}$-injector of G.



2.7 Generating Fitting classes

No discussion of examples of Fitting classes could end without some remarks on this topic. The important general problem of determining the smallest Fitting class containing a given set of groups seems a very thorny one indeed reflecting an inability to handle normal products. For instance, many more able minds than our own have been applied to a determination of the Fitting class generated by Σ_3, but without success. The only real progress in defining a Fitting class by a construction process starting with a certain group, was made by Dark in [11]. As we mentioned in 1.2, the importance of this paper is his construction of a group with a Fischer $\mathfrak{F}$-subgroup which is not an $\mathfrak{F}$-injector. Leaning heavily on Dark's techniques we now construct a family of Fitting classes which, because we are no longer trying to concoct simultaneously a nastily embedded subgroup, are less complicated than his class.

2.7.1 Proposition

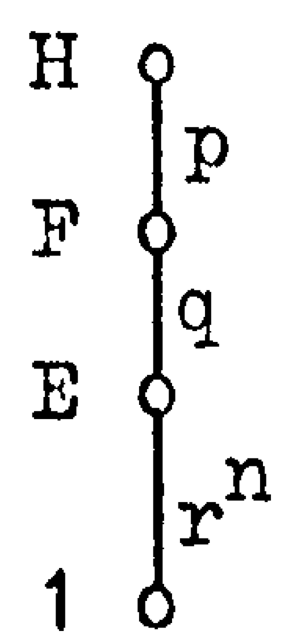
Let p, q and r be primes such that:

a) $p \mid n$ where n = order of r mod q, and

b) $1 < m < n$ and $m \nmid n$ where m = order of r mod p. Then:

i) The elementary Abelian group E of order r^n does not possess commuting automorphisms α and β of orders p and q respectively.

ii) There exists a group H with the following unique chief series. Furthermore E is a minimal normal subgroup of F.



Proof

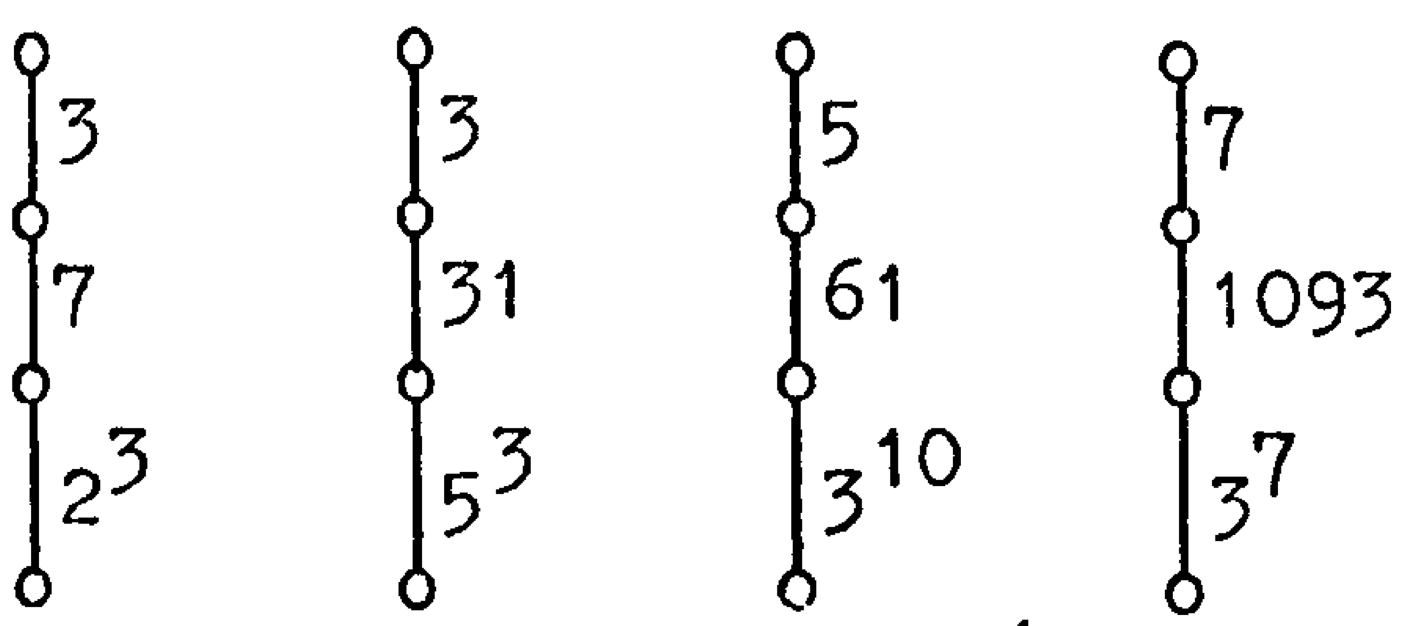
The following two statements are well-known results (see for example section II.3 of Huppert's book).

(1) Each non-trivial irreducible $GF(r)C_q$-module has dimension n and each non-trivial irreducible $GF(r)C_p$-module has dimension m .

(2) The set of all maps from $GF(r^n)$ to itself of the form $x \to \lambda x^{r^t}$ where $0 \neq \lambda \in GF(r^n)$ and $1 \leq t \leq N$, is a group of automorphisms of $GF(r^n)^+$ ($\cong E$) which is a split extension of a cyclic group B of order r^n-1 (the set of multiplications $x \to \lambda x$) by a cyclic group A of order n (generated by the Frobenius automorphism $x \to x^r$).

i) By assumption b) , (1) and Maschke's theorem, $C_E(\alpha) \neq 1$. The fact that α and β commute implies that β normalizes $C_E(\alpha)$, but by (1) β acts irreducibly on E , a contradiction , so such commuting automorphisms do not exist.

ii) Consider the group AB of automorphisms of E as in (2) . By definition of n , $q \mid r^n-1$, so the cyclic group B has a unique subgroup $< b >$ say of order q , which by (1) acts irreducibly on E . Since $p \mid n$ by supposition, the group A has a subgroup $< a >$ of order p , and $< a >$ will normalize $< b >$. By i) this last action is not trivial, and putting $H = E < b > < a >$, ii) is established. (Notice that $n \mid q-1$ since $r^{q-1} \equiv 1 \bmod q$, so certainly $p \mid q-1$.) $\square$



It may be verified that these are possible forms for H .

Now let H be a (fixed) group with the properties of 2.7.1 ii). Let X be a direct product of copies of H , and let Y be any subgroup of X whose index is a power of p and such that a Sylow p-subgroup Y_p of Y acts non-trivially on the normal subgroup F of each direct factor H of X . Equivalently , the image of Y_p under the natural projection onto any direct factor of X , is non-trivial. Let $\mathcal{Y}$ be the set of all such groups Y together with the identity group. Our aim is to prove that the set $\{ G : O^{p'}(G)/O_p(G) \in \mathcal{Y} \}$ is a Fitting class.

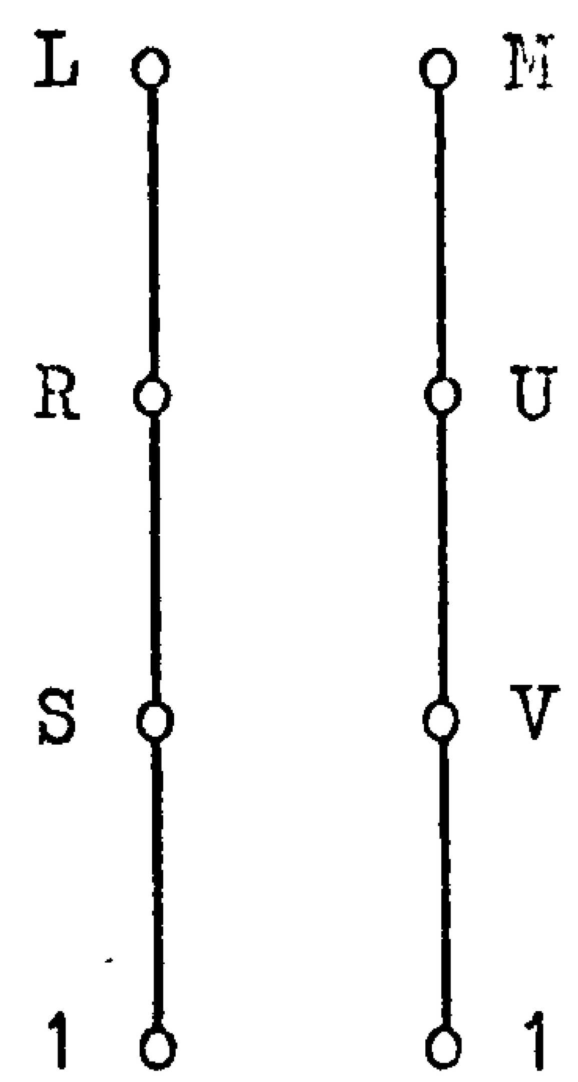
2.7.2 Lemma

If $\mathcal{Y} \ni L, M \lhd LM = G$ then $G/O_p(G) \in \mathcal{Y}$.

Proof

Certainly $1 = O_p(Y)$ for all $Y \in \mathcal{Y}$, so we may suppose $L, M \neq 1$. Let R and U be the $\mathcal{S}_{\{r,q\}}$-radicals and S and V the $\mathcal{S}_r$-radicals of L and M respectively. R is a direct product of copies of F , $R = F_1 \times F_2 \times \dots \times F_t$ say, and since F has trivial centre it follows by an easy argument that any normal subgroup of R isomorphic to F must be one of the F_i . So calling these copies of F occurring as direct factors of R and U ' normal factors ' we have :

(1) Any automorphism of R (respectively U) permutes its normal factors.



Let Σ be a fixed Sylow system of G , $\Sigma = \{ G_p, G_q, G_r, \dots \}$. Then Σ reduces into L, M, R, U, S, V, and the normal factors of R and U . By 1.2.7 $G_p = (L \cap G_p)(M \cap G_p)$ etc. , G_p normalizes G_q and G_q normalizes G_r . These facts will be used without mention.

(2) The normal factors of R and U are normal in RU .

Let F be a normal factor of U. Let $x \in S$ and suppose $F \neq F^x$, so by (1) F and F^x are distinct normal factors of U.
If $\langle b \rangle = F \cap G_q$ then $(b^{-1})^x.b = [x,b] \in S$. Now if c is a non-trivial r-element of ~~of~~ F, $(b^{-1})^x$ centralizes c since it lies in a different normal factor of U. Thus $F \ni c^{-1}c^b = c^{-1}c^{(b^{-1})^x.b} = [c,[x,b]] \in S$. But S is Abelian and b centralizes no non-trivial r-element of F, therefore $1 \neq [c,b] = [c,b]^x \in F \cap F^x = 1$, a contradiction.

Now let F_1 be a normal factor of R and put $\langle b_1 \rangle = F_1 \cap G_q$. Suppose b_1 does not normalize F, then $F, F^{b_1}, \ldots\ldots\ldots, F^{b_1^{q-1}}$ are pairwise distinct normal factors of U. $F_1 \lhd L \lhd LM$, so $[b,b_1,b_1] \in U \cap F_1$. Since $F \lhd U$ this implies that $[b,b_1,b_1]$ normalizes F. But

$$\begin{aligned}[b,b_1,b_1] &= b_1^{-1}b^{-1}b_1bb_1^{-1}b^{-1}b_1^{-1}bb_1b_1 \\ &= (b^{-1})^{b_1}b(b^{-1})^{b_1}b^{b_1^2} \\ &= b.(b^{-2})^{b_1}.b^{(b_1)^2}\end{aligned}$$

since F, F^{b_1}, $F^{b_1^2}$ commute in pairs. So $[b,b_1,b_1]$ is a non-trivial q-element of F_1 which normalizes F. Having shown that S normalizes F, this implies F_1 normalizes F. Then (2) is an immediate deduction.

(3) If F_1 and F are normal factors of R and U respectively and $F_1 \cap F \neq 1$, then $F_1 = F$.

By (2) F_1 and F are normal subgroups of RU, so $F_1 \cap F \lhd F, F_1$. Now F and F_1 have unique minimal normal subgroups (of order r^n) so $F_1 \cap F \neq 1$ implies $F_1 \cap G_r = F \cap G_r = E$ say.
First suppose $F \leqslant R$, and let $R = F_1 \times F_2 \times \ldots \times F_t$ be the decomposition of R as the product of its normal factors. Then $\langle b \rangle = F \cap G_q \leqslant R \cap G_q = (F_1 \cap G_q) \times (F_2 \cap G_q) \times \ldots \times (F_t \cap G_q)$ so for $j = 2,\ldots,t$ $[F_j \cap G_r, b] \leqslant F_j \cap G_r \cap F = F_j \cap E = 1$.
Therefore b is an element of $R \cap G_q$ which centralizes $(F_2 \times \ldots \times F_t) \cap G_r$ which implies $b \in F_1$, and hence $F = F_1$.

Now we shall assume $F \nleq R$, or equivalently $b \notin R$, and obtain a contradiction. By definition of the class $\mathcal{Y}$, there exists an element $a_1 \in L \cap G_p$ which acts non-trivially on F_1 . Then a_1 normalizes E , so $E \leq F^{a_1}$ which is a normal factor of U by (1) . Now E lies in a unique normal factor of U , so a_1 normalizes F , whence it normalizes $F \cap G_q = \langle b \rangle$, which implies $[b,a_1] \in F \cap G_q \cap L$. But $F \cap G_q \cap L = 1$ by our assumption, and therefore a_1 and b induce commuting automorphisms of E , against 2.7.1 i) , and the proof of (3) is complete.

(4) RU = (direct product of those normal factors of R which intersect U trivially) $\times$ (direct product of those normal factors of U which intersect R trivially) $\times$ (direct product of the normal factors of $R \cap U$) .

First suppose F_1 is a normal factor of R such that $F_1 \cap F = 1$ for all normal factors F of U . Then using (2) , $[F_1 , F] \leq F_1 \cap F = 1$ for all normal factors F of U , hence $[F_1 , U] = 1$ and so $F_1 \cap U \leq Z(U) = 1$. Thus for each normal factor F_1 of R , either $F_1 \cap U = 1$ and $F_1U = F_1 \times U$, or $F_1 \cap F \neq 1$ for some normal factor F of R and then $F_1 = F$ by (3). (4) now follows.

(5) The normal factors of R and U are normal subgroups of $LM = G$. For let $a \in M \cap G_p$, then by (1) a permutes the normal factors of R . If F_1 is a normal factor of R with $F_1 \nleq U$, then $F_1{}^a \nleq U^a = U$ also. Then $[F_1 , a] \leq F_1{}^aF_1 \cap M = F_1{}^aF_1 \cap U = 1$ by (4) , so a centralizes F_1 . If F_1 is a normal factor of R lying in U , then it must be a normal factor of U as we saw in (4) , so F_1 is normalized by a . Thus a normalizes each normal factor of R , and the deduction of (5) is now clear.

Suppose F is a normal factor of both R and U. By (5) $F \lhd G$.

By definition of the class $\mathcal{Y}$, there exist elements $a_1 \in L \cap G_p$ and $a \in M \cap G_p$ each of which acts non-trivially on F. Put $\langle b \rangle = F \cap G_q$ as usual. $\langle a, a_1 \rangle = P$ say, is a p-group which normalizes both $\langle b \rangle$ and $E = F \cap G_r$. If $a^* \in P$ centralizes $\langle b \rangle$, then a^* centralizes E by 2.7.1 i). So $C_P(F) = C_P\langle b \rangle$, but $P/C_P\langle b \rangle$ is certainly cyclic and neither a nor a_1 centralizes $\langle b \rangle$, so it follows that :

(6) $\langle a \rangle$ and $\langle a_1 \rangle$ induce the same automorphisms of F.

Now suppose F is a normal factor of R with $F \cap U = 1$. Then by (5) $[F, M \cap G_p] \leq F \cap M = F \cap U = 1$, so the automorphisms of F induced by G are merely those induced by $R \cap G_p$.

It follows from this and (6) that for any normal factor F of RU, the split extension $[F]G_p/C_{G_p}(F)$ is isomorphic to H. Also $C_{G_p}(F)$ is a maximal normal subgroup of G_p and hence $C_{G_p}(RU)$ is a normal subgroup of G_p with $G_p/C_{G_p}(RU)$ an elementary Abelian p-group. Of course $O_p(G)$ centralizes RU and hence $O_p(G) = C_{G_p}(RU)$. We have therefore established that $G/O_p(G) \in \mathcal{Y}$ as required. □

2.7.3 Lemma

If W sn $Y \in \mathcal{Y}$ and W is generated by its p-elements then $W \in \mathcal{Y}$.

Proof

Let $W \lhd W_1 \lhd W_2 \lhd \ldots \lhd W_n = Y \in \mathcal{Y}$, then by well-known results $[O_{p'}(Y), W_p] = [O_{p'}(Y), \underbrace{W_p, W_p, \ldots W_p}_{n \text{ terms}}] \leq W$, where W_p is a Sylow p-subgroup of W. From the structure of H it is clear that the normal subgroup $[O_{p'}(H), H_p]$ must coincide with $O_{p'}(H)$. It follows that $[O_{p'}(Y), W_p] = K$ say, is precisely the product of the normal factors of Y on which W_p acts non-trivially, in particular $KW_p \in \mathcal{Y}$. We have $KW_p \leq W$, on the other hand $KW_p \lhd Y$, so KW_p contains all the p-elements of W . Hence by hypothesis $KW_p = W$, and so $W \subset \mathcal{Y}$. $\square$

2.7.4 THEOREM

$\mathcal{J} = \{ G : O^{p'}(G)/O_p(G) \in \mathcal{Y} \}$ is a Fitting class.

PROOF

Suppose $\mathcal{J} \ni N_1, N_2 \lhd N_1N_2 = G$. Then by 2.5.1

$$O^{p'}(G)/O_p(G) = O^{p'}(N_1)O^{p'}(N_2)/O_p(G)$$
$$= O^{p'}(N_1)O_p(G)/O_p(G) \; . \; O^{p'}(N_2)O_p(G)/O_p(G)$$

and $O^{p'}(N_i)O_p(G)/O_p(G) \cong O^{p'}(N_i)/O^{p'}(N_i) \cap O_p(G)$

$= O^{p'}(N_i)/O_p(N_i) \in \mathcal{Y}$, so by 2.7.2 $O^{p'}(G)/O_p(G) \in \mathcal{Y}$.

Suppose $N \lhd G \in \mathcal{J}$. Then $O^{p'}(N)/O_p(N) = O^{p'}(N)/O_p(G) \cap N$

$\lhd O^{p'}(G) \cap N/O_p(G) \cap N \cong O_p(G)(O^{p'}(G) \cap N)/O_p(G) \lhd O^{p'}(G)/O_p(G)$.

Therefore we have $O^{p'}(N)/O_p(N) \in S_n \{O^{p'}(G)/O_p(G)\} \subseteq \mathcal{Y}$, and $O^{p'}(N)/O_p(N)$ is clearly generated by its p-elements, so by 2.7.3 it lies in $\mathcal{Y}$. $\square$

2.8 A problem of Gaschütz

As an application of 2.7.4 we now settle Test Problem 8.6 of [7] which asks : If $\mathcal{J}$ is a Fitting class, does $Z \leq Z(G) \cap G'$ and $G/Z \in \mathcal{J}$ imply $G \in \mathcal{J}$. Our answer is no. Since $Z(G) \cap G' \leq \phi(G)$ for each G , the question is an attempt to discover the rôle of the Frattini subgroup in Fitting class theory.

A p-group P is called extra-special when P/P' is elementary Abelian and $P' = Z(P) \cong C_p$. It is well-known (see for example page 204 of Gorenstein's book) that for arbitrary n and $p \neq 2$, there exists a unique (up to isomorphism) extra-special p-group P of exponent p and order p^{2n+1} .

P is generated by elements $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ subject to the relations

$$g^p = 1 \text{ for all } g \in P ,$$

$$* \quad [x_i , x_j] = [x_i , y_j] = [y_i , y_j] = 1 \text{ for all } i,j \in \{1,\ldots,n\} \text{ with } i \neq j ,$$

$$1 \neq [x_i , y_i] = z \text{ say for all } i \in \{1,\ldots,n\} \text{ and } z \in Z(P) .$$

Let $^-$ denote the natural epimorphism $P \to P/P'$. $\overline{P}$ can be regarded as a vector space V say, over $GF(p)$ with basis $\{\overline{x}_1, \overline{y}_1, \ldots \overline{x}_n, \overline{y}_n\}$ and the map $f : V \times V \to GF(p)$ defined by $[g_1 , g_2] = z^{f(\overline{g}_1,\overline{g}_2)}$ is a skew symmetric bilinear form on V . Indeed f is the form $\begin{bmatrix} 0 & 1 & 0 & 0 & \\ -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & -1 & 0 & \\ & & & & \ddots \end{bmatrix}$ with respect to the given basis of V . Automorphisms of P which centralize $P' = \langle z \rangle$ induce isometries of V , and therefore we have a homomorphism $h : C_A(P') \to S_p(2n,p)$, where $A = \text{Aut } P$. By a well-known result $C_A(P/P')$ is a p-group, so it will centralize P' , and so $C_A(P/P')$ is the kernel of the map h . Conversely if

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1\,2n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2\,2n} \\ \vdots & & & \\ a_{2n\,1} & & \cdots\cdots & a_{2n\,2n} \end{bmatrix}$$

is an element of $GL(2n,p)$ leaving the form f invariant, that is a member of $Sp(2n,p)$, then a study of the relations * reveals that

$$x_1 \to x_1^{a_{11}} y_1^{a_{12}} \cdots\cdots y_n^{a_{1\,2n}}$$

$$y_1 \to x_1^{a_{21}} y_1^{a_{22}} \cdots\cdots y_n^{a_{2\,2n}}$$

$$\vdots$$

$$y_n \to x_1^{\varepsilon_{2n\,1}} \cdots\cdots\cdots y_n^{\varepsilon_{2n\,2n}}$$

induces an automorphism of P centralizing P' . (N.B. our convention of right operators) Therefore we have a map $Sp(2n,p) \to C_A(P')$ which, though not necessarily a homomorphism, is certainly a left inverse for h , so it follows that $Sp(2n,p) \cong C_A(P')/C_A(P/P')$. Remembering that $C_A(P/P')$ is a p-group , an easy application of the Schur-Zassenhaus theorem (an unnecessarily stern measure in the case to follow) then shows that :

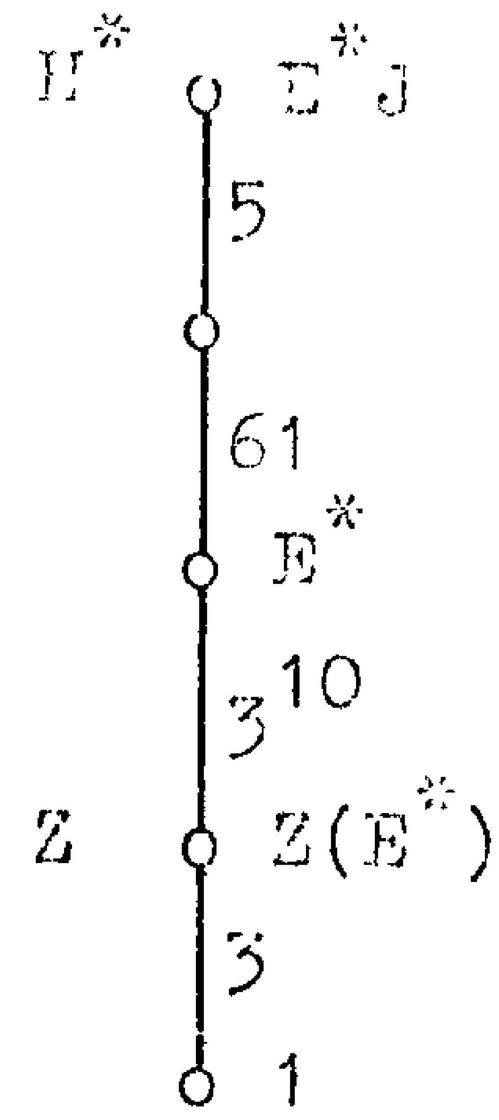
** Each p'-subgroup of $Sp(2n,p)$ is isomorphic to a subgroup of Aut P centralizing P' .

We now appeal to the theory of Chevalley groups in order to prove that $Sp(10,3)$ (the Chevalley group $C_5(3) = C$ say) has a subgroup of order 61.5 (suggested by its order), and we thank Professor Carter for the following argu~~e~~ment. We are informed that Chapters E and G of [13] contain the relevant facts.

The conjugacy classes of maximal tori in a finite Chevalley group correspond 1-1 with the conjugacy classes of elements in the Weyl group W . In our case, an anisotropic maximal torus T_W of order 3^5+1 corresponds to a Coxeter element w of order 10 . Now more of the general theory shows that $C_W(w)$ is isomorphic to a subgroup of $N_C(T_W)/T_W$, and so in particular there exists an element of order 5 in C which normalizes T_W . Now $3^5+1 = 61.4$ so T_W certainly has a characteristic subgroup of order 61 , and the existence of a subgroup of order 61.5 follows at once.

By ** this implies that $Aut(E^*)$ contains a subgroup J say , of order 61.5 , where E^* is the extra-special group of exponent 3 and order 3^{11}. J centralizes $Z(E^*) = Z$ say.

Put $H^* = E^*J$. J acts faithfully and irreducibly on E^*/Z so $H = H^*/Z$ is a group of the type considered in 2.7.1, and we may construct a Fitting class $\mathfrak{F}$ from this group as in 2.7. It is easy to check that $H^* \notin \mathfrak{F}$, while clearly $Z \leq Z(H^*) \cap (H^*)'$ and $H^*/Z \cong H \in \mathfrak{F}$.



We remark that the converse statement to Gaschütz's question does not hold either. For, the non-nilpotent group $G = [Q_8]C_3$ lies in the Fitting class $\mathfrak{F}_2$ of groups with central 2-socle whereas $G/Z(G) \cap G' \cong A_4 \notin \mathfrak{F}_2$.

In the first section of this chapter we develop the theory of pronormality, basing our approach on key theorems of Mann-Alperin and Fischer. This important topic receives scanty and ununified treatment in the literature. We then give our simpler presentation of work of Chambers [14], strengthening his results.

A subgroup H of G is said to be p-normally embedded in G if a Sylow p-subgroup of H is a Sylow p-subgroup of some normal subgroup of G. We shall call a subgroup H of G strongly pronormal in G, written H stropro G if it is p-normally embedded in G for all primes p. The importance of this concept in our study of Fitting classes is the previously mentioned fact that the $\mathfrak{F}$-injectors of G for a Fischer class $\mathfrak{F}$ are strongly pronormal in G.

3.1 Pronormal subgroups

3.1.1 Proposition

a) H pro G and $H \leqslant L \leqslant G \Rightarrow$ H pro L.

b) H pro G and $H \leqslant K \lhd G \Rightarrow G = KN_G(H)$. (Frattini argument)

c) H pro G and $N \lhd G \Rightarrow$ HN pro G, HN/N pro G/N and $N_G(HN) = N_G(H)N$.

d) H pro G and H sn G $\Leftrightarrow H \lhd G$.

e) H pro G $\Rightarrow N_G(H)$ is pronormal and self-normalizing in G.

f) Hall subgroups and maximal subgroups are pronormal.

We omit the elementary proofs.

The following result is very useful for verifying pronormality in actual examples.

3.1.2 Lemma (Gaschütz)

Suppose $N \lhd G$ and $H \leqslant G$ such that HN pro G and H pro $N_G(HN)$.

Then H pro G .

Proof

For arbitrary $x \in G$ we must show H and H^x are conjugate in their join T say. Since HN pro G and $< HN, (TN)^x > = TN$, there exist $t \in T$ and $n \in N$ such that $HN = (HN)^{xtn}$. Hence $xt \in N_G(HN)$, so by our hypothesis there exists $y \in < H, H^{xt} >$ such that $H^y = H^{xt}$ that is $H^{yt^{-1}} = H^x$. But $< H, H^{xt} > \leq < H, H^x >$, so $yt^{-1} \in T$ and the proof is complete. $\square$

3.1.3 THEOREM (Mann [15] , Alperin)

H pro $G \Leftrightarrow$ each Sylow system Σ of G reduces into a unique conjugate of H .

PROOF

We prove the implication $\Rightarrow$ by induction, supposing the result in this direction holds for groups of order less than $|G|$. Suppose the Sylow system Σ of G reduces into H and H^x . Let M be a minimal normal subgroup of G . Let $\bar{\ }$ denote the natural epimorphism $G \to G/M$. If $G_\pi \in \Sigma$ then by assumption $H \cap G_\pi$ is a Hall π-subgroup of H and so $(H \cap G_\pi)M/M$ is a Hall π-subgroup of HM/M . Now $(H \cap G_\pi)M \leq HM \cap G_\pi M$ and hence $HM/M \cap G_\pi M/M$ is a Hall π-subgroup of HM/M . Thus $\bar{\Sigma} \searrow \bar{H}$ and similarly $\bar{\Sigma} \searrow \overline{H^x}$. 3.1.1 c) implies $\bar{H}$, $\overline{H^x}$ pro $\bar{G}$ so by induction $\bar{H} = \overline{H^x}$ and hence $HM = H^xM$. Now H pro G so $H^x = H^m$ for some $m \in M$. By 1.2.7 , $\Sigma \searrow HM$ so $\Sigma \cap HM$ is a Sylow system of HM which reduces into its (by 3.1.1 a)) pronormal subgroups H and H^m . If $HM < G$ we may apply our induction hypothesis to deduce $H = H^m$. Thus we may assume $H < G$ and $HM = G$ for each minimal normal subgroup M of G . It follows at once that G is a primitive group in which H complements the unique minimal normal subgroup M of G . If M is a p-group then a minimal normal subgroup N/M of G/M is a q-group for some $q \neq p$, since M is self-centralizing in G . Now if $G_{p'} \in \Sigma$ our supposition implies $G_{p'} \leq H, H^m$ and so

$N \cap G_{p'} = N \cap H = N \cap H^m$ is a Sylow q-subgroup of N . $N_G(N \cap G_{p'})$ is a maximal subgroup of G complementing N but of course H and H^m normalize $H \cap N = N \cap H^m$, so $H = H^m$ as required.

Now conversely suppose each Sylow system Σ of G reduces into a unique conjugate of $H \leqslant G$. Given $x \in G$, pick Σ so that $\Sigma \searrow \langle H, H^x \rangle$, H (e.g. extend a system of H to a system of $\langle H, H^x \rangle$ to a system of G). There exists $y \in \langle H, H^x \rangle$ such that $\Sigma \searrow (H^x)^y$, so by supposition $H = H^{xy}$ and therefore H pro G

$\square$

3.1.4 THEOREM (Fischer , unpublished)

Suppose the Sylow system Σ of G reduces into each of the pronormal subgroups $H_1, H_2, \ldots, H_n$ of G . (So by 3.1.3 no two of the H_i are distinct and conjugate.) Put $H = \langle H_1, H_2, \ldots, H_n \rangle$. Then H^G is the set of minimal (w.r.t. inclusion) elements of $\mathcal{J} = \{ \langle H_1^{x_1}, H_2^{x_2}, \ldots, H_n^{x_n} \rangle : x_i \in G \}$ and $\Sigma \searrow H$ pro G .

PROOF

Let $T = \langle H_1^{x_1}, H_2^{x_2}, \ldots, H_n^{x_n} \rangle$ be an arbitrary member of $\mathcal{J}$. Suppose $\Sigma^g \searrow T$. Certainly $\Sigma^g \searrow H_i^g$ for all i . $\Sigma^g \cap T$ reduces into some conjugate of $H_i^{x_i}$ in T , say $H_i^{x_i t_i}$ where $t_i \in T$. But then $\Sigma^g \searrow H_i^{x_i t_i}$, so by 3.1.3 $H_i^g = H_i^{x_i t_i}$. Therefore $H^g = \langle H_1^g, H_2^g, \ldots, H_n^g \rangle \leqslant T$ and we have proved the first part of the statement.

Putting $x_1 = x_2 = \ldots = x_n = 1$, we have $H = T$ and so $H^g \leqslant T$ implies $H = H^g$. Therefore $\Sigma^g \searrow H = H^g$ and hence $\Sigma \searrow H$. If also $\Sigma \searrow H^x$, then the above argument yields $H_i \leqslant H^x$ for all i and so $H \leqslant H^x$. Therefore $H = H^x$ and by 3.1.3 H pro G . $\square$

As an immediate corollary of 1.2.7 we have :

3.1.5 Proposition

If the Sylow system Σ of G reduces into subgroups A and B , and $A \perp B$, then $\Sigma \searrow AB$, $A \cap B$.

3.1.6 Proposition

Suppose $A, B < G$ and $A \perp B$. Then there exists a Sylow system Σ of G which reduces into A and B (and hence also into AB and $A \cap B$ by 3.1.5). Moreover if A, B pro G, then AB pro G.

Proof

Let Σ_0 be a Sylow system of AB which reduces into A. Then for some $ab \in AB$, $\Sigma_0{}^{ab} \searrow B$ so $\Sigma_0{}^{a} \searrow B^{b^{-1}} = B$. But $\Sigma_0{}^{a} \searrow A^{a} = A$, so extending $\Sigma_0{}^{a}$ to a system Σ of G we have $\Sigma \searrow A, B$.

If A, B pro G, then by 3.1.4, $AB = \langle A, B \rangle$ pro G. $\square$

3.1.7 Remarks

a) 3.1.3 and 3.1.6 show that no two distinct conjugates of a pronormal subgroup permute.

b) There exists a non-soluble group G with a pair of permutable pronormal subgroups A and B such that AB is not pronormal in G. An example in a paper of Philip Hall cited by Chambers in [14] exhibits this. The simple group $L = PSL(2,7)$ of order $168 = 7.8.3$ has two distinct conjugacy classes of subgroups of order 24 which are interchanged by an involutory automorphism x of L. Let H and H^x be representatives of these classes of subgroups of L. Put $G = \langle L, x \rangle$. H is a product of Sylow 2- and 3-subgroups of L and these are of course pronormal in G. However, as has been stated H and H^x are not conjugate in L, so neither are they conjugate in its subgroup $\langle H, H^x \rangle$.

c) An important consequence of 3.1.3 is that a fixed Sylow system of G determines a unique member of each conjugacy class of pronormal subgroups of G, and in many investigations it suffices to focus attention on these representatives. For instance, suppose A and B are (distinct) pronormal subgroups of G determined in this way, then if conjugates A^x and B^y say, of A and B permute, it follows that A and B themselves permute. For by 3.1.6 $\Sigma^g \searrow A^x$, B^y for some $g \in G$, but then $A^x = A^g$ and $B^y = B^g$ by 3.1.3 , and $A^g \mid B^g \Rightarrow A \mid B$.

d) Of course such a pair of pronormal subgroups A and B do not necessarily permute. For example, with the aid of 3.1.2 it is easy to show that $< (1234) >$ and $< (123) >$ are pronormal subgroups of Σ_4 , and certainly there exists a Sylow system which reduces into them both but they do not permute.

e) If A ; B pro G and $A \mid B$, then it does not follow that $A \cap B$ is pronormal in G . For as in d) $< (1234) >$ pro Σ_4 and $< (12)(34),(13)(24) > \lhd \Sigma_4$, although their intersection $< (13)(24) >$ is subnormal but not normal, and hence by 3.1.1 d) cannot be pronormal.

3.1.8 Proposition

As in 3.1.4 , suppose $\Sigma \searrow H_1, H_2, \ldots, H_n$ pro G and $H = < H_1, H_2, \ldots, H_n >$. Assume further that for some $x_i \in G$ $H_1^{x_1}, H_2^{x_2}, \ldots, H_n^{x_n}$ permute in pairs and let K be their product. Then K and H are conjugate and in particular K pro G by 3.1.4 .

Proof

By remark 3.1.7 c) $H_1, H_2, \ldots, H_n$ permute in pairs, so $H = H_1H_2 \ldots H_n$. If $n = 2$, then by 3.1.6 there exists $g \in G$ such that $\Sigma^g \searrow H_1^{x_1}, H_2^{x_2}$, so as before $H_1^{x_1} = H_1^g$ and $H_2^{x_2} = H_2^g$ by 3.1.3 . Hence $H^g = K$. Now suppose $n > 2$ and proceed by induction on n . So $H_1^{x_1}H_2^{x_2} \ldots H_{n-1}^{x_{n-1}} = (H_1H_2 \ldots H_{n-1})^x$ for some $x \in G$. But now $\Sigma \searrow (H_1H_2 \ldots H_n)$, H_n and $(H_1H_2 \ldots H_{n-1})^x H_n^{x_n} = K$ so again by induction K is conjugate to H as required. $\square$

3.1.9 Remarks

a) It can happen that a group G has no Sylow system which reduces into every member of a given set of pairwise permutable pronormal subgroups of G .

b) If the product H of a set of pronormal subgroups of a group G is actually a subgroup, it is not necessarily pronormal.

3.1.3 has the following simple but useful corollary.

3.1.10 Proposition

If $\Sigma \searrow H$ pro G then $\Sigma \searrow N_G(H)$ and $D = N_G(\Sigma)$ normalizes H.

Proof

Certainly $\Sigma \searrow (N_G(H))^g$ for some $g \in G$.
But $H^g \lhd N_G(H^g) = (N_G(H))^g$, so $\Sigma \searrow H^g$ and hence $H = H^g$ by 3.1.3 giving $\Sigma \searrow N_G(H)$.
If $d \in D$ then $\Sigma = \Sigma^d \searrow H^d$, so again by 3.1.3 $H = H^d$. $\square$

3.2 Strongly pronormal subgroups

Suppose H is a p-normally embedded subgroup of G. So a Sylow p-subgroup H_p of H is a Sylow p-subgroup of some normal subgroup N say, of G (N could clearly be taken to be $< H_p{}^G >$) and is therefore pronormal in G. If X/Y is a p-chief factor of G then N (being normal in G) either covers or avoids it. In case N covers X/Y then so does its Sylow p-subgroup H_p, and hence H covers X/Y. If N avoids X/Y, so too does H_p and therefore H must also. We have shown that H covers or avoids each p-chief factor of G.

Now let H stropro G, so every Sylow subgroup of H is pronormal in G. Suppose further that $K \leq G$ such that, for each p, a Sylow p-subgroup of K is conjugate to a Sylow p-subgroup of H, in G. (We shall call such K locally conjugate to H in G.) By their orders H and K are minimal elements of $\{ < H_{p_1}{}^{x_1}, \dots, H_{p_n}{}^{x_n} > : x_i \in G \}$ where $\{ p_1, \dots, p_n \}$ are the primes dividing $|H|$. Thus by 3.1.4 H and K are conjugate pronormal subgroups of G.
Again suppose H stropro G and this time that K is a subgroup of G which covers each chief factor of G which H covers and avoids

each chief factor which H avoids. So, for each p, K_p covers all the p-chief factors of G below $< H_p^G >$ and avoids those above. It follows that K_p is a Sylow p-subgroup of $< H_p^G >$ for each p, and hence that H and K are locally conjugate in G. From what we have seen above, H and K are actually conjugate in G.

Collecting these results and adding some easily proved facts we have :

3.2.1 Proposition (Chambers [14])

If H stropro G and $K \leqslant G$ then

a) H covers or avoids each chief factor of G.

b) H pro G

c) K locally conjugate to H in G $\Rightarrow$ K conjugate to H in G

d) K covers or avoids chief factors in the same manner as H $\Rightarrow$ K and H are conjugate in G

e) $H \leqslant T \leqslant G \Rightarrow H$ stropro $\cancel{G}$ T

f) $N \lhd G \Rightarrow HN$ stropro G and HN/N stropro G/N

g) $N \lhd G$, $N \leqslant T \leqslant G$ and T/N stropro $G/N \Rightarrow T$ stropro G.

3.2.2 Remarks

a) The example in 3.1.7 b) shows that H stropro G does not imply H pro G for non-soluble groups, so our terminology is misleading outside $\mathcal{S}$ and is best confined to this class.

b) As we remarked in 3.1.7 d), $< (1234) >$ pro Σ_4 but this 2-subgroup is not strongly pronormal in Σ_4.

c) The strongly pronormal p-subgroups of a group G are exactly the Sylow p-subgroups of the normal subgroups of G.

3.2.3 Lemma

Let P_1, P_2 stropro G such that $P_1, P_2 \leqslant G_p$ a Sylow p-subgroup of G. Then $P_1 \perp P_2$ and P_1P_2, $P_1 \cap P_2$ stropro G.

Proof

Let $N_i = < P_i^G >$ so $P_i = G_p \cap N_i$ for $i = 1,2$. Put $N = N_1N_2$.

Then $P_1 \cap P_2 = G_p \cap (N_1 \cap N_2)$ so $P_1 \cap P_2$ stropro G .

$|N \cap G_p| = |N|_p = |N_1|_p |N_2|_p / |N_1 \cap N_2|_p$

$= |N_1 \cap G_p| |N_2 \cap G_p| / |N_1 \cap N_2 \cap G_p| = |P_1| |P_2| / |P_1 \cap P_2|$

$= |P_1 P_2|$

But $P_1 P_2 \subseteq N \cap G_p$, therefore $P_1 \perp P_2$ and $P_1 P_2$ stropro G □

3.2.4 Lemma

Let Σ be a Sylow system of G . Suppose P, Q stropro G with $P \leqslant G_p \in \Sigma$ and $Q \leqslant G_q \in \Sigma$, and $p \nmid q$. Then $P \perp Q$ and PQ stropro G .

Proof

The hypotheses imply P is a Sylow p-subgroup of $< P^G > G_{p'}$, which has p-power index in G , and Q is a Sylow q-subgroup of $< Q^G > G_{q'}$, which has q-power index in G , where $G_{p'}, G_{q'} \in \Sigma$. Hence $G_{\{p,q\}} \cap < P^G > G_{p'} \cap < Q^G > G_{q'}$ is a subgroup of G of order $|P||Q|$ which contains the subset PQ of G , where $G_{\{p,q\}} \in \Sigma$. Therefore $P \perp Q$ and PQ stropro G follows at once . □

3.2.5 THEOREM

Suppose the Sylow system Σ of G reduces into the strongly pronormal subgroups A and B of G . Then $A \perp B$ and AB , $A \cap B$ stropro G .

PROOF

Suppose $p_1, p_2, \ldots, p_n$ are the primes dividing $|G|$. For $i = 1,\ldots,n$ let $A_{p_i} = A \cap G_{p_i}$, $B_{p_i} = B \cap G_{p_i}$ where $G_{p_i} \in \Sigma$. By 3.2.3 and 3.2.4 the strongly pronormal subgroups $A_{p_1},\ldots,A_{p_n}$, $B_{p_1},\ldots,B_{p_n}$ of G are pairwise permutable and (by order) $A_{p_i} B_{p_i}$ is a Sylow p_i-subgroup of their product AB and $A_{p_i} B_{p_i}$ stropro G. So AB stropro G . By 3.1.5 $G_p \cap (A \cap B) = A_p \cap B_p$ is a Sylow p-subgroup of $A \cap B$, and by 3.2.3 $A_p \cap B_p$ stropro G.. Thus $A \cap B$ stropro G . □

Anticipating 3.3.1 we have an omnibus corollary :

3.2.6 Corollary

Let Σ be a Sylow system of G . Then the set of strongly pronormal subgroups of G into which Σ reduces is a lattice where join is product, which includes all normal subgroups of G and an injector for each Fischer class.

3.3 Fitting classes with strongly pronormal injectors

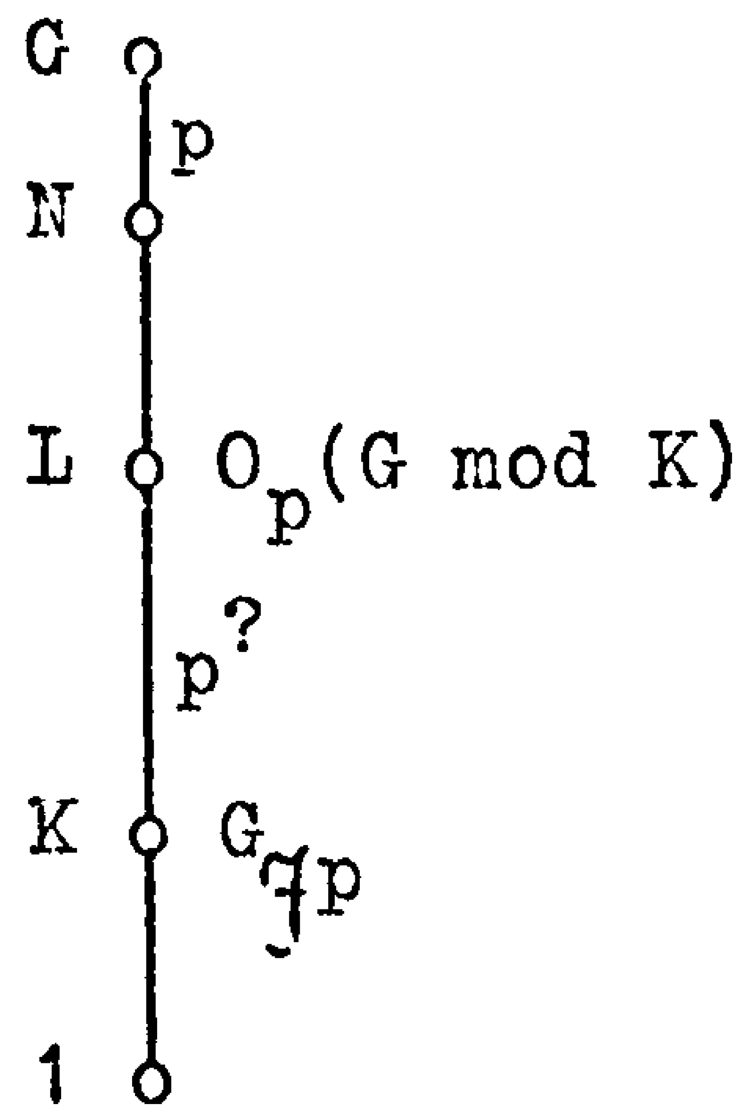
3.3.1 THEOREM (Fischer)

If $\mathfrak{F}$ is a Fischer class and V is an $\mathfrak{F}$-injector of G then V stropro G .

PROOF

We choose and fix a prime p , and show that the hypotheses imply V is p-normally embedded in G . Let G be a group of minimal order for which this is false. Put $K = G_{\mathfrak{F}^p}$. By definition of $\mathfrak{F}^p$, the $\mathfrak{F}$-injector $V \cap K$ of K contains a Sylow p-subgroup of K and hence $K < G$. Let $L = O_p(G \bmod K)$. Again by definition of $\mathfrak{F}^p$, we have $O_{p'}(G/K) = 1$, so $L/K = F(G/K) \neq 1$.

First suppose $L = G$, so G/K is a p-group and hence KV sn G . But of course V pro G , so KV pro G by 3.1.1 c) . Then 3.1.1 d) shows $KV \lhd G$ and it follows that V contains a Sylow p-subgroup of KV against the assumption that G is a counterexample. Therefore $L \neq G$ and we let N be a maximal normal subgroup of G containing L .



If V_p is a Sylow p-subgroup of V, then $V_p \cap N$ is a Sylow p-subgroup of $V \cap N$ which is an $\mathfrak{F}$-injector of N. By the minimality of G, $V_p \cap N$ is a Sylow p-subgroup of $< (V_p \cap N)^N >$. Now it is clear that the Sylow p-subgroups of the $\mathfrak{F}$-injectors of an arbitrary group form a characteristic conjugacy class from which it follows that $(V_p \cap N)^N = (V_p \cap N)^G$. By assumption V is not p-normally embedded in G, so $V_p \nleq N$.

The fact that $\mathfrak{F}$ is a Fischer class implies $V_p(K \cap V) \in \mathfrak{F}$. By 1.1.4 a) $V_p(K \cap V)$ is an $\mathfrak{F}$-injector of V_pK and hence $V_pK \in \mathfrak{F}^p$. Now V_pK sn V_pL because V_pL/K is a p-group. So $V_pK \leq (V_pL)_{\mathfrak{F}^p}$ which implies $[V_pK , L] \leq (V_pL)_{\mathfrak{F}^p} \cap L = L_{\mathfrak{F}^p} = L \cap G_{\mathfrak{F}^p} = K$. This means that V_p centralizes $L/K = F(G/K)$ which forces $V_p \leq L$ as usual. This final contradiction shows no such G exists. So $\mathfrak{F}$-injectors are p-normally embedded for all primes p and the theorem is established. □

As an immediate consequence of 3.2.1 d) we have :

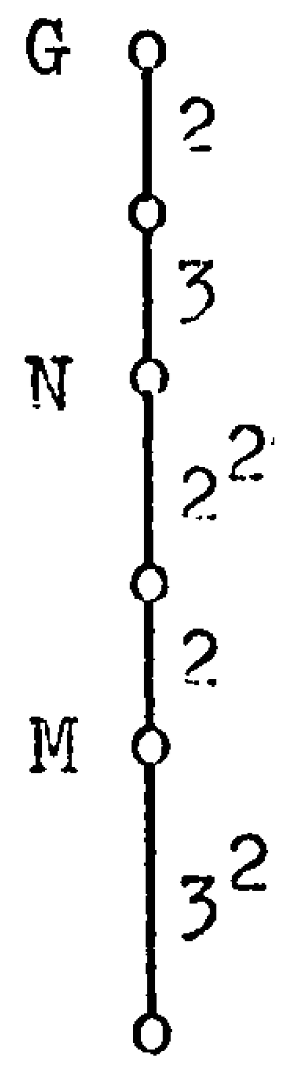
<u>3.3.2 Corollary</u> (Chambers [14])

The $\mathfrak{F}$-injectors of a group G for a Fischer class $\mathfrak{F}$ are characterized by their covering and avoidance of the chief factors of G.

Unfortunately this does not hold for all Fitting classes.

<u>3.3.3 Example</u>

Let $M \cong C_3 \times C_3$ be the natural module (under right action) for $GL(2,3) \doteq H$ say, and put $G = MH$.



$H \cong [Q_8]\Sigma_3$ and G has the unique chief series shown. Let $A = < \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}2&0\\0&1\end{bmatrix} >$

and $B = < \begin{bmatrix}1&1\\0&1\end{bmatrix} \begin{bmatrix}1&0\\0&2\end{bmatrix} >$, subgroups of H.

$\Sigma_3 \cong A$, B and both complement N in G.

$C_M(A) = < (0,1) >$ is the unique minimal normal subgroup of MA whereas $C_M(B)$ is trivial.

Therefore $MA \in \mathfrak{F}_3$ but $MB \notin \mathfrak{F}_3$ where $\mathfrak{F}_3$ is the class of groups with central 3-socle. (c.f. Example 2.3.2)

Using 1.1.4 b) we see that MA is an $\mathcal{Y}_2$-injector of G, whereas MB certainly is not. Clearly MA and MB cover (and avoid) the same chief factors of G in the series shown and these are the only chief factors G has. Notice too that MA is not 2-normally embedded in G.

The following theorem is little more than Hartley's lemma 3 of [4]. There he uses it to derive 3.3.1 and here we shall give some further corollaries of this powerful result.

<u>3.3.4 THEOREM</u> (Hartley)

Suppose P is a p-subgroup of G such that

* $\quad N \lhd G \Rightarrow G = N_G(P \cap N)N$.

Then the following are equivalent

a) P stropro G

b) P pro G

c) $P \lhd G_p$ for some Sylow p-subgroup G_p of G

d) P centralizes each p-chief factor of G which it avoids.

<u>PROOF</u>

The implications a) $\Rightarrow$b) $\Rightarrow$c) are clear in view of 3.1.1 and 3.2.1 b). Suppose c) holds and $P \lhd G_p$.

If H/K is a p-chief factor of G avoided by P, then

$[P\ ,\ H \cap G_p] \leqslant P \cap H \cap G_p = P \cap H \leqslant K$. But $H = K(H \cap G_p)$ and so P centralizes H/K, therefore d) holds.

Observe that the condition * implies that P covers or avoids each p-chief factor of G. For if H/K is such a factor, we have $K(P \cap H) \lhd G$ by * , which yields the statement.

Now suppose d) holds. We show that, for any $X \lhd G$, the hypotheses * and d) carry over to $P \cap X$, and then deduce a) by induction on $|P|$. Thus by *, $N \lhd G \Rightarrow G = N_G(P \cap (X \cap N))(X \cap N)$ so certainly $G = N_G((P \cap X) \cap N)N$ and therefore $P \cap X$ satisfies condition * .

By our observation above $P \cap X$ must cover or avoid each p-chief factor H/K of G. Suppose $P \cap X$ avoids H/K. In the case that P also avoids H/K, then P (and hence also $P \cap X$) centralizes H/K since we are assuming d) holds. Therefore assume further that P covers H/K. Now $[P \cap H\ ,\ P \cap X] \leq P \cap X \cap H \leq K$, so again $P \cap X$ centralizes H/K, and $P \cap X$ satisfies condition d).

Let $T = < P^G >$, then by * $T = < P^{N_G(P)T} > = < P^T >$. Thus T is generated by p-elements so, putting $L = T^{\mathfrak{N}} \lhd G$, T/L is a non-trivial p-group. Then since PL sn $T = < P^T >$, it follows that $PL = T$. So $P \cap L < P$ and therefore by our induction hypothesis $P \cap L$ is a Sylow p-subgroup of $< (P \cap L)^G > = R$ say.

P avoids all the p-chief factors of G between R and L , so by supposition centralizes them. It follows that $T = < P^G >$ centralizes them too, and hence that L/R is p-nilpotent. But of course L/R has no non-trivial p-quotient since T/L is a p-group, so L/R must be a p'-group. This means $P \cap L$ is a Sylow p-subgroup of L and from $PL = T$ it follows that P is a Sylow p-subgroup of T, so P stropro G and a) holds. $\square$

Subgroups P with property * arise very naturally in our theory. For suppose $\mathfrak{F}$ and $\mathfrak{H}$ are arbitrary Fitting classes and G a group. Let P be a Sylow p-subgroup of an $\mathfrak{H}$-injector W of an $\mathfrak{F}$-injector V of G. Since Sylow subgroups and injectors for a Fitting class form characteristic conjugacy classes, it follows easily that

P^G is a characteristic conjugacy class of G. (1)

Furthermore if $N \lhd G$, then

$P \cap N$ is a Sylow p-subgroup of an $\mathfrak{H}$-injector $W \cap N$ of an $\mathfrak{F}$-injector $V \cap N$ of N . (2)

For certainly $V \cap N$ is an $\mathfrak{F}$-injector of N , but then $V \cap N \lhd V$ so $W \cap (V \cap N) = W \cap N$ is an $\mathfrak{H}$-injector of $V \cap N$. Also $W \cap N \lhd W$ so $P \cap (W \cap N) = P \cap N$ is a Sylow p-subgroup of $W \cap N$.

Combining (1) and (2) we see that for any $N \lhd G$, $(P \cap N)^N$ is a characteristic conjugacy class of N so by the Frattini argument $G = N_G(P \cap N)N$ and P does indeed satisfy condition * .

In the same vein, it is not difficult to extend the given arguments to show that, for any Fitting classes $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ a Sylow subgroup of an $\mathfrak{F}_1$-injector of an $\mathfrak{F}_2$-injector of an ... ... of an $\mathfrak{F}_n$-injector of a group G also satisfies * . In fact we can even vary the process by taking the $\mathfrak{F}_i$-radical instead of an $\mathfrak{F}_i$-injector at the ith stage, for various i , and still obtain a group with property * . Of course the very case we have dealt with in (1) and (2) is itself an extension of the observation that a Sylow subgroup of an $\mathfrak{F}$-injector of a group G has property * .

In view of this final remark, 3.3.4 immediately implies :

3.3.5 Corollary

If P is a Sylow p-subgroup of an $\mathfrak{F}$-injector of a group G and P pro G then P stropro G .

3.3.6 Corollary

Let W be an $\mathfrak{H}$-injector of an $\mathfrak{F}$-injector V of a group G (where $\mathfrak{F}$ and $\mathfrak{H}$ are arbitrary Fitting classes). If V is p-normally embedded in G and W is p-normally embedded in V , then W is p-normally embedded in G .

Proof

Let $W_p \leq V_p \leq G_p$ be Sylow subgroups of W, V, and G respectively. By hypothesis V_p is a Sylow p-subgroup of $\langle V_p{}^G \rangle = K$ say. By our remarks above $\langle W_p{}^G \rangle = \langle W_p{}^K \rangle$ since W_p satisfies * in G. Also W_p is a Sylow p-subgroup of the $\mathfrak{H}$-injector $W \cap K$ of the $\mathfrak{F}$-injector $V \cap K$ of K , so W_p satisfies * in the group K . By hypothesis W_p stropro V so in particular W_p pro V_p and hence

$W_p \lhd V_p$, which we already know is a Sylow subgroup of K . By 3.3.4 it follows that W_p stropro K and from $< W_p^{\,G} > = < W_p^{\,K} >$ we deduce the result. $\square$

3.3.7 Corollary

If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are both Fitting classes for which the injectors of an arbitrary group are strongly pronormal, then the same is true for $\mathfrak{F}_1\mathfrak{F}_2$.

Proof

We use the terminology of 2.6.4 . Suppose $T = W \cap G_{\mathfrak{F}_1\pi_2}$, where W is an $\mathfrak{F}_1$-injector of G . By hypothesis W stropro G , so using 3.2.5 T stropro G and hence TG_{π_2} stropro G . It is clear from 2.6.4 that TG_{π_2} is an $\mathfrak{F}_1\mathfrak{S}_{\pi_2}$-injector of G . By definition V/T is an $\mathfrak{F}_2$-injector of TG_{π_2}/T , so by hypothesis and 3.2.1 g) V stropro TG_{π_2} . Certainly V is an $\mathfrak{F}_1\mathfrak{F}_2$-injector of TG_{π_2} . For each $p \mid |G|$, we have established the hypotheses of 3.3.6 , and its conclusion shows V stropro G as desired. $\square$

3.3.8 Remarks

a) Suppose $\mathfrak{F}$ is a Fitting class of characteristic π such that if V is an $\mathfrak{F}$-injector of a π-group H , then V stropro H . We show this actually implies that an $\mathfrak{F}$-injector of an arbitrary group G is strongly pronormal in G. For as in 1.1.6 d) an $\mathfrak{F}$-injector V of an $\mathfrak{S}_\pi$-injector G_π of G is an $\mathfrak{F}$-injector of G . By supposition V stropro G_π and clearly G_π stropro G , therefore V stropro G by 3.3.6 . For example, if $\mathfrak{X}$ is a normal Fitting class and π is a set of primes, an $\mathfrak{X}_\pi$-injector of a π-group H is of course an $\mathfrak{X}$-injector of H and is therefore normal and hence strongly pronormal in H . So it follows that an $\mathfrak{X}_\pi$-injector of any group G is strongly pronormal in G .

b) Normal Fitting classes are not necessarily Fischer classes, so this last observation is not immediate from 3.3.1 . For instance, the normal Fitting class described in 2.4.2 for the prime 3 contains $G = [C_3 \times C_3]C_2$ (where each involution inverts each 3-element) but not the subgroup Σ_3 of G . However this subgroup is an extension of a normal subgroup of G by a 2-group, and therefore lies in each Fischer class to which G belongs.

c) The strong pronormality of the injectors for a Fischer class is exploited in work of Makan [16] . There he shows that in each group G an injector for a Fischer class permutes with a prefrattini subgroup, and he establishes covering and avoidance properties of their product.

In the light of the previous chapter we now investigate further the class $\mathfrak{F}^{\pi}$ with a view to substantiating remark 2.6.2 c) . Extending our terminology we shall call a Fitting class $\mathfrak{F}$ strongly pronormal if the $\mathfrak{F}$-injectors of each group G are strongly pronormal in G . Further we say $\mathfrak{F}$ is permutable if an $\mathfrak{F}$-injector V of an arbitrary group G permutes with each member of any Sylow system Σ of G which reduces into V . We write $V \perp \Sigma$ in this case. (By remark 3.1.7 c) this holds if, for each π, V permutes with some Hall π-subgroup of G.)

In the second section of this chapter we examine an interesting lemma of Fischer, and establish conditions equivalent to the permutability of a Fitting class, involving system normalizers.

4.1 Permutable Fitting classes

4.1.1 Proposition

If $\mathfrak{F}$ is strongly pronormal (for example a Fischer class or normal Fitting class) then $\mathfrak{F}$ is permutable.

Proof

Suppose the Sylow system Σ of G reduces into an $\mathfrak{F}$-injector V of G. Certainly each $G_{\pi} \in \Sigma$ is strongly pronormal in G and clearly $\Sigma \searrow G_{\pi}$. So if V stropro G , then by 3.2.3 $V \perp G_{\pi}$ and the result follows. $\square$

4.1.2 Proposition

Suppose $\mathfrak{F}$ is a permutable Fitting class and G a group with Sylow system Σ . Let V be the unique $\mathfrak{F}$-injector of G into which Σ reduces. Then

a) $VG_{\pi'}$ is an $\mathfrak{F}^{\pi}$-injector of G and $\mathfrak{F}^{\pi}$ is permutable.

b) $VG_{\pi_1'} \cap VG_{\pi_2'} = VG_{(\pi_1 \cap \pi_2)'}$ is an $\mathfrak{F}^{\pi_1 \cup \pi_2}$-injector of G and $\mathfrak{F}^{\pi_1 \cup \pi_2} = \mathfrak{F}^{\pi_1} \cap \mathfrak{F}^{\pi_2}$.

c) $VG_{\pi_1'}VG_{\pi_2'} = VG_{(\pi_1\cap\pi_2)'}$ is an $\mathfrak{F}^{\pi_1\cap\pi_2}$-injector of G

and $(\mathfrak{F}^{\pi_1})^{\pi_2} = \mathfrak{F}^{\pi_1\cap\pi_2} = (\mathfrak{F}^{\pi_2})^{\pi_1}$.

(Here $G_\omega \in \Sigma$ for each set of primes ω.)

Proof

a) The first part is 2.6.3 . By 3.1.5 $\Sigma \searrow VG_{\pi'}$, so this is the unique $\mathfrak{F}^{\pi}$-injector of G into which Σ reduces. Clearly $(VG_{\pi'})G_\omega = VG_{\pi'\cup\omega}$, which is a subgroup of G by the permutability of $\mathfrak{F}$, so we have shown $\mathfrak{F}^{\pi}$ is permutable.

b) $(\pi_1'\cap\pi_2') = (\pi_1\cup\pi_2)'$ so certainly $VG_{\pi_1'} \cap VG_{\pi_2'} \geq VG_{(\pi_1\cup\pi_2)'}$. On the other hand it is easily seen that $|VG_{(\pi_1\cup\pi_2)'}| = \gcd(|VG_{\pi_1'}| , |VG_{\pi_2'}|)$, so equality of the subgroups follows. Bearing a) in mind the rest is obvious.

c) $(\pi_1'\cup\pi_2') = (\pi_1\cap\pi_2)'$ so by a) $VG_{\pi_1'}VG_{\pi_2'} = VG_{(\pi_1\cap\pi_2)'}$ is an $\mathfrak{F}^{\pi_1\cap\pi_2}$-injector of G . Also by a) $\mathfrak{F}^{\pi_1}$ is permutable so the unique $(\mathfrak{F}^{\pi_1})^{\pi_2}$-injector of G into which Σ reduces = (the unique $\mathfrak{F}^{\pi_1}$-injector of G into which Σ reduces)$G_{\pi_2'}$ = $VG_{\pi_1'}G_{\pi_2'} = VG_{(\pi_1\cap\pi_2)'}$ which is an $\mathfrak{F}^{\pi_1\cap\pi_2}$-injector of G. Coincidence of injectors in all groups clearly implies the coincidence of the classes, so by symmetry the whole statement follows. □

4.1.3 Proposition

$\mathfrak{F}_1$ and $\mathfrak{F}_2$ are permutable Fitting classes implies $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2$ is permutable.

Proof

Let Σ be a fixed Sylow system of the group G . Our proof is another illustration of the principle mentioned in 3.1.7 c) . Let T be the unique $\mathfrak{F}_1$-injector of $K = G_{\mathfrak{F}_1\pi_2}$ into which Σ reduces. By 3.1.10 and the Frattini argument T is normalized by $G_{\pi_2} \in \Sigma$. Theorem 2.6.4 said that if V/T is an $\mathfrak{F}_2$-injector of TG_{π_2}/T , then

V is an $\mathfrak{F}$-injector of G and since $\Sigma \searrow TG_{\pi_2}$ (by 3.1.5), V may be taken to be the unique $\mathfrak{F}$-injector of G into which Σ reduces.

For any π we must show that $V \perp G_\pi \subset \Sigma$. By 2.6.5 VK/K is an $\mathfrak{F}_2$-injector of G/K and the system $\bar{\Sigma} = \{ \ldots, G_\pi K/K, \ldots \}$ of G/K reduces into VK/K , so by the assumed permutability of $\mathfrak{F}_2$ $VK/K \perp G_\pi K/K$, that is VKG_π is a subgroup of G . $\mathfrak{F}_1$ is also permutable by hypothesis so $V \cap K = T \perp G_\pi \cap K = K_\pi$ say. Now by 3.1.6 TK_π pro G and has $(\pi_2 \cup \pi)'$-index in K , so by the Frattini argument applied to VKG_π, TK_π is normalized by a Hall $\pi_2 \cup \pi$-subgroup of VKG_π , which by 3.1.10 again may be taken to be $VKG_\pi \cap G_{\pi_2} G_\pi$ since $\Sigma \searrow VKG_\pi$. By the modular law

$$
\begin{aligned}
TK_\pi(VKG_\pi \cap G_{\pi_2} G_\pi) &= VKG_\pi \cap TG_{\pi_2} G_\pi \\
&= VG_\pi(K \cap TG_{\pi_2} G_\pi) \\
&= VG_\pi T(K \cap G_{\pi_2} G_\pi) \\
&= VG_\pi T(K \cap G_{\pi_2})(K \cap G_\pi) \\
&= VG_\pi \qquad \text{(since } K \cap G_{\pi_2} \leq T \leq V) \qquad \square
\end{aligned}
$$

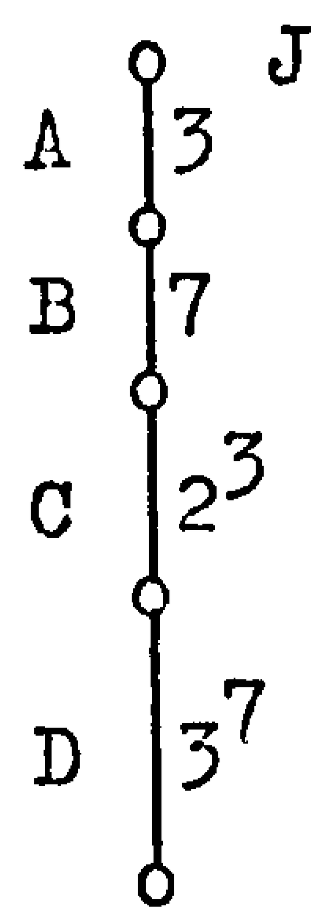
4.1.4 Remarks

a) 2.6.1 shows that for arbitrary $\mathfrak{F}$, an $\mathfrak{F}^p$-injector V of a group G has p-power index in G , so by an obvious order argument $V \perp \Sigma$ if $\Sigma \searrow V$ (each product is either V or G). Thus $\mathfrak{F}^p$ is a permutable class. With this in mind we can use example 3.3.3 to deny the converse 4.1.1 , at the same time showing that the injectors for a permutable Fitting class are not necessarily characterized by their covering and avoidance properties. For (in the notation of the example) MA is an $(\mathfrak{F}_3)^{\{2\}}$-injector of G by 1.1.4 b) while MB is not, and as we saw MA is not 2-normally embedded.

b) $\mathfrak{F}^\pi$ is clearly the set of all those groups which are the product of an $\mathfrak{F}$-injector and a Hall π'-subgroup, and we have just seen that when $\mathfrak{F}$ is strongly pronormal, an $\mathfrak{F}^\pi$-injector of

an arbitrary group is such a product too. Thus it is reasonable to conjecture that the set of groups which are the product of an $\mathfrak{F}_1$-injector and an $\mathfrak{F}_2$-injector (for given classes $\mathfrak{F}_1$ and $\mathfrak{F}_2$) form a Fitting class $\mathfrak{X}$ say, and hopefully an $\mathfrak{X}$-injector of any group G is also such a product. (By 3.2.6 we may ensure that this product always exists by taking $\mathfrak{F}_1$ and $\mathfrak{F}_2$ to be strongly pronormal classes.) Our expectations are strengthened by the elegant dual result (for Schunck classes) due to Blessenohl (Theorem 7.3.8 of [7]). However the following example shows that this 'join' $\mathfrak{X}$ of Fitting classes $\mathfrak{F}_1$ and $\mathfrak{F}_2$ need not be Fitting. Let $\mathfrak{F}_1$ be the normal class described in 2.4.2 for the prime 3 and $\mathfrak{F}_2$ that for the prime 5 .
Put $G = D_6 \times D_{10}$, then $G_{\mathfrak{F}_1} = C_3 \times D_{10}$ and $G_{\mathfrak{F}_2} = D_6 \times C_5$, so $G = G_{\mathfrak{F}_1} G_{\mathfrak{F}_2}$. But G has a normal subgroup $N \cong [C_3 \times C_5]C_2$ (where each involution inverts each 2'-element) and $N_{\mathfrak{F}_1} = C_3 \times C_5 = N_{\mathfrak{F}_2}$, so $N \neq N_{\mathfrak{F}_1} N_{\mathfrak{F}_2}$.

c) It seems to be very difficult in general to decide whether a non-strongly pronormal Fitting class is permutable or not and in particular we have been unable to determine whether the class $\mathfrak{F}_p$ of groups with central p-socle enjoys this property, though we suspect it does not. Thus to show that there exist classes which are not permutable we fall back on the work of Dark, and our example is a simple ammendment of his.



Dark constructs (by methods akin to 1.2.4) a group ABCD = J say with a unique chief series, in which each of the subgroups A, B, C, and D covers the chief factor shown, and A normalizes B, C, and D , B normalizes C and D , and C normalizes D . We now examine J in more detail. (Indeed the arguments apply to any group with the stated properties of J .)

In Dark's notation, $A = \langle a' \rangle$. B does not centralize D , and since 3 has order 6 mod 7 it follows that $D = [D , B] \times C_D(B)$ is the decomposition of D into an irreducible component of order 3^6 and a trivial component of order 3 , under the action of B . Let $C_D(B) = \langle d \rangle$. A normalizes B and D , so it normalizes both $[D , B]$ and $\langle d \rangle$, hence a' and d commute. Let $a = a'd$.

We show $\langle a \rangle$ normalizes no Sylow 2-subgroup of J. (1)

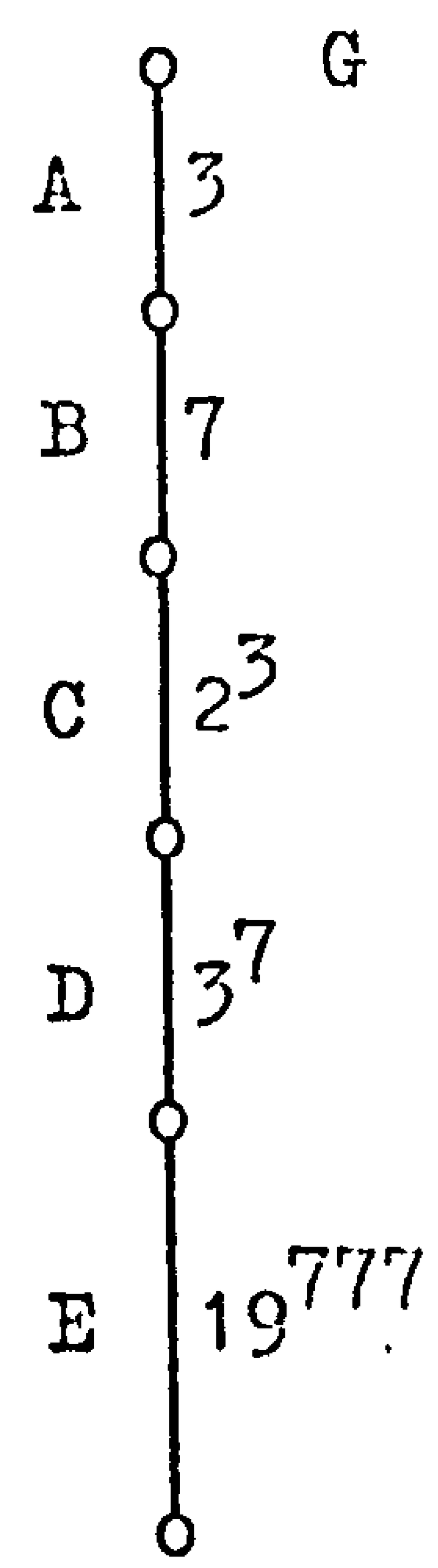
Suppose it does, then we may assume a normalizes C^{d_o} where $d_o \in D$. Then $a^{d_o^{-1}} = (a'd)^{d_o^{-1}}$ normalizes C . But a' normalizes C , so $(a'd)^{d_o^{-1}}(a')^{-1} = d_o a' d_o^{-1}(a')^{-1}d = [d_o^{-1},(a')^{-1}]d = x$ say , normalizes C too. Now A centralizes $D/[D , B]$, so $[d_o^{-1},(a')^{-1}] \in [D , B]$ and hence $x \neq 1$. Then $[C , x] \leqslant C \cap D$, so x is a non-trivial fixed point of C in D . Now $C_D(C)$ is normalized by A, B, C and D , that is by J , and D is a minimal normal subgroup of J , so C centralizes D , a contradiction.

Notice that $B\langle a' \rangle$ and $B\langle a \rangle$ are not conjugate in J. (2)

For if they are, it follows from Sylow's theorem that $\langle a' \rangle$ and $\langle a \rangle$ are conjugate in J and hence that $\langle a \rangle$ normalizes a Sylow 2-subgroup of J , against (1) .

(Notice too that we have filled in some of the details of 1.1.6.)

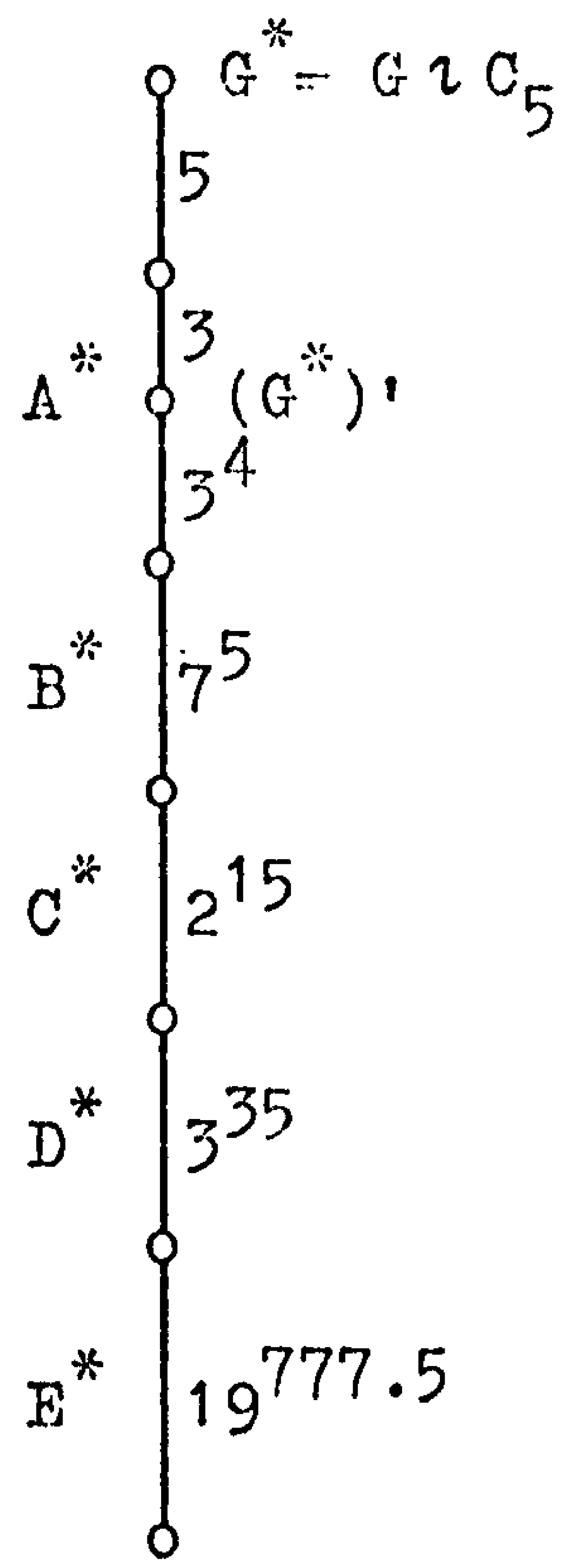
Dark then constructs a metabelian group E of order 19^{777} on which J acts, and writes G for the corresponding split extension. His Fitting class $\mathfrak{X}$ is constructed from the group $H = \langle a \rangle BE$ (the construction we imitated in chapter 2) and satisfies $\mathfrak{S}_{3'} \subseteq \mathfrak{X}$, $H \in \mathfrak{X}$, DE, $ABCE \notin \mathfrak{X}$. (Notice that $J = ABCD$ is a primitive group , so any complement in G of 'the chief factor D' is a conjugate of $ABCE$ in G .)



In view of these facts (and 1.1.4 b) of course), it is clear that BCE is an $\mathfrak{X}$-injector of G , and the importance of Dark's work is that he proves H is a Fischer $\mathfrak{X}$-subgroup of G .

Now let $\mathfrak{Y} = \mathfrak{X} \cap \mathfrak{S}_{19}\mathfrak{S}_{7}\mathfrak{S}_{3}$. Further application of 1.1.4 b) shows that H is an $\mathfrak{Y}$-injector of G and obviously the Sylow system generated by {E, AD, C, B} reduces into H. However H cannot permute with C since by (1) above, $< a >$ fails to normalize C . Thus $\mathfrak{Y}$ is not a permutable Fitting class. (Observe that ABE $\notin \mathfrak{Y}$, for if not ABE is (by 1.1.4 b)) an $\mathfrak{Y}$-injector of G and therefore conjugate to H against (2). So in particular ABE $= < a' >$ BE $\nsim < a >$ BE = H .)

d) More gymnastics with Dark's group show that $(\mathfrak{Y}^{\pi_1})^{\pi_2} = (\mathfrak{Y}^{\pi_2})^{\pi_1}$ can be false when $\mathfrak{Y}$ is not permutable. This time we let $\mathfrak{Y} = \mathfrak{X} \cap \mathfrak{S}_{19}\mathfrak{S}_{7}\mathfrak{S}_{5}\mathfrak{S}_{3}$ and consider $G \wr C_5 = G^*$ say, where $\mathfrak{X}$ and G are as above. Let $B^* = B \times B \times B \times B \times B$ etc.



Since 3 has order 4 mod 5 , and with the aid of the lemma in [17] , it follows that the factors of G^* above E^* as shown, are chief factors. Remembering that injectors cover or avoid chief factors it is not difficult to verify that $E^*C^*B^*C_5$ is an $\mathfrak{Y}^{\{3,5\}}$-injector of G^* and hence $G^* \in (\mathfrak{Y}^{\{3,5\}})^{\{2,5\}}$. Also $E^*D^*B^*A^*$ is an $\mathfrak{Y}^{\{2,5\}}$-injector of G^* which fails to cover the 5-chief factor, so $G^* \notin (\mathfrak{Y}^{\{2,5\}})^{\{3,5\}}$.

e) Remark 3.3.8 a) said that a Fitting class of characteristic π is strongly pronormal if it is 'strongly pronormal in $\mathfrak{S}_{\pi}$ '. We conjecture that similarly, such a Fitting class is permutable if it is permutable in $\mathcal{S}_\pi$. Unfortunately the equivalences established in the following section seem to be of little help in proving this.

4.2 Fischer's lemma and conditions equivalent to permutability

Suppose now that $\mathfrak{F}$ is a Fitting class and G a group with a normal subgroup N such that $G/N \in \mathfrak{N}$, and $N_{\mathfrak{F}} = F$ say, is $\mathfrak{F}$-maximal in N . F char $N \trianglelefteq G$ so $F \triangleleft G$. Let V be an $\mathfrak{F}$-maximal subgroup of G containing F .

By 1.1.4 a) V is an $\mathfrak{F}$-injector of G and therefore V pro G . (1)

As a corollary of the proof of Hartley's lemma 1.1.3 we have $V = (FC)_{\mathfrak{F}}$ for some Carter subgroup C of G . (2)

Our first aim is a new proof of the following sharper version of (2) , known to Fischer, whose long argument appears in [2] .

4.2.1 THEOREM (Fischer)

$V = (FD)_{\mathfrak{F}}$ for some system normalizer D of G . (Specifically this holds for the normalizer of any system which reduces into V.)

PROOF

We need the following lemma, inspired by Fischer's approach.

4.2.2 Lemma

Let P be a pronormal p-subgroup of a group H which complements $K \triangleleft H$. Then P normalizes each p-complement of K .

Proof

If $J \triangleleft H$ and $J \leq K$, then PJ/J is a pronormal (by 3.1.1 c)) complement of $K/J \triangleleft H/J$, so the hypotheses carry over to the quotient H/J , and we argue by induction on $|H|$. If $O_{p'}(K) \neq 1$, then by the induction hypothesis, $PO_{p'}(K)/O_{p'}(K)$ normalizes each p-complement of $K/O_{p'}(K)$, which yields the desired conclusion. If $O_{p'}(K) \neq 1$, then $1 \neq O_p(K) = K_{\mathfrak{N}}$ the Fitting

subgroup of K . Now from our hypothesis P pro $O_p(K)P$, but obviously P sn $O_p(K)P$, therefore $P \lhd O_p(K)P$ by 3.1.1 d) . This implies that $[O_p(K), P] \leqslant P \cap K = 1$. $K_{\mathfrak{N}}$ contains its centralizer in K so it follows that
$P \lhd Z(O_p(K)) \times P = C_H(O_p(K)) \lhd H$. Again P pro H implies $P \lhd H$, so $H = K \times P$ and the result obviously holds in this case. $\square$

Suppose the Sylow system $\Sigma = \{ G_p, .., G_\pi, .., G_{p'}, .. \}$ of G reduces into V . Whenever $\Sigma \searrow T \leqslant G$ we write $T_\pi = T \cap G_\pi$ for each $G_\pi \in \Sigma$. Let $\overline{G} = G/F$, then $\overline{\Sigma}$ (the images of the members of Σ under the natural epimorphism) reduces into $\overline{V}$, and $\overline{G}_p \cap \overline{V} = \overline{G_p \cap V} = \overline{V}_p$ etc. is easily shown. Now $G/N \in \mathfrak{N}$, so $FV_p \lhd V$ and hence FV_p is $\mathfrak{F}$-maximal in NV_p . It follows as in (1) that FV_p is an $\mathfrak{F}$-injector of NV_p , therefore FV_p pro NV_p . Thus $\overline{V}_p$ is a pronormal p-subgroup of $\overline{NV}_p$ complementing $\overline{N}$, so by 4.2.2 $\overline{V}_p$ normalizes each p-complement of $\overline{N}$, and in particular normalizes $\overline{N}_{p'}$. Now $\Sigma \searrow V$, so $\overline{V}_{p'} \leqslant \overline{G}_{p'}$ and hence $\overline{V}_{p'}$ normalizes $\overline{N} \cap \overline{G}_{p'} = \overline{N}_{p'}$. We have therefore shown that $\overline{V} \leqslant N_{\overline{G}}(\overline{\Sigma} \cap \overline{N}) = E/F$ say, a relative system normalizer of $\overline{N}$ in $\overline{G}$.

First suppose that $E = G$. Clearly this holds iff $\overline{N} = N/F \in \mathfrak{N}$, so in this case $G/F \in \mathfrak{N}^2$. A result of long standing due to Carter states that Carter subgroups and system normalizers are the same thing in $\mathfrak{N}^2$-groups, so by the homomorphism invariance of both of these classes of subgroups, the product FC coincides with FD , where D is any system normalizer of G lying in C , and then the statement of the theorem follows from (2) .

We shall now assume that $E < G$, and suppose inductively that the theorem holds for groups of order less than $|G|$.
Certainly V is an $\mathfrak{F}$-maximal subgroup of E containing F , and F is $\mathfrak{F}$-maximal and normal in $N \cap E$. Moreover $N \cap E \lhd E$ and $E/N \cap E \cong NE/N \leqslant G/N \in \mathfrak{N}$. Thus the conditions of the theorem

are satisfied for the group $\overline{E}$, and from our induction hypothesis we deduce that $\overline{V} \leq \overline{F}\overline{B}$, where $\overline{B}$ is a system normalizer of $\overline{E}$, so also $\overline{B}$ is a system normalizer of $\overline{T}$. Now one of the many celebrated results of Philip Hall shows that $\overline{B}$ is actually a system normalizer of $\overline{G}$ and so has the form FD/F for some system normalizer D of G . We conclude that $V \leq FD$ and $V = (FD)_{\mathfrak{F}}$ follows at once.

Let $D^* = N_G(\Sigma)$, then it is well-known that $\Sigma \searrow D^*$. Using 3.1.5 we see $\Sigma \searrow (FD^*)_{\mathfrak{F}}$, but of course $(FD^*)_{\mathfrak{F}}$ is a conjugate of V which is pronormal in G , so by 3.1.3 $V = (FD^*)_{\mathfrak{F}}$ and the proof is complete. $\square$

Suppose next that :

$\mathfrak{F}$ is a Fitting class and G a group with a Sylow system $\Sigma =$

* $\{ G_p, .., G_\pi, .., G_{p'}, .. \}$ reducing into the $\mathfrak{F}$-injector V

of G , and $D = N_G(\Sigma)$.

If $N \lhd G$ with $G/N \in \mathfrak{N}$, then we have just seen that $V \cap N \lhd G$ implies $V \leq (V \cap N)D$. In general we only know that $V \cap N$ pro G (1.1.7 a)), so certainly D normalizes $V \cap N$ by 3.1.5 and 3.1.10 . We ask under what circumstances is the inclusion $V \leq (V \cap N)D$ still true, and find that this is related to the permutability of the Fitting class $\mathfrak{F}$.

4.2.3 Lemma

If $H \leq G$, $N \lhd G$ and $H \perp G_\pi$, a Hall π-subgroup of G , then $HG_\pi \cap N = (H \cap N)(G_\pi \cap N)$ so $(H \cap N) \perp (G_\pi \cap N)$.

Proof

Let $H_{\pi'}$ be a Hall π'-subgroup of H , then $H_{\pi'}G_\pi = HG_\pi$ by an order argument. An element g of HG_π is therefore uniquely expressible as $g = xy$ where $x \in H_{\pi'}$ and $y \in G_\pi$. If $xy \in N$, then x is a π'-element of NG_π , so $x \in N$ and hence also $y \in N$. Therefore $HG_\pi \cap N \leq (H \cap N)(G_\pi \cap N)$. The converse is clear and the lemma proven. $\square$

4.2.4 Lemma

If * holds and $N \lhd G$ with $G/N \in \mathcal{S}_p$, then $V \perp G_{p'}$ if and only if $(V \cap N) \perp G_{p'}$ and $V \leqslant (V \cap N)D$.

Proof

First suppose $V \perp G_{p'}$, then certainly $(V \cap N) \perp (G_{p'} \cap N)$ by 4.2.3 and $(G_{p'} \cap N) = G_{p'}$ since G/N is a p-group. $V_p = V \cap G_p$ is a Sylow p-subgroup of $G_{p'}V$, so $V_p \cap N_{G_{p'}V}(G_{p'}) = P$ say, is a Sylow p-subgroup of $N_{G_{p'}V}(G_{p'})$. By the Frattini argument $N_{G_{p'}V}(G_{p'})$ covers $G_{p'}V/G_{p'}V \cap N = G_{p'}V/G_{p'}(V \cap N)$ (by 4.2.3), which is a p-group, therefore P covers this quotient too. So by the modular law

$$V = V \cap (V \cap N)G_{p'}P = (V \cap N)(V \cap G_{p'}P) = (V \cap N)(V \cap G_{p'})P$$
$$= (V \cap N)P .$$

Now $P = V_p \cap N_{G_{p'}V}(G_{p'}) \leqslant G_p \cap N_G(G_{p'})$ = the unique Sylow p-subgroup of D. Thus $V \leqslant (V \cap N)D$.

Conversely if $V \leqslant (V \cap N)D$ and $(V \cap N) \perp G_{p'}$, then V normalizes $(V \cap N)G_{p'}$ since by 3.1.10 D normalizes both $V \cap N$ and $G_{p'}$. So $V(V \cap N)G_{p'}$ is a subgroup of G, that is $V \perp G_{p'}$. □

4.2.5 Lemma

If * holds and $N \lhd G$ with $G/N \in \mathcal{S}_\pi$, then $V \perp G_\pi$ if and only if $(V \cap N) \perp (G_\pi \cap N)$.

Proof

By 4.2.3 the implication $\Rightarrow$ is clear. By 1.1.7 a) $V \cap N$ and $G_\pi \cap N$ are pronormal in G, so by 3.1.6 $(V \cap N)(G_\pi \cap N) = T$ say, is pronormal in G. Therefore the Frattini argument implies that $N_G(T)$ covers G/N, and by 3.1.10 $\Sigma \searrow N_G(T)$. We deduce that G_π normalizes T, so $(V \cap N) \perp G_\pi$. Now $\Sigma \searrow V$, so G_π contains a Hall π-subgroup of V and hence $V \leqslant (V \cap N)G_\pi$. It then follows that $V \perp G_\pi$. □

4.2.6 Remarks

a) 4.2.4 cannot be improved to the statement :

If * holds and $N \lhd G$ with $G/N \in \mathfrak{N}_\pi$, then

$V \perp G_{\pi'} \Leftrightarrow (V \cap N) \perp G_{\pi'}$ and $V \leq (V \cap N)D$.

The implication $\Leftarrow$ certainly holds , the proof being as in the lemma. However, the example in 4.1.3 c) shows that $\Rightarrow$ is false. For let $\pi = 19'$ and let N be the normal subgroup of G of index 3 . As we saw $H = \langle a \rangle BE$ is an $\mathfrak{F}$-injector of G so $H \cap N = BE$. Obviously $H \perp G_{\pi'} = G_{19} = E^{*}$. Now $A = \langle a' \rangle$ is clearly a system normalizer of J (since B, C and D cover eccentric chief factors) and therefore $(H \cap N)D = \langle a' \rangle BE$ in this case. However $H \neq \langle a' \rangle BE$ so $H \nleq (H \cap N)D$.

b) It follows by the argument used in the proof of 4.1.2 b) that if $T \leq G$ and $T \perp G_{\pi_1}, G_{\pi_2} \in \Sigma$, then $T \perp G_{\pi_1} \cap G_{\pi_2}$. Thus $T \perp G_{p'} \in \Sigma$ for all p implies $T \perp \Sigma$.

4.2.7 THEOREM

The following conditions on the Fitting class $\mathfrak{F}$ are equivalent.

a) $\mathfrak{F}$ is permutable.

b) For any group G , * implies $V \leq (V \cap G^{\mathfrak{N}})D$.

c) For any group G , * implies $V \leq G_{\mathfrak{F}} N_G(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}})$.

PROOF

a) $\Rightarrow$ b) : If * holds and $V \perp \Sigma$, then by 4.2.4 $V \leq (V \cap O^p(G))D$ for all $p \mid |G|$. Therefore $D \cap V$ supplements, in V , each of its normal subgroups $V \cap O^p(G)$ for $p \mid |G|$. Now these have pairwise coprime indices in V , so it follows that $D \cap V$ supplements their intersection $V \cap G^{\mathfrak{N}}$ and hence $V \leq (V \cap G^{\mathfrak{N}})D$.

b) $\Rightarrow$ a) : Assuming b) holds we establish that $* \Rightarrow V \perp \Sigma$ by induction on $|G|$. So let N be a maximal normal subgroup of G of index p say. $V \cap N$ is the unique $\mathfrak{F}$-injector of N into which the system $\Sigma \cap N$ reduces, so by induction $V \cap N \perp \Sigma \cap N$.

Now certainly $G^{\mathfrak{N}} \leq N$. so by assumption $V \leq (V \cap N)D$. We have $V \cap N \perp G_{p'}$, so by 4.2.4 $V \perp G_{p'}$, and for $p \subset \pi$, $V \perp G_{\pi}$ by 4.2.5 , giving $V \perp \Sigma$.

To prove the equivalence of b) and c) we ~~again~~ use induction on $|G|$.

b) $\Rightarrow$ c) : If $G \in \mathfrak{F}\mathfrak{N}\mathfrak{N}$, then $N_G(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}}) = D$, and $G^{\mathfrak{N}} \subset \mathfrak{F}\mathfrak{N}$ so that $V \cap G^{\mathfrak{N}} \leq G_{\mathfrak{F}}$. By assumption $V \leq (V \cap G^{\mathfrak{N}})D$ therefore $V \leq G_{\mathfrak{F}} N_G(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}})$, and c) holds in this case.

We now assume $G \notin \mathfrak{F}\mathfrak{N}\mathfrak{N}$ and let N be a maximal normal subgroup of G containing $G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}}$. Then $N_{\mathfrak{F}\mathfrak{N}\mathfrak{N}} = G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}}$, $N_{\mathfrak{F}} = G_{\mathfrak{F}}$ and $\Sigma \cap N$ is a system of N reducing into $V \cap N$ so by our induction hypothesis $V \cap N \leq G_{\mathfrak{F}} N_N(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}})$. Certainly $D \leq N_G(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}})$ so D normalizes $N_N(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}})$. We are assuming b) holds so $V \leq (V \cap G^{\mathfrak{N}})D$, therefore $V \leq (V \cap N)D \leq G_{\mathfrak{F}} N_N(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}})D \leq G_{\mathfrak{F}} N_G(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}})$.

c) $\Rightarrow$ b) : If $G \in \mathfrak{F}\mathfrak{N}$, then $G^{\mathfrak{N}} \in \mathfrak{F}$ so $(V \cap G^{\mathfrak{N}})D = G^{\mathfrak{N}}D = G$ and b) holds trivially. When $G \notin \mathfrak{F}\mathfrak{N}$, $G_{\mathfrak{F}} N_G(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}}) \neq G$. By Hall's celebrated results, $\Sigma \searrow N_G(\Sigma \cap G_{\mathfrak{F}\mathfrak{N}\mathfrak{N}}) = T$ say , and furthermore $N_T(\Sigma \cap T)$, a system normalizer of T , is actually D . Let $D_0 = N_{G_{\mathfrak{F}}T}(\Sigma \cap G_{\mathfrak{F}}T)$, then of course $D \leq D_0$. Homomorphism invariance implies $D_0 G_{\mathfrak{F}}/G_{\mathfrak{F}}$ is a system normalizer of $G_{\mathfrak{F}}T/G_{\mathfrak{F}} \cong T/T \cap G_{\mathfrak{F}}$ of which $D(T \cap G_{\mathfrak{F}})/(T \cap G_{\mathfrak{F}})$ is a system normalizer. Hence $D_0 G_{\mathfrak{F}}/G_{\mathfrak{F}} \cong D(T \cap G_{\mathfrak{F}})/(T \cap G_{\mathfrak{F}}) \cong D/D \cap (T \cap G_{\mathfrak{F}}) = D/D \cap G_{\mathfrak{F}} \cong DG_{\mathfrak{F}}/G_{\mathfrak{F}}$. Then $D \leq D_0$ forces $D_0 G_{\mathfrak{F}} = DG_{\mathfrak{F}}$. c) holds so $V \leq G_{\mathfrak{F}}T$, and we are assuming that $G_{\mathfrak{F}}T < G$, so the induction hypothesis gives $V \leq (V \cap (G_{\mathfrak{F}}T)^{\mathfrak{N}})D_0$. Now clearly $(G_{\mathfrak{F}}T)^{\mathfrak{N}} \leq (G^{\mathfrak{N}} \cap G_{\mathfrak{F}}T)$, and so $V \leq (V \cap_G G^{\mathfrak{N}} \cap G_{\mathfrak{F}}T)D_0 = (V \cap G^{\mathfrak{N}})D_0 \leq (V \cap G^{\mathfrak{N}})D_0 G_{\mathfrak{F}} = (V \cap G^{\mathfrak{N}})DG_{\mathfrak{F}}$. $G_{\mathfrak{F}}/(G^{\mathfrak{N}})_{\mathfrak{F}} \cong G_{\mathfrak{F}}G^{\mathfrak{N}}/G^{\mathfrak{N}}$, so D covers all the (central) chief factors of G between $(G^{\mathfrak{N}})_{\mathfrak{F}}$ and $G_{\mathfrak{F}}$, and therefore $G_{\mathfrak{F}} \leq (G^{\mathfrak{N}})_{\mathfrak{F}} D \leq (V \cap G^{\mathfrak{N}})D$. With the previous inequlity, this implies $V \leq (V \cap G^{\mathfrak{N}})D$ and b) holds. $\square$

5.1 Strong containment

In [18] Cline introduces a partial ordering $\ll$ on the collection of saturated formations. If $\mathfrak{E}$ and $\mathfrak{F}$ are Fitting classes then analogously we say $\mathfrak{E}$ is strongly contained in $\mathfrak{F}$ (written $\mathfrak{E} \ll \mathfrak{F}$) provided that, in each group G, an $\mathfrak{F}$-injector of G contains some $\mathfrak{E}$-injector. In view of 1.1.4 c) this is equivalent to the requirement that an $\mathfrak{E}$-injector of an $\mathfrak{F}$-injector of a group G is an $\mathfrak{E}$-injector of G. Clearly $\mathfrak{E} \ll \mathfrak{F}$ implies $\mathfrak{E} \subseteq \mathfrak{F}$. This terminology applies nicely to the work of previous chapters. For instance, $\mathfrak{F}$ is a permutable Fitting class $\Leftrightarrow$ $\mathfrak{F} \ll \mathfrak{F}^\pi$ for all π, is obvious in the light of chapter 4 .

Cline attacks the problem of determining the saturated formations maximal with respect to $\ll$, producing some which are (for example $\mathfrak{N}^r$) and proving that certain others are not. Here we take a preliminary look at the corresponding problem for Fitting classes and show that a strongly pronormal class, maximal in this sense, is necessarily normal.

5.1.1 Lemma

Suppose $\mathfrak{F}$ is a Fitting class such that $\mathfrak{X} = Q\{ G/G_{\mathfrak{F}} : G \in \mathfrak{S} \}$ $\neq \mathfrak{S}$. Then $\mathfrak{F}$ is normal.

Proof

Let H be an arbitrary group and $1 \neq K \in \mathfrak{S} - \mathfrak{X}$. Set $G = H \wr K$ and let B be the base group of the wreath product.
Then $G_{\mathfrak{F}} \nleq B$, otherwise K is a quotient of $G/G_{\mathfrak{F}}$ and therefore a member of $\mathfrak{X}$, contrary to its choice. Let $G_0 = BG_{\mathfrak{F}}$, $K_0 = K \cap G_0$ and $r = |G : G_0|$. Then $G_0 = BK_0 \cong H_0 \wr K_0$ where $H_0 = \prod_r H$, the direct product of r copies of H, and $1 \neq K_0$.

Now $G_0 \lhd G$, so $G_{\mathfrak{F}} = (G_0)_{\mathfrak{F}}$ is a normal subgroup of G_0 which supplements its base group B. By lemma 8.2 of [19] it follows that $G_{\mathfrak{F}} \geq B'$, whence $H' \in \mathfrak{F}$. We have thus shown that $\mathfrak{D} \subseteq \mathfrak{F}$ and so $\mathfrak{F}$ is normal. □

5.1.2 THEOREM

If the Fitting class $\mathfrak{F}\ (\neq \mathfrak{S})$ is strongly pronormal and maximal with respect to $\ll$, then $\mathfrak{F}$ is normal and for each G, $G/G_{\mathfrak{F}} \in \mathfrak{S}_p$ for some fixed prime p.

PROOF

By 4.1.1 and our remark above, the hypothesis implies $\mathfrak{F} \ll \mathfrak{F}^q$ for all primes q. Now clearly $\mathfrak{F}^q = \mathfrak{S}$ for all q forces $\mathfrak{F} = \mathfrak{S}$. So by maximality there exists a prime p such that $\mathfrak{F} = \mathfrak{F}^p$, that is (by 2.6.1 c)) the $\mathfrak{F}$-injectors of any G have p-power index in G . In particular p is unique. Let V be an $\mathfrak{F}$-injector of an arbitrary group G. By hypothesis V stropro G and since $|G : V|$ is a power of p, it follows that $< V_p^{\,G} > \leq V$ so in fact $V_p \leq G_{\mathfrak{F}}$ and therefore $V = G_{\mathfrak{F}}G_{p'}$ for any p-complement $G_{p'}$ of G lying in V. Now by 2.6.5' a) and the remark following it $O_p(G \bmod G_{\mathfrak{F}})G_{p'}$ is an $\mathfrak{F}\mathfrak{S}_p\mathfrak{S}_{p'}$-injector of G, so $\mathfrak{F} \ll \mathfrak{F}\mathfrak{S}_p\mathfrak{S}_{p'}$. Clearly $\mathfrak{F} = \mathfrak{F}\mathfrak{S}_p\mathfrak{S}_{p'}$ implies $\mathfrak{F} = \mathfrak{S}$, so the other possibility $\mathfrak{F}\mathfrak{S}_p\mathfrak{S}_{p'} = \mathfrak{S}$, must hold. Since $\mathfrak{S}_p\mathfrak{S}_{p'} = Q\mathfrak{S}_p\mathfrak{S}_{p'} \neq \mathfrak{S}$, lemma 5.1.1 gives us the desired conclusion. □

Although we have seen that $\mathfrak{F} \ll \mathfrak{F}^p$ is not in general true, we would be surprised if there were a Fitting class $\mathfrak{F}$, maximal with respect to $\ll$, which did not satisfy $\mathfrak{F} = \mathfrak{F}^p$ for some prime p . Indeed we suspect that the maximal classes are among those described in the conclusion of 5.1.2, that is ' normal classes of prime power index '. Nor can we decide whether such classes are actually maximal. Clearly $\ll$ and $\subseteq$ coincide for normal classes but the example of Remark 4.1.4 b) shows that the intuitively hopeful process of forming the ' join ' of two normal Fitting classes, does not yield a Fitting class.

5.2 Normal Fitting classes and $\mathfrak{F}^*$

In this section we expand on an earlier remark that the radical of the direct product of groups A and B need not be the direct product of their radicals. More specifically we examine the radical of a direct power of a group G and show that for a normal Fitting class $\mathfrak{X}$, the radical is never the corresponding direct power of G, (unless of course $G \in \mathfrak{X}$). This investigation puts the ~~set~~ family of normal Fitting classes in a new setting, and indeed we shall see that around each Fitting class $\mathfrak{F}$ there is a spectrum of Fitting classes (partly induced by the set of normal classes) with properties close to those of $\mathfrak{F}$.

Let $\mathfrak{F}$ be a Fitting class and G a group. Let $X = \prod_n G = \{ (g_1, g_2, \ldots, g_n) : g_i \in G \}$ be the direct product of n copies of G. Let $G^* = \{ g_1 : (g_1, g_2, \ldots, g_n) \in X_{\mathfrak{F}} \}$, the image of $X_{\mathfrak{F}}$ under the natural projection of X onto its first direct factor. Now $X_{\mathfrak{F}}$ char X and, in the obvious way, X admits both $\prod_n \mathrm{Aut}\, G$ and Σ_n as groups of automorphisms. It follows that G^* char G and G^* is the image of X under the natural projection of X onto any of its direct factors. Clearly $G_{\mathfrak{F}} \leq G^*$ and $\prod_n G_{\mathfrak{F}} \leq (\prod_n G)_{\mathfrak{F}} \leq \prod_n G^*$.

5.2.1 Lemma

With the preceding notation

a) $G^*/G_{\mathfrak{F}}$ is Abelian

b) $(\prod_n G)_{\mathfrak{F}} = \{ (g_1, g_2, \ldots, g_n) : g_i \in G^*$ for all i and $g_1 g_2 \cdots g_n \in G_{\mathfrak{F}} \}$,

and $(G \times G)_{\mathfrak{F}} = (G_{\mathfrak{F}} \times G_{\mathfrak{F}}) \langle (g, g^{-1}) : g \in G^* \rangle$, so the definition of G^* is independent of n .

c) Aut G^* centralises $G^*/G_{\mathfrak{J}}$.

<u>Proof</u>

a) If G is any direct factor of X, $[G, X_{\mathfrak{J}}] \leq G \cap X_{\mathfrak{J}} = G_{\mathfrak{J}}$ and so $(\prod_n G)_{\mathfrak{J}}/\prod_n G_{\mathfrak{J}} \leq Z(\prod_n G/\prod_n G_{\mathfrak{J}}) = \prod_n Z(G/G_{\mathfrak{J}})$. Therefore $G^*/G_{\mathfrak{J}}$ is a homomorphic image of an Abelian group, hence is itself Abelian.

b) Suppose $(g_1, g_2, \ldots, g_n) \in (\prod_n G)_{\mathfrak{J}}$, then certainly $g_i \in G^*$ for each i by the definition of G^*.

Furthermore $(g_1, g_2, \ldots, g_n) \in (\prod_n G)_{\mathfrak{J}}$

$\Rightarrow (g_1, g_2, \ldots, g_n, 1) \in (\prod_{n+1} G)_{\mathfrak{J}}$ (by considering the obvious embedding $\prod_n G \lhd \prod_n G \times G = \prod_{n+1} G$)

$\Rightarrow (1, g_2, \ldots, g_n, g_1) \in (\prod_{n+1} G)_{\mathfrak{J}}$ (since $(\prod_{n+1} G)_{\mathfrak{J}}$ is invariant under the natural action of Σ_{n+1})

$\Rightarrow (g_1, 1, \ldots, 1, g_1^{-1}) \in (\prod_{n+1} G)_{\mathfrak{J}}$

$\Rightarrow (g_1, 1, \ldots, g_1^{-1}, 1) \in (\prod_{n+1} G)_{\mathfrak{J}}$ (action of Σ_{n+1} again)

$\Rightarrow (g_1, 1, \ldots, g_1^{-1}) \in (\prod_n G)_{\mathfrak{J}}$ ($\prod_n G \lhd \prod_{n+1} G$ again)

Similarly $(1, \ldots, g_i, \ldots, g_i^{-1}) \in (\prod_n G)_{\mathfrak{J}}$ for $i = 2, \ldots, n-1$,

and so $(g_1, g_2, \ldots, g_{n-1}, g_{n-1}^{-1} \cdots g_2^{-1} g_1^{-1}) \in (\prod_n G)_{\mathfrak{J}}$,

hence $(1, 1, \ldots, (g_1 g_2 \cdots g_n)^{-1}) \in (\prod_n G)_{\mathfrak{J}}$

which implies $g_1 g_2 \cdots g_n \in G_{\mathfrak{J}}$.

Now if $g_i \in G^*$, then $(g_1, \ldots, g_i, \ldots, g_n) \in (\prod_n G)_{\mathfrak{J}}$ for suitable ~~some~~ $g_j \in G^*$ $(j \neq i)$ by the definition of G^*. Then as above this implies $(1, \ldots, g_i, \ldots, g_j^{-1}) \in (\prod_n G)_{\mathfrak{J}}$. So conversely if $g_1, \ldots, g_n \in G^*$ and $g_1 g_2 \cdots g_n \in G_{\mathfrak{J}}$ we have

$$(g_1, g_2, \ldots, g_{n-1}, g_{n-1}^{-1} \cdots g_2^{-1} g_1^{-1}) \in (\prod_n G)_{\mathfrak{J}}$$

$$\text{and } (1, 1, \ldots\ldots, g_1 g_2 \cdots g_n) \in (\prod_n G)_{\mathfrak{J}}$$

$$\text{therefore } (g_1, g_2, \ldots, g_{n-1}, g_n) \in (\prod_n G)_{\mathfrak{J}}$$

These manoeuvres show $g \in G^* \Leftrightarrow (g, 1, \ldots, g^{-1}) \in (\prod G)_{\mathfrak{J}}$

$$\Leftrightarrow (g, g^{-1}) \in (G \times G)_{\mathfrak{J}}.$$

Loosely speaking $(\prod_n G)_{\mathfrak{J}}/\prod_n G_{\mathfrak{J}}$ is the subgroup of $\prod_n G^*/\prod_n G_{\mathfrak{J}}$ (which by a) is an Abelian group), comprising the elements with trivial product of components. In particular $(G \times G)_{\mathfrak{F}}/G_{\mathfrak{F}} \times G_{\mathfrak{F}}$ is the 'skew diagonal' of $G^*/G_{\mathfrak{F}} \times G^*/G_{\mathfrak{F}}$, and so $(G \times G)_{\mathfrak{F}} = (G_{\mathfrak{F}} \times G_{\mathfrak{F}})\langle (g,g^{-1}) : g \in G^* \rangle$.

c) Certainly $(G \times G)_{\mathfrak{F}} = (G^* \times G^*)_{\mathfrak{F}}$ char $(G^* \times G^*)$. Let $a \in \mathrm{Aut}\, G^*$. By its action on the first direct factor, a induces an automorphism of $G^* \times G^*$. Therefore $(g^a, g^{-1}) \in (G^* \times G^*)_{\mathfrak{F}}$ for all $g \in G^*$. Hence $(g^{-1}g^a, 1) \in (G^* \times G^*)_{\mathfrak{F}}$, which implies $g^{-1}g^a \in G_{\mathfrak{F}}$ for all $g \in G^*$, that is a centralizes $G^*/G_{\mathfrak{F}}$. $\square$

Now define $\mathfrak{F}^* = \{ G : G = G^* \}$, so given a Fitting class $\mathfrak{F}$,
$\mathfrak{F}^* = \{ G : (G \times G)_{\mathfrak{F}} = (G_{\mathfrak{F}} \times G_{\mathfrak{F}})\langle (g,g^{-1}) : g \in G \rangle \}$.

<u>5.2.2 THEOREM</u>

If $\mathfrak{F}$ is a Fitting class then

a) $\mathfrak{F}^*$ is a Fitting class

b) in the notation of lemma 5.2.1 $G^* = G_{\mathfrak{F}^*}$

c) $\mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}^*)^* \subseteq \mathfrak{F}\mathfrak{A}$ (the last class is not necessarily Fitting of course)

d) $\mathfrak{F}$ is normal $\Leftrightarrow \mathfrak{F}^* = \mathfrak{S}$.

<u>PROOF</u>

a) Suppose $N \lhd G \in \mathfrak{F}^*$, then $(G \times G)_{\mathfrak{F}} = (G_{\mathfrak{F}} \times G_{\mathfrak{F}})\langle (g,g^{-1}) : g \in G \rangle$, therefore $(N \times N)_{\mathfrak{F}} = (N \times N) \cap (G \times G)_{\mathfrak{F}} \geq (N_{\mathfrak{F}} \times N_{\mathfrak{F}})\langle (n,n^{-1}) : n \in N \rangle$. So the image of $(N \times N)_{\mathfrak{F}}$ under the projection of $N \times N$ onto a direct factor is the whole of N, that is $N = N^*$ and $N \in \mathfrak{F}^*$.

Now suppose $\mathfrak{F}^* \ni N_1, N_2 \lhd G = N_1N_2$. For $i = 1,2$, G induces automorphisms of N_i, so by 5.2.1 c) and our supposition, G centralizes $N_i/(N_i)_{\mathfrak{F}}$, that is $[N_i, G] \leq (N_i)_{\mathfrak{F}} \leq G_{\mathfrak{F}}$. It follows that $G' = [N_1N_2, G] \leq G_{\mathfrak{F}}$. Now let $n_1n_2 = g \in G$ where $n_i \in N_i$. Then $(g,g^{-1}) = (n_1n_2, n_2^{-1}n_1^{-1}) = (n_1,n_1^{-1})(n_2,n_2^{-1})(1,[n_2^{-1},n_1^{-1}])$. Thus by supposition and our observation $(g,g^{-1}) \in (G \times G)_{\mathfrak{F}}$. Projecting onto a direct factor gives $G = G^*$, so $G \in \mathfrak{F}^*$.

b) $(G \times G)_{\mathfrak{F}} = (G_{\mathfrak{F}} \times G_{\mathfrak{F}}) \langle (g,g^{-1}) : g \in G^* \rangle$ Therefore $(G^* \times G^*)_{\mathfrak{F}} = ((G^*)_{\mathfrak{F}} \times (G^*)_{\mathfrak{F}}) \langle (g,g^{-1}) : g \in G^* \rangle$, since $G_{\mathfrak{F}} \lhd G^* \lhd G$, so certainly $G^* \leq G_{\mathfrak{F}^*}$. On the other hand $(G \times G)_{\mathfrak{F}} \geq (G_{\mathfrak{F}^*} \times G_{\mathfrak{F}^*})_{\mathfrak{F}} = ((G_{\mathfrak{F}^*})_{\mathfrak{F}} \times (G_{\mathfrak{F}^*})_{\mathfrak{F}}) \langle (g,g^{-1}) : g \in G_{\mathfrak{F}^*} \rangle$
$= (G_{\mathfrak{F}} \times G_{\mathfrak{F}}) \langle (g,g^{-1}) : g \in G_{\mathfrak{F}^*} \rangle$, since obviously $\mathfrak{F} \subseteq \mathfrak{F}^*$. From this it follows that $G_{\mathfrak{F}^*} \leq G^*$, and so we have equality.

c) In view of 5.2.1 a) $\mathfrak{F} \subseteq \mathfrak{F}^* \subseteq \mathfrak{F}\mathfrak{A}$ is clear, and hence also $\mathfrak{F}^* \subseteq (\mathfrak{F}^*)^*$, so it remains to prove $\mathfrak{F}^* \geq (\mathfrak{F}^*)^*$. Suppose $G \in (\mathfrak{F}^*)^*$, then by definition $(G \times G)_{\mathfrak{F}^*} =$ $(G_{\mathfrak{F}^*} \times G_{\mathfrak{F}^*}) \langle (g,g^{-1}) : g \in G \rangle$, therefore by b)

$$(G \times G \times G \times G)_{\mathfrak{F}} = ((G \times G)_{\mathfrak{F}} \times (G \times G)_{\mathfrak{F}}) \langle (g_1, g_2, g_1^{-1}, g_2^{-1}) : (g_1, g_2) \in (G \times G)_{\mathfrak{F}^*} \rangle$$
$$\geq ((G \times G)_{\mathfrak{F}} \times (G \times G)_{\mathfrak{F}}) \langle (g, g^{-1}, g^{-1}, g) : g \in G \rangle .$$

Thus $(G \times G \times G \times G)_{\mathfrak{F}}$ projects <u>onto</u> each direct factor, so $G \in \mathfrak{F}^*$.

d) If $\mathfrak{F}$ is a normal class then by 2.4.3 $G' \leq G_{\mathfrak{F}} \leq G$ for all G. In particular $(G \wr C_2)' \leq (G \wr C_2)_{\mathfrak{F}}$. Letting $C_2 = \langle i \rangle$ in the wreath product we have $[(g,1), i] = (g^{-1}, g)$, so $(g^{-1}, g) \in (G \times G) \cap (G \wr C_2)_{\mathfrak{F}} = (G \times G)_{\mathfrak{F}}$. Therefore $G = G^*$, and so $\mathfrak{F}^* = \mathfrak{S}$. If $\mathfrak{F}^* = \mathfrak{S}$, then by c) $\mathfrak{F}\mathfrak{A} = \mathfrak{S}$ so $\mathfrak{F}$ is normal. $\square$

<u>5.2.3 Remarks</u>

a) $\mathfrak{F} = Q\mathfrak{F} \Rightarrow \mathfrak{F} = \mathfrak{F}^*$. For if $G \in \mathfrak{F}^*$, then by definition G is a homomorphic image of $(G \times G)_{\mathfrak{F}}$, so $G \in \mathfrak{F}$.

b) $\mathfrak{F} = S\mathfrak{F} \Rightarrow \mathfrak{F} = \mathfrak{F}^*$. For suppose $\mathfrak{F} \subset \mathfrak{F}^*$ and let G be of minimal order in $\mathfrak{F}^* - \mathfrak{F}$. Then G has a unique maximal normal subgroup N of index p say, and $N \in \mathfrak{F}$. Furthermore by 5.2.1 b) and the definition of $\mathfrak{F}^*$, $(\prod_p G)_{\mathfrak{F}} = \{ (g_1, g_2, \ldots, g_p) : g_i \in G$ and $g_1 g_2 \ldots g_p \in N \}$, which contains the diagonal $\{ (g, g, \ldots, g) : g \in G \}$, a subgroup isomorphic to G. So $G \in S\mathfrak{F} = \mathfrak{F}$ giving a contradiction, and we deduce $\mathfrak{F} = \mathfrak{F}^*$.

c) $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \Rightarrow \mathfrak{F}_1^* \subseteq \mathfrak{F}_2^*$. For if $G \in \mathfrak{F}_1^*$ then $(G \times G)_{\mathfrak{F}_1} = (G_{\mathfrak{F}_1} \times G_{\mathfrak{F}_1}) \langle (g, g^{-1}) : g \in G \rangle \leq (G \times G)_{\mathfrak{F}_2}$ so $G \in \mathfrak{F}_2^*$.

d) If $\{\mathfrak{F}_i\}_{i \in I}$ are arbitrary Fitting classes, then $\bigcap_{i \in I} \mathfrak{F}_i^* = (\bigcap_{i \in I} \mathfrak{F}_i)^*$.

First notice that $G \in \mathfrak{F}^* \Leftrightarrow (G' \times G') \langle (g, g^{-1}) : g \in G \rangle \in \mathfrak{F}$.

For by 5.2.1 a) $G \in \mathfrak{F}^*$ implies $G/G_{\mathfrak{F}} \in \mathfrak{A}$, so $G' \leq G_{\mathfrak{F}}$, therefore $(G' \times G') \langle (g, g^{-1}) : g \in G \rangle \leq (G \times G)_{\mathfrak{F}}$ and $(G' \times G') \langle (g, g^{-1}) : g \in G \rangle \in \mathfrak{F}$ follows. Conversely this last inclusion implies $G^* = G$ that is $G \in \mathfrak{F}^*$.

So $G \in \bigcap_{i \in I} \mathfrak{F}_i^* \Leftrightarrow (G' \times G') \langle (g, g^{-1}) : g \in G \rangle \in \bigcap_{i \in I} \mathfrak{F}_i \Leftrightarrow G \in (\bigcap_{i \in I} \mathfrak{F}_i)^*$.

Given $\mathfrak{F}$, define $\underline{\mathfrak{F}_*} = \bigcap \{\mathfrak{H} : \mathfrak{H}^* = \mathfrak{F}^*\}$ so $\mathfrak{F}_* \subseteq \mathfrak{F} \subseteq \mathfrak{F}^*$.

e) $\mathfrak{F}_* \subseteq \mathfrak{K} \subseteq \mathfrak{F}^*$ implies $\mathfrak{K}^* = \mathfrak{F}^*$ and $\mathfrak{K}_* = \mathfrak{F}_*$.

For by d) $(\mathfrak{F}_*)^* = \mathfrak{F}^*$, so by c) and 5.2.2 c) $\mathfrak{F}^* = (\mathfrak{F}_*)^* \subseteq \mathfrak{K}^* \subseteq (\mathfrak{F}^*)^* = \mathfrak{F}^*$, giving $\mathfrak{K}^* = \mathfrak{F}^*$, so by definition $\mathfrak{K}_* = \mathfrak{F}_*$.

Notice that by 5.2.2 d) $\mathfrak{S}_*$ = the unique smallest normal Fitting class.

<u>5.2.4 THEOREM</u>

For each Fitting class $\mathfrak{F}$ and group G, $\mathfrak{F} \ll \mathfrak{F}^*$ and if V^* is an $\mathfrak{F}^*$-injector of G then $(V^*)_{\mathfrak{F}}$ is an $\mathfrak{F}$-injector of G.

<u>PROOF</u>

Since $\mathfrak{F} \subseteq \mathfrak{F}^* \subseteq \mathfrak{F}\mathfrak{A}$, the second assertion follows at once from the first. Let N be a maximal normal subgroup of G and Σ a Sylow system of G, and put $D = N_G(\Sigma)$. Let V, V^* be the unique $\mathfrak{F}$- and $\mathfrak{F}^*$-injector respectively of G into which Σ reduces. Then $W = V \cap N$ and $W^* = V^* \cap N$ are the unique $\mathfrak{F}$- and $\mathfrak{F}^*$-injectors of N into which $\Sigma \cap N$ reduces. We assume inductively that for a group H of order less than $|G|$ with Sylow system Ω:

the $\mathfrak{F}$-injector of H into which Ω reduces $\leq$ the $\mathfrak{F}^*$-injector of H into which Ω reduces.

So by induction $W < W^*$, which implies $W = (W^*)_{\mathfrak{F}}$ by our initial remark. Since $(W^*)_{\mathfrak{F}}$ char $W^* \lhd V^*$ it follows that $V, V^* \leq N_G(W)$. Now by 3.1.10 $\Sigma \to N_G(W)$, so V and V^* are the $\mathfrak{F}$- and $\mathfrak{F}^*$-injectors of $N_G(W)$ into which $\Sigma \cap N_G(W)$ reduces. Thus if $W \ntriangleleft G$, the induction hypothesis implies $V \leq V^*$, so we may assume $W \lhd G$. Then 4.2.1 implies $(WD)_{\mathfrak{F}} = V$. Now by 3.1.10 again, D normalizes V^*, and already $W \leq V^*$ so V normalizes V^*. By 3.1.5 $\Sigma \to VV^*$, so we may assume $VV^* = G$. Then $V^* \lhd G$ so $V^* = G_{\mathfrak{F}^*}$.
If $V \ntriangleleft G$ then $W = G_{\mathfrak{F}}$ since $|V : W| = p$ in this case. By 5.2.1 c) Aut $G_{\mathfrak{F}^*}$ centralizes $G_{\mathfrak{F}^*}/G_{\mathfrak{F}}$, so in particular G centralizes V^*/W, but $G/V^* = VV^*/V^* \cong V/V^* \cap V$ is a quotient of $V/W \cong C_p$, so G/W is 'central by cyclic', in particular, Abelian. Therefore $V \lhd G$ after all, so certainly $\mathfrak{F} \subseteq \mathfrak{F}^*$ implies $V = G_{\mathfrak{F}} \leq G_{\mathfrak{F}^*} = V^*$.

$\square$

5.2.5 Corollary

$\mathfrak{F}_* \subseteq \mathfrak{H} \subseteq \mathfrak{K} \subseteq \mathfrak{F}^*$ implies $\mathfrak{H} \ll \mathfrak{K}$, that is, for Fitting classes between $\mathfrak{F}_*$ and $\mathfrak{F}^*$, $\subseteq$ coincides with $\ll$.

Proof

Put $\mathfrak{J} = \mathfrak{F}^* = \mathfrak{H}^* = \mathfrak{K}^*$ by 5.2.3 e). Let V be an $\mathfrak{J}$-injector of the group G. Then 5.2.4 states that $V_{\mathfrak{H}}$ and $V_{\mathfrak{K}}$ are $\mathfrak{H}$- and $\mathfrak{K}$-injectors of G respectively. Then $\mathfrak{H} \subseteq \mathfrak{K}$ implies $V_{\mathfrak{H}} \leq V_{\mathfrak{K}}$, so $\mathfrak{H} \ll \mathfrak{K}$ follows.

$\square$

5.2.6 Corollary

If $\mathfrak{F}$ is any Fitting class, $\mathfrak{X}$ is a normal Fitting class and V is an $\mathfrak{F}$-injector of G then $V_{\mathfrak{X}}$ is an $(\mathfrak{F} \cap \mathfrak{X})$-injector of G.

Proof

By 5.2.3 d) and 5.2.2 d) $(\mathfrak{F} \cap \mathfrak{X})^* = \mathfrak{F}^* \cap \mathfrak{X}^* = \mathfrak{F}^*$.
So by 5.2.2 e) and 5.2.5 $\mathfrak{F} \cap \mathfrak{X} \ll \mathfrak{F}$. $V_{\mathfrak{X}} \lhd V$ therefore $V_{\mathfrak{X}} \in \mathfrak{F}$ and hence is $(\mathfrak{F} \cap \mathfrak{X})$-maximal in V. It follows that $V_{\mathfrak{X}}$ is an $(\mathfrak{F} \cap \mathfrak{X})$-injector of V and we are done.

$\square$

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