# SPECTRAL EXPANSIONS OF OVERCONVERGENT MODULAR FUNCTIONS

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ABSTRACT. The main result of this paper is an instance of the conjecture made by Gouvêa and Mazur in [GM95], which asserts that for certain values of $r$ the space of $r$-overconvergent $p$-adic modular forms of tame level $N$ and weight $k$ should be spanned by the finite slope Hecke eigenforms. For $N = 1$, $p = 2$ and $k = 0$ we show that this follows from the combinatorial approach initiated by Emerton [Eme98] and Smithline [Smi00], using the classical $LU$ decomposition and results of Buzzard–Calegari [BC05]; this implies the conjecture for all $r \in (\frac{5}{12}, \frac{7}{12})$. Similar results follow for $p = 3$ and $p = 5$ with the assumption of a plausible conjecture, which would also imply formulae for the slopes analogous to those of [BC05].

We also show that (for general $p$ and $N$) the space of weight 0 overconvergent forms carries a natural inner product with respect to which the Hecke action is self-adjoint. When $N = 1$ and $p \in \{2, 3, 5, 7, 13\}$, combining this with the combinatorial methods allows easy computations of the $q$-expansions of small slope overconvergent eigenfunctions; as an application we calculate the $q$-expansions of the first 20 eigenfunctions for $p = 5$, extending the data given in [GM95].

## 1. BACKGROUND

Let $S_k(\Gamma_1(N))$ denote the space of classical modular cusp forms of weight $k$ and level $N$. It has long been known that these objects satisfy many interesting congruence relations. One very powerful method for studying the congruences obeyed by modular forms modulo powers of a fixed prime $p$ is to embed this space into the $p$-adic Banach space $\mathcal{S}_k(\Gamma_1(N), r)$ of $r$-overconvergent $p$-adic cusp forms, defined as in [Kat73] using sections of $\omega^{\otimes k}$ on certain affinoid subdomains of $X_1(N)$ obtained by removing discs of radius $p^{-r}$ around the supersingular points; this space has been used to great effect by Coleman and others ([Col96, Col97]).

It is known that there is a Hecke action on $\mathcal{S}_k(\Gamma_1(N), r)$, as with the classical spaces, and these operators are continuous; and moreover, at least for $0 < r < \frac{p}{p+1}$, the Atkin-Lehner operator $U$ is compact. There is a rich spectral theory for compact operators on $p$-adic Banach spaces (see [Ser62]), and this is a powerful tool for studying the spaces $\mathcal{S}_k(\Gamma_1(N), r)$. In this paper, we shall attempt to make this spectral theory explicit in the case $N = 1$, $k = 0$, for certain small primes $p$.

## 2. A USEFUL BASIS

In all the computations in this paper, we shall restrict to the case of tame level 1; hence we shall write $\mathcal{S}_k(r)$ for $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}), r)$, regarded as a Banach space over $\mathbb{C}_p$.

Recall that if $\psi$ is any lifting of the mod $p$ Hasse invariant to a modular form in characteristic 0, and $E$ is any elliptic curve over $\mathbb{C}_p$ such that $|\psi(E)| > p^{-\frac{p}{p+1}}$, then $E$ has a canonical $p$-subgroup; hence, for $0 < r < \frac{p}{p+1}$, the $r$-overconvergent locus $X_0(1)_{\geq p^{-r}}$ is isomorphic to a certain subregion of $X_0(p)$. (This is proved in [Kat73], using the theory of the Newton polygon.)

If $p$ is one of the primes 2, 3, 5, 7, or 13, then $X_0(p)$ has genus 0. We shall pick an explicit uniformiser for this curve, and identify in terms of this uniformiser the image of $X_0(1)_{\ge p^{-r}}$ under the canonical subgroup map, and hence obtain a basis for our space $\mathcal{S}_k(r)$.

**Theorem 1.** *Let $p$ be one of the primes 2, 3, 5, 7, or 13. Let $f_p$ be the function*

$$\left[\frac{\Delta(pz)}{\Delta(z)}\right]^{\frac{1}{p-1}}.$$

*Then $f_p$ is a rational function on the modular curve $X_0(p)$, and the forgetful functor gives an isomorphism between the region of the modular curve $X_0(p)$ where $|f_p| \le 1$ and the ordinary locus $X_0(1)_{\mathrm{ord}}$. Moreover, for any $r \in [0, \frac{p}{p+1})$, this extends to an isomorphism between the region where $|f_p| \le p^{\frac{12r}{p-1}}$ and $X_0(1)_{\ge p^{-r}}$.*

*Proof.* That $f_p$ is a rational function on $X_0(p)$ is clear from the fact that $\Delta(z)$ and $\Delta(pz)$ are both classical modular forms of weight 12 and level $p$, and $\Delta$ has no zeros on $X_0(p)$. It has a zero of order 1 at $z = \infty$ by inspection of its $q$-expansion, and no other zeros as $\Delta$ does not vanish on the complex upper half-plane; so it is a uniformiser for $X_0(p)$.

It remains to prove that the subsets defined by $|f_p| \le p^{\frac{12r}{p-1}}$ agree with the $r$-overconvergent locus as defined in [Kat73] using lifts of the Hasse invariant. For $p = 2$ this is proved in [BC05, §4]; for $p \ge 5$ it is [Smi01, Prop 3.5]. In the remaining case $p = 3$ Smithline uses a different measure of supersingularity and it is not immediately obvious this agrees with the valuation of the Hasse invariant; we show that the two do in fact agree below, in §7. ☐

**Corollary 2.** *For any $0 \le r < \frac{p}{p+1}$, the space $\mathcal{S}_0(r) = \mathcal{S}_0(\mathrm{SL}_2(\mathbb{Z}), r)$ of $r$-overconvergent $p$-adic tame level 1 cuspidal modular functions (modular forms of weight 0) has an orthonormal basis $(cf_p, (cf_p)^2, (cf_p)^3, \dots)$ where $c$ is any element of $\mathbb{C}_p$ with $|c| = p^{\frac{12r}{p-1}}$.*

(This follows as we have given an isomorphism between this space and a $p$-adic closed disc, and the algebra of rigid-analytic functions on a $p$-adic closed disc with uniformising parameter $x$ is the Tate algebra $\mathbb{C}_p\langle x\rangle$.)

**Theorem 3.** *Let $U$ be the Atkin-Lehner operator acting on $\mathcal{S}_0(r)$, and let $u_{ij}^{(r)}$ be the matrix coefficients of $U$ with respect to the basis defined above. Then the following results hold:*

1. $u_{ij}^{(r)} = c^{j-i} u_{ij}^{(0)}$.
2. *There is a $p \times p$ matrix $M^{(r)}$, which is 'skew upper triangular' (that is, $M_{ij}^{(r)} = 0$ if $i + j > p + 1$), with the property that*

$$u_{ij} = \sum_{a,b=1}^{p} M_{ab}^{(r)} u_{i-a,j-b}^{(r)}$$

   *for all $i, j > p$.*
3. $u_{ij}^{(r)} = 0$ *if $i > pj$ or $j > pi$, so in particular $U(f_p^k)$ is a polynomial in $f_p$ of degree at most $pk$.*

*Proof.* Part (1) is an elementary manipulation. Given this, it is clearly sufficient to prove the existence of $M$ when $r = 0$. This result is well-known for $p = 2$, and may be found in Emerton's thesis [Eme98]; it is apparently initially due to Kolberg. The same approach may be used for the other values of $p$, or alternatively one may deduce the result from [Smi00, Lemma 3.3.2], where it is shown that there is a polynomial $I_p(x, y)$ of degree $p$ in each variable such that $I_p(V(f_p), \frac{1}{f_p}) = 0$, where $V$ is the operator induced by $q \mapsto q^p$. Smithline produces this identity by noting that there exists a polynomial $H_p$ of degree $p+1$ with integer coefficients such that $\frac{H_p(f_p)}{f_p}$ is the level 1 $j$-invariant, and thus we have

$$\frac{H_p(p^{-12/(p-1)}/f_p)}{p^{-12/(p-1)}/f_p} = \frac{H_p(V(f_p))}{V(f_p)}$$

since both sides are equal to $V(j)$. Clearing denominators and cancelling the factor $V(f_p) - p^{-12/(p-1)}/f_p$ (which is clearly not identically zero) gives $I_p$, and it is thus clear that $I_p$ has integer coefficients, total degree $p+1$, constant coefficient equal to 1 and all linear terms zero. Multiplying by $f_p^j$, applying $U$ and using "Coleman's trick" — the identity $U(fV(g)) = gU(f)$ — gives the required recurrence, with $M_{ab}$ being the coefficient of $x^a y^b$ in $-I_p(x, y)$. So part (2) of the theorem follows.

Finally, since $U(1) = 1$ and coefficients of the recurrence are polynomials in $f_p$ of degree at most $p$, it follows by induction that $U(f_p^j)$ must be a polynomial of degree at most $pj$ in $f_p$; thus $u_{ij} = 0$ if $i > pj$. On the other hand, it is immediate from the $q$-expansion that if $j > pi$, $U(f_p^j)$ must vanish to degree $i$ at the origin, so $u_{ij} = 0$ in this region as well. □

The polynomials $H_p$ are easy to compute by comparing $q$-expansions, and hence we can easily determine the polynomials $I_p$ explicitly (they are tabulated in [Smi00, §3.3]) and thus the matrices $M$. For example, when $p = 2$ we find that

$$M^{(0)} = \begin{pmatrix} 48 & 1 \\ 2^{12} & 0 \end{pmatrix},$$

and when $p = 3$,

$$M^{(0)} = \begin{pmatrix} 270 & 36 & 1 \\ 26244 & 729 & 0 \\ 531441 & 0 & 0 \end{pmatrix}.$$

**Corollary 4.** *The operator $U$ is an "operator of rational generation" in Smithline's sense; that is, there exists a rational function $R(x, y)$ whose Taylor series expansion is equal to $\sum_{i,j} u_{ij} x^i y^j$. The function $R$ is equal to*

$$-\frac{y}{p}\frac{\partial}{\partial y} \log I_p(x, y).$$

## 3. Computations of slopes

If $X$ is any compact operator acting on a $p$-adic Banach space, it has a (possibly empty!) countable set of nonzero eigenvalues, for each of which the generalised eigenspace $\bigcup_{k=1}^{\infty} \operatorname{Ker}\left[(U - \lambda_i)^k\right]$ is finite-dimensional. The $p$-adic valuations of these eigenvalues are known as the *slopes*. The finite slope eigenvalues occur as the inverses of roots of the characteristic power series $\det(I - tX)$.

In our case, it is known that $U$ is compact for $r \in (0, \frac{p}{p+1})$. Given the values of $u_{ij}^{(r)}$ for $1 \leq i, j \leq N$, it is easy to calculate the characteristic power series of this $N \times N$ matrix (since the entries are rational); and the general theory of compact operators tells us that this will converge rapidly to the characteristic power series of $U$. So we can easily calculate approximations to the eigenvalues, and in particular we can determine the slopes. The results obtained will be independent of $r$, since it is known that any overconvergent $U$-eigenform of finite slope must extend to a function on $X_0(1)_{\geq p^{-r}}$ for all $r < \frac{p}{p+1}$ (see [Buz03]).

The slopes of $U$ are somewhat mysterious; the complete list of slopes is known only for $p = 2$, tame level 1 and weight 0 by [BC05], and for 2-adic, 3-adic and

5-adic weights near the boundary of weight space by [BK05], [Jac03] and [Kil06] respectively. There are conjectures ([Buz05], [Cla05]) for a general weight, prime and level, but these appear to be rather inaccessible at present.

In the approach of [BC05], the next step would be to attempt to decompose the $U$ operator as $U = ADB$ where $A$ is lower triangular, $B$ is upper triangular, $D$ is diagonal, and both $A$ and $B$ have all diagonal entries 1. If this factorisation exists (which is the case if none of the top left $r \times r$ minors are singular) then it is unique, and can be calculated rapidly by Gaussian elimination; usefully, the $i, j$ entry of each of $A, B, D$ is determined by $u_{mn}$ for $mn \le \max(i, j)$, so in our case the entries of these matrices are rational and can be calculated exactly using our algorithm for calculating $U$.

**Conjecture 5.** *For $p \in \{2, 3, 5\}$ and all $r$ in some open interval containing $\frac{1}{2}$, the $U$ operator acting on $\mathcal{S}_k(r)$ has a factorisation $U^{(r)} = A^{(r)} D B^{(r)}$, where $A^{(r)}$ and $B^{(r)}$ have entries in $\mathcal{O}_{\mathbb{C}_p}$ and are congruent to the identity modulo $p$, and the entries of $D$ are given by the following formulae:*

| $p$ | $D_{ii}$ | $\nu_p D_{ii}$ |
|---|---|---|
| 2 | $\dfrac{2^{4i+1}(3i)!^2 i!^2}{3 \cdot (2i)!^4}$ | $1 + 2\nu_2\left(\frac{(3i)!}{i!}\right)$ |
| 3 | $\dfrac{3^{3i}(6i)!(2i)!i!}{2 \cdot (3i)!^3}$ | $2i + 2\nu_3\left(\frac{(2i)!}{i!}\right)$ |
| 5 | $\dfrac{5^{2i}(10i)!(3i)!^2 i!}{3 \cdot (5i)!^3 (2i)!}$ | $i + 2\nu_5\left(\frac{(3i)!}{i!}\right)$ |

This is known in the case $p = 2$, by [BC05] (for $r = \frac{1}{2}$, but we extend the result to all $r \in (\frac{5}{12}, \frac{7}{12})$ below). For $p = 3$ and $p = 5$ it is open, but a calculation of $U_{ij}$ for $1 \le i, j \le 100$ suggests that the conjecture holds for $r \in (\frac{1}{3}, \frac{2}{3})$ in both cases. However, the same computation suggests that the entries of $A$ and $B$ are not given by any hypergeometric term (as they are divisible by too many large primes).

If this conjecture is true, then lemma 5 of [BC05] would tell us that the Newton polygon of $ADB$ is the same as that of $D$, so the $i$th slope would be equal to the valuation of the $i$th diagonal entry of $D$. Indeed, Frank Calegari has conjectured formulae for the slopes for $p = 3$ and $p = 5$ (cited in [Smi04]), and these agree with those given in the third column above. Furthermore, these formulae also appear to agree with the combinatorial recipe of [Buz05]; but without a concise formula for $A_{ij}$ and $B_{ij}$, there does not seem to be any chance of proving these results by this method.

For $p = 7$ and $p = 13$ the pattern is much less clear; there still appears to be an $ADB$ factorisation with $A$ and $B$ congruent to the identity, but the entries of $D$ do not appear to be given by any simple hypergeometric form. It is interesting to note that in these cases, there are several distinct "slope modules" in the conjectural picture of [Cla05], so one would not expect all the slopes to be given by a single simple formula.

## 4. Computations of eigenfunctions

If $M$ is an $n \times n$ matrix over a $p$-adic field, then calculating the eigenvalues and eigenvectors of $M$ to any desired degree of accuracy is computationally very easy, as Hensel's lemma allows easy calculation of the eigenvalues. More generally, if $M$ is the matrix of a compact operator and $M_n$ is the $n \times n$ truncation, then one can calculate the eigenvectors of $M$ using $M_n$: if $\lambda$ is an eigenvalue of $M$, and $n$ is sufficiently large compared to the slope of $\lambda$, then there will be an eigenvalue $\lambda_n$ of $M_n$ which is highly congruent to $\lambda$, and and as $n \to \infty$, $\lambda_n$ will converge to $\lambda$ and the associated eigenvectors $v_n$ will converge to an eigenvector of $M$.

Let us do this in the case $p = 5$ (for comparison with the calculations in [GM95]). We begin by fixing a value of $r$; in this case, it is convenient to choose $r = \frac{1}{3}$, since in this case we may take $c = p$ and the $u_{ij}$ are all rational. We now take an $N \times N$ truncation of the matrix of $U$ and diagonalise this using the PARI/GP functions `polrootspadic()` and `matker()`; this gives an approximate $U$-eigenfunction. As it is necessary to divide by entries of the matrix in this computation, the resulting eigenvector is known to slightly less precision than the eigenvalue; but this is not a serious problem as calculating the roots of $p$-adic polynomials is computationally very easy – working modulo $5^{300}$ is no problem on current machines.

If we take $N = 3$, we obtain three eigenvalues of slopes $\sigma_1 = 1$, $\sigma_2 = 4$ and $\sigma_3 = 5$, and three corresponding approximate eigenfunctions $\phi_1$, $\phi_2$ and $\phi_3$. Repeating the calculation for a range of $N$, it seems that changing $N$ does not change $\phi_1 \bmod 5^8$, so the value obtained for $N = 3$ is apparently already correct to this precision; moreover, taking $N = 4$ is enough to give it mod $5^{10}$, and $N = 5$ gives it mod $5^{16}$. So the functions obtained appear to be converging very rapidly in the $q$-expansion topology (or, equivalently, in the supremum norm on $X_0(1)_{\mathrm{ord}}$). The first 30 terms of the $q$-expansion of the first few $\phi_i$ is given modulo $5^{15}$ in §8.

## 5. Spectral expansions

It is a standard consequence of the spectral theory that for each nonzero eigenvalue $\lambda_i$ of $U$, there is a projection $\pi_i$ onto the corresponding generalised eigenspace, and this projection commutes with $U$. Since for any $x \geq 0$, the set $I_x$ of indices $i$ such that $\lambda_i$ has slope $\leq x$ is finite, one can form for any $h \in S_k(r)$ the series

$$e_x(h) = \sum_{i \in \Lambda_x} \pi_i(h).$$

This is known as the *asymptotic $U$-spectral expansion of $h$*. This will not generally converge as $x \to \infty$; but it is uniquely determined by the property that for any $x$ there exists $\epsilon > 0$ with $\nu_p\left(\left\|U^k(h - e_x(h))\right\|\right) \geq (x + \epsilon)k$ for all $k \gg 0$.

For $p = 2, 3, 5$, all the generalised eigenspaces are conjecturally one-dimensional, spanned by eigenfunctions $\phi_i$, so we should obtain a sequence of constants $c_i(h) = \pi_i(f)/\phi_i$. In principle, the spectral theory gives an explicit form for the spectral projections $\pi_i$. The first projection $\pi_1$ is easy, as one simply iterates the process of applying $U$ and dividing by the eigenvalue $\lambda_1$. One can then consider $h' = h - \pi_1(h)$ and iterate $U$ on this; the same process of iterating and dividing by $\lambda_2$ should converge to the second projection $\pi_2$, but this is unstable with regard to small errors in the calculation of $\pi_1(h)$ – such errors will inevitably grow at a rate of $(\lambda_1/\lambda_2)^k$ until they swamp the desired answer. So this method is not really usable in practice.

However, the symmetry properties of $U$ provide us with an alternative approach. Let $g = p^{6/(p-1)} f$, so $(g, g^2, g^3, \dots)$ are a basis for $\mathcal{S}_0(\frac{1}{2})$.

**Theorem 6.** *Define the symmetric bilinear form $\langle , \rangle$ on $\mathcal{S}_0(\frac{1}{2})$ by*

$$\langle g^i, g^j \rangle = \begin{cases} i & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

*Then $U$ is self-adjoint with respect to this form; and for all $i$ such that the $\lambda_i$ eigenspace is 1-dimensional and $\langle \phi_i, \phi_i \rangle \neq 0$, the spectral projection operators $\pi_i$ are given by $\pi_i(h) = c_i(h)\phi_i$ where*

$$c_i(h) = \frac{\langle h, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle}.$$

*Furthermore, the same formula in fact gives us a pairing* $\mathcal{S}_0(r) \times \mathcal{S}_0(1-r) \to \mathbb{C}_p$ *for any* $r \in (\frac{1}{p+1}, \frac{p}{p+1})$.

*Proof.* If $p \in \{2, 3, 5, 7, 13\}$, then we can show that $U$ is self-adjoint with respect to this bilinear form by proving that $u_{ij}^{(1/2)} = \frac{j}{i} u_{ji}^{(1/2)}$. This follows from Corollary 4 above; the generating function $R(x, y)$ is $\frac{y}{p} \frac{\partial}{\partial y} \log I_p(x, y)$, and from the construction of $I_p$ we see that it satisfies

$$I_p(x, y) = I_p(p^{-12/(p-1)} y, p^{12/(p-1)} x),$$

so after an appropriate rescaling we see that $x \frac{\partial}{\partial x} R(x, y)$ is symmetric in $x$ and $y$, implying the result.

However, one can prove this in general – without the assumption that $X_0(p)$ have genus 0 – by using the theory of residues of $p$-adic differential forms. This theory is developed in [FvdP04]; for a general rigid space $X/k$ we can construct sheaves of finite differentials $\Omega^f_{X/k}$, and the notion of residue of a differential at a point can be defined in a consistent way. Now, if $\alpha$ and $\beta$ are in $\mathcal{S}_0(\frac{1}{2})$, and $w$ denotes the Atkin-Lehner involution on $X_0(p)$, then the differential

$$w^*(\alpha).\mathrm{d}\beta$$

is defined on the annulus $|A| = p^{-1/2}$ (a "ring domain") and thus has a residue at the cusp $\infty$. It is readily seen that if we define

$$\langle \alpha, \beta \rangle = \mathrm{Res}_{z=\infty} w^*(\alpha).\mathrm{d}\beta$$

then this agrees with the above definition when $p \in \{2, 3, 5, 7, 13\}$ (it is sufficient to check the result when $\alpha$ and $\beta$ are powers of $f$; in this case it is immediate from the fact that $w^*(g) = \frac{1}{g}$.)

Let $\Phi_1$ and $\Phi_2$ be the two canonical maps $X_0(p^2) \to X_0(p)$, namely $\Pi_1 : (E, C) \mapsto (E, C[p])$ and $\Pi_2 : (E, C) \mapsto (E/C[p], C/C[p])$; this gives a symmetric correspondence on $X_0(p)$, and the operator on functions corresponding to the trace of this correspondence is $U$. So we may write

$$\begin{aligned}
\langle U\alpha, \beta \rangle &= \mathrm{Res}_{\infty \in X_0(p)}\, w^*(U\alpha)\ \mathrm{d}\beta \\
&= \mathrm{Res}_{\infty \in X_0(p)}\, U(w^*\alpha)\ \mathrm{d}\beta \\
&= \mathrm{Res}_{\infty \in X_0(p)}\, \Phi_{2*}\Phi_1^* w^*\alpha\ \mathrm{d}\beta \\
&= p\, \mathrm{Res}_{\infty \in X_0(p^2)}\, \Phi_1^* w^*\alpha\ \mathrm{d}\Phi_2^*\beta \\
&= \mathrm{Res}_{\infty \in X_0(p)}\, w^*\alpha\ d\Phi_{1*}\Phi_2^*\beta \\
&= \langle \alpha, U\beta \rangle.
\end{aligned}$$

It now follows that any two eigenfunctions with different eigenvalues must be orthogonal, and the explicit form for the spectral projection operators is immediate. □

(Exactly the same argument also shows that the operators $T_\ell$ are self-adjoint for $\ell \neq p$.)

This pairing allows us to calculate spectral expansions extremely easily for functions $h$ that are at least $\frac{1}{2}$-overconvergent, given sufficiently accurate knowledge of the eigenfunctions themselves. As in the previous section, we shall take $p = 5$. Then the function $h = \frac{1}{j}$ is $r$-overconvergent for all $r < \frac{5}{6}$, and the constants $c_i$

turn out to be:

| $i$ | $c_i$ |
|---|---|
| 1 | 8295001 |
| 2 | $5^4 \times 7540786$ |
| 3 | $5^4 \times 2165317$ |
| 4 | $5^8 \times 8075994$ |
| 5 | $5^9 \times 4502966$ |
| 6 | $5^{10} \times 4930721$ |
| 7 | $5^{12} \times 7120582$ |
| 8 | $5^{14} \times 7314891$ |
| 9 | $5^{18} \times 2324226$ |
| 10 | $5^{22} \times 1076376$ |
| $\vdots$ | $\vdots$ |

Here, as in the tables of eigenfunctions in §8, we use a relative precision of $O(5^{10})$ – that is, we write a general element of $\mathbb{Z}_5$ in the form $5^a b$ where $b \in (\mathbb{Z}/5^{10}\mathbb{Z})^\times$. These numbers appear to be tending 5-adically to zero extremely rapidly, suggesting that the $U$-spectral expansion is in fact convergent, at least in the (rather feeble) $q$-expansion topology.

One might optimistically make the following conjecture:

**Conjecture 7** (Gouvêa-Mazur spectral expansion conjecture, strong form)**.** *Let $h$ be any $r$-overconvergent modular function, where $r \in (\frac{1}{p+1}, \frac{p}{p+1})$. Then the spectral expansion of $h$ converges to $h$, in the supremum norm of $X_0(1)_{\ge p^{-r}}$.*

One cannot expect this to work for $r \le \frac{1}{p+1}$, for two reasons. Firstly, since the eigenfunctions themselves are not necessarily any more than $\frac{p}{p+1}$-overconvergent, we cannot guarantee that the linear functional $\langle \cdot, \phi_i \rangle$ even makes sense. More seriously, if $r < \frac{1}{p+1}$ then there exist nonzero functions in the kernel of $U$; the spectral expansion of any such form is always zero.

## 6. The spectral expansion conjecture

Let us now suppose either that $p = 2$, or that $p = 3$ or 5 and Conjecture 5 above holds. We shall show that this implies the spectral expansion conjecture.

Let $A^{(r)}$ and $B^{(r)}$ be the matrices occurring in the $LDU$ factorisation of $U^{(r)}$. ($D$ is clearly independent of $r$.)

**Lemma 8.** *For $p = 2$, Conjecture 5 holds for all $r \in \left(\frac{5}{12}, \frac{7}{12}\right)$; that is, for any $r$ in this range, $A^{(r)}$ and $B^{(r)}$ have entries in $\mathcal{O}_{\mathbb{C}_2}$ and their reductions modulo the maximal ideal are equal to the identity matrix.*

*Proof.* Since by construction $A$ is lower triangular, $B$ is upper triangular and their diagonal entries are 1, it is sufficient to prove that $A^{(\frac{7}{12})}$ and $B^{(\frac{5}{12})}$ have entries in $\mathcal{O}_{\mathbb{C}_2}$. Conveniently, we may choose $c$ to be an integer power of $p$ in these cases, so the matrices have entries in $\mathbb{Q}_p$. Suppose $2j \ge i > j \ge 0$. Then we shall show the stronger statement that $a_{ij}^{(7/12)}/j = b_{ji}^{(5/12)}/i \in \mathbb{Z}_2$. From [BC05] we know that

$$a_{ij}^{(\frac{7}{12})} = 2^{j-i} a_{ij}^{(\frac{1}{2})} = 2^{j-i} \cdot 6ij \left(\frac{(2j)!}{2^j j!}\right)^2 \left(\frac{2^i i!}{(2i)!}\right)^2 \frac{(2i-1)!}{(i+j)!} \frac{(2j+i-1)!}{(3j)!} \binom{j}{i-j}.$$

The first two bracketed terms are clearly in $\mathbb{Z}_2^\times$, so we can safely ignore them. If we put $i = j + t$, what is left is

$$2^{1-t} \cdot 3ij \left(\frac{(2j+2t-1)!}{(2j+t)!}\right) \left(\frac{(3j+t-1)!}{(3j)!}\right) \binom{j}{t}.$$

If $t$ is odd, we are safe, as the two factorial terms each simplify to products of $t-1$ consecutive integers, and each product contains $\frac{t-1}{2}$ even integers which cancel all the factors of 2 in the denominator. If $t$ is even, then we are in slightly more trouble. The first product always ends on an odd integer so it has $\frac{t}{2}-1$ even terms, and the second one depends on $j$; if $3j+1$ is even, we get $\frac{t}{2}$ even factors, but if $(3j+1)$ is odd, then we are one short. However, this occurs only if $j$ is even, and consequently $i$ is even; so $a_{ij}/j \in \mathbb{Z}_2$, as claimed. □

**Theorem 9.** *Let $K$ be a field complete with respect to a non-archimedean valuation, with ring of integers $\mathcal{O}_K$ and maximal ideal $\mathfrak{M}_K$. Let $S$ be the space of sequences over $K$ with entries tending to zero. Then if $M$ is any operator on $S$ given by a matrix of the form $ADB$ where $D$ is diagonal with strictly increasing valuations and $A$,$B$ have entries in $\mathcal{O}_K$ congruent to the identity modulo $\mathfrak{M}_K$, then we can find a matrix $C$, also with integral entries congruent to the identity, such that $C^{-1}MC$ is diagonal.*

*Proof.* The statement is not affected by conjugating $M$ by any matrix congruent to the identity, so we conjugate by $B^{-1}$, allowing us to assume without loss of generality that $M = AD$. It is known (see [BC05]) that $M$ has the same Newton polygon as $D$. Hence, for every $j$ there is an eigenvector $v_j$ such that $Mv_j = \mu_j v_j$ with $\frac{\mu_j}{D_{jj}} \in \mathcal{O}_K^\times$, and $v_j$ is unique up to scalars. We normalise $v_j$ so it is integral with norm 1.

Suppose $Dv_j = \eta_j w_j$, where $w_j$ has norm 1 and $\eta_j \in K$. Then since $A = \mathrm{Id} \bmod \mathfrak{M}_K$, $\mu_j v_j = ADv_j = \eta_j A w_j$. Comparing norms, we see that $\varepsilon_j = \eta_j^{-1}\mu_j \in \mathcal{O}_K^\times$, and reducing mod $\mathfrak{M}_K$ we have $\overline{\varepsilon_j}\ \overline{v_j} = \overline{A}\ \overline{w_j}$. But $\overline{A}$ is the identity, and consequently $\overline{\varepsilon_j}\ \overline{v}_j = \overline{w}_j$. This is impossible unless $\overline{v_j}$ has all its components zero outside the $j$th.

Now if $C$ is the matrix whose $j$th column is $v_j$, then we evidently have $MC = CE$ where $E$ is the diagonal matrix with $E_{ii} = \mu_i$, and since $C$ is congruent to the identity, it is necessarily invertible (since the series $(1+T)^{-1} = 1 - T + T^2 + \dots$ converges whenever $|T| < 1$). □

**Corollary 10** (Spectral expansion theorem)**.** *For any $r \in \left(\frac{5}{12}, \frac{7}{12}\right)$, the finite slope eigenfunctions form an orthonormal basis of the space $\mathcal{S}_0(r)$; that is, for all $h \in \mathcal{S}_0(r)$, the sum*

$$\sum_{i=1}^{\infty} \pi_i(h)$$

*converges to $h$, and $\|h\| = \sup_i \|\pi_i(h)\|$.*

Note in particular that this implies that the kernel of $U$ is zero for all $r > \frac{5}{12}$; it is in fact known that the kernel is zero for $r \geq \frac{1}{p+1}$, by Lemma 6.13 of [BC06].

## 7. Appendix A: Overconvergent forms at small level

In this appendix, we finish off the proof of Theorem 1 in order to show that the space we work with really is the same as the space of $r$-overconvergent $p$-adic modular forms, for each $p \in \{2, 3, 5, 7, 13\}$. Since we work only with weight zero forms, the problem of whether or not the sheaf $\omega^{\otimes k}$ descends does not arise, and hence the problem is reduced to identifying in terms of our chosen uniformiser the region of $X_0(p)$ corresponding to the $r$-overconvergent locus. For $p \geq 5$, the Hasse invariant lifts to level 1 via the classical level 1 Eisenstein series $E_{p-1}$, so we can measure overconvergence directly using this form; the argument is given in [Smi01, Prop 3.5]. However, for $p = 2$ and $p = 3$, the Hasse invariant does not lift to characteristic 0 in level 1, so we need to introduce auxiliary level structure. The case $p = 2$ is covered in [BC05, §4], using a weight 1 $\theta$ series of level 3 as a Hasse lifting, so we are left with the case $p = 3$. Smithline shows that in this case the region where $|f_3| \leq 3^{6r}$ coincides with the region where $|E_6| \geq 3^{-3r}$, for all $r < \frac{3}{4}$; so we must compare the valuations of $E_6$ and the Hasse invariant.

Consider the 2-stabilised Eisenstein series $E_2' = 2E_2(2z) - E_2(z)$, which is a modular form of weight 2 and level $\Gamma_0(2)$. Since $E_2(z) \equiv E_2(2z) \equiv 1 \bmod 3$, $E_2'$ is a lift of the mod 3 Hasse invariant. Using our parameter $f_2$ on $X_0(2)$, we have the identities

$$\frac{E_2'^6}{\Delta} = \frac{(1 + 2^6 f_2)^3}{f_2}$$

and

$$\frac{E_6^2}{\Delta} = \frac{(1 + 2^6 f_2)(1 - 2^9 f_2)^2}{f_2}.$$

The supersingular region corresponds to $|1 + 2^6 f_2| < 1$; in this region $|f_2| = 1$, so if $|1 + 2^6 f_2| > 3^{-2}$, then $|1 + 2^6 f_2| = |1 + 2^6 f_2 - 9.2^6 f_2| = |1 - 2^9 f_2|$. Since supersingular curves have good reduction, $|\Delta| = 1$ also, hence

$$\begin{aligned} |E_2'| \geq 3^{-r} &\iff \left|\frac{E_2'^6}{\Delta}\right| \geq 3^{-6r} \\ &\iff \left|\frac{E_6^2}{\Delta}\right| \geq 3^{-6r} \\ &\iff |E_6| \geq 3^{-3r} \end{aligned}$$

for all $r < 1$, and the result follows.

## 8. Appendix B: $q$-expansions of small slope 5-adic eigenfunctions

The following list gives the first 20 terms of the $q$-expansions of the 20 smallest slope 5-adic eigenforms, with the coefficients given to a relative precision of $O(5^{10})$. This computation took less than 1 minute on a standard laptop PC.

$$\begin{aligned} \phi_1 = q &+ 8528631q^2 + 8596652q^3 + 2788848q^4 + 5 \times 610813q^5 + 6727787q^6 \\ &+ 2747331q^7 + 5 \times 3412617q^8 + 6989312q^9 + 5 \times 4155753q^{10} + 538817q^{11} \\ &+ 9643146q^{12} + 6371187q^{13} + 5536986q^{14} + 5 \times 9298076q^{15} + 8198461q^{16} \\ &+ 3226656q^{17} + 5179372q^{18} + 5 \times 9335108q^{19} + 5 \times 7582174q^{20} + O(q^{21}) \end{aligned}$$

$$\begin{aligned} \phi_2 = q &+ 441709q^2 + 2550713q^3 + 4301618q^4 + 5^4 \times 2356503q^5 + 2966642q^6 \\ &+ 3223594q^7 + 5 \times 9703174q^8 + 7251077q^9 + 5^4 \times 9677377q^{10} + 3828592q^{11} \\ &+ 5453634q^{12} + 4410268q^{13} + 3763396q^{14} + 5^4 \times 1117889q^{15} + 1692896q^{16} \\ &+ 2395464q^{17} + 4642468q^{18} + 5 \times 2705229q^{19} + 5^4 \times 8143729q^{20} + O(q^{21}) \end{aligned}$$

$$\begin{aligned} \phi_3 = q &+ 7123391q^2 + 727387q^3 + 8909193q^4 + 5^5 \times 6386403q^5 + 6931192q^6 \\ &+ 3140781q^7 + 5 \times 2842166q^8 + 3306102q^9 + 5^5 \times 3855698q^{10} + 1486467q^{11} \\ &+ 1481191q^{12} + 909182q^{13} + 3295871q^{14} + 5^5 \times 5659586q^{15} + 2077746q^{16} \\ &+ 7148211q^{17} + 2935007q^{18} + 5 \times 6743039q^{19} + 5^5 \times 1590279q^{20} + O(q^{21}) \end{aligned}$$

$$\phi_4 = q + 2764444q^2 + 5364423q^3 + 7074448q^4 + 5^8 \times 6938782q^5 + 8303937q^6 + 2059419q^7 + 5 \times 5835813q^8 + 6128137q^9 + 5^8 \times 9032833q^{10} + 9024817q^{11} + 9297879q^{12} + 3774838q^{13} + 3966786q^{14} + 5^8 \times 3159036q^{15} + 908886q^{16} + 1286194q^{17} + 2888953q^{18} + 5 \times 3751388q^{19} + 5^8 \times 5567336q^{20} + O(q^{21})$$

$$\phi_5 = q + 5791436q^2 + 3059457q^3 + 3403033q^4 + 5^9 \times 8921438q^5 + 6832127q^6 + 3955981q^7 + 5 \times 3439059q^8 + 6952557q^9 + 5^9 \times 7517468q^{10} + 9760342q^{11} + 7351831q^{12} + 8002297q^{13} + 231841q^{14} + 5^9 \times 79791q^{15} + 4456166q^{16} + 7616646q^{17} + 5698727q^{18} + 5 \times 7110866q^{19} + 5^9 \times 6515204q^{20} + O(q^{21})$$

$$\phi_6 = q + 6831044q^2 + 1698148q^3 + 2950248q^4 + 5^{10} \times 6825297q^5 + 6519012q^6 + 8819044q^7 + 5 \times 5659178q^8 + 8713237q^9 + 5^{10} \times 7635693q^{10} + 4926567q^{11} + 6568829q^{12} + 5335163q^{13} + 6117561q^{14} + 5^{10} \times 3121831q^{15} + 9149661q^{16} + 3456869q^{17} + 7282553q^{18} + 5 \times 82178q^{19} + 5^{10} \times 464281q^{20} + O(q^{21})$$

$$\phi_7 = q + 8461691q^2 + 7744062q^3 + 4618543q^4 + 5^{13} \times 9616002q^5 + 8166342q^6 + 9150156q^7 + 5 \times 7971386q^8 + 1468177q^9 + 5^{13} \times 860632q^{10} + 5105092q^{11} + 4044791q^{12} + 5464782q^{13} + 1658171q^{14} + 5^{13} \times 1617624q^{15} + 6957796q^{16} + 2187611q^{17} + 8154182q^{18} + 5 \times 4201019q^{19} + 5^{13} \times 4662586q^{20} + O(q^{21})$$

$$\phi_8 = q + 9458634q^2 + 1415388q^3 + 310018q^4 + 5^{14} \times 7929152q^5 + 341242q^6 + 8941094q^7 + 5 \times 5522594q^8 + 6133252q^9 + 5^{14} \times 1385868q^{10} + 1356842q^{11} + 6694484q^{12} + 1201868q^{13} + 8361846q^{14} + 5^{14} \times 4325351q^{15} + 165471q^{16} + 8543864q^{17} + 8163393q^{18} + 5 \times 8748199q^{19} + 5^{14} \times 6016611q^{20} + O(q^{21})$$

$$\phi_9 = q + 1036606q^2 + 8499877q^3 + 6100798q^4 + 5^{19} \times 9288232q^5 + 7872462q^6 + 6770081q^7 + 5 \times 8252407q^8 + 2114087q^9 + 5^{19} \times 4598717q^{10} + 7406442q^{11} + 7211221q^{12} + 9554887q^{13} + 6194461q^{14} + 5^{19} \times 1422464q^{15} + 9065311q^{16} + 5385831q^{17} + 659347q^{18} + 5 \times 9351018q^{19} + 5^{19} \times 2209136q^{20} + O(q^{21})$$

$$\phi_{10} = q + 8935814q^2 + 2184043q^3 + 7194158q^4 + 5^{20} \times 9176128q^5 + 844127q^6 + 1292144q^7 + 5 \times 1755091q^8 + 8018557q^9 + 5^{20} \times 1173192q^{10} + 9267217q^{11} + 6670794q^{12} + 8784078q^{13} + 1023341q^{14} + 5^{20} \times 3438004q^{15} + 9735791q^{16} + 7839479q^{17} + 9681648q^{18} + 5 \times 9158266q^{19} + 5^{20} \times 2941474q^{20} + O(q^{21})$$

$$\phi_{11} = q + 8097156q^2 + 5482427q^3 + 4624273q^4 + 5^{21} \times 3090372q^5 + 6130737q^6 + 9435206q^7 + 5 \times 3663802q^8 + 7112412q^9 + 5^{21} \times 1525782q^{10} + 9588067q^{11} + 2822446q^{12} + 9371737q^{13} + 4796011q^{14} + 5^{21} \times 4517844q^{15} + 9306236q^{16} + 2578856q^{17} + 6765897q^{18} + 5 \times 5575723q^{19} + 5^{21} \times 1143306q^{20} + O(q^{21})$$

$$\phi_{12} = q + 9675784q^2 + 6753913q^3 + 116218q^4 + 5^{24} \times 8946888q^5 + 8811542q^6 + 8069219q^7 + 5 \times 6507279q^8 + 2269902q^9 + 5^{24} \times 1463317q^{10} + 3569092q^{11} + 4386034q^{12} + 8715668q^{13} + 4467696q^{14} + 5^{24} \times 9032119q^{15} + 3147446q^{16} + 1255689q^{17} + 5281293q^{18} + 5 \times 2446659q^{19} + 5^{24} \times 4273334q^{20} + O(q^{21})$$

$$\phi_{13} = q + 852841q^2 + 6464712q^3 + 8669718q^4 + 5^{25} \times 9222513q^5 + 2306167q^6 + 7752656q^7 + 5 \times 1741296q^8 + 4498152q^9 + 5^{25} \times 5178183q^{10} + 8249092q^{11} + 3428716q^{12} + 4365957q^{13} + 5551946q^{14} + 5^{25} \times 1789381q^{15} + 4399821q^{16} + 7853311q^{17} + 7277957q^{18} + 5 \times 2773209q^{19} + 5^{25} \times 4930084q^{20} + O(q^{21})$$

$$\phi_{14} = q + 3696344q^2 + 5088573q^3 + 4864773q^4 + 5^{28} \times 7513547q^5 + 4948987q^6 + 9082919q^7 + 5 \times 5387723q^8 + 4212787q^9 + 5^{28} \times 7887793q^{10} + 5486817q^{11} + 6445179q^{12} + 264638q^{13} + 9163761q^{14} + 5^{28} \times 9742181q^{15} + 5608361q^{16} + 3782269q^{17} + 5653853q^{18} + 5 \times 2678998q^{19} + 5^{28} \times 1361081q^{20} + O(q^{21})$$

$$\phi_{15} = q + 5997936q^2 + 2852832q^3 + 6767908q^4 + 5^{29} \times 3494278q^5 + 239127q^6 + 8242231q^7 + 5 \times 5754659q^8 + 1331682q^9 + 5^{29} \times 8544583q^{10} + 7742217q^{11} + 9202956q^{12} + 5295922q^{13} + 6847716q^{14} + 5^{29} \times 4110921q^{15} + 7441416q^{16} + 6452396q^{17} + 9423977q^{18} + 5 \times 2768516q^{19} + 5^{29} \times 6717924q^{20} + O(q^{21})$$

$$\phi_{16} = q + 7855519q^2 + 4239748q^3 + 3954673q^4 + 5^{30} \times 8731987q^5 + 1047337q^6 + 7593044q^7 + 5 \times 6656568q^8 + 7374337q^9 + 5^{30} \times 2676878q^{10} + 5407692q^{11} + 536154q^{12} + 2961238q^{13} + 6487961q^{14} + 5^{30} \times 8403651q^{15} + 524436q^{16} + 8063044q^{17} + 6134653q^{18} + 5 \times 7095743q^{19} + 5^{30} \times 4787751q^{20} + O(q^{21})$$

$$\phi_{17} = q + 3058366q^2 + 808487q^3 + 3957143q^4 + 5^{35} \times 4332043q^5 + 2667867q^6 + 2677656q^7 + 5 \times 9265831q^8 + 2140627q^9 + 5^{35} \times 8177988q^{10} + 8770592q^{11} + 1797641q^{12} + 6220257q^{13} + 4023221q^{14} + 5^{35} \times 7870816q^{15} + 1693096q^{16} + 9074636q^{17} + 4429232q^{18} + 5 \times 7074024q^{19} + 5^{35} \times 3867524q^{20} + O(q^{21})$$

$$\phi_{18} = q + 4792184q^2 + 9735438q^3 + 3075793q^4 + 5^{36} \times 9618893q^5 + 6310342q^6 + 6556094q^7 + 5 \times 1549289q^8 + 6307052q^9 + 5^{36} \times 7085437q^{10} + 8972592q^{11} + 2599209q^{12} + 5715468q^{13} + 956796q^{14} + 5^{36} \times 5570759q^{15} + 552671q^{16} + 2538389q^{17} + 2900318q^{18} + 5 \times 6364319q^{19} + 5^{36} \times 1100899q^{20} + O(q^{21})$$

$$\phi_{19} = q + 3408581q^2 + 217102q^3 + 1581998q^4 + 5^{39} \times 2535503q^5 + 9752262q^6 + 1937831q^7 + 5 \times 7503797q^8 + 7627362q^9 + 5^{39} \times 6413743q^{10} + 6787817q^{11} + 7664171q^{12} + 3969712q^{13} + 21561q^{14} + 5^{39} \times 3787931q^{15} + 4478661q^{16} + 4153256q^{17} + 4630822q^{18} + 5 \times 6899078q^{19} + 5^{39} \times 8331244q^{20} + O(q^{21})$$

$$\phi_{20} = q + 7376064q^2 + 5111168q^3 + 6655533q^4 + 5^{40} \times 1816457q^5 + 9314002q^6 + 8378394q^7 + 5 \times 6422316q^8 + 8376307q^9 + 5^{40} \times 2303998q^{10} + 9013467q^{11} + 8230044q^{12} + 8742078q^{13} + 48716q^{14} + 5^{40} \times 7907401q^{15} + 463666q^{16} + 6617104q^{17} + 6593773q^{18} + 5 \times 2535366q^{19} + 5^{40} \times 7084706q^{20} + O(q^{21})$$

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