AUTHOR: **Ayşe Altıntaş** DEGREE: **Ph.D.**

TITLE: **Multiple Point Spaces and Finitely Determined Map-germs**

DATE OF DEPOSIT: ....

I agree that this thesis shall be available in accordance with the regulations governing the University of Warwick theses.

I agree that the summary of this thesis may be submitted for publication.

I **agree** that the thesis may be photocopied (single copies for study purposes only).

Theses with no restriction on photocopying will also be made available to the British Library for microfilming. The British Library may supply copies to individuals or libraries. subject to a statement from them that the copy is supplied for non-publishing purposes. All copies supplied by the British Library will carry the following statement:

> "Attention is drawn to the fact that the copyright of this thesis rests with its author. This copy of the thesis has been supplied on the condition that anyone who consults it is understood to recognise that its copyright rests with its author and that no quotation from the thesis and no information derived from it may be published without the author's written consent."

AUTHOR'S SIGNATURE: ....

USER'S DECLARATION

1. I undertake not to quote or make use of any information from this thesis without making acknowledgement to the author.
2. I further undertake to allow no-one else to use this thesis while it is in my care.

DATE SIGNATURE ADDRESS

....

....

....

....

....

Multiple Point Spaces and Finitely Determined Map-germs

by

Ayşe Altıntaş

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Department of Mathematics

February 2011

THE UNIVERSITY OF WARWICK

Contents

List of Tables

List of Figures

Acknowledgments

I would like to express my sincere gratitude to my supervisor Professor David Mond who supported me during the course of my studies and taught me a lot. Without his guidance and enthusiasm, this work would not be possible.

I would like to thank Meral Tosun for her encouragement, helpful suggestions and for lending me an ear whenever I needed. I would also like to acknowledge Kevin Houston and Paul Cadman's suggestions.

I am indebted to my parents, Hanife and Mustafa, for their constant support and looking after me even when I was miles away from home. Muharrem, Sezai, Zeynep, Nazlı and Gülay for their moral support. I am grateful to Thomas'ım for his patience and love, and for giving me reasons to smile everyday.

It is almost impossible to list everyone whom I owe an honest gratitude. But I cannot forget to thank Thomas Cuschieri, Anita, Stephen and Mirela for their friendship and for helping me keep my sanity; Azadeh, Shengtian, Yang, Michele and Sara for their company and lunchtime crossword sessions.

It is a pleasure to thank the staff at the Mathematics Institute for their assistance and warm welcome for all these years.

This research was sponsored by a Vice Chancellor's scholarship at the University of Warwick.

Declarations

I declare that this thesis and the work presented in it are my own and represent my own original research. Wherever contributions of others are included, I have endeavoured to ensure that this is stated clearly and attributed with explicit references. No part of this thesis has been submitted for a degree or any other qualification at Warwick University or any other institution.

Abstract

This thesis is mainly based on the study of singularities of holomorphic map-germs from n-space to p-space with $n < p$.

We show that a minimal resolution of the kernel of the multiplication map $\mu : \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} \to \mathcal{O}_{\mathbb{C}^n,0}$, $a \otimes b \mapsto ab$, is given by

$$0 \longrightarrow \mathcal{O}^r_{\mathbb{C}^n,0} \xrightarrow{f^*\lambda^1_1} \mathcal{O}^r_{\mathbb{C}^n,0} \longrightarrow \ker(\mu) \longrightarrow 0$$

where λ^1_1 is the matrix obtained from λ, a symmetric matrix presenting $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, by deleting the first row and the first column (Proposition 2.5.2). We prove that if f is a corank 1 map-germ with finite $\mathcal{A}_e$-codimension, then there exists a resolution of $\mathcal{O}_{D^k(f)}$ over $\mathcal{O}_{D^{k-1}(f)}$ given by

$$0 \to \mathcal{O}^{r-k+1}_{D^k(f)} \xrightarrow{\gamma} \mathcal{O}^{r-k+1}_{D^k(f)} \to \mathcal{O}_{D^{k+1}(f)} \to 0$$

in which γ is equal to $f^*\lambda^k_k$, the pullback of the matrix λ^k_k obtained from λ by deleting the rows $1, \ldots, k$ and the columns $1, \ldots, k$ (Theorem 2.6.6). As a corollary, we show that $\det f^*\lambda^{k-1}_{k-1} \cdot \det f^*\lambda^k_k$ defines a free divisor in $D^k(f)$ (Proposition 2.8.4).

We investigate finitely $\mathcal{A}$-determined map-germs from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ ($n \geq 3$) of corank ≥ 2 satisfying the Mond conjecture. We provide geometric criteria on finite determinacy for $n = 3$ (Theorem 4.4.1). We give two sets of examples of finitely $\mathcal{A}$-determined corank 2 map-germs from $\mathbb{C}^3$ to $\mathbb{C}^4$ which satisfy the conjecture. For the dimensions (n, p) with $n < p$, we prove a criterion which yields finitely $\mathcal{A}$-determined map-germs from the known ones (Theorem 5.1.2). We prove the existence of three *series* of finitely $\mathcal{A}$-determined map-germs of corank 2 from $\mathbb{C}^4$ to $\mathbb{C}^5$ which also support the conjecture.

We include a program code for a SINGULAR command that calculates $\mathcal{A}_e$-codimension, and a classification of 2-jets of corank 2 map-germs from 3-space to 4-space.

Introduction

The main topic of this thesis is the theory of singularities of holomorphic map-germs from n-space to p-space with $n < p$ and (n, p) are in the range of Mather's nice dimensions ([Mat70], or Remark 1.3.17). In particular, the interaction between multiple point spaces and finite $\mathcal{A}$-determinacy of map-germs is studied.

A map-germ is said to be finitely determined with respect to an equivalence relation if it is determined by its Taylor series expansion up to a degree k. The most important contribution was made by Mather in a series of papers titled "Stability of C^∞-mappings I-VI" where he introduced equivalence relations $\mathcal{R}$, $\mathcal{C}$, $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{A}$, and proved necessary and sufficient conditions on finite determinacy with respect to those relations among other results. Over the years, many others have contributed to the subject. We suggest [Wal81] to the reader for a survey.

In this thesis, we consider $\mathcal{A}$-equivalence which is a relation induced by an action of groups of local diffeomorphisms on the space of holomorphic map-germs. This action basically allows coordinate changes by local diffeomorphisms on the source and on the target space.

One of the intriguing problems in Singularity theory is to relate algebraic properties of map-germs with topological properties of the images of stable perturbations of those map-germs. In the theory of isolated hypersurface singularities and more generally of complete intersection singularities, we meet several important results regarding this type of relations. By the results of Milnor ([Mil68]), in the case of hypersurface singularities, and Hamm ([Ham71]), for isolated complete intersection singularities, the Milnor fibre of an n-dim isolated complete intersection singularity has the homotopy type of a wedge of n-spheres. The number of such spheres is called the *Milnor number* μ. Greuel proved that μ is equal to the dimension τ of the base of a miniversal deformation of the singularity for weighted homogeneous isolated complete intersection singularities ([Gre80]); and in [LS85], Looijenga and Steenbrink showed that $\mu \geq \tau$ holds for arbitrary isolated complete intersection singularities. From the works of Mond and others we see the evidence of a similar phenomenon in the theory of holomorphic mappings. In [Mon91], Mond proved that the image of a stable perturbation of a finitely $\mathcal{A}$-determined map-germ from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ has the homotopy type of a wedge of n-spheres for $n \leq 6$. The number μ_I of the spheres is an $\mathcal{A}$-invariant of a map-germ, and it is referred to as the *image Milnor number*. Pellikaan and de Jong (unpublished) then de Jong and van Straten ([dJvS91]) and later Mond ([Mon91]) proved that for any finitely $\mathcal{A}$-determined map-germ f from surface to 3-space, $\mathcal{A}_e$-codim$(f) \leq \mu_I(f)$ and with the equality if f is weighted homogeneous. In [Mon91], Mond conjectured that the same statement holds for map-germs from n-space to $(n+1)$-space with $n \leq 6$ (Conjecture 3.1.5). This thesis mainly focuses on finding new examples of finitely $\mathcal{A}$-determined corank 2 map-germs supporting the conjecture.

Classification of finitely $\mathcal{A}$-determined map-germs, or even finding examples, becomes harder as the dimensions and the corank get higher. However, one can consider a geometric interpretation of finite determinacy: by the results of ([Mat73]) and Gaffney ([Gaf75]), a finite map-germ is finitely $\mathcal{A}$-determined if and only if it has an isolated instability at the origin; that is, the multi-germ of the map-germ at the preimage of any point in a punctured neighbourhood of the origin is $\mathcal{A}$-stable (cf. Theorem 1.3.20). This theorem suggest a natural passage to the notion of multiple point spaces. The kth multiple point space on the target is the set of points which have k or more preimages. On the other hand, the kth multiple point space D^k is the closure of the set of k-tuple points having the same image under the map-germ and distinct components. For instance, Marar and Mond characterised finite determinacy and stability in terms of multiple point spaces for corank 1 map-germs: f is finitely $\mathcal{A}$-determined if and only if each multiple point space of dimension 1 or greater is an isolated complete intersection singularity; moreover, f is stable if and only if each multiple point space of dimension 1 or greater is a smooth space ([MM89, Theorem 2.14]). For corank ≥ 2 map-germs we run into difficulties since there is no description for the analytic structure of D^k for $k \geq 3$. However, one can consider their projections D^k_1 into the source as they, set theoretically, coincide with the preimages of multiple point spaces on the target. We follow this consideration in Chapter 4.

The outline of this thesis is as follows.

Background and conventions are explained in Chapter 1. We provide proofs where we could not find a direct reference. Hence, we do not claim originality. Mond and Pellikaan's work on presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ is also recalled in that chapter.

In Chapter 2 we study multiple point spaces in detail. Sections 2.1-2.4 contain background information and some tools that we will need in the rest of the thesis. Our contribution in Section 2.3 is the following. If $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is finite and generically one-to-one, and if f is a pullback of $F\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N+1}, 0)$ by $g\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{N+1}, 0)$, then a resolution of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ can be deduced from a resolution of $\mathcal{O}_{\mathbb{C}^N,0}$ over $\mathcal{O}_{\mathbb{C}^{N+1},0}$ via g (Proposition 2.3.5). Consequently, one gets $\mathrm{Fitt}_j(f_*\mathcal{O}_{\mathbb{C}^n,0}) = g^*\mathrm{Fitt}_j(F_*\mathcal{O}_{\mathbb{C}^N,0})$ (Corollary 2.3.6). For corank 1 map-germs, one can also recover $\mathcal{O}_{D^k(f)}$ from $\mathcal{O}_{D^k(F)}$ by the same pullback diagram that yields f. This fact was stated by Goryunov in Section 3.5 of [Gor95] but without a proof. We give the details that were omitted in his paper (Proposition 2.3.7).

In Section 2.4, we study certain iterations involving the maps $\pi^k_{k-1}\colon D^k \to D^{k-1}$ and the map-germ itself. In [Gor95], it was hinted that $D^s(\pi^k_{k-1}(f)) \cong D^{k+s-1}(f)$ for corank 1 stable map-germs ($k, s \geq 1$). Later Goryunov and Mond showed a set theoretical isomorphism in [GM93]. We recall their proof in Proposition 2.4.1. By another result of [Gor95], we prove that Goryunov and Mond's isomorphism also induces an analytic isomorphism (Corollary 2.4.4). We generalise it for finitely $\mathcal{A}$-determined map-germs (Proposition 2.4.6).

In the rest of Chapter 2, we consider finite map-germs from n-space to $(n+1)$-space. We study the multiplication map

$$\begin{array}{rcl}\mu\colon \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} & \to & \mathcal{O}_{\mathbb{C}^n,0} \\ a \otimes b & \mapsto & ab\end{array}$$

and prove that a resolution of its kernel over $\mathcal{O}_{\mathbb{C}^n,0}$ is given by the pullback of a submatrix of the matrix resolving $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ (Proposition 2.5.2). As a consequence, we observe that $D^2_1(f)$ coincides with the support of the kernel of μ in $\mathbb{C}^n$ (Corollary 2.5.3). In [KLU92], Kleiman, Lipman and Ulrich proved that $\ker(\mu) \cong \mathcal{O}_{D^2(f)}$ for corank 1 map-germs. By this equality and Proposition 2.5.2, we prove inductively that if f is $\mathcal{A}$-finite and if

$$0 \to \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{G} \mathcal{O}_{\mathbb{C}^n,0} \to 0$$

is exact with λ symmetric and $G = \begin{bmatrix} 1 & y & \dots & y^r \end{bmatrix}$, then there exists a resolution

$$0 \to \mathcal{O}_{D^k(f)}^{r-k+1} \xrightarrow{\gamma} \mathcal{O}_{D^k(f)}^{r-k+1} \to \mathcal{O}_{D^{k+1}(f)} \to 0$$

in which $\gamma = f^*\lambda_k^k$, the pullback by f of the matrix obtained from λ by deleting the rows $1, \dots, k$ and the columns $1, \dots, k$ (Theorem 2.6.6). In Section 2.7, we study maps between the resolutions of $\mathcal{O}_{ID^2}$, the ring of the idiot's double point space, and $\mathcal{O}_{D^2}$ over $\mathcal{O}_{\mathbb{C}^n,0}$. By a result of [MS10] and Theorem 2.6.6, we observe new examples of free divisors (Proposition 2.8.4).

In Chapter 3, we review the Mond conjecture and a result by Damon and Mond (Theorem 3.2.1) which allows us to rephrase the conjecture as Conjecture 3.2.14. Their theorem requires Damon's theory on non-linear sections and $\mathcal{K}_V$-equivalence which also provides an alternative method for calculating $\mathcal{A}_e$-codimension (Theorem 3.2.10). We finish the chapter by an algorithm for calculating $\mathcal{A}_e$-codimension based on Damon's theory (Algorithm 3.2.15).

In Chapter 4, we investigate examples of finitely $\mathcal{A}$-determined map-germs from 3-space to 4-space which also support the conjecture. We list the stable map-germs in these dimensions. We prove that $f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^3,0}) = \mathrm{Fitt}_0((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ and that $f^*\mathrm{Fitt}_2(f_*\mathcal{O}_{\mathbb{C}^3,0})$ and $\mathrm{Fitt}_1((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ agree outside the origin for corank ≥ 2 map-germs in $\mathcal{E}_{3,4}^0$ (Proposition 4.2.1). We show that D^2 is a normalisation of its image D_1^2 in $\mathbb{C}^n$ (Proposition 4.3.1).

In [HK99], Houston and Kirk gave a classification of corank 1 map-germs in $\mathcal{E}_{3,4}^0$ with the strata of $\mathcal{A}_e$-codimension ≤ 4, and showed that all the examples in their list satisfy the Mond conjecture. However, a classification for corank 2 map-germs is costly. Still, for a start, we classify 2-jets of corank 2 map-germs in these dimensions (Appendix C). We translate Mather and Gaffney's criterion on finite $\mathcal{A}$-determinacy into geometric criteria (Theorem 4.4.1) that can easily be checked by a computer program such as SINGULAR ([DGPS10]) and Macaulay2 ([GS]). We prove that there are no finitely $\mathcal{A}$-determined map-germs of corank 2 in $\mathcal{E}_{3,4}^0$ with degrees $(1, 2, 2, 2)$ (Proposition 4.5.1). Moreover, there are no $\mathcal{A}$-finite map-germs of the form $(x, y, z) \mapsto (x, y^2 + xz, z^2 + xy, P_{2k}(x, y, z))$ where $P_{2k}(x, y, z)$ is a homogeneous polynomial of degree $2k$ for $k \geq 1$ (Proposition 4.5.2). Finally, we give two sets of $\mathcal{A}$-finite map-germs of corank 2:

$$\tilde{A}_k\colon (x,y,z)\mapsto (x, y^k+xz+x^{2k-2}y, yz, z^2+y^{2k-1}) \qquad \text{for } k=2,3,4,5,6,$$
$$\tilde{B}^{\pm}_{2k+1}\colon (x,y,z)\mapsto (x, y^2+xz, z^2+xy, y^{2k+1}\pm y^{2k-1}z^2+z^{2k+1}) \text{ for } k=1,2,4,5,$$
$$\tilde{B}^{-}_{2k+1}\colon (x,y,z)\mapsto (x, y^2+xz, z^2+xy, y^{2k+1}-y^{2k-1}z^2+z^{2k+1}) \text{ for } k=3,6.$$

We check that they satisfy the conjecture (Proposition 4.5.3 and 4.5.5). We rephrase Marar's formula for the image Milnor number of map-germs in $\mathcal{E}^0_{3,4}$ so that it only involves D^2, D^k_1 for $k=2,3,4$ and the ramification locus (Corollary 4.6.5).

In Chapter 5, we define the notion of reductions which is a procedure for generating map-germs in lower dimensions from a given one. We prove a criterion on finite $\mathcal{A}$-determinacy of reductions (Theorem 5.1.2). We give an algorithm for producing examples of $\mathcal{A}$-finite map-germs from old (Algorithm 5.1.3).

From one of the examples of finitely $\mathcal{A}$-determined corank 2 map-germs in $\mathcal{E}^0_{2,3}$ that were given by Bruce and Marar ([BM96]), we deduce a "series" of $\mathcal{A}$-finite map-germs:

$$f\colon (t,x,y)\mapsto (t, x^2+t^ky, y^2-t^kx, x^3+x^2y+xy^2-y^3)$$

(Proposition 5.1.4). For $k\leq 20$, it has $\mathcal{A}_e$-codimension $45k-12$ and satisfies the conjecture. Similarly, we find an $\mathcal{A}$-finite map-germ

$$(x,y,z)\mapsto (x^2+yz, xy+y^3, x^2+y^5+xz, z)$$

out of Marar and Nuño-Ballesteros' list of corank 2 map-germs in $\mathcal{E}^0_{2,3}$ ([MNB08]). We also produce three series of $\mathcal{A}$-finite map-germs from $\tilde{A}_2$ and $\tilde{B}^+_3$:

$$\begin{aligned}
\tilde{C}_k\colon (x,y,z,t) &\mapsto (x, t, y^2+xz+x^2y, yz+t^k, z^2+y^3)\\
\tilde{D}_k\colon (x,y,z,t) &\mapsto (x, t, y^2+xz, z^2+xy, y^3+yz^2+z^3+t^kz)\\
\tilde{E}_k\colon (x,y,z,t) &\mapsto (x, t, y^2+xz+x^2y+t^ky, yz, z^2+y^3+t^{2k}y)
\end{aligned}$$

(see Proposition 5.2.1, 5.2.2 and 5.2.4 respectively). We prove that $\mathcal{A}_e$-codim$(\tilde{C}_k) = 33k-18$ and $\mathcal{A}_e$-codim$(\tilde{D}_k)=51k-33$. We check that $\tilde{E}_k$, for $k\leq 20$, satisfies the formula $\mathcal{A}_e$-codim$(\tilde{E}_k)=45k-18$. We also confirm that all examples satisfy the conjecture for $k\leq 20$.

In Appendix A, we discuss Noether, Kähler and Dedekind differents for

$\mathcal{O}_{\mathbb{C}^n,0}$, when considered an $\mathcal{O}_{\mathbb{C}^p,0}$ algebra via a finite map-germ $f \in \mathcal{E}^0_{n,p}$ (Proposition A.0.15).

In Appendix B, we list a code for a SINGULAR library which calculates $\mathcal{A}_e$-codimension of a given map-germ.

In Appendix C, we show our calculations for a classification of 2-jets of corank 2 map-germs from 3-space to 4-space.

In Appendix D, we show Groebner basis calculations for the modules $N\mathcal{K}_{V,e}g_k$ for the series $\tilde{C}_k$ and $\tilde{D}_k$. We use these bases in the proof of $\mathcal{A}_e$-codimension formulae in Proposition 5.2.1 and 5.2.2.

Chapter 1

Preliminaries

This section contains the definitions and some results from the literature that we will need in this thesis. Most of them are well-known. We recall them here in order to introduce our conventions and notations. We also give proofs when we could not find a direct reference.

1.1 Modules

In this thesis, we will work with modules over commutative Noetherian rings with identity. Let R be a such ring and M an R-module.

Definition 1.1.1 (§2, [Mat89])**.** Let N and N' be submodules of M. The set $(N : N') := \{r \in R \mid rN' \subset N\}$ is an ideal of A, and it is referred to as the *quotient* of N by N'. Similarly, if $I \subset A$ is an ideal then $(N : I) := \{x \in M \mid Ix \in N\}$ is a submodule of M. The ideal $(0 : M)$ is called the *annihilator* of M and denoted by $\mathrm{Ann}(M)$. The *dimension* of M is defined to be $\dim M = \dim(R/\mathrm{Ann}M)$.

Local rings and localisations.

Definition 1.1.2 ([Mat89])**.** A ring R with only one maximal ideal is called a *local ring*. It is customary to say that $(R, \mathfrak{m})$ is a local ring in order to indicate that R is a local ring with maximal ideal $\mathfrak{m}$. If $(R, \mathfrak{m})$ is a local ring then the field $k := R/\mathfrak{m}$ is called the *residue field* of R.

A characterisation of local rings is as follows.

Lemma 1.1.3 (p. 3, [Mat89]). *Let R be a ring and $\mathfrak{m} \subset R$ be an ideal. Then R is local with maximal ideal $\mathfrak{m}$ if and only if $R \setminus \mathfrak{m}$ is the set of the units in R.*

Let S be a *multiplicative* subset of R, that is, (i) if $x, y \in S$ then $xy \in S$, and (ii) $1 \in S$. The *localisation* of R with respect to S is defined to be the ring $S^{-1}R := \left\{ \frac{r}{s} \mid r \in R, s \in S \right\}$ with the following property:

$$\frac{r_1}{s_1} = \frac{r_2}{s_2} \text{ if and only if } s(r_1 s_2 - r_2 s_1) = 0 \text{ for some } s \in S.$$

Remark 1.1.4 (Example 2, [Mat89]). Let $\mathfrak{p} \subset R$ be a prime ideal. We refer to the localisation of R with respect to $R \setminus \mathfrak{p}$ as the localisation of R at $\mathfrak{p}$ and denote it by $R_{\mathfrak{p}}$. Then $R_{\mathfrak{p}}$ is a local ring with the maximal ideal $\mathfrak{p}R_{\mathfrak{p}} := \left\{ \frac{a}{b} \mid a \in \mathfrak{p}, b \notin \mathfrak{p} \right\}$.

The localisation M_S of M is defined in the same way as R_S: We have

$$M_S := \left\{ \frac{m}{s} \mid m \in M, s \in S \right\},$$

and $\frac{m_1}{s_1} = \frac{m_2}{s_2}$ if and only if $s(m_1 s_2 - m_2 s_1) = 0$ for some $s \in S$.

Definition 1.1.5. Let $\mathrm{Spec}(R)$ denote the set of all prime ideals of R. The *support* of M is the set $\mathrm{Supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$.

If M is finitely generated then $\mathrm{Supp}(M)$ coincides with the closed subset $V(\mathrm{Ann}(M))$ of $\mathrm{Spec}(R)$ ([Mat89, p.26]). The *dimension* of M is set to be the dimension of $\mathrm{Supp}(M)$.

Fitting Ideals. If W is a set of generators of an R-module M which is minimal, in the sense that any proper subset of W does not generate M, then W is said to be a *minimal basis* of M ([Mat89, p. 8]).

Definition 1.1.6 (p. 12 and p.153, [Mat89]). A *presentation* of M is a short exact sequence

$$G \xrightarrow{\alpha} F \xrightarrow{\beta} M \to 0 \tag{1.1}$$

of R-modules, in which F and G are free R-modules. In case R is a local ring with the maximal ideal $\mathfrak{m}$ and M is finitely generated, a presentation as in (1.1) is called *minimal* if (i) G and F are finite free R-modules, (ii) the rank of F is equal to the minimal number of generators of M, that is, $\dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$, and (iii) $\mathrm{image}(\alpha) \subset \mathfrak{m} \cdot F$.

Definition 1.1.7 (§20.2, [Eis95])**.** Let $\alpha : G \to F$ be a map of free modules represented by a matrix. Define the ideal $\bigwedge^j \alpha$ generated by minors (that is, determinants of submatrices) of size of j. We make the convention that the determinant of 0×0-matrix (like a product with no factors) is 1. In particular, $I_0(\alpha) = R$ and we set $I_j(\alpha) = R$ if $j \leq 0$.

These ideals of minors define famous invariants for modules, named after Fitting due to his work [Fit36]. Here, we refer to Eisenbud's book as it is more accessible.

Proposition-Definition 1.1.8 (Corollary-Definition 20.4, [Eis95])**.** Let M be a finitely generated module over a ring R, and let $\mathbf{L} : G \xrightarrow{\alpha} F \to M \to 0$ and $\mathbf{L}' : G' \xrightarrow{\alpha'} F' \to M \to 0$ be two presentations, with F and F' finitely generated free modules of rank r and r'. For each number i with $i \geq 0$, we have $I_{r-i}(\alpha) = I_{r'-i}(\alpha')$, and we define the *$i$-th Fitting invariant* of M to be the ideal

$$\mathrm{Fitt}_i(M) := I_{r-i}(\alpha) \subset R.$$

Proposition 1.1.9. *Let M be a finite R-module. Then*

$$\mathrm{Supp}(M) = V(\mathrm{Fitt}_0(M)).$$

Proof. By [Eis95, Proposition 20.7], $\mathrm{Fitt}_0(M) \subset \mathrm{Ann}(M)$, moreover, if M can be generated by n elements then $(\mathrm{Ann}(M))^n \subset \mathrm{Fitt}_0(M)$. Therefore, $\sqrt{\mathrm{Fitt}_0(M)} = \sqrt{\mathrm{Ann}(M)}$ whence the result. □

Tensor products. We suggest [Mat89] and [Eis95] for details on tensor products. Here, we recall some of properties of tensor products.

Remark 1.1.10. (1) The tensor product $M \otimes_R N$ can be given an R-module structure on the left by the multiplication $r \cdot m \otimes n := rm \otimes n$ or an R-module structure on the right by $r \cdot m \otimes n := m \otimes rn$, for all $m \in M$, $n \in N$ and $r \in R$.

(2) Let S be an R-algebra. Let M be an R, N be an S-module. Then $M \otimes_R N$ has an R-module structure set by $r \cdot m \otimes n := rm \otimes n$ and an S-module structure set by $s \cdot m \otimes n := m \otimes sn$, for $m \in M$, $n \in N$, $r \in R$ and $s \in S$.

Proposition 1.1.11. *Let M and N be nonzero R-modules. We have*

$$\mathrm{Supp}(M \otimes_R N) \cong \mathrm{Supp}(M) \cap \mathrm{Supp}(N).$$

Proof. By Theorem 4.4 of [Mat89], $R_\mathfrak{p} \cong R \otimes_R R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, furthermore,

$$(M \otimes_R N)_\mathfrak{p} \cong (M \otimes_R N) \otimes_R R_\mathfrak{p}.$$

By the standard properties of tensor products (see, for instance, Appendix A of [Mat89]),

$$(M \otimes_R N) \otimes_R R_\mathfrak{p} \cong M \otimes_R (N \otimes_R R_\mathfrak{p}) \cong M \otimes_R N_\mathfrak{p},$$

or

$$(M \otimes_R N) \otimes_R R_\mathfrak{p} \cong R_\mathfrak{p} \otimes_R (M \otimes_R N) \cong (R_\mathfrak{p} \otimes_R M) \otimes_R N \cong M_\mathfrak{p} \otimes_R N.$$

Hence,

$$(M \otimes_R N)_\mathfrak{p} \cong M \otimes_R N_\mathfrak{p} \cong M_\mathfrak{p} \otimes_R N.$$

We conclude that $(M \otimes_R N)_\mathfrak{p} \neq 0$ if and only if $M_\mathfrak{p} \neq 0$ and $N_\mathfrak{p} \neq 0$. The result follows from the definition of the support of a module. □

Cohen-Macaulay modules.

Definition 1.1.12 (§16, [Mat89])**.** An element $r \in R$ is said to be *M-regular* if $rm \neq 0$ for all $0 \neq m \in M$. A sequence $r_1, \ldots, r_n$ of elements of R is an *M-sequence* (or an *M-regular sequence*) if the following two conditions hold:

(1) r_1 is M-regular, r_2 is (M/r_1M)-regular, …, r_n is $(M/\sum_{i=1}^{n-1} r_iM)$-regular,

(2) $M/\sum_{i=1}^{n} r_iM \neq 0$.

Definition 1.1.13 (§16, [Mat89])**.** The length of a maximal M-sequence in an ideal I of R is called the *I-depth* of M and denoted by $\mathrm{depth}(I, M)$. In particular, for a Noetherian local ring $(R, \mathfrak{m})$, we call $\mathrm{depth}(\mathfrak{m}, M)$ simply the *depth* of M and write $\mathrm{depth}(M)$.

Remark 1.1.14. Some authors refer to $\mathrm{depth}(I, M)$ as *grade of M* in I following Rees' original work [Ree57]. We will only use the term grade for the ideals and set

$$\mathrm{grade}(I) = \mathrm{grade}(I, R).$$

Definition 1.1.15 (Chapter 19,[Eis95])**.** The *projective dimension* of M over R is defined to be the minimum of the lengths of projective resolutions of M over R, and denoted by $\mathrm{proj.dim}_R M$ or $\mathrm{proj.dim} M$ when the ring is clear from context.

Definition 1.1.16 (§17, [Mat89])**.** Let $(R, \mathfrak{m})$ be Noetherian local ring and M a finite R-module. We say that M is a *Cohen-Macaulay module* if $M \neq 0$ and $\operatorname{depth}(M) = \dim M$, or if $M = 0$. If R itself is a Cohen-Macaulay module we say that R is a *Cohen-Macaulay ring.*

1.2 Analytic geometry

Holomorphic functions. Let $U \subset \mathbb{C}^n$ be open and connected, $f: U \to \mathbb{C}$ be a function. We will say that f is *holomorphic* if f is complex differentiable in p for all $p \in U$. It is well-known that one-variable functions are holomorphic if and only if they are locally represented by their Taylor series expansion. A similar statement also holds for multi-variable functions. Before stating the theorem, we need some definitions.

Let $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{C}^n$ and ν_i, for $i = 1, \ldots, n$, be non-ngegative integers. Let us set $\nu := (\nu_1, \ldots, \nu_n)$ and $\mathbf{x}^\nu := \prod_{i=1}^n x_i^{\nu_i}$. The expression

$$\sum_{\nu=0}^{\infty} a_\nu (\mathbf{x} - p)^\nu$$

with $a_\nu \in \mathbb{C}$, is called a *formal power series* in the variables $x_1, \ldots, x_n$ around p. The set of formal power series has a ring structure given by the usual addition and multiplication. The ring of formal power series around $p = 0$ is denoted by $\mathbb{C}[[\mathbf{x}]]$.

Definition 1.2.1 (Definition 3.1.3, [dJP00])**.** (1) Let $U \subset \mathbb{C}^n$ and $f : U \to \mathbb{C}$ be a function. The formal power series $\sum_{\nu=0}^{\infty} a_\nu (\mathbf{x} - p)^\nu$ is said to be *converging* on U with *limit* f if for all $q \in U$, the formal power series $\sum_{\nu=0}^{\infty} a_\nu (\mathbf{x} - p)^\nu$ converges in q with limit $f(q)$.

(2) Let $U \subset \mathbb{C}^n$ be open and connected, $f: U \to \mathbb{C}$ be a function. Then f is called *analytic* if for all $q \in U$ there exists a neighbourhood V of q in U, and a power series $\sum_{\nu=0}^{\infty} a_\nu (\mathbf{x} - p)^\nu$ which converges on V to f. For a fixed q, the set of formal power series that converge on a neighbourhood of q form a ring, called *convergent power series ring.* The ring of convergent power series at p is denoted by $\mathcal{O}_{\mathbb{C}^n,p}$, or sometimes by $\mathbb{C}\{\mathbf{x}\}$ when $p = 0$.

It is known that an analytic function is holomorphic (see, for example, [dJP00, Theorem 3.1.6]). There is stronger relation for continuous functions.

Theorem 1.2.2 (Theorem 3.1.7, [dJP00])**.** *Let $U \subset \mathbb{C}^n$ be open and $f\colon U \to \mathbb{C}$ be a continuous function. Then, f is holomorphic if and only if it is analytic.*

Analytic sets.

Definition 1.2.3 (Definition 3.1.12, [dJP00])**.** A set or space $X \subset \mathbb{C}^n$ is called *analytic* if for any point $p \in X$, there exists an open subset V of p in $\mathbb{C}^n$ and finitely many holomorphic functions $f_1, \ldots, f_s$ defined on V such that

$$X \cap V = \{x \in V \mid f_1(x) = \cdots = f_s(x) = 0\}.$$

Definition 1.2.4 (Definition 3.3.8, [dJP00])**.** (1) Let $f_1, \ldots, f_n$ be holomorphic functions on an open subset U. Let $p \in U$, and suppose $f_1(p) = \cdots = f_n(p) = 0$. The set $\{f_1, \ldots, f_n\}$ is called a set of *coordinate functions* at p if $\det\big(\frac{\partial f_j}{\partial x_i}(p)\big) \neq 0$.

(2) A subset $X \subset \mathbb{C}^n$ is a *complex submanifold* of $\mathbb{C}^n$ of codimension k if for every $x \in X$ there exists an open subset $U \in \mathbb{C}^n$ and coordinate functions $w_1, \ldots, w_n$ of x such that $X \cap U = \{y \in U \mid w_1(y) = \cdots = w_k(y) = 0\}$ for some $k \leq n$.

By Inverse Function Theorem (see, for instance, [dJP00, Corollary 3.3.7]), a complex submanifold $X \subset \mathbb{C}^n$ can be interpreted as an open subset of $\mathbb{C}^k$ for some k. This motivates the following definition of smooth subspaces.

Definition 1.2.5 (Definition 3.3.11, [dJP00])**.** Let $U \subset \mathbb{C}^n$ be an open subset and $X \subset U$ an analytic subset. A point $x \in X$ *regular*, or X is called *smooth* at x, if there exists an open subset V in $\mathbb{C}^n$ containing x such that $X \cap V$ is a complex submanifold of $\mathbb{C}^n$. A point in X is called *singular* if it is not regular. We denote the set of singular points of X by $\mathrm{Sing}(X)$.

These definitions direct us to the following criterion for singular points.

Definition 1.2.6 (Jacobian Criterion)**.** Let $X \subset \mathbb{C}^n$ be an analytic set of codimension k given as the zero set of holomorphic functions $w_1, \ldots, w_m$ on an open subset $U \in \mathbb{C}^n$ $(m \leq n)$. Then,

$$\mathrm{Sing}(X) \cap U = \left\{ x \in X \cap U \mid \mathrm{rank}\big(\frac{\partial w_j}{\partial x_i}(x)\big) < k \right\}.$$

As we work in analytic category, we need to consider analytic tensor products which we define below.

Definition 1.2.7 (Definition 7.3.6,[dJP00])**.** Let $A := \mathbb{C}\{x\}/I$ and $B := \mathbb{C}\{y\}/J$ be analytic algebras. Then we define the *analytic tensor product* $A\hat{\otimes}B$ by

$$A\hat{\otimes}B := \mathbb{C}\{x, y\}/I + J.$$

In case A and B are reduced, and are the local rings of $(X, 0)$ and $(Y, 0)$ respectively, the analytic tensor product is the local ring of the product $(X \times Y, 0)$.

Germs of analytic sets and maps.

Definition 1.2.8 (Definition 3.4.1, [dJP00])**.** Let X be a topological space and $p \in X$. Subsets A and B of X are called *equivalent* at p if there exists a neighbourhood U of p such that $A \cap U = B \cap U$. It is easy to check that this forms an equivalence relation. The equivalence class of A at p is called the *germ* of A at p, and A is called a *representative* of the germ. We write (A, p) for the germ of A at p.

Definition 1.2.9. A *germ of an analytic space* (X, x) is a germ at x of a locally analytic subset of $\mathbb{C}^n$ and is of the form $(V(I), x)$ for some ideal $I = (f_1, \dots, f_s) \subset \mathcal{O}_{\mathbb{C}^n,x}$. In that case, we call

$$\mathcal{O}_{X,x} := \frac{\mathcal{O}_{\mathbb{C}^n,x}}{I\mathcal{O}_{\mathbb{C}^n,x}}$$

the *coordinate ring* of (X, x).

Definition 1.2.10. Let (X, x) and (Y, y) be germs of topological spaces, and let U and V be open neighbourhoods of x in X. If $f\colon (U, x) \to (Y, y)$ and $g\colon (V, x) \to (Y, y)$ are continuous maps, then f and g are *equivalent* if there exists an open neighbourhood $W \subset U \cap V$ of x in X such that $f\mid_W = g\mid_W$. The equivalence class of f is called the *germ* of f. A *representative* of $f\colon (X, x) \to (Y, y)$ is, then, of the form $f\colon (U, x) \to (Y, y)$.

Finite map-germs.

Definition 1.2.11 (Chapter I, [GR04])**.** Let $f\colon X \to Y$ be a map. One says that f is *closed* if the image by f of every closed set in X is again closed in Y. If f is continuous, closed and the fibre $f^{-1}(y)$ is finite for all all $y \in Y$, then f is called a *finite map*.

Definition 1.2.12. Let $f\colon (X, x) \to (Y, y)$ be an analytic map-germ. Let $f_1, \dots, f_p$ denote its components. The *local algebra* $Q(f)$ of f is an analytic $\mathbb{C}$-algebra defined by

$$Q(f) := \frac{\mathcal{O}_{X,x}}{f^*\mathfrak{m}_{Y,y}} = \frac{\mathcal{O}_{X,x}}{(f_1, \ldots, f_p)\mathcal{O}_{X,x}}.$$

Definition 1.2.13. The *multiplicity* of f is the $\mathbb{C}$-vector space dimension

$$q(f) := \dim_{\mathbb{C}} Q(f).$$

A fundamental result from local analytic geometry:

Theorem 1.2.14 (Theorem 1.11, [Fis76])**.** *If $f\colon X \to Y$ is a holomorphic map, $x \in X$ and $y := f(x)$, then the following conditions are equivalent:*

(i) $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,y}$, i.e. it is a finitely generated $\mathcal{O}_{Y,y}$-module,

(ii) $q(f) < \infty$,

(iii) x is an isolated point of its fibre $M_{f(x)}$, i.e. $f^{-1}f(x) = \{x\}$.

Corollary 1.2.15 (Lemma 3.2, [Fis76])**.** *Let $f\colon X \to Y$ be a holomorphic map. Let p be an isolated point of its fibre $X_{f(x)}$. Then there are open neighbourhoods $U \subset X$ of x and $V \subset Y$ of $f(x)$ such that $f|_U\colon U \to V$ is finite.*

Definition 1.2.16 (Definition 3.4.9, [dJP00])**.** Let (X, x) and (Y, y) be two germs of topological spaces. Let $f\colon (X, x) \to (Y, y)$ be a germ of a continuous map. Then f is called *finite* if there exists a representative $\tilde{f}\colon X \to Y$ which is finite. Assuming f is finite, the *image* of f is defined to be $\operatorname{Im}(f) := (\tilde{f}(X), y)$ where $\tilde{f}\colon X \to Y$ is a representative such that $\tilde{f}^{-1}(f(x)) = \{x\}$ and $\tilde{f}$ is finite.

1.3 Equivalence relations on map-germs

We recall some definitions and theorems on Right-Left ($=\mathcal{A}$) equivalence and Contact ($=\mathcal{K}$) equivalence of map-gems. For more details, we recommend Martinet's book [Mar82] or Wall's survey [Wal81].

We consider holomorphic map-germs $f\colon X \to Y$ between smooth (or complex analytic) manifolds X and Y of dimension n and p, respectively, defined on some neighbourhood of a point $\mathbf{x} \in X$. Here, we will take X and Y to be complex spaces of finite dimensions and consider the spaces

$$\mathcal{E}_{n,p} := \{f\colon (\mathbb{C}^n, \mathbf{x}) \to (\mathbb{C}^p, \mathbf{y}) \mid f \text{ is holomorphic}\}$$

and

$$\mathcal{E}^0_{n,p} := \{f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0) \mid f \text{ is holomorphic}\}\,.$$

Let us denote the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,0}$ by $\mathfrak{m}_{\mathbb{C}^n}$. Then, $\mathfrak{m}_{\mathbb{C}^n,\mathbf{x}} = (x_1,\ldots,x_n)$ and $\mathfrak{m}_{\mathbb{C}^p,\mathbf{y}} = (y_1,\ldots,y_p)$ for some chosen coordinate systems (see Proposition 1, p.2 of [Mar82]). If $f = (f_1,\ldots,f_p)$ then f is holomorphic if and only if each f_i is. So, we can identify $\mathcal{E}_{n,p}$ (resp. $\mathcal{E}^0_{n,p}$) with $(\mathcal{O}_{\mathbb{C}^n,\mathbf{x}})^p$ (resp. $(\mathcal{O}_{\mathbb{C}^n,0})^p$).

Definition 1.3.1. Let $f \in \mathcal{E}^0_{n,p}$. The *corank* of f is defined to be $\dim_{\mathbb{C}}(\ker(\mathrm{d}f(0)))$. We denote the space of map-germs with corank k by Σ^k.

Definition 1.3.2. A map-germ $f \in \mathcal{E}^0_{n,p}$ is said to be *weighted homogeneous* with a weight vector $w = (w_1,\ldots,w_n)$ for a chosen coordinate system $(x_1,\ldots,x_n)$ on $(\mathbb{C}^n,0)$ if

$$\lambda^{k_i} f_i = f_i(\lambda^{w_1}x_1,\ldots,\lambda^{w_n}x_n)$$

for some $k_1,\ldots,k_p \in \mathbb{N}$ and for all λ, $i = 1,\ldots,p$.

Definition 1.3.3. (1) The *k-jet* of a germ $f \in \mathcal{E}_{n,p}$ at a point $a \in \mathbb{C}^n$, $j^k f(a)$, is the power series expansion of f at a up to degree k. These k-jets form the vector space $J^k(n,p)$ of the k-jets from n-space to p-space. (2) We have natural maps $j^k f\colon \mathbb{C}^n \to J^k(n,p)$ given by $a \mapsto j^k f(a)$.

$\mathcal{A}$-equivalence.

We consider the group $\mathrm{Diff}(\mathbb{C}^n,0)$ of local diffeomorphisms at the origin of $\mathbb{C}^n$ with the properties

(i) $\phi(0) = 0$, and

(ii) ϕ defines a diffeomorphism between an open subset U' of a neighbourhood U of $0 \in \mathbb{C}^n$ and $\phi(U')$, e.g. $\mathrm{d}\phi(0)$ is invertible, for a $\phi \in \mathrm{Diff}(\mathbb{C}^n,0)$.

An action of the group $\mathcal{A} := \mathrm{Diff}(\mathbb{C}^n,0) \times \mathrm{Diff}(\mathbb{C}^p,0)$ on $\mathcal{E}^0_{n,p}$ is defined by

$$\begin{array}{rcl} \mathrm{Diff}(\mathbb{C}^n,0) \times \mathrm{Diff}(\mathbb{C}^p,0) \times \mathcal{E}^0_{n,p} & \to & \mathcal{E}^0_{n,p} \\ (\phi,\psi,f) & \mapsto & \psi \circ f \circ \phi^{-1} \end{array} \tag{1.2}$$

This action gives rise to one of the equivalence relations on $\mathcal{E}^0_{n,p}$:

Definition 1.3.4. A map-germ $g \in \mathcal{E}^0_{n,p}$ is *$\mathcal{A}$-equivalent* (or *Right-Left equivalent*) to $f \in \mathcal{E}^0_{n,p}$ if $g \in \mathcal{A}\cdot f$. In that case, we write $f \sim_{\mathcal{A}} g$.

Definition 1.3.5. A map-germ $f \in \mathcal{E}^0_{n,p}$ is k-$\mathcal{A}$-*determined* if whenever $j^k f(0) = j^k g(0)$ for a $g \in \mathcal{E}^0_{n,p}$, we have $f \sim_{\mathcal{A}} g$.
If f is k-$\mathcal{A}$-determined for some $k < \infty$ then it is said to be finitely $\mathcal{A}$-*determined.*

Definition 1.3.6 ([MM94])**.** An *unfolding* of $f\colon X \to P$ over a space germ S is a map $F\colon \mathcal{X} \to P \times S$ together with a flat projection $\pi\colon \mathcal{X} \to S$ such that if $\pi_2\colon P \times S \to S$ is the cartesian projection, then

(i) $\pi_2 \circ F = \pi$,

(ii) there is an isomorphism $j\colon X \to \pi^{-1}(0)$, and $F \circ j\colon X \to P \times \{0\} = P$ is equal to f.

If F is a parametrised unfolding, that is, if $\mathcal{X} = X \times S$ then F has a nice formulation.

Definition 1.3.7. A *d-parameter unfolding* F of $f \in \mathcal{E}^0_{n,p}$ is a differentiable map-germ

$$\begin{array}{rcl} F\colon (\mathbb{C}^n \times \mathbb{C}^d, 0) & \to & (\mathbb{C}^p \times \mathbb{C}^d, 0) \\ (\mathbf{x}, \mathbf{t}) & \mapsto & (f_{\mathbf{t}}(\mathbf{x}), \mathbf{t}) \end{array} \tag{1.3}$$

with $f_0(\mathbf{x}) = f(\mathbf{x})$ and $\mathbf{t} = (t_1, \ldots, t_d)$.

Definition 1.3.8. Two unfoldings F and G (with the same number of parameters) of f are *isomorphic* if there exist germs of diffeomorphisms $\phi \in \mathrm{Diff}(\mathbb{C}^n \times \mathbb{C}^d, 0)$ and $\psi \in \mathrm{Diff}(\mathbb{C}^p \times \mathbb{C}^d, 0)$ such that $G = \psi \circ F \circ \phi^{-1}$. Note that in this case ϕ and ψ are given by

$$\phi(\mathbf{x}, \mathbf{t}) = (\tilde{\phi}_{\mathbf{t}}(\mathbf{x}), \mathbf{t}) \text{ and } \psi(y, \mathbf{t}) = (\tilde{\psi}_{\mathbf{t}}(\mathbf{y}), \mathbf{t}) \tag{1.4}$$

with $\tilde{\phi}_{\mathbf{t}} \in \mathrm{Diff}(\mathbb{C}^n, 0)$ and $\tilde{\psi}_{\mathbf{t}} \in \mathrm{Diff}(\mathbb{C}^p, 0)$. An unfolding is called *trivial* if it is isomorphic to the constant unfolding $(\mathbf{x}, \mathbf{t}) \mapsto (f(\mathbf{x}), \mathbf{t})$.

Definition 1.3.9. Let $F\colon (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ be a d-parameter unfolding of a map-germ $f \in \mathcal{E}^0_{n,p}$, and let $h\colon (\mathbb{C}^q, 0) \to (\mathbb{C}^d, 0)$ be given by $\mathbf{v} \mapsto \mathbf{u} := h(\mathbf{v})$. We define the *base change* operation induced by h to be

$$\begin{array}{rcl} h^*F\colon (\mathbb{C}^n \times \mathbb{C}^q, 0) & \to & (\mathbb{C}^p \times \mathbb{C}^q, 0) \\ (\mathbf{x}, \mathbf{v}) & \mapsto & (\mathbf{v}, f_{h(\mathbf{v})}(\mathbf{x})). \end{array}$$

Definition 1.3.10. Two unfoldings F and G of f are *equivalent* if G is isomorphic to h^*F for a germ h of diffeomorphism between the parameter spaces of F and G. An unfolding F of f is $\mathcal{A}_e$*-versal* if every unfolding of f is equivalent to F.

Let $\Theta(f)$ denote the space of *vector fields along* f, that is, the space of vector fields $\xi\colon \mathbb{C}^n \to T\mathbb{C}^p$ satisfying $pr \circ \xi = f$ where $pr\colon T\mathbb{C}^p \to \mathbb{C}^p$ is the natural projection. If we fix coordinate systems on $\mathbb{C}^n$ and $\mathbb{C}^p$ we can easily identify $\Theta(f)$ with $(\mathcal{O}_{\mathbb{C}^n,0})^p$.

Let us consider a 1-parameter unfolding F of $f \in \mathcal{E}^0_{n,p}$ given by $F(\mathbf{x}, t) = (f_t(\mathbf{x}), t)$ with $t \in \mathbb{C}$. Then the map $x \mapsto f_t$ defines a path in $\mathcal{E}_{n,p}$ with the initial point f. Hence we get a tangent vector $\frac{\partial f_t(\mathbf{x})}{\partial t} |_{t=0} \in T_{f(\mathbf{x})}\mathbb{C}^p$. So, the map

$$\hat{f} = \frac{\partial f_t}{\partial t}\colon x \mapsto (f(\mathbf{x}), \frac{\partial f_t(\mathbf{x})}{\partial t} |_{t=0}) \tag{1.5}$$

is a vector field along f. If F is a trivial unfolding, i.e. for any t, $f_t(\mathbf{x}) = \tilde{\psi}_t \circ f \circ \tilde{\phi}_t^{-1}$ for $\tilde{\phi}_t \in \mathrm{Diff}(\mathbb{C}^n, 0)$ and $\tilde{\psi}_t \in \mathrm{Diff}(\mathbb{C}^p, 0)$, then by differentiation

$$\frac{\partial f_t}{\partial t} = -\mathrm{d}f \cdot \frac{\partial \tilde{\phi}_t}{\partial t} + \frac{\partial \tilde{\psi}_t}{\partial t} \circ f \in \Theta(f).$$

It is clear that $X_t = \frac{\partial \tilde{\phi}_t}{\partial t}$ belongs to the germ $\Theta_{\mathbb{C}^n,0}$ of vector fields on $(\mathbb{C}^n, 0)$, and $Y_t = \frac{\partial \tilde{\phi}_t}{\partial t}$ belongs to $\Theta_{\mathbb{C}^p,0}$. This motivates the following definition.

Definition 1.3.11. The *space of trivial unfoldings* (or the *extended tangent space*) of $f \in \mathcal{E}^0_{n,p}$ is defined by

$$T\mathcal{A}_e f := \{\mathrm{d}f \cdot X + Y \circ f \mid X \in \Theta_{\mathbb{C}^n,0} \text{ and } Y \in \Theta_{\mathbb{C}^p,0}\}.$$

By definition, $T\mathcal{A}_e f$ is a subspace of $\Theta(f)$. In fact, one can define maps

$$\begin{aligned} tf\colon \Theta_{\mathbb{C}^n,0} &\to \Theta(f) \\ X &\mapsto \mathrm{d}f \cdot X \end{aligned} \qquad \text{and} \qquad \begin{aligned} wf\colon \Theta_{\mathbb{C}^p,0} &\to \Theta(f) \\ Y &\mapsto Y \circ f \end{aligned}$$

and then set $T\mathcal{A}_e f = tf(\Theta_{\mathbb{C}^n,0}) + wf(\Theta_{\mathbb{C}^p,0})$. If $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p}$ is the coordinate system on $T_0\mathbb{C}^p$ then

$$T\mathcal{A}_e f = \mathrm{d}f \cdot \mathcal{O}_{\mathbb{C}^n,0} + f^*(\mathcal{O}_{\mathbb{C}^p,0}) \cdot \left\{ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p} \right\}$$

where $f^*(\mathcal{O}_{\mathbb{C}^p,0})$ is the set of elements of the form $\varphi \circ f_i$ for all $\varphi \in \mathcal{O}_{\mathbb{C}^p,0}$.

Definition 1.3.12. The *deformation space* (or *the normal space*) is the quotient

$$N\mathcal{A}_e f := \frac{\Theta(f)}{tf(\Theta_{\mathbb{C}^n,0}) + wf(\Theta_{\mathbb{C}^p,0})}.$$

We define $\mathcal{A}_e$-*codimension* of f to be $\mathcal{A}_e\text{-codim}(f) := \dim_{\mathbb{C}} N\mathcal{A}_e f$.

Theorem 1.3.13 ([Mar82], Chapter XIV)**.** *A d-parameter unfolding F of $f \in \mathcal{E}^0_{n,p}$ is $\mathcal{A}_e$-versal if and only if its* initial speeds $\left\{\frac{\partial f_t}{\partial t_1}|_{t=0}, \ldots, \frac{\partial f_t}{\partial t_d}|_{t=0}\right\}$ *generate $N\mathcal{A}_e f$ as a $\mathbb{C}$-vector space.*

Definition 1.3.14. A map-germ f is called *stable* if any $\mathcal{A}_e$-versal unfolding of f is trivial.

Theorem 1.3.15 ([Mat69a])**.** *A germ f is stable if and only if $T\mathcal{A}_e f = \Theta(f)$, i.e. f is infinitesimally stable.*

Remark 1.3.16. Theorem 1.3.15 can be deduced from Theorem 1.3.13.

Remark 1.3.17. Not all map-germs admit a stable perturbation, e.g. a stable deformation by an arbitrarily small amount. This actually depends on the dimensions n and p. In [Mat70], Mather gave the necessary and sufficient conditions on the pair (n, p) under which the proper stable mappings are dense in the set of proper maps from n-space to p-space, and referred to such pairs as "nice dimensions". These are the same dimension pairs for which every finitely $\mathcal{A}$-determined map-germ has a stable perturbation. For example, if $(n, n + 1)$ is in nice dimensions if $n \leq 6$.

$\mathcal{K}$-equivalence.

Now, we recall another equivalence relation on map-germs, namely the contact equivalence. This is induced by the action of the group

$$\mathcal{K} := \{\Phi \in \text{Diff}(\mathbb{C}^n \times \mathbb{C}^p, 0) \mid \exists \varphi \in \text{Diff}(\mathbb{C}^n, 0) \text{ with } \Phi \circ i = i \circ \varphi \text{ and } \pi \circ \Phi = \varphi \circ \pi\}$$

where $i \colon \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^p$ is the inclusion $i(\mathbf{x}) = (\mathbf{x}, 0)$ and $\pi \colon \mathbb{C}^n \times \mathbb{C}^p \to \mathbb{C}^n$ is the projection $\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ ([Mat68]). So, for a diffeomorphism $\phi \in \mathcal{K}$ of the form $\phi(\mathbf{x}, \mathbf{y}) = (h(\mathbf{x}), \psi(\mathbf{x}, \mathbf{y}))$, and for $f, g \in \mathcal{E}^0_{n,p}$ with $g = \phi \cdot f$, we have $g \circ h(\mathbf{x}) = \psi(\mathbf{x}, f(\mathbf{x}))$, i.e. ϕ transforms the graph of f into the graph of g.

We can define the notions of $\mathcal{K}$-equivalence, $\mathcal{K}$-determinacy, $\mathcal{K}_e$-versal unfolding in the same way above simply by replacing the group $\mathcal{A}$ with $\mathcal{K}$. We only note that the extended $\mathcal{K}$-tangent space is given by

$$T\mathcal{K}_e f := tf(\theta_{\mathbb{C}^n,0}) + f^*\mathfrak{m}_{\mathbb{C}^p,0} \cdot \Theta(f).$$

Finite determinacy of map-germs. We state the following criterion on finite determinacy due to Mather ([Mat68]) and Gaffney ([Gaf79]).

Theorem 1.3.18. *For $f \in \mathcal{E}^0_{n,p}$ and $\mathcal{G}$ (=one of Mather's groups), the following are equivalent:*

(i) f is finitely $\mathcal{G}$-determined,

(ii) for some k, $T\mathcal{G}_e f \supseteq \mathfrak{m}^k_{\mathbb{C}^n,0}\theta(f)$,

(iii) $\mathcal{G}_e - \operatorname{codim}(f) < \infty$.

Proposition 1.3.19. *If $f \in \mathcal{E}^0_{n,p}$, $n < p$, is finitely $\mathcal{K}$-determined then there exist neighbourhoods U of the origin in $\mathbb{C}^n$ and V of the origin in $\mathbb{C}^p$ with $f(U) \subseteq V$ such that the restriction $f|_U \colon U \to V$ is a finite map.*

Proof. See [Gaf75, p. 66], or [Mar89, Lemma 1.1]. □

Mather's characterisation of stable multi-germs. The following geometric characterisation of finite determinacy was first stated by Mather in [Mat73, p. 241] but without a proof. Then, Gaffney presented the statement with a proof in his thesis [Gaf75].

Theorem 1.3.20 ([Gaf75])**.** *Let $f \in \mathcal{E}^0_{n,p}$ be a finitely $\mathcal{K}$-determined holomorphic map-germ. Then f is finitely $\mathcal{A}$-determined if and only if for any representative of f, there exist neighbourhoods U of 0 in $\mathbb{C}^n$ and V of 0 in $\mathbb{C}^p$, with $f(U) \subseteq V$, such that for all $\mathbf{y} \in V - \{0\}$, the multi-germ of f at $f^{-1}(\mathbf{y}) \cap \Sigma_f \cap U$ is $\mathcal{A}$-stable.*

Here Σ_f is the set of critical points of f, i.e. the set of points at which the differential of f has rank less than the minimum of n and p. For $n < p$, clearly $\Sigma_f = U$. Moreover, finite $\mathcal{K}$-determinacy implies finiteness by Proposition 1.3.19.

Chapter 2

Multiple point spaces

2.1 Background

In this section, we talk about the notion of multiple point spaces for map-germs which will occupy a large part of this thesis.

Definition 2.1.1 (Set theoretic definition of multiple point spaces)**.** Given a map-germ $f \in \mathcal{E}^0_{n,p}$, the kth multiple point space $D^k(f)$ of f is defined to be the set

$$D^k(f) := \text{closure}\left\{(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in (\mathbb{C}^n)^k \mid f(\mathbf{x}_1) = \cdots = f(\mathbf{x}_k), \mathbf{x}_i \neq \mathbf{x}_j \text{ for } i \neq j\right\}$$

where $\mathbf{x}_i := (x_{i1}, \ldots, x_{in})$ for all $i = 1, \ldots, k$.

However, the sets $D^k(f)$ may not be reduced, which is a property that Definition 2.1.1 misses out. Moreover, when the expected dimension of $D^k(f)$ is zero, this definition may not describe the right structure. For example, $f\colon x \mapsto (x^2, x^3)$ has $D^2(f) = \emptyset$. But for a perturbation $f_t\colon x \mapsto (x^2, x^3 + tx)$, $t \neq 0$, of f we find $D^2(f_t) \neq \emptyset$. We would like to refine the definition so that $D^k(f)$ behaves well under deformations. That is, if $F\colon \mathbb{C}^n \times S \to \mathbb{C}^p \times S$ is a level preserving family of maps defined as $F(\mathbf{x}, \mathbf{t}) = (f_{\mathbf{t}}(x), \mathbf{t})$ and $f_{\mathbf{t}} \in \mathcal{E}^0_{n,p}$ is a deformation of f then the diagram

$$\begin{array}{ccc} D^k(f_{\mathbf{t}}) & \longrightarrow & D^k(F) \\ \downarrow & & \downarrow \\ \{\mathbf{t}\} & \longrightarrow & S \end{array}$$

should be a fibre square. In order to obtain this property we would like to equip

$D^k(f)$ with a suitable analytic structure. When the corank of f is 1, an analytic structure for $D^k(f)$ is known for all values of k. However, if corank ≥ 2, we only have an explicit description for this analytic structure for $k = 2$. Hence we must consider the corank 1 and corank ≥ 2 cases separately.

2.1.1 Ideals defining multiple point spaces for corank 1 map-germs.

For the moment we want to distinguish the set theoretic description of the multiple point spaces from the analytic definition. So we will denote the latter by $\tilde{D}^k(f)$. In other words, $\tilde{D}^k(f)$ will denote the variety given by an ideal sheaf $\mathcal{I}_k(f)$ that we will shortly define.

Let $f \in \mathcal{E}^0_{n,p}$ be of corank 1 and $n < p$. Then, after suitable coordinate changes on the domain and the target, f can be written as

$$f\colon (\mathbf{x}, y) \mapsto (\mathbf{x}, f_n(\mathbf{x}, y), \ldots, f_p(\mathbf{x}, y))$$

for $\mathbf{x} \in \mathbb{C}^{n-1}, y \in \mathbb{C}$. In this case, $D^k(f)$ can be embedded in $\mathbb{C}^{n-1} \times \mathbb{C}^k$. Moreover, $\mathcal{I}_k(f)$ is generated by $(k-1)(p-n+1)$ functions R^j_i, for $i = 1, \ldots, k-1$ and $j = n, \ldots, p$, which are defined iteratively as follows

$$\begin{aligned} R^j_1(\mathbf{x}, y_1, y_2) &= \frac{f_j(\mathbf{x}, y_2) - f_j(\mathbf{x}, y_1)}{y_2 - y_1} \qquad \text{and} \\ R^j_i(\mathbf{x}, y_1, \ldots, y_{i+1}) &= \frac{R^j_{i-1}(\mathbf{x}, y_1, \ldots, y_{i-1}, y_{i+1}) - R^j_{i-1}(\mathbf{x}, y_1, \ldots, y_{i-1}, y_i)}{y_{i+1} - y_i} \end{aligned} \tag{2.1}$$

([Mon87]). The following proposition provides S_k-invariant generators for $\mathcal{I}_k(f)$.

Proposition 2.1.2 ([Mon87],[MM89])**.** *The ideal $\mathcal{I}_k(f)$ is also generated by*

$$\mathcal{H}^k_{i,j} := \frac{\begin{vmatrix} 1 & y_1 & \ldots & y_1^{i-1} & f_j(\mathbf{x}, y_1) & y_1^{i+1} & \ldots & y_1^{k-1} \\ 1 & y_2 & \ldots & y_2^{i-1} & f_j(\mathbf{x}, y_2) & y_2^{i+1} & \ldots & y_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_k & \ldots & y_k^{i-1} & f_j(\mathbf{x}, y_k) & y_k^{i+1} & \ldots & y_k^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & \ldots & y_1^{k-1} \\ 1 & y_2 & \ldots & y_2^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_k & \ldots & y_k^{k-1} \end{vmatrix}} \tag{2.2}$$

for $1 \leq i \leq k-1$ *and* $n \leq j \leq p$*.*

Proposition 2.1.3 (Proposition 2.16, [MM89])**.** *Let* $f \in \mathcal{E}^0_{n,p}$ *be a stable map-germ of corank 1. Then* $D^k(f) = \tilde{D}^k(f)$*, that is, the set-theoretic definition coincides with the analytical one.*

Thus we adopt the following definition for multiple point spaces.

Proposition-Definition 2.1.4 (cf. Proposition 2.5, [Hou10] and [Gaf83])**.** Let $f \in \mathcal{E}^0_{n,p}$ be a finitely $\mathcal{A}$-determined map-germ of corank 1. Let $F\colon (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ be a $\mathcal{A}_e$-versal unfolding of f given by $F(\mathbf{x}, y, \mathbf{u}) := (\tilde{F}_{\mathbf{u}}(\mathbf{x}, y), \mathbf{u})$ with $\tilde{F}_0(\mathbf{x}, y) = f(\mathbf{x}, y)$. Then we set

$$D^k(f) = \tilde{D}^k(F) \cap \{\mathbf{u}_1 = 0, \ldots, \mathbf{u}_k = 0\}$$

where $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are the coordinates in $(\mathbb{C}^n \times \mathbb{C}^d)^k$ corresponding to the unfolding parameters.

Proof. We only need to show that this definition is independent of the chosen unfolding. Assume that F' is another $\mathcal{A}_e$-versal unfolding of f. Then $\tilde{D}^k(F) \cong \tilde{D}^k(F')$ by an S_k-equivariant isomorphism which is induced from the $\mathcal{A}$-action giving $F \sim_{\mathcal{A}} F'$ as we show below.

Since both unfoldings are $\mathcal{A}_e$-versal, the number of parameters are the same. So we have $F'\colon (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$, $F'(\mathbf{x}, y, \mathbf{t}) := (\tilde{F}'_{\mathbf{t}}(\mathbf{x}, y), \mathbf{t})$ with $\tilde{F}'_0(\mathbf{x}, y) = f(\mathbf{x}, y)$. Then, there exist germs of diffeomorphisms $\phi \in \mathrm{Diff}(\mathbb{C}^n \times \mathbb{C}^d, 0)$ and $\psi \in \mathrm{Diff}(\mathbb{C}^p \times \mathbb{C}^d, 0)$ defined as in (1.4) such that $F' = \psi \circ F \circ \phi^{-1}$. If we introduce coordinates into this equality we get

$$F'(\mathbf{x}, y, \mathbf{u}) = (\tilde{\psi}_{\mathbf{t}}(F_1(\tilde{\phi}^{-1}_{\mathbf{t}}(\mathbf{x}, y), \mathbf{u}), \ldots, F_p(\tilde{\phi}^{-1}_{\mathbf{t}}(\mathbf{x}, y), \mathbf{u})), \mathbf{u}). \tag{2.3}$$

As ϕ and ψ are invertible they induce an isomorphism $\tilde{D}^k(F) \cong \tilde{D}^k(F')$. On the other hand, the S_k-action on $D^k(F)$ is induced from the map $\sigma \cdot F(\mathbf{x}, y, \mathbf{u}) := F(\mathbf{x}, \sigma(y), \mathbf{u})$, where $\sigma \in S_k$ (cf. Proposition 2.1.2). We also have

$$F'(\mathbf{x}, \sigma(y), \mathbf{u}) = (\tilde{\psi}_t(F_1(\tilde{\phi}^{-1}_{\mathbf{t}}(\mathbf{x}, \sigma(y)), \mathbf{u}), \ldots, F_p(\tilde{\phi}^{-1}_{\mathbf{t}}(\mathbf{x}, \sigma(y)), \mathbf{u})), u)$$

for all $\sigma \in S_k$ by (2.3). Hence it also follows that the isomorphism is S_k-invariant. □

Projection maps.

We recall the maps between the multiple point spaces. We set $D^1(f) := (\mathbb{C}^n, 0)$, $D^0(f) := (\mathbb{C}^p, 0)$ and $\pi_0^1 := f$. For any $k \geq 1$ and for each $i = 1, \ldots, k$, there exists a map $\pi_{k,i}^{k+1}(f)\colon D^{k+1}(f) \to D^k(f)$ induced by the projection $(\mathbb{C}^n)^{k+1} \to (\mathbb{C}^n)^k$ which forgets the i-th factor. Clearly, the images of $\pi_{k,i}^{k+1}$ for different i and the same k are equal to each other. Here, we will fix $i = k+1$ and write π_k^{k+1} instead of $\pi_{k,k+1}^{k+1}$. We define $D_j^{k+1}(f) := \pi_j^{j+1} \circ \pi_{j+1}^{j+2} \circ \cdots \pi_k^{k+1}(D^{k+1}(f))$; moreover

$$\epsilon^k := \pi_k^{k+1} \circ \cdots \circ \pi_1^2 \circ f\colon D^{k+1}(f) \to (\mathbb{C}^p, 0).$$

The algebra of a finite map-germ of corank 1 has a special property:

Proposition 2.1.5. *Let $f \in \mathcal{E}_{n,p}^0$ be a finite analytic map-germ of corank 1 given by $f(\mathbf{x}, y) = (\mathbf{x}, f_n(\mathbf{x}, y), \ldots, f_p(\mathbf{x}, y))$. Then*

$$Q(f) \cong \frac{\mathcal{O}_{\mathbb{C},0}}{(y^{r+1})\mathcal{O}_{\mathbb{C},0}}$$

for some $r < \infty$. In such case we have

$$Q(\pi_k^{k+1}) \cong \frac{\mathcal{O}_{\mathbb{C},0}}{(y^{r-k+1})\mathcal{O}_{\mathbb{C},0}} \quad \text{and} \quad q(\pi_k^{k+1}) = r - k + 1$$

for $k = 1, \ldots, r+1$.

Proof. If f is finite, clearly $Q(f) \cong \frac{\mathcal{O}_{\mathbb{C},0}}{(y^{r+1})\mathcal{O}_{\mathbb{C},0}}$ for some $r < \infty$. The local algebra of π_k^{k+1} is by definition

$$\begin{aligned} Q(\pi_k^{k+1}) = \frac{\mathcal{O}_{D^{k+1}(f)}}{(\pi_k^{k+1})^*\mathfrak{m}_{D^k(f)}} &= \frac{\mathcal{O}_{D^{k+1}(f)}}{(\overline{\mathbf{x}}, \overline{y}_1, \ldots, \overline{y}_k)\mathcal{O}_{D^{k+1}(f)}} \\ &\cong \frac{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k+1},0}}{\mathcal{I}_{k+1}(f) + (\mathbf{x}, y_1, \ldots, y_k)\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k+1},0}} \end{aligned}$$

where $(\overline{\mathbf{x}}, \overline{y}_1, \ldots, \overline{y}_k)$ is the ideal generated by the classes of $\mathbf{x}, y_1, \ldots, y_k$ in $\mathcal{O}_{D^{k+1}(f)}$. Here, $\mathcal{I}_{k+1}(f)$ is the ideal generated by the polynomials

$$\begin{array}{llll}
R_1^n(\mathbf{x},y_1,y_2), & R_1^{n+1}(\mathbf{x},y_1,y_2), & \dots, & R_1^p(\mathbf{x},y_1,y_2),\\
R_2^n(\mathbf{x},y_1,y_2,y_3), & R_2^{n+1}(\mathbf{x},y_1,y_2,y_3), & \dots, & R_2^p(\mathbf{x},y_1,y_2,y_3),\\
R_3^n(\mathbf{x},y_1,y_2,y_3,y_4), & R_3^{n+1}(\mathbf{x},y_1,y_2,y_3,y_4), & \dots, & R_3^p(\mathbf{x},y_1,y_2,y_3,y_4),\\
\vdots & \vdots & & \vdots\\
R_k^n(\mathbf{x},y_1,\dots,y_{k+1}), & R_k^{n+1}(\mathbf{x},y_1,\dots,y_{k+1}), & \dots, & R_k^p(\mathbf{x},y_1,\dots,y_{k+1})
\end{array}$$

by (2.1). Now R_i^j belongs to the ideal $(\mathbf{x},y_1,\dots,y_{i+1})\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k+1},0}$ for all $i = 1,\dots,k-1$ and $j = n,\dots,p$. Therefore,

$$\begin{aligned}
\mathcal{I}_{k+1}(f) + (\mathbf{x},y_1,\dots,y_k) =& (R_k^n(\mathbf{x},y_1,\dots,y_{k+1}),\dots,R_k^p(\mathbf{x},y_1,\dots,y_{k+1}))+\\
&+(\mathbf{x},y_1,\dots,y_k)\\
=&(R_k^n(0,\dots,0,y_{k+1}),\dots,R_k^p(0,\dots,0,y_{k+1}))+(\mathbf{x},y_1,\dots,y_k).
\end{aligned}$$

We have $R_1^j(0,0,y_{k+1}) = \frac{f_j(0,y_{k+1})}{y_{k+1}}$ for all $j = 1,\dots,p$ (cf. (2.1)). If we assume that

$$R_{s-1}^j(0,\dots,0,y_{k+1}) = \frac{f_j(0,y_{k+1})}{y_{k+1}^{s-1}}$$

then we get

$$R_s^j(0,\dots,0,y_{k+1}) = \frac{f_j(0,y_{k+1})}{y_{k+1}^{s}}$$

for all $j = 1,\dots,p$. Hence, by induction, $R_k^j(0,\dots,0,y_{k+1}) = \frac{f_j(0,y_{k+1})}{y_{k+1}^k}$. So,

$$\begin{aligned}
Q(\pi_k^{k+1}) \cong& \frac{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k+1},0}}{\Big(R_k^n(\mathbf{x},y_1,\dots,y_{k+1}),\dots,R_k^p(\mathbf{x},y_1,\dots,y_{k+1}),\mathbf{x},y_1,\dots,y_k\Big)\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k+1},0}}\\
=& \frac{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k+1},0}}{\Big(R_k^n(0,\dots,0,y_{k+1}),\dots,R_k^p(0,\dots,0,y_{k+1}),\mathbf{x},y_1,\dots,y_k\Big)\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k+1},0}}\\
\cong& \frac{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k+1},0}}{\Big(\frac{f_n(0,y_{k+1})}{y_{k+1}^k},\dots,\frac{f_p(0,y_{k+1})}{y_{k+1}^k},\mathbf{x},y_1,\dots,y_k\Big)\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k+1},0}}\\
\cong& \frac{\mathcal{O}_{\mathbb{C},0}}{\Big(\frac{f_n(0,y_{k+1})}{y_{k+1}^k},\dots,\frac{f_p(0,y_{k+1})}{y_{k+1}^k}\Big)\mathcal{O}_{\mathbb{C},0}} \cong \frac{\mathcal{O}_{\mathbb{C},0}}{(y^{r-k+1})\mathcal{O}_{\mathbb{C},0}}. \qquad (2.4)
\end{aligned}$$

This concludes the proof. □

Corollary 2.1.6. *Let $f \in \mathcal{E}^0_{n,p}$ be finite and of corank 1. Then $\pi^{k+1}_k(f)$ is also finite for all $k \geq 1$.*

Proof. This is an easy consequence of Proposition 2.1.5 and Theorem 1.2.14. □

2.1.2 An ideal defining the double point space for corank 2 map-germs

Independent of corank, we define an ideal sheaf

$$\mathcal{I}'_2(f) := (f \times f)^* I_{\Delta_p} + \bigwedge^n \alpha \subseteq \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n,0} \tag{2.5}$$

where I_{Δ_p} is the ideal defining the diagonal $\Delta_p := \{(\mathbf{z}_1, \mathbf{z}_2) \in \mathbb{C}^p \times \mathbb{C}^p \mid \mathbf{z}_1 = \mathbf{z}_2\}$ and $\alpha := (\alpha_{ij})$ is the matrix with entries coming from the equations

$$f_i(\mathbf{x}_1) - f_i(\mathbf{x}_2) = \sum_{i=1}^{p} \alpha_{ij}(\mathbf{x}_1, \mathbf{x}_2) \cdot (x_{1j} - x_{2j})$$

for $i = 1, \ldots, p$ ([Mon87]). Let $\tilde{D}^2(f)'$ be the variety defined by $\mathcal{I}'_2(f)$ and $\tilde{\pi}^2_1(f)\colon \tilde{D}^2(f)' \to (\mathbb{C}^n, 0)$ the projection $(\mathbf{x}_1, \mathbf{x}_2) \mapsto \mathbf{x}_1$. It is easy to see that if f has corank 1 then $\mathcal{I}'_2(f)$ coincides with the description in (2.1) when $k = 2$. We will show that $\mathcal{I}'_2(f)$ gives the right analytic structure for $D^2(f)$, i.e. $\tilde{D}^2(f)' = D^2(f)$, if f is finitely $\mathcal{A}$-determined. First we need some properties.

Proposition 2.1.7. *Let $f \in \mathcal{E}^0_{n,p}$ be finite. Then $\tilde{\pi}^2_1(f)$ is also finite.*

Proof. We have

$$\begin{aligned}
Q(\tilde{\pi}^2_1(f)) &= \frac{\mathcal{O}_{\tilde{D}^2(f)'}}{(\tilde{\pi}^2_1)^* \mathfrak{m}_{\mathbb{C}^n,0}} = \frac{\mathcal{O}_{\tilde{D}^2(f)'}}{(\mathbf{x}_1)\mathcal{O}_{\tilde{D}^2(f)'}} \cong \frac{\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n,0}}{\mathcal{I}'_2(f) + (\mathbf{x}_1)\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n,0}}. \\
&\cong \frac{\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n,0}}{(f_1(\mathbf{x}_1) - f_1(\mathbf{x}_2), \ldots, f_p(\mathbf{x}_1) - f_p(\mathbf{x}_2)) + \bigwedge^n \alpha + (\mathbf{x}_1)\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n,0}} \\
&\cong \frac{\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n,0}}{(f_1(\mathbf{x}_2), \ldots, f_p(\mathbf{x}_2)) + \bigwedge^n \alpha_0 + (\mathbf{x}_1)\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n,0}} \\
&\cong \frac{\mathcal{O}_{\mathbb{C}^n,0}}{(f_1(\mathbf{x}_2), \ldots, f_p(\mathbf{x}_2)) + \bigwedge^n \alpha_0}
\end{aligned}$$

where α_0 is the matrix with entries $\alpha_{ij}(0, \mathbf{x}_2)$. Since

$$(f_1(\mathbf{x}_2), \ldots, f_p(\mathbf{x}_2)) + \bigwedge^n \alpha_0 \supseteq (f_1(\mathbf{x}_2), \ldots, f_p(\mathbf{x}_2)),$$

it follows that $Q(\tilde{\pi}_1^2(f))$ is a finite $\mathbb{C}$-vector space since $Q(f)$ is. □

Corollary 2.1.8. *Let $f \in \mathcal{E}^0_{n,p}$ be a finite map-germ of corank ≥ 2. Then*

$$q(f) \geq q(\tilde{\pi}_1^2).$$

Example 2.1.9. For $f\colon (x, y, z) \mapsto (x, y^2 + xz, z^2 + xy, y^3 + y^2z + z^3)$, we find $Q(f) = \mathbb{C} \cdot \{1, y, z, yz\}$ and $Q(\tilde{\pi}_1^2) = \mathbb{C} \cdot \{1, y_2, z_2\}$.

Example 2.1.10. For $f\colon (x, y, z) \mapsto (x, y^2 + xz + x^2y, yz, z^2 + y^3)$, we have $Q(f) = \mathbb{C} \cdot \{1, y, z\}$ and $Q(\tilde{\pi}_1^2) = \mathbb{C} \cdot \{1, y_2, z_2\}$.

Proposition 2.1.11. *Let $f \in \mathcal{E}^0_{n,n+1}$ be a finite and generically one-to-one map-germ. Then $\tilde{D}^2(f)'$ is a Cohen-Macaulay space of codimension $n + 1$.*

Proof. Let $A := (a_{ij})$ be a generic $(n + 1) \times n$-matrix over the ring $R := S[a_{ij} \mid 1 \leq i \leq n + 1, 1 \leq j \leq n]$. The quotient $R[s_1, \ldots, s_n]/\bigwedge^n A + (t_1, \ldots, t_{n+1})$ is Cohen-Macaulay of codimension $n+1$ where $t_1, \ldots, t_{n+1}$ are defined by the equality

$$\begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n+1} \end{bmatrix} = A \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$$

by [DCS81, Theorem 2.7] – there one needs to make the following substitutions $n_0 = n + 1$, $n_1 = n$, $n_j = 1$ for $j \geq 2$, $X^{(1)} = A$, $X^{(2)} = \begin{bmatrix} s_1 & \ldots & s_n \end{bmatrix}^t$, and $k_1 = n - 1$ so that their $\mathcal{E}(k_1)$ equals to $\bigwedge^n A + (t_1, \ldots, t_{n+1})$.

Let $(\mathbf{x}_1, \mathbf{x}_2) = ((x_{11}, \ldots, x_{1n}), (x_{21}, \ldots, x_{2n}))$ be the chosen coordinate system on $X := \mathbb{C}^n \times \mathbb{C}^n$, $Y := \operatorname{Spec}(\mathbb{C}[s_1, \ldots, s_n, a_{ij} \mid 1 \leq i \leq n + 1, 1 \leq j \leq n])$ and $Z := V(\bigwedge^n A + (t_1, \ldots, t_{n+1}))$. Define $F\colon X \to Y$ by

$$F\colon (\mathbf{x}_1, \mathbf{x}_2) \mapsto (\alpha_{11}, \ldots, \alpha_{n+1,n}, x_{11} - x_{21}, \ldots, x_{1n} - x_{2n})$$

where α_{ij} are the entries of the matrix α satisfying

$$\begin{bmatrix} f_1(\mathbf{x}_1) - f_1(\mathbf{x}_2) \\ f_2(\mathbf{x}_1) - f_2(\mathbf{x}_2) \\ \vdots \\ f_{n+1}(\mathbf{x}_1) - f_{n+1}(\mathbf{x}_2) \end{bmatrix} = \alpha \cdot \begin{bmatrix} x_{11} - x_{21} \\ x_{12} - x_{22} \\ \vdots \\ x_{1n} - x_{2n} \end{bmatrix}.$$

Then $F^{-1}(Z) = \tilde{D}^2(f)'$ by the definition of $\mathcal{I}_2'(f)$ (see (2.5)).

As f is finite and generically one-to-one, the image $\tilde{D}_1^2(f)'$ of $\tilde{\pi}_1^2$ has codimension 1 in $(\mathbb{C}^n, 0)$. Moreover, $\tilde{\pi}_1^2$ is also finite by Proposition 2.1.7. Hence, $\tilde{D}^2(f)'$ has codimension $n+1$. On the other hand, Z is Cohen-Macaulay of codimension $n+1$. It follows from $F^{-1}(Z) \cong (X \times Z) \cap \mathrm{graph}(F)$ that

$$\mathrm{codim}_Y Z \geq \mathrm{codim}_X F^{-1}(Z).$$

Since we have the equality, graph(F) is defined by a regular sequence ([Mat89, Theorem 17.4]). Hence $\tilde{D}^2(f)'$ is Cohen-Macaulay of codimension $n+1$. □

In the case where $p = n+1$ and $n < 6$, stable map-germs have corank 1.

Proposition 2.1.12. *If $f \in \mathcal{E}^0_{n,n+1}$ is a stable map-germ of corank ≥ 2 then $n \geq 6$.*

Proof. If f is stable and of corank 2 then $j^1 f$ is transverse to $\Sigma^2 \subset J^1(n, n+1)$ at 0 by [Mar82, §1.2, Chapter XV]. The definition of the transversality states that

$$\Sigma^2 + \mathrm{d}f(T_0\mathbb{C}^n) = J^1(n, n+1).$$

So we must have codim $\Sigma^2 \leq n$. On the other hand, Σ^2 is a vector space of codimension 6 by [Mar82, §5.1, Chapter VII]. Therefore, under these conditions, $n \geq 6$. □

Proposition 2.1.13. *Let $f \in \mathcal{E}^0_{n,n+1}$, $n < 6$, be a finitely $\mathcal{A}$-determined map-germ of corank ≥ 2. Then $\tilde{D}^2(f)'$ is of dimension $n-1$ and has at most an isolated singularity at the origin.*

Proof. Let us consider a point $(\mathbf{x}_1, \mathbf{x}_2) \in \tilde{D}^2(f)' \setminus \{0\}$ such that $f(\mathbf{x}_1) = f(\mathbf{x}_2) = \mathbf{y}$. Let $\bar{f} \colon (\mathbb{C}^n, \{\mathbf{x}_1, \mathbf{x}_2\}) \to (\mathbb{C}^{n+1}, \mathbf{y})$ be a representative of f at $S := \{\mathbf{x}_1, \mathbf{x}_2\}$. Since f is finitely $\mathcal{A}$-determined, the multi-germ $\bar{f}$ is stable by Theorem 1.3.20. Moreover, Proposition 1.6 of [Mat69b] implies that $f^{(1)}$ and $f^{(2)}$, considered as mono-germs, are also stable. In these dimensions, $f^{(i)}$ has corank 1 by Proposition 2.1.12. So, the statement follows from [MM89, Proposition 2.13] where it was proven that the kth multiple point space of a stable multi-germ is nonsingular. □

Proposition 2.1.14. *If $F \in \mathcal{E}^0_{N,N+1}$ is a stable map-germ with $N < 6$, then $D^2(F) = \tilde{D}^2(F)'$.*

Proof. By Proposition 2.1.13, $\tilde{D}^2(F)'$ is smooth, hence reduced, away from the origin. In fact, it is reduced everywhere by the Cohen-Macaulay property (Proposition 2.1.11). So

$$\tilde{D}^k(F)' \cap U = D^k(F) \cap U$$

where $U := \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{C}^n \times \mathbb{C}^n \mid \mathbf{x}_1 \neq \mathbf{x}_2\}$. Now it remains to show that $\tilde{D}^2(F)'$ is equal to the closure of $\tilde{D}^2(F)' \cap U$. Since F is stable, the image $\tilde{D}^2_1(F)'$ of $\tilde{\pi}^2_1$ is a reduced hypersurface in the domain. Moreover, F is finite. So π^2_1 is also finite by Proposition 2.1.7. It follows that the image of any irreducible component of $\tilde{D}^2(F)'$ is closed in $\tilde{D}^2_1(F)'$. As $\tilde{D}^2(F)'$ is equidimensional, the image of those components must coincide with $\tilde{D}^2_1(F)'$. But, $\tilde{\pi}^2_1$ is only $2\colon 1$ over the singular locus of $\tilde{D}^2_1(F)'$. So $\tilde{D}^2(F)'$ itself is irreducible which implies that the only closed subvariety of $\tilde{D}^2(F)'$ is $\tilde{D}^2(F)' \cap \{\mathbf{x}_1 = \mathbf{x}_2\}$ which has codimension 1 in $\tilde{D}^2(F)'$. Therefore, $\tilde{D}^2(F)'$ is the closure of $\tilde{D}^2(F)' \cap U$. □

Definition 2.1.15 (cf. Definition 2.1.4)**.** Let $f \in \mathcal{E}^0_{n,n+1}$ be a finitely $\mathcal{A}$-determined map-germ. Let $F\colon (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}^d, 0)$ be a $\mathcal{A}_e$-versal unfolding of f given by $F(\mathbf{x}, \mathbf{u}) := (\tilde{F}_{\mathbf{u}}(\mathbf{x}), \mathbf{u})$. Then we set

$$D^2(f) = \tilde{D}^2(F)' \cap \{\mathbf{u}_1 = \mathbf{u}_2 = 0\},$$

and write $\pi^2_1(f)$ instead of $\tilde{\pi}^2_1(f)$.

Proposition 2.1.16. *Let $f \in \mathcal{E}^0_{n,p}$ be a map-germ. Let R_f denote the ramification ideal of f generated by all $(\min\{n,p\}) \times (\min\{n,p\})$-minors of $\mathrm{d}f$. Then*

$$R_f = (\pi^2_1)_*(\mathcal{I}'_2(f) + I_{\Delta_n}) \tag{2.6}$$

where I_{Δ_n} is the ideal of $\Delta_n := \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{C}^n \times \mathbb{C}^n \mid \mathbf{x}_1 = \mathbf{x}_2\}$.

Proof. Notice that $(f \times f)^*\Delta_p \subset I_{\Delta_n}$. So we need to relate α_{ij} with the derivatives of the components of f. Let us consider the Taylor series expansion of $f_i(\mathbf{x}_2)$ at $\mathbf{x}_1$:

$$f_i(\mathbf{x}_2) = f_i(\mathbf{x}_1) + \frac{\partial f_i}{\partial x_{21}}(\mathbf{x}_1) \cdot (x_{21} - x_{11}) + \cdots + \frac{\partial f_i}{\partial x_{2n}}(\mathbf{x}_1) \cdot (x_{2n} - x_{1n}) + h.o.t.$$

for all $i = 1, \ldots, p$. So we have

$$\sum_{j=1}^{p} \alpha_{ij}(x_{2j} - x_{1j}) = \frac{\partial f_i}{\partial x_{21}}(\mathbf{x}_1) \cdot (x_{21} - x_{11}) + \cdots + \frac{\partial f_i}{\partial x_{2n}}(\mathbf{x}_1) \cdot (x_{2n} - x_{1n}) + h.o.t.$$

for all $i = 1, \ldots, p$. Now, let us put $x_{1j} = x_{2j}$ for all $j \in \left\{1, \ldots, \hat{k}, \ldots, n\right\}$, where $\hat{k}$ means that some $k \in \{1, \ldots, n\}$ is omitted. Then we get

$$\alpha_{ik}(x_{2k} - x_{1k}) = \frac{\partial f_i}{\partial x_{2k}}(\mathbf{x}_1) \cdot (x_{2k} - x_{1k}) + \frac{1}{2} \frac{\partial^2 f_i}{\partial x_{2k}^2}(\mathbf{x}_1) \cdot (x_{2k} - x_{1k})^2 + \cdots$$

If we divide each side by $x_{2k} - x_{1k}$ and then put $x_{2k} = x_{1k}$ we obtain $\alpha_{ik} = \frac{\partial f_i}{\partial x_{2k}}(\mathbf{x}_1)$. Since this equality holds for all $i = 1, \ldots, p$ and $k = 1, \ldots, n$ we get $\alpha = \mathrm{d}f$ along the diagonal. So the result follows. □

Lemma 2.1.17. *Let $f \in \mathcal{E}^0_{n,p}$ be a map-germ. Then, away from the diagonal Δ_n, we have $\bigwedge^n \alpha \subset (f \times f)^* I_{\Delta_p}$.*

Proof. Let $(\mathbf{x}_1, \mathbf{x}_2) \in \tilde{D}^2(f)' \setminus \tilde{D}^2(f)' \cap \Delta_n$. Let us assume that $x_{11} - x_{21} \neq 0$ in $\tilde{D}^2(f)'$. In this case, we can choose $\alpha_{i1} = \frac{f_i(\mathbf{x}_1) - f_i(\mathbf{x}_2)}{x_{11} - x_{21}}$ and $\alpha_{ij} = 0$ for all $i = 1, \ldots, p$ and $j = 2, \ldots, n$ so that

$$f_i(\mathbf{x}_1) - f_i(\mathbf{x}_2) = \sum_{j=1}^{p} \alpha_{ij}(x_{1j} - x_{2j}). \tag{2.7}$$

Then the matrix α admits the form

$$\begin{bmatrix} \frac{f_1(\mathbf{x}_1) - f_1(\mathbf{x}_2)}{x_{11} - x_{21}} & 0 & \cdots & 0 \\ \frac{f_2(\mathbf{x}_1) - f_2(\mathbf{x}_2)}{x_{11} - x_{21}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \frac{f_p(\mathbf{x}_1) - f_p(\mathbf{x}_2)}{x_{11} - x_{21}} & 0 & \cdots & 0 \end{bmatrix}. \tag{2.8}$$

So, we get $\bigwedge^n \alpha = 0$.

If any two of the components of $\mathbf{x}_1$ and $\mathbf{x}_2$ are different, say $x_{11} \neq x_{21}$ and $x_{12} \neq x_{22}$, then we have an option to choose $\alpha_{i1} = \frac{f_i(\mathbf{x}_1) - f_i(\mathbf{x}_2)}{x_{11} - x_{21}}$ or $\alpha_{i2} = \frac{f_i(\mathbf{x}_1) - f_i(\mathbf{x}_2)}{x_{12} - x_{22}}$, and $\alpha_{ij} = 0$ for all $i = 1, \dots, p$ and $j = 3, \dots, n$ so that they satisfy $f_i(\mathbf{x}_1) - f_i(\mathbf{x}_2) = \sum_{j=1}^{p} \alpha_{ij}(x_{1j} - x_{2j})$. Then, α has the form

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \cdots & 0 \\ \alpha_{31} & \alpha_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{p1} & \alpha_{p2} & 0 & \cdots & 0 \end{bmatrix}.$$

Then, again, $\bigwedge^n \alpha = 0$.

Similarly, if among all the components of $\mathbf{x}_1$ and $\mathbf{x}_2$ only two of them are equal to each other then we get $\bigwedge^n \alpha = 0$. However, when $x_{1j} \neq x_{2j}$ for all $j = 1, \dots, n$ α can be chosen as

$$\begin{bmatrix} \frac{f_1(\mathbf{x}_1) - f_1(\mathbf{x}_2)}{x_{11} - x_{21}} & 0 & \cdots & 0 \\ 0 & \frac{f_2(\mathbf{x}_1) - f_2(\mathbf{x}_2)}{x_{21} - x_{22}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{f_p(\mathbf{x}_1) - f_p(\mathbf{x}_2)}{x_{1n} - x_{2n}} \end{bmatrix}$$

in which case $\bigwedge^n \alpha \subset (f \times f)^* \Delta_p$. □

2.2 Presentations of the push-forwards

In this section we recall the works of Mond and Pellikaan in [MP89].

Let (X, x) be a germ of an irreducible Cohen-Macaulay variety of dimension n, and $f\colon (X, x) \to (\mathbb{C}^{n+1}, 0)$ be a finite analytic map-germ. By Weierstrass Division theorem (see, for example, [dJP00, Corollary 3.2.12]), $\mathcal{O}_{X,x}$ is a finite $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module. Moreover, if $Q(f)$ is spanned by the classes of some $g_0, g_1, \dots, g_r \in \mathcal{O}_{X,x}$ as a vector space over $\mathbb{C} = \mathcal{O}_{\mathbb{C}^{n+1},0}/\mathfrak{m}_{\mathbb{C}^{n+1},0}$, then $g_0, \dots, g_r$ generate $\mathcal{O}_{X,x}$ as an $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module. Clearly, one can assume that $g_0 = 1$.

By Lemma 2.1 of [MP89], there exists a resolution of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ of length 1, i.e. an exact sequence

$$0 \to \mathcal{O}_{\mathbb{C}^{n+1},0}^{r+1} \xrightarrow{\lambda} \mathcal{O}_{\mathbb{C}^{n+1},0}^{r+1} \xrightarrow{G} \mathcal{O}_{X,x} \to 0 \tag{2.9}$$

where $G := \begin{bmatrix} 1 & g_1 & \dots & g_r \end{bmatrix}$ and λ is a $(r+1) \times (r+1)$ matrix with entries $\lambda_{ij} \in$

$\mathcal{O}_{\mathbb{C}^{n+1},0}$ for $i, j = 0, \ldots, r$. If, in addition, (X, x) is a Gorenstein variety[1] then λ can be chosen to be a symmetric matrix ([MP89, Proposition 2.5]).

Algorithm 2.2.1 (§2, [MP89])**.** INPUT. A finite map-germ $f\colon (X, x) \to (\mathbb{C}^{n+1}, 0)$ where X is a Cohen-Macaulay space of dimension n.

STEP 1. Choose a projection $\nu\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ such that $\tilde{f} := \nu \circ f$ is still finite. After a suitable coordinate change, put f into the form $f(x) = (\tilde{f}(x), f_{n+1}(x))$. Let Y_{n+1} denote the last component of the coordinate system on $\mathbb{C}^{n+1}$ so that $Y_{n+1} \circ f = f_{n+1}$.

STEP 2. Assume that $Q(\tilde{f}) = \mathbb{C} \cdot \{1, g_1 \ldots, g_r\}$ for some finite $r \geq 0$ so that $\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{C}^{n+1},0} \cdot \{1, g_1, \ldots, g_r\}$.

Find elements $\alpha_{ij} \in \mathcal{O}_{\mathbb{C}^n}$, $0 \leq i, j \leq r$, satisfying

$$g_j f_{n+1} = \sum_{i=0}^{r} (\alpha_{ij} \circ \tilde{f}) g_i.$$

STEP 3. Form a matrix $\lambda = (\lambda_{ij})$ by setting

$$\lambda_{ij} = \alpha_{ij} \circ \nu \qquad \text{for } i \neq j \text{ and}$$
$$\lambda_{ii} = \alpha_{ii} \circ \nu - Y_{n+1}.$$

OUTPUT. A short exact sequence of the form

$$0 \longrightarrow \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{G} \mathcal{O}_{X,x} \longrightarrow 0$$

where $G = \begin{bmatrix} 1 & g_1 & \ldots & g_r \end{bmatrix}$.

Corollary 2.2.2. *If f is a finite multi-germ $f\colon (X, S) \to (\mathbb{C}^{n+1}, 0)$ with branches $f^{(i)}$, $i = 1, \ldots, k$, then we have*

$$0 \longrightarrow \mathcal{O}^{l}_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^{l}_{\mathbb{C}^{n+1},0} \longrightarrow \mathcal{O}_{X,S} \longrightarrow 0$$

where $\lambda = \begin{bmatrix} \lambda^1 & 0 & \ldots & 0 \\ 0 & \lambda^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda^k \end{bmatrix}$ *and λ^i is a presentation matrix of the branch $f^{(i)}$*

[1] see, for example, [BH93] for a definition

calculated as above.

2.2.1 Multiple point spaces on the target.

It is also useful to study multiple points in the target for finite map-germs. These are the spaces $M_k(f)$ consisting of the points which have at least k preimages in the source. By Proposition 1.5 of [MP89], $M_k(f)$ is the zero locus of the ideal $\mathrm{Fitt}_{k-1}(f_*\mathcal{O}_{\mathbb{C}^n,0})$. Set theoretically, we have $M_k(f) = \epsilon^k(D^k(f))$. So, $D^k_1(f) = V(f^*\mathrm{Fitt}_{k-1}(f_*\mathcal{O}_{\mathbb{C}^n,0}))$. When $(n,p) = (n, n+1)$, these Fitting ideals can be read from a short exact sequence of the form (2.9).

2.3 Pullbacks of map-germs

Definition 2.3.1 (Definition 1.46, [GLS07]). Let $f\colon X \to T$, $g\colon Y \to T$ be two morphisms of complex spaces. Then the *(analytic) fibre product* of X and Y over T is a triple $(X\times_T Y, p_X, p_Y)$ consisting of a complex space $X\times_T Y$ and two morphisms $p_X\colon X\times_T Y \to X$, $p_Y\colon X\times_T Y \to Y$ such that $f\circ p_X = g\circ p_Y$, satisfying the following *universal property*: for any complex space Z, and given morphisms $\nu\colon Z \to X$ and $\eta\colon Z \to Y$ there exists a unique morphism $\theta\colon Z \to X\times_S Y$ such that $\nu = p_X \circ \theta$ and $\eta = p_Y \circ \theta$.

For a proof of the existence of the fibre product see, for example, [GLS07, p. 49].

Notice that for $Z = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$, θ is an isomorphism and ν, η are induced from the natural projections $X\times Y \to X$, $X\times Y \to Y$, respectively. So, we take $X\times_S Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$.

In particular, if $T = \{pt\}$ is a reduced point then $X\times_{\{pt\}} Y$ is called the *Cartesian product* $X\times Y$ of X and Y. If $\mathcal{O}_{X,x} = \mathbb{C}\{\mathbf{x}\}/I$, $\mathcal{O}_{Y,y} = \mathbb{C}\{\mathbf{y}\}/J$ for $x \in X$, $y \in Y$, then

$$\mathcal{O}_{X\times Y,(x,y)} = \mathbb{C}\{\mathbf{x},\mathbf{y}\}/(I\mathbb{C}\{\mathbf{x},\mathbf{y}\} + J\mathbb{C}\{\mathbf{x},\mathbf{y}\}).$$

This local ring is the analytic tensor product $\mathcal{O}_{X,x}\hat{\otimes}\mathcal{O}_{Y,y}$ of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ by Definition 1.2.7. We consider the analytic tensor rather than the algebraic one because two definitions may not coincide at stalks. The analytic tensor product usually contains the algebraic tensor product as a proper subring. For example,

$$\mathcal{O}_{\mathbb{C}\times\mathbb{C},(0,0)} = \mathbb{C}\{x,y\} \subsetneq \mathbb{C}\{x\} \otimes_{\mathbb{C}} \mathbb{C}\{y\} = \mathcal{O}_{\mathbb{C},0} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0}$$

(see [GLS07] for details).

However, the following lemma shows that when g is a finite morphism then we may choose to study with the algebraic tensor products instead.

Lemma 2.3.2 (Lemma 1.89, [GLS07])**.** *Let f and g be as above. Assume that g is finite. Then there is a natural isomorphism*

$$p_X^{-1}\mathcal{O}_X \otimes_{p_Y^{-1}g^{-1}\mathcal{O}_T} p_Y^{-1}\mathcal{O}_Y \xrightarrow{\cong} \mathcal{O}_{X\times_T Y}$$

induced by the map $a \otimes b \mapsto ab := p_X(a) \cdot p_Y(b)$.

Corollary 2.3.3. *Let f and g be as above. Assume that g is finite, and $(a,b) \in X \times_T Y$. Then*

$$\mathcal{O}_{X,a} \otimes_{\mathcal{O}_{T,g(b)}} \mathcal{O}_{Y,b} \cong \mathcal{O}_{X\times_Y X,(a,b)}.$$

Proof. This follows from the definition of the topological preimage sheaf: If, say, $\varphi\colon \mathcal{X} \to \mathcal{Y}$ is a continuous map of topological spaces and $\mathcal{G}$ is a sheaf on $\mathcal{Y}$ then the *topological preimage* sheaf $\varphi^{-1}\mathcal{G}$ is defined to be the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{V\supset\varphi(U)} \mathcal{G}(V), \qquad U \subset \mathcal{X} \text{ open }.$$

For a point $x \in \mathcal{X}$ we have $(\varphi^{-1}\mathcal{G})_x = \mathcal{G}_{\varphi(x)}$. $\square$

By Mather's results ([Mat70]), if $f \in \mathcal{E}^0_{n,p}$ is a $\mathcal{K}$-finite, then there exists a stable germ $F \in \mathcal{E}^0_{N,P}$ and a germ of an immersion $g \in \mathcal{E}^0_{p,P}$ with g transverse to F, such that f completes the following commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^N,0) & \xrightarrow{F} & (\mathbb{C}^P,0) \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle g} \\ (\mathbb{C}^n,0) \cong (\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p,0) & \xrightarrow{f} & (\mathbb{C}^p,0) \end{array} \tag{2.10}$$

and $P - N = p - n$. Here the transversality condition guarantees that the fibre product $(\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p, 0)$ is a smooth space of codimension P, e.g $\mathbb{C}^n \cong \mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p$.

Moreover, f is $\mathcal{A}$-equivalent to the projection $(\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ and i is an immersion which is defined to be the projection $(\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p, 0) \to (\mathbb{C}^N, 0)$.

Definition 2.3.4. A map-germ f defined by the above setup is called the *pullback of F* by g.

A $\mathcal{K}_e$-versal unfolding of a map-germ is $\mathcal{A}$-stable (see [Wal81, Part I: 3]). In other words, any $\mathcal{K}$-finite map-germ admits a $\mathcal{A}$-stable unfolding. So, we focus on $\mathcal{K}$-finite map-germs.

Proposition 2.3.5. *Let $f \in \mathcal{E}^0_{n,n+1}$ be defined as a pullback of $F \in \mathcal{E}^0_{N,N+1}$ by $g \in \mathcal{E}^0_{n+1,N+1}$. In addition, assume that F is finite and $\mathbb{C}^N \times_{\mathbb{C}^{N+1}} \mathbb{C}^{n+1}$ has the expected dimension n. If $\mathcal{O}_{\mathbb{C}^N,0}$ has a resolution*

$$0 \to \mathcal{O}^{r+1}_{\mathbb{C}^{N+1},0} \xrightarrow{\lambda} \mathcal{O}^{r+1}_{\mathbb{C}^{N+1},0} \to \mathcal{O}_{\mathbb{C}^N,0} \to 0 \tag{2.11}$$

then there exists a resolution of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ given by

$$0 \to \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{g^*\lambda} \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \to \mathcal{O}_{\mathbb{C}^n,0} \to 0$$

which is minimal; that is, $q(f) = q(F) = r + 1$.

Proof. Since F is finite with $q(F) = r + 1$, there exists a minimal resolution of the form (2.11). We tensor the sequence (2.11) by $\mathcal{O}_{\mathbb{C}^{n+1},0}$ to get

$$\begin{array}{ccccccc}
\mathcal{O}^{r+1}_{\mathbb{C}^{N+1},0} \otimes \mathcal{O}_{\mathbb{C}^{n+1},0} & \xrightarrow{\lambda\otimes 1} & \mathcal{O}^{r+1}_{\mathbb{C}^{N+1},0} \otimes \mathcal{O}_{\mathbb{C}^{n+1},0} & \longrightarrow & \mathcal{O}_{\mathbb{C}^N,0} \otimes_{\mathcal{O}_{\mathbb{C}^{N+1},0}} \mathcal{O}_{\mathbb{C}^{n+1},0} & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
\mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} & \xrightarrow{g^*\lambda} & \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} & \longrightarrow & \mathcal{O}_{\mathcal{O}_{\mathbb{C}^N \times_{\mathbb{C}^{N+1}} \mathbb{C}^{n+1},0}} & \longrightarrow & 0.
\end{array}$$

The determinant of λ is not identically zero. Therefore $\det(g^*\lambda) = 0$ if and only if $g(\mathbb{C}^{n+1}) \subseteq \mathrm{im}(F)$. However, $g(\mathbb{C}^{n+1}) \subseteq \mathrm{im}(F)$ contradicts the assumption on the dimension of the fibre product. So we must have $\det(g^*\lambda) \neq 0$. Hence the sequence

$$0 \to \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{g^*\lambda} \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \to \mathcal{O}_{\mathbb{C}^N \times_{\mathbb{C}^{N+1}} \mathbb{C}^{n+1},0} \cong \mathcal{O}_{\mathbb{C}^n,0} \to 0 \tag{2.12}$$

is exact.

By construction, $\lambda_{ij} \in \mathfrak{m}_{\mathbb{C}^{N+1},0}$ for all $i, j = 1, \dots, r + 1$ (cf. [MP89]). As $g(0) = 0$, all the components of g belongs to the maximal ideal $\mathfrak{m}_{\mathbb{C}^{n+1},0}$. So $g^*\lambda_{ij} \in$

$\mathfrak{m}_{\mathbb{C}^{n+1},0}$ whence (2.12) is minimal. It follows that $q(f) = q(F)$. $\square$

Consequently,

Corollary 2.3.6. *Let $f \in \mathcal{E}^0_{n,n+1}$ be defined as a pullback of $F \in \mathcal{E}^0_{N,N+1}$ by $g \in \mathcal{E}^0_{n+1,N+1}$. In addition, assume that $\mathbb{C}^N \times_{\mathbb{C}^{N+1}} \mathbb{C}^{n+1}$ has the expected dimension n. Then $\mathrm{Fitt}_j(f_*\mathcal{O}_{\mathbb{C}^n,0}) = g^*\mathrm{Fitt}_j(F_*\mathcal{O}_{\mathbb{C}^N,0})$.*

The following property was stated by Goryunov in Section 3.5 of [Gor95] but without a proof. Here we will present it with a detailed proof.

Proposition 2.3.7. *Let $f \in \mathcal{E}^0_{n,p}$ be a corank 1 map-germ defined as a pullback of $F \in \mathcal{E}^0_{N,P}$ by $g \in \mathcal{E}^0_{p,P}$. Let, in addition, g be transverse to F and $i \in \mathcal{E}^0_{n,N}$ be the map-germ making the pullback diagram commutative. Then*

$$\mathcal{O}_{D^k(f)} \cong (\underbrace{i \times \cdots \times i}_{k \text{ times}})^*\mathcal{O}_{D^k(F)}. \tag{2.13}$$

Proof. We have a natural map

$$\underbrace{i \times i \times \cdots \times i}_{k \text{ times}} \colon (\mathbb{C}^n \times \cdots \times \mathbb{C}^n, 0) \to (\mathbb{C}^N \times \cdots \times \mathbb{C}^N, 0) \tag{2.14}$$

induced by $i \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^N, 0)$. So, we need to show that $(\underbrace{i \times \cdots \times i}_{k \text{ times}})^*\mathcal{I}_k(F) \cong \mathcal{I}_k(f)$ as $\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^k,0}$-modules. First of all, we will put f into a standard form.

Clearly, F is $\mathcal{A}$-equivalent to the map-germ

$$\begin{aligned} F' \colon (\mathbb{C}^N, 0) &\to (\mathbb{C}^P, 0) \\ (\mathbf{x}, y) := (x_1, \ldots, x_{N-1}, y) &\mapsto (x_1, \ldots, x_{N-1}, F_N(\mathbf{x}, y), \ldots, F_P(\mathbf{x}, y)) \end{aligned}$$

Without the loss of generality, we can assume that $F = F'$. The transversality condition implies that g is equivalent to the map-germ

$$\begin{aligned} g \colon (\mathbb{C}^p, 0) &\to (\mathbb{C}^P, 0) \\ \mathbf{z} := (z_1, \ldots, z_p) &\mapsto (g_1(\mathbf{z}), \ldots, g_{N-1}(\mathbf{z}), z_n + \alpha_n(\mathbf{z}), \ldots, z_p + \alpha_p(\mathbf{z})) \end{aligned}$$

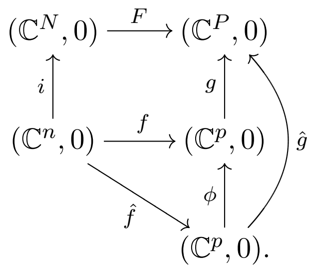
where $\alpha_j(\mathbf{z}) \in \mathcal{O}_{\mathbb{C}^p,0}$ for $j = n, \dots, p$. A diffeomorphism

$$\begin{aligned} \phi\colon (\mathbb{C}^p, 0) &\rightarrow (\mathbb{C}^p,) \\ z_j &\mapsto z_j + \alpha_j(\mathbf{z}) =: \hat{z}_j \qquad \text{for } j = n, \dots, p \end{aligned}$$

yields

$$g \sim_{\mathcal{A}} \hat{g}\colon \hat{\mathbf{z}} := (\hat{z}_1, \dots, \hat{z}_p) \mapsto (\hat{g}_1(\hat{\mathbf{z}}), \dots, \hat{g}_{N-1}(\hat{\mathbf{z}}), \hat{z}_n, \dots, \hat{z}_p)$$

and also $f \sim_{\mathcal{A}} \hat{f}$ where $\hat{f}$ is defined as a fibre product of F and $\hat{g}$:



(2.15)

Since $\mathcal{A}$-equivalent maps have isomorphic multiple point spaces, we can study $\hat{f}$, $\hat{g}$ instead of f, g, respectively. For simplicity we will drop the symbol "ˆ" from now on. In other words, we will consider the map-germs

$$F(\mathbf{x}, y) = (\mathbf{x}, F_N(\mathbf{x}, y), \dots, F_P(\mathbf{x}, y))$$

and

$$g(\mathbf{z}) = (g_1(\mathbf{z}), \dots, g_{N-1}(\mathbf{z}), z_n, \dots, z_p).$$

In this case, $\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p$ is defined by the set of equations

$$x_1 = g_1(\mathbf{z}), \dots, x_{N-1} = g_{N-1}(\mathbf{z}), \dots F_N(\mathbf{x}, y) = z_n, \dots, F_P(\mathbf{x}, y) = z_p. \qquad (2.16)$$

Over $\mathcal{O}_{\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p, (0,0)}$ it is possible to solve the variables $x_1, \dots, x_{N-1}$ and $z_n, \dots, z_p$ in terms of $z_1, \dots, z_{n-1}$ and y from (2.16). Let us denote the coordinates $(z_1, \dots, z_{n-1})$ by $\hat{\mathbf{z}}$. Then,

$$x_1 = g_1(\hat{\mathbf{z}}, h_n(\hat{\mathbf{z}}, y), \dots, h_p(\hat{\mathbf{z}}, y)), \dots, x_{N-1} = g_{N-1}(\hat{\mathbf{z}}, h_n(\hat{\mathbf{z}}, y), \dots, h_p(\hat{\mathbf{z}}, y)),$$
$$z_n = h_n(\hat{\mathbf{z}}, y), \dots, z_p = h_p(\hat{\mathbf{z}}, y)$$

for some functions $h_i(\hat{\mathbf{z}}, y) \in \mathcal{O}_{\mathbb{C}^n,0}$ $(i = n, \ldots, p)$. It follows that $\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p$ is given by the ideal

$$\begin{aligned}\mathcal{I} := \big(&x_1 - g_1(\hat{\mathbf{z}}, h_n(\hat{\mathbf{z}}, y), \ldots, h_p(\hat{\mathbf{z}}, y)), \ldots, x_{N-1} - g_{N-1}(\hat{\mathbf{z}}, h_n(\hat{\mathbf{z}}, y), \ldots, h_p(\hat{\mathbf{z}}, y)),\\ &z_n - h_n(\hat{\mathbf{z}}, y), \ldots, z_p - h_p(\hat{\mathbf{z}}, y)\big).\end{aligned}$$

Hence, we have an isomorphism

$$\begin{array}{rcl}(\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p, 0) & \to & (\mathbb{C}^n, 0)\\ (\mathbf{x}, y, z_1, \ldots, z_p) & \mapsto & (y, \hat{\mathbf{z}})\end{array}$$

with the inverse

$$\begin{array}{rcl}(\mathbb{C}^n, 0) & \to & (\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p, 0)\\ (y, \hat{\mathbf{z}}) & \mapsto & \big(g_1(\hat{\mathbf{z}}, h_n(\hat{\mathbf{z}}, y), \ldots, h_p(\hat{\mathbf{z}}, y)), \ldots, g_{N-1}(\hat{\mathbf{z}}, h_n(\hat{\mathbf{z}}, y), \ldots, h_p(\hat{\mathbf{z}}, y)), y, \hat{\mathbf{z}}\big).\end{array}$$

Therefore f has the form $f\colon (y, \hat{\mathbf{z}}) \mapsto \big(\hat{\mathbf{z}}, h_n(\hat{\mathbf{z}}, y), \ldots, h_p(\hat{\mathbf{z}}, y)\big)$ with $\hat{\mathbf{z}} \in \mathbb{C}^{n-1}, y \in \mathbb{C}$. Moreover,

$$i\colon (y, \hat{\mathbf{z}}) \mapsto \big(g_1(\hat{\mathbf{z}}, h_n(\hat{\mathbf{z}}, y), \ldots, h_p(\hat{\mathbf{z}}, y)), \ldots, g_{N-1}(\hat{\mathbf{z}}, h_n(\hat{\mathbf{z}}, y), \ldots, h_p(\hat{\mathbf{z}}, y)), y\big).$$

We consider the definition of the generators $\mathcal{H}^k_{l,j}$ for $\mathcal{I}_k(F)$ given by Proposition 2.1.2. Then

$$(\underbrace{i \times i \times \cdots \times i}_{k \text{ times}})^* \mathcal{H}^k_{l,j} = \frac{\begin{vmatrix} 1 & y_1 & \ldots & y_1^{l-1} & i^*F_{N+j}(\mathbf{x}, y_1) & y_1^{l+1} & \ldots & y_1^{r-1}\\ 1 & y_2 & \ldots & y_2^{l-1} & i^*F_{N+j}(\mathbf{x}, y_2) & y_2^{l+1} & \ldots & y_2^{r-1}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 1 & y_k & \ldots & y_k^{l-1} & i^*F_{N+j}(\mathbf{x}, y_k) & y_k^{l+1} & \ldots & y_k^{k-1}\end{vmatrix}}{\begin{vmatrix} 1 & y_1 & \ldots & y_1^{k-1}\\ 1 & y_2 & \ldots & y_2^{k-1}\\ \vdots & \vdots & & \vdots\\ 1 & y_k & \ldots & y_k^{k-1}\end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 1 & y_1 & \dots & y_1^{l-1} & h_{n+j}(\hat{\mathbf{z}}, y_1) & y_1^{l+1} & \dots & y_1^{k-1} \\ 1 & y_2 & \dots & y_2^{l-1} & h_{n+j}(\hat{\mathbf{z}}, y_2) & y_2^{l+1} & \dots & y_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_k & \dots & y_k^{l-1} & h_{n+j}(\hat{\mathbf{z}}, y_k) & y_k^{l+1} & \dots & y_k^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & \dots & y_1^{k-1} \\ 1 & y_2 & \dots & y_2^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_k & \dots & y_k^{k-1} \end{vmatrix}} := \overline{\mathcal{H}}_{l,j}^k$$

for $1 \leq l \leq k-1$ and $0 \leq j \leq P-N = p-n$. Again by Proposition 2.1.2, the polynomials $\overline{\mathcal{H}}_{l,j}^k$ generates the ideal $\mathcal{I}_k(f)$. Hence $(\underbrace{i \times \cdots \times i}_{k \text{ times}})^* \mathcal{I}_k(F) \cong \mathcal{I}_k(f)$. This concludes the proof. □

2.4 Principle of iteration

The setup of multiple point spaces allows us to define certain *iterations* involving the projections π_{k-1}^k and the map-germ itself. In [Kle81], Kleiman defined $(k+1)$-th multiple point scheme to be the double point scheme of π_{k-1}^k (see Remark 2.6.4 (2)). In [Gor95] it was hinted that there was a generalisation for corank 1 stable map-germs; namely, an isomorphism $D^s(\pi_{k-1}^k(f)) \cong D^{k+s-1}(f)$. Later, Goryunov and Mond showed the explicit isomorphism in [GM93]. Below, we recall their result in detail as we will need the technique later. By another result of [Gor95], we will prove an analytic isomorphism and then generalise it for finitely $\mathcal{A}$-determined map-germs.

Proposition 2.4.1 (Remark 2.7 (iii), [GM93])**.** *Let $f \in \mathcal{E}_{n,p}^0$ be finite. Then, set theoretically, we have*

$$D^s(\pi_{k-1}^k(f)) \cong D^{k+s-1}(f)$$

for $k, s \geq 1$.

Proposition 2.4.1 is referred to as *the Principle of iteration.*

Proof. Let $\mathbf{x}_j^i := (x_{j1}^i, \dots, x_{jn}^i)$ denote a point in $(\mathbb{C}^n, 0)$. Let us write π_{k-1}^k instead of $\pi_{k-1}^k(f)$ for simplicity. By definition,

$$D^s(\pi^k_{k-1}) = \text{closure}\big\{((\mathbf{x}^1_1,\dots,\mathbf{x}^1_k),\dots,(\mathbf{x}^s_1,\dots,\mathbf{x}^s_k)) \in (D^k(f))^s \mid \pi^k_{k-1}(\mathbf{x}^1_1,\dots,\mathbf{x}^1_k) = \cdots = \pi^k_{k-1}(\mathbf{x}^s_1,\dots,\mathbf{x}^s_k), (\mathbf{x}^i_1,\dots,\mathbf{x}^i_k) \neq (\mathbf{x}^j_1,\dots,\mathbf{x}^j_k) \text{ for } i \neq j\big\}.$$

Equivalently,

$$\begin{aligned} D^s(\pi^k_{k-1}) &= \text{closure}\big\{((\mathbf{x}^1_1,\dots,\mathbf{x}^1_k),\dots,(\mathbf{x}^s_1,\dots,\mathbf{x}^s_k)) \in (D^k(f))^s \mid \\ &\qquad (\mathbf{x}^1_1,\dots,\mathbf{x}^1_{k-1}) = \cdots = (\mathbf{x}^s_1,\dots,\mathbf{x}^s_{k-1}), \mathbf{x}^i_k \neq \mathbf{x}^j_k \text{ for } i \neq j\big\} \\ &\cong \text{closure}\big\{((\mathbf{x}^1_1,\dots,\mathbf{x}^1_{k-1},\mathbf{x}^1_k),\dots,(\mathbf{x}^1_1,\dots,\mathbf{x}^1_{k-1},\mathbf{x}^s_k)) \in (D^k(f))^s \mid \\ &\qquad \mathbf{x}^i_k \neq \mathbf{x}^j_k \text{ for } i \neq j\big\} \\ &\cong \text{closure}\big\{((\mathbf{x}^1_1,\dots,\mathbf{x}^1_{k-1},\mathbf{x}^1_k),\dots,(\mathbf{x}^1_1,\dots,\mathbf{x}^1_{k-1},\mathbf{x}^s_k)) \in (\mathbb{C}^n)^{ks} \mid \\ &\qquad f(\mathbf{x}^1_1) = \cdots = f(\mathbf{x}^1_k) = \cdots = f(\mathbf{x}^s_k), \mathbf{x}^1_i \neq \mathbf{x}^1_j, \mathbf{x}^i_k \neq \mathbf{x}^j_k \\ &\qquad \text{for } i \neq j\big\} \end{aligned} \tag{2.17}$$

Let us define

$$\begin{aligned} \phi^n_{s,k} \colon (\mathbb{C}^n)^{ks} &\to (\mathbb{C}^n)^{k+s-1} \\ ((\mathbf{x}^1_1,\dots,\mathbf{x}^1_k),\dots,(\mathbf{x}^s_1,\dots,\mathbf{x}^s_k)) &\mapsto (\mathbf{x}^1_1,\dots,\mathbf{x}^1_{k-1},\mathbf{x}^1_k,\dots,\mathbf{x}^s_k)) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \psi^n_{k,s} \colon (\mathbb{C}^n)^{k+s-1} &\to (\mathbb{C}^n)^{ks} \\ (\mathbf{y}_1,\dots,\mathbf{y}_{k+s-1}) &\mapsto ((\mathbf{y}_1,\dots,\mathbf{y}_{k-1},\mathbf{y}_k),\dots,(\mathbf{y}_1,\dots,\mathbf{y}_{k-1},\mathbf{y}_{k+s-1})) \end{aligned} \tag{2.19}$$

where $\mathbf{y}_i = (y_{i1},\dots,y_{in}) \in \mathbb{C}^n$. We see that $\phi^n_{s,k}(D^s(\pi^k_{k-1})) \subseteq D^{k+s-1}(f)$ and $\psi^n_{k,s}(D^{k+s-1}(f)) \subseteq D^s(\pi^k_{k-1})$. In fact, $\phi^n_{s,k}|_{D^s(\pi^k_{k-1})}$ and $\psi^n_{k,s}|_{D^{k+s-1}(f)}$ are bijections, and $\phi^n_{s,k}|_{D^s(\pi^k_{k-1})} \circ \psi^n_{k,s}|_{D^{k+s-1}(f)} = 1$, $\psi^n_{k,s}|_{D^{k+s-1}(f)} \circ \phi_{s,k}|_{D^s(\pi^k_{k-1})} = 1$. This concludes the proof. □

Remark 2.4.2. The result of Proposition 2.4.1 also applies to the projections $\pi^k_{k-1}(f)$ for a finite $f \in \mathcal{E}^0_{n,p}$. To be more precise,

$$D^s(\pi^r_{r-1}(\pi^k_{k-1})) \cong D^{r+s-1}(\pi^k_{k-1})$$

for $r, s, k \geq 1$.

Notice that $\phi^n_{s,k}$ and $\psi^n_{k,s}$, defined in (2.18) and (2.19), do not necessarily induce analytic isomorphisms. One reason is that the multiple point spaces may not be reduced. We will show that they do induce analytic isomorphisms in the case of corank 1 map-germs which are stable or finitely $\mathcal{A}$-determined.

If $f \in \mathcal{E}^0_{n,p}$ is stable and of corank 1 then it is $\mathcal{A}$-equivalent to

$$\begin{aligned} \varphi_{c,r+1} \colon (\mathbb{C}^n,0) &\to (\mathbb{C}^p,0) \\ (u_1,\dots,u_{n-1},z) &\mapsto (u_1,\dots,u_{n-1},\lambda_1,\dots,\lambda_c) \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= z^{r+1} + u_{r-1}z^{r-1} + \cdots + u_1 z, \\ \lambda_2 &= u_{2r-1}z^r + u_{2r-2}z^{r-1} + \cdots + u_r z, \\ &\vdots \\ \lambda_c &= u_{cr-1}z^r + u_{cr-2}z^{r-1} + \cdots + u_{cr-r} z, \end{aligned}$$

and $n \geq cr$, $p = c + n - 1$ for some particular choice of $c > 1$ and $r = q(f) - 1$ ([Gor95]). In particular, for $p = n+1$, f is $\mathcal{A}$-equivalent to a trivial unfolding of the map-germ

$$\begin{aligned} \varphi_{r+1} \colon (\mathbb{C}^{2r},0) &\to (\mathbb{C}^{2r+1},0) \\ (u_1,\dots,u_{r-1},v_1,\dots,v_r,z) &\mapsto (\mathbf{u},\mathbf{v},z^{r+1} + \sum_{i=1}^{r-1} u_i z^i, \sum_{i=1}^{r} v_i z^i). \end{aligned} \tag{2.20}$$

(see also [Mor65]).

In [Gor95], Goryunov presented the following proposition in a greater generality, namely for corank 1 stable map-germs $f \in \mathcal{E}^0_{n,p}$. Here we will spell out some details which were omitted from the proof in [Gor95].

Proposition 2.4.3. *Let $f \in \mathcal{E}^0_{n,n+1}$ be a stable map-germ of corank 1. Then*

$$\pi^k_{k-1} \sim_{\mathcal{A}} \varphi_{r-k+2} \times 1 \tag{2.21}$$

where 1 *denotes the identity map on* $\mathbb{C}^{k-1}$.

Proof. Let us assume that $q(f) = r+1$ so that f is $\mathcal{A}$-equivalent to a trivial unfolding of the map-germ φ_{r+1} defined by (2.20). We have

$$\mathcal{I}_k(\varphi_{r+1})\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^k} \cong \mathcal{I}_k(f)\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^k}$$

in which the isomorphism is induced by the same maps giving the $\mathcal{A}$-equivalence. Therefore, we can assume that f is $\mathcal{A}$-equivalent to φ_{r+1} for simplicity. We will prove the statement by induction on k.

The initial step: $k = 2$. That is $\pi_1^2\colon D^2(\varphi_{r+1}) \to D^1(\varphi_{r+1})$ is $\mathcal{A}$-equivalent to $\varphi_1 \times 1\colon (\mathbb{C}^{2r-2} \times \mathbb{C}, 0) \to (\mathbb{C}^{2r-1} \times \mathbb{C}, 0)$. Let us show that.

We find that $\mathcal{I}_2(\varphi_{r+1})$ is generated by the polynomials

$$z_1^r + z_1^{r-1}z_2 + \cdots + z_2^r + u_{r-1}(z_1^{r-2} + z_1^{r-3}z_2 + \cdots + z_2^{r-2}) + \cdots + u_2(z_1 + z_2) + u_1$$

and

$$v_r(z_1^{r-1} + z_1^{r-2}z_2 + \cdots + z_2^{r-1}) + \cdots + v_2(z_1 + z_2) + v_1.$$

Let us put

$$\begin{aligned} h_1^1 :=& z_1^r + z_1^{r-1}z_2 + \cdots + z_2^r + u_{r-1}(z_1^{r-2} + z_1^{r-3}z_2 + \cdots + z_2^{r-2}) + \cdots + \\ &+ u_2(z_1 + z_2), \\ h_2^1 :=& v_r(z_1^{r-1} + z_1^{r-2}z_2 + \cdots + z_2^{r-1}) + \cdots + v_2(z_1 + z_2). \end{aligned}$$

Notice that h_1 depends on the variables $u_2, \ldots, u_{r-1}, z_1, z_2$ and h_2 depends on $v_2, \ldots, v_r, z_1$ and z_2. Therefore the projection

$$\begin{array}{rcl} \eta\colon D^2(\varphi_{r+1}) & \to & \mathbb{C}^{2r-1} \\ (u_1, \ldots, u_{r-1}, v_1, \ldots, v_r, z_1, z_2) & \mapsto & (u_2, \ldots, u_{r-1}, v_2, \ldots, v_r, z_1, z_2) \end{array}$$

composed with

$$\begin{array}{rcl} \xi\colon \mathbb{C}^{2r-1} & \to & D^2(\varphi_{r+1}) \\ (u_2, \ldots, u_{r-1}, v_2, \ldots, v_r, z_1, z_2) & \mapsto & (-h_1, u_2, \ldots, u_{r-1}, -h_2, v_2, \ldots, v_r, z_1, z_2) \end{array}$$

is the identity on $D^2(\varphi_{r+1})$ whence $D^2(\varphi_{r+1}) \cong \mathbb{C}^{2r-1}$. It also follows that π_1^2 is $\mathcal{A}$-equivalent to the map-germ

$$
\begin{array}{rcl}
g\colon (\mathbb{C}^{2r-1},0) & \to & (\mathbb{C}^{2r},0)\\
(u_2,\dots,u_{r-1},v_2,\dots,v_r,z_1,z_2) & \mapsto & (-h_1,u_2,\dots,u_{r-1},-h_2,v_2,\dots,v_r,z_1)
\end{array}
$$

Claim. g is $\mathcal{A}$-stable.
Proof of the claim. We will prove the claim by using Theorem 2.14(i) of [MM89], i.e. by showing that all multiple point spaces of g are smooth if non-empty.

Let us introduce the notation

$$
P_m(z_1,\dots,z_l) := \sum_{\substack{a_1+\dots+a_l=m\\ a_i\geq 0}} z_1^{a_1}\cdots z_l^{a_l}.
$$

For example, $h_1 = P_r(z_1,z_2) + u_{r-1}P_{r-2}(z_1,z_2) + \dots + u_2P_1(z_1,z_2)$ and
$h_2 = v_rP_{r-1}(z_1,z_2) + v_{r-1}P_{r-2}(z_1,z_2) + \dots + v_2P_1(z_1,z_2)$.

Recall the defining equations for the ideal $\mathcal{I}_k$ (see (2.1)). We have

$$
\begin{array}{rcl}
R_1^1 & = & \dfrac{1}{z_3-z_2}\big(h_1(u_2,\dots,u_{r-1},z_1,z_3) - h_1(u_2,\dots,u_{r-1},z_1,z_2)\big)\\
& = & P_{r-1}(z_1,z_2,z_3) + u_{r-1}P_{r-3}(z_1,z_2,z_3) + \dots + u_3P_1(z_1,z_2,z_3) + u_2\\
& = & R_1^1(u_2,\dots,u_{r-1},z_1,z_2,z_3)
\end{array}
$$

and

$$
\begin{array}{rcl}
R_2^1 & = & \dfrac{1}{z_3-z_2}\big(h_2(v_2,\dots,v_r,z_1,z_3) - h_2(v_2,\dots,v_r,z_1,z_2)\big)\\
& = & v_rP_{r-2}(z_1,z_2,z_3) + v_{r-1}P_{r-2}(z_1,z_2,z_3) + \dots + v_3P_1(z_1,z_2,z_3) + v_2\\
& = & R_2^1(v_2,\dots,v_r,z_1,z_2,z_3).
\end{array}
$$

Assume that

$$
\begin{aligned}
R_i^1 &= P_{r-i}(z_1,\dots,z_{i+2}) + u_{r-1}P_{r-i-2}(z_1,\dots,z_{i+2}) + \dots + u_{i+2}P_1(z_1,\dots,z_{i+2}) +\\
&\quad + u_{i+1}\\
&= R_i^1(u_{i+1},\dots,u_{r-1},z_1,\dots,z_{i+2})
\end{aligned}
$$

and

$$\begin{aligned}
R_i^2 &= v_r P_{r-i-1}(z_1, \ldots, z_{i+2}) + v_{r-1} P_{r-i-2}(z_1, \ldots, z_{i+2}) + \cdots + \\
&+ v_{i+2} P_1(z_1, \ldots, z_{i+2}) + v_{i+1} \\
&= R_i^2(v_{i+1}, \ldots, v_r, z_1, \ldots, z_{i+2}).
\end{aligned}$$

Then we find

$$\begin{aligned}
R_{i+1}^1 &= \frac{R_i^1(u_{i+1}, \ldots, u_{r-1}, z_1, \ldots, z_{i+1}, z_{i+3}) - R_i^1(u_{i+1}, \ldots, u_{r-1}, z_1, \ldots, z_{i+1}, z_{i+2})}{z_{i+3} - z_{i+2}} \\
&= P_{r-i-1}(z_1, \ldots, z_{i+3}) + u_{r-1} P_{r-i-3}(z_1, \ldots, z_{i+3}) + \cdots + \\
&+ u_{i+3} P_1(z_1, \ldots, z_{i+3}) + u_{i+2} \\
&= R_{i+1}^1(u_{i+2}, \ldots, u_{r-1}, z_1, \ldots, z_{i+3}), \\
R_{i+1}^2 &= \frac{R_i^2(v_{i+1}, \ldots, v_r, z_1, \ldots, z_{i+1}, z_{i+3}) - R_i^2(v_{i+1}, \ldots, v_r, z_1, \ldots, z_{i+1}, z_{i+2})}{z_{i+3} - z_{i+2}} \\
&= v_r P_{r-i-2}(z_1, \ldots, z_{i+3}) + \cdots + v_{i+3} P_1(z_1, \ldots, z_{i+3}) + v_{i+2} \\
&= R_{i+1}^2(v_{i+2}, \ldots, v_r, z_1, \ldots, z_{i+3}).
\end{aligned}$$

These equations holds for all $i \leq r-1$. Therefore, $\mathcal{I}_k(g)$ defines a smooth analytic space for all $k \leq r$ and an empty set for $k > r$. This concludes the proof of the claim.

As g is stable and of corank 1 with $q(g) = r$, and $q(\varphi_r) = r$, g is $\mathcal{A}$-equivalent to a trivial unfolding of $\varphi_r \colon \mathbb{C}^{2r-2} \to \mathbb{C}^{2r-1}$, i.e. to the map-germ

$$\varphi_r \times 1 \colon (\mathbb{C}^{2r-2} \times \mathbb{C}, 0) \to (\mathbb{C}^{2r-1} \times \mathbb{C}, 0)$$

(see [Mat69b, Theorem A]). So, we have a commutative diagram

$$\begin{array}{ccc}
D^2(\varphi_{r+1}) & \xrightarrow{\cong} & D^1(\varphi_r \times 1) \\
\downarrow \pi_1^2 & & \downarrow \varphi_r \times 1 \\
D^1(\varphi_{r+1}) & \xrightarrow{\cong} & D^0(\varphi_r \times 1)
\end{array} \tag{2.22}$$

Step 2. Assume that $\pi_{k-2}^{k-1} \colon D^{k-1}(\varphi_{r+1}) \to D^{k-2}(\varphi_{r+1})$ is $\mathcal{A}$-equivalent to

$\varphi_{r-k+3}\times 1\colon (\mathbb{C}^{2r-2k+4}\times\mathbb{C}^{k-2},0)\to(\mathbb{C}^{2r-2k+5}\times\mathbb{C}^{k-2},0)$. So

$$\begin{array}{ccc} D^2(\pi^{k-1}_{k-2}) & \xrightarrow{\cong} & D^2(\varphi_{r-k+3}\times 1) \\ \downarrow{\scriptstyle \hat{\pi}^2_1} & & \downarrow \\ D^1(\pi^{k-1}_{k-2}) & \xrightarrow{\cong} & D^1(\varphi_{r-k+3}\times 1) \end{array} \tag{2.23}$$

where $\hat{\pi}^2_1$ is a restriction of the projection $D^{k-1}(\varphi_{r+1})\times D^{k-1}(\varphi_{r+1})\to D^{k-1}(\varphi_{r+1})$. If we repeat the initial step for $\varphi_{r-k+3}\times 1$ we obtain

$$\begin{array}{ccc} D^2(\varphi_{r-k+3}\times 1) & \longrightarrow & D^1(\varphi_{r-k+2}\times 1) \\ \downarrow{\scriptstyle \hat{\pi}^2_1} & & \downarrow{\scriptstyle \varphi_{r-k+2}\times 1} \\ D^1(\varphi_{r-k+3}\times 1) & \longrightarrow & D^0(\varphi_{r-k+2}\times 1) \end{array} \tag{2.24}$$

Since π^{k-1}_{k-2} is stable and of corank 1, all of its multiple point spaces are smooth and reduced if non-empty, by Theorem 2.14(i) of [MM89]. In this case $\phi^{2r-k+2}_{2,k-1}|_{D^2(\pi^{k-1}_{k-2})}$, defined by (2.18), is an analytic isomorphism. So we get

$$\begin{array}{ccccccc} D^k(\varphi_{r+1}) & \xrightarrow{\cong} & D^2(\pi^{k-1}_{k-2}) & \xrightarrow{\cong} & D^2(\varphi_{r-k+3}\times 1) & \longrightarrow & D^1(\varphi_{r-k+2}\times 1) \\ \downarrow{\scriptstyle \pi^k_{k-1}} & & \downarrow{\scriptstyle \hat{\pi}^2_1} & & \downarrow & & \downarrow{\scriptstyle \varphi_{r-k+2}\times 1} \\ D^{k-1}(\varphi_{r+1}) & \xrightarrow{\cong} & D^1(\pi^{k-1}_{k-2}) & \xrightarrow{\cong} & D^1(\varphi_{r-k+3}\times 1) & \longrightarrow & D^0(\varphi_{r-k+2}\times 1) \end{array} \tag{2.25}$$

Therefore, $\pi^k_{k-1}\sim_{\mathcal{A}}\varphi_{r-k+2}\times 1$. □

Corollary 2.4.4. *Let $f\in\mathcal{E}^0_{n,n+1}$ be a stable map-germ of corank 1. Let $\psi^n_{k,s}$ be given by (2.19). Then $\psi^n_{k,s}$ induces $\mathcal{O}_{D^s(\pi^k_{k-1})}\otimes_{\mathcal{O}_{(\mathbb{C}^n)^{ks},0}}\mathcal{O}_{(\mathbb{C}^n)^{k+s-1},0}\cong\mathcal{O}_{D^{k+s-1}(f)}$.*

Proof. By Proposition 2.4.3, π^k_{k-1} is stable and of corank 1. So the multiple point spaces of π^k_{k-1} are smooth hence reduced spaces of the right dimensions by Theorem 2.14 of [MM89]. Therefore $\psi^n_{k,s|D^s(\pi^k_{k-1})}$ also induces an isomorphism

$$(\psi^n_{k,s|D^s(\pi^k_{k-1})})^*\colon\mathcal{O}_{D^s(\pi^k_{k-1})}\to\mathcal{O}_{D^{k+s-1}(f)}.$$

□

Lemma 2.4.5. *Let $f\in\mathcal{E}^0_{n,n+1}$ be a finite map-germ of corank 1. Let $F\in\mathcal{E}^0_{n+d,n+d+1}$ be a parametrised unfolding of f. Then $\pi^k_{k-1}(F)$ is an unfolding of $\pi^k_{k-1}(f)$.*

Proof. By definition, we have

$$\mathcal{O}_{D^k(f)} \cong \mathcal{O}_{D^k(F)} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^d\times\mathbb{C}^k,0}} \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^d\times\mathbb{C}^k,0}/\mathfrak{m}_{\mathbb{C}^d,0}.$$

So the result follows from Definition 1.3.6. □

We can extend the statement of Corollary 2.4.4 to finitely $\mathcal{A}$-determined map-germs of corank 1.

Proposition 2.4.6. *Let $f \in \mathcal{E}^0_{n,n+1}$ be a finitely $\mathcal{A}$-determined map-germ of corank 1. Then $\mathcal{O}_{D^s(\pi^k_{k-1})} \cong \mathcal{O}_{D^{k+s-1}(f)}$.*

Proof. Let us assume that f has the form $f(\mathbf{x}, y) = (\mathbf{x}, f_1(\mathbf{x}, y), f_2(\mathbf{x}, y))$ for $\mathbf{x} \in \mathbb{C}^{n-1}$ and $y \in \mathbb{C}$. Let $F\colon (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}^d, 0)$ be a stable unfolding of f given by

$$F(\mathbf{x}, y, \mathbf{u}) = (\mathbf{x}, F_1(\mathbf{x}, y, \mathbf{u}), F_2(\mathbf{x}, y, \mathbf{u}), \mathbf{u})$$

with the property $F_j(\mathbf{x}, y, 0) = f_j(\mathbf{x}, y)$ for $j = 1, 2$.

We have $\mathcal{O}_{D^s(\pi^k_{k-1})} \cong \mathcal{O}_{D^{k+s-1}(f)}$ by Corollary 2.4.4. Now,

$$\mathcal{O}_{D^{k+s-1}(f)} \cong \mathcal{O}_{D^{k+s-1}(F)} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^d\times\mathbb{C}^{k+s-1},0}} \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^d\times\mathbb{C}^{k+s-1},0}/\mathfrak{m}_{\mathbb{C}^d,0}.$$

By Lemma 2.4.5, $\pi^k_{k-1}(F)$ is a stable unfolding of $\pi^k_{k-1}(f)$. In fact, $\pi^k_{k-1}(F)$ is a stable corank 1 map-germ from $D^k(F) \cong \mathbb{C}^{n-k+1}$ to $D^{k-1}(F) \cong \mathbb{C}^{n-k}$ ([MM89, Theorem 2.14]). We have

$$\mathcal{O}_{D^s(\pi^k_{k-1}(f))} \cong \mathcal{O}_{D^s(\pi^k_{k-1}(F))} \otimes_{\mathcal{O}_{\mathbb{C}^{n-k}\times\mathbb{C}^d\times\mathbb{C}^s,0}} \mathcal{O}_{\mathbb{C}^{n-k}\times\mathbb{C}^d\times\mathbb{C}^s,0}/\mathfrak{m}_{\mathbb{C}^d,0}.$$

By Theorem 2.14 of [MM89], $D^{k+s-1}(F) \cong \mathbb{C}^{n+d-k-s+2}$ and $D^{k+s-1}(f)$ is a complete intersection singularity of dimension $n-k-s+2$. As $D^{k+s-1}(f)$ is the zero fibre of $\rho\colon D^{k+s-1}(F) \to (\mathbb{C}^d, 0)$, ρ is flat at 0 ([Eis95, Theorem 18.16]). On the other hand, $D^s(\pi^k_{k-1}(f))$ is the zero fibre of $\sigma\colon D^s(\pi^k_{k-1}(F)) \to (\mathbb{C}^d, 0)$. So, by a similar argument, we also see that σ is flat at 0. It follows that

$$\begin{aligned}\mathcal{O}_{D^{k+s-1}(f)} &\cong \mathcal{O}_{D^s(\pi^k_{k-1}(F))} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^d\times\mathbb{C}^{k+s-1},0}} \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^d\times\mathbb{C}^{k+s-1},0}/\mathfrak{m}_{\mathbb{C}^d,0}\\ &\cong \mathcal{O}_{D^s(\pi^k_{k-1}(f))}.\end{aligned}$$

□

2.5 An alternative interpretation of the double point in the source

In this section, we consider finite analytic map-germs $f \in \mathcal{E}^0_{n,n+1}$ of any corank and show that a resolution (over $\mathcal{O}_{\mathbb{C}^n,0}$) of the kernel of the multiplication (or *diagonal*) map

$$\begin{aligned} \mu\colon \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} &\rightarrow \mathcal{O}_{\mathbb{C}^n,0} \\ a \otimes b &\mapsto ab \end{aligned} \tag{2.26}$$

can be deduced from the presentation of $\mathcal{O}_{\mathbb{C}^n,0}$ as an $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module.

We consider the set

$$ID^k(f) := \Big\{(x_1, \ldots, x_k) \in (\mathbb{C}^n)^k \mid f(x_1) = \cdots = f(x_k)\Big\} \tag{2.27}$$

which was referred to as *'idiot's definition of multiple point spaces* first by Houston in [Hou99]. It is easy to see from the definitions that $D^k(f)$ and $ID^k(f)$ agree outside the diagonal of $(\mathbb{C}^n)^k$ and that $D^k(f) \subset ID^k(f)$. By definition, $ID^2(f)$ is equal to the fibre product $(\mathbb{C}^n \times_{\mathbb{C}^{n+1}} \mathbb{C}^n, 0)$ defined by the commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \xrightarrow{f} & (\mathbb{C}^{n+1}, 0) \\ {\scriptstyle pr_1}\uparrow & & \uparrow{\scriptstyle f} \\ (\mathbb{C}^n \times_{\mathbb{C}^{n+1}} \mathbb{C}^n, 0) & \xrightarrow{pr_2} & (\mathbb{C}^n, 0) \end{array} \tag{2.28}$$

where pr_1 (resp. pr_2) is the projection to the first (resp. the second) factor. Moreover, the ring of holomorphic functions on $(\mathbb{C}^n \times_{\mathbb{C}^{n+1}} \mathbb{C}^n, 0)$ is isomorphic to $\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0}$ (Corollary 2.3.3). Also, $\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0}$ can be equipped with two $\mathcal{O}_{\mathbb{C}^n,0}$-module structures: The left (resp. right) $\mathcal{O}_{\mathbb{C}^n,0}$-module structure is induced from pr_1 (resp. pr_2). In other words, $\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0}$ is considered as an $\mathcal{O}_{\mathbb{C}^n,0}$-module via multiplication on the left

$$c \cdot a \otimes b = ((pr_1)^* c) a \otimes b = c \otimes 1 \cdot a \otimes b = ac \otimes b$$

and an $\mathcal{O}_{\mathbb{C}^n,0}$-module via multiplication on the right

$$c \cdot a \otimes b = ((pr_2)^* c) a \otimes b = 1 \otimes c \cdot a \otimes b = a \otimes bc$$

for all $a, b, c \in \mathcal{O}_{\mathbb{C}^n,0}$ (cf. Remark 1.1.10). Here we will assume that it has $\mathcal{O}_{\mathbb{C}^n,0}$-module structure via multiplication on the right.

We want to define the generators of $\ker(\mu)$ over $\mathcal{O}_{\mathbb{C}^n,0}$. In order to do this, we need the following property.

Lemma 2.5.1. *Let R be a commutative ring, S be a subring of R, and $\ker(\mu)$ be the kernel of the multiplication map*

$$\begin{array}{rcl} \mu\colon R \otimes_S R & \to & R \\ a \otimes b & \mapsto & ab. \end{array} \tag{2.29}$$

If R is generated by the elements $r_1, \ldots, r_n$ as an S-module then $\ker(\mu)$ is generated by $r_1 \otimes 1 - 1 \otimes r_1, \ldots, r_n \otimes 1 - 1 \otimes r_n$ over R.

Proof. Let us assume $R \otimes_S R$ is an R-module with respect to multiplication on the right. Each element in $R \otimes_S R$ has the form $\sum_{i \in I} a_i \otimes b_i$ where I is finite and $a_i, b_i \in R$ for all $i \in I$. Let $\sum_I a_i \otimes b_i \in \ker(\mu)$, i.e. $\mu(\sum_I a_i \otimes b_i) = \sum_I a_i b_i = 0$. Then, we have

$$\begin{aligned} \sum_i a_i \otimes b_i &= \sum_i b_i \cdot a_i \otimes 1 \\ &= \sum_i b_i \cdot a_i \otimes 1 - 1 \otimes (\sum_i a_i b_i) \\ &= \sum_i b_i \cdot a_i \otimes 1 - \sum_i 1 \otimes a_i b_i \\ &= \sum_i b_i \cdot a_i \otimes 1 - \sum_i b_i \cdot 1 \otimes a_i \\ &= \sum_i b_i \cdot (a_i \otimes 1 - 1 \otimes a_i). \end{aligned} \tag{2.30}$$

As $r_1, \ldots, r_n$ generate R, there exist elements $\alpha_{ij} \in S$, for $i \in I$ and $j = 1, \ldots, n$, such that $a_i = \sum_{j=1}^n \alpha_{ij} r_j$. If we plug this equality in (2.30) we get

$$\begin{aligned} \sum_i b_i \cdot (a_i \otimes 1 - 1 \otimes a_i) &= \sum_i b_i \cdot (\sum_{j=1}^n \alpha_{ij} r_j \otimes 1 - 1 \otimes \sum_{j=1}^n \alpha_{ij} r_j) \\ &= \sum_i b_i \cdot (\sum_{j=1}^n \alpha_{ij} (r_j \otimes 1 - 1 \otimes r_j)) \\ &= \sum_{j=1}^n \sum_i b_i \alpha_{ij} \cdot (r_j \otimes 1 - 1 \otimes r_j). \end{aligned} \tag{2.31}$$

Therefore $\sum_i a_i \otimes b_i = \sum_{j=1}^n c_j \cdot (r_j \otimes 1 - 1 \otimes r_j)$ where $c_j = \sum_i b_i \alpha_{ij} \in R$ for all $j = 1, \dots, n$. $\square$

Let us assume that $Q(f)$ is the vector space spanned by the classes of some $1, g_1, \dots, g_r \in \mathcal{O}_{\mathbb{C}^n,0}$. Then, the kernel of (2.26) is generated by the elements $g_1 \otimes 1 - 1 \otimes g_1, \dots, g_r \otimes 1 - 1 \otimes g_r$ over $\mathcal{O}_{\mathbb{C}^n,0}$ by Lemma 2.5.1. Assume that there exists an exact sequence

$$\mathbf{L}_\bullet : 0 \to \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{G} \mathcal{O}_{\mathbb{C}^n,0} \to 0 \tag{2.32}$$

where $G := \begin{bmatrix} 1 & g_1 & \dots & g_r \end{bmatrix}$ and λ is a symmetric $(r+1) \times (r+1)$ matrix.

Proposition 2.5.2. *Let $f \in \mathcal{E}^0_{n,n+1}$ be a $\mathcal{K}$-finite and generically one-to-one map-germ. Let λ be defined as above. Then the following sequence is exact:*

$$0 \longrightarrow \mathcal{O}^r_{\mathbb{C}^n,0} \xrightarrow{f^*\lambda^1_1} \mathcal{O}^r_{\mathbb{C}^n,0} \xrightarrow{H} \ker(\mu) \longrightarrow 0 \tag{2.33}$$

where λ^1_1 is the matrix obtained from λ by deleting the first row and the first column and H is defined by

$$\hat{e}_i \mapsto g_i \otimes 1 - 1 \otimes g_i \qquad \text{for } i = 1, \dots, r. \tag{2.34}$$

Proof. We tensor $\mathbf{L}_\bullet$ with $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ to get

$f^*\mathbf{L}_\bullet$:

$$\mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} \xrightarrow{\lambda\otimes 1} \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} \xrightarrow{G\otimes 1} \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} \to 0.$$

Since $\mathcal{O}_{\mathbb{C}^{n+1},0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} \cong \mathcal{O}_{\mathbb{C}^n,0}$ and $\mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} \cong \mathcal{O}^{r+1}_{\mathbb{C}^n,0}$, $f^*\mathbf{L}_\bullet$ is isomorphic to the sequence

$$\mathcal{O}^{r+1}_{\mathbb{C}^n,0} \xrightarrow{f^*\lambda} \mathcal{O}^{r+1}_{\mathbb{C}^n,0} \xrightarrow{G'} \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} \to 0 \tag{2.35}$$

where $f^*\lambda$ is the matrix with entries $\lambda_{ij} \circ f$ for $i, j = 0, \dots, r$.

The columns of λ generate the module of relations among $1, g_1, \dots, g_r$ by the exactness of $\mathbf{L}_\bullet$, i.e. there exist relations of the form

$$G \cdot \begin{bmatrix} \lambda_{i0} \circ f \\ \lambda_{i1} \circ f \\ \vdots \\ \lambda_{ir} \circ f \end{bmatrix} = \lambda_{i0} \circ f + \sum_{j=1}^{r} g_j(\lambda_{ij} \circ f) = 0 \qquad \text{for all } i = 0, \ldots, r. \tag{2.36}$$

Recall that λ is a symmetric matrix. So, we get a relation among the entries of each row of $f^*\lambda$ as well, that is,

$$\begin{bmatrix} \lambda_{0i} & \lambda_{1i} & \cdots & \lambda_{ri} \end{bmatrix} \begin{bmatrix} 1 \\ g_1 \\ \vdots \\ g_r \end{bmatrix} = 0 \tag{2.37}$$

for all $i = 0, \ldots, r$. Let us put $G^t = \begin{bmatrix} 1 \\ g_1 \\ \vdots \\ g_r \end{bmatrix}$. Then, $G^t \colon \mathcal{O}_{\mathbb{C}^n,0} \to \mathcal{O}_{\mathbb{C}^n,0}^{r+1}$ is a map into the kernel of $f^*\lambda$.

Claim. The image of G^t is equal to the kernel of $f^*\lambda$.

Proof of the claim. Let us assume that there exists another map $B \colon \mathcal{O}_{\mathbb{C}^n,0} \to \mathcal{O}_{\mathbb{C}^n,0}^{r+1}$ such that $\operatorname{image}(B) \subseteq \ker(f^*\lambda)$. Let B be given by a matrix $\begin{bmatrix} b_0 & b_1 & \cdots & b_r \end{bmatrix}^t$ for some $b_0, b_1, \ldots, b_r \in \mathcal{O}_{\mathbb{C}^n,0}$. Then we have

$$0 = f^*\lambda \cdot \Big(\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_r \end{bmatrix} - b_0 \begin{bmatrix} 1 \\ g_1 \\ \vdots \\ g_r \end{bmatrix} \Big) = f^*\lambda \cdot \begin{bmatrix} 0 \\ b_1 - b_0 g_1 \\ \vdots \\ b_r - b_0 g_r \end{bmatrix}. \tag{2.38}$$

We want to show that $b_i - b_0 g_i$ is equal to zero for all $i = 1, \ldots, r$.

Let as assume the contrary. By Theorem 3.4 of [MP89], the first Fitting ideal $\operatorname{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^n,0})$ of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ is generated by the maximal minors of the matrix λ^1 which is obtained by deleting the first row of λ. Let $\hat{\lambda}_i$ denote the i-th column of λ. Then, we have $f^*\hat{\lambda}_1 = -\sum_{i=1}^{r} g_i f^*\hat{\lambda}_{i+1}$ since $\begin{bmatrix} 1 & g_1 & \cdots & g_r \end{bmatrix}^t$

is in the kernel of $f^*\lambda$. Hence, $f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^n,0})$ is generated by the determinant of the matrix $f^*\lambda_1^1$. But, (2.38) implies that the determinant of $f^*\lambda_1^1$ is equal to zero unless $b_i - b_0 g_i$ is equal to zero for all $i = 1, \dots, r$. This is a contradiction since $f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^n,0})$ defines $D_1^2(f)$ and f is generically one-to-one. Therefore, we must have $b_i - b_0 g_i = 0$ for all $i = 1, \dots, r$. □

It is clear to see that G^t is one-to-one. Thus, we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^n,0} \xrightarrow{G^t} \mathcal{O}_{\mathbb{C}^n,0}^{r+1} \xrightarrow{f^*\lambda} \mathcal{O}_{\mathbb{C}^n,0}^{r+1} \xrightarrow{G'} \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow 0\,. \tag{2.39}$$

Now we construct the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{O}_{\mathbb{C}^n,0} & \xrightarrow{G^t} & \mathcal{O}_{\mathbb{C}^n,0}^{r+1} & \xrightarrow{f^*\lambda} & \mathcal{O}_{\mathbb{C}^n,0}^{r+1} & \xrightarrow{G'} & \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} & \longrightarrow 0 \\
 & \| & & \downarrow A_1 & & \downarrow A_0 & & \| & \\
\mathbf{K}_\bullet : 0 \longrightarrow & \mathcal{O}_{\mathbb{C}^n,0} & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{C}^n,0}^{r+1} & \xrightarrow{\alpha} & \mathcal{O}_{\mathbb{C}^n,0}^{r+1} & \xrightarrow{G''} & \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} & \longrightarrow 0
\end{array} \tag{2.40}$$

by the lifting maps

$$A_0 = \left[\begin{array}{c|ccc} 1 & g_1 & \dots & g_r \\ \hline 0 & & & \\ \vdots & & \mathrm{Id}_{r\times r} & \\ 0 & & & \end{array}\right] \text{ and } A_1 = \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline -g_1 & & & \\ \vdots & & \mathrm{Id}_{r\times r} & \\ -g_r & & & \end{array}\right],$$

and the horizontal maps which are given by

$$\alpha = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & f^*\lambda_1^1 & \\ 0 & & & \end{array}\right], \beta = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } G'' = e_i \mapsto \begin{cases} 1 \otimes 1 & \text{if } i = 0, \\ g_i \otimes 1 - 1 \otimes g_i & \text{if } i = 1, \dots, r \end{cases}$$

where $\mathrm{Id}_{r\times r}$ denotes the $r \times r$ identity matrix. Therefore we can minimalise $\mathbf{K}_\bullet$ to the exact sequence

$$\mathbf{K}'_\bullet : 0 \to \mathcal{O}_{\mathbb{C}^n,0}^{r} \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{C}^n,0}^{r+1} \xrightarrow{G''} \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} \to 0 \tag{2.41}$$

where α_1 is the matrix obtained from α by deleting the first column.

It is clear to see from $\mathbf{K}'_\bullet$ that $\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0}$ splits into the sum $\mathcal{O}_{\mathbb{C}^n,0} \oplus \ker(\mu)$ and $f^*\lambda_1^1$ gives a presentation of $\ker(\mu)$ (cf. Proposition 2.5.1). So we have got an exact sequence

$$0 \longrightarrow \mathcal{O}^r_{\mathbb{C}^n,0} \xrightarrow{f^*\lambda_1^1} \mathcal{O}^r_{\mathbb{C}^n,0} \xrightarrow{H} \ker(\mu) \longrightarrow 0 \tag{2.42}$$

where H is induced from G'', i.e. is defined by

$$\hat{e}_i \mapsto g_i \otimes 1 - 1 \otimes g_i \qquad \text{for } i = 1, \ldots, r.$$

This concludes the proof. □

Corollary 2.5.3. *Let $f \in \mathcal{E}^0_{n,n+1}$ be finite and generically one-to-one. Then*

$$\operatorname{Supp}(\ker(\mu)) = D^2_1(f).$$

Proof. We have $\operatorname{Fitt}_0(\ker(\mu))\mathcal{O}_{\mathbb{C}^n,0} = (\det(f^*\lambda_1^1))$ by Proposition 2.5.2. On the other hand, $D^2_1(f) = f^{-1}(M_2(f))$; that is, the variety defined by $f^*\operatorname{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^n,0}) = \operatorname{Fitt}_1(\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0})$. By the exactness of (2.41), that ideal is also generated by $\det(f^*\lambda_1^1)$. So, the result follows from Proposition 1.1.9. □

2.6 Presentation of $\mathcal{O}_{D^{k+1}(f)}$ over $\mathcal{O}_{D^k(f)}$ for corank 1 map-germs

For corank 1 map-germs, there is a stronger relation between $\ker(\mu)$ and $\mathcal{O}_{D^2_1(f)}$. Namely, $\ker(\mu) \cong \mathcal{O}_{D^2_1(f)}$. This was proven by Kleiman, Lipman and Ulrich in [KLU92]. Here, we will recall their works and use that isomorphism to determine a presentation of $\mathcal{O}_{D^{k+1}(f)}$ over $\mathcal{O}_{D^k(f)}$.

We recall a construction of double point spaces from algebro-geometric point of view. This terminology was first introduced by Ronga [Ron73], and then used by Laksov [Lak78], Fulton and Laksov [FL77], Kleiman [Kle81], and others to study the multiple point formulas and higher order multiple point spaces.

Definition 2.6.1 ([Lak78], [FL77])**.** Let Y be a closed quasi-projective subscheme of a quasi-projective scheme X. Assume that X is purely n-dimensional and $\dim Y < n$. Let D be a closed subscheme of Y. If D is a Cartier divisor on X, then its ideal

$I(D)$ is invertible, moreover there exists a closed subscheme Z defined by an ideal $I(Z) := I(Y)I(D)^{-1}$ so that $Y = Z \cup D$. In that case Z is called the *residual scheme* to D in Y.

Note that in general, i.e. for an arbitrary closed subscheme D of Y, there is no method to find a closed subscheme Z satisfying $Y = D \cup Z$.

Let $\hat{f}\colon X \to Y$ be a finite morphism from a purely n-dimensional scheme X to a non-singular p-dimensional scheme Y. Let us consider the morphism $\hat{f} \times \hat{f}\colon X \times X \to Y \times Y$ induced by $\hat{f}$. Then the diagonal Δ_Y is a local complete intersection of codimension p in $Y \times Y$, and $(\hat{f} \times \hat{f})^{-1}(\Delta_Y)$ contains Δ_X as a subscheme.

Let $\sigma\colon \widetilde{X \times X} \to X \times X$ be the blow-up of $X \times X$ along Δ_X (see Definition IV-16 and Proposition IV-22 of [EH01] for details). Let $E(X) := \sigma^{-1}(\Delta_X)$ be the exceptional divisor. Then $E(X)$ is a Cartier divisor ([Lak78]). The residual scheme to $E(X)$ in $\sigma^{-1}(\hat{f} \times \hat{f})^{-1}(\Delta_Y)$ is called the double point scheme of f, and denoted by X_2 ([FL77]).

Note that $(\hat{f} \times \hat{f})^{-1}(\Delta_Y)$ is equal to the fibre product $X \times_Y X$ $(= ID^2(\hat{f}))$. So the residual scheme lives in $\sigma^{-1}(X \times_Y X)$.

On the other hand, the diagonal Δ_X in $X \times_Y X$ is locally defined as follows. Let $U := \mathrm{Spec}(A)$ and $V := \mathrm{Spec}(B)$ be open sets such that $\hat{f}(U) \subset V$. Then $\Delta_X \cap U \times_V U$ is defined by the ideal $I(\Delta_X)|U \times_V U$ generated by all elements of the form $a \otimes 1 - 1 \otimes a$ for $a \in A$ ([EH01, p. 93]. Hence, $I(\Delta_X)$ coincides with $\ker(\mu)$ of ours.

Lemma 2.6.2 (Lemma 3.2, [KLU92]). *Let $\hat{f}$ and σ be as above, but $p = n + 1$, so that*

$$\nu := \sigma_|\colon \widetilde{X \times_Y X} := X_2 \to X \times_Y X$$

is the blow-up of $X \times_Y X$ along Δ_X. Let U be an open subset of $X \times_Y X$ on which $I(\Delta_X)$ is generated by a single element of $\mathcal{O}_{\Delta_X}(U)$. Then the restriction $\nu_|\colon \nu^{-1}U \to U$ is a closed embedding, its ideal is $\mathrm{Ann}(I(\Delta_X))|U$, and there is an isomorphism of $\mathcal{O}_U$-modules,

$$\nu_*\mathcal{O}_{X_2}|U \cong I(\Delta_X)|U. \tag{2.43}$$

In particular, $\nu\colon X_2 \to X \times_Y X$ is a closed embedding off the set $\bar{\sum}^2$ of points at which $\hat{f}$ has corank greater than or equal to 2 by Proposition 3.4 of [KLU92]. Hence $\nu_*\mathcal{O}_{X_2} = I(\Delta_X)$ at every point only if $\hat{f}$ is of corank 1. (We can also see that

$I(\Delta_X)$ is generated by one element in case of corank 1 map-germs $f \in \mathcal{E}^0_{n,n+1}$ if we consider the embedding $ID^2(f) \hookrightarrow \mathbb{C}^{n-1} \times \mathbb{C}^2$.)

As a result, we get

$$\mathrm{Spec}(\nu_* \mathcal{O}_{X_2}) \cong \mathrm{Spec}(I(\Delta_X)) = \mathrm{Supp}(I(\Delta_X)).$$

Therefore, $\nu(X_2)$ is the complement of Δ_X in $X \times_Y X$ whence $\nu(X_2) = D^2(\hat{f})$ according to our notation. In fact,

$$pr_1 \circ \nu = \pi^2_1$$

for $pr_1 \colon X \times_Y X \to X$ is the projection $(\mathbf{x}_1, \mathbf{x}_2) \mapsto \mathbf{x}_1$.

Proposition 2.6.3 (cf. Proposition 3.4(2), [KLU92])**.** *Let $f \in \mathcal{E}^0_{n,n+1}$ be finite and generically one-to-one map-germ of corank 1. Then $\mathcal{O}_{D^2(f)} \cong \ker(\mu)$.*

Remark 2.6.4. (1) If $\hat{f}$ has corank greater than or equal to 2, we still have

$$D^2_1(\hat{f}) = pr_i \circ \nu(X_2)$$

where $pr_i \colon X \times_Y X \to X$ is the projection $(\mathbf{x}_1, \mathbf{x}_2) \mapsto \mathbf{x}_i$, for $i = 1, 2$ ([Lak78, p.85]).

(2) In [Kle81], Kleiman constructed higher multiple point schemes X_r by an iteration based on X_2. His construction was as follows: Let $f_0 \colon X_1 \to X_0$ be $\hat{f} \colon X \to Y$. Assume f_{r-1} is defined. Consider $X_r \times_{X_{r-1}} X_r$ and its diagonal subscheme Δ_{X_r} which is closed since f_{r-1} is seperated. Then X_{r+1} is defined to be the corresponding residual scheme, and $f_r := pr \circ \nu'$ where ν' is defined in a similar way as ν.

Lemma 2.6.5 (Lemma 3.9, [KLU96])**.** *Let $f \in \mathcal{E}^0_{n,n+1}$ be finite and generically one-to-one map-germ of corank 1. Then*

$$\mathrm{Fitt}_k((\pi^2_1)_* \mathcal{O}_{D^2(f)}) = f^* \mathrm{Fitt}_{k+1}(f_* \mathcal{O}_{\mathbb{C}^n,0})$$

for $k \geq 0$.

Proof. We give an alternative proof following our terminology. Assume that a resolution of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ is given by a symmetric $(r+1) \times (r+1)$-matrix λ. As

Fitting ideals commute with the base change,

$$f^*\mathrm{Fitt}_{k+1}(f_*\mathcal{O}_{\mathbb{C}^n,0}) = \mathrm{Fitt}_{k+1}(f^*f_*\mathcal{O}_{\mathbb{C}^n,0}) = \bigwedge^{r-k} f^*\lambda.$$

By (2.41), $\bigwedge^{r-k} f^*\lambda$ is the ideal of $(r-k)\times(r-k)$-minors of

$$\alpha_1 = \left[\begin{array}{ccc} 0 & \dots & 0 \\ \hline & f^*\lambda_1^1 & \end{array}\right]$$

which equals to the $(r-k)\times(r-k)$-minors of $f^*\lambda_1^1$. Since the size of $f^*\lambda_1^1$ is r, the result follows by Proposition 2.5.2 and Corollary 2.6.3. □

Recall that we set $D^1(f) = (\mathbb{C}^n, 0)$. So we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^r_{\mathbb{C}^n,0} & \xrightarrow{f^*\lambda_1^1} & \mathcal{O}^r_{\mathbb{C}^n,0} & \longrightarrow & \mathcal{O}_{D^2(f)} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathcal{O}^r_{D^1(f)} & \xrightarrow{f^*\lambda_1^1} & \mathcal{O}^r_{D^1(f)} & \longrightarrow & \mathcal{O}_{D^2(f)} & \longrightarrow & 0 \end{array} \tag{2.44}$$

The following theorem states a generalisation of this fact to higher multiple point spaces.

Theorem 2.6.6. *Let $f \in \mathcal{E}^0_{n,n+1}$ be finitely $\mathcal{A}$-determined map-germ of corank 1. Let*

$$0 \to \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{G} \mathcal{O}_{\mathbb{C}^n,0} \to 0$$

be a resolution of $\mathcal{O}_{\mathbb{C}^n,0}$ where λ is symmetric and $G = \begin{bmatrix} 1 & y & \cdots & y^r \end{bmatrix}$. Then for any $k = 1,\dots,n$, there is an exact sequence

$$0 \to \mathcal{O}^{r-k+1}_{D^k(f)} \xrightarrow{\phi} \mathcal{O}^{r-k+1}_{D^k(f)} \to \mathcal{O}_{D^{k+1}(f)} \to 0 \tag{2.45}$$

in which ϕ is equal to $(\epsilon^{k+1})^*\lambda_k^k$. Moreover, $(\epsilon^{k+1})^*\lambda_k^k = f^*\lambda_k^k$.

Here, λ_k^k is the matrix obtained from λ by deleting the rows $1,\dots,k$ and the columns $1,\dots,k$, and $\epsilon^{k+1} = f \circ \pi_1^2 \circ \cdots \pi_k^{k+1}$.

Proof. We will prove the theorem by induction on k. For $k = 1$, the statement of the theorem is obtained by combining Proposition 2.5.2 and 2.6.3. For $k > 1$, first, we need to prove the existence of the sequence (2.45).

Let f be of the form $f(\mathbf{x}, y) = (\mathbf{x}, f_n(\mathbf{x}, y), f_{n+1}(\mathbf{x}, y))$ where $\mathbf{x} \in \mathbb{C}^{n-1}$ and $y \in \mathbb{C}$. Let $F\colon (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}^d, 0)$ be a stable unfolding of f given by

$$F(\mathbf{x}, y, u_1, \ldots, u_d) = (\mathbf{x}, F_n(\mathbf{x}, y, u_1, \ldots, u_d), F_{n+1}(\mathbf{x}, y, u_1, \ldots, u_d), u_1, \ldots, u_d)$$

with the conditions $F_n(\mathbf{x}, y, 0, \ldots, 0) = f_n(\mathbf{x}, y)$ and $F_{n+1}(\mathbf{x}, y, 0, \ldots, 0) = f_{n+1}(\mathbf{x}, y)$. By Theorem 2.14 of [MM89], $D^{k+1}(F)$ is a smooth variety of dimension $n + d - k$ for $1 \leq k \leq n$. We have

$$\begin{aligned} q(F) &= \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^d, 0}}{F^* \mathfrak{m}_{\mathbb{C}^{n+1} \times \mathbb{C}^d, 0}} \\ &= \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^d, 0}}{(\mathbf{x}, F_n(\mathbf{x}, y, u_1, \ldots, u_d), F_{n+1}(\mathbf{x}, y, u_1, \ldots, u_d), u_1, \ldots, u_d)\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^d, 0}} \\ &\cong \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n, 0}}{f^* \mathfrak{m}_{\mathbb{C}^{n+1}, 0}} \\ &= q(f). \end{aligned} \tag{2.46}$$

By Proposition 2.1.5 and (2.46), $q(\pi_k^{k+1}(F)) = r - k + 1 = q(\pi_k^{k+1}(f))$. Now $\pi_k^{k+1}(F)$ is finite, its source is a $(n+d-k)$-dimensional smooth space and its target is a smooth space of dimension $(n + d - k + 1)$. So there exists a presentation

$$0 \to \mathcal{O}_{D^k(F)}^{r-k+1} \xrightarrow{\phi_u} \mathcal{O}_{D^k(F)}^{r-k+1} \xrightarrow{\varphi} \mathcal{O}_{D^{k+1}(F)} \to 0 \tag{2.47}$$

where ϕ_u is a symmetric $(r-k+1) \times (r-k+1)$-matrix and $\varphi = \begin{bmatrix} 1 & y_{k+1} & \ldots & y_{k+1}^{r-k+1} \end{bmatrix}$ (cf. (2.4)). Notice that $\mathcal{O}_{D^k(f)}$ is an $\mathcal{O}_{D^k(F)}$-module via $\hat{\imath}\colon D^k(f) \to D^k(F)$ given by $(\mathbf{x}, y_1, \ldots, y_k) \mapsto (\mathbf{x}, 0, y_1, \ldots, y_k)$ (cf. Remark 2.3.7). We tensor (2.47) with $\mathcal{O}_{D^k(f)}$ over $\mathcal{O}_{D^k(F)}$ to get the following exact sequence:

$$\mathcal{O}_{D^k(F)}^{r-k+1} \otimes \mathcal{O}_{D^k(f)} \xrightarrow{\phi_u \otimes 1} \mathcal{O}_{D^k(F)}^{r-k+1} \otimes \mathcal{O}_{D^k(f)} \xrightarrow{\varphi \otimes 1} \mathcal{O}_{D^{k+1}(F)} \otimes_{\mathcal{O}_{D^k(F)}} \mathcal{O}_{D^k(f)} \to 0. \tag{2.48}$$

We have

$$\begin{aligned} \mathcal{O}_{D^{k+1}(f)} = \frac{\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^k, 0}}{\mathcal{I}_{k+1}(f)} &= \frac{\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^d \times \mathbb{C}^{k+1}, 0}}{\mathcal{I}_{k+1}(f) + \mathfrak{m}_{\mathbb{C}^d, 0}} \\ &\cong \frac{\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^d \times \mathbb{C}^{k+1}, 0}}{\mathcal{I}_{k+1}(F) + \mathfrak{m}_{\mathbb{C}^d, 0}} \end{aligned}$$

$$\begin{aligned}
&\cong \frac{\mathcal{O}_{D^{k+1}(F)}}{\bar{\mathfrak{m}}_{\mathbb{C}^d,0}} \otimes_{\mathcal{O}_{D^k(F)}} \mathcal{O}_{D^k(F)} \\
&\cong \mathcal{O}_{D^{k+1}(F)} \otimes_{\mathcal{O}_{D^k(F)}} \frac{\mathcal{O}_{D^k(F)}}{\bar{\mathfrak{m}}_{\mathbb{C}^d,0}} \\
&\cong \mathcal{O}_{D^{k+1}(F)} \otimes_{\mathcal{O}_{D^k(F)}} \mathcal{O}_{D^k(f)}
\end{aligned}$$

where $\bar{\mathfrak{m}}_{\mathbb{C}^d,0}$ is the ideal generated by the classes of the generators of $\mathfrak{m}_{\mathbb{C}^d,0}$ modulo the ideal $\mathcal{I}_k(F)$. Hence (2.48) is isomorphic to the sequence

$$\mathcal{O}^{r-k+1}_{D^k(f)} \xrightarrow{\phi} \mathcal{O}^{r-k+1}_{D^k(f)} \xrightarrow{\varphi} \mathcal{O}_{D^{k+1}(f)} \to 0 \tag{2.49}$$

in which ϕ is the matrix ϕ_u modulo the ideal $\mathfrak{m}_{\mathbb{C}^d,0}$.

As $\mathcal{O}_{D^{k+1}(F)}$ (and resp. $\mathcal{O}_{D^{k+1}(f)}$) is a Cohen-Macaulay module of dimension $n+d-k$ (resp. $n-k$), by Theorem 17.4 of [Mat89], we see that height$(\bar{\mathfrak{m}}_{\mathbb{C}^d,0}) = d$ and that $\bar{u}_1, \ldots, \bar{u}_d$ is an $\mathcal{O}_{D^{k+1}(F)}$-regular sequence as well. Therefore, by Lemma 1.3.5 of [BH93], we find

$$\begin{aligned}
\text{proj.dim}_{\mathcal{O}_{D^k(F)}} \mathcal{O}_{D^{k+1}(F)} &= \text{proj.dim}_{\mathcal{O}_{D^k(F)}/\mathfrak{m}_{\mathbb{C}^d,0}} \mathcal{O}_{D^{k+1}(F)}/\mathfrak{m}_{\mathbb{C}^d,0} \\
&= \text{proj.dim}_{\mathcal{O}_{D^k(f)}} \mathcal{O}_{D^{k+1}(f)} = 1.
\end{aligned} \tag{2.50}$$

W conclude that the sequence

$$0 \to \mathcal{O}^{r-k+1}_{D^k(f)} \xrightarrow{\phi} \mathcal{O}^{r-k+1}_{D^k(f)} \xrightarrow{\varphi} \mathcal{O}_{D^{k+1}(f)} \to 0 \tag{2.51}$$

is exact. Notice that ϕ is symmetric since ϕ_u is.

Now, we will prove that ϕ in (2.51) is equal to $(\epsilon^{k+1})^*\lambda^k_k$ and that $(\epsilon^{k+1})^*\lambda^k_k = f^*\lambda^k_k$ by an induction on k. The initial step is already given in 2.44. For the $(k-1)$-th step, we consider the projection $\pi^k_{k-1}(f)$ for which $q(\pi^k_{k-1}(f)) = r-k+2$. We assume that $\mathcal{O}_{D^k(f)}$ is generated by $1, y_k, \ldots, y_k^{r-k+2}$ over $\mathcal{O}_{D^{k-1}(f)}$ and that there is a presentation

$$\mathbf{D}_\bullet : 0 \to \mathcal{O}^{r-k+2}_{D^{k-1}(f)} \xrightarrow{\hat{\phi}} \mathcal{O}^{r-k+2}_{D^{k-1}(f)} \xrightarrow{\hat{\varphi}} \mathcal{O}_{D^k(f)} \to 0 \tag{2.52}$$

where $\hat{\phi} = (\epsilon^k)^*\lambda^{k-1}_{k-1}$ and $\hat{\varphi} = \begin{bmatrix} 1 & y_k & \cdots & y_k^{r-k+2} \end{bmatrix}$. Since π^j_{j-1} is defined by $(\mathbf{x}, y_1, \ldots, y_j) \mapsto (\mathbf{x}, y_1, \ldots, y_{j-1})$, for any $\alpha \in \mathcal{O}_{D^{j-1}(f)}$ we may take $(\pi^j_{j-1})^*\alpha = \alpha$. So, let $(\epsilon^k)^*\lambda^{k-1}_{k-1} = f^*\lambda^{k-1}_{k-1}$.

We have $\mathcal{I}_{k+1}(f) \cong \mathcal{I}_2(\pi^k_{k-1})$ by Proposition 2.4.6. Therefore, we can go through the same steps that we followed to prove Proposition 2.5.2 to conclude the kth step, i.e. the result. To be more precise, first, we tensor the sequence $\mathbf{D}_\bullet$ with $\mathcal{O}_{D^k(f)}$ to get

$$\begin{array}{l}\mathbf{D}_\bullet \otimes \mathcal{O}_{D^k(f)} : \ \mathcal{O}^{r-k+2}_{D^{k-1}(f)} \otimes_{\mathcal{O}_{D^{k-1}(f)}} \mathcal{O}_{D^k(f)} \xrightarrow{\hat{\phi}\otimes 1} \mathcal{O}^{r-k+2}_{D^{k-1}(f)} \otimes_{\mathcal{O}_{D^{k-1}(f)}} \mathcal{O}_{D^k(f)} \\ \qquad\qquad\qquad \Big\downarrow \hat{\varphi}\otimes 1 \\ \qquad\qquad\qquad \mathcal{O}_{D^k(f)} \otimes_{\mathcal{O}_{D^{k-1}(f)}} \mathcal{O}_{D^k(f)} \longrightarrow 0\end{array}$$

which is equivalent to

$$(\pi^k_{k-1})^*\mathbf{D}_\bullet : \mathcal{O}^{r-k+2}_{D^k(f)} \xrightarrow{(\pi^k_{k-1})^*\hat{\phi}} \mathcal{O}^{r-k+2}_{D^k(f)} \xrightarrow{\hat{\varphi}'} \mathcal{O}_{D^k(f)} \otimes \mathcal{O}_{D^k(f)} \longrightarrow 0$$

where

$$\hat{\varphi}' \colon e_i \mapsto \begin{cases} 1 \otimes 1 & \text{if } i = 0, \\ y_k^i \otimes 1 & \text{if } i = 1, \dots, r-k+2. \end{cases}$$

Again, we take $(\pi^k_{k-1})^*\hat{\phi} = \hat{\phi}$.

As we did before, we can complete $\mathbf{D}_\bullet \otimes \mathcal{O}_{D^k(f)}$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_{D^k(f)} \xrightarrow{\hat{\varphi}^t} \mathcal{O}^{r-k+2}_{D^k(f)} \xrightarrow{\hat{\phi}} \mathcal{O}^{r-k+2}_{D^k(f)} \xrightarrow{\hat{\varphi}'} \mathcal{O}_{D^k(f)} \otimes \mathcal{O}_{D^k(f)} \longrightarrow 0.$$

Here π^k_{k-1}, $\hat{\varphi}$, $\hat{\phi}$ and $\hat{\varphi}'$ play the roles of f, G, λ and G', respectively, in Proposition 2.5.2. Then we get a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{O}_{D^k(f)} & \xrightarrow{\hat{\varphi}^t} & \mathcal{O}^{r-k+2}_{D^k(f)} & \xrightarrow{\hat{\phi}} & \mathcal{O}^{r-k+2}_{D^k(f)} & \xrightarrow{\hat{\varphi}'} & \mathcal{O}_{D^k(f)} \otimes \mathcal{O}_{D^k(f)} & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow A_1 & & \Big\downarrow A_0 & & \Big\| & & \\ 0 & \longrightarrow & \mathcal{O}_{D^k(f)} & \xrightarrow{\beta} & \mathcal{O}^{r-k+2}_{D^k(f)} & \xrightarrow{\alpha} & \mathcal{O}^{r-k+2}_{D^k(f)} & \xrightarrow{\hat{\varphi}''} & \mathcal{O}_{D^k(f)} \otimes \mathcal{O}_{D^k(f)} & \longrightarrow & 0 \end{array}$$

where

$$A_0 = \left[\begin{array}{c|ccc} 1 & y_k & \dots & y_k^{r-k+1} \\ \hline 0 & & & \\ \vdots & & \mathrm{Id}_{(r-k+1)\times(r-k+1)} & \\ 0 & & & \end{array}\right], \quad A_1 = \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline -y_k & & & \\ \vdots & & \mathrm{Id}_{r\times r} & \\ -y_k^{r-k+1} & & & \end{array}\right],$$

and

$$\alpha = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & \hat{\phi}_1^1 & \\ 0 & & & \end{array}\right], \qquad \beta = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\hat{\varphi}'' \colon e_i \mapsto \begin{cases} 1 \otimes 1 & \text{if } i = 0, \\ y_k^i \otimes 1 - 1 \otimes y_k^i & \text{if } i = 1, \dots, r-k+2 \end{cases}$$

(here $\mathrm{Id}_{(r-k+1)\times(r-k+1)}$ denotes the identity matrix). As the kernel of the multiplication map

$$\mu_k \colon \mathcal{O}_{D^k(f)} \otimes_{\mathcal{O}_{D^{k-1}(f)}} \mathcal{O}_{D^k(f)} \to \mathcal{O}_{D^k(f)}$$

is generated by $\left\{ y_k^i \otimes 1 - 1 \otimes y_k^i \mid i = 1, \dots, r-k+1 \right\}$, we get a presentation

$$0 \longrightarrow \mathcal{O}_{D^k(f)}^{r-k+1} \xrightarrow{\hat{\phi}_1^1} \mathcal{O}_{D^k(f)}^{r-k+1} \xrightarrow{\hat{H}} \ker(\mu_k) \longrightarrow 0$$

where H is defined by $\hat{e}_i \mapsto y_k^i \otimes 1 - 1 \otimes y_k^i$ for $i = 1, \dots, r-k+2$. By Proposition 2.4.6 and 2.6.3, $\ker(\mu_k) \cong \mathcal{O}_{D^2(\pi_{k-1}^k)} \cong \mathcal{O}_{D^{k+1}(f)}$. Therefore we get

$$0 \to \mathcal{O}_{D^k(f)}^{r+k-1} \xrightarrow{\phi} \mathcal{O}_{D^k(f)}^{r+k-1} \to \mathcal{O}_{D^{k+1}(f)} \to 0$$

where $\phi := \hat{\phi}_1^1 = (f^*\lambda_{k-1}^{k-1})_1^1 = f^*\lambda_k^k$. This concludes the proof of the theorem. □

2.7 Lifting of presentations

In this section, we look at the relation between D^2 and the idiot's double point space ID^2 for map-germs in $\mathcal{E}_{n,n+1}^0$. By definition, D^2 and ID^2 agree outside of the diagonal in $\mathbb{C}^n \times \mathbb{C}^n$, and $D^2 \subset ID^2$. So there is a natural map $\mathcal{O}_{ID^2} \to \mathcal{O}_{D^2}$.

In Section 2.5, we have seen that $\mathcal{O}_{ID^2(f)} \cong \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0}$ splits into

$$\mathcal{O}_{ID^2(f)} \cong \mathcal{O}_{\mathbb{C}^n,0} \oplus \ker(\mu)$$

when considered as an $\mathcal{O}_{\mathbb{C}^n,0}$-module (cf. (2.41)). In the case of corank 1 map-germs we have $\ker(\mu) \cong \mathcal{O}_{D^2(f)}$ by a result of [KLU92] (Proposition 2.6.3). We can form an isomorphism explicitly:

$$\begin{aligned} T\colon \ker(\mu) \to \quad & (y_2-y_1)\mathcal{O}_{D^2(f)} & \to \mathcal{O}_{D^2(f)} \\ y\otimes 1 - 1\otimes y \mapsto \quad & y_2 - y_1 & \\ & h & \mapsto h(y_2-y_1)^{-1} \end{aligned}$$

with an inverse given by

$$\begin{aligned} T^{-1}\colon \mathcal{O}_{D^2(f)} \to \quad & (y_2-y_1)\mathcal{O}_{D^2(f)} & \to \ker(\mu) \\ h \mapsto \quad & (y_2-y_1)h & \\ & y_2 - y_1 & \mapsto y\otimes 1 - 1\otimes y. \end{aligned}$$

So, $\hat{T}\colon \mathcal{O}_{ID^2} \to \mathcal{O}_{D^2}$ is the composition of T with the projection $\mathcal{O}_{\mathbb{C}^n,0} \oplus \ker(\mu) \to \ker(\mu)$.

In corank ≥ 2 case, there is no longer an isomorphism between $\ker(\mu)$ and $\mathcal{O}_{D^2(f)}$. One reason is that $\ker(\mu)$ is Gorenstein (it has a presentation given by a symmetric matrix, cf. [KU97, Theorem 2.3]) but $\mathcal{O}_{D^2(f)}$ is not. Still, we can form a map between the presentations of $\mathcal{O}_{ID^2}$ and $\mathcal{O}_{D^2}$ over $\mathcal{O}_{\mathbb{C}^n,0}$.

Remark 2.7.1. Let $f \in \mathcal{E}^0_{n,n+1}$ be of corank ≥ 2 with $Q(f) = \mathbb{C}\cdot\{1, g_1, \ldots, g_r\}$. Suppose that $q(f) = q(\pi^2_1)$. Then we may take $Q(\pi^2_1) = \mathbb{C}\cdot\{1, g_1(x_2), \ldots, g_r(x_2)\}$ (cf. Proposition 2.1.7). We have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^r_{\mathbb{C}^n,0} & \xrightarrow{f^*\lambda^1} & \mathcal{O}^{r+1}_{\mathbb{C}^n,0} & \xrightarrow{G'} & \mathcal{O}_{\mathbb{C}^n,0}\otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}}\mathcal{O}_{\mathbb{C}^n,0} & \longrightarrow & 0 \\ & & \downarrow B & & \| A & & \downarrow \hat{T} & & \\ 0 & \longrightarrow & \mathcal{O}^{r+1}_{\mathbb{C}^n,0} & \longrightarrow & \mathcal{O}^{r+1}_{\mathbb{C}^n,0} & \xrightarrow{M} & \mathcal{O}_{D^2(f)} & \longrightarrow & 0 \end{array}$$

where $G' = \begin{bmatrix} 1\otimes 1 & g_1\otimes 1 - 1\otimes g_1 & \cdots & g_r\otimes 1 - 1\otimes g_r \end{bmatrix}$,
$M = \begin{bmatrix} 1 & g_1(x_2) & \cdots & g_r(x_2) \end{bmatrix}$, B is the lift of A and $\hat{T}$ is the map $g_i\otimes 1 \mapsto g_i(x_2)$.

Remark 2.7.2. Let $f \in \mathcal{E}^0_{n,n+1}$ be of corank ≥ 2 with $Q(f) = \mathbb{C}\cdot\{1, g_1, \ldots, g_r\}$. Suppose that $q(f) \geq q(\pi^2_1)$. We may assume that $Q(\pi^2_1) = \mathbb{C}\cdot\{1, g_1(x_2), \ldots, g_s(x_2)\}$ so that $\mathcal{O}_{D^2(f)} = \mathcal{O}_{\mathbb{C}^n,0}\cdot\{1, g_1(x_2), \ldots, g_s(x_2)\}$ (cf. Proposition 2.1.7). Then there exist $a_{ij} \in \mathcal{O}_{\mathbb{C}^n,0}$, $i, j = 0, \ldots, s$, satisfying $g_{s+i}(x_2) = a_{i0}(x_1) + \sum_{j=1}^{s} a_{ij}(x_1)\cdot g_j(x_2)$

for all $i = 0, \ldots, r-1$. We have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^r_{\mathbb{C}^n,0} & \xrightarrow{f^*\lambda^1} & \mathcal{O}^{r+1}_{\mathbb{C}^n,0} & \xrightarrow{G'} & \mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n+1},0}} \mathcal{O}_{\mathbb{C}^n,0} & \longrightarrow & 0 \\ & & \downarrow B & & \downarrow A & & \downarrow \hat{T} & & \\ 0 & \longrightarrow & \mathcal{O}^s_{\mathbb{C}^n,0} & \longrightarrow & \mathcal{O}^s_{\mathbb{C}^n,0} & \xrightarrow{M} & \mathcal{O}_{D^2(f)} & \longrightarrow & 0 \end{array}$$

where $G' = \begin{bmatrix} 1 \otimes 1 & g_1 \otimes 1 - 1 \otimes g_1 & \cdots & g_r \otimes 1 - 1 \otimes g_r \end{bmatrix}$,

$M = \begin{bmatrix} 1 & g_1(x_2) & \cdots & g_{r-1}(x_2) \end{bmatrix}$, $A = \left[\begin{array}{c|c} \mathrm{Id}_{r\times r} & (a_{ij}) \end{array} \right]$, B is the lift of A and $\hat{T}$ is the map $g_i \otimes 1 \mapsto g_i(x_2)$.

Corollary 2.7.3. *The support of D^2 in the source is D^2_1 for a finite and generically one-to-one map-germ. However, the support of ID^2 in the source is the whole domain.*

2.8 Examples of free divisors

Let X be a nonsingular n-dimensional complex manifold, and let $D \subseteq X$ be a hypersurface with reduced defining ideal I_X. Let Der(-log D) denote the module of logarithmic vector fields that are tangent to D at its regular points, i.e. the module of vector fields $\xi \in \theta_X$ such that $\xi(I_X) \subset I_X$.

Definition 2.8.1. The hypersurface $D \subseteq X$ is called a *free divisor* (or a *Saito divisor*) Der(-log D) is a locally free $\mathcal{O}_X$-module.

Theorem 2.8.2 (Saito's Criterion - Theorem 1.8 (ii), [Sai80])**.** *The hypersurface $D \subset X$ is a free divisor in the neighbourhood of a point x if and only if there are germs of vector fields $\xi_1, \ldots, \xi_n \in$ Der(-log $D)_x$, such that the determinant of the matrix of coefficients $[\xi_1, \ldots, \xi_n]$ with respect to some, or any, $\mathcal{O}_{X,x}$-basis of* $\mathrm{Der}_{X,x}$ *is a reduced equation for D at x. In this case, $\xi_1, \ldots, \xi_n$ form a basis for* Der(-log $D)_x$.

Another characterisation of free divisors can be found in Terao and Aleksandrov's works ([Ter80],[Ale90]): D is a free divisor if and only if the singular set of D is a Cohen-Macaulay space of codimension 1 in D.

Finding examples of free divisors has been a quest for singularity theorists. Most famously known examples are discriminants of versal unfoldings of hypersurface and complete intersection singularities ([Sai80] and [Loo84]), bifurcation sets of (versal unfoldings of) isolated hypersurface singularities ([Bru85] and [Ter83]) and Coxeter arrangements ([Ter80]). Discriminants (=the set of non-rigid representations) of Dynkin quivers with "real" roots are also linear free divisors ([MB06]).

More recently, Mond and Schulze gave the following example.

Theorem 2.8.3 ([MS10])**.** *If $f \in \mathcal{E}^0_{n,n+1}$ is a stable corank 1 map-germ and λ is a matrix resolving $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, then* $\det\lambda \cdot \det\lambda^1_1$ *defines a free divisor in* $\mathbb{C}^{n+1}$.

Now, we have

Proposition 2.8.4. *Let $f \in \mathcal{E}^0_{n,n+1}$ be a stable map-germ of corank 1. Let*

$$0 \to \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{\lambda} \mathcal{O}^{r+1}_{\mathbb{C}^{n+1},0} \xrightarrow{H} \mathcal{O}_{\mathbb{C}^n,0} \to 0$$

be a resolution of $\mathcal{O}_{\mathbb{C}^n,0}$ where λ is symmetric and $H = \begin{bmatrix} 1 & y & \dots & y^r \end{bmatrix}$. Then

$$S := \left\{ \det(f^*\lambda^{k-1}_{k-1}) \cdot \det(f^*\lambda^k_k) = 0 \right\}$$

is a free divisor in $D^k(f)$ for $k = 1, \dots, r+1$ ($\lambda^0_0 := \lambda$).

Proof. It is trivial for $k = 1$: Since $\det(f^*\lambda) = 0$, $S = (\mathbb{C}^n, 0)$. Assume that $k \geq 2$. We have observed that $\mathcal{O}_{D^{k+1}(f)}$ admits a minimal resolution of length 1 over $\mathcal{O}_{D^k(f)}$ given by the matrix $f^*\lambda^k_k$. Since f is stable, the projection maps $\pi^{k+1}_k \colon D^{k+1}(f) \to D^k(f)$ are also stable and of corank 1. Moreover, $D^k(f)$ is a smooth space of dimension $n - k + 1$ or empty for all k. Therefore, the statement follows by applying Theorem 2.8.3 to π^{k+1}_k. □

Remark 2.8.5. For stable corank 1 map-germs φ_{r+1} defined by (2.20), ($r = 2, 3, 4$), we observe that

$$\left\{ \det(\lambda) \cdot \det(\lambda^1_1) \cdot \det(\lambda^2_2) = 0 \right\}$$

is a free divisor but

$$\left\{ \det(\lambda) \cdot \det(\lambda^1_1) \cdots \det(\lambda^3_3) = 0 \right\}$$

is not. We are yet to check this phenomenon for $r \geq 5$.

Chapter 3

A conjecture of Mond

In this chapter, we review a conjecture of Mond which asks a relation between the topology of the image of a stable perturbation of a finitely $\mathcal{A}$-determined map-germ $f \in \mathcal{E}^0_{n,n+1}$ and its $\mathcal{A}_e$-codimension; this is a relation mostly referred to as a $\mu = \tau$-*type relation* in Singularity theory. We will recall the results of de Jong, van Straten, Mond *et al.* which answers the question for $n = 2$. We will also recall Damon and Mond's work on a similar phenomenon for map-germ in dimensions (n, p) with $n \geq p$. together with Damon's theory of nonlinear sections of a germ of a variety $(V, 0)$.

3.1 A review of the conjecture

Let $f \in \mathcal{E}^0_{n,p}$ be a finitely $\mathcal{A}$-determined map-germ with $2 \leq n \leq p$. Let $F \in \mathcal{E}^0_{n+d,p+d}$, given by $F(\mathbf{x}, \mathbf{t}) = (f_{\mathbf{t}}(\mathbf{x}), \mathbf{t})$, be an $\mathcal{A}_e$-versal unfolding of f and let $\pi\colon \mathbb{C}^p \times \mathbb{C}^d \to \mathbb{C}^d$ be the natural projection. If f is not stable then there exists an analytic set $B \subset \mathbb{C}^d$, called the bifurcation set, such that for $\mathbf{t} \in B$ the map $f_{\mathbf{t}}$ is not stable. The bifurcation set is a proper subset if and only if f has a stable perturbation and in particular if (n, p) are nice dimensions. Moreover,

Theorem 3.1.1 ([Mar91], Theorem 1.1)**.** *There exists an $\varepsilon > 0$, neighbourhoods U of the origin of $\mathbb{C}^n \times \mathbb{C}^d$ and T of the origin of $\mathbb{C}^d$ and a proper, finite to one representative of the unfolding $F\colon U \to B_\varepsilon \times T$ such that, if $X = F(\bar{U})$ then the stable-type stratified mapping $F\colon F^{-1}(\bar{B}_\varepsilon \times (T - B)) \to X \cap (\bar{B}_\varepsilon \times (T - B))$ is locally trivial over $T - B$ with respect to the stratified submersion $\pi\colon \bar{B}_\varepsilon \times (T - B) \to T - B$.*

This provides a fibration of the mapping $F\colon \bar{U} \to \text{image}(F) =: X$ whose fibre over a parameter $\mathbf{t} \in T - B$ is the stable map $f_{\mathbf{t}} : U_{\mathbf{t}} \to X_{\mathbf{t}}$ with $X_{\mathbf{t}} := X \cap (\bar{B}_\varepsilon \times \{\mathbf{t}\})$

where $U_{\mathbf{t}} = \{\mathbf{x} \in \mathbb{C}^n \mid (\mathbf{x}, \mathbf{t}) \in U\}$. Also a topological fibration of the image X of F is locally trivial over $T - B$, whose fibre over $\mathbf{t} \in T - B$ is the image $X_{\mathbf{t}}$ of the stable mapping $f_{\mathbf{t}}$. Such $X_{\mathbf{t}}$ is called a *disentanglement* of the image of f.

Mond proved the following results for the case $p = n + 1$:

Lemma 3.1.2 (Lemma 1.3, [Mon91]). *The topology of the disentanglement is independent of the choice of the stabilisation* $f_{\mathbf{t}}$.

Theorem 3.1.3 (Theorem 1.4, [Mon91]). *Let* $f \in \mathcal{E}^0_{n,n+1}$ *be a finitely* $\mathcal{A}$*-determined map-germ, with* $(n, n+1)$ *nice dimensions (i.e.* $n \leq 6$*), and let* $X_{\mathbf{t}}$ *be a disentanglement of the image of* f*. Then* $X_{\mathbf{t}}$ *has the homotopy type of a wedge of* n*-spheres.*

The number of spheres in the wedge is an $\mathcal{A}$-invariant of f. Mond called this number the *image Milnor number*. It is denoted by $\mu_I(f)$.

Pellikaan and de Jong (unpublished) then de Jong and van Straten ([dJvS91]) and later Mond ([Mon91]) proved the following theorem:

Theorem 3.1.4. *For map germs* $f \in \mathcal{E}^0_{2,3}$ *of finite* $\mathcal{A}_e$*-codimension, we have*

$$\mathcal{A}_e\text{-codim}(f) \leq \mu_I(f) \tag{3.1}$$

and the equality if f *is weighted homogeneous.*

In [Mon91], Mond conjectured that a similar statement holds for all $n \geq 3$:

Conjecture 3.1.5 (the Mond conjecture). Let $f \in \mathcal{E}^0_{n,n+1}$ be a map germ of finite $\mathcal{A}_e$-codimension and suppose that $(n, n+1)$ are in Mather's range of nice dimensions ([Mat70]), i.e. $n \leq 6$. Then $\mathcal{A}_e$-codim$(f) \leq \mu_I(f)$ and the equality holds if f is weighted homogeneous.

For particular examples, there are ways to check whether the conjecture is satisfied or not. One of them is to calculate the image Milnor number (at least in corank 1 case) and $\mathcal{A}_e$-codimension separately and then compare them. Another method (which works for any corank) can be deduced from Damon and Mond's work [DM91] in which they prove a theorem with a similar nature to Theorem 3.1.4. Their method relies on the relation between $\mathcal{A}$-equivalence and $\mathcal{K}_V$ (and $\mathcal{K}_H$) equivalences which were introduced by Damon. In the following section we recall the main results of Damon and Mond and also [Dam91] which provide an alternative method for calculating $\mathcal{A}_e$-codimension.

3.2 $\mathcal{K}_V$ and $\mathcal{K}_H$-equivalences

In [DM91], Damon and Mond considered the map germs $f \in \mathcal{E}^0_{n,p}$ with $n \geq p$. In this dimensions, similar results also hold if we replace "image" in the above statements by "discriminant". When (n, p) are nice dimensions, the discriminant $D(f_{\mathbf{t}})$ intersected with a Milnor ball B_ε about 0 (i.e. a ball with radius ε such that for ε' with $0 < \varepsilon' \leq \varepsilon$, $D(f)$ is stratified transverse to $\partial B_{\varepsilon'}$), has the homotopy type of a wedge of $(p-1)$-dimensional spheres. Damon and Mond called the number of spheres in the wedge the *discriminant Milnor number* and denoted it by $\mu_\Delta(f)$. They proved the following important result:

Theorem 3.2.1 (Theorem 1 + Theorem 2, [DM91])**.** *Let $f \in \mathcal{E}^0_{n,p}$ be a map-germ of finite $\mathcal{A}$-codimension. Suppose (n, p) is in the range of nice dimensions with $n \geq p$. Then*

$$\mathcal{A}_e\text{-codim}(f) \leq \mu_\Delta(f) \tag{3.2}$$

and with the equality if f is weighted homogeneous.

One of the key parts of the proof of Theorem 3.2.1 was based on the relation between $\mathcal{A}$-equivalence and $\mathcal{K}_V$ and $\mathcal{K}_H$-equivalences which was introduced by Damon as subgroups of the contact group $\mathcal{K}$ in [Dam87] and studied further in [Dam91]. This relation not only aids the proof of their theorem it also provides an alternative method for calculating $\mathcal{A}_e$-codimension of a map-germ (see Corollary 3.2.12). Now we recall the definitions of the groups $\mathcal{K}_V$ and $\mathcal{K}_H$.

If $g \in \mathcal{E}^0_{s,m}$ and $(V, 0) \subset (\mathbb{C}^p, 0)$ is a germ of a variety then the subgroup $\mathcal{K}_V$ of $\mathcal{K}$ is defined as

$$\mathcal{K}_V := \{\Phi \in \mathcal{K} \mid \Phi(\mathbb{C}^n \times V) \subseteq \mathbb{C}^n \times V\}$$

and similarly for unfoldings. This group yields $\mathcal{K}_V$-equivalence which captures the isomorphism classes of the germs of varieties $g^{-1}(V)$. In other words, $g \sim_{\mathcal{K}_V} g'$ if and only if $g^{-1}(V)$ is isomorphic to $g'^{-1}(V)$.

Moreover, given a map-germ $H\colon (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$, we have

$$\mathcal{K}_H := \{\Phi \in \mathcal{K} \mid H \circ pr_2 \circ \Phi = H \circ pr_2\}$$

where $pr_2\colon \mathbb{C}^s \times \mathbb{C}^m \to \mathbb{C}^m$ is the natural projection. This group measures the $\mathcal{K}$-equivalence of the germs by an equivalence which preserves all of the level sets of H.

In order to define the tangent spaces of $g \in \mathcal{E}^0_{s,m}$ with respect to $\mathcal{K}_V$ and $\mathcal{K}_H$-equivalences we need to consider Der(-log V), the module of vector fields tangent to V, and Der(-log H), the module of vector fields tangent to the level sets of H. Namely,

$$\text{Der(-log } V) = \{\xi \in \theta_{\mathbb{C}^m,0} \mid \xi(I_V) \subseteq I_V\}$$

where I_V is the ideal of V, and

$$\text{Der(-log } H) = \{\xi \in \theta_{\mathbb{C}^m,0} \mid \xi(H) = 0\}\,.$$

We have

$$T\mathcal{K}_{V,e}g = tg(\theta_{\mathbb{C}^s,0}) + g^*(\text{Der(-log } V)) \quad \text{and} \quad T\mathcal{K}_{H,e}g = tg(\theta_{\mathbb{C}^s,0}) + g^*(\text{Der(-log } H)).$$

If $\{\xi_i\}_{i=1}^m$ is a set of generators for Der(-log V) then

$$T\mathcal{K}_{V,e}g = tg(\theta_{\mathbb{C}^s,0}) + \mathcal{O}_{\mathbb{C}^m,0}\,\{\xi_1 \circ g, \ldots, \xi_m \circ g\}\,.$$

A similar definition holds for Der(-log H). If $\{\eta_i\}_{i=1}^l$ is a set of generators for Der(-log H) then

$$T\mathcal{K}_{H,e}g = tg(\theta_{\mathbb{C}^s,0}) + \mathcal{O}_{\mathbb{C}^m,0}\,\{\eta_1 \circ g, \ldots, \eta_l \circ g\}\,.$$

As for $\mathcal{A}$ and $\mathcal{K}$, basic deformation and determinacy theorems hold for $\mathcal{K}_V$ and $\mathcal{K}_H$ as well. Deformation spaces (or normal spaces) are

$$N\mathcal{K}_{V,e}g = \frac{\theta(g)}{T\mathcal{K}_{V,e}} \quad \text{and} \quad N\mathcal{K}_{H,e}g = \frac{\theta(g)}{T\mathcal{K}_{H,e}}.$$

By definition

$$\mathcal{K}_{V,e}\text{-codim}(g) = \dim_{\mathbb{C}} N\mathcal{K}_{V,e}g \quad \text{and} \quad \mathcal{K}_{H,e}\text{-codim}(g) = \dim_{\mathbb{C}} N\mathcal{K}_{H,e}g.$$

Definition 3.2.2 (p. 703, [Dam87])**.** A map-germ $g \in \mathcal{E}^0_{s,m}$ is said to be algebraically transverse to a germ $(V,0) \subset (\mathbb{C}^m,0)$ at x_0 if

$$\mathrm{d}_{x_0}g(T_{x_0}\mathbb{C}^s) + \mathbb{C}\cdot\left\{\xi_1|_{g(x_0)}, \ldots, \xi_m|_{g(x_0)}\right\} = T_{g(x_0)}\mathbb{C}^m.$$

Here $\xi_i|_{g(x_0)}$ means the evaluation of ξ_i at the point $g(x_0)$.

Proposition 3.2.3 (Proposition 2.2, [Dam87])**.** *Let g and V be as above. Then g is finitely $\mathcal{K}_V$-determined if and only if g is algebraically transverse to V in a punctured neighbourhood of* 0.

By the results of [Mat68] or [Dam87], a similar argument holds for $\mathcal{K}_H$-equivalence: Assuming $\{\eta_i\}_{i=1}^{l}$ is a set of generators for Der(-log H), g is finitely $\mathcal{K}_H$-determined if and only if

$$\mathrm{d}_{x_0}g(T_{x_0}\mathbb{C}^s) + \mathbb{C}\cdot\left\{\eta_1|_{g(x_0)},\ldots,\eta_l|_{g(x_0)}\right\} = T_{g(x_0)}\mathbb{C}^m$$

holds for any $x_0 \neq 0$ in a neighbourhood of 0.

Let $\mathcal{N}\mathcal{K}_V g$ and $\mathcal{N}\mathcal{K}_H g$ denote the associated normal sheaves to the normal spaces $N\mathcal{K}_{V,e}g$ and $N\mathcal{K}_{H,e}g$. By Nakayama's Lemma ([Mat89]), g is finitely $\mathcal{K}_V$-determined if and only if

$$\mathrm{Supp}(N\mathcal{K}_{V,e}g) := \mathrm{Supp}(\mathcal{N}\mathcal{K}_V g) = \{0\} \text{ or } \emptyset.$$

Similarly, g has finite $\mathcal{K}_H$-codimension if and only if

$$\mathrm{Supp}(N\mathcal{K}_{H,e}g) := \mathrm{Supp}(\mathcal{N}\mathcal{K}_H g) = \{0\} \text{ or } \emptyset.$$

One can also extend the notion of $\mathcal{K}_V$ to the group of unfolding-equivalences $\mathcal{K}_{V/}$ which is also a *geometric subgroup* of $\mathcal{A}$ and $\mathcal{K}$ (see [Dam87]). The associated extended tangent space is given by the following definition.

Definition 3.2.4 ([Dam98])**.** Let $(V,0) \subset (\mathbb{C}^m,0)$ be a germ of a variety and $G\colon (\mathbb{C}^s \times \mathbb{C}^d,0) \to (\mathbb{C}^m,0)$ be a deformation of g given by $G = G(\mathbf{y},\mathbf{u})$. The relative extended tangent space is defined by

$$\begin{aligned} T\mathcal{K}_{V,e/\mathbb{C}^d}G &= tG(\Theta_{\mathbb{C}^s\times\mathbb{C}^d/\mathbb{C}^d,0}) + G^*(\mathrm{Der}(\text{-log } V)) \\ &= (\frac{\partial G}{\partial y_1},\ldots,\frac{\partial G}{\partial y_s})\mathcal{O}_{\mathbb{C}^s\times\mathbb{C}^d,0} + G^*(\mathrm{Der}(\text{-log } V)) \end{aligned}$$

and the relative normal space by

$$N\mathcal{K}_{V,e/\mathbb{C}^d}G = \frac{\Theta(G)}{T\mathcal{K}_{V,e/\mathbb{C}^d}G}.$$

Definition 3.2.5 (Definition 2.1, [Dam98])**.** The $\mathcal{K}_V$-critical set of G is given by

$$C_V(G) = \operatorname{Supp}(N\mathcal{K}_{V,e/\mathbb{C}^d}G) = \operatorname{Supp}(\mathcal{N}\mathcal{K}_V G)$$

and the $\mathcal{K}_V$-discriminant of G by $D_V(G) = \pi_|(C_V(G))$ where $\pi\colon \mathbb{C}^s \times \mathbb{C}^d \to \mathbb{C}^d$ is the cartesian projection.

Remark 3.2.6. By the same method of Lemma 2.2 of [Dam98], we can deduce that if G is a deformation of a map-germ g of finite $\mathcal{K}_V$-codimension then $\pi_|\colon C_V(G) \to D_V(G)$ is a finite map, and $C_V(G)$, $D_V(G)$ are analytic subsets of the same dimension. Moreover, $C_V(G)$ consists of points (y_0, u_0), with $z_0 = G(y_0, u_0)$, such that the germ $G(-, u_0)\colon (\mathbb{C}^s, y_0) \to (\mathbb{C}^m, z_0)$ is not algebraically transverse to V at y_0.

Remark 3.2.7. By [Dam98, Theorem 1], $D_V(G)$ is a free divisor if V is.

Definition 3.2.8 (Definition 3.1, [DM91])**.** Given a hypersurface $(V, 0) \subset (\mathbb{C}^m, 0)$, a germ $H\colon (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is said to be a *good defining equation* for V if $H^{-1}(0) = V$ and there exists a germ of a vector field e such that $e(H) = H$.

Clearly, an example of a good defining equation is the case where H is a weighted homogeneous germ of a function.

Lemma 3.2.9 (Lemma 3.3, [DM91])**.** *If H is a good defining equation for V then*

$$\operatorname{Der}(\text{-log } V) = \operatorname{Der}(\text{-log } H) \oplus \mathcal{O}_{\mathbb{C}^p,0}\{e\}.$$

If V is a free divisor then $\operatorname{Der}(\text{-log } H)$ *is a free $\mathcal{O}_{\mathbb{C}^p,0}$-module on $t-1$ generators.*

In particular, if H is a good defining equation for V and weighted homogeneous of positive weight, and if $g \in \mathcal{E}^0_{s,m}$ has finite $\mathcal{K}_V$-codimension and weighted homogeneous for the same weights, then $N\mathcal{K}_{V,e}g = N\mathcal{K}_{H,e}g$ ([DM91, Lemma 3.4]).

In [Dam91], Damon relates $\mathcal{A}$-equivalence with $\mathcal{K}_V$-equivalence as follows. Assume $f \in \mathcal{E}^0_{n,p}$ is obtained by a pullback diagram of the form

$$\begin{array}{ccc} (\mathbb{C}^N, 0) & \xrightarrow{F} & (\mathbb{C}^P, 0) \\ \uparrow & & \uparrow{\scriptstyle g} \\ (\mathbb{C}^n, 0) & \xrightarrow{f} & (\mathbb{C}^p, 0) \end{array} \tag{3.3}$$

where F is a stable and parametrised unfolding of f, g is a map-germ transverse to F. Let $\Sigma(F) = \left\{\mathbf{x} \in (\mathbb{C}^N, 0) \mid \operatorname{rank}(\mathrm{d}F(\mathbf{x})) < P\right\}$ denote the critical set of F

and $D(F) := F(\Sigma(F))$ the discriminant of F. Set $V := D(F)$. One of the most important results of [Dam91] is the following.

Theorem 3.2.10 (Theorem 2,[Dam91]). *Let $N\mathcal{A}_e f$ and $N\mathcal{K}_{V,e} g$ be both (or either) be finite dimensional. Then there exists an isomorphism of $\mathcal{O}_{\mathbb{C}^p,0}$-modules*

$$\hat{\Psi} : N\mathcal{A}_e f \xrightarrow{\cong} N\mathcal{K}_{V,e} g$$

induced by

$$\begin{array}{rcl} \Psi\colon \theta(g) & \to & \theta(f) \\ \eta & \mapsto & -\eta_1 \circ f + \mathrm{d}_{\mathbf{u}}\bar{F}(\mathbf{x},0)(\eta_2) \circ f. \end{array}$$

where $F(\mathbf{x},\mathbf{u}) = (\bar{F}(\mathbf{x},\mathbf{u}),\mathbf{u}) = (\mathbf{y},\mathbf{u})$ with respect to a chosen local coordinate system and $\eta = \eta_1 + \eta_2$ relative to the decomposition $\mathbb{C}^p_{\mathbf{y}} \times \mathbb{C}^{P-p}_{\mathbf{u}}$.

Remark 3.2.11. Following [Dam91], alternative proofs without the condition that $N\mathcal{A}_e f$ (and/or $N\mathcal{K}_{V,e} g$) has finite $\mathbb{C}$-dimension have been given by other mathematicians. See [Mon00, §8] for one of them and other references.

Corollary 3.2.12 (Corollary 3.18, [DM91]). *If f is weighted homogeneous and finitely $\mathcal{A}$-determined, so that F, g, V and the defining equation H can be chosen to be weighted homogeneous for the same weights, then*

$$N\mathcal{A}_e f \cong N\mathcal{K}_{V,e} g \cong N\mathcal{K}_{H,e} g.$$

For the proof of Theorem 3.2.1, the authors of [DM91] considered the relative deformation space

$$N\mathcal{K}_{H,e/\mathbb{C}} G := \frac{\theta(G)}{tG(\theta_{\mathbb{C}^p\times\mathbb{C}/\mathbb{C}}) + G^*(\mathrm{Der}(\text{-log } H))},$$

for a 1-parameter deformation $G(\mathbf{y},t) = (g_t(\mathbf{y}),t)$ of g. They proved that $N\mathcal{K}_{H,e/\mathbb{C}} G$ is a Cohen-Macaulay module of dimension 1 by using the fact that Der(-log H) is free $\mathcal{O}_{\mathbb{C}^{p+1},0}$-module (which follows easily as V is the discriminant hypersurface whence a free divisor, see [Loo84]). Therefore, in this setting, $\mathcal{K}_H$-equivalence has a *free deformation theory*, that is, the length of $N\mathcal{K}_{H,e/\mathbb{C}} G$ counts the number of the points of $\mathcal{K}_H$-instability points of g_t that the origin splits into after deforming g into

g_t. Since

$$\dim_{\mathbb{C}}(N\mathcal{K}_{H,e}g_t)_{\mathbf{y}} = \dim_{\mathbb{C}}\mathcal{O}_{\mathbb{C}^p,\mathbf{y}}/J_h =: \mu(h_t;\mathbf{y})$$

where $h = H \circ g_t$, $\mathbf{y}$ a point such that H is nonsingular at $g_t(\mathbf{y})$ and J_h is the jacobian ideal of h ([DM91, Lemma 5.6]), the only contribution to $N\mathcal{K}_{H,e}g$ comes from the points $y \in \mathbb{C}^p$ with $g_t(\mathbf{y}) \notin V$. And the milnor numbers $\mu(h_t;\mathbf{y})$ for those $\mathbf{y}$ sum up to $\mu_\Delta(f)$ whence the relation between $\mathcal{A}_e$-codimension and the image Milnor number.

Remark 3.2.13. We can adopt all the steps of the proof of Theorem 3.2.1 for the case $(n,p) = (n, n+1)$ except the one where they proved that $N\mathcal{K}_{H,e/\mathbb{C}}G$ is Cohen-Macaualay module. This is because the discriminant (which equals to the image in these dimensions) of a stable map-germ is no longer a free divisor. However we have no evidence that $N\mathcal{K}_{H,e/\mathbb{C}}G$ cannot be a Cohen-Macaulay module. So, Conjecture 3.1.5 becomes a corollary to the following problem.

Conjecture 3.2.14. Let $F \in \mathcal{E}^0_{N,N+1}$ be a stable map-germ and $g \in \mathcal{E}^0_{n+1,N+1}$ be an immersion transverse to F. Let G be a 1-parameter deformation of g, H be the defining equation of the image of F. Then $N\mathcal{K}_{H,e/\mathbb{C}}G$ is a Cohen-Macaulay module of dimension 1.

In Chapter 4 and 5, we will provide new examples supporting Conjecture 3.2.14, or equaivalently Conjecture 3.1.5.

3.2.1 A SINGULAR library for calculating $\mathcal{A}_e$-codimension

We construct an algorithm for calculating $\mathcal{A}_e$-codimension of a map-germ $f \in \mathcal{E}^0_{n,p}$ based on Damon's theory.

Algorithm 3.2.15 (Algorithm for $\mathcal{A}_e$-codimension)**.** INPUT. A stable unfolding $F \in \mathcal{E}^0_{n+d,p+d}$ of f.

STEP 1. Define $g\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ by $g(\mathbf{y}) = (\mathbf{y}, 0)$.

STEP 2. Find the defining equation for the image V of F.

STEP 3. Calculate the module Der(-log V) (or Der(-log H) if the maps are weighted homogeneous for the same weights).

STEP 4. Calculate $T\mathcal{K}_{V,e}g = \mathrm{d}g \cdot \mathcal{O}_{\mathbb{C}^p,0} + g^*(\text{Der(-log } V))$.

OUTPUT. $\mathcal{A}_e$-codim$(f) = \mathcal{K}_{V,e}$-codim$(g) = \dim_{\mathbb{C}} N\mathcal{K}_{V,e}$ (or $\mathcal{A}_e$-codim$(f) = \mathcal{K}_{H,e}$-codim$(g) = \dim_{\mathbb{C}} N\mathcal{K}_{H,e}$ if f is weighted homogenous).

SINGULAR Example 3.2.16. Let $f \in \mathcal{E}^0_{2,3}$ be defined by $f(x,y) = (x, y^3, xy + y^5)$. We find a stable unfolding of f to be $F\colon (x,y,u,v) \mapsto (x, y^3 + uy, xy + y^5 + vy^2, u, v)$.

Let X, Y, Z, U, V denote the local coordinates on $(\mathbb{C}^3 \times \mathbb{C}^2, 0)$. Let $g\colon (X,Y,Z) \mapsto (X,Y,Z,0,0)$ so that f is equal to the fibre product of F and g. The following code calculates $\mathcal{A}_e$-codim$(f) = 2$.

```
LIB "matrix.lib";       //need this library for the command "concat"
ring t=0,(X,Y,Z),dp;                   //ring of the target of f
ring T=0,(X,Y,Z,U,V),dp;               //ring of the target of F
ring S=0,(x,y,u,v),dp;                 //ring of the domain of F
ideal p=0;
map F=T,x,y^3+uy,xy+y^5+vy^2,u,v;
setring T;
ideal H=preimage(S,F,p);
ideal JH=jacob(H);                     //Jacobian ideal of H
module derlogv=modulo(JH,H);
setring t;
map g=T,X,Y,Z,0,0;
def gderloghv=g(derloghv);
ideal ig=X,Y,Z;
matrix Jg=jacob(ig);
def TKVeg=concat(Jg,gderlogv); //combine the generators of the modules
vdim(std(TKVeg));       //vector dimension of the complement of TKVeg
//-> 2
```

Remark 3.2.17. We have written a library `Ae.lib` for SINGULAR with a procedure called `Aecodim()` based on Algorithm 3.2.15 (see Appendix B). The command `Aecodim()` basically reads the map (which needs to be input as an ideal for technical reasons), calculates $\mathcal{K}_e$-stable unfolding, changes the ordering of the ring to weighted reverse lexicographical ordering if the map is weighted homogeneous, follows Algorithm 3.2.15 and outputs the multiplicity, $\mathcal{K}_e$-codimension and finally $\mathcal{A}_e$-codimension of the map-germ. We prefer a weighted ordering as the processing times are significantly shorter than those for non-weighted homogeneous map-germs. During the process the command `Aecodim()` does not change (the ordering of) the ring defined by the user. In the following example we calculate the $\mathcal{A}_e$-codimension of the map in Example 3.2.16 by using the command `Aecodim()`.

SINGULAR Example 3.2.18. `LIB "Ae.lib";`

```
ring r=0,(x,y),dp;     //define the ring with any ordering
ideal f=x,y^3,xy+y^5;
Aecodim(f);
//->Multiplicity = 3
//->K_e-codim = 2
//->A_e-codimension = 2
basering;       //see the active ring
//   characteristic : 0
//   number of vars : 2
//        block   1 : ordering lp
//                  : names    x y
//        block   2 : ordering C
```

Chapter 4

Corank 2 map-germs from $\mathbb{C}^3$ to $\mathbb{C}^4$

4.1 Stable mono-germs

In these dimensions, all stable mono-germs have corank 1 by Proposition 2.1.12. By Theorem A of [Mat69b], two stable map-germs defined in the same dimensions are $\mathcal{A}$-equivalent if and only if their algebras are isomorphic. According to Mather's classification in [Mat69b], a stable corank 1 map-germ from n-space to p-space has a local algebra isomorphic to $\mathbb{C}\{z\}/(z^{l+1})$ where $l \leq n/(p-n+1)$. In this particular case, i.e. $n = 3$ and $p = 4$, l becomes less than or equal to 2. So, we meet only two cases.

Proposition 4.1.1. *A stable map-germ $f \in \mathcal{E}^0_{3,4}$ of corank 1 is an immersion or $\mathcal{A}$-equivalent to a constant 1-parameter unfolding of a cross-cap.*

Proof. Clearly, f is equivalent to an immersion when $l = 1$.

For $l = 2$, we can obtain a standard form for a stable map as follows. We consider the map-germ

$$\begin{array}{rcl} g\colon (\mathbb{C},0) & \to & (\mathbb{C}^2,0) \\ z & \mapsto & (z^2,0) \end{array}$$

which has $Q(g) = \mathbb{C}\{z\}/(z^2)$. We calculate its $\mathcal{A}_e$-versal (hence stable) unfolding as

$$\begin{aligned} G\colon (\mathbb{C}\times\mathbb{C},0) &\rightarrow (\mathbb{C}\times\mathbb{C}^2,0)\\ (y,z) &\mapsto (y,z^2,yz)\end{aligned}$$

that is the map-germ named as the cross-cap. A trivial unfolding of G is then of the form $1\times G\colon (x,y,z)\mapsto (x,y,z^2,yz)$. We find $Q(1\times G)\cong \mathbb{C}\{z\}/(z^2)$. Therefore f must be $\mathcal{A}$-equvialent to $1\times G$ by Mather's result. □

4.2 A comparison of the Fitting ideals of $\mathcal{O}_{\mathbb{C}^3,0}$ and $\mathcal{O}_{D^2(f)}$

In this section we study the relation between the Fitting ideals of $\mathcal{O}_{\mathbb{C}^3,0}$ and $\mathcal{O}_{D^2(f)}$.

Assume that $f\in\mathcal{E}^0_{3,4}$ is finite and generically one-to-one and we are given a resolution

$$0\rightarrow \mathcal{O}^{r+1}_{\mathbb{C}^4,0}\xrightarrow{\lambda}\mathcal{O}^{r+1}_{\mathbb{C}^4,0}\rightarrow\mathcal{O}_{\mathbb{C}^3,0}\rightarrow 0.$$

Since f is finite, $\pi^2_1(f)$ is also finite by Proposition 2.1.7. So $\mathcal{O}_{D^2(f)}$ is a finite $\mathcal{O}_{\mathbb{C}^3,0}$-module and has a resolution of length 1 over $\mathcal{O}_{\mathbb{C}^3,0}$ by the results of [MP89]. Moreover the resolution is given by an $s\times s$-matrix for some $s\leq r+1$.

Proposition 4.2.1. *Let $f\in\mathcal{E}^0_{3,4}$ be a finitely $\mathcal{A}$-determined map-germ. Then*

$$\begin{aligned} f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^3,0}) &= \mathrm{Fitt}_0((\pi^2_1)_*\mathcal{O}_{D^2(f)}) \qquad \text{and}\\ (f^*\mathrm{Fitt}_2(f_*\mathcal{O}_{\mathbb{C}^3,0}))_{\mathfrak{p}} &= \mathrm{Fitt}_1((\pi^2_1)_*\mathcal{O}_{D^2(f)})_{\mathfrak{p}}\end{aligned}$$

for all $\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_{\mathbb{C}^3,0})\setminus\mathfrak{m}_{\mathbb{C}^3,0}$.

Proof. We will work this out locally. First, we will prove that

$$(f^*\mathrm{Fitt}_k(f_*\mathcal{O}_{\mathbb{C}^3,0}))_{\mathfrak{m}_{\mathbb{C}^3,\mathbf{x}}} = \mathrm{Fitt}_{k-1}((\pi^2_1)_*\mathcal{O}_{D^2(f)})_{\mathfrak{m}_{\mathbb{C}^3,\mathbf{x}}} \tag{4.1}$$

for all $0\neq\mathbf{x}\in\mathbb{C}^3,0$, and $k=1,2$. Then we will discuss the extension of the equality to the stalks at the origin.

We consider a multi-germ $\bar{f}\colon(\mathbb{C}^3,S)\rightarrow(\mathbb{C}^4,\mathbf{y})$ where $y\neq 0$ and $S:=f^{-1}(\mathbf{y})$. Here, we can assume that $S=\{\mathbf{x}_1,\mathbf{x}_2\}$ since we will only deal with D^2. Let $f^{(i)}\colon(\mathbb{C}^3,0^{(i)})\rightarrow(\mathbb{C}^4,\mathbf{y})$ be the i-th branch of $\bar{f}$. Since f is stable, each branch $f^{(i)}$, $i=1,2$, is also stable by [Mat69b, Proposition 1.6]. Hence, $f^{(i)}$ is either an immersion or $\mathcal{A}$-equivalent to a trivial unfolding of a cross-cap by Proposition 4.1.1. Having both of them equivalent to a trivial unfolding of a cross-cap contradicts the finite determinacy of f since it will imply that $\bar{f}$ forms a quadruple point away from the origin (see the proof of Theorem 4.4.1). So we only need to study two cases: both $f^{(1)}$ and $f^{(2)}$ are immersions or $f^{(1)}$ is an immersion and $f^{(2)}$ is equivalent to a cross-cap.

Notice that

$$(f^*\mathrm{Fitt}_k(f_*\mathcal{O}_{\mathbb{C}^3,0}))_{\mathfrak{m}_{\mathbb{C}^3,\mathbf{x}_i}} = f^*(\mathrm{Fitt}_k(f_*\mathcal{O}_{\mathbb{C}^3,0})_{\mathfrak{m}_{\mathbb{C}^4,\mathbf{y}}}) = f^{(i)*}\mathrm{Fitt}_k(f_*\mathcal{O}_{\mathbb{C}^3,S})$$

for $i = 1, 2$.

On the other hand, for a bi-germ $\bar{f}$ we have

$$D^2(\bar{f}) = D^2(f^{(1)}) \cup D^2(f^{(2)}) \cup D^2(f^{(1)}, f^{(2)})$$

where $D^2(f^{(1)}, f^{(2)}) = (f^{(1)} \times f^{(2)})^{-1}(\Delta_{2,4}) \cup (f^{(2)} \times f^{(1)})^{-1}(\Delta_{2,4})$. Hence there are two projection maps that we need to consider:

$$\begin{array}{rcl} \pi_1\colon D_1 := D^2(f^{(1)}) \cup D^2(f^{(1)}, f^{(2)}) & \to & (\mathbb{C}^3, x_1) \\ \pi_2\colon D_2 := D^2(f^{(3)}) \cup D^2(f^{(1)}, f^{(2)}) & \to & (\mathbb{C}^3, x_2). \end{array}$$

Therefore $\mathrm{Fitt}_{k-1}((\pi_1^2)_*\mathcal{O}_{D^2(f)})_{\mathfrak{m}_{\mathbb{C}^3,\mathbf{x}_i}} = \mathrm{Fitt}_{k-1}((\pi_i)_*\mathcal{O}_{D_i})$.

Recall that $D^2(f)$ is a Cohen-Macaulay space of dimension 2 by Proposition 2.1.11, and π_1^2 is finite. In the following we will use Algorithm 2.2.1 and Corollary 2.2.2 to find a presentation of $\mathcal{O}_{\mathbb{C}^3,S}$ and $\mathcal{O}_{D^2(\bar{f})}$ over $\mathcal{O}_{\mathbb{C}^4,y}$ and $\mathcal{O}_{\mathbb{C}^3,S}$, respectively.

Case 1. We consider the first case, that is, both $f^{(1)}$ and $f^{(2)}$ are immersions. Let us assume that $f^{(1)}\colon (x, y, z) \mapsto (x, y, z, 0)$. Clearly, the iso-singular locus A_1 of $f^{(1)}$ is the whole domain $(\mathbb{C}^3, x_1)$. By [Mat69b, Proposition 1.6], $f^{(1)}|_{A_1}$ is transverse to $f^{(2)}|_{A_2}$. This is equivalent to saying that $p \circ f^{(2)}$ is a submersion where $p\colon \mathbb{C}^4 \to \mathbb{C}$ is the projection $(Y_1, \ldots, Y_4) \mapsto Y_4$. So $f^{(2)}$ is of the form

$$f^{(2)}\colon (t, u, v) \mapsto (a(t, u, v), b(t, u, v), c(t, u, v), v + d(t, u, v))$$

for some $a, b, c, d \in \mathcal{O}_{\mathbb{C}^3,x_2}$. Since $f^{(2)}$ has corank 0, without the loss of generality we can assume that $a = t + a'(t, u, v)$ and $b = u + b'(t, u, v)$ for some $a', b', c', d' \in \mathcal{O}_{\mathbb{C}^3,x_2}$.

So

$$f^{(2)}\colon (t,u,v) \mapsto (t+a'(t,u,v), u+b'(t,u,v), c(t,u,v), v+d(t,u,v)).$$

The following coordinate change

$$\begin{array}{rcl} t & \mapsto & t+a'(t,u,v) \\ u & \mapsto & u+b'(t,u,v) \\ v & \mapsto & v+d(t,u,v) \end{array}$$

followed by

$$Y_3 \mapsto Y_3 - c(Y_1,Y_2,Y_4)$$

yields

$$f^{(2)} \sim_{\mathcal{A}} (t,u,v) \mapsto (t,u,0,v).$$

The latter also effects $f^{(1)}$: We have

$$f^{(1)} \sim_{\mathcal{A}} (x,y,z) \mapsto (x-c(y,z,0),y,z,0).$$

We apply $x \mapsto x - c(y,z,0)$ on $(\mathbb{C}^3, x_1)$ to get $f^{(1)} \sim_{\mathcal{A}} (x,y,z) \mapsto (x,y,z,0)$ again. Therefore we can assume $\bar{f}$ has the form

$$\bar{f}\colon \begin{cases} f^{(1)}\colon (x,y,z) \mapsto (x,y,z,0) \\ f^{(2)}\colon (t,u,v) \mapsto (t,u,0,v) \end{cases} .$$

We find $Q(f^{(i)}) = \mathbb{C}$ for $i=1,2$. So the size of the presentation matrix is 1 in each case. Clearly $\tilde{f}^{(1)}\colon (x,y,z) \mapsto (x,y,z)$ is finite. We choose $f_4 = 0$ which gives us $\alpha_{11} = 0$. Hence $\lambda^1 = \lambda^1_{11} = -Y_4$. Similarly we find $\lambda^2 = -Y_3$. Therefore a presentation for $\bar{f}$ is given by

$$0 \longrightarrow \mathcal{O}^2_{\mathbb{C}^4,\mathbf{y}} \xrightarrow[\begin{bmatrix} -Y_4 & 0 \\ 0 & -Y_3 \end{bmatrix}]{} \mathcal{O}^2_{\mathbb{C}^4,\mathbf{y}} \longrightarrow \mathcal{O}_{\mathbb{C}^3,S} \longrightarrow 0 .$$

From this exact sequence we find $\mathrm{Fitt}_1(\bar{f}_*\mathcal{O}_{\mathbb{C}^3,S}) = (Y_3,Y_4)$. Therefore

$$f^{(1)*}\mathrm{Fitt}_1(\bar{f}_*\mathcal{O}_{\mathbb{C}^3,S}) = (z)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1},$$
$$f^{(2)*}\mathrm{Fitt}_1(\bar{f}_*\mathcal{O}_{\mathbb{C}^3,S}) = (v)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_2}.$$

On the other hand, $D^2(\bar{f}) = D^2(f^{(1)}, f^{(2)})$ since $D^2(f^{(i)}) = \emptyset$ for $i = 1, 2$. We find $D^2(\bar{f}) = V(x - t, y - u, z, v)$. So the projection

$$\begin{array}{rcl} \pi_1 \colon D^2(\bar{f}) & \to & (\mathbb{C}^3, \mathbf{x}_1) \\ (x, y, z, t, u, v) & \mapsto & (x, y, z) \\ \pi_2 \colon D^2(\bar{f}) & \to & (\mathbb{C}^3, \mathbf{x}_2) \\ (x, y, z, t, u, v) & \mapsto & (t, u, v) \end{array}$$

are mono-germs. We choose $\tilde{\pi}_1 \colon (x, y, z, t, u, v) \mapsto (x, y)$ and find

$$Q(\tilde{\pi}_1) = \frac{\mathcal{O}_{D^2(\bar{f})}}{(x, y)} \cong \frac{\mathcal{O}_{\mathbb{C}^6}}{(x - t, y - u, z, v,) + (x, y)} \cong \mathbb{C}.$$

Since $z = 0$ on $D^2(\bar{f})$ we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^3, \mathbf{x}_1} \xrightarrow{\ z\ } \mathcal{O}_{\mathbb{C}^3, \mathbf{x}_1} \longrightarrow (\pi_1)_* \mathcal{O}_{\mathcal{O}_{D^2(\bar{f})}} \longrightarrow 0\ .$$

For π_2, we choose $\tilde{\pi}_2 \colon (x, y, z, t, u, v) \mapsto (t, u)$ so that $Q(\tilde{\pi}_2) \cong \mathbb{C}$. We also have $v = 0$ on $D^2(\bar{f})$ so

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^3, \mathbf{x}_2} \xrightarrow{\ v\ } \mathcal{O}_{\mathbb{C}^3, \mathbf{x}_2} \longrightarrow (\pi_2)_* \mathcal{O}_{\mathcal{O}_{D^2(\bar{f})}} \longrightarrow 0\ .$$

Clearly, $\mathrm{Fitt}_0((\pi_1)_* \mathcal{O}_{D^2(\bar{f})}) = (z)\mathcal{O}_{\mathbb{C}^3, \mathbf{x}_1}$ and $\mathrm{Fitt}_0((\pi_2)_* \mathcal{O}_{D^2(\bar{f})}) = (v)\mathcal{O}_{\mathbb{C}^3, \mathbf{x}_2}$. So the equality of the ideals follows.

Case 2. Let $f^{(1)}$ be an immersion and $f^{(2)}$ equivalent to a cross-cap. Let us assume that after suitable coordinate changes $f^{(2)}$ has the form

$$\begin{array}{rcl} f^{(2)} \colon (\mathbb{C}^3, x_2) & \to & (\mathbb{C}^4, y) \\ (t, u, v) & \mapsto & (t, u, v^2, uv). \end{array}$$

Then any map-germ given by $(t', u, v) \mapsto (t', u, v^2, uv)$ is $\mathcal{A}$-equivalent to $f^{(2)}$. Hence the iso-singular locus A_2 of $f^{(2)}$ is $\mathbb{C} \times \{0\} \times \{0\}$. Again, by the transversality condition, the composition $p \circ f^{(1)}$ must be a submersion where $p \colon \mathbb{C}^4 \to \mathbb{C}^3$ is the projection $(Y_1, \ldots, Y_4) \mapsto (Y_2, Y_3, Y_4)$. It follows that $f^{(1)}$ is of the form $f^{(1)} \colon (x, y, z) \mapsto (A(x, y, z), x + B(x, y, z), y + C(x, y, z), z + D(x, y, z))$ for some

$A, B, C, D \in \mathcal{O}_{\mathbb{C}^3,x_2}$. Moreover, the following coordinates changes

$$\begin{array}{rcl} x & \mapsto & x + B(x,y,z) \\ y & \mapsto & y + C(x,y,z) \\ z & \mapsto & z + D(x,y,z) \end{array}$$

followed by

$$Y_1 \mapsto Y_1 - A(Y_2, Y_3, Y_4)$$

yield

$$f^{(1)} \sim_{\mathcal{A}} (x,y,z) \mapsto (0,x,y,z)$$

and

$$f^{(2)} \sim_{\mathcal{A}} (t,u,v) \mapsto (t - A(u,v^2,uv), u, v^2, uv).$$

We apply $t \mapsto t - A(u,v^2,uv)$ on $(\mathbb{C}^3, \mathbf{x}_2)$ to get $f^{(2)} \sim_{\mathcal{A}} (t,u,v) \mapsto (t,u,v^2,uv)$. Therefore we can assume that $\bar{f}$ is given by

$$\bar{f} \colon \begin{cases} f^{(1)} \colon (x,y,z) \mapsto (0,x,y,z) \\ f^{(2)} \colon (t,u,v) \mapsto (t,u,v^2,uv) \end{cases} .$$

Let us calculate a presentation of $\bar{f}_* \mathcal{O}_{\mathbb{C}^3,S}$. We have $Q(f^{(1)}) = \frac{\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1}}{(x,y,z)} \cong \mathbb{C}$ and $Q(f^{(2)}) = \frac{\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_2}}{(t,u,v^2,uv)} = \mathbb{C} \cdot \{1, v\}$. It follows that $\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1} = \mathcal{O}_{\mathbb{C}^4,\mathbf{y}} \cdot \{1\}$, and $\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1}$, as an $\mathcal{O}_{\mathbb{C}^4,\mathbf{y}}$-module, is isomorphic to $\mathcal{O}_{\mathbb{C}^4,\mathbf{y}}/(Y_1)\mathcal{O}_{\mathbb{C}^4,\mathbf{y}}$. So we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^4,\mathbf{y}} \xrightarrow{Y_1} \mathcal{O}_{\mathbb{C}^4,\mathbf{y}} \longrightarrow \mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1} \longrightarrow 0 .$$

For $f^{(2)}$ we follow the algorithm to find a presentation matrix. We choose

$$\begin{array}{rcl} \nu \colon \mathbb{C}^4 & \to & \mathbb{C}^3 \\ (Y_1, \dots, Y_4) & \mapsto & (Y_1, Y_2, Y_3) \end{array}$$

so that $\tilde{f}^{(2)} = \nu \circ f^{(2)} = (t,u,v^2)$. Since $Q(\tilde{f}^{(2)}) = \mathbb{C} \cdot \{1, v\}$ we need to find functions $\alpha_{ij} \in \mathcal{O}_{\mathbb{C}^4,y}$, $1 \leq i,j \leq 2$, satisfying

$$uv \cdot 1 = \alpha_{11}(t,u,v^2) + \alpha_{21}(t,u,v^2)v$$

$$uv \cdot v = \alpha_{12}(t,u,v^2) + \alpha_{22}(t,u,v^2)v.$$

A solution is given by

$$\alpha_{11} = 0, \alpha_{21} = Y_2,$$

$$\alpha_{12} = Y_2 Y_3, \alpha_{22} = 0.$$

Therefore

$$0 \longrightarrow \mathcal{O}^2_{\mathbb{C}^4,\mathbf{y}} \xrightarrow{\begin{bmatrix} -Y_4 & Y_2Y_3 \\ Y_2 & -Y_4 \end{bmatrix}} \mathcal{O}^2_{\mathbb{C}^4,\mathbf{y}} \xrightarrow{\begin{bmatrix} 1 & v \end{bmatrix}} \mathcal{O}_{\mathbb{C}^3,\mathbf{x}_2} \longrightarrow 0\ .$$

Thus,

$$0 \longrightarrow \mathcal{O}^3_{\mathbb{C}^4,\mathbf{y}} \xrightarrow{\begin{bmatrix} Y_1 & 0 & 0 \\ 0 & -Y_4 & Y_2Y_3 \\ 0 & Y_2 & -Y_4 \end{bmatrix}} \mathcal{O}^3_{\mathbb{C}^4,\mathbf{y}} \longrightarrow \mathcal{O}_{\mathbb{C}^3,S} \longrightarrow 0\ .$$

We find $\mathrm{Fitt}_1(\bar{f}_*\mathcal{O}_{\mathbb{C}^3,S}) = (Y_1Y_4, Y_1Y_2, Y_4^2 - Y_2^2Y_3)$ and $\mathrm{Fitt}_2(\bar{f}_*\mathcal{O}_{\mathbb{C}^3,S}) = (Y_1, Y_2, Y_4)$ which yield

$$f^{(1)*}\mathrm{Fitt}_1(\bar{f}_*\mathcal{O}_{\mathbb{C}^3,S}) = (z^2 - x^2y)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1}, \quad f^{(2)*}\mathrm{Fitt}_1(\bar{f}_*\mathcal{O}_{\mathbb{C}^3,S}) = (tu)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_2},$$

and

$$f^{(1)*}\mathrm{Fitt}_2(\bar{f}_*\mathcal{O}_{\mathbb{C}^3,S}) = (x,z)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1}, \qquad f^{(2)*}\mathrm{Fitt}_2(\bar{f}_*\mathcal{O}_{\mathbb{C}^3,S}) = (t,u)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_2}.$$

On the other hand, we have $D^2(\bar{f}) = D^2(f^{(2)}) \cup D^2(f^{(1)}, f^{(2)})$ since $D^2(f^{(1)}) = \emptyset$. We find $D_1 := D^2(f^{(1)}, f^{(2)}) = V(t, x-u, y-v^2, z-uv)$. For the ideal $\mathcal{I}_2$ defining $D^2(f^{(2)})$, we consider the coordinate system $((x,y,z),(t,u,v))$ on $\mathbb{C}^3 \times \mathbb{C}^3$ and use the definition (2.5) of $\mathcal{I}_2(f)$ to find $D^2(f^{(2)}) = V(x-t, y-u, v+z, u)$. Now we construct a presentation of $\mathcal{O}_{D_1}$ over $\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1}$ via

$$\begin{aligned} \pi_1 \colon D_1 &\to (\mathbb{C}^3, \mathbf{x}_1) \\ (x,y,z,t,u,v) &\mapsto (x,y,z) \end{aligned}$$

We choose $\tilde{\pi}_1\colon (x,y,z,t,u,v) \mapsto (x,y)$ which has

$$Q(\tilde{\pi}_1) = \frac{\mathcal{O}_{D_1}}{(x,y)} \cong \frac{\mathcal{O}_{\mathbb{C}^6}}{(t, x-u, y-v^2, z-uv)+(x,y)} \cong \mathbb{C}\cdot\{1,v\}\,.$$

We need functions $\alpha_{ij} \in \mathbb{C}\{x,y\}$ satisfying

$$z = \alpha_{11}(x,y) + \alpha_{21}(x,y)v,$$

$$zv = \alpha_{12}(x,y) + \alpha_{22}(x,y)v.$$

On D_1 we have $z = uv$, $x = u$ and $y = v^2$, so $z = xv, zv = xy$, i.e. $\alpha_{11} = 0, \alpha_{21} = x, \alpha_{12} = xy, \alpha_{22} = 0$. Therefore,

$$0 \longrightarrow \mathcal{O}^2_{\mathbb{C}^3,\mathbf{x}_1} \xrightarrow[\begin{bmatrix} -z & xy \\ x & -z \end{bmatrix}]{} \mathcal{O}^2_{\mathbb{C}^3,\mathbf{x}_1} \xrightarrow[\begin{bmatrix} 1 & v \end{bmatrix}]{} (\pi_1)_*\mathcal{O}_{D_1} \longrightarrow 0\,. \tag{4.2}$$

Let us calculate a presentation matrix for $(\pi_2)_*\mathcal{O}_{D^2(\bar{f})}$. This time, π_2 is a bi-germ with branches

$$\begin{aligned} \pi_2^{(1)}\colon D^2(f^{(2)}) &\to (\mathbb{C}^3, \mathbf{x}_2) \\ (x,y,z,t,u,v) &\mapsto (t,u,v) \\ \pi_2^{(2)}\colon D_1 &\to (\mathbb{C}^3, \mathbf{x}_2) \\ (x,y,z,t,u,v) &\mapsto (t,u,v) \end{aligned}$$

We choose $\tilde{\pi}_1^{(1)} = (u,v)$ so that $Q(\tilde{\pi}_1^{(1)}) \cong \mathbb{C}$, and $\tilde{\pi}_1^{(2)} = (t,v)$ so that $Q(\tilde{\pi}_1^{(2)}) \cong \mathbb{C}$. Hence, both branches are immersions. We find

$$0 \longrightarrow \mathcal{O}^2_{\mathbb{C}^3,\mathbf{x}_2} \xrightarrow[\begin{bmatrix} t & 0 \\ 0 & u \end{bmatrix}]{} \mathcal{O}^2_{\mathbb{C}^3,\mathbf{x}_2} \longrightarrow (\pi_2)_*\mathcal{O}_{D^2(\bar{f})} \longrightarrow 0\,. \tag{4.3}$$

From (4.2) and (4.3) we obtain

$$\begin{aligned} &\mathrm{Fitt}_0((\pi_1)_*\mathcal{O}_{D^2(\bar{f})}) = (z^2 - x^2y)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1}, && \mathrm{Fitt}_0((\pi_2)_*\mathcal{O}_{D^2(\bar{f})}) = (tu)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_2}, \\ &\mathrm{Fitt}_1((\pi_1)_*\mathcal{O}_{D^2(\bar{f})}) = (x,z)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_1}, && \mathrm{Fitt}_1((\pi_2)_*\mathcal{O}_{D^2(\bar{f})}) = (t,u)\mathcal{O}_{\mathbb{C}^3,\mathbf{x}_2}. \end{aligned}$$

This ends the proof of $(f^*\mathrm{Fitt}_k(f_*\mathcal{O}_{\mathbb{C}^3,0}))_{\mathfrak{m}_{\mathbb{C}^3,\mathbf{x}}} = \mathrm{Fitt}_{k-1}((\pi_1^2)_*\mathcal{O}_{D^2(f)})_{\mathfrak{m}_{\mathbb{C}^3,\mathbf{x}}}$ for $0 \neq \mathbf{x} \in (\mathbb{C}^3, 0)$.

By Proposition 3.5 of [MP89], given that f is finite, and of degree 1 onto its image, e.g. if f is finitely $\mathcal{A}$-determined, $f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^3,0})$ is a principal ideal. Moreover it defines the double locus $D_1^2(f)$ of f in the source, and it is a reduced hypersurface in the case that f is finitely $\mathcal{A}$-determined (cf. Theorem 4.4.1). Hence $\mathcal{O}_{\mathbb{C}^3,0}/f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^3,0})$ is a Cohen-Macaulay module of dimension 2. On the other hand, $\mathrm{Fitt}_0((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ is a principal ideal as it is the determinant of a certain square matrix. So $\mathcal{O}_{\mathbb{C}^3,0}/\mathrm{Fitt}_0((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ is also a Cohen-Macaulay module of dimension 2. As $\mathrm{Fitt}_0((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ agrees with $f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^3,0})$ at every maximal ideal away from the origin, the quotient is reduced at those points. In fact it is also reduced at the origin by the Cohen-Macaulay property. Therefore the equality of the ideals $\mathrm{Fitt}_0((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ and $f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^3,0})$ extends to the origin. $\square$

Remark 4.2.2. Notice that π_1^2 is also finite and $D^2(f)$ is a Cohen-Macaulay space by Proposition 2.1.11. Also, π_1^2 is generically one-to-one as its the double point locus on its target space coincides with $D_1^3(f)$ (see Section 3.3). Hence we can apply Theorem 3.4 of [MP89] to conclude that $\mathrm{Fitt}_1((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ is the ideal generated by all $(s-1)\times(s-1)$-minors of an $(s-1)\times s$-matrix, and that it is a determinantal ideal. So $\mathcal{O}_{\mathbb{C}^3,0}/\mathrm{Fitt}_1((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ is a Cohen-Macaulay module of dimension 1. From the calculations above we have seen that $\mathcal{O}_{\mathbb{C}^3,0}/\mathrm{Fitt}_1((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ is reduced away from the origin. Hence it is reduced everywhere. It follows that $f^*\mathrm{Fitt}_2(f_*\mathcal{O}_{\mathbb{C}^3,0}) \subseteq \mathrm{Fitt}_1((\pi_1^2)_*\mathcal{O}_{D^2(f)})$.

Examples suggest that the equality of $f^*\mathrm{Fitt}_2(f_*\mathcal{O}_{\mathbb{C}^3,0})$ and $\mathrm{Fitt}_1((\pi_1^2)_*\mathcal{O}_{D^2(f)})$ holds at the origin as well. We leave this as an open problem.

Conjecture 4.2.3. Let $f \in \mathcal{E}_{3,4}^0$ be a finitely $\mathcal{A}$-determined map-germ. Then $f^*\mathrm{Fitt}_2(f_*\mathcal{O}_{\mathbb{C}^3,0}) = \mathrm{Fitt}_1((\pi_1^2)_*\mathcal{O}_{D^2(f)})$.

4.3 D^2 is a normalisation of D_1^2

In this dimensions, we observe another property of the double point spaces.

Proposition 4.3.1. *Let $f \in \mathcal{E}_{3,4}^0$ be a finitely $\mathcal{A}$-determined map-germ of corank ≥ 2. Then $D^2(f)$ is a normal surface and $\pi_1^2\colon D^2(f) \to D_1^2(f)$ is a normalisation of $D_1^2(f)$.*

Proof. We have already shown that $D^2(f)$ is a Cohen-Macaulay space (see Proposition 2.1.11). As $D^2(f)$ is a surface with at most an isolated singularity by Proposition 2.1.13, it is normal by Serre's criterion on normal rings (see [Ser00, Theorem IV.D.11]). On the other hand, the map $\pi_1^2\colon D^2(f) \to D_1^2(f)$ is finite and onto whence a normalisation. □

4.4 Geometric criteria for finite determinacy

By the results of Proposition 4.2.1, it is more convenient to make the following assumptions. We let $D_1^2(f) = V(f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^3,0}))$, $D_1^3(f) = V(\mathrm{Fitt}_1((\pi_1^2)_*\mathcal{O}_{D^2(f)}))$. Moreover, we assume that $D_1^4(f) = V(f^*\mathrm{Fitt}_3(f_*\mathcal{O}_{\mathbb{C}^3,0}))$.

Theorem 4.4.1. *Let $f \in \mathcal{E}_{3,4}^0$ be a $\mathcal{K}$-finite analytic map-germ. Then f is finitely $\mathcal{A}$-determined if and only if*

(i) $D^2(f)$ *has at most an isolated singularity at the origin,*

(ii) $D_1^k(f)$ *is of dimension* $4-k$ *or empty for* $1 \le k \le 4$,

(iii) $(\mathrm{Jac}(D_1^2(f)) : I_{D_1^3(f)}) \supseteq \mathfrak{m}_{\mathbb{C}^3,0}^{l_1}$ *for some* $l_1 < \infty$, *and* $D_1^3(f)$ *has at most an isolated singularity at the origin,*

(iv) $R_f + \mathrm{Jac}(D_1^2(f)) \supseteq \mathfrak{m}_{\mathbb{C}^3,0}^{l_2}$ *for some* $l_2 < \infty$.

Proof. The proof is mainly based on Gaffney and Mather's geometric criterion on finitely $\mathcal{A}$-determined map-germs (see Theorem 1.3.20).

Let us assume that f is finitely $\mathcal{A}$-determined. We have already observed in Proposition 2.1.13 that (i) holds.

(ii) - (iii) By Proposition 3.5 of [MP89], given that f is finite and of degree 1 onto its image, $f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^3,0})$ is a principal ideal. We get $D_1^2(f) = \mathbb{C}^n$ only when f fails to be degree 1 onto its image. In that case f is nowhere stable so it cannot be finitely $\mathcal{A}$-determined. Hence we must have dim $D_1^2(f) = 2$.

Let us consider the kth multiple point spaces on the target, namely $M_k(f) = f(D_1^k(f))$. By set-up,

$$\mathrm{Sing}(M_k(f)) = f(\mathrm{Sing}(D_1^k(f))) \cup M_{k+1}(f)$$

for $k \ge 1$. A similar equality applies to any finite map. Hence,

$$\mathrm{Sing}(D_1^2(f)) = \pi_1^2(\mathrm{Sing}(D^2(f))) \cup \pi_1^2(D_1^2(\pi_1^2))$$

for $D_1^2(f)$ is the image of $\pi_1^2 := \pi_1^2(f)$. By Proposition 2.4.1, $D^2(\pi_1^2) \cong D^3(f)$. Therefore $\pi_1^2(D_1^2(\pi_1^2)) \cong D_1^3(f)$. So

$$\mathrm{Sing}(D_1^2(f)) = \pi_1^2(\mathrm{Sing}(D^2(f))) \cup D_1^3(f) = \{0\} \cup D_1^3(f). \tag{4.4}$$

By Proposition 4.2.1, $D_1^3(f)$ is 1-dimensional. It follows from (4.4) that $D_1^2(f)$ is a reduced hypersurface and that Sing $D_1^2(f)$ and $D_1^3(f)$ agree outside the origin.

As for the singular set of $D_1^3(f)$, if $0 \neq \mathbf{x} \in \mathrm{Sing}(D_1^3(f))$ then it has multiplicity ≥ 2 as a point in $D_1^3(f)$. In that case one can find a small perturbation f_t of f for which x splits into two points $\mathbf{x}_1, \mathbf{x}_2 \in D_1^3(f_t)$. However, this also results a change in the topology of the image of f. So f cannot be stable. Therefore, we must have $\mathrm{Sing}(D_1^3(f)) = \{0\}$ or $\emptyset$.

We claim that $D_1^4(f) = \{0\}$ or empty. If f, as a mono-germ, forms a quadruple point away from the origin then there is a line of quadruple points in the image, since f is holomorphic. But a line of quadruple points in these dimensions is not stable. Hence we must have $D_1^4(f) = \{0\}$ or empty.

(v) Let us assume that there exists a point $0 \neq \mathbf{x} \in \mathrm{Sing}(D_1^2(f)) \cap V(R_f)$. By Proposition 2.1.16, $V(R_f) = \pi_1^2(D^2(f) \cap \Delta_{\mathbb{C}^3 \times \mathbb{C}^3})$. So we have $(\mathbf{x}, \mathbf{x}) \in (\pi_1^2)^{-1}(x)$. This implies that x is a point of multiplicity ≥ 2 on $\mathrm{Sing}(D_1^2(f)) = D_1^3(f)$. In this case f defines an unstable triple point at $f(\mathbf{x})$. Hence no such $\mathbf{x}$ exists.

This concludes the first part of the proof. Let us now assume that the conditions (i)-(iv) hold.

We assume $0 \neq y \in \mathbb{C}^4$ and consider the multi-germ $\bar{f} \colon (\mathbb{C}^3, S) \to (\mathbb{C}^4, \mathbf{y})$ where $S := f^{-1}(\mathbf{y})$. Note that in these dimensions $D^5 = \emptyset$. So there are only the following cases:

(c_1) $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$; multiplicity$(\mathbf{x}_i) = 1$, $i = 1, \ldots, 4$,

(c_2) $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$; multiplicity$(\mathbf{x}_i) \geq 2$ for some i,

(c_3) $S = \{\mathbf{x}_1, \mathbf{x}_2\}$; multiplicity$(\mathbf{x}_i) \geq 2$, $i = 1, 2$,

(c_4) $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$; multiplicity$(\mathbf{x}_i) = 1$, $i = 1, 2, 3$,

(c_5) $S = \{\mathbf{x}_1, \mathbf{x}_2\}$; multiplicity$(\mathbf{x}_i) \geq 2$ for some i,

(c_6) $S = \{\mathbf{x}\}$; multiplicity$(x) \geq 3$,

(c_7) $S = \{\mathbf{x}_1, \mathbf{x}_2\}$; multiplicity$(\mathbf{x}_i) = 1$, $i = 1, 2$,

(c_8) $S = \{\mathbf{x}\}$; multiplicity$(x) \geq 2$,

(c_9) $S = \{\mathbf{x}\}$; multiplicity$(x) = 1$.

Let us study $\bar{f}$ in each case:
Case (c_1). We get a point $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \in D^4(f)$. So $\mathbf{x}_i$ must be 0 for all $i = 1, 2, 3, 4$ by (iv).
Case (c_2). Let us assume that $\mathbf{x}_3 = \mathbf{x}_4$. Then $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in D^3(f)$ as well as $(\mathbf{x}_3, \mathbf{x}_3) \in D^2(f)$ and consequently $\mathbf{x}_3 \in V(R_f)$. So we must have $\mathbf{x}_3 = 0$ by (v). As $f^{-1}(0) = \{0\}$, we have $\mathbf{x}_i = 0$ for $i = 1, 2, 3, 4$.
Case (c_3). We have $(\mathbf{x}_1, \mathbf{x}_1), (\mathbf{x}_2, \mathbf{x}_2) \in D^2(f)$ and $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2) \in D^3(f)$, equivalently $\mathbf{x}_1, \mathbf{x}_2 \in D^3_1(f)$ and $\mathbf{x}_1, \mathbf{x}_2 \in \text{Sing } D^2_1(f)$ by (iii). But $\mathbf{x}_1, \mathbf{x}_2$ also belong to $V(R_f)$. So $\mathbf{x}_1, \mathbf{x}_2$ must be 0 by (v).
Case (c_4). We have $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in D^3_1(f)$ (equivalently $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \text{Sing } D^2_1(f)$). At each of these points, $\bar{f}$ is an immersion by (v). Therefore, $(\mathbb{C}^3, \mathbf{x}_1)$, $(\mathbb{C}^3, \mathbf{x}_2)$ and $(\mathbb{C}^3, \mathbf{x}_3)$ are mapped immersively into $\mathbb{C}^4$ to form a line of triple points. In other words, the image of f at y is locally of the form $\{XYZ = 0\} \times \mathbb{C}$. Hence $\bar{f}$ is $\mathcal{A}$-stable.
Case (c_5). In this case, $(\mathbf{x}_1, \mathbf{x}_1, \mathbf{x}_3) \in D^3(f, (2, 1))$, equivalently, $\mathbf{x}_1 \in V(R_f) \cap D^3_1(f)$. So $\mathbf{x}_1$ must be 0. We also get $\mathbf{x}_2 = 0$ since $f^{-1}(0) = \{0\}$.
Case (c_6). Now, $(\mathbf{x}_1, \mathbf{x}_1, \mathbf{x}_1) \in D^3(f) \cap \Delta_{2,3}$. But $D^3(f) \cap \Delta_{2,3}$ is the origin. So $\mathbf{x}_1$ must be 0.
Case (c_7). We have $\mathbf{x}_1, \mathbf{x}_2 \in D^2_1(f) \setminus \text{Sing}(D^2_1(f))$. At each of these points $\bar{f}$ is an immersion, again by (v). Therefore, $(\mathbb{C}^3, \mathbf{x}_1)$ and $(\mathbb{C}^3, \mathbf{x}_2)$ are mapped immersively into $\mathbb{C}^4$ to define a surface which is locally of the form $\{XY = 0\} \times \mathbb{C}^2$. Hence $\bar{f}$ is $\mathcal{A}$-stable.
Case (c_8). This is equivalent to saying that $(\mathbf{x}_1, \mathbf{x}_2) \in D^2(f) \cap \Delta_1$. But $(\mathbf{x}_1, \mathbf{x}_1) \in D^2(f)$ if and only if $\mathbf{x}_1 \in V(R_f)$ by Lemma 2.1.16. If $\mathbf{x}_1 \in \text{Sing } D^2_1(f)$ then $\mathbf{x}_1$ must be 0. If not, f is equivalent to a trivial unfolding of a cross-cap whence stable.
Case (c_9). f is one-to-one, so it is stable.

This concludes the proof. $\square$

Corollary 4.4.2. *Stable map-germs from $\mathbb{C}^3$ to $\mathbb{C}^4$ are*

1. *An immersion,*
2. *A line of triple points,*
3. *Normal crossing of two hyperplanes in $\mathbb{C}^4$,*
4. *An immersion $\cup$ a cross-cap,*
5. *A trivial unfolding of a cross-cap.*

4.5 Examples and non-examples

We are interested in finding new examples of finitely $\mathcal{A}$-determined and corank 2 map germs in $\mathcal{E}^0_{3,4}$. We note that our aim is not to obtain a full classification of corank 2 map-germs in these dimensions. Although, we have found it reasonable to classify the 2-jets for our purposes. See Table C.1 in Appendix C for a list of 2-jets with their $\mathcal{A}^2$-codimension.

We have observed the following cases where no examples exist.

Proposition 4.5.1. *There are no finitely $\mathcal{A}$-determined homogeneous map-germs of corank 2 in $\mathcal{E}^0_{3,4}$ with degrees $(1,2,2,2)$.*

Proof. By classifications methods, we can assume that a finite homogeneous map-germ with degrees $(1,2,2,2)$ has the form

$$f\colon (x,y,z) \mapsto (x, y^2 + axz, bxy + cxz + dyz, z^2 + exy)$$

for some $a,b,c,d,e \in \mathbb{C}$ (see Appendix C.3 and equation (C.17)). We will prove the claim by showing that f forms a non-transversal intersection in its image.

We have

$$\mathrm{d}f = \begin{bmatrix} 1 & 0 & 0 \\ az & 2y & ax \\ by + cz & dz + bx & dy + cx \\ ey & ex & 2z \end{bmatrix} : \mathbb{C}^3 \times T\mathbb{C}^3 \to T\mathbb{C}^4.$$

At a point of the form $(0, y, \alpha y)$, $\alpha \in \mathbb{C} \setminus \{0\}$, $\mathrm{d}f$ induces

$$\mathrm{d}_{(0,y,\alpha y)} f = \begin{bmatrix} 1 & 0 & 0 \\ a\alpha y & 2y & 0 \\ (b + c\alpha)y & d\alpha y & dy \\ ey & 0 & 2\alpha y \end{bmatrix} : T_{(0,y,\alpha y)}\mathbb{C}^3 \to T_{f(0,y,\alpha y)}\mathbb{C}^4.$$

If we apply the linear transformation

$$\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} - \frac{1}{2}a\alpha\frac{\partial}{\partial y} + (b + c\alpha - \frac{1}{2}ad\alpha^2)\frac{\partial}{\partial z}$$

on $T_{(0,y,\alpha y)}\mathbb{C}^3$ then $\mathrm{d}_{(0,y,\alpha y)} f$ takes the form

$$\mathrm{d}_{(0,y,\alpha y)} f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2y & 0 \\ 0 & d\alpha y & dy \\ (e - 2b\alpha - 2c\alpha^2 + ad\alpha^3)y & 0 & 2\alpha y \end{bmatrix}.$$

Similarly, at a point $(0, -y, -\alpha y)$ we obtain

$$\mathrm{d}_{(0,-y,-\alpha y)} f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2y & 0 \\ 0 & -d\alpha y & -dy \\ -(e - 2b\alpha - 2c\alpha^2 + ad\alpha^3)y & 0 & -2\alpha y \end{bmatrix}.$$

Let $\alpha_1 \in \mathbb{C}$ be a solution of the equation $e - 2b\alpha - 2c\alpha^2 + ad\alpha^3 = 0$. Then we get

$$\mathrm{d}_{(0,y,\alpha_1 y)} f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2y & 0 \\ 0 & d\alpha_1 y & dy \\ 0 & 0 & 2\alpha_1 y \end{bmatrix} \text{ and } \mathrm{d}_{(0,-y,-\alpha_1 y)} f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2y & 0 \\ 0 & -d\alpha_1 y & -dy \\ 0 & 0 & -2\alpha_1 y \end{bmatrix}.$$

Now $f(0, y, \alpha y) = f(0, -y, -\alpha y)$ and

$$\mathrm{d}_{(0,y,\alpha_1 y)} f(u, v, w) = (u, 2yv, d\alpha_1 yv + dyw, 2\alpha_1 yw) = \mathrm{d}_{(0,-y,-\alpha_1 y)} f(u, -v, -w).$$

So, $T_{(0,y,\alpha_1 y)}\mathbb{C}^3$ is mapped onto the image of $T_{(0,-y,-\alpha_1 y)}\mathbb{C}^3$ whence the claim. □

Proposition 4.5.2. *Let $P_{2k}(x, y, z)$ be a homogeneous polynomial of degree $2k$ for $k \geq 1$. Let $f \in \mathcal{E}^0_{3,4}$ be of the form $(x, y, z) \mapsto (x, y^2 + xz, z^2 + xy, P_{2k}(x, y, z))$. Then f is not finitely $\mathcal{A}$-determined.*

Proof. The statement for $k = 1$ follows from Proposition 4.5.1. For $k \geq 2$, we will prove the statement by showing that f forms a line of quadruple points in the image.

Let us consider two lines $L_+ : t \mapsto (0, t, \alpha t)$ and $L_- : t \mapsto (0, t, -\alpha t)$ in $\mathbb{C}^3$ for $\alpha \in \mathbb{C}$. We have

$$\begin{aligned} f(0, t, \alpha t) &= (0, t^2, \alpha^2 t^2, P_{2k}(0, t, \alpha t)) \quad \text{and} \\ f(0, t, -\alpha t) &= (0, t^2, \alpha^2 t^2, P_{2k}(0, t, -\alpha t)). \end{aligned}$$

Then L_+ and L_- share the same image by f if and only if $P_{2k}(0, t, \alpha t) = P_{2k}(0, t, -\alpha t)$. We have $P_{2k}(x, y, z) = \sum_{l+m+n=2k} a_{lmn} x^l y^m z^n$ for $a_{lmn} \in \mathbb{C}$, for all l, m, n. So

$$P_{2k}(0, t, \alpha t) = \sum_{m+n=2k} a_{0mn} t^m (\alpha t)^n = \sum_{m+n=2k} a_{0mn} \alpha^n t^{m+n} = \sum_{m+n=2k} a_{0mn} \alpha^n t^{2k}.$$

Similarly, $P_{2k}(0, t, -\alpha t) = \sum_{m+n=2k} a_{0mn} (-\alpha)^n t^{2k}$. It follows that

$$\begin{aligned} P_{2k}(0, t, \alpha t)) = P_{2k}(0, t, -\alpha t)) &\Leftrightarrow \sum_{m+n=2k} a_{0mn} \alpha^n = \sum_{m+n=2k} a_{0mn} (-\alpha)^n \\ &\Leftrightarrow \sum_{\substack{m+2i-1=2k \\ i \geq 1}} 2a_{0mn} \alpha^{2i-1} = 0. \end{aligned}$$

Since $\mathbb{C}$ is algebraically closed, we can find at least one solution for the equaiton $\sum_{\substack{m+2i-1=2k \\ i \geq 1}} 2a_{0mn} \alpha^{2i-1} = 0$. Let α_1 be a solution. On the other hand we have $f(0, t, \alpha t) = f(0, -t, -\alpha t)$ and $f(0, t, -\alpha t) = f(0, -t, \alpha t)$. Therefore

$$f(0, t, \alpha_1 t) = f(0, t, -\alpha_1 t) = f(0, -t, -\alpha_1 t) = f(0, -t, \alpha_1 t)$$

whence the result. □

By experiment, we have found two set of examples of finitely $\mathcal{A}$-determined weighted homogeneous map-germs of corank 2. We have calculated $\mathcal{A}_e$-codimension of the examples using SINGULAR (see Remark 3.2.17). However, at some point, it takes huge amount of time to get a result on a computer. There, we have used Theorem 4.4.1 to justify finite determinacy. In order to see the conjecture is satisfied we have used the ideas of [DM91].

Proposition 4.5.3. *Let $\tilde{A}_k \in \mathcal{E}^0_{3,4}$ be defined by*

$$\tilde{A}_k\colon (x,y,z) \mapsto (x, y^k + xz + x^{2k-2}y, yz, z^2 + y^{2k-1})$$

for $k \geq 2$. Then, it is finitely $\mathcal{A}$-determined for $k = 2, \ldots, 6$ with the following data:

	$\mathcal{A}_e$-codimension	Weights	the Mond conjecture
$\tilde{A}_2$	18	$(1,2,3)$	Yes
$\tilde{A}_3$	186	$(1,2,5)$	Yes
$\tilde{A}_4$	844	$(1,2,7)$	Yes
$\tilde{A}_5$	$< \infty$	$(1,2,9)$	?
$\tilde{A}_6$	$< \infty$	$(1,2,11)$	?

Table 4.1: The first set of examples

Proof. We use the command `Aecodim()` from the library `Ae.lib` in SINGULAR to calculate the $\mathcal{A}_e$-codimensions (see Remark 3.2.17). We apply Theorem 4.4.1 to justify finite determinacy for the last two map-germs in the list. Recall that the conjecture holds if $\mathcal{K}_H$ has a free deformation theory, that is, $N\mathcal{K}_{H,e/\mathbb{C}}G$ is a Cohen-Macaulay module of dimension 1 in the context (cf. Remark 3.2.13 and Conjecture 3.2.14). To show this, it is sufficient to check that $N\mathcal{K}_{H,e/\mathbb{C}}G$ has dimension 1. Because $N\mathcal{K}_{H,e/\mathbb{C}}G \otimes \mathcal{O}_{\mathbb{C}^4\times\mathbb{C},0}/(t)\mathcal{O}_{\mathbb{C}^4\times\mathbb{C},0} \cong N\mathcal{K}_{H,e}g$ and the right-hand side has dimension 0. So the Cohen-Macaulay property follows from [Mat89, Theorem 17.4]. For $\tilde{A}_2$,

```
LIB "matrix.lib";
ring T=0,(X,Y,Z,W,A,B,C),(wp(1,4,5,6,2,3,4));
ring S=0,(x,y,z,a,b,c),(wp(1,2,3,2,3,4));
ideal p=0;
map F=T,x,y2+xz+x2y+ay,yz+by,z2+y3+cy,a,b,c;    // A stable unfolding of Ã1
setring T;
ideal H=preimage(S,F,p);    // the ideal of the image of F
ideal jH=jacob(H);
module derH=syz(jH);    // derH = Der(-log H)
ring T0=0,(X,Y,Z,W,A),dp;
```

```
map G=T,X,Y,Z,W,A,0,0;     // a 1-parameter unfolding of
// g(X,Y,Z,W) = (X,Y,Z,W,0,0,0) which induces Ã_1 from F.
matrix jG[7][4]=1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1;  //im(jG) = tG(Θ_{C^4×C/C})
def tkhG=concat(G(derH),jG);    // tkhG = TK_{H,e/C}G
dim(std(tkhG));
//-> 1
```

confirms the conjecture. For $\tilde{A}_3$ and $\tilde{A}_4$, we only need to change the rings `T`, `S` and the unfolding `F` accordingly. Then, we get the desired output, which is 1. We have yet to check the conjecture for $\tilde{A}_5$ and $\tilde{A}_6$. □

Conjecture 4.5.4. $\tilde{A}_k\colon (x,y,z) \mapsto (x, y^k + xz + x^{2k-2}y, yz, z^2 + y^{2k-1})$ is finitely determined for all $k \geq 7$.

Proposition 4.5.5. *Let $\tilde{B}^{\pm}_{2k+1} \in \mathcal{E}^0_{3,4}$ be a map-germ defined by*

$$\tilde{B}^{\pm}_{2k+1}\colon (x,y,z) \mapsto (x, y^2 + xz, z^2 + xy, y^{2k+1} \pm y^{2k-1}z^2 + z^{2k+1})$$

for $k \geq 1$. Then, $\tilde{B}^{\pm}_3, \tilde{B}^{\pm}_5, \tilde{B}^{-}_7, \tilde{B}^{\pm}_9, \tilde{B}^{\pm}_{11}$ and $\tilde{B}^{-}_{13}$ are finitely $\mathcal{A}$-determined with the following data:

	$\mathcal{A}_e$-codimension	Weights	the Mond conjecture
$\tilde{B}^{\pm}_3$	33	$(1,1,1)$	Yes
$\tilde{B}^{\pm}_5$	252	$(1,1,1)$	Yes
$\tilde{B}^{-}_7$	837	$(1,1,1)$	Yes
$\tilde{B}^{\pm}_9$	1968	$(1,1,1)$	Yes
$\tilde{B}^{\pm}_{11}$	3825	$(1,1,1)$	Yes
$\tilde{B}^{-}_{13}$	6588	$(1,1,1)$	?

Table 4.2: The second set of examples

Proof. We use the command `Aecodim()` from the library `Ae.lib` in SINGULAR to calculate the $\mathcal{A}_e$-codimensions (see Remark 3.2.17). For $\tilde{B}^{-}_7$ and $\tilde{B}^{-}_{13}$, D^2 is not an isolated singularity. Hence those two are not finitely $\mathcal{A}$-determined by Theorem 4.4.1. We confirm that map-germs in the table satisfy the Mond conjecture on SINGULAR by checking that $\mathcal{K}_H$ has a free deformation theory (cf. Remark 3.2.13 and Conjecture 3.2.14). Computer runs out of memory for $\tilde{B}^{-}_{13}$. □

Conjecture 4.5.6. $\tilde{B}^-_{2k+1}\colon (x,y,z)\mapsto (x, y^2+xz, z^2+xy, y^{2k+1}\pm y^{2k-1}z^2+z^{2k+1})$ is finitely $\mathcal{A}$-determined for all k, and $\tilde{B}^+_{2k+1}$ is finitely $\mathcal{A}$-determined only for those k which are not divisible by 3.

4.6 A formula for the Image Milnor Number

In [Mar89], Marar gave a formula for the Euler characteristic of the image of a stable perturbation of a map-germ in $\mathcal{E}^0_{n,n+1}$ in terms of the Euler characteristics of the multiple point spaces. For $n=3$ it translates into the following formula.

Proposition 4.6.1 (cf. Example (ii), p. 35, [Mar89])**.** *Let $f\in\mathcal{E}^0_{3,4}$ be a finitely $\mathcal{A}$-determined map-germ. Let f_t be a stable perturbation of f. Then the image Milnor number of f is given by*

$$\mu_I(f)=\frac{1}{2}(\mu(D^2(f))+\mu(D^2(f,(2)))+|D^3(f_t,(2,1))|)+\frac{1}{6}(\mu(D^3(f))+\frac{1}{4}|D^4(f_t)|-1) \tag{4.5}$$

if none of the multiple point spaces is empty.

We want to reformulate this formula in a way that it will only consist of data which we are able to calculate for map-germs with corank ≥ 2.

Lemma 4.6.2. *Let f_t be a stable perturbation. Then $|D^4(f_t)|=6|D^4_1(f_t)|$.*

Proof. Consider a point $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)\in D^4(f)$. Then the points $(\mathbf{x}_1,\mathbf{x}_i,\mathbf{x}_j,\mathbf{x}_k)$, where i,j,k is a permutation of the indices $2,3,4$, also belong to $D^4(f_t)$. All of these 6 points projects down to $\mathbf{x}_1\in D^4_1(f_t)$. The relation follows. □

Lemma 4.6.3. *Let f_t be a stable perturbation. Then*

$$D^3(f_t,(2,1))\cong D^3_1(f_t)\cap V(R_{f_t}).$$

Proof. By definition,

$$D^3(f_t,(2,1))=D^3(f_t)\cap\{\mathbf{x}_1=\mathbf{x}_2\}\cong D^3(f_t,(1,2))=D^3(f_t)\cap\{\mathbf{x}_2=\mathbf{x}_3\}.$$

Clearly, it is also isomorphic to the image of the projection

$$\begin{aligned}\pi^3_2|_{D^3(f_t,(2,1))}\colon D^3(f_t,(2,1)) &\to D^2(f_t)\\ (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &\mapsto (\mathbf{x}_1,\mathbf{x}_2).\end{aligned}$$

(Equivalently, we can consider the projection $\pi^3_2|_{D^3(f_t,(1,2))}\colon D^3(f_t,(1,2)) \to D^2(f_t)$ since it defines the same image.) So, we have $D^3(f_t,(2,1)) \cong D^3_2(f_t) \cap \{\mathbf{x}_1 = \mathbf{x}_2\}$ and that is contained in $D^2(f_t,(2))$. On the other hand, $\pi^2_1\colon D^2(f_t,(2)) \to \mathbb{C}^3$ is a bijection onto its image $V(R_{f_t})$ by Proposition 2.1.16. So, the result follows. □

Lemma 4.6.4. *Let $f \in \mathcal{E}^0_{3,4}$ be finitely $\mathcal{A}$-determined map-germ, and let f_t be a stable perturbation of f. Then*

$$\mu(D^3(f)) = 2\mu(D^3_1(f)) - 8|D^4_1(f_t)| + |D^3(f_t,(2,1))| - 1.$$

Proof. First, we prove that $D^3(f)$ has at most an isolated singularity. Let us consider the projection $\pi^3_1 := \pi^2_1 \circ \pi^3_2\colon D^3(f) \to (\mathbb{C}^3, 0)$. By Proposition 2.1.7, π^2_1 is finite. By Proposition 2.4.1, $D^s(\pi^3_2) \cong D^{2+s}(f)$ for $s \geq 1$. Since f is finitely $\mathcal{A}$-determined, $D^4(f) = \{0\}$ or $\emptyset$, and $D^k(f) = \emptyset$ for $k \geq 5$ by Theorem 4.4.1. So π^3_2 is finite. It follows that π^3_1 is also finite. In that case

$$\mathrm{Sing}(D^3_1(f)) = \pi^3_1(\mathrm{Sing}(D^3(f)) \cap M_2(\pi^3_1).$$

By Proposition 2.4.1, $D^2(\pi^3_1) \cong D^5(f)$. So, $M_2(\pi^3_1)$ is empty. Hence, $\mathrm{Sing}(D^3_1(f)) = \pi^3_1(\mathrm{Sing}(D^3(f))$. Finally, by Theorem 4.4.1(iii), $D^3(f)$ has at most an isolated singularity at the origin.

At each point $\mathbf{x}$ of $D^4_1(f_t)$, $D^3_1(f_t)$ has a triple point: Since f_t is stable, f_t has an ordinary quadruple point at $f_t(\mathbf{x})$. In other words, there are four copies of $\mathbb{C}^3$ such that each one is locally based at $\mathbf{x}_i \in f^{-1}f(\mathbf{x})$ (i=1,...,4), and mapped immersively into $\mathbb{C}^4$ to form the quadruple point. The image of each of these 3-spaces intersects the other three transversally. So $M_3(f_t)$, the locus of triple points on the target, is locally just the union of the "coordinate axes" at $f_t(\mathbf{x})$. The inverse image of $M_3(f_t)$ by f_t gives us four ordinary triple point in $D^3_1(f)$.

When we deform $D^3_1(f)$ to $D^3_1(f_t)$ (this is equivalent to taking a stable perturbation f_t of f and calculating its triple point locus on the source), the first Betti number of $D^3_1(f)$ increases by 2 at each triple point of $D^3_1(f_t)$, i.e. at each point of $D^4_1(f)$. Therefore, we have the equality

$$\mu(D^3_1(f)) = b_1(D^3_1(f_t)) + 2|D^4_1(f_t)|.$$

Let us consider the projection $\pi^3_1\colon D^3(f_t) \twoheadrightarrow D^3_1(f_t)$, $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \mapsto \mathbf{x}_1$. It is clealy a finite map. By construction, for any generic point $\mathbf{x}_1 \in D^3_1(f_t)$, there are two distinct points $\mathbf{x}_2, \mathbf{x}_3 \in D_1^3(f_t)$ that are mapped $f_t(\mathbf{x}_1)$. So, the preimage of $\mathbf{x}_1$ by π_1^3 consists of two points: $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, $(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_2)$. However, if $\mathbf{x}_2 = \mathbf{x}_3 \in \pi_1^2 \circ \pi_2^3(D^3(f_t,(2,1))$, then $(\pi_1^2 \circ \pi_2^3)^{-1}(\mathbf{x}_1) = \{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2)\}$. By Lemma 4.6.3 and Theorem 4.4.1, $(D^3(f_t,(2,1))$ is a finite set of points. Moreover, $\pi_1^2 \circ \pi_2^3$ is generically 2-to-1 and ramified on $D^3(f_t,(2,1))$. By The Riemann-Hurwitz formula, or by assigning a triangulation on both $D^3(f_t)$ and $D_1^3(f_t)$ and then counting the vertices, we obtain

$$\chi(D^3(f_t)) = 2\chi(D^3(f_t)) + 4|D_1^4(f_t)| - |D^3(f_t,(2,1))| \tag{4.6}$$

and equivalently

$$1 - b_1(D^3(f_t)) = 2(1 - b_1(D_1^3(f_t))) + 4|D_1^4(f_t)| - |D^3(f_t,(2,1))|. \tag{4.7}$$

Since f_t is stable, $D^3(f_t)$ is nonsingular. So $\mu(D^3(f)) = b_1(D^3(f_t))$. So, the result follows. □

These lemmas lead to

Corollary 4.6.5. *The formula (4.5) is equivalent to*

$$\mu_I(f) = \frac{1}{2}\mu(D^2(f)) + \frac{1}{3}\mu(D_1^3(f)) + \frac{1}{2}\mu(V(R_f)) + \frac{2}{3}|D_1^3(f_t) \cap V(R_{f_t})| - \frac{13}{12}|D_1^4(f_t)| - \frac{1}{3}.$$

Remark 4.6.6. In general, neither $D_1^3(f)$ or $V(R_f)$ will not be a complete intersection singularity. However, for a finitely $\mathcal{A}$-determined map-germ, we will have isolated space curves in hand. In that case, we can use the following formula due to Buchweitz and Greuel ([BG80]):

$$\mu(X,x) = \dim_{\mathbb{C}}(\omega_{X,x}/\mathrm{d}\mathcal{O}_{X,x})$$

where (X,x) is a germ of an isolated curve singularity and $\omega_{X,x}$ is its dualising sheaf. Alternatively, we can calculate the Tjurina number, τ, for the curves by using the command `T_1` from `sing.lib`, and the Cohen-Macaulay type $t := \dim_{\mathbb{C}}(\omega_{X,x}/\mathfrak{m}_{X,x}\omega_{X,x})$ of the curve by `CMtype()` from `spcurve.lib` on SINGULAR ([DGPS10]), and then deduce the Milnor number from formula $\mu = \tau - t + 1$ due to Greuel's ([Gre83]).

Remark 4.6.7. We claim that the number of elements in $D^3(f_t,(2,1))$ and $D_1^4(f_t)$ are given by $\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^3,0}}{I_{D_1^3(f)} + R_f}$ and $\mathrm{Fitt}_3(f_*\mathcal{O}_{\mathbb{C}^3,0})$, respectively. For specific examples we can show that the modules $M := \frac{\mathcal{O}_{\mathbb{C}^3\times\mathbb{C}^d,0}}{I_{D^3_1(F)}+R_F}$ and $N := \frac{\mathcal{O}_{\mathbb{C}^4\times\mathbb{C}^d,0}}{\mathrm{Fitt}_3(F_*\mathcal{O}_{\mathbb{C}^3\times\mathbb{C}^d,0})}$, where $F\colon(\mathbb{C}^3\times\mathbb{C}^d,0)\to(\mathbb{C}^4\times\mathbb{C}^d,0)$, $(\mathbf{x},\mathbf{t})\mapsto(f_t(\mathbf{x}),\mathbf{t})$ is a stable unfolding, satisfy the principle of conservation ([dJP00, Theorem 6.4.7]). That is, M_0, the stalk of M at 0 is a free $\mathcal{O}_{\mathbb{C}^d,0}$-module of finite rank so that for all sufficiently small neighbourhoods U of 0 in $(\mathbb{C}^3,0)$ there exists an open neighbourhood V of 0 in $(\mathbb{C}^d,0)$ such that for all $t\in V$

$$\dim_{\mathbb{C}}((M|_{\mathbb{C}^3\times\{0\}})_0)=\sum_{p\in U\times\{t\}}\dim_{\mathbb{C}}((M|_{\mathbb{C}^3\times\{t\}})_p)$$

where $(M|_{\mathbb{C}^3\times\{0\}})_0\cong\frac{\mathcal{O}_{\mathbb{C}^3,0}}{I_{D^3_1(f)}+R_f}$ and $(M|_{X_t})_p\cong\frac{\mathcal{O}_{U,p'}}{I_{D^3_1(f_t)}+R_{f_t}}$ with $p=(p',t)$. A similar statement also holds for N. It is still open question to give a general proof to the fact that M and N are free modules over the ring of the parameter space.

Example 4.6.8. We have the following data for the examples from the previous section.

f	$\mu(D^2)$	$\mu(D^3_1)$	$\mu(V(R_f))$	$\lvert D^3_1(f_t)\cap V(R_{f_t})\rvert$	$\lvert D^4_1(f_t)\rvert$	$\mu_I(f)$
$\tilde{A}_3$	?	323	32	68	32	186
$\tilde{A}_4$	?	2506	72	222	336	844
$\tilde{B}^{\pm}_3$	?	45	8	16	4	33
$\tilde{B}^{\pm}_5$	?	497	32	64	48	252
$\tilde{B}^{-}_7$	?	1837	72	144	180	837

Table 4.3: Invariants for the map-germs $\tilde{A}_k$ and $\tilde{B}^{\pm}_{2k+1}$.

Here, for the image Milnor number, we used the fact that the Mond conjecture, i.e. the equality $\mu_I(f)=\mathcal{A}_e\text{-codim}(f)$, holds for those map-germs (cf. Proposition 4.5.3 and 4.5.5). Corollary 4.6.5 suggests the following values for $\mu(D^2)$.

	$\mu(D^2)$
$\tilde{A}_3$	104
$\tilde{A}_5$	378
$\tilde{B}^{\pm}_3$	16
$\tilde{B}^{\pm}_5$	160
$\tilde{B}^{-}_7$	576

Table 4.4: Milnor numbers of the double point spaces.

Chapter 5

Generating new examples from old

5.1 Reductions of map-germs

It is natural to ask whether we can generate examples of finitely determined map-germs by looking at 1 (or more)-parameter unfoldings of the known ones. An answer was given by Cooper his thesis ([Coo94]). He proved that the augmentation (which we will recall shortly) of a map-germ of $\mathcal{A}_e$-codimension 1 has also got $\mathcal{A}_e$-codimension 1 ([CMA02, Theorem 2.5]). This definition is a generalisation of the one given by Goryunov in [Gor84].

> Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a multi-germ of $\mathcal{A}_e$-codimension 1 where S is a finite subset of $\mathbb{C}^n$. Let $F\colon (\mathbb{C} \times \mathbb{C}^n, \{0\} \times S) \to (\mathbb{C} \times \mathbb{C}^p, (0,0))$, $F(x,u) := (u, f_u(x))$ be an $\mathcal{A}_e$-versal unfolding of f. Define a map-germ $A_F\colon (\mathbb{C}\times\mathbb{C}^n, \{0\}\times S) \to (\mathbb{C}\times\mathbb{C}^p, (0,0))$ by $A_F(\lambda, x) = (\lambda, f_{\lambda^2}(x))$. Then the equivalence class of A_F is called the *augmentation* of f ([CMA02, Section 2] or [Coo94]).

In [Hou98], Houston considered a more general definition:

> Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be unfolded by the map $F\colon (\mathbb{C}^n\times\mathbb{C}, S\times\{0\}) \to (\mathbb{C}^p \times \mathbb{C}, (0,0))$, $F(x,t) := (\tilde{F}(x,t), t)$, where S is a finite subset of $\mathbb{C}^n$. Let $g\colon (\mathbb{C}^q, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function. Then the *augmentation* of f by F and g, denoted $A_{F,g}(f)$, is the map-germ

$A_{F,g}(f)\colon (\mathbb{C}^n \times \mathbb{C}^q, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^q, (0,0))$ defined by $A_{F,g}(f)(x,z) = (\tilde{F}(x, g(z)), z)$ ([Hou98, Definition 3.1]).

Notice that Houston's definition coincides with Cooper's when $g(z) = z^2$. Houston proved that if the unfolding F is stable as a map-germ, then $\mathcal{A}_e\text{-codim}(f)\tau(g) \leq \mathcal{A}_e\text{-codim}(A_{F,g}(f))$, and if, moreover, F or g is weighted homogeneous then there is an equality ([Hou98, Theorem 3.3]). It follows that if F is in the nice dimensions or g is weighted homogeneous then $A_{F,g}(f)$ is finitely $\mathcal{A}$-determined if and only if g defines an isolated singularity ([Hou98, Corollary 3.5]).

Now we consider a special base change operation for unfoldings which will yield finitely $\mathcal{A}$-determined map-germs under some conditions (cf. Definition 1.3.9).

Definition 5.1.1. Let $F \in \mathcal{E}^0_{n+d,p+d}$ be a map-germ given by $F(\mathbf{x}, \mathbf{u}) = (F_{\mathbf{u}}(\mathbf{x}), \mathbf{u})$ and $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$ be a holomorphic map-germ. We define the *reduction* $R_\gamma(F)$ of F by γ as

$$\begin{aligned} R_\gamma(F) : (\mathbb{C}^n \times \mathbb{C}, 0) &\to (\mathbb{C}^p \times \mathbb{C}, 0) \\ (\mathbf{x}, t) &\mapsto (F_{\gamma(t)}(\mathbf{x}), t). \end{aligned}$$

The reduction $R_\gamma(F)$ coincides with Houston's definition of augmentations in a certain case: If F is a 1-parameter unfolding of $f \in \mathcal{E}^0_{n,p}$ and $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ then $R_\gamma(F) = A_{F,\gamma}(f)$. So, in that case, Houston's results apply. But more generally,

Theorem 5.1.2. *Let $F \in \mathcal{E}^0_{n+d,p+d}$ be a parametrised stable unfolding of a finite $f \in \mathcal{E}^0_{n,p}$ and V be the image of F. Let G be the identity map on $(\mathbb{C}^p \times \mathbb{C}^d, 0)$. Assume that $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$ is a non-constant map-germ parametrising a curve which intersects the $\mathcal{K}_V$-discriminant $D_V(G)$ only at the origin. Then, $R_\gamma(F)$ is finitely $\mathcal{A}$-determined if f is.*

Proof. Assume that f is finitely $\mathcal{A}$-determined. Notice that f can be obtained as pullback of F by the inclusion $g_0\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$, $\mathbf{Y} \mapsto (\mathbf{Y}, 0)$, which is transverse to F ([Dam87]). So, $N\mathcal{A}_e f \cong N\mathcal{K}_{V,e} g_0$ by Theorem 3.2.10.

Consider $R_\gamma(F)$ as a pullback of F by

$$\begin{aligned} \hat{g}\colon (\mathbb{C}^p \times \mathbb{C}, 0) &\to (\mathbb{C}^p \times \mathbb{C}^d, 0) \\ (\mathbf{Y}, t) &\mapsto (\mathbf{Y}, \gamma_1(t), \dots, \gamma_d(t)) \end{aligned}$$

i.e. the following is a commutative diagram:

$$\begin{array}{ccc}
(\mathbb{C}^n \times \mathbb{C}^d, 0) & \xrightarrow{F} & (\mathbb{C}^p \times \mathbb{C}^d, 0) \\
\uparrow & & \uparrow \hat{g} \\
(\mathbb{C}^n \times \mathbb{C}, 0) & \xrightarrow{R_\gamma(F)} & (\mathbb{C}^p \times \mathbb{C}, 0).
\end{array}$$

Notice that $\hat{g}$ is transverse to F since g is. We will prove the claim by showing that the support of $N\mathcal{K}_{V,e}\hat{g}$ consists of the origin at most (cf. Proposition 3.2.3). For this, we will relate the support of $N\mathcal{K}_{V,e}\hat{g}$ with that of $N\mathcal{K}_{V,e/\mathbb{C}^d}G$.

We have

$$\begin{aligned}
N\mathcal{K}_{V,e/\mathbb{C}^d}G &= \frac{\Theta(G)}{tG(\Theta_{\mathbb{C}^p\times\mathbb{C}^d/\mathbb{C}^d,0}) + G^*\text{Der(-log } V)} \\
&= \frac{(\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d,0})^{p+d}}{(\frac{\partial G}{\partial Y_1}, \ldots, \frac{\partial G}{\partial Y_p})\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d,0} + \text{Der(-log } V)}.
\end{aligned}$$

Suppose that Der(-log V) is generated by $\xi_1, \ldots, \xi_m$. Let us write ξ_i as $\xi_i = \xi_i^1 + \xi_i^2$ relative to the decomposition $\mathbb{C}^{p+d} = \mathbb{C}^p \times \mathbb{C}^d$ and let Der(-log $V)_2$ denote the module generated by $\xi_1^2, \ldots, \xi_m^2$. Since $\frac{\partial G}{\partial Y_i} = \frac{\partial}{\partial Y_i}$, the projection onto the last d components induces

$$N\mathcal{K}_{V,e/\mathbb{C}^d}G \cong \frac{(\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d,0})^d}{\text{Der(-log } V)_2}.$$

On the other hand,

$$\begin{aligned}
N\mathcal{K}_{V,e}\hat{g} &= \frac{\Theta(\hat{g})}{t\hat{g}(\Theta_{\mathbb{C}^p\times\mathbb{C},0}) + \hat{g}^*\text{Der(-log } V)} \\
&\cong \frac{(\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0})^{p+d}}{(\frac{\partial \hat{g}}{\partial Y_1}, \ldots, \frac{\partial \hat{g}}{\partial Y_p}, \frac{\partial \hat{g}}{\partial t})\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0} + \hat{g}^*\text{Der(-log } V)}.
\end{aligned}$$

We also have $\frac{\partial \hat{g}}{\partial Y_i} = \frac{\partial}{\partial Y_i}$. So, the projection onto the last d components induces

$$N\mathcal{K}_{V,e}\hat{g} \cong \frac{(\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0})^d}{\left(\begin{bmatrix} \frac{\partial\gamma_1}{\partial t} \\ \vdots \\ \frac{\partial\gamma_d}{\partial t} \end{bmatrix}\right)\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0} + \hat{g}^*\text{Der(-log } V)_2}. \tag{5.1}$$

Consider the extension

$$M := N\mathcal{K}_{V,e/\mathbb{C}^d} G \otimes_{\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d,0}} \mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d\times\mathbb{C},0} \cong \frac{(\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d\times\mathbb{C},0})^d}{\text{Der(-log }V)_2}.$$

We get

$$\begin{aligned} M \otimes_{\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d\times\mathbb{C},0}} \frac{\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d\times\mathbb{C},0}}{(U_1-\gamma_1(t),\dots,U_d-\gamma_d(t))} &\cong \frac{M}{(U_1-\gamma_1(t),\dots,U_d-\gamma_d(t))M} \\ &\cong \frac{(\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0})^d}{\hat{g}^*\text{Der(-log }V)_2}. \end{aligned} \tag{5.2}$$

in which the last isomorphism is induced by the maps

$$(\mathbb{C}^p\times\mathbb{C}^d\times\mathbb{C},0) \underset{}{\overset{\rho}{\rightleftarrows}} (\mathbb{C}^p\times\mathbb{C},0)$$

$$(Y_1,\dots,Y_p,U_1,\dots,U_d,t) \overset{\rho}{\longmapsto} (Y_1,\dots,Y_p,t)$$

$$(Y_1,\dots,Y_p,\gamma_1(t),\dots,\gamma_d(t),t) \longleftarrow\!\!\dashv (Y_1,\dots,Y_p,t).$$

By (5.2) and Proposition 1.1.11,

$$\begin{aligned} \text{Supp}\Big(\frac{(\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0})^d}{\hat{g}^*\text{Der(-log }V)_2}\Big) &\cong \rho\Big(\text{Supp}(M)\cap\text{Supp}\Big(\frac{\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d\times\mathbb{C},0}}{(U_1-\gamma_1(t),\dots,U_d-\gamma_d(t))}\Big)\Big) \\ &\cong \rho\big((\text{Supp}(N\mathcal{K}_{V,e/\mathbb{C}^d}G)\times\mathbb{C})\cap(\mathbb{C}^p\times\text{graph}(\gamma))\big). \end{aligned} \tag{5.3}$$

Recall that $\text{Supp}(N\mathcal{K}_{V,e/\mathbb{C}^d}G)$ is the set of points (Y,U) such that $g_U := G(-,U)\colon(\mathbb{C}^p,Y)\to(\mathbb{C}^p\times\mathbb{C}^d,G(Y,U))$ is not algebraically transverse to V at Y (Remark 3.2.6). Since f is finitely $\mathcal{A}$-determined, g_0 is algebraically transverse to V in a punctured neighbourhood of 0. From $N\mathcal{K}_{V,e/\mathbb{C}^d}G\otimes_{\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d,0}}\mathcal{O}_{\mathbb{C}^p,0} = N\mathcal{K}_{V,e}g_0$ it follows that

$$\text{Supp}(N\mathcal{K}_{V,e/\mathbb{C}^d}G)\cap\mathbb{C}^p\times\{0\} = \{0\}\,.$$

By assumption, $\text{image}(\gamma)\cap D_V(G)=\{0\}$. So, $\text{Supp}(\frac{(\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0})^d}{\hat{g}^*\text{Der(-log }V)_2}) = \{0\}$ by (5.3).

We use (5.1) to form the following presentation of $N\mathcal{K}_{V,e}\hat{g}$:

$$\mathcal{O}^d_{\mathbb{C}^p\times\mathbb{C},0} \xrightarrow{\chi:=\begin{bmatrix}\frac{\partial\gamma}{\partial t} & \hat{g}^*\xi_1^2 & \cdots & \hat{g}^*\xi_m^2\end{bmatrix}} \mathcal{O}^d_{\mathbb{C}^p\times\mathbb{C},0} \to N\mathcal{K}_{V,e}\hat{g}\to 0.$$

Hence,

$$\mathrm{Fitt}_0(N\mathcal{K}_{V,e}\hat{g}) = \mathrm{Fitt}_0(\frac{\mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0})^d}{\hat{g}^*\mathrm{Der}(\text{-log } V)_2}) + J$$

where J is the ideal generated by the minors of ξ not containing the first column. Therefore, $\mathrm{Supp}(N\mathcal{K}_{V\times\mathbb{C},e}\hat{g}) \cong \{0\} \cap V(J)$. So $\mathrm{Supp}(N\mathcal{K}_{V,e}\hat{g})$ is $\{0\}$ or $\emptyset$ which means $\hat{g}$ has finite $\mathcal{K}_V$-codimension; equivalently, $R_\gamma(F)$ is finitely $\mathcal{A}$-determined by Theorem 3.2.10. This concludes the proof. □

We formulate an algorithm based on the ideas of Theorem 5.1.2.

Algorithm 5.1.3. INPUT. A finitely $\mathcal{A}$-determined map-germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$.

STEP 1. Form the commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^d, 0) & \xrightarrow{F} & (\mathbb{C}^p \times \mathbb{C}^d, 0) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle g} \\ (\mathbb{C}^n, 0) & \xrightarrow{f} & (\mathbb{C}^p, 0) \end{array}$$

where F is a parametrised stable unfolding of f and $g\colon \mathbf{Y} \mapsto (\mathbf{Y}, \mathbf{0})$.

STEP 2. Calculate $N\mathcal{K}_{V,e/\mathbb{C}^d}G$ where G is the identity on $\mathbb{C}^p \times \mathbb{C}^d$ and V is the image of F.

STEP 3. Let $D^* := D_V(G) \setminus \{0\}$. Choose a parametrised curve $\gamma\colon t \mapsto (\gamma_1(t), \dots, \gamma_d(t))$ which lives in $\mathbb{C}^d \setminus D^*$.

OUTPUT. A finitely $\mathcal{A}$-determined map-germ $R_\gamma(F)$ defined by the following diagram.

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^d, 0) & \xrightarrow{F} & (\mathbb{C}^p \times \mathbb{C}^d, 0) \\ \uparrow & & \uparrow{\scriptstyle \hat{g}} \\ (\mathbb{C}^n \times \mathbb{C}, 0) & \xrightarrow{R_\gamma(F)} & (\mathbb{C}^p \times \mathbb{C}, 0) \end{array}$$

where

$$\hat{g}\colon (\mathbf{Y}, t) \mapsto (\mathbf{Y}, \gamma_1(t), \dots, \gamma_d(t)).$$

In [BM96], Bruce and Marar gave examples of finitely $\mathcal{A}$-determined map-germs from $\mathbb{C}^2$ to $\mathbb{C}^3$. Those are

$$(x, y) \mapsto (x^2, y^2, x^{2r+1} + xy + y^{2s+1}), \qquad (5.4)$$

$$(x, y) \mapsto (x^2, y^2, x^3 + x^2y + xy^2 - y^3), \quad \text{with } \mathcal{A}_e\text{-codim} = 12. \qquad (5.5)$$

From (5.5), we deduce the following examples from $\mathbb{C}^3$ to $\mathbb{C}^4$.

Proposition 5.1.4. *Let* $f \in \mathcal{E}^0_{3,4}$ *be given by*

$$f\colon (t,x,y) \mapsto (t, x^2 + t^k y, y^2 - t^k x, x^3 + x^2 y + xy^2 - y^3) \tag{5.6}$$

is finitely $\mathcal{A}$*-determined for* $k \geq 1$*.*

Proof. We observe that f is a reduction of

$$F(x,y,a,b,c,d,e) := (x^2 + ay, y^2 + bx + cy, x^3 + x^2y + xy^2 - y^3 + dx + ey, a, \ldots, e)$$

by $\gamma\colon t \mapsto (t^k, -t^k, 0, 0, 0)$. The image of γ is defined by $\{A + B = C = D = E = 0\}$ and cuts $D_V(G)$ only at the origin (see p. 99 for an example of SINGULAR code calculating $D_V(G)$). Therefore, f is finitely $\mathcal{A}$-determined by Theorem 5.1.2. □

Remark 5.1.5. For $k \leq 20$, $\mathcal{A}_e$-codimension of (5.6) has the formula $45k - 12$; moreover, it satisfies the Mond conjecture.

Remark 5.1.6. Marar and Nuño-Ballesteros ([MNB08]) listed the following examples of corank 2 $\mathcal{A}$-finite map-germs

$$\begin{aligned}
(x,y) &\mapsto (x^2, y^3 + xy, x^2y + y^5)\\
(x,y) &\mapsto (x^2, y^3 + xy, x^5 + 2x^4y^2 + y^{10})\\
(x,y) &\mapsto (x^2, y^3 + xy, x^6 + y^{13})\\
(x,y) &\mapsto (x^2, y^3 + xy, x^7 + 2x^6y^2 + y^{14}).
\end{aligned}$$

By Theorem 5.1.2, it is possible to generate examples from their list. But computer runs out of memory before concluding the calculations. However, by experiment (and Theorem 2.6.6), we find that

$$(x,y,z) \mapsto (x^2 + yz, y^3 + xy, x^2 + y^5 + xz, z)$$

is finitely $\mathcal{A}$-determined. Because of the technical reasons, we have not been able to check whether it satisfies the conjecture.

5.2 Series of finitely determined map-germs from $\mathbb{C}^4$ to $\mathbb{C}^5$

In this section we prove the existence of three series of finitely $\mathcal{A}$-determined map-germs from $\mathbb{C}^4$ to $\mathbb{C}^5$. Moreover, we calculate their $\mathcal{A}_e$-codimensions.

Proposition 5.2.1. *The series*

$$\begin{aligned} \tilde{C}_k\colon (\mathbb{C}^4,0) &\rightarrow (\mathbb{C}^5,0) \\ (x,y,z,t) &\mapsto (x,t,y^2+xz+x^2y,yz+t^ky,z^2+y^3) \end{aligned}$$

has $\mathcal{A}$-codimension $30k-18$ for $k \geq 1$. Moreover, $\tilde{C}_k$ satisfies the Mond conjecture (Conjecture 3.1.5) for $k \leq 20$.

Proof. Notice that $\tilde{C}_k$ can be considered as a reduction of

$$F(x,y,z,a,b,c) := (x,y^2+xz+x^2y+ay,yz+by,z^2+y^3+cy,a,b,c)$$

by $\gamma\colon t \mapsto (0,t^k,0)$. And F is a stable unfolding of

$$(x,y,z) \mapsto (x,y^2+xz+x^2y,yz,z^2+y^3)$$

which has $\mathcal{A}$-codimension 18 by Proposition 4.5.3. Hence, we may check the finite determinacy of $\tilde{C}_k$ using Theorem 5.1.2. Let G be the identity on $(\mathbb{C}^4\times\mathbb{C}^3,0)$. The following SINGULAR code calculates the ideal I_D defining $D_V(G)$.

```
LIB "matrix.lib";
ring T=0,(X,Y,Z,W,A,B,C),(wp(1,4,5,6,2,3,4));
ring s=0,(x,y,z,a,b,c),(wp(1,2,3,2,3,4));
ideal p=0;
map F=T,x,y2+xz+x2y+ay,yz+by,z2+y3+cy,a,b,c;
setring T;
ideal h=preimage(s,F,p);          \\ Defining equation for the image V of F
ideal jh=jacob(h);
module derv=modulo(jh,h);         \\ derv := Der(-log V)
def tkv=submat(derv,5..7,1..ncols(derv));    \\ tkv := Der(-log V)_2
ideal sup=std(minor(tkv,3));      \\ Annihilator of NK_{V,e}G
ideal ID=eliminate(sup,XYZW);     \\ Ideal of the discriminant
```

```
ideal intersect=ID+(A,C); \\ Intersection of the discriminant and im(γ)
radical(intersect);
//-> _[1]=A
//-> _[2]=B
//-> _[3]=C
```

So any curve $\gamma\colon t \mapsto (0, \gamma_2(t), 0)$, where $\gamma_2(t)$ is not constant, intersects $D_V(G)$ only at the origin. It follows by Theorem 5.1.2 that $\tilde{C}_k$ is also finitely $\mathcal{A}$-determined. Now we will show that $N\mathcal{A}_e\tilde{C}_k \cong N\mathcal{K}_{V,e}\hat{g}_k$ has dimension $30k - 18$ over $\mathbb{C}$. Here $\hat{g}_k\colon (X, Y, Z, W, t) \mapsto (X, Y, Z, W, 0, t^k, 0)$. We consider the isomorphism

$$N\mathcal{K}_{V,e}\hat{g}_k \cong \frac{(\mathcal{O}_{\mathbb{C}^5,0})^3}{(\begin{bmatrix} 0 \\ t^{k-1} \\ 0 \end{bmatrix})\mathcal{O}_{\mathbb{C}^5,0} + \hat{g}_k^*\text{Der(-log } V)_2}$$

(cf. (5.1)). Let us set $N := \left(\begin{bmatrix} 0 & t^{k-1} & 0 \end{bmatrix}^t\right)\mathcal{O}_{\mathbb{C}^5,0} + \hat{g}_k^*\text{Der(-log } V)_2$. Since $N\mathcal{K}_{V,e}\hat{g}_k$ is a finite dimensional vector space,

$$\dim_{\mathbb{C}} N\mathcal{K}_{V,e}\hat{g}_k = \dim_{\mathbb{C}} \frac{\mathcal{O}^3_{\mathbb{C}^5,0}}{(\text{LT}(N))\mathcal{O}_{\mathbb{C}^5,0}}$$

where $(\text{LT}(N))$ is the module generated by the leading terms (with respect to $>$) of the elements in N ([Eis95, Theorem 15.3], see also Appendix D).

We fix "$>$" to be a reverse lexicographic order with priority given to the coefficients (cf. Definition D.0.2). We will deduce a Groebner basis for N from a Groebner basis of $N_0 := \hat{g}_1^*\text{Der(-log } V)_2$. Following the SINGULAR code above,

```
ring T0=1,(X,Y,Z,W,t),rp;       // here rp= " > "
map g_1=T,X,Y,Z,W,0,t,0;
def dv=std(g(tkv));
```

compute a Groebner basis for N_0. The generators, up to some column operations, are as follows.

$$n_1 := \begin{bmatrix} X^8 \\ 0 \\ 0 \end{bmatrix}, \quad \text{LT}(n_1) = \begin{bmatrix} X^8 \\ 0 \\ 0 \end{bmatrix}; \tag{5.7}$$

$$
\begin{aligned}
n_2 &:= \begin{bmatrix} 0 \\ X^9 \\ 0 \end{bmatrix}, & \mathrm{LT}(n_2) &= \begin{bmatrix} 0 \\ X^9 \\ 0 \end{bmatrix}; \\
n_3 &:= \begin{bmatrix} -277/229X^7 \\ -25/458X^8 \\ X^9 \end{bmatrix}, & \mathrm{LT}(n_3) &= \begin{bmatrix} 0 \\ 0 \\ X^9 \end{bmatrix}; \\
n_4 &:= \begin{bmatrix} 5/2X^2 \\ -1/4X^3 \\ Y - 1/2X^4 \end{bmatrix}, & \mathrm{LT}(n_4) &= \begin{bmatrix} 0 \\ 0 \\ Y \end{bmatrix}; \\
n_5 &:= \begin{bmatrix} X^2Y + 74/39X^6 \\ -61/312X^7 \\ -59/156X^8 \end{bmatrix}, & \mathrm{LT}(n_5) &= \begin{bmatrix} X^2Y \\ 0 \\ 0 \end{bmatrix}; \\
n_6 &:= \begin{bmatrix} -8/7XY - 36/7X^5 \\ X^2Y - 4/7X^6 \\ 40/7X^7 \end{bmatrix}, & \mathrm{LT}(n_6) &= \begin{bmatrix} 0 \\ X^2Y \\ 0 \end{bmatrix}; \qquad (5.8) \\
n_7 &:= \begin{bmatrix} Y^2 + 3439/32X^4Y \\ -19473/256X^5Y \\ -82731/128X^6Y \end{bmatrix}, & \mathrm{LT}(n_7) &= \begin{bmatrix} Y^2 \\ 0 \\ 0 \end{bmatrix}; \\
n_8 &:= \begin{bmatrix} 147/2X^3Y - 12939/16X^7 \\ Y^2 - 817/16X^4Y + 4331/32X^8 \\ -3483/8X^5Y - 1173/16X^9 \end{bmatrix}, & \mathrm{LT}(n_8) &= \begin{bmatrix} 0 \\ Y^2 \\ 0 \end{bmatrix}; \\
n_9 &:= \begin{bmatrix} Z + 29/2XY - 2549/16X^5 \\ -159/16X^2Y + 861/32X^6 \\ -693/8X^3Y - 243/16X^7 \end{bmatrix}, & \mathrm{LT}(n_9) &= \begin{bmatrix} Z \\ 0 \\ 0 \end{bmatrix}; \\
n_{10} &:= \begin{bmatrix} 2Y + 29/4X^4 \\ Z + 7/4XY - 1/8X^5 \\ -3/2X^2Y - 9/4X^6 \end{bmatrix}, & \mathrm{LT}(n_{10}) &= \begin{bmatrix} 0 \\ Z \\ 0 \end{bmatrix}; \\
n_{11} &:= \begin{bmatrix} 17/2X^3 \\ 1/2Y - 3/4X^4 \\ Z + 3XY - 3/2X^5 \end{bmatrix}, & \mathrm{LT}(n_{11}) &= \begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix};
\end{aligned}
$$

$$n_{12} := \begin{bmatrix} W + 189/40XZ + 201/55X^2Y - 4853/80X^6 \\ -15/176X^2Z - 177/44X^3Y + 927/110X^7 \\ -4491/880X^3Z - 3087/110X^4Y \end{bmatrix}, \quad \mathrm{LT}(n_{12}) = \begin{bmatrix} W \\ 0 \\ 0 \end{bmatrix};$$

$$n_{13} := \begin{bmatrix} -Z + 1677/88XY - 18877/32X^5 \\ W + 305/88XZ - 3215/88X^2Y + \frac{3623}{44}X^6 \\ -11609/352X^2Z - 5919/22X^3Y \end{bmatrix}, \quad \mathrm{LT}(n_{13}) = \begin{bmatrix} 0 \\ W \\ 0 \end{bmatrix};$$

$$n_{14} := \begin{bmatrix} 7/33Y + 55/24X^4 \\ 20/33Z - 19/132XY - 13/33X^5 \\ W - 47/88XZ + 41/22X^2Y \end{bmatrix}, \quad \mathrm{LT}(n_{14}) = \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix}; \quad (5.9)$$

$$n_{15} := \begin{bmatrix} t + 57/4X^3 \\ Y - 2X^4 \\ 3/4Z + 6XY \end{bmatrix}, \quad \mathrm{LT}(n_{15}) = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix};$$

$$n_{16} := \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}, \quad \mathrm{LT}(n_{16}) = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix};$$

$$n_{17} := \begin{bmatrix} -3/2X \\ -1/4X^2 \\ t + 3/2X^3 \end{bmatrix}, \quad \mathrm{LT}(n_{17}) = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}.$$

We check that LT(`dv`) = LT(`N_0`) by

```
print(lead(dv));
print(lead(N_0));
```

It follows that those n_i's form a Groebner basis for N_0. We define

$$m\colon (X, Y, Z, W, t) \mapsto (X, Y, Z, W, t^k)$$

so that $\hat{g}_k = \hat{g}_1 \circ m$. In that case $\hat{g}_k^*\mathrm{Der}(\text{-log } V)_2 = m^*\hat{g}_1^*\mathrm{Der}(\text{-log } V)_2$. Then a basis for N is given by $n_1, \ldots, n_{14}$ and

$$\tilde{n}_{15} := m^*n_{15} = \begin{bmatrix} t^k + 57/4X^3 \\ Y - 2X^4 \\ 3/4Z + 6XY \end{bmatrix} \quad \text{with } \mathrm{LT}(\tilde{n}_{15}) = \begin{bmatrix} t^k \\ 0 \\ 0 \end{bmatrix}; \qquad (5.10)$$

$$\tilde{n}_{16} := m^* n_{16} = \begin{bmatrix} 0 \\ t^k \\ 0 \end{bmatrix} \quad \text{with } \mathrm{LT}(\tilde{n}_{16}) = \begin{bmatrix} 0 \\ t^k \\ 0 \end{bmatrix};$$

$$\tilde{n}_{17} := m^* n_{17} = \begin{bmatrix} -3/2X \\ -1/4X^2 \\ t^k + 3/2X^3 \end{bmatrix} \quad \text{with } \mathrm{LT}(\tilde{n}_{17}) = \begin{bmatrix} 0 \\ 0 \\ t^k \end{bmatrix}; \tag{5.11}$$

$$n_{18} := \begin{bmatrix} 0 \\ t^{k-1} \\ 0 \end{bmatrix} \quad \text{with } \mathrm{LT}(n_{18}) = \begin{bmatrix} 0 \\ t^{k-1} \\ 0 \end{bmatrix}.$$

Clearly we can discard $\tilde{n}_{16}$. By Buchberger's Algorithm, we calculate that $n_1, \ldots, n_{14}$, $\tilde{n}_{15}$, $\tilde{n}_{17}$, n_{18} together with the following of elements form a Groebner basis for N (see Appendix D.1 for details).

$$n_{21} := \begin{bmatrix} Yt^{k-1} + 11/2X^4t^{k-1} \\ 0 \\ -3/2X^6t^k \end{bmatrix}, \quad \mathrm{LT}(n_{21}) = \begin{bmatrix} Yt^{k-1} \\ 0 \\ 0 \end{bmatrix} \quad \text{(cf. (D.7))};$$

$$n_{23} := \begin{bmatrix} 2/7X^5t^{k-1} \\ 0 \\ X^7t^{k-1} \end{bmatrix}, \quad \mathrm{LT}(n_{23}) = \begin{bmatrix} 0 \\ 0 \\ X^7t^{k-1} \end{bmatrix} \quad \text{(cf. (D.9))};$$

$$n_{24} := \begin{bmatrix} X^5t^{k-1} \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{LT}(n_{24}) = \begin{bmatrix} X^5t^{k-1} \\ 0 \\ 0 \end{bmatrix} \quad \text{(cf. (D.10))}.$$

Therefore,

$$\mathrm{LT}(N) = (\begin{bmatrix} X^8 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ X^9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ X^9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Y \end{bmatrix}, \begin{bmatrix} X^2Y \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ X^2Y \\ 0 \end{bmatrix}, \begin{bmatrix} Y^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ Y^2 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} Z \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ Z \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix}, \begin{bmatrix} W \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ W \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix}, \begin{bmatrix} t^k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ t^k \end{bmatrix}, \begin{bmatrix} 0 \\ t^{k-1} \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} Yt^{k-1} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ X^7t^{k-1} \end{bmatrix}, \begin{bmatrix} X^5t^{k-1} \\ 0 \\ 0 \end{bmatrix}).$$

Consequently,

$$
\begin{aligned}
\dim_{\mathbb{C}} N\mathcal{K}_{V,e}\hat{g}_k &= \dim_{\mathbb{C}} \frac{\mathcal{O}^3_{\mathbb{C}^5,0}}{(\mathrm{LT}(N))\mathcal{O}_{\mathbb{C}^5,0}} \\
&= \dim_{\mathbb{C}} \Big(\frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^8, X^2Y, Y^2, Z, W, X^5t^{k-1}, Yt^{k-1}, t^k)\mathcal{O}_{\mathbb{C}^5,0}} \\
&\oplus \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^9, X^2Y, Y^2, Z, W, t^{k-1})\mathcal{O}_{\mathbb{C}^5,0}} \\
&\oplus \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^9, Y, Z, W, X^7t^{k-1}, t^k)\mathcal{O}_{\mathbb{C}^5,0}} \Big) \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^8, X^2Y, Y^2, Z, W, X^5t^{k-1}, Yt^{k-1}, t^k)\mathcal{O}_{\mathbb{C}^5,0}} + \\
&+ \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^9, X^2Y, Y^2, Z, W, t^{k-1})\mathcal{O}_{\mathbb{C}^5,0}} + \\
&+ \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^9, Y, Z, W, X^7t^{k-1}, t^k)\mathcal{O}_{\mathbb{C}^5,0}}.
\end{aligned} \tag{5.12}
$$

It is known that $\mathbb{C}$-vector space dimension of the variety of a zero dimensional monomial ideal is equal to the number of monomials in the complement of that ideal (see, for example, [CLO05]). Hence, for the summands in (5.12), we need to count the monomials that do not belong to the denominators.

The complement of the ideal $(X^8, X^2Y, Y^2, Z, W, X^5t^{k-1}, Yt^{k-1}, t^k)$ in $\mathcal{O}_{\mathbb{C}^5,0}$ consists of $10k-5$ monomials, those are

$$
\begin{gathered}
1, X, \ldots, X^7, Y, XY \\
t, Xt, \ldots, X^7t, Yt, XYt \\
\vdots \\
t^{k-2}, Xt^{k-2}, \ldots, X^7t^{k-2}, Yt^{k-2}, XYt^{k-2} \\
t^{k-1}, Xt^{k-1}, \ldots, X^4t^{k-1}.
\end{gathered}
$$

The complement of the ideal $(X^9, X^2Y, Y^2, Z, W, t^{k-1})$ in $\mathcal{O}_{\mathbb{C}^5,0}$ consists of $11k-11$

monomials, those are

$$\begin{gathered}1, X, \dots, X^8, Y, XY\\ t, Xt, \dots, X^8 t, Yt, XYt\\ \vdots\\ t^{k-2}, Xt^{k-2}, \dots, X^8 t^{k-2}, Yt^{k-2}, XYt^{k-2}.\end{gathered}$$

Furthermore, the complement of the ideal $(X^9, Y, Z, W, X^7 t^{k-1}, t^k)$ in $\mathcal{O}_{\mathbb{C}^5,0}$ consists of the following monomials

$$\begin{gathered}1, X, \dots, X^8\\ t, Xt, \dots, X^8 t,\\ \vdots\\ t^{k-2}, Xt^{k-2}, \dots, X^8 t^{k-2}\\ t^{k-1}, Xt^{k-1}, \dots, X^6 t^{k-1}.\end{gathered}$$

These all add up to $\dim_{\mathbb{C}} N\mathcal{K}_{V,e}\hat{g}_k = 30k - 18$.

As for the Mond conjecture, we have not got a proof for general case. However, we have checked that for each $k \leq 20$, $N\mathcal{K}_{H,e/\mathbb{C}}G$ is a 1-dimensional Cohen-Macaulay module by a calculation on SINGULAR (here G is a 1-parameter unfolding of $\hat{g}_k$ and H is the defining equation of the image of F). So, $\tilde{C}_k$ satisfies the Mond conjecture for $k \leq 20$ (cf. Remark 3.2.13). □

Proposition 5.2.2. *The series*

$$\begin{array}{rcl}\tilde{D}_k\colon (\mathbb{C}^4,0) & \to & (\mathbb{C}^5,0)\\ (x,y,z,t) & \mapsto & (x, t, y^2+xz, z^2+xy, y^3+yz^2+z^3+t^k z)\end{array}$$

has $\mathcal{A}$-codimension $51k-33$ for $k \geq 1$. Moreover, $\tilde{D}_k$ satisfies the Mond conjecture for $k \leq 20$.

Proof. Notice that $\tilde{D}_k$ can be considered as a reduction of

$$F(x,y,z,a,b,c,d) := (x, y^2+xz+az, z^2+xy+bz, y^3+yz^2+z^3+cy+dz, a, b, c, d)$$

by $\gamma\colon t \mapsto (0,0,0,t^k)$. On the other hand, F is a stable unfolding of

$$(x,y,z) \mapsto (x, y^2+xz, z^2+xy, y^3+yz^2+z^3)$$

which has $\mathcal{A}$-codimension 33 by Proposition 4.5.5. So we check the finite determinacy of $\tilde{D}_k$ using Theorem 5.1.2. So, let G be the identity on $(\mathbb{C}^4 \times \mathbb{C}^4, 0)$. We calculate the ideal I_D defining $D_V(G)$.

```
LIB "matrix.lib";
ring T=31991,(X,Y,Z,W,A,B,C,D),(wp(1,2,2,3,1,1,2,2));
ring s=31991,(x,y,z,a,b,c,d),(wp(1,1,1,1,1,2,2));
ideal p=0;
map F=T,x,y2+xz+az,z2+xy+bz,y3+yz2+z3+cy+dz,a,b,c,d;
setring T;
ideal h=preimage(s,F,p);      \\ Defining equation for the image V of F
ideal jh=jacob(h);
module derv=modulo(jh,h);     \\ derv := Der(-log V)
def tkv=submat(derv,5..8,1..ncols(derv));    \\ tkv := Der(-log V)_2
ideal ann=std(minor(tkv,4));   \\ Annihilator of NK_{V,e}G
ideal ID=eliminate(sup,XYZW);
ideal intersect=ID+(A,B,C);
radical(intersect);
//-> _[1]= A
//-> _[2]= B
//-> _[3]= C
//-> _[4]= D
```

For this example, the elimination process takes a very long time. An alternative method for checking that γ cuts the discriminant $D_V(G)$ only at 0 can be given as follows. By Remark 3.2.6, $\pi_|\colon \mathrm{Supp}(N\mathcal{K}_{V,e}G) \to D_V(G)$ is finite. So

$$\begin{aligned}(\pi_|)^{-1}(\mathrm{im}(\gamma) \cap D_V(G)) &= (\pi_|)^{-1}(D_V(G)) \cap (\mathbb{C}^4 \times \mathrm{im}(\gamma)) \\ &= \mathrm{Supp} N\mathcal{K}_{V,e}G \cap (\mathbb{C}^4 \times \mathrm{im}(\gamma)). \qquad (5.13)\end{aligned}$$

As the image of γ is defined by the ideal (A,B,C), (5.13) can be calculated by

```
ideal i=ann+(A,B,C);
```

For the claim, we only need to check that this intersection has dimension 0:

```
dim(std(i));
//-> 0
```

Thus, $\mathrm{im}(\gamma) \cap D_V(G)$ consists of only the origin. Hence, $\tilde{D}_k$ is finitely determined for all $k \geq 1$.

Next, we will show that $N\mathcal{K}_{V,e}\hat{g}_k$ has dimension $51k - 33$ over $\mathbb{C}$. Here, $\hat{g}_k \colon (X, Y, Z, W, t) \mapsto (X, Y, Z, W, 0, 0, 0, t^k)$. Consider the isomorphism

$$N\mathcal{K}_{V,e}\hat{g}_k \cong \frac{(\mathcal{O}_{\mathbb{C}^5,0})^4}{(\begin{bmatrix} 0 \\ 0 \\ 0 \\ t^{k-1} \end{bmatrix})\mathcal{O}_{\mathbb{C}^5,0} + \hat{g}_k^*\mathrm{Der}(\text{-log } V)_2}$$

and set $N := (\begin{bmatrix} 0 & 0 & 0 & t^{k-1} \end{bmatrix}^t)\mathcal{O}_{\mathbb{C}^5,0} + \hat{g}_k^*\mathrm{Der}(\text{-log } V)_2$. Since $N\mathcal{K}_{V,e}\hat{g}_k$ is a finite dimensional vector space,

$$\dim_{\mathbb{C}} N\mathcal{K}_{V,e}\hat{g}_k = \dim_{\mathbb{C}} \frac{\mathcal{O}^4_{\mathbb{C}^5,0}}{(\mathrm{LT}(N))\mathcal{O}_{\mathbb{C}^5,0}}.$$

We fix "$>$" to be a reverse lexicographic order with priority given to the coefficients (cf. Definition D.0.2). We will deduce a Groebner basis for N from a Groebner basis of $N_0 := \hat{g}_1^*\mathrm{Der}(\text{-log } V)_2$.

We carry out our calculations over an algebraically closed field with characteristic 31991 because, over characteristic 0, we experience high digit coefficients in the standard basis calculations for N_0. For example, we see 56 digits with respect to $>$, 56 with lexicographic module ordering and 78 with degree reverse lexicographic ordering.

Following the SINGULAR code above,

```
ring T0=31991,(X,Y,Z,W,t),rp;      // rp= " > "
map g_1=T,X,Y,Z,W,0,t,0;
def dv=std(g(tkv));
```

calculates a Groebner basis for N_0. After some column operations we find the following generators for N_0:

$$
\begin{aligned}
n_1 &:= \begin{bmatrix} X^7 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & \mathrm{LT}(n_1) &= \begin{bmatrix} X^7 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \\
n_2 &:= \begin{bmatrix} -3945X^6 \\ X^6 \\ 0 \\ 0 \end{bmatrix}, & \mathrm{LT}(n_2) &= \begin{bmatrix} 0 \\ X^6 \\ 0 \\ 0 \end{bmatrix}; \\
n_3 &:= \begin{bmatrix} -7323X^6 \\ 0 \\ X^7 \\ 0 \end{bmatrix}, & \mathrm{LT}(n_3) &= \begin{bmatrix} 0 \\ 0 \\ X^7 \\ 0 \end{bmatrix}; \\
n_4 &:= \begin{bmatrix} 5243X^6 \\ 0 \\ 0 \\ X^7 \end{bmatrix}, & \mathrm{LT}(n_4) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ X^7 \end{bmatrix}; \\
n_5 &:= \begin{bmatrix} X^4Y - 14878X^6 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & \mathrm{LT}(n_5) &= \begin{bmatrix} X^4Y \\ 0 \\ 0 \\ 0 \end{bmatrix}; \\
n_6 &:= \begin{bmatrix} -2026X^6 \\ X^4Y \\ 0 \\ 0 \end{bmatrix}, & \mathrm{LT}(n_6) &= \begin{bmatrix} 0 \\ X^4Y \\ 0 \\ 0 \end{bmatrix}; \\
n_7 &:= \begin{bmatrix} 29X^6 \\ 0 \\ X^5Y \\ 0 \end{bmatrix}, & \mathrm{LT}(n_7) &= \begin{bmatrix} 0 \\ 0 \\ X^5Y \\ 0 \end{bmatrix}; \\
n_8 &:= \begin{bmatrix} 1078X^3Y - 9461X^5 \\ -13858X^3Y - 8640X^5 \\ 10532X^4Y - 4023X^6 \\ X^4Y - 2715X^6 \end{bmatrix}, & \mathrm{LT}(n_8) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ X^4Y \end{bmatrix};
\end{aligned} \tag{5.14}
$$

$$n_9 := \begin{bmatrix} XY^2 + 11853X^3Y + 1810X^5 \\ 4423X^3Y - 8177X^5 \\ -11702X^4Y - 9074X^6 \\ -9456X^6 \end{bmatrix}, \qquad \mathrm{LT}(n_9) = \begin{bmatrix} XY^2 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$n_{10} := \begin{bmatrix} 10603X^3Y - 7324X^5 \\ XY^2 - 2427X^3Y - 13307X^5 \\ -6843X^4Y + 9265X^6 \\ 6637X^6 \end{bmatrix}, \qquad \mathrm{LT}(n_{10}) = \begin{bmatrix} 0 \\ XY^2 \\ 0 \\ 0 \end{bmatrix};$$

$$n_{11} := \begin{bmatrix} -12479Y^2 - 14763X^2Y + 9824X^4 \\ -4939Y^2 + 7505X^2Y + 11322X^4 \\ XY^2 + 4208X^3Y + 946X^5 \\ 14377X^3Y - 10776X^5 \end{bmatrix}, \quad \mathrm{LT}(n_{11}) = \begin{bmatrix} 0 \\ 0 \\ XY^2 \\ 0 \end{bmatrix};$$

$$n_{12} := \begin{bmatrix} 11561Y^2 - 10372X^2Y - 9271X^4 \\ 7291Y^2 - 9820X^2Y - 14665X^4 \\ 15816X^3Y - 15158X^5 \\ XY^2 - 5758X^3Y - 13221X^5 \end{bmatrix}, \quad \mathrm{LT}(n_{12}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ XY^2 \end{bmatrix}; \tag{5.15}$$

$$n_{13} := \begin{bmatrix} Y^3 - 7011X^6 \\ -8630X^6 \\ 0 \\ 2367X^5Y \end{bmatrix}, \qquad \mathrm{LT}(n_{13}) = \begin{bmatrix} Y^3 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$n_{14} := \begin{bmatrix} 12633X^6 \\ Y^3 - 10469X^6 \\ 0 \\ 10311X^5Y \end{bmatrix}, \qquad \mathrm{LT}(n_{14}) = \begin{bmatrix} 0 \\ Y^3 \\ 0 \\ 0 \end{bmatrix};$$

$$n_{15} := \begin{bmatrix} -3429X^3Y + 5519X^5 \\ 6298X^3Y + 8944X^5 \\ Y^3 + 3129X^4Y - 5885X^6 \\ 6451X^6 \end{bmatrix}, \qquad \mathrm{LT}(n_{15}) = \begin{bmatrix} 0 \\ 0 \\ Y^3 \\ 0 \end{bmatrix};$$

$$n_{16} := \begin{bmatrix} 1640X^3Y + 12031X^5 \\ -7573X^3Y - 252X^5 \\ 15985X^4Y + 346X^6 \\ Y^3 - 13733X^6 \end{bmatrix}, \qquad \mathrm{LT}(n_{16}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Y^3 \end{bmatrix};$$

$$n_{17} := \begin{bmatrix} -13710Z + 9139Y - 4568X^2 \\ Z - 4568Y - 4573X^2 \\ 13705XY + 2290X^3 \\ 6857XY - 12568X^3 \end{bmatrix}, \qquad \mathrm{LT}(n_{17}) = \begin{bmatrix} 0 \\ Z \\ 0 \\ 0 \end{bmatrix};$$

$$n_{18} := \begin{bmatrix} -3X \\ 3X \\ Z + 3Y + 15991X^2 \\ X^2 \end{bmatrix}, \qquad \mathrm{LT}(n_{18}) = \begin{bmatrix} 0 \\ 0 \\ Z \\ 0 \end{bmatrix};$$

$$n_{19} := \begin{bmatrix} 3X \\ -X \\ -2Y - 15994X^2 \\ Z \end{bmatrix}, \qquad \mathrm{LT}(n_{19}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Z \end{bmatrix};$$

$$n_{20} := \begin{bmatrix} X^2Z + 12090Y^2 + 2199X^2Y - 7992X^4 \\ -621Y^2 + 487X^2Y - 8680X^4 \\ -3443X^3Y - 11359X^5 \\ -10907X^3Y + 12592X^5 \end{bmatrix}, \qquad \mathrm{LT}(n_{20}) = \begin{bmatrix} X^2Z \\ 0 \\ 0 \\ 0 \end{bmatrix}; \tag{5.16}$$

$$n_{21} := \begin{bmatrix} YZ + 11410Y^2 - 10021X^2Y + 15621X^4 \\ 12180Y^2 - 260X^2Y - 11722X^4 \\ -14804X^3Y + 11766X^5 \\ -12496X^3Y - 13068X^5 \end{bmatrix}, \qquad \mathrm{LT}(n_{21}) = \begin{bmatrix} YZ \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$n_{22} := \begin{bmatrix} Z^2 - 4574X^2Z - 6880Y^2 + 12373X^2Y - 10835X^4 \\ -2002Y^2 + 12265X^2Y - 458X^4 \\ 422X^3Y - 195X^5 \\ -15167XY^2 + 12940X^3Y - 11538X^5 \end{bmatrix}, \mathrm{LT}(n_{22}) = \begin{bmatrix} Z^2 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$n_{23} := \begin{bmatrix} W - 9142XZ + 11426XY + 2268X^3 \\ 2281XY - 13700X^3 \\ Y^2 - 14830X^2Y - 9157X^4 \\ -15994Y^2 + 12564X^2Y - 1713X^4 \end{bmatrix}, \qquad \mathrm{LT}(n_{23}) = \begin{bmatrix} W \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$n_{24} := \begin{bmatrix} -4565XZ - 2299XY - 14855X^3 \\ W - 6841XY + 1137X^3 \\ 11Y^2 + 12548X^2Y - 8550X^4 \\ Y^2 + 2302X^2Y + 13135X^4 \end{bmatrix}, \qquad \mathrm{LT}(n_{24}) = \begin{bmatrix} 0 \\ W \\ 0 \\ 0 \end{bmatrix};$$

$$
\begin{aligned}
n_{25} &:= \begin{bmatrix} 4569Z - 13713Y + 6873X^2 \\ -9136Y + 6839X^2 \\ W - 4594XY - 11400X^3 \\ -2283XY + 6852X^3 \end{bmatrix}, & \mathrm{LT}(n_{25}) &= \begin{bmatrix} 0 \\ 0 \\ W \\ 0 \end{bmatrix}; \\
n_{26} &:= \begin{bmatrix} -13710Z + 9143Y + 11414X^2 \\ -4573Y + 11436X^2 \\ 13726XY + 10266X^3 \\ W + 6854XY - 12566X^3 \end{bmatrix}, & \mathrm{LT}(n_{26}) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ W \end{bmatrix}; \\
n_{27} &:= \begin{bmatrix} t + Z - 5X^2 \\ 3X^2 \\ 4XY \\ -15991XY + 15995X^3 \end{bmatrix}, & \mathrm{LT}(n_{27}) &= \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix}; \\
n_{28} &:= \begin{bmatrix} -13711Z + 9142Y - 4577X^2 \\ t - 4574Y - 4563X^2 \\ 13723XY - 13719X^3 \\ 6855XY - 12565X^3 \end{bmatrix}, & \mathrm{LT}(n_{28}) &= \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix}; \\
n_{29} &:= \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix}, & \mathrm{LT}(n_{29}) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix}; \\
n_{30} &:= \begin{bmatrix} -4571Z + 13709Y + 9154X^2 \\ 9142Y + 9128X^2 \\ Xt + 4554XY + 11445X^3 \\ -13710XY + 9139X^3 \end{bmatrix}, & \mathrm{LT}(n_{30}) &= \begin{bmatrix} 0 \\ 0 \\ Xt \\ 0 \end{bmatrix}; \\
n_{31} &:= \begin{bmatrix} -4569XZ - 2287XY - 6849X^3 \\ -6852XY + 9132X^3 \\ Yt - 3440X^2Y - 8550X^4 \\ 10290X^2Y + 13136X^4 \end{bmatrix}, & \mathrm{LT}(n_{31}) &= \begin{bmatrix} 0 \\ 0 \\ Yt \\ 0 \end{bmatrix}; \\
n_{32} &:= \begin{bmatrix} 4592XZ - 13725XY + 6642X^3 \\ -9143XY + 7015X^3 \\ t^2 + 33Y^2 + 11637X^2Y + 4360X^4 \\ 3Y^2 - 2249X^2Y + 6872X^4 \end{bmatrix}, & \mathrm{LT}(n_{20}) &= \begin{bmatrix} 0 \\ 0 \\ t^2 \\ 0 \end{bmatrix}.
\end{aligned} \tag{5.17}
$$

It is easy to check that $\{\mathrm{LT}(n_i),\dots,\mathrm{LT}(n_{32})\}$ generate $\mathrm{LT}(N_0)$, in other words, that these n_i's form a Groebner basis for N_0. Let $m\colon (X,Y,Z,W,t)\mapsto (X,Y,Z,W,t^k)$ so that $\hat{g}_k=\hat{g}_1\circ m$; furthermore $\hat{g}_k^*\mathrm{Der}(\text{-log } V)_2 = m^*\hat{g}_1^*\mathrm{Der}(\text{-log } V)_2$. Then a basis for N is given by $n_1,\dots,n_{26}$ and

$$\begin{aligned}
\tilde{n}_{27} &:= m^*n_{27} = \begin{bmatrix} t^k+Z-5X^2 \\ 3X^2 \\ 4XY \\ -15991XY+15995X^3 \end{bmatrix}, & \mathrm{LT}(\tilde{n}_{27}) &= \begin{bmatrix} t^k \\ 0 \\ 0 \\ 0 \end{bmatrix};\\
\tilde{n}_{28} &:= m^*n_{28} = \begin{bmatrix} -13711Z+9142Y-4577X^2 \\ t^k-4574Y-4563X^2 \\ 13723XY-13719X^3 \\ 6855XY-12565X^3 \end{bmatrix}, & \mathrm{LT}(\tilde{n}_{28}) &= \begin{bmatrix} 0 \\ t^k \\ 0 \\ 0 \end{bmatrix};\\
\tilde{n}_{29} &:= m^*n_{29} = \begin{bmatrix} 0 & 0 & 0 & t^k \end{bmatrix}^t, \\
\tilde{n}_{30} &:= m^*n_{30} = \begin{bmatrix} -4571Z+13709Y+9154X^2 \\ 9142Y+9128X^2 \\ Xt^k+4554XY+11445X^3 \\ -13710XY+9139X^3 \end{bmatrix}, & \mathrm{LT}(\tilde{n}_{30}) &= \begin{bmatrix} 0 \\ 0 \\ Xt^k \\ 0 \end{bmatrix};\\
\tilde{n}_{31} &:= m^*n_{31} = \begin{bmatrix} -4569XZ-2287XY-6849X^3 \\ -6852XY+9132X^3 \\ Yt^k-3440X^2Y-8550X^4 \\ 10290X^2Y+13136X^4 \end{bmatrix}, & \mathrm{LT}(\tilde{n}_{31}) &= \begin{bmatrix} 0 \\ 0 \\ Yt^k \\ 0 \end{bmatrix};\\
\tilde{n}_{32} &:= m^*n_{32} = \begin{bmatrix} 4592XZ-13725XY+6642X^3 \\ -9143XY+7015X^3 \\ t^{2k}+33Y^2+11637X^2Y+4360X^4 \\ 3Y^2-2249X^2Y+6872X^4 \end{bmatrix}, & \mathrm{LT}(\tilde{n}_{33}) &= \begin{bmatrix} 0 \\ 0 \\ t^{2k} \\ 0 \end{bmatrix};\\
n_{33} &:= \begin{bmatrix} 0 & 0 & 0 & t^{k-1} \end{bmatrix}^t.
\end{aligned} \tag{5.18}$$

It is clear that we can discard $\tilde{n}_{29}$ as it is a multiple of n_{33}. In fact, the elements $n_1,\dots,n_{26},\tilde{n}_{27},\tilde{n}_{28},\tilde{n}_{30},\tilde{n}_{31},\tilde{n}_{32},n_{33}$ together with the following elements form a

Groebner basis for N (see Appendix D.2 for details).

$$n_{36} = \begin{bmatrix} 3718Y^2t^{k-1} + 1565X^2Yt^{k-1} - 6969X^4t^{k-1} \\ Y^2t^{k-1} - 10646X^2Yt^{k-1} - 8308X^4t^{k-1} \\ -9098X^3Yt^{k-1} + 11349X^5t^{k-1} \\ 0 \end{bmatrix}, \mathrm{LT}(\tilde{n}_{36}) = \begin{bmatrix} 0 \\ Y^2t^{k-1} \\ 0 \\ 0 \end{bmatrix} \quad \text{(cf. (D.19));}$$

$$n_{38} = \begin{bmatrix} 15994Xt^{k-1} \\ -15995Xt^{k-1} \\ Yt^{k-1} + 7997X^2t^{k-1} \\ 0 \end{bmatrix}, \quad \mathrm{LT}(\tilde{n}_{38}) = \begin{bmatrix} 0 \\ 0 \\ Yt^{k-1} \\ 0 \end{bmatrix} \quad \text{(cf. (D.21));}$$

$$n_{39} = \begin{bmatrix} Zt^{k-1} + 10670Yt^{k-1} + 15969X^2t^{k-1} \\ 10657Yt^{k-1} - 5307X^2t^{k-1} \\ 10700XYt^{k-1} - 8037X^3t^{k-1} \\ 0 \end{bmatrix}, \quad \mathrm{LT}(\tilde{n}_{39}) = \begin{bmatrix} Zt^{k-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(cf. (D.22));}$$

$$n_{43} = \begin{bmatrix} Y^2t^{k-1} + 2579X^2Yt^{k-1} - 10708X^4t^{k-1} \\ -3812X^2Yt^{k-1} - 9495X^4t^{k-1} \\ -14648X^5t^{k-1} \\ 0 \end{bmatrix}, \quad \mathrm{LT}(\tilde{n}_{43}) = \begin{bmatrix} Y^2t^{k-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(cf. (D.33));}$$

$$n_{45} = \begin{bmatrix} -9141XYt^{k-1} + 14849X^3t^{k-1} \\ XYt^{k-1} + 10282X^3t^{k-1} \\ -15994X^4t^{k-1} \\ 0 \end{bmatrix}, \quad \mathrm{LT}(\tilde{n}_{45}) = \begin{bmatrix} 0 \\ XYt^{k-1} \\ 0 \\ 0 \end{bmatrix} \quad \text{(cf. (D.35));}$$

$$n_{52} = \begin{bmatrix} X^5t^{k-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{LT}(\tilde{n}_{52}) = \begin{bmatrix} X^5t^{k-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(cf. (D.42));}$$

$$n_{53} = \begin{bmatrix} X^2Yt^{k-1} - 3619X^4t^{k-1} \\ 1398X^4t^{k-1} \\ -5830X^5t^{k-1} \\ 0 \end{bmatrix}, \quad \mathrm{LT}(\tilde{n}_{53}) = \begin{bmatrix} XY^2t^{k-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(cf. (D.43));}$$

$$n_{54} = \begin{bmatrix} -11989X^4t^{k-1} \\ 10664X^4t^{k-1} \\ X^5t^{k-1} \\ 0 \end{bmatrix}, \quad \mathrm{LT}(\tilde{n}_{54}) = \begin{bmatrix} 0 \\ 0 \\ X^5t^{k-1} \\ 0 \end{bmatrix} \quad \text{(cf. (D.44));}$$

$$n_{55} = \begin{bmatrix} 4285X^4t^{k-1} \\ X^4t^{k-1} \\ 0 \\ 0 \end{bmatrix}, \mathrm{LT}(\tilde{n}_{55}) = \begin{bmatrix} 0 \\ X^4t^{k-1} \\ 0 \\ 0 \end{bmatrix} (\text{cf. (D.45)}).$$

It follows that

$$\begin{aligned}
&\dim_{\mathbb{C}} N\mathcal{K}_{V,e}\hat{g}_k = \dim_{\mathbb{C}} \frac{\mathcal{O}^4_{\mathbb{C}^5,0}}{(\mathrm{LT}(N))\mathcal{O}_{\mathbb{C}^5,0}} \\
&= \dim_{\mathbb{C}} \Big(\frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^7, X^4Y, XY^2, Y^3, X^2Z, YZ, Z^2, W, X^5t^{k-1}, X^2Yt^{k-1}, Y^2t^{k-1}, Zt^{k-1}, t^k)\mathcal{O}_{\mathbb{C}^5,0}} \\
&\oplus \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^6, X^4Y, XY^2, Y^3, Z, W, X^4t^{k-1}, XYt^{k-1}, Y^2t^{k-1}, t^k)\mathcal{O}_{\mathbb{C}^5,0}} \\
&\oplus \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^7, X^5Y, XY^2, Y^3, Z, W, X^5t^{k-1}, Yt^{k-1}, Xt^k, t^{2k})\mathcal{O}_{\mathbb{C}^5,0}} \\
&\oplus \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^7, X^4Y, XY^2, Y^3, Z, W, t^{k-1})\mathcal{O}_{\mathbb{C}^5,0}} \Big) \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^7, X^4Y, XY^2, Y^3, X^2Z, YZ, Z^2, W, X^5t^{k-1}, X^2Yt^{k-1}, Y^2t^{k-1}, Zt^{k-1}, t^k)\mathcal{O}_{\mathbb{C}^5,0}} + \\
&+ \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^6, X^4Y, XY^2, Y^3, Z, W, X^4t^{k-1}, XYt^{k-1}, Y^2t^{k-1}, t^k)\mathcal{O}_{\mathbb{C}^5,0}} + \qquad (5.19) \\
&+ \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^7, X^5Y, XY^2, Y^3, Z, W, X^5t^{k-1}, Yt^{k-1}, Xt^k, t^{2k})\mathcal{O}_{\mathbb{C}^5,0}} + \\
&+ \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^7, X^4Y, XY^2, Y^3, Z, W, t^{k-1})\mathcal{O}_{\mathbb{C}^5,0}}.
\end{aligned}$$

The complement of the ideal

$$(X^7, X^4Y, XY^2, Y^3, X^2Z, YZ, Z^2, W, X^5t^{k-1}, X^2Yt^{k-1}, Y^2t^{k-1}, Zt^{k-1}, t^k)$$

consists of $14k-7$ monomials, those are

$$\begin{gathered}
1, X, \ldots, X^6, Y, XY, X^2Y, X^3Y, Y^2, Z, XZ, \\
t, Xt, \ldots, X^6t, Yt, XYt, X^2Yt, X^3Yt, Y^2t, Zt, XZt, \\
\vdots \\
t^{k-2}, Xt^{k-2}, \ldots, X^6t^{k-2}, Yt^{k-2}, XYt^{k-2}, X^2Yt^{k-2}, X^3Yt^{k-2}, Y^2t^{k-2}, Zt^{k-2}, XZt^{k-2}, \\
t^{k-1}, Xt^{k-1}, X^2t^{k-1}, X^3t^{k-1}, X^4t^{k-1}, Yt^{k-1}, XYt^{k-1}.
\end{gathered}$$

The complement of $(X^6, X^4Y, XY^2, Y^3, Z, W, X^4t^{k-1}, XYt^{k-1}, Y^2t^{k-1}, t^k)$ consists of $11k-6$ monomials, those are

$$\begin{gathered}1, X, \dots, X^5, Y, XY, X^2Y, X^3Y, Y^2,\\ t, Xt, \dots, X^5t, Yt, XYt, X^2Yt, X^3Yt, Y^2t,\\ \vdots\\ t^{k-2}, Xt^{k-2}, \dots, X^5t^{k-2}, Yt^{k-2}, XYt^{k-2}, X^2Yt^{k-2}, X^3Yt^{k-2}, Y^2t^{k-2},\\ t^{k-1}, Xt^{k-1}, X^2t^{k-1}, X^3t^{k-1}, Yt^{k-1}.\end{gathered}$$

The complement of the ideal $(X^7, X^5Y, XY^2, Y^3, Z, W, X^5t^{k-1}, Yt^{k-1}, Xt^k, t^{2k})$ consists of $14k-8$ monomials, those are

$$\begin{gathered}1, X, \dots, X^6, Y, XY, X^2Y, X^3Y, X^4Y, Y^2,\\ t, Xt, \dots, X^6t, Yt, XYt, X^2Yt, X^3Yt, X^4Y, Y^2t,\\ \vdots\\ t^{k-2}, Xt^{k-2}, \dots, X^6t^{k-2}, Yt^{k-2}, XYt^{k-2}, X^2Yt^{k-2}, X^3Yt^{k-2}, X^4t^{k-2}, Y^2t^{k-2},\\ t^{k-1}, Xt^{k-1}, X^2t^{k-1}, X^3t^{k-1}, t^k, \dots, t^{2k-1}.\end{gathered}$$

Finally, the complement of the ideal $(X^7, X^4Y, XY^2, Y^3, Z, W, t^{k-1})$ is spanned by the following $12(k-1)$ monomials

$$\begin{gathered}1, X, \dots, X^6, Y, XY, X^2Y, X^3Y, Y^2,\\ t, Xt, \dots, X^6t, Yt, XYt, X^2Yt, X^3Yt, Y^2t,\\ \vdots\\ t^{k-1}, Xt^{k-1}, \dots, X^6t^{k-1}, Yt^{k-1}, XYt^{k-1}, X^2Yt^{k-1}, X^3Yt^{k-1}, Y^2t^{k-1}.\end{gathered}$$

All these sum up to $51k-33$. Hence, this concludes the proof of $\mathcal{A}_e$-codim$\tilde{D}_k = 51k-33$. As for the Mond conjecture, we have not got a proof for general case. However, we have checked that for each $k \leq 20$, $N\mathcal{K}_{H,e/\mathbb{C}}G$ is a 1-dimensional Cohen-Macaulay module by a calculation on SINGULAR over characteristic 0 (here G is a 1-parameter unfolding of $\hat{g}_k$ and H is the defining equation of the image of F). Therefore, $\tilde{D}_k$ satisfies the conjecture for $k \leq 20$ (cf. Remark 3.2.13).

$\square$

Remark 5.2.3. We have checked for $k \leq 30$ that $\mathcal{A}_e$-codim$(\tilde{D}_k) = 51k - 33$ over characteristic 0 as well.

Proposition 5.2.4. *Let $\tilde{E}_k \in \mathcal{E}^0_{4,5}$ be given by*

$$\begin{array}{rcl} \tilde{E}_k \colon (\mathbb{C}^4, 0) & \to & (\mathbb{C}^5, 0) \\ (x, y, z, t) & \mapsto & (x, t, y^2 + xz + x^2 y + t^k y, yz, z^2 + y^3 + t^{2k} y) \end{array}$$

for $k \geq 1$. Then $\tilde{E}_k$ is finitely $\mathcal{A}$-determined.

Proof. We can easily check that $\tilde{E}_k$ is a reduction of

$$F(x, y, z, a, b, c) := (x, y^2 + xz + x^2 y + ay, yz + by, z^2 + y^3 + cy, a, b, c)$$

by $\gamma \colon t \mapsto (t^k, 0, t^{2k})$. Then the image of γ is given by $\left\{C - A^2 = 0, B = 0\right\}$. One can check that, the image cuts $D_V(G)$ only at the origin. Hence, $\tilde{E}_k$ is finitely $\mathcal{A}$-determined by Theorem 5.1.2. $\square$

Remark 5.2.5. By the same method of Proposition 5.2.1 and 5.2.2, it is possible to show that $\mathcal{A}_e$-codimension of $\tilde{E}_k$ is, in fact, $45k - 18$.

Remark 5.2.6. It is possible to prove that $N\mathcal{K}_{H,e/\mathbb{C}}G$ is a Cohen-Macaulay module of dimension 1 for a 1-parameter deformation of g_k for all the examples. One needs to consider a Groebner basis with respect to a *graded reverse lexicographic order* for the module in question and use a property by Bayer and Stillman (see, for instance, [Eis95, Theorem 15.13]).

Conclusion

The main motivation for this thesis has been the Mond conjecture (Conjecture 3.1.5). We studied the interaction between multiple point spaces and finite determinacy of map-germs. We found new examples supporting the conjecture and possible leads that could assist a further study.

We introduced the notion of reduction of map-germs – a special base change operation on unfoldings – and proved a criterion on finite determinacy of reductions (Theorem 5.1.2). This criterion provides an algorithm for producing examples from old (Algorithm 5.1.3). We were able to find three series of $\mathcal{A}$-finite map-germs from 4-space to 5-space:

$$\begin{aligned} \tilde{C}_k : (x,y,z,t) &\mapsto (x,t,y^2+xz+x^2y, yz+t^k, z^2+y^3) \\ \tilde{D}_k : (x,y,z,t) &\mapsto (x,t,y^2+xz, z^2+xy, y^3+yz^2+z^3+t^k z) \\ \tilde{E}_k : (x,y,z,t) &\mapsto (x,t,y^2+xz+x^2y+t^k y, yz, z^2+y^3+t^{2k}y) \end{aligned}$$

(see Proposition 5.2.1, 5.2.2 and 5.2.4 respectively). By an application of Groebner bases, we proved that $\mathcal{A}_e$-codim$(\tilde{C}_k) = 33k-18$ and $\mathcal{A}_e$-codim$(\tilde{D}_k) = 51k-33$. We carried out the proof for the latter over characteristic 31991 since the number of the digits of coefficients in the basis reach up to 50 over characteristic 0. We confirmed that for $k \leq 30$, the formula also holds over characteristic 0. For $k \leq 20$, $\mathcal{A}_e$-codimension of $\tilde{E}_k$ satisfy the formula $\mathcal{A}_e$-codim$(\tilde{E}_k) = 45k-18$ (a general proof by a Groebner basis calculation can be given). We left this lengthy calculation for a further study. All the series satisfy the Mond conjecture for $k \leq 20$. For $k > 20$, we are yet to check the conjecture or prove a general statement.

These series were induced from the examples $\tilde{A}_2, \tilde{B}_3^+ \in \mathcal{E}_{3,4}^0$ (Proposition 4.5.3 and 4.5.5). We had started with a classification of 2-jets of corank 2 map-germs from

3-space to 4-space (Appendix C). We noticed that there are no $\mathcal{A}$-finite map-germs of corank 2 in $\mathcal{E}^0_{3,4}$ with degrees $(1,2,2,2)$ (Proposition 4.5.1). Moreover, there are no $\mathcal{A}$-finite map-germs of the form $(x,y,z) \mapsto (x, y^2+xz, z^2+xy, P_{2k}(x,y,z))$ where $P_{2k}(x,y,z)$ is a homogeneous polynomial of degree $2k$ for $k \geq 1$ (Proposition 4.5.2). Then, by experiment, we found $\tilde{A}_k$ for $k = 2,3,4,5,6$, and $\tilde{B}^{\pm}_{2k+1}$ for $k = 1,2,4,5$ and $\tilde{B}^{-}_{2k+1}$ for $k = 3,6$ as examples of finitely $\mathcal{A}$-determined map-germs (Proposition 4.5.3 and 4.5.5). We do not have a general proof for finite determinacy of those. However, we calculated $\mathcal{A}_e$-codimensions for some values of k by our command `Aecodim()` from `Ae.lib` on SINGULAR (Section 3.2.1). When k got bigger, e.g. $k \geq 5$ for $\tilde{A}_k$, computer ran out memory before concluding the calculations. There we used Theorem 4.4.1 which gives conditions on isolated instability of map-germs in $\mathcal{E}^0_{3,4}$. We confirmed that all of these but $\tilde{A}_5, \tilde{A}_6$ and $\tilde{B}^{-1}_{13}$ satisfy the conjecture. For $\tilde{A}_5, \tilde{A}_6$ and $\tilde{B}^{-1}_{13}$ we are yet to check.

We proved that for corank ≥ 2 map-germs in $\mathcal{E}^0_{3,4}$, $f^*\mathrm{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^3,0}) = \mathrm{Fitt}_0((\pi^2_1)_*\mathcal{O}_{D^2(f)})$, and $f^*\mathrm{Fitt}_2(f_*\mathcal{O}_{\mathbb{C}^3,0})$ and $\mathrm{Fitt}_1((\pi^2_1)_*\mathcal{O}_{D^2(f)})$ agree outside the origin (Proposition 4.2.1).

We rephrased Marar's formula for the image Milnor number of map-germs in $\mathcal{E}^0_{3,4}$ (Corollary 4.6.5). In the formula, there is only one entry that we could not manage calculating directly for corank ≥ 2 map-germs; namely, Milnor number of the double point space. This is due to the fact that D^2 is no longer an ICIS.

We showed that $D^2_1(f)$ coincides with support of $\ker(\mu)$ in $\mathbb{C}^n$ by constructing a presentation of $\ker(\mu)$ for any corank. Kleiman, Lipman and Ulrich had proved that $\ker(\mu)$ and $\mathcal{O}_{D^2(f)}$ are isomorphic for corank 1 map-germs. We used this fact and the construction of a presentation of $\ker(\mu)$ to prove that if f is $\mathcal{A}$-finite, a resolution of $\mathcal{O}_{D^{k+1}(f)}$ over $\mathcal{O}_{D^k(f)}$ is a short exact sequence which can be deduced inductively from the resolution of $\mathcal{O}_{\mathbb{C}^n,0}$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$ (Theorem 2.6.6). We studied the lifting maps between the resolutions of $\mathcal{O}_{ID^2}$ and $\mathcal{O}_{D^2}$ over $\mathcal{O}_{\mathbb{C}^n,0}$. Combining Theorem 2.6.6 with a result of [MS10], we deduced new examples of free divisors living in D^k (Proposition 2.8.4).

We also compared Noether, Kähler and Dedekind differents for $\mathcal{O}_{\mathbb{C}^n,0}$, when considered as an $\mathcal{O}_{\mathbb{C}^p,0}$ algebra via a finite $f \in \mathcal{E}^0_{n,p}$ (Example A.0.15).

Questions for further studies.

1. Prove Mond's conjecture (Conjecture 3.1.5) or equivalently, prove that $N\mathcal{K}_{V,e/\mathbb{C}}G$ is a Cohen-Macaulay module of dimension 1 (cf. Conjecture 3.2.14).

2. Suppose that $f \in \mathcal{E}^0_{n,p}$ is $\mathcal{A}$-finite and $R_\gamma(F)$ is constructed as in Theorem 5.1.2. Is it true that $R_\gamma(F)$ satisfies Mond conjecture if f does?

3. Is it possible to modify Theorem 5.1.2 to involve map-germs $G\colon (\mathbb{C}^{p'}, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ rather than just the identity?

4. Investigate Theorem 2.8.3 and Proposition 2.8.4 further to see if there is a relation to the Mond conjecture.

5. Prove that $\det(\lambda) \cdot \det(\lambda^1_1) \cdot \det(\lambda^2_2)$ defines a free divisor in the case of corank 1 stable map-germs (cf. Remark 2.8.5).

6. Let $f \in \mathcal{E}^0_{n,n+1}$ be of corank ≥ 2. Prove that $f^*\mathrm{Fitt}_i(f_*\mathcal{O}_{\mathbb{C}^n,0})$ coincides with $\mathrm{Fitt}_{i-1}((\pi^2_1)_*\mathcal{O}_{D^2(f)})$ for $i \geq 2$ (cf. Conjecture 4.2.3).

7. Prove that $\tilde{A}_k : (x, y, z) \mapsto (x, y^k + xz + x^{2k-2}y, yz, z^2 + y^{2k-1})$ is finitely $\mathcal{A}$-determined for all $k \geq 7$ (Conjecture 4.5.4).

8. Prove that $\tilde{B}^-_{2k+1} : (x, y, z) \mapsto (x, y^2 + xz, z^2 + xy, y^{2k+1} \pm y^{2k-1}z^2 + z^{2k+1})$ is finitely $\mathcal{A}$-determined for all k, and $\tilde{B}^+_{2k+1}$ is finitely $\mathcal{A}$-determined for those k which are not divisible by 3 (Conjecture 4.5.6).

9. Expand the list of examples of finitely $\mathcal{A}$-determined map-germs.

10. For an $f \in \mathcal{E}^0_{3,4}$, $D^2(f)$ is a Cohen-Macaulay space and a normalisation of $D^2_1(f)$ (Proposition 4.3.1). Could this fact be used to calculate $\mu(D^2(f))$?

11. Prove that $\mathrm{Fitt}_3(f_*\mathcal{O}_{\mathbb{C}^3,0})$ gives the right structure for $M_4(f)$, that is, $|M_4(f_t)| = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^4,0}}{\mathrm{Fitt}_3(f_*\mathcal{O}_{\mathbb{C}^3,0})}$ (cf. Remark 4.6.7).

12. Prove that $|D^3_1(f_t) \cap V(R_{f_t})| = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^3,0}}{I_{D^3_1(f)} + R_f}$ (cf. Remark 4.6.7).

13. Find geometric conditions on the finite $\mathcal{A}$-determinacy of map-germs in $\mathcal{E}^0_{4,5}$.

14. Study the geometry of the series $\tilde{C}_k$, $\tilde{D}_k$ and $\tilde{E}_k$.

15. Are there any finitely $\mathcal{A}$-determined weighted homogeneous corank 3 map-germs?

16. Is it only a coincidence that we meet the so-called *bracelet* equation in the classification of 2-jets (cf. Remark C.3.1)?

Appendix A

Differents

In Chapter 2, we have seen that the multiplication map (2.26) plays a role in the theory of double point spaces. It also directs us to the notion of the Noether different of an algebra over a commutative ring. We will recall the definitions of three differents: Noether, Kähler and Dedekind, and show that those three coincide for $\mathcal{O}_{\mathbb{C}^n,0}$ when considered as an $\mathcal{O}_{\mathbb{C}^p,0}$-algebra via a finite map-germ. For details, we suggest [Kun86] or [Yos90] to the reader.

Noether (Homological) Different. The Noether different was first introduced by Noether in [Noe50] for commutative rings. Later, Auslander and Goldman ([AG60]) used this theory in the study of Brauer groups.

The setup for the definition is as follows. Let S and R be commutative Noetherian rings. Suppose $S = (S, +, .)$. Let S^o be the *opposite ring* of S; that is, $S^o = (S, +, *)$ with the multiplication $*$ given by $a * b = b.a$. Clearly, when S is commutative $S = S^o$. The algebra $S^e := S \otimes_R S^o$ is called *the enveloping algebra* of S over R. We consider the multiplication map

$$\begin{aligned} \mu\colon S^e &\to S \\ a \otimes b &\mapsto ab, \end{aligned}$$

and its kernel $\ker(\mu)$.

Definition A.0.7. The *Noether different* of $R \to S$ is defined by

$$\vartheta_N(S/R) := \mu(\mathrm{Ann}_{S^e}(\ker(\mu))).$$

It can easily be shown that $\vartheta_N(S/R)$ is an ideal of S.

Definition A.0.8. The ring S is a *separable* R-algebra if μ splits, i.e. if there exists an S^e-homomorphism $i\colon S \to S^e$ such that μi is the identity map on S.

The Noether different measures the separability of S over R.

Proposition A.0.9 (Lemma 2.1, [AB59])**.** *Let S be an R-algebra. Then the following statements are equivalent:*

(i) S is S^e-projective,

(ii) μ splits,

(iii) $\vartheta_N(S/R) = S$.

Another important characteristic of $\vartheta_N(S/R)$ is the following.

Lemma A.0.10 (Lemma 6.8, [Yos90])**.** *Let S be an R-algebra as above and let M be any S^e-module. Then $\vartheta_N(S/R)$ annihilates the ith Hochschild cohomology $H^i_R(S, M)$ for $i > 0$.*

How do we calculate $\vartheta_N(S/R)$?

Suppose S is a finite R-algebra. Then there exists a presentation of S over R as

$$S = R[X_1, \dots, X_n]/a(X)$$

where a is an ideal of $R[X_1, \dots, X_n]$. In that case we have

$$\begin{aligned} S^e &\cong R[X_1, \dots, X_n]/a(X) \otimes_R R[X_1, \dots, X_n]/a(X) \\ &\cong R[X_1, \dots, X_n, x_1, \dots, x_n]/a(X) + a(x). \end{aligned}$$

So μ is the map $x_i \mapsto X_i$. Then $\ker(\mu)$ is generated by $X_1 - x_1, \dots, X_n - x_n$. Moreover,

$$\vartheta_N(S/R) = \mu\left(0 :_{S^e} (X_1 - x_1, \dots, X_n - x_n)\right).$$

Kähler Different.

Definition A.0.11. Let S and R be given as above. *The Kähler different* is, then

$$\vartheta_K(S/R) := \mathrm{Fitt}_0(\Omega^1_{S/R}).$$

Let us assume that S is a finite R-algebra. If $S = R[X_1, \dots, X_n]/a$ with $a = (f_1, \dots, f_p)$ then there exists an exact sequence

$$S^p \xrightarrow{J} S^n \cong SdX_1 \oplus SdX_n \longrightarrow \Omega^1_{S/R} \longrightarrow 0$$

where

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_p}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_p}{\partial x_n} \end{bmatrix}$$

and x_i is the image of X_i in S. Hence $\vartheta_K(S/R)$ equals to the ramification ideal.

Proposition A.0.12 (Proposition 10.18, [Kun86])**.** *Let S be an R algebra of finite type. Let k be the number of generators of $\Omega^1_{S/R}$. Then*

$$\vartheta_N(S/R)^k \subset (\mathrm{Ann}\Omega^1_{S/R})^k \subset \vartheta_K(S/R) \subset \vartheta_N(S/R) \subset \mathrm{Ann}\Omega^1_{S/R}.$$

Dedekind Different.

Definition A.0.13. Let S and R be as before. Let $Q(S)$ and $Q(R)$ denote the total rings of fractions of S and R respectively. Then *the Dedekind different* of $R \to S$ is

$$\vartheta_D(S/R) := \{\varphi(Tr) \in S \mid \varphi \in \mathrm{Hom}_S(\mathrm{Hom}_R(S, R), S)\}$$

where Tr is the trace map of $Q(S)$ over $Q(R)$, that is, the map sending $s \in S$ to the trace of the $Q(R)$-linear endomorphism of $Q(S)$ induced by multiplication by s.

The Dedekind different is also equal to the inverse ideal of *the complementary module* $C^S := \{x \in Q(S) \mid Tr(xS) \subset R\}$.

It is known that in certain cases some (or all) the differents coincide. The following states one of those cases.

Lemma A.0.14 (Chapter 6, [Yos90])**.** *If S is reduced and Cohen-Macaulay then the Noether different is equal to the Dedekind different.*

Now, we will show another example.

Example A.0.15. Let $f \in \mathcal{E}^0_{n,p}$, $n < p$, be a finite analytic map-germ which induces the $\mathcal{O}_{\mathbb{C}^p,0}$-algebra structure on $\mathcal{O}_{\mathbb{C}^n,0}$. Then

$$\vartheta_N(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{O}_{\mathbb{C}^p,0}) = \vartheta_D(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{O}_{\mathbb{C}^p,0}) \quad \text{and} \quad \sqrt{\vartheta_N(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{O}_{\mathbb{C}^p,0})} = \sqrt{\vartheta_K(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{O}_{\mathbb{C}^p,0})}.$$

Proof. The equality $\vartheta_N(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{O}_{\mathbb{C}^p,0}) = \vartheta_D(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{O}_{\mathbb{C}^p,0})$ is just Lemma A.0.14. So we only need to show the second equality as follows.

Consider the isomorphism $(\mathbb{C}^n, 0) \cong \text{graph}(f) = \{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{C}^n \times \mathbb{C}^p\}$ which induces

$$\mathcal{O}_{\mathbb{C}^n,0} \cong \mathcal{O}_{\mathbb{C}^p,0}\{\mathbf{x}\}/(Y_1 - f_1(\mathbf{x}), \dots, Y_p - f_p(\mathbf{x})).$$

Hence,

$$\begin{aligned}\mathcal{O}^e_{\mathbb{C}^n,0} &= \mathcal{O}_{\mathbb{C}^p,0}\{\mathbf{x}\}/(Y_1 - f_1(\mathbf{x}), \dots, Y_p - f_p(\mathbf{x})) \otimes_{\mathcal{O}_{\mathbb{C}^p,0}} \\ &\qquad \mathcal{O}_{\mathbb{C}^p,0}\{\mathbf{x}\}/(Y_1 - f_1(\mathbf{x}), \dots, Y_p - f_p(\mathbf{x})) \\ &\cong \frac{\mathcal{O}_{\mathbb{C}^p,0}\{\mathbf{x}, \mathbf{X}\}}{(Y_1 - f_1(\mathbf{x}), \dots, Y_p - f_p(\mathbf{x}), Y_1 - f_1(\mathbf{X}), \dots, Y_p - f_p(\mathbf{X}))} \\ &\cong \frac{\mathcal{O}_{\mathbb{C}^p,0}\{\mathbf{x}, \mathbf{X}\}}{(Y_1 - f_1(\mathbf{x}), \dots, Y_p - f_p(\mathbf{x}), f_1(\mathbf{X}) - f_1(\mathbf{x}), \dots, f_p(\mathbf{X}) - f_p(\mathbf{x}))}.\end{aligned}$$

In this case, μ is equivalent to $X_i \mapsto x_i$ for all $i = 1, \dots, n$. So,

$$\ker(\mu) = (x_1 - X_1, \dots, x_n - X_n)$$

and $\text{Ann}_{\mathcal{O}^e_{\mathbb{C}^n,0}}(\ker(\mu)) = \left\{h \in \mathcal{O}^e_{\mathbb{C}^n,0} \mid h \cdot (x_i - X_i) = 0 \text{ for all } i = 1, \dots, n\right\}$. On the other hand, there exist $\hat{\alpha}_{ij}(x, X)$, $1 \le i \le n, 1 \le j \le p$, satisfying

$$f_j(x) - f_j(X) = \sum_{i=1}^{n} \hat{\alpha}_{ij} \cdot (x_i - X_i)$$

for all $j = 1, \dots, p$. But $f_j(\mathbf{x}) - f_j(\mathbf{X}) \equiv 0$ for all $j = 1, \dots, p$. Hence we obtain the exact sequence

$$(\mathcal{O}^e_{\mathbb{C}^n,0})^p \xrightarrow{\hat{\alpha} := (\hat{\alpha}_{ij})} (\mathcal{O}^e_{\mathbb{C}^n,0})^n \longrightarrow \ker(\mu) \longrightarrow 0.$$

It follows that

$$\text{Fitt}_0(\ker(\mu)) = \bigwedge^n \hat{\alpha} \mod (f \times f)^* I_{\Delta_p}.$$

Since $\hat{\alpha}$ coincides with the matrix α in the definition of $\mathcal{I}_2(f)$ (cf. (2.5)), by Lemma 2.1.16, we deduce that $\vartheta_N(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{O}_{\mathbb{C}^p,0}) = R_f = \vartheta_K(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{O}_{\mathbb{C}^p,0})$. This concludes the proof. □

Appendix B

SINGULAR code for Ae.lib

```
///////////////////////////////////////////////////////////////////////////////
// Singular-library
//////////////////////////////////////////////////////////////////////////////
version="$Id: Ae.lib, v 1.00  February, 2011 $";
category="Singularities";
//////////////////////////////////////////////////////////////////////////////

proc Aecodim(ideal f)
{
     if (not(defined(concat)))
          {
          LIB "matrix.lib";
          }
     def esk=basering;
     int i,j;
     int qf=vdim(std(f));
     if (qf==-1)
          {
          return("map is not K-finite");
          }
/// Check if the map is weighted homogeneous:
     def k0=1;
     def K0=1;
     intvec av0=qhweight(f);
     string vr=varstr(esk);
     execute("ring rhom=\r0,("+vr+"),(wp(av0));");
```

```
     def f=fetch(esk,f);
     for (i=1; i<=size(f); i++)
          {
          K0=homog(f[i]);
          if (K0==0)
               {"Map is not weighted homogeneous.";
                    k0=0;}
          }
     setring esk;
     int sf=size(f)*size(f);
     matrix jf=jacob(f);
     module T1=jacob(f)+f*freemodule(size(f));         //  T1:=T\mathcal{K}_ef
     matrix l=kbase(std(T1));
     "Generators for NK_ef:";
     print(l);
/// Eliminate constant vectors from the basis:
     intvec k;
     for (i=1; i<=nrows(l); i++)
          {
          for (j=1; j<=ncols(l); j++)
               {
               if (l[i,j]==1)
                    { k=k,j;}
               else {k=k;}
               }
          }
     intvec K;
     if (size(k)>1)
          {
          for (i=1;i<=ncols(l); i++)
               {
               int h;
               for (j=2; j<=size(k);j++)
                    {if (k[j]!=i) {h++;}
                     }
               if (h==size(k)-1) {K=K,i;}
               kill h;
               }
          K=K[2..size(K)];
```

```
        def l2=submat(l,1..nrows(l),K);
        }
    else {l2=l;}
    kill k;  kill K;
    int vs=nvars(esk);
    int kecod=ncols(l);
/// Define the domain of the stable unfolding F(x,a):=(Fa(x),a):
    execute("ring r2=\r0,("+vr+",a(1..ncols(l2))),dp;");
    def f=fetch(esk,f);
    matrix Fa[size(f)][1]=f;
    def l2=fetch(esk,l2);
    for (i=1;i<=ncols(l2);i++)
        {
        Fa=Fa+a(i)*submat(l2,1..size(f),i);
        }
    ideal fa=Fa;
    intvec av=qhweight(fa);
/// If the map is not weighted homogeneous:
    if (k0==0 || k0<0)
        {
        execute("ring targr=\r0,(X(1..size(fa)),a(1..ncols(l2))),dp;");
        setring r2;
        ideal p=0;
        map F=targr,fa[1..size(fa)],a(1..ncols(l2));
        setring targr;
        def fa=fetch(r2,fa);
        ideal h=preimage(r2,F,p);
        matrix jh=jacob(h);
        module derv=modulo(jh,h);
        execute("ring targr0=\r0,(X(1..size(fa))),dp;");
        def fa=fetch(r2,fa);
        map inc=targr,X(1..size(fa)),0;
        ideal ix=X(1..size(fa));
        module im=jacob(ix)+inc(derv);
        int aecod=vdim(std(im));
        }
/// If the map is weighted homogeneous:
    else
        {
```

```
          execute("ring r3=\r0,("+vr+",a(1..ncols(l2))),(wp(av));");
          def Fa=fetch(r2,Fa);
          ideal fa=Fa;
          def l2=fetch(esk,l2);
          intvec WV;
          intvec WP;
          for (i=1; i<=nrows(Fa);i++)
               {WV[i]=deg(Fa[i,1]); }
          for (i=vs+1;i<=nvars(r3);i++)
               {WP[i-vs]=av[i]; }
      execute("ring targr=\r0,(X(1..size(fa)),a(1..ncols(l2))),(wp(WV,WP));");
          setring r3;
          ideal p=0;
          map F=targr,fa[1..size(fa)],a(1..ncols(l2));
          setring targr;
          def fa=fetch(r3,fa);
          ideal h=preimage(r3,F,p);
          matrix jh=jacob(h);
          ideal xy=submat(jh,1,1..size(fa));
          ideal uv=submat(jh,1,size(fa)+1..nvars(targr));
          module m=modulo(uv,xy);
          execute("ring targr0=\r0,(X(1..size(fa))),(wp(WV));");
          def fa=fetch(r3,fa);
          map inc=targr,X(1..size(fa)),0;
          module im=std(inc(m));
          int aecod=vdim(im);
          }
     "Multiplicity =",qf;      // Print the results:
     if (kecod==-1)
          {
          return("map is not K-finite");
          }
     "K_e-codim =",kecod;
     if (aecod==-1)
          {
          return("map is not A-finite");
          }
     return("A_e-codimension =",aecod);
}
```

Appendix C

A classification of 2-jets of corank 2 map-germs from $\mathbb{C}^3$ to $\mathbb{C}^4$

We claim that there are twenty nine $\mathcal{A}^2$-orbits in $J^2(3,4)$ of corank 2 which we list in Table C.1. In this appendix, we show our calculations in detail. Our method mostly consists of elementary coordinate transformations. For more details on classification techniques, we suggest D. Mond's article [Mon87] or J.W. Bruce's survey [Bru01].

We will denote the coordinates on $\mathbb{C}^3$ and $\mathbb{C}^4$ by x, y, z and X, Y, Z, W, respectively. We will write a coordinate change in a single line, e.g.

$$(x, y, z) \mapsto \big(\psi_1(x, y, z),\ \psi_2(x, y, z),\ \psi_3(x, y, z)\big).$$

However, if a transformation fixes some of the coordinates we will only express its effect on those it changes. For example, $y \mapsto y + az$ means $(x, y, z) \mapsto (x, y + az, z)$, and $y \mapsto y + az,\ z \mapsto z + by$ means $(x, y, z) \mapsto (x, y + az, z + by)$. Furthermore, we will keep the same notation after every coordinate change for simplicity. The effect of the transformations will be clear in the context.

Definition C.0.16. A *scaling* is a transformation of the type

$$(x, y, z) \mapsto (ax, by, cz)$$

for some $a, b, c \in \mathbb{C} \setminus \{0\}$.

2-jet	$\mathcal{A}^2$-codimension	Weights	Reference
$(x, 0, 0, 0)$	21		B.1.2.a
$(x,\ y^2, 0, 0)$	16		B.1.1.a.i
$(x,\ y^2 + xz, 0, 0)$	14		B.1.1.a.ii
$(x,\ yz, 0, 0)$	14		B.1.1.b
$(x,\ xy, 0, 0)$	17		B.1.1.c.i
$(x,\ y^2,\ z^2, 0)$	11		B.2.1.a.i.α_1
$(x,\ y^2 + xz,\ z^2, 0)$	10	$(2b - c, b, c)$	B.2.1.a.i.α_2
$(x,\ y^2 + xz,\ z^2 + xy, 0)$	9	$(1, 1, 1)$	B.2.1.a.i.α_4
$(x,\ y^2 + xz,\ yz, 0)$	10	$(2b - c, b, c)$	B.2.1.a.i.β_1
$(x,\ y^2,\ xz + yz, 0)$	11	(a, a, c)	B.2.1.a.i.β_2
$(x,\ y^2,\ yz, 0)$	12		B.2.1.a.i.β_3
$(x,\ y^2,\ xz, 0)$	12		B.2.1.b.ii.α_1
$(x,\ xy + xz,\ yz, 0)$	11	(a, b, b)	B.2.1.b.ii.α_1
$(x,\ xz,\ yz, 0)$	12		B.2.1.b.ii.α_2
$(x,\ y^2 + xz,\ xy, 0)$	13	$(2b - c, b, c)$	B.2.1.b.ii.α_2
$(x,\ xy,\ xz, 0)$	15		B.2.1.b.ii.α_2
$(x,\ y^2,\ xy, 0)$	14		B.2.1.b.ii.α_2
$(x,\ y^2,\ yz + xz + xy,\ z^2)$	6	$(1, 1, 1)$	B.3.1.a.i.α
$(x,\ y^2,\ yz + xz,\ z^2)$	7	(a, a, c)	B.3.1.a.i.β
$(x,\ y^2 + xz,\ yz,\ z^2)$	8	$(2b - c, b, c)$	B.3.1.a.i.γ
$(x,\ y^2,\ yz,\ z^2)$	10		B.3.1.a.i.δ
$(x,\ y^2,\ xy + xz,\ z^2)$	7	(a, b, b)	B.3.1.a.ii.α_1
$(x,\ y^2 + xz,\ xy,\ z^2)$	8	$(2b - c, b, c)$	B.3.1.a.ii.α_2
$(x,\ y^2,\ xz,\ z^2)$	9		B.3.1.a.ii.α_2
$(x,\ y^2,\ xz,\ yz)$	8		B.3.1.a.ii.β_1
$(x,\ y^2 + xz,\ xy,\ yz)$	9	$(2b - c, b, c)$	B.3.1.a.ii.β_2
$(x,\ y^2,\ xy,\ yz)$	10		B.3.1.a.ii.β_2
$(x,\ y^2,\ xy,\ xz)$	11		B.3.1.b.ii.α
$(x,\ xy,\ xz,\ yz)$	10		B.3.1.b.ii.β

Table C.1: A list of the classes of 2-jets of corank 2 map-germs from $\mathbb{C}^3$ to $\mathbb{C}^4$.

Remark C.0.17. If a coefficient is non-zero in one of f_i's, then we can scale by dividing the corresponding coordinate in the target by that coefficient to get a *monic* series out of that component. In that case, taking that coefficient to be 1 does not effect the nature of (the jet of) the map-germ. So, in what follows, we will assume that some coefficients are equal to 1 when it is evident. For example, $Z = ay^2 + byz$ is equivalent to $Z = y^2 + \frac{b}{a}yz$ by scaling if $a \neq 0$. So we may take $Z = y^2 + byz$.

Definition C.0.18. We say that two jets (or map-germs) are *symmetric* if they are equivalent by the transformation $(x, y, z) \mapsto (x, z, y)$ or by $(X, Y, Z, W) \mapsto (X, \sigma(Y, Z, W), \sigma(Y, Z, W), \sigma(Y, Z, W))$ where σ is a permutation of Y, Z and W.

A 2-jet of a generic corank 2 map-germ f from $\mathbb{C}^3$ to $\mathbb{C}^4$, up to a permutation of the components, is of the form

$$j^2 f(0) = \big(x + \alpha(x, y, z),\ f_2(x, y, z),\ f_3(x, y, z),\ f_4(x, y, z)\big)$$

where

$$\begin{aligned} f_2(x, y, z) &= a_1y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy + a_6x^2 \\ f_3(x, y, z) &= b_1y^2 + b_2yz + b_3z^2 + b_4xz + b_5xy + b_6x^2 \\ f_4(x, y, z) &= c_1y^2 + c_2yz + c_3z^2 + c_4xz + c_5xy + c_6x^2 \end{aligned}$$

for some $a_i, b_i, c_i \in \mathbb{C}$, $i = 1, \ldots, 6$, and $\alpha(x, y, z) \in \mathfrak{m}_{\mathbb{C}^3,0}/\mathfrak{m}^3_{\mathbb{C}^3,0}$. However we can simplify its structure. First of all, the transformation $x \mapsto x + \alpha(x, y, z)$ yields $j^2 f(0) \sim_{\mathcal{A}} \big(x,\ f_2(x, y, z),\ f_3(x, y, z),\ f_4(x, y, z)\big)$. Furthermore, x^2 terms in $j^2 f(0)$ can be cancelled by

$$(X, Y, Z, W) \mapsto (X, Y - a_6X^2, Z - b_6X^2, W - c_6X^2).$$

Thus we can simply take

$$\begin{aligned} j^2 f(0) = (x,\ a_1y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy, b_1y^2 + b_2yz + b_3z^2 + b_4xz + b_5xy, \\ c_1y^2 + c_2yz + c_3z^2 + c_4xz + c_5xy) \end{aligned}$$

Let us consider the *coefficients* matrix

$$C := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix}.$$

Now, depending on the rank of C we can list three cases:
Case 1. If $\operatorname{rank}(C) = 1$ then $j^2 f(0) \sim_{\mathcal{A}} (x,\ a_1y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy, 0, 0)$ by an appropriate linear coordinate change on the target.
Case 2. If $\operatorname{rank}(C) = 2$ then $j^2 f(0) \sim_{\mathcal{A}} (x,\ a_1y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy,\ b_1y^2 + b_2yz + b_3z^2 + b_4xz + b_5xy, 0)$ by a suitable linear coordinate change on the target.

Case 3. If $\text{rank}(C) = 3$ then

$$j^2f(0) = (x,\ a_1y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy, b_1y^2 + b_2yz + b_3z^2 + b_4xz + b_5xy,$$
$$c_1y^2 + c_2yz + c_3z^2 + c_4xz + c_5xy)$$

where none of the components are identically zero.

C.1 Case 1. $f_3 = f_4 = 0$

Let $j^2f(0) = (x,\ a_1y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy, 0, 0)$ for some $a_1, \ldots, a_5 \in \mathbb{C}$. Then we meet the subcases we list below (see Figure C.1 for a *tree* of the subcases).

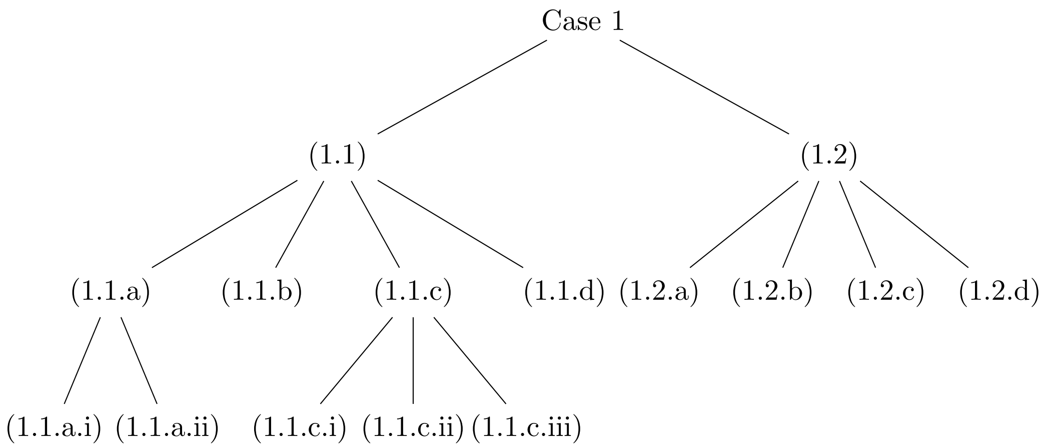


Figure C.1: Tree of the subcases under Case 1.

(1.1) If $a_1 \neq 0$ then we can assume that $a_1 = 1$ by Remark C.0.17. If, moreover, $a_5 \neq 0$ we apply

$$(x, y, z) \mapsto (x, y + \frac{a_5}{2}x, z)$$

and then

$$(X, Y, Z, W) \mapsto (X, Y - \frac{a_5}{4}X^2, Z, W)$$

to get $\big(x,\ y^2 + a_2yz + a_3z^2 + (a_4 - \frac{a_2a_5}{2})xz, 0, 0\big)$. Now,

(1.1.a) Assume that $a_2 = a_3 = 0$.

(1.1.a.i) If $a_4 - \frac{a_2a_5}{2} = 0$ we get $\boxed{(x,\ y^2, 0, 0)}$.

(1.1.a.ii) If $a_4 - \frac{a_2a_5}{2} \neq 0$, by $x \mapsto (a_4 - \frac{a_2a_5}{2})x$ we get $\boxed{(x,\, y^2 + xz, 0, 0)}$.

(1.1.b) Assume that $a_2 \neq 0, a_3 = 0$. Apply $z \mapsto a_2z + y$ so that

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, yz + \frac{1}{a_2}(a_4 - \frac{a_2a_5}{2})xz - \frac{1}{a_2}(a_4 - \frac{a_2a_5}{2})xy, 0, 0\big).$$

If $a_4 - \frac{a_2a_5}{2} = 0$ we get $\boxed{(x, yz, 0, 0)}$. If not,

$$y \mapsto y - \frac{1}{a_2}(a_4 - \frac{a_2a_5}{2})x$$

followed by

$$Y \mapsto Y + \frac{1}{a_2^2}(a_4 - \frac{a_2a_5}{2})^2X^2$$

yields

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, yz - \frac{1}{a_2}(a_4 - \frac{a_2a_5}{2})xy, 0, 0\big).$$

Now $z \mapsto z + \frac{1}{a_2}(a_4 - \frac{a_2a_5}{2})x$ cancels the term xy. So again $j^2f(0) \sim_{\mathcal{A}} (x,\, yz, 0, 0)$.

(1.1.c) Assume that $a_2 = 0,\, a_3 \neq 0$. Apply

$$(x, y, z) \mapsto (x, y + \alpha z, z + \beta y) \tag{C.1}$$

for some $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta \neq 1$, to get

$$\big(x,\, (1+a_3\beta^2)y^2 + (-2\alpha - 2a_3\beta)yz + (a_3 + \alpha^2)z^2 + (a_4 - \frac{a_2a_5}{2})(xz - \beta xy), 0, 0\big). \tag{C.2}$$

(1.1.c.i) Let us choose $\alpha = \sqrt{-a_3}$ and $\beta = -\frac{1}{\sqrt{-a_3}}$. In that case

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, (a_4 - \frac{a_2a_5}{2})xz + \frac{1}{\sqrt{-a_3}}(a_4 - \frac{a_2a_5}{2})xy, 0, 0\big).$$

If $a_4 - \frac{a_2a_5}{2} \neq 0$, by scaling and $x \mapsto \frac{1}{\sqrt{-a_3}}x$ we get $(x, xz, 0, 0)$ which is symmetric to $\boxed{(x,\, xy, 0, 0)}$.

(1.1.c.ii) If we choose $\alpha = \sqrt{-a_3}$ but $\beta \neq \mp\frac{1}{\sqrt{-a_3}}$ then

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, (1 + a_3\beta^2)y^2 + (-2\sqrt{-a_3} - 2a_3\beta)yz + (a_4 - \frac{a_2a_5}{2})(xz - \beta xy), 0, 0\big)$$

$$\sim_{\mathcal{A}} \big(x,\, (-2\sqrt{-a_3} - 2a_3\beta)y(\frac{1+a_3\beta^2}{-2\sqrt{-a_3}-2a_3\beta}y + z) + (a_4 - \frac{a_2a_5}{2})(xz - \beta xy),\, 0, 0\big).$$

Let us set $a := -2\sqrt{-a_3} - 2a_3\beta$, $b := \frac{1+a_3\beta^2}{-2\sqrt{-a_3}-2a_3\beta}$ and $c := a_4 - \frac{a_2a_5}{2}$ for simplicity. Note that $a \neq 0$, $b \neq 0$. We have

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, ay(by+z) + c(xz - \beta xy), 0, 0\big).$$

Apply $z \mapsto z + by$ to get

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, ayz + cxz + c(-\beta - b)xy, 0, 0\big). \tag{C.3}$$

By $y \mapsto y + \frac{c}{a}x$, $Y \mapsto Y + \frac{c^2}{a}(\beta + b)X^2$

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, ayz + c(-\beta - b)xy, 0, 0\big).$$

If $-\beta - b = 0$ we get the jet $(x,\, yz, 0, 0)$ by scaling. If not, we still get the same jet by $z \mapsto z + \frac{c}{a}(-\beta - b)x$ and scaling.

(1.1.c.iii) If $\alpha \neq \mp\sqrt{-a_3}$ and $\beta = -\frac{1}{\sqrt{-a_3}}$ then

$$\begin{aligned} j^2f(0) &\sim_{\mathcal{A}} \big(x,\, (-2\alpha + 2\sqrt{-a_3})yz + (a_3 + \alpha^2)z^2 + (a_4 - \frac{a_2a_5}{2})(xz - \beta xy), 0, 0\big) \\ &\sim_{\mathcal{A}} \big(x,\, (-2\alpha + 2\sqrt{-a_3})z(y + \frac{a_3+\alpha^2}{-2\alpha+2\sqrt{-a_3}}z) + (a_4 - \frac{a_2a_5}{2})(xz - \beta xy), 0, 0\big). \end{aligned}$$

Let us set $d := -2\alpha + 2\sqrt{-a_3}$, $e := \frac{a_3+\alpha^2}{-2\alpha+2\sqrt{-a_3}}$ and keep $c = a_4 - \frac{a_2a_5}{2}$ to simplify the notation. So we have

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, dz(y + ez) + cxz - \beta cxy, 0, 0\big).$$

As $d \neq 0$, $e \neq 0$ we can apply $y \mapsto y + ez$ to get

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, dyz + (c + \beta ce)xz - \beta cxy, 0, 0\big)$$

whence a jet of the form (C.3).

(1.1.d) Assume that $a_2a_3 \neq 0$. By a transformation of the form (C.1),

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\ (1-a_2\beta + a_3\beta^2)y^2 + (-2\alpha + a_2(1+\alpha\beta) - 2a_3\beta)yz + \\ + (\alpha^2 - a_2\alpha + a_3)z^2 + (a_4 - \frac{a_2a_5}{2})(xz - \beta xy), 0, 0\big)$$

for some $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta \neq 1$. Then, we can cancel any of y^2, yz or z^2 by choosing appropriate values for α and β. In any of those cases we go back to the case (1.1.c).

(1.2) If $a_1 = 0$ then we have $j^2f(0) = (x,\ a_2yz + a_3z^2 + a_4xz + a_5xy, 0, 0)$.

(1.2.a) Assume that $a_2 = a_3 = 0$. If $a_4 = a_5 = 0$ then we find $\boxed{(x, 0, 0, 0)}$. If $a_4 \neq 0, a_5 = 0$ then by scaling we get $(x,\ xz, 0, 0)$. If $a_4 = 0, a_5 \neq 0$ then by scaling and symmetry we again find $(x,\ xz, 0, 0)$. If $a_4 \neq 0, a_5 \neq 0$ then by $y \mapsto y + \frac{a_4}{a_5}z$ we get the same jet.

(1.2.b) If $a_2 \neq 0$, $a_3 = 0$ we get a jet equivalent to (C.3).

(1.2.c) Assume that $a_2 = 0$, $a_3 \neq 0$. We have

$$j^2f(0) \sim_{\mathcal{A}} (x,\ z^2 + \frac{a_4}{a_3}xz + \frac{a_5}{a_3}xy, 0, 0).$$

If $a_4 \neq$ by $z \mapsto z + \frac{a_4}{2a_3}x$, $Y \mapsto Y + \frac{a_4^2}{4a_3^2}X^2$ we get

$$j^2f(0) \sim_{\mathcal{A}} (x,\ z^2 + \frac{a_5}{a_3}xy, 0, 0) \sim_{\mathcal{A}} (x,\ y^2 + \frac{a_5}{a_3}xz, 0, 0)$$

which is a jet we studied in (1.1.a).

(1.2.d) Assume that $a_2a_3 \neq 0$. We have

$$j^2f(0) \sim_{\mathcal{A}} (x,\ yz + \frac{a_3}{a_2}z^2 + \frac{a_4}{a_2}xz + \frac{a_5}{a_2}xy, 0, 0).$$

By $y \mapsto y + \frac{a_3}{a_2}z$ we get

$$j^2f(0) \sim_{\mathcal{A}} (x,\ yz + \frac{a_4 - a_3a_5}{a_2}xz + \frac{a_5}{a_2}xy, 0, 0)$$

which is a jet of the form (C.3). This ends the calculations for Case 1.

C.2 Case 2. $f_4 = 0$

We consider

$$j^2f(0) = (x,\ a_1y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy,\ b_1y^2 + b_2yz + b_3z^2 + b_4xz + b_5xy, 0)$$

where neither of f_2 and f_3 are non-zero. First, we separate this case into two subcases determined by the value of a_1. See Figure C.2 for a tree of the all subcases.

(2.1) Assume that $a_1 = 1$. If $b_1 \neq 0$ then by $Z \mapsto Z - b_1Y$ we can cancel the term y^2 in f_2:

$$j^2f(0) \sim_{\mathcal{A}} (x,\ y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy,\ b_2yz + b_3z^2 + b_4xz + b_5xy, 0).$$

(2.1.a) Consider the case $b_2 = 1$. We have

$$j^2f(0) \sim_{\mathcal{A}} (x,\ y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy,\ yz + b_3z^2 + b_4xz + b_5xy, 0). \quad (C.4)$$

(2.1.a.i) If $a_3 \neq 0$ we apply $Y \mapsto Y + \theta Z$ to get

$$j^2f(0) \sim_{\mathcal{A}} \big(x, y^2 + (a_2+\theta)yz + (a_3+\theta b_3)z^2 + (a_4+\theta b_4)xz + (a_5+\theta b_5)xy,$$
$$yz + b_3z^2 + b_4xz + b_5xy, 0\big).$$

Let us choose θ such that $a_2 + \theta = 2\sqrt{a_3 + \theta b_3}$. Then, by $y \mapsto y + \frac{1}{2}(a_3 + \theta b_3)z$,

$$j^2f(0) \sim_{\mathcal{A}} (x,\ y^2 + kxz + lxy, yz + mz^2 + nxz + b_5xy) \quad (C.5)$$

where $k = a_4 + \theta b_4 - \frac{1}{2}(a_3 + \theta b_3)(a_5 + \theta b_5)$, $l = a_5 + \theta b_5$, $m = b_3 - \frac{1}{2}(a_3 + \theta b_3)$, $n = b_4 - \frac{1}{2}b_5(a_3 + \theta b_3)$.

(2.1.a.i.α) If $m \neq 0$ we apply $Z \mapsto \frac{1}{m}Z + \frac{1}{4m^2}Y$ to get

$$\big(x,\ y^2 + kxz + lxy,\ z^2 + \frac{1}{m}yz + \frac{1}{4m^2}y^2 + (\frac{n}{m} + \frac{k}{4m^2})xz + (\frac{b_5}{m} + \frac{l}{4m^2})xy, 0\big). \quad (C.6)$$

Now, by $z \mapsto z + \frac{1}{2m}y$

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\ y^2 + kxz + \frac{2lm-k}{2m}xy,\ z^2 + \frac{4mn+k}{4m^2}xz + (\frac{4b_5m+l}{4m^2} - \frac{4mn+k}{8m^3})xy, 0\big).$$

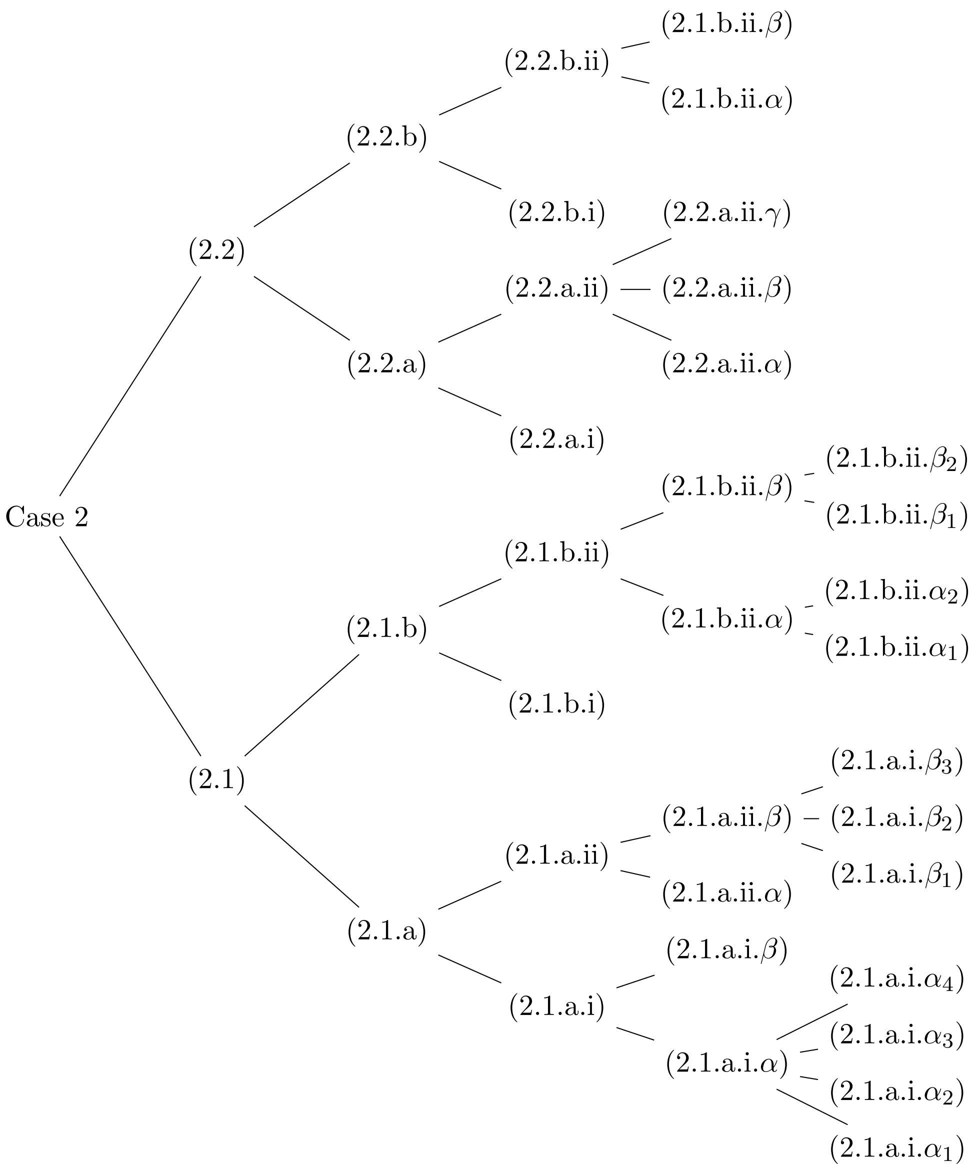


Figure C.2: Tree of the subcases under Case 2.

For simplicity, let $l' := \frac{2lm-k}{4m}$, $n' := \frac{4mn+k}{8m^2}$ and $b_5' := \frac{4b_5m+l}{4m^2} - \frac{4mn+k}{8m^3}$. So

$$j^2f(0) \sim_{\mathcal{A}} (x,\ y^2 + kxz + 2l'xy,\ z^2 + 2n'xz + b_5'xy, 0). \tag{C.7}$$

If $l' \neq 0$ then $y \mapsto y + l'x$ followed by $(X,Y,Z,W) \mapsto (X, Y + l'^2X^2, Z + b_5'l'X^2, W)$ yields $j^2f(0) \sim_{\mathcal{A}} (x,\ y^2+kxz,\ z^2+n'xz+b_5'xy, 0)$. Moreover, if $n' \neq 0$, by $z \mapsto z+n'x$

and then

$$(X,Y,Z,W)\mapsto(X,Y+kn'X^2,Z+n'^2X^2,W)$$

returns

$$(x,\ y^2+kxz,\ z^2+b_5'xy,0).$$

(2.1.a.i.α_1) If both k and b_5' are zero we get $\boxed{(x,\ y^2,\ z^2,0)}$.

(2.1.a.i.α_2) If $k\neq 0$ and $b_5'=0$, by scaling we get $\boxed{(x,\ y^2+xz,\ z^2,0)}$.

(2.1.a.i.α_3) If $k=0$ and $b_5'\neq 0$ we get $(x,y^2,z^2+xy,0)$ which is symmetric to $(x,\ y^2+xz,\ z^2,0)$.

(2.1.a.i.α_4) If $k,b_5'\neq 0$ then, by scaling, we get $\boxed{(x,\ y^2+xz,\ z^2+xy,0)}$.

(2.1.a.i.β) If $m=0$ then we have

$$j^2f(0)\sim_{\mathcal{A}}(x,\ y^2+kxz+lxy,\ yz+nxz+b_5xy,0) \tag{C.8}$$

We can cancel the term lxy by $y\mapsto y+\frac{l}{2}x$ and then apply

$$(X,Y,Z,W)\mapsto(X,Y+\frac{l^2}{4}X^2,Z+\frac{lb_5}{2}X^2,W)$$

to get $j^2f(0)\sim_{\mathcal{A}}\big(x,\ y^2+kxz,\ yz+(n-\frac{l}{2})xz+b_5xy,0\big)$. Now, $z\mapsto z+b_5x$ followed by

$$(X,Y,Z,W)\mapsto(X,Y+kb_5X^2,Z+b_5(n-\frac{l}{2})X^2,W)$$

yields

$$j^2f(0)\sim_{\mathcal{A}}\big(x,\ y^2+kxz,\ yz+(n-\frac{l}{2})xz,0\big).$$

Here we will apply Mather's Lemma [Mat69b, Lemma 3.I] to determine the orbits in the set $S:=\big\{(x,y^2+kxz,yz+nxz,0)\mid k,n\in\mathbb{C}\big\}$. Let us set $g_{k,n}:=(x,y^2+kxz,yz+nxz,0)$. It is clear that S is a connected submanifold of $J^2(3,4)$. By Mather's Lemma, S is contained in a single orbit by the action of $\mathcal{A}^2$ if and only if the dimension of

$$T\mathcal{A}^2g_{k,n}=tg_{k,n}(\mathfrak{m}_{\mathbb{C}^3,0}\Theta_{\mathbb{C}^3,0})+g_{k,n}^*(\mathfrak{m}_{\mathbb{C}^4,0})+\mathfrak{m}_{\mathbb{C}^3,0}^3\Theta(g_{k,n}) \tag{C.9}$$

is independent of $g_{k,n}$, i.e. independent of k, n, and

$$T_{g_{k,n}}S = \mathbb{C} \cdot \left\{ \begin{bmatrix} 0 \\ xz \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ xz \\ 0 \end{bmatrix} \right\}$$

is contained in $T\mathcal{A}^2 g_{k,n}$ for all $k, n \in \mathbb{C}$. Here, $\mathcal{A}^2$ is the group acting on $J^2(3,4)$ by $(\phi, \psi, f) \mapsto j^2(\psi \circ f \circ \phi^{-1})$. Let $\{e_1 := \frac{\partial}{\partial X}, e_2 := \frac{\partial}{\partial Y}, e_3 := \frac{\partial}{\partial Z}, e_4 := \frac{\partial}{\partial W}\}$ be the standard basis for $\Theta(g_{k,n})$. We have

$$\mathrm{d}g_{k,n} = \begin{bmatrix} 1 & 0 & 0 \\ kz & 2y & kx \\ nz & z & y + nx \\ 0 & 0 & 0 \end{bmatrix}.$$

So, $T\mathcal{A}^2 g$ consists of the elements

$$\begin{cases} xe_i \text{ for } i = 1, \dots, 4, \\ x^2 e_i \text{ for } i = 1, \dots, 4, \\ (y^2 + kxz)e_i \text{ for } i = 1, \dots, 4, \\ (yz + nxz)e_i \text{ for } i = 1, \dots, 4, \\ xe_1 + kxze_2 + nxze_3, \\ ye_1 + kyze_2 + nyze_3, \\ ze_1 + kz^2 e_2 + nz^2 e_3, \\ 2xye_2 + xze_3, \\ 2y^2 e_2 + yze_3, \\ 2yze_2 + z^2 e_3, \\ kx^2 e_2 + xye_3 + nx^2 e_3, \\ kxye_2 + y^2 e_3 + nxye_3, \\ kxze_2 + yze_3 + nxze_3. \end{cases}$$

and all the elements of the form he_i where $h \in \mathfrak{m}^3_{\mathbb{C}^3,0}$ and $i = 1, \dots, 4$. Therefore, the dimension of $T\mathcal{A}^2 g_{k,n}$ is determined by the solution of the system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & k & 0 & 0 & 0 & 0 & n & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & k & 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & k & 0 & 0 & 0 & 0 & n & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & 0 & 0 & n & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k & 0 & 0 & 0 & 0 & n & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} ye_1 \\ ze_1 \\ xye_2 \\ xze_2 \\ yze_2 \\ y^2e_2 \\ z^2e_2 \\ xye_3 \\ xze_3 \\ yze_3 \\ y^2e_3 \\ z^2e_3 \end{bmatrix} = 0. \qquad \text{(C.10)}$$

Let us denote the coefficient matrix of (C.10) by M. We see that the codimension of $T\mathcal{A}^2 g_{k,n}$ equals to the minimum value, which is 10, if and only if M has maximal rank, or equivalently, if and only if $\bigwedge^{12} M$ does not vanish. We find that $\bigwedge^{12} M = (k^2, kn)$, $\bigwedge^{11} M = (k, n)$ whereas $\bigwedge^{10} M = 1$.

(2.1.a.i.β_1) If $k \neq 0$ then M has maximal rank and the dimension of $T\mathcal{A}^g_{k,n}$ is constant. Moreover, xze_2, xze_3 belongs to $T\mathcal{A}^g_{k,n}$. Therefore, $\{(x,\, y^2 + kxz,\, yz + nxz, 0) \mid k, n \in \mathbb{C}, k \neq 0\}$ is contained in a single orbit. We choose $\boxed{(x,\, y^2 + xz,\, yz, 0)}$ as a representative.

(2.1.a.i.β_2) If $k = 0$ and $n \neq 0$ then the codimension of $T\mathcal{A}^g_{k,n}$ is still constant but grows by 1 (relative to the case (2.1.a.i.β_1)). By scaling and symmetry we find $\boxed{(x,\, y^2,\, xz + yz, 0)}$.

(2.1.a.i.β_3) If $k = n = 0$ then by scaling $j^2 f(0) \sim_{\mathcal{A}} \boxed{(x,\, y^2,\, yz, 0)}$.

(2.1.a.ii) If $a_3 = 0$ then

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, y^2 + a_2 yz + a_4 xz + a_5 xy,\, yz + b_3 z^2 + b_4 xz + b_5 xy, 0).$$

(2.1.a.ii.α) Let us assume that $b_3 = 0$, that is,

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, y^2 + a_2 yz + a_4 xz + a_5 xy,\, yz + b_4 xz + b_5 xy, 0).$$

If $a_2 \neq 0$, apply $Y \mapsto Y - a_2 Z$ to get

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\ y^2 + (a_4 - a_2 b_4)xz + (a_5 - a_2 b_5)xy,\ yz + b_4 xz + b_5 xy, 0\big).$$

This is a jet of the form (C.8).

(2.1.a.ii.β) Let us assume that $b_3 \neq 0$. If $a_2 = 0$ then we are in the case C.2. So we assume that $a_2 \neq 0$. Apply $Y \mapsto Y + \theta Z$ for some $\theta \in \mathbb{C}$ to get

$$\begin{aligned}&\big(x, y^2 + (a_2 + \theta)yz + b_3\theta z^2 + (a_4 + b_4\theta)xz + (a_5 + b_5\theta)xy,\\ &\quad yz + b_3 z^2 + b_4 xz + b_5 xy, 0\big).\end{aligned}$$

Let us choose θ such that $a_2 + \theta = 2\sqrt{b_3\theta}$. Then, by $y \mapsto y + \frac{1}{2}(a_2 + \theta)z$ we get

$$\begin{aligned}j^2 f(0) \sim_{\mathcal{A}} &\big(x, y^2 + (a_4 + b_4\theta - \frac{1}{2}(a_5 + b_5\theta)(a_2 + \theta))xz + (a_5 + b_5\theta)xy,\\ &\quad yz + (b_3 - \frac{a_2 + \theta}{2})z^2 + (b_4 - \frac{b_5(a_2 + \theta)}{2})xz + b_5 xy, 0\big).\end{aligned}$$

This is a jet of the form (C.5).

(2.1.b) If $b_2 = 0$ then

$$j^2 f(0) \sim_{\mathcal{A}} (x,\ y^2 + a_2 yz + a_3 z^2 + a_4 xz + a_5 xy,\ b_3 z^2 + b_4 xz + b_5 xy, 0).$$

(2.1.b.i) Assume that $b_3 = 1$. If $a_3 \neq 0$ we cancel the term $a_3 z^2$ by $Y \mapsto Y - a_3 Z$. So we can assume

$$j^2 f(0) \quad \sim_{\mathcal{A}} \quad (x,\ y^2 + a_2 yz + a_4 xz + a_5 xy,\ z^2 + b_4 xz + b_5 xy, 0)$$

If $a_2 = 0$ then we have a jet of the form (C.7). If $a_2 \neq 0$ then by $Y \mapsto Y + \frac{1}{4}a_2^2 Z$

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\ y^2 + a_2 yz + \frac{1}{4}a_2^2 z^2 + (a_4 + \frac{1}{4}a_2^2 b_4)xz + (a_5 + \frac{1}{4}a_2^2 b_5)xy,\ z^2 + b_4 xz + b_5 xy, 0\big).$$

Now, $y \mapsto y + \frac{1}{2}a_2 z$ yields the jet

$$\big(x,\ y^2 + (a_4 + \frac{1}{4}a_2^2 b_4 - \frac{1}{2}a_2(a_5 + \frac{1}{4}a_2^2 b_5))xz + (a_5 + \frac{1}{4}a_2^2 b_5)xy,\ z^2 + (b_4 - \frac{1}{2}a_2 b_5)xz + b_5 xy, 0\big).$$

Again, this is of the form (C.7).

(2.1.b.ii) If $b_3 = 0$ then

$$j^2f(0) \sim_{\mathcal{A}} (x,\, y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy,\, b_4xz + b_5xy, 0). \tag{C.11}$$

(2.1.b.ii.α) If $b_4 \neq 0$ then we can assume that $b_4 = 1$ by Remark C.0.17. Moreover, if $a_4 \neq 0$, we can cancel the term a_4xz by $Y \mapsto Y - a_4Z$. So it is sufficient to consider

$$j^2f(0) \sim_{\mathcal{A}} (x,\, y^2 + a_2yz + a_3z^2 + a_5xy,\, xz + b_5xy, 0).$$

Now, by $z \mapsto z + b_5y$

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, (1 - a_2b_5 + a_3b_5^2)y^2 + (a_2 - 2a_3b_5)yz + a_3z^2 + a_5xy,\, xz, 0\big).$$

(2.1.b.ii.α_1) Let us assume that $1 - a_2b_5 + a_3b_5^2 \neq 0$. Then

$$j^2f(0) \sim_{\mathcal{A}} (x,\, y^2 + \frac{a_2 - 2a_3b_5}{1 - a_2b_5 + a_3b_5^2}yz + \frac{a_3}{1 - a_2b_5 + a_3b_5^2}z^2 + \frac{a_5}{1 - a_2b_5 + a_3b_5^2}xy,\, xz, 0). \tag{C.12}$$

If both a_5 and $a_2 - 2a_3b_5$ are non-zero then we can cancel the term xy by $y \mapsto y + \frac{a_5}{2(1-a_2b_5+a_3b_5^2)}x$ and

$$Y \mapsto Y + \frac{a_5^2}{4(1 - a_2b_5 + a_3b_5^2)^2}X^2 + \frac{(a_2 - 2a_3b_5)a_5}{2(1 - a_2b_5 + a_3b_5^2)^2}Z.$$

So,

$$j^2f(0) \sim_{\mathcal{A}} (x,\, y^2 + \frac{a_2 - 2a_3b_5}{1 - a_2b_5 + a_3b_5^2}yz + \frac{a_3}{1 - a_2b_5 + a_3b_5^2}z^2,\, xz, 0). \tag{C.13}$$

Now, if $a_2 - 2a_3b_5 = 0$ and $a_3 = 0$ we get $\boxed{(x, y^2, xz, 0)}$. If $a_2 - 2a_3b_5 \neq 0$ and $a_3 = 0$ then we obtain $(x,\, y^2 + yz,\, xz, 0)$ by scaling. This jet is also equivalent to $\boxed{(x,\, xy + xz,\, yz, 0)}$ by symmetry and scaling after $z \mapsto z + y$.

If $a_3 \neq 0$ then we again find $(x,\, y^2 + yz,\, xz, 0)$ by a transformation of the form $y \mapsto y + \alpha z$, $\alpha \in \mathbb{C}$, and scaling.

(2.1.b.ii.α_2) Assume that $1 - a_2b_5 + a_3b_5^2 = 0$. We have

$$j^2f(0) \sim_{\mathcal{A}} (x,\, (a_2 - 2a_3b_5)yz + a_3z^2 + a_5xy,\, xz, 0).$$

Furthermore, we can assume that at least one of $a_2 - 2a_3b_5$, a_3 and a_5 is non-zero.

If $a_2 - 2a_3b_5 \neq 0$ then

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, z(y + \frac{a_3}{a_2 - 2a_3b_5}z) + \frac{a_5}{a_2 - 2a_3b_5}xy,\, xz, 0\big).$$

By $y \mapsto y + \frac{a_3}{a_2-2a_3b_5}z$ and $Y \mapsto Y + \frac{a_3a_5}{(a_2-2a_3b_5)^2}Z$ we get

$$j^2f(0) \sim_{\mathcal{A}} (x,\, yz + \frac{a_5}{a_2 - 2a_3b_5}xy,\, xz, 0).$$

Now, $z \mapsto z + \frac{a_5}{a_2-2a_3b_5}x$ and $Z \mapsto Z + \frac{a_5}{a_2-2a_3b_5}X^2$ yield $(x,\, yz,\, xz, 0)$ which is symmetric to $\boxed{(x,\, xz,\, yz, 0)}$.

If $a_2 - 2a_3b_5 = 0$ and $a_3a_5 \neq 0$ then by scaling and symmetry we find $\boxed{(x,\, y^2 + xz,\, xy, 0)}$.

If $a_2 - 2a_3b_5 = 0$, $a_3 = 0$ and $a_5 \neq 0$ then we get $\boxed{(x,\, xy,\, xz, 0)}$.

Finally, if $a_2 - 2a_3b_5 = 0$, $a_5 = 0$ and $a_3 \neq 0$ then we get $\boxed{(x,\, y^2,\, xy, 0)}$ by symmetry.

(2.1.b.ii.β**)** If $b_5 \neq 0$ we may assume that $b_5 = 1$. So we consider

$$j^2f(0) \sim_{\mathcal{A}} (x,\, y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy,\, b_4xz + xy, 0).$$

By $y \mapsto y + b_4z$ and $Y \mapsto Y - a_5Z$ we get

$$\big(x,\, y^2 + (a_2 - 2b_4)yz + (b_4^2 - a_2b_4 - a_3)z^2 + (a_4 - a_5b_4)xz,\, xy, 0\big). \tag{C.14}$$

But that is equivalent to

$$\big(x,\, z^2 + (a_2 - 2b_4)yz + (b_4^2 - a_2b_4 - a_3)y^2 + (a_4 - a_5b_4)xy,\, xz, 0\big)$$

by symmetry. Let us set $p := a_2 - 2b_4$, $r := b_4^2 - a_2b_4 - a_3$ and $s := a_4 - a_5b_4$ for simplicity. So

$$j^2f(0) \sim_{\mathcal{A}} (x,\, z^2 + pyz + ry^2 + sxy,\, xz, 0).$$

If $r \neq 0$ then by $y \mapsto y + \frac{s}{2r}x$ and $Y \mapsto Y + \frac{s^2}{4r}X^2 + \frac{ps}{2r}Z$ we get

$$j^2f(0) \sim_{\mathcal{A}} (x,\, z^2 + pyz + ry^2,\, xz, 0)$$

which is of the form (C.13). On the other hand, if $r = 0$ we meet the cases below.

(2.1.b.ii.β_1) $p \neq 0$. By $y \mapsto y + \frac{1}{p}z$ and $Y \mapsto Y + \frac{s}{p}Z$ we get

$$j^2 f(0) \sim_{\mathcal{A}} (x,\ pyz + sxy,\ xz, 0).$$

If $s \neq 0$ we can cancel sxy by $z \mapsto z + \frac{s}{p}x$ and $Z \mapsto Z + \frac{s}{p}X^2$. Now by scaling

$$j^2 f(0) \sim_{\mathcal{A}} (x,\ yz,\ xz, 0).$$

(2.1.b.ii.β_2) $p = 0$, $s \neq 0$. By scaling and symmetry

$$j^2 f(0) \sim_{\mathcal{A}} (x,\ z^2 + xy,\ xz, 0) \sim_{\mathcal{A}} (x,\ y^2 + xz,\ xy, 0).$$

(2.2) Assume that $a_1 = 0$. Then

$$j^2 f(0) = (x,\ a_2yz + a_3z^2 + a_4xz + a_5xy,\ b_1y^2 + b_2yz + b_3z^2 + b_4xz + b_5xy, 0).$$

(2.2.a) Assume that $a_2 = 1$. If $b_1 \neq 0$ then we have a jet of the form (C.4). So we take $b_1 = 0$, i.e. consider

$$j^2 f(0) \sim_{\mathcal{A}} (x,\ yz + a_3z^2 + a_4xz + a_5xy,\ b_2yz + b_3z^2 + b_4xz + b_5xy, 0).$$

Moreover we can assume that $b_2 = 0$ since b_2yz can be cancelled by $Z \mapsto Z - b_2Y$. So we consider $(x,\ yz + a_3z^2 + a_4xz + a_5xy,\ b_3z^2 + b_4xz + b_5xy, 0)$.

(2.2.a.i) If $b_3 \neq 0$ we can cancel the term a_3z^2 by $Y \mapsto Y - \frac{a_3}{b_3}Z$ so that

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\ yz + (a_4 - \frac{a_3b_4}{b_3})xz + (a_5 - \frac{a_3b_5}{b_3})xy,\ b_3z^2 + b_4xz + b_5xy, 0\big).$$

This is a jet of the form (C.8) up to symmetry.

(2.2.a.ii) Assume we have $j^2 f(0) = (x,\ yz + a_3z^2 + a_4xz + a_5xy,\ b_4xz + b_5xy, 0)$. Notice that if $a_3 \neq 0$ we can cancel a_3z^2 by $y \mapsto y + a_3z$. So we may assume that $a_3 = 0$.

(2.2.a.ii.α) If $b_4b_5 \neq 0$ then by $z \mapsto z + \frac{b_5}{b_4}y$ and $Y \mapsto Y - \frac{a_4}{b_4}Z$, $Z \mapsto \frac{1}{b_4}Z$

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, yz - \frac{b_5}{b_4}y^2 + (a_5 - \frac{a_4b_5}{b_4})xy,\, xz, 0\big). \tag{C.15}$$

This is equivalent to (C.12) (for $a_3 = 0$).

(2.2.a.ii.β) If $b_4 = 0$ and $b_5 \neq 0$, by symmetry and scaling

$$j^2f(0) \sim_{\mathcal{A}} (x,\, yz + a_4xy + a_5xz,\, xz, 0)$$

Cancel a_5xz by $Y \mapsto Y - a_5Z$. Then by $z \mapsto z + a_4x$ and $Z \mapsto Z + a_4X^2$

$$j^2f(0) \sim_{\mathcal{A}} (x,\, yz,\, xz, 0).$$

(2.2.a.ii.γ) If $b_4 \neq 0$, $b_5 = 0$ then we have a jet symmetric to (2.2.a.ii.β).

(2.2.b) If $a_2 = 0$ then

$$j^2f(0) = (x,\, a_3z^2 + a_4xz + a_5xy,\, b_1y^2 + b_2yz + b_3z^2 + b_4xz + b_5xy, 0).$$

We have already studied the case $b_1 \neq 0$ in (2.a.ii). So we can assume that $b_1 = 0$. Moreover we can assume that $b_2 = 0$ as we have already considered the case $b_2 \neq 0$ in (2.2.a). So we have

$$j^2f(0) = (x,\, a_3z^2 + a_4xz + a_5xy,\, b_3z^2 + b_4xz + b_5xy, 0).$$

(2.2.b.i) If $a_3 \neq 0$ then we cancel b_3z^2 by $Z \mapsto Z - \frac{b_3}{a_3}Y$:

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, a_3z^2 + a_4xz + a_5xy,\, (b_4 - \frac{a_4b_3}{a_3})xz + (b_5 - \frac{a_5b_3}{a_3})xy, 0\big),$$

moreover

$$\sim_{\mathcal{A}} \big(x,\, z^2 + \frac{a_4}{a_3}xz + \frac{a_5}{a_3}xy,\, (b_4 - \frac{a_4b_3}{a_3})xz + (b_5 - \frac{a_5b_3}{a_3})xy, 0\big)$$
$$\sim_{\mathcal{A}} \big(x,\, y^2 + \frac{a_4}{a_3}xy + \frac{a_5}{a_3}xz,\, (b_4 - \frac{a_4b_3}{a_3})xy + (b_5 - \frac{a_5b_3}{a_3})xz, 0\big).$$

This is a special case under (2.1.b.i).

(2.2.b.ii) Let $a_3 = 0$. We can assume that $b_3 = 0$, otherwise we find a jet symmetric to (2.2.b.i). So we take

$$j^2 f(0) = (x,\ a_4xz + a_5xy,\ b_4xz + b_5xy, 0).$$

Notice that, by assumption, the rank of the matrix $\begin{bmatrix} a_4 & b_4 \\ a_5 & b_5 \end{bmatrix}$ is 2.

(2.2.b.ii.α) Assume $a_4 \neq 0$. If $a_5 \neq 0$ then we cancel a_5xy by $z \mapsto z + \frac{a_5}{a_4}$:

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\ a_4xz,\ b_4xz + (b_5 - \frac{a_5b_4}{a_4})xy, 0\big).$$

Now, by $Y \mapsto \frac{1}{a_4}Y$, $Z \mapsto Z - \frac{b_4}{a_4}Y$

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\ xz,\ (b_5 - \frac{a_5b_4}{a_4})xy, 0\big).$$

That yields $(x,\ xz,\ xy, 0)$.

(2.2.b.ii.β) The case where $a_5 \neq 0$ is symmetric to (2.2.b.ii.α).
This concludes the calculations for Case 2.

C.3 Case 3.

We study

$$j^2 f(0) = (x,\ a_1y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy, b_1y^2 + b_2yz + b_3z^2 + b_4xz + b_5xy, \\ c_1y^2 + c_2yz + c_3z^2 + c_4xz + c_5xy)$$

First, we get two subcases depending on the value of a_1. We refer to Figure C.3 for a tree of the all subcases.

(3.1) Assume $a_1 \neq 0$. By Remark C.0.17 we can take $a_1 = 1$. Then we can also assume that $b_1 = c_1 = 0$ since the transformation $Z \mapsto Z - b_1Y$, $W \mapsto W - c_1Y$ cancels b_1y^2 and c_1y^2. So we consider

$$j^2 f(0) = (x,\ y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy, b_2yz + b_3z^2 + b_4xz + b_5xy, \\ c_2yz + c_3z^2 + c_4xz + c_5xy) \tag{C.16}$$

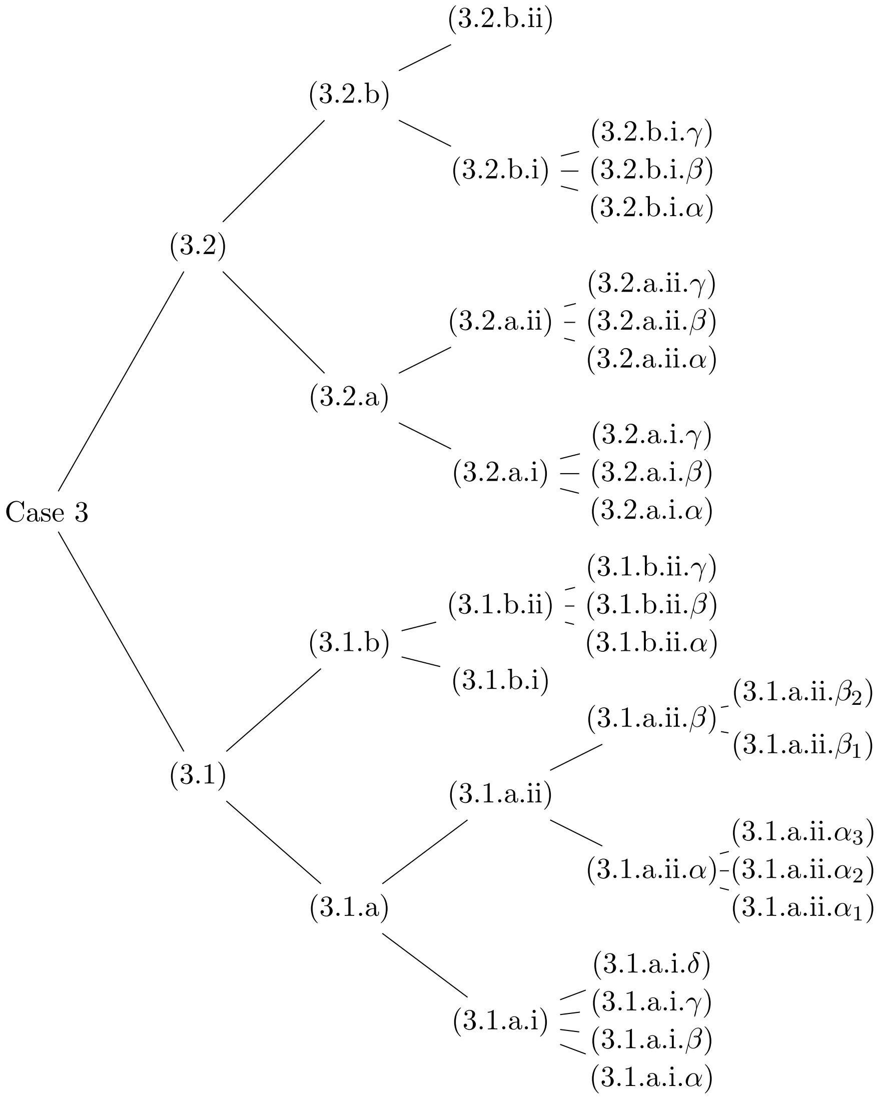


Figure C.3: Tree of the subcases under Case 3.

(3.1.a) If $b_2 \neq 0$ then we may take $b_2 = 1$ by Remark C.0.17. Moreover, by $Y \mapsto Y - a_2 Z$, $W \mapsto W - c_2 Z$ we get

$$\begin{aligned} j^2 f(0) = \big(&x, y^2 + (a_3 - a_2 b_3) z^2 + (a_4 - a_2 b_4) xz + (a_5 - a_2 b_5) xy, \\ &yz + b_3 z^2 + b_4 xz + b_5 xy, \\ &(c_3 - b_3 c_2) z^2 + (c_4 - b_4 c_2) xz + (c_5 - b_5 c_2) xy\big). \end{aligned}$$

Let us set $a'_3 := a_3 - a_2b_3$, $a'_4 := a_4 - a_2b_4$, $a'_5 := a_5 - a_2b_5$, $c'_3 := c_3 - b_3c_2$, $c'_4 := c_4 - b_4c_2$ and $c'_5 := c_5 - b_5c_2$ so that

$$j^2f(0) = (x,\, y^2 + a'_3z^2 + a'_4xz + a'_5xy,\, yz + b_3z^2 + b_4xz + b_5xy,\, c'_3z^2 + c'_4xz + c'_5xy).$$

We meet the following cases.

(3.1.a.i) Suppose $c'_3 \neq 0$. We scale on Z by $Z \mapsto \frac{1}{c'_3}Z$ to get

$$j^2f(0) \sim_{\mathcal{A}} (x,\, y^2 + a'_3z^2 + a'_4xz + a'_5xy,\, yz + b_3z^2 + b_4xz + b_5xy,\, z^2 + \frac{c'_4}{c'_3}xz + \frac{c'_5}{c'_3}xy).$$

Apply $Y \mapsto Y - a'_3W$, $Z \mapsto Z - b_3W$ so that

$$\begin{aligned} j^2f(0) \sim_{\mathcal{A}} \big(x, y^2 + (a'_4 - \frac{a'_3c'_4}{c'_3})xz + (a'_5 - \frac{a'_3c'_5}{c'_3})xy,& \\ yz + (b_4 - \frac{b_3c'_4}{c'_3})xz + (b_5 - \frac{b_3c'_5}{c'_3})xy,\, z^2 + \frac{c'_4}{c'_3}xz + \frac{c'_5}{c'_3}xy\big).& \end{aligned}$$

By $y \mapsto y + \frac{1}{2}(a'_5 - \frac{a'_3c'_5}{c'_3})x$, $z \mapsto z + \frac{c'_5}{2c'_3}x$ and adding X^2 with appropriate coefficients to Y, Z and W, respectively, we get

$$\big(x,\, y^2+(a'_4-\frac{a'_3c'_4}{c'_3})xz,\, yz+(b_4-\frac{b_3c'_4}{c'_3}-\frac{1}{2}(a'_5-\frac{a'_3c'_5}{c'_3}))xz+(b_5-\frac{b_3c'_5}{c'_3}-\frac{c'_5}{2c'_3})xy,\, z^2+\frac{c'_4}{c'_3}xz\big).$$

Let $a := a'_4 - \frac{a'_3c'_4}{c'_3}$, $b := b_4 - \frac{b_3c'_4}{c'_3} - \frac{1}{2}(a'_5 - \frac{a'_3c'_5}{c'_3})$, $c := b_5 - \frac{b_3c'_5}{c'_3} - \frac{c'_5}{2c'_3}$ and $d := \frac{c'_4}{c'_3}$. We have

$$j^2f(0) \sim_{\mathcal{A}} (x,\, y^2 + axz,\, yz + bxz + cxy,\, z^2 + dxy). \tag{C.17}$$

Here we will apply Mather's Lemma [Mat69b, Lemma 3.I] to determine the orbits in the set $S := \big\{(x,\, y^2 + axz,\, yz + bxz + cxy,\, z^2 + dxy) \mid a, b, c, d \in \mathbb{C}\big\}$. Let us set $g_{\mathbf{a}} := (x,\, y^2 + axz,\, yz + bxz + cxy,\, z^2 + dxy)$. It is clear that S is a connected submanifold of $J^2(3,4)$. By Mather's Lemma, S is contained in a single orbit by the action of $\mathcal{A}^2$ if and only if the dimension of

$$T\mathcal{A}^2g_{\mathbf{a}} = tg_{\mathbf{a}}(\mathfrak{m}_{\mathbb{C}^3,0}\Theta_{\mathbb{C}^3,0}) + g^*_{\mathbf{a}}(\mathfrak{m}_{\mathbb{C}^4,0}) + \mathfrak{m}^3_{\mathbb{C}^3,0}\Theta(g_{\mathbf{a}}) \tag{C.18}$$

is independent of $g_{\mathbf{a}}$, i.e. independent of a, b, c, d, and

$$T_{g_{\mathbf{a}}}S = \mathbb{C} \cdot \left\{ \begin{bmatrix} 0 \\ xz \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ xz \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ xy \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ xy \end{bmatrix} \right\}$$

is contained in $T\mathcal{A}^2 g_{\mathbf{a}}$ for all $g_{\mathbf{a}}$. Here, $\mathcal{A}^2$ is the group acting on $J^2(3,4)$ by $(\phi, \psi, f) \mapsto j^2(\psi \circ f \circ \phi^{-1})$. Now we study $T\mathcal{A}^2 g_{\mathbf{a}}$. Let us set $e_1 = \frac{\partial}{\partial X}$, $e_2 = \frac{\partial}{\partial Y}$, $e_3 = \frac{\partial}{\partial Z}$ and $e_4 = \frac{\partial}{\partial W}$ for the basis of $\Theta(g_{\mathbf{a}})$. We have

$$\mathrm{d}g_{\mathbf{a}} = \begin{bmatrix} 1 & 0 & 0 \\ az & 2y & ax \\ bz + cy & z + cx & y + bx \\ dy & dx & 2z \end{bmatrix}.$$

So, $T\mathcal{A}^2 g_{\mathbf{a}}$ consists of the elements

$$\begin{cases} axze_2 + bxze_3 + cxye_3 + dxye_4, \\ ye_1 + ayze_2 + byze_3 + cy^2e_3 + dy^2e_4, \\ ze_1 + az^2e_2 + bz^2e_3 + cyze_3 + dyze_4, \\ 2xye_2 + xze_3, \\ 2y^2e_2 + yze_3 + cxye_3 + dxye_4, \\ 2yze_2 + z^2e_3 + cxze_3 + dxze_4, \\ xye_3 + 2xze_4, \\ axye_2 + y^2e_3 + bxye_3 + 2yze_4, \\ axze_2 + yze_3 + bxye_3 + 2z^2e_4, \\ (y^2 + axz)e_2, \\ (y^2 + axz)e_3, \\ (z^2 + dxy)e_3, \\ (z^2 + dxy)e_4, \\ (yz + bxz + cxy)e_2, \\ (yz + bxz + cxy)e_3, \\ (yz + bxz + cxy)e_4. \end{cases} \tag{C.19}$$

Notice that elements of the form he_1, for $h \in \mathfrak{m}^2_{\mathbb{C}^3,0}$, are contained in $T\mathcal{A}^2 g_{\mathbf{a}}$: We have $h\frac{\partial g_{\mathbf{a}}}{\partial x} = he_1 + ahze_2 + bhze_3 + chye_3 + dhye_4$ and $hz, hy \in \mathfrak{m}^3_{\mathbb{C}^3,0}$, so $h\frac{\partial g_{\mathbf{a}}}{\partial x} = he_1$ modulo $\mathfrak{m}^3_{\mathbb{C}^3,0}\Theta(g_{\mathbf{a}})$. Therefore, the dimension of $T\mathcal{A}^2 g_{\mathbf{a}}$ is determined by the solution of the following system formed by the data in (C.19):

$$\begin{bmatrix}
0 & 0 & 0 & a & 0 & 0 & 0 & c & b & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & b & c & 0 & 0 & 0 & 0 & d & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & a & 0 & 0 & c & 0 & b & 0 & 0 & d & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & c & 0 & 1 & 0 & 0 & d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & c & 0 & 0 & 1 & 0 & d & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & b & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & 0 & b & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 1 & 0 \\
0 & 0 & c & b & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & b & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & b & 1 & 0 & 0 \\
0 & 0 & d & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix} ye_1 \\ ze_2 \\ xye_2 \\ xze_2 \\ yze_2 \\ y^2e_2 \\ z^2e_2 \\ xye_3 \\ xze_3 \\ yze_3 \\ y^2e_3 \\ z^2e_3 \\ xye_4 \\ xze_4 \\ yze_4 \\ y^2e_4 \\ z^2e_4 \end{bmatrix} = 0. \quad \text{(C.20)}$$

For example, all the elements in (C.20) belong to $T\mathcal{A}^2 g_{\mathbf{a}}$ if and only if the coefficient matrix of (C.20) has the maximal rank, i.e. 17.

Let us denote the coefficient matrix in (C.20) by M. We find that the maximal minors of M is generated by

$$\Lambda(a,b,c,d) := 27a^2d^2 - 72abcd - 32ac^3 - 32b^3d - 16b^2c^2 \quad \text{(C.21)}$$

(See Remark C.3.1 for other topics where we meet the same polynomial).

Before working out the orbits over $\mathbb{C}^4_{a,b,c,d}$, we will show that the term dxy in $g_{\mathbf{a}}$ can be cancelled.

Assume $c \neq 0$. By $W \mapsto W + \theta Y + \eta Z$, for some $\theta, \eta \in \mathbb{C}$,

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, y^2 + axz,\, yz + bxz + cxy,\, z^2 + (d + \eta c)xy + \theta y^2 + \eta yz + (a\theta + b\eta)xz\big)$$

and by $z \mapsto z + \frac{1}{2}(a\theta + b\eta)x$

$$j^2 f(0) \sim_{\mathcal{A}} \big(x, y^2 + axz,\, yz + bxz + (c - \frac{1}{2}(a\theta + b\eta))xy,$$
$$z^2 + (d + \eta c - \frac{1}{2}\eta(a\theta + b\eta))xy + \theta y^2 + \eta yz\big).$$

We solve θ and η from

$$d + \eta c - \frac{1}{2}\eta(a\theta + b\eta) = 0, \qquad \text{and} \qquad \theta = \frac{\eta^2}{4}.$$

So, $z \mapsto z + \frac{\eta}{2}y$ yields

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, y^2 + axz - \frac{\eta}{2}xy,\, yz - \frac{\eta}{2}y^2 + bxz + c'xy,\, z^2).$$

Now we complete the square in Y by adding $\frac{\eta^2}{16}X^2$ and apply $y \mapsto y - \frac{\eta}{4}x$ to get

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, y^2 + axz,\, yz - \frac{\eta}{2}y^2 + (b + \frac{\eta}{4})xz + (c' - \frac{\eta^2}{4})xy,\, z^2\big).$$

Finally, we cancel $-\frac{\eta}{2}y^2$ by $Z \mapsto Z + \frac{\eta}{2}Y$.

On the other hand, if $c = 0$ we do the following transformations. First, $y \mapsto y - x$ yields

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, y^2 + axz + 2xy,\, yz + (b+1)xz,\, z^2 + dxy\big).$$

Then, by $W \mapsto W + \theta Y + \eta Z$, for some $\theta, \eta \in \mathbb{C}$,

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, y^2{+}axz{+}2xy,\, yz{+}(b{+}1)xz,\, z^2{+}(d{+}2\theta)xy{+}(a\theta{+}\eta(b{+}1))xz{+}\eta yz{+}\theta y^2\big).$$

Now, we apply $z \mapsto z + \frac{1}{2}(a\theta + \eta(b+1))x$ to get

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, y^2 + axz + 2xy, yz + (b+1)xz - \frac{1}{2}(a\theta + \eta(b+1))xy,$$
$$z^2 + (d + 2\theta - \frac{1}{2}(a\theta + \eta(b+1)))xy + \eta yz + \theta y^2\big).$$

We choose θ and η such that $d+2\theta-\frac{1}{2}(a\theta+\eta(b+1))=0$ and $\theta=\frac{\eta^2}{4}$. So, $z\mapsto z+\frac{\eta}{2}y$ yields

$$j^2f(0)\sim_{\mathcal{A}}\big(x,\,y^2+axz+(2-\frac{a\eta}{2})xy,\,yz-\frac{\eta}{2}y^2+(b+1)xz-\frac{1}{2}(a\theta+2\eta(b+1))xy,\,z^2\big).$$

We complete the square in Y and do $y\mapsto y+(1-\frac{a\eta}{4})x$ to cancel $(2-\frac{a\eta}{2})xy$. Next, $Z\mapsto Z+\frac{\eta}{2}Y$ yields a jet of the form $(x,\,y^2+axz,\,yz+b'xz+c'xy,\,z^2)$. Thus we can take $d=0$, i.e. consider

$$g_{\mathbf{a}}=(x,y^2+axz,yz+bxz+cxy,z^2).$$

It is clear that the rank of M determines the dimension of $T\mathcal{A}^2g_{\mathbf{a}}$. So we look at the minors of M. The ideal of the maximal minors of M is generated by

$$\Lambda':=\Lambda(a,b,c,0)=c^2(b^2+2ac).$$

Thus, Λ' defines the set of triples at which $T\mathcal{A}^2g_{\mathbf{a}}$ fails include any of the monomials present in D.

We have

$$\bigwedge^{16}M:=(c^3,bc^2,ac^2,b^2c,abc,b^3,2ab^2+3a^2c).$$

Since we are only interested in the zero set of $\bigwedge^{16}M$, it is sufficient to consider its radical

$$\sqrt{\bigwedge^{16}M}=(b,c).$$

We find the radical of the ideal of 15×15-minors of M to be

$$\sqrt{\bigwedge^{17}M}=(a,b,c).$$

(3.1.a.i.α) If Λ' is non-zero then M has the maximal rank, so all the monomials involved in D belong to $T\mathcal{A}^2g_{\mathbf{a}}$. By using a computer program, we see that the complement of $T\mathcal{A}^2g_{\mathbf{a}}$ in $\mathfrak{m}_{\mathbb{C}^3,0}\Theta(g_{\mathbf{a}})$ can be generated, as a $\mathbb{C}$-vector space, by ye_2,ye_3,ye_4,ze_2,ze_3 and ze_4. It follows that $T_{g_{\mathbf{a}}}S$ is contained in $T\mathcal{A}^2g_{\mathbf{a}}$. Hence $\Lambda'\neq 0$ determines one orbit. We choose a representative $g_{\mathbf{a}}$ for which $\Lambda'\neq 0$, e.g. $\boxed{(x,\,y^2,\,yz+xz+xy,\,z^2)}$.

(3.1.a.i.β) Assume $\Lambda' = 0$ but $c \neq 0$. In other words, let $b^2 + 2ac = 0$, $c \neq 0$. In that case, M has rank 16. So, the codimension of $T\mathcal{A}^2 g_{\mathbf{a}}$ grows by one. In fact, by using a computer program, we find that the complement of $T\mathcal{A}^2 g_{\mathbf{a}}$ in $\mathfrak{m}_{\mathbb{C}^3,0}\Theta(g_{\mathbf{a}})$ can be generated by $ye_2, ye_3, ye_4, ze_2, ze_3,$ ze_4 and yze_4. Hence, by Mather's Lemma, we conclude that there is a single orbit for this case. We choose $\boxed{(x,\, y^2,\, yz + xy,\, z^2)}$ as a representative.

In the case where $\Lambda' = 0$ and $b \neq 0$, we obtain the same orbit by a similar discussion.

(3.1.a.i.γ) If $\Lambda' = 0$, $b = c = 0$ but $a \neq 0$ we find $\boxed{(x,\, y^2 + xz,\, yz,\, z^2)}$ by scaling.

(3.1.a.i.δ) If $a = b = c = d = 0$ then we find $\boxed{(x,\, y^2,\, yz,\, z^2)}$.

(3.1.a.ii) If $c_3' = 0$ then

$$j^2 f(0) = (x,\, y^2 + a_3' z^2 + a_4' xz + a_5' xy,\, yz + b_3 z^2 + b_4 xz + b_5 xy,\, c_3' z^2 + c_4' xz + c_5' xy).$$

If $b_3 \neq 0$ we cancel $b_3 z^2$ by $y \mapsto y + b_3 z$. We get

$$\begin{aligned} j^2 f(0) \sim_{\mathcal{A}} \big(x, y^2 - 2b_3 yz + (b_3^2 + a_3')z^2 + (a_4' - a_5' b_3)xz + a_5' xy, \\ yz + (b_4 - b_3 b_5)xz + b_5 xy,\, (c_4' - b_3 c_5')xz + c_5 xy\big). \end{aligned}$$

So we may assume that $b_3 = 0$ and consider

$$j^2 f(0) = (x,\, y^2 + a_3' z^2 + a_4' xz + a_5' xy,\, yz + b_4 xz + b_5 xy,\, c_3' z^2 + c_4' xz + c_5' xy).$$

Now we want to complete $y^2 + a_3' z^2$ to a form $(\alpha y + z)^2$ and then make the coordinate change $z \mapsto z + \alpha y$ as changing y will introduce z^2 back in f_2. Scale Y by $Y \mapsto \frac{1}{a_3'} Y$ so that

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, z^2 + \frac{1}{a_3'} y^2 + \frac{a_4'}{a_3'} xz + \frac{a_5'}{a_3'} xy,\, yz + b_4 xz + b_5 xy,\, c_4' xz + c_5 xy).$$

Now, $Y \mapsto Y + \frac{2}{\sqrt{a_3'}} Z$ followed by $z \mapsto z + \frac{1}{\sqrt{a_3'}} y$ yields a jet of the form

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, z^2 + axz + bxy,\, yz + cy^2 + dxz + exy,\, hxz + ixy) \tag{C.22}$$

for some $a, b, c, d, e, h, i \in \mathbb{C}$.

(3.1.a.ii.α) Assume that $c \neq 0$. By $Z \mapsto \frac{1}{c}Z + \frac{1}{4c^2}Y$ we get

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, z^2+axz+bxy,\, y^2+\frac{1}{c}yz+\frac{1}{4c^2}z^2+(d+\frac{a}{4c^2})xz+(e+\frac{b}{4c^2})xy,\, hxz+ixy\big).$$

Now, by $y \mapsto y + \frac{1}{2c}z$

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, z^2 + (a - \frac{b}{2c})xz + bxy, y^2 + (d - \frac{e}{2c} + \frac{a}{4c^2} - \frac{b}{8c^3})xz + (e + \frac{b}{4c^2})xy,\\ (h - \frac{i}{2c})xz + ixy\big). \tag{C.23}$$

Let $j := a - \frac{b}{2c}$, $k := d + \frac{a}{4c^2} - \frac{e}{2c} - \frac{b}{8c^3}$, $l := e + \frac{b}{4c^2}$ and $m := h - \frac{i}{2c}$. Then

$$j^2f(0) \sim_{\mathcal{A}} (x,\, z^2 + jxz + bxy,\, y^2 + kxz + lxy,\, mxz + ixy).$$

(3.1.a.ii.α_1) $mi \neq 0$. By $Y \mapsto Y - \frac{b}{i}W$, $Z \mapsto Z - \frac{k}{m}W$,

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, z^2 + (j - \frac{bm}{i})xz,\, y^2 + (l - \frac{ik}{m})xy,\, mxz + ixy\big).$$

Moreover, by $y \mapsto y + \frac{1}{2}(l + \frac{ik}{m})x$, $z \mapsto z + \frac{1}{2}(j - \frac{bm}{i})x$ and by adding X^2 to Y, Z and W, respectively, with appropriate coefficients we obtain

$$j^2f(0) \sim_{\mathcal{A}} (x,\, z^2,\, y^2,\, mxz + ixy).$$

Next, we scale by $y \mapsto iy$, $z \mapsto mz$ and then $Y \mapsto \frac{1}{m^2}Y$, $Z \mapsto \frac{1}{i^2}Y$ to get $(x,\, z^2,\, y^2,\, xz + xy) \sim_{\mathcal{A}} \boxed{(x,\, y^2,\, xy + xz,\, z^2)}$.

(3.1.a.ii.α_2) $m \neq 0$, $i = 0$. This time, we apply $Y \mapsto Y - \frac{j}{m}W$, $Z \mapsto Z - \frac{k}{m}W$ to get

$$j^2f(0) \sim_{\mathcal{A}} (x,\, z^2 + bxy,\, y^2 + lxy,\, mxz).$$

Now, $y \mapsto y + \frac{l}{2}x$ and $Y \mapsto Y + \frac{bl}{2}X^2$, $Z \mapsto Z + \frac{l^2}{4}X^2$, $W \mapsto W + \frac{lm}{2}X^2$ yield

$$j^2f(0) \sim_{\mathcal{A}} (x,\, z^2 + bxy,\, y^2,\, mxz)$$

which is, up to scaling and symmetry, equivalent to $\boxed{(x,\, y^2 + xz,\, xy,\, z^2)}$ (if $b \neq 0$)

or $\boxed{(x,\, y^2,\, xz,\, z^2)}$ (if $b = 0$).

(3.1.a.ii.α_3) $m = 0$, $i \neq 0$. This is symmetric to (3.1.a.ii.α_2).

(3.1.a.ii.β) $c = 0$. We have

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, z^2 + axz + bxy,\, yz + dxz + exy,\, hxz + ixy)$$

If $h = i = 0$ then it is equivalent to (C.8).

(3.1.a.ii.β_1) $i \neq 0$. By $y \mapsto y + \frac{h}{i} z$ and $W \mapsto \frac{1}{i} W$,

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, z^2 + (a - \frac{bh}{i})xz + bxy,\, yz - \frac{h}{i} z^2 + (d - \frac{eh}{i})xz + exy,\, xy\big).$$

If we apply $z \mapsto z + \frac{1}{2}(a - \frac{bh}{i})x$ (and add appropriate multiples of X^2 to Y and Z) then we get

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, z^2 + bxy,\, yz - \frac{h}{i} z^2 + (d - \frac{eh}{i} + \frac{h}{2i}(a - \frac{bh}{i}))xz + (e - \frac{1}{2}(a - \frac{bh}{i}))xy,\, xy\big).$$

Moreover, we cancel z^2 in Z by $Z \mapsto Z + \frac{h}{i} Y$, and xy in Y and Z by adding the right multiples of W so that

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, z^2,\, yz + (d - \frac{eh}{i} + \frac{h}{2i}(a - \frac{bh}{i}))xz,\, xy\big).$$

Now, $y \mapsto y + (d - \frac{eh}{i} + \frac{h}{2i}(a - \frac{bh}{i}))x$ and $W \mapsto W + (d - \frac{eh}{i} + \frac{h}{2i}(a - \frac{bh}{i}))X^2$ yield $j^2 f(0) \sim_{\mathcal{A}} (x,\, z^2,\, yz,\, xy)$ which is equivalent to $\boxed{(x,\, y^2,\, xz,\, yz)}$ by symmetry.

(3.1.a.ii.β_2) $i = 0$, $h \neq 0$. We can assume that $h = 1$ by Remark C.0.17. So

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, z^2 + axz + bxy,\, yz + dxz + exy,\, xz).$$

We cancel xz terms in Y and Z by $Y \mapsto Y - aW$, $Z \mapsto Z - dW$:

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, z^2 + bxy,\, yz + exy,\, xz). \tag{C.24}$$

If $e \neq 0$ then, by $z \mapsto z + ex$,

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, z^2 - 2exz + e^2x^2 + bxy,\, yz,\, xz).$$

By $Y \mapsto Y + 2eW - e^2X^2$, $j^2f(0) \sim_{\mathcal{A}} (x, z^2 + bxy, yz, xz)$. Up to a scaling and symmetry, we get $\boxed{(x,\ y^2 + xz,\ xy,\ yz)}$ (if $b \neq 0$) or $\boxed{(x,\ y^2,\ xy,\ yz)}$ (if $b = 0$).

(3.1.b) Assume that $b_2 = 0$. We have

$$j^2f(0) = (x,\ y^2 + a_3z^2 + a_4xz + a_5xy,\ b_3z^2 + b_4xz + b_5xy,\ c_2yz + c_3z^2 + c_4xz + c_5xy).$$

If $c_2 \neq 0$ then we have a case equivalent to (3.1.a). So we take $c_2 = 0$.

(3.1.b.i) If $b_3 \neq 0$ then we can take b_3 to be 1 by Remark C.0.17. Then $W \mapsto W - c_3Z$ yields

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\ y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy,\ z^2 + b_4xz + b_5xy,\ (c_4 - b_4c_3)xz + (c_5 - b_5c_3)xy\big).$$

If $a_2 \neq 0$ then $Y \mapsto Y + (\frac{a_2^2}{4} - a_3)Z$ and $y \mapsto y + \frac{a_2}{2}z$ returns

$$\begin{aligned} j^2f(0) \sim_{\mathcal{A}} \big(x,\ y^2 + (a_4 - \frac{a_2a_4}{2})xz + a_5xy, z^2 + (b_4 - \frac{a_2b_5}{2})xz + b_5xy,& \\ (c_4 - b_4c_3 - \frac{a_2}{2}c_5 - b_5c_3)xz + (c_5 - b_5c_3)xy\big).& \end{aligned}$$

which is of the form (C.23). If $a_2 = 0$ then by $Y \mapsto Y - a_3Z$ we find ourselves in a similar situation.

(3.1.b.ii) Let $b_3 = 0$. If $c_3 \neq 0$ then we are back to the case (3.1.b.i). So assume $c_3 = 0$. We have

$$j^2f(0) \sim_{\mathcal{A}} (x,\ y^2 + a_2yz + a_3z^2 + a_4xz + a_5xy,\ b_4xz + b_5xy,\ c_4xz + c_5xy).$$

Here we assume that $\begin{vmatrix} b_4 & b_5 \\ c_4 & c_5 \end{vmatrix} \neq 0$ as otherwise we have a jet equivalent to (C.11). In fact, we only need to consider the following case as the others will be equivalent up to a linear transformation involving only Z and W.

Suppose $b_4 \neq 0$. By $y \mapsto y + \frac{b_4}{b_5}z$ and $Z \mapsto \frac{1}{b_5}Z$,

$$\begin{aligned} j^2f(0) \sim_{\mathcal{A}} \big(x, y^2 + (a_2 - \frac{2b_4}{b_5})yz + (a_3 - \frac{b_4^2}{b_5^2})z^2 + (a_4 - \frac{a_5b_4}{b_5})xz + a_5xy,\ xy,& \\ (c_4 - \frac{b_4c_5}{b_5})xz + c_5xy\big).& \end{aligned}$$

Moreover, $Y \mapsto Y - a_5 Z$, $W \mapsto W - c_5 Z$ yields

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, y^2 + (a_2 - \frac{2b_4}{b_5})yz + (a_3 - \frac{b_4^2}{b_5^2})z^2 + (a_4 - \frac{a_5 b_4}{b_5})xz,\, xy,\, (c_4 - \frac{b_4 c_5}{b_5})xz\big).$$

Now we assume that $c_4 - \frac{b_4 c_5}{b_5} \neq 0$ as otherwise we have jet of the form (C.14). So by scaling on Z and $Y \mapsto Y - (a_4 - \frac{a_5 b_4}{b_5})W$,

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, y^2 + ayz + bz^2,\, xy,\, xz)$$

for some $a, b \in \mathbb{C}$.

(3.1.b.ii.α) If $a = b = 0$ we get $\boxed{(x,\, y^2,\, xy,\, xz)}$.

(3.1.b.ii.β) If $a \neq 0$ and $b = 0$ then $z \mapsto z + \frac{1}{a}y$, $W \mapsto W + \frac{1}{a}Z$ yield $\boxed{(x,\, xy,\, xz,\, yz)}$.

(3.1.b.ii.γ) If $b \neq 0$ then by $y \mapsto y + \varphi z$, $Z \mapsto Z + \varphi W$ for some $\varphi \in \mathbb{C}$ we find

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, y^2 + (a - 2\varphi)yz + (b - a\varphi + \varphi^2)z^2,\, xy,\, xz\big).$$

We choose φ such that $b - a\varphi + \varphi^2 = 0$. In that case, if $a - 2\varphi$ is also zero then $j^2 f(0) \sim_{\mathcal{A}} (x,\, y^2,\, xy,\, xz)$. If not, by $z \mapsto z - \frac{1}{a-2\varphi}y$ and $W \mapsto W + \frac{1}{a-2\varphi}Z$ we find $(x,\, yz,\, xy,\, xz)$.

(3.2) Suppose $a_1 = 0$. We have

$$\begin{aligned} j^2 f(0) = (x,\, a_2 yz + a_3 z^2 + a_4 xz + a_5 xy, b_1 y^2 + b_2 yz + b_3 z^2 + b_4 xz + b_5 xy,\\ c_1 y^2 + c_2 yz + c_3 z^2 + c_4 xz + c_5 xy) \end{aligned}$$

If $b_1 \neq 0$ or $c_1 \neq 0$ we have a jet equivalent to (C.16). So we assume $b_1 = c_1 = 0$.

(3.2.a) Let $a_2 = 1$. So

$$j^2 f(0) = (x,\, yz + a_3 z^2 + a_4 xz + a_5 xy,\, b_2 yz + b_3 z^2 + b_4 xz + b_5 xy,\, c_2 yz + c_3 z^2 + c_4 xz + c_5 xy).$$

If b_2 is non-zero then $Z \mapsto Z - b_2 Y$ cancels $b_2 yz$. Similarly, if $c_2 \neq 0$ then we apply $W \mapsto W - c_2 Y$ to cancel $c_2 yz$. Hence we may assume that $b_2 = c_2 = 0$. So we consider

$$j^2f(0) = (x,\, yz + a_3z^2 + a_4xz + a_5xy,\, b_3z^2 + b_4xz + b_5xy,\, c_3z^2 + c_4xz + c_5xy).$$

(3.2.a.i) Assume $b_3 = 1$. By $Y \mapsto Y - a_3Z$, $W \mapsto W - c_3Z$,

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, yz + (a_4 - a_3b_4)xz + (a_5 - a_3b_5)xy,\, z^2 + b_4xz + b_5xy,\\ (c_4 - b_4c_3)xz + (c_5 - b_5c_3)xy\big).$$

If $b_4 \neq 0$ then we can cancel b_4xz by $z \mapsto z + \frac{b_4}{2}x$. This coordinate change will only introduce x^2 in f_i's and we can cancel those by adding X^2 to Y, Z and W, respectively, with appropriate coefficients. So we assume that $b_4 = 0$ without the loss of generality. Let us set $a := a_4 - a_3b_4$, $b := a_5 - a_3b_5$, $c := b_5$, $d := c_4 - b_4c_3$ and $e := c_5 - b_5c_3$ for simplicity, i.e. consider

$$j^2f(0) \sim_{\mathcal{A}} (x,\, yz + axz + bxy,\, z^2 + cxy,\, dxz + exy).$$

(3.2.a.i.α) If $de \neq 0$ then by $y \mapsto y + \frac{d}{e}z$ and $W \mapsto \frac{1}{e}W$,

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, yz - \frac{d}{e}z^2 + (a - \frac{bd}{e})xz + bxy,\, z^2 + cxy - \frac{cd}{e}xz,\, xy\big).$$

Now, by $z \mapsto z - \frac{cd}{2e}x$ and adding X^2 to Y, Z and W, respectively, with appropriate coefficients we find a jet of the form

$$j^2f(0) \sim_{\mathcal{A}} (x,\, yz + kz^2 + lxz + mxy,\, z^2 + nxy,\, xy) \tag{C.25}$$

for some $k, l, m, n \in \mathbb{C}$. Then, we cancel mxy and nxy by $Y \mapsto Y - mW$, $Z \mapsto Z - nW$:

$$j^2f(0) \sim_{\mathcal{A}} (x,\, yz + kz^2 + lxz,\, z^2,\, xy).$$

Moreover, $Y \mapsto Y - kZ$ cancels kz^2. By $y \mapsto y + lx$ and $W \mapsto W + lX^2$ yield $(x,\, yz,\, z^2,\, xy)$ which is equivalent to $(x,\, y^2,\, xz,\, yz)$ by symmetry.

(3.2.a.i.β) If $d = 1$, $e = 0$ then

$$j^2f(0) \sim_{\mathcal{A}} (x,\, yz + axz + bxy,\, z^2 + cxy,\, xz).$$

Now $Y \mapsto Y - aW$ cancels the term axz. But this is a jet of the form (C.24).

(3.2.a.i.γ) Let $d = 0$, $e = 1$. This is equivalent to the jet (C.25).

(3.2.a.ii) If $b_3 = 0$ then

$$j^2 f(0) = (x,\, yz + a_3 z^2 + a_4 xz + a_5 xy,\, b_4 xz + b_5 xy,\, c_3 z^2 + c_4 xz + c_5 xy).$$

If $c_3 \neq 0$ then we are back to the case C.3. So assume $c_3 = 0$.

(3.2.a.ii.α) Assume $b_4 b_5 \neq 0$. By $y \mapsto y + \frac{b_4}{b_5} z$ and $Z \mapsto \frac{1}{b_5} Z$,

$$j^2 f(0) = (x,\, yz + (a_3 - \frac{b_4}{b_5}) z^2 + (a_4 - \frac{a_5 b_4}{b_5}) xz + a_5 xy,\, xy,\, (c_4 - \frac{b_4 c_5}{b_5}) xz + c_5 xy).$$

Now, $Y \mapsto Y - a_5 Z$, $W \mapsto W - c_5 Z$ yields

$$j^2 f(0) = (x,\, yz + (a_3 - \frac{b_4}{b_5}) z^2 + (a_4 - \frac{a_5 b_4}{b_5}) xz,\, xy,\, (c_4 - \frac{b_4 c_5}{b_5}) xz).$$

If $c_4 - \frac{b_4 c_5}{b_5} = 0$ then we have a jet equivalent to (C.15). So assume $c_4 - \frac{b_4 c_5}{b_5} \neq 0$. By scaling

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, yz + (a_3 - \frac{b_4}{b_5}) z^2 + (a_4 - \frac{a_5 b_4}{b_5}) xz,\, xy,\, xz\big).$$

Then, $Y \mapsto Y - (a_4 - \frac{a_5 b_4}{b_5}) W$ yields

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, yz + (a_3 - \frac{b_4}{b_5}) z^2,\, xy,\, xz\big). \tag{C.26}$$

Here, if $a_3 - \frac{b_4}{b_5} = 0$ then we get (x, xy, xz, yz). If $a_3 - \frac{b_4}{b_5} \neq 0$ then we apply $y \mapsto y + (a_3 - \frac{b_4}{b_5}) z$ to get

$$j^2 f(0) = \big(x,\, yz,\, xy - (a_3 - \frac{b_4}{b_5}) xz,\, xz\big).$$

So $Z \mapsto Z + (a_3 - \frac{b_4}{b_5}) W$ again returns (x, xy, xz, yz).

(3.2.a.ii.β) Assume $b_4 = 1$, $b_5 = 0$. We have

$$j^2f(0) \sim_{\mathcal{A}} (x,\, yz + a_3z^2 + a_4xz + a_5xy,\, xz,\, c_4xz + c_5xy).$$

By $Y \mapsto Y - a_4Z$, $W \mapsto W - c_4Z$,

$$j^2f(0) \sim_{\mathcal{A}} (x,\, yz + a_3z^2 + a_5xy,\, xz,\, c_5xy).$$

We assume that $c_5 = 1$ as $c_5 = 0$ is from Case 2. Then, $Y \mapsto Y - a_5W$ returns a jet equivalent to (C.26).

(3.2.a.ii.γ) If $b_4 = 0$, $b_5 = 1$ then we have a case similar to (3.2.a.ii).

(3.2.b) If $a_2 = 0$ then

$$j^2f(0) \sim_{\mathcal{A}} (x,\, a_3z^2+a_4xz+a_5xy,\, b_2yz+b_3z^2+b_4xz+b_5xy,\, c_2yz+c_3z^2+c_4xz+c_5xy).$$

Notice that if $b_2 \neq 0$ or $c_2 \neq 0$ then it is a jet we have studied in Remark C.3. So assume $b_2 = c_2 = 0$.

(3.2.b.i) Suppose $a_3 = 1$. Then

$$j^2f(0) \sim_{\mathcal{A}} (x,\, z^2 + a_4xz + a_5xy,\, b_3z^2 + b_4xz + b_5xy,\, c_3z^2 + c_4xz + c_5xy).$$

By $Z \mapsto Z - b_3Y$, $W \mapsto W - c_3Y$,

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, z^2 + a_4xz + a_5xy, (b_4 - a_4b_3)xz + (b_5 - a_5b_3)xy,$$
$$(c_4 - a_4c_3)xz + (c_5 - a_5c_3)xy\big).$$

(3.2.b.i.α) Let $(b_4 - a_4b_3)(b_5 - a_5b_3) \neq 0$. Scale by $Z \mapsto \frac{1}{b_5-a_5b_3}Z$, set $b := \frac{b_4-a_4b_3}{b_5-a_5b_3}$ so that

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, z^2 + a_4xz + a_5xy,\, bxz + xy,\, (c_4 - a_4c_3)xz + (c_5 - a_5c_3)xy\big). \quad \text{(C.27)}$$

Then, by $y \mapsto y + bz$

$$j^2f(0) \sim_{\mathcal{A}} \big(x,\, z^2+(a_4-a_5b)xz+a_5xy,\, xy,\, (c_4-a_4c_3-b(c_5-a_5c_3))xz+(c_5-a_5c_3)xy\big).$$

Now, $Y \mapsto Y - a_5 Z$, $W \mapsto W - (c_5 - a_5 c_3)Z$ yields

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, z^2 + (a_4 - a_5 b)xz,\, xy,\, (c_4 - a_4 c_3 - b(c_5 - a_5 c_3))xz\big).$$

Clearly, if $c_4 - a_4 c_3 - b(c_5 - a_5 c_3) = 0$ then we have a jet from Case 2. So assume it is not equal to zero. By scaling and then $Y \mapsto Y - (a_4 - a_5 b)W$ we get $j^2 f(0) \sim_{\mathcal{A}} (x,\, z^2,\, xz,\, xy)$ which is symmetric to (x, y^2, xy, xz).

(3.2.b.i.β) Let $b_4 - a_4 b_3 = 0$, $b_5 - a_5 b_3 = 1$. This case is equivalent to the previous one.

(3.2.b.i.γ) Let $b_4 - a_4 b_3 = 1$, $b_5 - a_5 b_3 = 0$. We have

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, z^2 + a_4 xz + a_5 xy,\, xz,\, (c_4 - a_4 c_3)xz + (c_5 - a_5 c_3)xy).$$

So, $Y \mapsto Y - a_4 Z$, $W \mapsto W - (c_4 - a_4 c_3)Z$ yields

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, z^2 + a_5 xy,\, xz,\, (c_5 - a_5 c_3)xy\big).$$

As before, we assume that $c_5 - a_5 c_3 \neq 0$. By scaling W and then $Z \mapsto Z - s_5 W$ we find $(x,\, z^2,\, xz,\, xy)$ which is equivalent to $(x,\, y^2,\, xy,\, xz)$.

(3.2.b.ii) If $a_3 = 0$ then

$$j^2 f(0) = (x,\, a_4 xz + a_5 xy,\, b_3 z^2 + b_4 xz + b_5 xy,\, c_3 z^2 + c_4 xz + c_5 xy).$$

If $b_3 \neq 0$ or $c_3 \neq 0$ then it is equivalent to (C.27). So assume $b_3 = c_3 = 0$. Then

$$j^2 f(0) \sim_{\mathcal{A}} (x,\, a_4 xz + a_5 xy,\, b_4 xz + b_5 xy,\, c_4 xz + c_5 xy).$$

If $a_4 a_5 \neq 0$ then $y \mapsto y + \frac{a_4}{a_5} z$ yields

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, a_5 xy,\, (b_4 - \frac{a_4 b_5}{a_5})xz + b_5 xy,\, (c_4 - \frac{a_4 c_5}{a_5})xz + c_5 xy\big).$$

By scaling Y and then $Z \mapsto Z - b_5 Y$, $W \mapsto W - c_5 Y$ we find

$$j^2 f(0) \sim_{\mathcal{A}} \big(x,\, xy,\, (b_4 - \frac{a_4 b_5}{a_5})xz,\, (c_4 - \frac{a_4 c_5}{a_5})xz\big).$$

Then we get either $(x,\, xy,\, xz, 0)$ or $(x,\, xy, 0, 0)$. The other cases, i.e. $a_4 = 0$ or $a_5 = 0$, result the same jet.
This finalises the calculations for the classification of 2-jets.

Remark C.3.1. We meet the polynomial Λ (C.21) in other parts of mathematics, as well. It is sometimes referred as a *bracelet*. For example, it is isomorphic to the defining equation of the discriminant in the space of binary cubics $\left\{ax^3 + bx^2 + cx + d \mid a \neq 0\right\}$, that is,

$$18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 \tag{C.28}$$

([Fis01, A.1.2]). Also in [MWA03], Mond and Wik Atique define a *discriminant* in the space of hypersurfaces given by

$$\left\{8kl^3 + 3l^2m^2 + 24klmn + 8m^3n - 16k^2n^2 = 0\right\}$$

that characterises the transversal sections for which one can obtain $\mathcal{A}_e$-codimension 1 map-germs from the stable map-germ $F\colon (A, B, C, D, x, y) \mapsto (A, B, C, D, x^2 + Ay, xy+Bx+Cy, y^2+Dx)$ by pull-back diagrams. Their equation is also isomorphic to (C.21). Moreover, one can check that the $\mathcal{K}_V$-discriminant of the identity map G on $(\mathbb{C}^7, 0)$, for $V := \text{image}(F)$, is defined to be

$$\left\{27A^2D^2 - 72ABCD - 32AB^3 - 32C^3D - 16B^2C^2 = 0\right\}$$

which is isomorphic to Λ by $B \mapsto c, C \mapsto b$ (see Definition 3.2.5 for the definition of $\mathcal{K}_V$-discriminant). See also [SK77, Proposition 6, §5] for the place of this equation in representation theory.

Appendix D

Groebner Basis calculations

In this section we calculate Groebner bases of $\mathcal{K}_V$-tangent spaces which play a key part in the proof of Proposition 5.2.1 and 5.2.2. The terminology and definitions for Groebner bases are quite standard. One can consult, for example, [CLO05] or [Eis95] for details. Groebner bases calculations are implemented in computer algebra programs such as SINGULAR and Macaulay. Here, we do the calculations by hand as we want to find an explicit set of generators for $\mathcal{K}_V$-tangent spaces which are defined for series of map-germs.

Definition D.0.2. Let F be a free module with basis $e_1, \ldots, e_n$. A *reverse lexicographic order* with priority given to the coefficients is defined with the following property: $x^\alpha e_i > x^\beta e_j$ if $x^\alpha >_{revlex} x^\beta$, that is, $\exists i \in \{1, \ldots, n\}$ such that $\alpha_{i+1} = \beta_{i+1}, \ldots, \alpha_n = \beta_n$, $\alpha_i > \beta_i$, or $x^\alpha = x^\beta$ and $e_i > e_j$. We denote the leading term of an element f by $\mathrm{LT}(f)$.

A set $B = \{b_1, \ldots, b_s\}$ of generators for a module N is a Groebner basis if

$$\mathrm{LT}(N) := \{\mathrm{LT}(n) \mid n \in N\} = (\mathrm{LT}(b_1), \ldots, \mathrm{LT}(b_s))$$

([CLO05, Definition 2.6]). Buchberger's Criterion ([CLO05, Theorem 2.9]) provides an algorithm (famously known as Buchberger's algorithm, [CLO05, Theorem 2.11]) for producing Groebner bases out of a given set of generators for a module. The algorithm principally involves the notion of syzygy polynomials: The *syzygy polynomial* (or *S-polynomial*) of two elements, say f and g, is defined to be

$$S(f, g) := \frac{\mathrm{LCM}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(f)} f - \frac{\mathrm{LCM}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(g)} g$$

where LCM stands for the *least common multiple.* If $\mathrm{LT}(f) = x^\alpha e_i$ and $\mathrm{LT}(g) = x^\beta e_j$ for $i \neq j$ then $\mathrm{LCM}(\mathrm{LT}(f), \mathrm{LT}(g)) = 0$ whence $S(f,g) = 0$.

Buchberger's algorithm works as follows: If one has a set of generators $B = \{n_1, \ldots, n_s\}$, then B can be completed to a Groebner basis by successively adding nonzero remainders of $S(n_i, n_j)$ on division by B to B. This is the method we will use in the rest of this section.

D.1 The series $\tilde{C}_k$

We fix ">" to be a reverse lexicographic order with priority given to the coefficients. We aim to find a Groebner basis for $N = (\begin{bmatrix}0 & t^{k-1} & 0\end{bmatrix}^t)\mathcal{O}_{\mathbb{C}^5,0} + \hat{g}_k^*\mathrm{Der}(\text{-log}V)_2$ defined in p. 100. A set of generators is given by $B := \{n_1, \ldots, n_{14}, \tilde{n}_{15}, \tilde{n}_{17}, n_{18}\}$ where n_i and $\tilde{n}_{15}, \tilde{n}_{17}$ defined by (5.8)-(5.11).

Step 1. Notice that $n_1, \ldots, n_{14}$ already satisfy the Buchberger's Criterion as they form a part of a Groebner basis for N_0 and they do not depend on the variable t. So we consider the S-polynomials only involving $\tilde{n}_{15}$, $\tilde{n}_{17}$ and n_{18}. We investigate the remainders of the S-polynomials on division by B. The only non-zero S-polynomials are listed below.

$$S(n_1\tilde{n}_{15}) = \begin{bmatrix} -\frac{57}{4}X^{11} \\ 2X^{12} - X^8Y \\ -\frac{3}{4}X^8Z - 6X^9Y \end{bmatrix}, \quad S(n_5, \tilde{n}_{15}) = \begin{bmatrix} \frac{74}{39}X^6t^k - \frac{57}{4}X^5Y \\ -\frac{61}{312}X^7t^k - X^2Y^2 + 2X^6Y \\ -\frac{59}{156}X^8t^k - \frac{3}{4}X^2YZ - 6X^3Y^2 \end{bmatrix},$$

$$S(n_3, \tilde{n}_{17}) = \begin{bmatrix} \frac{3}{2}X^{10} - \frac{277}{229}X^7t^k \\ \frac{1}{4}X^{11} - \frac{25}{458}X^8t^k \\ -\frac{3}{2}X^{12} \end{bmatrix}, \quad S(n_6, n_{18}) = \begin{bmatrix} -\frac{8}{7}XYt^{k-1} - \frac{36}{7}X^5t^{k-1} \\ -\frac{4}{7}X^6t^{k-1} \\ \frac{40}{7}X^7t^{k-1} \end{bmatrix},$$

$$S(n_4, \tilde{n}_{17}) = \begin{bmatrix} \frac{5}{2}X^2t^k + \frac{3}{2}XY \\ -\frac{1}{4}X^3t^k + \frac{1}{4}X^2Y \\ -\frac{1}{2}X^4t^k - \frac{3}{2}X^3Y \end{bmatrix}, \quad S(n_7, \tilde{n}_{15}) = \begin{bmatrix} \frac{3439}{32}X^4Yt^k - \frac{57}{4}X^3Y^2 \\ -\frac{19473}{256}X^5Yt^k - Y^3 + 2X^4Y^2 \\ -\frac{82731}{128}X^6Yt^k - \frac{3}{4}Y^2Z - 6XY^3 \end{bmatrix},$$

$$S(n_8, n_{18}) = \begin{bmatrix} \frac{147}{2}X^3Yt^{k-1} - \frac{12939}{16}X^7t^{k-1} \\ -\frac{817}{16}X^4Yt^{k-1} + \frac{4331}{32}X^8t^{k-1} \\ -\frac{3483}{8}X^5Yt^{k-1} - \frac{1173}{16}X^9t^{k-1} \end{bmatrix},$$

$$S(n_9,\tilde{n}_{15}) = \begin{bmatrix} \frac{29}{2}XYt^k - \frac{2549}{16}X^5t^k - \frac{57}{4}X^3Z \\ -\frac{159}{16}X^2Yt^k + \frac{861}{32}X^6t^k - YZ + 2X^4Z \\ -\frac{693}{8}X^3Yt^k - \frac{243}{16}X^7t^k - \frac{3}{4}Z^2 - 6XYZ \end{bmatrix},$$

$$S(n_{10},n_{18}) = \begin{bmatrix} 2Yt^{k-1} + \frac{29}{4}X^4t^{k-1} \\ \frac{7}{4}XYt^{k-1} - \frac{1}{8}X^5t^{k-1} \\ -\frac{3}{2}X^2Yt^{k-1} - \frac{9}{4}X^6t^{k-1} \end{bmatrix}, S(n_{11},\tilde{n}_{17}) = \begin{bmatrix} \frac{17}{2}X^3t^k + \frac{3}{2}XZ \\ \frac{1}{2}Yt^k - \frac{3}{4}X^4t^k + \frac{1}{4}X^2Z \\ 3XYt^k - \frac{3}{2}X^5t^k - \frac{3}{2}X^3Z \end{bmatrix},$$

$$S(n_{12},\tilde{n}_{15}) = \begin{bmatrix} \frac{189}{40}XZt^k + \frac{201}{55}X^2Yt^k - \frac{4853}{80}X^6t^k - \frac{57}{4}X^3W \\ -\frac{15}{176}X^2Zt^k - \frac{177}{44}X^3Yt^k + \frac{927}{110}X^7t^k - YW + 2X^4W \\ -\frac{4491}{880}X^3Zt^k - \frac{3087}{110}X^4Yt^k - \frac{3}{4}ZW - 6XYW \end{bmatrix},$$

$$S(n_{13},n_{18}) = \begin{bmatrix} -Zt^{k-1} + \frac{1677}{88}XYt^{k-1} - \frac{18877}{32}X^5t^{k-1} \\ \frac{305}{88}XZt^{k-1} - \frac{3215}{88}X^2Yt^{k-1} + \frac{3623}{44}X^6t^{k-1} \\ -\frac{11609}{352}X^2Zt^{k-1} - \frac{5919}{22}X^3Yt^{k-1} \end{bmatrix},$$

$$S(n_{14},\tilde{n}_{17}) = \begin{bmatrix} \frac{7}{33}Yt^k + \frac{55}{24}X^4t^k + \frac{3}{2}XW \\ \frac{20}{33}Zt^k - \frac{19}{132}XYt^k - \frac{13}{33}X^5t^k + \frac{1}{4}X^2W \\ -\frac{47}{88}XZt^k + \frac{41}{22}X^2Yt^k - \frac{3}{2}X^3W \end{bmatrix}.$$

We apply the division algorithm for modules ([CLO05, Theorem 2.5]) on these generators and find the following equalities.

$$S(n_1,\tilde{n}_{15}) = \frac{5}{916}X^3n_1 + \frac{9}{458}X^3n_2 + \frac{47}{156}X^4n_3 - \frac{15}{4}X^9n_4 - \frac{5}{7}X^5n_5 - \frac{5}{8}X^6n_6 - \frac{3}{4}X^8n_{11}$$

$$\begin{aligned} S(n_3,\tilde{n}_{17}) = &- \frac{67337}{209764}X^2n_1 + \frac{7570}{52441}X^2n_2 - \frac{66605}{35724}X^3n_3 + \frac{4155}{916}X^8n_4 + \\ &+ \frac{1385}{1603}X^4n_5 + \frac{1385}{1832}X^5n_6 + \frac{831}{916}X^7n_{11} - \frac{277}{229}X^7\tilde{n}_{15} - \frac{25}{458}X^8tn_{18}, \end{aligned}$$

$$S(n_4,\tilde{n}_{17}) = -\frac{87}{8}X^3n_4 - \frac{21}{16}n_6 - \frac{15}{8}X^2n_{11} + \frac{5}{2}X^2\tilde{n}_{15} - \frac{1}{2}X^4\tilde{n}_{17} - \frac{1}{4}X^3tn_{18},$$

$$\begin{aligned} S(n_5,\tilde{n}_{15}) = &\frac{1902483335}{2000544}Xn_1 - \frac{318058601}{2286336}Xn_2 + \frac{232251841}{2725632}X^2n_3 + \frac{74}{39}X^6\tilde{n}_{15} + \\ &- \left(\frac{5453739}{11648}X^7 + \frac{15}{4}X^3Y\right)n_4 + \frac{89909}{7644}X^3n_5 - \frac{5}{7}Xn_7 - \frac{59}{156}X^8\tilde{n}_{17} + \\ &- \left(\frac{3870131}{69888}X^4 + \frac{5}{8}Y\right)n_6 + \left(\frac{37}{26}X^6 - \frac{3}{4}X^2Y\right)n_{11} - \frac{61}{312}X^7tn_{18}, \end{aligned}$$

$$S(n_6,n_{18}) = -\frac{4}{7}X^6n_{18} + \begin{bmatrix} -\frac{8}{7}XYt^{k-1} - \frac{36}{7}X^5t^{k-1} & 0 & \frac{40}{7}X^7t^{k-1} \end{bmatrix}^t, \qquad (D.1)$$

$$
\begin{aligned}
S(n_7, \tilde{n}_{15}) =& -\frac{14800935743}{234496}X^3 n_1 + \frac{8664452355}{937984}X^3 n_2 - \frac{724322321}{159744}X^4 n_3 + \\
& -\left(\frac{12078969}{1024}X^9 + \frac{86799}{128}X^5 Y + \frac{15}{4}XY^2\right) n_4 - \frac{380785}{1792}X^5 n_5 + \\
& -\frac{3107803}{2048}X^6 n_6 + \frac{759}{16}X^3 n_7 - \left(\frac{5}{8}Y + \frac{25237}{256}X^4\right) n_8 - \frac{19473}{256}X^5 Y t n_{18} + \\
& -\left(\frac{3}{4}Y^2 + \frac{10317}{128}X^4 Y\right) n_{11} + \frac{3439}{32}X^4 Y \tilde{n}_{15} - \frac{82731}{128}X^6 Y \tilde{n}_{17},
\end{aligned}
$$

$$
\begin{aligned}
S(n_8, n_{18}) =& -\frac{27373}{104}t^{k-1} n_3 - \frac{3483}{8}X^5 t^{k-1} n_4 + \frac{147}{2}X t^{k-1} n_5 - \frac{817}{16}X^4 Y n_{18} \\
& + \frac{24277}{916}X^8 n_{18} + \left[-\tfrac{326247}{1832}X^7 t^{k-1} \quad 0 \quad 0\right]^t ,
\end{aligned} \tag{D.2}
$$

$$
\begin{aligned}
S(n_9, \tilde{n}_{15}) =& -\frac{48449981}{29312}n_1 + \frac{55538793}{234496}n_2 - \frac{43089}{1024}X n_3 + \frac{4797}{56}X^2 n_5 + \frac{5}{4}n_7 + \\
& -\left(\frac{1424307}{512}X^6 + \frac{705}{16}X^2 Y + \frac{693}{8}X^3 t^k\right) n_4 - \frac{332937}{1024}X^3 n_6 + \\
& -\frac{195}{32}X n_8 - \frac{63}{8}X^3 n_9 + \left(\frac{23}{16}X^4 - \frac{5}{8}Y\right) n_{10} - \frac{117}{2}X^7 \tilde{n}_{17} + \\
& -\left(\frac{3}{4}Z + \frac{117}{8}XY + \frac{705}{16}X^5\right) n_{11} + \left(\frac{229}{4}X^5 + \frac{29}{2}XY\right) \tilde{n}_{15} + \\
& + \left(-\frac{159}{16}X^2 Y t + \frac{21}{4}X^6 t\right) n_{18},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{10}, n_{18}) =& -\frac{3}{2}X^2 t^{k-1} n_4 + \left(\frac{7}{4}XY - \frac{1}{2}X^5\right) n_{18} + \\
& + \left[2Y t^{k-1} + 11X^4 t^{k-1} \quad 0 \quad -3X^6 t^{k-1}\right]^t ,
\end{aligned} \tag{D.3}
$$

$$
\begin{aligned}
S(n_{11}, \tilde{n}_{17}) =& \frac{1575}{16}X^4 n_4 - \frac{39}{7}n_5 + \frac{341}{32}X n_6 + \frac{3}{2}X n_9 + \frac{1}{4}X^2 n_{10} - \frac{63}{8}X^3 n_{11} \\
& + \frac{17}{2}X^3 \tilde{n}_{15} + \left(3XY - \frac{3}{2}X^5\right) \tilde{n}_{17} + \left(\frac{1}{2}Y t - \frac{3}{4}X^4 t\right) n_{18},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{12}, \tilde{n}_{15}) =& \frac{131576772273}{22570240}X n_1 - \frac{21999004647}{25794560}X n_2 + \frac{4696408581}{10250240}X^2 n_3 + \\
& -\left(\frac{60313203}{394240}X^7 + \frac{187353}{704}X^3 Y\right) n_4 + \frac{4099869}{86240}X^3 n_5 - \frac{2697}{616}X n_7 + \\
& -\left(\frac{8585769}{788480}X^4 + \frac{53211}{1408}Y\right) n_6 - \left(\frac{3087}{110}X^4 Y + \frac{4491}{880}X^3 Z\right) \tilde{n}_{17} + \\
& + \left(\frac{1213}{352}XY + \frac{5}{11}Z - \frac{2231}{704}X^5\right) n_{10} - \frac{57}{4}X^3 n_{12} + \left(2X^4 - Y\right) n_{13} +
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{3471}{880}XZ+\frac{93919}{1760}X^2Y-\frac{73031}{1760}X^6\right)n_{11}-\left(\frac{3}{4}Z+6XY\right)n_{14}+\\
&+\left(\frac{201}{55}X^2Y-\frac{4853}{80}X^6+\frac{189}{40}XZ\right)\tilde{n}_{15}+\left(\frac{46279}{1760}X^4-\frac{7}{4}Y\right)n_9+\\
&-\left(\frac{15}{176}X^2Zt+\frac{177}{44}X^3Yt-\frac{927}{110}X^7t\right)n_{18},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{13},n_{18})=&-\frac{8049}{32}X^3t^{k-1}n_4-t^{k-1}n_9+\frac{305}{88}Xt^{k-1}n_{10}-\frac{11609}{352}X^2t^{k-1}n_{11}+\\
&+\left(\frac{353}{16}X^6-\frac{2307}{64}X^2Y\right)n_{18}+\\
&+\left[\tfrac{213}{8}XYt^{k-1}+\tfrac{2157}{16}X^5t^{k-1}\quad 0\quad -\tfrac{1461}{8}X^7t^{k-1}\right]^t, \qquad \text{(D.4)}
\end{aligned}
$$

$$
\begin{aligned}
S(n_{14},\tilde{n}_{17})=&\frac{16717}{1716}n_3-\left(\frac{939045}{1408}X^5+\frac{35}{44}XY\right)n_4+\frac{365}{14}Xn_5-\frac{215711}{2816}X^2n_6+\\
&-\frac{35}{264}n_8-\frac{3361}{440}X^2n_9+\frac{13}{352}X^3n_{10}+\left(\frac{99837}{7040}X^4-\frac{7}{44}Y\right)n_{11}+\\
&+\frac{3}{2}Xn_{12}+\frac{1}{4}X^2n_{13}-\frac{3}{2}X^3n_{14}+\left(\frac{55}{24}X^4+\frac{7}{33}Y\right)\tilde{n}_{15}+\\
&+\left(\frac{41}{22}X^2Y-\frac{47}{88}XZ\right)\tilde{n}_{17}+\left(\frac{20}{33}Zt-\frac{19}{132}XYt-\frac{13}{33}X^5t\right)n_{18}.
\end{aligned}
$$

From (D.1)-(D.4), respectively, we read the following elements.

$$
n_{19}:=\begin{bmatrix} XYt^{k-1}+\frac{9}{2}X^5t^{k-1}\\ 0\\ -5X^7t^{k-1}\end{bmatrix}, \qquad \text{(D.5)}
$$

$$
n_{20}:=\begin{bmatrix} X^7t^{k-1}\\ 0\\ 0\end{bmatrix}, \qquad \text{(D.6)}
$$

$$
n_{21}:=\begin{bmatrix} Yt^{k-1}+\frac{11}{2}X^4t^{k-1}\\ 0\\ -\frac{3}{2}X^6t^{k-1}\end{bmatrix}, \qquad \text{(D.7)}
$$

$$
n_{22}:=\begin{bmatrix} XYt^{k-1}+\frac{719}{142}X^5t^{k-1}\\ 0\\ -\frac{487}{71}X^7t^{k-1}\end{bmatrix}. \qquad \text{(D.8)}
$$

Notice that $n_{19} = Xn_{21} + \begin{bmatrix} -X^5t^{k-1} & 0 & -\frac{7}{2}X^7t^{k-1} \end{bmatrix}^t$. So let

$$n_{23} := -\frac{2}{7}\begin{bmatrix} -X^5t^{k-1} & 0 & -\frac{7}{2}X^7t^{k-1} \end{bmatrix} = \begin{bmatrix} \frac{2}{7}X^5t^{k-1} & 0 & X^7t^{k-1} \end{bmatrix}. \tag{D.9}$$

Replacing n_{19} by n_{23} does not effect the output of the algorithm; it only shortens the process. So, we discard n_{19} from the basis. Now,

$$n_{22} = Xn_{21} - \frac{761}{142}n_{23} + \begin{bmatrix} \frac{544}{497}X^5t^{k-1} & 0 & 0 \end{bmatrix}^t.$$

We replace n_{22} by

$$n_{24} := \begin{bmatrix} X^5t^{k-1} & 0 & 0 \end{bmatrix}^t. \tag{D.10}$$

Moreover, $n_{20} = X^2n_{24}$. So we may exclude n_{20} from the basis as well. Therefore, by the end of the first step we extend the set of generators to

$$B' = \{n_1, \ldots, n_{14}, \tilde{n}_{15}, \tilde{n}_{17}, n_{18}, n_{21}, n_{23}, n_{24}\}.$$

Step 2. Let us set $B := B'$. We consider the S-polynomials involving only for n_{21}, n_{23} and n_{24}. The non-zero ones are given as follows.

$$S(n_{21}, n_1) = \begin{bmatrix} \frac{11}{2}X^{12}t^{k-1} \\ 0 \\ -\frac{3}{2}X14t^{k-1} \end{bmatrix}, \qquad S(n_{21}, n_5) = \begin{bmatrix} \frac{281}{78}X^6t^{k-1} \\ \frac{61}{312}X^7t^{k-1} \\ -\frac{175}{156}X^8t^{k-1} \end{bmatrix},$$

$$S(n_{21}, n_7) = \begin{bmatrix} -\frac{3263}{32}X^4Yt^{k-1} \\ \frac{19473}{256}X^5Yt^{k-1} \\ \frac{82539}{128}X^6Yt^{k-1} \end{bmatrix}, \quad S(n_{21}, \tilde{n}_{15}) = \begin{bmatrix} \frac{11}{2}X^4t^k - \frac{57}{4}X^3Y \\ -Y^2 + 2X^4Y \\ -\frac{3}{2}X^6t^k - \frac{3}{4}YZ - 6XY^2 \end{bmatrix},$$

$$S(n_{21}, n_9) = \begin{bmatrix} \frac{11}{2}X^4Zt^{k-1} - \frac{29}{2}XY^2t^{k-1} + \frac{2549}{16}X^5Yt^{k-1} \\ \frac{159}{16}X^2Y^2t^{k-1} - \frac{861}{32}X^6Yt^{k-1} \\ -\frac{3}{2}X^6Zt^{k-1} + \frac{693}{8}X^3Y^2t^{k-1} + \frac{243}{16}X^7Yt^{k-1} \end{bmatrix},$$

$$S(n_{21}, n_{12}) = \begin{bmatrix} \frac{11}{2}X^4Wt^{k-1} - \frac{189}{40}XYZt^{k-1} - \frac{201}{55}X^2Y^2t^{k-1} + \frac{4853}{80}X^6Yt^{k-1} \\ \frac{15}{176}X^2YZt^{k-1} + \frac{177}{44}X^3Y^2t^{k-1} - \frac{927}{110}X^7Yt^{k-1} \\ -\frac{3}{2}X^6Wt^{k-1} + \frac{4491}{880}X^3YZt^{k-1} + \frac{3087}{110}X^4Y^2t^{k-1} \end{bmatrix},$$

$$S(n_{21},n_{24})=\begin{bmatrix}\frac{11}{2}X^9t^{k-1}\\0\\-\frac{3}{2}X^{11}t^{k-1}\end{bmatrix},\qquad S(n_{23},n_3)=\begin{bmatrix}\frac{2397}{1603}X^7t^{k-1}\\\frac{25}{458}X^8t^{k-1}\end{bmatrix},$$

$$S(n_{23},n_4)=\begin{bmatrix}\frac{2}{7}X^5Yt^{k-1}-\frac{5}{2}X^9t^{k-1}\\\frac{1}{4}X^{10}t^{k-1}\\\frac{1}{2}X^{11}t^{k-1}\end{bmatrix},$$

$$S(n_{23},n_{11})=\begin{bmatrix}\frac{2}{7}X^5Zt^{k-1}-\frac{17}{2}X^{10}t^{k-1}\\-\frac{1}{2}X^7Yt^{k-1}+\frac{3}{4}X^{11}t^{k-1}\\-3X^8Yt^{k-1}+\frac{3}{2}X^{12}t^{k-1}\end{bmatrix},$$

$$S(n_{23},n_{14})=\begin{bmatrix}\frac{2}{7}X^5Wt^{k-1}-\frac{7}{33}X^7Yt^{k-1}-\frac{55}{24}X^{11}t^{k-1}\\-\frac{20}{33}X^7Zt^{k-1}+\frac{19}{132}X^8Yt^{k-1}+\frac{13}{33}X^{12}t^{k-1}\\\frac{47}{88}X^8Zt^{k-1}-\frac{41}{22}X^9Yt^{k-1}\end{bmatrix},$$

$$S(n_{23},n_{17})=\begin{bmatrix}\frac{2}{7}X^5t^k+\frac{3}{2}X^8\\\frac{1}{4}X^9\\-\frac{3}{2}X^{10}\end{bmatrix},\qquad S(n_{24},n_5)=\begin{bmatrix}-\frac{74}{39}X^9t^{k-1}\\\frac{61}{312}X^{10}t^{k-1}\\\frac{59}{156}X^{11}t^{k-1}\end{bmatrix},$$

$$S(n_{24},n_7)=\begin{bmatrix}-\frac{3439}{32}X^9Yt^{k-1}\\\frac{19473}{256}X^{10}Yt^{k-1}\\\frac{82731}{128}X^{11}Yt^{k-1}\end{bmatrix},\qquad S(n_{24},\tilde{n}_{15})=\begin{bmatrix}-\frac{57}{4}X^8\\-X^5Y+2X^9\\-\frac{3}{4}X^5Z-6X^6Y\end{bmatrix},$$

$$S(n_{24},n_{12})=\begin{bmatrix}-\frac{189}{40}X^6Zt^{k-1}-\frac{201}{55}X^7Yt^{k-1}+\frac{4853}{80}X^{11}t^{k-1}\\\frac{15}{176}X^7Zt^{k-1}+\frac{177}{44}X^8Yt^{k-1}-\frac{927}{110}X^{12}t^{k-1}\\\frac{4491}{880}X^8Zt^{k-1}+\frac{3087}{110}X^9Yt^{k-1}\end{bmatrix},$$

$$S(n_{24},n_9)=\begin{bmatrix}-\frac{29}{2}X^6Yt^{k-1}+\frac{2549}{16}X^{10}t^{k-1}\\\frac{159}{16}X^7Yt^{k-1}-\frac{861}{32}X^{11}t^{k-1}\\\frac{693}{8}X^8Yt^{k-1}+\frac{243}{16}X^{12}t^{k-1}\end{bmatrix}.$$

The results of division by B are listed below.

$$S(n_{21},n_1)=-\frac{3}{2}X^7n_{23}+\frac{83}{14}X^7n_{24},$$

$$S(n_{21},n_5)=\frac{61}{312}X^7n_{18}-\frac{175}{156}Xn_{23}+\frac{51}{13}Xn_{24},$$

$$S(n_{21},n_7) = \frac{82539}{128}X^6t^{k+1}n_4 + \left(\frac{82539}{512}X^9 + \frac{19473}{256}X^5Y\right)n_{18} - \frac{3263}{32}X^4n_{21} + \\ + \frac{43383}{256}X^3n_{23} - \frac{1970627}{1792}X^3n_{24},$$

$$S(n_{21},n_9) = \frac{693}{8}X^3Yt^{k+1}n_4 + \frac{159}{16}Yt^{k+1}n_6 - \frac{22}{7}Xt^{k+1}n_7 + \frac{11}{2}X^4t^{k+1}n_9 + \\ - \frac{3}{2}X^6t^{k+1}n_{11} - \left(\frac{164175}{896}X^6Y + \frac{9543}{64}X^{10}\right)n_{18} + \\ + \left(\frac{2601}{32}X^4 - \frac{693813}{448}Y\right)n_{23} + \left(\frac{1088739}{1568}Y + \frac{193927}{224}X^4\right)n_{24},$$

$$S(n_{21},n_{12}) = -\frac{25371}{64}X^4Yt^{k+1}n_4 - \frac{189}{40}XYt^{k+1}n_9 + \left(\frac{47991}{1760}Z - \frac{13989}{1760}XY\right)n_{23} + \\ + \frac{4491}{880}X^3Yt^{k+1}n_{11} + \frac{11}{2}X^4t^{k+1}n_{12} - \frac{3}{2}X^6t^{k+1}n_{14} + \frac{1035}{16}X^2Yn_{21} + \\ + \left(\frac{7981}{176}X^7Y - \frac{5841}{128}X^3Y^2 - \frac{10327}{220}X^{11} + \frac{485}{352}X^6Z\right)n_{18} + \\ + \frac{15}{176}X^2Yt^{k+1}n_{10} + \left(\frac{53933}{160}X^5 - \frac{104037}{3080}Zn_{24} - \frac{91109}{770}XY\right)n_{24},$$

$$S(n_{21},\tilde{n}_{15}) = \frac{229}{52}n_3 + \left(\frac{15}{4}XY - \frac{18927}{64}X^5\right)n_4 + \frac{213}{28}Xn_5 - \frac{4461}{128}X^2n_6 - \frac{5}{8}n_8 + \\ - \left(\frac{3}{4}Y + \frac{33}{8}X^4\right)n_{11} + \frac{11}{2}X^4\tilde{n}_{15} - \frac{3}{2}X^6\tilde{n}_{17},$$

$$S(n_{21},n_{24}) = -\frac{3}{2}X^4n_{23} + \frac{83}{14}X^4n_{24}, \qquad S(n_{23},n_3) = \frac{25}{458}X^8n_{18} + \frac{2397}{1603}X^2n_{24},$$

$$S(n_{23},n_4) = \frac{1}{4}X^{10}n_{18} + \frac{1}{2}X^4n_{23} + \left(\frac{2}{7}Y - \frac{37}{14}X^4\right)n_{24},$$

$$S(n_{23},n_{11}) = \left(\frac{3}{4}X^{11} - \frac{1}{2}X^7Y\right)n_{18} + \left(\frac{3}{2}X^5 - 3XY\right)n_{23} + \\ + \left(\frac{2}{7}Z + \frac{6}{7}XY - \frac{125}{14}X^5\right)n_{24},$$

$$S(n_{23},n_{14}) = \left(\frac{19}{132}X^8Y + \frac{13}{33}X^{12} - \frac{20}{33}X^7Z\right)n_{18} + \left(\frac{47}{88}XZ - \frac{41}{22}X^2Y\right)n_{23} + \\ + \left(\frac{2}{7}W - \frac{47}{308}XZ + \frac{74}{231}X^2Y - \frac{55}{24}X^6\right)n_{24},$$

$$S(n_{23},\tilde{n}_{17}) = -\frac{72}{229}n_1 + \frac{77}{458}n_2 - \frac{3}{2}Xn_3 + \frac{2}{7}tn_{24},$$

$$S(n_{24},n_5) = \frac{61}{312}X^{10}n_{18} + \frac{59}{156}X^4 n_{23} - \frac{365}{182}X^4 n_{24},$$

$$S(n_{24},n_7) = \frac{82731}{128}X^4 Y n_{23} + \frac{19473}{256}X^{10}Y n_{18} - \frac{130877}{448}X^4 Y n_{24},$$

$$S(n_{24},n_9) = \left(\frac{159}{16}X^7Y - \frac{861}{32}X^{11}\right)n_{18} + \left(\frac{693}{8}XY + \frac{243}{16}X^5\right)n_{23} + \\ + \left(\frac{17357}{112}X^5 - \frac{157}{4}XY\right)n_{24},$$

$$S(n_{24},n_{12}) = \left(\frac{15}{176}X^7Z - \frac{927}{110}X^{12} + \frac{177}{44}X^8Y\right)n_{18} + \left(\frac{4491}{880}XZ + \frac{3087}{110}X^2Y\right)n_{23} + \\ + \left(\frac{4853}{80}X^6 - \frac{642}{55}X^2Y n_{24} - \frac{4761}{770}XZ\right)n_{24},$$

$$S(n_{24},\tilde{n}_{15}) = \frac{5}{916}n_1 + \frac{9}{458}n_2 + \frac{47}{156}Xn_3 - \frac{15}{4}X^6n_4 - \frac{5}{7}X^2n_5 - \frac{5}{8}X^3n_6 - \frac{3}{4}X^5n_{11}.$$

Therefore, the algorithm terminates here. Hence,

$$B = \{n_1, \ldots, n_{14}, \tilde{n}_{15}, \tilde{n}_{17}, n_{18}, n_{21}, n_{23}, n_{24}\}$$

is a Groebner basis for N by Buchberger's criterion.

D.2 The series $\tilde{D}_k$

We fix ">" to be a reverse lexicographic order with priority given to the coefficients. We calculate a Groebner basis for $N = (\begin{bmatrix} 0 & 0 & 0 & t^{k-1}\end{bmatrix}^t)\mathcal{O}_{\mathbb{C}^5,0} + \hat{g}_k^*\mathrm{Der}(\text{-log}V)_2$ defined in p. 107. A set of generators is given by

$$B := \{n_1, \ldots, n_{26}, \tilde{n}_{27}, \tilde{n}_{28}, \tilde{n}_{30}, \tilde{n}_{31}, \tilde{n}_{32}, n_{33}\}$$

where n_i and $\tilde{n}_{27}, \ldots, \tilde{n}_{32}$ and n_{33} are defined by (5.14) - (5.18).

Step 1. Notice that $n_1, \ldots, n_{14}$ already satisfy the Buchberger's Criterion as they form a part of a Groebner basis for N_0 and not depend on the variable t. So it is sufficient to consider the S-polynomials involving only $\tilde{n}_{27}, \ldots, \tilde{n}_{32}$ and n_{33}. We apply the division algorithm on these S-polynomials by B and find the forms below.

$$S(\tilde{n}_{27},n_1) = \big(Z - 1544X^2 - 10297Y\big)n_1 + 3X^3n_2 + 4XYn_3 + \big(15995X^3 - 15991XY\big)n_4,$$

$$
\begin{aligned}
S(\tilde{n}_{27}, n_5) =& - 5701Xn_1 + \big(6892Y - 12643X^2\big)n_2 + 6578Yn_3 - 1621Yn_6 + \\
&+ \big(7439X^2 + 9248Y\big)n_4 + \big(2655X^2 - 4851Y + Z\big)n_5 - 15399Yn_7 + \\
&- 15991XYn_8 + 14878X^6\tilde{n}_{27},
\end{aligned}
$$

$$
\begin{aligned}
S(\tilde{n}_{27}, n_9) =& 12465n_1 - 15570Xn_2 + 14456Xn_3 - 10754Xn_4 + 7781Xn_5 + 9981Xn_7 \\
&+ 11214Xn_6 + \big(4293Y - 15102X^2\big)n_8 + \big(9074X^5 + 11702X^3Y\big)\tilde{n}_{30} + \\
&+ \big(4984Y - 870X^2\big)n_{10} + \big(10135X^3 + 4XY\big)n_{11} - 15991XYn_{12} + \\
&+ \big(5120X^3 - 8065XY\big)n_{20} + XYn_{21} - \big(1810X^5 + 11853X^3Y\big)\tilde{n}_{27} + \\
&+ \big(8177X^5 - 4423X^3Y\big)\tilde{n}_{28} - \big(13672X^2 + 241Y\big)n_9 + 9456X^6tn_{33},
\end{aligned}
$$

$$
\begin{aligned}
S(\tilde{n}_{27}, n_{13}) =& 3563Xn_1 + \big(1759X^2 - 8179Y - Z - t^k\big)n_2 + \big(7969X^2Y + 4Y^2\big)n_{11} + \\
&+ \big(3751Y + 10877Z + 10877t^k\big)n_3 - \big(15602X^2Y + 15991Y^2\big)n_{12} + \\
&- \big(5625X^2 + 11892Y - 3786Z - 3786t^k\big)n_4 + 4344XYn_9 + \\
&+ \big(14884X^2 + 15407Y - 7654Z - 7654t^k\big)n_5 - 13671XYn_{10} + \\
&+ \big(678Y - 11111Z - 11111t^k\big)n_6 + \big(13887Y + Z\big)n_{13} + \\
&- \big(13063Y - 8255Z - 8255t^k\big)n_7 + 2942Yn_{14} - 11250X^6\tilde{n}_{27} + \\
&+ \big(15216XY - 2367XZ - 2367Xt^k\big)n_8,
\end{aligned}
$$

$$
\begin{aligned}
S(\tilde{n}_{27}, n_{20}) =& - 12236n_2 + 7079n_3 - 4679n_4 + 2200n_5 + 8666n_6 + 15162n_7 + \\
&+ 12966Xn_8 - 13507Xn_9 + 10823Xn_{10} + \big(2793Y - 11659X^2\big)n_{11} + \\
&+ \big(13159X^2 - 15123Y\big)n_{12} + \big(8683X^4 - 487X^2Y + 621Y^2\big)n_{17} + \\
&- 8079n_{13} - 12210n_{14} + 9229Yn_{21} + \big(10907X^3Yt - 12592X^5t\big)n_{33} + \\
&+ \big(11359X^5 + 3447X^3Y\big)n_{18} + \big(8680X^4 - 487X^2Y + 621Y^2\big)\tilde{n}_{28} + \\
&+ \big(10970X^2 - 15399Y + Z\big)n_{20} + \big(11359X^4 + 3443X^2Y\big)\tilde{n}_{30} + \\
&+ \big(3403X^5 - 5084X^3Y\big)n_{19} + \big(7992X^4 - 2199X^2Y - 12090Y^2\big)\tilde{n}_{27},
\end{aligned}
$$

$$
\begin{aligned}
S(\tilde{n}_{27}, n_{21}) =& - 2470n_2 - 5244n_3 + 1639n_4 - 15826n_5 + 9828n_6 + 5348n_7 + \\
&+ 1296Xn_8 + 6997Xn_9 - 7184Xn_{10} + \big(4Z - 1234X^2 - 6774Y\big)n_{11} + \\
&- \big(9576X^2 + 12461Y + 15991Z\big)n_{12} + 1570n_{13} - 15558n_{14} + \\
&+ \big(14433X^2Y - 15560X^4 - 9238Y^2\big)n_{17} + \big(6316Y + Z\big)n_{21} + \\
&- \big(11321X^5 + 9218X^3Y\big)n_{18} + \big(14804X^2Y - 11766X^4\big)\tilde{n}_{30} + \\
&+ \big(11830Y - 1150X^2\big)n_{20} + \big(10021X^2Y - 15621X^4 - 11410Y^2\big)\tilde{n}_{27} + \\
&+ \big(11722X^4 + 260X^2Y - 12180Y^2\big)\tilde{n}_{28} + \big(3698X^5 - 3106X^3Y\big)n_{19} +
\end{aligned}
$$

$$+\big(13068X^5t+12496X^3Yt\big)n_{33},$$

$$\begin{aligned}S(\tilde n_{27},n_{22})=&6593n_2+14574n_3-1573n_4+11382n_5-13513n_6-7641n_7+\\&+4256Xn_8+2608Xn_9+\big(13027X^2+12854Y-3Z\big)n_{11}+\\&+\big(14454Y-1931X^2+15167Z+15167t^k\big)n_{12}+4100n_{13}+717n_{14}+\\&+\big(1699X^4+8118X^2Y+3X^2Z+537Y^2\big)n_{17}+\\&+\big(11201X^5+8251X^3Y+4XYZ\big)n_{18}-\big(15054X^4+4466X^2Y\big)\tilde n_{30}+\\&+\big(542X^5-7006X^3Y+15995X^3Z-15991XYZ\big)n_{19}+13133Xn_{10}+\\&+\big(9346X^2-406Y+13708Z+4574t^k\big)n_{20}+14076Yn_{21}+Zn_{22}+\\&+\big(5547Y^2-7421X^4-10589X^2Y\big)\tilde n_{28}+\big(2849X^5t-2135X^3Yt\big)n_{33}+\\&+\big(13342X^4-13048X^2Y-15348Y^2\big)\tilde n_{27},\end{aligned}$$

$$\begin{aligned}S(\tilde n_{27},n_{23})=&5272n_8+8298n_9-6164n_{10}-1312Xn_{11}-1616Xn_{12}+964Xn_{21}+\\&-\big(13726X^3+2281XY\big)n_{17}+\big(9157X^4+10253X^2Y-Y^2\big)n_{18}+\\&+\big(6284X^4+10279X^2Y+15994Y^2\big)n_{19}+\big(9157X^3+14830XY\big)\tilde n_{30}+\\&+\big(Z-5X^2\big)n_{23}+4XYn_{25}+\big(15995X^3-15991XY\big)n_{26}-Y\tilde n_{31}+\\&-\big(2268X^3+11426XY-9142XZ\big)\tilde n_{27}+\big(13700X^3-2281XY\big)\tilde n_{28}+\\&+\big(1713X^4t-12564X^2Yt+15994Y^2t\big)n_{33}-3591Xn_{20}+3X^2n_{24},\end{aligned}$$

$$\begin{aligned}S(\tilde n_{28},n_2)=&-12097Xn_1-\big(1597X^2+11681Y\big)n_2+\big(15347X^2-8765Y\big)n_3+\\&-\big(7695X^2+15065Y\big)n_4-\big(5417X^2+14903Y\big)n_5+8537Yn_6+\\&+14335X^4n_{20}+3945X^6\tilde n_{27},\end{aligned}$$

$$\begin{aligned}S(\tilde n_{28},n_6)=&-563Xn_1+\big(11649X^2-11688Y\big)n_2-\big(15659X^2+12377Y\big)n_3+\\&+\big(15642Y-13853X^2\big)n_4-14520Yn_6-11441Yn_7+6855XYn_8+\\&-\big(12302X^2-6390Y+13711Z\big)n_5+12323X^4n_{20}+2026X^6\tilde n_{27},\end{aligned}$$

$$\begin{aligned}S(\tilde n_{28},n_{10})=&15666n_1+14534Xn_2+13824Xn_3+9324Xn_4+11357Xn_5-3719Xn_6\\&+\big(4627X^2+13946Y\big)n_8+\big(12570Y-6925X^2-13711Z\big)n_9+\\&-\big(15267X^2+12660Y\big)n_{10}+\big(13723XY-9834X^3\big)n_{11}-4689Xn_7+\\&+\big(13608X^5-11183X^3Y\big)n_{17}-\big(615X^6+11257X^4Y\big)n_{18}+\\&+8307X^6n_{19}-\big(7889X^3+7140XY\big)n_{20}+6855XYn_{12}+\\&+\big(7324X^5-10603X^3Y\big)\tilde n_{27}+\big(13307X^5+2427X^3Y\big)\tilde n_{28}+\\&+\big(6843X^3Y-9265X^5\big)\tilde n_{30}-6637X^6tn_{33},\end{aligned}$$

$$S(\tilde{n}_{28},n_{14})=\big(9434X^2-341Y+13711Z\big)n_2+\big(815X^2+11507Y+7495Z\big)n_3+$$
$$+\big(2136X^2-2888Y+11547Z-2240t^k\big)n_4+\big(1688Y-13711Z\big)n_{13}+$$
$$+\big(1350Y-14227X^2+13514Z+14381t^k\big)n_5+6683X^6\tilde{n}_{27}+$$
$$+\big(9750Y+1779Z+13959t^k\big)n_6+\big(511Y-147Z-13993t^k\big)n_7+$$
$$-\big(8453XY-15063XZ+10311Xt^k\big)n_8+4472XYn_9+6522Yn_{14}+$$
$$+\big(14462X^2Y+13723Y^2\big)n_{11}+\big(6548X^2Y+6855Y^2\big)n_{12}+$$
$$+2017Xn_1-12174X^4n_{20}-13627X^6\tilde{n}_{28}+3397XYn_{10}+11174X^6\tilde{n}_{30},$$

$$S(\tilde{n}_{28},n_{17})=-15620n_{11}-12288n_{12}-\big(13702X^2+4574Y\big)n_{17}-3278n_{20}+$$
$$-\big(13719X^3+9126XY\big)n_{18}+\big(9142XY-5710X^3\big)n_{19}-2606n_{21}+$$
$$+\big(4568X^2-9139Y+13710Z\big)\tilde{n}_{27}+\big(4573X^2+4568Y\big)\tilde{n}_{28}+$$
$$+4570n_{22}-\big(2290X^2+13705Y\big)\tilde{n}_{30}+\big(12568X^3t-6857XYt\big)n_{33},$$

$$S(\tilde{n}_{28},n_{24})=9755n_8+15429n_9-3070n_{10}+8910Xn_{11}+15005Xn_{12}+9181n_{15}+$$
$$-\big(3243X^3+12407XY\big)n_{17}+\big(6855XY-12565X^3\big)n_{26}-187Xn_{21}+$$
$$+\big(13048X^4+14397X^2Y+13711Y^2\big)n_{18}-9789Xn_{22}-11Y\tilde{n}_{31}+$$
$$+\big(12729X^4-11078X^2Y+4571Y^2\big)n_{19}-10526Xn_{20}+$$
$$+\big(9142Y-4577X^2-13711Z\big)n_{23}+\big(13723XY-13719X^3\big)n_{25}+$$
$$+\big(14855X^3+2299XY+4565XZ\big)\tilde{n}_{27}-9139n_{16}+$$
$$+\big(6841XY-1137X^3\big)\tilde{n}_{28}+\big(8550X^3-12548XY\big)\tilde{n}_{30}+$$
$$-\big(4563X^2+4574Y\big)n_{24}-\big(13135X^4t+2302X^2Yt+Y^2t\big)n_{33},$$

$$S(\tilde{n}_{30},n_3)=14680Xn_1+\big(14187X^2+11149Y\big)n_2+\big(5492X^2+4691Y\big)n_3+$$
$$-\big(15809X^2+3030Y\big)n_4-\big(1841X^2+1085Y\big)n_5+3747Yn_6+$$
$$-11894X^4n_{20}+7323X^6\tilde{n}_{27},$$

$$S(\tilde{n}_{30},n_7)=8202Xn_1-\big(7581X^2+1614Y\big)n_2+\big(3720X^2+10848Y\big)n_3+$$
$$+\big(6554X^2-4598Y\big)n_4+\big(11180X^2+10217Y-4571Z\big)n_5+$$
$$+6380Yn_6-9100Yn_7-13710XYn_8+5557X^4n_{20}-29X^6\tilde{n}_{27},$$

$$S(\tilde{n}_{30},n_{11})=15067n_2-1476n_3-5704n_4-1422n_5-2259n_6-6069n_7+2181Xn_8+$$
$$+4816Xn_9+13819Xn_{10}-\big(35X^2+2339Y\big)n_{11}-3390n_{13}+$$
$$-\big(5146X^2+255Y\big)n_{12}-9313n_{14}+\big(10776X^5t-14377X^3Yt\big)n_{33}+$$
$$-\big(11167X^2+8948Y\big)n_{20}+8623Yn_{21}-\big(946X^4+4208X^2Y\big)\tilde{n}_{30}+$$

$$+\big(14763X^2Y-9824X^4+12479Y^2\big)\tilde{n}_{27}+$$
$$-\big(11322X^4+7505X^2Y-4939Y^2\big)\tilde{n}_{28},$$

$$S(\tilde{n}_{30},n_{15})=-15475Xn_1+\big(4571Z-9749X^2-7350Y-8944t^k\big)n_2-590Yn_{14}+$$
$$+\big(7631X^2+2257Y-4753Z+5885t^k\big)n_3-13200XYn_9+$$
$$+\big(1325Z-14100X^2-2161Y-6451t^k\big)n_4-7484XYn_{10}+$$
$$+\big(6850X^2-14152Y-11720Z+3429t^k\big)n_5+15209X^4n_{20}+$$
$$+\big(1949Y-13327Z-6298t^k\big)n_6-\big(706Y-15775Z+3129t^k\big)n_7+$$
$$+\big(15329XY+6599XZ\big)n_8+\big(13184X^2Y+4554Y^2\big)n_{11}+$$
$$+\big(1475X^2Y-13710Y^2\big)n_{12}+\big(12964Y-4571Z\big)n_{13}-996X^6\tilde{n}_{27},$$

$$S(\tilde{n}_{30},n_{18})=-9783n_{11}+2667n_{12}+\big(9128X^2+9142Y\big)n_{17}+15031n_{20}+$$
$$+\big(11445X^3+4554XY\big)n_{18}+\big(9139X^3-13710XY\big)n_{19}-X^3tn_{33}+$$
$$-3922n_{21}-4571n_{22}+3X^2\tilde{n}_{27}-3X^2\tilde{n}_{28}-\big(15991X^2+3Y\big)\tilde{n}_{30},$$

$$S(\tilde{n}_{30},n_{25})=-15776n_8-988n_9-1891n_{10}-10440Xn_{11}-1686Xn_{12}+$$
$$+13693n_{15}-13710n_{16}-\big(2606X^3+2615XY\big)n_{17}+$$
$$+\big(4571Y^2-12419X^4-12716X^2Y\big)n_{18}-10645Xn_{21}+$$
$$+\big(10764X^2Y-10608X^4-9139Y^2\big)n_{19}-5176Xn_{20}-3267Xn_{22}+$$
$$+\big(9154X^2+13709Y-4571Z\big)n_{23}+\big(9128X^2+9142Y\big)n_{24}+$$
$$+\big(11445X^3+4554XY\big)n_{25}+\big(9139X^3-13710XY\big)n_{26}+$$
$$+\big(13713XY-6873X^3-4569XZ\big)\tilde{n}_{27}+\big(9136XY-6839X^3\big)\tilde{n}_{28}+$$
$$+\big(11400X^3+4594XY\big)\tilde{n}_{30}+\big(2283X^2Yt-6852X^4t\big)n_{33},$$

$$S(\tilde{n}_{30},\tilde{n}_{31})=4554n_{11}-13710n_{12}+4569n_{20}-4571n_{21},$$

$$S(\tilde{n}_{30},\tilde{n}_{32})=-312n_{11}+13011n_{12}+13713X^2n_{17}-13707XYn_{18}+$$
$$+\big(13710X^3+4574XY\big)n_{19}-9821n_{20}+13699n_{21}+4571n_{22}+$$
$$+\big(9154X^2+13709Y-4571Z\big)\tilde{n}_{27}+\big(9128X^2+9142Y\big)\tilde{n}_{28}+$$
$$+\big(11445X^2+4554Y\big)\tilde{n}_{30}+\big(9139X^3t-13710XYt\big)n_{33},$$

$$S(\tilde{n}_{31},n_3)=\big(5495X^3+2671XY-9752Xt^k\big)n_1+\big(10872X^2Y-12939X^4\big)n_2+$$
$$+\big(13572X^4+5065X^2Y\big)n_3+\big(-5825X^4-13806X^2Y\big)n_4+$$
$$+\big(-9247X^2Y+7323X^2t^k\big)n_5+9850X^2Yn_6-4569X^6n_{20},$$

$$
\begin{aligned}
S(\tilde{n}_{31}, n_7) = & -15851Xn_1 + \big(14931X^2 - 3251Y\big)n_2 - \big(8918X^2 - 9055Y\big)n_3 + \\
& -\big(3111X^2 + 1287Y\big)n_4 + \big(781X^2 - 7956Y\big)n_5 - 4132Yn_6 + \\
& -4540X^4 n_{20} - 29X^6\tilde{n}_{27},
\end{aligned}
$$

$$
\begin{aligned}
S(\tilde{n}_{31}, n_{11}) = & 15067n_2 - 1476n_3 - 5704n_4 - 1422n_5 - 2259n_6 - 6069n_7 + \\
& +2181Xn_8 + 4816Xn_9 + 13819Xn_{10} - \big(35X^2 + 6893Y\big)n_{11} + \\
& +\big(13455Y - 5146X^2\big)n_{12} + \big(10776X^5 t - 14377X^3 Yt\big)n_{33} + \\
& -\big(11167X^2 + 13517Y\big)n_{20} - \big(946X^4 + 4208X^2Y\big)\tilde{n}_{30} + \\
& +\big(14763X^2Y - 9824X^4 + 12479Y^2\big)\tilde{n}_{27} - 3390n_{13} + 13194Yn_{21} + \\
& +\big(4939Y^2 - 11322X^4 - 7505X^2Y\big)\tilde{n}_{28} - 9313n_{14},
\end{aligned}
$$

$$
\begin{aligned}
S(\tilde{n}_{31}, n_{15}) = & -4156n_1 + 2852Xn_2 + 3438Xn_3 + 8751Xn_4 - 6501Xn_5 - 1579Xn_7 + \\
& -\big(1667X^2 + 11047Y\big)n_8 + \big(2170X^2 + 9731Y - 4569Z\big)n_9 + \\
& -\big(2171X^2 + 4214Y\big)n_{10} - \big(9893X^3 + 3440XY\big)n_{11} + 15377X^6 n_{19} + \\
& +15066Xn_6 + 10290XYn_{12} + \big(4775X^5 - 9625X^3Y\big)n_{17} + \\
& +\big(1230X^6 - 9477X^4Y\big)n_{18} - \big(12373X^3 + 14561XY\big)n_{20} + \\
& +\big(3429X^3Y - 5519X^5\big)\tilde{n}_{27} - \big(8944X^5 + 6298X^3Y\big)\tilde{n}_{28} + \\
& +\big(5885X^5 - 3129X^3Y\big)\tilde{n}_{30} - 6451X^6 tn_{33},
\end{aligned}
$$

$$
\begin{aligned}
S(\tilde{n}_{31}, n_{18}) = & 3193n_8 - 10150n_9 + 4203n_{10} - 9663Xn_{11} - 14984Xn_{12} + 3605Xn_{20} + \\
& +\big(9132X^3 - 6852XY\big)n_{17} - \big(8550X^4 + 3440X^2Y\big)n_{18} - 3918Xn_{21} + \\
& +\big(13136X^4 + 10290X^2Y\big)n_{19} - 4569Xn_{22} + 3XY\tilde{n}_{27} - 3XY\tilde{n}_{28} + \\
& -15991XY\tilde{n}_{30} - 3Y\tilde{n}_{31} - X^2Ytn_{33},
\end{aligned}
$$

$$
\begin{aligned}
S(\tilde{n}_{31}, n_{25}) = & -14892n_2 + 12089n_3 + 11604n_4 - 5972n_5 - 9180n_6 - 15798n_7 + \\
& -12631Xn_8 + \big(11050Y - 7111X^2 + 4569Z + 4594t^k\big)n_{11} + \\
& -14295Xn_9 - 2490n_{14} + \big(2396Y - 4115X^2 - 9142Z + 2283t^k\big)n_{12} + \\
& -2275Xn_{10} - 2685n_{13} + \big(11322X^4 - 12471X^2Y - 8251Y^2\big)n_{17} + \\
& +\big(5204X^5 + 4039X^3Y\big)n_{18} - \big(6115X^5 + 7975X^3Y\big)n_{19} + \\
& +\big(4170X^2 - 7649Y + 10448Z\big)n_{20} + \big(13549X^3Yt - 13580X^5t\big)n_{33} + \\
& -\big(6849X^3 + 2287XY + 4569XZ\big)n_{23} + \big(9132X^3 - 6852XY\big)n_{24} + \\
& -\big(8550X^4 + 3440X^2Y\big)n_{25} + \big(13136X^4 + 10290X^2Y\big)n_{26} + \\
& +\big(139Y^2 - 4076X^4 - 7463X^2Y\big)\tilde{n}_{27} - \big(6139Y + 4569t^k\big)n_{21} +
\end{aligned}
$$

$$- \big(15168X^4 + 9415X^2Y + 6670Y^2\big)\tilde{n}_{28} + \big(10278X^4 + 1821X^2Y\big)\tilde{n}_{30},$$

$$\begin{aligned}
S(\tilde{n}_{31}, \tilde{n}_{32}) =& 11102n_8 + 4138n_9 - 1481n_{10} + 14412Xn_{11} + 2068Xn_{12} + \\
& - 33n_{15} - 3n_{16} + 13707X^3n_{17} + \big(13711X^4 + 4565X^2Y\big)n_{19} + \\
& - 13715X^2Yn_{18} + 7353Xn_{20} - 9169Xn_{21} + 4569Xn_{22} + \\
& - \big(6849X^3 + 2287XY + 4569XZ\big)\tilde{n}_{27} + \big(9132X^3 - 6852XY\big)\tilde{n}_{28} + \\
& - \big(8550X^3 + 3440XY\big)\tilde{n}_{30} + \big(13136X^4t + 10290X^2Yt\big)n_{33},
\end{aligned}$$

$$\begin{aligned}
S(\tilde{n}_{32}, n_3) =& \big(9028X^3 - 9250XY - 4350Xt^k\big)n_1 - 4861Yt^kn_6 + 7323X^6t^k\tilde{n}_{27} + \\
& + \big(4789X^4 - 6077X^2Y + 12499X^2t^k + 4433Y^2 + 15300Yt^k\big)n_2 + \\
& + \big(33Y^2 - 12433X^4 - 13652X^2Y - 5357X^2t^k - 1482Yt^k\big)n_3 + \\
& + \big(820X^2t^k - 7855X^4 - 15211X^2Y + 3Y^2 - 7392Yt^k\big)n_4 + \\
& + \big(10197X^2Y + 702X^2t^k + 15973Yt^k\big)n_5 + \big(4592X^6 - 7323X^4t^k\big)n_{20},
\end{aligned}$$

$$\begin{aligned}
S(\tilde{n}_{32}, n_7) =& \big(1504XY - 13000XZ + 4329Xt^k\big)n_1 + \big(9583Y^2 - 13982Yt^k\big)n_6 + \\
& + \big(7015X^2Y - 4121X^2t^k - 15214Y^2 - 14123Yt^k\big)n_2 + 428Y^2n_7 + \\
& + \big(4360X^2Y + 9501X^2t^k - 8285Y^2 + 3990Yt^k\big)n_3 - 29X^6t^k\tilde{n}_{27} + \\
& + \big(6872X^2Y + 2714X^2t^k + 5896Y^2 + 12519Yt^k\big)n_4 + 3XY^2n_8 + \\
& + \big(215X^2Y + 4592X^2Z - 1890X^2t^k - 3234Y^2 + 1291Yt^k\big)n_5 + \\
& + 29X^4t^kn_{20},
\end{aligned}$$

$$\begin{aligned}
S(\tilde{n}_{32}, n_{11}) =& - 11505Xn_1 + \big(2502X^2 - 6476Y - 8650Z - 2118t^k\big)n_2 + \\
& + \big(15526Z - 1051X^2 - 15601Y - 7584t^k\big)n_3 + 13194Yt^kn_{21} + \\
& + \big(10165Z - 14979X^2 - 6885Y + 14355t^k\big)n_4 + \\
& + \big(1776X^2 + 5202Y - 12285Z + 14644t^k\big)n_5 + \\
& + \big(6194Y + 3869Z + 15582t^k\big)n_6 - \big(10286Y + 9296Z + 3704t^k\big)n_7 + \\
& - \big(5730XY + 572Xt^k\big)n_8 + \big(4592XZ - 3243XY + 5060Xt^k\big)n_9 + \\
& - \big(5547XY + 11986Xt^k\big)n_{10} - \big(946X^4t^k + 4208X^2Yt^k\big)\tilde{n}_{30} + \\
& - \big(14720X^2Y + 5100X^2t^k - 33Y^2 + 6893Yt^k\big)n_{11} + \\
& + \big(8660X^2t^k - 11542X^2Y + 3Y^2 + 13455Yt^k\big)n_{12} + \\
& + \big(13150Y + 12828t^k\big)n_{14} + \big(10776X^5t^{k+1} - 14377X^3Yt^{k+1}\big)n_{33} + \\
& + \big(4768X^4 - 11167X^2t^k - 8948Yt^k\big)n_{20} - \big(6768Y - 5857t^k\big)n_{13} + \\
& + \big(14763X^2Yt^k - 14869X^6 - 9824X^4t^k + 12479Y^2t^k\big)\tilde{n}_{27} +
\end{aligned}$$

$$+\big(4939Y^2t^k-11322X^4t^k-7505X^2Yt^k\big)\tilde n_{28},$$

$$\begin{aligned}S(\tilde n_{32},n_{15})=&\big(4559t^k-3287Y-5261Z\big)n_1+\big(11473XY+7017XZ+3775Xt^k\big)n_2+\\&+\big(1476XZ-14309XY-5587Xt^k\big)n_3-2249XY^2n_{12}+\\&+\big(9762XZ-10852XY-11542Xt^k\big)n_4+33Y^2n_{15}+\\&+\big(4426XY-1828XZ-12896Xt^k\big)n_5+3Y^2n_{16}+\\&+\big(1923XY-10812XZ-3576Xt^k\big)n_6-6451X^6t^{k+1}n_{33}+\\&+\big(359X^2Z-2881X^2Y-871X^2t^k-10725Y^2-935Yt^k\big)n_8+\\&+\big(14288X^2Z-9786X^2Y+5530X^2t^k-10745Y^2+4592YZ+\\&-10227Yt^k\big)n_9+\big(10449Xt^k-5191XY-6134XZ\big)n_7+\\&-\big(5651X^2Y+1990X^2Z+2260X^2t^k+3760Y^2+6373Yt^k\big)n_{10}+\\&-\big(7777X^3Y+9296X^3Z-13108X^3t^k-11637XY^2\big)n_{11}+\\&-\big(8258X^3t^k+14280XYt^k\big)n_{20}-\big(5519X^5t^k-3429X^3Yt^k\big)\tilde n_{27}+\\&-\big(8944X^5t^k+6298X^3Yt^k\big)\tilde n_{28}+\big(5885X^5t^k-3129X^3Yt^k\big)\tilde n_{30},\end{aligned}$$

$$\begin{aligned}S(\tilde n_{32},n_{18})=&-\big(2133X^3+9143XY\big)n_{17}+\big(4360X^4-11224X^2Y+33Y^2\big)n_{18}+\\&-93n_{15}-\big(2267X^4+15971X^2Y-3Y^2\big)n_{19}-14367Xn_{20}-3t^k\tilde n_{31}+\\&+15842n_{10}+3945Xn_{21}-\big(7420X^4t-1148X^2Yt+X^2t^{k+1}\big)n_{33}+\\&-6685Xn_{11}+\big(6861XY-11474X^3+13713XZ+3Xt^k\big)\tilde n_{27}+\\&+8259Xn_{12}-9121Xn_{22}+\big(4613X^3-11435XY-3Xt^k\big)\tilde n_{28}+\\&+13224n_8-12229n_9-\big(6341X^3-10344XY+15991Xt^k\big)\tilde n_{30},\end{aligned}$$

$$\begin{aligned}S(\tilde n_{32},n_{25})=&12477n_2+2968n_3-558n_4+\big(6872X^4-2249X^2Y+3Y^2\big)n_{26}+\\&+15987n_5+\big(9754X^2+6532Y-4592Z-4138t^k\big)n_{11}-12695n_{13}+\\&+3585n_6+\big(7507Y-2498X^2-6888Z-3646t^k\big)n_{12}+14045n_{14}+\\&-11655n_7+\big(594X^4-10572X^2Y+13707X^2t^k-9842Y^2\big)n_{17}+\\&-8367Xn_8-6843Xn_9+\big(9265X^3Y-12442X^5-13715XYt^k\big)n_{18}+\\&+\big(11432X^3Y-982X^5+13711X^3t^k+4565XYt^k\big)n_{19}+4569t^kn_{22}+\\&+\big(9630Y-13695t^k\big)n_{21}+\big(6642X^3-13725XY+4592XZ\big)n_{23}+\\&+\big(7015X^3-9143XY\big)n_{24}+\big(4360X^4+11637X^2Y+33Y^2\big)n_{25}+\\&+\big(15332X^4+8665X^2Y-6873X^2t^k-15050Y^2+13713Yt^k+\\&-4569Zt^k\big)\tilde n_{27}-\big(2124X^2+13129Y-7872Z+7500t^k\big)n_{20}+\end{aligned}$$

$$
\begin{aligned}
&- \big(8616X^4 + 5365X^2Y + 6839X^2t^k + 282Y^2 - 9136Yt^k\big)\tilde{n}_{28}+ \\
&+ \big(11400X^2t^k - 1301X^4 - 11498X^2Y + 4594Yt^k\big)\tilde{n}_{30} - 11742Xn_{10}+ \\
&+ \big(13749X^3Yt - 14560X^5t - 6852X^3t^{k+1} + 2283XYt^{k+1}\big)n_{33},
\end{aligned}
$$

$$
S(n_{33}, n_4) = \begin{bmatrix} 5243X^6t^{k-1} & 0 & 0 & 0 \end{bmatrix}^t, \tag{D.11}
$$

$$
S(n_{33}, n_8) = 2715X^6n_{33} + \begin{bmatrix} -1078X^3Yt^{k-1} + 9461X^5t^{k-1} \\ 13858X^3Yt^{k-1} + 8640X^5t^{k-1} \\ -10532X^4Yt^{k-1} + 4023X^6t^{k-1} \\ 0 \end{bmatrix}, \tag{D.12}
$$

$$
\begin{aligned}
S(n_{33}, n_{12}) =& \big(5758X^3Y + 13221X^5\big)n_{33}+ \\
&+ \begin{bmatrix} 10372X^2Yt^{k-1} - 11561Y^2t^{k-1} + 9271X^4t^{k-1} \\ 9820X^2Yt^{k-1} - 7291Y^2t^{k-1} + 14665X^4t^{k-1} \\ -15816X^3Yt^{k-1} + 15158X^5t^{k-1} \\ 0 \end{bmatrix},
\end{aligned} \tag{D.13}
$$

$$
S(n_{33}, n_{16}) = 13733X^6n_{33} + \begin{bmatrix} -1640X^3Yt^{k-1} - 12031X^5t^{k-1} \\ 7573X^3Yt^{k-1} + 252X^5t^{k-1} \\ -15985X^4Yt^{k-1} - 346X^6t^{k-1} \\ 0 \end{bmatrix}, \tag{D.14}
$$

$$
S(n_{33}, n_{19}) = \begin{bmatrix} -3Xt^{k-1} \\ Xt^{k-1} \\ 2Yt^{k-1} + 15994X^2t^{k-1} \\ 0 \end{bmatrix}, \tag{D.15}
$$

$$
\begin{aligned}
S(n_{33}, n_{26}) =& \big(12566X^3 - 6854XY\big)n_{33}+ \\
&+ \begin{bmatrix} 13710Zt^{k-1} - 9143Yt^{k-1} - 11414X^2t^{k-1} \\ 4573Yt^{k-1} - 11436X^2t^{k-1} \\ -13726XYt^{k-1} - 10266X^3t^{k-1} \\ 0 \end{bmatrix}.
\end{aligned} \tag{D.16}
$$

By (D.11) - (D.16), we add the following vectors to B.

$$
n_{34} := \begin{bmatrix} X^6t^{k-1} & 0 & 0 & 0 \end{bmatrix}^t, \tag{D.17}
$$

$$n_{35} := \begin{bmatrix} 14489X^3Yt^{k-1} - 9080X^5t^{k-1} \\ -5888X^3Yt^{k-1} + 13024X^5t^{k-1} \\ X^4Yt^{k-1} - 3533X^6t^{k-1} \\ 0 \end{bmatrix}, \tag{D.18}$$

$$n_{36} := \begin{bmatrix} 3718Y^2t^{k-1} + 1565X^2Yt^{k-1} - 6969X^4t^{k-1} \\ Y^2t^{k-1} - 10646X^2Yt^{k-1} - 8308X^4t^{k-1} \\ -9098X^3Yt^{k-1} + 11349X^5t^{k-1} \\ 0 \end{bmatrix}, \tag{D.19}$$

$$n_{37} := \begin{bmatrix} -15390X^3Yt^{k-1} + 14088X^5t^{k-1} \\ 3768X^3Yt^{k-1} + 24X^5t^{k-1} \\ X^4Yt^{k-1} - 12220X^6t^{k-1} \\ 0 \end{bmatrix}, \tag{D.20}$$

$$n_{38} := \begin{bmatrix} 15994Xt^{k-1} \\ -15995Xt^{k-1} \\ Yt^{k-1} + 7997X^2t^{k-1} \\ 0 \end{bmatrix}, \tag{D.21}$$

$$n_{39} := \begin{bmatrix} Zt^{k-1} + 10670Yt^{k-1} + 15969X^2t^{k-1} \\ 10657Yt^{k-1} - 5307X^2t^{k-1} \\ 10700XYt^{k-1} - 8037X^3t^{k-1} \\ 0 \end{bmatrix} \tag{D.22}$$

Notice that $\mathrm{LT}(n_{35})$ is divisible by $\mathrm{LT}(n_{38})$, and $\mathrm{LT}(n_{35}) = \mathrm{LT}(n_{37})$. So, we can simplify the generating set by another division on n_{35} and n_{37} by B. We have

$$\begin{aligned} n_{37} &= n_{35} + \begin{bmatrix} 2112X3Yt^{k-1} - 8823X5t^{k-1} \\ 9656X3Yt^{k-1} - 13000X5t^{k-1} \\ -8687X6t^{k-1} \\ 0 \end{bmatrix}, \\ n_{35} &= X^4 n_{38} - 5888 n_{40} + \begin{bmatrix} 10847X^3Yt^{k-1} - 15585X^5t^{k-1} \\ 12902X^5t^{k-1} \\ -7365X^6t^{k-1} \\ 0 \end{bmatrix}. \end{aligned}$$

Hence, we replace n_{37} and n_{35} by

$$n_{40} \quad := \quad \frac{1}{9656}\big(n_{37} - n_{35}\big) = \begin{bmatrix} -1272X^3Yt^{k-1} + 12221X^5t^{k-1} \\ X^3Yt^{k-1} + 2013X^5t^{k-1} \\ 506X^6t^{k-1} \\ 0 \end{bmatrix}$$

and

$$n_{41} \quad := \quad \frac{1}{10847}\big(n_{35} - X^4 n_{38} + 5888 n_{40}\big) = \begin{bmatrix} X^3Yt^{k-1} - 8519X^5t^{k-1} \\ 2048X^5t^{k-1} \\ -331X^6t^{k-1} \\ 0 \end{bmatrix},$$

respectively. Therefore, we extend the set of generators to

$$B' = B \cup \{n_{34}, n_{36}, n_{38}, n_{39}, n_{40}, n_{41}\}.$$

Step 2. We set $B = B'$ and study the remainders of the non-zero S-polynomials involving only n_{34}, n_{36}, n_{38}, n_{39}, n_{40} and n_{41}.

$$S(n_{34}, n_5) = 14878Xt^{k-1}n_1,$$

$$\begin{aligned} S(n_{34}, n_9) =& \big(8177X^4t^{k-1} - 4423X^2Yt^{k-1}\big)n_2 - \big(10627X^3t^{k-1} + 3645XYt^{k-1}\big)n_1 + \\ &+ \big(9074X^4t^{k-1} + 11702X^2Yt^{k-1}\big)n_3 + 9456X^4t^{k-1}n_4, \end{aligned}$$

$$S(n_{34}, n_{13}) = \big(13937X^5t^{k-1} - 2327X^3Yt^{k-1}\big)n_1 + 8630X^6t^{k-1}n_2 - 2367X^4Yt^{k-1}n_4,$$

$$\begin{aligned} S(n_{34}, n_{20}) =& -6646Xt^{k-1}n_1 + \big(8680X^2t^{k-1} - 487Yt^{k-1}\big)n_2 + \\ &+ \big(11359X^2t^{k-1} + 3443Yt^{k-1}\big)n_3 + \big(10907Yt^{k-1} - 12592X^2t^{k-1}\big)n_4 + \\ &+ \big(3613X^2t^{k-1} - 12090Yt^{k-1}\big)n_5 + 621Yt^{k-1}n_6, \end{aligned}$$

$$\begin{aligned} S(n_{34}, n_{21}) =& \big(12497XYt^{k-1} - 783X^3t^{k-1}\big)n_1 - \big(11766X^4t^{k-1} - 14804X^2Yt^{k-1}\big)n_3 + \\ &+ \big(11722X^4t^{k-1} + 260X^2Yt^{k-1} - 12180Y^2t^{k-1}\big)n_2 + \\ &+ \big(13068X^4t^{k-1} + 12496X^2Yt^{k-1}\big)n_4 - 11028X^2Yt^{k-1}n_5, \end{aligned}$$

$$\begin{aligned} S(n_{34}, n_{22}) =& \big(15786X^3t^{k-1} - 7330XYt^{k-1} + 4574XZt^{k-1}\big)n_1 + 12038X^2Yt^{k-1}n_5 + \\ &+ \big(458X^4t^{k-1} - 12265X^2Yt^{k-1} + 2002Y^2t^{k-1}\big)n_2 - \big(422X^2Yt^{k-1} + \\ &- 195X^4t^{k-1}\big)n_3 + \big(11538X^4t^{k-1} - 12940X^2Yt^{k-1} + 15167Y^2t^{k-1}\big)n_4, \end{aligned}$$

$$
\begin{aligned}
S(n_{34},n_{23})=&\big(12929Yt^{k-1}-8771X^2t^{k-1}+9142Zt^{k-1}\big)n_1+1617XYt^{k-1}n_5+\\
&+\big(13700X^3t^{k-1}-15241XYt^{k-1}\big)n_2+11204XYt^{k-1}n_6+\\
&+\big(9157X^3t^{k-1}-7200XYt^{k-1}\big)n_3+15797XYt^{k-1}n_7+\\
&+\big(1713X^3t^{k-1}-641XYt^{k-1}\big)n_4+15994X^2Yt^{k-1}n_8,\\
S(n_{34},\tilde{n}_{27})=&2057Xn_1+\big(487Y-8683X^2\big)n_2-\big(11359X^2+3447Y\big)n_3-X^4n_{20}+\\
&-\big(3403X^2-5084Y\big)n_4+\big(6684X^2+12090Y\big)n_5-621Yn_6,\\
S(n_{34},n_{39})=&13853Xt^{k-1}n_1+\big(5307X^2t^{k-1}-10657Yt^{k-1}\big)n_2+\\
&+\big(8037X^2t^{k-1}-10700Yt^{k-1}\big)n_3+5489X^2t^{k-1}n_5,\\
S(n_{34},n_{41})=&15479Xt^4n_1-2048X^2t^4n_2+331X^2t^4n_3,\\
S(n_{36},n_2)\ =&\big(4855X^3t^{k-1}+14067XYt^{k-1}\big)n_1-\big(8308X^4t^{k-1}+10646X^2Yt^{k-1}\big)n_2+\\
&+\big(11349X^4t^{k-1}-9098X^2Yt^{k-1}\big)n_3+7663X^2Yt^{k-1}n_5,\\
S(n_{36},n_6)\ =&-14762Xt^{k-1}n_1+\big(-8308X^2t^{k-1}-10646Yt^{k-1}\big)n_2+\\
&+\big(11349X^2t^{k-1}-9098Yt^{k-1}\big)n_3+\big(3718Yt^{k-1}-6123X^2t^{k-1}\big)n_5,\\
S(n_{36},n_{10})=&-6637X^6n_{33}-2255X^4n_{38}-8219n_{40}-2549n_{41}+\\
&+\begin{bmatrix}3718XY^2t^{k-1}+12585X^5t^{k-1}\\ 1450X^5t^{k-1}\\ 12417X^6t^{k-1}\\ 0\end{bmatrix}, \qquad \text{(D.23)}\\
S(n_{36},n_{14})=&6181t^{k-1}n_2-9504t^{k-1}n_3+12490t^{k-1}n_4-11498t^{k-1}n_5+\\
&-3682t^{k-1}n_6-479t^{k-1}n_7+9446Xt^{k-1}n_8+3682Xt^{k-1}n_9+\\
&+1687Xt^{k-1}n_{10}-9098X^2t^{k-1}n_{11}+3718t^{k-1}n_{13}+10740n_{34},\\
S(n_{36},n_{17})=&228t^{k-1}n_3-14319t^{k-1}n_2-6785t^{k-1}n_4-1012t^{k-1}n_5-9198t^{k-1}n_6+\\
&-12583t^{k-1}n_7+4670Xt^{k-1}n_8-5639Xt^{k-1}n_9+14108Xt^{k-1}n_{10}+\\
&+\big(7318X^2t^{k-1}-13705Yt^{k-1}\big)n_{11}+\big(10725X^2t^{k-1}-6857Yt^{k-1}\big)n_{12}\\
&-12186t^{k-1}n_{13}-8909t^{k-1}n_{14}-\big(8308X^4t^{k-1}+10646X^2Yt^{k-1}\big)n_{17}+\\
&+\big(11349X^5t^{k-1}-9098X^3Yt^{k-1}\big)n_{18}+13547n_{34}+\\
&+\big(10302X^2t^{k-1}-12153Yt^{k-1}\big)n_{20}-14563Yt^{k-1}n_{21},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{36},n_{24})=&12809t^{k-1}n_1-11567Xt^{k-1}n_2-15639Xt^{k-1}n_3-7823Xt^{k-1}n_4+\\
&-14538Xt^{k-1}n_5+5274Xt^{k-1}n_6-\big(2958X^2t^{k-1}+11390Yt^{k-1}\big)n_8+\\
&-3337Xt^{k-1}n_7+\big(1879Yt^{k-1}-13787X^2t^{k-1}\big)n_{10}-5578Yt^{k-1}n_{16}+\\
&+\big(3313X^2t^{k-1}+1712Yt^{k-1}-11912Zt^{k-1}\big)n_9-3729Yt^{k-1}n_{15}+\\
&-\big(8823X^3t^{k-1}+7524XYt^{k-1}\big)n_{11}+\big(736XYt^{k-1}-6667X^3t^{k-1}\big)n_{20}+\\
&+\big(8171X^5t^{k-1}-2401X^3Yt^{k-1}\big)n_{17}+439X^6t^{k-1}n_{19}+\\
&+15900XYt^{k-1}n_{12}+\big(8101X^6t^{k-1}-9437X^4Yt^{k-1}\big)n_{18}+\\
&+\big(1565X^2Yt^{k-1}-6969X^4t^{k-1}+3718Y^2t^{k-1}\big)n_{23}-\big(8308X^4t^{k-1}+\\
&+10646X^2Yt^{k-1}\big)n_{24}+\big(11349X^5t^{k-1}-9098X^3Yt^{k-1}\big)n_{25},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{36},\tilde n_{28})=&5335n_2-1988n_3-11169n_4+4188n_5-13155n_6+6523n_7-7828n_{13}+\\
&+1653Xn_8+15148Xn_9-7603Xn_{10}+\big(7434X^2+3396Y\big)n_{11}+\\
&+\big(6954X^2+8405Y\big)n_{12}-1761n_{14}+\big(2809X^2+7404Y\big)n_{20}+\\
&+9993Yn_{21}+\big(1565X^2Y-6969X^4+3718Y^2\big)\tilde n_{27}+\\
&-\big(8308X^4+10646X^2Y\big)\tilde n_{28}+\big(11349X^4-9098X^2Y\big)\tilde n_{30},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{36},n_{40})=&7306t^{k-1}n_1+6397Xt^{k-1}n_2-8647Xt^{k-1}n_3-1285Xt^{k-1}n_4+\\
&-5767Xt^{k-1}n_5-9639Xt^{k-1}n_6-199Xt^{k-1}n_7+4990X^2t^{k-1}n_9,
\end{aligned}
$$

$$
S(n_{38},n_3)=10720Xt^{k-1}n_1-15995X^2t^{k-1}n_2+7997X^2t^{k-1}n_3+7323X^2t^{k-1}n_5,
$$

$$
S(n_{38},n_7)=-15995t^{k-1}n_2+7997t^{k-1}n_3-11548n_{34},
$$

$$
\begin{aligned}
S(n_{38},n_{11})=&\big(-14377X^3Y+10776X^5\big)n_{33}+4939n_{36}-8544X^3n_{38}+\\
&+\begin{bmatrix}12111Y^2t^{k-1}+11053X^2Yt^{k-1}+6926X^4t^{k-1}\\ -4119X^2Yt^{k-1}+13700X^4t^{k-1}\\ -11833X^5t^{k-1}\\ 0\end{bmatrix},
\end{aligned}
\tag{D.24}
$$

$$
\begin{aligned}
S(n_{38},n_{15})=&-6451X^6n_{33}-15995Xn_{36}+\big(7997X^2Y-580X^4\big)n_{38}+\\
&+11022n_{40}-9423n_{41}+\begin{bmatrix}14135XY^2t^{k-1}-13779X^5t^{k-1}\\ -14292X^5t^{k-1}\\ -10813X^6t^{k-1}\\ 0\end{bmatrix},
\end{aligned}
\tag{D.25}
$$

$$S(n_{38}, n_{18}) = -15995Xt^{k-1}n_{17} + 7997X^2t^{k-1}n_{18} + \big(12566X^2Y - 1713X^4\big)n_{33} + \\ - \big(15926X^2 + 3Y\big)n_{38} - 9142Xn_{39} + \\ + \begin{bmatrix} 10XYt^{k-1} - 15940X^3t^{k-1} \\ -14XYt^{k-1} - 15984X^3t^{k-1} \\ -21X^4t^{k-1} \\ 0 \end{bmatrix}, \tag{D.26}$$

$$S(n_{38}, n_{25}) = 4590t^{k-1}n_{11} - 5713t^{k-1}n_{12} - 6t^{k-1}n_{20} - 4569t^{k-1}n_{21} + \\ + 15994Xt^{k-1}n_{23} - 15995Xt^{k-1}n_{24} + 7997X^2t^{k-1}n_{25} + \\ + \big(7681X^3Y - 1982X^5\big)n_{33} + 13073n_{36} + 13946X^3n_{38} + \\ + \begin{bmatrix} -556Y^2t^{k-1} + 11822X^2Yt^{k-1} - 12222X^4t^{k-1} \\ -8972X^2Yt^{k-1} - 2224X^4t^{k-1} \\ -10340X^5t^{k-1} \\ 0 \end{bmatrix}, \tag{D.27}$$

$$S(n_{38}, \tilde{n}_{30}) = -4554n_{11} + 13710n_{12} - 4569n_{20} + 4571n_{21} + 15994X^2\tilde{n}_{27} + \\ - 15995X^2\tilde{n}_{28} + 7997X^2\tilde{n}_{30},$$

$$S(n_{38}, \tilde{n}_{31}) = 15994X\tilde{n}_{27} - 15995X\tilde{n}_{28} + 7997X\tilde{n}_{30},$$

$$S(n_{38}, \tilde{n}_{32}) = 11102n_8 + 4138n_9 - 1481n_{10} + 14412Xn_{11} + 2068Xn_{12} - 33n_{15} + \\ - 3n_{16} + 13707X^3n_{17} - 13715X^2Yn_{18} + \big(13711X^4 + 4565X^2Y\big)n_{19} + \\ + 7353Xn_{20} - 9169Xn_{21} + \big(9132X^3 - 6852XY - 15995Xt^k\big)\tilde{n}_{28} + \\ + 4569Xn_{22} + \big(15994Xt^k - 6849X^3 - 2287XY - 4569XZ\big)\tilde{n}_{27} + \\ - \big(8550X^3 + 3440XY - 7997Xt^k\big)\tilde{n}_{30} + \big(13136X^4t + 10290X^2Yt\big)n_{33},$$

$$S(n_{39}, n_1) = \big(10457X^2t^{k-1} - 5489Yt^{k-1}\big)n_1 + \big(10657XYt^{k-1} - 5307X^3t^{k-1}\big)n_2 + \\ + \big(-8037X^3t^{k-1} + 10700XYt^{k-1}\big)n_3,$$

$$S(n_{39}, n_5) = -909Xt^{k-1}n_1 + \big(11064Yt^{k-1} - 6627X^2t^{k-1}\big)n_2 + 10700Yt^{k-1}n_7 + \\ - \big(9251X^2t^{k-1} + 674Yt^{k-1}\big)n_3 + \big(15994Yt^{k-1} - 4480X^2t^{k-1}\big)n_4 + \\ + \big(3485X^2t^{k-1} - 10948Yt^{k-1}\big)n_5 + 4496Yt^{k-1}n_6 + 14878X^4t^{k-1}n_{20},$$

$$S(n_{39}, n_9) = 3420t^{k-1}n_1 - 10303Xt^{k-1}n_2 - 14512Xt^{k-1}n_3 - 10419Xt^{k-1}n_4 + \\ - 2610Xt^{k-1}n_5 - 14296Xt^{k-1}n_6 + \big(8396X^3t^{k-1} + 3753XYt^{k-1}\big)n_{20} +$$

$$
\begin{aligned}
&- \big(14449X^2t^{k-1} + 2647Yt^{k-1}\big)n_8 + \big(7606X^2t^{k-1} - 4996Yt^{k-1}\big)n_9 + \\
&+ \big(4095Yt^{k-1} - 197X^2t^{k-1}\big)n_{10} + \big(10700XYt^{k-1} - 5562X^3t^{k-1}\big)n_{11} + \\
&+ 11377Xt^{k-1}n_7 + \big(8177X^5t^{k-1} - 4423X^3Yt^{k-1}\big)n_{17} + \\
&+ \big(9074X^6t^{k-1} + 11702X^4Yt^{k-1}\big)n_{18} + 9456X^6t^{k-1}n_{19},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{39}, n_{13}) =& 7460Xt^{k-1}n_1 - \big(13468X^2t^{k-1} + 8364Yt^{k-1} + Zt^{k-1}\big)n_2 + \\
&+ \big(15295X^2t^{k-1} - 4044Yt^{k-1} + 10877Zt^{k-1}\big)n_3 + 2393XYt^{k-1}n_{10} + \\
&+ \big(3852X^2t^{k-1} - 9571Yt^{k-1} + 3786Zt^{k-1}\big)n_4 + 5536Yt^{k-1}n_{13} + \\
&+ \big(15797X^2t^{k-1} + 14013Yt^{k-1} - 7654Zt^{k-1}\big)n_5 + 8825Yt^{k-1}n_{14} + \\
&- \big(7840Yt^{k-1} + 11111Zt^{k-1}\big)n_6 + \big(8255Zt^{k-1} - 2663Yt^{k-1}\big)n_7 + \\
&+ \big(10686XYt^{k-1} - 2367XZt^{k-1}\big)n_8 - 9120XYt^{k-1}n_9 + \\
&+ \big(9691X^2Yt^{k-1} + 10700Y^2t^{k-1}\big)n_{11} + 10819X^2Yt^{k-1}n_{12} + \\
&- 11250X^4t^{k-1}n_{20},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{39}, n_{20}) =& \big(10907X^3Y - 12592X^5\big)n_{33} + 621n_{36} + 1594X^3n_{38} + \\
&+ \begin{bmatrix} 14375Y^2t^{k-1} - 3664X^2Yt^{k-1} + 3325X^4t^{k-1} \\ -801X^2Yt^{k-1} + 11293X^4t^{k-1} \\ 10804X^5t^{k-1} \\ 0 \end{bmatrix},
\end{aligned} \tag{D.28}
$$

$$
\begin{aligned}
S(n_{39}, n_{21}) =& \big(12496X^3Y + 13068X^5\big)n_{33} + \big(10700XY + 10641X^3\big)n_{38} + \\
&- 1523n_{36} + \begin{bmatrix} -633Y^2t^{k-1} - 5781X^2Yt^{k-1} - 8430X^4t^{k-1} \\ -4818X^2Yt^{k-1} + 5758X^4t^{k-1} \\ -2396X^5t^{k-1} \\ 0 \end{bmatrix},
\end{aligned} \tag{D.29}
$$

$$
\begin{aligned}
S(n_{39}, n_{22}) =& - 15379t^{k-1}n_{11} + 15243t^{k-1}n_{21} + 7562t^{k-1}n_{12} + 9107t^{k-1}n_{20} + \\
&+ \big(10657Yt^{k-1} - 5307X^2t^{k-1}\big)n_{17} + 9635n_{36} + 9126X^3n_{38} + \\
&+ \big(10700XYt^{k-1} - 8037X^3t^{k-1}\big)n_{18} - \big(14X^3Y + 14996X^5\big)n_{33} + \\
&+ \begin{bmatrix} -3422Y^2t^{k-1} - 13863X^2Yt^{k-1} + 10974X^4t^{k-1} \\ -2799X^2Yt^{k-1} - 7961X^4t^{k-1} \\ 2527X^5t^{k-1} \\ 0 \end{bmatrix},
\end{aligned} \tag{D.30}
$$

$$S(n_{39},n_{23})= - 925t^{k-1}n_8 - 4444t^{k-1}n_9 - 14229t^{k-1}n_{10} - 4730Xt^{k-1}n_{11}+$$
$$+ 10348Xt^{k-1}n_{12} + 67t^{k-1}n_{15} + 5329t^{k-1}n_{16} - 5338n_{40}+$$
$$+ \left(13700X^3t^{k-1} - 2281XYt^{k-1}\right)n_{17} + 8624Xt^{k-1}n_{20}+$$
$$+ \left(9157X^4t^{k-1} + 14830X^2Yt^{k-1} - Y^2t^{k-1}\right)n_{18} - 7315Xt^{k-1}n_{21}+$$
$$+ \left(1713X^4t^{k-1} - 12564X^2Yt^{k-1} + 15994Y^2t^{k-1}\right)n_{19} + 5452n_{41}+$$
$$+ \left(15969X^2t^{k-1} + 10670Yt^{k-1}\right)n_{23} + 10495X^4n_{38}+$$
$$+ 9142Xt^{k-1}n_{22} + \left(10657Yt^{k-1} - 5307X^2t^{k-1}\right)n_{24}+$$
$$+ \left(10700XYt^{k-1} - 8037X^3t^{k-1}\right)n_{25} + 8400X^6n_{33}+$$
$$+ \begin{bmatrix}-5830X^5t^{k-1} & 2918X^5t^{k-1} & 12204X^6t^{k-1} & 0\end{bmatrix}^t, \tag{D.31}$$

$$S(n_{39},\tilde{n}_{27})=648n_{11} + 783n_{12} - 3X^2n_{17} - 4XYn_{18} + \left(15991XY - 15995X^3\right)n_{19}+$$
$$+ 6103n_{20} + 21n_{21} - n_{22} + \left(15969X^2 + 10670Y\right)\tilde{n}_{27}+$$
$$+ \left(10657Y - 5307X^2\right)\tilde{n}_{28} + \left(10700Y - 8037X^2\right)\tilde{n}_{30},$$

$$S(n_{39},n_{41})=13300t^{k-1}n_1 + 9057Xt^{k-1}n_2 + 4213Xt^{k-1}n_3 - 8698Xt^{k-1}n_4+$$
$$- 3690Xt^{k-1}n_5 - 7968Xt^{k-1}n_6 + 4927Xt^{k-1}n_7 + 9845X^2t^{k-1}n_8+$$
$$- 9839X^2t^{k-1}n_9 + 3542X^2t^{k-1}n_{10} + 10700X^3t^{k-1}n_{11}+$$
$$- 2048X^5t^{k-1}n_{17} + 331X^6t^{k-1}n_{18} - 13454X^3t^{k-1}n_{20},$$

$$S(n_{40},n_2) = - 13690Xt^{k-1}n_1 + 2013X^2t^{k-1}n_2 + 506X^2t^{k-1}n_3 + 2673X^2t^{k-1}n_5,$$

$$S(n_{40},n_6) =2013t^{k-1}n_2 + 506t^{k-1}n_3 - 1272t^{k-1}n_5 - 1889n_{34},$$

$$S(n_{40},n_{10})=219t^{k-1}n_1 + 9238Xt^{k-1}n_2 - 2642Xt^{k-1}n_3 - 6053Xt^{k-1}n_4+$$
$$+ 10873Xt^{k-1}n_5 + 80Xt^{k-1}n_6 - 1780Xt^{k-1}n_7 - 1272X^2t^{k-1}n_9,$$

$$S(n_{40},n_{14})=\left(12721Yt^{k-1} - 12809X^2t^{k-1}\right)n_1 + \left(10469X^3t^{k-1} - 4069XYt^{k-1}\right)n_2+$$
$$+ 6623XYt^{k-1}n_3 - 9727XYt^{k-1}n_4 - 10515XYt^{k-1}n_5+$$
$$- 2347XYt^{k-1}n_6 - 8623XYt^{k-1}n_7 - 1272X^2Yt^{k-1}n_9,$$

$$S(n_{40},n_{17})= - 5509t^{k-1}n_1 - 5262Xt^{k-1}n_2 - 10197Xt^{k-1}n_3 - 14744Xt^{k-1}n_4+$$
$$- 672Xt^{k-1}n_5 + 13878Xt^{k-1}n_6 - 12307Xt^{k-1}n_7 + 506X^6t^{k-1}n_{18}+$$
$$+ \left(12569Yt^{k-1} - 8738X^2t^{k-1}\right)n_8 + \left(14271Yt^{k-1} - 8924X^2t^{k-1}\right)n_9+$$
$$+ \left(14873X^2t^{k-1} + 14167Yt^{k-1}\right)n_{10} - 5297X^3t^{k-1}n_{11}+$$
$$+ 2013X^5t^{k-1}n_{17} + \left(2218X^3t^{k-1} + 12438XYt^{k-1}\right)n_{20},$$

$$
\begin{aligned}
S(n_{40},n_{24})=&-1711Xt^{k-1}n_1+\big(6839X^2t^{k-1}+6263Yt^{k-1}\big)n_2+2013X^5t^{k-1}n_{24}+\\
&-\big(10904X^2t^{k-1}+10234Yt^{k-1}\big)n_3-\big(15597X^2t^{k-1}+2572Yt^{k-1}\big)n_4+\\
&+\big(9424X^2t^{k-1}-10331Yt^{k-1}\big)n_5+8265Yt^{k-1}n_7+12910XYt^{k-1}n_8+\\
&+1279XYt^{k-1}n_9+6556XYt^{k-1}n_{10}+1261X^2Yt^{k-1}n_{11}+\\
&+1907X^2Yt^{k-1}n_{12}+\big(11176X^4t^{k-1}-11326X^2Yt^{k-1}\big)n_{20}+\\
&-7412Yt^{k-1}n_6+506X^6t^{k-1}n_{25}+\big(12221X^5t^{k-1}-1272X^3Yt^{k-1}\big)n_{23},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{40},\tilde{n}_{28})=&-6513n_1-328Xn_2+9668Xn_3+2811Xn_4-2617Xn_5+4436Xn_6+\\
&+11266Xn_7+\big(8901X^2+8422Y\big)n_8+\big(3148X^2-11428Y\big)n_9+\\
&-\big(14373X^2+4938Y\big)n_{10}+\big(14983XY-10637X^3\big)n_{20}+\\
&+2567X^3n_{11}+\big(12221X^5-1272X^3Y\big)\tilde{n}_{27}+2013X^5\tilde{n}_{28}+506X^5\tilde{n}_{30},
\end{aligned}
$$

$$
S(n_{41},n_1)=-15479X^2t^{k-1}n_1+2048X^3t^{k-1}n_2-331X^3t^{k-1}n_3,
$$

$$
S(n_{41},n_5)=2048t^{k-1}n_2-331t^{k-1}n_3-601n_{34},
$$

$$
\begin{aligned}
S(n_{41},n_9)=&-13467t^{k-1}n_1+8177Xt^{k-1}n_2+9074Xt^{k-1}n_3+9456Xt^{k-1}n_4+\\
&+11619Xt^{k-1}n_5-2375Xt^{k-1}n_6+11371Xt^{k-1}n_7,
\end{aligned}
$$

$$
\begin{aligned}
S(n_{41},n_{13})=&\big(13937X^2t^{k-1}+350Yt^{k-1}\big)n_1+8630X^3t^{k-1}n_2-2367XYt^{k-1}n_4+\\
&-8519XYt^{k-1}n_5+2048XYt^{k-1}n_6-331XYt^{k-1}n_7,
\end{aligned}
$$

$$
\begin{aligned}
S(n_{41},n_{20})=&3199t^{k-1}n_1-10410Xt^{k-1}n_2+12583Xt^{k-1}n_3+15755Xt^{k-1}n_4+\\
&+7998Xt^{k-1}n_5+5289Xt^{k-1}n_7+\big(12933X^2t^{k-1}+10907Yt^{k-1}\big)n_8+\\
&+9436Xt^{k-1}n_6-\big(15817X^2t^{k-1}+12090Yt^{k-1}\big)n_9+\\
&+\big(4919X^2t^{k-1}+621Yt^{k-1}\big)n_{10}-7678X^3t^{k-1}n_{11}+\\
&+2048X^5t^{k-1}n_{17}-331X^6t^{k-1}n_{18}+13454X^3t^{k-1}n_{20},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{41},n_{21})=&-15425t^{k-1}n_1+2699Xt^{k-1}n_2-13246Xt^{k-1}n_3+12669Xt^{k-1}n_4+\\
&+12766Xt^{k-1}n_5+6131Xt^{k-1}n_6-12694Xt^{k-1}n_7+13470X^2t^{k-1}n_8+\\
&+3965X^2t^{k-1}n_9+6897X^2t^{k-1}n_{10}+2048X^5t^{k-1}n_{17}+\\
&-331X^6t^{k-1}n_{18}+13454X^3t^{k-1}n_{20},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{41},n_{22})=&\big(637Yt^{k-1}+5059Zt^{k-1}\big)n_1-\big(4823XYt^{k-1}+10161XZt^{k-1}\big)n_2+\\
&+\big(13740XYt^{k-1}+12985XZt^{k-1}\big)n_3-7781X^3Yt^{k-1}n_{11}+\\
&+\big(7171XYt^{k-1}+5326XZt^{k-1}\big)n_4-\big(1049XYt^{k-1}+3850XZt^{k-1}\big)n_5+
\end{aligned}
$$

$$
\begin{aligned}
&- \big(4538XYt^{k-1} + 11861XZt^{k-1}\big)n_6 + 2048X^5Zt^{k-1}n_{17} + \\
&+ \big(11070XYt^{k-1} + 1962XZt^{k-1}\big)n_7 - 331X^6Zt^{k-1}n_{18} + \\
&+ \big(-12042X^2Yt^{k-1} + 974X^2Zt^{k-1} + 15167Y^2t^{k-1}\big)n_8 + \\
&+ \big(15375X^2Zt^{k-1} - 2159X^2Yt^{k-1}\big)n_9 + 13454X^3Zt^{k-1}n_{20} + \\
&+ \big(5283X^2Zt^{k-1} - 3750X^2Yt^{k-1}\big)n_{10},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{41}, n_{23}) = &11008Xt^{k-1}n_1 + \big(3636X^2t^{k-1} + 11763Yt^{k-1}\big)n_2 - 331X^6t^{k-1}n_{25} + \\
&+ \big(2345X^2t^{k-1} + 12697Yt^{k-1}\big)n_3 - X^2Yt^{k-1}n_{11} - 8519X^5t^{k-1}n_{23} + \\
&+ \big(12660X^2t^{k-1} + 8349Yt^{k-1}\big)n_4 + 2048X^5t^{k-1}n_{24} + \\
&+ \big(7534X^2t^{k-1} - 12335Yt^{k-1} + 9142Zt^{k-1}\big)n_5 + 1389Yt^{k-1}n_6 + \\
&- 4649Yt^{k-1}n_7 - 12089XYt^{k-1}n_8 - 11133XYt^{k-1}n_9 + \\
&- 9998XYt^{k-1}n_{10} + 15994X^2Yt^{k-1}n_{12} - 9150X^4t^{k-1}n_{20},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{41}, \tilde{n}_{27}) = &- 4983n_1 - 11411Xn_2 - 7095Xn_3 - 2543Xn_4 + \\
&- 10322Xn_5 - 11545Xn_6 + 10494Xn_7 + \big(5084Y - 677X^2\big)n_8 + \\
&+ \big(-828X^2 + 12090Y\big)n_9 + \big(6116X^2 - 621Y\big)n_{10} - 8914X^3n_{11} + \\
&- \big(8875X^3 + XY\big)n_{20} - 8519X^5\tilde{n}_{27} + 2048X^5\tilde{n}_{28} - 331X^5\tilde{n}_{30}.
\end{aligned}
$$

By (D.23)-(D.31), we add the following terms to B'.

$$
n_{42} := \begin{bmatrix} XY^2t^{k-1} + 9907X^5t^{k-1} \\ 1205X^5t^{k-1} \\ -12516X^6t^{k-1} \\ 0 \end{bmatrix}, \tag{D.32}
$$

$$
n_{43} := \begin{bmatrix} Y^2t^{k-1} + 2579X^2Yt^{k-1} - 10708X^4t^{k-1} \\ -3812X^2Yt^{k-1} - 9495X^4t^{k-1} \\ -14648X^5t^{k-1} \\ 0 \end{bmatrix} \tag{D.33}
$$

$$
n_{44} := \begin{bmatrix} XY^2t^{k-1} + 4693X^5t^{k-1} \\ 9611X^5t^{k-1} \\ -3287X^6t^{k-1} \\ 0 \end{bmatrix}, \tag{D.34}
$$

$$n_{45} := \begin{bmatrix} -9141XYt^{k-1} + 14849X^3t^{k-1} \\ XYt^{k-1} + 10282X^3t^{k-1} \\ -15994X^4t^{k-1} \\ 0 \end{bmatrix}, \tag{D.35}$$

$$n_{46} := \begin{bmatrix} Y^2t^{k-1} - 2668X^2Yt^{k-1} - 1474X^4t^{k-1} \\ -7809X^2Yt^{k-1} + 4X^4t^{k-1} \\ 11296X^5t^{k-1} \\ 0 \end{bmatrix}, \tag{D.36}$$

$$n_{47} := \begin{bmatrix} Y^2t^{k-1} - 3719X^2Yt^{k-1} + 14577X^4t^{k-1} \\ 10651X^2Yt^{k-1} + 12746X^4t^{k-1} \\ -4715X^5t^{k-1} \\ 0 \end{bmatrix}, \tag{D.37}$$

$$n_{48} := \begin{bmatrix} Y^2t^{k-1} + 4406X^2Yt^{k-1} + 13962X^4t^{k-1} \\ -15154X^2Yt^{k-1} + 4034X^4t^{k-1} \\ 12689X^5t^{k-1} \\ 0 \end{bmatrix}, \tag{D.38}$$

$$n_{49} := \begin{bmatrix} Y^2t^{k-1} + 13625X^2Yt^{k-1} - 14344X^4t^{k-1} \\ -4982X^2Yt^{k-1} + 13455X^4t^{k-1} \\ 15733X^5t^{k-1} \\ 0 \end{bmatrix}, \tag{D.39}$$

$$n_{50} = \begin{bmatrix} 12776X^5t^{k-1} \\ 2868X^5t^{k-1} \\ X^6t^{k-1} \\ 0 \end{bmatrix} \tag{D.40}$$

Therefore we can expand B to $B' = B \cup \{n_{42}, \dots, n_{50}\}$. However, B' is not "minimal". Because,

$$n_{42} = 3812n_{40} + 15644n_{41} + Xn_{43} - 11658n_{50} + \begin{bmatrix} 2301X^5t^{k-1} \\ 3612X^5t^{k-1} \\ 0 \\ 0 \end{bmatrix},$$

$$n_{44} = 3812n_{40} + 15644n_{41} + Xn_{43} - 2429n_{50} - 197n_{51} + \begin{bmatrix} 13110X^5t^{k-1} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$n_{46} = n_{43} - 3997Xn_{45} + \begin{bmatrix} -8102X^2Yt^{k-1} - 14609X^4t^{k-1} \\ -1782X^4t^{k-1} \\ 15944X^5t^{k-1} \\ 0 \end{bmatrix},$$

$$n_{47} = n_{43} + 14463Xn_{45} + 13173n_{53} + \begin{bmatrix} -5713X^4t^{k-1} \\ -13186X^4t^{k-1} \\ -7567X^5t^{k-1} \\ 0 \end{bmatrix},$$

$$n_{48} = n_{43} - 11342Xn_{45} + 7436n_{53} - 15557n_{54} + \begin{bmatrix} 10523X^4t^{k-1} \\ -11144X^4t^{k-1} \\ 0 \\ 0 \end{bmatrix}.$$

So we replace $n_{42}, n_{44}, n_{46}, n_{47}, n_{48}$ by

$$n_{51} := \begin{bmatrix} 11240X^5t^{k-1} & X^5t^{k-1} & 0 & 0 \end{bmatrix}^t, \tag{D.41}$$

$$n_{52} := \begin{bmatrix} X^5t^{k-1} & 0 & 0 & 0 \end{bmatrix}^t, \tag{D.42}$$

$$n_{53} := \begin{bmatrix} X^2Yt^{k-1} - 3619X^4t^{k-1} & 1398X^4t^{k-1} & -5830X^5t^{k-1} & 0 \end{bmatrix}^t, \tag{D.43}$$

$$n_{54} := \begin{bmatrix} -11989X^4t^{k-1} & 10664X^4t^{k-1} & X^5t^{k-1} & 0 \end{bmatrix}^t \tag{D.44}$$

$$n_{55} := \begin{bmatrix} 4285X^4t^{k-1} & X^4t^{k-1} & 0 & 0 \end{bmatrix}^t, \tag{D.45}$$

respectively. Then,

$$\tilde{n}_{31} = tn_{38} - 7997X\tilde{n}_{30} + 15995X\tilde{n}_{28} - 15994X\tilde{n}_{27},$$

$$n_{34} = Xn_{52},$$

$$n_{40} = X^2n_{45} - 7408n_{52} + 7869Xn_{53} - 14315Xn_{54} - 10119Xn_{55},$$

$$n_{41} = 3737n_{52} + Xn_{53} + 5499Xn_{54} - 1183Xn_{55},$$

$$n_{49} = n_{43} - 1170Xn_{45} + 1070n_{53},$$

$$n_{50} = 30n_{52} + Xn_{54} - 7796Xn_{55},$$

$$n_{51} = 6955n_{52} + Xn_{55}.$$

Thus, we may exclude $\tilde{n}_{31}, n_{34}, n_{40}, n_{41}$ from the basis and add $n_{43}, n_{45}, n_{52}, \ldots, n_{55}$ instead. Therefore, we obtain

$$B' = \{n_1, \ldots, n_{26}, \tilde{n}_{27}, \tilde{n}_{28}, \tilde{n}_{30}, \tilde{n}_{32}, n_{33}, n_{36}, n_{38}, n_{39}, n_{43}, n_{45}, n_{52}, \ldots, n_{55}\}.$$

Step 3. We apply the division algorithm on the S-polynomials only involving $n_{43}, n_{45}, n_{52}, \ldots, n_{55}$. Non-zero ones are listed below.

$$S(n_{43}, n_1) = \big(9X^2Yt^{k-1} - 8503X^4t^{k-1}\big)n_1 - \big(9495X^5t^{k-1} + 3812X^3Yt^{k-1}\big)n_2 + \\ - 14648X^5t^{k-1}n_3,$$

$$S(n_{43}, n_5) = 524Xt^{k-1}n_1 - 9495X^2t^{k-1}n_2 - 14648X^2t^{k-1}n_3 - 14534X^2t^{k-1}n_5 + \\ - 3812X^2t^{k-1}n_6,$$

$$S(n_{43}, n_9) = 9456X^6n_{33} + 11702X^4n_{38} - 8235X^2n_{45} + 3315n_{52} - 10586Xn_{53} + \\ + 9814Xn_{54} - 9625Xn_{55},$$

$$S(n_{43}, n_{13}) = -14136t^{k-1}n_2 - 4573t^{k-1}n_3 + 9031t^{k-1}n_4 + 10150t^{k-1}n_5 + \\ - 13061t^{k-1}n_6 - 7299t^{k-1}n_7 - 2367Xt^{k-1}n_8 + 2579Xt^{k-1}n_9 + \\ - 3812Xt^{k-1}n_{10} + 12598Xn_{52},$$

$$S(n_{43}, n_{20}) = -14172Xt^{k-1}n_1 + \big(5842X^2t^{k-1} + 15787Yt^{k-1} - 9495Zt^{k-1}\big)n_2 + \\ + \big(12886X^2t^{k-1} + 10592Yt^{k-1} - 14648Zt^{k-1}\big)n_3 - 6208XYt^{k-1}n_8 + \\ + \big(8125X^2t^{k-1} + 10993Yt^{k-1}\big)n_4 + 11780XYt^{k-1}n_9 + 621Yt^{k-1}n_{14} + \\ + \big(5024Yt^{k-1} - 5947X^2t^{k-1} + 2579Zt^{k-1}\big)n_5 - 8033XYt^{k-1}n_{10} + \\ - \big(8627Yt^{k-1} + 3812Zt^{k-1}\big)n_6 - 10390Yt^{k-1}n_7 + 3443X^2Yt^{k-1}n_{11} + \\ + 10907X^2Yt^{k-1}n_{12} - 12090Yt^{k-1}n_{13} - 8631X^4t^{k-1}n_{20},$$

$$S(n_{43}, n_{21}) = 8605t^{k-1}n_3 + 2583t^{k-1}n_4 - 10672t^{k-1}n_5 + 5337t^{k-1}n_6 - 4957t^{k-1}n_7 +$$

$$+\,13151Xt^{k-1}n_8+9794Xt^{k-1}n_9+7225Xt^{k-1}n_{10}-11718X^2t^{k-1}n_{11}+$$
$$+\,881X^2t^{k-1}n_{12}+9397t^{k-1}n_{13}-5636t^{k-1}n_{14}-$$
$$+\,\big(9495X^4t^{k-1}+3812X^2Yt^{k-1}\big)n_{17}-14648X^5t^{k-1}n_{18}+$$
$$+\,\big(13353Yt^{k-1}-15779X^2t^{k-1}\big)n_{20}-7653Xn_{52}-3286X^2n_{55},$$

$$S(n_{43},n_{22})=11570Xt^{k-1}n_1+\big(5155X^2t^{k-1}+12772Yt^{k-1}+4453Zt^{k-1}\big)n_2+$$
$$+\,\big(3277Yt^{k-1}-1049X^2t^{k-1}-12888Zt^{k-1}\big)n_3+$$
$$+\,\big(10606Zt^{k-1}\ -12933X^2t^{k-1}-6177Yt^{k-1}\big)n_4+$$
$$+\,\big(245X^2t^{k-1}-4545Yt^{k-1}+4919Zt^{k-1}\big)n_5+$$
$$-\,\big(9754Yt^{k-1}+10893Zt^{k-1}\big)n_6-\big(13948Yt^{k-1}-13108Zt^{k-1}\big)n_7+$$
$$+\,\big(6914XYt^{k-1}-1521XZt^{k-1}\big)n_8+\big(3864Yt^{k-1}-11184Zt^{k-1}\big)n_{13}+$$
$$+\,\big(8156XZt^{k-1}-7142XYt^{k-1}\big)n_9+\big(12292Yt^{k-1}+6544Zt^{k-1}\big)n_{14}+$$
$$+\,\big(9569XYt^{k-1}-12588XZt^{k-1}\big)n_{10}+$$
$$+\,\big(5469X^2Zt^{k-1}-13176X^2Yt^{k-1}\big)n_{11}+$$
$$+\,\big(15207X^2Yt^{k-1}-11615X^2Zt^{k-1}+15167Y^2t^{k-1}\big)n_{12}+$$
$$-\,\big(9495X^4Zt^{k-1}+3812X^2YZt^{k-1}\big)n_{17}-14648X^5Zt^{k-1}n_{18}+$$
$$+\,\big(13353YZt^{k-1}-8421X^4t^{k-1}-15779X^2Zt^{k-1}\big)n_{20},$$

$$S(n_{43},n_{23})=-\,8692t^{k-1}n_1+13049Xt^{k-1}n_2-14318Xt^{k-1}n_3-13143Xt^{k-1}n_4+$$
$$+\,11435Xt^{k-1}n_5-7484Xt^{k-1}n_6-\big(12398X^2t^{k-1}+1602Yt^{k-1}\big)n_8+$$
$$+\,\big(15331X^2t^{k-1}+849Yt^{k-1}\big)n_{10}-\big(1301X^3t^{k-1}+9799XYt^{k-1}\big)n_{11}+$$
$$+\,\big(9142Zt^{k-1}-11303X^2t^{k-1}-5280Yt^{k-1}\big)n_9+3375XYt^{k-1}n_{12}+$$
$$-\,Yt^{k-1}n_{15}+15994Yt^{k-1}n_{16}+\big(1558X^3Yt^{k-1}-8833X^5t^{k-1}\big)n_{17}+$$
$$+\,\big(1845X^6t^{k-1}+1780X^4Yt^{k-1}\big)n_{18}+7070X^6t^{k-1}n_{19}+$$
$$+\,\big(14417X^3t^{k-1}-15242XYt^{k-1}\big)n_{20}+12552Xt^{k-1}n_7+$$
$$+\,\big(2579X^2Yt^{k-1}-10708X^4t^{k-1}\big)n_{23}+$$
$$-\,\big(9495X^4t^{k-1}+3812X^2Yt^{k-1}\big)n_{24}-14648X^5t^{k-1}n_{25},$$

$$S(n_{43},\tilde{n}_{27})=-\,14818n_2+13504n_3+13595n_4+14984n_5-273n_6+11443n_7+$$
$$-\,14092Xn_8-8010Xn_9-8415Xn_{10}-\big(3390X^2+4Y\big)n_{11}+$$
$$+\,\big(12936X^2+15991Y\big)n_{12}-14648X^4\tilde{n}_{30}-15851n_{13}+$$
$$+\,11846n_{14}+\big(4383Y-2703X^2\big)n_{20}-Yn_{21}+$$
$$+\,\big(2579X^2Y-10708X^4\big)\tilde{n}_{27}-\big(9495X^4+3812X^2Y\big)\tilde{n}_{28},$$

$$
\begin{aligned}
S(n_{43},n_{39})=&-15004t^{k-1}n_2-13642t^{k-1}n_3+2000t^{k-1}n_4+542t^{k-1}n_5+\\
&-2775t^{k-1}n_6+11766t^{k-1}n_7-12207Xt^{k-1}n_8+12702Xt^{k-1}n_9+\\
&-14981Xt^{k-1}n_{10}-\big(4222X^2t^{k-1}+10700Yt^{k-1}\big)n_{11}+\\
&+9557X^2t^{k-1}n_{12}+15271t^{k-1}n_{13}-2281t^{k-1}n_{14}+\\
&-\big(9495X^4t^{k-1}+3812X^2Yt^{k-1}\big)n_{17}-14648X^5t^{k-1}n_{18}+\\
&+\big(13353Yt^{k-1}-15779X^2t^{k-1}\big)n_{20}-2479Xn_{52},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{43},n_{52})=&\big(9Yt^{k-1}-8503X^2t^{k-1}\big)n_1-\big(9495X^3t^{k-1}+3812XYt^{k-1}\big)n_2+\\
&-14648X^3t^{k-1}n_3,
\end{aligned}
$$

$$
\begin{aligned}
S(n_{43},n_{53})=&-9495t^{k-1}n_2-14648t^{k-1}n_3+6198t^{k-1}n_5-5210t^{k-1}n_6+\\
&+5830t^{k-1}n_7-266Xn_{52},
\end{aligned}
$$

$$
S(n_{45},n_2)\;=-8092Xt^{k-1}n_1+10282X^2t^{k-1}n_2-15994X^2t^{k-1}n_3-5196X^2t^{k-1}n_5,
$$

$$
S(n_{45},n_6)\;=10282t^{k-1}n_2-15994t^{k-1}n_3-9141t^{k-1}n_5+3709Xn_{52},
$$

$$
\begin{aligned}
S(n_{45},n_{10})=&-9141t^{k-1}n_9-4251X^6n_{33}+771X^4n_{38}+6728X^2n_{45}-9501n_{52}+\\
&+12948Xn_{53}-15472Xn_{54}-9386Xn_{55},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{45},n_{14})=&245t^{k-1}n_1-10375Xt^{k-1}n_2-9286Xt^{k-1}n_3+8402Xt^{k-1}n_4+\\
&-5880Xt^{k-1}n_5+7046Xt^{k-1}n_6+10126Xt^{k-1}n_7-14220X^2t^{k-1}n_8+\\
&-\big(8195X^2t^{k-1}+9141Yt^{k-1}\big)n_9-9740X^2t^{k-1}n_{10}-6072X^3t^{k-1}n_{11},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{45},n_{17})=&-251t^{k-1}n_8+4406t^{k-1}n_9+908t^{k-1}n_{10}-13705Xt^{k-1}n_{11}+\\
&-6857Xt^{k-1}n_{12}+10282X^3t^{k-1}n_{17}-15994X^4t^{k-1}n_{18}+\\
&-3268Xt^{k-1}n_{20}+4569Xt^{k-1}n_{21}-169X^6n_{33}+15534X^4n_{38}+\\
&-15737X^2n_{45}-14336n_{52}+14492Xn_{53}-15453Xn_{54}+13065Xn_{55},
\end{aligned}
$$

$$
\begin{aligned}
S(n_{45},n_{24})=&8385t^{k-1}n_2+587t^{k-1}n_3+14268t^{k-1}n_4-9216t^{k-1}n_5-11031t^{k-1}n_6+\\
&+3145t^{k-1}n_7-5008Xt^{k-1}n_8+1974Xt^{k-1}n_9+3268Xt^{k-1}n_{10}+\\
&+\big(12771X^2t^{k-1}+9130Yt^{k-1}\big)n_{11}+\big(3012X^2t^{k-1}-2285Yt^{k-1}\big)n_{12}+\\
&-6825t^{k-1}n_{13}+5768t^{k-1}n_{14}-\big(4571X^2t^{k-1}+1965Yt^{k-1}\big)n_{20}+\\
&+\big(14849X^3t^{k-1}-9141XYt^{k-1}\big)n_{23}+10282X^3t^{k-1}n_{24}+\\
&-15994X^4t^{k-1}n_{25}+15203Xn_{52},
\end{aligned}
$$

$$S(n_{45},\tilde{n}_{28})=2057n_8+725n_9+6555n_{10}-9150Xn_{11}-13707Xn_{12}-9139Xn_{21}+ \\ +168Xn_{20}+\big(14849X^3-9141XY\big)\tilde{n}_{27}+10282X^3\tilde{n}_{28}-15994X^3\tilde{n}_{30},$$

$$S(n_{45},n_{36})=-12859t^{k-1}n_9+3087X^6n_{33}+2750X^4n_{38}-15704X^2n_{45}-1436n_{52}+ \\ -12751Xn_{53}-4361Xn_{54}-11032Xn_{55},$$

$$S(n_{45},n_{55})=10282t^{k-1}n_2-15994t^{k-1}n_3-13426t^{k-1}n_5+7516Xn_{52},$$

$$S(n_{52},n_5)\;=14878t^{k-1}n_1,$$

$$S(n_{52},n_9)\;=\big(-10627X^2t^{k-1}-3645Yt^{k-1}\big)n_1+\big(8177X^3t^{k-1}-4423XYt^{k-1}\big)n_2+ \\ +\big(9074X^3t^{k-1}+11702XYt^{k-1}\big)n_3+9456X^3t^{k-1}n_4,$$

$$S(n_{52},n_{13})=\big(13937X^4t^{k-1}-2327X^2Yt^{k-1}\big)n_1+8630X^5t^{k-1}n_2-2367X^3Yt^{k-1}n_4,$$

$$S(n_{52},n_{20})=-13739t^{k-1}n_1+1903Xt^{k-1}n_2-4138Xt^{k-1}n_3-5697Xt^{k-1}n_4+ \\ +1516Xt^{k-1}n_5+11943Xt^{k-1}n_6-7678Xt^{k-1}n_7+10907X^2t^{k-1}n_8+ \\ -12090X^2t^{k-1}n_9+621X^2t^{k-1}n_{10},$$

$$S(n_{52},n_{21})=\big(11880Yt^{k-1}-783X^2t^{k-1}\big)n_1+\big(11722X^3t^{k-1}+260XYt^{k-1}\big)n_2+ \\ +\big(14804XYt^{k-1}-11766X^3t^{k-1}\big)n_3-11410XYt^{k-1}n_5+ \\ +\big(13068X^3t^{k-1}+12496XYt^{k-1}\big)n_4-12180XYt^{k-1}n_6,$$

$$S(n_{52},n_{22})=\big(15786X^2t^{k-1}+12299Yt^{k-1}+4574Zt^{k-1}\big)n_1+ \\ +\big(458X^3t^{k-1}-4521XYt^{k-1}\big)n_2+\big(195X^3t^{k-1}+9582XYt^{k-1}\big)n_3+ \\ +\big(11538X^3t^{k-1}-6952XYt^{k-1}\big)n_4+4255XYt^{k-1}n_5+ \\ +5418XYt^{k-1}n_6-7781XYt^{k-1}n_7+15167X^2Yt^{k-1}n_8,$$

$$S(n_{52},n_{23})=12176Xt^{k-1}n_1+\big(11345Yt^{k-1}-3411X^2t^{k-1}\big)n_2+ \\ +\big(10349X^2t^{k-1}-10438Yt^{k-1}\big)n_3-\big(10733X^2t^{k-1}+4794Yt^{k-1}\big)n_4+ \\ +\big(3742Yt^{k-1}-3728X^2t^{k-1}\big)n_5-6012Yt^{k-1}n_6+ \\ +15797Yt^{k-1}n_7+15994XYt^{k-1}n_8+9142X^4t^{k-1}n_{20},$$

$$S(n_{52},\tilde{n}_{27})=4951n_1-8795Xn_2+2030Xn_3-6520Xn_4+3335Xn_5-10322Xn_6+ \\ -8914Xn_7+5084X^2n_8+12090X^2n_9-621X^2n_{10}-X^3n_{20},$$

$$S(n_{52},n_{39})=5849t^{k-1}n_1+5307Xt^{k-1}n_2+8037Xt^{k-1}n_3-10670Xt^{k-1}n_5+ \\ -10657Xt^{k-1}n_6-10700Xt^{k-1}n_7,$$

$$S(n_{52},n_{53})=8057t^{k-1}n_1-1398Xt^{k-1}n_2+5830Xt^{k-1}n_3,$$

$$S(n_{53},n_1)=-8057X^2t^{k-1}n_1+1398X^3t^{k-1}n_2-5830X^3t^{k-1}n_3,$$

$$S(n_{53},n_5)=1398t^{k-1}n_2-5830t^{k-1}n_3+6821Xn_{52},$$

$$\begin{aligned}S(n_{53},n_9)=&8177t^{k-1}n_2+9074t^{k-1}n_3+9456t^{k-1}n_4-15472t^{k-1}n_5-3025t^{k-1}n_6+\\&+5872t^{k-1}n_7+7482Xn_{52},\end{aligned}$$

$$\begin{aligned}S(n_{53},n_{13})=&-2387Xt^{k-1}n_1+8630X^2t^{k-1}n_2-2367Yt^{k-1}n_4+\\&-\big(10692X^2t^{k-1}+3619Yt^{k-1}\big)n_5+1398Yt^{k-1}n_6-5830Yt^{k-1}n_7,\end{aligned}$$

$$\begin{aligned}S(n_{53},n_{20})=&1921t^{k-1}n_2+553t^{k-1}n_3+13299t^{k-1}n_4-7346t^{k-1}n_5+1040t^{k-1}n_6+\\&-369t^{k-1}n_7+5350Xt^{k-1}n_8+10903Xt^{k-1}n_9-13378Xt^{k-1}n_{10}+\\&+3443X^2t^{k-1}n_{11}+10907X^2t^{k-1}n_{12}-12090t^{k-1}n_{13}+621t^{k-1}n_{14}+\\&+1398X^4t^{k-1}n_{17}-5830X^5t^{k-1}n_{18}+352X^2t^{k-1}n_{20}+1236Xn_{52},\end{aligned}$$

$$\begin{aligned}S(n_{53},n_{21})=&11470t^{k-1}n_2-1173t^{k-1}n_3-11146t^{k-1}n_4+2968t^{k-1}n_5+\\&+10305t^{k-1}n_6-4635t^{k-1}n_7-7937Xt^{k-1}n_8-12287Xt^{k-1}n_9+\\&+14466Xt^{k-1}n_{10}+1398X^4t^{k-1}n_{17}-5830X^5t^{k-1}n_{18}+\\&+352X^2t^{k-1}n_{20}+1236Xn_{52},\end{aligned}$$

$$\begin{aligned}S(n_{53},n_{22})=&1716Xt^{k-1}n_1+\big(6637X^2t^{k-1}-7605Yt^{k-1}-1390Zt^{k-1}\big)n_2+\\&+\big(2888X^2t^{k-1}+14606Yt^{k-1}-6933Zt^{k-1}\big)n_3+\\&+\big(7474Yt^{k-1}-7181X^2t^{k-1}+13502Zt^{k-1}\big)n_4+\\&+\big(14642X^2t^{k-1}+15947Yt^{k-1}-13648Zt^{k-1}\big)n_5+\\&-\big(10820Yt^{k-1}+7687Zt^{k-1}\big)n_6+\big(10021Zt^{k-1}-3292Yt^{k-1}\big)n_7+\\&+\big(15207XYt^{k-1}+11558XZt^{k-1}\big)n_8+1398X^4Zt^{k-1}n_{17}+\\&+\big(3864XYt^{k-1}-877XZt^{k-1}\big)n_9-5830X^5Zt^{k-1}n_{18}+\\&+\big(12292XYt^{k-1}-5345XZt^{k-1}\big)n_{10}+15167X^2Yt^{k-1}n_{12}+\\&+\big(352X^2Zt^{k-1}-10271X^4t^{k-1}\big)n_{20},\end{aligned}$$

$$\begin{aligned}S(n_{53},n_{23})=&-9389t^{k-1}n_1-15808Xt^{k-1}n_2-11000Xt^{k-1}n_3+10359Xt^{k-1}n_4+\\&+8626Xt^{k-1}n_5-4154Xt^{k-1}n_6+9659Xt^{k-1}n_7+\\&+\big(13261X^2t^{k-1}+9049Yt^{k-1}\big)n_8-\big(13002X^2t^{k-1}+9008Yt^{k-1}\big)n_9+\\&+\big(4777Yt^{k-1}-13661X^2t^{k-1}\big)n_{10}-\big(14167X^3t^{k-1}+XYt^{k-1}\big)n_{11}+\end{aligned}$$

$$+ 15994XYt^{k-1}n_{12} + \big(9142XYt^{k-1} - 1776X^3t^{k-1}\big)n_{20} +$$
$$- 3619X^4t^{k-1}n_{23} + 1398X^4t^{k-1}n_{24} - 5830X^5t^{k-1}n_{25},$$

$$S(n_{53},\tilde{n}_{27}) = 12049n_2 + 12309n_3 - 10081n_4 + 13415n_5 - 13763n_6 + 15347n_7 +$$
$$+ 13426Xn_8 - 5281Xn_9 + 10966Xn_{10} - 3447X^2n_{11} + 5084X^2n_{12} +$$
$$+ 12090n_{13} - 621n_{14} + \big(8561X^2 - Y\big)n_{20} - 3619X^4\tilde{n}_{27} + 1398X^4\tilde{n}_{28} +$$
$$- 5830X^4\tilde{n}_{30},$$

$$S(n_{53},n_{39}) = - 286t^{k-1}n_2 + 7860t^{k-1}n_3 - 15117t^{k-1}n_4 - 6108t^{k-1}n_5 +$$
$$+ 12142t^{k-1}n_6 + 3132t^{k-1}n_7 + 739Xt^{k-1}n_8 - 6413Xt^{k-1}n_9 +$$
$$- 14170Xt^{k-1}n_{10} - 10700X^2t^{k-1}n_{11} + 1398X^4t^{k-1}n_{17} +$$
$$- 5830X^5t^{k-1}n_{18} + 352X^2t^{k-1}n_{20} + 3361Xn_{52},$$

$$S(n_{54},n_3) = 10664t^{k-1}n_2 - 3351Xn_{52},$$

$$S(n_{54},n_7) = - 11989t^{k-1}n_5 + 10664t^{k-1}n_6 - 11207Xn_{52},$$

$$S(n_{54},n_{11}) = - 8888Xt^{k-1}n_1 - \big(11322X^2t^{k-1} + 7505Yt^{k-1}\big)n_2 +$$
$$- \big(946X^2t^{k-1} + 4208Yt^{k-1}\big)n_3 + \big(10776X^2t^{k-1} - 14377Yt^{k-1}\big)n_4 +$$
$$+ \big(19X^2t^{k-1} + 490Yt^{k-1}\big)n_5 + 15603Yt^{k-1}n_6,$$

$$S(n_{54},n_{15}) = \big(8558X^3t^{k-1} - 10351XYt^{k-1}\big)n_1 - \big(8944X^4t^{k-1} + 6298X^2Yt^{k-1}\big)n_2 +$$
$$+ \big(5885X^4t^{k-1} - 3129X^2Yt^{k-1}\big)n_3 - 6451X^4t^{k-1}n_4 +$$
$$- \big(11178X^2Yt^{k-1} + 11989Y^2t^{k-1}\big)n_5 + 10664Y^2t^{k-1}n_6,$$

$$S(n_{54},n_{18}) = - 7854t^{k-1}n_2 + 11206t^{k-1}n_3 - 10538t^{k-1}n_4 - 10104t^{k-1}n_5 +$$
$$+ 7797t^{k-1}n_6 - 2051t^{k-1}n_7 - 5416Xt^{k-1}n_8 - 7054Xt^{k-1}n_9 +$$
$$- 495Xt^{k-1}n_{10} + 10664X^4t^{k-1}n_{17} - 7419X^2t^{k-1}n_{20} - 11512Xn_{52},$$

$$S(n_{54},n_{25}) = 7641t^{k-1}n_1 + 7063Xt^{k-1}n_2 - 7008Xt^{k-1}n_3 + 4339Xt^{k-1}n_4 +$$
$$+ 4724Xt^{k-1}n_5 - 3917Xt^{k-1}n_6 - 4838Xt^{k-1}n_7 + 9294X^2t^{k-1}n_9 +$$
$$+ \big(11714X^2t^{k-1} - 8676Yt^{k-1}\big)n_8 - 15706X^2t^{k-1}n_{10} - 6X^3t^{k-1}n_{11} +$$
$$- 15983X^3t^{k-1}n_{20} - 11989X^4t^{k-1}n_{23} + 10664X^4t^{k-1}n_{24},$$

$$S(n_{54},\tilde{n}_{30}) = - 15397n_2 + 169n_3 + 10757n_4 - 4778n_5 - 15159n_6 + 12689n_7 +$$
$$+ 12107Xn_8 + 14396Xn_9 + 5627Xn_{10} - 197X^2n_{20} - 11989X^4\tilde{n}_{27} +$$
$$+ 10664X^4\tilde{n}_{28},$$

$$\begin{aligned}S(n_{54},\tilde{n}_{32})=&\big(15596X^2-9514Y-15106t^k\big)n_3-1771Xn_1-\big(3810X^2+5215Y\big)n_2+\\&+\big(1070X^2+7855Y+3092t^k\big)n_4+\big(1572X^2-6816Y-11111t^k\big)n_5+\\&+\big(5222Y+12536t^k\big)n_6+\big(5600t^k-428Y\big)n_7+\\&-\big(3XY+15522Xt^k\big)n_8-2662Xt^kn_9+14235Xt^kn_{10}+\\&-\big(7858X^4+4768X^2t^k\big)n_{20}-\big(12190X^6+11989X^4t^k\big)\tilde{n}_{27}+\\&+\big(4943X^6+10664X^4t^k\big)\tilde{n}_{28},\end{aligned}$$

$$S(n_{54},n_{38})=15995t^{k-1}n_2-7997t^{k-1}n_3-11989t^{k-1}n_5+10664t^{k-1}n_6+341Xn_{52},$$

$$S(n_{55},n_2)=8230Xn_{52},\qquad S(n_{55},n_6)=4285t^{k-1}n_5-3807Xn_{52},$$

$$\begin{aligned}S(n_{55},n_{10})=&-10801Xt^{k-1}n_1+\big(13307X^2t^{k-1}+2427Yt^{k-1}\big)n_2-6637X^2t^{k-1}n_4+\\&+\big(6843Yt^{k-1}-9265X^2t^{k-1}\big)n_3+\big(6153X^2t^{k-1}+4285Yt^{k-1}\big)n_5,\end{aligned}$$

$$\begin{aligned}S(n_{55},n_{14})=&\big(-12809X^3t^{k-1}+3992XYt^{k-1}\big)n_1+10469X^4t^{k-1}n_2+\\&-10311X^2Yt^{k-1}n_4+\big(-5833X^2Yt^{k-1}+4285Y^2t^{k-1}\big)n_5,\end{aligned}$$

$$\begin{aligned}S(n_{55},n_{17})=&-10451t^{k-1}n_2-21t^{k-1}n_3-3776t^{k-1}n_4-15117t^{k-1}n_5+\\&-7721t^{k-1}n_6+4051t^{k-1}n_7-177Xt^{k-1}n_8+11241Xt^{k-1}n_9+\\&+10036Xt^{k-1}n_{10}-13996X^2t^{k-1}n_{20}+4267Xn_{52},\end{aligned}$$

$$\begin{aligned}S(n_{55},n_{24})=&939t^{k-1}n_1+3325Xt^{k-1}n_2+6557Xt^{k-1}n_3+8978Xt^{k-1}n_4+\\&-4811Xt^{k-1}n_5-6822Xt^{k-1}n_6+11Xt^{k-1}n_7-14156X^2t^{k-1}n_9+\\&+\big(5519X^2t^{k-1}+9567Yt^{k-1}\big)n_8+7534X^2t^{k-1}n_{10}+7710X^3t^{k-1}n_{11}+\\&-10940X^3t^{k-1}n_{20}+4285X^4t^{k-1}n_{23},\end{aligned}$$

$$\begin{aligned}S(n_{55},\tilde{n}_{28})=&10895n_2-6781n_3-11591n_4+4842n_5-8410n_6-10463n_7+\\&+12157Xn_8-8398Xn_9-807Xn_{10}+9426X^2n_{20}+4285X^4\tilde{n}_{27},\end{aligned}$$

$$\begin{aligned}S(n_{55},n_{36})=&-1714Xt^{k-1}n_1+\big(8308X^2t^{k-1}+10646Yt^{k-1}\big)n_2+\\&+\big(9098Yt^{k-1}-11349X^2t^{k-1}\big)n_3+\big(2316X^2t^{k-1}+567Yt^{k-1}\big)n_5.\end{aligned}$$

Since all the remainders are zero, the algorithm terminates here. Therefore,

$$B=\{n_1,\ldots,n_{26},\tilde{n}_{27},\tilde{n}_{28},\tilde{n}_{30},\tilde{n}_{32},n_{33},n_{36},n_{38},n_{39},n_{43},n_{45},n_{52},\ldots,n_{55}\}$$

is a Groebner basis for N.

Bibliography

[AB59] M. Auslander and D.A. Buchsbaum, *On ramification theory in Noetherian rings*, American Journal of Mathematics **81** (1959), no. 3, 749–765.

[AG60] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Transactions of the American Mathematical Society **97** (1960), no. 3, 367–409.

[Ale90] A. G. Aleksandrov, *Nonisolated Saito singularities*, Theory of singularities and its applications, American Mathematical Society, RI, 1990, pp. 211–246.

[BG80] R.-O. Buchweitz and G.-M. Greuel, *The Milnor number and deformations of complex curve singularities*, Invent. Math. **58** (1980), 241–281.

[BH93] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, 1993.

[BM96] J. W. Bruce and W. L. Marar, *Images and varieties*, J. Math. Sci. **82** (1996), no. 5, 3633–3641.

[Bru85] J. W. Bruce, *Vector fields on discriminants and bifurcation varieties*, J. London Math. Soc. (1985), no. 17, 257–262.

[Bru01] ———, *Classifications in Singularity theory and their applications*, New developments in Singularity theory (Cambridge, 2000), NATO Science Series II: Mathematics, Physics and Chemistry, vol. 21, Kluwer Academic Publications, Dordrecht, 2001, pp. 3–33.

[CLO05] D. A. Cox, J. Little, and D. O'Shea, *Using algebraic geometry*, second ed., Graduate Text in Mathematics, no. 185, Springer, 2005.

[CMA02] T. Cooper, D. Mond, and R.W. Atique, *Vanishing topology of codimension 1 multi-germs over* $\mathbb{R}$ *and* $\mathbb{C}$, Compositio Mathematica **131** (2002), no. 2, 121–160.

[Coo94] T. Cooper, *Map germs of* $\mathcal{A}_e$*-codimension one*, Ph.D. thesis, University of Warwick, 1994.

[Dam87] J. Damon, *Deformations of sections of singularities and Gorenstein surface singularities*, American Journal of Mathematics **109** (1987), 695–721.

[Dam91] ———, $\mathcal{A}$*-equivalence and the equivalence of sections of images and discriminants*, Singularity Theory and Applications (Warwick 1989) (D. Mond and J. Montaldi, eds.), Lecture Notes in Mathematics, vol. 1462, Springer, 1991, pp. 93–121.

[Dam98] ———, *On the legacy of free divisors: Discriminants and Morse-type singularities*, American Journal of Mathematics **120** (1998), no. 3, 453–492.

[DCS81] C. De Concini and E. Strickland, *On the variety of complexes*, Adv. Math **41** (1981), 57–77.

[DGPS10] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, Singular *3-1-2 — A computer algebra system for polynomial computations*, http://www.singular.uni-kl.de.

[dJP00] T. de Jong and G. Pfister, *Local analytic geometry*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 2000.

[dJvS91] T. de Jong and D. van Straten, *Disentanglements*, Singularity Theory and Applications (Warwick 1989) (D. Mond and J. Montaldi, eds.), Lecture Notes in Mathematics, vol. 1462, Springer, 1991.

[DM91] J. Damon and D. Mond, $\mathcal{A}$*-codimension and the vanishing topology of discriminants*, Invent. Math. **106** (1991), 217–242.

[EH01] D. Eisenbud and J. Harris, *The geometry of schemes*, Graduate Text in Mathematics, vol. 197, Springer, New York, 2001.

[Eis95] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Text in Mathematics, vol. 150, Springer, New York, 1995.

[Fis76] G. Fischer, *Complex analytic geometry*, Lecture Notes in Mathematics, vol. 538, Springer-Verlag, Berlin, 1976.

[Fis01] ———, *Plane algebraic curves*, Student Mathematical Library, no. 15, American Mathematical Society, 2001.

[Fit36] H. Fitting, *Die determinantenideale eines moduls*, Jahresbericht der Deutschen Math.-Vereinigung **46** (1936), 195–229.

[FL77] W. Fulton and D. Laksov, *Residual intersections and the double point formula*, Real and Complex Singularities (Oslo 1976) (P. Holm, ed.), Sijthoff and Noordhooff, 1977.

[Gaf75] T. Gaffney, *Properties of finitely determined germs*, Ph.D. thesis, Bandeis University, 1975.

[Gaf79] ———, *A note on the order of determination of a finitely determined germ*, Invent. Math. **52** (1979), 127–130.

[Gaf83] ———, *Multiple points and associated loci*, Singularities, Part 1, Proc. Sympos. Pure Math., no. 40, American Mathematical Society, RI, 1983, pp. 429–437.

[GLS07] G.-M. Greuel, C. Lossen, and E. Shustin, *Introduction to singularities and deformations*, Springer-Verlag, Berlin, 2007.

[GM93] V. Goryunov and D. Mond, *Vanishing cohomology of singularities of mappings*, Compositio Mathematica **89** (1993), 45–80.

[Gor84] V. Goryunov, *Singularities of projections of full intersections*, J. Soviet Math **27** (1984), 2785–2811.

[Gor95] ———, *Semi-simplicial resolutions and homology of images and discriminants of mappings*, Proc. London Math. Soc. **70** (1995), no. 3, 363–385.

[GR04] H. Grauert and R. Remmert, *Theory of Stein spaces*, Springer-Verlag, Berlin, 2004.

[Gre80] G.-M. Greuel, *Dualität in der lokalen kohomologie isolierter singularitäten*, Math. Ann. **250** (1980), 157–173.

[Gre83] ———, *On deformation of curves and a formula of deligne*, Algebraic Geometry (J.M. Aroca, R.-O. Buchweitz, M. Giusti, and M. Merle, eds.), Lecture Notes in Mathematics, no. 961, Springer, 1983, pp. 141–168.

[GS] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, http://www.math.uiuc.edu/Macaulay2/.

[Ham71] H. Hamm, *Lokale topologische eigenschaften komplexer räume*, Math. Ann. **191** (1971), 235–252.

[HK99] K. Houston and N. Kirk, *On the classification and geometry of corank 1 map-germs from three-space to four-space*, Singularity Theory (Liverpool 1996), London Maths. Soc. Lecture Notes Series, vol. 263, Cambridge University Press, 1999.

[Hou98] K. Houston, *On singularities of foldings maps and augmentations*, Math. Scand. **82** (1998), 191–206.

[Hou99] ———, *An introduction to the image computing spectral sequence*, Singularity Theory (Liverpool 1996), London Maths. Soc. Lecture Notes Series, vol. 263, Cambridge University Press, 1999.

[Hou10] ———, *Stratification of unfoldings of corank 1 singularities*, Quart. J. Math. **61** (2010), 413–435.

[Kle81] S. Kleiman, *Multiple-point formulas I: Iterations*, Acta Math. **147** (1981), 13–49.

[KLU92] S. Kleiman, J. Lipman, and B. Ulrich, *The source double-point cycle of a finite map of codimension one*, Complex Projective Varieties (Ellingsrud G., C. Peskine, G. Sacchiero, and S. A. Stromme, eds.), London Maths. Soc. Lecture Notes Series, vol. 179, Cambridge University Press, 1992, pp. 199–212.

[KLU96] ———, *The multiple-point schemes of a finite curvilinear map of codimension one*, Ark. Mat. **34** (1996), 285–326.

[KU97] S. Kleiman and B. Ulrich, *Gorenstein algebras, symmetric matrices, self-linked ideals, and symbolic powers*, Transactions of the American Mathematical Society **349** (1997), no. 12, 4973–5000.

[Kun86] E. Kunz, *Kähler differentials*, Friedr. Vieweg & Sohn, Braunschweig, 1986.

[Lak78] D. Laksov, *Residual intersections and Todd's formula for the double locus of a morphism*, Acta. Math. **140** (1978), 75–92.

[Loo84] E. J. N. Looijenga, *Isolated singular points on complete intersections*, Cambridge University Press, 1984.

[LS85] E. J. N. Looijenga and J. H. M. Steenbirk, *Milnor numbers and tjurina numbers of complete intersections*, Math. Ann. **271** (1985), 121–124.

[Mar82] J. Martinet, *Singularities of smooth functions and maps*, Cambridge University Press, 1982.

[Mar89] W. L. Marar, *Mapping fibrations and multiple point schemes*, Ph.D. thesis, University of Warwick, 1989.

[Mar91] ———, *The Euler characteristic of the disentanglement of the image of a corank 1 map-germ*, Singularity Theory and Applications (Warwick 1989) (D. Mond and J. Montaldi, eds.), Lecture Notes in Mathematics, vol. 1462, Springer, New York, 1991.

[Mat68] J. Mather, *Stability of* C^∞*-mappings* III*: Finitely determined map germs*, Inst. Hautes Etudes Sci. Publ. Math. **35** (1968), 279–308.

[Mat69a] ———, *Stability of* C^∞*-mappings* II*: Infinitesimal stability implies stability*, Ann. of Math. **89** (1969), 259–291.

[Mat69b] ———, *Stability of* C^∞*-mappings* IV*: Classification of stable germs by* r*-algebras*, Inst. Hautes Etudes Sci. Publ. Math. **37** (1969), 223–248.

[Mat70] ———, *Stability of* C^∞*-mappings* VI*: The nice dimensions*, Proc. Liverpool Singularities Symposium Vol. 1 (C. T. C. Wall, ed.), Lecture Notes in Mathematics, vol. 192, Springer, Berlin, 1970, pp. 207–253.

[Mat73] ———, *Generic projections*, Ann. of Math. **98** (1973), 226–245.

[Mat89] H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1989.

[MB06] D. Mond and R.-O. Buchweitz, *Linear free divisors and quiver representations*, Singularities and computer algebra, London Maths. Soc. Lecture Notes Series, no. 324, Cambridge University Press, 2006, pp. 41–77.

[Mil68] J. Milnor, *Singular points on complex hypersurfaces*, vol. 61, Ann. Math. Stud., 1968.

[MM89] W. L. Marar and D. Mond, *Multiple point schemes for corank 1 maps*, J. London Math. Soc. **39** (1989), no. 2, 553–567.

[MM94] D. Mond and J. Montaldi, *Deformations of maps on complete intersections, Damon's $\mathcal{K}_V$-equivalence and bifurcations*, Singularities (Lille 1991), London Maths. Soc. Lecture Notes Series, vol. 201, Cambridge University Press, 1994.

[MNB08] W. L. Marar and J. J. Nuño-Ballesteros, *A note on finite determinacy for corank 2 map germs from surfaces to 3-space*, Math. Proc. Cambridge Philos. Soc. **145** (2008), no. 1, 153–163.

[Mon87] D. Mond, *Some remarks on the geometry and classification of germs of maps from surfaces to 3-spaces*, Topology **26** (1987), 361–383.

[Mon91] ______, *Vanishing cycles for analytic maps*, Singularity Theory and Applications (Warwick 1989) (D. Mond and J. Montaldi, eds.), Lecture Notes in Mathematics, vol. 1462, Springer, New York, 1991.

[Mon00] ______, *Differential forms on free and almost free divisors*, Proc. London Math. Soc. **81** (2000), no. 3, 587–617.

[Mor65] B. Morin, *Formes canoniques des singularités d'une application differérentiable*, C. R. Acad. Sci. Paris **260** (1965), 5662–5665 and 6503–6506.

[MP89] D. Mond and R. Pellikaan, *Fitting ideals and multiple points of analytic mappings*, Algebraic Geometry and Complex Analysis (Pátzcuaro, 1987), Lecture Notes in Mathematics, vol. 1414, Springer, Berlin, 1989.

[MS10] D. Mond and M. Schulze, *Adjoint divisors and free divisors*, arXiv:1001.1095v3, 2010.

[MWA03] D. Mond and R. Wik Atique, *Not all codimension 1 germs have good real pictures*, Real and Complex Singularities, Lecture Notes in Pure and Appl. Math., vol. 232, Dekker, New York, 2003, pp. 189–200.

[Noe50] E. Noether, *Idealdifferentiation and differente*, J.reine ang. Math. **188** (1950), 1–21.

[Ree57] D. Rees, *The grade of an ideal or module*, Proc. Cambridge Phil. Soc. **53** (1957), 28–42.

[Ron73] F. Ronga, *"Le calcul des classes duales aux points doubles" d'une application*, Compositio Mathematica **27** (1973), no. 2, 223–232.

[Sai80] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. Math. **27** (1980), 265–291.

[Ser00] J.-P. Serre, *Local algebra*, Springer Monograms in Mathematics, Springer-Verlag, Berlin, 2000.

[SK77] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. (1977), no. 65, 1–155.

[Ter80] H. Terao, *Arrangements of hyperplanes and their freeness i, ii*, J. Fac. Sci. Univ. Tokyo Sect. Math. (1980), no. 27, 293–320.

[Ter83] ———, *The bifurcation set and logarithmic vector fields*, Math. Ann. (1983), no. 263, 313–321.

[Wal81] C. T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. Lond. Math. Soc. **13** (1981), 481–539.

[Yos90] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Maths. Soc. Lecture Notes Series, no. 146, Cambridge University Press, 1990.