Sequential Auctions with Informational Externalities and Aversion to Price Risk: Decreasing and Increasing Price Sequences*

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Abstract

A large body of empirical research has shown that prices of identical goods sold sequentially sometimes increase and often decline across rounds. This paper introduces a tractable form of risk aversion, called aversion to price risk, and shows that declining prices arise naturally when bidders are averse to price risk. When there are informational externalities, there is a countervailing effect which pushes prices to raise along the path of a sequential auction, even if bidder's signals are independent. The paper shows how to decompose the effect of aversion to price risk from the effect of informational externalities.

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1 Introduction

Sequential auctions are a common way of selling multiple lots of the same or similar goods. How should we expect prices to vary from one round to the next when there is no significant delay between rounds? A plausible answer is that we should expect the law of one price to hold; controlling for any difference between the lots, on average prices ought to be the same in different rounds. If they were not, if prices in round 2 were lower than in round 1, then we should expect, roughly speaking, demand to shift from round 1 to round 2 until prices are equalized.

This simple answer turns out to have solid game theoretical foundations. As first shown by Weber (1983) and Milgrom and Weber (1982), in the basic model with symmetric, risk-neutral, unit-demand bidders having independent private values, the price sequence of any standard auction is a martingale (the expected value of P_{k+1}, the price in round $k+1$, conditional on P_k, the price in round k, is equal to P_k).

The validity of the answer, however, does not seem to be supported by the data. A large body of empirical research has shown, in different settings, that prices sometimes increase and often decline across rounds. Sequential auctions where prices have been shown to decline include wine (Ashenfelter, 1989, McAfee and Vincent, 1993), flowers (van den Berg et al., 2001), livestock (Buccola, 1982), gold jewellery (Chanel et al., 1996), china from shipwrecks (Ginsburgh and van Ours, 2007), stamps (Thiel and Petry, 1995), Picasso prints (Pesando and Shum, 1996), art (Beggs and Graddy, 1997), condominiums (Ashenfelter and Genesove, 1992), commercial real estate (Lusht, 1994). There is also experimental evidence of declining prices (Burns, 1985, and Keser and Olson, 1996). Ashenfelter and Graddy (2003) contains a general survey that focuses on art auctions. While declining prices are more frequent, increasing prices have also been documented. For example, they were found for library books by Deltas and Kosmopolou (2001), watches by Chanel et al. (1996), wool by Jones et al. (2004), and Israeli cable tv by Gandal (1997).

An extension of the basic model generates increasing prices. Maintaining all the other assumptions, Milgrom and Weber (1982) showed that if the bidders' signals are affiliated rather than statistically independent and there are informational externalities, then the price sequence is a submartingale (the expected value of P_{k+1} conditional on P_k is higher than P_k). There are informational externalities (or interdependent types), if a bidder's payoff from winning an object directly depends on the signals (or types) of the other bidders. One difficulty of the model with affiliated signals, however, is that it is not very tractable. Milgrom and Weber (1982) was circulated as a working paper for a long time and only published in 2000. In a foreword and bracketed comments in their published version, Milgrom and Weber motivated the publishing delay to the proofs of many of their results having "refused to come together" (p.179) and added that some of the results "should be regarded as being in doubt" (p.188).

Ashenfelter (1989) informally proposed risk aversion as a possible explanation of declining prices. The

most commonly used model of risk averse bidders (e.g., Matthews, 1983, and McAfee and Vincent, 1993) assume that a bidder has a monetary value for the object, so that risk aversion is defined on the difference between the monetary value of the object and its price. McAfee and Vincent (1993) demonstrated that this model of risk averse bidders does not yield a convincing explanation of declining prices. They studied two-round, private-value, first-price and second-price auctions, and showed that prices decline only if bidders display increasing absolute risk aversion, which seems implausible. Under the more plausible assumption of decreasing absolute risk-aversion, an equilibrium in pure strategies does not exist and average prices need not decline. Because of this difficulty to explain declining prices with a simple generalization of the basic model, they have become a sort of a puzzle, referred in the literature as the *afternoon effect* (because after a morning auction, often the second round takes place in the afternoon), or the *declining price anomaly.*

The first contribution of this paper is to introduce a special case of risk aversion, called *aversion to price risk*, and to show that without informational externalities a pure strategy equilibrium exists and prices decline, when bidders are averse to price risk. An important advantage of the notion of aversion to price risk is that it yields a model that is as tractable as the model with risk neutral bidders. Indeed, the proofs deriving the equilibrium bidding functions are straightforward modifications of the standard proofs in the risk neutral case. In contrast, as we already mentioned, the commonly used model of risk averse bidders is not very tractable.

Aversion to price risk is a useful conceptual innovation. It is a special case of the general model of risk averse bidders studied by Maskin and Riley (1984). Contrary to the model analysed by McAfee and Vincent (1993), it assumes separability of a bidder's payoff between utility from winning an object and utility from the bidder's monetary wealth (or disutility from paying the price). A bidder that is averse to price risk prefers to win an object at a certain price, rather than at a random price with the same expected value. Budget constraints with costly financing are a natural way in which aversion to price risk arises. Suppose a bidder values an object \$21, only has a budget of \$10 and must pay an interest rate of 10% on any amount borrowed to finance a price above \$10. Such a bidder strictly prefers a certain price of \$10 which gives him a payoff of \$11, to a random price of \$20 or \$0 with equal probability. Because it costs an additional \$1 to finance a purchase at a price of \$20, the random price gives the bidder an expected payoff of \$10.5. As we shall see, a budget constraint is not the only possible explanation for aversion to price risk.

It is easiest to explain the intuition for why aversion to price risk generates declining prices in the case of a two-round, second-price auction with private values and unit-demand bidders. The price in the last round will be determined by the second highest bid. The crucial observation is that in the first round each bidder chooses his optimal bid assuming that he will win and will be the price setter; that is, he assumes that his bid is tied with the highest bid of his opponents. This is because a small change in his bid only matters when the bidder wins and is the price setter. The fundamental implication of this observation

is that in choosing his optimal bid, a bidder views the first-round price as certain (equal to his bid) and the second-round price as a random variable (equal to the second highest, second-round bid). Optimality requires that the bidder be indifferent between winning in the first or in the second round. Aversion to price risk then implies that the expected second-round price (conditional on the first-round price) must be lower than the first-round price. The difference is the risk premium that the bidder must receive to be indifferent between winning at a random, rather than a certain, price. The result that aversion to price risk generates a tendency for prices to decline, called an *aversion to price risk effect*, and its intuition, is general. It holds for auctions with more than two rounds, for second-price, first-price, and English auctions. It also holds if there are informational externalities (i.e., if values are not purely private).

An advantage of assuming statistically independent, rather than affiliated, bidders' signals is that the model remains very tractable when adding informational externalities. Another contribution of this paper is to show that affiliated signals are not needed to explain increasing prices. Informational externalities alone, even with independent signals, push prices to increase across rounds.[1] The intuition for this result is the following. In any but the last round of a sequential auction, it is optimal for a bidder to bid so as to be indifferent between barely winning (being tied with an opponent) and winning in the next round. For a risk-neutral bidder, this amounts to bidding the expected price in the next round conditional on barely winning. On the other hand, the winner never barely wins. Thus, with information externalities the winning price in the current round conveys, on average, good news to bidders and raises the next round price. More precisely, the expected price in round $k+1$ conditional on the realized price in round k is higher than the expected price in round $k+1$ conditional on barely winning in round k.

The paper also studies the case with both informational externalities and aversion to price risk, and shows how to separate the aversion to price risk effect, which reduces prices from one round to the next, from the informational effect, which increases prices from round to round. The combined presence of the two effects may help explaining the more complex price paths, with prices increasing between some rounds and decreasing between others, that we sometimes observe in the data (e.g., see Jones et al., 2004).

Most of the paper studies the first-price and second-price sequential auctions with unit-demand bidders, but Section 8 shows that the results extend to English auctions.

Before proceeding, I should add that there are other models in the literature that generate declining prices. They include winners having the option to buy additional units (Black and de Meza, 1992), heterogeneity of objects (Bernhardt and Scoones, 1994, Engelbrecht-Wiggans, 1994, Gale and Hausch, 1994), ordering of the objects for sale by declining value (Beggs and Graddy, 1997), absentee bidders (Ginsburgh, 1998), an unknown number of objects for sale (Jeitschko, 1999), asymmetry among bidders

[1] Stochastic scale effects (Jeitschko and Wolfstetter, 1998) and uncertainty about the number of rounds (Feng and Chatterjee, 2005) may also generate increasing prices.

(Gale and Stegeman, 2001), etc. I view aversion to price risk as complementary to the other explanations given in the literature. The explanation based on aversion to price risk has the advantage of applying very generally, without requiring any additional modification of the basic auction model. This is an important advantage because, as the empirical evidence suggests, declining prices have been found to prevail even with no buyers' option to buy additional objects, with identical objects, etc.

The paper is organized as follows. Section 2 introduces the model. Section 3 presents the equilibria of the first-price and second-price auction. Section 4 presents the afternoon effect when there are no informational externalities. Section 5 looks at risk neutral bidders with informational externalities. Section 6 studies the general model with aversion to price risk and informational externalities. It defines and discusses the aversion to price risk effect and the informational effect. Section 7 presents a calibrated example that shows how the data from empirical studies can be reproduced for plausible values of an aversion to price risk and an informational externality parameter. Section 8 discusses extensions of the model and the robustness of the main results. Section 9 concludes. Most of the proofs and additional technical results are in the appendices.

2 The Model

There are K identical objects to be auctioned and N symmetric bidders, $N > K$. Each bidder has unit demand. Bidder i observes the realization x_i of a signal X_i, a random variable with support $[\underline{x}, \overline{x}]$. I assume that the signals are i.i.d. random variables with density f and distribution F. Let the vector $x = (x_1, ..., x_N) \in [\underline{x}, \overline{x}]^N$ be the profile of signals of all bidders, and x_{-i} the vector of signals of all bidders except i. If i wins an object, the price he pays at auction, P, is, from bidder i's point of view, a random variable which depends on the bids of all bidders. If a bidder does not win an object, he pays nothing. Let p be a price realization. Bidder i's payoff when winning the object is

$$u(x_i, x_{-i}, -p) = V(x_i, x_{-i}) - \ell(p). \tag{1}$$

The function $V(x_i, x_{-i})$ is the payoff or value from obtaining one object. *Informational externalities* are allowed and V may depend on the signals of all bidders. When the utility of bidder i only depends on x_i there are no information externalities, or values are private. I make some standard assumptions: $V(x_i, x_{-i})$ is positive, twice differentiable, the same for all bidders i, symmetric in x_j, $j \neq i$, and increasing in all its arguments with $\partial V(x_i, x_{-i})/\partial x_i \geq \partial V(x_i, x_{-i})/\partial x_j \geq 0$. The latter assumption (which is commonly made when there are interdependent valuations, e.g., see Dasgupta and Maskin, 2000) guarantees the allocational efficiency of the equilibrium.

The function $\ell(p)$ is the bidder's cost, or loss, when paying price p. I will assume that ℓ is a convex function. When ℓ is linear, bidder i is risk neutral and the model reduces to the standard model (with

informational externalities); when ℓ is strictly convex, the bidder is *averse to price risk*. I normalize ℓ by assuming $\ell(0) = 0$.

A first interpretation of bidders that are averse to price risk is that they do not have an equivalent monetary value for the good on sale, but the good's quality $V(x)$ contributes additively to the utility of money. A second interpretation is that $V(x)$ is the monetary value of the object, but bidders are financially constrained and must borrow to finance the purchase (e.g., see Che and Gale, 1998). In such a case $\ell(p)$ is the cost the winner incurs when the price is p. An example is $\ell(p) = p\,(1 + r\,(p))$, where the interest rate r must satisfy the condition $2r'(p) + pr''(p) \geq 0$ to ensure convexity of ℓ.[2]

There are two main advantages of my formulation of the utility function u. The first is that it lends itself to a clean and novel interpretation of the behaviour of risk averse bidders. For example, consider a static, single-item, auction. A simple intuition for why bidders that are averse to price risk bid higher in a first-price than in a second-price auction is that a first-price auction insures them against any price risk; price when winning is certain in a first-price and random in a second-price auction. The standard explanation of revenue ranking with risk averse bidders is less transparent, having to appeal to the risk of losing the object.[3] As we shall see, the intuition for the results concerning sequential auctions is similarly transparent. The second advantage of my formulation is tractability. The model with bidders that are averse to price risk yields pure strategy equilibria independently of the degree of aversion to price risk. On the other hand, McAfee and Vincent (1993) considered two-item, sequential, first-price and second-price auctions with private values, and demonstrated that an equilibrium in pure strategies does not exists when utility is of the form $U(x_i - p)$ and it displays decreasing absolute risk aversion; that is $-U''(m)/U'(m)$ is decreasing in m.

It is worth pointing out that (1) is a special case of the utility function used in the general, single-item, auction design model with risk averse bidders of Maskin and Riley (1984). They postulate that bidder i's utility when he wins an object and pays p can be written as $u(x, -p)$, and as an example of the general model mention the case in which $u(x, -p) = U\,(V(x) - \ell(p))$. Most of the literature on auctions with risk averse bidders, however, has focused on the subcase with $V(x) = x_i$ and $\ell(p) = p$.

3 Sequential Auctions: Equilibrium

In this section, I present the equilibrium bidding strategies of the sequential first-price and second-price auctions, in which one object is sold in each of K successive rounds to the highest bidder, at a price equal

[2] While I will assume, for simplicity, that ℓ is differentiable everywhere, note that when the winning bidder pays a fixed interest r for any amount above his budget M, then the loss function is convex: $\ell(p) = p + r \max\{0, p - M\}$.

[3] A potentially interesting idea, that I will not pursue here, is that aversion to price risk on the part of some bidders may help to explain the use of "buy-it-now" options in online auctions.

to the highest and second highest bid respectively. Recall that because of unit demand winners drop out of the auction. In round $k \leq K$, the bidding function of each of the $N-(k-1)$ remaining bidders depends not only on his type, but also on the common history of announced prices and bids from previous rounds. I will assume that at the end of each round only the bid of the winner is announced and I will look for symmetric equilibria in which the bid of a player is an increasing function of his type in each round k. Thus, in both auctions, revealing the winning bid is equivalent to revealing the signal of the winning bidder.

The bidding functions of the sequential first-price and second-price auctions were first derived by Weber (1983) and Milgrom and Weber (1982) for the case of risk neutral bidders.[4] The next two propositions extend Milgrom and Weber's results to the case of aversion to price risk. The proofs are straightforward extensions of the risk-neutral case and are relegated to Appendix A.

Let the random variable $Y_j^{(n)}$, an order statistic, be the j-th highest type of bidder out of n. The expected value of the object as a function of i's type x and the k highest types of bidder i's opponents, $k = 0, ..., N-1$ will be written as:

$$v_k(x, y_k, ..., y_1) = E\left[V(X_i, X_{-i})|X_i = x, Y_k^{(N-1)} = y_k, ..., Y_1^{(N-1)} = y_1\right]. \tag{2}$$

To understand the bidding functions in each round of a sequential first-price auctions with K objects we need to make a few observations. First, recall that in a standard, single-item, first-price auction with risk neutral bidders and private values, the bid of player i is equal to the expectation, conditional on winning, of the item's value to his closest competitor, the competitor with the highest signal. Second, with informational externalities the expected value of i's closest competitor j must be computed as if i's signal were equal to j's signal. Third, in an auction with K objects, bidder i's closest competitor is the bidder with the K-th highest signal among his opponents; this bidder would be the marginal winner if i were not around. Fourth, from the point of view of bidder i, the type of the closest competitor is the random variable $Y_K^{(N-1)}$. Thus, in round k the expected value to i's closest competitor, conditional on the closest competitor assuming he has the same type as i and on the types of past winners is

$$v_k(Y_K^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ..., y_1) = E\left[V(X_i, X_{-i})|Y_K^{(N-1)} = X_i, Y_{k-1}^{(N-1)} = y_{k-1}, ..., Y_1^{(N-1)} = y_1\right].$$

Fifth, type x of bidder i conditioning on winning in round k amounts to conditioning on $Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}$. Finally, when bidders are averse to price risk it is useful to think of $L = \ell\left(\beta_k^{S1}(x; ...)\right)$ as the *bid loss* associated with the bid $\beta_k^{S1}(x; ...)$. Defining $\phi = \ell^{-1}$ as the inverse of the loss function ℓ, $\phi(L)$ is the bid that generates a bid loss L to the bidder. The function ϕ is strictly increasing and concave.

[4] Milgrom and Weber (1982) worked with affiliated types. As they say in the foreword added to the published version, because of affiliation their proofs have to be considered in doubt; see Mezzetti et al. (2008), for some recent progress.

Proposition 1 *In each round k of a sequential first-price auction, the bid loss of bidder i with signal x is equal to the expected value of the item to his closest competitor conditional on (i) the closest competitor assuming that he has the same signal as bidder i, (ii) the history of the winning types prior to round k, and (iii) bidder i winning in round k:*

$$\beta_k^{S1}(x; y_{k-1}, ..., y_1) = \phi\left(E\left[v_k\left(Y_K^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ...\right)|Y_k^{(N-1)} \leq x \leq y_{k-1}\right]\right).$$

β_k^{S1} is an increasing function of all its arguments when there are informational externalities. With private values $v_k(Y_K^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ...) = V\left(Y_K^{(N-1)}\right)$ and β_k^{S1} does not depend on the types of the winners of previous rounds (equivalently, it does not depend on the price history).

To understand the bidding functions in a sequential second-price auction, say that bidder i is pivotal in round k if he has the same signal as the k-th highest of his opponents, and hence he is in a tie with one opponent as the remaining bidder with the highest signal. The difference between the bids in a first-price and a second-price auction is that in a second-price auction a bidder assumes that he is pivotal, rather than assuming he is just the winner. Thus, for example, in a single-item, second-price auction with private values and risk neutral bidders, bidder i's bid is equal to his expected value for the object or, equivalently, the expected value of the object to his closest opponent, conditional on being pivotal.

Proposition 2 *In each round k of a sequential second-price auction, the bid loss of bidder i with signal x is equal to the expected value of the item to his closest competitor conditional on (i) the closest competitor assuming that he has the same signal as bidder i, (ii) the history of the winning types prior to round k, and (iii) bidder i being pivotal in round k:*

$$\beta_k^{S2}(x; y_{k-1}, ..., y_1) = \phi\left(E\left[v_k(Y_K^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ...)|Y_k^{(N-1)} = x\right]\right).$$

Note that with a single object, $K = 1$, private values and risk neutrality, the formulas in Propositions 1 and 2 reduce to the well known bid functions $\beta_1^{S1}(x) = E\left[V\left(Y_1^{(N-1)}\right)|Y_1^{(N-1)} \leq x\right]$ and $\beta_1^{S2}(x) = E\left[V\left(Y_1^{(N-1)}\right)|Y_1^{(N-1)} = x\right] = V(x)$.

I conclude this section with a remark. In a first-price auction, announcing the bid of the winner, as I have assumed, is equivalent to announcing the selling price, the standard practice in real auctions. This is not so in a second-price auction, where announcing the winning price amounts to revealing the type of the highest loser, a bidder who will be present in the next round. I will study the case of announcing the winning price in a sequential second-price auction in Section 8.

4 The Afternoon Effect

In this section, I assume that there are no informational externalities. With private values, i's payoff when winning the object is $V(x_i)$; to simplify notation, I will renormalize the type space and write $V(x_i) = x_i$.

The next proposition shows that declining prices are a natural consequence of aversion to price risk.

Proposition 3 *When there are no informational externalities, the price sequences in a sequential first-price and in a sequential second-price auction are a supermartingale: The expected price in round $k+1$ conditional on the price in round k is lower than the price in round k.*

Proof. If there are no informational externalities, then announcing either the winning bids or prices has no direct effect on the bidding functions. Consider first a sequential first-price auction. Given private values, it is

$$\beta_k^{S1}(x) = \phi\left(E\left[Y_K^{(N-1)} | Y_k^{(N-1)} \leq x\right]\right).$$

Suppose type x of bidder i wins auction $k < K$. Then it must be $P_k^{S1} = \beta_k^{S1}(x)$ and $Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}$. It follows that

$$\begin{aligned}
E\left[P_{k+1}^{S1} | P_k^{S1}\right] &= E\left[P_{k+1}^{S1} | \beta_k^{S1}(x)\right] \\
&= E\left[\beta_{k+1}^{S1}(Y_k^{(N-1)}) | Y_k^{(N-1)} \leq x\right] \\
&= E\left[\phi\left(\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)})\right)\right) | Y_k^{(N-1)} \leq x\right] \\
&< \phi\left(E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)})\right) | Y_k^{(N-1)} \leq x\right]\right) \\
&= \phi\left(E\left[E\left[Y_K^{(N-1)} | Y_{k+1}^{(N-1)} \leq Y_k^{(N-1)}\right] | Y_k^{(N-1)} \leq x\right]\right) \\
&= \phi\left(E\left[Y_K^{(N-1)} | Y_k^{(N-1)} \leq x\right]\right) \\
&= \beta_k^{S1}(x) \\
&= P_k^{S1},
\end{aligned}$$

where the inequality follows from Jensen's inequality, given that ϕ is a strictly concave function. This shows that the price sequence in a sequential first-price auction is a supermartingale.

Consider now a sequential second-price auction. Given private values, it is

$$\beta_k^{S2}(x) = \phi\left(E\left[Y_K^{(N-1)} | Y_k^{(N-1)} = x\right]\right)$$

Suppose in round k the winner is the bidder with signal $Y_k^{(N-1)}$, and bidder i of type x is the price setter; that is, $P_k^{S2} = \beta_k^{S2}(x)$. In round $k+1$, bidder i of type x wins the auction, and the price setter is the

bidder with the signal $Y_{k+1}^{(N-1)}$. It follows that

$$\begin{aligned}
E\left[P_{k+1}^{S2}|P_k^{S2}\right] &= E\left[P_{k+1}^{S2}|\beta_k^{S2}(x)\right] \\
&= E\left[\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)})|Y_k^{(N-1)}=x\right] \\
&= E\left[\phi\left(\ell\left(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)})\right)\right)|Y_k^{(N-1)}=x\right] \\
&< \phi\left(E\left[\ell\left(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)})\right)|Y_k^{(N-1)}=x\right]\right) \\
&= \phi\left(E\left[E\left[Y_K^{(N-1)})|Y_{k+1}^{(N-1)}=Y_{k+1}^{(N-1)}\right]|Y_k^{(N-1)}=x\right]\right) \\
&= \phi\left(E\left[Y_K^{(N-1)})|Y_k^{(N-1)}=x\right]\right) \\
&= \beta_k^{S2}(x) \\
&= P_k^{S2}.
\end{aligned}$$

Thus, the price sequence in a sequential second-price auction is also a supermartingale. ■

The intuition for the afternoon effect is essentially the same in a first-price and a second-price sequential auction. In each round before the last, conditional on having the highest remaining type and being the price setter, a bidder must be indifferent between winning now and winning in the next round.[5] But if a bidder is the price setter, then he knows the current price, while next round's price is random. Because of aversion to price risk, next round's expected price must then be lower than the price now. The difference is the risk premium the bidder must receive to be indifferent between the certain price now and the random price in the next round.

To understand this intuition in more detail, consider round $k < K$ of the second-price auction. Suppose type x of bidder i has lost all preceding auctions. Suppose also that in round k bidder x considers raising his bid by a small amount ε above $\beta_k^{S2}(x)$. This will only make a difference if, after the deviation, he wins in round k, while he would have otherwise lost and won in round $k+1$. For this to happen, it must be that $Y_k^{(N-1)} \simeq x$; that is, we must be in the event in which bidder i with signal x is at the margin between winning and losing in round k (i.e., he must be pivotal, his signal must be tied with the signal of another bidder). Conditional on this event, the *marginal cost* of the deviation is the loss incurred in period k when bidding according to the deviation,

$$\ell(\beta_k^{S2}(x)+\varepsilon),$$

while the *marginal benefit* is the expected loss avoided in period $k+1$,

$$E\left[\ell(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}))|Y_k^{(N-1)} \simeq x \leq Y_{k-1}^{(N-1)})\right].$$

[5] In a sequential first-price auction, if a bidder has the highest signal in round k and he bids according to the equilbrium bidding function, then he is automatically the price setter. In a sequential second-price auction, conditioning on the highest-signal bidder *also* being the price setter amounts to requiring that his signal is tied with the signal of another bidder (i.e., the bidder is pivotal).

Equating marginal cost and marginal benefit (and sending ε to zero) gives the following *indifference condition* (see equation (23) in Appendix A):

$$\ell(\beta_k^{S2}(x)) = E\left[\ell(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}))|Y_k^{(N-1)} = x)\right].$$

The indifference condition says that the certain loss when winning in period k at a price $\beta_k^{S2}(x)$ must be equal to the expected loss when winning in period $k+1$. Since bidders are averse to price risk, it must then be the case that the expected price in round $k+1$ is less than the price $\beta_k^{S2}(x)$ in round k. For the marginal bidder to be indifferent between winning at a certain price now, or at an uncertain price in the next round, it must be the case that the next round's expected price (conditional on the current price) is lower than the current price. Hence prices must decline from one period to the next.

Now consider a sequential first-price auction. Suppose bidder i wins in round k if he bids as a type x; that is, suppose $Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}$. Bidder i can also always bid so low so as to lose in round k. With a losing bid, bidder i discovers the value of $Y_k^{(N-1)}$ (the signal of the winner when bidder i bids low). Bidder i can then win for sure in round $k+1$ by bidding $\beta_{k+1}^{S1}(Y_k^{(N-1)})$ (i.e., by bidding as if his type were $Y_k^{(N-1)}$). The indifference condition for sequential first-price auctions states that bidder i must be indifferent between winning in round k and in round $k+1$ (see equation (15) in Appendix A):

$$\ell(\beta_k^{S1}(x)) = E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)})\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}\right].$$

The left hand side is the certain loss associated with the period k price; the right hand side is the expected loss associated with the random price in period $k+1$. Thus the price sequence must also be decreasing in a sequential first-price auction.

Another way to understand Proposition 3 is the following. We know from Weber (1983) and Milgrom and Weber (1982) that if bidders are risk neutral and values are private, then the price sequence is a martingale. When bidders are averse to price risk, the loss $\ell(p)$ plays the same role for a bidder as the price p in the case of risk neutrality. Hence, the loss sequence associated with the price setting bids will be a martingale. Since ℓ is a convex function, the price sequence must be a supermartingale; that is, price declines over time.

5 The Effect of Informational Externalities

In the literature on sequential auctions with unit demand, an increasing price sequence has only been derived in the model with affiliated signals by Milgrom and Weber (1982).[6,7] In this section, I show that affiliated signals are not needed to generate increasing price sequences. With independent signals, risk neutral bidders (i.e., ℓ is the identity function) and informational externalities, prices *increase* along the equilibrium path of a sequential auction. The price sequence is a martingale only if there are no informational externality (i.e., in the common terminology, values are private).

Proposition 4 *In a sequential first-price and in a sequential second-price auction with announcement of the winning bids, if bidders are risk neutral and there are informational externalities, then the price sequence is a submartingale: The expected price in round $k+1$ conditional on the price in round k is higher than the price in round k.*

Proof. Recall that ℓ and ϕ coincide with the identity function. Consider first a sequential first-price auction. Suppose type x of bidder i wins auction $k < K$. It must be $Y_k^{(N-1)} \leq x < y_{k-1} < ... < y_1$, and $P_k^{S1} = \beta_k^{S1}(x; y_{k-1}, ..., y_1) = E\left[v_k\left(Y_K^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ...\right)|Y_k^{(N-1)} \leq x \leq y_{k-1}\right]$. It follows that

$$\begin{aligned}
E\left[P_{k+1}^{S1}|P_k^{S1}\right] &= E\left[P_{k+1}^{S1}|\beta_k^{S1}(x; y_{k-1}, ..., y_1)\right] \qquad (3)\\
&= E\left[\beta_{k+1}^{S1}(Y_k^{(N-1)}; x, y_{k-1}, .., y_1)|Y_k^{(N-1)} \leq x \leq y_{k-1}, ...\right]\\
&> E\left[\beta_{k+1}^{S1}(Y_k^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, .., y_1)|Y_k^{(N-1)} \leq x \leq y_{k-1}, ...\right]\\
&= E\left[v_{k+1}\left(Y_K^{(N-1)}, Y_{k+1}^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ...\right)|Y_k^{(N-1)} \leq x \leq y_{k-1}, ...\right]\\
&= E\left[v_k\left(Y_K^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ...\right)|Y_k^{(N-1)} \leq x \leq y_{k-1}, ...\right]\\
&= \beta_k^{S1}(x; y_{k-1}, .., y_1)\\
&= P_k^{S1},
\end{aligned}$$

where the inequality follows from $\beta_{k+1}^{S1}(\cdot)$ being a strictly increasing function of all its arguments and $Y_k^{(N-1)} \leq x$. Thus, the price sequence in a sequential first-price auction is a submartingale.

[6] However, as I pointed out in the introduction, in the published version of their working paper, Milgrom and Weber (1982, 2000) acknowledge that the proofs of their results with affiliated signals "should be regarded as being in doubt" (p. 188). Mezzetti et al. (2008) were able to provide a proof for the case of a sequential second-price auction in which the winning bids are announced.

[7] When bidders have multi-unit demand, many new technical issues arise and only special versions of the model have been solved. For example, in a repeated game set-up with common-values, one informed and one uninformed player, Jamison and Horner (2008) have shown that the price jumps up after a bid of the informed player reveals that the common value is high. Virag (2007) obtains a similar conclusion assuming that bids are not revealed, but the value of the item is discovered by the uninformed party after winning once.

Now consider a sequential second-price auction. Suppose that in round k the winner is the bidder with signal $Y_k^{(N-1)}$, and bidder i of type x is the price setter; that is, $P_k^{S2} = \beta_k^{S2}(x; y_{k-1}, ..., y_1)$. In round $k+1$, bidder i of type x wins the auction, and the price setter is the bidder with signal $Y_{k+1}^{(N-1)}$. It follows that

$$\begin{aligned}
E\left[P_{k+1}^{S2}|P_k^{S2}\right] &= E\left[P_{k+1}^{S2}|\beta_k^{S2}(x; y_{k-1}, .., y_1)\right] \qquad (4)\\
&= E\left[\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|Y_{k+1}^{(N-1)} \leq x \leq Y_k^{(N-1)} \leq y_{k-1}, ..\right]\\
&> E\left[\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|Y_k^{(N-1)} = x \leq y_{k-1}, ..\right]\\
&= E\left[E\left[v_{k+1}(Y_K^{(N-1)}, Y_{k+1}^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ...)|Y_{k+1}^{(N-1)} = Y_{k+1}^{(N-1)}\right]|Y_k^{(N-1)} = x\right]\\
&= E\left[v_{k+1}(Y_K^{(N-1)}, Y_{k+1}^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ...)|Y_k^{(N-1)} = x\right]\\
&= E\left[v_k(Y_K^{(N-1)}, Y_k^{(N-1)}, y_{k-1}, ...)|Y_k^{(N-1)} = x\right]\\
&= \beta_k^{S2}(x; y_{k-1}, .., y_1)\\
&= P_k^{S2}.
\end{aligned}$$

Thus, the price sequence in a sequential second-price auction is also a submartingale. ■

To see why with informational externalities and risk neutral bidders prices tend to increase from one round to the next even in the case of independent signals, consider a sequential second-price auction (the reasoning for a first-price auction is similar). First, recall that in each round k, the price is equal to the bid of the loser with the highest signal. Second, observe that in each round k before the last, the price-setter, as any other bidder, chooses a bid that makes him indifferent between winning and winning in the next round, conditional on being pivotal in round k; that is, conditional on being tied with the winner as the remaining bidder with the highest signal. Thus, the price in round k is equal to the expected price in the next round conditional on the the price-setter in round k being pivotal. With informational externalities, this is lower than the next round expected price conditional on the current price. This is because with probability one the price-setter is not pivotal; that is, round k winner has a higher, not the same, signal than the price-setter, and the value of an object directly depends in a positive way on the signals of all bidders. Hence price tends to increase over time.

6 Aversion to Price Risk and Informational Externalities: Effect Decomposition

While aversion to price risk pushes prices to decline over time, informational externalities introduce a tendency for prices to increase. If bidders are both averse to price risk and there are informational externalities, it is possible to decompose the two countervailing effects on the price sequence.

Recall that, given a bid loss L, we can think of $\phi(L)$ as the bid, or *implicit price*, that generates L. Let P_k^{Sj} be the price in round k of a first-price (for $j=1$) or a second-price (for $j=2$) auction. In defining the aversion to price risk effect and the informational externality effect, we always condition on the price in round k. This amount to conditioning on the types $y_{k-1},...,y_1$ of the winners in the first $k-1$ rounds and on the type x of the price setter in round k.

We define the aversion to price risk effect as the difference between the price expected in round $k+1$ and the implicit price associated with the expected loss in round $k+1$:

$$A_{k+1}^{Sj}(x,y_{k-1},...,y_1) = E\left[P_{k+1}^{Sj}|P_k^{Sj}\right] - \phi\left(E\left[\ell\left(P_{k+1}^{Sj}\right)|P_k^{Sj}\right]\right).$$

The aversion to price risk effect is non-positive. It is immediate that in the case of risk neutrality $A_{k+1}^{Sj}=0$. That $A_{k+1}^{Sj}(\cdot)<0$ when bidders are averse to price risk follows immediately from the convexity of ℓ, which implies that $\ell\left(E\left[P_{k+1}^{Sj}|P_k^{Sj}\right]\right) < E\left[\ell\left(P_{k+1}^{Sj}\right)|P_k^{Sj}\right]$.

We define the informational externality effect as the difference between the implicit price associated with the expected loss in round $k+1$ and the implicit price associated with the loss in round k:

$$I_{k+1}^{Sj}(x,y_{k-1},...,y_1) = \phi\left(E\left[\ell\left(P_{k+1}^{Sj}\right)|P_k^{Sj}\right]\right) - \phi\left(\ell\left(P_k^{Sj}\right)\right)$$

The informational externality effect is zero with private values and strictly positive with informational externalities. This follows from the bid loss sequence being a martingale with private values and a sub-martingale with informational externalities; that is, $E\left[\ell\left(P_{k+1}^{Sj}\right)|P_k^{Sj}\right] \geq \ell\left(P_k^{Sj}\right)$ with strict inequality when there are informational externalities.[8]

The next proposition, whose proof follows immediately from the definitions of A_{k+1}^{Sj} and I_{k+1}^{Sj}, shows that in both sequential auctions the expected price in round $k+1$, conditional on the price in round k, is equal to the sum of the price in round k, the aversion to price risk effect, and the informational externality effect.

Proposition 5 *Suppose the price setter in round k has signal x and the history of winners' signals up to round $k-1$ is $y_{k-1},...,y_1$. In the sequential first-price auction ($j=1$) and the sequential second-price auction ($j=2$) with announcement of the winning bids we have:*

$$E\left[P_{k+1}^{Sj}|P_k^{Sj}\right] = P_k^{Sj} + A_{k+1}^{Sj}(x,y_{k-1},...,y_1) + I_{k+1}^{Sj}(x,y_{k-1},...,y_1). \tag{5}$$

Furthermore, it is $A_{k+1}^{Sj}(\cdot)<0$ if bidders are averse to price risk and $A_{k+1}^{Sj}(\cdot)=0$ if bidders are risk neutral; it is $I_{k+1}^{Sj}(\cdot)>0$ if there are informational externalities and $I_{k+1}^{Sj}(\cdot)=0$ if there are private values.

[8] The proof of this statement mirrors the proof of Proposition 4. The only modification that is needed is to replace P_{k+1}^{Sj}, P_k^{Sj}, β_{k+1}^{Sj}, and β_k^{Sj} in (3) and (4) with $\ell\left(P_{k+1}^{Sj}\right)$, $\ell\left(P_k^{Sj}\right)$, $\ell\left(\beta_{k+1}^{Sj}\right)$ and $\ell\left(\beta_k^{Sj}\right)$.

7 A Calibrated Example

Can aversion to price risk explain the declining price sequences we observe in the data? What is the degree of aversion to price risk that is needed? May prices decline even with informational externalities? This section provides some answers to these questions. I introduce a simple parametric example, and show that its predictions match the data from a sample of empirical studies, for reasonable specifications of the parameters. The empirical reference points for the discussion in this section are the papers of Ashenfelter (1989) and McAfee and Vincent (1993) on sequential (mostly two-round) auctions of identical bottles of wine sold in equal lot sizes. Ashenfelter's (1989) data set included auctions between August 1985 and December 1987 in four different location (Christies's London and Chicago, Sotheby's London and Buttersfield's San Francisco). McAfee and Vincent (1993) looked at auctions held at Christies's in Chicago in 1987. They both found evidence of declining prices; the price in the second auction was twice more likely to decrease than to increase. The average price ratio P_2/P_1 they found is displayed in Table 1.

	Mean Ratio P_2/P_1
Ashenfelter (1989): Christies's London	.9943
Ashenfelter (1989): Sotheby's London	.9875
Ashenfelter (1989): Christies's Chicago	.9884
Ashenfelter (1989): Butterfield's San Francisco	.9663
McAfee and Vincent (1993): Christies's Chicago	.9922

Table 1: *Price ratio in the data*

I will make the following simplifying assumptions to the model. The quality valuation of an object to bidder i is $x_i + b\sum_{j\neq i} x_j$, with $b \in [0,1]$. If $b = 0$, there are no informational externalities. The random variables X_i are distributed on $[0,1]$ with distribution function $F(x) = x^a$, with $a > 0$. The loss function is:

$$\ell(p) = \frac{p^{1+r}}{1+r},$$

We can interpret $r = p\ell''/\ell'$ as a coefficient of relative price-risk aversion. The inverse of ℓ is

$$\phi(z) = (1+r)^{\frac{1}{1+r}}\, z^{\frac{1}{1+r}}.$$

I will restrict attention to the second-price auction and the case of two rounds, $K = 2$. In Appendix B, I compute the bidding functions, the expected price in round 2 conditional on the first-round price P_1, and the ratio of the conditional expected second-round price to the first-round price.

In all computations reported in this section, I will set $r = 2$, a commonly used value for relative risk aversion in computational macroeconomics (e.g., see Ljungqvist and Sargent, 2000); it implies that a bidder is willing to pay a price about 1% higher to avoid a 50-50 gamble of a 10% increase or a 10% decrease in price. The results do not seem overly sensitive to the value of r.

Assume first that there are no informational externalities, that is $b = 0$. Figure 1 displays the price ratio $E[P_2|P_1]/P_1$ as a function of $A = a(N-2)$.

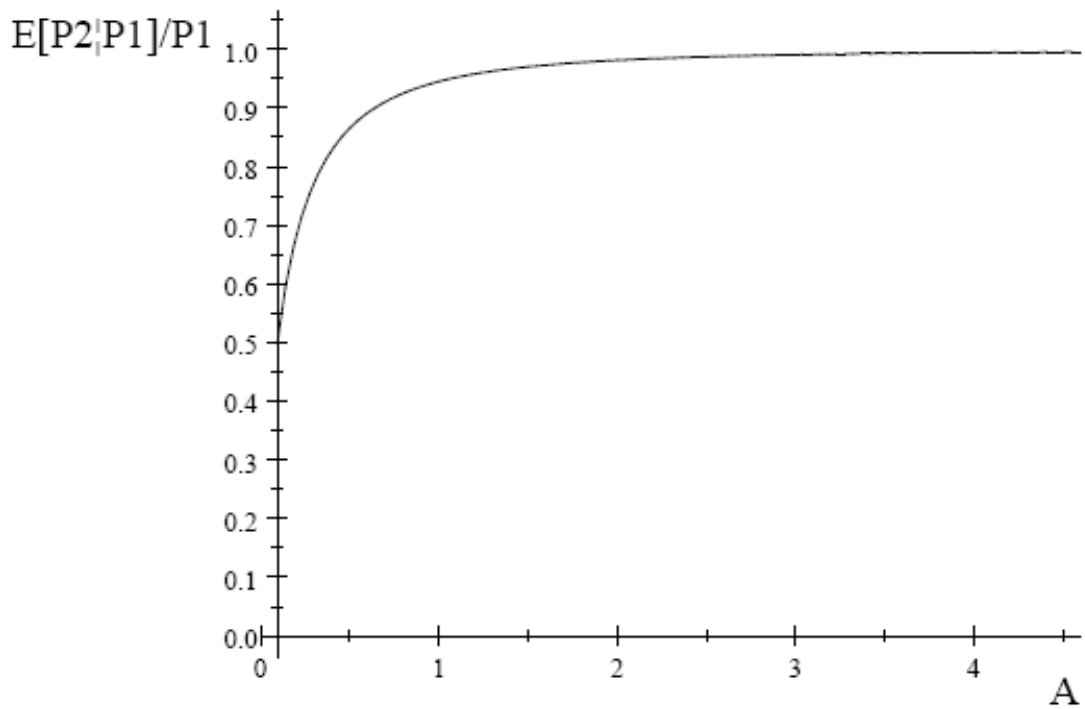

Fig. 1: *Price ratio with no informational externalities as a function of* $A = a(N-2)$

The average price ratio in the data in Table 1 ranges from 0.9663 to 0.9943, which correspond to values of A from 1.3903 to 3.9788. In most of the auctions considered the number of bidders was relatively small, typically well below 20. If we take $N = 10$, this gives values of a between 0.1738 and 0.4973 as those consistent with the data.

Suppose now that there are informational externalities, $b > 0$. Lower values of the parameter a are needed to match the data. Setting $r = 2$ as before, $N = 10$, and $a = 0.1$ yields the relationship between the price ratio and the informational externality parameter b shown in Figure 2. The range of the average price ratio in the data in Table 1 corresponds to values of the informational externality parameter between 0.0827 and 0.1260. Prices may decline even if there are informational externalities. In fact, the presence of both aversion to price risk and informational externalities could help explaining why in some auctions prices decline and in other they increase; it could also help explaining why in some multiple round auctions

prices decline between some rounds and increase between other rounds (e.g., see Jones et al. 2004).

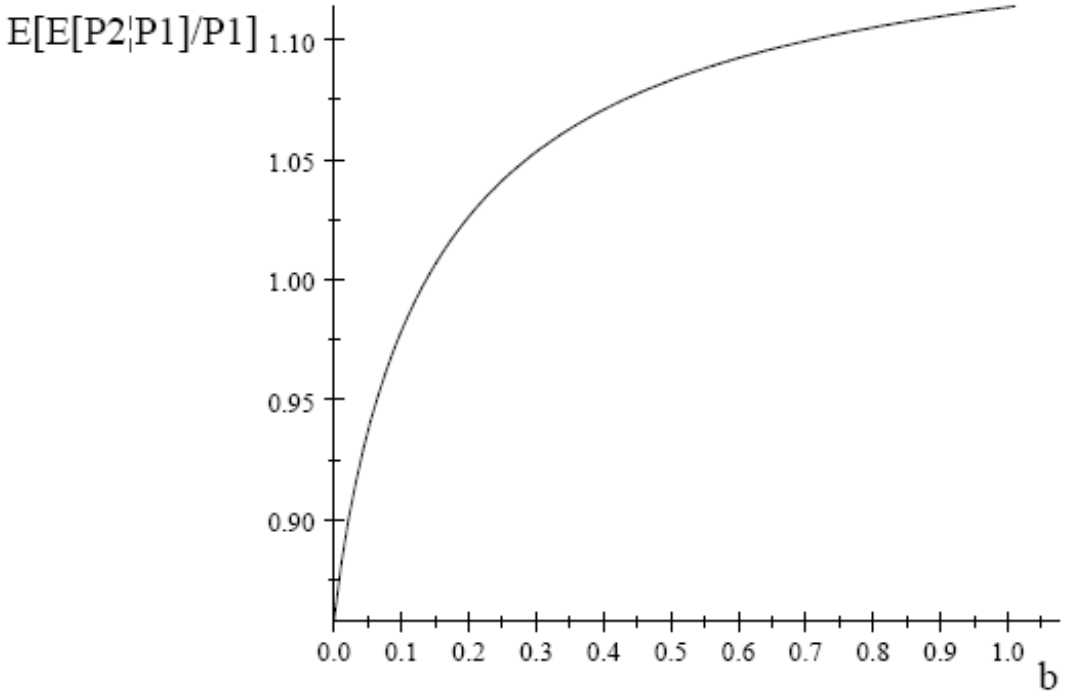

Fig. 2: *Price ratio as a function of the information externality parameter b*

8 Extensions

In this section, first I show that the aversion to price risk effect is present even when the auction format is an oral ascending auction. Then I study sequential second-price auctions in which the winning price is announced.

8.1 The English Auction

Sequential auctions are often run using an English, or oral ascending, format. The most commonly used model to study a static, single unit, English auction is the so-called Japanese version, in which a price clock moves continuously and bidders can only decide when to quit. Once a bidder quits, he cannot re-enter. When the second to last bidder quits, the clock stops and the last bidder standing wins at the current clock price. Using the Japanese format for sequential auctions is problematic. As Milgrom and Weber (1982) first pointed out, the equilibrium is the same as in the static English auction for multiple items. This is because the Japanese format forces all losing bidders to reveal their types during the first round and hence eliminates all uncertainty from future rounds.

Such a counterintuitive conclusion is a by-product of the extreme nature of the Japanese format. In practical ascending auctions, it is not the case that at the end of the first round all bidders in the room

know the identity of all future winners and the types of all losing bidders. Bidders often stay silent at the beginning and only start bidding towards the end of a round. Some bidders stay silent throughout a round. To deal with these issues in a static, single-unit environment, Harstad and Rothkopf (2000) have introduced an *Alternating Recognition* (AR) model of the English auction, which better fits practice in many auctions.

I will use a two-round version of the AR English auction and will assume that there are no informational externalities, so that $V(x_i, x_{-i}) = x_i$. Bidders are averse to price risk and have unit demand. In each round of the auction, the price moves up continuously as in the Japanese format. The difference is that at each point in time there are only two active bidders. At the beginning of each round, the auctioneer selects randomly (with uniform probability) from the pool of bidders who have not yet won an object and asks them sequentially to be active. Bidders can accept or reject to enter (be active). The clock starts after two bidders enters. Each active bidder decides when to exit. When a bidder exits, the clock stops and the auctioneer randomly selects from the pool of non active bidders in search for a replacement. The round ends when only one bidder is active and all others have exited or refused to enter. The standing active bidder wins an object at the current clock price. Bidders observe all decisions to accept to be active and to exit.

In the following proposition (the proof is in Appendix C), I will construct a monotone equilibrium in which in each round k the price β_k^E below which a bidder is willing to enter and at which he will exit when active is an increasing function of his type x. In such an equilibrium, the first-round exit price reveals a bidder's type and a bidder rejects to be active if and only if a higher type has already exited.

Proposition 6 *There is an equilibrium of the two-round, AR English auction, with no informational externalities, in which:* (1) *In the second round, bidder i of type x accepts to enter when the current clock price is below $\beta_2^E(x) = \phi(x)$; if active, bidder i exits at price $\phi(x)$.* (2) *In the first round, let $M \leq N-2$ be the number of bidders that have either exited or rejected to be active and $y_1^{(M)}$ the highest type among them (if $M = 0$, let $y_1^{(0)} < \underline{x}$). Bidder i of type x accepts to be active when $x > y_1^{(M)}$ and exits when the clock price reaches*

$$\beta_1^E(x; M, y_1^{(M)}) = \phi\left(E\left[\max\left\{Y_2^{(N-1-M)}, Y_1^{(M)}\right\} | Y_1^{(N-1-M)} = x, Y_1^{(M)} = y_1^{(M)}\right]\right). \tag{6}$$

After M bidders have exited in the first round, $\max\left\{Y_2^{(N-1-M)}, Y_1^{(M)}\right\}$ is the type of bidder i's closest competitor. Thus, in the AR English auction with private values, after M bidders have exited the first round, the bid loss of bidder i with signal x is equal to the expected value of the item to his closest competitor conditional on the highest type of the M bidders that have exited and on bidder i being pivotal.[9]

[9] Since active bidders have higher types than the bidders that have exited, when $M \leq N-2$, it is $Y_1^{(N-1-M)} > y_1^{(M)}$.

The winners of the auction are the two highest valuation bidders. There are two possible outcomes. First, all bidders become active in the first round. In this case, the types of all losing bidders become known by the end of the first round; the prices in the first and in the second round are the same and equal to $\phi(Y_3^{(N)})$. This is the same outcome that would obtain in the first round of the Japanese auction, but it can only happen if either the highest or the second highest bidder are selected to be active after the other $N-2$ bidders have already exited. The second possible outcome is that the two highest bidders (say with types x_1 and $x_2 < x_1$) are active together in the first round when only $M < N-2$ of the other bidders have already exited. In this case the first round price is $P_1 = \beta_1^E(x_2; M, y_1^{(M)})$. The second round price is random; its expectation, conditional on P_1, M and $y_1^{(M)}$ is $E\left[\phi\left(\max\left\{Y_2^{(N-1-M)}, Y_1^{(M)}\right\}\right)|Y_1^{(N-1-M)} = x_2, Y_1^{(M)} = y_1^{(M)}\right]$, which is less than P_1 by Jensen's inequality. Since the second outcome will occur with strictly positive probability, we have proved the following proposition.

Proposition 7 *With strictly positive probability, the expected price in round* 2 *of the two-round AR English auction with no informational externalities, conditional on the price and bid history in round* 1*, is lower than the price in round* 1*.*

8.2 Announcing the Winning Prices in a Sequential Second-Price Auction

A delicate issue in sequential auctions is what information is revealed from round to round. The natural candidate is to reveal the winning price in each round. Unlike in a first-price auction, however, announcing the winning price in a second-price auction amounts to revealing the type of the highest loser, a bidder who will be present in the next round. As a consequence, existence of an equilibrium with increasing bidding functions is problematic.

I will now show that in the case of no informational externalities increasing equilibrium bidding functions exist and are the same as under the policy of revealing the bid of the winner.

Proposition 8 *When there are no informational externalities, $V(x_i, x_{-i}) = x_i$, on the equilibrium path of the symmetric equilibrium of the sequential second-price auction with announcement of the winning prices, the bidding functions can be written as*

$$\beta_k^{S2p}(x) = \phi\left(E\left[Y_K^{(N-1)}|Y_k^{(N-1)} = x\right]\right).$$

This proposition, whose proof is in Appendix C, can be understood as follows. Without informational externalities, equilibrium bids do not depend on the past history of bids, no matter whether the winning bids, or the winning prices, are announced. In both cases, in the last round it is a dominant strategy to place a bid-loss equal to the item's value. The bids in earlier rounds are then determined recursively, via

the indifference condition (23). History does not matter because in each round k a bidder bids as if he were pivotal (i.e., as if $Y_k^{(N-1)} = x$).

When, on the other hand, there are informational externalities and the winning price is revealed in each round, an equilibrium of the sequential second-price auction with an increasing bidding function does not exist. The reason is simple: a bidder who, based on the history of prices, knows that he will lose in round k, but will almost certainly win in round $k+1$, has an incentive to deviate and make a very low bid in round k. By doing so he will avoid being the price setter in round k. The price setter in round k will be a bidder with a lower type, and hence in round $k+1$ all other bidders (including the future price setter) will have lower estimates of an object's conditional value and will make lower bids. As a result, the deviating bidder will profit by winning at a lower price in round $k+1$. This is made clear by Example 1 in Appendix C.

9 Conclusions

The classic model of risk neutral bidders assumes additive separability in a bidder's preferences over objects and money. The special case of risk averse bidders that I have studied in this paper maintains additive separability, but postulates that a bidder prefers a certain price to an equivalent (on average) random price. Additive separability of preferences makes the model very tractable. As I show in Lemma 1 in Appendix A, it implies that bidder-payoff equivalence holds: All auction mechanisms with the same allocation rule and which give the same payoff to the lowest bidder type are bidder-payoff equivalent.

Without additive separability, the effects due to risk aversion and private information interact in a complex way; for example, as shown by Maskin and Riley (1984), a risk neutral auctioneer choosing an optimal mechanism will not want to fully insure bidders. The simple model of aversion to price risk I studied in this paper, on the other hand, is very tractable. As a result, it yields a simple explanation of declining prices in sequential auctions. In effect, declining prices allow the bidder to insure himself in the current round against next round price randomness.

The paper also uncovers an informational externality effect. When there is no aversion to price risk, but there are informational externalities, prices increase between rounds of sequential auctions even when signals are independent. In essence, optimal equilibrium behaviour leads the current price setter to underestimate the signal of his highest opponent, and hence next round price.

Several empirical implications can be drawn from this paper. First, the more important a concern is price risk for bidders, the more we should expect prices to decline between rounds. Thus, for example, if there is a serious possibility that new bidders may enter in the next round, then price risk is more severe and we should expect prices to decline more.

Second, when informational externalities, or value interdependencies, are not very important, but bidders are averse to price risk, then prices are likely to decline. When value interdependencies are more important than price risk, then we should expect prices to increase between rounds. For example, if the auctioneer publishes all the information at his disposal (as the professional auction houses typically do), including value estimates of the objects for sale, then interdependencies are reduced and it is more likely that we see prices decline (as the data broadly suggests), rather than increase between rounds. If bidders are professionals, buying the goods for resale, and little information is provided about resale value by the auctioneer before the auction, then it is more likely that prices will increase between rounds.[10]

Third, the less information about bidders' values transpires during a round, the more we should expect prices to decline. Thus, if each round is an oral ascending auction, the larger the number of bidders that remain silent during the initial rounds, the higher future price randomness, and hence the more likely are prices to decline between rounds.

More generally, the interaction between the aversion to price risk effect and the informational externality effect could help to explain the complex price paths we sometimes observe in the data.

[10] This seems broadly consistent with Deltas and Kosmopoulou (2005) study of an auction of rare library books, in which price estimates were not published and a lower bound on the number of professionals in the auction is estimated by the authors at about 25%.

Appendix A

This appendix contains the proofs of Propositions 1 and 2. First, it is it is useful to derive a lemma showing that bidders' payoffs are the same in every auction having the same outcome function and yielding the same payoff to the lowest type of bidder.

Suppose that k objects have already been sold to the k highest type bidders, $y_1, ..., y_k$; suppose also that the winners' types have been revealed. Consider a mechanism in which $\pi_{k+1}(x', y_{N-1}, ..., y_{k+1})$ is i's probability of winning one of the remaining objects and $p_{k+1}(x', y_{N-1}, ..., y_{k+1})$ is i's payment when he behaves as a type x'. Then, bidder i's expected payoff when his type is x, but he behaves as if his type were x' is

$$U_{k+1}(x'; x; y_k, ..., y_1) = \int_{\underline{x}}^{y_k} ... \int_{\underline{x}}^{y_{N-2}} \big[V(x, y_{N-1}, ..., y_1)\pi_i(x', y_{N-1}, ..., y_{k+1}) - \ell(p_i(x_i', y_{N-1}, ..., y_{k+1}))\big] f(y_{N-1}, ..., y_{k+1}|Y_k^{(N-1)} = y_k) dy_{N-1}...dy_{k+1},$$

where $f(y_{N-1}, ..., y_{k+1}|y_k)$ is the density of the order statistics $Y_{N-1}^{(N-1)}, ..., Y_{k+1}^{(N-1)}$ conditional on $Y_k^{(N-1)} = y_k$. (By independence, it is not necessary to condition on the order statistics $Y_h^{(N-1)}$ with $h < k$.)

Letting $U_{k+1}^*(x; y_k, ..., y_1) = U_{k+1}(x; x; y_k, ..., y_1)$ be the expected payoff in equilibrium of type x, and using a standard envelope argument yields

$$\frac{\partial U_{k+1}^*(x; y_k, ..., y_1)}{\partial x} = \int_{\underline{x}}^{y_k} ... \int_{\underline{x}}^{y_{N-2}} \frac{\partial V(x, y_{N-1}, ..., y_1)}{\partial x} \pi_i(x, y_{N-1}, ..., y_{k+1}) f(y_{N-1}, ..., y_{k+1}|Y_k^{(N-1)} = y_k) dy_{N-1}...dy_{k+1}. \tag{7}$$

Denote the distribution and density function of the order statistic $Y_j^{(n)}$ as $F_j^{(n)}$ and $f_j^{(n)}$. If the mechanism is efficient, as the sequential auctions studied in this paper, then (7) becomes

$$\frac{\partial U_{k+1}^*(x; y_k, .., y_1)}{\partial x} = E\left[\frac{\partial V(x, Y_{N-1}^{(N-1)}, .., Y_{k+1}^{(N-1)}, y_k, .., y_1)}{\partial x} \Big| x > Y_K^{(N-1)}\right] F_K^{(N-1)}(x|Y_k^{(N-1)} = y_k). \tag{8}$$

Equation (7), combined with $U_{k+1}^*(\underline{x}; ...) = \underline{u}$, yields the bidder-payoff equivalence lemma.

Lemma 1 *Suppose $k < K$ objects have already been sold to the highest type bidders, and the winning types have been announced. Bidders' payoffs are the same in any mechanism $\langle \pi_{k+1}, p_{k+1} \rangle$ having the same outcome function π_{k+1} and yielding the same payoff to the lowest type. Equation (7) (equation (8) if the mechanism is efficient) and the boundary condition $U_{k+1}^*(\underline{x}; ...) = \underline{u}$ determine a bidder's payoff.*

We are now ready to prove propositions 1 and 2.

Proof of Proposition 1. Let $\beta_k^{S1}(x; y_{k-1}, ..., y_1)$ be round k equilibrium bidding function. Recall that, assuming that β_k^{S1} is increasing in x, on the equilibrium path the true types of the winning bidders are

revealed. Suppose that if the winning bid in round k is higher than the highest equilibrium bid, then all bidders believes that the winning bidder's type is the same as the type of the previous round's winner; if the observed winning bid in round k is below the lowest equilibrium bid, then bidders believe that the winner's type is the lowest possible type.

Let $U_k^*(x; y_{k-1}, ...)$ be the expected payoff for a type x of bidder in the continuation equilibrium beginning in round k (i.e., the payoff conditional on having lost all previous auctions and on the history up to round k). In writing a bidder's payoff, I will use the function v_k, defined in (2). Suppose that all the other bidders follow the equilibrium strategies, while bidder i is considering deviating in round k (only). First note that, given his beliefs, it is not profitable for bidder i to bid above the highest possible bid of the other bidders $\beta_k^{S1}(y_{k-1}; \cdot)$. Bidding below the lowest possible bid is equivalent to bidding the lowest bid; in both cases winning is a zero probability event. Hence if there is a profitable deviation, there is a profitable deviation with a bid in the range of possible bids. The payoff of bidder i of type x when he bids $b = \beta_k^{S1}(z; y_{k-1}, ...)$ (i.e., he bids like a type z) in round k is:

$$\begin{aligned} U_k(z; x; y_{k-1}, ...) = & \int_{\underline{x}}^{z} \left[v_k(x, y_k, ...) - \ell(\beta_k^{S1}(z; y_{k-1}, ...)) \right] f_k^{(N-1)} \left(y_k | Y_{k-1}^{(N-1)} = y_{k-1} \right) dy_k \\ & + \int_{z}^{\overline{x}} U_{k+1}^*(x; y_k, y_{k-1}, ..) f_k^{(N-1)} \left(y_k | Y_{k-1}^{(N-1)} = y_{k-1} \right) dy_k. \end{aligned} \tag{9}$$

Differentiating with respect to z yields the first order condition

$$\begin{aligned} & v_k(x, z, y_{k-1}, ..) f_k^{(N-1)} \left(z | Y_{k-1}^{(N-1)} = y_{k-1} \right) - \frac{d \left(\ell(\beta_k^{S1}(z; y_{k-1}, ...)) F_k^{(N-1)} \left(z | Y_{k-1}^{(N-1)} = y_{k-1} \right) \right)}{dz} \\ & - U_{k+1}^*(x; z, y_{k-1}, ..,) f_k^{(N-1)} \left(z | Y_{k-1}^{(N-1)} = y_{k-1} \right) = 0. \end{aligned} \tag{10}$$

Since on the equilibrium path it is $x \leq y_{k-1}$, and $z = x$ must be optimal, we obtain the following necessary condition for equilibrium:

$$\begin{aligned} & v_k(x, x, y_{k-1}, ..) f_k^{(N-1)} \left(x | Y_{k-1}^{(N-1)} = y_{k-1} \right) - \frac{d \left(\ell(\beta_k^{S1}(x; y_{k-1}, ..)) F_k^{(N-1)} \left(x | Y_{k-1}^{(N-1)} = y_{k-1} \right) \right)}{dx} \\ & - U_{k+1}^*(x; x, y_{k-1}, ..,) f_k^{(N-1)} \left(x | Y_{k-1}^{(N-1)} = y_{k-1} \right) = 0. \end{aligned} \tag{11}$$

Observe that if the signal of the winner in round $k < K$ is x, then in round $k+1$ bidder i with signal x wins with probability 1; hence, it is

$$U_{k+1}^*(x; x, y_{k-1}, ..,) = v_k(x, x, y_{k-1}, ...) - \ell(\beta_{k+1}^{S1}(x; x, y_{k-1}, ...)). \tag{12}$$

Since $U_{K+1}^*(x; x, y_{K-1}, ..,) = 0$, equation (12) also holds for $k = K$, provided we define

$$\beta_{K+1}^{S1}(x; x, y_{K-1}, ...) = \phi \left(v_K(x, x, y_{K-1}, ...) \right). \tag{13}$$

Using (12), equation (11) can be written as

$$\ell(\beta_{k+1}^{S1}(x;x,y_{k-1},...))f_k^{(N-1)}\left(x|Y_{k-1}^{(N-1)}=y_{k-1}\right)-\frac{d\left(\ell(\beta_k^{S1}(x;\cdot))F_k^{(N-1)}\left(x|Y_{k-1}^{(N-1)}=y_{k-1}\right)\right)}{dx}=0. \quad (14)$$

Integrating (14) we obtain

$$\begin{aligned}\ell(\beta_k^{S1}(x;y_{k-1},..,y_1)) &= \int_{\underline{x}}^{x}\ell\left(\beta_{k+1}^{S1}(\widetilde{x};\widetilde{x},y_{k-1},..,y_1)\right)\frac{f_k^{(N-1)}\left(\widetilde{x}|Y_{k-1}^{(N-1)}=y_{k-1}\right)}{F_k^{(N-1)}\left(x|Y_{k-1}^{(N-1)}=y_{k-1}\right)}d\widetilde{x} \quad (15)\\ &= E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)};Y_k^{(N-1)},y_{k-1},...)\right)|Y_k^{(N-1)}\leq x\leq Y_{k-1}^{(N-1)}=y_{k-1}\right].\end{aligned}$$

By (13), for $k=K$, this yields

$$\ell(\beta_K^{S1}(x;y_{K-1},...))=E\left[v_K(Y_K^{(N-1)},Y_K^{(N-1)},y_{K-1},...)|Y_K^{(N-1)}\leq x\leq y_{K-1}\right].$$

Working backwards, (15) yields

$$\ell\left(\beta_k^{S1}(x;y_{k-1},...)\right)=E\left[v_k\left(Y_K^{(N-1)},Y_k^{(N-1)},y_{k-1},...\right)|Y_k^{(N-1)}\leq x\leq y_{k-1}\right],$$

and hence on the equilibrium path the bidding function must satisfy

$$\beta_k^{S1}(x;y_{k-1},..,y_1)=\phi\left(E\left[v_k\left(Y_K^{(N-1)},Y_k^{(N-1)},y_{k-1},...\right)|Y_k^{(N-1)}\leq x\leq y_{k-1}\right]\right). \quad (16)$$

Note from (16) that if values are private $v_k\left(Y_K^{(N-1)},Y_k^{(N-1)},y_{k-1},...\right)=V\left(Y_K^{(N-1)}\right)$, and β_k^{S1} is independent of $y_1,...,y_{k-1}$.

It remains to show that the first order condition (11) is a sufficient condition for equilibrium. Using (11) to replace the second term on the left hand side of equation (10) we obtain

$$\begin{aligned}\frac{\partial U_k}{\partial z} &= \left[v_k(x,z,y_{k-1},...)-v_k(z,z,y_{k-1},...)\right]f_k^{(N-1)}\left(z|Y_{k-1}^{(N-1)}=y_{k-1}\right) \quad (17)\\ &+\left[U_{k+1}^*(z;z,y_{k-1},..,)-U_{k+1}^*(x;z,y_{k-1},..,)\right]f_k^{(N-1)}\left(z|Y_{k-1}^{(N-1)}=y_{k-1}\right).\end{aligned}$$

Since v_k is increasing in x and $U_{K+1}^*=0$, for $k=K$ the sign of $\frac{\partial U_k}{\partial z}$ is the same as $x-z$; hence $z=x$ is optimal.

Now suppose $k<K$; take first the case $z\leq x$. Note that

$$U_{k+1}^*(x;z,y_{k-1},..,)=v_k(x,z,y_{k-1},...)-\ell(\beta_{k+1}^{S1}(z;z,y_{k-1},...)),$$

because in this case bidder i wins for sure in round k, and bids $\beta_{k+1}^{S1}(\min\{x,z\};z,y_{k-1},...)=\beta_{k+1}^{S1}(z;z,y_{k-1},...)$. It follows that $\frac{\partial U_k}{\partial z}=0$ for $z\leq x$ and bidder i has no incentive to bid less than the equilibrium strategy in round k.

Now suppose that $z > x$. By Lemma 1, equation (8), we have

$$\begin{aligned}\frac{\partial U^*_{k+1}(x;z,y_{k-1},..,y_1)}{\partial x} &= E\left[\frac{\partial V(x,Y^{(N-1)}_{N-1},..,Y^{(N-1)}_{k+1},z,y_{k-1},..,y_1)}{\partial x}\Big| x > Y^{(N-1)}_K\right] F^{(N-1)}_K(x|Y^{(N-1)}_k = z)\\ &< E\left[\frac{\partial V(x,Y^{(N-1)}_{N-1},..,Y^{(N-1)}_{k+1},z,y_{k-1},..,y_1)}{\partial x}\right]\\ &= \frac{\partial v_k(x,z,y_{k-1},...)}{\partial x}. \end{aligned} \tag{18}$$

Integrating between x and z, it follows that

$$U^*_{k+1}(z;z,y_{k-1},..,) - U^*_{k+1}(x;z,y_{k-1},..,) < v_k(z,z,y_{k-1},...) - v_k(x,z,y_{k-1},...),$$

and hence that $\frac{\partial U_k}{\partial z} < 0$ for $z > x$; bidder i has no incentive to bid more than the equilibrium strategy in round k. This concludes the proof of the proposition. ■

Proof of Proposition 2. Let $\beta^{S2}_k(x;y_{k-1},...,y_1)$ be round k equilibrium bidding function. Suppose that if the winning bid in round k is higher than the highest equilibrium bid, then all bidders believe that the winning bidder's type is the same as the type of the previous round's winner; if the observed winning bid in round k is below the lowest equilibrium bid, then bidders believe that the winner's type is the lowest possible type.

Let $U^*_k(x;y_{k-1},...)$ be the expected payoff for a type x of bidder at the beginning of round k. Suppose that all the other bidders follow the equilibrium strategies, while bidder i is considering deviating in round k. As for the case of a sequential first-price auction, if there is a profitable deviation, there is a profitable deviation with a bid in the range of possible bids. Recalling (2), the payoff of bidder i of type x when he bids $b = \beta^{S2}_k(z;y_{k-1},...)$ (i.e., he bids like a type z) in auction k can be written as:

$$\begin{aligned}U_k(z;x;y_{k-1},...) = &\int_{\underline{x}}^{z}\left[v_k(x,y_k,...) - \ell(\beta^{S2}_k(y_k;y_{k-1},...))\right] f^{(N-1)}_k\left(y_k|Y^{(N-1)}_{k-1} = y_{k-1}\right) dy_k\\ &+ \int_{z}^{\overline{x}} U^*_{k+1}(x;y_k,y_{k-1},..) f^{(N-1)}_k\left(y_k|Y^{(N-1)}_{k-1} = y_{k-1}\right) dy_k.\end{aligned}$$

Differentiating with respect to z yields the first order condition

$$\begin{aligned}&\left[v_k(x,z,y_{k-1},...) - \ell(\beta^{S2}_k(z;y_{k-1},...))\right] f^{(N-1)}_k\left(z|Y^{(N-1)}_{k-1} = y_{k-1}\right)\\ &- U^*_{k+1}(x;z,y_{k-1},..,) f^{(N-1)}_k\left(z|Y^{(N-1)}_{k-1} = y_{k-1}\right) = 0.\end{aligned} \tag{19}$$

Since on the equilibrium path $z = x$ must be optimal, the following is a necessary condition for equilibrium:

$$v_k(x,x,y_{k-1},...) - \ell(\beta^{S2}_k(x;y_{k-1},...)) - U^*_{k+1}(x;x,y_{k-1},..,) = 0. \tag{20}$$

If $k = K$, then $U^*_{k+1}(x;\cdot) = 0$, and (20) yields that on the equilibrium path the bidding function must satisfy

$$\beta^{S2}_K(x; y_{K-1}, ..., y_1) = \phi\left(v_K(x, x, y_{K-1}, ..., y_1)\right). \tag{21}$$

If the signal of the winner in round $k < K$ is x, then in round $k+1$ bidder i with signal x wins with probability 1; hence, it is

$$U^*_{k+1}(x; x, y_{k-1}, ..,) = v_k(x, x, y_{k-1}, ...) - \int_{\underline{x}}^{x} \ell(\beta^{S2}_{k+1}(y_{k+1}; x, y_{k-1}, ...)) f^{(N-1)}_{k+1}\left(y_{k+1}|Y^{(N-1)}_k = x\right). \tag{22}$$

Thus (20) can be written as

$$\begin{aligned}\ell(\beta^{S2}_k(x; y_{k-1}, ...)) &= \int_{\underline{x}}^{x} \ell(\beta^{S2}_{k+1}(y_{k+1}; x, y_{k-1}, ...)) f^{(N-1)}_{k+1}\left(y_{k+1}|Y^{(N-1)}_k = x\right) \\ &= E\left[\ell(\beta^{S2}_{k+1}(Y^{(N-1)}_{k+1}; Y^{(N-1)}_k, ...))|Y^{(N-1)}_k = x \leq Y^{(N-1)}_{k-1} = y_{k-1}, ...\right].\end{aligned} \tag{23}$$

Recalling (21) and working backwards we obtain

$$\ell(\beta^{S2}_k(x; y_{k-1}, ...)) = E\left[v_k(Y^{(N-1)}_K, x, y_{k-1})\right].$$

Thus, we have shown that on the equilibrium path the bidding function must satisfy

$$\beta^{S2}_k(x; y_{k-1}, .., y_1) = \phi\left(E\left[v_k(Y^{(N-1)}_K, Y^{(N-1)}_k, y_{k-1}, ...)|Y^{(N-1)}_k = x\right]\right). \tag{24}$$

It remains to show that the first order condition (20) is a sufficient condition for equilibrium. Using (20) to replace the second term on the left hand side of equation (19) we obtain

$$\begin{aligned}\frac{\partial U_k}{\partial z} &= \left[v_k(x, z, y_{k-1}, ...) - v_k(z, z, y_{k-1}, ...)\right] f^{(N-1)}_k\left(z|Y^{(N-1)}_{k-1} = y_{k-1}\right) \\ &\quad + \left[U^*_{k+1}(z; z, y_{k-1}, ..,) - U^*_{k+1}(x; z, y_{k-1}, ...)\right] f^{(N-1)}_k\left(z|Y^{(N-1)}_{k-1} = y_{k-1}\right).\end{aligned} \tag{25}$$

Consider $k = K$; since v_k is increasing in x and $U^*_{K+1} = 0$, the sign of $\frac{\partial U_K}{\partial z}$ is the same as $x - z$; hence $z = x$ is optimal.

Now suppose $k < K$; take first the case $z \leq x$. Note that

$$U^*_{k+1}(x; z, y_{k-1}, ..,) = v_k(x, z, y_{k-1}, ...) - \int_{\underline{x}}^{z} \ell(\beta^{S2}_{k+1}(y_{k+1}; z, y_{k-1}, ...)) f^{(N-1)}_{k+1}\left(y_{k+1}|Y^{(N-1)}_k = z\right),$$

because in this case bidder i wins for sure in round k. It follows that $\frac{\partial U_k}{\partial z} = 0$ for $z \leq x$ and bidder i has no incentive to bid less than the equilibrium strategy in round k.

Now take the case $z > x$. As shown in (18), by Lemma 1 we have

$$\frac{\partial U^*_{k+1}(x; z, y_{k-1}, ..., y_1)}{\partial x} < \frac{\partial v_k(x, z, y_{k-1}, ...)}{\partial x}.$$

Integrating between x and z, it follows that

$$v_k(z,z,y_{k-1},...) - v_k(x,z,y_{k-1},...) > U^*_{k+1}(z;z,y_{k-1},..,) - U^*_{k+1}(x;z,y_{k-1},..,),$$

and hence that $\frac{\partial U_k}{\partial z} < 0$ for $z > x$; bidder i has no incentive to bid more than the equilibrium strategy in round k. This concludes the proof of the proposition. ■

Appendix B

In this appendix, I compute the bidding functions and price ratios for the example discussed in Section 7.

Recalling that

$$\phi(z) = (1+r)^{\frac{1}{1+r}} z^{\frac{1}{1+r}},$$

we can use Proposition 2 to calculate the bidding functions in the sequential second-price auction:

$$\beta_2^{S2}(x;y_1) = (1+r)^{\frac{1}{1+r}} \left(by_1 + \left(1 + b + (N-3)\frac{a}{a+1}b\right) x \right)^{\frac{1}{1+r}};$$

$$\beta_1^{S2}(x) = (1+r)^{\frac{1}{1+r}} \left(\frac{a\,(N-2)}{a\,(N-2)+1} x + b\left(x + (N-2)\,\frac{a}{a+1} x \right) \right)^{\frac{1}{1+r}}.$$

The expected price in round 2, conditional on the first-round price P_1 is:

$$\begin{aligned}
&E\left[P_2 | P_1 = \beta_1^{S2}(x)\right] \\
&= (1+r)^{\frac{1}{1+r}} E\left[\left(b\frac{a}{a+1}\frac{1-x^{a+1}}{1-x^a} + \left(1+b+(N-3)\frac{a}{a+1}b\right) Y_2^{(N-1)} \right)^{\frac{1}{1+r}} | Y_2^{(N-1)} \leq x \right] \\
&= (1+r)^{\frac{1}{1+r}} \int_0^x \left(b\frac{a}{a+1}\frac{1-x^{a+1}}{1-x^a} + \left(1+b+(N-3)\frac{a}{a+1}b\right) z \right)^{\frac{1}{1+r}} a(N-2)\frac{z^{a(N-2)-1}}{x^{a(N-2)}} dz \\
&= (1+r)^{\frac{1}{1+r}} x^{\frac{1}{1+r}} \int_0^1 \left(b\frac{a}{a+1}\frac{1-x^{a+1}}{x-x^{a+1}} + \left(1+b+(N-3)\frac{a}{a+1}b\right) z \right)^{\frac{1}{1+r}} a(N-2) z^{a(N-2)-1} dz.
\end{aligned}$$

It follows that the ratio of the conditional expected second-round price to the first-round price is:

$$\frac{E\left[P_2|P_1\right]}{P_1} = \frac{\int_0^1 \left(b\frac{a}{a+1}\frac{1-x^{a+1}}{x-x^{a+1}} + \left(1+b+(N-3)\frac{a}{a+1}b\right) z \right)^{\frac{1}{1+r}} a(N-2) z^{a(N-2)-1} dz}{\left(\frac{a(N-2)}{a(N-2)+1} + b\left(1 + (N-2)\,\frac{a}{a+1}\right)\right)^{\frac{1}{1+r}}}. \tag{26}$$

In the case of no informational externalities, that is $b = 0$, this becomes:

$$\frac{E\left[P_2|P_1\right]}{P_1} = \frac{\int_0^1 z^{\frac{1}{1+r}} a(N-2) z^{a(N-2)-1} dz}{\left(\frac{a(N-2)}{a(N-2)+1}\right)^{\frac{1}{1+r}}} = \frac{(a(N-2))^{\frac{r}{1+r}} \left(a\,(N-2)+1\right)^{\frac{1}{1+r}}}{\left(a(N-2) + \frac{1}{1+r}\right)}. \tag{27}$$

If there are informational externalities, $b > 0$, $\frac{E[P_2|P_1]}{P_1}$ depends on the signal x of the first-round price setter. Since x is the value of the second order statistic out of N draws, the expected value of the price ratio is

$$E\left[\frac{E\left[P_2|P_1\right]}{P_1}\right] =$$

$$\int_0^1 \frac{\int_0^1 \left(b\frac{a}{a+1}\frac{1-x^{a+1}}{x-x^{a+1}} + \left(1+\frac{b}{a+1}\left(1+(N-2)a\right)\right)z\right)^{\frac{1}{1+r}} a(N-2)z^{a(N-2)-1}dz}{\left(\frac{a(N-2)}{a(N-2)+1} + b\left(1+(N-2)\frac{a}{a+1}\right)\right)^{\frac{1}{1+r}}} N(N-1)a\left(1-x^a\right)x^{a(N-1)-1}dx.$$

Appendix C

In this appendix, I prove the propositions for the sequential AR English auction and the sequential second-price auction with announcement of the winning prices presented in Section 8. I also present Example 1, which shows that when there are informational externalities and the winning price is revealed in each round, an equilibrium of the sequential second-price auction with an increasing bidding function does not exist. I start with the AR English auction.

Proof of Proposition 6. Given that values are private, in the second round it is a weakly dominant strategy for a type x bidder to enter when called and to stay in if the current price is below the implicit price associated with x. Hence $\beta_2^E(x) = \phi(x)$ and bidding behaviour in the second round does not depend on the information revealed in the first round.[11] Let $\beta_1^E(x; M, y_1^{(M)})$ be the first round exit price (bid) of an active bidder i of type $x \geq y_1^{(M)}$ when M other bidders have either exited or rejected a call to be active and $y_1^{(M)}$ is the highest type among them. (This type is revealed by the bidding behaviour if all bidders in M follow the equilibrium bidding strategy.) Let $f_{1,2}^{(N-2-M)}(y,w)$ be the joint density of the first and second order statistic out of $N-2-M$ bidders. The payoff of a type x active bidder who behaves like a type z when M bidders have already exited is the following

$$\begin{aligned}
U_k(z; x, M, y_1^{(M)}) = &\int_{y_1^{(M)}}^{z}\int_0^{y_A}\left[x-\ell\left(\beta_1\left(y_A; M, y_1^{(M)}\right)\right)\right] f_1^{(N-2-M)}(y)\, f\left(y_A|y_A \geq y_1^{(M)}\right) dy dy_A \\
&+ \int_{y_1^{(M)}}^{x}\int_{y_A}^{1}\int_0^{y_A}\left[x-y_A\right] f_{1,2}^{(N-2-M)}(y,w)\, f\left(y_A|y_A \geq y_1^{(M)}\right) dw dy dy_A \\
&+ \int_z^1\int_{y_1^{(M)}}^{\min\{x,y_A\}}\left[x-y\right] f_1^{(N-2-M)}(y)\, f\left(y_A|y_A \geq y_1^{(M)}\right) dy dy_A \\
&+ \int_z^1\int_0^{y_1^{(M)}}\left[x-y_1^{(M)}\right] f_1^{(N-2-M)}(y)\, f\left(y_A|y_A \geq y_1^{(M)}\right) dy dy_A.
\end{aligned}$$

[11] Harstad and Rothkopf (2000) showed that the equilibrium of the single-item AR English auction with risk neutral bidders differs from the equilibrium of the second-price auction when types are affiliated and there are informational externalities. Their equilibrium reduces to the equilibrium of the second-price auction when there are independent private values.

Differentiating with respect to z we obtain

$$\begin{aligned}
&\int_0^z \left[x - \ell\left(\beta_1\left(z; M, y_1^{(M)}\right)\right)\right] f_1^{(N-2-M)}(y)\, f\left(z|z \geq y_1^{(M)}\right) dy \\
&- \int_{y_1^{(M)}}^{\min\{x,z\}} [x-y]\, f_1^{(N-2-M)}(y)\, f\left(z|z \geq y_1^{(M)}\right) dy \\
&- \int_0^{y_1^{(M)}} \left[x - y_1^{(M)}\right] f_1^{(N-2-M)}(y)\, f\left(z|z \geq y_1^{(M)}\right) dy.
\end{aligned} \tag{28}$$

Equating to zero, setting $z = x$ (as it must be in equilibrium) and simplifying yields

$$0 = \int_0^x \left[\max\left\{y, y_1^{(M)}\right\} - \ell\left(\beta_1\left(x; M, y_1^{(M)}\right)\right)\right] f_1^{(N-2-M)}(y)\, dy,$$

which can be written as

$$\begin{aligned}
\beta_1\left(x; M, y_1^{(M)}\right) &= \phi\left(\int_0^x \max\left\{y, y_1^{(M)}\right\} \frac{f_1^{(N-2-M)}(y)}{F_1^{(N-2-M)}(x)} dy\right) \\
&= \phi\left(E\left[\max\left\{Y_2^{(N-1-M)}, Y_1^{(M)}\right\} | Y_1^{(N-1-M)} = x; Y_1^{(M)} = y_1^{(M)}\right]\right)
\end{aligned}$$

Using this formula, we can see that (28) is zero for $z \leq x$ and negative for $z > x$. This concludes the proof of the proposition. ■

Now consider the sequential second-price auction with announcement of the winning prices.

Proof of Proposition 8. As in a static second-price auction, it is clear that in round K bidding according to the equilibrium strategy is a weakly dominant strategy; a bidder wins if and only if he obtains a positive payoff and the price he pays does not depend on his bid.

Now consider round $k < K$; suppose that all the other bidders follow their equilibrium strategies, as described in the proposition, while bidder i is considering deviating. Suppose first that bidder i of type x is the price setter in round $k-1$ and hence the bidder with the k-th highest signal (this implies that the k-th highest signal among his $N-1$ opponents is less than, or equal to, x). Note first that bidding as a type $z > x$ yields the same payoff as bidding as a type x (he wins for sure). If he deviates in round k (only) and bids as if he were a type $z \leq x$, he either wins in round k, or in round $k+1$; he obtains a payoff

$$\begin{aligned}
x - \int_{\underline{x}}^z E\left[Y_K^{(N-1)} | Y_k^{(N-1)} = y\right] f_k^{(N-1)}\left(y | Y_k^{(N-1)} \leq x\right) dy \\
- \int_z^x \int_{\underline{x}}^y E\left[Y_K^{(N-1)} | Y_{k+1}^{(N-1)} = t\right] f_{k+1}^{(N-1)}\left(t | Y_k^{(N-1)} = y\right) dt f_k^{(N-1)}\left(y | Y_k^{(N-1)} \leq x\right) dy.
\end{aligned}$$

Differentiating with respect to z yields

$$\begin{aligned}
&- E\left[Y_K^{(N-1)} | Y_k^{(N-1)} = z\right] f_k^{(N-1)}\left(z | Y_k^{(N-1)} \leq x\right) \\
&\qquad + \int_{\underline{x}}^z E\left[Y_K^{(N-1)} | Y_{k+1}^{(N-1)} = t\right] f_{k+1}^{(N-1)}\left(t | Y_k^{(N-1)} = z\right) dt f_k^{(N-1)}\left(z | Y_k^{(N-1)} \leq x\right),
\end{aligned}$$

which is equal to zero for all values of z. It follows that type x has no incentive to deviate in round k.

Now consider a type $x < y_k$, the price setter in round $k-1$. If in round k he bids as if he were a type $z < y_k$, then he loses and obtains the same (expected, future) payoff independently of his bid. It follows that he may as well bid as a type x; by equation (8), doing so gives him the (equilibrium) payoff

$$\left\{x - E\left[Y_K^{(N-1)} | Y_K^{(N-1)} < x, Y_k^{(N-1)} = y_k\right]\right\} \Pr\left[Y_K^{(N-1)} < x | Y_k^{(N-1)} = y_k\right], \tag{29}$$

where $\Pr\left[Y_K^{(N-1)} < x | Y_k^{(N-1)} = y_k\right]$ is the probability that $Y_K^{(N-1)} < x$ conditional on $Y_k^{(N-1)} = y_k$. If type x bids in round k as if he were a type $z = y_k$, then he ties with the round k winner and he may as well raise his bid and win for sure, or lower his bid and lose for sure. If he bids above the bid of the y_k type, so that he wins for sure, type x obtains a payoff

$$\begin{aligned}
& x - E\left[Y_K^{(N-1)} | Y_k^{(N-1)} = y_k\right] \\
& = \left\{x - E\left[Y_K^{(N-1)} | Y_K^{(N-1)} < x, Y_k^{(N-1)} = y_k\right]\right\} \Pr\left[Y_K^{(N-1)} < x | Y_k^{(N-1)} = y_k\right] \\
& + \left\{x - E\left[Y_K^{(N-1)} | Y_K^{(N-1)} \geq x, Y_k^{(N-1)} = y_k\right]\right\} \Pr\left[Y_K^{(N-1)} \geq x | Y_k^{(N-1)} = y_k\right] \\
& < \left\{x - E\left[Y_K^{(N-1)} | Y_K^{(N-1)} < x, Y_k^{(N-1)} = y_k\right]\right\} \Pr\left[Y_K^{(N-1)} < x | Y_k^{(N-1)} = y_k\right].
\end{aligned}$$

It follows from (29) that it is not profitable for a bidder of type x to deviate and bid more than a type y_k. This concludes the proof of the proposition. ■

The next example shows that with informational externalities and the winning price being revealed in each round, an equilibrium of the sequential second-price auction with an increasing bidding function does not exist.

Example 1

There are four bidders, three objects, and the common quality of an object is $V = x_1 + x_2 + x_3 + x_4$. Without loss of generality, let $x_1 > x_2 > x_3 > x_4$ (bidders, of course, only know their own signals). Suppose there exists an increasing equilibrium. Then bidder 1 wins the first round and announcing the price reveals x_2, the signal of bidder 2. Suppose $x_3 = x_2 - \varepsilon$, with ε "arbitrarily small". At the beginning of the second round, bidder 3 knows that if he bids according to the equilibrium strategy, then with probability "arbitrarily close" to 1 he will be the price setter in round 2 and win an object in round 3. The price he will pay in round 3 is the bid of bidder 4. Since this is the last round, it is a weakly dominant strategy for bidder 4 to bid $b = \phi\left(E[X_1 | X_1 \geq x_2] + x_2 + x_3 + x_4\right)$ (recall that x_2 and x_3 have been revealed by the price announcements, but x_1 has not). Now consider a deviation by bidder 3 in round 2; suppose he bids zero. Then the price setter in round 2 is bidder 4 and his signal is revealed. In round 3 bidder 4's weakly dominant bid is $\widehat{b} = \phi\left(E[X_1 | X_1 \geq x_2] + x_2 + 2x_4\right)$, since bidder 4 assumes he is pivotal; that is, he assumes $x_3 = x_4$.

After having deviated in round 2, in round 3 bidder 3's weakly dominant strategy is to use a bid-loss equal to the conditional expected value of the object; that is, he will bid $\phi\left(E[X_1|X_1 \geq x_2] + x_2 + x_3 + x_4\right)$. It follows that by deviating bidder 3 will win in the third round and pay a price $\widehat{b}$ which is less than the price b he would pay if he followed the equilibrium strategy. Hence we have a contradiction; bidder 3 of type $x_3 = x_2 - \varepsilon$ has a profitable deviation in round 2 from the supposed increasing equilibrium.

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