Networks and Poisson line patterns: fluctuation asymptotics

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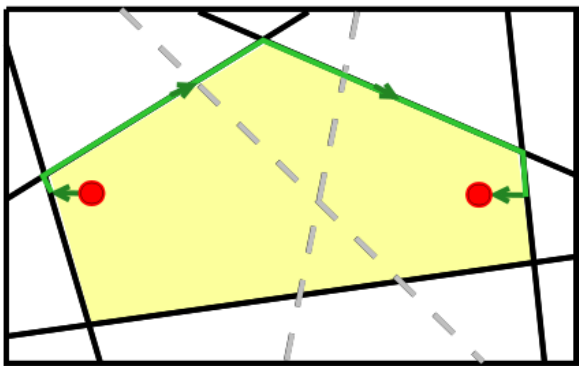
(joint work with David Aldous)

In [1] it was shown how to construct networks connecting arbitrary configurations $\mathbf{x}^n$ of n cities in a square of area n which for example (i) involve only εn more total network length than the Euclidean Steiner tree connecting all n cities, and yet (ii) establish a network connection length between two randomly chosen cities which is on average only $O(\log n)$ more than average Euclidean connection length [1, Theorem 3, version (b)]. Moreover, under a certain quantitative equidistribution condition on the city locations $\mathbf{x}^n$ (which can be phrased either analytically or in terms of a truncated Wasserstein coupling between a randomly chosen city and the uniform distribution on the square), a complementary result shows that for $O(n)$ total connection length the average network connection length must have an excess over the average Euclidean connection length of at least $\Omega(\sqrt{\log n})$ [1, Theorem 5]. The methods of proof involve stochastic geometry: in the case of the lower bound [1, Theorem 5] the idea is to associate the uniform choice of two cities with approximately uniform random lines, and then to use simple ideas from stereology. In the case of the upper bound [1, Theorem 3] one augments the Euclidean Steiner tree by a sparse Poisson line process, to obtain good long-distance communication, and adds an additional relatively infinitesimal amount of additional connectivity, to ensure efficient passage from Steiner tree to line process network.

The key calculation for [1, Theorem 3] concerns construction of an augmented Poisson line network connecting two planar points at $(-\frac{n}{2}, 0)$ and $(\frac{n}{2}, 0)$. Let $\mathcal{C}_n$ be the cell containing the two planar points formed by the tessellation Π_n^* of all lines from a unit intensity Poisson line process Π which do not separate $(-\frac{n}{2}, 0)$ from $(\frac{n}{2}, 0)$. We can consider connecting $(-\frac{n}{2}, 0)$ to $(\frac{n}{2}, 0)$ as in the figure, by first proceeding from $(-\frac{n}{2}, 0)$ towards $(-\infty, 0)$ until the perimeter $\partial\mathcal{C}_n$ is met, then proceeding around $\partial\mathcal{C}_n$ either clockwise or anti-clockwise (according to taste), until encountering the ray from $(n, 0)$ to $(\infty, 0)$, then finally proceeding back to $(\frac{n}{2}, 0)$ along this ray.



Line process properties and Palm distribution theory are used in [1] to express the mean length of the perimeter $\partial\mathcal{C}_n$ as a double integral: analysis shows that this is asymptotic to $2n + \frac{8}{3}(\log n + \gamma + \frac{5}{3}) + o(1)$ (this agrees with similar higher-dimensional results in [2, Theorem 1.3], also compare [6, Satz 5]). The result provides an asymptotic upper bound for the mean network length between $(-\frac{n}{2}, 0)$ and $(\frac{n}{2}, 0)$ in the network formed by augmenting Π by two Exponential(1) random segments required to connect $(-\frac{n}{2}, 0)$ and $(\frac{n}{2}, 0)$ to Π as above.

This augmented Poisson line network has an intrinsic interest, and various natural questions arise. For example, how far will the clockwise path following $\partial\mathcal{C}_n$ deviate laterally from the Euclidean path running directly from $(-\frac{n}{2}, 0)$ to $(\frac{n}{2}, 0)$? and where will the maximum lateral deviation occur? How much random variation in the length of the path should one expect to see? To what extent will the true network geodesic deviate from one of the two paths running clockwise or anti-clockwise around the cell?

The question of vertical deviation is best addressed by using the methods used to evaluate the mean length of $\partial\mathcal{C}_n$. Almost surely a point (x, y) of $\partial\mathcal{C}_n$ of maximal y-coordinate must be an intersection of two lines of Π^*_n, for which one line has positive and the other has negative slope, and for which no further lines of Π^*_n separate (x, y) from $(-\frac{n}{2}, 0)$ and $(\frac{n}{2}, 0)$. Almost surely there is exactly one such point, and Palm distribution arguments then show that its probability density is as follows, for $-\infty < x < \infty$ and $y > 0$:

$$\text{(1)} \quad \frac{1}{4}\left(\sin\alpha + \sin\beta - \sin(\alpha+\beta)\right) \times \\ \times \exp\left(-\tfrac{1}{2}\left(\sqrt{(x-\tfrac{n}{2})^2 + y^2} + \sqrt{(x+\tfrac{n}{2})^2 + y^2} - n\right)\right) \mathrm{d}x\mathrm{d}y\,.$$

Here α $\beta \in (0, \pi)$ are the interior angles at $(-\frac{n}{2}, 0)$, and $(\frac{n}{2}, 0)$ of the triangle formed by (x, y), $(-\frac{n}{2}, 0)$, and $(\frac{n}{2}, 0)$. Using new coordinates $u = \frac{2}{n}x$ and $v = y/\sqrt{n}$, it follows that the limiting density for large n in (u, v) coordinates is given by

$$\text{(2)} \qquad \frac{v^3}{(1-u^2)^2} \exp\left(-\frac{v^2}{1-u^2}\right) \mathrm{d}u\mathrm{d}v\,.$$

Hence asymptotically the point of maximal y-coordinate has x-coordinate distributed uniformly over $(-\frac{n}{2}, \frac{n}{2})$, and has y-coordinate which has conditional distribution the length of a Gaussian 4-vector of zero mean and variance parameter $\frac{n}{2}(1 - \frac{4x^2}{n^2})$.

Questions of random variation can be addressed by reformulating the problem in terms of a simple growth process. Slightly abusing notation, let Π^*_∞ be the tessellation obtained from the Poisson line process by deleting all lines intersecting the positive x-axis. Construct the cell $\mathcal{C}_\infty$ containing the origin formed by Π^*_∞, and consider the path from the origin to $(\infty, 0)$ formed as above, and proceeding clockwise round the cell. The y-coordinate of this path grows to infinity, and we can understand the asymptotic random variation of the length of $\partial\mathcal{C}_n$ by investigating the stochastic dynamics of this growth.

Let the path be parametrized by τ, the excess of arc-length S over x-coordinate X. Let Θ be the angle that the path makes with the positive x-axis, so that $\Theta_0 = \pi$ and Θ decreases with increasing τ. Poisson line process computations show that in τ-time the angle Θ changes at instants of a Poisson point process of intensity $\frac{1}{2}$; moreover the jump in angle $\Theta - \Theta_-$ is such that

$$\text{(3)} \qquad \mathbb{P}\left[\Theta_- - \Theta \leq \phi \mid \Theta_-\right] \quad = \quad \frac{1 - \cos\phi}{1 - \cos\Theta_-} \qquad \text{for } 0 \leq \phi \leq \Theta_- \,.$$

Using Rebolledo's martingale central limit theorem [5] for a compensated version of $-\log\Theta$ as a function of excess τ, and applying $\mathrm{d}\tau = (\sec\Theta - 1)\mathrm{d}X$, we may obtain asymptotic expressions using Brownian motion B. Omitting analytical details which actually require careful attention, the argument runs as follows,

$$\text{(4)} \qquad X_\tau \quad \approx \quad 2\int_0^\tau \exp\left(\tfrac{3}{2}u - \sqrt{7}B_u\right)\mathrm{d}u\,.$$

This leads to the existence of a field of Brownian motions $\{\widetilde{B}^\tau_u : u \geq 0\}$ (related to B by time-reversal) parametrized by τ and such that the excess τ at x-coordinate X satisfies

$$\text{(5)} \qquad \tau \;\approx\; \tfrac{2}{3}\left(\log X_\tau + \sqrt{7}B_\tau - \log 2 - \log\int_0^\infty \exp\left(-\tfrac{3}{2}u + \sqrt{7}\widetilde{B}^\tau_u\right)\mathrm{d}u\right)\,.$$

Hence it follows that the mean excess of τ at distance x is asymptotically $\frac{2}{3}\log x$ (in fact suggesting yet another approach to the asymptotics of [1]) and the excess has random variation of order $O(\sqrt{\log x})$ when $X = x$, agreeing with simulations.

Note that the integral $\int_0^\infty \exp\left(-\frac{3}{2}u + \sqrt{7}\widetilde{B}^\tau_u\right)\mathrm{d}u$ is of Dufresne type, hence is proportional to the reciprocal of a Gamma random variable [4, 7]; see also [3]. However approximation errors will be of the same order as the contribution of this term.

A fully rigorous argument, with applications to the behaviour of true geodesics in the augmented network (the third of the questions mentioned above), will be published elsewhere.

References

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