On embeddings and dimensions of global attractors associated with dissipative partial differential equations

by

Eleonora Pinto de Moura

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Warwick Mathematics Institute

September 2010

THE UNIVERSITY OF WARWICK

Contents

Acknowledgments

First of all I would like to thank my supervisor, Dr. James C. Robinson, for his encouragement and enthusiasm. All the results in this thesis would not have been possible without his guidance and help. I also would like to thank him for his incredible support and generosity with his time and effort.

I would like to thank Dr. Jaime J. Sánchez-Gabites for many enjoyable and useful discussions and for his amazing patience. The results in Chapter 5 of this thesis are joint work with him.

I also would like to thank my examiners Professor Colin Sparrow and Dr. Eric Olson for their insightful commentary and corrections to the manuscript.

I also would like to thank Professor Colin Rourke, Professor José Sanjurjo, Dr. Masoumeh Dashti and Dr. Ricardo Rosa for their help and support.

I would like to thank Tim, Kostas, Jorge, Sofia, Michael D., Carlos, Maité, Zé, Jaime, Masoumeh, Sarah, Michael A., Matthew, Mirela, Mikolaj, Lisa, Umar, Bruno, James B., Azadeh, Andy, Will, Stephen, Nick, Sara, Shengtian, Yang, Ayse and David. You have all made these years at Warwick unforgettable. I also would like to thank Clara, Henrique, Cacá, Clarisse, Rémy, Julia and Daniel for all their love and support during all this years.

I would like specially to thank Tim Hobson for his constant support and belief in me.

Finally I would like to thank my father for all his unwavering support and encouragement. I would like to dedicate this thesis to him.

My research was fully funded by a PhD grant from the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), a Brazilian government agency.

Declarations

I declare that, to the best of my knowledge, the material contained in this thesis is my own original work except where otherwise indicated, cited or commonly known.

The results in Chapter 5 are joint work with Dr. Jaime J. Sánchez-Gabites. The research was done under the supervision of Dr. James C. Robinson during the period of 2006-2010. The material contained in this thesis is submitted to the University of Warwick for the Degree of Doctor of Philosophy and has not been submitted to any other university or for any other degree.

The results in Chapter 2 and Chapter 3 of this thesis have resulted in the papers Pinto de Moura and Robinson (2010a) and Pinto de Moura and Robinson (2010b), respectively. Two papers based in Chapter 4 and 5 have been submitted for publication.

Abstract

Hunt and Kaloshin (1999) proved that it is possible to embed a compact subset X of a Hilbert space with upper box-counting dimension d into $\mathbb{R}^N$ for any $N > 2d+1$, using a linear map L whose inverse is Hölder continuous with exponent $\alpha < (N-2d)/N(1+\tau(X)/2)$, where $\tau(X)$ is the 'thickness exponent' of X. More recently, Ott et al. (2006) conjectured that "many of the attractors associated with the evolution equations of mathematical physics have thickness exponent zero".

In Chapter 2 we study orthogonal sequences in a Hilbert space H, whose elements tend to zero, and similar sequences in the space c_0 of null sequences. These examples are used to show that Hunt and Kaloshin's result, and a related result due to Robinson (2009) for subsets of Banach spaces, are asymptotically sharp. An analogous argument shows that the embedding theorems proved by Robinson (2010), in terms of the Assouad dimension, for the Hilbert and Banach space case are asymptotically sharp.

In Chapter 3 we introduce a variant of the thickness exponent, the Lipschitz deviation $\mathrm{dev}(X)$. We show that Hunt and Kaloshin's result and Corollary 3.9 in Ott et al. (2006) remain true with the thickness replaced by the Lipschitz deviation. We then prove that $\mathrm{dev}(X) = 0$ for the attractors of a wide class of semilinear parabolic equations, thus providing a partial answer to the conjecture of Ott, Hunt, & Kaloshin.

In Chapter 4 we study the regularity of the vector field on the global attractor associated with parabolic equations. We show that certain dissipative equations possess a linear term that is log-Lipschitz continuous on the attractor. We then prove that this property implies that the associated global attractor $\mathcal{A}$ lies within a small neighbourhood of a smooth manifold, given as a Lipschitz graph over a finite number of Fourier modes. This provides an alternative proof that the global attractor $\mathcal{A}$ has zero Lipschitz deviation.

In Chapter 5 we use shape theory and the concept of cellularity to show that if $\mathcal{A}$ is the global attractor associated with a dissipative partial differential equation in a real Hilbert space H and the set $\mathcal{A}-\mathcal{A}$ has finite Assouad dimension d, then there is an ordinary differential equation in $\mathbb{R}^{m+1}$, with $m > d$, that has unique solutions and reproduces the dynamics on $\mathcal{A}$. Moreover, the dynamical system generated by this new ordinary differential equation has a global attractor $\mathcal{X}$ arbitrarily close to $L\mathcal{A}$, where L is a homeomorphism from $\mathcal{A}$ into $\mathbb{R}^{m+1}$.

Chapter 1

Introduction

The main focus of this thesis is to examine the regularity of embeddings of finite-dimensional compact subsets of an infinite-dimensional Hilbert space into a Euclidean space, motivated by applications to the theory of global attractors.

1.1 Semigroups and Global Attractors

There are many interesting partial differential equations whose solutions generate an infinite-dimensional dynamical system. The evolution of such a dynamical system can be described by a continuous semigroup $\{S(t)\}_{t\geq 0}$ of solution operators.

Definition 1.1.1 (Ladyzhenskaya, 1991)**.** *A* continuous semigroup *on a Hilbert space H is a family of continuous linear operators on H, $\{S(t)\}_{t\geq 0}$, satisfying*

(i) $S(0) = I$*,*

(ii) $S(t)S(s) = S(s)S(t) = S(s+t)$*, for all* $t, s \geq 0$*, and such that*

(iii) the mapping given by $(t, x) \mapsto S(t)x$ *from* $[0, \infty] \times H$ *to* H *is continuous.*

In this case, the family of operators is defined by

$$S(t)u_0 = u(t; u_0), \quad \text{for all} \quad t \geq 0,$$

where $u(t; u_0)$ is the solution of the differential equation with initial condition u_0.

A large number of nonlinear dissipative evolution equations have been shown to possess global attractors (see Hale (1988), Ladyzhenskaya (1991), Babin and Vishik (1992), Temam (1997) and Robinson (2001), for more detail).

Definition 1.1.2 (Robinson, 2001)**.** *Let H be a Hilbert space and let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup defined on H. A* global attractor *$\mathcal{A} \subset H$ is a compact invariant set, i.e.*

$$S(t)\mathcal{A} = \mathcal{A} \quad \text{for all} \quad t \geq 0 \tag{1.1}$$

that attracts all bounded sets, i.e.

$$\operatorname{dist}(S(t)X, \mathcal{A}) \to 0 \quad \text{as} \quad t \to \infty, \tag{1.2}$$

for any bounded set $X \subset H$. If the global attractor $\mathcal{A}$ exists, then it is unique.

The distance in (1.2) is the Hausdorff semidistance between two non-empty subsets $X, Y \subset H$,

$$\operatorname{dist}(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|,$$

where $\|\cdot\|$ denotes the norm in H.

It follows from the definition that $\mathcal{A}$ contains all stationary points, all periodic orbits, and all almost-periodic orbits. In fact, the global attractor $\mathcal{A}$ is composed of all complete bounded orbits (see Hale, 1988). One can also characterize global attractors as the union of the omega-limit sets of all bounded sets X, where

$$\begin{aligned} \omega(X) &= \Big\{ y \in X : \; y = \lim S(t_n)x_n, \text{ for some } t_n \to \infty \text{ and } \{x_n\} \in X \Big\} \\ &= \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)X}. \end{aligned}$$

If a system possesses a global attractor, the study of the asymptotic behaviour of the system essentially reduces to the analysis of the dynamics on the attractor, as shown by Langa and Robinson (1999) for example. Global attractors are complicated objects to describe, and usually, as in Hale (1988), the study is focused on their geometric properties as subsets of the infinite-dimensional phase space in which they lie.

The complexity of the flow on the attractor depends in part on the dimension of $\mathcal{A}$. Many dissipative equations, such as the Kuramoto-Sivashinsky and the 2D Navier-Stokes equations, possess finite-dimensional global attractors (see Constantin and Foias (1988), Eden et al. (1994), Temam (1997) and Robinson (2001) for a more detailed study). It is therefore natural to seek a finite-dimensional dynamical system whose asymptotic behaviour reproduces that of the original flow.

A necessary step towards solving this problem relies on proving the existence of an embedding of the global attractor into some Euclidean space with a sufficiently regular inverse. A map ϕ between metric spaces (X,d) and $(Y,\tilde{d})$ is an *embedding* if it is a homeomorphism onto its image. In this thesis we are particularly interested in embeddings of compact subsets of Hilbert and Banach spaces into Euclidean spaces.

1.2 Notions of Dimension

In order to study embeddings of sets with complex geometric structure, such as attractors, it is necessary to introduce precise measures of dimension, such as the Hausdorff and the upper box-counting dimension. In this section we introduce some important notions of dimension for subsets of a Hilbert space H. We restrict ourselves to those dimensions and their properties that are especially useful in the study of the asymptotic behaviour of dynamical systems generated by dissipative partial differential equations.

1.2.1 The Hausdorff dimension

Definition 1.2.1 (Falconer, 1990)**.** *The* Hausdorff dimension *of a set X is defined by*

$$\dim_{\mathrm{H}}(X) = \inf\{d : \mathcal{H}^d(X) = 0\} = \sup\{d : \mathcal{H}^d(X) = \infty\},$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure

$$\mathcal{H}^d(X) = \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=1}^{\infty} |U_i|^d : \{U_i\} \text{ is a } \delta\text{−cover of } X \right\} \right).$$

Here $|U_i| = \sup\{|x-y| : x, y \in U_i\}$ is the diameter of the set U_i and a δ-cover is a cover $\{U_i\}$ such that $|U_i| \le \delta$ for all i. The following general result estimates the effect of general transformations on the Hausdorff dimension of sets.

Proposition 1.2.2 (Falconer, 1990)**.** *Let X be a compact subset of a Hilbert space H. Suppose that $f : X \to H$ is Hölder continuous with exponent η, i.e. there exist $c > 0$ and $\eta > 0$ such that*

$$\|f(x) - f(y)\| \le c\|x-y\|^{\eta}, \quad \text{for all} \quad x, y \in X.$$

Then $\dim_{\mathrm{H}} f(X) \le (1/\eta)\dim_{\mathrm{H}}(X)$.

It follows from Proposition 1.2.2 that if $f : X \to H$ is a bi-Lipschitz transformation, i.e,

$$c_1\|x-y\| \le \|f(x) - f(y)\| \le c_2\|x-y\|, \quad \text{for all} \quad x, y \in X,$$

where $0 < c_1 \leq c_2 < \infty$, then $\dim_{\mathrm{H}} f(X) = \dim_{\mathrm{H}}(X)$.

1.2.2 The upper box-counting dimension

Definition 1.2.3 (Falconer, 1990)**.** *Let X be a compact subset of a Hilbert space H. The* upper box-counting dimension $\dim_{\mathrm{box}}(X)$ *of X is given by*

$$\dim_{\mathrm{box}}(X) = \limsup_{\varepsilon \to 0} \frac{\log \mathcal{N}_X(\varepsilon)}{-\log \varepsilon},$$

where $\mathcal{N}_X(\varepsilon)$ denotes the minimum number of ε-balls in H necessary to cover X.

It follows from the definition that the upper box-counting dimension of X, $\dim_{\mathrm{box}}(X)$, is the infimum over all d for which there exists K such that

$$\mathcal{N}_X(\varepsilon) \leq K(1/\varepsilon)^d, \quad \text{for } 0 < \varepsilon < 1.$$

Roughly speaking, $N_X(\varepsilon)$ grows at the rate of $\varepsilon^{-\dim_{\mathrm{box}}(X)}$ as $\varepsilon \to 0$. More precisely, $\mathcal{N}_X(\varepsilon)\varepsilon^d$ tends to infinity if $d < \dim_{\mathrm{box}}(X)$, and $\mathcal{N}_X(\varepsilon)\varepsilon^d$ tends to zero if $d > \dim_{\mathrm{box}}(X)$.

It is important to note that, for every $X \subset H$, $\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{box}}(X)$. In addition, the upper box-counting dimension shares the following property with the Hausdorff dimension.

Proposition 1.2.4. *Suppose that $f : X \to H$ is Hölder continuous with exponent η, then $\dim_{\mathrm{box}} f(X) \leq (1/\eta) \dim_{\mathrm{box}}(X)$.*

Hence, the upper box-counting dimension, like the Hausdorff dimension, is invariant under bi-Lipschitz transformations. For more information on the Hausdorff and upper box-counting dimension, see Falconer (1990).

1.2.3 The Assouad dimension

Another useful measure, the Assouad dimension, was introduced by Assouad (1983) in the context of bi-Lipschitz embeddings of abstract metric spaces, and generalizes the dimensional order of Bouligand (1928). One may conveniently define this dimension in terms of homogeneous spaces.

Definition 1.2.5. *A metric space (X, d) is said to be (M, s)-*homogeneous *(or simply* homogeneous*) if any ball of radius r can be covered by at most $M(r/\rho)^s$ smaller balls of radius ρ, for some $M \geq 1$ and $s \geq 0$.*

Using this auxiliary notion, one can characterized the Assouad dimension of X, $\dim_{\mathrm{A}}(X)$, similarly to the box-counting dimension.

Definition 1.2.6 (Luukkainen, 1998)**.** *The* Assouad dimension *of* X, $\dim_{\mathrm{A}}(X)$, *is the infimum of all* s *such that* (X, d) *is* (M, s)*-homogeneous, for some* $M \geq 1$.

It is clear that if X is homogeneous then it has finite Assouad dimension. So, as any metric dimension, the Assouad dimension depends on the metric used.

Olson (2002) proved that X has finite Assouad dimension if it satisfies the following *doubling property*.

Lemma 1.2.7 (Olson, 2002)**.** *Let* $\mathcal{N}_X(r, r/2)$ *be the number of* $r/2$*-balls required to cover any* r*-ball in* X*. If there exists* K *such that* $\mathcal{N}_X(r, r/2) < K$ *holds for all* $r < 1$*, then* X *has finite Assouad dimension. Moreover,* $\dim_{\mathrm{A}}(X) \leq \log_2 K$.

Since the Assouad dimension is scale invariant, one may think of it as a measure of the 'size' of a metric space in all scales.

It follows from the definition that $\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{box}}(X) \leq \dim_{\mathrm{A}}(X)$, if (X, d) is a compact metric space. Moreover,

Lemma 1.2.8 (Movahedi-Lankarani, 1992)**.** *The Assouad dimension has the following properties:*

(i) If $X_1 \subset X_2$*, then* $\dim_{\mathrm{A}}(X_1) \leq \dim_{\mathrm{A}}(X_2)$.

(ii) If X *is an open subset of* $\mathbb{R}^n$*, then* $\dim_{\mathrm{A}}(X) = n$.

(iii) If (X_1, d_1) *and* (X_2, d_2) *are bi-Lipschitz isomorphic, then* $\dim_{\mathrm{A}}(X_1) = \dim_{\mathrm{A}}(X_2)$.

It follows from (ii) and (iii) that a metric space (X, d) has necessarily finite Assouad dimension if there exists a bi-Lipschitz embedding of (X, d) into a Euclidean space. However, the converse is not true (see Lang and Plaut (2001) or Heinonen (2001) for discussion). In fact, Lipschitz maps, even if they are homeomorphisms, can raise the Assouad dimension. In section 2.1.2, we discuss some properties of the Assouad dimension of orthogonal sequences. For a comprehensive treatment of the Assouad dimension see Luukkainen (1998) and Olson (2002).

1.3 Embeddings of finite-dimensional sets

Motivated by the theory of infinite-dimensional dynamical systems, the regularity of embeddings of finite-dimensional sets into Euclidean spaces has been studied in various papers. Mañé (1981) showed that if X is a compact subset of a Banach space with upper box-counting dimension d, then for any $n \geq 2d+1$ a generic set of projections of X onto any n-dimensional subspace is injective on X. Considerable work on orthogonal projections followed this result. Ben-Artzi et al. (1993) proved that given a compact subset $X \subset \mathbb{R}^N$ with finite upper box-counting dimension, there exists an orthogonal projection P such that P is injective on X and its inverse is Hölder continuous when restricted to $P(X)$. Furthermore, they established a sharp upper bound on the Hölder exponent using an approach that will be adopted in Chapter 2. Eden et al. (1994) gave a constructive proof of the existence of such projections in a Hilbert space and showed again that the inverse is Hölder continuous if $X \subset \mathbb{R}^N$. Foias and Olson (1996) showed the existence of a dense set of projections with Hölder continuous inverse when X is a compact subset of an infinite-dimensional Hilbert space, but did not obtain a bound for the Hölder exponent.

The most powerful current results are based on an argument due to Hunt and Kaloshin (1999), who provided an explicit bound on the Hölder exponent. Their theorem is expressed in terms of prevalence[1], which generalizes the notion of 'Lebesgue almost every' from finite to infinite-dimensional spaces.

Definition 1.3.1 (Hunt et al., 1992). *Let V be a normed linear space. A Borel subset $S \subset V$ is* prevalent *if there exists a compactly supported probability measure μ on V such that $\mu(S + x) = 1$, for all $x \in V$. In particular, if S is prevalent then S is dense in V.*

In other words, a subset $S \subset V$ is *prevalent* if there exists a compact $Q \subset V$ supporting a probability measure μ such that $x + \pi \in S$, for μ-almost every $\pi \in Q$ and for all $x \in V$. In practice, one can think of Q as a probe set consisting of perturbations π, and prove the prevalence of S by constructing Q in such a way that $\mu\big(Q \setminus (S + x)\big) = 0$, for all $x \in V$.

As a component of the proof of their embedding result, Hunt and Kaloshin (1999) introduced the 'thickness exponent', a quantity that measures how well a compact set $X \subset H$ can be approximated by finite-dimensional subspaces.

Definition 1.3.2 (Hunt and Kaloshin, 1999). *Let X be a compact subset of a Hilbert space H. For $\varepsilon > 0$, let $\mathrm{d}_H(X, \varepsilon)$ be the minimum dimension of all finite-dimensional*

[1] The term prevalence was coined by Hunt, Sauer and Yorke, who introduced the concept in their 1992 paper. Essentially the same notion was used earlier by Christensen (1973) in a study of the differentiability of Lipschitz mappings between infinite-dimensional spaces. Let G be a Abelian group with a complete separable metric. Christensen (1973) define a Borel set A to be a *Haar null set* if there is a Borel probability measure μ on G such that $\mu(A + x) = 0$ for every $x \in G$. Hence, a *prevalent set* is a set whose complement is a Haar null set. See Benyamini and Lindenstrauss (1973) for more details.

subspaces $U \subset H$ such that every point of X lies within a distance ε of U; if no such U exists, then $\mathrm{d}_H(X,\varepsilon)=\infty$. *The* thickness exponent $\tau(X)$ *of X is defined as*

$$\tau(X)=\limsup_{\varepsilon\to 0}\frac{\log \mathrm{d}_H(X,\varepsilon)}{-\log\varepsilon}.$$

The following lemma shows that the thickness exponent is bounded above by the upper box-counting dimension.

Lemma 1.3.3 (Lemma 3.5 in Hunt and Kaloshin, 1999)**.** *Let $X \subset H$ be a compact set with upper box-counting dimension* $\dim_{\text{box}}(X)$. *Then* $\tau(X) \leq \dim_{\text{box}}(X)$.

Proof. If one covers X by $\mathcal{N}(X,\varepsilon)$ balls of radius ε, X clearly lies within ε of the linear subspace spanned by the centres of these balls. Hence, $\mathrm{d}_H(X,\varepsilon)\leq\mathcal{N}(X,\varepsilon)$. It follows from this inequality that $\tau(X)\leq\dim_{\text{box}}(X)$ as claimed. □

Using the concept of the thickness exponent, Hunt and Kaloshin (1999) established the following abstract result concerning general embeddings of compact subsets of a Hilbert space into a Euclidean space of sufficiently large dimension. Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and a scalar product $(\cdot,\cdot)$. We will write $|\cdot|$ for the norm on any Euclidean space.

Theorem 1.3.4 (Hunt and Kaloshin, 1999)**.** *Let X be a compact subset of a real Hilbert space H with upper box-counting dimension* $\dim_{\text{box}}(X)=d$ *and let $\tau(X)$ be the thickness exponent of X. Let $N>2d$ be an integer and let ζ be a real number with*

$$0<\zeta<\frac{N-2d}{N(1+\tau(X)/2)}. \tag{1.3}$$

Then for a prevalent set of bounded linear maps $L: H\to\mathbb{R}^N$ there exists a $C_L>0$ such that

$$C_L|L(x)-L(y)|^{\zeta}\geq\|x-y\| \quad \text{for all} \quad x,y\in X. \tag{1.4}$$

In particular, these maps are injective on X.

Note that the thickness exponent is used to bound explicitly the Hölder exponent of the inverse of L restricted to the image of X. The regularity of the inverse of such a linear map is important for understanding how it may distort a compact set, despite being injective on it.

Inspired by the work of Hunt and Kaloshin (1999), Olson and Robinson (2010) introduced the notion of the *m-Lipschitz deviation.* This quantity provides a measure of the extent to which a compact subset $X\subset H$ can be approximated by graphs of Lipschitz functions defined over finite-dimensional subspaces of H. In Chapter 3 we will discuss this notion in more detail.

An analogous result to Theorem 1.3.4 is also true for Banach spaces. In order to study the embeddings of finite-dimensional subsets of a Banach space B, Robinson (2009) introduced the concept of the dual thickness exponent $\tau^*(X)$ of a compact subset $X \subset B$, for which one can show that $\tau^*(X) \leq \dim_{\text{box}}(X)$.

Definition 1.3.5 (Robinson, 2009)**.** *Given $\theta > 0$, let $n_\theta(X, \varepsilon)$ denote the lowest dimension of any linear subspace V of B^* such that for any $x, y \in X$ with $\|x-y\| \geq \varepsilon$ there exists an element $\psi \in V$ such that $\|\psi\| = 1$ and*

$$|\psi(x-y)| \geq \varepsilon^{1+\theta}.$$

Set

$$\tau^*_\theta(X) = \limsup_{\varepsilon \to 0} \frac{\log n_\theta(X, \varepsilon)}{-\log \varepsilon},$$

and define the dual thickness $\tau^*(X)$ by

$$\tau^*(X) = \lim_{\theta \to 0} \tau^*_\theta(X).$$

Using the dual thickness, Robinson (2009) proved the following embedding theorem for subsets of Banach spaces.

Theorem 1.3.6 (Robinson, 2009)**.** *Let X be a compact subset of a Banach space B with finite upper box-counting dimension $\dim_{\text{box}}(X) = d$ and $\tau^*(X) = \tau^*$. Then for any integer $N > 2d$ and any θ with*

$$0 < \theta < \frac{N - 2d}{N(1 + \tau^*)} \tag{1.5}$$

there exists a prevalent set of bounded linear maps $L : B \to \mathbb{R}^N$ such that, for some $c_L > 0$.

$$c_L |L(x) - L(y)|^\theta \geq \|x - y\| \quad \text{for all} \quad x, y \in X. \tag{1.6}$$

In particular, these maps are injective on X.

In order to show that the limiting Hölder exponent in (1.3) is sharp, Hunt and Kaloshin (1999) considered a infinite binary tree and associated each node to the corresponding element of the standard basis of ℓ^2 with their indices written in base 2. Then, they constructed an intricate example consisting of certain linear combinations of those basis elements corresponding to certain infinite branches of the binary tree. In Section 2.1.1, we show that the class of orthogonal sequences in a Hilbert space H, whose elements tend to zero, as considered by Ben-Artzi et al. (1993), have thickness exponent equal to their box-counting dimension. This result provides a much simpler example than that given by Hunt and Kaloshin to demonstrate that the bound (1.3) on the Hölder exponent in Theorem 1.3.4 is sharp as $N \to \infty$. In Section 2.2, we study a general set of sequences in c_0 that enables us to use methods derived from Ben-Artzi et al. (1993) in the Banach space case. We then apply the results obtained to show that (1.5) is sharp.

More recently, Ott et al. (2006) also utilized the thickness exponent to show how the Hausdorff dimension of a subset of an infinite-dimensional space is affected by mappings into finite-dimensional spaces. It follows from standard properties of the Hausdorff dimension (see Falconer (1990) or Section 1.2.1) that for any bounded linear map L that satisfies the inequality (1.4),

$$\frac{N-2d}{N(1+\tau(X)/2)}\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{H}}\big(L(X)\big) \leq \dim_{\mathrm{H}}(X).$$

But more is true, as Ott et al. (2006) showed:

Theorem 1.3.7 (Ott et al., 2006)**.** *Let H be a real Hilbert space. Let $X \subset H$ be a compact set with thickness $\tau(X)$. For a prevalent set of linear maps $L : H \to \mathbb{R}^N$,*

$$\dim_{\mathrm{H}}\big(L(X)\big) \geq \min\left\{N, \frac{\dim_{\mathrm{H}}(X)}{1+\tau(X)/2}\right\}.$$

Note that in both Theorem 1.3.4 and Theorem 1.3.7, sets with zero thickness have a special place. In the first result, when $\tau(X) = 0$ one can make the Hölder exponent of the embedding as close to one as required by taking N sufficiently large; while in the second, the Hausdorff dimension of sets with zero thickness is preserved by 'most' linear maps. It is therefore a natural question when one can show that a set X has zero thickness exponent.

Much of the interest in the above results comes from their application to the attractors of infinite-dimensional dynamical systems associated to certain dissipative partial differential equations. It is known that the thickness exponent of many attractors of infinite-dimensional systems is significantly smaller than their box-counting dimension: Friz and Robinson (1999) proved, in particular, that if an attractor is uniformly bounded in the Sobolev space $H^s(\Omega)$, with $s > 0$ and Ω a sufficiently smooth bounded domain in $\mathbb{R}^n$, then its thickness in $L^2(\Omega)$ is at most n/s. Ott et al. (2006) (see also Ott and Yorke, 2005) conjecture that, under certain dynamical hypotheses, the attractor of a sufficiently dissipative and smooth flow has zero thickness.

We are not aware of any explicit examples of attractors of natural models that can be proved to have non-zero thickness, but there is also no proof currently available that 'many attractors' do have zero thickness exponent. The result of Friz and Robinson (1999) implies that any global attractor of a PDE that is bounded in $H^1(\Omega)$, where Ω is a d-dimensional domain, has thickness exponent (in $L^2(\Omega)$) no larger than d. But this result can only guarantee that global attractors have zero thickness when they are uniformly bounded in the Sobolev spaces H^s for all $s > 0$.

Let us consider the 2D Navier–Stokes equations as an illustrative example; we write the equations in their functional form

$$\mathrm{d}u/\mathrm{d}t + \nu Au + B(u, u) = f,$$

see Constantin and Foias (1988), for example. It is known (Temam, 1997) that if the forcing function f is 'smooth', i.e. $f \in H^k$ for any k, then its attractor is also smooth, and so has zero thickness. But one can easily construct a forcing $f \in L^2$ for which the attractor is not bounded in H^3, by choosing any $u^* \in H^2 \setminus H^3$ and setting

$$f = \nu Au^* + B(u^*, u^*).$$

For such a forcing, u^* is a stationary solution of the equation, and so must be an element of the attractor. Since $u^* \notin H^3$, the attractor cannot be bounded in H^3. This does not demonstrate that the corresponding attractor has non-zero thickness, but does show that one cannot use the idea that 'thickness is inversely proportional to smoothness' to prove a satisfactory general result in this context.

In Chapter 3 we prove a version of the conjecture of Ott, Hunt, and Kaloshin. In Section 3.1 we define a new quantity, based on ideas in Olson and Robinson (2010), called the Lipschitz deviation, and show that Theorems 1.3.4 and 1.3.7 remain true with the thickness replaced by the Lipschitz deviation. Then, in Section 3.2, we show that the global attractors of a large class of semilinear parabolic equations have zero Lipschitz deviation: in particular, if one considers the two-dimensional Navier-Stokes equations, our results imply that if $f \in L^2$ the attractor has zero Lipschitz deviation (as remarked above, it is only known that the attractor has zero thickness if $f \in H^s$ for every s).

The proof of this result uses ideas from the theory of approximate inertial manifolds (Foias et al., 1988a), and relies on the fact, proved by Eden et al. (1994), that the solutions of these semilinear equations satisfy the geometric 'squeezing property', introduced by Foias and Temam (1979). (In fact we show that the class of equations we consider has a family of approximate inertial manifolds of 'exponential order', cf. Debussche and Temam (1994) and Rosa (1995)).

A related approach to the study of the regularity of embeddings into a Euclidean space is to consider those conditions under which a metric space (X, d) admits a bi-Lipschitz embedding into a Euclidean space. Since the Assouad dimension is preserved under bi-Lipschitz mappings and is finite for subsets of Euclidean spaces, $\dim_{\mathrm{A}}(X) < \infty$ is a necessary condition for a subset $X \subset H$ to be embedded in a bi-Lipschitz way into $\mathbb{R}^N$. Hence, if $\mathcal{A}$ is a global attractor of a dynamical system possessing an inertial manifold, which we define in Section 3.2.1, then its Assouad dimension must be finite. Therefore many dissipative equations, such as the Kuramoto-Sivashinsky equation, have global attractors with finite Assouad dimension. However, there is still no general method to bound the Assouad dimension of global attractors associated with dissipative equations.

Movahedi-Lankarani (1992) was the first to introduce the Assouad dimension in the study of projections. In order to obtain an example of a set X with finite fractal dimension which cannot be bi-Lipschitz embedded into any Euclidean space, he constructed a set X with infinite Assouad dimension. Therefore, if one is interested in whether X may be nicely embedded into a finite-dimensional space, it seems reasonable to investigate what happens if X has finite Assouad dimension.

Indeed, Olson and Robinson (2010) used this dimension to obtain greater regularity for the inverse of the embedding map.

Theorem 1.3.8 (Olson and Robinson, 2010). *Let X be a compact subset of a real Hilbert space H such that the set $X - X$ of differences between elements of X has Assouad dimension* $\dim_{\mathrm{A}}(X - X) < s < N$*, where $N \in \mathbb{N}$. If*

$$\gamma > \frac{2 + 3N}{2(N - s)}, \tag{1.7}$$

then there exists a prevalent set of bounded linear maps $\pi : H \to \mathbb{R}^N$ that are injective on X and γ-almost bi-Lipschitz, that is, there exist $\delta_\pi > 0$, $c_\pi > 0$ such that

$$\frac{1}{c_\pi} \frac{\|u - v\|}{(-\log \|u - v\|)^\gamma} \leq |\pi(u) - \pi(v)| \leq c_\pi \|u - v\|, \tag{1.8}$$

for all $u, v \in X$ with $\|u - v\| \leq \delta_\pi$.

Note that, for any $\gamma > 3/2$ we can choose N large enough to obtain a γ-almost bi-Lipschitz embedding into $\mathbb{R}^N$. However, the exponent $\gamma > 3/2$ is not asymptotically sharp. Moreover, it is important to remark that, although reasonable, the hypothesis $\dim_{\mathrm{A}}(X - X) < \infty$ is quite restrictive. Olson (2002), for instance, gave an example of a set X for which $\dim_{\mathrm{A}}(X) = 0$, but $\dim_{\mathrm{A}}(X - X) = \infty$.

In order to improve the exponent γ, Robinson (2010) used a result due to Ball (1986), concerning the maximum volume of hyperplane slices of the unit cube, and obtained the following result.

Theorem 1.3.9 (Robinson, 2010)**.** *Let X be a compact subset of a real Hilbert space H such that* $\dim_{\mathrm{A}}(X - X) < s < N$*, where $N \in \mathbb{N}$. If*

$$\gamma > \frac{2+N}{2(N-s)}, \tag{1.9}$$

then there exists a prevalent set of bounded linear maps $\pi : H \to \mathbb{R}^N$ that are injective on X and γ-almost bi-Lipschitz, i.e. there exist $\delta_\pi > 0$, $c_\pi > 0$ such that

$$\frac{1}{c_\pi}\frac{\|u-v\|}{(-\log\|u-v\|)^\gamma} \leq |\pi(u) - \pi(v)| \leq c_\pi\|u-v\|, \tag{1.10}$$

for all $u, v \in X$ with $\|u - v\| \leq \delta_\pi$.

Note that, for any $\gamma > 1/2$ we can choose N large enough to obtain a γ-almost bi-Lipschitz embedding into $\mathbb{R}^N$. Hence, using the Assouad dimension, Robinson (2010) was able to obtain more regularity for the inverse of embeddings of finite-dimensional sets into Euclidean spaces. In section 2.1.3, we present an example of an orthogonal sequence in a Hilbert space H, whose norm decays exponentially, for which $\gamma > 1/2$. Therefore, the exponent of the logarithmic correction term γ in Theorem 1.3.9 is sharp as $N \to \infty$.

An analogous result to Theorem 1.3.9 is also true for Banach spaces.

Theorem 1.3.10 (Robinson, 2010)**.** *Let X be a compact subset of a Banach space B such that* $\dim_{\mathrm{A}}(X - X) < s < N$*, where $N \in \mathbb{N}$. If*

$$\gamma > \frac{N+1}{N-s}, \tag{1.11}$$

then there exists a prevalent set of bounded linear maps $\pi : B \to \mathbb{R}^N$ that are injective on X and γ-almost bi-Lipschitz, i.e. there exist $\delta_\pi > 0$, $c_\pi > 0$ such that

$$\frac{1}{c_\pi}\frac{\|u-v\|}{(-\log\|u-v\|)^\gamma} \leq |\pi(u) - \pi(v)| \leq c_\pi\|u-v\|, \tag{1.12}$$

for all $u, v \in X$ with $\|u - v\| \leq \delta_\pi$.

Note that the limiting value of γ as $N \to \infty$ is strictly greater than one (see Robinson (2010) for more details). In section 2.2, we present an example that shows that the limiting value $\gamma > 1$ is also sharp.

1.4 Finite-dimensional asymptotic dynamics

The existence of global attractors with finite upper box-counting dimension for a wide class of dissipative equations (see Babin and Vishik (1992), Foias and Temam (1979), Hale (1988), Temam (1997), for example) strongly suggests that it might be possible to construct a system of ordinary differential equations whose asymptotic dynamics reproduces the dynamics on the original attractor $\mathcal{A}$. However, because of the complexity of the flow on the attractor $\mathcal{A}$ and its irregular structure, the finite dimensionality of $\mathcal{A}$ alone is not immediately sufficient to guarantee the existence of such a system of ordinary differential equations.

To illustrate the problem, consider a dissipative differential equation written as an abstract evolution equation of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{G}(u), \quad u \in H, \tag{1.13}$$

defined on a real separable Hilbert space H and associated with a global attractor $\mathcal{A}$. Ideally, one would like to construct a finite-dimensional system of ordinary differential equations in some $\mathbb{R}^N$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{H}(x), \quad x \in \mathbb{R}^N, \tag{1.14}$$

such that

(i) the attractor $\mathcal{A}$ would be embedded in $\mathbb{R}^N$ via some homeomorphism $L : \mathcal{A} \longrightarrow L\mathcal{A} \subseteq \mathbb{R}^N$,

(ii) the solutions of the finite-dimensional system (1.14) would be unique,

(iii) the dynamics of (1.14) on $L\mathcal{A}$ would reproduce those of (1.13) on $\mathcal{A}$, i.e. $\mathcal{H}(x) = L\mathcal{G}(L^{-1}x)$, for every $x \in L\mathcal{A}$, and

(iv) the set $L\mathcal{A}$ would be the global attractor for (1.14).

Indeed, the existence of such a system of ordinary differential equations with analogous asymptotic dynamics has only been proved for dissipative partial differential equations that possess an inertial manifold. Introduced by Foias et al. (1985), inertial manifolds are positively invariant finite-dimensional Lipschitz manifold that contain the global attractor and attract all orbits exponentially (see Constantin and Foias (1988), Constantin et al. (1988), Foias et al. (1988a), Foias et al. (1988b), Temam (1997), for more details). All the methods available in the literature construct inertial manifolds as graphs of functions from a finite-dimensional eigenspace associated with the low Fourier modes into the complementary infinite-dimensional eigenspace corresponding to the high Fourier modes.

Foias et al. (1988b) showed that if a 'certain spectral gap condition' holds for a given system, then it will possesses an inertial manifold. Unfortunately, this sufficient condition is quite restrictive, and there are many equations, such as the 2D Navier-Stokes equations, that do not satisfy it. Nonetheless, Kukavica (2003) and Kukavica (2005) showed that the global attractor of certain dissipative equations, such as the Burgers equation in one space dimension, lies in a Lipschitz graph over a finite number of Fourier modes independently of the theory of inertial manifolds.

In the cases in which an inertial manifold has not been shown to exist, other approaches have been explored to reconstruct the dynamics on the attractor within a finite-dimensional system. They usually involve embedding the attractor $\mathcal{A}$ of an evolution equation and its dynamics into a Euclidean space of sufficiently high dimension and then extending the vector field to the whole space, in such a way that the image of $\mathcal{A}$ is the global attractor of the new system. For example, Eden et al. (1994) projected the original dissipative partial differential equation and obtained a finite system of ordinary differential equations which reproduce the dynamics on the attractor $\mathcal{A}$. The new system has as its global attractor the projection $P\mathcal{A}$ of $\mathcal{A}$. However, due to the lack of regularity of the vector field on $P\mathcal{A}$ (which need not even to be continuous), there is no guarantee of uniqueness of the solutions of the finite-dimensional system.

Romanov (2000) discussed the problem of a finite-dimensional description of the asymptotic behaviour of dissipative equations more abstractly. He defined the dynamics on the attractor $\mathcal{A}$ to be 'finite-dimensional' if there exists a bi-Lipschitz map $\Pi : \mathcal{A} \to \mathbb{R}^N$, for some N, and an ordinary differential equation with a Lipschitz vector field on $\mathbb{R}^N$ such that the dynamics on $\mathcal{A}$ and $\Pi(\mathcal{A})$ are conjugated under Π. He then showed that this property is equivalent to the attractor being contained in a finite-dimensional Lipschitz manifold, given as a graph over a sufficiently large number of Fourier modes. Hence, his definition and that of an inertial manifold are much more similar than they first appear. In Section 4.2, we investigate other possible ways to define when the asymptotic dynamics of solutions of parabolic equations are 'finite-dimensional'. We discuss conditions under which an attractor is a subset of a Lipschitz manifold given as a graph over a finite-dimensional space; in particular, we give a concise proof of an important part of Romanov's result.

Now suppose there exists a linear map $L : H \to \mathbb{R}^N$ that is injective on $\mathcal{A}$.

In order to study the smoothness of the embedded equation on $L\mathcal{A}$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = h(x) = L\mathcal{G}L^{-1}(x), \quad x \in L\mathcal{A}, \tag{1.15}$$

one needs to consider the continuity of the vector field on $\mathcal{A}$ and the continuity of the inverse of the embedding L restricted to $L\mathcal{A}$. If one would like a system of ordinary differential equations with unique solutions that generates a flow $\{S_t\}_{t\geq 0}$, then the embedded vector field h in $L\mathcal{A}$ does not need to be Lipschitz; it is sufficient for h to be 1-log-Lipschitz[2]. Hence, one needs to show that there exist

(i) an exponent $\eta > 0$ such that the vector field on the attractor $\mathcal{A}$ is η-log-Lipschitz in H, and

(ii) an exponent $\gamma > 0$ such the inverse of linear embedding $L : H \to \mathbb{R}^N$ is γ-log-Lipschitz when restricted to $L\mathcal{A}$,

for which the inequality $\eta + \gamma \leq 1$ holds so that the solutions are unique. It is, therefore, reasonable to consider separately the problem of the regularity of the vector field $\mathcal{G}$ on the global attractor $\mathcal{A}$ associated with certain parabolic equations and the one of the smoothness of the inverse of linear embeddings L restricted to $L\mathcal{A}$.

In Chapter 4, we will focus our discussion on the regularity of the vector field $\mathcal{G}$ in (1.13). If we assume the very strong condition that L is a bi-Lipschitz embedding, then we would only need the vector field to be 1-log-Lipschitz to guarantee existence and uniqueness of solutions of the embedded equation. In Section 4.3, we specifically show that the linear operator $A : H \to H$ from the 2D Navier-Stokes equations is 1-log-Lipschitz continuous using methods developed by Kukavica (2007).

In Section 4.4, we prove that the 1-log-Lipschitz continuity of the linear term implies that there exists a family of Lipschitz manifolds $\mathcal{M}_N$ such that the distance between the N-dimensional manifold $\mathcal{M}_N$ and the attractor $\mathcal{A}$ is exponentially small in N. We then apply the methods developed in Section 3.2 to show that, for certain dissipative equations, one can then make the Hölder exponent in (1.4) as close to one as required by taking N sufficiently large.

In Chapter 5, we continue to discuss the problem of a finite-dimensional description of the asymptotic dynamics of dissipative equations such as (1.13), but now focussing on the consequences of the existence of bounded linear embeddings

[2]It is, then, possible to extend $h : L\mathcal{A} \to \mathbb{R}^N$ to 1-log-Lipschitz function $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}^N$ (see McShane (1934) for details)

$L|_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbb{R}^N$ with a sufficiently regular inverse (namely, Lipschitz with a logarithmic correction). In Section 5.1, we construct a system of ordinary differential equations in $\mathbb{R}^N$ that reproduces the dynamics on $\mathcal{A}$ in $L\mathcal{A}$ without discussing whether the new system possess a global attractor. We then use the regularity of L^{-1} restricted to $L\mathcal{A}$ to guarantee uniqueness of solutions for the system of ordinary differential equations.

Ideally, one would like to construct a smooth system of equations that generates a dynamical system with a prescribed global attractor possessing non-trivial dynamics. Garay (1991) recognized cellularity[3] as a distinctive property of attractors for flows and showed that, given a cellular compact subset of a Euclidean space, there exists a dynamical system whose attractor is the given set. However, the dynamics on this compact set is trivial. Garay's proof was based on a paper of McCoy (1973) on cells and cellularity of normed linear spaces of infinite dimension and on results from infinite-dimensional topology, in particular from shape theory. Introduced by Borsuk (1975), shape theory is a powerful tool in the study of topological dynamics (see Günther and Segal (1993), Günther (1995), Sanjurjo (1995), Robinson (1999)).

It is important to note that Theorem 2.7 in Garay (1991) does not guarantee, as would be desirable, the existence of a smooth system of ordinary differential equations whose attractor is the image of the original attractor, but only of a flow. The existence of such a system of ordinary differential equations was proved by Günther (1995), generalising a result obtained by Günther and Segal (1993). Using smoothing results from piecewise linear topology to replace general flows by flows arising from differential equations, Günther (1995) showed that, for any finite-dimensional compact set $\mathcal{A}$ with the shape of a finite polyhedron, there is a differentiable flow in a finite-dimensional Euclidean space with an attractor homeomorphic to the given set. However, once again the flow constructed is stationary on the attractor. Therefore, the general question of whether the dynamics on the global attractor $\mathcal{A}$ may be embedded into a finite-dimensional system of ordinary differential equations remains an open problem. It is nevertheless clear that, in order to preserve the original dynamics, certain continuity properties of the embedding L are crucial.

Following Garay (1991) and Günther (1995), we exploit in Section 5.2 certain topological properties of the global attractor $\mathcal{A}$ to construct a system of ordinary differential equations with the prescribed compact set $L\mathcal{A} \times \{0\}$ as global attractor. However, the dynamics on $L\mathcal{A} \times \{0\}$ is trivial. In Section 5.3 we combine our previous results in such a way that we obtain a new dynamical system generated by

[3]Cellularity and other topological properties are defined in Section 5.2.1 and in Appendix B.

an ordinary differential equation that has a global attractor $\mathcal{X}$ arbitrarily close to $L\mathcal{A} \times \{0\}$.

We conclude this introduction by remarking on two outstanding problems. First, in contrast to the Hausdorff and box-counting dimensions, there are no general techniques to bound the Assouad dimension of invariant sets of infinite-dimensional dynamical systems. And, finally, the smoothness of the original vector field on the attractor $\mathcal{A}$ is, in general, known only to be 1-log-Lipschitz. The question of whether the vector field is in fact η-log-Lipschitz with $\eta < 1/2$ remains open. Such regularity would, when combined with the sharp bound $\gamma > 1/2$ given by Theorem 1.3.9, guarantee that $\gamma + \eta \leq 1$, which implies the existence of unique solutions to the ODE on the projected attractor.

Chapter 2

Orthogonal sequences and regularity of embeddings into finite-dimensional spaces

2.1 Orthogonal Sequences in Hilbert Spaces

A general class of orthogonal sequences $\mathcal{A}$ in a Hilbert space H, whose elements tend to zero, was considered by Ben-Artzi et al. (1993) to show that the existence of a finite rank projection P and a constant $C > 0$ such that

$$\|x - y\| \leq C\|Px - Py\|^{\zeta}, \quad \text{for all} \quad x, y \in \mathcal{A},$$

impose a restriction on the Hölder exponent ζ:

$$\zeta \leq (1 + \dim_{\text{box}}(\mathcal{A})/2)^{-1}.$$

In this section we consider the same class of orthogonal sequences in a real separable Hilbert space H in order to show that the upper bound on the Hölder exponent in Hunt and Kaloshin's embedding (Theorem 1.3.4) is sharp as $N \to \infty$. Let $\{a_n : n = 1, 2, ...\}$ be an orthogonal sequence in $H \setminus \{0\}$ with $\lim_{n\to\infty} \|a_n\| = 0$. Throughout this chapter we consider $\mathcal{A}$ to be the compact set

$$\mathcal{A} = \{a_1, a_2, ...\} \cup \{0\} \tag{2.1}$$

and assume without loss of generality that $\|a_n\| \geq \|a_{n+1}\| > 0$, for $n = 1, 2, ...$ Next we shall study the thickness and the Assouad dimension of $\mathcal{A}$. The upper box-counting dimension of such sets $\mathcal{A}$ has already been discussed in Ben-Artzi et al. (1993).

2.1.1 Hölder exponent when $\dim_{\text{box}}(\mathcal{A}) < \infty$

In their Lemma 3.1, Ben-Artzi et al. (1993) showed that the upper box-counting dimension of $\mathcal{A}$, $\dim_{\text{box}}(\mathcal{A})$, is given by

$$\dim_{\text{box}}(\mathcal{A}) = \limsup_{n\to\infty} \frac{\log n}{\log(1/\|a_n\|)} = \inf\left\{\nu : \sum_{n=1}^{\infty} \|a_n\|^{\nu} < \infty\right\}. \tag{2.2}$$

In order to show that the thickness exponent of $\mathcal{A}$ is given by the same expression, we use the following lemma. This result, due to M. Doré (personal communication), gives an explicit lower bound on $\mathrm{d}_H(X,\varepsilon)$ for finite-dimensional subspaces X that approximate orthogonal sets in Hilbert spaces.

Lemma 2.1.1. *Let $X = \{v_1, \ldots, v_n\}$ be an orthogonal set of nonzero vectors in a Hilbert space H. Then*

$$\mathrm{d}_H(X,\varepsilon) \geq n(1 - \varepsilon^2/M^2),$$

where $M = \min\{\|v_1\|, \ldots, \|v_n\|\}$ and $\mathrm{d}_H(X,\varepsilon)$ is as in Definition 1.3.2.

Proof. If $\mathrm{d}_H(X,\varepsilon) = d$ then there exist $v_i' \in H$ such that $\|v_i' - v_i\| < \varepsilon$, and such that the space spanned by $\{v_1', \ldots, v_n'\}$ has dimension d. Let P be the orthogonal projection onto U, the n-dimensional space spanned by $\{v_1, \ldots, v_n\}$ and let $v_i'' = Pv_i'$. Since $Pv_i = v_i$, we still have the inequality $\|v_i'' - v_i\| < \varepsilon$ and clearly the dimension of the linear span of $\{v_1', \ldots, v_n'\}$ is at least that of the linear span of $\{v_1'', \ldots, v_n''\}$.

Suppose that the linear span of $\{v_1'', \ldots, v_n''\}$ has dimension $n - r$. We can write any element of U in terms of the $\{v_j''\}_{j=1}^n$ and an orthonormal basis for their r-dimensional orthogonal complement in U, $\{u_1, \ldots, u_r\}$. So

$$\begin{aligned} n\varepsilon^2 &\geq \sum_{i=1}^{n} \|v_i'' - v_i\|^2 \geq \sum_{i=1}^{n}\sum_{j=1}^{r} |(v_i, u_j)|^2 \\ &= \sum_{j=1}^{r}\sum_{i=1}^{n} \|v_i\|^2 \left|\left(u_j, \frac{v_i}{\|v_i\|}\right)\right|^2 \\ &\geq M^2 \sum_{j=1}^{r}\sum_{i=1}^{n} \left|\left(u_j, \frac{v_i}{\|v_i\|}\right)\right|^2 = M^2 r. \end{aligned}$$

It follows that $\mathrm{d}_H(X,\varepsilon) \geq n(1 - \varepsilon^2/M^2)$ as claimed. □

Now we prove a result analogous to (2.2) for the thickness exponent of $\mathcal{A}$.

Lemma 2.1.2. *The thickness* $\tau(\mathcal{A})$ *is given by*

$$\tau(\mathcal{A}) = \limsup_{n\to\infty} \frac{\log n}{\log(1/\|a_n\|)} \tag{2.3}$$

Proof. We know from Lemma 1.3.3 that $\tau(\mathcal{A}) \leq \dim_{\text{box}}(\mathcal{A})$. Therefore, it follows from equality (2.2) that

$$\tau(\mathcal{A}) \leq \limsup_{n\to\infty} \frac{\log n}{\log(1/\|a_n\|)}.$$

The argument leading to the reverse inequality is similar to the proof of Lemma 3.1 in Ben-Artzi et al. (1993), but makes use of our Lemma 2.1.1. Let n be a positive integer, large enough so that $\|a_n\| < 1$. Denote by n' the unique integer such that $n' \geq n$ and

$$\|a_n\| = \|a_{n+1}\| = \ldots = \|a_{n'}\| > \|a_{n'+1}\|,$$

and define $\varepsilon_n > 0$ via $\varepsilon_n^2 = (\|a_{n'}\|^2 + \|a_{n'+1}\|^2)/4$. Note that since $\|a_{n'}\|^2 > 2\varepsilon_n^2$,

$$1 - \frac{\varepsilon_n^2}{\|a_{n'}\|^2} > 1/2$$

and, consequently, it follows from Lemma 2.1.1 that

$$\mathrm{d}_H(\mathcal{A}, \varepsilon_n) \quad \geq \quad n'\left(1 - \frac{\varepsilon_n^2}{\|a_{n'}\|^2}\right) > \frac{n'}{2}$$

Combining this inequality with $2\varepsilon_n > \|a_{n'}\|$, $n' \geq n$, and $\|a_n\| = \|a_{n'}\|$, we obtain

$$\tau(\mathcal{A}) \geq \limsup_{n\to\infty} \frac{\log \mathrm{d}_H(\mathcal{A}, \varepsilon_n)}{-\log \varepsilon_n} \geq \limsup_{n\to\infty} \frac{\log(n/2)}{\log(2/\|a_n\|)} = \limsup_{n\to\infty} \frac{\log n}{-\log\|a_n\|}.$$

□

Next we will see that this example shows that the upper bound on the Hölder exponent in Hunt and Kaloshin's Theorem 1.3.4 is asymptotically sharp ($N \to \infty$), and hence provides a simpler alternative for that in Hunt and Kaloshin (1999). The following decomposition lemma allows us to work with orthogonal projections with Hölder continuous inverse, instead of general linear maps.

Lemma 2.1.3 (Theorem 3.5, Roman, 2007)**.** *Let* B *be a Banach space and let* $L : B \to \mathbb{R}^N$ *be a surjective linear map with kernel* V*. Then the quotient space*

$U = B/V$ has dimension N, and L can be decomposed uniquely as MP, where P is a projection onto U and $M : U \to \mathbb{R}^N$ is an invertible linear map.

Let $\{a_n : n = 1, 2, ...\}$ be the orthogonal sequence defined above. Consider the compact set $\mathcal{A} = \{a_1, a_2, ...\} \cup \{0\}$ in (2.1) with upper box-counting dimension d and thickness exponent τ. If there exists an orthogonal projection $P : H \to \mathbb{R}^N$ such that

$$\|a - \bar{a}\| \leq C\|P(a) - P(\bar{a})\|^{\zeta} \quad \text{for all} \quad a, \bar{a} \in \mathcal{A},$$

where $C > 0$ is a constant, then

$$\|a\| \leq C\|Pa\|^{\zeta} \quad \text{for every} \quad a \in \mathcal{A},$$

since $0 \in \mathcal{A}$. In their Theorem 3.2, Ben-Artzi et al. (1993) showed that, for $0 < \zeta < 1$, if there exists a finite-dimensional projection P in H and a positive number C such that $\|a\| \leq C\|Pa\|^{\zeta}$, for each $a \in \mathcal{A}$, then

$$\sum_{n=1}^{\infty} \|a_n\|^{2(1/\zeta - 1)} < \infty.$$

We will prove a similar result in Section 2.1.3.

It follows from Corollary 3.3 in Ben-Artzi et al. (1993) that the existence of such a projection implies that $\zeta \leq (1 + \dim_{\text{box}}(\mathcal{A})/2)^{-1}$. Therefore,

$$\zeta \leq (1 + \tau(\mathcal{A})/2)^{-1},$$

since we have shown that $\dim_{\text{box}}(\mathcal{A}) = \tau(\mathcal{A})$. Finally, as $N \to \infty$, the upper bound on the Hölder exponent ζ in (1.3) tends to $(1 + \tau(\mathcal{A})/2)^{-1}$. Hence, we recover the upper bound given by this example.

2.1.2 Properties of the Assouad dimension of orthogonal sequences

The Assouad dimension of orthogonal sequences particularly differs from its box-counting dimension. Due to the scale invariance property, the Assouad dimension is sensitive to inhomogeneities in the set. For example, Olson (2002) showed that the number $\mathcal{N}(r, r/2)$ of $r/2$-balls required to cover any r-ball in the the compact set

$$\widehat{\mathcal{A}} = \{0\} \cup \{e_n/n^{\alpha} : n \in \mathbb{N}\}$$

is unbounded as $r \to \infty$, where $\{e_n\}$ is an orthonormal sequence. Therefore, $\widehat{\mathcal{A}}$ has infinite Assouad dimension while $\dim_{\text{box}} \widehat{\mathcal{A}} = 1/\alpha$. Hence, a set $\mathcal{A}$ has finite Assouad dimension only if it has the scaling property needed for $\mathcal{N}(r, r/2)$ to be bounded.

Proposition 2.1.4 (Fact 4.2, Olson, 2002)**.** *Let* $\mathcal{A} = \{0\} \cup \{a_n e_n : n \in \mathbb{N}\}$. *If there exist* $K \geq 1$ *and* $0 < \alpha < 1$ *such that*

$$(1/K)\alpha^n \leq a_n \leq K\alpha^n, \tag{2.4}$$

then $\dim_{\mathrm{A}}(\mathcal{A}) = 0$.

Here $\dim_{\mathrm{A}}$ is the Assouad dimension given in Definition 1.2.6. The example presented in Section 2.1.3 relies on this result, which appears without proof in Olson (2002).

Proof. Let r and ρ be such that $0 < \rho < r$. Since we are considering orthogonal sequences in H, whose terms are converging to zero, the inhomogeneities of $\mathcal{A}$ are localized near the origin. So, the number of ρ-balls required to cover any r-ball in $\mathcal{A}$, $\mathcal{N}_{\mathcal{A}}(r, \rho)$, is bounded above by the the number of ρ-balls required to cover the ball of radius r centered at the origin.

So, consider the ball of radius $r = K\alpha^m$ centered at the origin, where $m \in \mathbb{N}$,

$$B(0, r) \cap \{a \in \mathcal{A} : \|a\| < r\} = \{0\} \cup \{a_n e_n : n > m\}.$$

Cover $B(0, r)$ by ρ balls. Each point a distance more than ρ from the origin will require a separate ball. As $0 < \alpha < 1$, the estimates on n are:

$$\frac{\log r + \log K}{\log \alpha} < n < \frac{\log \rho - \log K}{\log \alpha}. \tag{2.5}$$

Hence, the number of terms a_n such that $\rho < \|a_n\| < r$ is less than or equal to

$$\frac{\log \rho - \log K}{\log \alpha} - \left(\frac{\log r + \log K}{\log \alpha}\right).$$

Therefore,

$$\mathcal{N}_{\mathcal{A}}(r, \rho) \leq -\frac{1}{\log \alpha} \log\left(\frac{r}{\rho}\right) - \frac{2 \log K}{\log \alpha} + 1.$$

So, it follows that, for any $s > 0$, there exists $C > 0$ such that

$$\mathcal{N}_{\mathcal{A}}(r, \rho) \leq C\left(\frac{r}{\rho}\right)^s,$$

which implies that $\dim_{\mathrm{A}}(\mathcal{A}) = 0$. □

Moreover, the terms in any compact set $\mathcal{A}$ with finite Assouad dimension must decay at least exponentially.

Lemma 2.1.5. *If* $\dim_{\mathrm{A}}(\mathcal{A}) < \infty$*, then there exist* $C > 0$ *and* $\lambda > 0$ *such that* $\|a_n\| < C\mathrm{e}^{-n\lambda}$*.*

Proof. First note that if $\dim_{\mathrm{A}}(\mathcal{A}) < \infty$, then $\mathcal{A}$ has the doubling property, i.e., given $r > 0$ there exists $M > 0$ such that $\mathcal{N}(r, r/2) \leq M$. Hence, the number of points $a_i \in \mathcal{A}$ with $r/2 < \|a_i\| \leq r$ is smaller or equal to M. In particular, if $r = \|a_1\|$, then the number of points a_i with $\|a_1\|/2 < \|a_i\| \leq \|a_1\|$ is smaller or equal to M. Hence,

$$\|a_{M+1}\| \leq \|a_1\|/2 = 2^{-1}\|a_1\|.$$

Using induction, we can show that

$$\|a_{Mj+1}\| \leq 2^{-j}\|a_1\|, \quad \text{for all} \quad j \in \mathbb{N}.$$

Next, if $Mj + 1 \leq n \leq M(j+1)$, then

$$\|a_n\| \leq \|a_{Mj+1}\| \leq 2^{-j}\|a_1\| \leq 2^{-(n/M-1)}\|a_1\| = 2\|a_1\|2^{-n/M}.$$

So,

$$\|a_n\| \leq 2\|a_1\|\mathrm{e}^{-n(\log 2/M)} = 2\|a_1\|\mathrm{e}^{-n\lambda},$$

where $\lambda = \log 2/M > 0$. Consequently, if $\dim_{\mathrm{A}}(\mathcal{A}) < \infty$, then there exist constants $C > 2\|a_1\| > 0$ and $\lambda > 0$ such that $\|a_n\| < C\mathrm{e}^{-n\lambda}$. $\square$

In particular, if $a_n = \mathrm{e}^{-n^\alpha}e_n$, for some $\alpha < 1$, then $\|a_n\|$ decreases more slowly than $\mathrm{e}^{-n\lambda}$, with $\lambda > 0$. Hence, if $\bar{\mathcal{A}} = \{0\} \cup \{\mathrm{e}^{-n^\alpha}e_n\}_{n+1}^{\infty}$, then $\dim_{\mathrm{A}}(\bar{\mathcal{A}})$ is infinity. Finally, it is interesting to remark that there are also compact sets $\mathcal{A}$ with terms converging arbitrarily fast to zero with infinite Assouad dimension (see Theorem 4.3 in Olson (2002) for details).

2.1.3 Logarithmic exponent when $\dim_{\mathrm{A}}(\mathcal{A} - \mathcal{A}) < \infty$

Next we show a proposition similar to Theorem 3.2 in Ben-Artzi et al. (1993), but with a logarithmic correction term. Note, however, that *(ii)* $\Rightarrow$ *(iii)* is the most important fact, since we will use it to obtain an asymptotically sharp bound for the exponent of the logarithmic term in Theorem 1.3.9.

Proposition 2.1.6. *For* $\mathcal{A}$ *defined in* (2.1) *and any* $\gamma > 0$*, the following conditions are equivalent:*

(i) There exists a rank one projection P in H and a positive constant C such that

$$\|Pa\| \geq \frac{C\|a\|}{(-\log\|a\|)^{\gamma}}, \text{ for each } a \in \mathcal{A}.$$

(ii) There exists a finite dimensional projection P in H and a positive constant C such that

$$\|Pa\| \geq \frac{C\|a\|}{(-\log\|a\|)^{\gamma}}, \text{ for each } a \in \mathcal{A}.$$

(iii) $\sum_{n=1}^{\infty}(-\log\|a_n\|)^{-2\gamma} < \infty$.

Proof. Clearly $(i) \Rightarrow (ii)$. Next we will show that $(ii) \Rightarrow (iii)$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Let P be an orthogonal projection with rank k such that, for $\gamma > 0$,

$$\|Pa\| \geq \frac{C\|a\|}{(-\log\|a\|)^{\gamma}},$$

for each $a \in \mathcal{A}$. So, there exists an orthonormal basis $\{u_1, ..., u_k\}$ for the range of P, such that

$$Pe_n = \sum_{i=1}^{k}(e_n, u_i)u_i,$$

for every $n = 1, 2, ...$. Hence,

$$\sum_{n=1}^{\infty}(Pe_n, e_n) = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{k}(e_n, u_i)u_i, e_n\right) = \sum_{i=1}^{k}\sum_{n=1}^{\infty}|(e_n, u_i)|^2 = \sum_{i=1}^{k}\|u_i\|^2 = k$$

Therefore,

$$\begin{aligned}
\operatorname{rank}(P) &\geq \sum_{n=1}^{\infty}(Pa_n, a_n)\|a_n\|^{-2} = \sum_{n=1}^{\infty}\|Pa_n\|^2\|a_n\|^{-2} \\
&\geq \sum_{n=1}^{\infty}\frac{C^2\|a_n\|^2}{(-\log\|a_n\|)^{2\gamma}}\|a_n\|^{-2} = \sum_{n=1}^{\infty}C^2(-\log\|a_n\|)^{-2\gamma}.
\end{aligned}$$

Since the rank of P is finite, $\sum_{n=1}^{\infty}(-\log\|a_n\|)^{-2\gamma} < \infty$.

Next we will prove $(iii) \Rightarrow (i)$. Assume $\sum_{n=1}^{\infty}(-\log\|a_n\|)^{-2\gamma} < \infty$. Define a vector a_0 in H by

$$a_0 = \sum_{n=1}^{\infty}(-\log\|a_n\|)^{-\gamma}\|a_n\|^{-1}a_n.$$

Note that a_0 is well defined, since $\{a_n\}_{n=1}^{\infty}$ is an orthogonal sequence and

$$\begin{aligned}\|a_0\|^2 &= \left(\sum_{n=1}^{\infty}(-\log\|a_n\|)^{-\gamma}\|a_n\|^{-1}a_n, \sum_{n=1}^{\infty}(-\log\|a_n\|)^{-\gamma}\|a_n\|^{-1}a_n\right)\\ &= \sum_{n=1}^{\infty}(-\log\|a_n\|)^{-2\gamma} < \infty.\end{aligned}$$

Set $P = \|a_0\|^{-2}a_0 \otimes a_0$. Hence, $Pa_n = (a_0, a_n)\|a_0\|^{-2}a_0$, and consequently

$$\begin{aligned}\|Pa_n\| &= |(a_0, a_n)|\|a_0\|^{-2}\|a_0\| = \left|\left(\sum_{k=1}^{\infty}(-\log\|a_k\|)^{-\gamma}\|a_k\|^{-1}a_k, a_n\right)\right|\|a_0\|^{-1}\\ &= (-\log\|a_n\|)^{-\gamma}\|a_n\|^{-1}\|a_n\|^2\|a_0\|^{-1} = (-\log\|a_n\|)^{-\gamma}\|a_n\|\|a_0\|^{-1}.\end{aligned}$$

Therefore, $\|Pa\| \geq C(-\log\|a\|)^{-\gamma}\|a\|$ for every $a \in \mathcal{A}$, where $C = \|a_0\|^{-1}$. □

It is interesting to note that one can obtain the following result parallel to Ben-Artzi et al. (1993, Theorem 3.4).

Proposition 2.1.7. *If $\sum_{n=1}^{\infty}(-\log\|a_n\|)^{-2\gamma} < \infty$, then there exists a rank one projection P and a constant C such that*

$$\|Pa - Pa'\| \geq C\frac{\|a - a'\|}{(-\log\|a - a'\|)^{3\gamma}},$$

for every a and a' in $\mathcal{A}$.

Proof. First, assume that $\|a_1\| < 2^{-1/2}$. Let $\eta_n = \sum_{j=n}^{\infty}(-\log\|a_j\|)^{-2\gamma}$ for $n = 1, 2, ...$ Define a vector $a_0 \in H$ by

$$a_0 = \sum_{n=1}^{\infty}\eta_n(-\log\|a_n\|)^{-\gamma}\|a_n\|^{-1}a_n.$$

The vector a_0 is well defined because

$$\begin{aligned}\|a_0\|^2 &= \sum_{n=1}^{\infty}\eta_n^2(-\log\|a_n\|)^{-2\gamma}\\ &\leq \eta_1^2\sum_{n=1}^{\infty}(-\log\|a_n\|)^{-2\gamma} = \eta_1^3 < \infty.\end{aligned}$$

Set $P = \|a_0\|^{-2} a_0 \otimes a_0$. Then

$$\begin{aligned}
Pa_n &= (a_0, a_n)\|a_0\|^{-2}a_0 \\
&= \left(\sum_{k=1}^{\infty} \eta_k(-\log\|a_k\|)^{-\gamma}\|a_k\|^{-1}a_k, a_n\right)\|a_0\|^{-2}a_0 \\
&= \eta_n(-\log\|a_n\|)^{-\gamma}\|a_n\|^{-1}(a_n, a_n)\|a_0\|^{-2}a_0 \\
&= \eta_n(-\log\|a_n\|)^{-\gamma}\|a_n\|^{-1}\|a_n\|^2\|a_0\|^{-2}a_0 \\
&= \eta_n(-\log\|a_n\|)^{-\gamma}\|a_n\|\|a_0\|^{-2}a_0.
\end{aligned}$$

Now let n and m be two natural numbers with $n < m$. Hence,

$$\eta_n(-\log\|a_n\|)^{-\gamma}\|a_n\| \geq \eta_{n+1}(-\log\|a_n\|)^{-\gamma}\|a_n\| \geq \eta_m(-\log\|a_m\|)^{-\gamma}\|a_m\|,$$

which implies that

$$\begin{aligned}
\|Pa_n - Pa_m\| &= \left|\eta_n(-\log\|a_n\|)^{-\gamma}\|a_n\| - \eta_m(-\log\|a_m\|)^{-\gamma}\|a_m\|\right|\|a_0\|^{-2}\|a_0\| \\
&\geq |\eta_n - \eta_{n+1}|(-\log\|a_n\|)^{-\gamma}\|a_n\|\|a_0\|^{-1} \\
&= \left|\sum_{j=n}^{\infty}(-\log\|a_j\|)^{-2\gamma} - \sum_{j=n+1}^{\infty}(-\log\|a_j\|)^{-2\gamma}\right|(-\log\|a_n\|)^{-\gamma}\|a_n\|\|a_0\|^{-1} \\
&= (-\log\|a_n\|)^{-2\gamma}(-\log\|a_n\|)^{-\gamma}\|a_n\|\|a_0\|^{-1} \\
&= (-\log\|a_n\|)^{-3\gamma}\|a_n\|\|a_0\|^{-1}.
\end{aligned}$$

Therefore

$$\|Pa_n - Pa_m\| \geq \|a_0\|^{-1}(-\log\|a_n\|)^{-3\gamma}\|a_n\|.$$

Since a_n and a_m are orthogonal vectors with $\|a_n\| \geq \|a_m\|$,

$$\|a_n - a_m\|^2 = \|a_n\|^2 + \|a_m\|^2 \leq 2\|a_n\|^2, \tag{2.6}$$

and consequently $\|a_n - a_m\| \leq \sqrt{2}\|a_n\|$. Hence,

$$\|Pa_n - Pa_m\| \geq \frac{\|a_0\|^{-1}}{\sqrt{2}}(-\log\|a_n\|)^{-3\gamma}\|a_n - a_m\|. \tag{2.7}$$

Next it follows from (2.6) that there exists a constant K such that

$$\begin{aligned} -\log\|a_n - a_m\| &\geq -\log\sqrt{2} - \log\|a_n\| = \left(\frac{\log\sqrt{2}}{\log\|a_n\|} + 1\right)\big(-\log\|a_n\|\big) \\ &\geq \left(\frac{\log\sqrt{2}}{\log\|a_1\|} + 1\right)\big(-\log\|a_n\|\big) = K\big(-\log\|a_n\|\big), \end{aligned}$$

since $\|a_1\| \geq \|a_n\|$ for every $n \in \mathbb{N}$. As $\|a_1\| < 2^{-1/2}$,

$$-1 < \frac{\log\sqrt{2}}{\log\|a_1\|} < 0 \qquad \text{implies that} \qquad 0 < K < 1.$$

So,

$$\Big(-\log\|a_n - a_m\|\Big)^{3\gamma} \geq K^{3\gamma}\Big(-\log\|a_n\|\Big)^{3\gamma},$$

and consequently

$$\Big(-\log\|a_n - a_m\|\Big)^{-3\gamma} \leq K^{-3\gamma}\Big(-\log\|a_n\|\Big)^{-3\gamma}.$$

Therefore, it follows from (2.7) that

$$\begin{aligned} \|Pa_n - Pa_m\| &\geq \frac{\|a_0\|^{-1}}{\sqrt{2}}(-\log\|a_n\|)^{-3\gamma}\|a_n - a_m\| \\ &\geq \frac{\|a_0\|^{-1}}{\sqrt{2}}K^{3\gamma}(-\log\|a_n - a_m\|)^{-3\gamma}\|a_n - a_m\|. \end{aligned}$$

Now let $C = \frac{\|a_0\|^{-1}}{\sqrt{2}}K^{3\gamma}$. Since n and m are two arbitrary natural numbers with $n < m$, the previous inequality implies that

$$\|Pa - Pa'\| \geq C\frac{\|a - a'\|}{(-\log\|a - a'\|)^{3\gamma}}.$$

□

Next, in order to show that exponent in Theorem 1.3.9 is sharp, we consider a particular orthogonal sequence in a Hilbert space H. Let $\tilde{\mathcal{A}} = \{a_1, a_2, ...\} \cup \{0\}$ with $a_n = \mathrm{e}^{-n}e_n$ for every $n \in \mathbb{N}$. Since there exist K and α such that $0 < \alpha < 1$ and $(1/K)\alpha^n \leq \mathrm{e}^{-n} \leq K\alpha^n$, it follows from Proposition 2.1.4 that $\dim_{\mathrm{A}}(\tilde{\mathcal{A}}) = 0$. Since $\tilde{\mathcal{A}}$ is an orthogonal sequence, it follows from Lemma 8.4 in Olson and Robinson (2010) that $\dim_{\mathrm{A}}(\tilde{\mathcal{A}} - \tilde{\mathcal{A}}) = 0$.

Now, if there exists an orthogonal projection with rank N such that

$$\|Pa - Pa'\| \geq \frac{\|a - a'\|}{(-\log \|a - a'\|)^{\gamma}},$$

for each distinct $a, a' \in \tilde{\mathcal{A}}$, then

$$\|Pa\| \geq \frac{\|a\|}{(-\log \|a\|)^{\gamma}},$$

since $0 \in \tilde{\mathcal{A}}$. Therefore, (iii) of Proposition 2.1.6

$$\sum_{n=1}^{\infty} (-\log |\mathrm{e}^{-n}|)^{-2\gamma} = \sum_{n=1}^{\infty} n^{-2\gamma} < \infty.$$

Since the rank of P is finite, $\sum_{n=1}^{\infty} n^{-2\gamma} < \infty$ implies that $\gamma > 1/2$. Hence, this example gives a lower bound for the exponent γ of $1/2$. Therefore, the limiting value (as $N \to \infty$) of the exponent γ in Theorem 1.3.9 is optimal.

2.2 Sequences in c_0

Motivated by the results obtained for orthogonal sequences in Section 2.1, we study in this section a general set of sequences $\mathbf{A}$ in c_0 that allow us to obtain results analogous to those shown by Ben-Artzi et al. (1993) but in the Banach space setting. The space c_0 consists of all scalar sequences $x = (x_1, x_2, ...)$ for which

$$\lim_{n \to \infty} x_n = 0$$

and the norm of an element $x \in c_0$ is

$$\|x\|_{\infty} = \sup_n |x_i| < \infty.$$

Note that c_0 is a separable Banach space with the ℓ^{∞}-norm and $c_0 \subset \ell^{\infty}$.

Let $\{e_i\}_{i=1}^{\infty}$ be the standard basis for c_0, where $e_i = (0, ..., 0, 1, 0, ...)$ is a vector with 1 in the ith place and 0 in every other place. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{R} \setminus \{0\}$ such that $|a_n| \geq |a_{n+1}| > 0$ and $\lim_{n \to \infty} |a_n| = 0$. Consider the compact set $\mathbf{A} = \{\alpha_1, \alpha_2, ...\} \cup \{0\}$ in c_0, where $\alpha_i = a_i e_i$ for every $i = 1, 2, ...$

Proposition 2.2.1. *The upper box-counting dimension* $\dim_{\text{box}}(\mathbf{A})$ *is given by*

$$\begin{aligned}\dim_{\text{box}}(\mathbf{A}) &= \limsup_{n\to\infty} \frac{\log n}{\log(1/\|\alpha_n\|_\infty)} \\ &= \inf\{D > 0 : \sup_n(\|\alpha_n\|_\infty n^{1/D}) < \infty\} \\ &= \inf\left\{\nu : \sum_{n=1}^{\infty} \|\alpha_n\|_\infty^\nu < \infty\right\}\end{aligned}$$

The proof of this proposition is similar to that of Lemma 3.1 in Ben-Artzi et al. (1993) provided we make some simple adjustments. Hence we will only prove the first equality.

Proof. For each $0 < \varepsilon < \|\alpha_1\|_\infty$, denote by $n(\varepsilon)$ the unique positive integer such that $\|\alpha_{n(\varepsilon)}\|_\infty > \varepsilon \geq \|\alpha_{n(\varepsilon)+1}\|_\infty$. Since an ε-ball centered at the origin covers all the points in X except the first $n(\varepsilon)$ points, it follows that $N(X, \varepsilon) \leq n(\varepsilon) + 1$. Therefore,

$$\begin{aligned}\dim_{\text{box}}(X) &= \limsup_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{-\log\varepsilon} \\ &\leq \limsup_{\varepsilon\to 0} \frac{\log n(\varepsilon) + 1}{-\log\|\alpha_{n(\varepsilon)}\|_\infty} = \limsup_{n\to\infty} \frac{\log n}{\log(1/\|\alpha_n\|_\infty)}.\end{aligned}$$

Next, we prove the reverse inequality. Let n be a positive integer, large enough so that $\|\alpha_n\|_\infty < 1$. Denote by n' the unique integer such that $n' \geq n$ and

$$\|\alpha_n\|_\infty = \|\alpha_{n+1}\|_\infty = ... = \|\alpha_{n'}\|_\infty > \|\alpha_{n'+1}\|_\infty,$$

and define $\varepsilon(n) > 0$ via $\varepsilon(n) = (\|\alpha_{n'}\|_\infty + \|\alpha_{n'+1}\|_\infty)/4$.

Since $\inf\big(\|\alpha_i - \alpha_j\|_\infty\big) = \inf\big(\|\alpha_i\|_\infty\big) \geq \|\alpha_{n'}\|_\infty$, for $i, j = 1, ..., n'$ with $i < j$, $\|\alpha_i - \alpha_j\|_\infty \geq \|\alpha_{n'}\|_\infty$. Hence the distance between two elements of the set $\{\alpha_1, ..., \alpha_{n'}\}$ is at least equal to $\|\alpha_{n'}\|_\infty > 2\varepsilon(n)$. This implies that $N(X, \varepsilon(n)) \geq n'$.

Now if we combine this inequality with $4\varepsilon(n) > \|\alpha_{n'}\|_\infty$, $n' \geq n$, and $\|\alpha_n\|_\infty = \|\alpha_{n'}\|_\infty$, we obtain

$$\frac{\log n}{\log(1/\|\alpha_n\|_\infty)} \leq \frac{\log n'}{\log(1/\|\alpha_{n'}\|_\infty)} \leq \frac{\log N(X, \varepsilon(n))}{\log(1/4\varepsilon(n))}.$$

Since $\varepsilon(n) \to 0$ as $n \to \infty$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} \frac{\log n}{\log(1/\|\alpha_n\|_\infty)} &\leq \limsup_{n\to\infty} \frac{\log N(X, \varepsilon(n))}{\log(1/4\varepsilon(n))} \\ &\leq \limsup_{\varepsilon\to 0} \frac{\log N(X, \varepsilon)}{\log(1/4\varepsilon)} \\ &= \dim_{\text{box}}(X).\end{aligned}$$

□

Next consider the canonical basis of the sequence space c_0 and $\ell^1 = (c_0)^*$ with $e_n = (0, ..., 0, 1, 0, ...) \in c_0$ and $e_n^* = (0, ..., 0, 1, 0, ...) \in \ell^1$, where there is a 1 in the nth place and 0 in every other place. We find that $\|e_n\|_\infty = \|e_n^*\|_1 = e_n^*(e_n) = 1$ and $e_n^*(e_m) = 0$, for every $n \neq m$. A pair of sequences $\{e_n; e_n^*\}_{n\in\mathbb{N}}$ satisfying these conditions is called a *normalized biorthogonal system* in $c_0 \times \ell^1$. For more details, see Hájek et al. (2008).

Next define the vectors $\alpha_i^* \in \ell^1$ by setting $\alpha_i^* = a_i e_i^*$ so that $\alpha_i^*(\alpha_i) = |a_i|^2$ and $\alpha_i^*(\alpha_j) = 0$ for every $i \neq j$. Finally note that a projection P in c_0 is a bounded linear operator in c_0, such that $P^2 = P$.

Proposition 2.2.2. *If there exists a finite-dimensional projection P in c_0 and $\theta \in (0, 1)$ such that*

$$\|\alpha\|_\infty \leq C\|P\alpha\|_\infty^\theta, \text{ for each } \alpha \in \mathbf{A},$$

then

$$\sum_{i=1}^{\infty} |a_i|^{(1/\theta-1)} < \infty. \tag{2.8}$$

As in the Theorem 3.2 in Ben-Artzi et al. (1993), the converse statement is also true. Moreover it is interesting to note that the same result remains valid for similar sequences in ℓ^p, with (2.9) replaced by $\sum_{i=1}^{\infty} |a_i|^{(q/\theta-1)} < \infty$, where $p^{-1} + q^{-1} = 1$.

Proof. Let P be a finite-dimensional projection in c_0 such that

$$\|\alpha\|_\infty \leq C\|P\alpha\|_\infty^\theta,$$

for each $\alpha \in \mathbf{A}$, where $C > 0$. Let U be the range of P. Since P is a finite-dimensional projection, U is a finite-dimensional subspace of c_0. By Lemma 10.5 in Meise and Vogt (1997), U has an Auerbach basis $\{u_1, ...u_n\}$ with the coefficient functionals $\{f_1^*, ...f_n^*\} \in U^*$, such that $|u_i|_U = 1$, $|f_i|_{U^*} = 1$ and $f_i^*(u_k) = \delta_{ik}$ for $1 \leq i, k \leq n$. It follows from the Hahn-Banach Theorem that each f_i can be extended to an element

$\phi_i \in \ell^1$ with $\|\phi_i\|_1 = 1$ and such that $\phi_i(u_k) = \delta_{ij}$ for $1 \le i, k \le n$. Hence,

$$\phi_i = \sum_{k=1}^{\infty} \lambda_{ik} e_k^*, \quad \text{for each} \quad i = 1, 2, ...$$

with $\sum_{k=1}^{\infty} |\lambda_{ik}| = 1$. Thus, for every element $x \in c_0$, we can write

$$Px = \sum_{i=1}^{n} \phi_i(x) u_i.$$

On the one hand, for each $j = 1, 2, ...,$

$$Pe_j = \sum_{i=1}^{n} \phi_i(e_j) u_i,$$

where each $\phi_i \in \ell^1$. Using the triangle inequality, for each $j = 1, 2, ...,$

$$\begin{aligned} \|Pe_j\|_\infty &= \left\| \sum_{i=1}^{n} \phi_i(e_j) u_i \right\|_\infty \\ &\le \sum_{i=1}^{n} \|\phi_i(e_j) u_i\|_\infty = \sum_{i=1}^{n} |\phi_i(e_j)| \\ &= \sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} \lambda_{ik} e_k^*(e_j) \right| = \sum_{i=1}^{n} |\lambda_{ij}|. \end{aligned}$$

Therefore,

$$\sum_{j=1}^{\infty} \|Pe_j\|_\infty \le \sum_{j=1}^{\infty} \sum_{i=1}^{n} |\lambda_{ij}| = \sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} |\lambda_{ij}| \right) = n.$$

On the other hand, using the fact that $\|P\alpha_j\|_\infty = |a_j| \|Pe_j\|_\infty$ and that $|a_j|^{1/\theta} = \|\alpha_j\|_\infty^{1/\theta} \le C^{1/\theta} \|P\alpha_j\|_\infty$, we have

$$\begin{aligned} C^{1/\theta} \sum_{j=1}^{\infty} \|Pe_j\|_\infty &= C^{1/\theta} \sum_{j=1}^{\infty} \frac{|a_j| \|Pe_j\|_\infty}{|a_j|} = C^{1/\theta} \sum_{j=1}^{\infty} \|P\alpha_j\|_\infty \|\alpha_j\|_\infty^{-1} \\ &\ge \sum_{j=1}^{\infty} |a_j|^{1/\theta} |a_j|^{-1} = \sum_{j=1}^{\infty} |a_j|^{(1/\theta - 1)}. \end{aligned}$$

Since $\sum_{j=1}^{\infty} \|Pe_j\|_\infty < \infty$, $\sum_{i=1}^{\infty} |a_i|^{(1/\theta-1)} < \infty$. $\square$

Using Lemma 2.1.3, one can show that Theorem 1.3.6 remains true when linear maps are replaced by finite-rank projections.

Consider the compact set $\mathbf{A}$ in c_0, with upper box-counting dimension d and dual thickness exponent $\tau^*(\mathbf{A})$ (see Chapter 1 for more details). If there exists a projection P in c_0 with rank N and a constant $c_P > 0$ such that

$$\|a - \bar{a}\|_\infty \leq c_P \|Pa - P\bar{a}\|_\infty^\theta \quad \text{for all} \quad a, \bar{a} \in \mathbf{A},$$

then

$$\|a\|_\infty \leq c_P \|Pa\|_\infty^\theta,$$

for every $a \in \mathbf{A}$, since $0 \in \mathbf{A}$. It follows from Proposition 2.2.1 and Proposition 2.2.2 that the existence of such a projection implies that

$$\theta \leq (1 + \dim_{\text{box}}(\mathbf{A}))^{-1}.$$

Therefore, this example gives an upper bound on the Hölder exponent of the inverse of P restricted to $P\mathbf{A}$.

In addition, we note that $\dim_{\text{box}}(\mathbf{A}) = \tau^*(\mathbf{A})$, since $\tau^*(\mathbf{A}) \leq \dim_{\text{box}}(\mathbf{A})$. If not, $\tau^*(\mathbf{A}) < \dim_{\text{box}}(\mathcal{A})$ which implies that as $N \to \infty$ the upper bound on the Hölder exponent θ in Robinson's Theorem 1.3.6 would tend to

$$(1 + \tau^*(\mathbf{A}))^{-1} > (1 + \dim_{\text{box}}(\mathbf{A}))^{-1},$$

which, as we have seen, is not possible. Hence, as $N \to \infty$, the upper bound on the Hölder exponent θ tends to $(1 + \tau^*(\mathbf{A}))^{-1}$. So, we recover the limiting exponent $1/(1+\tau^*(\mathbf{A}))$. Therefore, this example shows that the Hölder exponent in Theorem 1.3.6 is asymptotically sharp.

One can adapt the methods developed above to show a result similar to Proposition 2.2.2, but with a logarithmic correction.

Proposition 2.2.3. *If there exists a finite-dimensional projection P in c_0 and $\gamma > 0$ such that*

$$\|P\alpha\|_\infty \geq \frac{\|\alpha\|_\infty}{(-\log \|\alpha\|_\infty)^\gamma}, \quad \text{for each } \alpha \in \mathbf{A},$$

then

$$\sum_{i=1}^{\infty} (-\log |a_j|)^\gamma < \infty. \tag{2.9}$$

Proof. Let P be a finite-dimensional projection in c_0 such that, for $\gamma > 0$,

$$\|P\alpha\|_\infty \geq \frac{\|\alpha\|_\infty}{(-\log \|\alpha\|_\infty)^\gamma},$$

for each $\alpha \in \mathbf{A}$. On the one hand, we can bound $\sum_{j=1}^{\infty} \|Pe_j\|_\infty$ just as in the proof of Proposition 2.2.2. On the other hand, using the fact that $\|P\alpha_j\|_\infty = |a_j|\|Pe_j\|_\infty$ and that $\|P\alpha_j\|_\infty \geq \|\alpha\|_\infty/(-\log\|\alpha\|_\infty)^\gamma = |a_j|/(-\log|a_j|)^\gamma$, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \|Pe_j\|_\infty &= \sum_{j=1}^{\infty} \frac{|a_j|\|Pe_j\|_\infty}{|a_j|} = \sum_{j=1}^{\infty} \|P\alpha_j\|_\infty |a_j|^{-1} \\ &\geq \sum_{j=1}^{\infty} \frac{|a_j|}{(-\log|a_j|)^\gamma}|a_j|^{-1} = \sum_{j=1}^{\infty}(-\log|a_j|)^\gamma.\end{aligned}$$

Since $\sum_{j=1}^{\infty} \|Pe_j\|_\infty < n$, $\sum_{i=1}^{\infty}(-\log|a_j|)^\gamma < \infty$.

□

Next, in order to show that the limiting value of the exponent γ in the Banach space version of Theorem 1.3.9 is sharp, we consider a particular sequence in c_0. Let $\tilde{\mathbf{A}} = \{\alpha_1, \alpha_2, ...\} \cup \{0\}$ with $\alpha_n = \mathrm{e}^{-n}e_n$, for every $n \in \mathbb{N}$. Since there exists K and β such that $0 < \beta < 1$ and $(1/K)\beta^n \leq \mathrm{e}^{-n} \leq K\beta^n$, it follows from Fact 4.2 in Olson (2002) (Proposition 2.1.4) that $\dim_{\mathrm{A}}(\tilde{\mathbf{A}}) = 0$. Moreover, it follows from Lemma 8.4 in Olson and Robinson (2010) that $\dim_{\mathrm{A}}(\tilde{\mathbf{A}} - \tilde{\mathbf{A}}) = 0$.

Now, if there exists a projection P in c_0 with rank N such that

$$\|P\alpha - P\alpha'\|_\infty \geq \frac{\|\alpha - \alpha'\|}{(-\log\|\alpha - \alpha'\|_\infty)^\gamma},$$

for each distinct $\alpha, \alpha' \in \tilde{\mathbf{A}}$, then

$$\|P\alpha\|_\infty \geq \frac{\|\alpha\|_\infty}{(-\log\|\alpha\|_\infty)^\gamma},$$

since $0 \in \tilde{\mathbf{A}}$. Therefore, it follows from Proposition 2.2.3

$$\sum_{n=1}^{\infty}(-\log|\mathrm{e}^{-n}|)^{-\gamma} = \sum_{n=1}^{\infty} n^{-\gamma} < \infty,$$

implies that $\gamma > 1$. Hence, this example shows that the limiting value of the logarithmic exponent γ as $N \to \infty$ in the Banach space version of Theorem 1.3.9 is sharp ($\gamma > 1$).

Chapter 3

Lipschitz deviation and embeddings of global attractors

3.1 The Lipschitz deviation

3.1.1 Definition of Lipschitz deviation

Inspired by the work of Hunt and Kaloshin (1999), Olson and Robinson (2010) defined a new quantity that measures how well a compact set X in a Hilbert space H can be approximated by graphs of Lipschitz functions (with prescribed Lipschitz constant) defined over a finite-dimensional subspace of H.

Definition 3.1.1 (Olson and Robinson, 2010)**.** *Let X be a compact subset of a real Hilbert space H. Let $\delta_m(X,\varepsilon)$ be the smallest dimension of a linear subspace $U \subset H$ such that*

$$\operatorname{dist}\big(X, \mathbf{G}_U[\varphi]\big) < \varepsilon,$$

for some m-Lipschitz function $\varphi : U \to U^\perp$, i.e.

$$\big\|\varphi(u) - \varphi(v)\big\| \leq m\|u - v\| \quad \text{for all} \quad u, v \in U,$$

where $U^\perp$ is orthogonal complement of U in H and $\mathbf{G}_U[\varphi]$ is the graph of φ over U:

$$\mathbf{G}_U[\varphi] = \big\{\, u + \varphi(u) : \ u \in U \,\big\}.$$

*The m-*Lipschitz deviation $\operatorname{dev}_m(X)$ *of X is given by*

$$\operatorname{dev}_m(X) = \limsup_{\varepsilon \to 0} \frac{\log \delta_m(X,\varepsilon)}{-\log \varepsilon}.$$

Since this quantity is bounded and non-increasing in m, the limit as m tends to infinity exists and is equal to the infimum. We therefore make the following definition, which is independent of m.

Definition 3.1.2. *Let X be a compact subset of a real Hilbert space H. The* Lipschitz deviation $\mathrm{dev}(X)$ *of X is given by*

$$\mathrm{dev}(X) = \lim_{m\to\infty} \mathrm{dev}_m(X). \tag{3.1}$$

Since $\delta_m(X,\varepsilon) \le \delta_0(X,\varepsilon) = \mathrm{d}_H(X,\varepsilon)$ (where $\mathrm{d}_H(X,\varepsilon)$ was introduced in Definition 1.3.2), $\mathrm{dev}_m(X) \le \tau(X)$, for all $m > 0$. Therefore, the Lipschitz deviation is bounded above by the thickness exponent.

We now give an example of a set X, for which $\mathrm{dev}(X)$ is strictly smaller than $\tau(X)$. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for a Hilbert space H, and consider the compact set

$$X = \left\{\frac{1}{n}\, e_1 + \frac{1}{n^2}\, e_n : \; n \ge 2\right\} \cup \{0\}.$$

It is relatively easy to show that X is contained in the graph of a 3-Lipschitz function over the one-dimensional subspace E_1 spanned by e_1: define ϕ on the discrete set of points $\{e_1/n\}_{n\in\mathbb{N}} \cup \{0\}$ by

$$\phi(e_1/n) = \frac{e_n}{n^2}, \qquad n \ge 2 \qquad \text{and} \qquad \phi(0) = 0.$$

On its domain of definition, ϕ is Lipschitz: for $m > n$,

$$\Big|\phi(e_1/n) - \phi(e_1/m)\Big| = \left|\frac{e_n}{n^2} - \frac{e_m}{m^2}\right| = n^{-2} + m^{-2} \le n^{-2} + (n+1)^{-2} < \frac{3}{n(n+1)}$$

and

$$\left|\frac{e_1}{n} - \frac{e_1}{m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right| \ge \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)},$$

and so

$$\Big|\phi(e_1/n) - \phi(e_1/m)\Big| \le 3\left|\frac{e_1}{n} - \frac{e_1}{m}\right|.$$

This implies that $\delta_3(X,\varepsilon) = 1$, for every $\varepsilon > 0$. Standard results (see for example Wells and Williams, 1975), allows one to extend ϕ to a 3-Lipschitz function defined on the whole of E_1. It follows that $\mathrm{dev}_3(X) = 0$, and so in particular $\mathrm{dev}(X) = 0$. However, $\tau(X) \ge 1$. To show this result, we will use Lemma 2.1.1 that gives an explicit lower bound on $\mathrm{d}_H(X,\varepsilon)$ for finite-dimensional subspaces X that approximate orthogonal sets in Hilbert spaces.

The argument adopted here follows from the method developed by Ben-Artzi et al. (1993) for finding lower bounds on the box-counting dimension of orthogonal sequences. For $n \geq 1$ set

$$a_n = \frac{e_1}{n+1} + \frac{e_{n+1}}{(n+1)^2};$$

note that $\|a_n\| > \|a_{n+1}\|$ and $\lim_{n\to\infty} \|a_n\| = 0$. Let $\varepsilon_n^2 = \big(\|a_n\|^2 + \|a_{n+1}\|^2\big)/4$. Since $\|a_j\|^2 \geq \|a_n\|^2 > 2\varepsilon_n^2$ for $j = 1, \dots, n$, it follows from the Lemma 2.1.1 that

$$\mathrm{d}_H(X, \varepsilon_n) \geq d\big(\{a_1, \dots, a_n\}, \varepsilon_n\big) \geq n\left(1 - \frac{\varepsilon_n^2}{\|a_n\|^2}\right) > \frac{n}{2}.$$

Since $(n+1)^{-1} < \|a_n\| < 2\varepsilon_n$,

$$\tau(X) \geq \limsup_{n\to\infty} \frac{\log \mathrm{d}_H(X, \varepsilon_n)}{-\log \varepsilon_n} \geq \limsup_{n\to\infty} \frac{\log(n/2)}{\log 2(n+1)} = 1.$$

It is interesting to observe that, in contrast to the thickness exponent, the Lipschitz deviation is not preserved under bounded linear transformations that have a bounded linear inverse. This observation is a direct consequence of the fact that the orthogonal splitting of a Hilbert space is not preserved under such transformations. Nevertheless the Lipschitz deviation provides us with more freedom to approximate compact sets than the thickness, while still maintaining sufficient control to obtain results that parallel those obtained using the thickness, as we shall now see.

3.1.2 Hölder embedding of compact sets

The following result is stated without proof as Theorem 6.5 in Olson and Robinson (2010) in terms of the m-Lipschitz deviation (for some $m > 0$). Since the focus of this chapter is the Lipschitz deviation, it is worthwhile to present here a proof of this theorem.

Theorem 3.1.3. *Let X be a compact subset of a real Hilbert space H with box-counting dimension d and let $\mathrm{dev}(X)$ be the Lipschitz deviation of X. Let $N > 2d$ be an integer and let ζ be a real number with*

$$0 < \zeta < \frac{N - 2d}{N\big(1 + \mathrm{dev}(X)/2\big)}. \tag{3.2}$$

Then for a prevalent set of bounded linear function there exists a $C > 0$ such that

$$C\big|f(x) - f(y)\big|^{\zeta} \geq \|x - y\| \quad \text{for all} \quad x, y \in X. \tag{3.3}$$

In particular, these maps are injective on X.

The proof follows that of Hunt and Kaloshin (1999), but with some significant changes, since we are using the Lipschitz deviation rather than the thickness.

Proof. Note that if (3.2) holds, then it holds if $\operatorname{dev}(X)$ is replaced by $\operatorname{dev}_m(X)$, for m sufficiently large. So we will work with a fixed m and prove the theorem in terms of the m-Lipschitz deviation.

For $j = 1, 2, \ldots$, let $d_j = \delta_m(X, 2^{-j\zeta}/6m)$ be the dimension of a linear subspace $U_j \subset H$ such that

$$\operatorname{dist}\bigl(X, \mathbf{G}_{U_j}[\varphi_j]\bigr) < 2^{-j\zeta}/6m,$$

for some m-Lipschitz function $\varphi_j : U_j \to U_j^{\perp}$, where $\mathbf{G}_{U_j}[\varphi_j]$ is the graph of φ_j over U_j. By definition,

$$\operatorname{dev}_m(X) = \limsup_{\varepsilon \to 0} \frac{\log \delta_m(X, \varepsilon)}{-\log \varepsilon}. \tag{3.4}$$

Thus, for all $\sigma > \operatorname{dev}_m(X)$, there exists $C_1 > 0$, that depends only on X and σ, such that $d_j \leq C_1 2^{j\zeta\sigma}$.

Let S_j be the closed unit ball in the linear subspace U_j. For any $u \in H$, denote u^* the element of H^* given by $u^*(x) = (u, x)$ where $(\cdot, \cdot)$ is the inner product in H. Let $\mathscr{L}(H, \mathbb{R}^N)$ denote the space of bounded linear functions from H into $\mathbb{R}^N$. Every bounded linear function $\pi : H \to \mathbb{R}^N$ can be written $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$, where each π_n is a linear functional on H.

As required by the definition of prevalence, we need to construct a probability measure μ on a compact subset $Q \subset \mathscr{L}(H, \mathbb{R}^N)$ of perturbations. We define this 'probe set' Q as

$$Q = \left\{ \pi = (\pi_1, \ldots, \pi_N) : \ \pi_n = \sum_{j=1}^{\infty} j^{-2} \phi_{nj}^* \ \text{ with } \ \phi_{nj} \in S_j, \ \text{for all } j \right\}.$$

Under the obvious identification $\mathscr{L}(H, \mathbb{R}^N) = \mathscr{L}(H, \mathbb{R}) \times \ldots \times \mathscr{L}(H, \mathbb{R})$, which is an isomorphism of Banach spaces and in particular a homeomorphism, the set Q is mapped bijectively onto $C \times \ldots \times C$, where

$$C := \left\{ \sum_{j=1}^{\infty} j^{-2} \phi_j^* \ \text{ with } \ \phi_j \in S_j, \ \text{for all } j \right\}.$$

Hence Q is homeomorphic to a product of N copies of C. Therefore, in order to show that Q is compact, it suffices to show that C is compact.

First note that the map $S_j \ni \phi \mapsto \phi^* \in \mathscr{L}(H, \mathbb{R})$ is continuous, and in fact an isometry. Now consider $\prod_{j\in\mathbb{N}} S_j$ endowed with the product topology. It is well known that the product topology of a countable collection of metric spaces is metrizable; in our case a suitable metric is given by

$$d\big((\phi_j), (\psi_j)\big) = \sum_{j\in\mathbb{N}} j^{-2} \big\|\phi_j - \psi_j\big\|, \qquad \text{where} \quad (\phi_j), (\psi_j) \in \prod_{j\in\mathbb{N}} S_j.$$

Define the map

$$u : \prod_{j\in\mathbb{N}} S_j \to \mathscr{L}(H, \mathbb{R}) \qquad \text{such that} \qquad u\big(\phi_1, \phi_2, \dots\big) = \sum_{j\in\mathbb{N}} j^{-2} \phi_j^*$$

Let $\varepsilon > 0$. If $d\big((\phi_j), (\psi_j)\big) < \varepsilon$, then

$$\begin{aligned}
\Big\|u\big(\phi_1, \phi_2, \dots\big) - u\big(\psi_1, \psi_2, \dots\big)\Big\| &= \Big\|\sum_{j\in\mathbb{N}} j^{-2} \big(\phi_j^* - \psi_j^*\big)\Big\| \\
&\leq \sum_{j\in\mathbb{N}} j^{-2} \big\|\phi_j^* - \psi_j^*\big\| \\
&= \sum_{j\in\mathbb{N}} j^{-2} \big\|\phi_j - \psi_j\big\| = d\big((\phi_j), (\psi_j)\big) < \varepsilon.
\end{aligned}$$

Hence the map

$$(\phi_j) \mapsto \sum_{j\in\mathbb{N}} j^{-2} \phi_j^*$$

is continuous.

Since each sphere S_j is finite-dimensional, they are compact, so by Tychonoff's theorem $\prod_{j\in\mathbb{N}} S_j$ is compact as well. Consequently, the set C, which is the image of $\prod_{j\in\mathbb{N}} S_j$ under the map given above, is also compact. Therefore, Q is a compact subset of $\mathscr{L}(H, \mathbb{R}^N)$.

Since we can identify S_j with the unit ball B_{d_j} in $\mathbb{R}^{d_j}$, it is possible to define a probability measure λ_{d_j} on S_j that corresponds to the uniform probability measure on B_{d_j}. Let μ be the probability measure on Q that results from choosing each ϕ_{nj} randomly with respect to λ_{d_j}, such that

$$\mu := \underbrace{\bigotimes_{j=1}^{\infty} \lambda_{d_j} \ \cdots \ \bigotimes_{j=1}^{\infty} \lambda_{d_j}}_{N \text{ copies}},$$

with the product measure $\bigotimes_{j=1}^{\infty} \lambda_{d_j}$ defined on $\bigtimes_{j=1}^{\infty} S_j$. See Appendix A.2 for more details.

We want to show that the set S of bounded linear maps $f : H \to \mathbb{R}^N$, for which there exists a $C > 0$ such that

$$C\big|f(x) - f(y)\big|^{\zeta} \geq \|x - y\| \quad \text{for all} \quad x, y \in X$$

is prevalent. So, we need to show that $\mu(S + f) = 1$ for all $f \in \mathscr{L}(H, \mathbb{R}^N)$ or, equivalently, that $\mu\big(Q \setminus (S + f)\big) = 0$. Thus, given $f \in \mathscr{L}(H, \mathbb{R}^N)$, we want to verify that μ-almost every $\pi \in Q$ can be written in the form $s + f$, for some $s \in S$. Thus $\pi - f \in S$, for μ-almost every $\pi \in Q$. Since $S = -S$ and $Q = -Q$, this is equivalent to showing that $f + \pi \in S$ for μ-almost every $\pi \in Q$.

For $j \geq 1$, define

$$Z_j = \Big\{(x, y) \in X \times X : \|x - y\| \geq 2^{-j\zeta}\Big\},$$

with norm $\big\|(x, y)\big\| = \|x\| + \|y\|$. Let Q_j be the set of linear maps that includes those which fails to satisfy the required continuity property, i.e.

$$Q_j = \left\{\pi \in Q : \Big|f(x) + \pi(x) - \big(f(y) + \pi(y)\big)\Big| \leq 2^{-j}, \ \text{ for some } (x, y) \in Z_j\right\}.$$

Denote L the Lipschitz constant that is valid for all $f + \pi$, with $\pi \in Q$.

We want to find an upper bound for $\mu(Q_j)$. It follows from the definition of box-counting dimension that, if $\delta > d$, then, for all $j \geq 1$ such that $2^{-j} \in (0, 1)$, there exists $C_2 > 0$ depending only on X and δ such that X can be covered by at most $C_2 2^{j\delta}$ balls of radius $2^{-(j+1)}$. Hence $Z_j \subset X \times X$ can be covered by at most $M_j = {C_2}^2 2^{2j\delta}$ balls of radius 2^{-j}. Let Y be the intersection of Z_j with one of these balls.

Note that if $(x, y), (x_0, y_0) \in Y$, then $\big\|(x, y) - (x_0, y_0)\big\| \leq 2^{-j+1}$. So,

$$\Big|f(x_0) + \pi(x_0) - \big(f(y_0) + \pi(y_0)\big)\Big| \geq (4L + 1)2^{-j}$$

implies that $\Big|f(x) + \pi(x) - \big(f(y) + \pi(y)\big)\Big| \geq 2^{-j}$, for all $(x, y) \in Y$. Hence,

$$Q_j \subseteq \bigcup_{i=1}^{M_j} \left\{ \pi \in Q : \ \Big|f(x_i) + \pi(x_i) - \big(f(y_i) + \pi(y_i)\big)\Big| \leq (4L + 1)2^{-j} \right\},$$

where $(x_i, y_i) \in Y_i$ and Y_i is the intersection of Z_j with the i-th ball in the covering mentioned earlier. Therefore, we wish to bound the probability that $\pi \in Q$, chosen randomly with respect to μ, satisfies

$$\Big|f(x_0)+\pi(x_0)-\big(f(y_0)+\pi(y_0)\big)\Big| = \big|f(x_0)-f(y_0)+\pi(x_0-y_0)\big| \leq (4L+1)2^{-j}.$$

Define P_j as the orthogonal projection onto U_j and $(I-P_j)$ its complement. Let z be the orthogonal projection of x_0-y_0 onto U_j. Lemma 5.5 in Olson and Robinson (2010) - which provides a more explicit proof than the argument developed in Hunt and Kaloshin (1999) - guarantees that

$$\mu\Big\{\pi\in Q : \big|(f+\pi)(x_0-y_0)\big| < \varepsilon\Big\} \leq C_3\Big(j^2 d_j^{1/2}\varepsilon\big\|P_j(x_0-y_0)\big\|^{-1}\Big)^N,$$

where C_3 is a constant independent of f and j. For the sake of completeness, we include a simple proof of this lemma as Lemma A.2.1 in the Appendix A.2. It follows from this result that

$$\begin{aligned}\mu(Q_j) &\leq \sum_{i=1}^{M_j}\mu\Big\{\pi\in Q:\ \Big|f(x_0)+\pi(x_0)-\big(f(y_0)+\pi(y_0)\big)\Big| \leq (4L+1)2^{-j}\Big\}\\ &\leq M_j C_3\Big(j^2 d_j^{1/2}(4L+1)2^{-j}\|z\|^{-1}\Big)^N,\end{aligned}$$

where $\|z\| = \big\|P_j(x_0-y_0)\big\|$.

Now we have to show that the images of nearby points on X remain far apart, although distances between points on X are shrunk by f. Hence, to finish the proof, it is necessary to find a lower bound for $\|z\|$. So recall that $(x_0,y_0)\in X\times X$ and that every point of X lies within $2^{-j\zeta}/6m$ of $\mathbf{G}_{U_j}[\varphi_j]$. As in Olson and Robinson (2010), we have for all $x\in X$,

$$\begin{aligned}\big\|(I-P_j)x-\varphi_j(P_jx)\big\| &= \big\|(I-P_j)x-\varphi_j(u)+\varphi_j(u)-\varphi_j(P_jx)\big\|\\ &\leq \big\|(I-P_j)x-\varphi_j(u)\big\| + \big\|\varphi_j(u)-\varphi_j(P_jx)\big\|\\ &\leq \big\|(I-P_j)x-\varphi_j(u)\big\| + m\big\|u-P_jx\big\|\\ &\leq 2m\Big\|x-\big(u+\varphi_j(u)\big)\Big\|.\end{aligned}$$

It follows that, for all $x\in X$,

$$\big\|(I-P_j)x-\varphi_j(P_jx)\big\| \leq 2m\ \mathrm{dist}\big(x,\mathbf{G}_{U_j}[\varphi_j]\big).$$

So

$$\big\|(I-P_j)x-\varphi_j(P_jx)\big\| \leq 2m\ \frac{2^{-j\zeta}}{6m} = \frac{2^{-j\zeta}}{3}.$$

Thus we have

$$\begin{aligned}
\|x_0 - y_0\| &= \big\|P_j x_0 + (I-P_j)x_0 - P_j y_0 - (I-P_j)y_0\big\| \\
&\le \|P_j x_0 - P_j y_0\| + \big\|(I-P_j)x_0 - (I-P_j)y_0\big\| \\
&= \|z\| + \big\|(I-P_j)x_0 - \varphi_j(P_j x_0) + \varphi_j(P_j x_0) \\
&\quad -\varphi_j(P_j y_0) + \varphi_j(P_j y_0) - (I-P_j)y_0\big\| \\
&\le \|z\| + \big\|(I-P_j)x_0 - \varphi_j(P_j x_0)\big\| \\
&\quad + \big\|\varphi_j(P_j x_0) - \varphi_j(P_j y_0)\big\| + \big\|\varphi_j(P_j y_0) - (I-P_j)y_0\big\| \\
&\le \|z\| + 2^{-j\zeta}/3 + m\|z\| + 2^{-j\zeta}/3 \\
&\le 2m\|z\| + 2^{-j\zeta+1}/3.
\end{aligned}$$

Since $\|x_0 - y_0\| \ge 2^{-j\zeta}$, it follows that $\|z\| \ge 2^{-(j\zeta+1)}/3m$. Then

$$\begin{aligned}
\mu(Q_j) &\le M_j\Big(C_4 d_j^{1/2}(4L+1)j^2\ 2^{-j(1-\zeta)}\Big)^N \\
&= C_2^2\ 2^{2j\delta}\Big(C_4 d_j^{1/2}(4L+1)j^2\ 2^{-j(1-\zeta)}\Big)^N,
\end{aligned}$$

where $C_4 = C_3^{1/N}6m$. As $d_j \le C_1\ 2^{j\zeta\sigma}$,

$$\begin{aligned}
\mu(Q_j) &\le C_2^2\ 2^{2j\delta}C_4^N\ C_1^{N/2}\ 2^{j\zeta\sigma N/2}(4L+1)^N j^{2N}\ 2^{-j(1-\zeta)N} \\
&\le C_5 j^{2N}\ 2^{-j[N(1-\zeta(1+\sigma/2))-2\delta]},
\end{aligned}$$

where C_5 depends only on L, N, X, δ and σ. Since we can choose δ and σ sufficiently close to d and $\mathrm{dev}_m(X)$, respectively, we obtain that

$$\zeta < \frac{N-2d}{N\big(1+\mathrm{dev}_m(X)/2\big)}.$$

Thus the exponent of 2^{-j} in the bound on $\mu(Q_j)$ is positive. It follows that

$$\sum_{j=1}^{\infty}\mu(Q_j) < \infty.$$

By the Borel-Cantelli Lemma, μ-almost every $f+\pi$ is contained in a finite number of Q_j. Hence there exists a J such that, for all $j \ge J$, $|x-y| \ge 2^{-j\zeta}$ implies $\Big|f(x)+\pi(x) - \big(f(y)+\pi(y)\big)\Big| \ge 2^{-j}$.

Let $C = \max\,(2^\zeta, 2^{J\zeta}R)$, where $R = \sup\big\{\|x\| : x \in X\big\}$. Then

$$C\Big|f(x) - \pi(x) - \big(f(y) - \pi(y)\big)\Big|^\zeta \geq \|x - y\| \quad \text{for all} \quad (x, y) \in X \times X.$$

Therefore the conclusion of the theorem holds for a prevalent set S in the space of bounded linear functions from H into $\mathbb{R}^N$.

□

One can use the compact set $\mathcal{A} \subset H$ presented in Section 2.1.1 or the example given in Hunt and Kaloshin (1999) to show that the upper limit on the Hölder exponent (1.3) in terms of the thickness is sharp as $N \to \infty$, and so is the bound in Theorem 1.3.4. However, $\mathrm{dev}(\mathcal{A}) \leq \tau(\mathcal{A})$ implies that

$$\frac{1}{1 + \tau(\mathcal{A})/2} \leq \frac{1}{1 + \mathrm{dev}(\mathcal{A})/2}.$$

Thus we must have in this case equality between those two terms, and consequently the Lipschitz deviation of $\mathcal{A}$ is equal to its thickness. Therefore we obtain that the upper limit (3.2) is sharp in the limit $N \to \infty$.

Since the upper limit (3.2) tends to one when $\mathrm{dev}(X) = 0$, it would be interesting to have conditions guaranteeing that the the Lipschitz deviation is zero; we will give such conditions in Section 3.2.

3.1.3 Hausdorff dimension of compact sets and Lipschitz maps

We have just seen that a finite-dimensional compact subset X of a Hilbert space can be embedded into a Euclidean space of sufficiently high dimension, using linear maps whose inverse is Hölder continuous. It is interesting to study how the Hausdorff dimension of a compact set is affected by this embedding. In the Introduction we recalled (Theorem 1.3.7) a result of Ott et al. (2006) that provides a lower bound on the Hausdorff dimension of the image of a compact set by a prevalent set of linear mappings into finite-dimensional spaces, based on the thickness exponent. One can prove an analogous result using the Lipschitz deviation. One just needs to adapt the proof of Theorem 1.3.7 (see Ott et al., 2006, for details) by making similar modifications to the ones outlined above.

Let M be any subspace of the space of the Borel measurable functions from H to $\mathbb{R}^N$ that contains the bounded linear functions.

Theorem 3.1.4. *Let H be a Hilbert space. Let $X \subset H$ be a compact set with*

Lipschitz deviation $\mathrm{dev}(X)$. *For almost every function* $L \in M$*,*

$$\dim_{\mathrm{H}}(L(X)) \geq \min\left\{N, \frac{\dim_{\mathrm{H}}(X)}{1+\mathrm{dev}(X)/2}\right\}.$$

It is important to note that if $\mathrm{dev}(X) = 0$, then the Hausdorff dimension of X is preserved by 'typical' mappings L.

An analogue of Theorem 3.1.4 is not valid for the box-counting dimension, because there are examples, given by Järvenpää (1994), Falconer and Howroyd (1996) and Sauer and Yorke (1997), in which the box-counting dimension is not preserved under linear functions.

3.2 Approximate inertial manifolds and the Lipschitz deviation

In this section we will use the theory of approximate inertial manifolds to show that the global attractors for a variety of semilinear parabolic equations have Lipschitz deviation zero. This provides a partial answer to the conjecture of Ott, Hunt, and Kaloshin that many such attractors have zero thickness exponent.

3.2.1 Approximate inertial manifolds

Our aim in this section is to prove that the global attractors of some dynamical systems generated by a class of semilinear parabolic equations have zero Lipschitz deviation. Note that the same setting and notation will be used in Chapter 4, where more details will be given.

Following Eden et al. (1994) we consider a dissipative partial differential equation written as an abstract evolution equation of the form

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = R(u), \\ u(0) = u_0 \end{cases} \tag{3.5}$$

in a separable real Hilbert space H with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We suppose that A is a positive self-adjoint linear operator with compact inverse A^{-1} and dense domain $D(A) \subset H$. For each $\alpha \geq 0$, we denote by $D(A^\alpha)$ the domain of A^α, i.e. $D(A^\alpha) = \{u : A^\alpha u \in H\}$; these are Hilbert spaces with inner product $(u,v)_\alpha = (A^\alpha u, A^\alpha v)$ and norm $\|u\|_\alpha = \|A^\alpha u\|$.

Since A is self-adjoint and its inverse is compact, H has an orthonormal basis $(w_j)_{j\in\mathbb{N}}$ consisting of eigenfunctions of the operator A such that

$$\begin{cases} Aw_j = \lambda_j w_j, \qquad j = 1, ..., \\ 0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_j \le ..., \\ \lambda_j \to \infty \qquad \text{as} \qquad j \to \infty. \end{cases} \tag{3.6}$$

For $n \in \mathbb{N}$ fixed, we consider the projection P_n onto the space spanned by the first n eigenfunctions $w_1, ..., w_n$ of A, and set $Q_n = I - P_n$; H is the direct sum of the orthogonal spaces $P_n H$ and $Q_n H$.

We assume that for any initial condition $u_0 \in H$, there exists a unique solution given by $u(t; u_0) = S(t)u_0$, where $\{S(t)\}_{t \ge 0}$ is a continuous semigroup from H into itself. We suppose that the system is dissipative in the sense that there exists a compact absorbing set in H, i.e. a set that absorbs all bounded sets in finite time. Consequently, standard results guarantee that equation (3.5) possesses a global attractor $\mathcal{A}$ in H (see Hale (1988), Babin and Vishik (1992), Temam (1997), Robinson (2001), for example).

Moreover we assume that, for some $0 < \alpha \le 1/2$, the nonlinear term R restricted to $\mathcal{A}$ in (3.5) is uniformly Lipschitz into H such that,

$$\|R(u) - R(v)\| \quad \le \quad c\|A^\alpha(u - v)\|, \quad \text{for} \quad u, v \in \mathcal{A}, \tag{3.7}$$

where c is a constant depending only on $\mathcal{A}$. These assumptions are reasonable, because usually $\mathcal{A}$ is bounded in a more regular space than H. For example, if we consider the 2D Navier-Stokes equations with forcing $f \in L^2$, then $\mathcal{A}$ is bounded in H^2, so

$$\|B(u, u) - B(v, v)\| \le c\|u - v\|_{1/2}, \quad \text{for} \quad u, v \in \mathcal{A}$$

(see Constantin and Foias (1988), Temam (1997) for details). We can then consider a modified equation that provides the same asymptotic behaviour in a neighbourhood of the global attractor. This abstract setting includes the reaction-diffusion and the 2D Navier-Stokes equations, among others (see Eden et al. (1994) for example). Note that when the nonlinear term R is locally Lipschitz from $D(A^\alpha)$ into H standard results show that the initial value problem (3.5) defines a continuous semigroup $\{S(t)\}_{t \ge 0}$ from $D(A^\alpha)$ into itself, for $t \ge 0$ (see Henry (1981) for details). However, we are assuming here that solutions exist for every $u_0 \in H$ and using the abstract form (3.5) to deduce additional properties of the semigroup defined on H.

The theory of inertial manifolds was introduced as a convenient, although indirect, method to study the long-term behaviour of dissipative dynamical systems. The existence of an inertial manifold allows the construction of a finite-dimensional system of ordinary differential equations that determines the dynamics on the attractor.

Definition 3.2.1 (Foias et al., 1985)**.** *An* inertial manifold *for the system* (3.5) *is a finite-dimensional Lipschitz manifold* $\mathcal{M}$ *enjoying the following properties:*

(i) $\mathcal{M}$ *is positively invariant for the semigroup* $\{S(t)\}_{t\geq 0}$*, i.e.* $S(t)\mathcal{M} \subset \mathcal{M}, t \geq 0$;

(ii) $\mathcal{M}$ *attracts the orbits of* $\{S(t)\}_{t\geq 0}$ *at an exponential rate.*

Foias et al. (1985) also showed that, if a certain 'spectral gap condition' holds - if there exists an n such that $\lambda_{n+1} - \lambda_n > k\lambda_{n+1}^{\alpha}$, where k is a constant depending on R - then the system (3.5) possesses an inertial manifold $\mathcal{M}$. Unfortunately, this condition is very restrictive and there are many equations, such as the 2D Navier-Stokes equations, that do not satisfy it.

Consequently other approaches have been explored in the cases where an inertial manifold has not been shown to exist. In the context of the 2D Navier-Stokes equations, Foias et al. (1988a) introduced the concept of an approximate inertial manifold (AIM), which is a finite-dimensional Lipschitz manifold, whose neighbourhood contains the global attractor $\mathcal{A}$. Moreover, it is possible to obtain approximate inertial manifolds without the restrictive spectral gap condition.

For equations of the above form we will appeal to a result due to Eden et al. (1994) to prove the existence of a family of approximate inertial manifolds of 'exponential order', i.e. a family of Lipschitz manifolds that approximate the attractor at an exponential rate with respect to their dimension. Their result shows that such equations satisfy a version of the 'squeezing property' that was introduced for the Navier-Stokes equations by Foias and Temam (1979). This dichotomy property states that it is possible to split the phase space into a finite-dimensional subspace and its infinite-dimensional orthogonal complement in such a way that the flow is essentially characterized by a finite number of parameters.

Proposition 3.2.2 (Eden et al., 1994)**.** *Suppose that equation* (3.5) *satisfies the assumptions in* (3.6) *and* (3.7)*. Then, there exists a time* t^* *such that if* $S = S(t^*)$ *and* n *is sufficiently large, there exists a projection* P_n *of rank* n *such that for every* $u, v \in \mathcal{A}$ *either*

$$\|Q_n(Su - Sv)\| \leq \|P_n(Su - Sv)\| \tag{3.8}$$

(where $Q_n = I - P_n$*) or*

$$\|Su - Sv\| \leq \delta_n \|u - v\|, \tag{3.9}$$

with

$$\delta_n \leq c_0 \mathrm{e}^{-\sigma\lambda_{n+1}},$$

where c_0 and σ are constants depending only on c and α, defined by equation (3.7).

If an infinite-dimensional dynamical system satisfies this property, it is possible to prove that its global attractor lies within a small neighbourhood of an finite-dimensional Lipschitz manifold. The following result follows from an abstract version of an argument due to Foias et al. (1988a).

Proposition 3.2.3 (Robinson, 2001)**.** *If (3.8) and (3.9) hold then there exists a Lipschitz function $\Phi : P_n H \to Q_n H$,*

$$\|\Phi(p) - \Phi(\bar{p})\| \leq \|p - \bar{p}\| \quad \text{for all} \quad p, \bar{p} \in P_n H, \tag{3.10}$$

such that $\mathcal{A}$ lies within a $4\delta_n R_H$ neighbourhood of the graph of Φ

$$\mathbf{G}_{P_n H}[\Phi] = \{u \in H : u = p + \Phi(p), p \in P_n H\},$$

where R_H is such that $\|u\| \leq R_H$, for all $u \in \mathcal{A}$.

Therefore, if one considers the equation (3.5), that satisfies (3.6) and (3.7), it is possible to combine this general result with Proposition 3.2.2 in order to obtain a Lipschitz manifold $\mathcal{M}_n$ given as the graph of some 1-Lipschitz $\Phi : P_n H \to Q_n H$ such that

$$\operatorname{dist}(\mathcal{A}, \mathcal{M}_n) \leq K_0 \mathrm{e}^{-\sigma \lambda_{n+1}},$$

where $K_0 = 4c_0 R_H$ and σ are constants depending on the attractor $\mathcal{A}$ and on the non-linear term R. Note that we are only using the theory of approximate inertial manifolds to find a Lipschitz graph whose ε-neighbourhood contains the attractor. The possible dynamical interpretation of the AIM is unimportant.

We remark here that Debussche and Temam (1994) and Rosa (1995) developed constructive methods to obtain a family of Lipschitz functions, whose graphs $\mathcal{M}_n$ are n-dimensional smooth manifolds that approximate the attractor $\mathcal{A}$ at an exponential order, such that

$$\operatorname{dist}(\mathcal{A}, \mathcal{M}_n) \leq \bar{c_0} e^{-\bar{\sigma} \lambda_n^{1-\alpha}},$$

where $\bar{c_0}$, $\bar{c_1} > 0$ are constants. However, the non-constructive method of Proposition 3.2.3 is sufficient for our purposes. In Chapter 4, we present another method that is based on the continuity of the linear term A.

3.2.2 Zero Lipschitz deviation

In this section we use the above results to prove that the global attractors of models that can be written in the form (3.5) have zero Lipschitz deviation. Recent papers, such as Ott and Yorke (2005) and Ott et al. (2006), have highlighted the importance of obtaining such a result for the thickness exponent. In light of the results of Section 3.1, a bound on the Lipschitz deviation serves as well as a bound on the thickness.

Proposition 3.2.4. *Let $\mathcal{A}$ be the global attractor for a dynamical system generated by a dissipative partial differential equation of the form of* (3.5) *satisfying the assumptions in* (3.6) *and* (3.7). *Assume in addition that*

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0. \tag{3.11}$$

Then the Lipschitz deviation of $\mathcal{A}$ is zero.

As we want to take the limit superior through a sequence of ε_n that tends to zero (rather than as $\varepsilon \to 0$), we need hypothesis (3.11). Although this hypothesis may seem a little strong, many dissipative equations satisfy it because, for an elliptic equation of order $2p$ defined in $\Omega \subset \mathbb{R}^m$, $\lambda_n \sim n^{2p/m}$ (for more details, see Davies (1986)).

Proof. Note that $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence that tends to infinity as n tends to infinity and satisfies (3.11). Let $\varepsilon_n = K_0 \mathrm{e}^{-\sigma\lambda_{n+1}}$, where K_0, σ are constants and n is the rank of P_n.

It follows from Proposition 3.2.2 and 3.2.3 that the global attractor $\mathcal{A}$ is contained in an ε_n-neighbourhood of a finite-dimensional Lipschitz manifold $\mathcal{M}$, defined as a graph of $\Phi : P_n H \to Q_n H$, with

$$|\Phi(p) - \Phi(\bar{p})| \leq |p - \bar{p}| \quad \text{for all} \quad p, \bar{p} \in P_n H.$$

Therefore, $\delta_1(\mathcal{A}, \varepsilon_n) = n$ and by hypotheses (3.11)

$$\limsup_{n\to\infty} \frac{\log \delta_1(\mathcal{A}, \varepsilon_{n+1})}{-\log \varepsilon_n} = \limsup_{n\to\infty} \frac{\log(n+1)}{\sigma\lambda_{n+1} - \log K_0} = 0.$$

Now fix $\varepsilon > 0$ such that $\varepsilon_{n+1} \leq \varepsilon < \varepsilon_n$. Hence,

$$\frac{\log \delta_1(\mathcal{A}, \varepsilon)}{-\log \varepsilon} \leq \frac{\log \delta_1(\mathcal{A}, \varepsilon_{n+1})}{-\log \varepsilon_n}.$$

Then, it follows from

$$\limsup_{n\to\infty} \frac{\log \delta_1(\mathcal{A}, \varepsilon_{n+1})}{-\log \varepsilon_n} = 0$$

that

$$\mathrm{dev}_1(\mathcal{A}) = \limsup_{\varepsilon\to 0} \frac{\log \delta_1(\mathcal{A}, \varepsilon)}{-\log \varepsilon} = 0.$$

Therefore, any global attractor $\mathcal{A}$ for a dynamical system generated by a dissipative partial differential equation of the form (3.5) has $\mathrm{dev}(\mathcal{A}) = 0$. □

Hence, one can apply this result to show that the 2D Navier-Stokes equations with forcing function $f \in L^2$ has a global attractor with zero Lipschitz deviation. Meanwhile, the attractor of the 2D Navier-Stokes equations has zero thickness exponent only if the forcing function $f \in H^k$ for any k (see Constantin and Foias (1988), and Friz and Robinson (1999)).

3.2.3 Consequences for attractors with zero Lipschitz deviation

Given the results of Section 3.1, Proposition 3.2.4 has two immediate consequences. First, we can obtain embeddings of global attractors with zero Lipschitz deviation in $\mathbb{R}^N$ that have a Hölder continuous inverse whose exponent is arbitrarily close to 1 by taking N sufficiently large (Theorem 3.1.3). Furthermore, the Hausdorff dimension of these sets is preserved under typical Lipschitz mappings into $\mathbb{R}^N$ (Theorem 3.1.4).

We now discuss briefly one further consequence for global attractors. Robinson (2005, Theorem 5.1) showed that the dynamics on finite-dimensional attractors can be reconstructed using a sufficient number of observations at equally spaced times, providing an infinite-dimensional version of results of Takens (1981) and Sauer et al. (1991). Simply by using Theorem 3.1.3 instead of Theorem 1.3.4 in the argument developed in Robinson (2005) one can replace the thickness exponent of the original result with the Lipschitz deviation. The assumption that $\mathrm{dev}(\mathcal{A}) = 0$ is now natural, and the result in this form requires the same number of observations as are required by Sauer et al. (1991) for a finite-dimensional attractor in a finite-dimensional system:

Theorem 3.2.5. *Let $\mathcal{A}$ be a compact subset of a Hilbert space H that has upper box-counting dimension d, zero Lipschitz deviation, and is an invariant set for a Lipschitz map $\Phi : H \to H$. Choose an integer $k > 2d$, and suppose further that the set $\mathcal{A}_p$ of p-periodic points of Φ satisfies $d_F(\mathcal{A}_p) < p/2$. Then a prevalent set of Lipschitz maps $L : H \to \mathbb{R}$ make the k-fold observation map $D_k[L, \Phi] : H \to \mathbb{R}^k$ defined by $D_k[L, \Phi](u) = (L(u), L(\Phi(u)), ..., L(\Phi^{k-1}(u)))$, injective on $\mathcal{A}$.*

Chapter 4

Log-Lipschitz continuity of the vector field

In this chapter we discuss the conditions under which a global attractor $\mathcal{A}$ associated with a dissipative parabolic equation lies in a Lipschitz graph over a finite number of Fourier modes. We then study the smoothness of the vector field restricted to $\mathcal{A}$.

4.1 Notation and general setting

Consider a dissipative parabolic equation written as an evolution equation of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = F(u) \tag{4.1}$$

in a separable real Hilbert space H with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We suppose that A is a positive self-adjoint linear operator with compact inverse and dense domain $D_H(A) \subset H$. For each $\alpha \geq 0$, we denote by $D_H(A^\alpha)$ the domain of A^α, i.e.

$$D_H(A^\alpha) = \{u : A^\alpha u \in H\};$$

these are Hilbert spaces with inner product $(u,v)_\alpha = (A^\alpha u, A^\alpha v)$ and norm $\|u\|_\alpha = \|A^\alpha u\|$. We know that for $\alpha > \beta$, the embedding $D_H(A^\alpha) \subset D_H(A^\beta)$ is dense and continuous such that

$$\|u\|_\beta \leq \widetilde{C}(\alpha,\beta)\|u\|_\alpha, \quad \text{for } u \in D_H(A^\alpha) \tag{4.2}$$

(see Sell and You (2002), for details). We assume that, for some $0 < \alpha \leq 1/2$, the nonlinear term F is locally Lipschitz from $D_H(A^\alpha)$ into H, such that for $u, v \in$

$D_H(A^\alpha)$,

$$\big\|F(u) - F(v)\big\| \quad \leq \quad K(R)\big\|A^\alpha(u-v)\big\|, \quad \text{with} \quad \|A^\alpha u\|, \|A^\alpha v\| \leq R, \qquad (4.3)$$

where K is a constant depending only on R. This abstract setting includes, among others, the 2D Navier-Stokes equations and the original Burgers equation with Dirichlet boundary values (see Eden et al. (1994), Temam (1997) for example).

Since A is self-adjoint densely defined operator and its inverse is compact, H has an orthonormal basis $\{w_j\}_{j\in\mathbb{N}}$ consisting of eigenfunctions of A such that

$$Aw_j = \lambda_j w_j \quad \text{for all} \quad j \in \mathbb{N}$$

with $0 < \lambda_1 \leq \lambda_2, ...$ and $\lambda_j \to \infty$ as $j \to \infty$. With $n \in \mathbb{N}$ fixed, define the finite-dimensional orthogonal projections P_n and their orthogonal complements Q_n by

$$P_n u = \sum_{j=1}^{n}(u, w_j)w_j \quad \text{and} \quad Q_n u = \sum_{j=n+1}^{\infty}(u, w_j)w_j.$$

Hence, we can write $u = P_n u + Q_n u$, for all $u \in H$. The orthogonal projections P_n and Q_n are bounded on the Hilbert spaces $D_H(A^\alpha)$, for any $\alpha > 0$ (see (4.2)). Notice that $P_n H = P_n D_H(A^\alpha) \subset D_H(A^\alpha)$, since $P_n H$ is a finite-dimensional subspace generated by the eigenvectors of A corresponding to the first n eigenvalues of A. These spectral projections commute with the operators e^{-At} for $t > 0$, i.e., $P_n\mathrm{e}^{-At} = \mathrm{e}^{-At}P_n$ and $Q_n\mathrm{e}^{-At} = \mathrm{e}^{-At}Q_n$. Moreover, it follows from Henry (1981, Section 1.5) that

$$\|\mathrm{e}^{-At}Q_n u\|_\alpha \quad \leq \quad \sup_{j\geq n+1}\big\{\lambda_j^\alpha \mathrm{e}^{-\lambda_j t}\big\}\|Q_n u\| \leq b_{n,\alpha}(t)\|Q_n u\|,$$

where

$$b_{n,\alpha}(t) = \begin{cases} \left(\dfrac{\mathrm{e}t}{\alpha}\right)^{-\alpha}, & \text{for} \quad 0 < t \leq \alpha/\lambda_{n+1} \\ \\ \lambda_{n+1}^\alpha \mathrm{e}^{-\lambda_{n+1}t}, & \text{for} \quad t \geq \alpha/\lambda_{n+1} \end{cases}$$

Therefore,

$$\Big\|A^\alpha \mathrm{e}^{-At}Q_n\Big\|_{\mathscr{L}(H,H)} \leq b_{n,\alpha}(t). \qquad (4.4)$$

Within this general setting, one can prove the local existence and uniqueness of solutions of (4.1) (see Henry (1981) for details). In particular, it follows from Henry (1981, Lemma 3.3.2) that the solution of the nonlinear equation (4.1), with initial condition $u(t_0) = u_0 \in D_H(A^\alpha)$, is given by the variation of constants formula

$$u(t) = \mathrm{e}^{-A(t-t_0)}u_0 + \int_{t_0}^{t} \mathrm{e}^{-A(t-s)} F\big(u(s)\big)\ \mathrm{d}s, \tag{4.5}$$

for $t > t_0$.

Thus, we can define $\{\Phi_t\}_{t\geq 0}$ to be the semigroup in $D_H(A^\alpha)$ generated by (4.1) such that, for any initial condition $u_0 \in D_H(A^\alpha)$, there exists a unique solution given by $u(t; u_0) = \Phi_t u_0$. We assume that this system is dissipative, i.e. that there exists a compact invariant absorbing set. It follows from standard results that (4.1) possesses a global attractor $\mathcal{A}$, the maximal compact invariant set in $D_H(A^\alpha)$ that uniformly attracts the orbits of all bounded sets (see Hale (1988), Babin and Vishik (1992), Temam (1997), Robinson (2001)). So, if $u(0) = u_0 \in \mathcal{A}$, then there is a unique solution $u(t) = \Phi_t u_0 \in \mathcal{A}$ that is defined for all $t \in \mathbb{R}$.

4.2 Finite-dimensionality of flows

Inertial manifolds, as discussed in Definition 3.2.1 of Chapter 3, are a convenient, although indirect, method to obtain a system of ordinary differential equations that reproduces the asymptotic dynamics on the global attractor. Romanov (2000) considered a more general definition of what it means for a system to be asymptotically finite-dimensional. We will see that this definition implies the existence of a Lipschitz manifold that contains the attractor, but does not require it to be exponentially attracting. Romanov (2000) defined the dynamics on a global attractor $\mathcal{A}$ to be *finite-dimensional* if for some $N \geq 1$ there exist:

(i) an ordinary differential equation $\dot{x} = \mathcal{H}(x)$ with a Lipschitz vector field $\mathcal{H}(x)$ in $\mathbb{R}^N$,

(ii) a corresponding flow $\{S_t\}$ on $\mathbb{R}^N$ and

(iii) a bi-Lipschitz embedding $\Pi : \mathcal{A} \to \mathbb{R}^N$, such that $\Pi(\Phi_t u) = S_t \Pi(u)$ for any $u \in \mathcal{A}$ and $t \geq 0$.

It follows from this definition that the evolution operators Φ_t are injective on $\mathcal{A}$ for $t > 0$. If we set $\Phi_{-t} = \Pi^{-1} S_{-t} \Pi$, then we see that in fact Φ_t is Lipschitz on $\mathcal{A}$ even for $t < 0$. Hence, we obtain a Lipschitz flow $\{\Phi_t\}$ defined on $\mathcal{A}$ for all $t \in \mathbb{R}$. In particular, there exist $C \geq 1$ and $\mu > 0$ such that

$$\big\|\Phi_t u - \Phi_t v\big\|_\alpha \leq C\big\|u - v\big\|_\alpha \mathrm{e}^{\mu|t|}, \tag{4.6}$$

for every $t \in \mathbb{R}$.

Considering the general Banach space case, Romanov (2000) proved that the finite-dimensionality of the dynamics on the attractor $\mathcal{A}$ is equivalent to five different criteria. In this chapter, however, we are only interested in the consequences of the finite-dimensionality of the dynamics on $\mathcal{A}$. Since our setting is simpler than Romanov (2000), the arguments involved in the proof become more transparent. Hence, we include here a concise and self-contained proof that if the attractor $\mathcal{A}$ has 'finite-dimensional dynamics' in Romanov's sense, then it must lie on a finite-dimensional manifold, defined as the graph of a Lipschitz function over P_nH, for some $n < \infty$.

Theorem 4.2.1 (Romanov, 2000)**.** *If the dynamics on $\mathcal{A}$ is finite-dimensional, then given any γ with $\alpha \leq \gamma < 1$ there exists an n_0 such that for any $n \geq n_0$*

$$\big\|Q_n(u-v)\big\|_\gamma \leq c\big\|P_n(u-v)\big\|_\alpha \quad \text{for all} \quad u, v \in \mathcal{A}, \tag{4.7}$$

where $c = c(\mathcal{A}, n, \gamma, \alpha)$.

Proof. First consider the variation of constants formula (4.5) with $t = 0$ and $u(0) = u \in \mathcal{A}$. If we apply the projection operator Q_n to both sides of (4.5), then

$$Q_n u = Q_n \mathrm{e}^{At_0} u(t_0) + \int_{t_0}^{0} \mathrm{e}^{As} Q_n F\big(u(s)\big)\ \mathrm{d}s.$$

Now, since the compact set $\mathcal{A}$ is bounded in $D_H(A^\alpha)$ and $u(t) \in \mathcal{A}$, it follows from (4.4) that $\lim_{t_0 \to -\infty} \|Q_n \mathrm{e}^{At_0} u(t_0)\|_\alpha = 0$. Consequently, letting t_0 tend to $-\infty$ we obtain

$$Q_n u = \int_{-\infty}^{0} \mathrm{e}^{As} Q_n F(\Phi_s u)\ \mathrm{d}s,$$

which converges in $D_H(A^\alpha)$. It follows from (4.6) that, for $u, v \in \mathcal{A}$,

$$\begin{aligned}
\big\|Q_n u - Q_n v\big\|_\gamma &\leq \int_{-\infty}^{0} \Big\|\mathrm{e}^{As} Q_n\big(F(\Phi_s u) - F(\Phi_s v)\big)\Big\|_\gamma\ \mathrm{d}s \\
&\leq K \int_{-\infty}^{0} \Big\|A^\gamma \mathrm{e}^{As} Q_n\Big\|_{op} \big\|\Phi_s u - \Phi_s v\big\|_\alpha \mathrm{d}s \\
&\leq KC\big\|u - v\big\|_\alpha \int_{-\infty}^{0} \Big\|A^\gamma \mathrm{e}^{As} Q_n\Big\|_{op} \mathrm{e}^{\mu|s|} \mathrm{d}s.
\end{aligned}$$

Using estimate (4.4) with $t = -s$, we find that

$$\left\|Q_n u - Q_n v\right\|_\gamma \le KC\left\|u - v\right\|_\alpha \int_{-\infty}^{0} b_{n,\gamma}(-s)\mathrm{e}^{-\mu s}\mathrm{d}s$$

from which we obtain the inequality

$$\left\|Q_n u - Q_n v\right\|_\gamma \le \vartheta_n \left\|u - v\right\|_\alpha, \tag{4.8}$$

where

$$\vartheta_n := \frac{1}{KC}\left\{\left(\frac{\mathrm{e}}{\gamma}\right)^{-\gamma}\left(\frac{\gamma}{\lambda_{n+1}}\right)^{1-\gamma}\frac{1}{1-\gamma} + \frac{\lambda_{n+1}^{\gamma}}{\lambda_{n+1}-\mu}\,\mathrm{e}^{\frac{-\gamma(\lambda_{n+1}-\mu)s}{\lambda_{n+1}}}\right\},$$

can be obtained by simple algebraic manipulation.

Note that, since $\gamma < 1$ and λ_{n+1} tends to infinity as $n \to \infty$, one can choose n sufficiently large to ensure that $\vartheta_n < 1$. Since $P_n + Q_n = I$, it follows that

$$\begin{aligned} \|Q_n(u_0 - v_0)\|_\gamma &\le \vartheta_n \|P_n(u_0 - v_0)\|_\alpha + \vartheta_n \|Q_n(u_0 - v_0)\|_\alpha \\ &\le \frac{\vartheta_n}{1-\vartheta_n}\|P_n(u_0 - v_0)\|_\alpha. \end{aligned}$$

□

Under the assumption that the non-linear term F is in $C^2(D_H(A^\alpha), H)$, Romanov (2000) showed that the finite-dimensionality of the dynamics on $\mathcal{A}$ implies that the vector field $\mathcal{G}(u) = -Au + F(u)$ is Lipschitz[1]. However, it is not clear how to adapt his argument to prove that A is Lipschitz. Nevertheless, a simple argument shows that finite-dimensionality implies that the operator A^β is Lipschitz in $\mathcal{A}$, provided that $\alpha + \beta < 1$.

Corollary 4.2.1. *If the dynamics on $\mathcal{A}$ is finite-dimensional, then, for β with $\alpha + \beta < 1$, A^β is Lipschitz on $\mathcal{A}$, i.e.*

$$\left\|A^\beta(u - v)\right\|_\alpha \le M\left\|u - v\right\|_\alpha, \quad \text{for all} \quad u, v \in \mathcal{A},$$

where α is given by (4.3).

[1]If $F \in C^2\big(D_H(A^\alpha), H\big)$, then it follows from Henry (1981, Corollary 3.4.6) that the map $(u_0, t) \mapsto u(t)$ is also in $C^2\big(\mathbb{R}^+ \times D_H(A^\alpha), D_H(A^\alpha)\big)$. Hence, the function $(u_0, t) \mapsto \mathrm{d}u(t)/\mathrm{d}t$ is C^1 with respect to (u_0, t). Since $\mathrm{d}u(t)/\mathrm{d}t = \mathcal{G}\big(u(t)\big)$, for a fixed time (we choose $t = 1$), the map $u_0 \mapsto \mathcal{G}\big(u(1)\big)$ is also a C^1-function and, consequently, a Lipschitz function. The finite dimensionality of the dynamics on $\mathcal{A}$ implies that the map $u_0 \mapsto u(1)$ is bi-Lipschitz on $\mathcal{A}$. And, therefore, the map $u(1) \mapsto \mathcal{G}\big(u(1)\big)$ is Lipschitz continuous.

Proof. It follows from Theorem 4.2.1 that, for all $u, v \in \mathcal{A}$,

$$\begin{aligned} \|A^\beta(u-v)\|_\alpha &= \|u-v\|_{\alpha+\beta} \leq \|P_n(u-v)\|_{\alpha+\beta} + \|Q_n(u-v)\|_{\alpha+\beta} \\ &\leq \left(\lambda_n^\beta + \frac{\vartheta_n}{1-\vartheta_n}\right)\|P_n(u-v)\|_\alpha \leq M\|u-v\|_\alpha. \end{aligned}$$

□

Note, however, that the requirement in Romanov's definition that $\mathcal{A}$ admits a bi-Lipschitz embedding into some $\mathbb{R}^N$ is very strong and unlikely to be satisfied in general. A sensible way to weaken this definition would be to relax the bi-Lipschitz assumption and assume the embedded vector field $\mathcal{H}$ to be just log-Lipschitz. However, the argument used in the proof of Theorem 4.2.1 would, then, not work.

Another possible option would be to remove the assumption that the flow is generated by an ODE and obtain the following minimum requirement for the dynamics on the attractor to be finite-dimensional:

Definition 4.2.2. *The dynamics on a global attractor $\mathcal{A}$ is* finite-dimensional *if, for some $N \geq 1$, there exist an embedding $\Pi : \mathcal{A} \to \mathbb{R}^N$ that is injective on $\mathcal{A}$, a flow $\{S_t\}$ in $\mathbb{R}^N$ and a global attractor X, such that the dynamics on $\mathcal{A}$ and X are conjugate under Φ via $\Pi(\Phi_t u) = S_t\Pi(u)$, for any $u \in \mathcal{A}$ and $t \geq 0$.*

(One can weaken this definition by assuming $\mathcal{A}$ to be just an invariant set.) However, even in this weak sense, it is still an open problem whether the finite-dimensionality of the global attractor $\mathcal{A}$ implies that the dynamics on $\mathcal{A}$ is finite-dimensional.

4.3 Log-Lipschitz continuity of the vector field

In the last section, we showed that if the dynamics on the attractor is finite-dimensional, then A^β is Lipschitz on $\mathcal{A}$ provided that $\alpha + \beta < 1$, where α is given by (4.3). It is relatively easy to show that the converse is also true (see Robinson (2003)).

Proposition 4.3.1. *Suppose that A^β is Lipschitz continuous on the global attractor $\mathcal{A}$ from $D_H(A^\alpha)$ into itself, i.e.*

$$\|A^\beta u - A^\beta v\|_\alpha \leq M\|u-v\|_\alpha \quad \text{for all} \quad u, v \in \mathcal{A}$$

for some $M > 0$. Then, the attractor is a subset of a Lipschitz manifold given as a graph over $P_N H$ for some N.

The proof of this result follows from a similar argument to the one developed for the proof of Proposition 4.4.1 below.

Now consider the embedded vector field on $X = L\mathcal{A}$

$$\dot{x} = h(x) = L\mathcal{G}L^{-1}(x),\ x \in X.$$

As remarked in Chapter 1, we would like the inverse of the embedding L to be as smooth as possible and to obtain as much regularity as we can for $\mathcal{G}$. However, in general, the regularity of $\mathcal{G}$ is determined by the regularity of the linear term A, which can be related to the smoothness of functions on the attractor $\mathcal{A}$. For example, it follows from the standard interpolation inequality

$$\|Au - Av\| \leq \|u - v\|^{r/(1+r)}\|A^{1+r}(u - v)\|^{1/(1+r)}, \quad \text{for } u, v \in \mathcal{A}, \tag{4.9}$$

that, if $\mathcal{A}$ is bounded in $D_H(A^{1+r})$, A is Hölder continuous on $\mathcal{A}$. In this way, the continuity of F on $\mathcal{A}$ can be deduced from the regularity of solutions on the attractor.

As an example of how one can develop this approach, Foias and Temam (1979) showed that, in the two dimensional case, the solutions of the Navier-Stokes equations are analytic in time and that the attractor is bounded in $D_H(A^{1/2}\mathrm{e}^{\tau A^{1/2}})$. Now, if $u \in D_H(A^{1/2}\mathrm{e}^{\tau A^{1/2}})$, then there exists an uniform constant $M > 0$, such that $\|A^{1/2}\mathrm{e}^{\tau A^{1/2}}u\|^2 < M$. Hence, $\|A^k u\|^2 \leq M'(4k)!/(2\tau)^{4k}$, where M' is a constant depending uniquely on M. It follows from (4.9), that

$$\|A(u - v)\| \leq \left[\frac{M'(4j)!}{(2\tau)^{4j}}\right]^{1/2j} \|u - v\|^{1-1/j}.$$

If we minimise the right-hand side over all possible choices of j, using the estimate

$$\left[\frac{M'(4j)!}{(2\tau)^{4j}}\right]^{1/2j} \leq cj^2,$$

we obtain that $A : \mathcal{A} \to H$ is 2-log-Lipschitz (see Robinson (2003) for example).

This result relies only on the smoothness of solutions. But one can do much better by making use of the underlying equation. Indeed, Kukavica (2007) used the structure of the differential equation (4.1) and far less restrictive conditions on $\mathcal{A}$ than above to show that $A^{1/2} : \mathcal{A} \to H$ is 1/2-log-Lipschitz. We briefly outline his argument, which was primarily developed to study the problem of backwards uniqueness for nonlinear equations with rough coefficients, and then show that it can be used to prove that $A : \mathcal{A} \to H$ is 1-log-Lipschitz.

In what follows we will consider the same equation as in Section 5.1

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = F(u). \tag{4.10}$$

However, here, we will assume that $\alpha = 1/2$ such that the nonlinear term F is locally Lipschitz from $D_H(A^{1/2})$ into H, i.e.

$$\big\|F(u) - F(v)\big\| \quad \leq \quad K(R)\big\|A^{1/2}(u-v)\big\|, \quad \text{for all} \quad u, v \in D_H(A^{1/2}), \tag{4.11}$$

with $\|A^{1/2}u\|, \|A^{1/2}v\| \leq R$, where K is a constant depending only on R. Moreover, we assume that the maximal invariant set $\mathcal{A}$ is bounded in $D_H(A^{1/2})$. The argument that follows is simple – the key observation is that the result is sufficiently abstract that one can make a variety of choices of H (e.g. we will take $H = L^2$ and $H = H^1$).

Let $u(t)$ and $v(t)$ be solutions of (4.10). The equation for the evolution of the difference $w(t) := u(t) - v(t)$ can be expressed as

$$\frac{\mathrm{d}w}{\mathrm{d}t} + Aw = f, \tag{4.12}$$

where $f(t) := F\big(u(t)\big) - F\big(v(t)\big)$. Our assumptions imply that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(Aw, w) = (w_t, Aw) = -(Aw, Aw) + (f, Aw) \tag{4.13}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(Aw, Aw) = (w_t, A^2w) = -(Aw, A^2w) + (f, A^2w). \tag{4.14}$$

Moreover, it follows from (4.11) that

$$\|f\| \leq \|F(u) - F(v)\| \leq K_1\|A^{1/2}w\|, \tag{4.15}$$

where $K_1 = K(R)$ and R is the bound on $\mathcal{A}$ in $D_H(A^{1/2})$. Consequently,

$$(f, w) \geq -K_2\|w\|\|A^{1/2}w\| \tag{4.16}$$

for some $K_1, K_2 \geq 0$.

Under these mild regularity assumptions, Kukavica (2007) proved the backward uniqueness property, i.e. if $w : [T_0, 0] \to H$ is a solution of (4.12), then $w(0) = 0$ implies that $w(t) = 0$ for all $t \in [T_0, 0]$. His approach consists in establishing upper bounds for the log-Dirichlet quotient

$$\widetilde{Q}(t) = \frac{(Aw(t), w(t))}{\|w(t)\|^2 \Big(\log \frac{M^2}{\|w(t)\|^2} \Big)},$$

where M is a sufficiently large constant. This quantity is a variation of the standard Dirichlet quotient $Q(t) = \|A^{1/2}w\|^2/\|w\|^2$ (see Ogawa (1965), Bardos and Tartar (1973) for details). Kukavica showed that, for equations of the form of (4.12), the log-Dirichlet quotient is bounded for all $t \geq 0$ and, as an application of this result, stated the following theorem.

Theorem 4.3.2 (After Kukavica, 2007)**.** *Under the above assumptions on the equation* (4.10) *with $F : D_H(A^{1/2}) \to H$ and $\mathcal{A} \subset D_H(A^{1/2})$, there exists a constant $C > 0$ such that*

$$\|A^{1/2}(u-v)\|^2 \leq C\|u-v\|^2 \log(M^2/\|u-v\|^2), \quad \text{for all} \quad u, v \in \mathcal{A},\ u \neq v,$$

where $M = 4\sup_{u\in\mathcal{A}} \|u\|$.

We give a quick summary of Kukavica's proof, filling in some details in the closing part of the argument.

Proof. Let

$$L(\|w\|) = \log \frac{M^2}{\|w\|^2},$$

where M is any constant such that

$$M \geq 4 \sup_{u_0 \in \mathcal{A}} \|u_0\|.$$

Note that $L(\|w(t)\|) \geq 1$ for all $t \in [0, T_0]$. For $t \in [0, T_0]$, denote $\widetilde{L}(t) = L\big(\|w(t)\|\big)$. Define the log-Dirichlet quotient as

$$\widetilde{Q}(t) = \frac{Q(t)}{L\big(\|w\|\big)} = \frac{\|A^{1/2}w\|^2}{\|w\|^2 L\big(\|w\|\big)} = \frac{\|A^{1/2}w\|^2}{\|w\|^2 \widetilde{L}(t)}$$

where $Q(t) = \|A^{1/2}w\|^2/\|w\|^2$.

Using (4.13) and (4.14), Kukavica (2007) showed in the proof of his Theorem 2.1 that

$$\widetilde{Q}'(t) + K_3\widetilde{Q}(t)^2 \leq K_4, \tag{4.17}$$

with $K_3 = 1/2$ and $K_4 = 2K_1^4$. Applying a variant of Gronwall's inequality[2] proved in Temam (1997, Lemma 5.1) to (4.17), we obtain that there exists T such that

$$\widetilde{Q}(t) \leq C(K_3, K_4), \quad \text{for all} \quad t \geq T,$$

where $C(K_3, K_4)$ and T are constants independent of $\widetilde{Q}(0)$.

Now, consider $u_0, v_0 \in \mathcal{A}$. Since solutions in the attractor exist for all time, we know there exists $t \geq T$ such that $u_0 = S(t)u(-t)$ and $v_0 = S(t)v(-t)$ with $u_0 \neq v_0$. So, $u(-t) \neq v(-t)$. Moreover, $\widetilde{Q}(-t) < \infty$ implies that $\widetilde{Q}(0) \leq C(K_3, K_4)$. Hence,

$$\sup_{u_0, v_0 \in \mathcal{A},\ u_0 \neq v_0} \widetilde{Q}(t) \leq C(K_3, K_4).$$

□

We now show that this result can be used to show that $A : \mathcal{A} \to H$ is 1-log-Lipschitz. Write $w = u - v$. If (4.15) and (4.16) hold with $H = L^2$, then there exits a constant $C_0 > 0$ such that

$$\|A^{1/2}w\|_{L^2}^2 \leq C_0 \|w\|_{L^2}^2 \log\left(M_0^2 / \|w\|_{L^2}^2\right), \tag{4.20}$$

where

$$M_0 \geq 4 \sup_{u \in \mathcal{A}} \|u\|_{L^2}.$$

This is the result of Kukavica (2007, Theorem 3.1) for the 2D Navier-Stokes equation.

Now assume that $\mathcal{A}$ is bounded in $D_H(A)$. If (4.15) and (4.16) hold with with $H = D_{L^2}(A^{1/2})$, then there exits a constant $C_1 > 0$ such that

$$\|Aw\|_{L^2}^2 \leq C_1 \|A^{1/2}w\|_{L^2}^2 \log\left(M_1^2 / \|A^{1/2}w\|_{L^2}^2\right) \tag{4.21}$$

where

$$M_1 \geq 4 \sup_{u \in \mathcal{A}} \|A^{1/2}u_0\|_{L^2}.$$

So,

$$\|Aw\|_{L^2}^2 \leq C_0 C_1 \|w\|_{L^2}^2 \log\left(M_0^2 / \|w\|_{L^2}^2\right) \log\left(M_1^2 / \|w\|_{H^1}^2\right).$$

[2]Lemma 5.1 (p167 in Temam, 1997): *Let y be a positive absolutely continuous function on $(0, \infty)$, which satisfies*

$$y' + \gamma y^p \leq \delta \tag{4.18}$$

with $p > 1$, $\gamma > 0$, $\delta \geq 0$. Then, for $t > 0$

$$y(t) \leq \left(\frac{\delta}{\gamma}\right)^{1/p} + \left(\gamma(p-1)t\right)^{-1/(p-1)}. \tag{4.19}$$

Since $\|w\|_{L^2} \leq \|w\|_{H^1}$,

$$\|Aw\|_{L^2}^2 \leq C_0 C_1 \|w\|_{L^2}^2 \log\left(M_0^2/\|w\|_{L^2}^2\right) \log\left(M_1^2/\|w\|_{L^2}^2\right).$$

One can choose M_0 and M_1 such that $M_0 \leq M_1$. Hence,

$$\|Aw\|_{L^2} \leq C\|w\|_{L^2} \log\left(M_1^2/\|w\|_{L^2}^2\right), \tag{4.22}$$

where $C = \sqrt{C_0 C_1}$.

Corollary 4.3.1. *Under the above assumptions on the equation* (4.10)*, if $\mathcal{A}$ is bounded in $D_H(A)$, then there exists a constant $C > 0$ such that*

$$\|A(u-v)\| \leq C\|u-v\| \log(M_1^2/\|u-v\|^2), \quad \text{for all} \quad u, v \in \mathcal{A},\ u \neq v,$$

where $M_1 \geq 4 \sup_{u \in \mathcal{A}} \|A^{1/2}u\|$.

Unfortunately, this result is not strong enough to prove the existence of a smooth finite-dimensional invariant manifold that contains the attractor. Hence, it would be interesting to know whether, in such a general setting, the 1-log-Lipschitz continuity, obtained for the linear term A, is sharp or if it can be improved. Nevertheless, one can use Corollary 4.3.1 to show that there exists a family of approximating Lipschitz manifolds $\mathcal{M}_N$, given as Lipschitz graphs defined over a N-dimensional spaces, such that the global attractor $\mathcal{A}$ associated with equation (4.10) lies within an exponentially small neighbourhood of $\mathcal{M}_N$ without a making use of the squeezing property.

4.4 Family of Lipschitz manifolds

Using the inequality (4.22) obtained in Section 4.3, one can show, for a wide class of parabolic equations, the existence of a family of Lipschitz manifolds $\mathcal{M}_N$ such that

$$\operatorname{dist}(\mathcal{M}_N, \mathcal{A}) \leq C\mathrm{e}^{-k\lambda_{N+1}},$$

where $\mathcal{M}_N$ is an N-dimensional manifold and C and k are positive constants. To obtain this result, we just assume the log-Lipschitz continuity of linear term A.

Proposition 4.4.1. *Suppose that, for some $C > 0$,*

$$\|Aw\|_{L^2} \leq C\|w\|_{L^2} \log\left(M_1^2/\|w\|_{L^2}^2\right), \tag{4.23}$$

where $w = u - v$ for $u, v \in \mathcal{A}$. Then, under the above conditions on equation (4.12), *for each $n > 0$, there exists a Lipschitz function $\Phi_n : P_nH \to Q_nH$,*

$$\|\Phi_n(p_1) - \Phi_n(p_2)\|_{L^2} \leq \|p_1 - p_2\|_{L^2} \quad \text{for all} \quad p_1, p_2 \in P_nH,$$

such that $\mathcal{A}$ lies within a $2M_1^2\mathrm{e}^{-\lambda_{n+1}/\sqrt{2}C}$-neighbourhood of the graph Φ_n,

$$\mathbf{G}[\Phi_n] = \{u \in H : u = p + \Phi_n(p), p \in P_nH\}.$$

Note that the method developed in this proof can also be used to prove Proposition 4.3.1.

Proof. Let $w = u - v$, for $u, v \in \mathcal{A}$. We can split $w = P_nw + Q_nw$, and observe that

$$\begin{aligned}\|Aw\|_{L^2}^2 &= \|A(P_nw + Q_nw)\|_{L^2}^2 = \|A(P_nw)\|_{L^2}^2 + \|A(Q_nw)\|_{L^2}^2 \\ &\geq \lambda_{n+1}^2\|Q_nw\|_{L^2}^2.\end{aligned}$$

It follows from (4.23) that

$$\begin{aligned}\|Aw\|_{L^2}^2 &\leq C^2\|w\|_{L^2}^2\Big(\log\big(M_1^2/\|w\|_{L^2}^2\big)\Big)^2 \\ &\leq C^2\big(\|P_nw\|_{L^2}^2 + \|Q_nw\|_{L^2}^2\big)\Big(\log\big(M_1^2/\|Q_nw\|_{L^2}^2\big)\Big)^2.\end{aligned}$$

Since $\log\big(M_1^2/\|Q_nw\|_{L^2}^2\big) > 1$,

$$\frac{\lambda_{n+1}^2\|Q_nw\|_{L^2}^2}{\Big(\log\big(M_1^2/\|Q_nw\|_{L^2}^2\big)\Big)^2} \leq C^2\|P_nw\|_{L^2}^2 + C^2\|Q_nw\|_{L^2}^2$$

Consider a subset X of $\mathcal{A}$ that is maximal for the relation

$$\|Q_n(u - v)\|_{L^2} \leq \|P_n(u - v)\|_{L^2} \quad \text{for all} \quad u, v \in X. \tag{4.24}$$

Note that if the P_n components of u and v agree, so that $P_nu = P_nv$, then $Q_nu = Q_nv$. Hence, for every $u \in X$, we can define uniquely $\phi_n(P_nu) = Q_nu$ such that $u = P_nu + \phi_n(P_nu)$. Moreover, it follows from (4.24) that

$$\|\phi_n(p_1) - \phi_n(p_2)\|_{L^2} \leq \|p_1 - p_2\|_{L^2} \quad \text{for all} \quad p_1, p_2 \in P_nX.$$

Standard results (see Wells and Williams (1975), for example) allow one to extend ϕ_n to a function $\Phi_n : P_n H \to Q_n H$, that satisfies the same Lipschitz bound.

Now, if $u \in \mathcal{A}$ but $u \notin X$, it follows that

$$\|Q_n(u-v)\|_{L^2} \geq \|P_n(u-v)\|_{L^2},$$

for some $v \in X$. Thus, if $w = u - v$, then

$$\frac{\lambda_{n+1}^2 \|Q_n w\|_{L^2}^2}{\left(\log\left(M_1^2/\|Q_n w\|_{L^2}^2\right)\right)^2} \leq 2C^2 \|Q_n w\|_{L^2}^2.$$

Hence,

$$\|Q_n w\|_{L^2}^2 \leq M_1^2 \mathrm{e}^{-\lambda_{n+1}/\sqrt{2}C},$$

which implies that

$$\begin{aligned}\|w\|_{L^2}^2 = \|P_n w\|_{L^2}^2 + \|Q_n w\|_{L^2}^2 &\leq 2\|Q_n w\|_{L^2}^2 \\ &\leq 2M_1^2 \mathrm{e}^{-\lambda_{n+1}/\sqrt{2}C}.\end{aligned}$$

Therefore,

$$\mathrm{dist}(u, \mathbf{G}[\Phi_n]) \leq 2M_1^2 \mathrm{e}^{-\lambda_{n+1}/\sqrt{2}C}. \tag{4.25}$$

□

A similar statement would hold if one used the inequality (4.20) obtained by Kukavica (2007), involving $A^{1/2}$, rather than (4.23) that considers A. However, one would obtain a worse exponent in (4.25), since λ_{n+1} would be replaced by $\lambda_{n+1}^{1/2} \leq \lambda_{n+1}$.

To illustrate this result, we consider the incompressible Navier-Stokes equations

$$\begin{aligned}\partial_t u - \nu \triangle u + u \cdot \nabla u + \nabla p &= F, \\ \nabla \cdot u &= 0,\end{aligned}$$

with periodic boundary conditions on $\Omega = [0, 2\pi]^2$ and initial condition $u(x,0) = u_0(t)$. Here $u(x,t)$ is the velocity vector field, $p(x,t)$ the pressure scalar function, ν the kinematic viscosity and $F(x,t)$ represents the volume forces that are applied to the fluid.

We restrict ourselves to the space-periodic case for simplicity. Let $\mathcal{H}$ be the space of all the C^∞ periodic divergence-free functions that have zero average on Ω. Let H be the closure of $\mathcal{H}$ with scalar product $(\cdot,\cdot)_{L^2}$ and norm $\|\cdot\|_{L^2}$, and let V

be similarly the closure of $\mathcal{H}$ with scalar product $(\cdot,\cdot)_{H^1}$ and norm $\|\cdot\|_{H^1}$. Let A be the Stokes operator defined by

$$Au = -\triangle u,$$

for all u in the domain $D(A)$ of A in H. Now consider the Navier-Stokes equations written in its functional form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) = F, \tag{4.26}$$

using the operator A and the bilinear operator B from $V \times V$ into V' defined by

$$(B(u,v),w) = b(u,v,w), \quad \text{for all} \quad u,v,w \in V.$$

If $F \in H$ is independent of time, then the equation (4.26) possesses a global attractor

$$\mathcal{A} = \left\{ u_0 \in H : S(t)u_0 \text{ exists for all } t \in \mathbb{R},\ \sup_{t\in\mathbb{R}} \|S(t)u_0\|_{L^2_{\mathrm{per}}(\Omega)} < \infty \right\},$$

where $S(t)u_0$ denotes a solution starting at u_0 on its maximal interval of existence (cf. Constantin and Foias (1988)). Under these assumptions, the difference of solutions $w = u - v$ will satisfy

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu Aw = -\big[B(w,u) + B(v,w)\big].$$

So, in this case we use Kukavica's Theorem with $f = -\big[B(w,u) + B(v,w)\big]$. Note that

$$\|f\|_{H^1} \leq K_1 \|A^{1/2}w\|_{H^1},$$

and consequently

$$(f, Aw) \geq -K_2 \|w\|_{H^1} \|A^{1/2}w\|_{H^1}.$$

Hence, one can apply Proposition 4.4.1 to the two dimensional Navier-Stokes equation with forcing $F \in L^2$ to show the existence of a family of approximate inertial manifolds of exponential order.

It follows from the arguments developed in Chapter 3 that the existence of a family of approximating Lipschitz manifolds for a dissipative equation of the form of (4.10) implies that the associated global attractor have zero Lipschitz deviation. Proposition 4.3.1 implies that if the linear term A is Lipschitz continuous, the inverse of the projection operators P_n, for some n, is also Lipschitz continuous. However, if A is only log-Lipschitz continuous, the projection operators P_n has a Hölder continuous inverse, whose exponent can be made arbitrarily close to one by choosing n sufficiently large.

Chapter 5

Embedded vector field with non-trivial dynamics on the global attractor

The material in this chapter was produced in collaboration with Dr. Jaime J. Sánchez-Gabites.

5.1 Embedding the dynamics on the global attractor into a Euclidean space

In this section, we construct a system of ordinary differential equations with unique solutions that reproduces the dynamics on $\mathcal{A}$, under the assumption that $\dim_{\mathrm{A}}(\mathcal{A}-\mathcal{A})<\infty$. However, we postpone any discussion of the existence of an attractor for the new system until Section 5.3.

Proposition 5.1.1. *Suppose that the dissipative evolution equation*

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\mathcal{G}(u), \quad u \in H, \tag{5.1}$$

has a global attractor $\mathcal{A}$ such that $d:=\dim_{\mathrm{A}}(\mathcal{A}-\mathcal{A})<\infty$, where $\dim_{\mathrm{A}}$ denotes Assouad dimension. Assume that the vector field $\mathcal{G}$ is Lipschitz continuous on $\mathcal{A}$. Then, for any $m>2(1+d)$, there exist a system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=g(x), \quad x \in \mathbb{R}^m, \tag{5.2}$$

and a bounded linear map $L: H \longrightarrow \mathbb{R}^m$ such that:

1. the function $g : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is bounded and Lipschitz continuous except for a logarithmic correction term,

2. the ordinary differential equation (5.2) *has unique solutions,*

3. the restriction $L|_{\mathcal{A}} : \mathcal{A} \longrightarrow L\mathcal{A}$ is an embedding whose image is invariant under (5.2)*,*

4. for every solution $u(t)$ of (5.1) *on the global attractor $\mathcal{A}$, there exists a unique solution $x(t)$ of* (5.2) *such that*

$$u(t) = L^{-1}\big(x(t)\big). \tag{5.3}$$

Since the focus of our discussion is now the regularity of the inverse of the linear embedding L when restricted to $\mathcal{A}$, we assumed the very strong condition that the vector field $\mathcal{G}$ is Lipschitz continuous from $\mathcal{A}$ into itself. In the above proposition, one can relax the condition on $\mathcal{G}$ and assume that the vector field is log-Lipschitz with exponent $\alpha < 1/2$ (see Pinto de Moura et al., 2010), but this is stronger than the result we obtained in the previous chapter (1-log-Lipschitz).

Proof. It follows from Theorem 1.3.9 that there exists a bounded linear map L from H into $\mathbb{R}^m$, that is injective on $\mathcal{A}$ and has a Lipschitz continuous inverse on $L\mathcal{A}$ except for a logarithmic correction term with logarithmic exponent γ.

If $x(t) = Lu(t)$, where $u(t) \in \mathcal{A}$, then the embedded vector field on $L\mathcal{A}$ that reproduces the dynamics on $\mathcal{A}$ is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = L\mathcal{G}L^{-1}(x), \;\; x \in L\mathcal{A}.$$

The function $g_1 : L\mathcal{A} \longrightarrow \mathbb{R}^m$ such that $g_1(x) = L\mathcal{G}(L^{-1}(x))$ is certainly continuous and bounded, since $L\mathcal{A}$ is compact.[1]

Next we shall consider the modulus of continuity of g_1. Given any $u, v \in \mathcal{A}$, define $Lu = x$ and $Lv = y$. It follows from Theorem 1.3.9 that there exist C_L, γ and δ_L such that

$$C_L\|L^{-1}x - L^{-1}y\| \geq |x-y| \geq \frac{1}{C_L}\frac{\|L^{-1}x - L^{-1}y\|}{\Big(-\log\big(\|L^{-1}x - L^{-1}y\|\big)\Big)^{\gamma}},$$

[1]Note that Eden et al. (1994) are careful to show that the linear embedding L does not create artificial fixed points. In our construction this result follows from the fact that we have enough smoothness to guarantee the uniqueness of solutions.

for $\|u-v\| \leq \delta_L$. Consequently, since $|Lu - Lv| \leq C_L\|u-v\|$, for every $x, y \in L\mathcal{A}$,

$$\begin{aligned}
\|L^{-1}x - L^{-1}y\| &\leq C_L\Big(-\log\big(\|L^{-1}x - L^{-1}y\|\big)\Big)^{\gamma}|x-y| \\
&\leq C_L\left(\log\left(\frac{C_L}{|x-y|}\right)\right)^{\gamma}|x-y| \leq C_L\ f_1(|x-y|),
\end{aligned}$$

where

$$f_1(|x|) = |x|\left(\log\left(\frac{C_L}{|x|}\right)\right)^{\gamma}. \tag{5.4}$$

Since we assumed that $\mathcal{G}$ is Lipschitz continuous, it follows that

$$\begin{aligned}
\big|g_1(x) - g_1(y)\big| &= \Big|L\mathcal{G}\big(L^{-1}(x)\big) - L\mathcal{G}\big(L^{-1}(y)\big)\Big| \\
&\leq \|L\|_{\mathrm{op}}\Big|\mathcal{G}\big(L^{-1}(x)\big) - \mathcal{G}\big(L^{-1}(y)\big)\Big| \\
&\leq \|L\|_{\mathrm{op}}K\big\|L^{-1}(x) - L^{-1}(y)\big\| \\
&\leq C_L K\|L\|_{\mathrm{op}} f_1\big(|x-y|\big).
\end{aligned}$$

Hence g_1 is Lipschitz continuous except for a logarithmic correction term. The modulus of continuity ω of g_1 is therefore the convex continuous function defined by

$$\omega(r) = C_L K\|L\|_{\mathrm{op}} f_1(r) = C_0 r\Big(\log\big(C_L/r\big)\Big)^{\gamma}, \quad \text{for } r \geq 0,$$

where $C_0 = C_L K\|L\|_{\mathrm{op}}$ is a constant.

However g_1 is only defined on the compact set $L\mathcal{A}$. To extend g_1 to the whole of $\mathbb{R}^m$, maintaining essentially the same modulus of continuity, first note that

$$\omega(0) = 0 \tag{5.5}$$

and that

$$\omega(r) > 0, \quad \text{if } r > 0. \tag{5.6}$$

Furthermore,

$$\begin{aligned}
\omega(r+s) &= C_0(r+s)\Big(\log\big(C_L/(r+s)\big)\Big)^{\gamma} \\
&= C_0 r\Big(\log\big(C_L/(r+s)\big)\Big)^{\gamma} + C_0 s\Big(\log\big(C_L/(r+s)\big)\Big)^{\gamma} \\
&\leq C_0 r\Big(\log\big(C_L/r\big)\Big)^{\gamma} + C_0 s\Big(\log\big(C_L/s\big)\Big)^{\gamma} \\
&= \omega(r) + \omega(s)
\end{aligned}$$

Hence the modulus of continuity of g_1 is convex. Therefore, ω is a positive increasing function of r, and regular in the sense that:

(i) $\omega(r)/r = C_0\Big(\ln\big(C_L/r\big)\Big)^{\gamma}$ is increasing as $r \to 0$ and

(ii) $\omega(2r) \leq c\,\omega(r)$, for some constant $c > 0$.

As the modulus of continuity ω satisfy all the above conditions, we can use the extension theorem due to (McShane, 1934, Theorem 2) (see also Stein, 1970, Corrolary of Theorem VI.3) to extend the function g_1 to a function $g : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ that is Lipschitz continuous except for a logarithmic correction term such that

$$|g(x) - g(y)| \leq M\omega\big(|x - y|\big), \tag{5.7}$$

for some $M > 0$. It follows from (5.7) that there exists a $T > 0$ such that the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x), \quad x(0) = x_0 \tag{5.8}$$

has at least one solution on $[0, T]$.

Now assume that $x(t)$ and $y(t)$ are solutions of (5.8) with initial conditions $x(0) = x_0$ and $y(0) = y_0$, respectively. Let $r(t) = |x(t) - y(t)|$. Since the modulus of continuity $\omega(r)$ of g is continuous for $r \geq 0$, convex and verifies

$$\int_0^1 \frac{\mathrm{d}r}{\omega(r)} = \int_{\ln(C_L)}^{\infty} s^{-\gamma}\mathrm{d}s = +\infty, \quad \text{for } 0 < \gamma \leq 1, \tag{5.9}$$

we can use Osgood's Criterion (see Hartman, 1964, for example) to show that (5.8) has at most one solution on any interval $[0, T]$, if the exponent γ of the logarithmic term in (5.4) is no larger than one. Since g is continuous and bounded from $\mathbb{R}^m$ into $\mathbb{R}^m$, it follows that any solution of the initial value problem (5.8) exists for all time. Therefore the solution of (5.8) through $x_0 = Lu_0$ with $u_0 \in \mathcal{A}$ can be uniquely given by

$$x(t) = Lu(t).$$

$\square$

It is important to remark that the regularity of the embedding obtained for Banach spaces is not enough to construct a finite-dimensional dynamical system with the same asymptotical behaviour as the original infinite-dimensional system. Since the limiting value of the logarithmic correction term γ in Theorem 1.3.10 is strictly greater than one, uniqueness of solutions of the embedded equation cannot be guaranteed even if $\mathcal{G}$ is Lipschitz.

5.2 Construction of an ordinary differential equation with prescribed global attractor

In the previous section, we have embedded the global attractor $\mathcal{A}$ into some Euclidean space $\mathbb{R}^m$ via a linear map $L : H \longrightarrow \mathbb{R}^m$ and showed that there is a differential equation (5.2) in $\mathbb{R}^m$ that has unique solutions and reproduces the dynamics of $\mathcal{A}$ on $L\mathcal{A}$. Ideally we would like to construct a new system of ordinary differential equations that reproduces the dynamics of $\mathcal{A}$ on $L\mathcal{A}$ and has $L\mathcal{A}$ as a global attractor. However we do not know if one can construct such a system, because of a topological obstruction.

Although every global attractor in a Euclidean space has the *cellularity* property - whose definition is recalled below - one can not guarantee that $L\mathcal{A}$ is a cellular set in $\mathbb{R}^m$ (see Garay, 1991). So, we will have to modify L by increasing the dimension m of the target space by one in order to obtain a new linear map L' such that $L'\mathcal{A} := L\mathcal{A} \times \{0\}$ is indeed a cellular set in $\mathbb{R}^{m+1}$. Influenced by Garay (1991) and Günther (1995), we show in this section that every cellular set in a Euclidean space is a global attractor, comprised of equilibria, for an entirely new system of ordinary differential equations in $\mathbb{R}^{m+1}$ and then apply this result to $L'\mathcal{A}$.

In Section 5.3, we will use this new system to modify (5.2) in such a way that its solutions enter asymptotically any prescribed neighbourhood of $L'\mathcal{A}$. Consequently, we will obtain a dynamical system that reproduces the dynamics of $\mathcal{A}$ on $L'\mathcal{A}$ and has a global attractor $\mathcal{X}$ lying within any prescribed (arbitrarily small) neighbourhood of $L'\mathcal{A}$.

5.2.1 Improving the linear embedding L

Definition 5.2.1. *A set C is called a* m-cell *if there exists a homeomorphism from $B_{\mathbb{R}^m}(1)$ onto C, where $B_{\mathbb{R}^m}(1)$ is the closed unit ball centered at the origin in $\mathbb{R}^m$.*

Definition 5.2.2. *A subset $X \subseteq \mathbb{R}^m$ is* cellular *in $\mathbb{R}^m$ if there exists a cellular sequence for X, that is, a sequence $\{C_i\}_{i\in\mathbb{N}}$ of m-cells in $\mathbb{R}^m$ such that $C_{i+1} \subset \mathrm{Int}C_i$ and $\bigcap_{i=1}^{\infty} C_i = X$. Alternatively, $X \subset \mathbb{R}^m$ is* cellular *if and only if for each open set $U \supset X$ there exists a m-cell C such that $X \subset \mathrm{Int}C \subset C \subset U$.*

It is important to remark that whether a set X is cellular or not depends not only on its topological type[2], but also on how it is embedded in $\mathbb{R}^m$. For example, the unit ball in $\mathbb{R}^3$ can be embedded in the standard way and therefore be cellular in $\mathbb{R}^3$. Or it can be embedded as an Alexander's horned ball (cf. Alexander, 1924), which

[2]Two spaces have the same topological type if they are homeomorphic.

is not cellular in $\mathbb{R}^3$ as its complement is not simply connected (see Blankinship and Fox, 1950). However, it follows from Daverman (1986, Corollary 5A, p.145) that it is cellular in $\mathbb{R}^4$.

Proposition 5.2.3. *Let $\mathcal{A}$ be a global attractor in H and let $L : H \longrightarrow \mathbb{R}^m$ be a linear embedding. Then the map $L' : H \longrightarrow \mathbb{R}^{m+1}$ defined by $L'u = (Lu, 0)$ is a linear embedding whose image $L'\mathcal{A}$ is cellular in $\mathbb{R}^{m+1}$, provided $m \geq 3$.*

Note that, although $\mathcal{A}$ is cellular and homeomorphic to $L\mathcal{A}$, $L\mathcal{A}$ is not necessarily cellular because the cellularity of a set depends on the embedding. Hence, we need to use a different property of $\mathcal{A}$ which is invariant under homeomorphisms.

Introduced by Borsuk in 1968, shape theory is a weakening of homotopy theory which makes it extremely useful to deal with complicated sets, by roughly overlooking their local structure. If two spaces have the same homotopy type, then they will have the same shape. In particular, if two spaces are homeomorphic, then they will have the same shape. For detailed information on shape theory see Borsuk (1975), Mardešić and Segal (1982).

Proof of Proposition 5.2.3. It follows from Theorem 3.6 in Kapitanski and Rodnianski (2000, p. 233) that the global attractor $\mathcal{A}$ has the same shape as H. Since the map $H \times [0, 1] \ni (u, t) \longrightarrow (1 - t) \cdot u \in H$ provides a homotopy between the identity $\mathrm{id} : H \longrightarrow H$ and the constant map $0 : H \longrightarrow H$, it follows from Theorem B.1.1 that H has the homotopy type of a point. Therefore H has the shape of a point and consequently $\mathcal{A}$ also has the shape of a point. Since shape is invariant under homeomorphisms, $L\mathcal{A}$ has also the shape of a point. Thus[3] by Daverman (1986, Corollary 5A, p.145) the set $L\mathcal{A} \times \{0\}$ is cellular in $\mathbb{R}^{m+1}$, provided $m \geq 3$. But $L\mathcal{A} \times \{0\}$ is precisely $L'\mathcal{A}$. □

5.2.2 Cellular sets are global attractors for systems of ordinary differential equations

Next we will show that if X is a cellular subset of $\mathbb{R}^{m+1}$, then there exists a system of ordinary differential equations with X as its global attractor consisting entirely of fixed points. Günther (1995) proved a similar result for compact sets with the shape of a finite polyhedron, but did not need to control the size of the region of attraction (whereas we want it to be all of $\mathbb{R}^{m+1}$). By restricting ourselves to a less general setting and considering only compacts sets with the shape of a point, we

[3] Daverman uses the concept of cell-likeness instead of "having the shape of a point", but both are equivalent. See Section 15 in Daverman (1986).

are able to give a significantly simpler proof that does not involve piecewise linear topology.

Lemma 5.2.4. *Given a cellular subset X of $\mathbb{R}^{m+1}$, with $m > 5$, there is a mapping $\phi : \mathbb{R}^{m+1} \longrightarrow [0, +\infty)$ of class $\mathcal{C}^r$, where r can be chosen to be arbitrarily large, such that the system generated by*

$$\dot{x} = -\nabla\phi(x) \tag{5.10}$$

has X as a global attractor. Furthermore, the mapping ϕ can be chosen such that

(i) $\phi(x) = 0$ if $x \in X$ and $\phi(x) > 0$ if $x \notin X$;

(ii) ϕ is proper, i.e. $\phi^{-1}([s,t])$ is compact for any $s < t \in \mathbb{R}$.

It follows from Lemma 5.2.4 that $\nabla\phi(x) = 0$ if and only if $x \in X$, because the zeros of $\nabla\phi(x)$ are precisely the equilibria of (5.10) that cannot be outside of X. Conversely, if $\phi : \mathbb{R}^{m+1} \longrightarrow [0, +\infty)$ is any $\mathcal{C}^r$ mapping such that $\nabla\phi(x) = 0$ if and only if $x \in X$, and $\phi(x) = 0$ if and only if $x \in X$, then by Lyapunov's theorem (Theorem 2.2 in Bhatia and Szegö, 1992) X is a global attractor for $\dot{x} = -\nabla\phi(x)$. Thus we only need to construct such a ϕ, which we do first on $\mathbb{R}^{m+1}\backslash X$ and then extend to all of $\mathbb{R}^{m+1}$.

Since cellularity is purely a topological notion and we want to obtain a differentiable map ϕ, we will need two auxiliary propositions. Denote $\mathbb{S}^m$ the unit sphere in $\mathbb{R}^{m+1}$, that is, $\mathbb{S}^m = \{x \in \mathbb{R}^{m+1} : \|x\| = 1\}$.

Proposition 5.2.5. *Let X be a cellular subset of $\mathbb{R}^{m+1}$. There exists a homeomorphism $h : \mathbb{R}^{m+1}\backslash X \longrightarrow \mathbb{S}^m \times (0, +\infty)$ such that the second coordinate of $h(x)$ converges to zero when x tends to X.*

Proof. Let Q be a ball in $\mathbb{R}^{m+1}$ centered at the origin such that X is contained in the interior of Q. It follows from Theorem 1 in Brown (1960) that there exists a continuous map $c : Q \longrightarrow Q$ that is surjective, injective on $Q \setminus X$, collapses X to a single point p in the interior of Q and is the identity on the boundary of Q. It is easy to construct a homeomorphism of Q onto itself that takes p to 0 and is the identity on the boundary, so we can assume that $p = 0$.

The properties of c imply that $c|_{Q\backslash X} : Q\backslash X \longrightarrow Q\backslash\{0\}$ is a homeomorphism and if $x \longrightarrow X$ then $c(x) \longrightarrow 0$. Extend $c|_{Q\backslash X}$ to all of $\mathbb{R}^{m+1}\backslash X$ by letting it be the identity outside Q. Finally,

$$h(x) := \left(\frac{c(x)}{\|c(x)\|}, \|c(x)\|\right)$$

has the required properties. □

In order to make h differentiable it will be necessary to use some smoothing results for manifolds, rather than maps, from Kirby and Siebenmann (1977). In Appendix C, we recall the definitions and some useful properties of differential manifolds, differential structures and differential maps. For more details, see Kirby and Siebenmann (1977).

Proposition 5.2.6. *Let X be a cellular subset of $\mathbb{R}^{m+1}$, with $m \geq 5$. There exists a mapping $\psi : \mathbb{R}^{m+1} \backslash X \longrightarrow (0, +\infty)$ of class $\mathcal{C}^\infty$ such that:*

(i) $\nabla \psi(x) \neq 0$ *for every* $x \in \mathbb{R}^{m+1} \backslash X$,

(ii) $\psi(x) \longrightarrow 0$ *when* $x \longrightarrow X$ *and*

(iii) ψ *is proper.*

Proof. Consider the map h obtained in Proposition 5.2.5. We would like ψ to be the second coordinate of h, but this choice would not be differentiable in general. Thus we first have to smooth h out. Let Σ be the differentiable structure $\mathbb{R}^{m+1} \backslash X$ inherits from $\mathbb{R}^{m+1}$ as an open subset, and transport it via h to obtain a new differentiable structure $h\Sigma$ on $\mathbb{S}^m \times (0, +\infty)$; clearly by construction $h : \big(\mathbb{R}^{m+1} \backslash X\big)_\Sigma \longrightarrow (\mathbb{S}^m \times (0, +\infty))_{h\Sigma}$ is a diffeomorphism. It follows from Theorem 5.1 in Kirby and Siebenmann (1977, p. 31) and Remark 1 following this theorem that there is a diffeomorphism $q : \big(\mathbb{S}^m \times (0, +\infty)\big)_{h\Sigma} \longrightarrow (\mathbb{S}^m)_\sigma \times (0, +\infty)$, where σ is some suitable differentiable structure on $\mathbb{S}^m$ (we need the hypothesis $m > 5$ precisely for this theorem to work). By Remark 1 following Kirby and Siebenmann (1977, Theorem 5.1, p. 31) one can require, and it will be technically convenient to do so, that $\mathrm{dist}\big(y, q(y)\big) \leq 1$ for every $y \in \mathbb{S}^m \times (0, +\infty)$, where dist is the maximum of the distances in $\mathbb{S}^m$ and $(0, +\infty)$.

It follows from the definition of a product differentiable structure that the projection onto the second factor $\mathrm{pr}_2 : (\mathbb{S}^m)_\sigma \times (0, +\infty) \longrightarrow (0, +\infty)$ is a $\mathcal{C}^\infty$ mapping and its differential is never zero. Then define $\psi := \mathrm{pr}_2 \circ q \circ h$, which makes the diagram

$$\begin{tikzcd}
(\mathbb{R}^{m+1} \backslash X)_\Sigma \arrow[r, "h"] \arrow[rrd, "\psi"'] & (\mathbb{S}^m \times (0, +\infty))_{h\Sigma} \arrow[r, "q"] & (\mathbb{S}^m)_\sigma \times (0, +\infty) \arrow[d, "\mathrm{pr}_2"] \\
& & (0, +\infty)
\end{tikzcd}$$

commutative. Clearly ψ is $\mathcal{C}^\infty$, because it is a composition of $\mathcal{C}^\infty$ maps. Now we have to check that ψ satisfies all the properties in the statement of the proposition:

(i) It is clear that $\nabla\psi(x) \neq 0$, because q and h are diffeomorphisms (thus their differentials are invertible) and pr_2 satisfies $\nabla \mathrm{pr}_2(x) \neq 0$.

(iii) Let $s < t \in \mathbb{R}$, consider a sequence $(x_i)_{i\in\mathbb{N}} \subseteq \psi^{-1}([s,t])$ and denote by $(y_i, z_i) := q \circ h(x_i)$. By hypothesis $\big((y_i, z_i)\big)_{i\in\mathbb{N}} \subseteq \mathbb{S}^m \times [s,t]$, which is a compact set, so the sequence $\big((y_i, z_i)\big)_{i\in\mathbb{N}}$ must have a convergent subsequence. The pre-image of this subsequence under the homeomorphism $q \circ h$ is a convergent subsequence of $(x_i)_{i\in\mathbb{N}}$. This shows that $\psi^{-1}([s,t])$ is compact and ψ is proper.

(ii) Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{m+1}\backslash X$ converging to X. We first show that $\big(\psi(x_i)\big)_{i\in\mathbb{N}}$ converges either to 0 or $+\infty$. Suppose not. Then it has some subsequence $\big(\psi(x_{i_j})\big)_{j\in\mathbb{N}}$ that is contained in a compact interval and, since ψ is proper, $(x_{i_j})_{j\in\mathbb{N}}$ is contained in some compact subset of $\mathbb{R}^{m+1}\backslash X$. This contradicts the fact that (x_i) converges to X.

Since we required that q moves points no more than 1 unit, we have

$$\mathrm{dist}\big(q \circ h(x_i)), h(x_i)\big) < 1.$$

Given that we chose dist as the maximum of the distances in $\mathbb{S}^m$ and $(0,+\infty)$, this implies that

$$\mathrm{dist}\big(\psi(x_i), \mathrm{pr}_2 \circ h(x_i)\big) = \mathrm{dist}\big(\mathrm{pr}_2 \circ q \circ h(x_i), \mathrm{pr}_2 \circ h(x_i)\big) < 1$$

as well. Since $\psi(x_i)$ converges to either 0 or $+\infty$ and $\mathrm{pr}_2 \circ h(x_i) \longrightarrow 0$ as stated in Proposition 5.2.5, it follows that $\psi(x_i) \longrightarrow 0$.

□

Proof of Lemma 5.2.4. We shall construct by induction a sequence of maps $\psi_k : \mathbb{R}^{m+1} \to [0,+\infty)$ of class $\mathcal{C}^k$, such that $\phi := \psi_k$ proves the lemma for $r = k$. First extend the mapping ψ given by Proposition 5.2.6 to the whole of $\mathbb{R}^{m+1}$ by letting it assume the value 0 on X, and call it ψ_0. The mapping ψ_0 is continuous but not differentiable near X. Hence we will use an argument given in Günther (1995) to improve ψ_0 to ψ_k.

The idea is to define $\psi_1 := b \circ \psi_0$, where $b : [0,+\infty) \longrightarrow [0,+\infty)$ is a diffeomorphism of class $\mathcal{C}^1$ whose derivative near 0 is sufficiently small to overcome the “roughness” of ψ_0 near X. Formally, for $x \in \mathbb{R}^{m+1}\backslash X$,

$$\frac{\partial}{\partial x_i}(b \circ \psi_0)(x) = (b' \circ \psi_0)(x)\, \frac{\partial \psi_0}{\partial x_i}(x),$$

so we need to choose b such that $b'\big(\psi_0(x)\big)\frac{\partial \psi_0}{\partial x_i}(x)$ converges to zero as x tends to X.

When ψ_k is already constructed, we set $\psi_{k+1} := b \circ \psi_k$, where $b : [0, +\infty) \longrightarrow [0, +\infty)$ is a suitable $\mathcal{C}^{k+1}$ diffemorphism. However, we now need to impose conditions on the rate at which $b^{(l)}(t) \longrightarrow 0$ as $t \longrightarrow 0$ for every $0 \leq l \leq k+1$. Indeed, for any multi-index α with $|\alpha| = k+1$,

$$\frac{\partial^\alpha \psi_{k+1}}{\partial x^\alpha} = (b' \circ \psi_k)\frac{\partial^\alpha \psi_k}{\partial x^\alpha} + P\left(\frac{\partial^\beta \psi_k}{\partial x^\beta}, b^{(l)}\right)$$

on $\mathbb{R}^{m+1} \backslash X$, where P is a polynomial in partial derivatives of ψ_k of order $|\beta| \leq k$ and derivatives of b of order $l \leq k+1$. Hence, it suffices to choose b subject to the conditions $b^{(l)}(0) = 0$ for every $l \leq k+1$ and $b'\big(\psi_k(x)\big)\frac{\partial^\alpha \psi_k}{\partial x^\alpha}(x) \longrightarrow 0$ as x tends to X and for $|\alpha| = k+1$ (observing that x tends to X if and only if $\psi_k(x)$ tends to zero). The existence of b is not entirely obvious, details are given in Pinto de Moura et al. (2010).

$\square$

5.3 Modified vector field with non-trivial dynamics on the global attractor

In this final section we combine the previous results to obtain a system of ordinary differential equations (5.11) that reproduces on $L\mathcal{A}$ the dynamics on $\mathcal{A}$ and has a global attractor $\mathcal{X}$ as close to $L\mathcal{A}$ as required. More precisely, we prove the following

Theorem 5.3.1. *Suppose that the dissipative evolution equation*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{G}(u), \quad u \in H, \tag{5.1}$$

has a global attractor $\mathcal{A}$ such that

$$d := \dim_{\mathrm{A}}(\mathcal{A} - \mathcal{A}) < \infty,$$

where $\dim_{\mathrm{A}}$ denotes Assouad dimension. Assume that the vector field $\mathcal{G}$ is Lipschitz continuous on $\mathcal{A}$. Then, for any $m > \max\{3 + 2d, 6\}$ and any prescribed $\varepsilon > 0$, there exist a system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{H}(x), \quad x \in \mathbb{R}^m \tag{5.11}$$

and a bounded linear map $L : H \to \mathbb{R}^m$ such that:

1. *the ordinary differential equation* (5.11) *has unique solutions,*

2. *the restriction* $L|_{\mathcal{A}} : \mathcal{A} \longrightarrow L\mathcal{A}$ *is an embedding whose image* $L\mathcal{A}$ *is invariant under the dynamics of* (5.11)*,*

3. *for every solution* $u(t)$ *of* (5.1) *on the attractor* $\mathcal{A}$ *there exists a unique solution* $x(t)$ *of* (5.11) *such that*
$$u(t) = L^{-1}(x(t)),$$

4. *the ordinary differential equation* (5.11) *has a global attractor* $\mathcal{X}$ *that contains* $L\mathcal{A}$ *and is contained in the* ε*–neighbourhood of* $L\mathcal{A}$ *i.e.* $\mathrm{dist_H}(\mathcal{X}, L\mathcal{A}) \leq \varepsilon$*.*

Note that although item *4.* is not ideal, we do obtain uniqueness of solutions which is certainly desirable. The construction in Eden et al. (1994, Chapter 10), for example, has the projection of $\mathcal{A}$ as a global attractor, but the finite-dimensional system of ordinary differential equations obtained lacks uniqueness (in fact $\mathcal{H}$ is not even continuous).

Proof. First one can use Proposition 5.2.3 to replace the mapping L obtained in Proposition 5.1.1 by a new one $L' : H \to \mathbb{R}^{m+1}$ with the additional property that its image is cellular. To keep notation simple we rename L' as L and $m+1$ as m.

Then we can use Lemma 5.2.4 to obtain a $\mathcal{C}^r$ mapping $\phi : \mathbb{R}^m \to [0, +\infty)$ such that $L\mathcal{A}$ is a global attractor for $\dot{x} = -\nabla\phi$. Denote by $B_\varepsilon(L\mathcal{A})$ the ε-neighbourhood of $L\mathcal{A}$ in $\mathbb{R}^m$. Since ϕ is proper, one can prove by contradiction using compactness that there exists $\delta > 0$ such that $P := \{x \in \mathbb{R}^m : \phi(x) \leq \delta\} \subseteq B_\varepsilon(L\mathcal{A})$.

Let $P_n := \{x \in \mathbb{R}^m : \phi(x) \leq 1/n\}$. Suppose, by contradiction, that there exists $\varepsilon > 0$ such that $C \cap P_n \neq \emptyset$, for every $n \in \mathbb{N}$, where $C = \big(B_\varepsilon(L\mathcal{A})\big)^c$. Choose $\{x_n\}_{n=1}^\infty$ such that $x_n \in C \cap P_n$ for each n. Since $P_n \supset P_m$, for every $n \leq m$, $x_n \in C \cap P_1$, for every n. Since ϕ is proper, each P_n is compact, and since C is closed, $C \cap P_n$ is compact, for every n. So, $\{x_n\}_{n=1}^\infty \subset C \cap P_1$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$.

Let $x = \lim_{k\to\infty} x_{n_k} \in C \cap P_1$. Since ϕ is continuous,
$$\phi(x) = \phi\Big(\lim_{k\to\infty} x_{n_k}\Big) = \lim_{k\to\infty} \phi\big(x_{n_k}\big) = \lim_{k\to\infty} \frac{1}{n_k} = 0.$$
So, $x \in L\mathcal{A}$, since $\phi(x) = 0$, which is a contradiction as $x \in C \cap P_1$.

Finally, let $\theta : \mathbb{R}^m \to [0, 1]$ be a $\mathcal{C}^\infty$ cut-off function such that $\theta \equiv 1$ on $L\mathcal{A}$ and $\theta \equiv 0$ outside of P. Take the mapping g obtained in Proposition 5.1.1 and multiply it by θ to make it zero outside of P. We shall call $f := \theta g$; clearly $\dot{x} = f(x)$ still reproduces the dynamics of $\mathcal{A}$ on $L\mathcal{A}$.

Now consider the equations

$$\dot{x} = -\nabla\phi(x), \tag{5.10}$$
$$\dot{x} = f(x) - \nabla\phi(x). \tag{5.12}$$

Observe that the right hand sides of (5.10) and (5.12) coincide for $x \notin P$. Therefore, since $\mathbb{R}^m \backslash P$ is negatively invariant for (5.10), it is also negatively invariant for (5.12) and it follows from Bhatia and Szegö (1992, Theorem 1.4, p.13) that P is positively invariant for (5.12).

The sets $\overline{P \cdot [t, +\infty)}$ are compact (being closed subsets of P) and decreasing with increasing t. It is standard that

$$\mathcal{X} := \bigcap_{t \geq 0} \overline{P \cdot [t, +\infty)}$$

is invariant and attracts P, i.e. given any $\delta > 0$ there exists $T_\delta > 0$ such that $P \cdot [T_\delta, +\infty) \subseteq B_\delta(\mathcal{X})$ (see Ladyzhenskaya, 1991, Theorem 2.1). By construction, $\mathcal{X}$ is contained in $B_\varepsilon(L\mathcal{A})$.

(1) $\mathcal{X}$ is a global attractor: Fix a bounded set $B \subseteq \mathbb{R}^m$ and let

$$M := \sup_{x \in B} \phi(x) \text{ and } \mu := \inf_{x \in B \backslash P} \|\nabla\phi(x)\|^2.$$

Observe that $\mu > 0$ because $\nabla\phi$ only vanishes on $L\mathcal{A}$, of which P is a neighbourhood. Thus there exists $T > 0$ big enough so that $M - \mu T < \delta$ holds.

We now claim that $x \cdot [T, +\infty) \subseteq P$ for any $x \in B$. Since P is positively invariant it clearly suffices to show that $x \cdot t \in P$ for some $t \in [0, T]$. We reason by contradiction, so assume that $x \cdot [0, T] \subseteq \mathbb{R}^m \backslash P$. By the mean value theorem

$$\phi(x \cdot T) = \phi(x) + \left.\frac{\mathrm{d}}{\mathrm{d}s}\phi(x \cdot s)\right|_{s=\xi} T$$

for some $\xi \in [0, T]$. Now

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\phi(x \cdot s)\right|_{s=\xi} = \langle \nabla\phi(x \cdot \xi), \dot{x}(\xi) \rangle = -\|\nabla\phi(x \cdot \xi)\|^2 \leq -\mu,$$

where we have used the fact that $\dot{x}(\xi) = -\nabla\phi(x \cdot \xi)$ because $x \cdot \xi \notin P$ by assumption and $\|\nabla\phi(x \cdot \xi)\|^2 \geq \mu$ by the same token. With the above equation and the fact that $\phi(x) \leq M$ because $x \in P$,

$$\phi(x \cdot T) \leq M - \mu T < \delta$$

which is a contradiction since then $x \cdot T \in P$ by definition.

Thus we see that $B \cdot [T, +\infty) \subseteq P$. Since given any $\delta > 0$ there exists $T_\delta > 0$ such that $P \cdot [T_\delta, +\infty) \subseteq B_\delta(\mathcal{X})$, $B \cdot [T + T_\delta, +\infty) \subseteq P \cdot [T_\delta, +\infty) \subseteq B_\delta(\mathcal{X})$. Thus for $t \geq T + T_\delta$ one has $\mathrm{dist}(B \cdot t, \mathcal{X}) < \delta$. This implies that $\mathrm{dist}(B \cdot t, \mathcal{X}) \to 0$ as $t \to +\infty$.

(2) $\mathcal{X}$ contains $L\mathcal{A}$: Since $\nabla\phi$ vanishes on $L\mathcal{A}$ and $\theta \equiv 1$ on it, (5.12) reduces to $\dot{x} = g(x)$ when $x \in L\mathcal{A}$. Thus $L\mathcal{A}$ is invariant for (5.12) and it is an immediate consequence of the fact that $L\mathcal{A} \subseteq P$ and the expression for $\mathcal{X}$ that $L\mathcal{A} \subseteq \mathcal{X}$ (alternatively, since $\mathcal{X}$ is the maximal compact invariant set in $\mathbb{R}^m$, clearly $L\mathcal{A} \subseteq \mathcal{X}$).

□

Chapter 6

Conclusion

Following Ben-Artzi et al. (1993), we have presented in Chapter 2 a simpler example than the one given by Hunt and Kaloshin (1999) to show that linear embeddings of finite-dimensional subsets of a Hilbert space or a Banach space into a Euclidean space, as proved by Hunt and Kaloshin (1999) and Robinson (2009), have sharp asymptotic bounds for the Hölder exponent of their inverses. Similarly, we proved that the exponent of the logarithmic correction term in the embedding theorem proved by Robinson (2010) is sharp as $N \to \infty$.

In Chapter 3, we have refined the definition of the Lipschitz deviation, a quantity that measures how well a compact subset X of a Hilbert space H can be approximated by Lipschitz graphs over a finite-dimensional spaces. We have shown that the existing results that rely on the 'thickness exponent' remain valid when this is replaced by the Lipschitz deviation. Moreover, one can find simple examples for which the Lipschitz deviation is strictly less than the thickness exponent.

Furthermore, answering in part the conjecture of Ott et al. (2006) that 'many of the attractors associated with evolution equations of mathematical physics have thickness exponent zero', we have shown that the attractors of a large class of semi-linear parabolic equations have zero Lipschitz deviation. In particular, we showed that for the attractor of the 2D Navier-Stokes equations with forcing $f \in L^2$ the Lipschitz deviation is zero.

However, the definition of the Lipschitz deviation (and hence all the results presented in Chapter 3) is restricted to subsets of Hilbert spaces. In Banach spaces one can define the 'dual thickness exponent' (Robinson, 2009), a quantity that is bounded above by the Lipschitz deviation in Hilbert spaces. Using this definition one can prove a result parallel to Theorem 3.1.3 that is valid for subsets of Banach spaces. It would be interesting to see whether the corresponding version of Theorem

3.1.4 remains true in Banach spaces using this definition.

In Chapter 4, we studied conditions under which the global attractor $\mathcal{A}$ is a subset of a Lipschitz manifold given as a graph over a finite-dimensional eigenspace of the linear term A. Then, we showed that, since the linear term of a wide class of dissipative partial differential equations is 1-Log-Lipschitz continuous, the associated global attractor $\mathcal{A}$ lies within a small neighbourhood of a finite-dimensional Lipschitz manifold. Consequently, we are able to use the arguments developed in Chapter 3 to obtain linear embeddings of the attractor into $\mathbb{R}^N$, whose inverse is Hölder continuous with exponent arbitrarily close to one by choosing N sufficiently large.

The existence of a system of ordinary differential equation whose asymptotic behaviour reproduces the dynamics on an arbitrary finite-dimensional global attractor remains an interesting open problem. Nevertheless, if we are able to show that there exist exponents $\eta > 0$ and $\gamma > 0$ such that the vector field on the attractor $\mathcal{A}$ is η-log-Lipschitz and the inverse of linear embedding $L : H \to \mathbb{R}^N$ is γ-log-Lipschitz when restricted to $L\mathcal{A}$, then we will obtain an embedded equation $\dot{x} = h(x)$ with unique solution, provided $\eta + \gamma \leq 1$. Therefore, we would like to improve the exponent 1 in Corollary 4.3.1. Indeed, if the result for the linear term A is optimal, then we need a bi-Lipschitz embedding to guarantee uniqueness of solutions. However, that Romanov (2000) obtained a better regularity result for the vector field, than we obtained for the linear term A, suggests that it may be possible to improve the logarithmic exponent in our result.

In Chapter 5, we showed that if the compact $\mathcal{A} \subset H$ is the global attractor associated with a dissipative evolution equation in H such that the vector field $\mathcal{G}$ is Lipschitz continuous on $\mathcal{A}$ and $\dim_{\mathrm{A}}(\mathcal{A} - \mathcal{A}) = d$, then there is an ordinary differential equation in $\mathbb{R}^{m+1}$, with $m > d$, that has unique solutions and reproduces the dynamics on $\mathcal{A}$. Moreover, we proved that the dynamical system generated by this new ordinary differential equation has a global attractor $\mathcal{X}$ arbitrarily close to $L\mathcal{A}$, where L is a bounded linear map from H into $\mathbb{R}^{m+1}$ that is injective on $\mathcal{A}$.

The results presented in this thesis highlight the importance of finding a general method to bound the Assouad dimension of the set $\mathcal{A} - \mathcal{A}$, where $\mathcal{A}$ is a global attractor associated with a partial differential equation in H. However Eden et al. (1994, Lemma 2.1) showed that, for a large class of dissipative equations for which the squeezing property holds, there exists a constant $K > 0$, such that the set $S(T)[\mathcal{A} \cap B(x, r)]$ can be covered by K balls of radius θr, for some $T > 0$. Hence, given its similarity with the doubling property, it might be possible to use their result to bound $\dim_{\mathrm{A}}(\mathcal{A})$. In addition one might be able to combine the above result with the fact, proved by Romanov (2000), that the function $u \mapsto \mathcal{G}\big(u(T)\big)$ is Lipschitz on

$\mathcal{A}$, for any fixed $T > 0$, to construct a system of ODEs whose asymptotic behaviour reproduces that of the original flow.

Appendix A

Estimates of 'how many' vectors in a ball have their image landing in a given small ball

A.1 Estimate for finite-dimensional case

For every $j \in \mathbb{N}$, let B_{d_j} be the unit ball in $\mathbb{R}^{d_j}$ and denote λ_j the uniform probability measure on B_{d_j}.

Lemma A.1.1 (Lemma 5.4, Olson and Robinson (2010)). *If $x \in \mathbb{R}^{d_j}$ and $\eta \in \mathbb{R}$, then*

$$\lambda_j\big\{\omega \in B_{d_j} : |\eta + (\omega \cdot x)| < \varepsilon\big\} < c\ d_j^{1/2}\varepsilon\|x\|^{-1}, \tag{A.1}$$

where c is a constant that does not depend on η or j.

Proof. Let $\{e_i\}_{i=1}^{d_j}$ be the canonical basis of $\mathbb{R}^{d_j}$. Given $x \in \mathbb{R}^{d_j}$ such that $x = \sum_{i=1}^{d_j} x_i e_i$, let $M = [x_1 \quad \dots \quad \dots \quad x_{d_j}]$ be the corresponding $1 \times d_j$ matrix. Let F be the map from $\mathbb{R}^{d_j}$ into $\mathbb{R}$ defined by $F(\omega) := M\omega + \eta = (\omega \cdot x) + \eta$.

If $\omega \in B_{d_j}$, i.e. $\|\omega\| \leq 1$, then it follows using Cauchy-Schwartz inequality that

$$|M\omega| = |(\omega \cdot x)| \leq \|\omega\|\|x\| \leq \|x\|.$$

Hence, $M\big(B_{d_j}\big)$ lies in the interval $[-\|x\|, \|x\|]$ in $\mathbb{R}$.

The set

$$\{\omega \in B_{d_j} : |F(\omega)| \leq \varepsilon\} = \{\omega \in B_{d_j} : |\eta + M\omega| \leq \varepsilon\}$$

consists of the vectors in B_{d_j} whose image by M lands in an interval of length 2ε.

If $\omega \in B_{d_j}$, then $M\omega \in [-\varepsilon, \varepsilon]$ if and only if $\|\omega\| \leq \varepsilon/\|x\|$. Therefore, the set

$$F^{-1}([-\varepsilon, \varepsilon]) \cap B_{d_j} = \{\omega \in B_{d_j} : |F(\omega)| \leq \varepsilon\}$$

is a cylindrical subset of B_{d_j} with base the interval $[-\varepsilon\|x\|^{-1}, \varepsilon\|x\|^{-1}]$. This subset has a d_j-dimensional volume less than $2\varepsilon\|x\|^{-1}\Omega_{d_j-1}$, where $\Omega_r = \pi^{r/2}/\Gamma\big(1+\frac{r}{2}\big)$ denotes the volume of the r-dimensional unit ball. Since the volume of B_{d_j} is Ω_{d_j},

$$\lambda_j\big\{\omega \in B_{d_j} : |\eta + (\omega \cdot x)| < \varepsilon\big\} < \frac{2\varepsilon\|x\|^{-1}\Omega_{d_j-1}}{\Omega_{d_j}} < c\, d_j^{1/2}\varepsilon\|x\|^{-1}.$$

□

A.2 Estimate for the infinite-dimensional case

Let H be a real Hilbert space. Let $\{U_j\}_{j=1}^{\infty}$ be a sequence of finite-dimensional subspaces of H such that $\dim U_j = d_j$. Let S_j be the closed unit ball in the linear subspace U_j. For any $u \in H$, denote u^* the element of H^* given by $u^*(x) = (u, x)$. Now define

$$Q = \left\{\pi = (\pi_1, ..., \pi_N) : \text{ where } \pi_n = \sum_{j=1}^{\infty} j^{-2}\phi_{nj}^*, \quad \phi_{nj} \in S_j\right\}.$$

First note that Q is a compact subset of $\mathscr{L}(H, \mathbb{R}^N)$. Since one can identify S_j with the unit ball B_{d_j} in $\mathbb{R}^{d_j}$, denote λ_j be the probability measure on S_j that corresponds to the uniform probability measure on B_{d_j}. Let μ be the probability measure on Q that results from choosing each ϕ_{nj} randomly with respect to λ_{d_j}.

Lemma A.2.1. *Given $x \in H$, for any $f \in \mathscr{L}(H, \mathbb{R}^N)$ and for any $j \in \mathbb{N}$,*

$$\mu\big\{\pi \in Q : \ |(f+\pi)x| < \varepsilon\big\} < C\Big(j^2 d_j^{1/2}\varepsilon\|P_j x\|^{-1}\Big)^N,$$

where P_j denotes the orthogonal projection onto U_j and C is a constant independent of f and j.

Proof. For any $x \in H$ and $f \in \mathscr{L}(H, \mathbb{R}^N)$, we wish to bound the probability that $\pi \in Q$, chosen randomly with respect to μ, satisfies $|f(x) + \pi(x)| < \varepsilon$. It follows from the definition of μ that

$$\mu\{\pi \in Q : |f(x) + \pi(x)| < \varepsilon\}$$
$$= \bigotimes_{j=1}^{\infty} \lambda_{d_j} \Bigg\{ (\phi_{nj})_{j=1}^{\infty} \in \bigtimes_{j=1}^{\infty} S_j : \Bigg| \Bigg(f_1(x) + \sum_{j=1}^{\infty} j^{-2}(\phi_{1j}, x),$$
$$\ldots,\ f_N(x) + \sum_{j=1}^{\infty} j^{-2}(\phi_{Nj}, x) \Bigg) \Bigg| < \varepsilon \Bigg\}$$

Since every component of the vector $f(x)+\pi(x)$ must have modulus no larger than ε, it follows from Fubini's Theorem that

$$\mu\{\pi \in Q : |(f + \pi)x| < \varepsilon\}$$
$$\leq \bigotimes_{j=1}^{\infty} \lambda_{d_j} \Bigg\{ (\phi_{nj})_{j=1}^{\infty} \in \bigtimes_{j=1}^{\infty} S_j : \Bigg| f_n(x) + \sum_{j=1}^{\infty} j^{-2}(\phi_{nj}, x) \Bigg| < \varepsilon, \text{ for each } n = 1, ..., N \Bigg\}$$
$$= \prod_{n=1}^{N} \Bigg[\bigotimes_{j=1}^{\infty} \lambda_{d_j} \Bigg\{ (\phi_{nj})_{j=1}^{\infty} \in \bigtimes_{j=1}^{\infty} S_j : \Bigg| f_n(x) + \sum_{j=1}^{\infty} j^{-2}(\phi_{nj}, x) \Bigg| < \varepsilon \Bigg\} \Bigg].$$

Now consider

$$\bigotimes_{j=1}^{\infty} \lambda_{d_j} \Bigg\{ (\phi_{nj})_{j=1}^{\infty} \in \bigtimes_{j=1}^{\infty} S_j : \Bigg| f_n(x) + \sum_{j=1}^{\infty} j^{-2}(\phi_{nj}, x) \Bigg| < \varepsilon \Bigg\}$$
$$= \bigotimes_{j=1}^{\infty} \lambda_{d_j} \Bigg\{ (\phi_{nj})_{j=1}^{\infty} \in \bigtimes_{j=1}^{\infty} S_j : \Big| \eta_n(x) + j^{-2}(\phi_{nj}, x) \Big| < \varepsilon \Bigg\},$$

where

$$\eta_n(x) = f_n(x) + \sum_{i \neq j} i^{-2}(\phi_{ni}, x).$$

It follows from Lemma A.1.1 that, for $\eta_n(x) \in \mathbb{R}$ fixed,

$$\begin{aligned} \lambda_{d_j}\Big\{\phi_{nj} \in S_j : \big|\eta_n(x) + j^{-2}(\phi_{nj}, x)\big| < \varepsilon\Big\} &\leq \lambda_{d_j}\Big\{\phi_{nj} \in S_j : \big|j^{-2}(\phi_{nj}, x)\big| < \varepsilon\Big\} \\ &\leq \lambda_{d_j}\Big\{\phi_{nj} \in S_j : \big|j^{-2}(\phi_{nj}, P_j x)\big| < \varepsilon\Big\} \\ &\leq c j^2 d_j^{1/2} \varepsilon \|P_j x\|^{-1} \qquad \text{(A.2)} \end{aligned}$$

where P_j denotes the orthogonal projection onto U_j and c is a constant independent of $\eta_n(x)$ and j. Therefore, it follows from the product structure of the measure

$\bigotimes_{j=1}^{\infty} \lambda_{d_j}$ that

$$\bigotimes_{j=1}^{\infty} \lambda_{d_j} \Big\{ (\phi_{nj})_{j=1}^{\infty} \in \bigtimes_{j=1}^{\infty} S_j : \Big| f_n(x) + \sum_{j=1}^{\infty} j^{-2}(\phi_{nj}, x) \Big| < \varepsilon \Big\}$$
$$\leq \lambda_{d_j} \Big\{ \phi_{nj} \in S_j : \big| j^{-2}(\phi_{nj}, x) \big| < \varepsilon \Big\}.$$

Finally, since the bound obtained in (A.2) is independent of n

$$\prod_{n=1}^{N} \Bigg[\bigotimes_{j=1}^{\infty} \lambda_{d_j} \Big\{ (\phi_{nj})_{j=1}^{\infty} \in \bigtimes_{j=1}^{\infty} S_j : \big| f_n(x) + \textstyle\sum_{j=1}^{\infty} j^{-2}(\phi_{nj}, x) \big| < \varepsilon \Big\} \Bigg]$$
$$\leq C \Big(j^2 d_j^{1/2} \varepsilon \| P_j x \|^{-1} \Big)^N,$$

where C is a constant independent of f and j. $\square$

Appendix B

Notes on homotopy and shape theory

In this appendix, we give some formal definitions of concepts used in Chapter 5. For more details, see Singer and Thorpe (1967), Borsuk (1975) and Kapitanski and Rodnianski (2000).

B.1 Homotopy theory

A *homotopy* between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0, 1] \to Y$ from the product of the space X with the unit interval $[0, 1]$ to Y such that, if $x \in X$ then $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Two spaces X and Y have the same *homotopy type* if there exist continuous maps (called *homotopy equivalences*) $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map $\mathrm{id}_X : X \to X$ and $f \circ g$ is homotopic to $\mathrm{id}_Y : Y \to Y$. A continuous map $f : X \to Y$ for which these homotopy relations hold true for some $g : Y \to X$ is called a *homotopy equivalence.* In particular, two homeomorphic spaces are of the same homotopy type, but the converse is not true. For example, a solid disk is not homeomorphic to a single point, although the disk and the point are of the same homotopy type.

A topological space X is *contractible* if the identity map $\mathrm{id}_X : X \to X$ is homotopic to a constant map; that is, if $\mathrm{id}_X \simeq c$, where $c : X \to \{x_0\}$ for some $x_0 \in X$.

Theorem B.1.1 (Singer and Thorpe, 1967). *A topological space X is contractible if and only if X is of the same homotopy type as a single point.*

B.2 Shape theory

The notion of homotopy type is especially well adapted for the study of 'nice' spaces. Since attractors may have very complicated geometric structure, it may not be productive to study its homotopy type. Instead, it is more reasonable to work with the concept of shape, introduced by Borsuk (1975).

A compact set $X \subset \mathbb{R}^m$ has *trivial shape* (or the *shape of a point*) if for every neighbourhood V of X, there exists another neighbourhood U of X, $U \subset V$, such that the inclusion map $\iota : U \to V$ is homotopic to a constant map $c : U \to V$ in V. The concept of having the shape of a point is then equivalent to cell-likeness (Daverman (1986, p.120) for more details).

Theorem B.2.1 (Borsuk, 1975, p.25)**.** *If X and Y have the same homotopic type, then X and Y have the same shape.*

In particular, if X is contractible, then X has the shape of a point. The following result, used in the proof of Proposition 5.2.3, is an immediate consequence of Theorem B.2.1.

Corollary B.2.1 (Borsuk, 1975, p.25)**.** *If X and Y are homeomorphic, then X and Y have the same shape.*

There are several papers concerning the shape of attractors of a finite-dimensional (semi)-flow (see Garay (1991) Günther and Segal (1993), Günther (1995), Sanjurjo (1995)). Under very mild assumptions, Kapitanski and Rodnianski (2000) proved that the global attractor has the same shape as the state space.

Theorem B.2.2 (Theorem 3.6, Kapitanski and Rodnianski, 2000, p.233)**.** *Let X be a complete metric space and $\mathcal{E} : X \times \mathbb{R}_+ \to X$ a continuous semiflow. Let $\mathcal{A}$ be the global attractor associated with the semidynamical system $(X, \mathcal{E})$. Then, $\mathcal{A}$ and X have the same shape.*

This result, used in the proof of Proposition 5.2.3, has shown to have many applications to partial differential equations, functional differential equations, and to more general processes.

Appendix C

Manifolds, Maps and Differential Structures

In this appendix, we give some formal definitions of concepts used in Chapter 5. For more details, see Hirsch (1976).

C.1 Differential Structures

A topological space M is called an n-dimensional *manifold* if it is locally homeomorphic to $\mathbb{R}^n$. That is, there is an open cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ of M such that for each $i \in \Lambda$ there is a map $\varphi_i : U_i \to \mathbb{R}^n$ which maps U_i homeomorphically onto an open subset of $\mathbb{R}^n$. We call (φ_i, U_i) a *chart* (or *coordinate system*) with domain U_i; the set of charts $\Phi = \{\varphi_i, U_i\}_{i\in\Lambda}$ is an *atlas*.

Two charts (φ_i, U_i), (φ_j, U_j) are said to have C^r overlap if the *coordinate change*

$$\varphi_j\varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is of differentiability class C^r, and $\varphi_i\varphi_j^{-1}$ is also C^r map. An atlas Φ on M is called C^r if every pair of its charts has C^r overlap. In this case there is a unique maximal C^r atlas Ψ which contains Φ. In fact Ψ is the set of all charts which have C^r overlap with every chart in Φ.

A maximal C^r atlas α on M is a C^r *differential structure*; the pair M_α is called a *manifold of class C^r*. Note that to determine a C^r differential structure it suffices to give a single C^r atlas contained in it. A *differential manifold* is a topological manifold equipped with a C^∞ differential structure. If M is a topological manifold, a smoothing M_α of M is a C^∞ atlas in M.

If M_Φ and N_Ψ are manifolds, then their Cartesian product is the manifold

$(M \times N)_\Theta$, where Θ is the differential structure containing all charts of the form

$$(\varphi \times \psi, U \times V); (\varphi, U) \in \Phi, (\psi, V) \in \Psi.$$

Here $\varphi \times \psi$ maps $U \times V$ into $\mathbb{R}^m \times \mathbb{R}^n$, which we identify with $\mathbb{R}^{m+n}$. It follows from Kirby and Siebenmann (1977) that if Θ is a smooth structure on $M \times \mathbb{R}$, then there exists a unique smooth structure σ on M, such that $(M \times \mathbb{R})_\Theta$ and $M_\sigma \times \mathbb{R}$ are diffeomorphic.

If M_Φ is a manifold and $W \subset M$ is an open set the *induced* differential structure on W is

$$\Psi \mid W = \{(\varphi, U) \in \Phi : U \subset W\}.$$

Let M be a topological space, N_Φ a manifold and $h : M \to N$ a homeomorphism of M onto an open subset of N. The *induced* differential structure on M is

$$h^*\Phi = \{(\varphi \circ h, h^{-1}U) : (\varphi, U) \in \Phi \text{ and } U \subset h(M)\}.$$

C.2 Differentiable Maps

In this section we shall suppress notation for the differential structure on a manifold M. Let M and N be C^r manifolds and $f : M \to N$ a map. A pair of charts (φ, U) for M and (ψ, V) for N is *adapted* to f if $f(U) \subset V$. In this case the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is defined; we call it the *local representation* of f in the given charts, *at the point* x if $x \in U$.

The map f is called *differentiable* at x if it has a local representation at x which is differentiable. And, in this case, every local representation at x is differentiable. This definition makes sense since a local representation is a map between open sets in Cartesian spaces. Similarly, f is *differentiable of class* $\mathcal{C}^r$ if it has $\mathcal{C}^r$ local representations at all points.

Let $f : M \to N$ and $g : N \to P$ be $\mathcal{C}^r$ maps between $\mathcal{C}^r$ manifolds. It is easy to verify, using local representations, that the composition $g \circ f : M \to P$ is also $\mathcal{C}^r$. The identity map and all constant maps are $\mathcal{C}^r$.

A $\mathcal{C}^r$ *diffeomorphism* $f : M \to N$ is a $\mathcal{C}^r$ map between $\mathcal{C}^r$ manifolds M and N which is a homeomorphism, and whose inverse $f^{-1} : N \to M$ is also of class $\mathcal{C}^r$. If such a map exists we call M and N $\mathcal{C}^r$ *diffeomorphic* manifolds.

Bibliography

J. W. Alexander. An example of a simply connected surface bounding a region which is not simply connected. *Proceedings of the National Academy of Sciences*, 10(1):8–10, 1924.

P. Assouad. Plongements Lipschitziens dans $\mathbb{R}^n$. *Bull. Soc. Math. France*, 111:429–448, 1983.

A. V. Babin and M. I. Vishik. *Attractors of Evolution Equations.* North-Holland Publishing Co., Amsterdam, 1992.

K. Ball. Cube slicing in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 97:465–473, 1986.

C. Bardos and L. Tartar. Sur l'unicité rétrograde des équations paraboliques et quelques questions voisines. *Arch. Rational Mech. Anal.*, 50:10–25, 1973.

A. Ben-Artzi, A. Eden, C. Foias and B. Nicolaenko. Hölder continuity for the inverse of Mañé's projection. *J. Math. Anal. Appl.*, 178(1):22–29, 1993.

Y. Benyamini and J. Lindenstrauss. *Geometric nonlinear functional analysis.* American Mathematical Society Colloquium Publications vol. 48. American Mathematical Society, 2000.

N. P. Bhatia and G. P. Szegö. *Stability theory of dynamical systems.* Springer-Verlag, Berlin, 1970.

J. E. Billotti and J. P. LaSalle. Dissipative periodic processes. *Bull. Amer. Math. Soc.*, 77(6):1082–1088, 1971.

W. A. Blankinship and R. H. Fox. Remarks on certain pathological open subsets of 3-space and their fundamental groups. *Proc. Amer. Math. Soc.*, 1(5):618–624, 1950.

K. Borsuk. *Theory of Shape.* Monografie Matematyczne vol. 59. Polish Scientific Publishers, Warszawa, 1975.

G. Bouligand. Ensembles impropres et ordre dimensionnel. *Bull. Sci. Math.*, 52:320–344 and 361–376, 1928.

M. Brown. A proof of the generalized Schoenflies theorem. *Bull. Amer. Math. Soc.*, 66:74–76, 1960.

V. V. Chepyzhov and M. I. Vishik. *Attractors for equations of mathematical physics.* American Mathematical Society Colloquium Publications vol. 49. American Mathematical Society, 2002.

J. P. R. Christensen. Measure theoretic zero sets in infinite dimensional spaces and applications to differentiability of Lipschitz mappings. *Publ. Dép. Math. (Lyon)*, 10(2):29–39, 1973.

P. Constantin and C. Foias. *Navier-Stokes Equations.* University of Chicago Press, Chicago, 1988.

P. Constantin, C. Foias, B. Nicolaenko and R. Temam. Inertial manifolds for nonlinear evolutionary equations *J. Diff. Eq.*, 73:309–353, 1988.

P. Constantin, C. Foias, B. Nicolaenko and R. Temam. *Integral manifolds and inertial manifolds for dissipative partial differential equations.* Springer-Verlag, New York, 1989.

R. J. Daverman. *Decomposition of manifolds.* Academic Press inc., London, 1986.

E. B. Davies. *Spectral theory and differential operators,* Cambridge Studies in Advanced Mathematics vol. 42. Cambrigde University Press, Cambridge, 1995.

A. Debussche and R. Temam. Convergent families of approximate inertial manifolds. *J. Math. Pures Appl.*, 73:489–522, 1994.

C. H. Dowker. On countably paracompact spaces. *Canadian J. Math.*, 3:219–224, 1951.

A. Eden, C. Foias, B. Nicolaenko and R. Temam. *Exponential Attractors for Dissipative Evolution Equations.* Research in Applied Mathematics Series. John Wiley and Sons, New York, 1994.

K. J. Falconer. *Fractal geometry: Mathematical foundations and applications.* John Wiley and Sons, New York, 1990.

K. Falconer and J. D. Howroyd. Projection theorems for box and packing dimensions. *Math. Proc. Camb. Phil. Soc.*, 119:287–295, 1996.

C. Foias, O. Manley and R. Temam. Modelling of the interaction of small and large eddies in two-dimensional turbulent flows. *RAIRO Modél. Math. Anal. Numér.*, 22(1):93–118, 1988.

C. Foias and E. Olson. Finite fractal dimension and Hölder-Lipschitz parametrization. *Indiana Univ. Math. J.*, 45(3):603–616, 1996.

C. Foias, G. Sell and R. Temam. Variétés inertielles des équations différentielles dissipatives. *C. R. Acad. Sci. Paris I*, 301:139–141, 1985.

C. Foias, G. R. Sell and R. Temam. Inertial Manifolds for nonlinear evolutionary equations. *J. Diff. Eq.*, 73:309–353, 1988.

C. Foias and R. Temam. Some analytic and geometric properties of solutions of the Navier-Stokes equations. *J. Math. Pures Appl.*, 58:339–365, 1979.

P. K. Friz and J. C. Robinson. Smooth attractors have zero thickness. *J. Math. Anal. Appl.*, 240:37–46, 1999.

B. M. Garay. Strong cellularity and global asymptotic stability. *Fund. Math.*, 138:147–154, 1991.

B. Günther. Construction of differentiable flows with prescribed attractor. *Topol. Appl.*, 62:87–91, 1995.

B. Günther and J. Segal. Every attractor of a flow on a manifold has the shape of a finite polyhedron. *Proc. Am. Math. Soc*, 119:321–329, 1993.

J. K. Hale. *Asymptotic behavior of dissipative systems.* Mathematical Surveys and Monographs Number 25. American Mathematical Society, Providence, RI, 1988.

P. Hájek, V. M. Santalucía, J. Vanderwerff and V. Zizler *Biorthogonal Systems in Banach Spaces.* CMS Books in Mathematics vol. 26. Springer-Verlag, New York, 2001.

P. Hartman. *Ordinary differential equations.* John Wiley and Sons, New York, 1964.

J. Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Lecture Notes in Mathematics vol. 840. Springer-Verlag, New York, 1981.

M. W. Hirsch. *Differential topology.* Graduate Texts in Mathematics vol. 33. Springer-Verlag, New York, 1976.

B. R. Hunt and V. Y. Kaloshin. Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces. *Nonlinearity*, 12:1263–1275, 1999.

B. R. Hunt, T. Sauer and J. A. Yorke. Prevalence: a translation-invariant 'almost every' on infinite-dimensional spaces. *Bull. Amer. Math. Soc.*, 27(2):217–238, 1992.

W. Hurewicz and H. Wallman. *Dimension Theory.* Princeton University Press, Princeton, 1948.

M. Järvenpää. On the upper Minkowski dimension, the packing dimension, and the orthogonal projections. *Ann. Acad. Sci. Fenn. Math. Diss.*, 99:1–34, 1999.

L. Kapitanski and I. Rodnianski. Shape and Morse theory of attractors. *Comm. Pure Appl. Math.*, 53(2):218–242, 2000.

R. C. Kirby and L. C. Siebenmann *Foundational essays on topological manifolds, smoothings and triangulations.* Annals of Mathematical Studies 88. Princeton University Press, Princeton, 1977.

I. Kukavica. Fourier Parametrization of Attrators for Dissipative Equations in one space dimension. *J. of Dyn. Diff. Eqns.,* 15(2/3):473–484, 2003.

I. Kukavica. On Fourier Parametrization of Global Attrators for Equations in one space dimension. *Discrete and Continuous Dynamical Systems*, 13(3):553–560, 2005.

I. Kukavica. Log-log convexity and backward uniqueness. *Proc. Amer. Math. Soc.*, 135(8):2415–2421, 2007.

O. A. Ladyzhenskaya. *Attractors for Semigroups and Evolution Equations.* Cambridge University Press, Cambridge, 1991.

U. Lang and C. Plaut. Bilipschitz embeddings of metric spaces into space forms. *Geom. Dedicata* 87, 285-307, 2001.

J. A. Langa and J. C. Robinson. Determining asymptotic behavior from the dynamics on attracting sets. *J. Dyn. Diff. Eqns.*, 11:319–331, 1999.

J. Luukkainen. Assouad dimension: Antifractal metrization, porous sets, and homogeneous measures. *J. Korean Math. Soc.*, 35(1):23–76, 1998.

R. Mañé. On the dimension of the compact invariant sets of certain nonlinear maps. *Springer Lecture Notes in Math.*, 898:230–242, 1981.

S. Mardešić and J. Segal. *Shape Theory.* North-Holland Publishing Co., Amsterdam, 1982.

R. A. McCoy. Cells and cellularity in infinite-dimensional normed linear spaces *Trans. Amer. Math. Soc.*, 176:401–410, 1973.

E. J. McShane. Extension of the range of functions. *Bull. Am. Math. Soc.* 40:837–842, 1934.

R. Meise and D. Vogt. *Introduction to Functional Analysis.* Oxford Graduate Texts in Mathematics. Oxford University Press, 1997.

H. Movahedi-Lankarani. On the inverse of Mañé projection. *Proceeding of the American Mathematical Society*, 116(2):555–560, 1992.

H. Ogawa. Lower bounds for solutions of differential inequalities in Hilbert space. *Proc. Amer. Math. Soc.*, 16:1241–1243, 1965.

E. Olson. Bouligand dimension and almost Lipschitz embeddings. *Pacific Journal of Mathematics*, 202:459–474, 2002.

E. J. Olson and J. C. Robinson. Almost bi-Lipschitz embeddings and almost homogeneous sets. *Trans. Amer. Math. Soc.*, 362(1):145–168, 2010.

W. Ott, B. Hunt and V. Kaloshin. The effect of projections on fractal sets and measures in Banach spaces. *Ergod. Theor. Dynam. Sys.*, 26(3):869–891, 2006.

W. Ott and J. A. Yorke. Prevalence. *Bull. Amer. Math. Soc.*, 42(3):263–290, 2005.

E. Pinto de Moura and J. C. Robinson. Orthogonal sequences and regularity of embeddings into finite-dimensional spaces. *J. Math. Anal. Appl.*, 368:254–262, 2010a.

E. Pinto de Moura and J. C. Robinson. Lipschitz deviation and embeddings of global attractors. *Nonlinearity,* 23:1695–1708, 2010b.

E. Pinto de Moura and J. C. Robinson. Log-Lipschitz continuity of the vector field on the attractor of certain parabolic equations. *Submitted*, 2010c.

E. Pinto de Moura, J. C. Robinson and J. J. Sanches-Gabites. Embedding of global attractors and their dynamics. *Proc. Amer. Math. Soc.*, to appear, 2010.

J. C. Robinson. Global Attractors: Topology and finite-dimensional dynamics. *J. Dyn. Diff. Eqns.*, 11(3):557–581, 1999.

J.C. Robinson. *Infinite-dimensional dynamical systems.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001.

J. C. Robinson. *Attractors and finite-dimensional behaviour in the Navier Stokes equations.* Instructional Conference Mathematical Analysis of Hydrodynamics, June 2003.

J. C. Robinson. A topological delay embedding theorem for infinite-dimensional dynamical systems. *Nonlinearity*, 18:2135–2143, 2005.

J. C. Robinson. Linear embeddings of finite-dimensional subsets of Banach spaces into Euclidean spaces. *Nonlinearity*, 22:711–728, 2009.

J. C. Robinson. Log-Lipschitz embeddings of homogeneous sets with sharp logarithmic exponents and slicing the unit cube. *Submitted*, 2010.

S. Roman. *Advanced Linear Algebra*, 3rd edition. Graduate Texts in Mathematics. Springer-Verlag, 2007.

A. V. Romanov. Finite-dimensional limiting dynamics for dissipative parabolic equations. *Sbornik: Mathematics*, 191(3):415–429, 2000.

R. Rosa. Approximate inertial manifolds of exponential order *Discrete and continuous dynamical systems*, 1(3):421–448, 1995.

J. M. R. Sanjurjo. On the structure of uniform attractors *J. Math. Anal. Appl.*, 192:519–528, 1995.

T. Sauer and J. A. Yorke. Are the dimensions of a set and its image equal under typical smooth functions? *Ergod. Theor. Dynam. Sys.*, 71:941–956, 1997.

T. Sauer, J.A. Yorke and M. Casdagli. Embedology. *J. Stat. Phys.*, 65:579–616, 1991.

G. R. Sell and Y. You. *Dynamics of evolutionary equations.* Applied Mathematical Sciences vol. 143. Springer-Verlag, New York, 2002.

I. M. Singer and J. A. Thorpe. *Lecture notes on elementary topology and geometry.* Undergraduate texts in mathematics. Springer-Verlag, New York, 1967.

E. M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, NJ, 1982.

F. Takens. *Detecting strange attractors in turbulence.* Springer Lecture Notes in Math., 898:366–381, 1981.

R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Springer Applied Mathematical Sciences vol. 68. Springer-Verlag, Berlin, 2nd edition, 1997.

J. H. Wells and L. R. Williams. *Embeddings and extensions in analysis.* Springer-Verlag, Berlin, 1975.