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Vector fitting approximation of a cylinder nonreflecting boundary kernel

K. Bavelis and C. Mias

To employ the modal nonreflecting boundary condition (MNRBC) in cylindrical coordinates in the finite element time domain (FETD) method, a time domain kernel expression must be found that it is the inverse Laplace transform (ILT) of a known frequency domain function. The inverse Laplace transformation is achieved using a methodology based on the partial fraction expansion of the frequency domain function. However, to date, no FETD results have been published based on this MNRBC methodology. A simpler implementation of the methodology based on vector fitting (VF) is proposed. Using the VF approach, FETD-MNRBC results of plane wave scattering from a cylinder are presented for the first time.

Introduction: The modal nonreflecting boundary condition (MNRBC), in cylindrical coordinates, is a well established boundary condition in the two-dimensional (2D) finite element frequency domain (FEFD) method simulations [1]. This boundary condition is based on the fact that the scattered field on a circular (fictitious) boundary surrounding a cylinder of arbitrary cross-section can be expressed in terms of summation of modal functions, of integer modal orders \( n \), which are products of Hankel functions or modified Bessel functions and azimuthal function terms, see for example [2]. The time domain version of this boundary condition has not been employed in finite element time domain (FETD) method simulations although the general methodology for developing a time domain MNRBC has been presented by Alpert et al [3]. The methodology relies on finding, for each order \( n \), the time domain expression of a cylinder nonreflecting boundary kernel. This requires that for each order \( n \) the inverse Laplace transform (ILT) of a known function that appears in the MNRBC in the frequency domain is found. This is achieved by expressing the frequency domain function as a summation of partial fractions which via the ILT are expressed as a summation of exponential
terms in the time domain. Because of the complexity involved in implementing the partial fraction expansion in [3] and the fact that only a limited number of partial fraction coefficients that correspond to a few cylinder kernel modal orders \((n=1,2,3,4)\) were presented in [3], an alternative simpler approach of implementing the partial fraction expansion based on vector fitting (VF) is proposed using the publicly available software VECTFIT [4]. This VF approach may have a greater appeal among engineers. Through computations, it is demonstrated that the VF results are of comparable accuracy to those of Alpert et al [3]. In addition, FETD-MNRBC results, based on VF, are presented for the first time.

The VF approach: To demonstrate the proposed VF partial fraction expansion approach and its accuracy, the Laplace transform expression \(Q_n(s)\) of the time domain cylinder nonreflecting boundary kernel \(q_n(t)\), used by Alpert et al (eq. 2.13 in [3]), is employed

\[
Q_n(s) = \frac{s}{c} + \frac{1}{2\rho} + \frac{s}{c} \frac{K'_n(\rho s/c)}{K_n(\rho s/c)} = \frac{1}{\rho} \left[ v + \frac{1}{2} + \frac{K'_n(v)}{K_n(v)} \right], \quad v = \frac{s\rho}{c}
\]  

(1)

where \(K_n\) is the modified Bessel function of the second kind and \(n\)th order. The derivative \(K'_n\) of \(K_n\) is with respect to the argument \(\rho s/c\) where \(s\) is the Laplace domain variable, \(\rho\) is the radius of the fictitious circular nonreflecting boundary \((\rho > 0)\), \(c\) is the speed of light in the medium surrounding the cylinder \((c > 0)\), assumed here to be free space. From the scaling properties of the Laplace transform it is sufficient to expand the following expression in terms of partial functions

\[
U_n(s) = s + \frac{1}{2} + s \frac{K'_n(s)}{K_n(s)} = s + \frac{1}{2} - s \frac{K_{n+1}(s)}{K_n(s)} + n
\]  

(2)

In (2), the derivative \(K'_n\) of \(K_n\) is now with respect to the argument \(s\). Once the ILT of \(U_n(s)\) is found, denoted as \(u_n(t)\), then the ILT of (1) can be obtained, for any \(\rho\) and \(c\), as follows

\[
q_n(t) = \frac{c}{\rho^2} u_n\left(\frac{c}{\rho} t\right)
\]  

(3)
The last expression in (2) is obtained using the identities of modified Bessel functions in [5] and it is the expression employed in VF with \( s = j\omega \). In general, VF approximately expresses \( U_\omega(s) \) in (2) in the following form

\[
U_n(s) \approx U_{n,\text{app}}(s) = d_n + h_n s + \sum_{m=1}^{M} \frac{r_{m,n}}{s - a_{m,n}}
\]

(4)

where the parameters \( d_n, h_n, r_{m,n} \) (pole coefficient) and \( a_{m,n} \) (pole location) are computed by VECTFIT (version 1). The subscript ‘\( \text{app} \)’ indicates an approximation. To enable VECTFIT to produce the desirable results, the asymptotic value of \( U_n(s) \) as \( s \to 0 \) is required. From [5], the small argument approximation of \( U_n(s) \) is

\[
\lim_{s \to 0} U_n(s) \approx \frac{1}{2} - |n| \quad \forall n \in \mathbb{Z} \quad \text{since} \quad \lim_{s \to 0} K_n(s) \approx \frac{1}{2} \Gamma(n) \left( \frac{1}{2} \right)^{-n}
\]

(5)

for \( n > 0 \) where \( \Gamma(\ldots) \) is the Gamma function [5]. Furthermore, computing \( d_n \) and \( h_n \) is optional in VECTFIT and therefore knowledge of the large argument approximation of \( U_n(s) \) is beneficial. In this case, \( U_n(s) \to 0 \) as \( s \to \infty \), hence, \( d_n = h_n = 0 \). Therefore, after applying ILT to the partial fraction expansion in (4) one obtains the following approximate time domain expression for \( u_n(t) \),

\[
u_{n,\text{app}}(t) = \sum_{m=1}^{M} r_{m,n} a_{m,n} t
\]

(6)

A benefit from employing such a summation in a FETD-MNRBC formulation is that it will allow the convolution that appears in the boundary integral term of the formulation to be computed in a fast manner using a recursive approach as shown in [6] for 2D planar periodic structures.

Numerical Results: The input parameters used in VECTFIT were: (a) 4000 frequency samples; (b) a frequency range of \( 0 \leq f \leq f_{\text{max}} \) with \( f_{\text{max}} = 4 \) Hz; (c) iter = 20 iterations; and (d) asympflag = 1 (as \( d_n = h_n = 0 \)). Table 1 lists the computed partial fraction parameters, \( r_{m,n} \) and \( a_{m,n} \). Their values are truncated to six decimal places. For \( n \) and \( M \) the values tabulated in [3] are used. Figure 1 shows plots of the function \( U_\omega(s = j2\pi f) \) versus frequency as based on: (a) the exact equation (2);
(b) the partial fraction approximation of the VF using the parameters in table 1 (without truncation); and (c) the partial fraction approximation of Alpert et al using the tabulated parameters in [3]. The absolute error value $10 \log_{10} |e_n(s = j2\pi f)|$ versus frequency $f$ is also shown in figure 1 for both approximations, where

$$|e_n(s)| = \left| U_{n}(s) - \sum_{m=1}^{M} \frac{r_{m,n}}{s - a_{m,n}} \right|$$

(7)

The figure indicates that both approximations are of comparable accuracy and that $10 \log_{10} |e_n| < -50$ dB for $n = 1, 2, 3, 4$.

The VF accuracy enabled us to obtain, for the first time, FETD-MNRBC results (figure 2) based on the proposed VF approach. The plot in figure 2 shows the bistatic scattering width ($\sigma_2/\lambda$), as defined in [7], of a perfectly electrically conducting triangular cylinder. The cylinder is surrounded by free space. The FETD-MNRBC results are compared with the integral equation formulation results presented in [2].

**Conclusion:** A VF approximation of a cylinder nonreflecting boundary kernel is proposed and validated. FETD-MNRBC results, based on this VF approximation, are presented for the first time.
References


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Figure and Table Captions

Figure 1
Caption:
Plots of $|U_n(s = j2\pi f)|$ and $10\log_{10}|\epsilon_n(s = j2\pi f)|$ versus frequency $f$ for $n = 1, 2, 3, 4$.

Figure 2
Caption:
Bistatic scattering width, $10\log_{10}(\sigma_{2-D}/\lambda)$, of a triangular cylinder. $x_1 = 1.0\lambda$, $x_2 = -0.707\lambda$, $y_1 = 0.707\lambda$ and $y_2 = -y_1$. For simplicity, $\lambda = 1$ m. The incident plane wave, with the electric field vector in the $z$-direction, is propagating in the negative $x$-direction. Surface $\Omega$ represents the finite element region and $\Gamma$ represents the circular boundary on which the MNRBC is applied.

Table 1
Caption:
VF computed parameters
Figure 1
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