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A C^r unimodal map with an arbitrary fast growth of the number of periodic points

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Abstract. In this paper we present a surprising example of a C^r unimodal map of an interval $f : I \to I$ whose number of periodic points $P_n(f) = |\{x \in I : f^n x = x\}|$ grows faster than any ahead given sequence along a subsequence $n_k = 3^k$. This example also shows that 'non-flatness' of critical points is necessary for the Martens–de Melo–van Strien theorem [M. Martens, W. de Melo and S. van Strien. Julia–Fatou–Sullivan theory for real one-dimensional dynamics. *Acta Math.* **168**(3–4) (1992), 273–318] to hold.

1. *Introduction*

In this paper we investigate growth of the number of periodic points of C^r maps $f : I \to I$ of the interval $I = [-1, 1]$. Denote by $C^r(I, I)$ the space of such maps with the uniform C^r topology.

Definition 1. A map $f : I \to I$ is called Artin–Mazur (A-M) if for some $C > 0$ we have

$$P_n(f) \le \exp(Cn) \quad \text{for all } n \in \mathbb{Z}_+, \tag{1}$$

where $P_n(f)$ denotes the number of periodic points of f of period n.

Artin and Mazur [**AM**] proved that for any $0 \le r \le \infty$ in $C^r(I, I)$ the set of A-M maps is C^r dense (see also [**K1**]). It turns out that for one-dimensional maps much more can be said about A-M maps.

Definition 2. We say that $f \in C^r$ has a non-flat critical point c if there is a local C^r diffeomorphism ϕ with $\phi(c) = 0$ such that $f(x) = |\phi(x)|^a + f(c)$ for some $a \ge 2$.

If a map $f : I \to I$ is C^r and at a critical point one of the higher derivatives of f does not vanish, this critical point is non-flat.

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THEOREM 1. [**MMS**] *Let $r \geq 2$ and $f : I \to I$ be a C^r map with no flat critical points. Then for some $n_0 \in \mathbb{N}$ and $\gamma > 0$ for any periodic point $x = f^n x$ of period $n > n_0$ we have $|(f^n)' x| > 1 + \gamma$.*

COROLLARY 1. *In the setting above for an open dense set, we have $P_n(f) \leq |I|(2\gamma)^{-1}\|f\|^n_{C^2}$ and, therefore, such an f is an A-M map and A-M maps form an open dense set in $C^r(I, I)$ for any $r \geq 2$.*

Our main result is a counterpart of the Martens–de Melo–van Strien theorem.

THEOREM 2. *For any sequence $a = \{a_n\}_{n\in\mathbb{N}}$ and any $r \in \mathbb{N}$ there exists a C^r unimodal map $f : I \to I$ such that for any $k \in \mathbb{Z}_+$ we have $P_{3^k}(f) > a_{3^k}$. The map is C^∞ everywhere except for the critical point.*

In the proof of this theorem, we will construct an explicit example of such a map. We will start with a C^ω unimodal infinitely renormalizable map (in fact, it will be a fixed point of a tripling operator) and perturb it near some of its periodic points. The perturbed map f will still be infinitely renormalizable and certainly it will have a flat critical point in the sense of Definition 2. Otherwise, it would contradict Theorem 1.

There is nothing particularly special about the sequence 3^k: any sequence of the form m^k, $m \geq 2$, would work, except that for $m = 2$ the proof is slightly more complicated.

It turns out that superexponential growth of the number of periodic point for higher-dimensional maps not only exists, but is (Baire) generic in certain open sets in the space of C^r smooth maps of manifolds [**K2**], which in turn is based on [**GST**] (see also [**K3**]). In [**KS**] the same phenomenon of generic superexponential growth in certain open sets is found for three-dimensional volume-preserving diffeomorphisms. A related question is to estimate the growth of the number of intersections of two submanifolds of complementary dimensions where one of the submanifolds is iterated by a diffeomorphism and the other one is fixed. A C^ω example on a two-dimensional torus was constructed in [**K4**], where the growth is superexponential.

2. *Degenerate periodic points*

Let $f : I \to I$ be a C^r map. We say that a periodic point $x_0 = f^n x_0$ is *neutral of order* $k \leq r$ if $(f^n)'(x_0) = 1$, $(f^n)^{(s)}(x_0) = 0$ for $s = 1, \dots, k-1$ and $(f^n)^{(k)}(x_0) \neq 0$. Here we use the notation $f^{(s)}$ to denote the sth derivative of f.

If $f \in C^r$ has a neutral periodic point $x_0 = f^n x_0$ of order $k \leq r$, then a C^l-perturbation, $l < k$, can create arbitrarily many periodic points of period n close to x_0. Indeed, for simplicity let us assume that $x_0 = 0$ and $n = 1$ (so x_0 is a fixed point). Let $\phi \in C^\infty$ be a hat function (i.e. $\phi(x) = 0$ for $|x| > 1$ and $\phi(x) = 1$ for $|x| < 1/2$). Note that for small x one has $\|f(x) - x\|_{C^m} = O(x^{k-m})$ if $m < k$. This implies that

$$\|(f(x) - x)\phi(x/\epsilon)\|_{C^l_{[-\epsilon,\epsilon]}} = O(\epsilon^{k-l}).$$

Then the function

$$\tilde{f}(x) = x + \sin(Nx)\delta\phi(x/\epsilon) + (f(x) - x)(1 - \phi(x/\epsilon))$$

has many fixed points in the interval $(-\epsilon/2, \epsilon/2)$ if N is large (the number of fixed points is of order $N\epsilon$), and $\tilde{f}$ is C^l close to f if ϵ and δ are small.

Thus, if one can create a C^r unimodal map with an infinite number of neutral periodic points $\{p_k\}_{k\in\mathbb{N}}$ of periods 3^k whose orbits are isolated, then this proves Theorem 2.

3. *Fixed point of a renormalization operator with a degenerate critical point*

A *symmetric unimodal map* is an endomorphism of the interval I of the form $f = \phi \circ q_t$, where $\phi \in \text{Diff}^2_+(I)$ is an orientation-preserving C^2 diffeomorphism of I and $q_t : I \to I$, $t \in [0, 1]$ is defined by $q_t(x) = -2t|x|^\alpha + 2t - 1$. The exponent $\alpha > 1$ is called the *critical exponent* of f and in what follows it will be an even integer. The map q_t is called the *canonical folding map* with *peak value* $t \in [0, 1]$. The peak value determines the maximum $q_t(0) = 2t - 1$. The above form for the canonical folding map is not just a choice for convenience, it naturally arises [**Ma**]. The diffeomorphism ϕ is called the *diffeomorphic part* of f. Notice that $f(-1) = f(1) = -1$. The collection of unimodal maps with chosen critical exponent $\alpha > 1$ is denoted by $\mathcal{U}_\alpha$.

Let $\mathcal{U}_\alpha$ be the collection of unimodal maps whose peak value is high enough such that the unimodal map has a fixed point $p \in (0, 1)$. For every $f \in \mathcal{U}_\alpha$, we can consider the first return map to the interval $[-p, p]$. If the peak value is not too high, the first return map will be just $f^2|_{[-p,p]}$; the unimodal map f is called *renormalizable*. The unimodal map obtained by rescaling this first return map to $[-p, p]$ is called the *renormalization* of f. The operator defined in this way is called the *renormalization operator*. Lanford III [**L**] and, later, Sullivan [**S**] proved that there is a fixed point for the renormalization operator.

More generally, a unimodal map $f \in \mathcal{U}_\alpha$ is called *renormalizable* if and only if there exists an expanding periodic point $p \in (-1, 1)$ such that the first return map to the central interval $C = [-p, p]$ is of the form $f^q : C \to C$ with $f^q(p) = p$ and $q \geq 2$. The first return map to C will be, up to rescaling, a unimodal map. This unimodal map is a *renormalization* of f. Notice that a renormalization is completely determined by the periodic point p. In particular, when $q = 3$, the renormalization operator is well defined. By a theorem of Epstein [**E**], such an operator has a fixed point $f_\alpha \in \mathcal{U}_\alpha$†. Moreover, f_α is real analytic.

Let α be an even integer larger than r, $f_\alpha \in \mathcal{U}_\alpha$ denote a fixed point of the renormalization operator for $q = 3$, and 0 be the critical point of f_α. This means that there exists $\lambda \in (-1, 0)$ such that $f^3_\alpha|(x) = \lambda f_\alpha(x/\lambda)$ for all $x \in [\lambda, -\lambda]$. Note that this functional equality implies that $-\lambda$ is a periodic point of f_α of period three.

Any renormalizable map f_α has two fixed points, one of which is -1 and the other of which we will denote by p. The intervals $[\lambda, -\lambda]$, $f([\lambda, -\lambda])$, and $f^2([\lambda, -\lambda])$ are disjoint, and therefore cannot contain p. The forward orbit of the critical point of f_α belongs to the union of these three intervals; hence, there is a neighborhood (p^-, p^+) of p free from the forward orbit of 0. We can assume that the point p is in the center of I, i.e. $|p - p_-| = |p_+ - p|$. The functional equality implies that $f^3_\alpha(\lambda p) = \lambda f_\alpha(p) = \lambda p$, so λp is a periodic point of f_α of period three. Arguing in the same way, we can see that the point $p_k = \lambda^k p$, $k \in \mathbb{Z}_+$, is a periodic point of period 3^k and that there is an interval $I_k = (\lambda^k p^-, \lambda^k p^+)$ free from the forward orbit of 0. Notice that the intervals I_k are disjoint. It is also easy to compute the derivatives at points p_k using the functional

† See Martens [**Ma**] for more detailed analysis of such fixed points.

equality

$$(f_\alpha^{3^k})^{(s)}(p_k) = \lambda^{k(1-s)} f_\alpha^{(s)}(p). \tag{2}$$

The main idea. We shall prove that there is an arbitrarily small C^r-perturbation $\tilde{f}$ of f_α which coincides with f_α along the orbits of the p_k's and has neutral periodic points of order at least $r+1$ at p_k, where $k > k_0$ and k_0 is large enough. Moreover, since the orbits $O_k(f_\alpha) = \{f_\alpha^s(p_k)\}_{s=0}^{3^k-1}$ of the p_k's are isolated from each other, there are C^r small perturbations of $\tilde{f}$ around the p_k's, for each $k > k_0$, which can create explosions of the number of periodic points from the p_k's of periods 3^k.

4. *Calculations*

In this section we show that, indeed, for a large enough k by a small C^r perturbation inside of I_k we can obtain a neutral periodic point p_k of order at least $r+1$.

To make the calculation in the case of general r easier to comprehend, we discuss the case $r=1$ first. For every k, we define a C^∞ function $\delta_k : \mathbb{R} \to \mathbb{R}$ which satisfies the following properties:

(1) $\delta_k(x) = 0$ for $x \notin I_k$;
(2) $\delta_k(p_k) = 0$;
(3) $\delta_k'(p_k) = f_\alpha'(p_k)(1 - f_\alpha'(p))/f_\alpha'(p)$;
(4) $\|\delta_k\|_{C^1} \to 0$ as $k \to 0$.

It is obvious that a function satisfying (1)–(3) exists, so only adding condition (4) makes the problem less trivial. Since the order of the critical point is α, we know that $f_\alpha'(p_k) \asymp \lambda^{k(\alpha-1)}$ (i.e. there is a constant $C > 0$ independent of k such that $C^{-1} < f_\alpha'(p_k)\lambda^{-k(\alpha-1)} < C$ for all k). Therefore, $\delta_k'(p_k) = O(|\lambda|^{k(\alpha-1)})$. The size of I_k is $|I||\lambda|^k$, so it is easy to see that such a function δ_k exists. (We will show this in the more general case later.)

Define the function $\tilde{f}_k = f_\alpha + \delta_k$. A trivial computation shows that $(\tilde{f}_k^{3^k})'(p_k) = 1$. Indeed,

$$\begin{aligned}(\tilde{f}_k^{3^k})'(p_k) &= (f_\alpha^{3^k-1} \circ \tilde{f})'(p_k) \\ &= (f_\alpha^{3^k})'(p_k)\frac{f_\alpha'(p_k) + \delta_k'(p_k)}{f_\alpha'(p_k)} \\ &= f_\alpha'(p)\frac{f_\alpha'(p_k)f_\alpha'(p) + f_\alpha'(p_k)(1 - f_\alpha'(p))}{f_\alpha'(p_k)f_\alpha'(p)} \\ &= 1.\end{aligned}$$

The function $\tilde{f} = f_\alpha + \sum_{k=k_0}^\infty \delta_k$ is a C^1 function with periodic points $\{p_0, p_1, \ldots\}$. All periodic points starting with p_{k_0} are neutral of order at least two and $\tilde{f}$ is C^1 close to f_α if k_0 is large enough.

In the general case, we proceed in the same way. We will define C^∞ functions δ_k satisfying all the properties above and the following extra properties:

(3') if $\tilde{f}_k = f_\alpha + \delta_k$, then $(\tilde{f}^{3^k})^{(s)}(p_k) = 0$ for all $s = 2, \ldots, r$;
(4') $\|\delta_k\|_{C^r} \to 0$ as $k \to 0$.

To prove that such functions exist, we have to estimate the higher derivatives of δ_k at the point p_k. For that, we need the following proposition.

PROPOSITION 1. *For any $s \in \mathbb{N}$, there is a collection of polynomials with integer coefficients $\{P_{sl}(x_1, \ldots, x_{s-l+1})\}_{1 \le l \le s}$ such that*

$$(g \circ f)^{(s)}(x) = \sum_{l=1}^{s} g^{(l)}(f(x)) P_{sl}(f'(x), \ldots, f^{(s-l+1)}(x)),$$

where g, f are C^s functions. Moreover, the polynomials P_{sl} satisfy the following recurrent rule for $P_{(s+1)l}$:

$$\begin{aligned} &P_{(s+1)l}(f'(x), \ldots, f^{(s-l+2)}(x)) \\ &\quad = \frac{d}{dx} P_{sl}(f'(x), \ldots, f^{(s-l+1)}(x)) + f'(x) P_{s(l-1)}(f'(x), \ldots, f^{(s-l+2)}(x)) \end{aligned}$$

and $P_{11}(f'(x)) = f'(x)$, $P_{s0} = P_{s(s+1)} = 0$.

The proposition can be proved by an elementary direct computation.

Observe that the order of the critical point is α; therefore, $f_\alpha^{(s)}(x) \asymp x^{\alpha - s}$. The recurrent rule implies that

$$P_{sl}(f_\alpha'(x), \ldots, f_\alpha^{(s-l+1)}(x)) \asymp x^{\alpha l - s}, \tag{3}$$

which can be checked by induction.

Now we will use the proposition to estimate $(f_\alpha^{3^k - 1})^{(s)}(f_\alpha(p_k))$.

PROPOSITION 2. *For every $s \ge 1$ as $k \to \infty$ the following asymptotic holds:*

$$(f_\alpha^{3^k - 1})^{(s)}(f_\alpha(p_k)) \asymp p_k^{1 - \alpha s}.$$

Proof. The proposition will be proved by induction. When $s = 1$, using the chain rule, we get

$$\begin{aligned} (f_\alpha^{3^k - 1})'(f_\alpha(p_k)) &= \frac{(f_\alpha^{3^k})'(p_k)}{f_\alpha'(p_k)} \\ &= \frac{f_\alpha'(p)}{f_\alpha'(p_k)} \\ &\asymp p_k^{1-\alpha}. \end{aligned}$$

Here we used equality (2) and $f_\alpha'(p_k) \asymp p_k^{\alpha - 1}$.

Now we will make the induction step $s - 1 \Rightarrow s$. The recurrent rule implies that $P_{ss} = (f'(x))^s$. Apply the previous proposition to $g = f_\alpha^{3^k - 1}$ and $f = f_\alpha$ at the point p_k. We get

$$\begin{aligned} (f_\alpha^{3k})^{(s)}(p_k) &= (f_\alpha^{3^k - 1})^{(s)}(f_\alpha(p_k))(f_\alpha'(p_k))^s \\ &\quad + \sum_{l=1}^{s-1} (f_\alpha^{3^k - 1})^{(l)}(f_\alpha(p_k)) P_{sl}(f_\alpha'(p_k), \ldots, f_\alpha^{(s-l+1)}(p_k)). \end{aligned}$$

Equality (2) implies that

$$(f_\alpha^{3k})^{(s)}(p_k) = \lambda^{k(1-s)} f_\alpha^{(s)}(p) \asymp p_k^{1-s}.$$

The induction assumption and equality (3) imply that

$$(f_\alpha^{3^k - 1})^{(l)}(f_\alpha(p_k)) P_{sl}(f_\alpha'(p_k), \ldots, f_\alpha^{(s-l+1)}(p_k)) \asymp p_k^{1 - \alpha l} p_k^{\alpha l - s} = p_k^{1-s}.$$

Finally, $(f_\alpha'(p_k))^s \asymp p_k^{(\alpha - 1)s}$ and $(f_\alpha^{3^k - 1})^{(s)}(f_\alpha(p_k)) \asymp p_k^{1 - \alpha s}$, as required. □

In the next proposition, we estimate the derivatives of δ_k at p_k.

PROPOSITION 3. *Suppose that δ_k satisfies conditions (1)–(3) and (3'). Then, as $k \to \infty$,*

$$\delta_k^{(s)}(p_k) \asymp p_k^{\alpha-s}.$$

Notice that the exponents of p_k in the derivatives of δ_k at p_k exactly match the exponents of the corresponding derivatives of f_α.

Proof. The proof of this proposition is very similar to the proof of the previous proposition, so we will be brief. Again, we will use induction to prove it. The case $s = 1$ we already considered before; let us make the induction step. So, we can assume that $\tilde{f}_k^{(l)}(p_k) \asymp p_k^{\alpha-l}$ for all $l = 1, \ldots, s-1$, where as before $\tilde{f}_k = f_\alpha + \delta_k$.

Using the formula for the derivative of a composition, we get

$$(\tilde{f}_k^{3k})^{(s)}(p_k) = (\tilde{f}_k^{3^k-1})'(\tilde{f}_k(p_k))\tilde{f}_k^{(s)}(p_k) + \sum_{l=2}^{s-1}(\tilde{f}_k^{3^k-1})^{(l)}(\tilde{f}_k(p_k))P_{sl}(\tilde{f}_k'(p_k), \ldots, \tilde{f}_k^{(s-l+1)}(p_k)).$$

Here we used the fact that

$$P_{s1}(f'(x), \ldots, f^{(s)}(x)) = f^{(s)}(x).$$

The orbit of p_k is disjoint from I_k except for the point p_k itself, so

$$(\tilde{f}_k^{3^k-1})^{(s)}(\tilde{f}_k(p_k)) = (f_\alpha^{3^k-1})^{(s)}(f_\alpha(p_k)).$$

Taking this into account and recalling that we want $(\tilde{f}_k^{3k})^{(s)}(p_k) = 0$, we obtain

$$\begin{aligned}\tilde{f}_k^{(s)}(p_k) &= -\left[\sum_{l=2}^{s-1}(f_\alpha^{3^k-1})^{(l)}(f_k(p_k))P_{sl}(\tilde{f}_k'(p_k), \ldots, \tilde{f}_k^{(s-l+1)}(p_k))\right] \Big/ (f_\alpha^{3^k-1})'(f_\alpha(p_k)) \\ &\asymp \left[\sum_{l=2}^{s-1} p_k^{1-\alpha l} p_k^{\alpha l - s}\right] \Big/ p_k^{1-\alpha} \\ &= p_k^{\alpha-s}. \qquad \square\end{aligned}$$

Now, when we know all the derivatives of δ_k, we will show that a required function exists.

PROPOSITION 4. *For every $r \in \mathbb{N}$, there is a constant $C = C(r)$ such that for all d^i, $i = 0, \ldots, r$, there is a function $\phi \in C^\infty([-\infty, \infty])$ such that $\|\phi\|_{C^r} \le C \max(\{|d^i|, i = 0, \ldots, r\})$, $\phi^{(i)}(0) = d^i$ for $i = 0, \ldots, r$, and ϕ is zero outside the interval $(-1, 1)$.*

Proof. For every $i \in \mathbb{Z}_+$, there is a function $\phi_i \in C^\infty((-\infty, \infty))$ such that $\phi_i(x) = 0$ if $x \notin (-1, 1)$, $\phi_i^{(j)}(0) = 0$ for all $j \ne i$, and $\phi_i^{(i)}(0) = 1$. Fix such a set of functions. Then, given $d^0, \ldots, d^r$, let $\phi = \sum_{j=0}^{r} d^j \phi_j$. This function and its derivatives have the required values at zero and the norm of ϕ can be estimated:

$$\|\phi\|_{C^r} \le \sum_{i=0}^{r} |d^i| \|\phi_i\|_{C^r} \le (r+1) \max(\{\|\phi_i\|_{C^r}, i = 0, \ldots, r\}) \times \max(\{|d^i|, i = 0, \ldots, r\}). \qquad \square$$

To finish the construction of δ_k, we linearly rescale the interval I_k to the interval $[-1, 1]$ and denote the corresponding function by $\hat{\delta}_k$. On the boundary of $[-1, 1]$, the value of $\hat{\delta}_k$ and its derivatives are zero and at the point 0 we have

$$\hat{\delta}_k^{(s)}(0) = \lambda^{ks} |I|^s \delta_k^{(s)}(p_k) \asymp \lambda^{k\alpha}.$$

Due to the previous proposition, we can find a function $\hat{d}_k$ which has the required values and its norm is bounded:

$$\|\hat{\delta}_k\|_{C^r} \le C\lambda^{k\alpha}.$$

Rescaling back to the interval I_k, we get a function δ_k and its norm satisfies

$$\|\delta_k\|_{C^r} \le C\lambda^{k(\alpha - r)}.$$

By construction, $\alpha > r$; therefore, the norm δ_k tends to zero. As in the case $r = 1$, the function we were looking for is $f_\alpha + \sum_{k=k_0}^{\infty} \delta_k$. This function is C^r and has neutral periodic points of order at least $r + 1$ of periods $3^{k_0}, \ldots$. The theorem is proved. $\square$

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