A \( C^r \) unimodal map with an arbitrary fast growth of the number of periodic points

V. KALOSHIN†§ and O. S. KOZLOVSKI‡

† Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA
(e-mail: kaloshin@math.psu.edu)
‡ University of Warwick, Coventry, CV4 7AL, England
(e-mail: oleg@maths.warwick.ac.uk)

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Abstract. In this paper we present a surprising example of a \( C^r \) unimodal map of an interval \( f : I \to I \) whose number of periodic points \( P_n(f) = |\{x \in I : f^n x = x\}| \) grows faster than any ahead given sequence along a subsequence \( n_k = 3^k \). This example also shows that ‘non-flatness’ of critical points is necessary for the Martens–de Melo–van Strien theorem [M. Martens, W. de Melo and S. van Strien. Julia–Fatou–Sullivan theory for real one-dimensional dynamics. Acta Math. 168(3–4) (1992), 273–318] to hold.

1. Introduction
In this paper we investigate growth of the number of periodic points of \( C^r \) maps \( f : I \to I \) of the interval \( I = [-1, 1] \). Denote by \( C^r(I, I) \) the space of such maps with the uniform \( C^r \) topology.

Definition 1. A map \( f : I \to I \) is called Artin–Mazur (A-M) if for some \( 0 < r \leq \infty \) in \( C^r(I, I) \) the set of A-M maps is \( C^r \) dense (see also [K1]).

Artin and Mazur [AM] proved that for any \( 0 \leq r \leq \infty \) in \( C^r(I, I) \) the set of A-M maps is \( C^r \) dense (see also [K1]). It turns out that for one-dimensional maps much more can be said about A-M maps.

Definition 2. We say that \( f \in C^r \) has a non-flat critical point \( c \) if there is a local \( C^r \) diffeomorphism \( \phi \) with \( \phi(c) = 0 \) such that \( f(x) = |\phi(x)|^a + f(c) \) for some \( a \geq 2 \).

If a map \( f : I \to I \) is \( C^r \) and at a critical point one of the higher derivatives of \( f \) does not vanish, this critical point is non-flat.

§ On leave from: Department of Mathematics, University of Maryland, College Park, MD 20740, USA.
Theorem 1. [MMS] Let \( r \geq 2 \) and \( f : I \to I \) be a \( C^r \) map with no flat critical points. Then for some \( n_0 \in \mathbb{N} \) and \( \gamma > 0 \) for any periodic point \( x = f^n x \) of period \( n > n_0 \) we have \(|(f^n)'x| > 1 + \gamma|l\).

Corollary 1. In the setting above for an open dense set, we have \( P_n(f) \leq |I|(2\gamma)^{-1}\|f\|^n_{C^2} \) and, therefore, such an \( f \) is an A-M map and A-M maps form an open dense set in \( C^r(I, I) \) for any \( r \geq 2 \).

Our main result is a counterpart of the Martens–de Melo–van Strien theorem.

Theorem 2. For any sequence \( a = (a_n)_{n \in \mathbb{N}} \) and any \( r \in \mathbb{N} \) there exists a \( C^r \) unimodal map \( f : I \to I \) such that for any \( k \in \mathbb{Z}_+ \) we have \( P_{3k}(f) > a_{3k} \). The map is \( C^\infty \) everywhere except for the critical point.

In the proof of this theorem, we will construct an explicit example of such a map. We will start with a \( C^{\omega} \) unimodal infinitely renormalizable map (in fact, it will be a fixed point of a tripling operator) and perturb it near some of its periodic points. The perturbed map \( f \) will still be infinitely renormalizable and certainly it will have a flat critical point in the sense of Definition 2. Otherwise, it would contradict Theorem 1.

There is nothing particularly special about the sequence \( 3k \): any sequence of the form \( m^k, m \geq 2 \), would work, except that for \( m = 2 \) the proof is slightly more complicated.

It turns out that superexponential growth of the number of periodic point for higher-dimensional maps not only exists, but is (Baire) generic in certain open sets in the space of \( C^r \) smooth maps of manifolds [K2], which in turn is based on [GST] (see also [K3]). In [KS] the same phenomenon of generic superexponential growth in certain open sets is found for three-dimensional volume-preserving diffeomorphisms. A related question is to estimate the growth of the number of intersections of two submanifolds of complementary dimensions where one of the submanifolds is iterated by a diffeomorphism and the other one is fixed. A \( C^{\omega} \) example on a two-dimensional torus was constructed in [K4], where the growth is superexponential.

2. Degenerate periodic points

Let \( f : I \to I \) be a \( C^r \) map. We say that a periodic point \( x_0 = f^n x_0 \) is neutral of order \( k \leq r \) if \( (f^n)'(x_0) = 1 \), \( (f^n)^{(s)}(x_0) = 0 \) for \( s = 1, \ldots, k - 1 \) and \( (f^n)^{(k)}(x_0) \neq 0 \). Here we use the notation \( f^{(s)} \) to denote the \( s \)th derivative of \( f \).

If \( f \in C^r \) has a neutral periodic point \( x_0 = f^n x_0 \) of order \( k \leq r \), then a \( C^l \)-perturbation, \( l < k \), can create arbitrarily many periodic points of period \( n \) close to \( x_0 \). Indeed, for simplicity let us assume that \( x_0 = 0 \) and \( n = 1 \) (so \( x_0 \) is a fixed point). Let \( \phi \in C^\infty \) be a hat function (i.e. \( \phi(x) = 0 \) for \( |x| > 1 \) and \( \phi(x) = 1 \) for \( |x| < 1/2 \)). Note that for small \( x \) one has \( \|f(x) - x\|_{C^m} = O(x^{k-m}) \) if \( m < k \). This implies that

\[
\|(f(x) - x)\phi(x/\epsilon)\|_{C^l_{[-\epsilon,\epsilon]}} = O(\epsilon^{k-l}).
\]

Then the function

\[
\tilde{f}(x) = x + \sin(Nx)\delta\phi(x/\epsilon) + (f(x) - x)(1 - \phi(x/\epsilon))
\]

has many fixed points in the interval \((-\epsilon/2, \epsilon/2)\) if \( N \) is large (the number of fixed points is of order \( N\epsilon \)), and \( \tilde{f} \) is \( C^l \) close to \( f \) if \( \epsilon \) and \( \delta \) are small.
3. Fixed point of a renormalization operator with a degenerate critical point

A symmetric unimodal map is an endomorphism of the interval $I$ of the form $f = \phi \circ q_1$, where $\phi \in \text{Diff}^2_+(I)$ is an orientation-preserving $C^2$ diffeomorphism of $I$ and $q_1 : I \to I$, $t \in [0, 1]$ is defined by $q_1(x) = -2t|x|^{\alpha} + 2t - 1$. The exponent $\alpha > 1$ is called the critical exponent of $f$ and in what follows it will be an even integer. The map $q_1$ is called the canonical folding map with peak value $t \in [0, 1]$. The peak value determines the maximum $q_1(0) = 2t - 1$. The above form for the canonical folding map is not just a choice for convenience, it naturally arises [Ma]. The diffeomorphism $\phi$ is called the diffeomorphic part of $f$. Notice that $f(-1) = f(1) = -1$. The collection of unimodal maps with chosen critical exponent $\alpha > 1$ is denoted by $U_\alpha$.

Let $U_\alpha$ be the collection of unimodal maps whose peak value is high enough such that the unimodal map has a fixed point $p \in (0, 1)$. For every $f \in U_\alpha$, we can consider the first return map to the interval $[-p, p]$. If the peak value is not too high, the first return map will be just $f^2|_{[-p, p]}$; the unimodal map $f$ is called renormalizable. The unimodal map obtained by rescaling this first return map to $[-p, p]$ is called the renormalization of $f$. The operator defined in this way is called the renormalization operator. Lanford III [L] and, later, Sullivan [S] proved that there is a fixed point for the renormalization operator. More generally, a unimodal map $f \in U_\alpha$ is called renormalizable if and only if there exists an expanding periodic point $p \in (-1, 1)$ such that the first return map to the central interval $C = [-p, p]$ is of the form $f^q : C \to C$ with $f^q(p) = p$ and $q \geq 2$. The first return map to $C$ will be, up to rescaling, a unimodal map. This unimodal map is a renormalization of $f$. Notice that a renormalization is completely determined by the periodic point $p$. In particular, when $q = 3$, the renormalization operator is well defined. By a theorem of Epstein [E], such an operator has a fixed point $f_\alpha \in U_\alpha$. Moreover, $f_\alpha$ is real analytic.

Let $\alpha$ be an even integer larger than $r$, $f_\alpha \in U_\alpha$ denote a fixed point of the renormalization operator for $q = 3$, and 0 be the critical point of $f_\alpha$. This means that there exists $\lambda \in (-1, 0)$ such that $f_\alpha^3(x) = \lambda f_\alpha(x/\lambda)$ for all $x \in [\lambda, -\lambda]$. Note that this functional equality implies that $-\lambda$ is a periodic point of $f_\alpha$ of period three.

Any renormalizable map $f_\alpha$ has two fixed points, one of which is $-1$ and the other of which we will denote by $p$. The intervals $[\lambda, -\lambda]$, $f([\lambda, -\lambda])$, and $f^2([\lambda, -\lambda])$ are disjoint, and therefore cannot contain $p$. The forward orbit of the critical point of $f_\alpha$ belongs to the union of these three intervals; hence, there is a neighborhood $(p^-, p^+)$ of $p$ free from the forward orbit of 0. We can assume that the point $p$ is in the center of $I$, i.e., $|p - p_-| = |p_+ - p|$. The functional equality implies that $f_\alpha^3(\lambda p) = \lambda f_\alpha(p) = \lambda p$, so $\lambda p$ is a periodic point of $f_\alpha$ of period three. Arguing in the same way, we can see that the point $p_k = \lambda^k p$, $k \in \mathbb{Z}_+$, is a periodic point of period $3^k$ and that there is an interval $I_k = (\lambda^k p^-, \lambda^k p^+)$ free from the forward orbit of 0. Notice that the intervals $I_k$ are disjoint. It is also easy to compute the derivatives at points $p_k$ using the functional

† See Martens [Ma] for more detailed analysis of such fixed points.
equality
\[(f_α^{3^k})^{(s)}(p_k) = λ^{k(1-s)} f_α^{(s)}(p).\] (2)

The main idea. We shall prove that there is an arbitrarily small $C^r$-perturbation $\tilde{f}$ of $f_α$ which coincides with $f_α$ along the orbits of the $p_k$'s and has neutral periodic points of order at least $r + 1$ at $p_k$, where $k > k_0$ and $k_0$ is large enough. Moreover, since the orbits $O_k(f_α) = \{f_α^s(p_k)\}_{s=0}^{3^k-1}$ of the $p_k$'s are isolated from each other, there are $C^r$ small perturbations of $\tilde{f}$ around the $p_k$'s, for each $k > k_0$, which can create explosions of the number of periodic points from the $p_k$'s of periods $3^k$.

4. Calculations

In this section we show that, indeed, for a large enough $k$ by a small $C^r$ perturbation inside of $I_k$ we can obtain a neutral periodic point $p_k$ of order at least $r + 1$.

To make the calculation in the case of general $r$ easier to comprehend, we discuss the case $r = 1$ first. For every $k$, we define a $C^∞$ function $δ_k : \mathbb{R} → \mathbb{R}$ which satisfies the following properties:

1. $δ_k(x) = 0$ for $x \notin I_k$;
2. $δ_k(p_k) = 0$;
3. $δ_k'(p_k) = f_α'(p_k)(1 - f_α'(p))/f_α'(p)$;
4. $||δ_k||_{C^1} → 0$ as $k → 0$.

It is obvious that a function satisfying (1)–(3) exists, so only adding condition (4) makes the problem less trivial. Since the order of the critical point is $α$, we know that $f_α''(p_k) \asymp λ^{k(α-1)}$ (i.e. there is a constant $C > 0$ independent of $k$ such that $C^{-1} < f_α''(p_k)λ^{-k(α-1)} < C$ for all $k$). Therefore, $δ_k'(p_k) = O(\lambda^k|α|^{k(α-1)})$. The size of $I_k$ is $|I||λ|^k$, so it is easy to see that such a function $δ_k$ exists. (We will show this in the more general case later.)

Define the function $\tilde{f}_k = f_α + δ_k$. A trivial computation shows that $(f_α^{3^k})'(p_k) = 1$. Indeed,

\[(f_α^{3^k})'(p_k) = (f_α^{3^k-1} \circ \tilde{f})'(p_k)\]
\[= (f_α^{3^k})'(p_k) \frac{f_α'(p_k) + δ_k'(p_k)}{f_α'(p_k)}\]
\[= f_α'(p_k) \frac{f_α'(p_k) + f_α''(p_k)(1 - f_α'(p))}{f_α'(p_k) f_α'(p)} = 1.\]

The function $\tilde{f} = f_α + \sum_{k=k_0}^{∞} δ_k$ is a $C^1$ function with periodic points $\{p_0, p_1, \ldots\}$. All periodic points starting with $p_{k_0}$ are neutral of order at least two and $\tilde{f}$ is $C^1$ close to $f_α$ if $k_0$ is large enough.

In the general case, we proceed in the same way. We will define $C^∞$ functions $δ_k$ satisfying all the properties above and the following extra properties:

3′) if $\tilde{f}_k = f_α + δ_k$, then $(\tilde{f}_α^{3^k})^{(s)}(p_k) = 0$ for all $s = 2, \ldots, r$;
4′) $||δ_k||_{C^r} → 0$ as $k → 0$.

To prove that such functions exist, we have to estimate the higher derivatives of $δ_k$ at the point $p_k$. For that, we need the following proposition.
PROPOSITION 1. For any \( s \in \mathbb{N} \), there is a collection of polynomials with integer coefficients \( \{P_{sl}(x_1, \ldots, x_{s-l+1})\}_{1 \leq l \leq s} \) such that

\[
(g \circ f)^{(s)}(x) = \sum_{l=1}^{s} g^{(l)}(f(x)) P_{sl}(f'(x), \ldots, f^{(s-l+1)}(x)),
\]

where \( g, f \) are \( C^s \) functions. Moreover, the polynomials \( P_{sl} \) satisfy the following recurrent rule for \( P_{(s+1)l} \):

\[
P_{(s+1)l}(f'(x), \ldots, f^{(s-l+2)}(x)) = \frac{d}{dx} P_{sl}(f'(x), \ldots, f^{(s-l+1)}(x)) + f'(x) P_{s(l-1)}(f'(x), \ldots, f^{(s-l+2)}(x))
\]

and \( P_{11}(f'(x)) = f'(x) \), \( P_{s0} = P_{s(s+1)} = 0 \).

The proposition can be proved by an elementary direct computation.

Observe that the order of the critical point is \( \alpha \); therefore, \( f^{(s)}(x) \approx x^{\alpha-s} \). The recurrent rule implies that

\[
P_{sl}(f'(x), \ldots, f^{(s-l+1)}(x)) \approx x^{\alpha l-s}, \tag{3}
\]

which can be checked by induction.

Now we will use the proposition to estimate \( (f^{3^k-1})^{(s)}(f_\alpha(p_k)) \).

PROPOSITION 2. For every \( s \geq 1 \) as \( k \to \infty \) the following asymptotic holds:

\[
(f^{3^k-1})^{(s)}(f_\alpha(p_k)) \approx p_k^{1-\alpha s}.
\]

Proof. The proposition will be proved by induction. When \( s = 1 \), using the chain rule, we get

\[
(f^{3^k-1})'(f_\alpha(p_k)) = \frac{(f^{3^k})'(p_k)}{f_\alpha'(p_k)} = \frac{f_\alpha'(p_k)}{f_\alpha'(p_k)} \approx p_k^{1-\alpha}.
\]

Here we used equality (2) and \( f_\alpha'(p_k) \approx p_k^{\alpha-1} \).

Now we will make the induction step \( s - 1 \Rightarrow s \). The recurrent rule implies that \( P_{ss} = (f^{(s)})^s \). Apply the previous proposition to \( g = f^{3^k-1} \) and \( f = f_\alpha \) at the point \( p_k \). We get

\[
(f^{3^k})^{(s)}(p_k) = (f^{3^k-1})^{(s)}(f_\alpha(p_k))(f_\alpha'(p_k))^s + \sum_{l=1}^{s-1} (f^{3^k-1})^{(l)}(f_\alpha(p_k)) P_{sl}(f_\alpha'(p_k), \ldots, f^{(s-l+1)}(p_k)).
\]

Equality (2) implies that

\[
(f^{3^k})^{(s)}(p_k) = \lambda^{(1-s)} f^{(s)}(p_k) \approx p_k^{1-s}.
\]

The induction assumption and equality (3) imply that

\[
(f^{3^k-1})^{(l)}(f_\alpha(p_k)) P_{sl}(f_\alpha'(p_k), \ldots, f^{(s-l+1)}(p_k)) \approx p_k^{1-\alpha l} p_k^{\alpha l-s} = p_k^{1-s}.
\]

Finally, \( (f_\alpha'(p_k))^s \approx p_k^{(\alpha-1)s} \) and \( (f^{3^k-1})^{(s)}(f_\alpha(p_k)) \approx p_k^{1-\alpha s} \), as required.
In the next proposition, we estimate the derivatives of \( \delta_k \) at \( p_k \).

**Proposition 3.** Suppose that \( \delta_k \) satisfies conditions (1)–(3) and (3'). Then, as \( k \to \infty \),

\[
\delta_k^{(s)}(p_k) \asymp p_k^{\alpha-s}.
\]

Notice that the exponents of \( p_k \) in the derivatives of \( \delta_k \) at \( p_k \) exactly match the exponents of the corresponding derivatives of \( f_\alpha \).

**Proof.** The proof of this proposition is very similar to the proof of the previous proposition, so we will be brief. Again, we will use induction to prove it. The case \( s = 1 \) we already considered before; let us make the induction step. So, we can assume that \( \tilde{f}_k^{(l)}(p_k) \asymp p_k^{\alpha-l} \) for all \( l = 1, \ldots, s - 1 \), where as before \( \tilde{f}_k = f_\alpha + \delta_k \).

Using the formula for the derivative of a composition, we get

\[
(f_k^{3k})^{(s)}(p_k) = (f_k^{3k-1})'(f_k(p_k)) f_k^{(s)}(p_k) + \sum_{l=2}^{s-1} (f_k^{3k-1})'(f_k(p_k)) P_{s,l}(f_k'(p_k), \ldots, f_k^{(s-l+1)}(p_k)).
\]

Here we used the fact that

\[
P_{s,1}(f'(x), \ldots, f^{(s)}(x)) = f^{(s)}(x).
\]

The orbit of \( p_k \) is disjoint from \( I_k \) except for the point \( p_k \) itself, so

\[
(f_k^{3k-1})^{(s)}(f_k(p_k)) = (f_\alpha^{3k-1})^{(s)}(f_\alpha(p_k)).
\]

Taking this into account and recalling that we want \( (f_k^{3k})^{(s)}(p_k) = 0 \), we obtain

\[
\tilde{f}_k^{(s)}(p_k) = -\left[ \sum_{l=2}^{s-1} (f_\alpha^{3k-1})'(f_k(p_k)) P_{s,l}(f_k'(p_k), \ldots, f_k^{(s-l+1)}(p_k)) \right] \bigg/ (f_\alpha^{3k-1})'(f_\alpha(p_k))
\]

\[
\times \left[ \sum_{l=2}^{s-1} p_k^{1-\alpha l} p_k^{\alpha l-s} \right] \bigg/ p_k^{1-\alpha}
\]

\[
= p_k^{\alpha-s}.
\]

Now, when we know all the derivatives of \( \delta_k \), we will show that a required function exists.

**Proposition 4.** For every \( r \in \mathbb{N} \), there is a constant \( C = C(r) \) such that for all \( d^i, i = 0, \ldots, r \), there is a function \( \phi \in C^\infty((-\infty, \infty]) \) such that \( ||\phi||_{C^r} \leq C \max(||d^i|, i = 0, \ldots, r) \), \( \phi^{(i)}(0) = 0 \) for \( i = 0, \ldots, r \), and \( \phi \) is zero outside the interval \((-1, 1)\).

**Proof.** For every \( i \in \mathbb{Z}_+ \), there is a function \( \phi_i \in C^\infty((-\infty, \infty)) \) such that \( \phi_i(x) = 0 \) if \( x \notin (-1, 1) \), \( \phi_i^{(j)}(0) = 0 \) for all \( j \neq i \), and \( \phi_i^{(i)}(0) = 1 \). Fix such a set of functions. Then,
given \( d^0, \ldots, d^r \), let \( \phi = \sum_{j=0}^r d^j \phi_j \). This function and its derivatives have the required values at zero and the norm of \( \phi \) can be estimated:

\[
\|\phi\|_{C^r} \leq \sum_{i=0}^r |d^i| \|\phi_i\|_{C^r} \leq (r + 1) \max(\{\|\phi_i\|_{C^r}, i = 0, \ldots, r\}) \times \max(\{|d^i|, i = 0, \ldots, r\}).
\]

To finish the construction of \( \delta_k \), we linearly rescale the interval \( I_k \) to the interval \([-1, 1]\) and denote the corresponding function by \( \hat{\delta}_k \). On the boundary of \([-1, 1]\), the value of \( \hat{\delta}_k \) and its derivatives are zero and at the point 0 we have

\[
\hat{\delta}_k^{(s)}(0) = \lambda^k |I| \delta_k^{(s)}(p_k) \approx \lambda^{k\alpha}.
\]

Due to the previous proposition, we can find a function \( \hat{\delta}_k \) which has the required values and its norm is bounded:

\[
\|\hat{\delta}_k\|_{C^r} \leq C \lambda^{k\alpha}.
\]

Rescaling back to the interval \( I_k \), we get a function \( \delta_k \) and its norm satisfies

\[
\|\delta_k\|_{C^r} \leq C \lambda^{k(\alpha-r)}.
\]

By construction, \( \alpha > r \); therefore, the norm \( \delta_k \) tends to zero. As in the case \( r = 1 \), the function we were looking for is \( f_\alpha + \sum_{k-k_0}^{\infty} \delta_k \). This function is \( C^r \) and has neutral periodic points of order at least \( r + 1 \) of periods \( 3^{k_0}, \ldots \). The theorem is proved. \( \square \)

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