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Persistent Markups in Bidding Markets with Financial Constraints*

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Abstract

This paper studies the impact of financial constraints on the persistency of high markups in a class of markets, including most of public procurement, known by practitioners as bidding markets. We develop an infinite horizon model in which two firms optimally reinvest working capital and bid for a procurement contract each period. Working capital is constrained by the firm's cash from previous period and some exogenous cash flow, it is costly and it increases the set of acceptable bids. The latter is because the risk of non-compliance means that only bids that have secured financing are acceptable and less profitable bids have access to less external financing. We say that the firm is (severely) financially constrained if its working capital is such that only bids (substantially) above production cost are acceptable. We show that markups are positive (high) if and only if one firm is (severely) financially constrained. Our main result is that markups are persistently high because one firm is severely financially constrained most of the time.

JEL Classification Numbers: L13, D43, D44. Keywords: bidding markets, financial constraints, markups.

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1 Introduction

A long standing concern in industrial organization is the persistency of high markups. Little attention has been given to the role of financial constraints in spite of being very common in many markets where economists have struggled to explain the persistency of high markups. A prominent example is highway maintenance procurement where firms are notorious for facing financial distress.[1] This is one of the many examples of bidding markets[2] where financial constraints have an impact on prices:

> ”Offers submitted on Monday by Global Infrastructure Partners and a consortium led by Manchester Airport Group have been depressed [...] by the problems of raising the necessary bank finance.”
>
> *“Ferrovial receives depressed bids for Gatwick,” Financial Times, April 28th, 2009*

Our main result is that the impact of financial constraints persist and explain the persistency of markups in bidding markets. Our model predicts that market concentration should be greater for larger projects than for smaller projects, a prediction in line with the empirical evidence.[3]

A representative example of the institutional details of the bidding markets we are interested in is highway maintenance procurement. As Hong and Shum (2002) pointed out “many of the contractors in these auctions bid on many contracts overtime, and likely derive a large part of their revenues from doing contract work for the state.” Besides,

[1] The most frequent explanations for high markups in this market invokes either collusion or capacity constraints, see for instance Che (2008) or Jofre-Bonet and Pesendorfer (2003), respectively.

[2] Bidding markets are those in which trade is organized through bidding. For a general reference on bidding markets see Klemperer (2005). The most commonly cited example is public procurement, e.g. contracts for highway maintenance, rubbish collection, medical or military supplies and even prison construction and management, see Bajari and Ye (2003), Jofre-Bonet and Pesendorfer (2003), and OFT (2004). It also include public auctions like timber auctions or motorway concessions. Other examples involve private companies that also deal with suppliers/clients asking for quotes and then choosing the best option. This is the way many companies outsource IT services, how airlines buy engines for their airplanes, or how North Sea oil platforms contract their annual helicopter services, see OECD (2006).

[3] Porter and Zona (1993) explain that “the market for large jobs [in procurement of highway maintenance] was highly concentrated. Only 22 firms submitted bids on jobs over $1 million. On the 25 largest jobs, 45 percent of the 76 bids were submitted by the four largest firms.”

Porter and Zona (1993) explain that "The set of firms submitting bids on large projects was small and fairly stable[...] There may have been significant barriers to entry, and there was little entry in a growing market."[4] Motivated by these observations, we study an infinite horizon model in which two firms reinvest and bid optimally. The cash available for reinvestment at the beginning of each period is observable and equal to the remaining working capital plus the earnings in previous procurement contract and some exogenous cash flow. Holding working capital is costly.

Our model of financial constraints takes seriously the fact that contracting in procurement is usually characterized by the risk of non-compliance[5] and that the bid requires financing.[6] Typically, these features imply, as we show in a model with moral hazard and limited liability in Appendix B, that a bid is acceptable only if it is backed by a surety bond[7] and that the firm faces a borrowing limit that relaxes with the firm's potential profits. As a consequence, only bids above a certain threshold can get the required external financing and so are acceptable. The set of acceptable bids is thus increasing in the firm's working capital. In this sense, we can identify reinvesting working capital as relaxing the financial constraint. By a Bertrand argument, the price is equal to the minimum acceptable bid for the firm with less working capital and thus markups are high if and only if one firm holds little working capital.

[4]Moreover, it can be shown that in a model with many firms and entry the natural extension of the equilibrium we study has the feature that only the two firms with more cash enter the market.

[5]The risk of non compliance usually arises because the transformation of the contract into profits requires undertaking an entrepreneurial project whose development might be jeopardized, for instance, by moral hazard. This is obvious in the case of a procurement since the good has typically to be produced, and it is also common in auctions. For instance a company that wins the right to operate a motorway converts the contract into profits only after some years of operating the service.

[6]This is usually the case in procurement because the procurement price is only paid upon completion of the contract. The usual explanation is that the banks have more expertise assessing the financial viability of the project than the buyer, see Calveras, Ganuza, and Hauk (2004). In auctions, it is the auction price that needs to be financed.

[7]This is an instrument that plays two roles: first, it certifies that the proposed bid is not jeopardized by the technological and financial conditions of the firm, and second, it insures against the losses in case of non-compliance. Indeed, the Surety Information Office highlights that "the surety [...] may require a financial statement [that] typically includes [...]: Balance sheet-shows the assets, liabilities, and net worth of the business as of the date of the statement. This helps the surety company evaluate the working capital and overall financial condition of the company." See http://www.sio.org/html/Obtain.html#Financial.

The strategic considerations that shape equilibrium reinvestments are easier to understand in a one-period version in which firms start with different amounts of cash. We call leader the firm with more cash and laggard the other firm. Reinvesting resembles an all pay auction[8] in that only the firm that reinvests more wins but both firms bear the cost of working capital. In this sense our model is similar to Galenianos and Kircher (2008) though it differs in that their working capital is not bounded by cash.[9] As in the analysis of the all pay auction with complete information, the equilibrium verifies here the following two properties: (i) each firm plays a mixed strategy[10] in a connected support; and (ii) in the interior of this support firms use a density that equalizes the marginal cost and marginal benefit of reinvesting.

In what we call a symmetric scenario, a firm that reinvests the laggard's cash makes losses for any pair of strategies that verify (i) and (ii). This means that firms cash is sufficiently abundant to render it irrelevant for the equilibrium. Thus, firms randomize with equal probability of winning as in the equilibrium of the standard symmetric all pay auction, and its usual zero profit condition implies here that the expected markup just covers the expected cost of working capital. Consequently, the markup tends to zero as the cost of working capital becomes negligible.

Instead, in what we call the laggard-leader scenario, the leader finds it profitable to reinvests the laggard's cash when the strategies verify (i) and (ii). Actually, this is what the leader does in equilibrium with a probability that tends to one as the cost of working capital becomes negligible. Thus, the laggard's chances of winning tend to zero,[11] which

[8] In particular, it resembles Che and Gale's (1998a) model of an all pay auction with caps in that reinvestments are bounded by cash. Our model is more general in that they assume exogenous caps that are common to all agents. Another difference is that the size of the prize varies with the rival's action. To the best of our knowledge, the literature on all pay auctions, see Kaplan, Luski, Sela, and Wettstein (2002) and Siegel (2009), has only considered the case of prizes that vary with the agent's action.

[9] Consequently, the laggard-leader scenario that we describe below does not arise in their model.

[10] We expect our equilibrium to be the limit of the equilibrium of a game in which firms have private information about their costs, as Amann and Leininger (1996) have already shown for the standard all pay auction.

[11] Assuming a tie breaking rule that allocates to the leader in case of a tie in which the laggard is reinvesting all its cash. This guarantees that the leader's has a best response when the laggard reinvests all its cash, a condition necessary to ensure existence of an equilibrium. Intuitively, it gives the same outcome as the limit when the leader reinvest marginally more than the laggard's cash. Alternatively, we

explains why the laggard's probability of reinvesting also tends to zero and the markup becomes maximum.

In our infinite horizon model, these considerations also apply to the unique equilibrium that is the limit of the sequence of equilibria of models with an increasing number of periods.[12] At any period, reinvesting also resembles an all pay auction because reinvesting more than the rival gives not only the procurement profits, as above, but also the benefits of being leader next period, and it is still costly because working capital has no value for future competition. The latter is because when the firm wins, it becomes leader and the leader's cash does not affect future competition, and when the firm loses, it remains laggard and the laggard's continuation payoffs are pinned down by the zero profit condition that holds in both scenarios. Interestingly, it is the prospect of strong future competition that dumps current incentives to reinvest and, as a consequence, current competition.

On the equilibrium path, the frequency of each scenario depends on the severity of the financial constraint. We say that the financial constraint is tight (resp. loose) when exogenous cash flow is small (resp. large) relative to the working capital that renders financial constraints irrelevant for bidding.

Our main result that financial constraints explain persistent high markups holds because the laggard-leader scenario occurs most of the time, even as the cost of working capital becomes negligible, when the financial constraint is sufficiently tight. Another consequence is that the same firm tends to win consecutive procurement contracts.[13] On the contrary, firms win each contract with the same probability when the financial constraints are so loose that the symmetric scenario occurs every period. This explains our prediction of greater market concentration for larger projects to the extend that one can associate the tightness of the financial constraint to the project's size.[14]

could derive our results assuming the usual uniform random tie-breaking rule and that bids are chosen from a discrete grid with a sufficiently large number of elements.

[12] The uniqueness result is proved in the supplementary material.

[13] To the extend that joint profits are larger in the laggard-leader scenario than in the symmetric scenario, our result is related to the literature on increasing dominance due to efficiency effects (see Budd, Harris, and Vickers (1993), Cabral and Riordan (1994) and Athey and Schmutzler (2001).) The novelty of our model is that the underlying static game displays neither strategic complementarity nor strategic substitutability.

[14] As we show in Appendix B.

Che and Gale (1998b) already showed that markups can reflect financial constraints. Our results show that this is the typical situation once we allow firms to relax their financial constraints endogenously.[15] Note, however, that whereas Che and Gale (1998b) assume that firms do not differ ex ante in the severity of their financial constraints, our results show that this is seldom the case.

Our characterization of the dynamics resembles that of Kandori, Mailath, and Rob (1993) in that we study a Markov process in which two persistent scenarios occur infinitely often and we ask which of the two occur most of the time as the randomness vanishes. We want to underscore that while the transition function of their stochastic process are exogenous, ours stems from the equilibrium strategies of the infinite horizon game. As in Cabral (2011), a typical time series of market shares displays not only a lot of concentration but also, and more importantly, tipping, i.e. the system is very persistent but moves across extremely asymmetric states.

Other explanations for the persistency of markups are capacity constraints and collusion. Our model is empirically distinguishable from these models in that it predicts negative correlation between the laggard's cash and the bids (or the price). An alternative to distinguish our model from the model of capacity constraints when the laggard's cash is not observable by the econometrician is to use as a proxy either completion dates of past projects of the laggard or past bids.[16] These proxies do not explain the current bids in the models of capacity constraints once one controls by backlog and costs, see Bajari and Ye (2003) and Jofre-Bonet and Pesendorfer (2003).

Another way in which our model is empirically distinguishable from the models of collusion is that the time average of the price may decrease with patience[17] and winning today increases the probability of winning tomorrow. Collusive models predict that patience increases the time average of the price, see Athey and Bagwell (2001) and Green

[15]Che and Gale (1996, 2000), DeMarzo, Kremer, and Skrzypacz (2005) and Rhodes-Kropf and Viswanathan (2007) also studied the effect of some given financial constraints in auctions and Pitchik and Schotter (1988), Maskin (2000), Benoit and Krishna (2001) and Pitchik (2009) how bidders distribute a fixed budget in a sequence of auctions. The latter is not a concern in our setup because the profits are realized before the beginning of the next auction.

[16]The former holds true because projects are usually paid upon completion and the latter because the laggard's past and current cash are positively correlated.

[17]This is what happens when the financial constraint is sufficiently loose.

and Porter (1984), and that winning today has either no effect on winning tomorrow, see Athey, Bagwell, and Sanchirico (2004), or decreases its probability, see McAfee and McMillan (1992), Athey and Bagwell (2001) and Aoyagi (2003).

Section 2 defines our canonical model of procurement with financial constraints. Section 3 illustrates the strategic forces that shape our results in a static version of our main model. The latter is analyzed in Section 4. Section 5 concludes. We also include an appendix with the more technical proofs (Appendix A) and an extension of our model to endogenize financial constraints (Appendix B).

2 A Reduced Form Model of Procurement with Financial Constraints

In this section, we describe a model of procurement that we later embed as a stage of the models in Sections 3 and 4.

We model the market as two firms[18] competing for a procurement contract of common and known cost c in a first price auction:[19] Each firm submits a bid, and the firm who submits the lower bid gets the contract at a price equal to its bid. Thus, a firm that bids p has *potential profits* $p - c$,[20] and if it wins the *markup* is $p - c$.

Our model of financing assumes the following two features of reality, (A) there is a risk of non-compliance and (B) each firm has access to a credit line that weakly increases with the firm's potential profits. In Appendix B, we introduce a model in which feature (B) arises due to feature (A), limited liability and moral hazard and that is inspired by the observation already emphasized by Tirole (2006), page 114, that:

> The borrower must [...] keep a sufficient stake in the outcome of the project in order to have an incentive not to waste the money.

[18] As in all pay auctions, see Baye, Kovenock, and de Vries (1996), assuming more than two firms rises the problem of multiplicity of equilibria. It may be shown that there is always an equilibrium in which two firms play the strategies we propose below for our two-firm model and the other firms do not reinvest.

[19] The particular auction framework turns out to be of little relevance because we assume symmetric information between the two firms. We conjecture similar results in a second price auction.

[20] A sale auction of a good with common and known value V can be easily encompassed in our analysis assuming that $c = -V < 0$ and bids are negative numbers. See also Footnote 22.

The risk of non-compliance can be hedged with a surety bond. We assume that non-compliance implies arbitrarily large damages to the buyer and the firms are only liable up to its working capital. Consequently, a bid is *acceptable* only if accompanied by a surety bond and a surety bond is issued only if it induces compliance. Compliance requires financing the production cost c. If the firm's working capital w is less than c, it needs to borrow $c - w$. Note that (B) implies that a loan of a given size L requires some minimum potential profits $\hat{\pi}(L)$.[21] Therefore, a firm with working capital w can finance the production of the good only for bids with potential profits above a minimum profitability requirement $\pi(w) \equiv \hat{\pi}(c - w)$.[22] Hence, only bids above $c + \pi(w)$ are acceptable. Finally, we say that the firm is *financially constrained* if its working capital is such that only bids strictly above production cost are acceptable. We denote by θ the working capital that renders financial constraints irrelevant for bidding. Formally, θ is the supremum of the set of working capitals for which the firm is financially constrained and we assume that $\theta \in (0, \infty)$.

Remark 1. *Note that (B) implies that π is weakly decreasing. Our analysis in Sections 3 and 4 applies whenever the set of acceptable bids is defined as here and π is weakly decreasing regardless of whether (A) or (B) holds.*[23]

For simplicity, we assume that π is also continuously differentiable.

3 A Static Model

We consider in this section a one-period version of the model we analyze in Section 4 to illustrate the strategic trade-offs that shape our results.

[21] Implicitly, we are assuming that these minimum potential profits are well-defined. This would be true, for instance, if the set of potential profits that give rise to a credit line weakly larger than L is non-empty and closed.

[22] In the case of a sale auction, if a firm bids a price p greater than its working capital w, it has to borrow $p - w$, which in our model requires that the potential profits $V - p$ are weakly larger than $\hat{\pi}(p - w)$. This defines a function $\pi(w)$ that gives us the minimum potential profits for a bid to be acceptable.

[23] For instance, the model of auctions with budget constraints analyzed by Che and Gale (1998b) in Section 3.2 violates (B) since there is no credit line. However, it can be analyzed in our framework with $\pi(w) = V - w$ and the interpretation of our model as a sale auction (see Footnote 20).

Our game has two stages. In the first stage, *the morning*, each firm starts with some cash (publicly observable) that can be either reinvested in the firm as working capital or distributed to the shareholders. In the second stage, *the afternoon*, firms choose an acceptable bid for a market as described in Section 2. We assume that the firms' working capitals are publicly observable at the beginning of the afternoon to avoid the complexities of asymmetric first price auctions.

The firm cares about the sum of the morning payments to the shareholders and the working capital plus market profits discounted at rate $\beta < 1$. The working capital reinvested in the firm has a rate of return $r \in [1, 1/\beta)$ between the morning and the afternoon.[24] Hence, a firm has working capital w in the afternoon if it reinvested in the morning w/r out of its cash and one unit of working capital has an implicit cost of $1 - \beta r > 0$. Throughout this paper, we say that the *cost of working capital becomes negligible* when we let r tend to $r^* \equiv 1/\beta$ from below.

We restrict to the case in which both firms start with different cash[25] and call *leader* to the firm that starts with more cash and *laggard* to the other firm.

In this model, it is important we model explicitly the tie-breaking rule in the auction.[26] In case of a tie, the firm with more working capital wins, and if both firms have the same amount of working capital, a uniformly random tie-breaking rule is used except when the laggard was reinvesting all its cash. In this latter case, the leader wins.

We solve the game by backward induction. There is a unique equilibrium in the second stage as can be deduced from the standard arguments in Bertrand competition. In this equilibrium both firms bid c if acceptable for both, and otherwise, the minimum acceptable bid for the firm with less working capital. Leaving aside the case where firms reinvest at least θ/r, which is suboptimal, the firm that wins has profits equal to the minimum

[24] Later, we are interested in the limit in which $1 - \beta r$ tends to zero and while all our results hold true when we set $r = 1$ and make β tend to one, our arguments are more obscure because firms continuation values diverge to infinity. Besides, an additional complication that we discuss after Proposition 5 arises in the extension of our model in Appendix B.

[25] We endogenize the firms' cash in the model of next section and show that our interest in the case in which both firms start with different cash is motivated in that it holds almost surely along the equilibrium path because of the mixed strategies firms use every period.

[26] As it is usually the case in Bertrand games and all pay auctions, we deviate from the more natural uniformly random tie-breaking rule to guarantee the existence of an equilibrium, see also Footnote 11.

profitability requirement of the other firm which are strictly positive by definition of θ. The firm with more working capital wins except for the case of a tie where the rule described above applies.

The continuation play described above implies that reinvesting resembles an all pay auction in that only the firm that reinvests more wins the procurement contract but both firms bear the cost of working capital. As it is usually the case in all pay auctions, there is no pure strategy equilibrium in the morning. This can be easily understood when both firms' cash is weakly larger than θ/r. If both firms choose different reinvestments, the one reinvesting more has a strictly profitable deviation: decreasing marginally its reinvestment.[27] If both firms reinvests the same x, there is also a strictly profitable deviation: increasing marginally its reinvestment if $x < \theta/r$, and otherwise, no reinvestment.[28]

The usual indifference condition of a mixed strategy equilibrium is verified if firms randomize with a density $\tilde{f}^r(x) \equiv \frac{1-\beta r}{\beta\pi(rx)}$ in a support[29] $[0, \tilde{\nu}^r]$ for some $\tilde{\nu}^r < \theta/r$.[30] This is because $\tilde{f}^r(x)$ equalizes the marginal cost $1 - \beta r$ of reinvesting x with its marginal expected benefit $\tilde{f}^r(x)\beta\pi(rx)$. The latter arises as a marginal increase in reinvestment around x let the firm move from losing to winning the procurement contract when the rival reinvests close to x. This event occurs with a limit probability approximated by the density of the rival's randomization at x, i.e. $\tilde{f}^r(x)$, and the firm discounted profits in the procurement contract when this event occurs are equal to $\beta\pi(rx)$.

We refer to the case in which the laggard's cash is weakly greater than $\tilde{\nu}^r$ as the *symmetric scenario.*

Proposition 1. *There is a unique equilibrium in the symmetric scenario. In this equilibrium both firms randomize with density* $\tilde{f}^r(x) = \frac{1-\beta r}{\beta\pi(rx)}$ *in the support* $[0, \tilde{\nu}^r]$.

[27] It saves on the cost of working capital without affecting to the procurement outcome.

[28] In the former case, the deviation is profitable because $x < \theta/r$ implies that winning the procurement contract gives strictly positive profits and the deviation breaks the tie in favor of the deviating firm with an arbitrarily small increase in the cost of working capital. In the latter case, $x \geq \theta/r$ implies that there is no profit in the procurement contract and the deviation saves the cost of working capital.

[29] We use the definition of support of a probability measure in Stokey and Lucas (1999). This is the smallest closed set with probability one.

[30] That $\tilde{\nu}^r$ is well defined can be deduced from the fact that continuous differentiability of π implies that the differential of π is uniformly bounded below in $[0, \theta]$ by a constant $-\kappa$ and hence, $\pi(x) \leq \kappa(\theta - x)$, and $\lim_{x\uparrow\theta/r} \int_0^x \frac{1-\beta r}{\beta\pi(r\tilde{x})} d\tilde{x} = +\infty$.

In this equilibrium, each firm's expected profits in the second stage are equal to its costs of working capital.[31] This can be easily deduced from the fact that the support of the firms' randomization includes zero and in this case the firm loses and has zero cost of working capital.

Corollary 1. *In the symmetric scenario: (i) both the laggard and the leader have the same probability of winning the procurement contract, (ii) a marginal change in the initial distribution of cash has no effect on the equilibrium play; and (iii) as the cost of working capital becomes negligible, each firm's reinvestment converges*[32] *to* θ/r^* *and the markup converges to zero.*

The first two results are straightforward. The third result arises because the density of the equilibrium randomization of reinvestments collapses to zero and the probability mass concentrates around θ/r^*. Intuitively, as the cost of working capital becomes negligible financial constraints become irrelevant.

The above equilibrium strategy is not feasible for the laggard if its cash is strictly less than $\tilde{\nu}^r$. We refer to this case as the *laggard-leader scenario.*

Proposition 2. *There is a unique equilibrium in the laggard-leader scenario. In this equilibrium both firms use a truncation of the strategy in Proposition 1 at the laggard's cash. The laggard puts the remaining probability in an atom of probability at zero, and the leader at the laggard's cash.*[33]

Both firms put their atom of probability at points that do not upset the incentives of the rival to play its equilibrium randomization. There is only one such point for the laggard, whereas the leader's randomization takes the minimum working capital which ensures that it wins the procurement contract.

[31] Hence, the expected discounted markup is equal to the total expected costs of working capital in which the firms incur. That is:

$$2(1-\beta r)\int_0^{\tilde{\nu}^r} x\tilde{f}^r(x)dx.$$

[32] In what follows, convergence is always in distribution unless stated otherwise.

[33] Interestingly, this equilibrium has the same features as the equilibrium of an all pay auction in which both agents have the same cap but the tie-breaking rule allocates to one of the agents only. The latter model has been studied in an independent and simultaneous work by Szec (2010).

The leader makes discounted expected profits in the second stage strictly greater than its cost of working capital. This is because the laggard puts an atom of probability at zero, and hence the leader expects to win the procurement contract at a profit by reinvesting arbitrarily little.[34]

Corollary 2. *In the laggard-leader scenario: (i) the leader wins the procurement contract with strictly greater probability than the laggard; (ii) a marginal increase in the laggard's cash changes the equilibrium play, and in particular, decreases the expected markup; and (iii) as the cost of working capital becomes negligible, the laggard's and leader's equilibrium reinvestment converge to zero and to the laggard's cash, respectively, the probability that the leader wins the procurement contract converges to one, and the markup converges to* $\pi(0)$.

The first two results are a straightforward consequence of the atoms in the equilibrium distributions. The limit results arise because the density of the equilibrium randomization of reinvestments collapses to zero and the probability moves to the mass points. Intuitively, there is an increasing preemption effect since the laggard's chances of winning vanish as the leader's equilibrium randomization increases its mass at the laggard's cash.

4 The Dynamic Model

In the analysis of the previous section, the qualitative features of the equilibrium hinge on whether the symmetric or the laggard-leader scenario occurs which depends on the initial distribution of cash. In this section, we endogenize this distribution by assuming that it is derived from the past market outcomes and we show that the laggard-leader scenario occurs most of the time.

[34]Consequently, the leader's discounted expected profit minus its cost of working capital is equal to the probability that the laggard plays at its atom, $(1-\tilde{F}^r(m_{\min}))$ where $m_{\min}$ is the laggard's cash, times the discounted profits in the procurement contract when both firms carry no working capital, $\beta\pi(0)$. Indeed, one can show that the expected discounted markup is equal to:

$$(1-\beta r)\left(2\int_0^{m_{\min}} x\tilde{f}^r(x)dx + (1-\tilde{F}^r(m_{\min}))m_{\min}\right) + (1-\tilde{F}^r(m_{\min}))\beta\pi(0),$$

where $\tilde{F}^r$ denotes a cumulative distribution with density $\tilde{f}^r$ in $[0,\tilde{\nu}^r]$

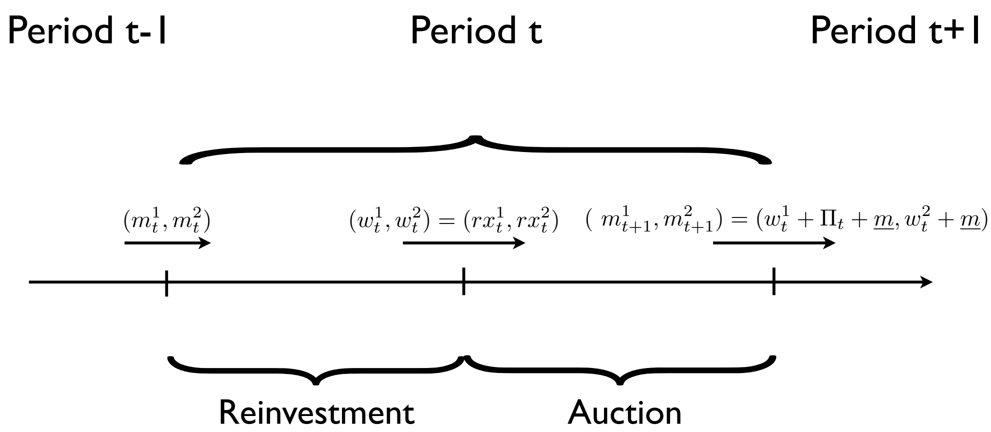

Figure 1: Time line when firms start with cash (m_t^1, m_t^2) reinvests (x_t^1, x_t^2), Firm 1 wins the procurement contract with profits Π_t, and firms start next period with cash (m_{t+1}^1, m_{t+1}^2).

4.1 The Game

We consider the infinite repetition of the time structure of the game in last section in which the cash in the first period is exogenous and afterwards, it is equal to its working capital in the previous period plus the profits in the procurement contract and some exogenous cash flow $\underline{m} > 0$.[35] We assume that $\underline{m}$ is constant across time and firms, and interpret it as derived from other activities of the firm. The firm pays to the shareholders only in the morning and maximizes the discounted value of these expected payments. The corresponding time-line is represented in Figure 1.

A Markov strategy consists of a reinvestment function, σ, and a bid function, b. The reinvestment function of a firm with cash m that faces a rival with cash m' is a randomization over the feasible reinvestments described by its cumulative distribution function $\sigma(.| m, m') \in \Delta(m)$, where $\Delta(m)$ denotes the set of cumulative distribution functions with support in $[0, m]$. The bid function of a firm with working capital w that faces a rival with working capital w' is an acceptable bid $b(w, w') \geq c + \pi(w)$.

We refer to the beginning of the period lifetime payoff of a firm that has cash m when

[35] All our results also hold true in the case $\underline{m} = 0$, though its analysis differs slightly.

its rival has m' as the firm's value function and denote it by $W(m, m')$.

Our interest is in a model with a tie-breaking rule as described in the previous section.[36] Here, however we shall make alternative assumptions. These assumptions do not affect the equilibrium strategies but let us avoid several technical complications that could distract from our main argument. These are the following: first, firms cannot reinvest all their cash and hence their reinvestments strategies are chosen from a subset $\underline{\Delta}(m)$ of distributions in $\Delta(m)$ with no atom of probability at m. Second, if both firms have the same working capital and submit the same bid p, they share the procurement profits, i.e. each gets $1/2(p-c)$. Third, in our equilibrium definition, see below, we do not put any constraint in information sets at the beginning of the morning in which both firms have the same amount of cash.[37]

We denote by $\phi(w, w', p, p')$ the cash of a firm that, in the previous afternoon, had working capital w and bid p against a rival that had working capital w' and bid p'. Formally:

$$\phi(w, w', p, p') \equiv w + \underline{m} + \begin{cases} p - c & \text{if } p < p', \text{ or } p = p' \text{ and } w > w', \\ \frac{1}{2}(p-c) & \text{if } p = p' \text{ and } w = w' \\ 0 & \text{o.w.} \end{cases}$$

Definition: A (symmetric)[38] Bidding and Investment (BI) equilibrium associated to π is a value function W, a bidding function b and a reinvestment function σ such that for every $w, w', m, m' \in \Re_+$, $m \neq m'$:

(a) $\sigma(\cdot| m, m')$ is a maximizer of:

$$\max_{\tilde{\sigma} \in \underline{\Delta}(m)} \int \left[m - x + \beta \int W\left(\phi^b(x, x'), \phi^b(x', x)\right) \sigma\left(dx' \middle| m', m\right) \right] \tilde{\sigma}(dx),$$

where $\phi^b(x, x') = \phi(rx, rx', b(rx, rx'), b(rx', rx))$.

[36] This is the model that we study in the supplementary material when we discuss the uniqueness of the equilibrium.

[37] The first assumption makes the cash irrelevant for the tie-breaking rule. The second assumption eliminates the need of introducing a new random variable to solve ties. The third assumption eliminates the analysis of information sets that are irrelevant for our equilibrium.

[38] As we show in the supplementary material, our restriction to symmetric equilibria is without loss of generality if one is interested in the limit of equilibria of the version of our model with finitely many periods.

(b) $b(w, w')$ is a maximizer of:

$$\max_{\tilde{b}\geq c+\pi(w)} W\left(\phi(w, w', \tilde{b}, b(w', w)), \phi(w', w, b(w', w), \tilde{b})\right).$$

(c) The value of the problem in (a) is equal to $W(m, m')$.

Conditions (a) and (b) are the usual one-shot deviation properties that must be verified in a Markov equilibrium. Condition (c) is the corresponding Bellman equation.

In what follows, we define a value function and a bidding and a reinvestment functions[39] and show that they are a BI equilibrium.

4.2 The Equilibrium Strategies

Let the bid function be defined by $b^*(w, w') \equiv c + \pi(\min\{w, w'\})^+$.[40] To define the reinvestment function, we use some auxiliary functions. Consider the following functional equation:

$$\hat{\Psi}(x) = \beta(1 - \hat{F}(x, \hat{\Psi}))\left(\pi(0) + \hat{\Psi}(\underline{m})\right), \tag{1}$$

for

$$\hat{F}(x, \hat{\Psi}) \equiv \min\left\{1, \int_0^x \frac{1-\beta r}{\beta((\pi(r\tilde{x}))^+ + \hat{\Psi}(r\tilde{x} + \underline{m}))} d\tilde{x}\right\}.$$

Lemma 1. *Equation (1) has a continuos decreasing solution $\Psi : \Re_+ \to \Re_+$ that verifies that $\hat{F}(\theta/r, \Psi) = 1$, and hence $\Psi(x) = 0$ for any $x \geq \theta/r$.*

Let W^* be a value function strictly increasing in the first argument and weakly decreasing in the second one that verifies:

$$W^*\left(m, m'\right) = \begin{cases} m + \frac{\beta}{1-\beta}\underline{m} & \text{if } m < m' \\ \\ m + \frac{\beta}{1-\beta}\underline{m} + \Psi\left(m'\right) & \text{if } m > m' \end{cases} \tag{2}$$

Thus, $\Psi(m')$ is the difference in continuation value between being leader or laggard when the other firm has cash m'.

[39] We show in the supplementary material that the version of our model with finitely many periods has a unique equilibrium and it is symmetric. The equilibrium studied here corresponds to the limit of that equilibrium.

[40] We adopt the convention that $a^+ = a$ if $a \geq 0$ and $a^+ = 0$ otherwise.

Let $\nu^r \in [0, \theta/r]$ be the solution in ν to $\hat{F}(\nu, \Psi) = 1$ and F^r a cumulative distribution function equal to $\hat{F}(., \Psi)$ in $[0, \nu^r]$ with density $f^r(x) \equiv \frac{1-\beta r}{\beta(\pi(rx)^+ + \Psi(rx+\underline{m}))}$. We define the reinvestment function $\sigma^*(.|m, m')$ in a very similar way as in Section 3. It is equal to F^r if both m and m' are weakly greater than ν^r, and otherwise a truncation of F^r at the minimum of m and m' that puts the remaining probability at either zero or the minimum between m and m' depending on whether $m < m'$ or $m > m'$.[41]

Proposition 3. (W^*, σ^*, b^*) *is a BI equilibrium.*

To understand why σ^* verifies condition (a) of a BI equilibrium note that it has the same qualitative features as the equilibrium reinvestment randomization in Section 3. In particular, its density equalizes the marginal cost of working capital, $(1 - \beta r)$, with the marginal benefit of reinvesting $f^r(x)\beta(\pi(rx)^+ + \Psi(rx + \underline{m}))$. The additional term $\Psi(rx + \underline{m})$ appears here because reinvesting more than the rival also allows the firm to become leader next period which means a jump in the continuation value of $\Psi(rx + \underline{m})$. The reason why b^* verifies condition (b) of a BI equilibrium is that as in Section 3, the firm finds it profitable to win at any price above the production cost c when the continuation payoffs are given by W^*.

To understand why the value function W^* satisfies condition (c) of a BI equilibrium consider the laggard and the leader separately. For the laggard, recall that the support of its randomization includes not reinvesting today which gives current payoffs of m plus the discounted value of being again laggard tomorrow with cash $\underline{m}$, which is equal to $\frac{\beta}{1-\beta}\underline{m}$. For the leader, note that the support of its randomization includes reinvesting arbitrarily little which gives limit current payoffs of m plus a continuation value that depends on whether the laggard reinvests or not. In the former case, which occurs with probability $F^r(m')$, the leader does not win today's procurement contract and gets the discounted

[41] This is:

- If $m < m'$:

$$\sigma^*(x|m, m') \equiv \begin{cases} F^r(x) + 1 - F^r(m) & \text{if } x \in [0, m) \\ 1 & \text{o.w.} \end{cases}$$

- If $m > m'$:

$$\sigma^*(x|m, m') \equiv \begin{cases} F^r(x) & \text{if } x < m' \\ 1 & \text{o.w.} \end{cases}$$

value of being laggard next period with cash $\underline{m}$, i.e. $\frac{\beta}{1-\beta}\underline{m}$. In the latter case, which occurs with probability $1-F^r(m')$, the leader wins today's procurement contract and gets the discounted value of being leader next period with cash $\underline{m}+\pi(0)$, which is equal to $\frac{\beta}{1-\beta}\underline{m}+\beta(\pi(0)+\Psi(\underline{m}))$. Thus, the continuation value is equal to:

$$\frac{\beta}{1-\beta}\underline{m}+(1-F^r(m'))\beta(\pi(0)+\Psi(\underline{m}))=\frac{\beta}{1-\beta}\underline{m}+\Psi(m'),$$

as can be deduced from the definition of Ψ in Equation (1) evaluated at $x=m'$ and $\hat{\Psi}=\Psi$.

We can also distinguish here between a symmetric and a laggard-leader scenarios, and an analogous version of Corollaries 1 and 2 holds true as well.[42] This is direct for the results described in the first two items, whereas those in the third item can be deduced from the following lemma.

Lemma 2. *As the cost of working capital becomes negligible:*

$$F^r(x)\to\begin{cases}0 & \textit{if } x<\theta/r^*,\\ 1 & \textit{if } x>\theta/r^*,\end{cases}\qquad \textit{and}\qquad \nu^r\to\theta/r^*.$$

Another interesting feature is how markups evolve. Denote by $w_{\min}$ the minimum of the firms' working capitals in a given period. Then, the equilibrium markup is equal to $\pi(w_{\min})^+$ and the laggard's cash in the morning next period is equal to $w_{\min}+\underline{m}$. We say that *the markup is increasing in an interval* if the expected difference between the markup

[42] In the symmetric scenario, reinvesting x has benefits that equals its cost $(1-\beta r)x$. The benefits are twofold: the expected discounted profits derived from winning in the afternoon and the jump in the continuation value when the firm becomes the leader. It follows that the expected discounted profits derived from winning are less than the cost of working capital. This means that the expected discounted markup is less than the total expected costs of working capital in which the firms incur. In particular, one can show that the expected markup is equal to:

$$(1-\beta r)2\int_0^{\nu^r}xf^r(x)dx-\beta\int_0^{\nu^r}\Psi(rx+\underline{m})dG(x),$$

where $G(x)\equiv F^r(x)+(1-F^r(x))F^r(x)$ is the distribution of the lower reinvestment. One can also show that the expected markup in the laggard-leader scenario is equal to:

$$(1-\beta r)\left(2\int_0^{m_{\min}}xf^r(x)dx+(1-F^r(m_{\min}))m_{\min}\right)-\beta\int_0^{m_{\min}}\Psi(rx+\underline{m})dG(x|m_{\min})+(1-F^r(m_{\min}))\beta(\pi(0)+\Psi(\underline{m})),$$

where $m_{\min}$ denotes the laggard's cash and $G(x|m_{\min})\equiv F^r(x)+(1-F^r(x))(F^r(x)+1-F^r(m_{\min}))$ the distribution of the lower reinvestment.

tomorrow and today conditional on $w_{\min}$ is non-negative for any realization of $w_{\min}$ in that interval and strictly positive in its interior. We also say that *the markup is decreasing in an interval* when the symmetric inequalities hold. We also refer to the *pre-laggard-leader scenario* to $[0, \theta/r^* - \underline{m})$ and to the *pre-symmetric scenario* to $(\theta/r^* - \underline{m}, \theta]$[43] and analyze the (point-wise) limit properties of the markup.

Proposition 4. *Suppose $\pi(0) > \pi(w)$ for any $w > 0$. As the cost of working capital becomes negligible, the markup is increasing in the pre-laggard-leader scenario and it is decreasing in the pre-symmetric scenario.*

4.3 The Equilibrium Dynamics

To study the frequency of the symmetric and the laggard-leader scenarios, we study the equilibrium stochastic process of the laggard's cash. Its state space is equal to $[\underline{m}, r\nu^r + \underline{m}]$ because the procurement profits are non negative and none of the firms reinvests more than ν^r. Moreover, the laggard's cash is determined by the equilibrium reinvestments and bidding in the previous period. The latter depends on the laggard's working capital in the previous period, which has a distributions that, as the equilibrium reinvestments, only depends on the laggard's cash in the previous period. Thus, the laggard's cash follows a Markov process. Its transition probabilities $Q^r : [\underline{m}, r\nu^r + \underline{m}] \times \mathcal{B} \to [0, 1]$, for $\mathcal{B}$ the Borel sets of $[\underline{m}, r\nu^r + \underline{m}]$, can be easily deduced from the equilibrium. In particular, they are defined by:[44]

$$Q^r\left(m, [\underline{m}, x]\right) = 1 - \left[1 - F^r\left(\frac{x - \underline{m}}{r}\right)\right] \cdot \left(F^r\left(m\right) - F^r\left(\frac{x - \underline{m}}{r}\right)\right)^+. \tag{3}$$

This expression is equal to one minus the probability that both the laggard and the leader reinvest strictly more than $\frac{x-\underline{m}}{r}$.

A distribution $\mu^r : \mathcal{B} \to [0, 1]$ is invariant if it verifies:

$$\mu^r\left(\mathcal{M}\right) = \int Q^r\left(m, \mathcal{M}\right) \mu^r\left(dm\right) \qquad \text{for all } \mathcal{M} \in \mathcal{B}. \tag{4}$$

[43] Note that $w_{\min}$ belongs to the pre-laggard-leader scenario if and only if $w_{\min} + \underline{m}$ belongs to the limit of the laggard-leader scenario as the cost of working capital becomes negligible. A similar statement also holds true for the pre-symmetric scenario and the symmetric scenario.

[44] As a convention, we denote by $[\underline{m}, \underline{m}]$ the singleton $\{\underline{m}\}$.

Standard arguments[45] can be used to show that there exists a unique invariant distribution and it is globally stable. A suitable law of large numbers can be applied to show that the fraction of the time that the Markov process spends on any set $\mathcal{M} \in \mathcal{B}$ converges (almost surely) to $\mu^r(\mathcal{M})$.

The invariant distribution puts positive probability in any open set in $[\underline{m}, r\nu^r + \underline{m}]$ if the transition probability gives positive probability to arriving to any such set in a finite number of periods. It is straightforward that this property is verified if one starts in the symmetric scenario. In the other scenario, it is sufficient to note that there is positive probability that the laggard's working capital increases at least $\underline{m}$ each period, and hence transits to the symmetric scenario in a finite number of periods.

To further characterize the invariant distribution, we distinguish two cases. We say that the *financial constraint is loose* when the ratio $\frac{r^* \underline{m}}{\theta}$ is weakly greater than one. Since $\nu^r \leq \frac{\theta}{r^*}$ for any $r \leq r^*$, this condition guarantees that the symmetric scenario occurs every period. We also say that the *financial constraint is tight* when the ratio $\frac{r^* \underline{m}}{\theta}$ is strictly less than $\frac{1}{1+r^*}$. This condition is equivalent to $(1 + r^*) r^* \underline{m} < \theta$ and hence it means a firm that begins a period with cash $\underline{m}$, faces a rate of return r^* and does not win the current procurement contract is still financially constrained next period.

Typically, the invariant distribution depends on a non trivial way on the transition probabilities. An exception is when the transition does not depend on the state which corresponds in our model to an exogenous cash flow sufficiently large to guarantee that only the symmetric scenario occurs, e.g. when the financial constraint is loose. In this case, the transition probability does not dependent on the laggard's current working capital and $\mu^r([\underline{m}, x)) = 1 - \left(1 - F^r\left(\frac{x - \underline{m}}{r}\right)\right)^2$. The properties of the equilibrium prices can be inferred from Corollary 1 and their limit from Lemma 2.[46]

More generally, the frequency of each scenario depends on the severity of the financial

[45] See Hopenhayn and Prescott (1992).

[46] In the more difficult case in which the transition probabilities depend on the state, the invariant distribution associated to the limit transition probabilities as the cost of working capital becomes negligible has an easy characterization. This is because the transition probabilities become degenerate and concentrate its probability in one point only, either $\underline{m}$ or $\theta + \underline{m}$, and thus any distribution with support in $\{\underline{m}, \theta + \underline{m}\}$ is an invariant distribution. Since there are multiple invariant distributions, we cannot apply a continuity argument to characterize what happens when the cost of working capital is small.

constraint. We say that the *extreme laggard-leader scenario occurs most of the time as the cost of working capital becomes negligible* when $\mu^r(\{\underline{m}\}) \to 1$.

Theorem 1. *If the financial constraint is tight, the extreme laggard-leader scenario occurs most of the time as the cost of working capital becomes negligible.*

This theorem characterizes the limit of the invariant distribution as the cost of working capital becomes negligible.[47] To understand its intuition let $A \equiv \{\underline{m}\}$, $B \equiv (\underline{m}, \tilde{m}]$ and $C \equiv (\tilde{m}, \theta + \underline{m}]$ for some $\tilde{m} \in (\underline{m}(1 + r^*), \theta/r^*)$. That such $\tilde{m}$ exists is a consequence of the financial constraint being tight. It follows from Lemma 2 that the probability of transiting to B vanishes and so does $\mu^r(B)$. Hence, the limit of the invariant distribution puts probability in at most A and C, and it is sufficient to argue that the ratio of $\frac{\mu^r(A)}{\mu^r(C)}$ diverges.

Recall that any invariant distribution equalizes the probability of being in a set times the probability of exiting from it with the probability of being one transition away from that set times the probability of entering to it. The definition of $\tilde{m}$ imply that one cannot transit from A to C in one period when r is close to r^*. Hence, $\frac{\mu^r(A)}{\mu^r(C)}$ satisfies:[48]

$$\underbrace{\frac{\int_A (1-Q^r(m,A))\mu^r(dm|A)}{\int_C (1-Q^r(m,C))\mu^r(dm|C)}}_{\textit{Exit ratio}} \cdot \frac{\mu^r(A)}{\mu^r(C)} = \sum_{S=B,C} \left(\underbrace{\frac{\int_S Q^r(m,A)\mu^r(dm|S)}{\int_B Q^r(m,C)\mu^r(dm|B)}}_{\textit{Entry ratios}} \cdot \frac{\mu^r(S)}{\mu^r(B)} \right). \tag{5}$$

The gist of the proof is to show that the exit ratio is uniformly bounded above and the first entry ratio diverges to infinity. To understand the first result note that to exit from A we need that the laggard reinvests in $(0, \underline{m})$ and to exit from C that either of the two firms reinvests in $[0, \frac{\tilde{m}-\underline{m}}{r}]$. The probability of the former event is less than the probability of the latter for r close to r^* because by definition of $\tilde{m}$ the former set is contained in the latter. For the second result note the laggard-leader scenario occurs in B and hence the probability of transiting to A, that is the laggard no reinvesting, goes to one.

The following corollary is a consequence of Theorem 1 and the properties of the laggard-leader scenario.

[47] In what follows, we let $Q^r(m, M) = Q^r(m, M \cap [\underline{m}, r\nu^r + \underline{m}])$ and $\mu^r(M) = \mu^r(M \cap [\underline{m}, r\nu^r + \underline{m}])$ for any Borel set M of the real line.

[48] We let $\mu^r(dm|\mathcal{M}) = \frac{\mu^r(dm)}{\mu^r(\mathcal{M})}$ if $\mu^r(\mathcal{M}) \neq 0$, and $\mu^r(dm|\mathcal{M}) = 1$ if $\mu^r(\mathcal{M}) = 0$ for $\mathcal{M} \in \{A, B, C\}$.

Corollary 3. *If the financial constraint is tight, as the cost of working capital becomes negligible: (i) the probability that the leader wins the procurement contract converges to one; and (ii) the fraction of time the markup is $\pi(0)$ converges to one (almost surely).*

Remark 2. *In this section we have focused on the case in which financial constraints are either tight or loose. It can be shown that as one moves $\frac{r^* m}{\theta}$ from the former case to the latter, the frequency of the extreme laggard-leader scenario and the symmetric scenario vary from 1 to 0 and from 0 to 1, respectively, as the cost of working capital becomes negligible.*

4.4 Numerical Solutions

Our analytical results apply to the extreme case where the cost of working capital, $1 - \beta r$, is close to zero. In order to explore what happens for values of $1 - \beta r$ farther from 0, we solve numerically for the invariant distribution for empirically grounded values of the parameters[49] and $\pi(w) = 1 - w$. The numerical results are useful, on the one hand, to show that mark-ups are largely as described in Theorem 1 and Corollary 3 and, on the other hand, to shed some light on the invariant distribution of market shares, which was not easy to characterize analytically.

For the case where the financial constraint is tight (we set $\frac{r^* m}{\theta} = 0.4$), our numerical solution shows that the extreme laggard-leader scenario occurs (i.e. $\mu(\underline{m})$ is) 99.52% of the periods and the largest markup ($\pi(0) = 1$) arises in 99.87% of the extreme laggard-leader scenarios. Therefore, the average markup is at least 0.9939. In contrast, for the case where the financial constraint is loose (i.e. $\frac{r^* m}{\theta} \geq 1$), where we are always in the symmetric scenario, the average markup is 0.1021.

[49] Early work of Hong and Shum (2002) suggest that firms that perform highway maintenance typically bid in 4 contracts per year:

> a data set of bids submitted in procurement contract auctions conducted by the NJDOT in the years 1989-1997 [...] firms which are awarded at least one contract bid in an average of 29-43 auctions.

See also Porter and Zona (1993). Therefore, we compute annual market shares for a year composed of four consecutive periods. We also choose $\beta = 0.9457$ and $r = 1$ so that the annual discount rate is 0.80, as in Jofre-Bonet and Pesendorfer (2003), implying an annual expected cost of working capital of 0.20.

The figure below plots the invariant distribution of annual market shares for our parametrization of tight and loose financial constraint. It shows that in the former case the same firm wins all the annual procurement contracts 98% of the years. Therefore, a typical time series of market shares displays not only a lot of concentration but also, and more importantly, tipping as in Cabral (2011), i.e. the system is very persistent but moves across extremely asymmetric states. On the contrary, in the case of a loose financial constraint there is little concentration in that each firms wins at least 25% of the annual procurement contracts in 85% of the years.

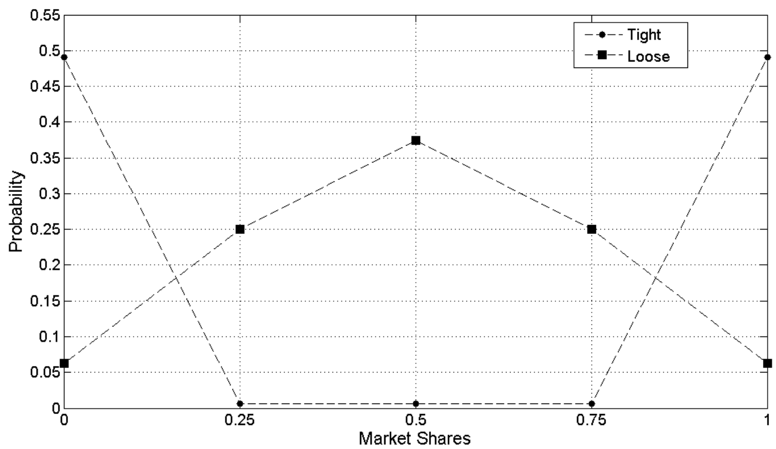

Figure 2: Invariant distribution of annual market shares (one year equals four periods).

5 Conclusions

We have studied a dynamic model of bidding markets with financial constraints. A key element of our analysis is that the stage at which firms choose how much working capital to reinvest resembles an all pay auction with caps.

The above features, and thus our results, seem pertinent for other models of investing under winner-take-all competition, like patent races. A more general approach should start by considering the robustness to our results to private information and to alternative models of winner-take-all competition with financial constraints. Existing results for all pay auctions and general contests[50] suggest this may be fruitful line of future research.

[50] Amann and Leininger (1996) study the relationship between the equilibrium of the all pay auction with and without private information, see also our Footnote 10, Alcalde and Dahm (2010) study the similarities

Finally, our analysis points out a tractable way to incorporate the dynamics of liquidity in Galenianos and Kircher's (2008) analysis of monetary policy.

Appendix

A Proofs

Proof of Proposition 1

To see why the proposed strategy is an equilibrium note that the expected payoff of Firm i with cash m^i when it reinvests x^i and the other firm randomizes its reinvestments according to a distribution with density $\tilde{f}^r$ is equal to:

$$m^i - (1-\beta r)x^i + \int_0^{x^i} \beta\pi(rx)\tilde{f}^r(x)\,dx,$$

which by definition of $\tilde{f}^r$ is equal to m^i for any $x^i \leq \tilde{\nu}^r$, and strictly less than m^i otherwise, as required.

As we argue in the main text, we can restrict to strategies with support in $[0, \theta/r]$. We prove two properties that any equilibrium (σ^1, σ^2) must verify. Later, we show that the proposed strategy is the only one that verifies them. These two properties also hold true in the more general case in which we do not restrict to $m^1, m^2 \geq \tilde{\nu}^r$.

Claim 1: *If $x \in (0, \theta/r]$ belongs to the support of σ^i, then $\sigma^j((x-\epsilon, x]) > 0$ ($j \neq i$) for any $\epsilon > 0$. In the particular case $x = \theta/r$, it must also be verified that $\sigma^j((x-\epsilon, x)) > 0$ ($j \neq i$) for any $\epsilon > 0$.*

Suppose that $x \in (0, \theta/r]$ belongs to the support of σ^i and $\sigma^j((x-\epsilon, w]) = 0$ for some $\epsilon > 0$. We argue that this contradicts the equilibrium. Firm i strictly prefers reinvesting $x - \epsilon$ to reinvesting x because the former saves on the cost of working capital without affecting the probability of winning at a profit. Thus, we can argue by continuity that Firm i can strictly improve by moving the probability that σ^i puts in the set $(x-\epsilon', x+\epsilon')$ to $x - \epsilon$ for $\epsilon' \in (0, \epsilon)$ and small enough. That Firm i's payoff are continuous at x is a

between the equilibrium outcome in an all pay auction and in some other models of contests.

consequence of the contradiction hypothesis that $\sigma^j((x-\epsilon, x]) = 0$ which rules out that Firm j puts an atom at x. The proof of the result for the particular case $x = \theta/r$ is similar. The only difference is that the contradiction hypothesis $\sigma^j((x-\epsilon, x)) = 0$ allows for an atom at θ. This, however, does not upset continuity because the procurement contract gives zero profits when the other firm picks working capital $x = \theta/r$.

Claim 2: *If for some* $x \in [0, \min\{\theta/r, m^j\})$, $\sigma^j((x-\epsilon, x]) > 0$ *for any* $\epsilon > 0$, *then* $\sigma^i(\{x\}) = 0$ *(*$i \neq j$*).*

By contradiction, suppose an $x \in [0, \min\{\theta/r, m^j\})$ for which $\sigma^j((x-\epsilon, x]) > 0$ for any $\epsilon > 0$ and $\sigma^i(\{x\}) > 0$. For $\epsilon' > 0$ small enough, Firm j can improve by moving the probability that its puts in $(x-\epsilon', x]$, to a point slightly above x. This deviation increases Firm j's cost of working capital marginally but allows the firm to win the procurement contract at a strictly positive profit if Firm i plays the atom at x.

Claim 1 and 2 imply that (i) the only point where there can be a mass point in the strategies is at zero, (ii) at most one of the firms' strategies can have an atom at zero, and (iii) the support of both σ^1 and σ^2 must be the same and equal to an interval $[0, \nu]$ for some $\nu \geq 0$. The usual indifference condition that must hold in a mixed strategy equilibrium implies that both σ^1 and σ^2 have density $\tilde{f}^r$ a.e. in $(0, \nu]$, for some $\nu > 0$, and this together with (i) and (ii) imply that neither σ^1 nor σ^2 puts an atom at zero and hence $\nu = \tilde{\nu}^r$ as desired. ■

Proof of Proposition 2

We start showing that the proposed strategies are an equilibrium. Let $m_{\min}$ be the laggard's cash. The leader has no incentive to deviate when the laggard plays the proposed strategy because, first, the leader is indifferent between any reinvestment in $(0, m_{\min}]$ by the same reasons as in the proof of Proposition 1, and, second, because neither zero reinvesting nor reinvesting strictly more than $m_{\min}$ improves its payoffs. The former deviation is not profitable because it has almost the same cost of working capital as a marginally small reinvestment but it differs in that the leader lose the profits of the procurement contract with strictly positive probability when the laggard reinvests zero. The latter deviation is not profitable because it lets the leader win in the afternoon in the same cases as $m_{\min}$ but has a greater cost of working capital. Similarly, the laggard has no incentive to deviate when the leader plays the proposed strategy because, first, the laggard is indifferent between any reinvestment in $[0, m_{\min})$, and, second, the laggard cannot improve by moving probability mass to $m_{\min}$ because the only difference with a working capital marginally lower occurs when the leader also carries working capital $m_{\min}$ but in this case our tie breaking rule implies that the leader wins the procurement contract.

To prove that there is no other equilibrium, note that Claims 1 and 2 in the proof of Proposition 1 also hold true here. They imply here that (i) the laggard's strategy can have a probability mass only at zero and the leader's only at either zero or $m_{\min}$, (ii) at most one of the firms' strategies can have an atom at zero, and (iii) the support of both σ^1 and σ^2 must be the same and equal to an interval $[0, \nu]$ for some $\nu \in [0, m_{\min}]$. Again, the usual indifference condition that must hold in a mixed strategy equilibrium implies that both σ^1 and σ^2 have density $\tilde{f}^r$ a.e. in $(0, \nu)$, and this together with (i)-(ii) and $m < \tilde{\nu}^r$ imply that $\nu = m_{\min}$ and that the laggard puts the remaining probability at zero and the leader at $m_{\min}$. ■

Proof of Proposition 3

We first show that $\sigma = \sigma^*$ verifies (a) in the definition of a BI equilibrium for the value function W^* and the bid function $b = b^*$. In case $m < m'$, σ^* has support $[0, m]$. Thus, it is sufficient to show that any degenerate reinvestment function in $\underline{\Delta}(m)$, i.e. any reinvestment $x \in [0, m)$, gives the same expected payoffs. The expected payoffs of reinvesting x are equal to:

$$m - x + \beta r x + \frac{\beta}{1-\beta}\underline{m} + \beta \int_0^x \left(\pi(rx') + \Psi(rx' + \underline{m})\right) f^r(x') dx'.$$

One can easily see after substituting f^r by its value that this expression is constant in x as desired. In case $m > m'$, σ^* has support $[0, m']$ and puts no atom of probability at zero. Thus, it is sufficient to show that any reinvestment $x \in (0, m']$ gives the same expected payoffs, and that these payoffs are weakly larger than either no reinvestment or reinvesting strictly more than m'. To prove so, note that a reinvestment $x \in (0, m)$ gives expected payoffs:

$$m-x+\beta rx+\frac{\beta}{1-\beta}\underline{m}+\beta\int_0^{\min\{x,m'\}}\big(\pi(rx')+\Psi(rx'+\underline{m})\big)\,f^r(x')dx'+\beta(1-F^r(m'))(\pi(0)+\Psi(\underline{m})), \tag{6}$$

One can easily see after substituting f^r by its value that this expression is constant in x up to m' and then decreasing as desired. Reinvesting $x=0$ gives payoffs:

$$m+\frac{\beta}{1-\beta}\underline{m}+\beta\frac{(1-F^r(m'))}{2}(\pi(0)+\Psi(\underline{m})),$$

which is strictly less than Equation (6) evaluated at $x=0$ as desired.

We next check that $b=b^*(w,w')$ verifies (b) of the definition of a BI equilibrium for the value function $W=W^*$. If $w,w'\geq\theta$ then $b^*(w,w')=b^*(w',w)=c$ and $\tilde{b}=c$ is optimal since any bid $\tilde{b}\geq c$ gives the same payoffs $W^*(w+\underline{m},w'+\underline{m})$, and any bid $\tilde{b}<c$ gives continuation payoffs $W^*(w+\tilde{b}-c+\underline{m},w'+\underline{m})<W^*(w+\underline{m},w'+\underline{m})$. If $w<w',\theta$, then $b^*(w,w')=b^*(w',w)=c+\pi(w)$, and $\tilde{b}=c+\pi(w)$ is optimal since any bid $\tilde{b}\geq c+\pi(w)$ gives payoffs $W^*(w+\underline{m},w'+\pi(w)+\underline{m})$. If $w'<w,\theta$, then $b^*(w,w')=b^*(w',w)=c+\pi(w')$, and $\tilde{b}=c+\pi(w')$ is optimal since any bid $\tilde{b}>c+\pi(w')$ gives payoffs $W^*(w+\underline{m},w'+\pi(w')+\underline{m})<W^*(w+\pi(w')+\underline{m},w'+\underline{m})$, a bid $\tilde{b}=c+\pi(w')$ gives payoffs $W^*(w+\pi(w')+\underline{m},w'+\underline{m})$ and any bid $\tilde{b}<c+\pi(w')$ gives payoffs $W^*(w+\tilde{b}-c+\underline{m},w'+\underline{m})<W^*(w+\pi(w')+\underline{m},w'+\underline{m})$. Finally, if $w=w'<\theta$, then $b^*(w,w')=b^*(w',w)=c+\pi(w)$ and $\tilde{b}=c+\pi(w)$ is optimal since it gives payoffs $W^*(w+\underline{m}+\pi(w)/2,w'+\pi(w)/2+\underline{m})$ and any bid $\tilde{b}>c+\pi(w)$ gives payoffs $W^*(w+\underline{m},w'+\pi(w)+\underline{m})<W^*(w+\underline{m}+\pi(w')/2,w'+\pi(w)/2+\underline{m})$.

The computations described above for the proof of condition (a) also show that the condition (c) in the definition of a BI equilibrium is verified. ■

Proof of Lemma 1

Let $\mathbb{C}$ be the set of bounded continuous and decreasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ and $\{\Psi_n\}\subset\mathbb{C}$ be a sequence defined recursively by $\Psi_0(x)=0$, and,

$$\Psi_{n+1}(x)=\beta\left(1-\hat{F}(x,\Psi_n)\right)(\pi(0)+\Psi_n(\underline{m}))\,. \tag{7}$$

This sequence verifies that if $\Psi_n\geq\Psi_{n-1}$, then:

$$\Psi_{n+1}(x)=\beta(1-\hat{F}(x,\Psi_n))\,(\pi(0)+\Psi_n(\underline{m}))\geq$$
$$\beta(1-\hat{F}(x,\Psi_{n-1}))\,(\pi(0)+\Psi_{n-1}(\underline{m}))=\Psi_n(x).$$

Moreover, $\Psi_1(x) \geq 0 = \Psi_0(x)$. Hence, we can argue by induction that our sequence is increasing and, thus, has a pointwise limit. We can use an adaptation of the argument in Footnote 30 to show that $\Psi_n(\theta/r) = 0$ implies that $\Psi_{n+1}(\theta/r) = 0$. Thus, we can also argue by induction that the pointwise limit of the sequence is equal to zero at $x = \theta/r$. Finally, one can deduce that the pointwise limit of the sequence verifies the limit equation in Equation (7) applying the monotone convergence theorem (Theorem 16.2 in Billingsley (1995)) to the sequence of integrals that corresponds to the sequence $\{\hat{F}(x, \Psi_n)\}$. ■

Proof of Lemma 2

For any $x < \theta/r^*$ that $F^r(x) \overset{r \to r^*}{\longrightarrow} 0$ is a direct consequence of the fact that for r sufficiently close to r^*, $x < \theta/r$, and hence $F^r(x) \leq f^r(x)x \leq \frac{(1-\beta r)}{\beta\pi(rx)}x$ since f^r is increasing and Ψ is non-negative. For any $x > \theta/r^*$, that $F^r(x) \overset{r \to r^*}{\longrightarrow} 1$ is a direct consequence of the fact that for r close to r^*, $x > \theta/r$ and hence $x > \nu^r$. This last fact also means that to prove that $\nu^r \overset{r \to r^*}{\longrightarrow} \theta/r^*$, it is sufficient to show that $\liminf_{r \to r^*} \nu^r \geq \theta/r^*$. We prove this last claim by contradiction. Suppose there exists an $\epsilon > 0$ and a sequence $\{r_n\}$ converging to r^* such that $\nu^{r_n} < \theta/r^* - \epsilon$ along the sequence. This means that $F^{r_n}(\theta/r^* - \epsilon) \geq F^{r_n}(\nu^{r_n}) = 1$ along the sequence, which contradicts the first limit of the lemma. ■

Proof of Proposition 4

In equilibrium, the markup is equal to π evaluated at r times the minimum reinvestment. The probability that the minimum reinvestment is weakly less than x when the laggard's cash is equal to m is equal to one minus the probability of the complementary event (that both firms reinvest strictly more than x):

$$1 - [1 - F^r(x)] \cdot (F^r(m) - F^r(x))^+ .$$

Lemma 2 implies that this distribution converges weakly to a mass point at 0 if $m < \theta/r^*$ and to a mass point at θ/r^* if $m > \theta/r^*$. This implies that the expected markup converges to $\pi(0)$ if the realization of $w_{\min}$ belongs to the pre-laggard-leader scenario and converges to $\pi(\theta) = 0$ if the realization of $w_{\min}$ belongs to the pre-symmetric scenario which implies the proposition. ■

Proof of Theorem 1

We show that as the cost of working capital becomes negligible: (i) $\mu^r(B)$ tends to 0 and (ii) $\frac{\mu^r(A)}{\mu^r(C)}$ diverges to infinity, where the sets A, B and C are defined after Theorem 1.

(i) can be deduced from the definition of an invariant distribution in Equation (4) applied to $\mathcal{M} = B$, Lemma 2, the definition of $\tilde{m}$ and the fact that for any $m \in [\underline{m}, r\nu^r + \underline{m}]$:

$$\begin{aligned}
Q^r(m, B) &= Q^r(m, [\underline{m}, \tilde{m}]) - Q^r(m, \{\underline{m}\}) \\
&= 1 - \left[1 - F^r\left(\tfrac{\tilde{m}-\underline{m}}{r}\right)\right] \cdot \left(F^r(m) - F^r\left(\tfrac{\tilde{m}-\underline{m}}{r}\right)\right)^+ - (1 - F^r(m)) \\
&= F^r(m) - \left[1 - F^r\left(\tfrac{\tilde{m}-\underline{m}}{r}\right)\right] \cdot \left(F^r(m) - F^r\left(\tfrac{\tilde{m}-\underline{m}}{r}\right)\right)^+ \\
&\leq 1 - \left[1 - F^r\left(\tfrac{\tilde{m}-\underline{m}}{r}\right)\right]^2,
\end{aligned}$$

where we have used Equation (3) in the second equality, and that the expression in the third line is increasing in m in the inequality.

To prove (ii), we use the following general property of an invariant distribution $\mu : \tilde{\mathcal{B}} \to [0, 1]$ associated to a transition probability $Q : M \times \tilde{\mathcal{B}} \to [0, 1]$ defined on an arbitrary interval M and $\tilde{\mathcal{B}}$ the Borel sets of M:

$$\int_{\mathcal{M}} (1 - Q(m, \mathcal{M}))\, \mu(dm) = \int_{M \setminus \mathcal{M}} Q(m, \mathcal{M})\mu(dm) \qquad \forall \mathcal{M} \in \tilde{\mathcal{B}}. \tag{8}$$

As a consequence:

$$\int_A (1 - Q^r(m, A))\, \mu^r(dm) = \sum_{S=B,C} \int_S Q^r(m, A)\mu^r(dm),$$

and hence:

$$\int_A (1 - Q^r(m, A))\, \mu^r(dm|A)\mu^r(A) = \sum_{S=B,C} \int_S Q^r(m, A)\mu^r(dm|S)\, \mu^r(S).$$

By a similarly argument:

$$\int_C (1 - Q^r(m, C))\, \mu^r(dm|C)\, \mu^r(C) = \int_B Q^r(m, C)\mu^r(dm|B)\, \mu^r(B),$$

since for r close to r^*, $Q^r(\underline{m}, C) = 0$ by definition of $\tilde{m}$.

Lemma 2 and our argument in the main text that the invariant distribution puts positive probability in any open set in $[\underline{m}, r\nu^r + \underline{m}]$ implies that $\mu^r(C) > 0$ for r close to r^*. Thus, the combination of the last two equations implies Equation (5) in the main text.

To complete the argument we show that the exit ratio in that equation is bounded above while the first entry ratio diverges to infinity.

Note that $Q^r(m, A) = Q^r(m, \{\underline{m}\})$ whereas $Q^r(m, C) = 1 - Q^r(m, [\underline{m}, \tilde{m}])$. Hence, an increase in m decreases the former expression and increases the latter, as can be deduced from Equation (3). This means that the first entry ratio is bounded below by:

$$\frac{Q^r(\tilde{m}, \{\underline{m}\})}{1 - Q^r(\tilde{m}, \{\underline{m}\})},$$

and the exit ratio is bounded above by:

$$\frac{1 - Q^r(\underline{m}, \{\underline{m}\})}{Q^r(r\nu^r + \underline{m}, [\underline{m}, \tilde{m}])}.$$

The former ratio diverges to infinity because $Q^r(\tilde{m}, \{\underline{m}\})$ tends to one as can be deduced from Equation (3) and Lemma 2. Equation (3) also means that the latter ratio is equal to:[51]

$$\frac{F^r(\underline{m})}{F^r\left(\frac{\tilde{m}-\underline{m}}{r}\right)\left(2 - F^r\left(\frac{\tilde{m}-\underline{m}}{r}\right)\right)},$$

which is weakly less than one for r close to r^* by definition of $\tilde{m}$. ■

B A Model of Financial Constraints

In this section, we study a model in which the risk of non-compliance arises endogenously due to moral hazard and limited liability. The main implication is that there is a decreasing function $\pi(w)$ that gives the minimum potential profits that guarantee that a firm with working capital w can finance the production of the good. In Section B.1, we revisit the model of Section 2 to endogenize the function π, and in Section B.2, we embed this revised model in the model of Section 4.

[51] An alternative proof is that:

$$\frac{F^r(\underline{m})}{F^r\left(\frac{\tilde{m}-\underline{m}}{r}\right)\left(2 - F^r\left(\frac{\tilde{m}-\underline{m}}{r}\right)\right)} = \frac{\int_0^{\underline{m}} \frac{1}{\pi(r\tilde{x})+\Psi(r\tilde{x}+\underline{m})}\, d\tilde{x}}{\int_0^{\frac{\tilde{m}-\underline{m}}{r}} \frac{1}{\pi(r\tilde{x})+\Psi(r\tilde{x}+\underline{m})}\, d\tilde{x}\left(2 - F^r\left(\frac{\tilde{m}-\underline{m}}{r}\right)\right)}$$
$$\leq \frac{\pi(0) + \Psi(\underline{m})}{\pi(r\underline{m}) + \Psi(\underline{m}(1+r))} \frac{\underline{m}}{\frac{\tilde{m}-\underline{m}}{r}},$$

where the inequality uses the fact that $F^r \leq 1$, $\pi + \Psi$ is decreasing and $\Psi(r\underline{m}) \geq 0$ since $\underline{m} < \theta/r$ by our hypothesis. The last expression is uniformly bounded above as desired since $\Psi(\underline{m}) \leq \frac{\beta}{1-\beta}\pi(0)$ as can be deduced from Equation (1) and $\pi(r\underline{m}) > 0$.

B.1 A Structural Model of Procurement with Financial Constraints

We revisit the model of Section 2 to provide foundations for assumptions (A) and (B). We assume that the firm that wins the procurement contract may choose between a *safe* project or a *risky* project. The safe project has cost c and succeeds with probability one, whereas the risky project has a lower cost $c(1-\varepsilon)$, $\varepsilon \in (0,1)$, but only succeeds with probability $\rho < 1$ and fails otherwise. In case of failure the firm cannot comply with the procurement contract, and hence only the safe technology ensures compliance. The surety bond that must accompany the bid and the required financing are supplied by banks.

B.2 A Dynamic Model with Endogenous Financial Constraints

We revisit the model of Section 4 and assume the model of procurement with financial constraints described in previous section for the afternoon stage of each period.

Banks are competitive, and hence do not charge for underwriting a bid that induces the firm to choose the safe technology. Thus, the role of banks is summarized by the set of bids that they are willing to underwrite. These are the acceptable bids for the firm. The bank decides to underwrite a bid upon observing the firm's working capital w. The bank, however, observes neither the other firm's working capital nor the history of past play.

We restrict to equilibria in which the set of bids underwritten by banks is symmetric and monotone. Symmetric in the sense that the banks' equilibrium strategy does not distinguish the identity of the firm and monotone in the sense that the banks underwrite any bid with potential profits above a certain minimum profitability requirement that we denote by $\pi(w)$.

We assume that a firm not complying with a procurement contract is banned from any other future procurement and it is replaced the following morning by another firm with working capital equal to $\underline{m}$. Since a firm banned from the procurement can still consume the exogenous revenue (which has a discounted value $\frac{\beta}{1-\beta}\underline{m}$) Firm i finds it profitable to choose the safe project if and only if,

$$\beta W(w + \Pi + \underline{m}, w' + \underline{m}) \geq \rho\beta W(w + \Pi + \varepsilon c + \underline{m}, w' + \underline{m}) + (1-\rho)\frac{\beta}{1-\beta}\underline{m}, \quad (9)$$

for W the continuation value in the morning of next period, (w, w') the distribution of firms' working capitals and Π the potential profits of the safe project. For a fixed distribution of working capitals, we say that the potential profits *induce the safe project* when the inequality in Equation (9) holds. If the opposite (weak) inequality holds, we say that the potential profits *induce the risky project.*

Equilibrium behavior by banks requires that the bids that are underwritten along the equilibrium path induce the safe project. Besides, sequential rationality requires that banks do not find it optimal to accept their rejected bids. This occurs when banks anticipate that their rejected bids, had they been accepted and won the procurement contract, would make the firm take the risky project. Formally, given a continuation value W, a bid function b and a distribution of working capitals, we say that the *banks anticipate that some potential profits implement the risky project* if they induce the risky project and the associated bid wins with positive probability when the rival bids according to b.

Definition: A financial, bidding and investment (FBI) equilibrium is a decreasing function π and a BI equilibrium (b, σ, W) associated to π such that:

(i) For any distribution of working capitals in which Firm i wins the procurement contract with positive probability in the BI equilibrium (b, σ, W), Firm's i potential profits in the procurement contract induce the safe project.

(ii) For any working capital of the firm, the banks anticipate that potential profits strictly below the minimum profitability requirement implement the risky project for some working capital of the other firm.

We are interested in an FBI equilibrium that generalizes the BI equilibrium in Proposition 3. Note, first, that the functions (b^*, σ^*, W^*) in the BI equilibrium are defined for some given π function. To make this dependence transparent, we shall slightly abuse of the notation and denote them by b^*_π, σ^*_π and W^*_π for a given decreasing function $\pi : \Re_+ \to \Re_+$. We define F^r_π, ν^r_π and Ψ_π in a similar way.

In what follows, we characterize sufficient conditions for $(W^*_\pi, \sigma^*_\pi, b^*_\pi, \pi)$ to be an FBI equilibrium. We start with condition (i) of an FBI equilibrium. If both firms use b^*_π, the winner of the procurement contract has more working capital than the loser. Besides, the inequality in Equation (9) for $W = W^*_\pi$, see Equation (2), and $w \geq w'$ is equivalent to:

$$\Pi \geq \hat{\theta} - w - \Psi_\pi(w'), \tag{10}$$

where $\hat{\theta} \equiv \frac{\rho}{1-\rho}\varepsilon c$. Since Ψ_π is decreasing and in a BI equilibrium the potential profits of the procurement contract are at least $\pi(w)$, the most stringent case for (i) is when $\Pi = \pi(w)$ and $w' = w$. Hence, a sufficient condition for (i) is:

$$\pi(w) \geq \hat{\theta} - w - \Psi_\pi(w). \tag{11}$$

Now, we shall argue that $\pi(w) \leq \hat{\theta} - w$ is a sufficient condition for (ii). Suppose a π that verifies this condition and consider a continuation value W^*_π, a bid function b^*_π, a a distribution of working capitals (w, w'), where w is arbitrary and w' is strictly greater than w and so large that $\Psi_\pi(w')$ is zero, and any potential profits $\Pi < \pi(w)$. First, if the bank underwrites a bid with potential profits Π to the firm with working capital w, $w' > w$ and $\Pi < \pi(w)$ means that this firm wins the procurement contract when the rival bids according to b^*_π.[52] Second, $\Pi < \pi(w)$ and $\pi(w) \leq \hat{\theta} - w$ implies that $\Pi < \hat{\theta} - w$ which means that the potential profits Π induce the risky project for the continuation value W^*_π and the distribution (w, w'). As a consequence, the banks anticipate that the potential profits Π implement the risky project. Therefore (and no further proof is necessary):

Proposition 5. *If π^* is a decreasing function that verifies:*

$$\hat{\theta} - w - \Psi_{\pi^*}(w) \leq \pi^*(w) \leq \hat{\theta} - w,$$

for any $w \geq 0$, then $(\pi^*, b^*_{\pi^*}, \sigma^*_{\pi^*}, W^*_{\pi^*})$ is an FBI equilibrium.

In this proposition is where the assumption $r > 1$ has some role. As the cost of working capital becomes small enough the financial constraints might disappear in that there exists an equilibrium in which banks are willing to underwrite any bid above cost independently of the firm's working capital, i.e. $\pi^*(w) \leq 0$. This is because the jump in continuation value when the firm becomes the leader, $\Psi_{\pi^*}(w)$, diverges as the cost of working capital becomes negligible. Intuitively, the prospect of winning an infinite number of periods at a strictly positive profit gives unboundedly large payoffs.

Remark 3. *For any π^* that verifies the conditions in Proposition 5, it can be shown that $\theta = \hat{\theta}$. Since $\hat{\theta} = \frac{\rho}{1-\rho}\varepsilon c$, c sufficiently large implies that the financial constraint is tight.*

[52] This argument does not generalize naturally to the second price auction as it hinges on both firms bidding the bid associated to the minimum profitability requirement of the firm with less working capital. For this case, a more elaborated argument can be used to show that the minimum of the π functions that verify the conditions of Proposition 5 is still an FBI equilibrium.

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Supplementary Material

A Finite Horizon Model

In this appendix, we show that the unique equilibrium of a finite period version of the model in Section 4 converges as the number of periods tend to infinity to the equilibrium that we analyze in Section 4. As Athey and Schmutzler (2001) have argued:

> If there are multiple equilibria, one equilibrium of particular interest (if it exists) is an equilibrium attained by taking the limit of first-period strategies as the horizon T approaches infinity.

Suppose a $T+1$ period model in which all the periods but the last one are as in the model of Section 4. In the last period, all the firm's cash is distributed to the shareholders. The dynamic link between periods and the firms' objective functions are also as in Section 4. The game starts in the morning of period 1 with both firms having the same amount of cash. We assume that in case of a tie in period t: the procurement contract is assigned to the firm with more working capital; if both firms have the same amount of working capital and one firm and only one reinvested all its cash, the contract is assigned to the firm that had more cash at the beginning of the period; in the remaining cases the contract is assigned randomly with equal probability between the two firms. We study the subgame perfect equilibria of the game.

To state our results, we use the functions Ψ_n defined in the proof of Lemma 1. To make the notation more easily readable, we let $\hat{\Psi}_t \equiv \Psi_{T+1-t}$ for $t \in \{1, 2, ..., T+1\}$. Let $F_t(x)$, $t \in \{1, 2, ..., T\}$, be a distribution function equal to $\hat{F}(x, \hat{\Psi}_{t+1})$ with density $f_t(x)$ and support $[0, \nu_t]$. To shorten the notation, we also denote $\hat{\pi}_t(x) \equiv \pi(x)^+ + \hat{\Psi}_{t+1}(x + \underline{m})$, hence $f_t(x) = \frac{1-\beta r}{\beta \hat{\pi}_t(rx)}$.

We start defining a function $W_t(m, m')$ that, as we show later, gives the expected continuation payoffs at the beginning of each period t for a firm with cash m that competes with a rival with cash m': $W_{T+1}(m, m') \equiv m$, and for $t \in \{1, 2, ..., T\}$,

$$W_t(m, m') \equiv m + \underline{m} \sum_{\tau=1}^{T+1-t} \beta^\tau + \begin{cases} \hat{\Psi}_t(m') & \text{if } m > m' \\ \hat{\hat{\Psi}}_t(m') & \text{if } m = m' < \gamma_t \\ 0 & \text{o.w.} \end{cases}$$

where

$$\hat{\hat{\Psi}}_t(m') \equiv \frac{\beta}{2}\hat{\pi}_t(rm') - (1-\beta r)m',$$

and γ_t is equal to the unique solution to $\hat{\hat{\Psi}}_t(\gamma_t) = 0$.

This continuation value implies that when both firms play b^* in the afternoon of period t and have continuation payoffs in the morning of period $t+1$ given by W_{t+1}, the expected continuation payoff at the beginning of the morning of period t of a firm with cash m that reinvests x when the other firm has cash m' and reinvests x' is equal to:

$$m + \underline{m} \sum_{\tau=1}^{T+1-t} \beta^{\tau} + \Gamma_t \beta \hat{\pi}_t(rx') - (1-\beta r)x, \tag{12}$$

where Γ_t denotes the probability that the firm wins the procurement contract in period t:

$$\Gamma_t = \begin{cases} 1 & \text{if } x > x' \text{ or } x = x' = m' < m \\ 0 & \text{if } x < x' \text{ or } x = x' = m < m' \\ 1/2 & \text{otherwise.} \end{cases}$$

Consequently, for the case in which the continuation payoffs in period $t+1$ are given by W_{t+1} and both firms play b^* in the afternoon of period t, $W_t(m, m')$ is equal to the value of no reinvesting plus an extra term $\hat{\Psi}_t(m')$ if the firm is leader, and $\hat{\hat{\Psi}}_t(m')$ if both firms have the same amount of cash and this is less than γ_t. The term $\hat{\Psi}_t(m')$ has the same interpretation as $\Psi(m')$ in the model of Section 4, whereas $\hat{\hat{\Psi}}_t(m')$ is the discounted increase in the continuation payoffs of each of the firms when they have the same amount of cash $m = m'$ and reinvest it all $x = x' = m = m'$. In this case, both firms have the same probability of winning the current procurement contract and becoming the leader next period.

We also define a distribution over reinvestments $\sigma_t^*(.|m, m')$ that, as we shall show later, characterizes the firms equilibrium reinvestments. Let $\sigma_t^*(.|m, m')$ be equal to $F_t(.)$ if both m and m' are weakly greater than ν_t. Let also $\sigma_t^*(.|m, m')$ be equal to a truncation of $F_t(.)$ at m that puts the remaining probability at 0 if $m < m', \nu_t$, and at m' if $m' < m, \nu_t$. If $m = m' \in [0, \gamma_t)$, let $\sigma_t^*(.|m, m')$ be a distribution that puts all the probability mass at m. Finally, if $m = m' \in [\gamma_t, \nu_t)$, let $\sigma_t^*(.|m, m')$ be a truncation of F_t at $\lambda_t(m)$ that puts the remaining probability at m, where $\lambda_t(m)$, $m \in (\gamma_t, \nu_t)$, is the unique value of

$\lambda \in (0, m)$ that solves:[53]

$$\int_0^{\lambda} \beta \hat{\pi}_t(r\tilde{x}) f_t(\tilde{x}) d\tilde{x} + \frac{\beta(1 - F_t(\lambda))}{2} \hat{\pi}_t(rm) - (1 - \beta r)m = 0. \tag{13}$$

This equation ensures that a firm is indifferent between reinvesting all its cash m and not reinvesting, when the payoffs are as in Equation (12) and the other firm uses σ_t^*.

Proposition 6. *There is a unique subgame perfect equilibrium of the game. In this equilibrium, and at any period $t \in \{1, 2, ..., T\}$, both firms use σ_t^* in the morning and b^* in the afternoon and have continuation payoffs at the beginning of the period given by W_t.*

Proof: We prove the proposition using backward induction. It is trivial that the continuation payoffs are equal to W_{T+1} in period $T + 1$. We can then apply recursively the following Lemmata: first, Lemma 4, then Lemma 5, and finally Lemma 3 to update the continuation payoffs.

Lemma 3. *The firm's continuation payoffs in the morning of period $t \in \{1, 2, ..., T\}$ are equal to W_t if in period t both firms play σ_t^* in the morning and b^* in the afternoon, and the continuation payoffs next period are given by W_{t+1}.*

Lemma 4. *There is a unique equilibrium in the game defined in the afternoon of period $t \in \{1, 2, ..., T\}$ by the continuation payoffs next period W_{t+1}. In this equilibrium, both firms play b^*.*

Lemma 5. *There is a unique equilibrium in the game defined in the morning of period $t \in \{1, 2, ..., T\}$ by both players using b^* in the afternoon and by continuation payoffs next period given by W_{t+1}. In this equilibrium, both firms play σ_t^*.*

Proof of Lemma 3: Under the conditions of the lemma, the payoffs are as described in Equation (12). We compute the expected payoffs of a firm with cash m when the rival has m' and both firms use σ_t^* distinguishing several cases.

[53] Existence and uniqueness of the solution follows from the properties of the function in the left hand side of the equation. This is increasing in λ by application of the Leibnitz rule, it is negative at $\lambda = 0$ by definition of γ and since $m > \gamma$, and it is strictly positive at $\lambda = m$. The latter can be checked substituting $f_t(\tilde{x})$ and using that $F_t(m) < 1$ since $m < \nu_t$.

If $m > m'$ then our firm randomization has support $[0, \min\{m', \nu_t\}]$ whereas the rival randomizes with density f_t in $(0, \min\{m', \nu_t\}]$ and puts the remaining probability at zero. The expected payoff of any reinvestment $x \in (0, \min\{m', \nu_t\}]$ is equal to:

$$m + \underline{m} \sum_{\tau=1}^{T+1-t} \beta^\tau + \beta(1 - F_t(\min\{m', \nu_t\}))\hat{\pi}_t(0) + \int_0^x \beta\hat{\pi}_t(r\tilde{x})f_t(\tilde{x})d\tilde{x} - (1 - \beta r)x.$$

If $m' < \nu_t$, this expression is equal to $m + \underline{m}\sum_{\tau=1}^{T+1-t} \beta^\tau + \hat{\Psi}_t(m')$ as desired by definition of f_t, $\hat{\pi}_t$ and $\hat{\Psi}_t$. Otherwise, this expression is equal to $m + \underline{m}\sum_{\tau=1}^{T+1-t} \beta^\tau$ as desired by definition of f_t.

The case $\nu_t \leq m \leq m'$ is as the case $m'_t \geq \nu_t$ above. The case $m < m', \nu_t$ is also similar. The difference is that $\sigma^*_t(.|m, m')$ puts an atom at zero and the other firm at m. Nevertheless, we can also argue as above that the expected payoffs of any reinvestment $x \in [0, m)$ is also equal to $m + \underline{m}\sum_{\tau=1}^{T+1-t} \beta^\tau$ as desired.

In the case $m = m' \in (\gamma_t, \nu_t)$, both firms randomize with density f_t in the interval $[0, \lambda_t(m)]$ and put the remaining density at m. The expected payoffs of any reinvestment $x \in [0, \lambda_t(m)]$ are also equal to $m + \underline{m}\sum_{\tau=1}^{T+1-t} \beta^\tau$ as desired by the same arguments as in the case $m > m' \geq \nu_t$ above. The expected payoffs of a reinvestment $x = m$ are equal to:

$$m + \underline{m} \sum_{\tau=1}^{T+1-t} \beta^\tau + \int_0^{\lambda_t(m)} \beta\hat{\pi}_t(r\tilde{x})f_t(\tilde{x})d\tilde{x} + \frac{\beta(1 - F_t(\lambda_t(m)))}{2}\hat{\pi}_t(rm) - (1 - \beta r)m,$$

which is equal to $m + \underline{m}\sum_{\tau=1}^{T+1-t} \beta^\tau$ as desired by definition of $\lambda_t(m)$.

Finally, if $m = m' \in [0, \gamma_t)$, then $\sigma^*_t(.|m, m')$ puts all the probability mass at $x = m$. Thus, the expected payoffs are equal to:

$$m + \underline{m} \sum_{\tau=1}^{T+1-t} \beta^\tau + \frac{\beta}{2}\hat{\pi}_t(rm') - (1 - \beta r)m',$$

which is equal to $m + \underline{m}\sum_{\tau=1}^{T+1-t} \beta^\tau + \hat{\hat{\Psi}}_t(m')$ as desired by definition of $\hat{\hat{\Psi}}_t$. $\square$

Proof of Lemma 4: The result follows from the usual arguments in Bertrand competition because W_t is strictly increasing in the first argument and weakly decreasing in its second argument. The monotonicity of W_t is direct except at $t < T + 1$ and $m = m' < \gamma_t$. For these points, we shall use that for any $y \leq \gamma_t$: (a) $\hat{\hat{\Psi}}_t(y) \geq 0$, and (b) $\hat{\Psi}_t(y) \geq \hat{\hat{\Psi}}_t(y)$. (a) is a direct consequence of the definition of γ_t and the fact that $\hat{\hat{\Psi}}_t$ is decreasing. To prove

(b) note that:

$$
\begin{aligned}
\hat{\Psi}_t(y) &= \beta(1-F_t(y))\hat{\pi}_t(0) \\
&= \beta(1-F_t(y))\hat{\pi}_t(0) + \beta\int_0^y \hat{\pi}_t(r\tilde{x})f_t(\tilde{x})d\tilde{x} - (1-\beta r)y \\
&\geq \beta\hat{\pi}_t(ry) - (1-\beta r)y \\
&\geq \hat{\hat{\Psi}}_t(y),
\end{aligned}
$$

where the first and last step are direct, the second equality follows from the definition of f_t and $\hat{\pi}_t$, and the first inequality from the fact that $\hat{\pi}_t(r\tilde{x})$ is decreasing in $\tilde{x}$.

To show that W_t is strictly increasing in its first argument at $m = m' = y < \gamma_t$ we have to show that: $W_t(y,y) - W_t(x,y) > 0$ for any $x < y$, and $W_t(x,y) - W_t(y,y) > 0$ for any $x > y$. These inequalities follow from (a) and (b) respectively, because their corresponding left hand sides are equal to $y - x + \hat{\hat{\Psi}}_t(y)$ and $x - y + \hat{\Psi}_t(y) - \hat{\hat{\Psi}}_t(y)$. Similarly, to show that W_t is decreasing in its second argument at $m = m' = y < \gamma_t$ we have to show that: $W_t(y,x) - W_t(y,y) \geq 0$ for any $x < y$, and $W_t(y,y) - W_t(y,x) \geq 0$ for any $x > y$. These inequalities follow from (b) and (a) respectively, because their corresponding left hand sides are equal to $\hat{\Psi}_t(y) - \hat{\hat{\Psi}}_t(y)$ and $\hat{\hat{\Psi}}_t(y)$. $\square$

Proof of Lemma 5: Under the conditions of the lemma, the payoffs are as described in Equation (12). If either $m, m' \geq \nu_t$ or $m \neq m'$, one can show that our proposed strategy is the unique equilibrium with identical arguments as in the proofs of Propositions 1 and 2. The difference is that one uses $\hat{\pi}_{t+1}$, f_t, F_t and ν_t instead of π, $\tilde{f}^r$, $\tilde{F}^r$ and $\tilde{\nu}^r$, respectively, and payoffs are scaled up by $\underline{m}\sum_{\tau=1}^{T+1-t}\beta^\tau$.

If $m = m' < \gamma_t$, the proposed strategy specifies that both firms reinvest all their cash m which gives expected payoffs $m + \underline{m}\sum_{\tau=1}^{T+1-t}\beta^\tau + \hat{\hat{\Psi}}_{t+1}(m')$ by definition of $\hat{\hat{\Psi}}_{t+1}$. To see why a deviation is not profitable, notice that $\hat{\hat{\Psi}}_{t+1}(m') \geq 0$ since $m' \leq \gamma_t$ and the best deviation is to choose $x = 0$ which gives a payoff of $m + \underline{m}\sum_{\tau=1}^{T+1-t}\beta^\tau$. Any other deviation gives lower payoffs as it implies a greater cost of working capital without affecting the firm's chances of winning.

If $m = m' \in [\gamma_t, \nu_t)$, the proposed strategy specifies that both firms randomize with density f_t in $[0, \lambda_t(m)]$ and put the remaining probability at m. As shown in the proof of Lemma 3, any reinvestment x in its support gives the same expected payoff. Reinvesting

$x = \lambda_t(m)$ gives a greater expected payoff than any reinvestment in $(\lambda_t(m), m)$, since they all win in the same instances but $\lambda_t(m)$ has less cost of working capital.

We next show that there is no other equilibrium in the case $m = m' < \nu_t$. Note, first, that Claims 1 and 2 in the proof of Proposition 1 also hold true here. Both claims imply here that: (a) any equilibrium distribution must be atomless in $(0, m)$; (b) only one firm's strategy can put an atom at 0 in equilibrium. Claim 1 and (a) imply that for any point $x < m$ contained in the support of a firm's strategy, the set $[0, x]$ must be contained in the support of both firm's strategies in equilibrium. In this case, the usual indifference condition implies that both equilibrium distributions must have density f_t in $(0, x)$. We refer to this equilibrium condition as (c).

Next, we show that (a), (b) and (c) imply that both firms use the same strategy in equilibrium. To see why, note that the only possible asymmetric equilibrium is that one of the firms, say 1, puts an atom of probability $\rho > 0$ at 0, density f_t in $(0, \overline{w})$ for some $\overline{w} \in [0, m]$ and the remaining probability, if any, in an atom at m, whereas the other firm's strategy has density f_t in $(0, \overline{w})$ and the remaining probability in an atom at m. Equilibrium requires that Firm 1 gets a weakly greater continuation value with a zero reinvestment than with a reinvestment of m and that Firm 2 gets weakly greater continuation value reinvesting m than with any reinvestment arbitrarily close to zero. The former condition requires that:

$$\int_0^{\overline{w}} \beta\hat{\pi}_t(r\tilde{x})f_t(\tilde{x})d\tilde{x} + \beta\frac{(1 - F_t(\overline{w}))}{2}\hat{\pi}_t(rm) - (1 - \beta r)m \leq 0,$$

whereas the latter condition requires that:

$$\beta\rho\hat{\pi}_t(0) + \int_0^{\overline{w}} \beta\hat{\pi}_t(r\tilde{x})f_t(\tilde{x})d\tilde{x} + \beta\frac{(1 - F_t(\overline{w}) - \rho)}{2}\hat{\pi}_t(rm) - (1 - \beta r)m \geq \beta\rho\hat{\pi}_t(0),$$

and hence both conditions are incompatible when $\rho > 0$ and $\hat{\pi}_t(rm) > 0$.

Consequently, the only possible equilibria are those in which both firms use density f_t in a set $[0, \overline{w})$ for some $\overline{w}$ in $[0, m]$ and put the remaining probability at m. To conclude the proof, we show that equilibrium requires that $\overline{w} = 0$, if $m \leq \gamma_t$ and that $\overline{w} = \lambda_t(m)$, otherwise. Note that the case $\overline{w} = 0$ requires that the firm weakly prefers to reinvest m to reinvest zero, i.e.:

$$\frac{\beta}{2}\hat{\pi}_t(rm) - (1 - \beta r)m \geq 0,$$

whereas the case $\overline{w} \in (0, m]$ requires indifference, i.e.:

$$\int_0^{\overline{w}} \beta \hat{\pi}_t(r\tilde{x}) f_t(\tilde{x}) d\tilde{x} + \frac{\beta(1 - F_t(\overline{w}))}{2} \hat{\pi}_t(rm) - (1 - \beta r)m = 0.$$

If $m \leq \gamma_t$, the former condition holds by definition of γ_t but the latter does not hold for any $\overline{w} \in (0, m]$, because it is increasing by application of Leibnitz rule and it is positive at $\overline{w} = 0$ by definition of γ_t. If $m > \gamma_t$, the former condition does not hold by definition of γ_t and the latter only holds for $\overline{w} = \lambda_t(m)$ by definition of λ_t. □

■

Finally, we show that the limit of the equilibrium of this finite game as the number of periods goes to infinity is equal to the equilibrium of the model in Section 4. To state this result, we abuse a little bit of the notation and denote by $\sigma^*_{t,T}$ and $W_{t,T}$ the functions σ^*_t and W_t, respectively, to make the dependence in the length of the time horizon of the game explicit.

Proposition 7. *For any $w \neq w'$, $\sigma^*_{t,T}(.|w, w')$ converges weakly to $\sigma^*(.|w, w')$ and $W_{t,T}(w, w')$ converges point-wise to $W^*(w, w')$ as T goes to infinity.*

Proof: Note that $\hat{\Psi}_{t+1} = \Psi_{T-t}$, and $\{\Psi_{T-t}\}_{T=t}^{\infty}$ is an increasing sequence that converges point-wise to Ψ, see proof of Lemma 1. This together with the monotone convergence theorem implies the weak convergence of F_t to F^r, and hence the weak convergence of $\sigma^*_{t,T}(.|w, w')$ to $\sigma^*(.|w, w')$, for $w \neq w'$. The point-wise convergence of $\hat{\Psi}_t$ to Ψ also implies the convergence of $W_{t,T}(w, w')$ to $W^*(w, w')$, for $w \neq w'$. ■