The Stochastic Elementary Formula Method and Approximate Travelling Waves for Semi-Linear Reaction Diffusion Equations

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The final and deep thanks are given to my wife, our parents and our son. Without their encouragement and sacrifice, I couldn't carry on my research.

DECLARATION

This thesis is based on my recent research on stochastic differential equations. Some of the results are contained in joint works with Professor K. D. Elworthy and Professor A. Truman and some in my independent works. Five papers arising from this thesis have been published or accepted. The first paper is my joint work with K. D. Elworthy which appeared in "Stochastics and Quantum Mechanics" edited by A. Truman and I.M. Davies, World Scientific, (1992) 298-316. The material in this paper does not appear in the thesis but many of the current ideas originated in that work. The second paper joint with K. D. Elworthy and A. Truman will appear in "Proceedings of the Royal Society of London" in September 1994. The third and fourth papers joint with K. D. Elworthy will be published in "Mathematical and Computer Modelling, An International Journal" and "Probability Theory and Mathematical Statistics: Proceedings of the Euler Institute Seminars Dedicated to the Memory of Kolmogorov" edited by I.A. Ibragimov and A.Y. Zaitsev, Gordon & Breach Science Publishers, respectively. The fifth paper is my independent work which has been published in "Proceedings of the Royal Society of Edinburgh", Vol. 124A(1994) , 273-299.

All the pictures in this thesis were produced by J. G. Gaines in Edinburgh.

No part of this thesis has been submitted for any degree at any other university.

Huaizhong Zhao
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SUMMARY OF THE THESIS:

The Stochastic Elementary Formula Method and Approximate Travelling Waves for Semi-Linear Reaction Diffusion Equations

by Huaizhong Zhao

In this thesis we consider approximate travelling wave solutions for stochastic and generalised KPP equations and systems by using the stochastic elementary formula method of Elworthy and Truman. We begin with the semi-classical analysis for generalised KPP equations. With a so-called "late caustic" assumption we prove that the global wave front is given by the Hamilton Jacobi function. We prove a Huygens principle on complete Riemannian manifolds without cut locus, with some bounds on their volume elements, in particular Cartan-Hadamard manifolds. Based on the semi-classical analysis we then consider the propagation of approximate travelling waves for stochastic generalised KPP equations. Three regimes of perturbation are considered: weak, mild, and strong. We show that weak perturbations have little effect on the wave like solutions of the unperturbed equations while strong perturbations essentially destroy the wave and force the solutions to decay rapidly. In the more difficult mild case we show the existence of a 'wave front', in front of which the solution is close to zero (of order $exp(-c_1\mu^{-2})$ as $\mu \sim 0$ for c_1 random) and behind which it has at least order $exp(-c_2\mu^{-1})$ for some random c_2 depending on the increment of the noise. For an alternative stochastic equation we classify the effect of the noise by the Lyapunov exponent of a corresponding stochastic ODE. Finally we study the asymptotic behaviour of reaction-diffusion systems with a small parameter by using the n-dimensional Feynman-Kac formula and Freidlin's large deviation theory. We obtain the travelling wave with nonlinear ergodic interactions and a special case with nonlinear reducible interactions.

Introduction

In recent years, nonlinear reaction diffusion equations have attracted great interest from the mathematical community. Such equations arise in the mathematical modelling of a number of social, biological and physical phenomena occurring in such diverse areas as population dynamics, population genetics, epidemiology, nerve impulse studies, electrical transmission line analysis, chemical reactor theory, neutron transport theory and the study of gas flow in porous media (Fitzgibbon (III) and Walker (1977), Winfree (1980), Segal (1980), Britton (1986) and Murray (1989)). Wave solutions were found in many natural phenomena (Murray (1989)). Many phenomena arising in various physical or biological contexts can be modelled by travelling waves, for example, shock waves, nerve impulses and various oscillating chemical reactions (Smoller (1983)). Classically, a travelling wave is taken to be a wave which travels without change of shape. A mathematical way of saying this is that if the solution is of the form

$$u(x,t) = U(x-\alpha t) = U(z), \qquad z = x - \alpha t, \tag{1}$$

then u(x,t) is a travelling wave, and it moves at a constant speed α in the positive x-direction. That x-αt is a constant (u is also a constant) means the coordinate system moves with speed α. The discovery, investigation and analysis of travelling waves in chemical reactions was first reported by Luther (1906). The classical and simplest case of the nonlinear reaction-diffusion equation which was studied by Fisher (1937) and Kolmogoroff, Petrovsky and Piscounoff (1937) is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + cu(1-u). \tag{2}$$

It is called the Fisher equation or KPP equation now. The books written by Fife (1979), Britton (1986) and Murray (1989) gave a full discussion of this equation. The basic method is that we seek a solution of the form (1), then from (1) and (2) we get a second order ordinary differential equation with a parameter α. If α is not smaller than $\alpha^* = \sqrt{2c}$, a nonnegative solution of the second order ordinary differential equation exists which satisfies

$$U(+\infty) = 0, U(-\infty) = 1. \tag{3}$$

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So if the speed of the propagation α is not smaller than the critical speed $\alpha^* = \sqrt{2c}$, the travelling wave solution exists. Generally speaking, the speed of the propagation of the wavefront solution depends on the explicit behaviour of the initial condition u(0,x) as $|x| \to +\infty$ (McKean (1975), Larson (1978), Monoranjan and Mitchell (1983), Mollison (1977) and Murray (1989), Zhao and Elworthy (1992)). On the other hand, if u(0,x) has compact support, that is the kind of initial condition used by Kolmogoroff et al (1937), Kolmogoroff et al (1937) showed the ultimate wave does not depend on the detailed form of u(0,x), the solution u(t,x) evolves to a travelling wave front with speed $\sqrt{2c}$.

The analytical method has been developed by Fife (1979), Smoller (1983), Murray (1989), Berestycki & Nirenberg (1992), Eckmann & Wayne (1994) etc.

When reaction-diffusion equations become complicated, for example, the diffusion terms depend on spatial variables or the nonlinear terms are very complicated, or several equations are coupled together, the method mentioned above will be difficult to use. Stochastic approaches have been used by Freidlin (1979, 1983, 1985, 1991, 1992), Gartner (1982), McKean (1975), Zhao and Elworthy (1992), Champneys, Harris, Toland, Warren & Williams (1993). Freidlin (1985) introduced a small parameter μ to consider the function $u(\frac{t}{\mu^2}, \frac{x}{\mu^2})$. For large t, the function is close to $U(\frac{x-\alpha t}{\mu^2})$. For small μ, $U(\frac{x-\alpha t}{\mu^2})$ is close to 1 as $x < \alpha t$, and is close to 0 as $x > \alpha t$. From (2), u^μ will satisfy

$$\frac{\partial u^\mu}{\partial t} = \frac{\mu^2}{2} \cdot \frac{\partial^2 u^\mu}{\partial x^2} + \frac{1}{\mu^2} c u^\mu (1 - u^\mu). \tag{4}$$

This equation can be treated by large deviation theory. Freidlin then considered the generalised KPP equation (Fisher equation)

$$\begin{cases} \dfrac{\partial u^\mu}{\partial t} = \dfrac{\mu^2}{2} \cdot \sum\limits_{i,j=1}^{r} \dfrac{\partial}{\partial x^i} \left(a^{ij}(x) \dfrac{\partial u^\mu}{\partial x^j} \right) + \dfrac{1}{\mu^2} f(x, u^\mu), \\ u^\mu(0,x) = g(x). \end{cases} \tag{5}$$

Suppose the initial function g(x) is nonnegative and its support $G_0 \neq R^r$ satisfies $[(G_0)] = [G_0]$ (where $[G_0]$ denotes the closure of the set G_0, and (G_0) the interior), $f(x, u^\mu) = c(x, u^\mu) u^\mu$ satisfies $c(x,u) \le c(x,0) = \bar{c}(x) \le \hat{c}$, $c(x,u) > 0$ for $0 \le u < 1$ and $c(x,u) < 0$ for $u > 1$. Freidlin (1985) defined a function V(t,x) and obtained that under a certain condition (N), as $\mu \to 0$, $u^\mu(t,x) \to 0$ for $(t,x) \in \{(t,x) : V(t,x) < 0\}$ and $u^\mu(t,x) \to 1$ for $(t,x) \in \{(t,x) : V(t,x) > 0\}$ by using the Feynman-Kac formula and large deviation theory. Freidlin generalised the work of Kolmogoroff et al (1937) and Fisher (1937) in three ways: (a). the Laplace operator becomes a very general elliptic, spatially dependent, second order partial differential operator, (b). $f(x,u)$ can be dependent on x and can be a very general nonlinear function satisfying the condition we mentioned above, (c). the wave speed was obtained in the general case. However Freidlin's waves are only approximate.

The method of semi-classical quantum mechanics of Elworthy and Truman resulted in the discovery of a new "Brownian -Riemannian" bridge process on manifolds, and an elegant new version of the Feynman-Kac formula in curved spaces geared to simplifying small-time and small-h asymptotics (see Elworthy & Truman (1981,1982)). They have also given rise to new results in quantum statistical mechanics e.g. a new proof of the result that the quantum mechanical partition function in curved space $Z_Q(h) \to Z_C$, the corresponding classical partition function, as $h \to 0$ (see Elworthy, Ndumu & Truman (1986)). A generalization of this "Brownian-Riemannian" bridge process has been obtained in Watling (1988,1992). Applications to Schrödinger operators of this bridge process were given in Watling (1988,1992), Elworthy, Truman & Watling (1985). Quite recently stochastic Hamilton Jacobi theory has been established in Truman & Zhao (1994). Certain stochastic partial differential equations have been solved explicitly and the fundamental solution and small time asymptotics have been obtained by using the stochastic Hamilton Jacobi theory.

We started to study the propagation of the travelling wave for generalised KPP equations by semi-classical analysis and Feynman-Kac formula, together with some ideas from Freidlin's pioneering work in this area, in Zhao and Elworthy (1992). In that paper, the travelling wave generated by $u_0^\mu(x) = T_0(x)\exp\{-\frac{1}{\mu^2}S_0(x)\}$ was studied before the time at which certain caustic points appear. We studied not only the speed, but also the shape of the trough of the travelling wave by employing classical mechanics, especially Hamilton-Jacobi theory. Ben Arous and Rouault (1992) also considered the shape of the trough of the travelling wave generated by the step function $\chi_{x\leq 0}$ using boundary layer analysis.

In chapter I, we first study the wave generated by $u_0^\mu = \sum_{j=1}^{N} T_{j,0}(x)\exp\{-\frac{1}{\mu^2}S_{j,0}(x)\}$ and various limiting cases. Here to cope with the superposition of a finite number of extended sources, we use N different Markov processes and construct N different classical mechanical paths. With a so called "late caustic" assumption we remove the "no caustic" condition of the earlier paper. We obtain the propagation of the global wave front and shape of the trough. We give our theorems under very weak conditions by introducing classical flow tubes, using the Maruyama-Girsanov-Cameron-Martin formula inside the tubes and appealing to Varadhan's theorem outside the tubes. We consider a generalised KPP equation with a deep well potential in which caustics do appear.

In chapter II, first we study the wave generated by δ -functions distributed at different points. Our theory shows clearly how the initial distributions contribute to the propagation of the travelling wave. The study of this problem is helpful in investigating the travelling wave on a Riemannian manifold generated by the step initial distribution which is approximated by the integral (infinite sum) of δ-functions. We prove a Huygens principle on complete Riemannian manifolds without cut locus, with some bounds on their volume elements, in particular Cartan-Hadamard manifolds.

The study of the evolution of physical and biological systems in random environment has been treated in many works (Anderson (1958), Ishimaru (1978), Jeffrey and Kawahara (1982), Papanicolaou (1988), Gartner and Molchanov (1990) for example). More recently there has been work by Tribe (1993) and also by Mueller & Sowers (1993) on certain nonlinear reaction diffusion equations with space time white noise perturbations. In chapters III, we study the propagation of the travelling waves for certain stochastic generalised KPP equations.

In chapter III we consider a stochastic generalised KPP equation with a multiplicative white noise of Itô's type

$$\begin{cases} du_t^\mu(x) = \left[\frac{1}{2}\mu^2\Delta u_t^\mu(x) + \frac{1}{\mu^2}c(x, u_t^\mu(x))u_t^\mu(x)\right] dt + \alpha(\mu)k(t,x)u_t^\mu(t)dw_t, \\ u_0^\mu(x) = T_0(x)e^{-\frac{1}{\mu^2}S_0(x)}. \end{cases} \tag{6}$$

We mainly concentrate on the study of the existence and propagation of approximate travelling waves of generalised KPP equations with seasonal multiplicative white noise perturbations of Itô type. Three regimes of perturbation are considered: weak, mild, and strong corresponding to $\alpha(\mu) = 1, \frac{1}{\mu}$ and $\frac{1}{\mu^2}$ respectively. We show that weak perturbations have little effect on the wave like solutions of the unperturbed equations while strong perturbations essentially destroy the wave and force the solutions to die down. In the more difficult mild case we show the existence of a 'wave front', in front of which the solution is close to zero (of order $exp(-c_1\mu^{-2})$ for c_1 random) and behind which it has at least order $exp(-c_2\mu^{-1})$ for some random c_2 depending on the increment of the noise. We can also give random upper bounds behind the front. However this is far from a complete picture.

Alternately in §9, we consider the following stochastic generalised KPP equation:

$$\begin{cases} du_t^\mu(x) = [\frac{\mu^2}{2}\Delta u_t^\mu(x) + \frac{1}{\mu^2}c(x, u_t^\mu(x))u_t^\mu(x)]dt + k(\frac{t}{\mu^2})u_t^\mu(x)dw_{\frac{t}{\mu^2}} \\ u_0^\mu(x) = T_0(x)e^{-\frac{S_0(x)}{\mu^2}}. \end{cases} \tag{7}$$

We classify the multiplicative white noise perturbations of generalised KPP equations and their effects on deterministic approximate travelling wave solutions by the behaviour of $\int_0^t k^2(s)ds$. If $\int_0^t k^2(s)ds < +\infty$, the solutions of the stochastic generalised KPP equations converge to deterministic approximate travelling waves and if $\frac{1}{2}\lim_{\sigma\to+\infty}\frac{1}{\sigma}\int_0^\sigma k^2(s)ds > \sup_{(t,x)\in D}\frac{1}{t}\int_0^t \bar{c}(\Phi_s(\Phi_t^{-1}(x)))ds$, where Φ is an associated classical mechanical flow, then the white noise perturbations kill the wave. For the case $a = \frac{1}{2}\lim_{\sigma\to+\infty}\frac{1}{\sigma}\int_0^\sigma k^2(s)ds \leq \sup_{(t,x)\in D}\frac{1}{t}\int_0^t \bar{c}(\Phi_s(\Phi_t^{-1}(x)))\ ds$ (supposing the existence of the limit) we show that there is a residual wave form but propagating at a different speed to that of the unperturbed equations. Note that $-a$ is the Lyapunov exponent of the stochastic ordinary differential equation $d\xi_t = k(t)\xi_t dw_t$.

In chapter IV, we study the asymptotic behaviour of reaction-diffusion systems with a small parameter by using the n-dimensional Feynman-Kac Formula (Stroock (1970), Babbit (1970)), and Freidlin's large deviation theory. The generalised solutions are introduced in §1. We obtain the travelling wave joining an unstable steady state and an asymptotically stable steady state of the diffusionless dynamical system in a reaction-diffusion system with nonlinear ergodic interactions and a special case with nonlinear reducible interactions. For this we use the well-known Frobenius theorem and appeal to a comparison theorem for ordinary differential systems (Lakshmikantham & Leela (1969)). We find that a comparison of DE_s (ODE_s, PDE_s, SDE_s) is always a useful technique to treat the very complicated equations herein.

Generally speaking, the travelling waves for the generalised KPP equations, especially stochastic generalised KPP equations, do not fit the definition given by (1). In fact the definition of the classical travelling wave is very narrow. The wave can travel along a straight line or a curve and the wave speed can be infinity. The crest of the travelling wave, especially for the stochastic KPP equation, can be a random process rather than constant 1. However, the behaviour of the solution in front of the wave front is quite different from that of the solution behind the wave front. Nevertheless if we consider the classical KPP equation, our travelling wave agrees with the classical one perfectly.

Chapter I. Semi-classical Analysis of Deterministic Generalised KPP Equations

In this chapter we consider reaction diffusion equations of the form

$$\frac{\partial u_t^\mu(x)}{\partial t} = \mu^2 L u_t^\mu(x) + \frac{1}{\mu^2} c(x, u_t^\mu(x)) u_t^\mu(x).$$

We will take L to be a general second order elliptic operator with $L1 = 0$ and smooth coefficients. The natural state space is a manifold M, e.g., $M = R^r$, and some of the phenomena, and conditions, are most easily described in geometrical terms. The operator L may be represented locally in a coordinate chart $x = (x^1, x^2, \cdots, x^r)$ by $L = \frac{1}{2} \sum_{i,j=1}^{r} a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b_i(x) \frac{\partial}{\partial x^i}$, where $a_{ij}, 1 \le i, j \le r$, $b_i, 1 \le i \le r$ are smooth scalar-valued functions on the domain of the chart and the $r \times r$ matrices $(a_{ij}(x))$ are symmetric and positive definite. As is well-known for a suitable choice of Riemannian metric on M, L takes the form

$$L = \frac{1}{2}\Delta + < b, \nabla >$$

for Δ the Laplace-Beltrami operator and b a smooth vector field. So we will take $L = \frac{1}{2}\Delta + < b(x), \nabla >$ as the differential operator in this chapter.

§1. Travelling Waves from a Finite Number of Extended Sources Under a Late Caustic Conditions

A. In this section we study the solution $u_t^\mu(x)$ of the following Cauchy problem with a small parameter μ

$$\begin{cases} \dfrac{\partial u_t^\mu(x)}{\partial t} = \dfrac{1}{2}\mu^2 \Delta u_t^\mu(x) + \mu^2 < b(x), \nabla u_t^\mu(x) > + \dfrac{1}{\mu^2} c(x, u_t^\mu(x)) u_t^\mu(x), \\ u_0^\mu(x) = \sum_{j=1}^{N} T_{j,0}(x) e^{-\frac{1}{\mu^2} S_{j,0}(x)}. \end{cases} \tag{1.1}$$

Here $t \in [0, \infty)$, $x \in M$, an r-dimensional complete Riemannian manifold with Δ its Laplace-Beltrami operator, b is a smooth vector field on M and $c : M \times [0, +\infty) \to R^1$. For given $T_{j,0}$, $S_{j,0} : M \to R^1$, let $u_t^\mu : M \to R^1$ denote the solution of the Cauchy problem.

We suppose that $T_{j,0}$ and $S_{j,0}$ are continuous and that $T_{j,0}$ is nonnegative. We always suppose that $c \in C^{3,3}(M \times [0, +\infty), R)$. First we consider the condition :

(I). For $x \in M$ and $u \geq 0$, $c(x, u) \leq \bar{c}(x) \leq \hat{c}$, where $\bar{c}$ is continuous, $\hat{c}$ is a constant.

By a semigroup argument as in Da Prato & Zabczyk (1988) and parabolic regularity we can prove the global existence and uniqueness of $C^{1,2}$ solutions for (1.1) under condition (I), given Lipschitz continuity of c in u, uniformly in x. See also Belopolskaya (1991) and Li & Zhao (1994).

Let $\Pi : O(M) \to M$ be the orthonormal frame bundle of M. The Levi-Civita connection of M determines a map

$$\mathcal{X} : O(M) \times R^r \to TO(M)$$

into the tangent space to $O(M)$, which trivializes the horizontal tangent bundle of $O(M)$. Suppose $\{A_t : 0 \leq t \leq T\}$ is a C^1 time dependent vector field on M. Let $\tilde{A}_t : O(M) \to TO(M)$ be its horizontal lift: so

$$T\Pi(\tilde{A}_t(y)) = A_t(\Pi(y)), \qquad 0 \leq t \leq T, y \in O(M),$$

where $T\Pi : TO(M) \to TM$ is the derivative map of Π. In particular, let $\tilde{b} : O(M) \to TO(M)$ be the horizontal lift of b.

For $x_0 \in M$ take $y_0 \in \Pi^{-1}(x_0)$ and consider the Stratonovich stochastic equation for $y : [0, \xi_A) \times \Omega \to O(M)$ with $y(0, \omega) = y_0$,

$$dy_s^\mu = \mu \mathcal{X}(y_s^\mu) \cdot dB_s + [\mu^2 \tilde{b}(y_s^\mu) + \tilde{A}_s(y_s^\mu)]ds, \tag{1.2}$$

where $\{B_s : s \geq 0\}$ is a r-dimensional Brownian motion defined on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and ξ_A is the explosion time of the solution: $0 < \xi_A \leq \infty$ for all $\omega \in \Omega$. Finally we get the Markov process X_t^μ on M by

$$X_t^\mu = \Pi(y_t^\mu).$$

By the above construction we may construct N Markov processes $X^{\mu}_{j,s}$, $j = 1, 2, \cdots, N$ on the Riemannian manifold M by taking $A_{j,s}(x) = \nabla Y_{j,s}(x)$ for suitable $Y_{j,s} : M \to R^1$. We let $\{B^{x,\mu}_s : s \geq 0\}$ be the solution of (1.2) with $A_{j,s} \equiv 0$.

Our approach is based on the use of the Maruyama-Girsanov-Cameron-Martin theorem in the following form:

Proposition 1.1. *Let $Z = C([0, +\infty), M)$ and $f : Z \to R$ be bounded and measurable, and let $\mathcal{K}$ be a compact set in $[0, +\infty) \times M$, with $\eta(\sigma)$ be the first exit time of any path $t \to (t, \sigma(t))$ from $\mathcal{K}$. Then*

$$\hat{E}\chi_{t<\eta(B^{x,\mu}_{\cdot})} f(B^{x,\mu}_{\cdot}) = \hat{E}\chi_{t<\eta(X^{x,\mu}_{j,\cdot})} \mathcal{M}_{j,t} f(X^{x,\mu}_{j,\cdot}), \tag{1.3}$$

where $X^{x,\mu}_{j,s}$ are defined from (1.2) with $A_{j,s} = \nabla Y_{j,s}$ and

$$\mathcal{M}_{j,t} = \exp\left\{ \frac{1}{\mu^2}(Y_{j,0}(x) - Y_{j,t}(X^{x,\mu}_{j,t})) + \frac{1}{2}\int_0^t \Delta Y_{j,s}(X^{x,\mu}_{j,s})ds \right.$$
$$\left. + \frac{1}{\mu^2}\int_0^t (\frac{\partial Y_{j,s}}{\partial s}(X^{x,\mu}_{j,s}) + \frac{1}{2}||A_{j,s}(X^{x,\mu}_{j,s})||^2)ds + \int_0^t < \nabla Y_{j,s}(X^{x,\mu}_{j,s}), b(X^{x,\mu}_{j,s}) > ds \right\}. \tag{1.4}$$

Proof. From the Maruyama-Girsanov-Cameron-Martin Formula (see Elworthy (1982)), we have (1.3) with

$$\mathcal{M}_{j,t} = \exp\left\{ -\frac{1}{\mu}\int_0^t < A_{j,s}(X^{x,\mu}_s), dB_s > -\frac{1}{2\mu^2}\int_0^t ||A_{j,s}(X^{x,\mu}_{j,s})||^2 ds \right\}.$$

The possibility of explosion of either of the two processes is irrelevant since the coefficients of (1.2) can be modified outside of a neighbourhood of $\mathcal{K}$ (to be time dependent) to give non-explosion, without affecting either side of (1.3). Applying Itô's Formula to $Y_{j,s}(X^{x,\mu}_{j,s})$ we have

$$Y_{j,t}(X^{x,\mu}_{j,t}) = Y_{j,0}(x) + \mu\int_0^t < \nabla Y_{j,s}(X^{x,\mu}_{j,s}), dB_s > + \int_0^t \left\{ \frac{1}{2}\mu^2 \Delta Y_{j,s}(X^{x,\mu}_{j,s}) \right.$$
$$\left. + < \nabla Y_{j,s}(X^{x,\mu}_{j,s}) + \mu^2 b(X^{x,\mu}_{j,s}), A_{j,s}(X^{x,\mu}_{j,s}) > + \frac{\partial Y_{j,s}(X^{x,\mu}_{j,s})}{\partial s} \right\} ds.$$

From this we get (1.4). ‡‡

B. Let $\bar{c}$ be C^2 and $S_{j,0}$ be C^1 and consider the classical mechanical systems $\Phi_{j,s} : M \to M \quad (j \in \{1, 2, \cdots, N\})$,

$$\begin{cases} \ddot{\Phi}_{j,s}(a) = -\nabla\bar{c}(\Phi_{j,s}(a)), \quad s \geq 0, \\ \dot{\Phi}_{j,0}(a) = \nabla S_{j,0}(a), \quad \Phi_{j,0}(a) = a. \end{cases} \tag{1.5}$$

For $a \in M$, set

$$\begin{aligned}
\Sigma_{j,a} =& \{t \geq 0 : \Phi_{j,t} \text{ gives a diffeomorphism of a neighbourhood of } a \text{ in } M \\
& \quad \text{on to a neighbourhood of } \Phi_{j,t}(a) \text{ and also } \Phi_{j,t}^{-1}(\Phi_{j,t}(a)) = \{a\}\}, \\
\sigma_j(a) =& \inf\{t \geq 0 : t \notin \Sigma_{j,a}\} \leq \infty, \\
\sigma_j^T(a) =& T \wedge \sigma_j(a), \text{ for } T \geq 0, \\
A_j =& \{(t,a) \in [0,\infty) \times M, t < \sigma_j(a)\}, \\
D_j =& \{(t, \Phi_{j,t}(a)) : t < \sigma_j(a)\}, \\
D_{j,t} =& \{x \in M : (t,x) \in D_j\} \\
D =& \bigcap_{j=1}^{N} D_j, \\
D_t =& \bigcap_{j=1}^{N} D_{j,t}.
\end{aligned}$$

Remark 1.1. *It is worth noting that in many cases of interest $D_t = M$ for $0 \leq t \leq T$ ('no caustics'), some $T > 0$, and in this case what follows is extremely simple for $t \in [0,T]$. In particular Theorem 1.1 is almost immediate given suitable bounds on ∇V_j and ΔV_j and assuming a non-explosion condition since in (1.12) below we can take $\eta_j \equiv +\infty$ and we have no need to appeal to Varadhan's theorem to control the second term, and so no need to assume that $\bar{c}$ is bounded (provided the integrals exist).*

Recall that a map is *proper* if the inverse image of each compact set is compact. Clearly $(s,a) \mapsto \Phi_{j,s}(a)$ is proper as a map $[0,t] \times M \to M$ if $\sup\{|\dot{\Phi}_{j,s}(a)| : 0 \leq s \leq t\} \leq \alpha_t d(a,x_0) + k$ for some $k \in R$, $\alpha_t < t^{-1}$ with $d(-,x_0)$ the distance from some point x_0. Some of the statement of our results simplify when D_j is known to be open, so we first give some lemmas on the lower semicontinuity of σ_j, though they are not explicitly needed.

Lemma 1.1. *If $\Phi_{j,.}(.)$ is proper on $[0,T] \times M$, then $A_j \cap [0,T) \times M$ is open in $[0,T) \times M$ and contains $\{0\} \times M$. In particular σ_j^T is lower semicontinuous and positive.*

In fact if $\mathcal{O}_j^T$ is an open set in M with $\{(t,a) \in [0,T] \times \mathcal{O}_j^T : \Phi_{j,t}(a) \in \mathcal{K}\}$ compact for each compact $\mathcal{K}$ in $\mathcal{O}_j^T$ then σ_j^T is lower semicontinuous on $\mathcal{O}_j^T$.

Proof. For $T, \mathcal{O}_j^T$ as described, suppose $(t,a) \in [0,T) \times \mathcal{O}_j^T$ with $t < \sigma_j(a)$. Suppose there exists $\{a_i\}_{i=1}^\infty$ in $\mathcal{O}_j^T$ with $\{t_i\}_{i=1}^\infty$ such that $t_i \to t, a_i \to a$ but $t_i \geq \sigma_j(a_i)$. There is a neighbourhood $U = (t-\delta, t+\delta) \times U_0$ of (t,a) in $[0,T) \times \mathcal{O}_j^T$ which is mapped by $(s,y) \to (s, \Phi_{j,s}(y))$ diffeomorphically onto a neighbourhood W of $(t, \Phi_{j,t}(a))$ in $[0,T) \times \mathcal{O}_j^T$, and we can modify (t_i, a_i) if necessary to assume that also $(t_i, a_i) \in U$ and $t_i \notin \Sigma_{j,a_i}$ for each i.

Then there exists $\{b_i\}_{i=1}^\infty$ in M with $b_i \neq a_i$ but $\Phi_{j,t_i}(b_i) = \Phi_{j,t_i}(a_i)$. By the process assumption we can assume there exists $b \in M$ with $b_i \to b$. Then $\Phi_{j,t}(a) = \lim_{i\to\infty} \Phi_{j,t_i}(a_i) = \lim_{i\to\infty} \Phi_{j,t_i}(b_i) = \Phi_{j,t}(b)$ giving $b = a$. But then $(t_i, b_i) \in U$ for sufficiently large i, contradicting our hypothesis.

Then $A_j \cap [0,T) \times \mathcal{O}_j^T$ is open in $[0,T) \times \mathcal{O}_j^T$ and σ_j^T is l.s.c. on $\mathcal{O}_j^T$. Strict positivity of σ_j^T on $\mathcal{O}_j^T$ follows by a similar argument. ‡‡

For $(t,x) \in D_j$ we define $V_j : D_j \to R$ by

$$V_j(t,x) = \int_0^t \bar{c}(\Phi_{j,s}(\Phi_{j,t}^{-1}(x)))ds - S_{j,0}(\Phi_{j,t}^{-1}(x)) - \frac{1}{2}\int_0^t |\dot{\Phi}_{j,s}(\Phi_{j,t}^{-1}(x))|^2 ds, \tag{1.6}$$

and with convention $V_j(t,x) = +\infty$ if $(t,x) \notin D_j$. As in Elworthy and Truman (1982), by computation we have that V_j satisfies the Hamilton-Jacobi equation

$$\frac{1}{2}||\nabla V_{j,t}(x)||^2 + \bar{c}(x) - \frac{\partial V_{j,t}}{\partial t}(x) = 0, \quad (t,x) \in D_j \tag{1.7}$$

with given initial function $-S_{j,0}$. Moreover

$$\dot{\Phi}_{j,t}(x) = -\nabla V_{j,t}(\Phi_{j,t}(x)). \tag{1.8}$$

For $x \in D_{j,t}$, set

$$z_{j,s}^t(x) = \Phi_{j,t-s}(\Phi_{j,t}^{-1}(x)). \tag{1.9}$$

Then we have

$$\frac{\partial z_{j,s}^t}{\partial s}(x) = \nabla V_{j,t-s}(z_{j,s}^t(x)). \tag{1.10}$$

Lemma 1.2.

(i). If $x \in D_{j,t}$, then $z_{j,s}^t(x) \in D_{j,t-s}$ for $0 \leq s \leq t$.

(ii). If a compact set $\mathcal{K}$ in $D_{j,t}$ has a neighbourhood in $D_{j,t}$, there is a neighbourhood $N_{j,\mathcal{K}}$ of

$$\{(s, z_{j,s}^t(x)) : 0 \leq s \leq t, x \in \mathcal{K}\} \text{ in } [0,t] \times M$$

with $\overline{N_{j,\mathcal{K}}} \subset D_j$.

Proof. (i). Set $a = \Phi_{j,t}^{-1}(x)$. Then $t < \sigma_j(a)$ and also $t - s < \sigma_j(a)$ for $0 \leq s \leq t$.

(ii). Let V_t be open in M with $\mathcal{K} \subset V_t \subset \overline{V_t} \subset D_{j,t}$. By definition of $D_{j,t}$ if $V_0 = \Phi_{j,t}^{-1}(V_t)$ then $\Phi_{j,s}$ is a diffeomorphism of V_0 onto an open set V_s of M for $0 \leq s \leq t$, with $\Phi_{j,s}^{-1}(V_s) = V_0$, and $(s, a) \to (s, \Phi_{j,s}(a))$ maps $[0, t] \times V_0$ diffeomorphically onto a neighbourhood of $\{(s, z_{j,s}^t(x)) : x \in \mathcal{K}, 0 \leq s \leq t\}$; this can be taken to be $N_{j,\mathcal{K}}$ shrunk if necessary to ensure $\overline{N_{j,\mathcal{K}}} \subset D$. ‡‡

C. Now for $(t, x) \in D$ define $V : D \to R$ by

$$V_t(x) = \max\{V_{j,t}(x) : 1 \leq j \leq N\}, \tag{1.11}$$

and adopt the convention $V_t(x) = +\infty$ if $(t, x) \notin D$.

Theorem 1.1. *Assume the condition (I), that* $\bar{c}$ *and* $S_{j,0}$ *are* C^2 *with* $\bar{c}$ *bounded above and* $S_{j,0}$ *bounded below, and that* $T_{j,0}$ *are bounded and measurable. Then*

$$\lim_{\mu \to 0} u_t^\mu(x) = 0,$$

uniformly on compacta in the interior of $\{(t, x) : V_t(x) < 0, t > 0\}$.

Proof. For compact $\mathcal{K}_0$ in the interior of $\{(t, x) : V_t(x) < 0\}$, by looking at the inverse image of a neighbourhood of $\mathcal{K}_0$ under $(s, a) \to (s, \Phi_{j,s}(a))$ and arguing as in the previous lemma we can obtain open sets $\mathcal{N}_{j,\mathcal{K}_0}$ of $[0, \infty) \times M$ with $\overline{\mathcal{N}_{j,\mathcal{K}_0}} \subset IntD$ and $(t - s, z_{j,s}^t(x)) \in \mathcal{N}_{j,\mathcal{K}_0}$ for $0 \leq s \leq t$ if $(t, x) \in \mathcal{K}_0$. Let $\eta_j(\sigma)$ be the first exit time of the path σ from $\mathcal{N}_{j,\mathcal{K}_0}$. Let $X_{j,s}^{x,\mu}$ for $0 \leq s < \eta_j(X_{j,\cdot}^{x,\mu})$ be the solution of (1.2) with $A_{j,s} = \nabla V_{j,t-s}$ up to exit time $\eta_j(X_{j,\cdot}^{x,\mu})$. Note that V_j is $C^{1,2}$ on $\mathcal{N}_{j,\mathcal{K}_0}$ by (1.8) since Φ_j^{-1} and $\dot{\Phi}_j$ are $C^{1,1}$. By the Feynman-Kac Formula, Proposition 1.1 and using (1.7) we have

$$\begin{aligned} u_t^\mu(x) = &\sum_{j=1}^N e^{\frac{1}{\mu^2} V_{j,t}(x)} \hat{E} \chi_{t < \eta_j(X_{j,\cdot}^{x,\mu})} T_{j,0}(X_{j,t}^{x,\mu}) e^{\frac{1}{2} \int_0^t \Delta V_{j,t-s}(X_{j,s}^{x,\mu}) ds} \\ &\cdot e^{\{\frac{1}{\mu^2} \int_0^t [c(X_{j,s}^{x,\mu}, u_{t-s}^\mu(X_{j,s}^{x,\mu})) - \bar{c}(X_{j,s}^{x,\mu})] ds + \int_0^t < \nabla V_{j,t-s}(X_{j,s}^{x,\mu}), b(X_{j,s}^{x,\mu}) > ds\}} \\ &+ \sum_{j=1}^N \hat{E} \chi_{t > \eta_j(B_\cdot^{x,\mu})} T_{j,0}(B_t^{x,\mu}) e^{-\frac{1}{\mu^2} S_{j,0}(B_t^{x,\mu}) + \frac{1}{\mu^2} \int_0^t c(B_s^{x,\mu}, u_{t-s}(B_s^{x,\mu})) ds}. \end{aligned} \tag{1.12}$$

By §3 of Elworthy & Truman (1981), especially Theorem 3C and its proof (i.e. Varadhan (1967)), the last term of the above has upper bound

$$\sum_{j=1}^{N} \hat{E}\chi_{t>\eta_j(B^{x,\mu}_\cdot)} T_{j,0}(B_t^{x,\mu}) e^{-\frac{1}{\mu^2}S_{j,0}(B_t^{x,\mu})+\frac{1}{\mu^2}\int_0^t \bar{c}(B_s^{x,\mu})ds}$$
$$=\sum_{j=1}^{N} e^{\frac{V_{j,t}(x)}{\mu^2}} \times [o(\mu^n)],$$

for any $n \geq 0$. See also §3 of Zhao & Elworthy (1992) for an explicit identification of $V_{j,t}(x)$ with the term giving the leading behaviour in Varadhan's theorem, denoted as $V_t(x)$ by Freidlin (1985). Finally we have

$$\begin{aligned} u_t^\mu(x) = \sum_{j=1}^{N} e^{\frac{1}{\mu^2}V_{j,t}(x)} \Big\{ &\hat{E}\chi_{t<\eta_j(X^{x,\mu}_{j,\cdot})} T_{j,0}(X_{j,t}^{x,\mu}) e^{\frac{1}{2}\int_0^t \Delta V_{j,t-s}(X^{x,\mu}_{j,s})ds} \\ &\cdot e^{\{\frac{1}{\mu^2}\int_0^t [c(X^{x,\mu}_{j,s}, u^\mu_{t-s}(X^{x,\mu}_{j,s})) - \bar{c}(X^{x,\mu}_{j,s})]ds + \int_0^t <\nabla V_{j,t-s}(X^{x,\mu}_{j,s}), b(X^{x,\mu}_{j,s})> ds\}} + o(\mu^n)\Big\}. \end{aligned} \tag{1.13}$$

Then the theorem follows. ‡‡

D. In the following we study the shape of the trough of the travelling wave.

For a compact subset $\mathcal{K}_0$ of D define $\mathcal{N}_{j,\mathcal{K}_0}$ as in the proof above with η_j the first exit time. Let

$$\tilde{X}^{x,\mu}_{j,s} = X^{x,\mu}_{j,s\wedge\eta_j}.$$

Lemma 1.3. *Assume all conditions of Theorem 1.1. Then if* $0 \leq \theta_i \leq \frac{t}{2}, i = 1, 2,$ *and* $\mathcal{K}_0$ *is a compact subset of* $\{(t,x) : V_{t-s}(z^t_{j_0,s}(x)) < 0, \theta_1 \leq s \leq t - \theta_1\}$

$$\overline{\lim_{\mu\to 0}}\, \mu^2 \sup_{\theta_1 \leq s \leq t-\theta_2} \log u^\mu_{t-s}(\tilde{X}^{x,\mu}_{j_0,s}) < 0, \tag{1.14}$$

in probability, uniformly in $(t,x) \in \mathcal{K}_0$.

Proof. As $\mu \to 0$, $\tilde{X}^{x,\mu}_{j_0,s}$ converges to $z^t_{j_0,s}(x)$ in probability uniformly in $0 \leq s \leq t$ uniformly for (t,x) on compact subsets of D by the fact that $\hat{P}(t < \eta_j(X^{x,\mu}_{j,\cdot})) \to 1$ as $\mu \to 0$ (which will be true by l.s.c. of $(x,\mu) \mapsto \eta_j(X^{x,\mu}_{j,\cdot})$ and compactness e.g. see Elworthy (1982), P138 and P214). Now V_{t-s} is continuous, so

$$V_{t-s}(\tilde{X}^{x,\mu}_{j_0,s}) \to V_{t-s}(z^t_{j_0,s}(x))$$

in probability. Therefore, for some $\delta^* > 0$, for any $\epsilon > 0$, there exists $\mu_0(\epsilon) > 0$ such that for $0 < \mu \leq \mu_0$,

$$\hat{P}\{V_{t-s}(\tilde{X}^{x,\mu}_{j_0,s}) \leq -\frac{1}{2}\delta^*, \theta_1 \leq s \leq t - \theta_2\} > 1 - \epsilon,$$

all $(t,x) \in \mathcal{K}_0$. From (1.13) it is easy to see that for some $\mu_1 > 0$, if $\mu \le \mu_1$, then

$$\hat{P}\left\{u^{\mu}_{t-s}(\tilde{X}^{x,\mu}_{j_0,s}) \le e^{-\frac{\delta^*}{4\mu^2}}\right\} > 1-\epsilon,$$

for all $(t,x) \in \mathcal{K}_0$. That proves the lemma. ‡‡

Now consider condition

(I'). $c(x,u) \le c(x,0) = \bar{c}(x) \le \hat{c}$, where $\bar{c}$ is continuous, $\hat{c}$ is a constant.

Also define $Q(t,x) = \{j \in \{1,2,\cdots,N\} : V_{j,t}(x) = V_t(x)\}$ and consider condition

(N^*). If $V_t(x) < 0$ assume that for all $j \in Q(t,x)$

$$V_{t-s}(z^t_{j,s}(x)) < 0, \quad 0 \le s \le t,$$

where $z^t_{j,s}$ is defined by (1.9).

Theorem 1.2. *Assume the conditions (I'), (N^*), and the conditions of Theorem 1.1, and $T_{j,0}$ is continuous. Then, for sufficiently small $\mu \ne 0$,*

$$u^{\mu}_t(x) = \sum_{j \in Q(t,x)} [\sqrt{\phi_{j,t}(x)}T_{j,0}(\Phi^{-1}_{j,t}(x))e^{\int_0^t <\nabla V_{j,t-s}(z^{x,\mu}_{j,s}),b(z^{x,\mu}_{j,s})>ds} + o(1)]e^{\frac{1}{\mu^2}V_t(x)}$$

uniformly for (t,x) in any compact subset of $\{(t,x) : V_t(x) < 0\}$. Here

$$\phi_j(s,x) = |\det T_x \Phi^{-1}_{j,s}(x)|.$$

Proof. For $\mathcal{K}_0$ compact in $\{(t,x) : V_t(x) < 0\}$ define η_j, $X^{x,\mu}_j$, $\tilde{X}^{x,\mu}_j$ as above for $j = 1,2,\cdots,N$. Then, as in the proof of Lemma 1.3, $\tilde{X}^{x,\mu}_{j,s}$ converges to $z^t_{j,s}(x)$ in probability, uniformly in $s \in [0,t]$, uniformly on $\mathcal{K}_0$. From the definition of (N^*) we know that for $j \in Q$, $V_{t-s}(z^t_{j,s}(x)) < 0$ for $0 \le s \le t$. Note that $V_{t-s}(z^t_{j,s}(x))$ are continuous functions of s by the definitions of V, (1.6) and (1.9) for $z^t_{j,s}(x)$. As $c(x,u)$ is $C^{1,1}$ in (x,u), for $j \in Q$, and $(t,x) \in \mathcal{K}_0$

$$\int_0^t [c(\tilde{X}^{x,\mu}_{j,s}, u^{\mu}_{t-s}(\tilde{X}^{x,\mu}_{j,s})) - \bar{c}(\tilde{X}^{x,\mu}_{j,s})]ds \le constant \cdot \int_0^t |u^{\mu}_{t-s}(\tilde{X}^{x,\mu}_{j,s})|ds. \ a.s. \quad (1.15)$$

So applying Lemma 1.3 and Lebesgue's dominated convergence theorem to (1.13), noting $\hat{P}\{t < \eta_j(X^{x,\mu}_{j,\cdot})\} \to 1$ as $\mu \to 0$, we have for $j \in Q$, and $(t,x) \in \mathcal{K}_0$

$$\begin{aligned}&\hat{E}\chi_{t<\eta_j(X^{x,\mu}_{j,\cdot})}T_{j,0}(X^{x,\mu}_{j,t})e^{\frac{1}{2}\int_0^t \Delta V_{j,t-s}(X^{x,\mu}_s)ds} \\ &\cdot e^{\frac{1}{\mu^2}\int_0^t\left[c(X^{x,\mu}_{j,s},u^{\mu}_{t-s}(X^{x,\mu}_{j,s}))-\bar{c}(X^{x,\mu}_{j,s})\right]ds+\int_0^t<\nabla V_{j,t-s}(X^{x,\mu}_{j,s}),b(X^{x,\mu}_{j,s})>ds} \\ =&T_{j,0}(z^t_{j,t})e^{\frac{1}{2}\int_0^t \Delta V_{j,t-s}(z^t_{j,s}(x))ds+\int_0^t<\nabla V_{j,t-s}(z^{x,\mu}_{j,s}),b(z^{x,\mu}_{j,s})>ds} + o(1).\end{aligned} \quad (1.16)$$

From the definition of $\phi_j(s,x)$ Elworthy and Truman (1981) noted that $\phi_j(s,x)$ satisfies the continuity equation, i.e.

$$\frac{\partial \phi_{j,s}}{\partial s} - div(\phi_{j,s} \nabla V_{j,s}) = 0, \tag{1.17}$$

and

$$\Delta V_{j,t} = \frac{\partial}{\partial t} \log \phi_{j,t} - < \nabla \log \phi_{j,t}, \nabla V_{j,t} > . \tag{1.18}$$

In particular we have

$$\Delta V_{j,t-s}(z^t_{j,s}(x)) = -\frac{\partial}{\partial s} \log \phi_{j,t-s}(z^t_{j,s}(x)), \quad (t,x) \in D. \tag{1.19}$$

By (1.12), (1.16) and (1.19), noting that for $i \notin Q(t,x)$,

$$\begin{aligned} & e^{\frac{1}{\mu^2} V_{i,t}(x)} \hat{E} \chi_{t<\eta_i(X^{x,\mu}_{i,\cdot})} T_{i,0}(X^{x,\mu}_{i,t}) e^{\frac{1}{2}\int_0^t \Delta V_{i,t-s}(X^{x,\mu}_{i,s})ds} \\ & \cdot e^{\frac{1}{\mu^2}\int_0^t \left[c(X^{x,\mu}_{i,s}, u^{\mu}_{t-s}(X^{x,\mu}_{i,s})) - \bar{c}(X^{x,\mu}_{i,s})\right]ds - \int_0^t <\nabla V_{i,t-s}(X^{x,\mu}_{i,s}), b(X^{x,\mu}_{i,s})>ds} \\ \le & \text{constant} \cdot e^{\frac{1}{\mu^2} V_t(x)} \cdot e^{\frac{1}{\mu^2}[V_{i,t}(x) - V_t(x)]} = \circ(1) \cdot e^{\frac{1}{\mu^2} V_t(x)}, \end{aligned}$$

we get the proof of the theorem. ‡‡

To obtain more detailed results we use a method of K. D. Watling. For simplicity we let $b \equiv 0$. For the more general situation, to obtain complete expansions see Watling (1992). For $(t,x) \in D_j$ define $\psi_j : D_j \to R$ by

$$\psi_j(t,x) = T_{j,0}(\Phi^{-1}_{j,t}(x)) \sqrt{\phi_{j,t}(x)}. \tag{1.20}$$

Then if $T_{j,0}$ are positive and C^1 we have

$$\frac{\partial}{\partial t} \log \psi_{j,t} = \frac{1}{2} \Delta V_{j,t} + < \nabla \log \psi_{j,t}, \nabla V_{j,t} > . \tag{1.21}$$

By (1.21), if we take $Y_j(s,x) = V_j(t-s,x) + \mu^2 \log \psi_j(t-s,x), \ \ 0 \le s \le t$, Proposition 1.1, together with some computations using (1.4), (1.7) and (1.21), and the same argument as (1.12), implies the following proposition:

Proposition 1.2. *Assume that $\bar{c}$ and $S_{j,0}$ are C^3 and $T_{j,0}$ are positive and C^2 with the conditions of Theorem 1.1. Then for sufficiently small $\mu \neq 0$ and (t,x) in a compact subset $\mathcal{K}_0 \subset D$, we have*

$$\begin{aligned} u^{\mu}_t(x) = \sum_{j=1}^{N} e^{\frac{1}{\mu^2} V_{j,t}(x)} \Big[& \psi_{j,t}(x) \cdot \hat{E} \chi_{t<\eta_j(X^{x,\mu}_{j,\cdot})} e^{\frac{1}{2}\mu^2 \int_0^t \psi^{-1}_{j,t-s}(X^{x,\mu}_{j,s}) \Delta \psi_{j,t-s}(X^{x,\mu}_{j,s})ds} \\ & \cdot e^{\frac{1}{\mu^2}\int_0^t \left[c(X^{x,\mu}_{j,s}, u^{\mu}_{t-s}(X^{x,\mu}_{j,s})) - \bar{c}(X^{x,\mu}_{j,s})\right]ds} + \circ(\mu^n) \Big], \end{aligned} \tag{1.22}$$

for any $n \geq 0$. *Here* $X_{j,s}^{x,\mu}$, $0 \leq s \leq \eta_j(X_{j,\cdot}^{x,\mu})$ *are defined by (1.2) up to the exit time* $\eta_j(X_{j,\cdot}^{x,\mu})$ *from* $\mathcal{N}_{j,\mathcal{K}_0}$ *defined as in the proof of Theorem 1.1 with*

$$A_{j,s} = \nabla V_{j,t-s} + \mu^2 \nabla \log \psi_{j,t-s}. \tag{1.23}$$

As in Lemma 1.3, we can prove:

Lemma 1.4. *Assume condition (I) and the hypotheses of Proposition 1.2. Let* $\mathcal{K}_0$ *be a compact subset of* $\{(t,x) : V_{t-s}(z_{j_0,s}(x)) < 0, 0 \leq s \leq t\}$ *and* $X_{j_0,s}^{x,\mu}$ *is defined by (1.2) with* $A_{j_0,s} = \nabla V_{j_0,t-s} + \mu^2 \nabla \log \psi_{j_0,t-s}$, *and stopped at* η_j *to give* $\tilde{X}_{j,s}^{x,\mu}$ *as before, then*

$$\overline{\lim_{\mu \to 0}} \mu^2 \sup_{0 \leq s \leq t} \log u_{t-s}^{\mu}(\tilde{X}_{j_0,s}^{x,\mu}) < 0,$$

in probability, uniformly on $\mathcal{K}_0$.

Theorem 1.3. *Assume conditions (I') and (N*) and all the conditions of Proposition 1.2. Then for sufficiently small* $\mu \neq 0$,

$$u_t^{\mu}(x) = e^{\frac{1}{\mu^2} V_t(x)} \sum_{j \in Q(t,x)} \psi_{j,t}(x)[1 + \frac{1}{2}\mu^2 \int_0^t \psi_{j,t-s}^{-1}(z_{j,s}^t(x)) \Delta \psi_{j,t-s}(z_{j,s}^t(x)) ds + \circ(\mu^2)], \tag{1.24}$$

uniformly in any compact subset of $\{(t,x) : V_t(x) < 0\}$.

Proof. Let $j \in Q(t,x)$. As in the proof of Lemma 1.3, as $\mu \to 0$, $\tilde{X}_{j,s}^{x,\mu}$, defined by (1.2) and (1.23), converges to $X_{j,s}^{x,0} = z_{j,s}^t(x)$ in probability, uniformly in $s \in [0,t]$. Applying Lebesgue's dominated convergence theorem to (1.22), as in the proof of Theorem 1.2 but now using Lemma 1.4, noting $P(t < \eta_j(X_{j,\cdot}^{x,\mu})) \to 1$ as $\mu \to 0$,

$$\begin{aligned}
&\hat{E}\chi_{t<\eta_j(X_{j,\cdot}^{x,\mu})} e^{\frac{1}{2}\mu^2 \int_0^t \psi_{j,t-s}^{-1}(X_{j,s}^{x,\mu}) \Delta \psi_{j,t-s}(X_{j,s}^{x,\mu}) ds + \frac{1}{\mu^2} \int_0^t \left[c(X_{j,s}^{x,\mu}, u_{t-s}^{\mu}(X_{j,s}^{x,\mu})) - \bar{c}(X_{j,s}^{x,\mu})\right] ds} \\
=&1 + \frac{1}{2}\mu^2 \int_0^t \psi_{j,t-s}^{-1}(z_{j,s}^t(x)) \Delta \psi_{j,t-s}(z_{j,t-s}^t(x)) ds + \circ(\mu^2).
\end{aligned}$$

This, together with (1.22), implies (1.24) if we note for any $k > 0$ and $i \notin Q(t,x)$,

$$\begin{aligned}
&\frac{1}{\mu^k} e^{\frac{1}{\mu^2} V_{i,t}(x)} \hat{E}\chi_{t<\eta_j(X_{j,\cdot}^{x,\mu})} e^{\frac{1}{2}\mu^2 \int_0^t \psi_{i,t-s}^{-1}(X_{i,s}^{x,\mu}) \Delta \psi_{i,t-s}(X_{i,s}^{x,\mu}) ds} \\
&\cdot e^{\frac{1}{\mu^2} \int_0^t \left[c(X_{i,s}^{x,\mu}, u_{t-s}^{\mu}(X_{i,s}^{x,\mu})) - \bar{c}(X_{i,s}^{x,\mu})\right] ds} \\
=& \circ(1) \cdot e^{\frac{1}{\mu^2} V_t(x)}.
\end{aligned}$$

‡‡

E. For the crest of the wave we consider the Cauchy problem (1.1) with positive $T_{j,0}$. Consider condition (I') and

(II). $c(x,u) > 0$ *for* $0 \leq u < 1$ *and* $c(x,u) < 0$ *for* $u > 1$,

and

(N^{**}). *If* $V_t(x) = 0$ *there exists* $j_0 \in Q(t,x)$ *with*

$$V_{t-s}(z^t_{j,s}(x)) < 0, \quad 0 < s < t.$$

Let $Z = \{(t,x) : V_t(x) = 0\}$. Recall our convention that $V_t(x) = \infty$ if $(t,x) \notin D$. Note that by the Hamilton-Jacobi equation (1.7) if $\bar{c}(x) > 0$ then $V_t(x)$ is strictly increasing in t, so for each x with $\bar{c}(x) > 0$, there is at most one $Z(x)$ with $(Z(x), x) \in Z$.

Lemma 1.5. *Assume conditions (I'), (II) , (N^{**}) and that $\bar{c}$ and $S_{j,0}$ are C^2 with $S_{j,0}$ are nonnegative, and that $T_{j,0}$ are positive and continuous. Then*

$$\varliminf_{\mu \to 0} \mu^2 \log u^\mu_t(x) \geq 0, \tag{1.25}$$

uniformly in any compact subset $\mathcal{K}_0$ of Z with $K_0 \in IntD$.

Proof. Take tubes $\mathcal{N}_{j,\mathcal{K}_0}$ in $[0,\infty) \times M$ as described above. Let $\eta_j(\sigma)$ be the first exit time of the path $(s, \sigma(s))$ from $\mathcal{N}_{j,\mathcal{K}_0}$. Let $X^{x,\mu}_{j,s}$ be defined by (1.2) with $A_{j,s} = \nabla V_{j,t-s}$ for $0 \leq s < \eta(X^{x,\mu}_{j,\cdot})$. Take $(t,x) \in \mathcal{K}_0$, and let j_0 be as given by (N^{**}). By (1.12)

$$\begin{aligned}
&u^\mu_t(x) \\
&\geq \sum_{j=1}^N e^{\frac{1}{\mu^2} V_{j,t}(x)} \hat{E} \chi_{t < \eta_j(X^{t,x}_{j,s})} T_{j,0}(X^{x,\mu}_{j,t}) e^{\frac{1}{2}\int_0^t \Delta V_{j,t-s}(X^{x,\mu}_{j,s}) ds} \\
&\quad \cdot e^{\{\frac{1}{\mu^2}\int_0^t [c(X^{x,\mu}_{j,s}, u^\mu_{t-s}(X^{x,\mu}_{j,s})) - \bar{c}(X^{x,\mu}_{j,s})] ds + \int_0^t < \nabla V_{j,t-s}(X^{x,\mu}_{j,s}), b(X^{x,\mu}_{j,s}) > ds\}} \\
&\geq \hat{E} \chi_{t < \eta_j(X^{t,x}_{j_0,s})} T_{j_0,0}(X^{x,\mu}_{j_0,t}) e^{\frac{1}{2}\int_0^t \Delta V_{j_0,t-s}(X^{x,\mu}_{j_0,s}) ds} \\
&\quad \cdot e^{\{\frac{1}{\mu^2}\int_0^t [c(X^{x,\mu}_{j_0,s}, u^\mu_{t-s}(X^{x,\mu}_{j_0,s})) - \bar{c}(X^{x,\mu}_{j_0,s})] ds + \int_0^t < \nabla V_{j_0,t-s}(X^{x,\mu}_{j_0,s}), b(X^{x,\mu}_{j_0,s}) > ds\}}.
\end{aligned}$$

Neglecting to write the suffix j_0, so $\eta = \eta_{j_0}$, $X^{x,\mu}_s = X^{x,\mu}_{j_0,s}$, etc. By Jensen's inequality, we get

$$\begin{aligned}
&\mu^2 \log u_t^\mu(x) \\
\geq &\mu^2 \log \hat{P}(t < \eta(X_\cdot^{x,\mu})) + \mu^2 \log \inf_{y:(s,y)\in\mathcal{N}_{\mathcal{K}_0}} T_0(y) \\
&+ \mu^2 t \inf_{(s,y)\in\mathcal{N}_{\mathcal{K}_0}} \{\frac{1}{2}\Delta V_s(y) + < \nabla V_s(y), b(y) >\} \\
&+ \hat{P}(\chi_{t<\eta(X_\cdot^{x,\mu})})^{-1} I,
\end{aligned} \tag{1.26}$$

where

$$I = \hat{E}_x^\mu \chi_{t<\eta(X_\cdot^{x,\mu})} \int_0^t [c(X_s^{x,\mu}, u_{t-s}(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})] ds.$$

Recall $\hat{P}(\chi_{t<\eta(X_\cdot^{x,\mu})}) \to 1$ uniformly on $\mathcal{K}_0$ as noted in the proof of Lemma 1.3. To estimate I we proceed as Zhao & Elworthy (1992): for $\delta > 0$, choose $0 < \theta \leq \frac{1}{4}t$ such that

$$8\theta\hat{c} \leq \delta.$$

By condition (N^{**}) there exists $\delta^* > 0$ such that

$$V_{t-s}(z_s^t(x)) < -\delta^* \text{ for } \theta \leq s \leq t - \theta.$$

By Lemma 1.3, $u_{t-s}^\mu(\tilde{X}_s^{x,\mu}) \to 0$, for $\theta \leq s \leq t-\theta$ uniformly in $\mathcal{K}_0$. Therefore applying Lebesgue's dominated convergence theorem, we have as $\mu \to 0$

$$\hat{E}_x^\mu \chi_{t<\eta(X_\cdot^{x,\mu})} \int_\theta^{t-\theta} [c(X_s^{x,\mu}, u_{t-s}(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})] ds \to 0$$

uniformly in $\mathcal{K}_0$. From

$$\begin{aligned}
&\hat{E}_x^\mu \chi_{t<\eta(X_\cdot^{x,\mu})} \int_0^t [c(X_s^{x,\mu}, u_{t-s}(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})] ds \\
=&\hat{E}_x^\mu \chi_{t<\eta(X_\cdot^{x,\mu})} \int_0^\theta [c(X_s^{x,\mu}, u_{t-s}(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})] ds \\
&+ \hat{E}_x^\mu \chi_{t<\eta(X_\cdot^{x,\mu})} \int_\theta^{t-\theta} [c(X_s^{x,\mu}, u_{t-s}(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})] ds \\
&+ \hat{E}_x^\mu \chi_{t<\eta(X_\cdot^{x,\mu})} \int_{t-\theta}^t [c(X_s^{x,\mu}, u_{t-s}(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})] ds
\end{aligned}$$

and (1.26) we know there exists $\mu_0(\delta) > 0$ such that if $0 < \mu \leq \mu_0$ then

$$\mu^2 \log u_t^\mu(x) \geq -\delta.$$

From this we get our lemma. ‡‡

Remark 1.2. *The assumption that $\bar{c}$ is bounded (which is part of (I')) is not needed if we have $D_t = M$, by Remark 1.1.*

Our basic "late caustic" assumption is essentially that caustics occur only after the wave front has passed: this can be formalized in the cases we are interested in as:

(DZ). *(i). If $(s,y) \in \partial(IntD)$, there is an open neighbourhood U_0 of y and $\epsilon > 0$ with $V_t(x) > 0$ on $IntD \cap (s-\epsilon, s] \times U_0$*
and
(ii). $\{0\} \times M \subset IntD$.

Here $(s,y) \in \partial(IntD)$ if every neighbourhood of (s,y) contains a point not in D, and a point which is in $IntD$.

Lemma 1.6. *Suppose $S_{j,0}(y) \geq 0$ for all j, and $\bar{c}(y) \geq 0$ for all $y \in M$, and assumption (DZ) holds. Then if $(t,x) \notin D$ or $(t,x) \in D$ with $V_t(x) > 0$, there exists $s \in [0,t]$ with $V_{t-s}(Y_s^{x,\mu}) = 0$. Here Y is any continuous process from x.*

Proof. Note $V(0, Y_t^{x,\mu}) = -\min_{1 \leq j \leq N} S_{j,0}(Y_t^{x,\mu}) \leq 0$ and $s \to V_{t-s}(Y_s^{x,\mu})$ is continuous on $(t-r, t]$ where

$$r = \inf\{\alpha : (\alpha, Y_{t-\alpha}^{x,\mu}) \notin IntD\}.$$

Since $(r, Y_{t-r}^{x,\mu}) \in \partial D$ and $(r-\epsilon, Y_{t-r+\epsilon}^{x,\mu}) \in IntD$ for $0 < \epsilon \leq r$ we can use assumption DZ(i) to conclude $V_{r-\epsilon}(Y_{t-r+\epsilon}^{x,\mu}) > 0$ for some $\epsilon > 0$. The result follows by continuity. ‡‡

Theorem 1.4. *Suppose that the process $B_s^{x,\mu}$ is non-explosive. Assume conditions (I'), (II), (N^{**}), (DZ), and that $\bar{c}$ and $S_{j,0}$ are C^2 with $S_{j,0}$ nonnegative, and that the $T_{j,0}$ are positive and bounded continuous. Then*

$$\lim_{\mu \to 0} u_t^{\mu}(x) = 1, \tag{1.27}$$

uniformly on any compacta in $\{(t,x) : V_t(x) > 0\} = \{(t,x) : t > Z(x)\}$.

Remark 1.3. *The non-explosion assumption can be replaced by the assumption that for any $t > 0$ as $x \to \infty$ in M $\underline{\lim} \sup_{0 \leq s \leq t} V_s(x) < 0$.*

Proof. Let $\mathcal{K}$ be a compact subset of $\{(t,x) : V_t(x) > 0\}$ and let λ be a small positive number. Define τ_1 as:

$$\tau_1 = \tau_1^{\mu,\lambda} = \inf\{s : u(t-s, B_s^{x,\mu}) \geq 1 - \lambda\},$$

with convention that $\tau_1 = \infty$ if the set is empty. Define

$$\tau_2 = \tau_2^\mu = \inf\{s : V(t-s, B_s^{x,\mu}) = 0\},$$

with convention that $V(r,y) = \infty$ if $(r,y) \notin D$. Note that τ_2 is defined and in $[0,t]$ if $t > Z(x)$ by Lemma 1.6. Set $\tau = \tau_1 \wedge \tau_2$. So we have

$$\begin{aligned} u^\mu(t,x) =& \hat{E}u^\mu(t-\tau, B_\tau^{x,\mu}) e^{\frac{1}{\mu^2}\int_0^\tau c(B_s^{x,\mu}, u^\mu(t-s, B_s^{x,\mu}))ds} \\ =& \hat{E}u^\mu(t-\tau_1, B_{\tau_1}^{x,\mu}) e^{\frac{1}{\mu^2}\int_0^{\tau_1} c(B_s^{x,\mu}, u^\mu(t-s, B_s^{x,\mu}))ds} \chi_{\tau_1 \le \tau_2} \\ &+ \hat{E}u^\mu(t-\tau_2, B_{\tau_2}^{x,\mu}) e^{\frac{1}{\mu^2}\int_0^{\tau_2} c(B_s^{x,\mu}, u^\mu(t-s, B_s^{x,\mu}))ds} \chi_{\tau_1 > \tau_2}. \end{aligned} \tag{1.28}$$

From the definition of τ_1 and condition (II) we know

$$\hat{E}u^\mu(t-\tau_1, B_{\tau_1}^{x,\mu}) e^{\frac{1}{\mu^2}\int_0^{\tau_1} c(B_s^{w,\mu}, u^\mu(t-s, B_s^{x,\mu}))ds} \chi_{\tau_1 \le \tau_2} \ge (1-\lambda) P\{\tau_1 \le \tau_2\}. \tag{1.29}$$

Write $V_0 = V(t,x) > 0$. Take $h(\mathcal{K}) > 0$ by assumption (DZ) such that

$$\{(s,y) : t-h \le s \le t \text{ and } y \in \bar{U}_h(x)\} \cap Z = \emptyset, (t,x) \in \mathcal{K}.$$

Here $U_h(x)$ is a neighbourhood of x with radius h. Let

$$\alpha(\mathcal{K}, \lambda) = \inf_{|y-x| \le h, 0 \le u \le 1-\lambda, (t,x) \in \mathcal{K}} h \cdot c(y,u).$$

It is evident that $\alpha > 0$. Let us select $\delta \in (0, \frac{1}{2}\alpha)$. From the definition of τ_2, we have

$$(t-\tau_2, B_{\tau_2}^{x,\mu}) \in Z \qquad a.s.$$

Therefore for any $\beta \in (0,1)$, there is a compact subset $\mathcal{K}_0$ in Z with $\hat{P}(\hat{\Omega}_0) > 1 - \frac{1}{2}\beta$ for

$$\hat{\Omega}_0 = \{\hat{\omega} \in \hat{\Omega} : (t-\tau_2, B_{\tau_2}^{x,\mu}) \in \mathcal{K}_0\}.$$

From Lemma 1.5 there exists $\mu_1(\mathcal{K}, \lambda, \beta) > 0$ such that for $0 < \mu < \mu_1$,

$$u^\mu(t-\tau_2, B_{\tau_2}^{x,\mu}) > e^{-\frac{\delta}{\mu^2}}, \qquad \text{on } \Omega_0.$$

Let

$$\tau_3 = \tau_3^\mu = \inf\{s : |B_s^{x,\mu} - x| = h\}.$$

It is clear that $\hat{P}\{\tau_3 < b\} \to 0$ as $\mu \downarrow 0$ for any $b > 0$. So there exists $\mu_2(\mathcal{K}) > 0$ such that for $0 < \mu < \mu_2$, $\hat{P}\{\tau_3 \le \tau_2\} \le \hat{P}\{\tau_3 \le t\} < \frac{1}{2}\beta$. Take $\mu_0 = \mu_1 \wedge \mu_2$. Then from the positivity of $c(x,u)$ for $0 \le u < 1$, for $0 < \mu < \mu_0$,

$$
\begin{aligned}
&\hat{E}\chi_{\tau_1 > \tau_2} u^\mu(t-\tau_2, B^{x,\mu}_{t-\tau_2}) e^{\frac{1}{\mu^2}\int_0^{\tau_2} c(B^{x,\mu}_s, u^\mu(t-s, B^{x,\mu}_s))ds} \\
\ge & e^{\frac{1}{\mu^2}(\alpha-\delta)} \hat{E}\chi_{\Omega_0}\chi_{\tau=\tau_2<\tau_3} \\
\ge & e^{\frac{\alpha}{2\mu^2}} \hat{E}\chi_{\Omega_0}\chi_{\tau=\tau_2<\tau_3} \\
\ge & (1-\lambda)\hat{P}\{\tau=\tau_2\} - \frac{1}{2}\beta - \hat{P}\{\tau_3 \le \tau_2\} \\
\ge & (1-\lambda)\hat{P}\{\tau=\tau_2\} - \beta.
\end{aligned}
\tag{1.30}
$$

By (1.28), (1.29), (1.30), $u^\mu(t,x) \ge 1-\lambda-\beta$. Thus $\varliminf\limits_{\mu\to 0} u^\mu_t(x) \ge 1$ uniformly on $\mathcal{K}$. By the same argument as Freidlin (1985) and as Lemma 2.1 in Zhao & Elworthy (1992) we can prove $\varlimsup\limits_{\mu\to 0} u^\mu_t(x) \le 1$. For completeness we include the proof given by Zhao and Elworthy (1992). It is easy to prove that under the condition (II), we have

$$u^\mu(t,x) \le \max\{1, ||u^\mu_0||\}.$$

Note $||u^\mu_0|| \le \sum\limits_{j=1}^N ||T_{j,0}||$. Now for any $\lambda > 0$ and $h > 0$, we set

$$\alpha = \inf\{-c(y,u) : ||u^\mu_0|| + 2 \ge u \ge 1+\lambda, y \in R^r, |y-x| \le h\}.$$

We introduce a Markov time

$$\tilde{\tau} = \tilde{\tau}^\mu = \inf\{s : u^\mu(t-s, w^{x,\mu}_s) \le 1+\lambda\}$$

with $\tilde{\tau} = \infty$ if $u^\mu(t-s, w^{x,\mu}_s) > 1+\lambda$ for $0 \le s \le t$. Then for μ small enough, we have

$$
\begin{aligned}
u^\mu(t,x) =& Eu^\mu(t-\tilde{\tau}\wedge t, w^{x,\mu}_{\tilde{\tau}\wedge t}) e^{\frac{1}{\mu^2}\int_0^{\tilde{\tau}\wedge t} c(w^{x,\mu}_s, u^\mu(t-s, w^{x,\mu}_s))ds} \\
=& Eu^\mu(t-\tilde{\tau}, w^{x,\mu}_{\tilde{\tau}}) e^{\frac{1}{\mu^2}\int_0^{\tilde{\tau}} c(w^{x,\mu}_s, u^\mu(t-s, w^{x,\mu}_s))ds} \chi_{\tilde{\tau}\le t} \\
&+ Eu^\mu_0(w^{x,\mu}_t) e^{\frac{1}{\mu^2}\int_0^{t} c(w^{x,\mu}_s, u^\mu(t-s, w^{x,\mu}_s))ds} \chi_{\tilde{\tau}> t} \\
\le & (1+\lambda)P\{\tilde{\tau} \le t\} + ||u^\mu_0|| e^{-\frac{1}{\mu^2}\alpha t} P\{\tilde{\tau} > t\} \\
&+ ||u^\mu_0|| P\{\inf\{s : |w^{x,\mu}_s - x| = h\} < t\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1+\lambda)[P\{\tilde{\tau} \leq t\} + P\{\tilde{\tau} > t\}] \\
&\quad + ||u_0^\mu|| P\{\inf\{s : |w_s^{x,\mu} - x| = h\} < t\} \\
&\leq 1 + \lambda + \sum_{j=1}^{N} ||T_{j,0}|| P\{\inf\{s : |w_s^{x,\mu} - x| = h\} < t\}.
\end{aligned}
$$

Note that $P\{\inf\{s : |w_s^{x,\mu} - x| = h\} < t\}$ is small when μ is small. This implies $\overline{\lim_{\mu\to 0}} u_t^\mu(x) \leq 1$. Finally we have the theorem. ‡‡

Remark 1.4. *For the simpler proof of this theorem with the no-caustic condition see Zhao & Elworthy (1992).*

F. For the KPP equation with initial condition $u_0^\mu(x) = e^{-\frac{x^2}{2\mu^2}}$, in Zhao & Elworthy (1992) we showed the wave front was given by $x^2 = ct(1+2t)$. A numerical simulation by J. Gaines is given in Figure I1. We can write down the wave front formula for the KPP equation with a superposition of two Gaussian distributions $u_0^\mu(x) = e^{-\frac{x^2}{2\mu^2}} + e^{-\frac{(x-1)^2}{2\mu^2}}$: a numerical simulation is shown in Figure I2.

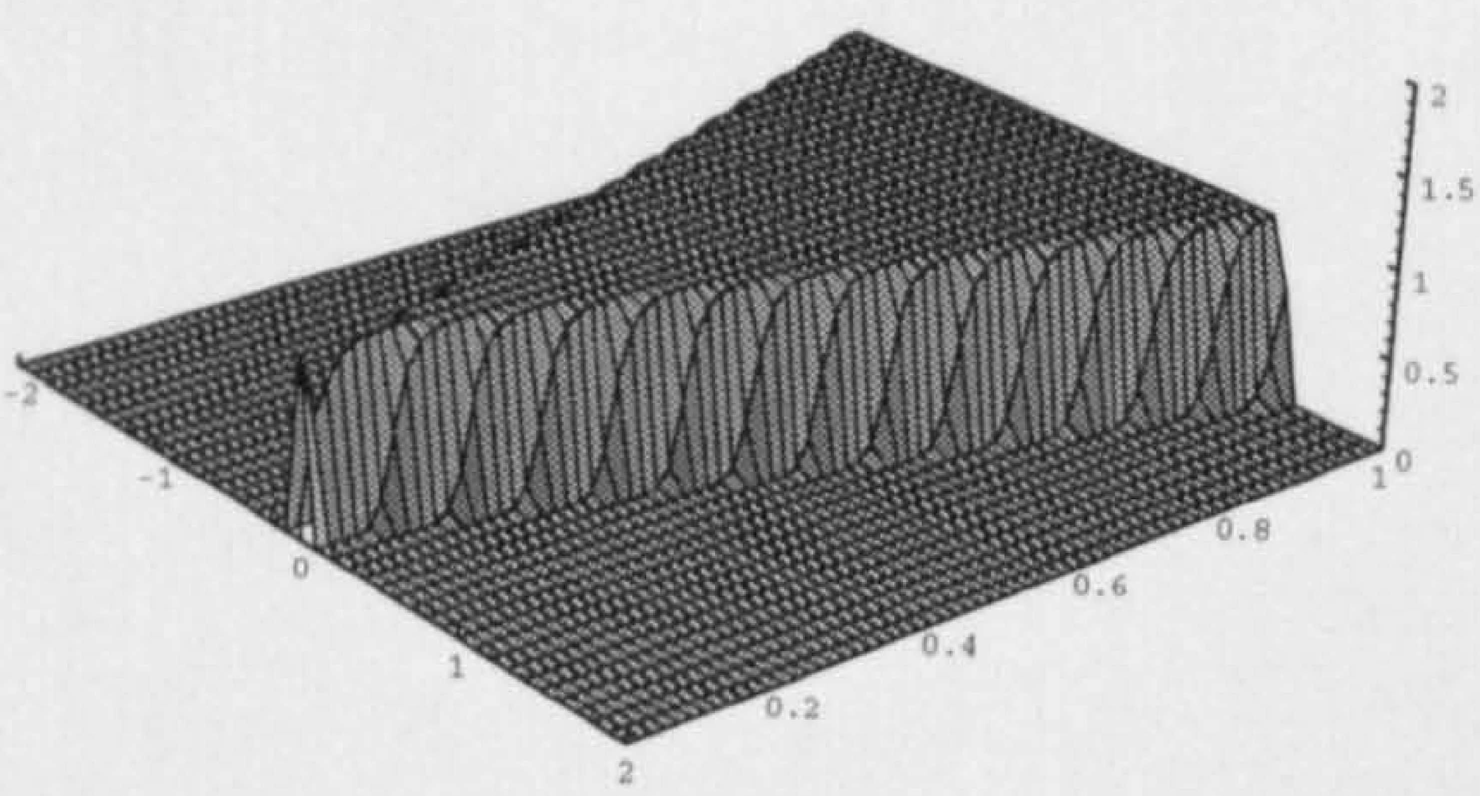

Figure I1

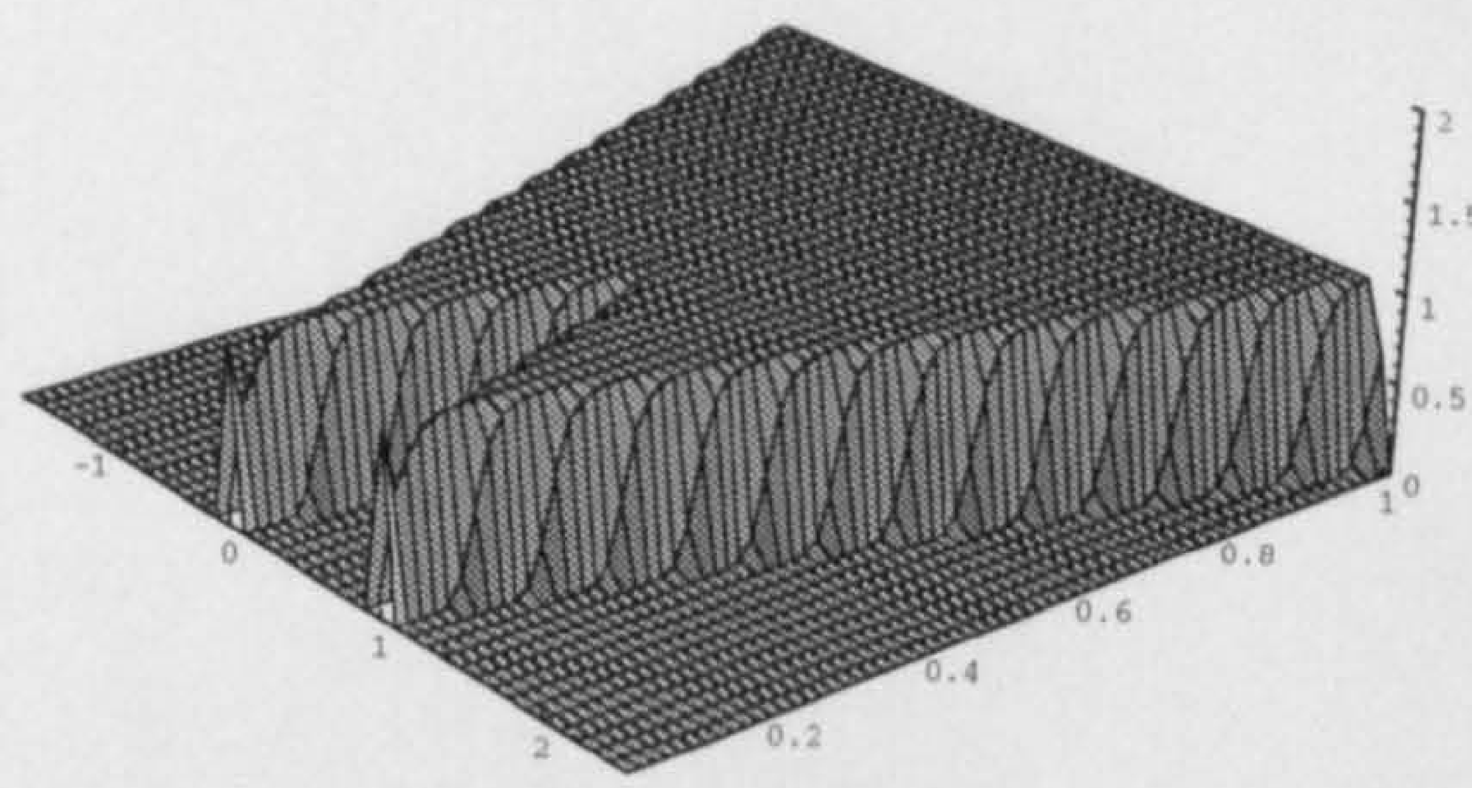

Figure I2

G. We can generalise the results of this section. Consider the equation with time dependent $c(t, x, u)$ impose a similar condition (I') on c and instead of (II) assume:

(II'). There exist bounded and continuous functions $\overline{H}_{t,x} \geq \underline{H}_{t,x} > 0$ for $0 \leq t \leq T$ for some $T > 0$ such that $c(t, x, u) > 0$ for $0 \leq u < \underline{H}_{t,x}$ and $c(t, x, u) < 0$ for $u > \overline{H}_{t,x}$

for $0 \le t \le T$, $x \in M$.

Note that Hamilton Jacobi theory is still true for the time dependent potential $\bar{c}(t,x)$ (Truman and Zhao [2]). Similarly we define $V_t(x)$ as before and the late caustic assumption and other conditions. It is quite obvious that Theorems 1.1-1.3 and Lemmas 1.5-1.6 are still true. The generalisation of Theorem 1.4 is given as follows:

Theorem 1.5. *Consider (1.1) with time dependent* $c(t,x,u)$ *and assume (I'), (II'),* (N^{**})*, (DZ) and that c is* $C^{1,2}$ *and* $S_{j,0}$ *are* C^2 *and nonnegative and* $T_{j,0}$ *are positive and bounded continuous. Suppose the process* $B_s^{x,\mu}$ *is non-explosive. Then*

$$\underline{H}_{t,x} \le \varliminf_{\mu\to 0} u_t^\mu(x) \le \varlimsup_{\mu\to 0} u_t^\mu(x) \le \overline{H}_{t,x}, \tag{1.31}$$

uniformly on any compact set in $\{(t,x) \in (0,T] \times M : V_t(x) > 0\}$.

Proof. The proof is similar to the proof of Theorem 1.4, but we need more analysis to deal with $\underline{H}_{t,x}$ and $\overline{H}_{t,x}$. Choose the same h as in the previous proof. Let $\mathcal{K}$ be a compact subset in $\{(t,x) \in (0,T] \times M : V_t(x) > 0\}$ and $\mathcal{K} \subset (0,\bar{t}] \times \mathcal{K}_2$ with a compact set $\mathcal{K}_2 \subset M$ and $(t,x) \in \mathcal{K}$. By the continuity and boundedness of $\underline{H}_{t,x}$ and the fact that $B_s^{x,\mu} \to x$ for $0 \le s \le \bar{t}$ as $\mu \to 0$ in $\hat{P}$ probability, for any $\lambda \in (0, \inf_{(t,x)\in\mathcal{K}} \underline{H}_{t,x})$ there exist $\mu_1^{(1)}(\mathcal{K},\lambda) > 0$, $\vartheta(\mathcal{K},\lambda) > 0$, and a subset $\hat{\Omega}_1 \subset \hat{\Omega}$ with $\hat{P}(\hat{\Omega}_1) > 1 - \frac{\lambda}{6 \sup_{0\le t\le \bar{t}} \sup_{x\in M} \underline{H}_{t,x}}$ such that for $0 < \mu < \mu_0^{(1)}$

$$\underline{H}_{t,y} \ge \underline{H}_{t,x} - \frac{1}{6}\lambda, \text{ for all } (t,x) \in \mathcal{K}, |y-x| < \theta,$$

$$\underline{H}_{t-s,x} \ge \underline{H}_{t,x} - \frac{1}{6}\lambda, \text{ for all } 0 \le s \le \vartheta, (t,x) \in \mathcal{K},$$

and

$$d(B_s^{x,\mu}, x) < \vartheta \text{ for all } x \in \mathcal{K}_2, 0 \le s \le \bar{t}, \hat{\omega} \in \hat{\Omega}_1.$$

Therefore if $\hat{\omega} \in \hat{\Omega}_1$ and $0 < \mu < \mu_1^{(1)}$,

$$\underline{H}_{t,B_s^{x,\mu}(\hat{\omega})} \ge \underline{H}_{t,x} - \frac{1}{6}\lambda, \text{ for all } x \in \mathcal{K}_2, 0 \le s,t \le \bar{t}.$$

Let $\mathcal{O}(\mathcal{K}_2) = \{x \in M : d(x,\mathcal{K}_2) \le \vartheta\}$. Then $\mathcal{O}(\mathcal{K}_2)$ is compact in M. Therefore condition (II') implies that there exists $\varepsilon^* > 0$ such that

$$c(t,x,u) > \varepsilon^*, \text{ for } 0 \le t \le T, x \in \mathcal{O}(\mathcal{K}_2) \text{ and } 0 \le u \le \underline{H}_{t,x} - \frac{1}{6}\lambda.$$

We can take smaller $\mu_0^{(1)} > 0$ if necessary to deduce if $0 < \mu < \mu_0^{(1)}$

$$[\inf_{(t,x)\in\mathcal{K}} \underline{H}_{t,x} - \frac{1}{3}\lambda]\exp\{\frac{\varepsilon^*\vartheta}{\mu^2}\} > \underline{H}_{t,x} - \frac{1}{2}\lambda, \text{ all } (t,x) \in \mathcal{K},$$

Define

$$\tau_1 = \tau_1^{\mu,\lambda} = \inf\{s \in [0,t] : u^\mu(t-s, B_s^{x,\mu}) \geq \underline{H}_{t-s,B_s^{x,\mu}} - \frac{1}{6}\lambda\}$$

with convention $\tau_1 = \infty$ if the set is empty, and

$$\tau_2 = \tau_2^\mu = \inf\{s : V(t-s, B_s^{x,\mu}) = 0\}$$

with convention that $V(r,y) = \infty$ if $(r,y) \notin D$. Note $\tau_2 \leq t$ if $V_t(x) > 0$.

Instead of (1.29) we have

$$\begin{aligned}
&\hat{E}u^\mu_{t-\tau_1}(B^{x,\mu}_{\tau_1})\exp\{\frac{1}{\mu^2}\int_0^{\tau_1} c(t-s, B_s^{x,\mu}, u^\mu_{t-s}(B_s^{x,\mu}))ds\}\chi_{\tau_1\leq\tau_2}\\
\geq&\hat{E}(\underline{H}_{t-\tau_1,B^{x,\mu}_{\tau_1}} - \frac{1}{6}\lambda)\exp\{\frac{\varepsilon^*\tau_1}{\mu^2}\}\cdot\chi_{\tau_1\leq\tau_2}\chi_{\hat{\Omega}_1}\\
\geq&\hat{E}(\underline{H}_{t-\tau_1,x} - \frac{1}{3}\lambda)\exp\{\frac{\varepsilon^*\tau_1}{\mu^2}\}\cdot\chi_{\tau_1\leq\tau_2}[\chi_{\tau_1\leq\vartheta}+\chi_{\tau_1>\vartheta}] - \frac{1}{6}\lambda\\
\geq&(\underline{H}_{t,x} - \frac{1}{2}\lambda)\hat{E}\chi_{\tau_1\leq\tau_2}\chi_{\tau_1\leq\vartheta} + \inf_{(t,x)\in\mathcal{K}}(\underline{H}_{t,x} - \frac{1}{3}\lambda)\exp\{\frac{\varepsilon^*\vartheta}{\mu^2}\}\hat{P}\{\vartheta\leq\tau_1\leq\tau_2\} - \frac{1}{6}\lambda\\
\geq&[\underline{H}_{t,x} - \frac{1}{2}\lambda]\hat{P}\{\tau_1\leq\tau_2\} - \frac{1}{6}\lambda.
\end{aligned} \tag{1.32}$$

Similarly to (1.30) it is easy to see that for any $\beta > 0$

$$\begin{aligned}
&\hat{E}\chi_{\tau_1>\tau_2}u^\mu(t-\tau_2, B^{x,\mu}_{\tau_2})\exp\{\frac{1}{\mu^2}\int_0^{\tau_2} c(t-s, B_s^{x,\mu}, u^\mu_{t-s}(B_s^{x,\mu}))ds\}\\
\geq&(\underline{H}_{t,x} - \lambda)\hat{P}\{\tau = \tau_2\} - \beta.
\end{aligned} \tag{1.33}$$

Therefore we have $\underline{\lim}_{\mu\to 0} u_t^\mu(x) \geq \underline{H}_{t,x}$. Similarly amending the last part of the proof of the Theorem 1.4 we can prove $\overline{\lim}_{\mu\to 0} u_t^\mu(x) \leq \overline{H}_{t,x}$. The theorem follows. ‡‡

Remark 1.5. *A special case of Theorem 1.5 is when we can choose $\overline{H}_{t,x} = \underline{H}_{t,x} = H_{t,x}$ say. In this case $\lim_{\mu\to 0} u_t^\mu(x) = H_{t,x}$ for $V_t(x) > 0$.*

§2. Nonlinearity and Caustics, An Example

For simplicity we will let the configuration manifold be R^1 in this section. Consider

$$\begin{cases} \dfrac{\partial u_t^{\lambda,\mu}(x)}{\partial t} = \dfrac{\mu^2}{2}\Delta u_t^{\lambda,\mu}(x) + \dfrac{1}{2\mu^2}x^2(1-u_t^{\lambda,\mu}(x))u_t^{\lambda,\mu}(x), \\ u_0^{\lambda,\mu}(x) = \dfrac{1}{\sqrt{2\pi\mu^{2+k}\lambda}}e^{-\frac{1}{2\mu^{2+k}\lambda}x^2}. \end{cases} \tag{2.1}$$

Here $k \geq 0$ is a constant. By the semigroup argument we have the global existence of the solution. Therefore we have Feynman-Kac formula.

We construct our classical mechanics by

$$\Phi_t^{\lambda,\mu}(x) = \frac{x}{\lambda\mu^k}\sin s + x\cos s. \tag{2.2}$$

So for $t < \mathrm{Arctg}(-\lambda\mu^k)$,

$$(\Phi_t^{\lambda,\mu}(x))^{-1} = \frac{\lambda\mu^k x}{\sin t + \lambda\mu^k\cos t}. \tag{2.3}$$

From this we get for $t < \mathrm{Arctg}(-\lambda\mu^k)$,

$$V_t^{\lambda,\mu}(x) = \frac{(-1+\lambda^2\mu^{2k})\sin 2t - 2\lambda\mu^k\cos 2t}{4(\sin t + \lambda\mu^k\cos t)^2}x^2, \tag{2.4}$$

with convention $V_t^{\lambda,\mu}(x) = \infty$ for $t \geq \mathrm{Arctg}(-\lambda\mu^k)$. It is evident as $\lambda \to 0$,

$$V_t^{\lambda,\mu}(x) \to V_t(x) = -\frac{\cos t}{2\sin t}x^2, t < \pi \tag{2.5}$$

and we use convention $V_t(x) = \infty$ for $t \geq \pi$. By (1.12), we get for $t < \mathrm{Arctg}(-\lambda\mu^k)$,

$$\begin{aligned} u_t^{\lambda,\mu}(x) = & \frac{1}{\sqrt{2\pi\mu^2(\sin t + \lambda\mu^k\cos t)}}e^{\frac{(-1+\lambda^2\mu^{2k})\sin 2t - 2\lambda\mu^k\cos 2t}{4\mu^2(\sin t+\lambda\mu^k\cos t)^2}x^2} \\ & \hat{E}e^{-\frac{1}{2\mu^2}\int_0^t (X_s^{t,\lambda,\mu}(x))^2 u_{t-s}(X_s^{t,\lambda,\mu}(x))ds}. \end{aligned} \tag{2.6}$$

Here $X_s^{t,\lambda,\mu}$ is the solution of (1.2) with $A = \nabla V_{t-s}^{\lambda,\mu}$ and $V_t^{\lambda,\mu}$ defined by (2.4). Note the stochastic ordinary differential equation here is a linear one for $s \leq t < \mathrm{Arctg}(-\lambda\mu^k)$. Therefore $\mathcal{M}_t$ in Proposition 1.1 is a martingale and the MGCM formula can be used. See Rogers and Williams (1987).

Theorem 2.1. *Let $u_t^{\lambda,\mu}(x)$ be the solution of (2.1). Then for any $k \geq 0$,*

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = \begin{cases} 0, & \text{for } 0 < t < \frac{\pi}{2}, \\ 1, & \text{for } \frac{\pi}{2} < t < +\infty, \end{cases} \tag{2.7}$$

uniformly in any compact subset of $\{(t,x) \in (0,\frac{\pi}{2}) \times (R - \{0\})\}$ *and* $\{(t,x) \in (\frac{\pi}{2}, +\infty) \times (R - \{0\})\}$ *respectively. Furthermore,*
for $k > 0$,

$$\lim_{\mu \to 0} u_t^{\lambda,\mu}(x) = \lim_{\mu \to 0} \lim_{\lambda \to 0} u_t^{\lambda,\mu}(x) \tag{2.8}$$

uniformly in any compact subset of $\{(t,x) \in (0,\frac{\pi}{2}) \times (R - \{0\})\}$ *and* $\{(t,x) \in (\frac{\pi}{2}, +\infty) \times (R - \{0\})\}$ *respectively, and if* $\lambda > 0$
for $k = 0$,

$$\lim_{\mu \to 0} u_t^{\lambda,\mu}(x) = \begin{cases} 0, & \textit{for } 0 \le t < \frac{1}{2}\mathrm{Arctg}\frac{2\lambda}{-1+\lambda^2}, \\ 1, & \textit{for } \frac{1}{2}\mathrm{Arctg}\frac{2\lambda}{-1+\lambda^2} < t < +\infty, \end{cases} \tag{2.9}$$

uniformly in any compact subset of $\{(t,x) \in [0, \frac{1}{2}\mathrm{Arctg}\frac{2\lambda}{-1+\lambda^2}) \times (R - \{0\})\}$ *and* $\{(t,x) \in (\frac{1}{2}\mathrm{Arctg}\frac{2\lambda}{-1+\lambda^2}, +\infty) \times (R - \{0\})\}$ *respectively. Here* Arctg$\cdot$ *denotes the smallest value of rm acrtg$\cdot$ bigger than* 0.

Note. $\lim_{\lambda \to 0} u_t^{\lambda,\mu}(x)$ *can be considered as the solution with a δ function at* 0 *as its initial distribution.*

Proof. For $\lambda > 0$, $D = [0, \mathrm{Arctg}(-\lambda\mu^k)) \times R$. Let $\mathcal{K}$ be a compact set in $(0, \frac{\pi}{2}) \times (R - \{0\})$. For sufficiently small λ (or μ if $k > 0$) $\mathcal{K} \subset D$. From (2.6) if $(t,x) \in \mathcal{K}$,

$$\begin{aligned} &\lim_{\lambda \to 0} u_t^{\lambda,\mu}(x) \\ =&\frac{1}{\sqrt{2\pi\mu^2 \sin t}} e^{-\frac{\cos t}{2\mu^2 \sin t}x^2} \hat{E} e^{-\frac{1}{2\mu^2}\int_0^t (X_s^{t,\mu}(x))^2 \lim_{\lambda \to 0} u_{t-s}^{\lambda,\mu}(X_s^{t,\mu}(x)) ds} \\ \le&\frac{1}{\sqrt{2\pi\mu^2 \sin t}} e^{-\frac{\cos t}{2\mu^2 \sin t}x^2} \\ \to& 0, \text{ as } \mu \to 0, \end{aligned} \tag{2.10}$$

where $X_s^{t,\mu}(x)$ is the Brownian bridge from x to 0.

Similarly we have the convergence to zero claimed in the other situations. The convergence to 1 before the caustic time comes from Theorem 1.4 when $\lambda > 0$, and for $\lim_{\mu \to 0} \lim_{\lambda \to 0} u_t^{\lambda,\mu}(x)$ by the same argument.

After the caustic time we can also use Theorem 1.4 with Remark 1.2 when $\lambda > 0$, $k = 0$: though strictly speaking, it does not satisfy the late caustic condition because of the wave front through 0 this does not affect the proof. Similar arguments can be used for the remaining cases. ‡‡

Remark 2.1. *Note that the non-linearity of the equation allows the solution to continue after the caustic time. For the analogue linear equation with $k = 0, \lambda = 1$, say*

$$\begin{cases} \dfrac{\partial u_t(x)}{\partial t} = \dfrac{\mu^2}{2}\Delta u_t(x) + \dfrac{1}{2\mu^2}x^2 u_t(x), \\ u_0(x) = \dfrac{1}{\sqrt{2\pi\mu^2}}e^{-\frac{1}{2\mu^2}x^2}, \end{cases} \tag{2.11}$$

if $0 < t < \frac{3}{4}\pi$, from Elworthy and Truman (1982), at time t, we find the solution of the linear equation can be represented by

$$u_t(x) = \frac{1}{\sqrt{2\pi\mu^2(\cos t + \sin t)}}e^{-\frac{\cos 2t}{2(\cos t+\sin t)^2\mu^2}x^2}, \tag{2.12}$$

which blows up at $t_0 = \frac{3}{4}\pi$. Note the Feynman-Kac formula holds before the solution blows up. See also Simon (1979).

In the following we will briefly consider the nonsymmetric case:

$$\begin{cases} \dfrac{\partial u_t^{\lambda,\mu}(x)}{\partial t} = \dfrac{\mu^2}{2}\Delta u_t^{\lambda,\mu}(x) + \dfrac{1}{2\mu^2}x^2(1 - u_t^{\lambda,\mu}(x))u_t^{\lambda,\mu}(x), \\ u_0^{\lambda,\mu}(x) = \dfrac{1}{\sqrt{2\pi\mu^{2+k}\lambda}}e^{-\frac{1}{2\mu^{2+k}\lambda}(x-a)^2}. \end{cases} \tag{2.13}$$

Without losing generality we assume $a > 0$. An elementary computation gives

$$V_t^{\lambda,\mu}(x) = \frac{\frac{1}{2}(-x^2 + (\lambda\mu^k)^2x^2 - a^2)\sin 2t - \lambda\mu^k x^2\cos 2t + 2ax\sin t + 2\lambda\mu^k ax\cos t - \lambda\mu^k\cos^2 t}{2(\sin t + \lambda\mu^k\cos t)^2}.$$

As $\lambda \to 0$,

$$V_t^{\lambda,\mu}(x) \to V_t(x) = \frac{(-x^2 - a^2)\cos t + 2ax}{2\sin t}. \tag{2.14}$$

So one can verify that for $t \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} &V_t(x) < 0, \text{ for } x > a\cdot\mathrm{ctg}(\frac{\pi}{4} - \frac{t}{2}), \text{ or } x < a\cdot\mathrm{tg}(\frac{\pi}{4} - \frac{t}{2}), \\ &V_t(x) > 0, \text{ for } a\cdot\mathrm{tg}(\frac{\pi}{4} - \frac{t}{2}) < x < a\cdot\mathrm{ctg}(\frac{\pi}{4} - \frac{t}{2}), \end{aligned}$$

and for $t \in [\frac{\pi}{2}, \pi)$,

$$\begin{aligned} &V_t(x) < 0, \text{ for } a\cdot\mathrm{ctg}(\frac{\pi}{4} - \frac{t}{2}) < x < a\cdot\mathrm{tg}(\frac{\pi}{4} - \frac{t}{2}), \\ &V_t(x) > 0, \text{ for } x > a\cdot\mathrm{tg}(\frac{\pi}{4} - \frac{t}{2}), \text{ or } x < a\cdot\mathrm{ctg}(\frac{\pi}{4} - \frac{t}{2}). \end{aligned}$$

We can prove as before:

Theorem 2.2. *Let $u_t^{\lambda,\mu}(x)$ be the solution of (2.13), then for $t \in [0, \frac{\pi}{2}]$,*

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = \begin{cases} 0, & \text{for } x > a\cdot\mathrm{ctg}(\frac{\pi}{4}-\frac{t}{2}), \text{ or } x < a\cdot\mathrm{tg}(\frac{\pi}{4}-\frac{t}{2}), \\ 1, & \text{for } a\cdot\mathrm{tg}(\frac{\pi}{4}-\frac{t}{2}) < x < a\cdot\mathrm{ctg}(\frac{\pi}{4}-\frac{t}{2}), \end{cases} \tag{2.15}$$

and for $t \in [\frac{\pi}{2}, \pi)$,

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = \begin{cases} 0, & \text{for } a\cdot\mathrm{ctg}(\frac{\pi}{4}-\frac{t}{2}) < x < a\cdot\mathrm{tg}(\frac{\pi}{4}-\frac{t}{2}), \\ 1, & \text{for } x > a\cdot\mathrm{tg}(\frac{\pi}{4}-\frac{t}{2}), \text{ or } x < a\cdot\mathrm{ctg}(\frac{\pi}{4}-\frac{t}{2}), \end{cases} \tag{2.16}$$

and for $t \in [\pi, \infty), x \neq -a$,

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = 1 \tag{2.17}$$

uniformly in any compact subset of each domain respectively.

Remark 2.2. *The constant $\dfrac{1}{(2\pi\lambda\mu^{2+k})^{\frac{r}{2}}}$ in u_0 can be omitted without changing the final answer as regards the limiting wave height and speed. However the effect of any function not identically 1 can be to make the approximation for finite μ less accurate, e.g. see (1.26) and Theorem 1.2.*

Remark 2.3. *The choice of $k = 2$ might be considered a suitable choice from the point of view of scaling.*

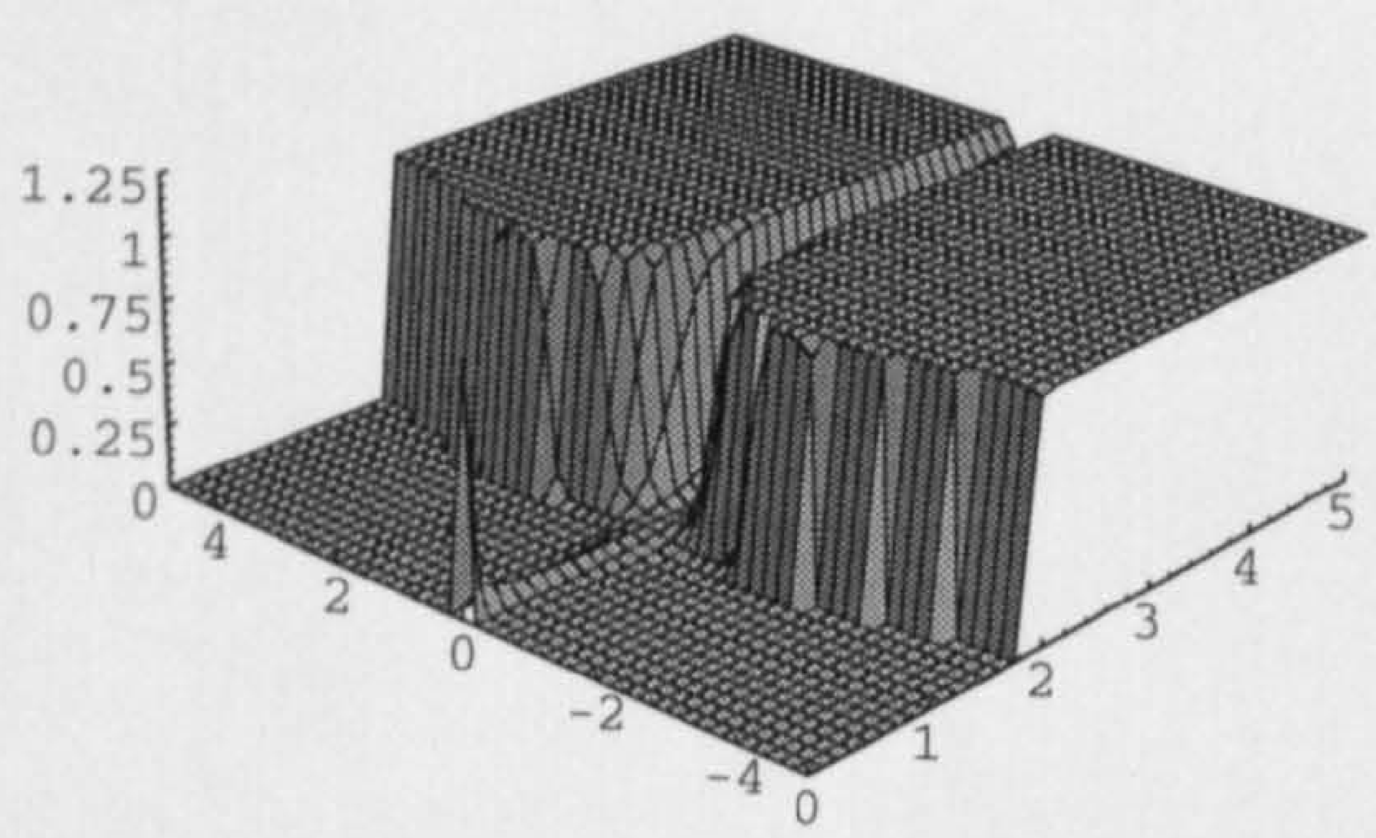

Figure I3

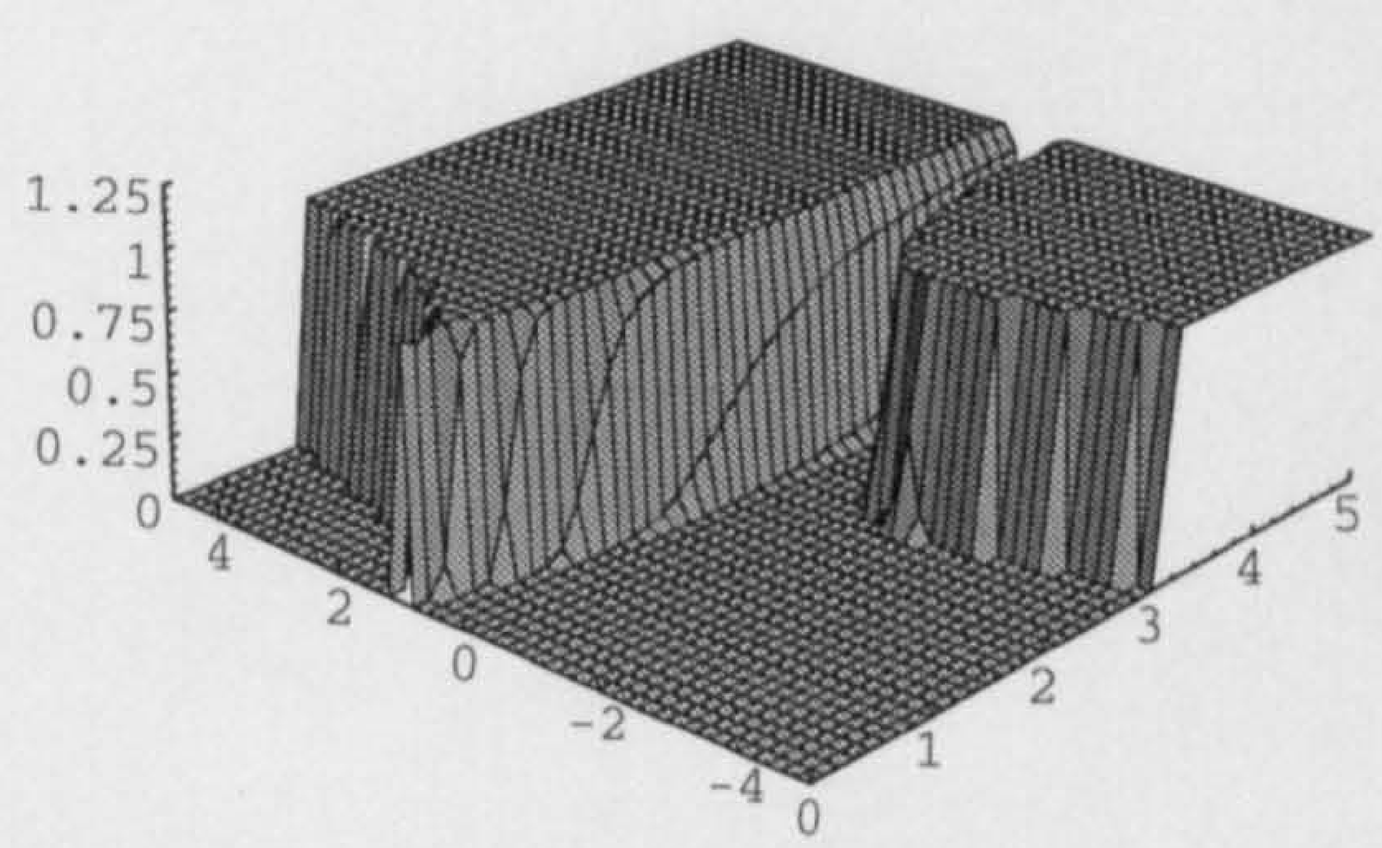

Figure I4

At the end of this section we show a numerical simulation by J. Gaines to equation (2.13) in Figures I3, I4 in which we take $a = 0$ and 1 respectively and $\mu = 0.1$.

It is not very easy to understand this phenomenon properly. At time $t = 0$, the mass is a point source at $x = a$. For $t > \frac{\pi}{2}$, the mass distributed behind $x = a\cdot\mathrm{ctg}(\frac{\pi}{4} - \frac{t}{2})$ disconnects the original mass. In order to understand this mathematically we compactify the configuration space by adding $\{\infty\}$, then the mass connects with the original mass via $x = \infty$. The numerical simulation supports our result perfectly.

§3. A Time Shift Approach

One can try different methods to prove the result of the last section. We show here in this section a time shift method as follows:

For $0 \le t \le t_0$ any t_0 in $[0, \pi)$ and small λ, it is obvious that the *no-caustic* condition, conditions (N^*) and (N^{**}) are satisfied. Note (2.5), by Theorem I.1.1 and Theorem I.1.4, we have for $x \neq 0$,

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = \begin{cases} 0, & \text{for } 0 < t < \dfrac{\pi}{2}, \\ 1, & \text{for } \dfrac{\pi}{2} < t < \pi. \end{cases} \tag{3.1}$$

Write $u^\mu_\cdot(\cdot) = \lim_{\lambda\to 0} u^{\lambda,\mu}_\cdot(\cdot)$. Note $u^\mu_{\frac{3}{4}\pi}(x) = 1 + \mathrm{o}(1)$. Let $\tilde{t} = t - \dfrac{3}{4}\pi$ and

$$\tilde{u}^\mu_{\tilde{t}}(x) = u^\mu_{\tilde{t}+\frac{3}{4}\pi}(x).$$

Then

$$\begin{cases} \dfrac{\partial}{\partial \tilde{t}}\tilde{u}^\mu_{\tilde{t}}(x) = \dfrac{\mu^2}{2}\Delta \tilde{u}^\mu_{\tilde{t}}(x) + \dfrac{1}{2\mu^2}x^2(1 - \tilde{u}^\mu_{\tilde{t}}(x))\tilde{u}^\mu_{\tilde{t}}(x), \\ \tilde{u}^\mu_0(x) = 1 + \mathrm{o}(1). \end{cases} \tag{3.2}$$

Now we construct a new classical mechanics corresponding to Cauchy problem (3.2) by

$$\begin{cases} \ddot{\tilde{\Phi}}_s(x) = -\tilde{\Phi}_s(x), \\ \dot{\tilde{\Phi}}_0(x) = 0, \quad \tilde{\Phi}_s(x) = x. \end{cases} \tag{3.3}$$

So we get $\tilde{\Phi}_{\tilde{s}}(x) = x\cos\tilde{s}$ which satisfies the no-caustic condition for $0 \le \tilde{s} < \frac{\pi}{2}$. Note $\Phi^{-1}_{\tilde{t}}(x) = \dfrac{x}{cos\tilde{t}}$ for $0 < \tilde{t} < \dfrac{\pi}{2}$. By the definition of $\tilde{z}^{\tilde{t}}_{\tilde{s}}(x)$, we get

$$\tilde{z}^{\tilde{t}}_{\tilde{s}}(x) = \tilde{\Phi}_{\tilde{t}-\tilde{s}}(\tilde{\Phi}^{-1}_{\tilde{t}}(x)) = \frac{\cos(\tilde{t} - \tilde{s})}{cos\tilde{t}}x, \quad 0 \le \tilde{s} \le \tilde{t} < \frac{\pi}{2}. \tag{3.4}$$

So

$$\tilde{V}_{\tilde{t}}(x) = \frac{1}{2}\int_0^{\tilde{t}} (\tilde{z}^{\tilde{t}}_{\tilde{s}}(x))^2 d\tilde{s} - \frac{1}{2}\int_0^{\tilde{t}} (\dot{\tilde{z}}^{\tilde{t}}_{\tilde{s}}(x))^2 d\tilde{s} = \frac{1}{2}x^2\mathrm{tg}\tilde{t}. \tag{3.5}$$

Before the caustic time $\tilde{t}_c = \frac{\pi}{2}$, $\tilde{V}_{\tilde{t}}(x) > 0$. So according to Theorem 1.4, we get as $\mu \to 0$

$$\tilde{u}^{\mu}_{\tilde{t}}(x) \to 1 \quad 0 < \tilde{t} < \frac{\pi}{2}.$$

The convergence is uniform in any compact subset of $\{(\tilde{t}, x) \in (0, \frac{\pi}{2}) \times (0, +\infty)\}$. We can extend the result to the whole time interval by repeating the above procedure. This, together with (3.1), implies (2.7). Note for $k > 0$, (2.5) is still true as $\mu \to 0$. Therefore, (2.8) can be proved by the same argument as above. For $k = 0$, $V^{\lambda}_t(x) = V^{\lambda,\mu}_t(x) < 0$ for $0 < t < \frac{1}{2}\mathrm{Arctg}\frac{2\lambda}{-1+\lambda^2}$, and $V^{\lambda}_t(x) = V^{\lambda,\mu}_t(x) > 0$ for $\frac{1}{2}\mathrm{Arctg}\frac{2\lambda}{-1+\lambda^2} < t < \mathrm{Arctg}(-\lambda)$. $\mathrm{Arctg}(-\lambda)$ is a caustic time. We can prove (2.9) by the same argument as above. ‡‡

Remark 3.1. *This method does not work to linear equations. In fact for the linear system in the case $k = 0$, $\lambda = 1$, $\frac{3}{4}\pi$ is the caustic time of its classical mechanics. Let $0 < t_0 < \frac{3}{4}\pi$, by (2.12)*

$$u^{\mu}_{t_0}(x) = \frac{1}{\sqrt{2\pi\mu^2(\cos t_0 + \sin t_0)}} e^{-\frac{\cos 2t_0}{2(\cos t_0 + \sin t_0)^2\mu^2}x^2}. \tag{3.6}$$

If we reconstruct a new classical mechanics from time $0 < t_0 < \frac{3}{4}\pi$, then it should be ($s' = s - t_0$)

$$\begin{cases} \ddot{\overline{\Phi}}_{s'}(x) = -\overline{\Phi}_{s'}(x) \\ \dot{\overline{\Phi}}_0(x) = \dfrac{\cos 2t_0}{1 + \sin 2t_0}x, \quad \overline{\Phi}_0(x) = x. \end{cases} \tag{3.7}$$

We get

$$\overline{\Phi}_{s'}(x) = x \cos s' + x\frac{\cos 2t_0}{1 + \sin 2t_0} \sin s'.$$

The new caustic time s'_c satisfies

$$\cos s'_c + \frac{\cos 2t_0}{1 + \sin 2t_0} \sin s'_c = 0.$$

From this, we get $s'_c = \frac{3}{4}\pi - t_0$, which corresponds to the caustic time $s_c = \frac{3}{4}\pi$ (the blow up time of the solution of the linear equation).

Chapter II. Huygens Principle

In this chapter we study the approximate travelling waves generated by δ-functions distributed at different points and the step initial distribution which is approximated by the integral (infinite sum) of δ-functions on complete Riemannian manifolds without cut locus, with some bounds on their volume elements, in particular Cartan-Hadamard manifolds.

§1. The Superposition of a Finite Number of Travelling Waves of the KPP Equation with Point Sources

Consider the following KPP equation on the Riemannian Manifold M, with metric d

$$\begin{cases} \dfrac{\partial u_t^{\lambda,\mu}(x)}{\partial t} = \dfrac{\mu^2}{2}\Delta u_t^{\lambda,\mu}(x) + \dfrac{\hat{c}}{\mu^2}(1 - u_t^{\lambda,\mu}(x))u_t^{\lambda,\mu}(x) \\ u_0^{\lambda,\mu}(x) = \dfrac{1}{(2\pi\mu^{2+k}\lambda)^{\frac{r}{2}}} \sum\limits_{j=1}^{N} e^{-\frac{d^2(x,a_j)}{2\mu^{2+k}\lambda}}, \end{cases} \tag{1.1}$$

where $x \in M$, $a_1, a_2, \cdots, a_N \in M$ and $k \geq 0$ a constant. We assume that each a_j can be taken as the centre for global geodesic coordinates on M, i.e. we assume that the exponential maps $exp_{a_j} : T_{a_j}M \to M$ are diffeomorphisms. This holds if M is simply connected with negative curvature. The classical mechanical flows $\Phi_{j,s}^{\lambda,\mu}(x)$ corresponding to $\bar{c}(x) = \hat{c}$ and $S_{j,0}^{\lambda,\mu}(x) = \dfrac{d^2(x,a_j)}{2\mu^k\lambda}$, $j = 1, 2, \cdots, N$ are

$$\Phi_{j,s}^{\lambda,\mu}(x) = x + \frac{x - a_j}{\mu^k\lambda}s,$$

in geodesic coordinates about a_j. Therefore, by definition the backward paths $z_{j,s}^{t,\lambda,\mu}(x)$ are given by

$$z_{j,s}^{t,\lambda,\mu}(x) = \Phi_{j,t-s}^{\lambda,\mu}((\Phi_{j,t}^{\lambda,\mu}))^{-1}) = x + \frac{s}{t + \lambda\mu^k}(a_j - x).$$

So

$$V_{j,t}^{\lambda,\mu}(x) = \hat{c}t - \frac{d^2(x,a_j)}{2(t + \lambda\mu^k)}. \tag{1.2}$$

As $\lambda \to 0$,

$$V_{j,t}^{\lambda,\mu}(x) \to V_{j,t}(x) = \hat{c}t - \frac{d^2(x,a_j)}{2t}. \tag{1.3}$$

Let $\theta_j(x)$ be the Jacobi determinant of the exponential maps: $\exp_{a_j} : T_{a_j}M \to M$ at $a_j \in M$: $\theta_j(x) = |det_M T_{v_j} \exp_{a_j}|$ with $x = \exp_{a_j}(v)$. Assume $\theta_j^{\frac{1}{2}} \Delta \theta_j^{-\frac{1}{2}}$ is bounded for each j. From (I.1.12), as Elworthy and Truman (1982), Elworthy (1982, 1988) and Ndumu (1991), we get

$$u_t^{\lambda,\mu}(x) = \frac{1}{(2\pi\mu^2(t+\lambda\mu^k))^{\frac{r}{2}}} \sum_{j=1}^{N} \theta_j^{-\frac{1}{2}}(x) e^{\frac{1}{\mu^2}V_{t,j}^{\lambda,\mu}(x)} \hat{E} e^{\frac{1}{2}\mu^2 \int_0^t \theta_j^{\frac{1}{2}}(X_{j,s}^{x,\lambda,\mu}) \Delta \theta_j^{-\frac{1}{2}}(X_{j,s}^{x,\lambda,\mu}) ds - \frac{\hat{c}}{\mu^2}\int_0^t u_{t-s}^{\lambda,\mu}(X_{j,s}^{x,\lambda,\mu}) ds}. \tag{1.4}$$

Here $X_{j,s}^{t,\lambda,\mu}$ is defined by (I.1.2) with $Y_{j,s} = -\dfrac{d^2(x,a_j)}{2(t-s+2\lambda\mu^k)} - \dfrac{1}{2}\mu^2 \log\theta_j(x)$. Taking the limit $\lambda \to 0$ as in Elworthy (1988), we obtain

$$\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = \frac{1}{(2\pi\mu^2 t)^{\frac{r}{2}}} \sum_{j=1}^{N} \theta_j^{-\frac{1}{2}}(x) e^{\frac{1}{\mu^2}V_{t,j}(x)} \cdot \hat{E} e^{\frac{1}{2}\mu^2 \int_0^t \theta_j^{\frac{1}{2}}(X_{j,s}^{x,0,\mu}) \Delta\theta_j^{-\frac{1}{2}}(X_{j,s}^{x,0,\mu}) ds - \frac{\hat{c}}{\mu^2}\int_0^t \lim_{\lambda\to 0} u_{t-s}^{\lambda,\mu}(X_{j,s}^{x,0,\mu}) ds}.$$

Here (the semi-classical bridge) $X_{j,s}^{t,0,\mu}$ satisfies (I.1.2) with $Y_{j,s} = -\dfrac{d^2(x,a_j)}{2(t-s)} - \dfrac{1}{2}\mu^2 \log\theta_j(x)$ for $0 \le s < t$ and has $X_t^{x,0,\mu}(x) = a_j$ almost surely.

Now by using the same method as in §1 of chapter I, we can prove:

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = \begin{cases} 0, & \text{for } \min\limits_{1\le j\le N} d(x,a_j) > \sqrt{2\hat{c}}t, \\ 1, & \text{for } \min\limits_{1\le j\le N} d(x,a_j) < \sqrt{2\hat{c}}t. \end{cases} \tag{1.5}$$

Furthermore, for $k > 0$,

$$\lim_{\mu\to 0} u_t^{\lambda,\mu}(x) = \lim_{\mu\to 0}\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = \begin{cases} 0, & \text{for } \min\limits_{1\le j\le N} d(x,a_j) > \sqrt{2\hat{c}}t, \\ 1, & \text{for } \min\limits_{1\le j\le N} d(x,a_j) < \sqrt{2\hat{c}}t, \end{cases} \tag{1.6}$$

and for $k = 0$,

$$\lim_{\mu\to 0} u_t^{\lambda,\mu}(x) = \begin{cases} 0, & \text{for } \min\limits_{1\le j\le N} d^2(x,a_j) > 2\hat{c}(\lambda+t)t, \\ 1, & \text{for } \min\limits_{1\le j\le N} d^2(x,a_j) < 2\hat{c}(\lambda+t)t. \end{cases} \tag{1.7}$$

The following figures are the numerical simulations of equation (1.1) by J. Gaines with $\hat{c} = 1$ and various initial conditions. In Figure II1 and Figure II2, the initial condition is an approximation to a point source. In Figure II1 we show the solution for $\mu = 0.4$, whereas in Figure II2, as well as in Figure II3, we take $\mu = 0.1$. We take a pair of point sources for Figure II3. We take $\hat{c} = 1$ throughout.

§2. The Decomposition of the Travelling Wave Generated by a Step Initial Distribution

In Kolmogoroff, Pitrovskii and Pisconoff's paper, they studied a nonlinear reaction diffusion equation which is called the KPP/Fisher equation today, with a step initial distribution $u_0^\mu(x) = \chi_{x\leq 0}$. Freidlin also considered the generalised KPP equation with a step initial distribution. In this section we treat the travelling wave generated by the characteristic function of certain subsets $\mathcal{A}$ on a Riemannian manifold.

Note the integral of δ-functions can approximate a step function in the sense that as $\lambda \to 0$, on a r-dimensional Riemannian manifold M,

$$\frac{1}{(4\pi\mu^4\lambda)^{\frac{r}{2}}}\int_{\mathcal{A}} e^{-\frac{d^2(x,p)}{4\mu^4\lambda}}dvol(p) \to \chi_{\mathcal{A}}(x). \tag{2.1}$$

Here $\mathcal{A}$ is a compact subset of M and $\chi_{\mathcal{A}}$ the characteristic function of $\mathcal{A}$, $dvol(p)$ denotes the volume element measure on M. By $u_t^{\lambda,\mu}(x)$ we denote the solution of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u_t^{\lambda,\mu}(x)}{\partial t} = \dfrac{\mu^2}{2}\Delta u_t^{\lambda,\mu}(x) + \dfrac{1}{\mu^2}c(u_t^{\lambda,\mu}(x))u_t^{\lambda,\mu}(x) \\ u_0^{\lambda,\mu}(x) = \dfrac{1}{(4\pi\mu^4\lambda)^{\frac{r}{2}}}\int_{\mathcal{A}} e^{-\frac{d^2(x,p)}{4\mu^4\lambda}}dvol(p). \end{cases} \tag{2.2}$$

Let $u_t^\mu(x)$ be the solution of the following problem

$$\begin{cases} \dfrac{\partial u_t^\mu(x)}{\partial t} = \dfrac{\mu^2}{2}\Delta u_t^\mu(x) + \dfrac{1}{\mu^2}c(u_t^\mu(x))u_t^\mu(x), \\ u_0^\mu(x) = \chi_{\mathcal{A}}(x). \end{cases} \tag{2.3}$$

We suppose $c(u)$ is Lipschitz continuous with respect to u and satisfies (I') and (II) of chapter I. By the Feynman-Kac formula, the boundedness of the solution and Gronwall's inequality we can prove:

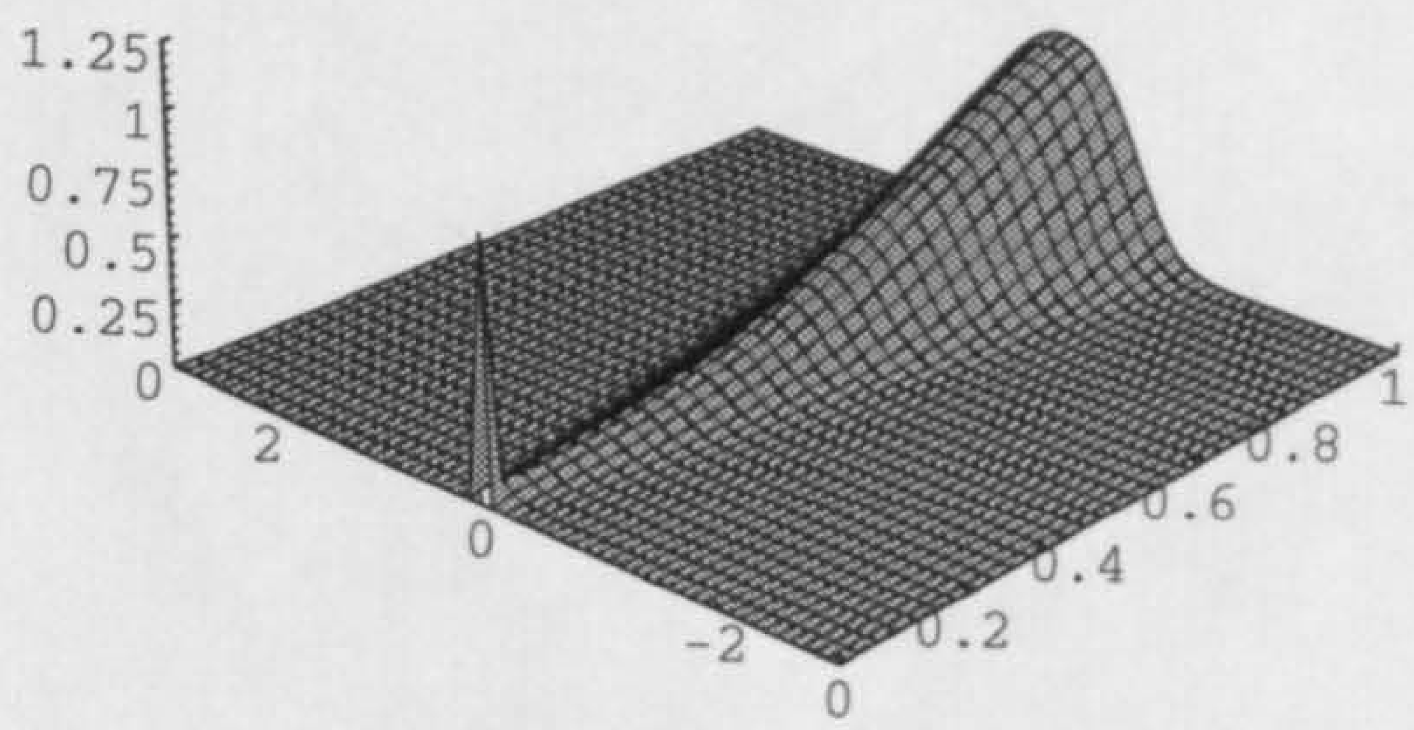

Figure II1

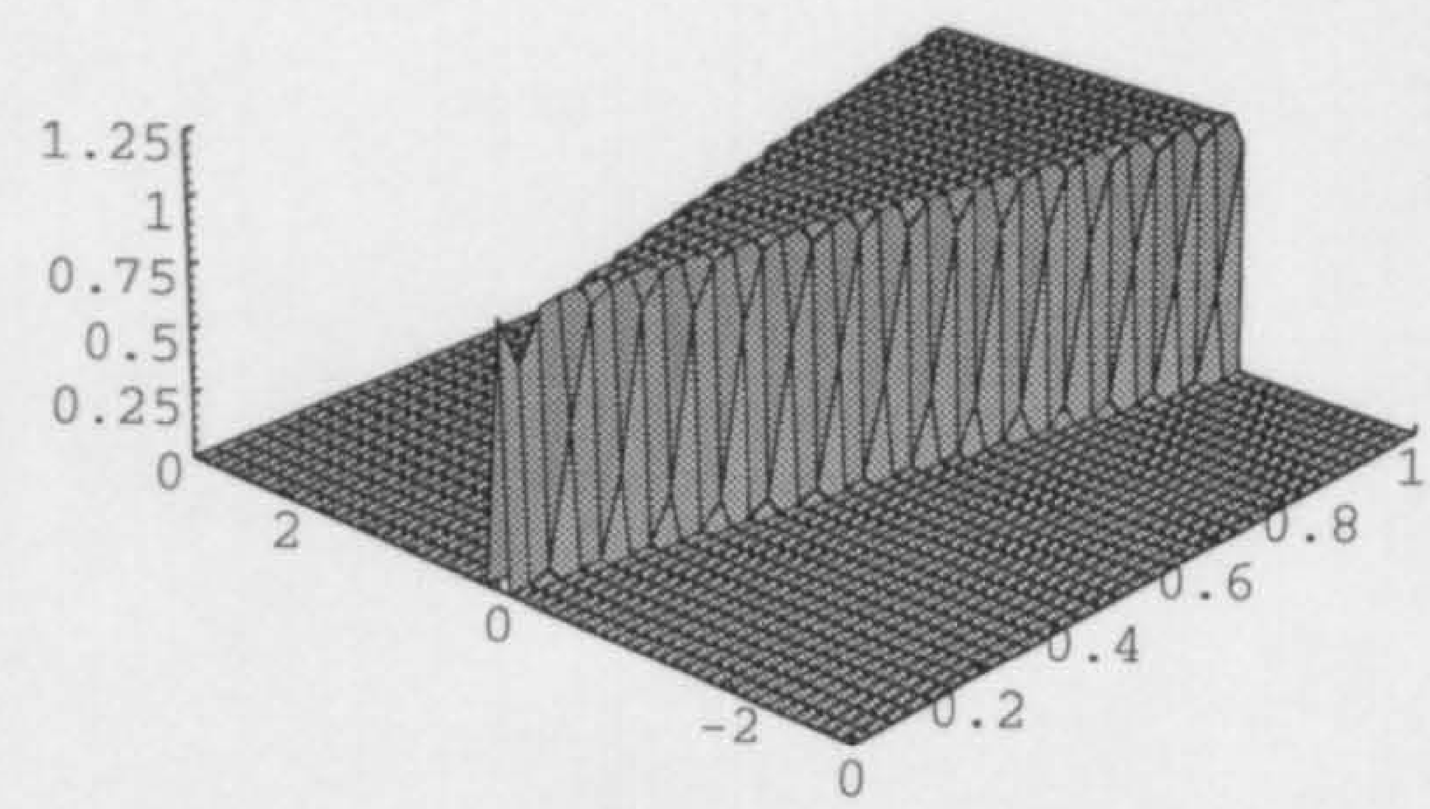

Figure II2

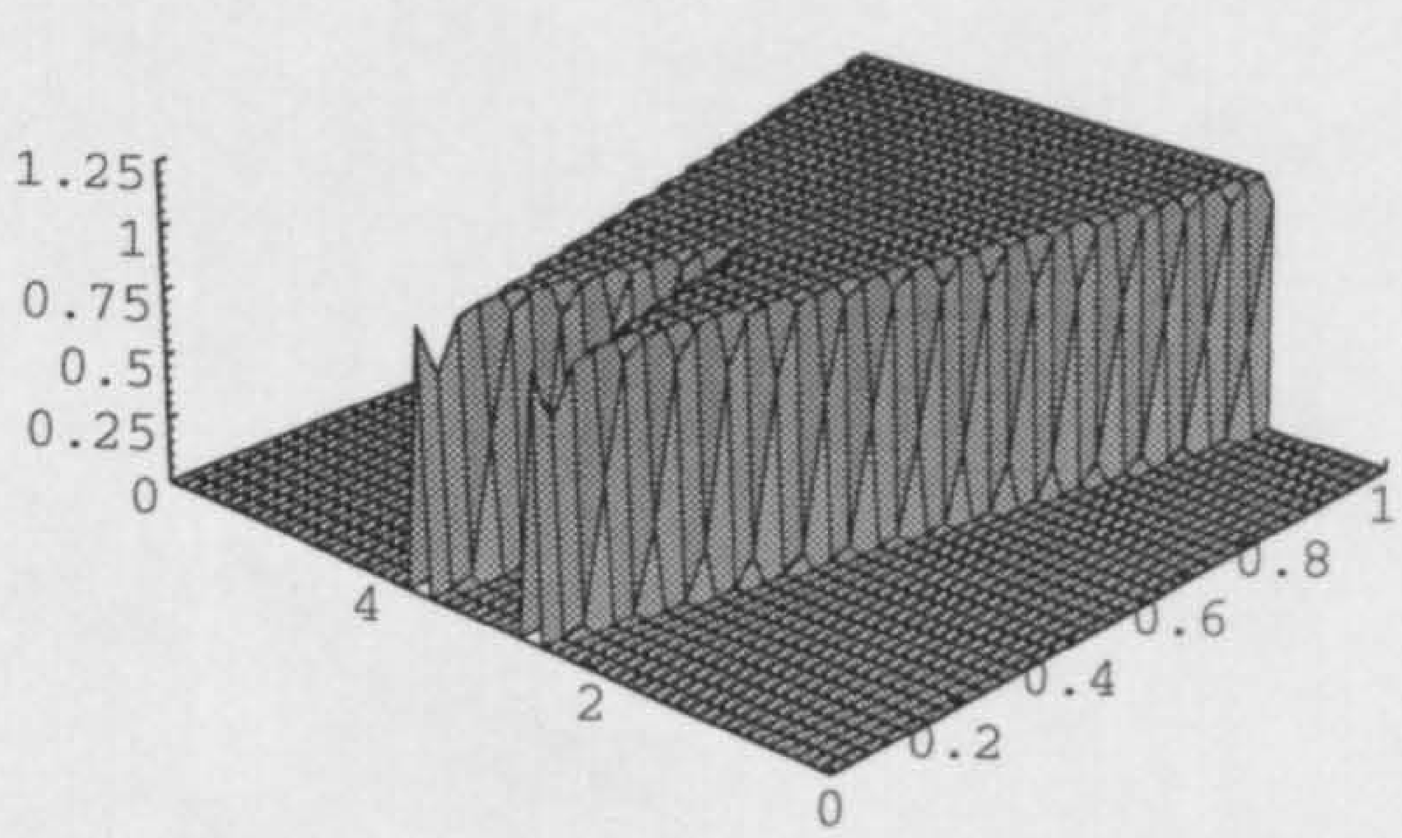

Figure II3

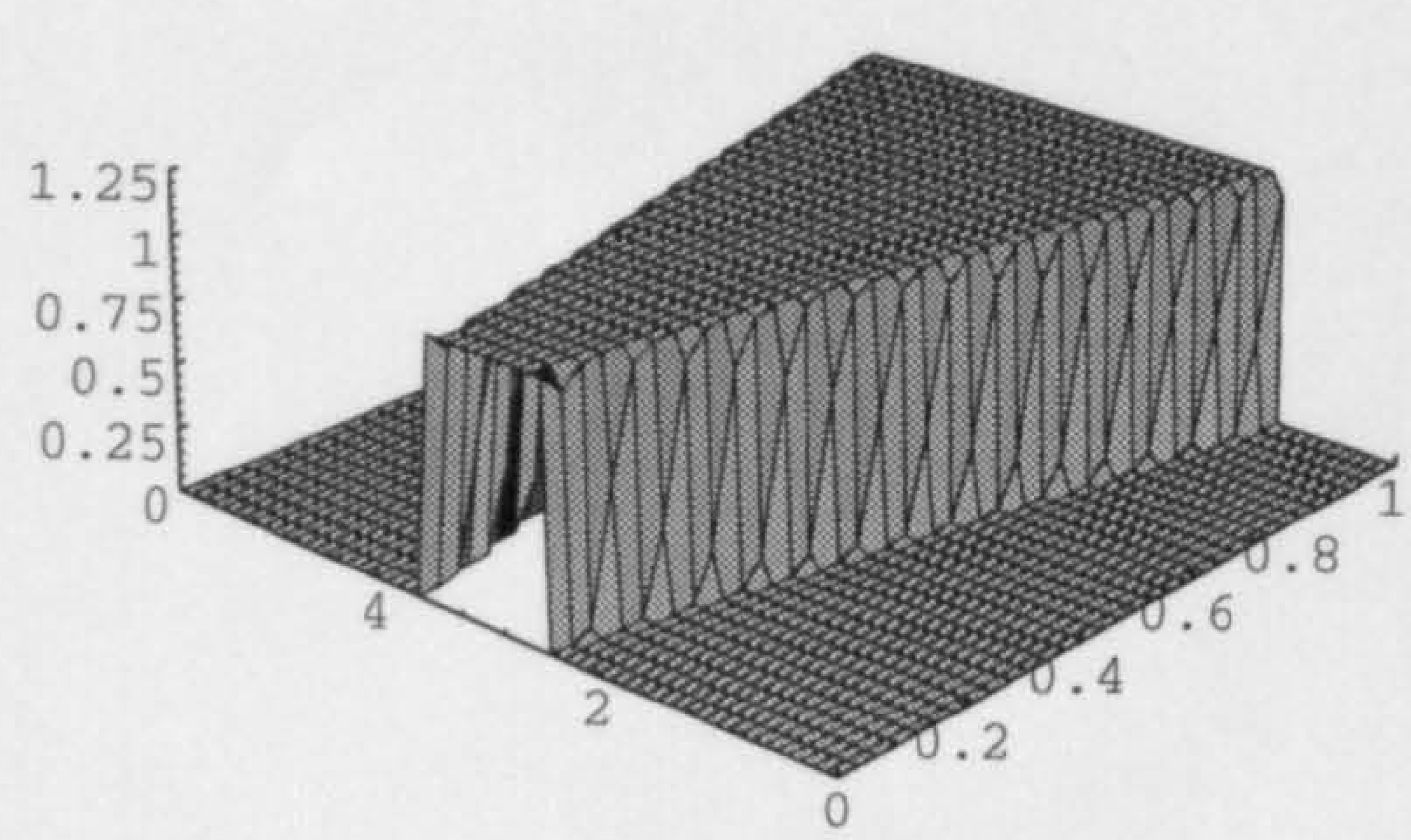

Figure II4

Proposition 2.1. *For any $\mu \neq 0$,*

$$\lim_{\lambda \to 0} u_t^{\lambda,\mu}(x) = u_t^{\mu}(x), \tag{2.4}$$

uniformly in any compact subset of $\{(t,x) \in [0,\infty) \times M\}$.

Now suppose that $exp_p : T_pM \to M$ is a diffeomorphism for each $p \in \mathcal{A}$ (as for $p = a_j$ in §1). Define $\Phi_{p,s}^{\lambda,\mu}(x)$, $\theta_p(x)$, etc just as $\Phi_{j,s}^{\lambda,\mu}(x)$, $\theta_j(x)$ in §1 with p instead of a_j. Assume $\theta_p^{\frac{1}{2}} \Delta \theta_p^{-\frac{1}{2}}$ is bounded on M for each p in $\mathcal{A}$. Arguing as in §1 we obtain the continuous version of (1.4):

$$\begin{aligned} &u_t^{\lambda,\mu}(x) \\ =&\frac{1}{(2\pi\mu^2(t+2\lambda\mu^2))^{\frac{r}{2}}} \int_{\mathcal{A}} \theta_p^{-\frac{1}{2}}(x) e^{-\frac{d^2(x,p)}{2\mu^2(t+2\lambda\mu^2)}} \\ &\hat{E} e^{\frac{1}{2}\mu^2 \int_0^t \theta_p^{\frac{1}{2}}(X_{s,p}^{t,\lambda,\mu}(x)) \Delta \theta_p^{-\frac{1}{2}}(X_{s,p}^{t,\lambda,\mu}(x)) ds + \frac{1}{\mu^2} \int_0^t c(u_{t-s}^{\lambda,\mu}(X_{s,p}^{t,\lambda,\mu}(x))) ds} dvol(p). \end{aligned} \tag{2.5}$$

As in §1, taking the limit as $\lambda \to 0$ we obtain

$$\begin{aligned} &u_t^{\mu}(x) \\ =&\frac{1}{(2\pi\mu^2 t)^{\frac{r}{2}}} \int_{\mathcal{A}} \theta_p^{-\frac{1}{2}}(x) e^{-\frac{d^2(x,p)}{2\mu^2 t}} \\ &\hat{E} e^{\frac{1}{2}\mu^2 \int_0^t \theta_p^{\frac{1}{2}}(X_{s,p}^{t,\mu}(x)) \Delta \theta_p^{-\frac{1}{2}}(X_{s,p}^{t,\mu}(x)) ds + \frac{1}{\mu^2} \int_0^t c(u_{t-s}^{\mu}(X_{s,p}^{t,\mu}(x))) ds} dvol(p), \end{aligned} \tag{2.6}$$

where $X_{s,p}^{t,\mu}(x)$ is now the semi-classical bridge from x to p. Based on (2.6) we prove the main result of this section. The case $M = R^r$ is due to Freidlin (1985).

Theorem 2.1. *Suppose conditions (I'), (II) and M is complete with global geodesic coordinate about each point p in $\mathcal{A}$. Assume $\theta_p^{\frac{1}{2}} \Delta \theta_p^{-\frac{1}{2}}$ is bounded on M for each p in $\mathcal{A}$. Then for $d(x,\mathcal{A}) > \sqrt{2\hat{c}}t$,*

$$\lim_{\mu \to 0} u_t^{\mu}(x) = 0, \tag{2.7}$$

uniformly in any compact subset of $\{(t,x) \in (0,\infty) \times M : d(x,\mathcal{A}) > \sqrt{2\hat{c}}t\}$, and for $d(x,\mathcal{A}) < \sqrt{2\hat{c}}t$,

$$\lim_{\mu \to 0} u_t^{\mu}(x) = 1, \tag{2.8}$$

uniformly in any compact subset of $\{(t,x) \in (0,\infty) \times M : d(x,\mathcal{A}) < \sqrt{2\hat{c}}t\}$.

Proof. That (2.7) holds is immediate from (2.6).

For (2.8), let q be a small positive number. Let $\mathcal{K}_1$ be a compact subset of $\{(t,x) \in (0,\infty) \times M : d(x,\mathcal{A}) = \sqrt{2(\hat{c}+q)t}\}$. For sufficiently small $\epsilon > 0$ and $(t,x) \in \mathcal{K}_1$, let

$$\mathcal{A}^{\epsilon}_{t,x} = \{p \in \mathcal{A} : d^2(x,p) < d^2(x,\mathcal{A}) + \epsilon\}. \tag{2.9}$$

Then

$$d(z^t_{s,p}(x), \mathcal{A}) > \sqrt{2\hat{c}}(t-s), 0 < s < t-\epsilon, \tag{2.10}$$

where $\{z^t_{s,p}(x), 0 \le s \le t\}$ is the geodesics from x to p in time t. Note that for $0 < s < t-\epsilon$, $X^{t,\mu}_{s,p}$ converges to $z^t_{s,p}(x)$ in probability as $\mu \to 0$. So by (2.7) there are $\mu^{(1)}_0(\mathcal{K}_1) > 0$ and $\epsilon^* > 0$ such that for $0 < \mu < \mu^{(1)}_0$ and $(t,x) \in \mathcal{K}_1$

$$\hat{P}\{u^{\mu}_{t-s}(X^{t,\mu}_{s,p}(x)) < e^{-\frac{\delta^*}{\mu^2}}, 0 < s < t-\epsilon\} > 1-\epsilon^*,$$

where δ^* is a small positive number. Let $\mu^{(2)}_0(\mathcal{K}_1)$ be a positive number such that for $0 < \mu < \mu^{(2)}_0$

$$\mu^2 \log \left(\frac{\theta_p^{-\frac{1}{2}}(x) vol \mathcal{A}^{\epsilon}_{t,x}}{(2\pi\mu^2 t)^{\frac{r}{2}}} \right) \ge -\epsilon, \tag{2.11}$$

and

$$\frac{1}{2}\mu^4 \int_0^t \theta_p^{\frac{1}{2}}(X^{t,\lambda,\mu}_{s,p}(x)) \Delta \theta_p^{-\frac{1}{2}}(X^{t,\lambda,\mu}_{s,p}(x)) ds \ge -\epsilon. \tag{2.12}$$

Let $\mu_0 = \min\{\mu^{(1)}_0, \mu^{(2)}_0\}$. Then for $0 < \mu < \mu_0$, from (2.6) we get

$$\begin{aligned} u^{\mu}_t(x) \ge & \frac{vol(\mathcal{A}^{\epsilon}_{t,x})}{(2\pi\mu^2 t)^{\frac{r}{2}}} \\ & \int_{\mathcal{A}^{\epsilon}_{t,x}} \theta_p^{-\frac{1}{2}}(x) e^{-\frac{d^2(x,p)}{2\mu^2 t}} \hat{E} e^{\frac{1}{\mu^2}\left[-\epsilon + \left\{\int_0^{t-\epsilon} + \int_{t-\epsilon}^t\right\} c(u^{\mu}_{t-s}(X^{t,\mu}_{s,p}(x))) ds\right]} d\frac{vol(p)}{vol(\mathcal{A}^{\epsilon}_{t,x})}. \end{aligned} \tag{2.13}$$

Using Jensen's inequality, we get for $0 < \mu < \mu_0$,

$$\begin{aligned} & \mu^2 \log u^{\mu}_t(x) \\ \ge & -\epsilon - \frac{d^2(x,\mathcal{A}) + \epsilon}{2t} - \epsilon + \hat{c}(t-\epsilon) - \epsilon\hat{c} \\ \ge & -2\epsilon - \frac{\epsilon}{2t} - 2\hat{c}\epsilon - qt. \end{aligned}$$

From this we get

$$\lim_{\mu \to 0} \mu^2 \log u^{\mu}_t(x) \ge -qt, \tag{2.14}$$

for $(t,x) \in \mathcal{K}_1$.

Next we give the analogue of Theorem I.1.4. Let $\mathcal{K}_2$ be a compact subset of $\{(t,x) \in (0,\infty) \times M : d(x,\mathcal{A}) < \sqrt{2\hat{c}}t\}$ and let $(t,x) \in \mathcal{K}_2$. Set $T = \sup\{t : (t,x) \in \mathcal{K}_2\}$. We choose $h > 0$ such that

$$\inf\{2\hat{c}s^2 - d^2(y,\mathcal{A}) : d(x,y) < h, |s-t| < h\} > \frac{1}{2}(2\hat{c}t^2 - d^2(x,\mathcal{A})).$$

For any $\beta > 0$, let $\alpha = \inf_{0 \le u \le 1-\beta} hc(u)$. Then $\alpha > 0$. Let us select $q \in (0, \frac{1}{4T}\alpha)$. Recall that $B_s^{x,\mu}$ is defined by (I.1.2) with $A \equiv 0$. We introduce two Markov times

$$\tau_1 = \inf\{s \ge 0 : u^\mu(t-s, B_s^{x,\mu}) \ge 1-\beta\},$$

and

$$\tau_2 = \inf\{s \ge 0 : d(B_s^{x,\mu},\mathcal{A}) = \sqrt{2(\hat{c}+q)}(t-s)\}.$$

Note τ_2 taken value in $[0,t]$. So it is evident that

$$(t-\tau_2, B_{\tau_2}^{x,\mu}) \in Z_q = \{(t,x) \in (0,\infty) \times M : d(x,\mathcal{A}) = \sqrt{2(\hat{c}+q)}t\}.a.s.$$

So there is a compact subset K in Z_q with $P(\Omega_0) > 1-\beta$ for $\Omega_0 = \{\omega \in \Omega : (t-\tau_2, B_{\tau_2}^{x,\mu}) \in K\}$. From (2.14) we know for sufficiently small $\mu \neq 0$

$$u^\mu(t-\tau_2, B_{\tau_2}^{x,\mu}) > e^{-\frac{\alpha}{2\mu^2}} \text{ on } \Omega_0. \tag{2.15}$$

Following the argument of Theorem I.1.4 we get $\underline{\lim}_{\mu\to 0} u_t^\mu(x) \ge 1$. The opposite direction of (2.8) can be obtained by the same methods as in the proof of Lemma 2.1 in Zhao & Elworthy (1992) or Freidlin (1985). So we have (2.8). ‡‡

For J. Gaines' numerical simulation of the travelling wave of this section see Figure II4.

Remark 2.1. *It is clear from this section that the travelling wave generated by a step function can be decomposed into the sum of an infinite number of fundamental travelling waves generated by a δ-function.*

Remark 2.2. *In this section we have proved a Huygens principle on complete Riemannian manifolds without cut locus, with some bounds on their volume elements, in particular Cartan-Hadamard manifolds..*

Chapter III. Stochastic Generalised KPP Equations

§1. Introduction to The Problem

The KPP equation with a 'mild' multiplicative white noise may be described by the stochastic Itô KPP equation,

$$du_t(x) = \left[\frac{1}{2}\Delta u_t(x) + \hat{c}(1-u_t(x))u_t(x)\right] dt + k u_t(x) dw_t, \tag{1.1}$$

where w_t is an R^1 Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. If we rescale to study the function $u_t^\mu(x) = u(\frac{t}{\mu^2}, \frac{x}{\mu^2})$, equation (1.1) becomes

$$du_t^\mu(x) = \left[\frac{1}{2}\mu^2 \Delta u_t^\mu(x) + \frac{1}{\mu^2}\hat{c}(1-u_t^\mu(x))u_t^\mu(x)\right] dt + \frac{1}{\mu} k u_t^\mu(x) d\mu w_{\frac{t}{\mu^2}}. \tag{1.2}$$

Here $\mu w_{\frac{t}{\mu^2}}$ is equivalent in law to w_t by the scaling property of Brownian motion. This suggests that we consider a stochastic generalised KPP equation with a multiplicative white noise of Itô's type

$$\begin{cases} du_t^\mu(x) = \left[\frac{1}{2}\mu^2 \Delta u_t^\mu(x) + \frac{1}{\mu^2} c(x, u_t^\mu(x)) u_t^\mu(x)\right] dt + \frac{1}{\mu} k(t,x) u_t^\mu(t) dw_t, \\ u_0^\mu(x) = T_0(x) e^{-\frac{1}{\mu^2} S_0(x)}. \end{cases} \tag{1.3}$$

Here $t \in [0, \infty)$, $x \in R^r$, Δ is the Laplacian operator on R^r and we are given T_0, $S_0 : R^r \to R^1$ and $c : R^r \times R^1 \to R^1$, with k a continuous function of t, x independent of μ and w_t is a one dimensional Brownian motion independent of μ defined on the probability space $(\Omega, \mathcal{F}, P)$. We are mainly concerned with seasonal noise, i.e. $k(t,x) \equiv k(t)$ except in §7. The function $u^\mu : [0, \infty) \times R^r \times \Omega \to R^1$ denotes the solution. If $k \equiv 0$ we get the deterministic generalised KPP equation which was studied in Zhao & Elworthy (1992) following Freidlin's earlier work. In this case for small μ, the solution u^μ has approximately the form of a travelling wave. We discuss the propagation of an approximate travelling wave solution of (1.3) in §2, §3 and §4 as the first equation of this chapter.

We always suppose that T_0 and S_0 are continuous and T_0 is nonnegative, and that $c(x,u)$ is $C^{3,3}$ in (x,u) and Lipschitz in u, uniformly in x. The choice of initial function of this form is justified in chapters I, II, where it is shown that this allows the consideration of superpositions of δ-functions as initial data and also the step functions considered by Freidlin. For simplicity in this chapter we consider a single extended source (except in §8) and R^r as our configuration space.

In fact the computer simulations in Gaines [1], [2] demonstrate that the waves should not just be considered as limiting objects in the large scale limit but as approximate, though rather strictly defined, features of the solutions to generalised KPP equations for the ratios of the parameters involved lying in particular bands. This justifies the study of (1.3) for small μ without recourse to considerations of scaling limits. It also suggests considerations of 'weak' noise perturbations given by

$$du_t^\mu(x) = [\frac{1}{2}\mu^2\Delta u_t^\mu(x) + \frac{1}{\mu^2}c(x,u_t^\mu(x))u_t^\mu(x)]dt + k(t,x)u_t^\mu(x)dw_t, \qquad (1.4)$$

and 'strong' noise perturbations, given by

$$du_t^\mu(x) = [\frac{1}{2}\mu^2\Delta u_t(x) + \frac{1}{\mu^2}c(x,u_t^\mu(x))u_t^\mu(x)]dt + \frac{1}{\mu^2}k(t,x)u_t^\mu(x)dw_t, \qquad (1.5)$$

again for small values of μ. In (1.2) these correspond to k of order μ and $\frac{1}{\mu}$ respectively.

One can think of (1.4) as a random perturbation of a deterministic KPP equation. In fact we show that solutions to (1.4) are well approximated for small μ by the approximate travelling wave of the unperturbed equation. However in the strong noise case (1.5), the solution decays to zero: the wave structure is destroyed. In the more difficult mild case we show the existence of a 'wave front', in front of which the solution is close to zero (of order $exp(-c_1\mu^{-2})$ for c_1 random) and behind which it has at least order $exp(-c_2\mu^{-1})$ for some random c_2 depending on the increment of the noise. We can also give random upper bounds behind the front. However this is far from a complete picture.

In §9 we study L^2 perturbation and non L^2 perturbation to the approximate travelling waves.

§2. The Random Trough and Classical Mechanics

A. In this section we consider the stochastic reaction diffusion equation defined by (1.3) with $k(t,x) \equiv k(t)$. If we assume condition (I) of §1A of Chapter I, by the regularity result of Flandoli (1992) we can prove the existence and regularity of the solution. By the stochastic Feynman-Kac formula in Brzezniak, Capinski and Flandoli (1990), the solution $u_t^\mu(x)$ to (1.3) satisfies

$$\begin{aligned} u_t^\mu(x) &= \hat{E}T_0(x+\mu B_t) \\ &\cdot e^{-\frac{1}{\mu^2}S_0(x+\mu B_t)+\frac{1}{\mu^2}\int_0^t c(x+\mu B_s, u_{t-s}^\mu(x+\mu B_s))ds-\frac{1}{2\mu^2}\int_0^t k^2(t-s)ds-\frac{1}{\mu}\int_0^t k(t-s)dw_{t-s}}, \end{aligned} \tag{2.1}$$

where B_t is an R^r Brownian motion defined on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ which is independent of w_t and the notation $\hat{E}$ represents the expectation over that probability space.

Denote by $X_s^{x,\mu}$ a Markov process defined by

$$\begin{cases} dX_s^{x,\mu} = \mu dB_s + A_s(X_s^{x,\mu})ds, \\ X_0^{x,\mu} = x, \end{cases} \tag{2.2}$$

where $A_s(x) = \nabla Y_s(x)$ for suitable $Y_s \in C^2(R^r \to R^1)$.

B. Let $\bar{c}$ and k be C^2 and S_0 be C^1 and as in Chapter I consider the classical mechanical system

$$\begin{cases} \ddot{\Phi}_s(x) = -\nabla\bar{c}(\Phi_s(x)), \quad s \geq 0, \\ \dot{\Phi}_0(x) = \nabla S_0(x), \quad \Phi_0(x) = x. \end{cases} \tag{2.3}$$

Recall some definitions from chapter I. For $a \in R^r$, let

$$\begin{aligned} \Sigma_a =& \{t \geq 0 : \Phi_t \text{ gives a diffeomorphism of a neighbourhood of } a \text{ in } R^r \\ & \quad \text{on to a neighbourhood of } \Phi_t(a) \text{ and also } \Phi_t^{-1}(\Phi_t(a)) = \{a\}\}, \\ \sigma(a) =& \inf\{t \geq 0 : t \notin \Sigma_a\} \leq +\infty, \\ \sigma^T(a) =& T \wedge \sigma(a), \\ D =& \{(t, \Phi_t(a)) : t < \sigma(a)\}, \\ D_t =& \{x \in R^r : (t,x) \in D\}. \end{aligned}$$

For $(t,x) \in D$, as in (1.6) define $V^k : D \mapsto R$ by

$$\begin{aligned} V_t^k(x) = & \int_0^t \bar{c}(\Phi_s(\Phi_t^{-1}(x)))ds - \frac{1}{2}\int_0^t k^2(s)ds \\ & - S_0(\Phi_t^{-1}(x)) - \frac{1}{2}\int_0^t |\dot{\Phi}_s(\Phi_t^{-1}(x))|^2 ds, \end{aligned} \tag{2.4}$$

and use the convention $V_t^k(x) = +\infty$ if $(t,x) \notin D$. As in (1.7), (1.8) V^k satisfies the Hamilton-Jacobi equation

$$\frac{1}{2}||\nabla V_t^k(x)||^2 + \bar{c}(x) - \frac{1}{2}k^2(t) - \frac{\partial V_t^k}{\partial t}(x) = 0, \quad (t,x) \in D, \tag{2.5}$$

with given initial function $-S_0$ and

$$\dot{\Phi}_t(x) = -\nabla V_t^k(\Phi_t(x)), (t,x) \in D. \tag{2.6}$$

For $x \in D_t$, set $z_s^t(x) = \Phi_{t-s}(\Phi_t^{-1}(x)),\ \ 0 \le s \le t$. Then, from (1.10) we have

$$\frac{\partial z_s^t}{\partial s}(x) = \nabla V_{t-s}^k(z_s^t(x)). \tag{2.7}$$

The following Lemma is valid for vector valued function k and Brownian motion w_t.

Lemma 2.1. *Assume $k : [0, +\infty) \mapsto R^n$ is bounded and continuous and w_s is a Brownian motion on R^n. Then for any $a, b > 0$, and $T > 0$,*

$$P\{\sup_{0\le t\le T}[\int_0^t < k(s), dw_s > -\frac{1}{2}a\int_0^t < k(s), k(s) > ds] > b\} \le e^{-ab}. \tag{2.8}$$

In particular, for any $\delta > 0$, $T > 0$, $\mu > 0$

$$P\{\mu \sup_{0\le t\le T}\int_0^t < k(s), dw_s >> \delta\} < e^{-\frac{\delta^2}{2\Gamma\mu^2}}. \tag{2.9}$$

Here $\Gamma = \sup_{0\le t\le T} \int_0^t ||k(s)||^2 ds$.

Proof. (2.8) is from McKean (1969) which can be proved by Doob's maximal theorem.

For (2.9) we use (2.8) directly, for any $0 \le t \le T$

$$\begin{aligned}
&P\{\mu\int_0^t < k(s), dw_s >> \delta\} = P\{\int_0^t < k(s), dw_s >> \frac{\delta}{\mu}\} \\
\le &P\{\int_0^t < k(s), dw_s > -\frac{\delta}{2\Gamma\mu}\int_0^t ||k(s)||^2 ds > \frac{\delta}{2\mu}\} \\
\le &e^{-\frac{\delta^2}{2\Gamma\mu^2}}.
\end{aligned}$$

‡‡

Theorem 2.1. *Assume the condition (I), that $\bar{c}, S_0$ are C^2 with S_0 bounded below, and that T_0 is bounded and measurable. Then for any compact subset $\mathcal{K}$ of $\{(t,x) : V_t^k(x) < 0, t > 0\}$, there exists $\mu_1(\mathcal{K}) > 0$ such that*

$$P\{\omega \in \Omega : \sup_{(t,x)\in\mathcal{K}} u_t^\mu(x,\omega) > e^{-\frac{\delta}{\mu^2}},\ for\ some\ \ 0 < \mu \le \mu_0\} < e^{-\frac{\delta^2}{8\Gamma\mu_0^2}}, \tag{2.10}$$

for any $0 < \mu_0 < \mu_1$, *where* $\delta = -\frac{1}{2}\sup\{V_t(x) : (t,x) \in \mathcal{K}\}$, $\Gamma = \sup_{t:(t,x)\in\mathcal{K}} \int_0^t k^2(s)ds$. *In particular, as* $\mu \to 0$, $u_t^\mu(x,\omega) \to 0$, *uniformly in* $\mathcal{K}$ *P-a.s..*

Proof. By looking at the inverse image of a neighbourhood of $\mathcal{K}$ under $(s,a) \mapsto (s, \Phi_s(a))$ and arguing as Lemma I.1.2, Theorem I.1.1 we obtain an open set $\mathcal{N}_\mathcal{K} \in IntD$ and $(t-s, z_s^t(x)) \in \mathcal{N}_\mathcal{K}$ for $0 \le s \le t$ if $(t,x) \in \mathcal{K}$. Let $\eta(\sigma)$ be the first exit time of a path σ from $\mathcal{N}_\mathcal{K}$. Let $X_s^{x,\mu}$ for $0 \le s < \eta(X^{x,\mu})$ be the solution of (2.2) with $A_s = \nabla V_{t-s}^k$ up to exit time $\eta(X^{x,\mu})$. Note that V^k is $C^{1,2}$ by (2.4) since Φ^{-1} and $\dot{\Phi}$ are $C^{1,1}$. By the Feynman-Kac formula (2.1), Proposition I.1.1 and using the Hamilton-Jacobi equation (2.5),

$$\begin{aligned} u_t^\mu(x) = {} & e^{\frac{1}{\mu^2}V_t^k(x)}\hat{E}\chi_{t<\eta(X^{x,\mu})}T_0(X_t^{x,\mu})e^{\frac{1}{2}\int_0^t \Delta V_{t-s}^k(X_s^{x,\mu})ds} \\ & \cdot e^{\frac{1}{\mu^2}\int_0^t\left[c(X_s^{x,\mu},u_{t-s}^\mu(X_s^{x,\mu}))-\bar{c}(X_s^{x,\mu})\right]ds+\frac{1}{\mu}\int_0^t k(s)dw_s} \\ & + \hat{E}\chi_{t>\eta(x+\mu B_\cdot)}T_0(x+\mu B_t)e^{-\frac{1}{\mu^2}S_0(x+\mu B_t)+\frac{1}{\mu^2}\int_0^t c(x+\mu B_s,u_{t-s}(x+\mu B_s))ds} \\ & \cdot e^{-\frac{1}{2\mu^2}\int_0^t k^2(s)ds+\frac{1}{\mu}\int_0^t k(s)dw_s}. \end{aligned} \tag{2.11}$$

Set $\Omega_0^\mu = \{\omega \in \Omega : \sup_{t:(t,x)\in\mathcal{K}} \mu\int_0^t k(s)dw_s(\omega) < \frac{1}{2}\delta\}$. Then, from Lemma 2.1, $P(\Omega_0^\mu) > 1 - e^{-\frac{\delta^2}{8\Gamma\mu^2}}$. On Ω_0^μ, by §3 of Elworthy & Truman (1981), especially Theorem 3C and its proof (i.e., Varadhan (1967)), as in the proof of Theorem I.1.1, the last term of (2.11) has upper bound

$$\begin{aligned} & \hat{E}\chi_{t>\eta(x+\mu B_\cdot)}T_0(x+\mu\hat{w}_t)e^{-\frac{1}{\mu^2}S_0(x+\mu B_t)+\frac{1}{\mu^2}\int_0^t[\bar{c}(x+\mu B_s)-\frac{1}{2}k^2(s)]ds+\frac{\frac{1}{2}\delta}{\mu^2}} \\ \le & e^{\frac{V_t^k(x)+\frac{1}{2}\delta}{\mu^2}} \times R(\mu), \end{aligned} \tag{2.12}$$

where $R(\mu) \le o(\mu^n)$ for any $n \ge 0$. On the other hand, there exists $\mu_1(\mathcal{K}) > 0$ such that for $0 < \mu < \mu_1$,

$$\mu^2 \sup_{(t,x)\in\mathcal{K}} \log[\hat{E}\chi_{t<\eta(X^{x,\mu})}T_0(X_t^{x,\mu})e^{\frac{1}{2}\int_0^t \Delta V_{t-s}^k(X_s^{x,\mu})ds} + R(\mu)] \le \frac{1}{2}\delta. \tag{2.13}$$

For any $0 < \mu_0 < \mu_1$, by (2.11), (2.12), (2.13), for $\omega \in \Omega_0^{\mu_0}$, $0 < \mu < \mu_0$,

$$\mu^2 \log u_t^\mu(x,\omega) < V_t(x) + \frac{1}{2}\delta + \frac{1}{2}\delta \le -\delta. \tag{2.14}$$

This, together with the definition of $\Omega_0^{\mu_0}$, implies the proof of the theorem. ‡‡

Corollary 2.1. *Assume all conditions of Theorem 2.1. If for every $(t,x) \in D$,*

$$\int_0^t \overline{c}(\Phi_s(\Phi_t^{-1}(x))ds < \frac{1}{2}\int_0^t k^2(s)ds, \tag{2.15}$$

then for any compact subset $\mathcal{K}$ of D, there exist $\delta(\mathcal{K}) > 0$ and $\mu_1(\mathcal{K}) > 0$ such that

$$P\{\omega \in \Omega : \sup_{(t,x)\in\mathcal{K}} u_t^\mu(x,\omega) > e^{-\frac{\delta}{\mu^2}}, \textit{for some } 0 < \mu \le \mu_0\} < e^{-\frac{\delta^2}{8\Gamma\mu_0^2}},$$

for any $0 < \mu_0 < \mu_1$, where $\Gamma = \sup_{t:(t,x)\in\mathcal{K}} \int_0^t k^2(s)ds$. In particular, as $\mu \to 0$, $u_t^\mu(x,\omega) \to 0$, uniformly in $\mathcal{K}$ P-a.s..

C. In the following we will give sharp estimates of the convergence to the trough. For a compact subset $\mathcal{K}$ of D define $\mathcal{N}_{\mathcal{K}}$ as in the proof above with η the first exit time. Let $\tilde{X}_s^{x,\mu} = X_{s\wedge\eta}^{x,\mu}$.

Lemma 2.2. *Assume all conditions of Theorem 2.1. Then if $0 \le \theta_i \le \frac{1}{2}t$, $i = 1,2$, for any compact subset $\mathcal{K}$ of $\{(t,x) : V_{t-s}^k(z_s^t(x)) < 0, \theta_1 \le s \le t-\theta_2\}$ and $\mu_1 > 0$, there exist $\mu_0 > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 1 - e^{-\frac{\delta_2^2}{8\Gamma\mu_0^2}}$ such that if $\omega \in \Omega_1$,*

$$\begin{aligned}\hat{P}\{\hat\omega \in \hat\Omega : \mu^2 \sup_{\theta_1\le s\le t-\theta_2,(t,x)\in\mathcal{K}} \log u_{t-s}^\mu(\tilde{X}_s^{x,\mu}(\hat\omega),\omega) < -\delta_2, \textit{ for all } \quad 0 < \mu \le \mu_0\}\\ > 1 - e^{-\frac{\delta_1}{\mu_1^2}},\end{aligned} \tag{2.16}$$

where $\Gamma = \sup_{t:(t,x)\in\mathcal{K}} \int_0^t k^2(s)ds$. In particular, for almost all $\omega \in \Omega$, and $\theta_1 \le s \le t-\theta_2$, as $\mu \to 0$,

$$\frac{u_{t-s}^\mu(\tilde{X}_s^{x,\mu},\omega)}{\mu^2} \to 0, \textit{ in } \quad \hat{P} \quad \textit{probability}\,. \tag{2.17}$$

Proof. As $\mu \to 0$, $\tilde{X}_s^{x,\mu}$ converges to $z_s^t(x)$ in $\hat{P}$ probability uniformly in $\theta_1 \le s \le t-\theta_2$ uniformly on $\mathcal{K}$ by the fact that $\hat{P}(t < \eta(X_\cdot^{x,\mu})) \to 1$ as $\mu \to 0$ as noted in chapter I (which will be true by l.s.c. of $(x,\mu) \mapsto \eta(X_\cdot^{x,\mu})$ and compactness e.g. see Elworthy (1982), P138 and P214). Now V_{t-s}^k is continuous, so $V_{t-s}^k(\tilde{X}_s^{x,\mu}) \to V_{t-s}^k(z_s^t(x))$, in $\hat{P}$-probability. Let $\delta_2 = -\frac{1}{4}\max\{V_{t-s}^k(z_s^t(x)) : \theta_1 \le s \le t-\theta_2, (t,x) \in \mathcal{K}\}$. Therefore, for any $\mu_1 > 0$, there is a subset $\hat\Omega_1^\mu \subset \hat\Omega$ with $\hat{P}(\hat\Omega_1) > 1 - e^{-\frac{\delta_1}{\mu_1^2}}$ with δ_1, $\mu_0 > 0$ such that for $0 < \mu < \mu_0$,

$$V_{t-s}(\tilde{X}_s^{x,\mu}) \le -2\delta_2, \theta_1 \le s \le t-\theta_2, \hat\omega \in \hat\Omega_1^\mu,$$

for $(t,x) \in \mathcal{K}$. From (2.11) it is easy to see changing μ_0 if necessary that

$$P\left\{\omega \in \Omega : u^{\mu}_{t-s}(\tilde{X}^{x,\mu}_s, \omega) \le e^{-\frac{\delta_2}{\mu^2}},\ \text{for } \hat{\omega} \in \hat{\Omega}_1, 0 < \mu \le \mu_0\right\} > 1 - e^{-\frac{\delta_2^2}{8\Gamma\mu_0^2}}.$$

That proves the lemma. ‡‡

Now we consider a modification of condition (N^*) of Chapter I

(N^{*k}). *If* $V^k_t(x) < 0$ *and* $z^t_s(x) = \Phi_{t-s}(\Phi_t^{-1}(x))$, *then*

$$V^k_{t-s}(z^t_s(x)) < 0, \quad 0 \le s \le t.$$

Theorem 2.2. *Assume the conditions (I') of Chapter I, (N^{*k}), and the conditions of Theorem 2.1. Then for sufficiently small* $\mu \ne 0$,

$$u^{\mu}_t(x,\omega) = e^{\frac{1}{\mu^2}V^k_t(x) + \frac{1}{\mu}\int_0^t k(s)dw_s(\omega)}\left[\sqrt{\phi_t(x)}T_0(\Phi_t^{-1}(x)) + R_1\right], \tag{2.18}$$

with $R_1 \to 0$ *as* $\mu \to 0$ *uniformly in any compact subset* $\mathcal{K}$ *of* $\{(t,x) : V_t(x) < 0\}$, *P-a.s.. Here*

$$\phi(s,x) = |\det D\Phi_s^{-1}(x)|.$$

Proof. For $(t,x) \in \mathcal{K}$, define $\mathcal{N}_{\mathcal{K}}$, $\eta, X^{x,\mu}_s, \tilde{X}^{x,\mu}_s$ as before. Then $\tilde{X}^{x,\mu}_s$ converges to $z^t_s(x)$ in $\hat{P}$−probability, uniformly in $s \in [0,t]$ uniformly on $\mathcal{K}$. From (N^{*k}) we know that $V_{t-s}(z^t_s(x)) < 0$ for $0 \le s \le t$. As $c(x,u)$ is $C^{3,3}$ in (x,u), for $(t,x) \in \mathcal{K}$

$$\int_0^t [c(\tilde{X}^{x,\mu}_s, u^{\mu}_{t-s}(\tilde{X}^{x,\mu}_s)) - \bar{c}(\tilde{X}^{x,\mu}_s)]ds \le constant \cdot \int_0^t |u^{\mu}_{t-s}(\tilde{X}^{x,\mu}_s)|ds, \hat{P} - a.s.. \tag{2.19}$$

So applying Lemma 2.2,

$$e^{\frac{1}{\mu^2}\int_0^t [c(\tilde{X}^{x,\mu}_s, u^{\mu}_{t-s}(\hat{X}^{x,\mu}_s)) - \bar{c}(\tilde{X}^{x,\mu}_s)]ds} \to 1, \tag{2.20}$$

in $L^p(\hat{\Omega}, \hat{P}, R)$ for each $1 \le p < +\infty$ and each $\omega \in \Omega$. Now using Lebesgue's dominated convergence theorem to (2.11), noting $\hat{P}\{t < \eta(X^{x,\mu}_{\cdot})\} \to 1$ as $\mu \to 0$, we have for sufficiently small $\mu \ne 0$,

$$\begin{aligned} &\hat{E}\chi_{t<\eta(X^{x,\mu}_{\cdot})}T_0(X^{x,\mu}_t)e^{\frac{1}{2}\int_0^t \Delta V^k_{t-s}(X^{x,\mu}_s)ds} \\ &\cdot e^{\frac{1}{\mu^2}\int_0^t [c(X^{x,\mu}_s, u^{\mu}_{t-s}(X^{x,\mu}_s)) - \bar{c}(X^{x,\mu}_s)]ds - \frac{1}{\mu}\int_0^t k(t-s)dw_{t-s}} \\ &= T_0(z^t_s(x))e^{\frac{1}{2}\int_0^t \Delta V^k_{t-s}(z^t_s(x))ds + \frac{1}{\mu}\int_0^t k(s)dw_s} + R_{11} \end{aligned} \tag{2.21}$$

with $R_{11} \to 0$ P. a.s. as $\mu \to 0$. As (2.12), the last term of (2.11) has upper bound

$$e^{\frac{V_t^k(x)}{\mu^2}+\frac{1}{\mu}\int_0^t k(s)dw_s} \times R_{12}, \tag{2.22}$$

where $R_{12} \leq o(\mu^n)$ for any $n \geq 0$. Applying (I.1.19)

$$\Delta V_{t-s}^k(z_s^t(x)) = -\frac{\partial}{\partial s}\log\phi_{t-s}(z_s^t(x)), (t,x) \in D,$$

and (2.11) (2.21), (2.22), we obtain (2.18) by letting $R_1 = R_{11} + R_{12}$. ‡‡

As before we define $\psi : D \to R$ for $(t,x) \in D$ by

$$\psi_t(x) = \sqrt{\phi_t(x)}T_0(\Phi_t^{-1}(x)).$$

So that if T_0 is positive and C^1, we have

$$\frac{\partial}{\partial t}\log\psi_t = \frac{1}{2}\Delta V_t^k + < \nabla\log\psi_t, \nabla V_t^k > . \tag{2.23}$$

Take $Y(s,x) = V^k(t-s,x) + \mu^2\log\psi(t-s,x)$, $0 \leq s \leq t$, The Feynman-Kac formula, Proposition I.1.1, with some computations using (I.1.4), (2.5), (2.12), (2.22) and (2.23) imply the following proposition:

Proposition 2.1. *Assume all the conditions of Theorem 2.1, that $\bar{c}$ and S_0 are C^3, T_0 is positive and C^2. Then for sufficiently small $\mu \neq 0$ and (t,x) in a compact subset $\mathcal{K} \subset D$,*

$$\begin{aligned} u_t^\mu(x) = e^{\frac{1}{\mu^2}V_t^k(x)} \cdot \Big[\psi_t(x) \cdot \hat{E}\chi_{t<\eta(X^{x,\mu}_\cdot)} e^{\frac{1}{2}\mu^2\int_0^t \psi_{t-s}^{-1}(X_s^{x,\mu})\Delta\psi_{t-s}(X_s^{x,\mu})ds}\cdot \\ e^{\frac{1}{\mu^2}\int_0^t[c(X_s^{x,\mu},u_{t-s}^\mu(X_s^{x,\mu}))-\bar{c}(X_s^{x,\mu})]ds+\frac{1}{\mu}\int_0^t k(s)dw_s} + [o(\mu^n)]\Big]. \end{aligned} \tag{2.24}$$

for any $n \geq 0$: Here $X_s^{x,\mu}$, $0 \leq s \leq \eta(X^{x,\mu}_\cdot)$ is defined by

$$dX_s^{x,\mu} = \mu dB_s + \nabla V_{t-s}^k(X_s^{x,\mu})ds + \mu^2\nabla\log\psi_{t-s}(X_s^{x,\mu})ds, \tag{2.25}$$

with $X_0^{x,\mu} = x$ up to exit time $\eta(X^{x,\mu}_\cdot)$ from $\mathcal{N}_{\mathcal{K}}$.

As Theorem I.1.3 and Theorem 2.2, we can easily formulate the following theorem.

Theorem 2.3. *Assume conditions (I'), (N^{*k}), all the conditions of Proposition 2.1. Then for sufficiently small $\mu \neq 0$,*

$$\begin{aligned} u_t^\mu(x,\omega) = e^{\frac{1}{\mu^2}V_t^k(x)+\frac{1}{\mu}\int_0^t k(s)dw_s(\omega)}\psi_t(x)\cdot \\ \left[1 + \frac{1}{2}\mu^2\int_0^t \psi_{t-s}^{-1}(z_s^t(x))\Delta\psi_{t-s}(z_s^t(x))ds + R_2\right], \end{aligned} \tag{2.26}$$

with $\frac{R_2}{\mu^2} \to 0$ *as* $\mu \to 0$ *uniformly in any compact subset of* $\{(t,x) : V_t^k(x) < 0\}$ *P-a.s..*

§3. The Random Crest

A. In this section we consider Cauchy problem (1.3) with $k(t,x) \equiv k(t)$ again and positive T_0. First we establish a comparison result. Let

$$< k, w >_t^\Delta = \sup_{0 \le s \le t} [-\int_0^s k(t-s) dw_{t-s}],$$
$$< k, w >_{\Delta,t} = \inf_{0 \le s \le t} [-\int_0^s k(t-s) dw_{t-s}].$$

Let $v_t^\mu(x)$ be the solution of the following Cauchy problem

$$\begin{cases} \frac{\partial v_t^\mu(x)}{\partial t} = \frac{1}{2}\mu^2 \Delta v_t^\mu(x) + \frac{1}{\mu^2}\left[c(x, v_t^\mu(x)) - \frac{1}{2}k^2(t)\right] v_t^\mu(x), \\ v_0^\mu(x) = u_0^\mu(x). \end{cases} \tag{3.1}$$

Lemma 3.1. *Assume* $c(x,-)$ *is decreasing. Then for all* t *and* ω*, with the increment* $< k, w >_t^\Delta$ *and* $< k, w >_{\Delta,t}$ *non-zero,*

$$v_t^\mu(x) e^{\frac{1}{\mu} < k,w >_{\Delta,t}} < u_t^\mu(x,\omega) < v_t^\mu(x) e^{\frac{1}{\mu} < k,w >_t^\Delta}. \tag{3.2}$$

Proof. Suppose $u_t^\mu(x,\omega) \ge v_t^\mu(x) e^{\frac{1}{\mu}<k,w>_t^\Delta}$. Then $u_t^\mu(x) > v_t^\mu(x)$. Therefore there exists $s_1 > 0$ such that $u_{t-s_1}^\mu(x + \mu B_{s_1}, \omega) = v_{t-s_1}^\mu(x + \mu B_{s_1})$ and for $0 \le s < s_1$, $u_{t-s}^\mu(x + \mu B_s, \omega) > v_{t-s}^\mu(x + \mu B_s)$. Therefore, by the Feynman-Kac formula and strong Markov property of Brownian motion, we know

$$\begin{aligned} & u_t^\mu(x,\omega) \\ =& \hat{E} u_{t-s_1}^\mu(x + \mu B_{s_1}, \omega) e^{\frac{1}{\mu^2}\int_0^{s_1} c(x+\mu B_s, u_{t-s}^\mu(x+\mu B_s))ds - \frac{1}{2\mu^2}\int_0^{s_1} k^2(t-s)ds - \frac{1}{\mu}\int_0^{s_1} k(t-s)dw_{t-s}} \\ <& \hat{E} v_{t-s_1}^\mu(x + \mu B_{s_1}) e^{\frac{1}{\mu^2}[\int_0^{s_1} c(x+\mu B_s, v_{t-s}^\mu(x+\mu B_s))ds - \frac{1}{2}\int_0^{s_1} k^2(t-s)ds] + \frac{1}{\mu}<k,w>_t^\Delta} \\ =& v_t^\mu(x) e^{\frac{1}{\mu}<k,w>_t^\Delta}. \end{aligned}$$

This is a contradiction. The other inequality goes similarly. ‡‡

Remark 3.1. *The proof of (2.18) when* $k = 0$*, e.g. as in chapter I allows us to substitute*

$$v_t^\mu(x) = e^{\frac{V_t^k(x)}{\mu^2}} [\sqrt{\phi_t(x)} T_0(\Phi_t^{-1}(x)) + o(\mu)]$$

in (3.2) to obtain a quick proof of convergence to the trough when $c(x,u)$ is decreasing in u.

B. Consider conditions:

(MN). There exists a bounded continuous positive function $H_{t,x}$ such that $c(x, H_{t,x}) = \frac{1}{2}k^2(t)$, $0 \le t \le T$, for some $T > 0$ and all $x \in R^r$.

(II^k). $c(x,u)$ is decreasing in u and for any $0 \le t \le T, x \in R^r$, $c(x,u) < c(x, H_{t,x})$, for $u > H_{t,x}$, and $c(x,u) > c(x, H_{t,x})$, for $u < H_{t,x}$.

(DZ^k). (Late caustic assumption (chapter I))

(i). If $(s,y) \in \partial(IntD)$, then there is an open neighbourhood U_0 of y and $\epsilon > 0$ with $V_t^k(x) > 0$ on $IntD \cap (s-\epsilon, s] \times U_0$

(ii). $\{0\} \times R^r \subset IntD$.

and

(N^{**k}). If $V_t^k(x) = 0$ and $z_s^t(x) = \Phi_{t-s}(\Phi^{-1}(x))$, then

$$V_{t-s}^k(z_s^t(x)) < 0, \quad 0 < s < t.$$

Let $Z^k = \{(t,x) : V_t^k(x) = 0\}$. Recall our convention that $V_t^k(x) = +\infty$ if $(t,x) \notin D$. Note that by the Hamilton-Jacobi equation if $\bar{c}(x) - \frac{1}{2}k^2(t) > 0$, then $V_t^k(x)$ is strictly increasing in t, so for each x, there is at most one $Z^k(x)$ with $(Z^k(x), x) \in Z^k$.

Theorem 3.1. *Assume conditions (I'), (MN), (II^k) and (N^{**k}), (DZ^k), and that $\bar{c}$ and S_0 are C^2 with S_0 nonnegative, and that T_0 is positive and bounded continuous. Then for any $\epsilon > 0$ and any compact subset $\mathcal{K}$ of $\{(t,x) : V_t^k(x) > 0, 0 < t \le T\}$, there exists $\mu_0(\mathcal{K}, \epsilon) > 0$, such that for $0 < \mu < \mu_0$, the solution $u_t^\mu(x)$ of Cauchy problem (1.3) satisfies,*

$$(H_{t,x} - \epsilon)e^{\frac{1}{\mu}<k,w>_{\Delta,t}} < u_t^\mu(x,\omega) < (H_{t,x} + \epsilon)e^{\frac{1}{\mu}<k,w>_t^\Delta}. \tag{3.3}$$

Proof. From Theorem I.1.5 we have $\mu_0(\mathcal{K}, \epsilon) > 0$ such that for $0 < \mu < \mu_0$, $(t,x) \in \mathcal{K}$, $H_{t,x} - \epsilon \le v_t^\mu(x) \le H_{t,x} + \epsilon$. Then the theorem follows from Lemma 3.1. ‡‡

The following is immediate from Theorem 3.1. With Theorem 2.1, it shows clearly the propagation of the wave front.

Corollary 3.1. *Assume conditions of Theorem 3.1. Then for* $V_t^k(x) > 0$, $0 < t \leq T$, *there exist random* $c_1 > 0$, $c_2 > 0$, *such that*

$$-c_1 < \underline{\lim}_{\mu \to 0} \mu \log \frac{u_t^\mu(x)}{H_{t,x}} \leq \overline{\lim}_{\mu \to 0} \mu \log \frac{u_t^\mu(x)}{H_{t,x}} < c_2, \quad P-a.s. \tag{3.4}$$

Figure III1 is a numerical simulation by J. Gaines for the travelling wave shown in Figure I1 perturbed by a mild multiplicative white noise $\frac{1}{\mu}u_t^\mu(x)dw_t$, i.e. $du_t^\mu(x) = [\frac{\mu^2}{2}\Delta u_t^\mu(x) + \frac{1}{\mu^2}(1-u_t^\mu(x))u_t^\mu(x)]dt + \frac{1}{\mu}u_t^\mu(x)dw_t$ with an initial Gaussian distribution $u_o^\mu(x) = e^{-\frac{x^2}{2\mu^2}}$. We take $\mu = 0.1$ in the calculation.

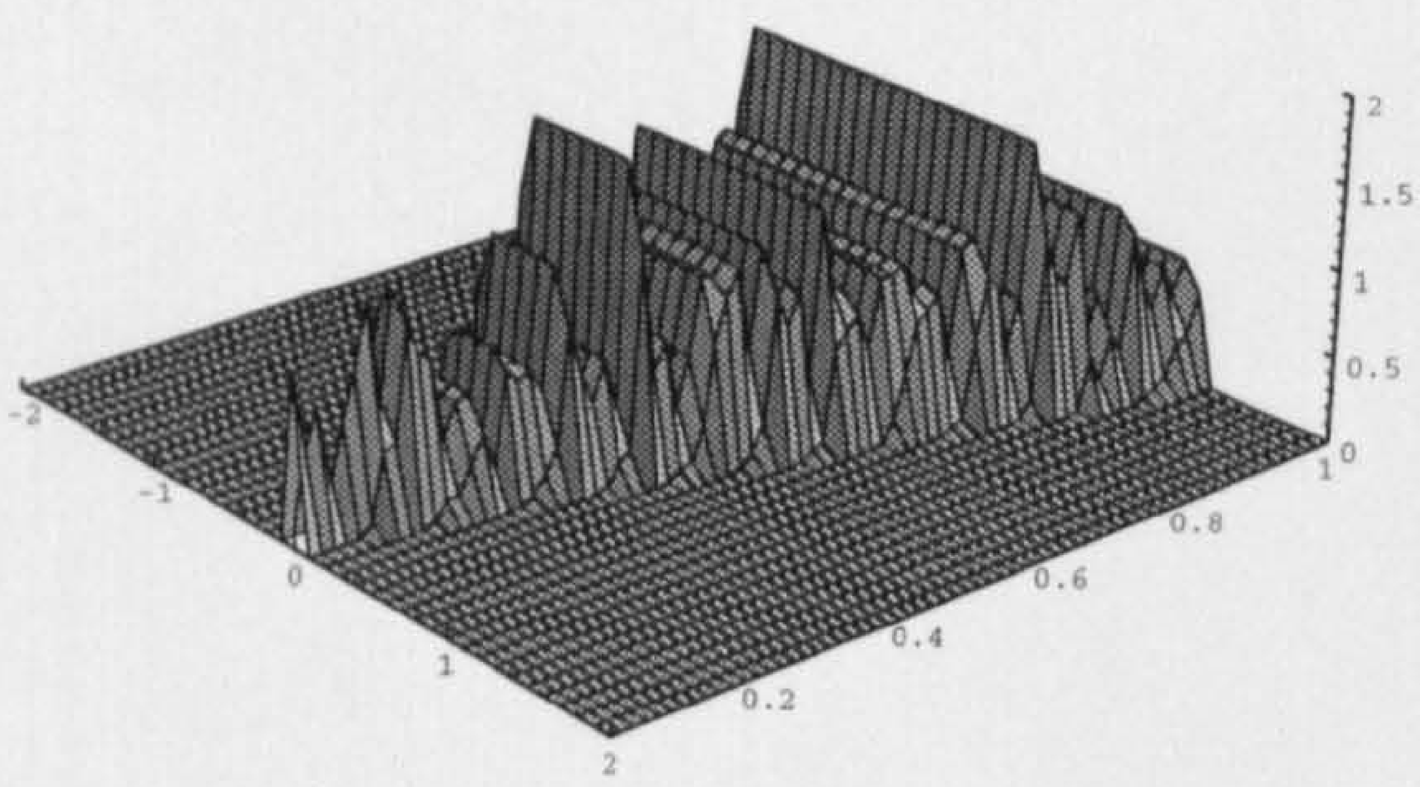

Figure III1

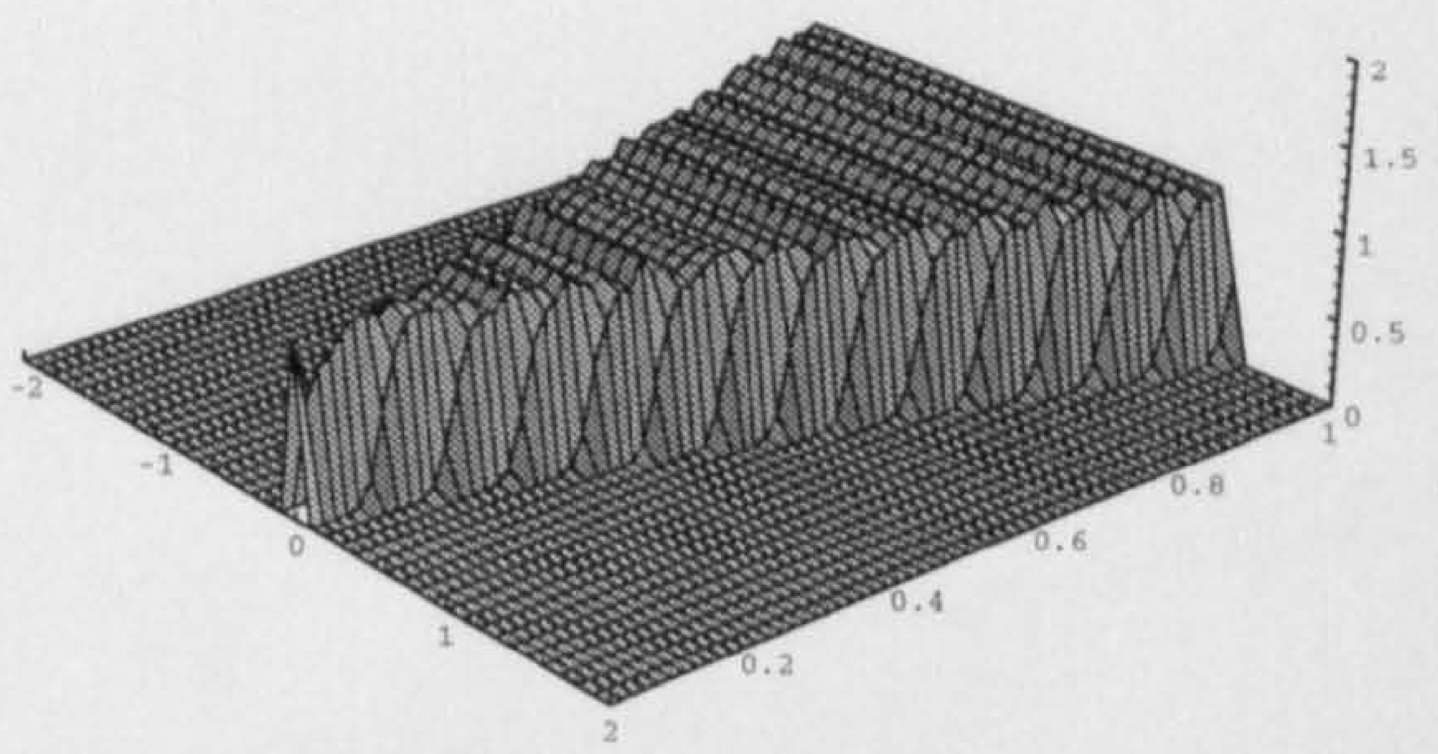

Figure III2

Remark 3.2. *The numerical work described in Figure III1 suggests that the height of the "crest" is rapidly fluctuating in time. Nevertheless another simulation which we will show in Figure III13 leads J. Gaines to suggest that the time average* $\frac{1}{t}\int_0^t u_s^\mu(x)ds$ *should have a straightforward description (see §8).*

§4. The Piecewise Upper Bound of the Travelling Wave

We continue studying equation (1.3) under condition (MN) of §3B. Define

$$G(v) = \sup_{x\in R^r}\{c(x,-)^{-1}(\frac{1}{2}k^2(t) - v) - H_{t,x}\}. \tag{4.1}$$

Concerning the upper bound of the crest of the wave for equation (1.3) with constant k, we have the following more precise result. The function $H_{t,x}$ is independent of t, i.e. $H_{t,x} \equiv H_x$. For simplicity we consider the one dimensional space case, i.e., $x \in R^1$. The infimum over independent ϵ, ν considerably strengthens the result. The coefficients of μ are chosen to make the theorem include a deterministic result by Freidlin.

Theorem 4.1. *Assume conditions (I'), (MN), (II^k) and that k is a constant. Then for $\mu > 0$, the solution $u_t^\mu(x)$ of Cauchy problem (1.3) satisfies,*

$$\begin{aligned} u_t^\mu(x) \le \inf_{\epsilon>0,\nu>0}\Bigg\{ &\left\{ e^{\frac{k}{\sqrt{k\wedge 1}}\epsilon}\cdot [H_x + G(\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s}) + \nu k\mu]\right\} \vee \|T_0\| \\ &+ e^{\frac{k}{\sqrt{k\wedge 1}}\epsilon}\cdot[\sup_{x\in R^1} H_x + G(\mu k \sup_{\sigma\le s\le t}\frac{w_t-w_{t-s}}{s})]\cdot e^{-\frac{\alpha^2(x,\nu,\mu)}{2T\mu^2}}\Bigg\}. \end{aligned} \tag{4.2}$$

For any compact subset $\mathcal{K}$ of $(0,T]\times R^1$, and $0 < \mu < \inf_{(t,x)\in\mathcal{K}} t\log^{-1}\left(\frac{\|T_0\|}{H_x}\vee 1\right)$,

$$\begin{aligned} u_t^\mu(x) \le \inf_{\epsilon>0,\nu>0}\Bigg\{ e^{\frac{k}{\sqrt{k\wedge 1}}\epsilon}\cdot\Bigg\{ &H_x + G(\mu k \sup_{\sigma\le s\le t}\frac{w_t-w_{t-s}}{s} + \mu) + \nu k\mu \\ &+ [\sup_{x\in R^1} H_x + G(\mu k\sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]\cdot e^{-\frac{\alpha^2(x,\nu,\mu)}{2T\mu^2}}\Bigg\}\Bigg\}. \end{aligned} \tag{4.3}$$

Here $\sigma = \inf\{s\in[0,t] : |w_t - w_{t-s}| \ge \frac{\mu\epsilon}{\sqrt{k\wedge 1}}\}$, and $\sigma = t$ if $|w_t - w_{t-s}| < \frac{\mu\epsilon}{\sqrt{k\wedge 1}}$ for all $0\le s\le t$; $\alpha(x,\nu,\mu) = \min\{|\sup\{y : y\in H^{-1}(\beta), \beta < H_x + \nu k\mu\} - x|, |\inf\{y : y \in H^{-1}(\beta), \beta < H_x + \nu k\mu\} - x|\}$.

Proof. We prove (4.3). For any $\epsilon > 0, \nu > 0$, let

$$\hat\Omega_0 = \{\hat\omega \in \hat\Omega : \sup_{0\le t\le T} H_{x+\mu B_t(\hat\omega)} \le H_x + \nu k\mu\}. \tag{4.4}$$

Then by an elementary estimate we have $\hat P(\hat\Omega_0) > 1 - e^{-\frac{\alpha^2(x,\nu,\mu)}{2T\mu^2}}$. For each $\omega\in\Omega$, we introduce a Markov time in the probability space $(\hat\Omega, \hat{\mathcal{F}}, \hat P)$ defined by

$$\tau = \tau^\mu = \inf\{s : u^\mu(t-s, x+\mu B_s, \omega) \le H_{x+\mu B_s} + G(\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)\}. \tag{4.5}$$

Then by the strong Markov property of Brownian motion, and the definition of τ, (4.5), we have for $0 < \mu < \inf\limits_{(t,x)\in\mathcal{K}} t\log^{-1}\left(\frac{\|T_0\|}{H_x} \vee 1\right)$, $\omega \in \Omega$, and $(t,x) \in \mathcal{K}$,

$$
\begin{aligned}
&u_t^\mu(x,\omega)\\
=&\hat{E}u^\mu_{t-\tau\wedge t}(x+\mu B_{\tau\wedge t},\omega)\\
&\cdot e^{\frac{1}{\mu^2}\int_0^{\tau\wedge t} c(x+\mu B_s, u^\mu_{t-s}(x+\mu B_s,\omega))ds - \frac{1}{2\mu^2}k^2 t\wedge\tau + \frac{1}{\mu}k(w_t - w_{t-t\wedge\tau})}\\
=&\hat{E}u^\mu_{t-\tau}(x+\mu B_\tau,\omega)\\
&\cdot e^{\frac{1}{\mu^2}\int_0^{\tau} c(x+\mu B_s, u^\mu_{t-s}(x+\mu B_s,\omega))ds - \frac{1}{2\mu^2}k^2 \tau + \frac{1}{\mu}k(w_t - w_{t-\tau})} \cdot \chi_{\tau\le\sigma}\chi_{\hat{\Omega}_0}\\
&+\hat{E}u^\mu_{t-\tau}(x+\mu B_\tau,\omega)\\
&\cdot e^{\frac{1}{\mu^2}\int_0^{\tau} c(x+\mu B_s, u^\mu_{t-s}(x+\mu B_s,\omega))ds - \frac{1}{2\mu^2}k^2 \tau + \frac{1}{\mu}k(w_t - w_{t-\tau})} \cdot \chi_{\sigma<\tau<t}\chi_{\hat{\Omega}_0}\\
&+\hat{E}u^\mu_{t-\tau}(x+\mu B_\tau,\omega)\\
&\cdot e^{\frac{1}{\mu^2}\int_0^{\tau} c(x+\mu B_s, u^\mu_{t-s}(x+\mu B_s,\omega))ds - \frac{1}{2\mu^2}k^2 \tau + \frac{1}{\mu}k(w_t - w_{t-\tau})} \cdot \chi_{\tau\le\sigma}\chi_{\hat{\Omega}\setminus\hat{\Omega}_0}\\
&+\hat{E}u^\mu_{t-\tau}(x+\mu B_\tau,\omega)\\
&\cdot e^{\frac{1}{\mu^2}\int_0^{\tau} c(x+\mu B_s, u^\mu_{t-s}(x+\mu B_s,\omega))ds - \frac{1}{2\mu^2}k^2 \tau + \frac{1}{\mu}k(w_t - w_{t-\tau})} \cdot \chi_{\sigma<\tau<t}\chi_{\hat{\Omega}\setminus\hat{\Omega}_0}\\
&+\hat{E}T_0(x+\mu B_t)\cdot e^{-\frac{1}{\mu^2}S_0(x+\mu B_t)}\\
&\cdot e^{\frac{1}{\mu^2}\int_0^{t} c(x+\mu B_s, u^\mu_{t-s}(x+\mu B_s,\omega))ds - \frac{1}{2\mu^2}k^2 t + \frac{1}{\mu}kw_t} \cdot \chi_{\tau\ge t}\\
\le&\hat{E}[H_{x+\mu B_\tau} + G(\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]\\
&\cdot e^{\frac{1}{\mu^2}\{\int_0^\tau[\frac12 k^2 - (\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]ds - \frac12 k^2\tau\} + \frac{k}{\sqrt{k\wedge 1}}\epsilon} \cdot \chi_{\tau\le\sigma}\chi_{\hat{\Omega}_0}\\
&+\hat{E}[H_{x+\mu B_\tau} + G(\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]\\
&\cdot e^{\frac{1}{\mu^2}\{\int_0^\tau[\frac12 k^2 - (\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]ds - \frac12 k^2\tau + \mu k\sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s}\tau\}} \cdot \chi_{\sigma<\tau<t}\chi_{\hat{\Omega}_0}\\
&+\hat{E}[H_{x+\mu B_\tau} + G(\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]\\
&\cdot e^{\frac{1}{\mu^2}\{\int_0^\tau[\frac12 k^2 - (\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]ds - \frac12 k^2\tau\} + \frac{k}{\sqrt{k\wedge 1}}\epsilon} \cdot \chi_{\tau\le\sigma}\chi_{\hat{\Omega}\setminus\hat{\Omega}_0}\\
&+\hat{E}[H_{x+\mu B_\tau} + G(\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]\\
&\cdot e^{\frac{1}{\mu^2}\{\int_0^\tau[\frac12 k^2 - (\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]ds - \frac12 k^2\tau + \mu k\sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s}\tau\}} \cdot \chi_{\sigma<\tau<t}\chi_{\hat{\Omega}\setminus\hat{\Omega}_0}\\
&+\|T_0\|e^{\frac{1}{\mu^2}\{\int_0^t[\frac12 k^2 - (\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu)]ds - \frac12 k^2 t + \mu k\sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s}t\}} \cdot \chi_{\tau\ge t}\\
\le&e^{\frac{k}{\sqrt{k\wedge 1}}\epsilon}\cdot[H_x + G(\mu k \sup_{\sigma\le s\le t}\frac{w_t - w_{t-s}}{s} + \mu) + \nu k\mu]\hat{P}\{\tau < t\}
\end{aligned}
$$

$$+ e^{\frac{k}{\sqrt{k\wedge 1}}\epsilon} \cdot [\sup_x H_x + G(\mu k \sup_{\sigma\le s\le t} \frac{w_t - w_{t-s}}{s} + \mu)] e^{-\frac{\alpha^2(x,\nu,\mu)}{2T\mu^2}}$$

$$+ H_x \hat{P}\{\tau \ge t\}$$

$$\le e^{\frac{k}{\sqrt{k\wedge 1}}\epsilon} \cdot \left\{ H_x + G(\mu k \sup_{\sigma\le s\le t} \frac{w_t - w_{t-s}}{s} + \mu) + \nu k \mu \right.$$

$$\left. + [\sup_x H_x + G(\mu k \sup_{\sigma\le s\le t} \frac{w_t - w_{t-s}}{s} + \mu)] \cdot e^{-\frac{\alpha^2(x,\nu,\mu)}{2T\mu^2}} \right\}.$$

So we have (4.3). The estimate (4.2) can be obtained by the same procedure to define a proper stopping time as (4.5) and doing an alternative calculation. ‡‡

The following two corollaries are direct implications of Theorem 4.1.

Corollary 4.1. *For the KPP equation with white noise perturbation, we have for* $\mu > 0$,

$$u_t^\mu(x) \le \inf_{\epsilon>0} \left\{ e^{\frac{k}{\sqrt{k\wedge 1}}\epsilon} \cdot \left[1 - \frac{k^2}{2\hat{c}} + \frac{1}{\hat{c}}(\mu k \sup_{\sigma\le s\le t} \frac{w_t - w_{t-s}}{s}) \right] \right\} \vee ||T_0||. \tag{4.6}$$

For any compact subset $\mathcal{K}$ *of* $(0,T] \times R^1$ *and* $0 < \mu < \inf_{(t,x)\in\mathcal{K}} t \log^{-1} \left(\frac{||T_0||}{1-\frac{k^2}{2\hat{c}}} \vee 1 \right)$,

$$u_t^\mu(x) \le \inf_{\epsilon>0} e^{\frac{k}{\sqrt{k\wedge 1}}\epsilon} \cdot \left\{ 1 - \frac{k^2}{2\hat{c}} + \frac{1}{\hat{c}}(\mu k \sup_{\sigma\le s\le t} \frac{w_t - w_{t-s}}{s} + \mu) \right\}. \tag{4.7}$$

Here $\sigma = \inf\{s \in [0,t] : |w_t - w_{t-s}| \ge \frac{\mu\epsilon}{\sqrt{k\wedge 1}}\}$, *and* $\sigma = t$ *if* $|w_t - w_{t-s}| < \frac{\mu\epsilon}{\sqrt{k\wedge 1}}$ *for all* $0 \le s \le t$.

Corollary 4.2. *(Freidlin (1985)) Consider deterministic generalised KPP equations, i.e.,* $k = 0$, $H_x \equiv 1$. *We have for any* $\mu > 0$,

$$u_t^\mu(x) \le 1 \vee ||T_0||, \tag{4.8}$$

and for any $0 < \mu < \inf_{(t,x)\in\mathcal{K}} t \log^{-1}(||T_0|| \vee 1)$,

$$u_t^\mu(x) \le 1 + G(\mu). \tag{4.9}$$

§5. The Propagation of the Travelling Wave in a Weak White Noise Potential

A. In this section we consider

$$\begin{cases} du_t^\mu(x) = \left[\frac{1}{2}\mu^2\Delta u_t^\mu(x) + \frac{1}{\mu^2}c(x,u_t^\mu(x))u_t^\mu(x)\right]dt + k(t)u_t^\mu(x)dw_t, \\ u_0^\mu(x) = T_0(x)e^{-\frac{1}{\mu^2}S_0(x)}. \end{cases} \tag{5.1}$$

Here $\Delta, c, k, T_0, S_0, w_t(\omega)$ are the same as before, and $u^\mu : [0,\infty) \times R^r \times \Omega \to R^1$ denotes the solution. As (2.1), by the Feynman-Kac formula $u_t^\mu(x)$ satisfies

$$\begin{aligned} u_t^\mu(x) &= \hat{E}T_0(x+\mu B_t) \\ &\cdot e^{-\frac{1}{\mu^2}S_0(x+\mu B_t)+\frac{1}{\mu^2}\int_0^t c(x+\mu B_s,u_{t-s}^\mu(x+\mu B_s))ds-\frac{1}{2}\int_0^t k^2(t-s)ds-\int_0^t k(t-s)dw_{t-s}}, \end{aligned} \tag{5.2}$$

where B_t is an R^r Brownian motion defined on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ independent of w_t.

Let $\Phi, z, \Sigma_a, \sigma(a), \sigma^T(a), D, D_t, \phi, \psi$ be defined as in §2. For $(t,x) \in D$, define

$$V_t(x) = \int_0^t \bar{c}(\Phi_s(\Phi_t^{-1}(x)))ds - S_0(\Phi_t^{-1}(x)) - \frac{1}{2}\int_0^t |\dot{\Phi}_s(\Phi_t^{-1}(x))|^2 ds \tag{5.3}$$

with convention $V_t(x) = +\infty$ if $(t,x) \notin D$.

A similar result to Theorem 2.1 can be obtained.

Theorem 5.1. *Assume the conditions of Theorem 2.1. Then for any compact subset $\mathcal{K}$ of $\{(t,x) : V_t(x) < 0, t > 0\}$, there exists $\mu_1(\mathcal{K}) > 0$ such that*

$$P\{\omega \in \Omega : \sup_{(t,x)\in\mathcal{K}} u_t^\mu(x,\omega) > e^{-\frac{\delta}{\mu^2}},\ some\ 0 < \mu \le \mu_0\} < e^{-\frac{\delta}{2\mu_0^2}}. \tag{5.4}$$

for all $0 < \mu_0 < \mu_1$, with $\delta = -\frac{1}{2}\sup\{V_t(x) : (t,x) \in \mathcal{K}\}$. In particular, as $\mu \to 0$, $u_t^\mu(x,\omega) \to 0$, uniformly in $\mathcal{K}$ P-a.s..

Proof. As in the proof of Theorem 2.1, we take an open set $\mathcal{N}_\mathcal{K} \subset IntD$ with $(t-s, z_s^t(x)) \in \mathcal{N}_\mathcal{K}$ for $0 \le s \le t$ if $(t,x) \in \mathcal{K}$. Let $X_s^{x,\mu}$ be the solution of (2.2) with $A_s = \nabla V_{t-s}$ up to the exit time $\eta(X_\cdot^{x,\mu})$ from $\mathcal{N}_\mathcal{K}$. Similarly to (2.11), from (5.2), we get

$$\begin{aligned} u_t^\mu(x) &= e^{\frac{1}{\mu^2}V_t(x)}\hat{E}\chi_{t<\eta(X_\cdot^{x,\mu})}T_0(X_t^{x,\mu})e^{\frac{1}{2}\int_0^t \Delta V_{t-s}(X_s^{x,\mu})ds} \\ &\cdot e^{\frac{1}{\mu^2}\int_0^t\left[c(X_s^{x,\mu},u_{t-s}^\mu(X_s^{x,\mu}))-\bar{c}(X_s^{x,\mu})\right]ds-\frac{1}{2}\int_0^t k^2(s)ds+\int_0^t k(s)dw_s} \\ &+ \hat{E}\chi_{t>\eta(x+\mu B_\cdot)}T_0(x+\mu\hat{w}_t)e^{-\frac{1}{\mu^2}S_0(x+\mu B_t)+\frac{1}{\mu^2}\int_0^t c(x+\mu B_s,u_{t-s}(x+\mu\hat{w}_s))ds} \\ &\cdot e^{-\frac{1}{2}\int_0^t k^2(s)ds+\int_0^t k(s)dw_s}. \end{aligned} \tag{5.5}$$

By (2.8) in Lemma 2.1, we immediately have that

$$P\{\omega \in \Omega : \mu_0^2 \sup_{0\le t\le T} [-\frac{1}{2}\int_0^t k^2(s)ds + \int_0^t k(s)dw_s(\omega)] > \frac{1}{2}\delta\} < e^{-\frac{\delta}{2\mu_0^2}}. \tag{5.6}$$

The theorem follows the same arguments as in the proof of Theorem 2.1. ‡‡

B. In this section we consider the following conditions called (N^*) and (N^{**}) instead of (N^{*k}) and (N^{**k}) of last two sections as with (N^{*k}) and (N^{**k}) they are clearly analogue of Freidlin's condition (N):

(N^*). *If* $V_t(x) < 0$, *then* $V_{t-s}(z_s^t(x)) < 0, \quad 0 \le s \le t,$

(N^{**}). *If* $V_t(x) = 0$, *then* $V_{t-s}(z_s^t(x)) < 0, \quad 0 < s < t.$

The following two theorems are proved in the same way as Theorem 2.2 and Theorem 2.3 respectively.

Theorem 5.2. *Assume the conditions (I'),* (N^*), *all conditions of Theorem 2.1, and* T_0 *is continuous. Then, for sufficiently small* $\mu \ne 0$,

$$u_t^\mu(x,\omega) = e^{\frac{1}{\mu^2}V_t(x) - \frac{1}{2}\int_0^t k^2(s)ds + \int_0^t k(s)dw_s(\omega)} \cdot [\sqrt{\phi_t(x)}T_0(\Phi_t^{-1}(x)) + R_1], \tag{5.7}$$

with $R_1 \to 0$ *as* $\mu \to 0$ *P-a.s. uniformly in any compact subset of* $\{(t,x) : V_t(x) < 0\}$.

Theorem 5.3. *Assume conditions (I'),* (N^*) *and all conditions of Proposition 2.1. Then for sufficiently small* $\mu \ne 0$,

$$\begin{aligned} u_t^\mu(x,\omega) &= \psi_t(x) e^{\frac{1}{\mu^2}V_t(x) - \frac{1}{2}\int_0^t k^2(s)ds + \int_0^t k(s)dw_s(\omega)} \\ &\cdot [1 + \frac{1}{2}\mu^2 \int_0^t \psi_{t-s}^{-1}(z_s^t(x))\Delta\psi_{t-s}(z_s^t(x))ds + R_2], \end{aligned} \tag{5.8}$$

with $\frac{R_2}{\mu^2} \to 0$ *as* $\mu \to 0$ *P-a.s. uniformly in any compact subset of* $\{(t,x) : V_t(x) < 0\}$.

C. Now we consider the condition:

(II). For any constant $\lambda > 0$, $\inf\{c(x,u) : x \in R^r, u \le 1-\lambda\} > 0$ *and* $\sup\{c(x,u) : x \in R^r, u \ge 1+\lambda\} < 0$.

Lemma 5.1. *Suppose* $T_0(x)$ *is a nonnegative bounded function,* $S_0(x)$ *is a nonnegative continuous function, and condition (II). Then for any* $0 < \epsilon < 1$ *and* $\bar{t} \ge \underline{t} > 0$, *there exist* $\delta(\bar{t},\underline{t},\epsilon) > 0$, $\mu_1(\bar{t},\underline{t},\epsilon) > 0$, *such that*

$$P\{\omega \in \Omega : \sup_{0<\mu\le\mu_0} \sup_{\underline{t}\le t\le\bar{t}, x\in R^r} u_t^\mu(x,\omega) > 1+\epsilon\} < e^{-\frac{\delta}{\mu_0}}, \tag{5.9}$$

for all $0 < \mu_0 < \mu_1$.

Proof. For any $\epsilon > 0$, we set $\alpha = -\sup\{c(x,u) : u \geq 1 + \frac{1}{3}\epsilon, x \in R^r\}$. By (II), $\alpha > 0$. Note that for any $\mu_0 > 0$ by (2.9) if $0 < \mu \leq \mu_0$ and

$$\Omega_1 = \{\omega \in \Omega : -\int_0^s k(t-s)dw_{t-s}(\omega) \leq \frac{1}{3}\epsilon, 0 \leq s \leq \mu\} \tag{5.10}$$

$P(\Omega_1) > 1 - e^{-\frac{\epsilon^2}{18\mu_0 \overline{k}_t^2}}$ where $\overline{k}_t = \sup\{k(s) : 0 \leq s \leq t\}$. By (2.8) in Lemma 2.1, there exists $0 < \mu_1(\overline{t}, \underline{t},, \epsilon) \leq 1$, if $0 < \mu_0 < \mu_1$, there is a subset $\Omega_2^{\mu_0}(\overline{t}, \underline{t},, \epsilon) \subset \Omega$ with $P(\Omega_2) > 1 - e^{-\frac{\alpha(1 \wedge \frac{1}{2}\underline{t})}{\mu_0}}$, such that for $0 < \mu < \mu_0$ and $\omega \in \Omega_2$

$$-\alpha + \mu[-\frac{1}{2}\int_0^s k^2(t-s)ds - \int_0^s k(t-s)dw_{t-s}(\omega)] \leq 0, \text{ for } \mu \leq s \leq t \leq \overline{t}, \tag{5.11}$$

and

$$\left\{\sup_{x \in R^r} T_0(x)\right\} e^{\frac{1}{\mu^2}[-\alpha t + \mu^2(-\frac{1}{2}\int_0^t k^2(t-s)ds - \int_0^t k(t-s)dw_{t-s}(\omega))]} \leq 1 + \epsilon, \underline{t} \leq t \leq \overline{t}. \tag{5.12}$$

Now for fixed $\omega \in \Omega_1 \cap \Omega_2$, we introduce a Markov time in the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ defined by

$$\tau = \tau^\mu = \inf\{s : u^\mu(t-s, x + \mu B_s, \omega) \leq 1 + \frac{1}{3}\epsilon\}. \tag{5.13}$$

Then by the strong Markov property of Brownian motion, (5.10), (5.11), (5.12) and the definition of τ, (5.13), we have if $0 < \mu_0 < \mu_1$, for $0 < \mu < \mu_0$ and $\omega \in \Omega_1 \cap \Omega_2$, and $\underline{t} \leq t \leq \overline{t}, x \in R^r$,

$$\begin{aligned}
&u_t^\mu(x,\omega) \\
=&\hat{E}u_{t-\tau\wedge t}^\mu(x + \mu B_{\tau\wedge t}, \omega) \\
&e^{\frac{1}{\mu^2}\int_0^{\tau\wedge t} c(x+\mu B_s, u_{t-s}^\mu(x+\mu B_s,\omega))ds - \frac{1}{2}\int_0^{t\wedge\tau} k^2(t-s)ds - \int_0^{t\wedge\tau} k(t-s)dw_{t-s}(\omega)} \\
=&\hat{E}u_{t-\tau}^\mu(x + \mu B_\tau, \omega) \\
&e^{\frac{1}{\mu^2}\int_0^{\tau} c(x+\mu B_s, u_{t-s}^\mu(x+\mu B_s,\omega))ds - \frac{1}{2}\int_0^{\tau} k^2(t-s)ds - \int_0^{\tau} k(t-s)dw_{t-s}(\omega)} \cdot \chi_{\tau<\mu}
\end{aligned}$$

$$+\hat{E}u^{\mu}_{t-\tau}(x+\mu B_\tau,\omega)$$
$$e^{\frac{1}{\mu^2}\int_0^\tau c(x+\mu B_s,u^{\mu}_{t-s}(x+\mu B_s,\omega))ds-\frac{1}{2}\int_0^\tau k^2(t-s)ds-\int_0^\tau k(t-s)dw_{t-s}(\omega)}\cdot\chi_{\mu\le\tau\le t}$$
$$+\hat{E}T_0(x+\mu B_t)$$
$$e^{-\frac{1}{\mu^2}S_0(x+\mu B_t)+\frac{1}{\mu^2}\int_0^t c(x+\mu B_s,u^{\mu}_{t-s}(x+\mu B_s,\omega))ds-\frac{1}{2}\int_0^t k^2(t-s)ds-\int_0^t k(t-s)dw_{t-s}(\omega)}\cdot\chi_{\tau>t}$$
$$\le(1+\frac{1}{3}\epsilon)\hat{E}e^{-\frac{1}{2}\int_0^\tau k^2(t-s)ds-\int_0^\tau k(t-s)dw_{t-s}(\omega)}\cdot\chi_{\tau<\mu}$$
$$+(1+\frac{1}{3}\epsilon)\hat{E}e^{\frac{1}{\mu^2}[-\alpha\mu+\mu^2(-\frac{1}{2}\int_0^\tau k^2(t-s)ds-\int_0^\tau k(t-s)dw_{t-s}(\omega))]}\chi_{\mu\le\tau\le t}$$
$$+||T_0||e^{\frac{1}{\mu^2}[-\alpha t+\mu^2(-\frac{1}{2}\int_0^t k^2(t-s)ds-\int_0^t k(t-s)dw_{t-s}(\omega))]}\hat{P}\{\tau>t\}$$
$$\le(1+\epsilon)[\hat{P}\{\tau<\mu\}+\hat{P}\{\mu\le\tau\le t\}+\hat{P}\{\tau>t\}]$$
$$=1+\epsilon.$$

The result follows by taking $\delta=\frac{1}{2}\{\frac{\epsilon^2}{18\bar{k}_t^2}\wedge[\alpha(1\wedge\frac{1}{2}\underline{t})]\}$ and changing $\mu_1\le\frac{\delta}{\log 2}$ if necessary. ‡‡

D. Set $Z=\{(t,x):V_t(x)=0\}$ and with convention $V_t(x)=\infty$ if $(t,x)\notin D$. The function $Z(x)$ is defined similarly as in §3. Also we have an alternative late caustic assumption (DZ) corresponding to $V_t(x)$.

In order to prove Lemma 5.3 in the sequel, we need the following lemma which can be proved similarly to Lemma 2.2.

Lemma 5.2. *Assume all conditions of Theorem 2.1. Then if* $0\le\theta_i\le\frac{1}{2}t$, $i=1,2$, *for any compact subset* $\mathcal{K}$ *of* $\{(t,x):V_{t-s}(z^t_s(x))<0,\theta_1\le s\le t-\theta_2\}$ *and* $\mu_1>0$, *there exist* $\mu_0>0$, $\delta_1>0$, $\delta_2>0$ *and* $\Omega_1\subset\Omega$ *with* $P(\Omega_1)>1-e^{-\frac{\delta_2}{2\mu_0^2}}$ *such that if* $\omega\in\Omega_1$,

$$\hat{P}\{\hat{\omega}\in\hat{\Omega}:\mu^2\sup_{\theta_1\le s\le t-\theta_2,(t,x)\in\mathcal{K}}\log u^{\mu}_{t-s}(\tilde{X}^{x,\mu}_s(\hat{\omega}),\omega)<-\delta_2,\ \textit{for all}\quad 0<\mu\le\mu_0\}$$
$$>1-e^{-\frac{\delta_1}{\mu_1^2}}. \tag{5.14}$$

In particular, for almost all $\omega\in\Omega$, *and* $\theta_1\le s\le t-\theta_2$, *as* $\mu\to 0$,

$$\frac{u^{\mu}_{t-s}(\tilde{X}^{x,\mu}_s,\omega)}{\mu^2}\to 0,\ in\quad \hat{P}\quad probability\ .$$

Lemma 5.3. *Assume conditions (I'), (II) , (N**), and that* $\bar{c}$ *and* S_0 *are* C^2 *with* S_0 *nonnegative and that* T_0 *is positive and continuous. Then for any* $\gamma>0$ *and any* *compact subset* $\mathcal{K}$ *of* $\{(t,x) : V_t(x) = 0\}$, *there exist* $\mu_1(\mathcal{K},\gamma) > 0$, $\delta(\mathcal{K},\gamma) > 0$ *such that*

$$P\{\omega \in \Omega : \mu^2 \log u_t^\mu(x,\omega) \geq -\gamma, \text{ all } 0 < \mu \leq \mu_0, (t,x) \in \mathcal{K}\} > 1 - e^{-\frac{\delta}{\mu_0^2}}, \tag{5.15}$$

for all $0 < \mu_0 < \mu_1$.

Proof. Let $(t,x) \in \mathcal{K}$ and $X_s^{x,\mu}$ be defined by (2.2) with $A_s = \nabla V_{t-s}$ up to exit time $\eta(X_\cdot^{x,\mu})$ from $\mathcal{N}_\mathcal{K}$. Based on (5.5), by Jensen's inequality, we have

$$\begin{aligned}
&\mu^2 \log u_t^\mu(x) \\
\geq &\mu^2 \left[\log \hat{P}(t < \eta(X_\cdot^{x,\mu})) + \log \inf_{x:(s,x)\in\mathcal{N}_\mathcal{K}} T_0(x) + \frac{1}{2} t \inf_{(s,y)\in\mathcal{N}_\mathcal{K}} \Delta V_s(y)\right] \\
&+ \hat{P}(t < \eta(X_\cdot^{x,\mu}))^{-1} \hat{E}_x^\mu \chi_{t<\eta(X_\cdot^{x,\mu})} \int_0^t [c(X_s^{x,\mu}, u_{t-s}^\mu(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})]\, ds \\
&+ \mu^2[-\frac{1}{2}\int_0^t k^2(t-s)ds - \int_0^t k(t-s)dw_{t-s}].
\end{aligned} \tag{5.16}$$

For any $\gamma > 0$, we choose $0 < \theta \leq \frac{1}{4}t$ such that $\theta\hat{c} \leq \frac{1}{5}\gamma$. Recall $\hat{P}(t < \eta(X_\cdot^{x,\mu})) \to 1$ uniformly for $\mathcal{K}$ as noted in §1 of chapter I. Therefore there exists $\mu_1^{(1)}(\mathcal{K}) > 0$ such that for $0 < \mu < \mu_1^{(1)}$ and $(t,x) \in \mathcal{K}$,

$$\mu^2 \left[\log \hat{P}(t < \eta(X_\cdot^{x,\mu})) + \log \inf_{x:(s,x)\in\mathcal{N}_\mathcal{K}} T_0(x) + \frac{1}{2} t \inf_{(s,y)\in\mathcal{N}_\mathcal{K}} \Delta V_s(y)\right] \geq -\frac{1}{5}\gamma.$$

By condition (N^{**}) we know there is $\delta^* > 0$ such that $V_{t-s}(z_s^t(x)) < -\delta^*$, for $\theta \leq s \leq t-\theta$. Note that $\tilde{X}_s^{x,\mu}$ converges to $z_s^t(x)$ in $\hat{P}$-probability uniformly in $s \in [0,t]$. So given $\epsilon^* > 0$, there exists $\mu_1^{(2)}(\mathcal{K}) > 0$, such that for $0 < \mu_0 < \mu_1^{(2)}$, there exists $\Omega_0^{\mu_0} \subset \Omega$ with $P(\Omega_0^{\mu_0}) > 1 - e^{-\frac{\delta^*}{4\mu_0^2}}$ such that for $0 < \mu < \mu_0$, $\omega \in \Omega_0$,

$$\hat{P}\{\hat{\omega} \in \hat{\Omega} : u_{t-s}^\mu(\tilde{X}_s^{x,\mu}) < e^{-\frac{\delta^*}{2\mu^2}} \text{ for all } (t,x) \in \mathcal{K}\} > 1 - \epsilon^*, \theta \leq s \leq t-\theta,$$

by Lemma 5.2. Therefore, applying Lebesgue's dominated convergence theorem, there exists $\mu_1^{(3)}(\mathcal{K}) > 0$ with $0 < \mu_1^{(3)} < \mu_1^{(2)}$ such that for $0 < \mu_0 < \mu_1^{(3)}$, $\omega \in \Omega_0^{\mu_0}$, $0 < \mu < \mu_0$,

$$\hat{P}(t < \eta(X_\cdot^{x,\mu}))^{-1} \hat{E}\chi_{t<\eta(X_\cdot^{x,\mu})} \int_\theta^{t-\theta} [c(X_s^{x,\mu}, u_{t-s}^\mu(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})]\, ds > -\frac{1}{5}\gamma,$$

for all $(t,x) \in \mathcal{K}$. Let $\mu_1 = \min\{\mu_1^{(1)}, \mu_1^{(3)}\}$. From

$$\int_0^t = \int_0^\theta + \int_\theta^{t-\theta} + \int_{t-\theta}^t,$$

and the fact that

$$P\{\omega \in \Omega : \mu^2 \sup_{0 \le t \le T} [-\frac{1}{2}\int_0^t k^2(t-s)ds - \int_0^t k(t-s)dw_{t-s}] < -\frac{1}{5}\gamma, \tag{5.17}$$
$$\text{some } 0 < \mu \le \mu_0\} < e^{-\frac{\gamma}{5\mu_0^2}},$$

we get the proof of the Lemma. ‡‡

Theorem 5.4. *Assume conditions (I'), (II), (N^{**}), (DZ), and that $\bar{c}$ and S_0 are C^2 with S_0 nonnegative, and that T_0 is positive, bounded, and continuous. Then for any $\epsilon > 0$ and any compact subset $\mathcal{K}$ of $\{(t,x) : V_t(x) > 0\}$, there exist $\mu_1(\mathcal{K},\epsilon) > 0$, $\delta(\mathcal{K},\epsilon) > 0$, such that*

$$P\{\omega \in \Omega : \sup_{0<\mu\le\mu_0} \sup_{(t,x)\in\mathcal{K}} |u_t^\mu(x,\omega) - 1| > \epsilon\} < e^{-\frac{\delta}{\mu_0}}, \tag{5.18}$$

for all $0 < \mu_0 < \mu_1$. In particular $u_t^\mu(x,\omega) \to 1$, uniformly in $\mathcal{K}$, P-a.s..

Proof. Define $\underline{t} = \inf\{t : (t,x) \in \mathcal{K}\}$, $\bar{t} = \sup\{t : (t,x) \in \mathcal{K}\}$. After Lemma 5.1 we only need to show for any $\epsilon > 0$, there exist $\mu_1(\epsilon,\mathcal{K}) > 0$, $\delta(\epsilon,\mathcal{K}) > 0$ such that

$$P\{\omega \in \Omega : u_t^\mu(x,\omega) < 1-\epsilon, \text{ some } 0 < \mu \le \mu_0, (t,x) \in \mathcal{K}\} < e^{-\frac{\delta}{\mu_0}},$$

for all $0 < \mu_0 < \mu_1$. For this we set $\alpha = \inf\{c(y,u) : u \le 1 - \frac{1}{4}\epsilon, y \in R^r\}$. By (II), $\alpha > 0$. Choose $h > 0$ by assumption (DZ) such that

$$\{(s,y) : t-h \le s \le t \text{ and } y \in \overline{B}_h(x)\} \cap Z = \emptyset, (t,x) \in \mathcal{K}. \tag{5.19}$$

As in the proof of Lemma 5.1, for any $\mu_0 \le \frac{\epsilon}{4\bar{k}_t^2}$ by (2.9) if $0 < \mu \le \mu_0$ and

$$\Omega_1 = \{\omega \in \Omega : -\frac{1}{2}\int_0^s k^2(t-s)ds - \int_0^s k(t-s)dw_{t-s}(\omega) \ge -\frac{1}{4}\epsilon, 0 < s < \mu\}, \tag{5.20}$$

then $P(\Omega_1) > 1 - e^{-\frac{\epsilon^2}{128\mu_0\bar{k}_t^2}}$ with $\bar{k}_t = \sup\{k(s), 0 \le s \le t\}$. By (2.8) of Lemma 2.1 for any $0 < \mu_0 < 1$, there is a subset $\Omega_2^{\mu_0} \subset \Omega$ with $P(\Omega_2) > 1 - e^{-\frac{\frac{1}{4}\alpha(1\wedge h)}{\mu_0}}$, such that for $0 < \mu < \mu_0$ and $\underline{t} \le t \le \bar{t}$ and $\omega \in \Omega_2$,

$$\frac{1}{2}\alpha + \mu[-\frac{1}{2}\int_0^s k^2(t-s)ds - \int_0^s k(t-s)dw_{t-s}(\omega)] \ge 0, \text{ for } \mu \le s \le t. \tag{5.21}$$

$$\frac{1}{2}\alpha h + \mu^2[-\frac{1}{2}\int_0^s k^2(t-s)ds - \int_0^s k(t-s)dw_{t-s}(\omega)] \ge 0, \text{ for } h \le s \le t. \tag{5.22}$$

Now for fixed $\omega \in \Omega_1 \cap \Omega_2$, we define two Markov times on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$:

$$\tau_1 = \tau_1^{\mu,\epsilon} = \inf\{s : u_{t-s}^{\mu}(x + \mu B_s, \omega) \geq 1 - \frac{1}{4}\epsilon\}, \tag{5.23}$$

with convention $\tau_1 = \infty$ if the set is empty, and

$$\tau_2 = \tau_2^{\mu} = \inf\{s : V_{t-s}(x + \mu B_s) = 0\}, \tag{5.24}$$

with convention $V(r, y) = \infty$ if $(r, y) \notin D$. Note that τ_2 is defined and in $[0, t]$ if $t > Z(x)$ by Lemma I.1.6 and assumption (DZ). Set $\tau = \tau_1 \wedge \tau_2$. From the definition of τ_2, we have $(t - \tau_2, x + \mu B_{\tau_2}) \in Z$, $\hat{P} - a.s.$. Therefore, there is a compact subset K in Z such that $\hat{P}(\hat{\Omega}_0) > 1 - \frac{1}{4}\epsilon$ when $\hat{\Omega}_0 = \{\hat{\omega} \in \hat{\Omega}, (t - \tau_2, x + \mu B_{\tau_2}) \in K\}$. From Lemma 5.3 we know there exists $\mu_1^{(1)}(\mathcal{K}, \epsilon) > 0$ such that if $0 < \mu_0 < \mu_1^{(1)}$, there is a subset $\Omega_3^{\mu} \subset \Omega$ with $P(\Omega_3) > 1 - e^{-\frac{\delta_1}{\mu_0}}$ with some $\delta_1 > 0$, such that for $0 < \mu < \mu_0$, $\omega \in \Omega_3$ and $(t, x) \in \mathcal{K}$,

$$u_{t-\tau_2}^{\mu}(x + \mu B_{\tau_2}, \omega) > e^{-\frac{\alpha h}{2\mu^2}}, \text{ on } \hat{\Omega}_0. \tag{5.25}$$

Let $\tau_3 = \inf\{s : |\mu \hat{w}_s| = h\}$. Then on $\{\hat{\omega} \in \hat{\Omega} : \tau_2 < \tau_3\}$, we have $\tau_2 > h$ by the definitions of τ_2 and h. But as $\mu \to 0$, $\hat{P}\{\tau_3 < b\} \to 0$ for any $b > 0$. So there exists $\mu_1^{(2)}(\epsilon, \mathcal{K}) > 0$ such that for $0 < \mu < \mu_1^{(2)}$,

$$\hat{P}\{\tau_2 < h\} \leq 1 - \hat{P}\{\tau_2 < \tau_3\} = \hat{P}\{\tau_3 < \tau_2\} < \frac{1}{4}\epsilon. \tag{5.26}$$

Take $\mu_1 = \min\{\mu_1^{(1)}, \mu_1^{(2)}, \frac{\epsilon}{4\bar{k}_t^2}, 1\}$. By the Feynman-Kac formula, strong Markov property of Brownian motion and (5.20)-(5.26) we have if $0 < \mu_0 < \mu_1$, then for all $0 < \mu < \mu_0$, $\omega \in \Omega_1 \cap \Omega_2^{\mu_0} \cap \Omega_3^{\mu_0}$ and $(t, x) \in \mathcal{K}$,

$$\begin{aligned}
& u_t^{\mu}(x, \omega) \\
=& \hat{E} u_{t-\tau}^{\mu}(x + \mu B_\tau, \omega) \\
& e^{\frac{1}{\mu^2}\int_0^\tau c(x+\mu B_s, u_{t-s}^{\mu}(x+\mu B_s,\omega))ds - \frac{1}{2}\int_0^\tau k^2(t-s)ds - \int_0^\tau k(t-s)dw_{t-s}(\omega)} \\
=& \hat{E} u_{t-\tau_1}^{\mu}(x + \mu B_{\tau_1}, \omega) \\
& e^{\frac{1}{\mu^2}\int_0^{\tau_1} c(x+\mu B_s, u_{t-s}^{\mu}(x+\mu B_s,\omega))ds - \frac{1}{2}\int_0^{\tau_1} k^2(t-s)ds - \int_0^{\tau_1} k(t-s)dw_{t-s}(\omega)} \chi_{\tau_1 < \mu} \chi_{\tau_1 \leq \tau_2}
\end{aligned}$$

$$+\hat{E}u^{\mu}_{t-\tau_1}(x+\mu B_{\tau_1},\omega)$$
$$e^{\frac{1}{\mu^2}\int_0^{\tau_1}c(x+\mu B_s,u^{\mu}_{t-s}(x+\mu B_s,\omega))ds-\frac{1}{2}\int_0^{\tau_1}k^2(t-s)ds-\int_0^{\tau_1}k(t-s)dw_{t-s}(\omega)}\chi_{\mu\le\tau_1\le\tau_2}$$
$$+\hat{E}u^{\mu}_{t-\tau_2}(x+\mu B_{\tau_2},\omega)$$
$$e^{\frac{1}{\mu^2}\int_0^{\tau_2}c(x+\mu B_s,u^{\mu}_{t-s}(x+\mu B_s,\omega))ds-\frac{1}{2}\int_0^{\tau_2}k^2(t-s)ds-\int_0^{\tau_2}k(t-s)dw_{t-s}(\omega)}\chi_{\tau_2<h}\chi_{\tau_2<\tau_1}$$
$$+\hat{E}u^{\mu}_{t-\tau_2}(x+\mu B_{\tau_2},\omega)$$
$$e^{\frac{1}{\mu^2}\int_0^{\tau_2}c(x+\mu B_s,u^{\mu}_{t-s}(x+\mu B_s,\omega))ds-\frac{1}{2}\int_0^{\tau_2}k^2(t-s)ds-\int_0^{\tau_2}k(t-s)dw_{t-s}(\omega)}\chi_{h\le\tau_2<\tau_1}$$
$$\ge(1-\frac{1}{4}\epsilon)\hat{E}e^{-\frac{1}{2}\int_0^{\tau_1}k^2(t-s)ds-\int_0^{\tau_1}k(t-s)dw_{t-s}(\omega)}\chi_{\tau_1<\mu}\chi_{\tau_1\le\tau_2}$$
$$+(1-\frac{1}{4}\epsilon)\hat{E}e^{\frac{1}{\mu^2}[\alpha\mu+\mu^2(-\frac{1}{2}\int_0^{\tau_1}k^2(t-s)ds-\int_0^{\tau_1}k(t-s)dw_{t-s}(\omega))]}\chi_{\mu\le\tau_1\le\tau_2}$$
$$+e^{-\frac{\alpha h}{2\mu^2}}\hat{E}e^{\frac{1}{\mu^2}[\alpha h+\mu^2(-\frac{1}{2}\int_0^{\tau_2}k^2(t-s)ds-\int_0^{\tau_2}k(t-s)dw_{t-s}(\omega))]}\chi_{\hat{\Omega}_0}\chi_{h\le\tau_2<\tau_1}$$
$$\ge(1-\frac{1}{2}\epsilon)[\hat{P}\{\tau_1\le\tau_2\}+\hat{E}\{\chi_{\hat{\Omega}_0}\chi_{h\le\tau_2<\tau_1}\}]$$
$$\ge(1-\frac{1}{2}\epsilon)\{\hat{P}\{\tau_1\le\tau_2\}+\hat{P}\{\tau_2<\tau_1\}\}-\hat{P}\{\tau_2<h\}-\frac{1}{4}\epsilon$$
$$\ge 1-\epsilon.$$

Furthermore we have $\delta(\epsilon,\mathcal{K})>0$ such that

$$P(\Omega_1\cap\Omega_2\cap\Omega_3)>1-e^{-\frac{\delta}{\mu_0}}.$$

The above two formulae, together with Lemma 5.1, give the proof of the theorem. ‡‡

As Figure III1, Figure III2 is a simulation by J. Gaines for the wave generated by the initial Gaussian distribution but perturbed by a weak multiplicative white noise udw_t ($\mu=0.1$ in the calculation). This agrees with the results in this section.

§6. The Generalised KPP Equations in a Strong Random Potential

A. In this section we consider

$$\begin{cases} du^{\mu}_t(x)=\left[\frac{1}{2}\mu^2\Delta u^{\mu}_t(x)+\frac{1}{\mu^2}c(x,u^{\mu}_t(x))u^{\mu}_t(x)\right]dt+\frac{k(t)}{\mu^2}u^{\mu}_t(x)dw_t, \\ u^{\mu}_0(x)=T_0(x)e^{-\frac{1}{\mu^2}S_0(x)}. \end{cases} \tag{6.1}$$

Here $\Delta,c,k,T_0,S_0,w_t(\omega)$ are the same as before and $u^{\mu}:[0,\infty)\times R^r\times\Omega\to R^1$ denotes the solution. As (2.1), by the Feynman-Kac formula, $u^{\mu}_t(x)$ satisfies

$$u^{\mu}_t(x)=\hat{E}T_0(x+\mu B_t)\cdot e^{-\frac{1}{\mu^2}S_0(x+\mu B_t)+\frac{1}{\mu^2}\int_0^t c(x+\mu B_s,u^{\mu}_{t-s}(x+\mu B_s))ds-\frac{1}{2\mu^4}\int_0^t k^2(t-s)ds-\frac{1}{\mu^2}\int_0^t k(t-s)dw_{t-s}}, \tag{6.2}$$

where B_t is a R^r Brownian motion defined on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. We define $V : [0,T] \times R^r \to R$ by (5.3). We consider conditions (I) or (I') and

(K). There exists a constant $\underline{k} > 0$ *such that* $|k(t)| \geq \underline{k} > 0$ *for all* x.

Theorem 6.1. *Assume conditions (I), (K) and that* S_0 *is nonnegative and* T_0 *is bounded. Then for any compact subset* $\mathcal{K}$ *of* $(0,+\infty) \times R^r$, $0 < \mu_1 < \left(\frac{-\hat{c}\bar{t}+\sqrt{(\hat{c}\bar{t})^2+\frac{1}{2}(\log\|T_0\|)\underline{k}^2\underline{t}}}{2\log\|T_0\|}\right)^{\frac{1}{2}}$ *if* $\|T_0\| > 1$ *and* $0 < \mu_1 < \frac{\underline{k}^2\underline{t}}{8\hat{c}\bar{t}}$ *if* $\|T_0\| \leq 1$, *we have*

$$P\{\omega \in \Omega : \sup_{(t,x)\in\mathcal{K}} u_t^\mu(x,\omega) > e^{-\frac{\underline{k}^2\underline{t}}{4\mu^4}}, \text{ some } 0 < \mu \leq \mu_0\} < e^{-\frac{\underline{k}^4\underline{t}^2}{128\Gamma\mu_0^4}},$$

for all $0 < \mu_0 < \mu_1$, *where* $\underline{t} = \inf\{t : (t,-) \in \mathcal{K}\}, \bar{t} = \sup\{t : (t,-) \in \mathcal{K}\}$, $\Gamma = \int_0^{\bar{t}} k^2(s)ds$. *In particular, as* $\mu \to 0$, $u_t^\mu(x,\omega) \to 0$, *uniformly in* $\mathcal{K}$ *P-a.s..*

Proof. From (6.2) and conditions (I), (K), we have

$$\mu^4 \log u_t^\mu(x) \leq -\frac{\underline{k}^2}{2}t + \mu^2\hat{c}t + \mu^4 log(\|T_0\| \vee 1) - \mu^2 \int_0^t k(t-s)dw_{t-s}. \tag{6.3}$$

Let μ_1 be as required by the theorem. For any $0 < \mu_0 < \mu_1$, by Lemma 2.1,

$$P\{\omega \in \Omega : \mu^2 \sup_{(t,x)\in\mathcal{K}} [-\int_0^t k(t-s)dw_{t-s}(\omega)] > \frac{\underline{k}^2\underline{t}}{8}, \text{ for some } 0 < \mu < \mu_0\} < e^{-\frac{\underline{k}^4\underline{t}^2}{128\Gamma\mu_0^4}}, \tag{6.4}$$

and if $0 < \mu < \mu_0$,

$$\mu^4 log(\|T_0\| \vee 1) + \mu^2\hat{c}\bar{t} \leq \frac{\underline{k}^2\underline{t}}{8}. \tag{6.5}$$

The result follows by (6.3), (6.4), (6.5). ‡‡

B. Under the condition (I'), we can get sharp estimates. Note that neither condition (N^{*k}) nor (N^*) is needed as in Theorem 2.2, 2.3, 5.2 and 5.3. Let ϕ, ψ be as in §2.

Theorem 6.2. *Assume condition (I'), the conditions of Theorem 2.1, and that* T_0 *is bounded continuous. Then for sufficiently small* $\mu \neq 0$,

$$u_t^\mu(x,\omega) = e^{\frac{1}{\mu^2}V_t(x) - \frac{1}{2\mu^4}\int_0^t k^2(s)ds + \frac{1}{\mu^2}\int_0^t k(s)dw_s}\left[\sqrt{\phi_t(x)}T_0(\Phi_t^{-1}(x)) + R_1\right], \tag{6.6}$$

with $R_1 \to 0$ *as* $\mu \to 0$ *P-a.s. uniformly in any compact subset of* $(0,+\infty) \times R^r$.

Theorem 6.3. *Assume condition (I'), the conditions of Proposition 2.1. Then for sufficiently small* $\mu \neq 0$,

$$\begin{aligned} u_t^\mu(x,\omega) &= e^{\frac{1}{\mu^2}V_t(x) - \frac{1}{2\mu^4}\int_0^t k^2(s)ds + \frac{1}{\mu^2}\int_0^t k(s)dw_s}\psi_t(x) \cdot \\ &\cdot \left[1 + \frac{1}{2}\mu^2 \int_0^t \psi_{t-s}^{-1}(z_s^t(x))\Delta\psi_{t-s}(z_s^t(x))ds + R_2\right], \end{aligned} \tag{6.7}$$

with $\frac{R_2}{\mu^2} \to 0$, *as* $\mu \to 0$ *P-a.s. uniformly in any compact subset of* $(0, +\infty) \times R^r$.

The numerical simulation will be shown in §8.

§7. Space Dependent Perturbations

A. Now we briefly discuss the space dependent noise perturbation: $\frac{1}{\mu}k(x)u_t^\mu(x)dw_t$. To do this we will impose stronger hypotheses on the coefficients than we have needed so far. The equation is:

$$\begin{cases} du_t^\mu(x) = \left[\frac{1}{2}\mu^2 \Delta u_t^\mu(x) + \frac{1}{\mu^2}c(x, u_t^\mu(x))u_t^\mu(x)\right] dt + \frac{1}{\mu}k(x)u_t^\mu(t)dw_t, \\ u_0^\mu(x) = T_0(x)e^{-\frac{1}{\mu^2}S_0(x)}. \end{cases} \tag{7.1}$$

Let $k \in C^2(R^r)$. If $X_s^{x,\mu}$ is defined by (2.2) for $0 \le s \le T$, then the following lemma is proved by integration by parts.

Lemma 7.1. *Suppose* Dk, A *are all bounded for* $0 \le t \le T$. *Then there exist* $c_1 > 0, c_2 > 0$ *such that for* $0 \le t \le T$

$$\hat{E}e^{-\frac{1}{\mu}\int_0^t k(X_s^{x,\mu})dw_{t-s}} \le e^{\frac{c_1}{\mu}|w|_t^* + \frac{1}{2}c_2(|w|_t^*)^2}, \tag{7.2}$$

where $|w|_t^* = \sup\limits_{0 \le s \le t} |w_s|$.

Proof. By integration by parts we have

$$-\frac{1}{\mu}\int_0^t k(X_s^{x,\mu})dw_{t-s} = \frac{1}{\mu}k(0)w_t - \frac{1}{\mu}\int_0^t Dk(X_s^{x,\mu})w_{t-s}dX_s^{x,\mu} \tag{7.3}$$

The second term of right hand side of (7.3) is

$$\begin{aligned} &-\frac{1}{\mu}\int_0^t Dk(X_s^{x,\mu})w_{t-s}(\mu dB_s + A_s(X_s^{x,\mu})ds) \\ =&-\int_0^t Dk(X_s^{x,\mu})w_{t-s}dB_s - \frac{1}{\mu}\int_0^t Dk(X_s^{x,\mu})A_s(X_s^{x,\mu})w_{t-s}ds \end{aligned} \tag{7.4}$$

By (7.3) and (7.4), there exists a constant $c_1 > 0$ such that

$$\hat{E}e^{-\frac{1}{\mu}\int_0^t k(X_s^{x,\mu})dw_{t-s}} \le e^{\frac{c_1}{\mu}|w|_t^*} \cdot J \tag{7.5}$$

Here

$$\begin{aligned} J =& \hat{E}e^{-\int_0^t Dk(X_s^{x,\mu})w_{t-s}dB_s} \\ =& \hat{E}e^{-\int_0^t Dk(X_s^{x,\mu})w_{t-s}dB_s - \frac{1}{2}\int_0^t |Dk(X_s^{x,\mu})w_{t-s}|^2 ds} \cdot e^{\frac{1}{2}\int_0^t |Dk(X_s^{x,\mu})w_{t-s}|^2 ds} \\ \le& e^{\frac{1}{2}c_2(|w|_t^*)^2} \end{aligned} \tag{7.6}$$

for some $c_2 > 0$. Therefore (7.2) follows. ‡‡

Let $\bar{c}$ and k be C^2 and S_0 be C^1 and consider the classical mechanical system

$$\begin{cases} \ddot{\Phi}_{k,s}(x) = -\nabla[\bar{c}(\Phi_{k,s}(x)) - \frac{1}{2}k^2(\Phi_{k,s}(x))], & s \geq 0, \\ \dot{\Phi}_{k,0}(x) = \nabla S_0(x), \quad \Phi_{k,0}(x) = x. \end{cases} \tag{7.7}$$

Assume a no-caustic condition: there exists $T > 0$ such that $\Phi_{k,s} : R^r \to R^r$ exists and is a diffeomorphism for $0 \leq s \leq T$. This is true if $\bar{c}, k, S_0$ have uniformly bounded first and second derivatives (See (Elworthy & Truman (1981)).

For $(t, x) \in [0, T] \times R^r$, define $\overline{V}^k(t, x)$ as before by

$$\begin{aligned} \overline{V}_t^k(x) = \int_0^t [\bar{c}(\Phi_{k,s}(\Phi_{k,t}^{-1}(x))) - \frac{1}{2}k^2(\Phi_{k,s}(\Phi_{k,t}^{-1}(x)))]ds \\ - S_0(\Phi_{k,t}^{-1}(x)) - \frac{1}{2}\int_0^t |\dot{\Phi}_{k,s}(\Phi_{k,t}^{-1}(x))|^2 ds. \end{aligned} \tag{7.8}$$

Again $\overline{V}^k$ satisfies the Hamilton-Jacobi equation

$$\frac{1}{2}||\nabla \overline{V}_t^k(x)||^2 + \bar{c}(x) - \frac{1}{2}k^2(x) - \frac{\partial \overline{V}_t^k(x)}{\partial t} = 0, \tag{7.9}$$

with given initial function $-S_0$. We have (2.6), (2.7) similarly. Let $X_s^{x,\mu}, 0 \leq s \leq t$ be defined by (2.2) by taking $A_s = D\overline{V}^k{}_{t-s}$. Then by the Feynman-Kac formula and Proposition I.1.1 (we allow $f : Z \times \Omega \to R$ bounded and measurable):

$$\begin{aligned} u_t^\mu(x) = e^{\frac{1}{\mu^2}\overline{V}_t^k(x)} \hat{E} T_0(X_t^{x,\mu}) e^{\frac{1}{2}\int_0^t \Delta \overline{V}_{t-s}^k(X_s^{x,\mu})ds} \\ \cdot e^{\frac{1}{\mu^2}\int_0^t [c(X_s^{x,\mu}, u_{t-s}^\mu(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})]ds - \frac{1}{\mu}\int_0^t k(X_s^{x,\mu})dw_{t-s}} \end{aligned} \tag{7.10}$$

By a similar argument to Theorem 2.1, using Lemma 7.1 and noting

$$P\{\mu|w|_t^* > \delta\} < e^{-\frac{\delta^2}{2\mu^2}} \tag{7.11}$$

for any $\delta > 0, \mu > 0$, we can prove

Theorem 7.1. *Suppose (I), that $\bar{c}, k$ and S_0 are c^2 with S_0 bounded below and that T_0 is bounded and measurable. Assume that the classical flow $\Phi_{k,t}$ given by (7.7) satisfies the no-caustic condition for $0 \leq t \leq T$ and $D^2\overline{V}^k{}_t(-), \Delta\overline{V}^k{}_t(-)$ are bounded above, $D\overline{V}_t^k(-), Dk(-)$ are all bounded for $0 \leq t \leq T$. Then for any compact subset $\mathcal{K}$ of $\{(t, x) : \overline{V}_t^k(x) < 0, 0 \leq t \leq T\}$, there exists $0 < \mu_1(\mathcal{K}) < 1$ such that*

$$P\{\omega \in \Omega : \sup_{(t,x)\in\mathcal{K}} u_t^\mu(x, \omega) > e^{-\frac{\delta_1}{\mu^2}}, \text{ some } 0 < \mu < \mu_0\} < e^{-\frac{\delta_2{}^2}{2\mu_0{}^2}} \tag{7.12}$$

for all $0 < \mu_0 < \mu_1$. *Here* $\delta_1 = -\frac{1}{2}\sup\{\overline{V}_t^k(x) : (t,x) \in K\}$, $\delta_2 = \frac{-c_1+\sqrt{c_1^2+2c_2\delta_1}}{c_2}$, c_1, c_2 *are defined in (7.2).*

B. As for Theorem 7.1, we can study the space dependent potential version of equation (5.1):

$$\begin{cases} du_t^\mu(x) = \left[\frac{1}{2}\mu^2\Delta u_t^\mu(x) + \frac{1}{\mu^2}c(x, u_t^\mu(x))u_t^\mu(x)\right] dt + k(x)u_t^\mu(t)dw_t, \\ u_0^\mu(x) = T_0(x)e^{-\frac{1}{\mu^2}S_0(x)}. \end{cases} \tag{7.13}$$

With all conditions of Lemma 7.1, we can prove there exist $c_1, c_2 > 0$

$$\hat{E}e^{-\int_0^t k(t-s, X_s^{x,\mu})dw_{t-s}} \leq e^{c_1|w|_t^* + \frac{1}{2}\mu^2 c_2(|w|_t^*)^2} \tag{7.14}$$

Theorem 7.2. *Assume the conditions of Theorem 7.1. Suppose that the classical flow* Φ_t *given by (2.3) satisfies the no-caustic condition for* $0 \leq t \leq T$ *and* $D^2\overline{V}_t(-), \Delta V_t(-)$ *are bounded above with* DV_t, Dk *are all bounded for* $0 \leq t \leq T$. *Then for any compact subset* $\mathcal{K}$ *of* $\{(t,x) : \overline{V}_t(x) < 0, 0 \leq t \leq T\}$, *there exists* $0 < \mu_1(\mathcal{K}) < 1$ *such that*

$$P\{\omega \in \Omega : \sup_{(t,x)\in\mathcal{K}} u_t^\mu(x,\omega) > e^{-\frac{\delta_1}{\mu^2}},\ some\ 0 < \mu < \mu_0\} < e^{-\frac{\delta_2^2}{2\mu_0^2}} \tag{7.15}$$

for all $0 < \mu_0 < \mu_1$. *Here* $\delta_1 = -\frac{1}{2}\sup\{V_t(x) : (t,x) \in k\}$, $\delta_2 = \frac{-c_1+\sqrt{c_1^2+2c_2\delta_1}}{c_2}$, *with* c_1, c_2 *as in (7.14).*

C. For the strong noise case, we get a complete picture:

$$\begin{cases} du_t^\mu(x) = \left[\frac{1}{2}\mu^2\Delta u_t^\mu(x) + \frac{1}{\mu^2}c(x, u_t^\mu(x))u_t^\mu(x)\right] dt + \frac{1}{\mu^2}k(x)u_t^\mu(t)dw_t, \\ u_0^\mu(x) = T_0(x)e^{-\frac{1}{\mu^2}S_0(x)}. \end{cases} \tag{7.16}$$

By a method similar to the proof of Theorem 6.1, using an alternative version of (7.2):

$$\hat{E}e^{-\frac{1}{\mu^2}\int_0^t k(x+\mu B_s)dw_{t-s}} \leq e^{\frac{c_1}{\mu^2}|w|_t^* + \frac{c_2}{2\mu^2}(|w|_t^*)^2} \tag{7.17}$$

for some $c_1, c_2 > 0$, we can prove

Theorem 7.3. *Assume conditions (I), (K) and that* S_0 *is nonnegative* T_0 *is bounded. Then for any compact subset* $\mathcal{K}$ *of* $(0, +\infty) \times R^r$, *with* $0 < \mu_1 <$

$\left(\frac{-\hat{c}\bar{t}+\sqrt{(\hat{c}\bar{t})^2+\frac{1}{2}(\log\|T_0\|)\underline{k}^2\underline{t}}}{2\log\|T_0\|}\right)^{\frac{1}{2}} \wedge 1$ *if* $\|T_0\| > 1$ *and* $0 < \mu_1 < \frac{\underline{k}^2\underline{t}}{8\hat{c}\bar{t}} \wedge 1$ *if* $\|T_0\| \le 1$, *we have*

$$P\{\omega \in \Omega : \sup_{(t,x)\in\mathcal{K}} u_t^{\mu}(x,\omega) > e^{-\frac{\underline{k}^2\underline{t}}{4\mu^4}},\ some\ 0 < \mu \le \mu_0\} < e^{-\frac{\delta^2}{2\mu_0^4}},$$

for all $0 < \mu_0 < \mu_1$, *where* $\underline{t} = \inf\{t : (t,-) \in \mathcal{K}\}$, $\bar{t} = \sup\{t : (t,-) \in \mathcal{K}\}$, $\delta = \frac{-c_1+\sqrt{c_1^2+\frac{1}{4}c_2\underline{k}^2\underline{t}}}{c_2}$ *for* c_1, c_2 *defined in (7.17). In particular, as* $\mu \to 0$, $u_t^{\mu}(x,\omega) \to 0$, *uniformly in* $\mathcal{K}$ *P-a.s..*

§8. Numerical Simulations, Conclusions and Remarks

We consider the propagation of the travelling wave of KPP equations in mild, weak and strong random potentials in this section. We take an approximate δ-function to be our initial distribution (see chapter II). Our equation is described as

$$\begin{cases} du_t^{\lambda,\mu}(x) = \left[\frac{1}{2}\mu^2\Delta u_t^{\lambda,\mu}(x) + \frac{1}{\mu^2}\hat{c}(1-u_t^{\lambda,\mu}(x))u_t^{\lambda,\mu}(x)\right]dt + F_\mu(w)_t u_t^{\lambda,\mu}(x), \\ u_0^{\lambda,\mu}(x) = \frac{1}{\sqrt{2\pi\mu^4\lambda}}e^{-\frac{x^2}{2\mu^4\lambda}}, \end{cases} \tag{8.1}$$

where $x \in R^1$, $\hat{c} > 0$, and $F_\mu(w)$ is a random potential term, Let w_t be a one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. First we consider $F_\mu(w)_t = \frac{1}{\mu}\hat{k}dw_t$ with d Itô's differentiation and constant $\hat{k}$. By definition we have

$$V_t^{k,\lambda,\mu}(x) = (\hat{c} - \frac{1}{2}\hat{k}^2)t - \frac{x^2}{2(t+\lambda\mu^2)} \to V_t^k(x) = (\hat{c} - \frac{1}{2}\hat{k}^2)t - \frac{x^2}{2t},$$

as $\lambda \to 0$. The following theorem can be obtained as in §2, §3 and chapter II in the deterministic case, especially §1.

Theorem 8.1. *Let* $u_t^{\lambda,\mu}(x)$ *be the solution of (8.1) with* $F_\mu(w)_t = \frac{1}{\mu}\hat{k}dw_t$ *with constant* $\hat{k}$. *If* $2\hat{c} > \hat{k}^2$, *for* $x > t\sqrt{2\hat{c}-\hat{k}^2}$, $\lim_{\mu\to 0}\mu^2\log u_t^{\lambda,\mu}(x) = \lim_{\mu\to 0}\lim_{\lambda\to 0}\mu^2\log u_t^{\lambda,\mu}(x) < 0$ *and for any compact subset* $\mathcal{K}$ *of* $\{(t,x) : x < t\sqrt{2\hat{c}-\hat{k}^2}\}$, *there exists* $c_1 > 0, c_2 > 0$,

$$-c_1 < \varliminf_{\mu\to 0}\mu\log u_t^{\lambda,\mu}(x) \le \varlimsup_{\mu\to 0}\mu\log u_t^{\lambda,\mu}(x) < c_2. \quad P-a.s.$$

If $2\hat{c} < \hat{k}^2$, *then for* $(t,x) \in (0,+\infty)\times R^r$, $\lim_{\mu\to 0}\mu^2\log u_t^{\lambda,\mu}(x) = \lim_{\mu\to 0}\lim_{\lambda\to 0}\mu^2\log u_t^{\lambda,\mu}(x) < 0$, *uniformly in any compact subset of* $(0,+\infty)\times R^r$, *P-a.s..*

For the case of a weak Itô's noise perturbation we take $F_\mu(w)_t = \hat{k}dw_t$. By definition we get

$$V_t^{\lambda,\mu}(x) = \hat{c}t - \frac{x^2}{2(t+\lambda\mu^2)} \to V_t(x) = \hat{c}t - \frac{x^2}{2t},$$

as $\lambda \to 0$. Therefore

Theorem 8.2. *Let $u_t^{\lambda,\mu}(x)$ be the solution of (8.1) with $F_\mu(w)_t = \hat{k}dw_t$. Then*

$$\lim_{\mu\to 0} u_t^{\lambda,\mu}(x) = \lim_{\mu\to 0}\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = \begin{cases} 0, & P-a.s.\ for\ x > \sqrt{2\hat{c}}t, \\ 1, & P-a.s.\ for\ x < \sqrt{2\hat{c}}t \end{cases}$$

The last theorem concerns the strong noise case.

Theorem 8.3. *Let $u_t^\mu(x)$ be the solution of (8.1) with $F_\mu(w_t)e = \frac{1}{\mu^2}\hat{k}edw_t$ with nonzero constant $\hat{k}$. Then*

$$\lim_{\mu\to 0} u_t^{\lambda,\mu}(x) = \lim_{\mu\to 0}\lim_{\lambda\to 0} u_t^{\lambda,\mu}(x) = 0,$$

uniformly in any compact subset of $(0,+\infty)\times R^r$, *P-a.s..*

In order to illustrate some of the theoretical results of this chapter, we describe J. Gaines' numerical simulations for equation (8.1) with various perturbations and initial conditions. We take $\hat{c} = 1$ throughout.

In Figures III3-III6 we show numerical simulations in the case as described in Theorem 8.1 with decreasing values of μ=: 0.3, 0.2, 0.1, 0.0751 respectively. The initial condition is a point source at $x = 0$ and $k = 1$.

The strong noise is illustrated in Figure III7. The parameter values used are $k = 1$, $\mu = 0.2$. Even with this quite large value of μ the wave dies away almost immediately (see Theorem 8.3 and §6 in general). Figure III8 is the weak perturbation shown in Theorem 8.2. We take parameter values $k = 3$ and $\mu = 0.075$.

Figures III9-III12 show mild noise case with fixed $\mu = 0.075$ varying the value at $k = 0,\ 0.1,\ 0.75,\ 1$ respectively. The pictures show clearly how the speed of the propagation of the wave decreases as k increases. See the wave front formula in Theorem 8.1.

For the mild noise case, Gaines calculated numerically the value of

$$I(k) = \int_0^1 u_t^\mu(x)dt$$

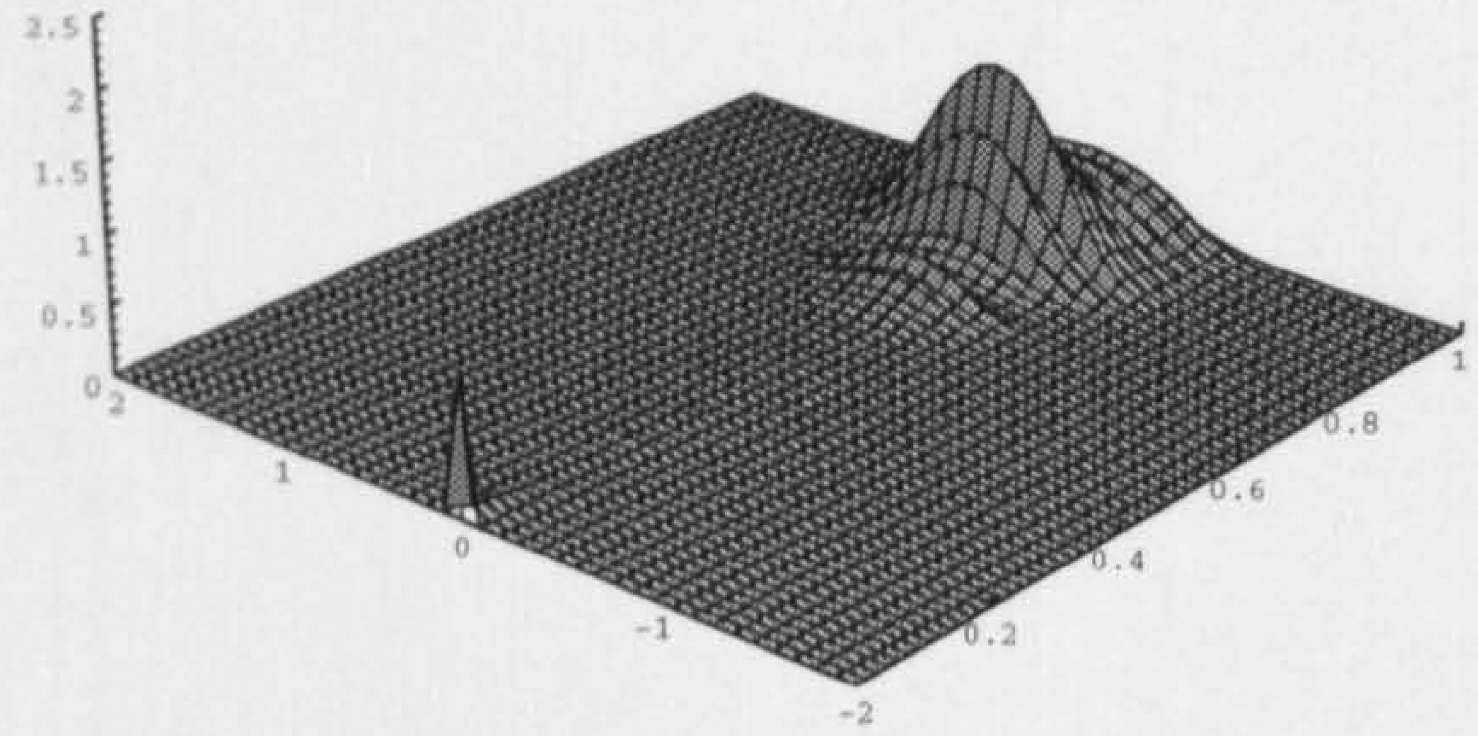

Figure III3

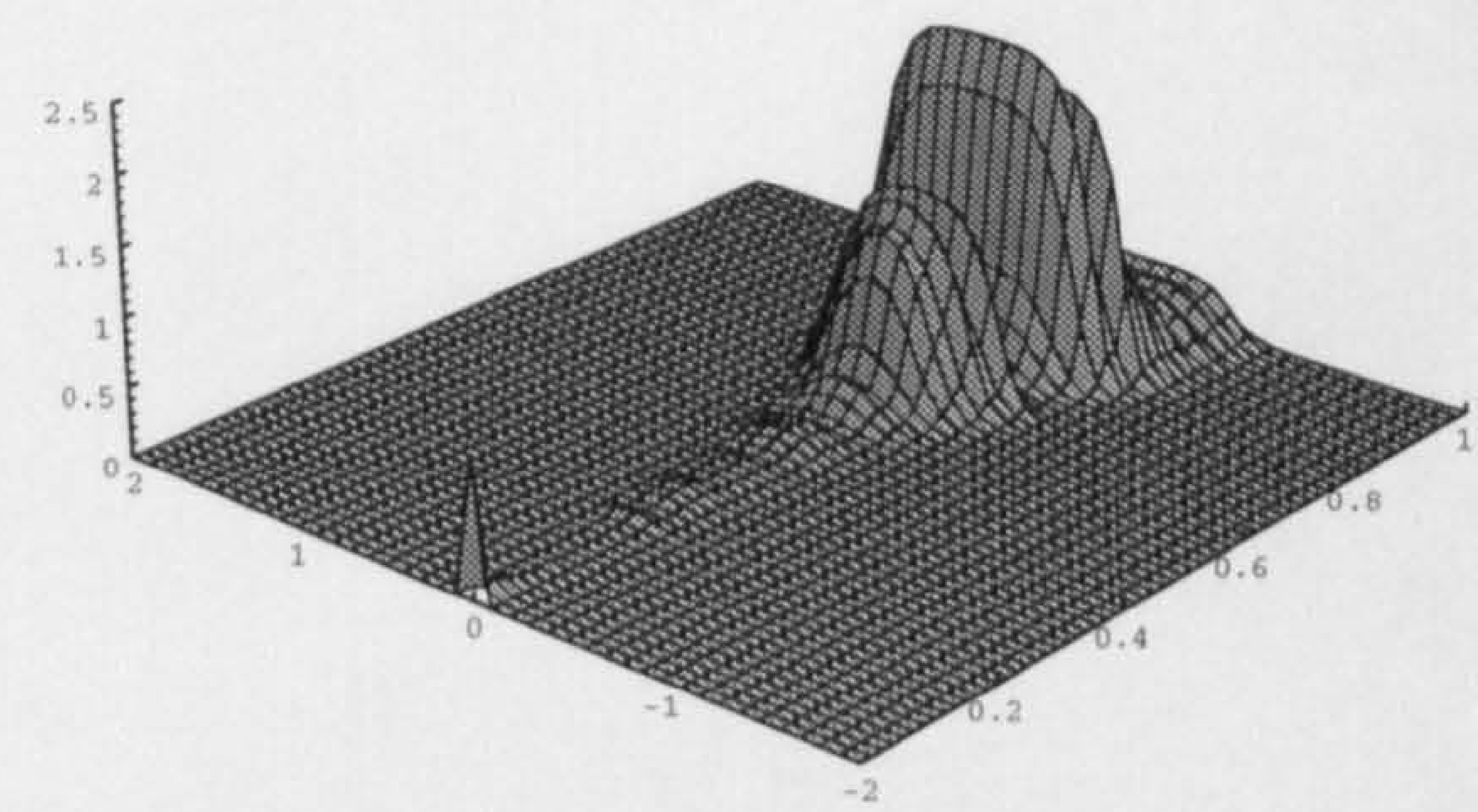

Figure III4

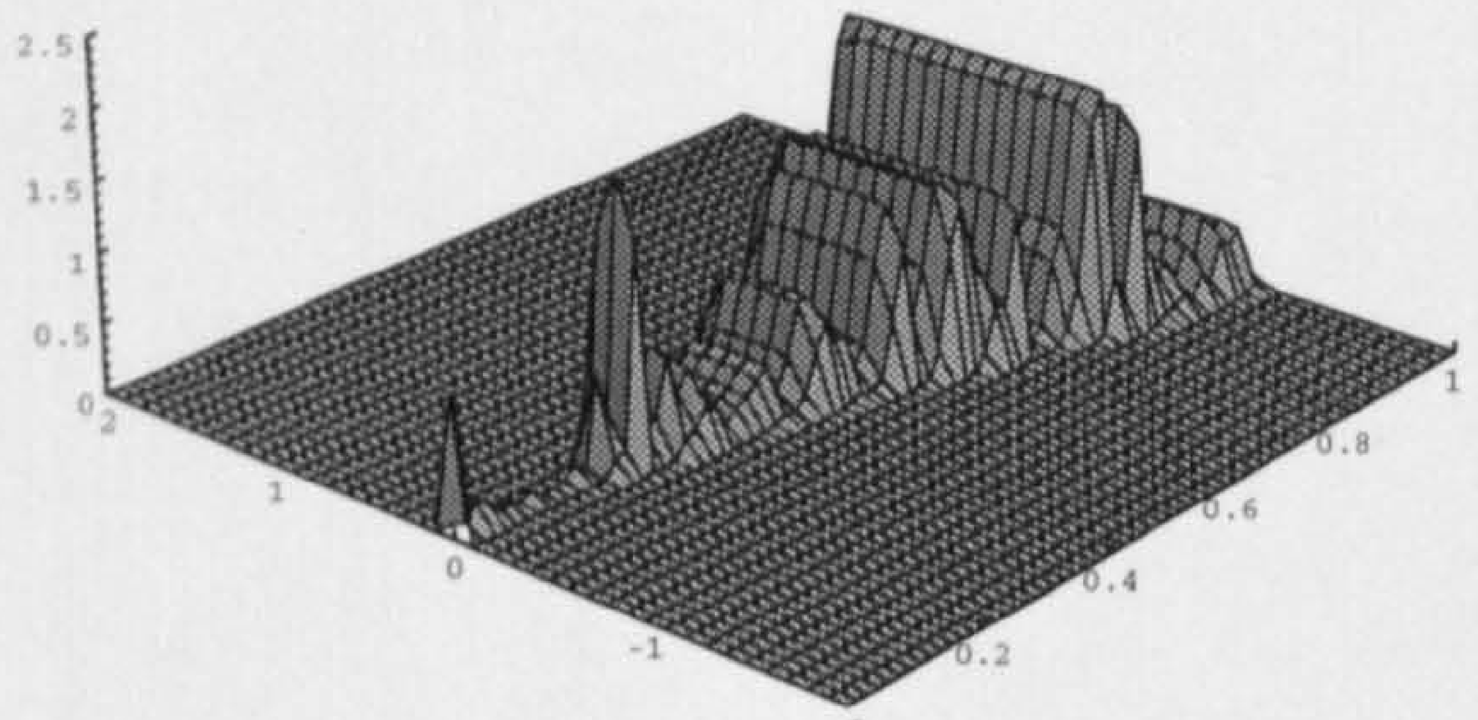

Figure III5

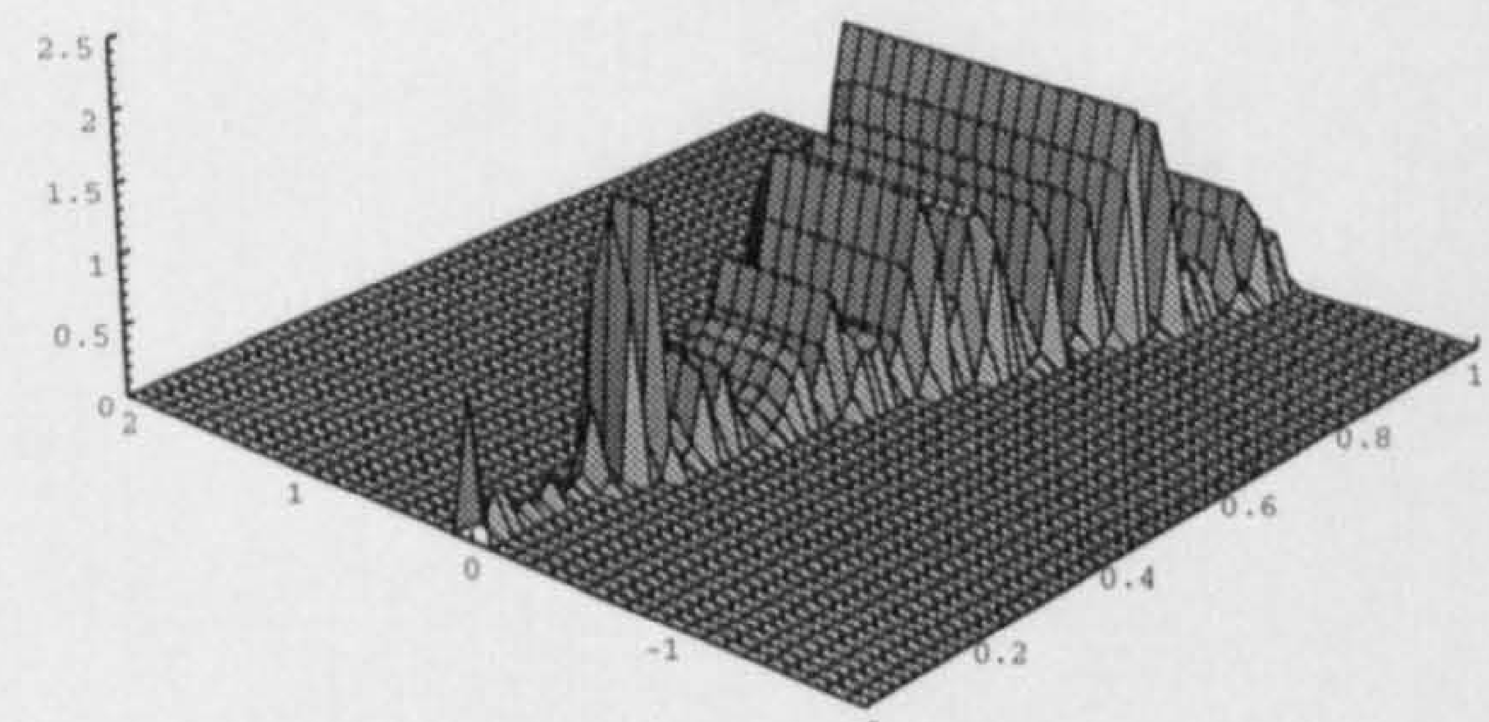

Figure III6

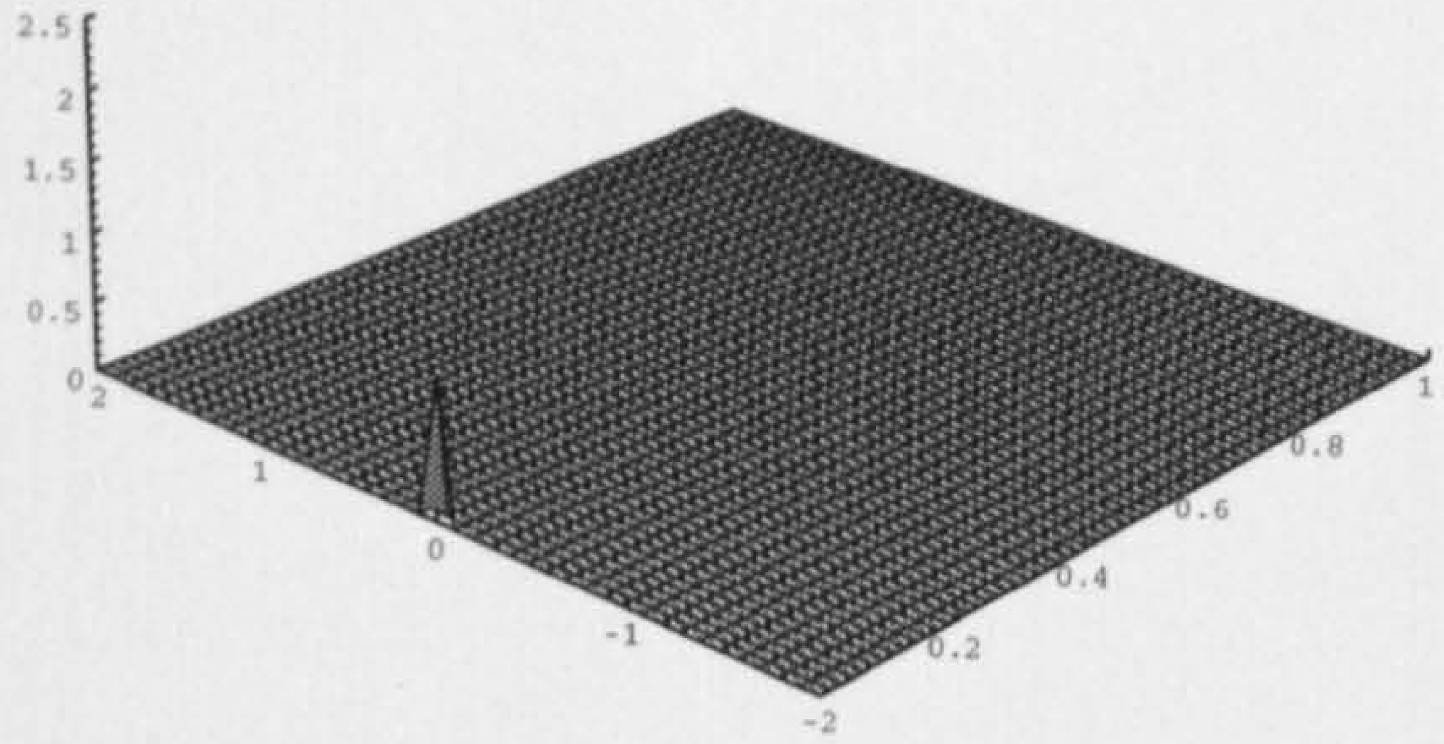

Figure III7

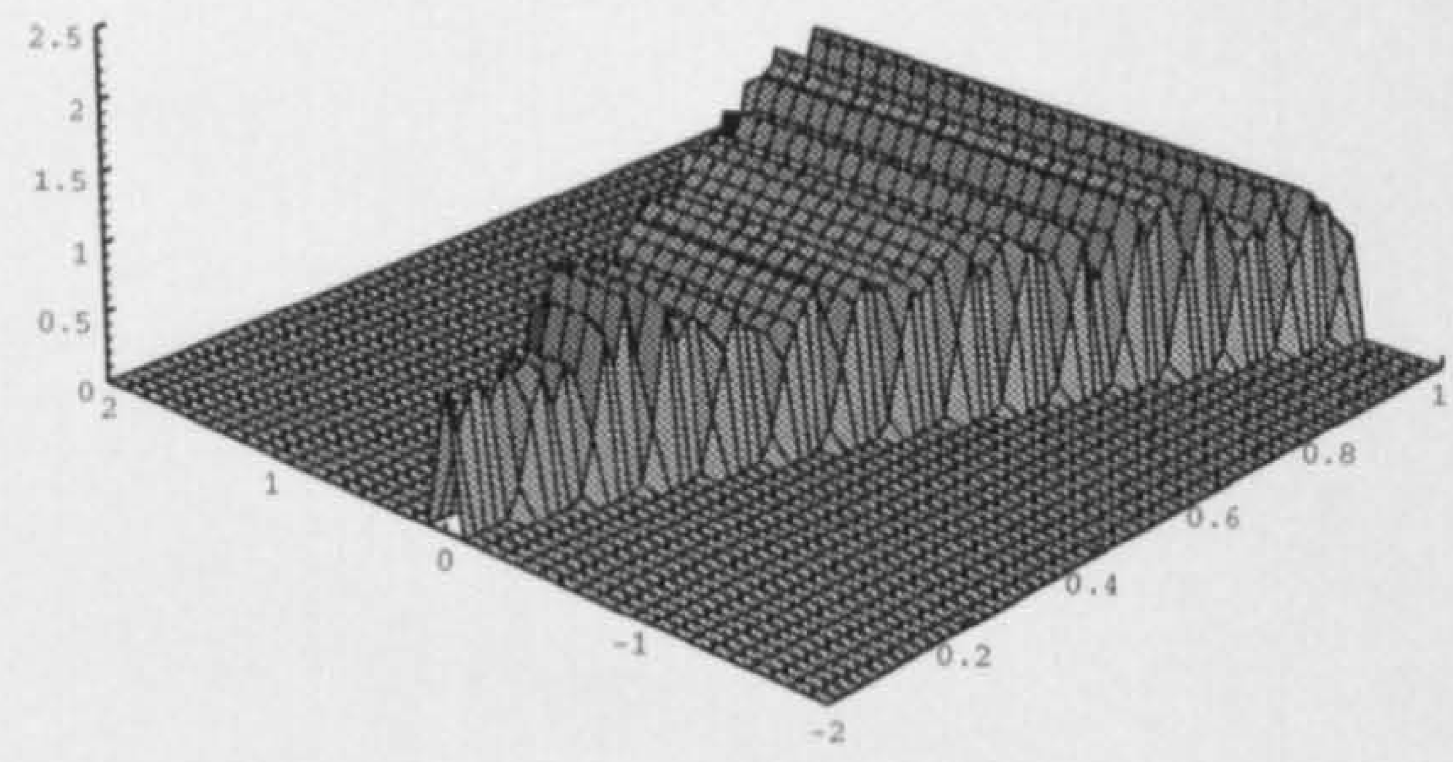

Figure III8

for ten different values of k. We fixed $\mu = 0.1$ and used a point source at 0 as the initial condition. In Figure III13 she plotted both the values of $I(k)$ obtained and the quadratic curve that is the best fit for these values. As can be seen, the fit is extremely good. See also Remark 3.2.

One can consider initial condition as a superposition of extended sources or point sources as well as step functions. Having proved the results of chapters I, II, III, we can easily treat the more complicated situations. We would not like to go to any details. We only demonstrate Figure III14 and Figure III15 where we take $k = 1$ and $\mu = 0.1$.

Remark 8.1. *Let the travelling wave of the deterministic KPP equation which was studied in chapters I, II be the basic travelling wave. Theorems 8.1, 8.2, and 8.3 show us clearly that the white noise potential reduces the velocity of the wave. When the randomness is strong enough, its velocity decreases to zero, i.e., the travelling wave no longer exists. A weak white noise potential does not change its velocity and amplitude and a very strong white noise potential kills the travelling wave with high probability. In other words, the travelling wave can not propagate in strong and very strong random potentials. Summarizing we can say that the Itô white noise potentials hinder the propagation of the travelling waves.*

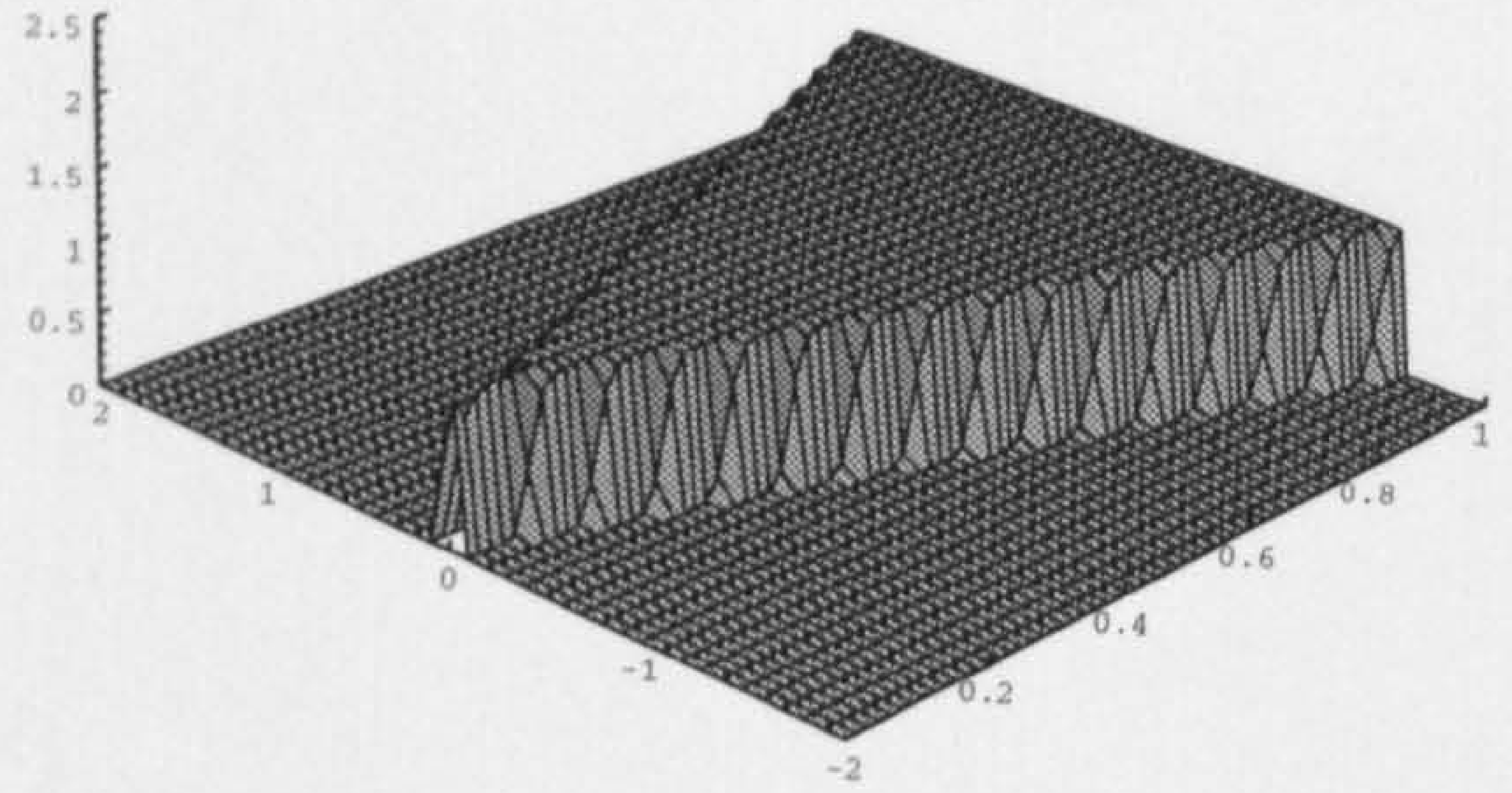

Figure III9

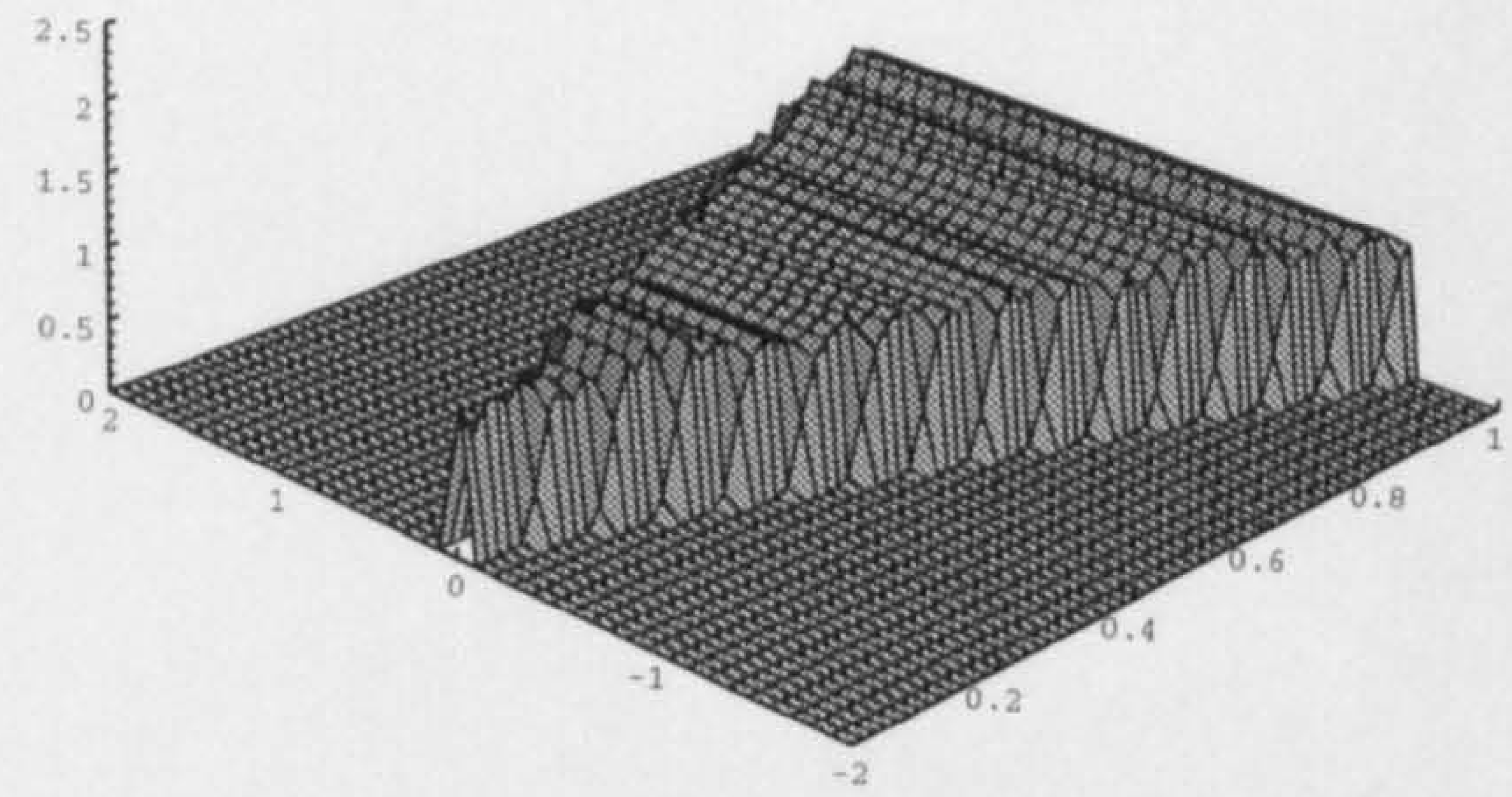

Figure III10

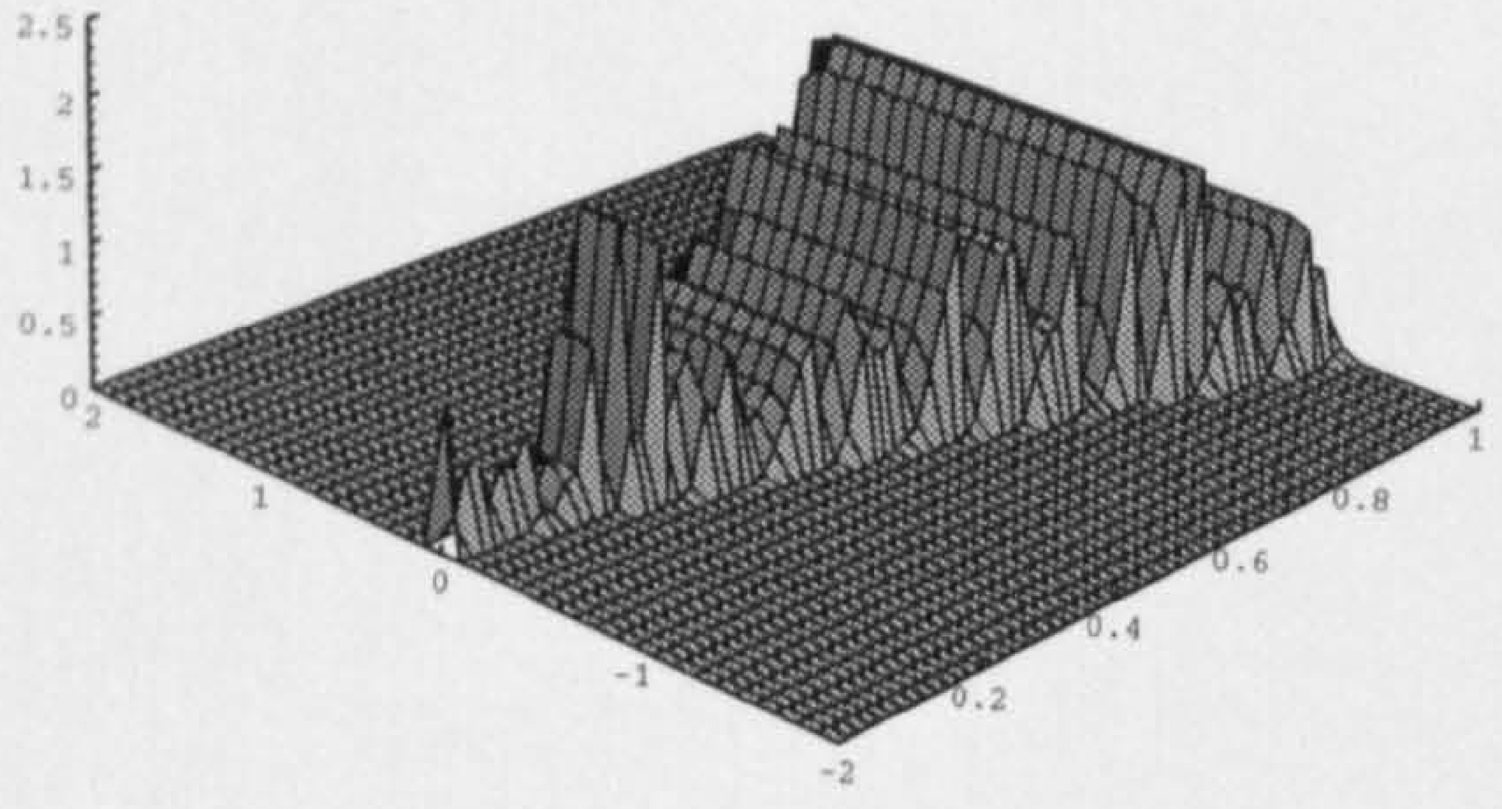

Figure III11

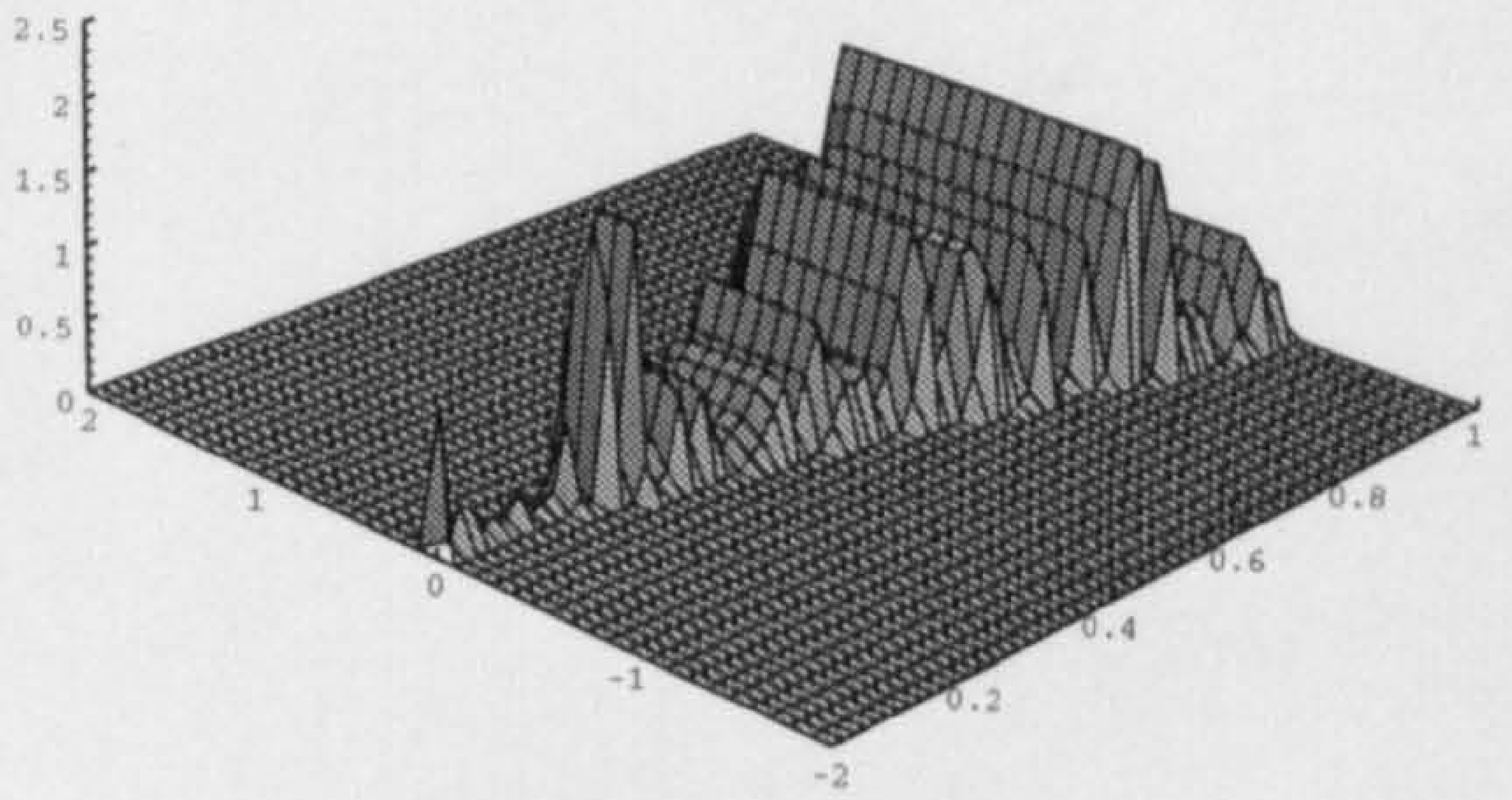

Figure III12

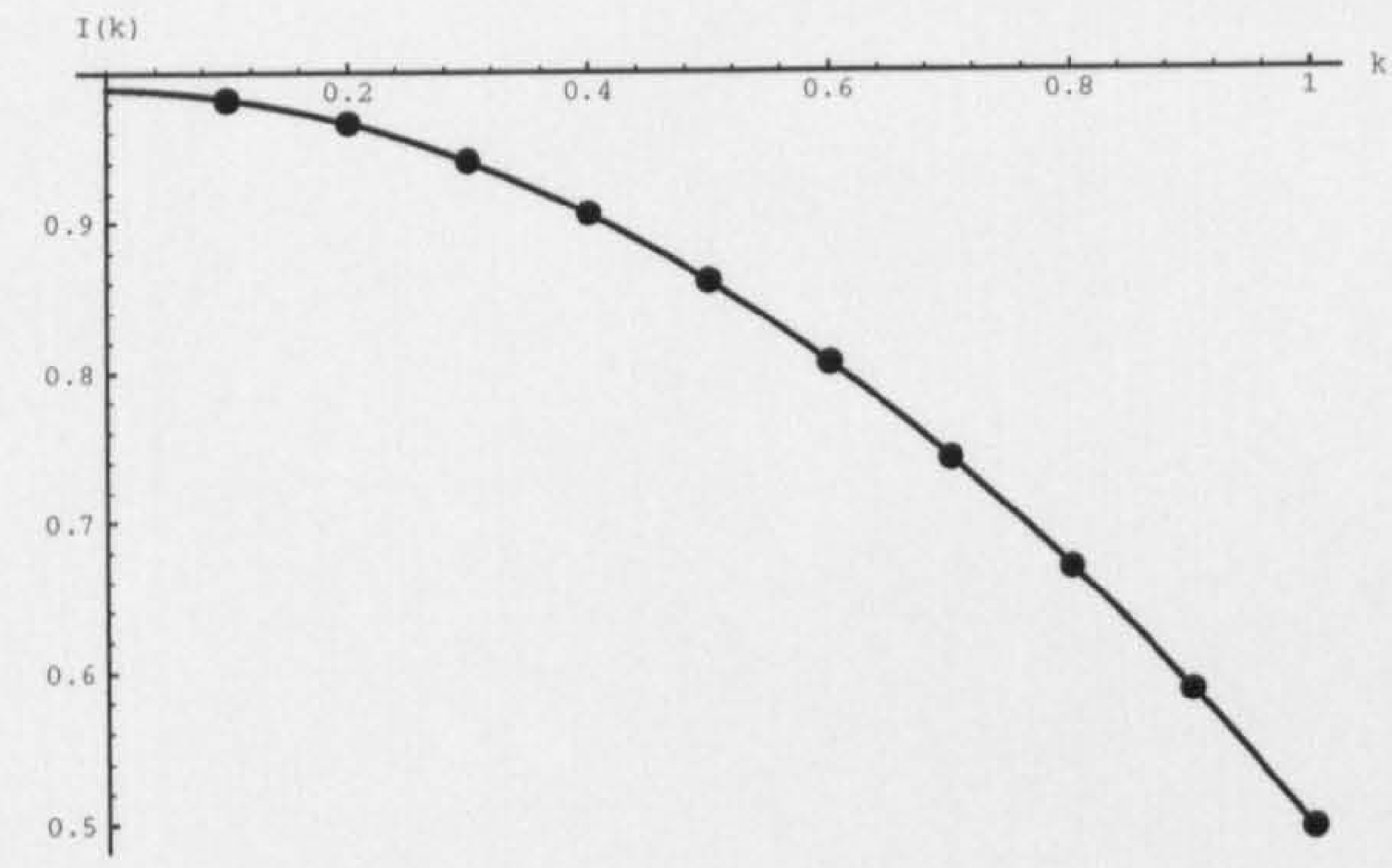

Figure III13

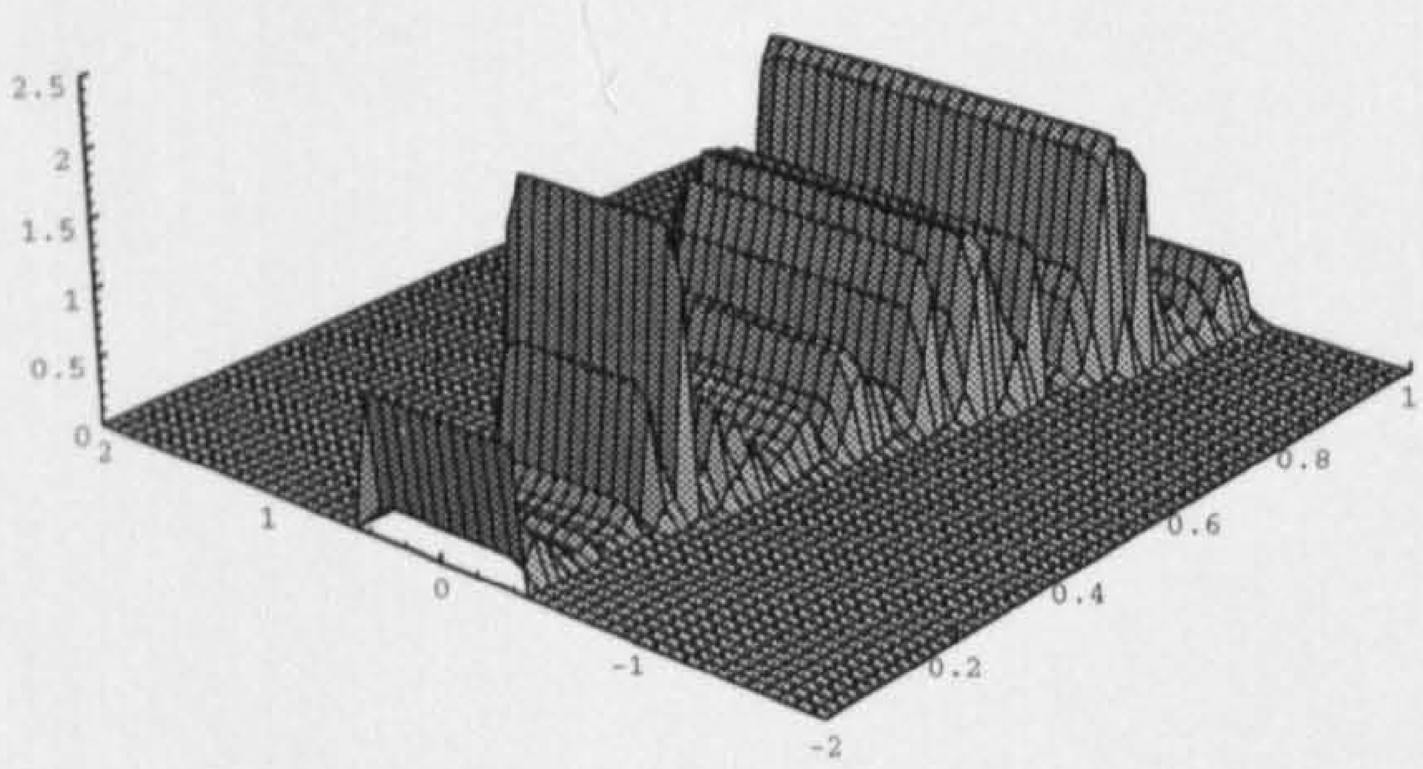

Figure III14

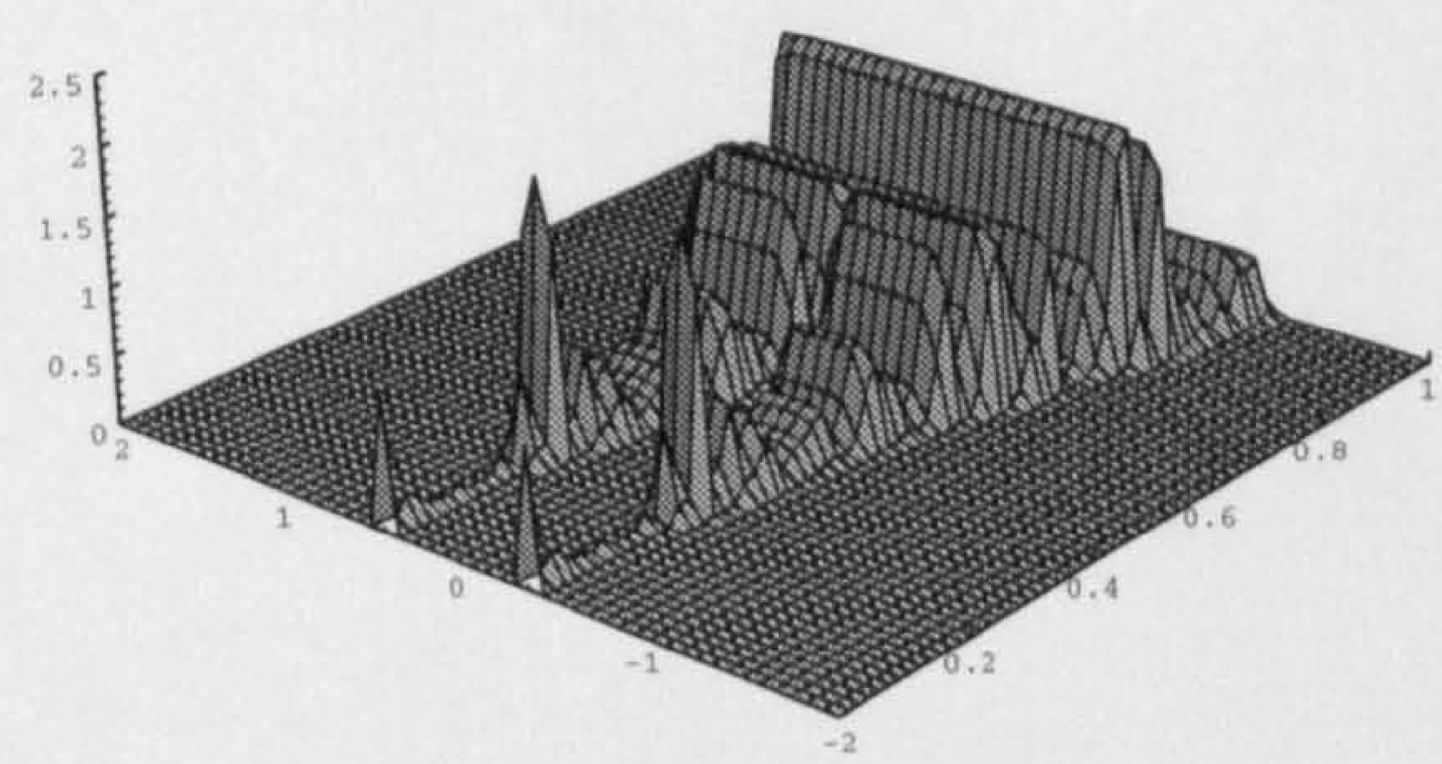

Figure III15

Remark 8.2. *$\lim_{\lambda \to 0} u_t^{\lambda,\mu}(x)$ might be considered as the solution with a δ-function at 0 as its initial distribution. The term μ^4 in u_0 can be replaced by μ^{2+h} for any $h \geq 0$ without changing the results. The case $h = 2$ might be considered a reasonable choice from the point of view of rescaling. More details about δ initial distributions can be found in chapter II.*

Remark 8.3. *Our results can be extended to equations where the vector valued function k: $R^1 \to R^n$ and n-dimensional Brownian motion w_t are concerned.*

§9. The Non L^2 and L^2 Perturbations and Lyapunov Exponents

A. In this section we study the following stochastic generalised KPP equation:

$$\begin{cases} du_t^\mu(x) = [\frac{\mu^2}{2}\Delta u_t^\mu(x) + \frac{1}{\mu^2}c(x, u_t^\mu(x))u_t^\mu(x)]dt + k(\frac{t}{\mu^2})u_t^\mu(x)dw_{\frac{t}{\mu^2}} \\ u_0^\mu(x) = T_0(x)e^{-\frac{S_0(x)}{\mu^2}}. \end{cases} \tag{9.1}$$

Let c, k, T_0, S_0 be the same as before. Travelling waves arise as a result of the large time and large space behaviours. It is intrinsic that the effect of a random perturbation depends on the mean value of the random noise. In fact we classify the random perturbation by the mean value $a = \lim_{\sigma \to \infty} \frac{1}{2\sigma} \int_0^\sigma k^2(s)ds$. Note that $-a$ is the Lyapunov exponent of the following stochastic ordinary differential equation:

$$\dot{\xi}_t = k(t)\xi_t dw_t.$$

Write

$$a_* = \frac{1}{2} \varliminf_{\sigma \to +\infty} \frac{1}{\sigma} \int_0^\sigma k^2(s)ds, \tag{9.2}$$

$$a^* = \frac{1}{2} \varlimsup_{\sigma \to +\infty} \frac{1}{\sigma} \int_0^\sigma k^2(s)ds. \tag{9.3}$$

By the Feynman-Kac formula, the solution $u_t^\mu(x)$ of equation (9.1) can be represented by

$$\begin{aligned} u_t^\mu(x) &= \hat{E} T_0(x + \mu B_t) \\ &\cdot e^{-\frac{1}{\mu^2}S_0(x+\mu B_t) + \frac{1}{\mu^2}\int_0^t c(x+\mu B_s, u_{t-s}^\mu(x+\mu B_s))ds - \frac{1}{2}\int_0^t k^2(\frac{t-s}{\mu^2})d\frac{s}{\mu^2} - \int_0^t k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}}}, \end{aligned} \tag{9.4}$$

where B_t is a R^r Brownian motion defined on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$.

Let $\bar{c}$ be C^2 and S_0 be C^1 and Φ and the region D be the same as in §2. For $(t,x) \in D$, define $\tilde{V}^{\mu,k} : D \mapsto R$ by

$$\begin{aligned}\tilde{V}_t^{\mu,k}(x) = &\int_0^t \bar{c}(\Phi_s(\Phi_t^{-1}(x)))ds - S_0(\Phi_t^{-1}(x)) \\ &- \frac{1}{2}\int_0^t |\dot{\Phi}_s(\Phi_t^{-1}(x))|^2 ds - \frac{1}{2}\mu^2 \int_0^{\frac{t}{\mu^2}} k^2(s)ds,\end{aligned} \tag{9.5}$$

and use the convention $\tilde{V}_t^{\mu,k}(x) = +\infty$ if $(t,x) \notin D$. Then the following Hamilton Jacobi equation is satisfied:

$$\frac{1}{2}||\nabla \tilde{V}_t^{\mu,k}(x)||^2 + \bar{c}(x) - \frac{1}{2}\mu^2 k(\frac{t}{\mu^2}) - \frac{\partial \tilde{V}_t^{\mu,k}}{\partial t}(x) = 0, \quad (t,x) \in D, \tag{9.6}$$

For any compact set $\mathcal{K}$ in D, as in the proof of Theorem 2.1 define $\mathcal{N}_{\mathcal{K}}$, $\eta(\sigma)$. Let $X_s^{x,\mu}$ for $0 \le s < \eta(X_\cdot^{x,\mu})$ be the solution of (1.2) with $A_s = \nabla \tilde{V}_{t-s}^{\mu,k}$ up to exit time $\eta(X_\cdot^{x,\mu})$. By the Feynman-Kac formula (9.4), Proposition I.1.1 and using the Hamilton-Jacobi equation (9.6),

$$\begin{aligned}u_t^\mu(x) = &e^{\frac{1}{\mu^2}\tilde{V}_t^{\mu,k}(x)} \hat{E}\chi_{t<\eta(X_\cdot^{x,\mu})} T_0(X_t^{x,\mu}) e^{\frac{1}{2}\int_0^t \Delta \tilde{V}_{t-s}^{\mu,k}(X_s^{x,\mu})ds} \\ &\cdot e^{\frac{1}{\mu^2}\int_0^t \left[c(X_s^{x,\mu}, u_{t-s}^\mu(X_s^{x,\mu})) - \bar{c}(X_s^{x,\mu})\right]ds - \int_0^t k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}}} \\ &+ \hat{E}\chi_{t>\eta(x+\mu B_\cdot)} T_0(x+\mu B_t) e^{-\frac{1}{\mu^2}S_0(x+\mu B_t) + \frac{1}{\mu^2}\int_0^t c(x+\mu B_s, u_{t-s}(x+\mu B_s))ds} \\ &\cdot e^{-\frac{1}{2}\int_0^t k^2(\frac{t-s}{\mu^2})d\frac{s}{\mu^2} - \int_0^t k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}}}.\end{aligned} \tag{9.7}$$

As (2.12), there exists a $R(\mu) = o(\mu^n)$ for any $n \ge 0$ such that

$$\begin{aligned}&\hat{E}\chi_{t>\eta(x+\mu B_\cdot)} T_0(x+\mu B_t) e^{-\frac{1}{\mu^2}S_0(x+\mu B_t) + \frac{1}{\mu^2}\int_0^t c(x+\mu B_s, u_{t-s}(x+\mu B_s))ds - \frac{1}{2}\int_0^t k^2(\frac{t-s}{\mu^2})d\frac{s}{\mu^2}} \\ \le &e^{\frac{\tilde{V}_t^{\mu,k}(x)}{\mu^2}} \times R(\mu).\end{aligned} \tag{9.8}$$

B. For simplicity we suppose the existence of $\lim_{\sigma \to +\infty} \frac{1}{\sigma}\int_0^\sigma k^2(s)ds$. Therefore $a = a^* = a_* \ge 0$. With this assumption for t in any compact set of $(0, +\infty)$, define

$$\tilde{V}_t^k(x) = \lim_{\mu \to 0} \tilde{V}_t^{\mu,k}(x). \tag{9.9}$$

Note $\tilde{V}_t^k(x) = V_t(x) - at$ where $V_t(x)$ is defined by (5.3).

Theorem 9.1. *Assume the condition (I), $a = \frac{1}{2}\lim_{\sigma\to+\infty}\frac{1}{\sigma}\int_0^\sigma k^2(s)ds$ (implying the existence), that $\bar{c}, S_0$ are C^2 with S_0 bounded below, and that T_0 is bounded and measurable. Then for any compact subset $\mathcal{K}$ of $\{(t,x) : \tilde{V}_t^k(x) < 0, t > 0\}$, there exists $\mu_1(\mathcal{K}) > 0$ such that*

$$P\{\omega \in \Omega : \sup_{(t,x)\in\mathcal{K}} u_t^\mu(x,\omega) > e^{-\frac{\delta}{\mu^2}}, \textit{ for some } \; 0 < \mu \le \mu_0\} < e^{-\frac{\delta^2}{16(a+1)\bar{t}\mu_0^2}}, \tag{9.10}$$

for any $0 < \mu_0 < \mu_1$, *where* $\delta = -\frac{1}{3}\sup\{\tilde{V}_t^k(x) : (t,x) \in \mathcal{K}\}$, $\bar{t} = \sup\{t : (t,x) \in \mathcal{K}\}$. *In particular, as* $\mu \to 0$, $u_t^\mu(x,\omega) \to 0$, *uniformly in* $\mathcal{K}$ *P-a.s..*

Proof. By the definition of $\tilde{V}^k$ and a there exists $\mu_1(\mathcal{K}) > 0$ such that for $(t,x) \in \mathcal{K}$, $0 < \mu < \mu_1$,

$$\tilde{V}_t^{\mu,k} \le \tilde{V}_t^k(x) + \delta \le -2\delta, \tag{9.11}$$

$$\frac{1}{2\frac{t}{\mu^2}} \int_0^{\frac{t}{\mu^2}} k^2(s)ds \le a + 1, \tag{9.12}$$

and (2.13) holds for the $R(\mu)$ in (9.8). For any $0 < \mu_0 < \mu_1$, set $\Omega_0^{\mu_0} = \{\omega \in \Omega : \sup_{t:(t,x)\in\mathcal{K}} -\mu^2 \int_0^t k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}} < \frac{1}{2}\delta$ for any $0 < \mu < \mu_0\}$. Then $P(\Omega_0^{\mu_0}) > 1 - e^{-\frac{\delta^2}{8\mu_0^4 \int_0^{\frac{t}{\mu_0^2}} k^2(s)ds}} > 1 - e^{-\frac{\delta^2}{16(a+1)\bar{t}\mu_0^2}}$. The rest of the proof proceeds as in the proof of Theorem 2.1. ‡‡

Remark 9.1. *If* $\lim_{\sigma\to+\infty} \frac{1}{\sigma}\int_0^\sigma k^2(s)ds$ *does not exist, define* $\overline{V}_t^k(x) = \overline{\lim_{\mu\to 0}}\, \tilde{V}_t^{\mu,k}(x)$. *Then Theorem 9.1 is still valid for* $\overline{V}_t^k(x)$ *if* $a^* < \infty$ *and taking* $a = a^*$.

C. We study the case $0 \le a = a^* = a_* \le \sup_{(t,x)\in D} \frac{1}{t}\int_0^t \bar{c}(\Phi_s(\Phi_t^{-1}(x)))ds$. We just proved a trough theorem (Theorem 9.1). For the crest we first establish a comparison result. Let

$$< k, w >_t^\Delta = \sup_{0\le\sigma\le t} [-\int_0^\sigma k(t-s)dw_{t-s}], \tag{9.13}$$

$$< k, w >_{\Delta,t} = \inf_{0\le\sigma\le t} [-\int_0^\sigma k(t-s)dw_{t-s}]. \tag{9.14}$$

Let $v_t^\mu(x)$ be the solution of the following Cauchy problem

$$\begin{cases} \frac{\partial v_t^\mu(x)}{\partial t} = \frac{1}{2}\mu^2 \Delta v_t^\mu(x) + \frac{1}{\mu^2}\left[c(x, v_t^\mu(x)) - \frac{1}{2}k^2(\frac{t}{\mu^2})\right] v_t^\mu(x), \\ v_0^\mu(x) = u_0^\mu(x). \end{cases} \tag{9.15}$$

By using the same method as Lemma 3.1 we have

Lemma 9.1. *Assume* $c(x,-)$ *is decreasing. Then for all* t *and* ω, *with the increment* $< k, w >_{\frac{t}{\mu^2}}^\Delta$ *and* $< k, w >_{\Delta,\frac{t}{\mu^2}}$ *non-zero,*

$$v_t^\mu(x) e^{< k,w >_{\Delta,\frac{t}{\mu^2}}} < u_t^\mu(x,\omega) < v_t^\mu(x) e^{< k,w >_{\frac{t}{\mu^2}}^\Delta}. \tag{9.16}$$

Suppose k is bounded and define

$$\overline{a}(t) = \frac{1}{2} \sup_{\mu \geq 0} k^2\left(\frac{t}{\mu^2}\right), \tag{9.17}$$

$$\underline{a}(t) = \frac{1}{2} \inf_{\mu \geq 0} k^2\left(\frac{t}{\mu^2}\right). \tag{9.18}$$

Consider conditions

$(\widetilde{MN})$. *$0 \leq a = a^* = a_* \leq \sup_{(t,x) \in D} \frac{1}{t} \int_0^t \overline{c}(\Phi_s(\Phi_t^{-1}(x)))ds$ (implying the existence of a).*

$(\widetilde{II})$. *$c(x,u)$ is decreasing in u and let bounded continuous positive functions $\overline{H}_{t,x}$ and $\underline{H}_{t,x}$ be defined by $c(x, \overline{H}_{t,x}) = \underline{a}(t)$, $c(x, \underline{H}_{t,x}) = \overline{a}(t)$ $0 \leq t \leq T$, for some $T > 0$ and all $x \in R^r$ and for any $0 \leq t \leq T, x \in R^r$, $c(x,u) < c(x, \overline{H}_{t,x})$, for $u > \overline{H}_{t,x}$, $c(x,u) > c(x, \underline{H}_{t,x})$, for $u < \underline{H}_{t,x}$.*

$(\widetilde{DZ}^k)$ and $(\tilde{N}^{**k})$ are similar to (DZ^k) and (N^{**k}) in §3 but with $\tilde{V}^k$ instead of V^k.

Theorem 9.2. *Assume conditions (I'), $(\widetilde{MN})$, $(\widetilde{II})$ and $(\tilde{N}^{**k})$, $(\widetilde{DZ}^k)$, and that $\bar{c}$ and S_0 are C^2 with S_0 nonnegative, and that T_0 is positive and bounded continuous. Then for any $\epsilon > 0$, and any compact subset $\mathcal{K}$ of $\{(t,x) : \tilde{V}_t^k(x) > 0, 0 < t \leq T\}$, there exists $\mu_0(\mathcal{K}, \epsilon) > 0$, such that for $0 < \mu < \mu_0$, the solution $u_t^\mu(x)$ of Cauchy problem (9.1) satisfies,*

$$(\underline{H}_{t,x} - \epsilon) e^{<k,w>_{\Delta, \frac{t}{\mu^2}}} \leq u_t^\mu(x, \omega) \leq (\overline{H}_{t,x} + \epsilon) e^{<k,w>^{\Delta}_{\frac{t}{\mu^2}}}. \tag{9.19}$$

Proof. Let $\overline{v}_t^\mu(x)$ be the solution of the Cauchy problem

$$\begin{cases} \frac{\partial \overline{v}_t^\mu(x)}{\partial t} = \frac{1}{2}\mu^2 \Delta \overline{v}_t^\mu(x) + \frac{1}{\mu^2}\left[c(x, \overline{v}_t^\mu(x)) - \underline{a}(t)\right] \overline{v}_t^\mu(x), \\ \overline{v}_0^\mu(x) = u_0^\mu(x), \end{cases}$$

and $\underline{v}_t^\mu(x)$ be the solution of the Cauchy problem

$$\begin{cases} \frac{\partial \underline{v}_t^\mu(x)}{\partial t} = \frac{1}{2}\mu^2 \Delta \underline{v}_t^\mu(x) + \frac{1}{\mu^2}\left[c(x, \underline{v}_t^\mu(x)) - \overline{a}(t)\right] \underline{v}_t^\mu(x), \\ \underline{v}_0^\mu(x) = u_0^\mu(x). \end{cases}$$

Then $\underline{v}_t^\mu(x) \leq v_t^\mu(x) \leq \overline{v}_t^\mu(x)$ can be proved by Feynman-Kac formula because $c(x, -)$ is decreasing. By Theorem I.1.5, for any $\epsilon > 0$, there exists $\mu_0(\mathcal{K}, \epsilon) > 0$ such that for $(t,x) \in \mathcal{K}$ and $0 < \mu < \mu_0$, $\overline{v}_t^\mu(x) \leq \overline{H}_{t,x} + \epsilon$ and $\underline{v}_t^\mu(x) \geq \underline{H}_{t,x} - \epsilon$. The theorem follows from Lemma 9.1. ‡‡

Remark 9.2. *Even if the positive function $\underline{H}_{t,x}$ in conditions $(\widetilde{MN})$ does not exist then Theorem 9.2 is still true in the following form:*

$$0 \leq u_t^\mu(x,\omega) \leq (\overline{H}_{t,x} + \epsilon)e^{<k,w>^{\Delta}_{\frac{t}{\mu^2}}}. \tag{9.20}$$

Corollary 9.1. *Assume conditions of Theorem 9.2 and $\underline{H}_{t,x} > 0$. Then for $\tilde{V}_t^k(x) > 0$, there exist $c_1 > 0, c_2 > 0$ such that*

$$-c_1 \leq \underline{\lim}_{\mu\to 0} \frac{\mu}{\sqrt{\log\log\frac{1}{\mu^2}}} \log u_t^\mu(x) \leq \overline{\lim}_{\mu\to 0} \frac{\mu}{\sqrt{\log\log\frac{1}{\mu^2}}} \log u_t^\mu(x) \leq c_2, P - a.s.. \tag{9.21}$$

Proof. The result follows by taking the logarithm on both sides of (9.20) and multiplying $\frac{\mu}{\sqrt{\log\log\frac{1}{\mu^2}}}$ and noting that if $0 \leq a < \infty$ there exists $c_1 > 0, c_2 > 0$ such that $\overline{\lim}_{\mu\to 0} \frac{\mu}{\sqrt{\log\log\frac{1}{\mu^2}}} < k,w >^{\Delta}_{\frac{t}{\mu^2}} \leq c_2$ and $\underline{\lim}_{\mu\to 0} \frac{\mu}{\sqrt{\log\log\frac{1}{\mu^2}}} < k,w >_{\Delta,\frac{t}{\mu^2}} \geq -c_1$ P-a.s. by the iterated logarithm law of Brownian motion. ‡‡

Remark 9.3. *From Theorem 9.1 and Corollary 9.1 we see clearly the wave front propagation for the stochastic generalised KPP equations. The wave front is given by $\tilde{Z} = \{(t,x) : \tilde{V}_t^k(x) = 0\}$ if we suppose the conditions in Corollary 9.1. In front of $\tilde{Z}$, $\overline{\lim}_{\mu\to 0} \mu^2 \log u_t^\mu(x) \leq -c$ $P-a.s.$ for certain constant $c > 0$ and behind $\tilde{Z}$, $\underline{\lim}_{\mu\to 0} \frac{\mu}{\sqrt{\log\log\frac{1}{\mu^2}}} \log u_t^\mu(x) \geq -c_1$ $P-a.s.$ for certain constants $c_1 > 0$. An immediate example is that of $k =$ constant which was examined in §8. The following example is interesting to see:*

Example 9.1. Consider the following stochastic KPP equation with a point source approximated by $\exp\{-\frac{x^2}{2\lambda\mu^4}\}$ in the limit $\lambda \to 0$

$$du_t^\mu(x) = [\frac{\mu^2}{2}\Delta u_t^\mu(x) + \frac{1}{\mu^2}(1 - u_t^\mu(x))u_t^\mu(x)]dt + \sin\frac{t}{\mu^2}u_t^\mu(x)dw_{\frac{t}{\mu^2}}. \tag{9.22}$$

This is equivalent to the following unscaled stochastic KPP equation

$$du_t(x) = [\frac{1}{2}\Delta u_t(x) + (1 - u_t(x))u_t(x)]dt + \sin t u_t(x)dw_t. \tag{9.23}$$

By the definition of $V_t^k(x)$ noting $a = \lim_{\sigma \to +\infty} \frac{1}{2\sigma} \int_0^\sigma k^2(s)ds = \frac{1}{4}$, and the result of §$II$.1 of chapter II, we have

$$\tilde{V}_t^k(x) = \frac{3}{4}t - \frac{x^2}{2t}. \tag{9.24}$$

Furthermore $\overline{a}(t) = \frac{1}{2}\sup_{\mu \geq 0} k^2(\frac{t}{\mu^2}) = \frac{1}{2}\sup_{\mu \geq 0} \sin^2(\frac{t}{\mu^2}) = \frac{1}{2}$ and $\underline{a}(t) = \frac{1}{2}\inf_{\mu \geq 0} k^2(\frac{t}{\mu^2}) = 0$. Therefore by definitions $\overline{H}_{t,x} = 1 - \underline{a}(t) \leq 1$ and $\underline{H}_{t,x} = 1 - \overline{a}(t) = \frac{1}{2}$. Thus by Theorem 9.1 and Theorem 9.2 the wave front is given by $x = \sqrt{\frac{3}{2}}t$. Note that the wave front of the unperturbed KPP equation is given by $x = \sqrt{2}t$.

D. Case $\sup_{(t,x) \in D} \frac{1}{t} \int_0^t \overline{c}(\Phi_s(\Phi_t^{-1}(x)))ds < a_* \leq a^* < +\infty$. For a compact subset $\mathcal{K}$ of D by (9.2), (9.3) we have $\mu_1(\mathcal{K}) > 0$ such that for $0 < \mu \leq \mu_1$, $(t,x) \in \mathcal{K}$

$$\frac{a_* + c^*}{2} \leq \frac{1}{2\frac{t}{\mu^2}} \int_0^{\frac{t}{\mu^2}} k^2(\sigma)d\sigma \leq \frac{5}{4}a^*, \tag{9.25}$$

and

$$\mu^2 \sup_{(t,x) \in \mathcal{K}} \log[\hat{E}\chi_{t < \eta(X_\cdot^{x,\mu})} T_0(X_t^{x,\mu}) e^{\frac{1}{2}\int_0^t \Delta V_{t-s}^{\mu,k}(X_s^{x,\mu})ds} + R(\mu)] \leq \frac{a_* - c^*}{8}\underline{t}. \tag{9.26}$$

Here $c^* = \sup_{(t,x) \in \mathcal{K}} \frac{1}{t} \int_0^t \overline{c}(\Phi_s(\Phi_t^{-1}(x)))ds$, $\underline{t} = \inf\{t : (t,-) \in \mathcal{K}\}$. For $0 < \mu_0 \leq \mu_1$, let

$$\Omega_0^{\mu_0} = \{\omega \in \Omega : -\mu^2 \int_0^t k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}} < \frac{a_* - c^*}{8}, \text{ for all } 0 < \mu < \mu_0, (t,-) \in \mathcal{K}\}. \tag{9.27}$$

Then $P(\Omega_0^{\mu_0}) > 1 - e^{-\frac{(a_* - c^*)^2}{128 \int_0^{\frac{t}{\mu_0^2}} k^2(\sigma)d\sigma \mu_0^4}} > 1 - e^{-\frac{(a_* - c^*)^2}{320a^*\overline{t}\mu_0^2}}$ by (9.25). Here $\overline{t} = \sup\{t : (t,-) \in \mathcal{K}\}$. Suppose S_0 is nonnegative. Therefore for $0 < \mu_0 \leq \mu_1$ and $\omega \in \Omega_0^{\mu_0}$, $0 < \mu < \mu_0$, taking the logarithm on both sides of (9.7) in which we use (9.8) to control the second term and multiplying μ^2

$$\begin{aligned} &\mu^2 \log u_t^\mu(x) \\ \leq& c^*t - \frac{a_* + c^*}{2}t + \frac{a_* - c^*}{8}t + \frac{a_* - c^*}{8}t \\ =& -\frac{a_* - c^*}{4}t. \end{aligned} \tag{9.28}$$

From this we obtain the following theorem:

Theorem 9.3. *Assume condition (I) and* $\sup_{(t,x) \in D} \frac{1}{t}\int_0^t \overline{c}(\Phi_s(\Phi_t^{-1}(x)))ds < a_* \leq a^* < +\infty$, *and that* $\overline{c}, S_0$ *are* C^2 *with* S_0 *nonnegative and that* T_0 *is bounded and measurable.*

Then for any compact subset $\mathcal{K}$ of D, there exists $\mu_1(\mathcal{K}) > 0$ such that

$$P\{\omega \in \Omega : u_t^\mu(x) > e^{-\frac{a_*-c^*}{4\mu^2}t},\ \textit{some}\ (t,x) \in \mathcal{K}, 0 < \mu < \mu_0\} < e^{-\frac{(a_*-c^*)^2}{320a^*\bar{t}\mu_0^2}}, \tag{9.29}$$

for all $0 < \mu_0 < \mu_1$ where $\bar{t} = \sup\{t : (t,-) \in \mathcal{K}\}$, a_*, a^* are defined by (9.2), (9.3). In particular as $\mu \to 0$, $u_t^\mu(x) \to 0$ uniformly in $\mathcal{K}$ P-a.s..

If $a^* \geq a_* > \hat{c} > 0$, we can take the logarithm on both sides of (9.4). Therefore

Corollary 9.2. *Assume condition (I) and $a^* \geq a_* > \hat{c} > 0$, and that S_0 is nonnegative and T_0 is bounded and measurable. Then for any $\bar{t} > \underline{t} > 0$, there exists $\mu_1(\underline{t}, \bar{t}) > 0$ such that*

$$P\{\omega \in \Omega : u_t^\mu(x) > e^{-\frac{a_*-\hat{c}}{4\mu^2}t},\ \textit{some}\ \underline{t} < t < \bar{t}, x \in R^r, 0 < \mu < \mu_0\} < e^{-\frac{(a_*-\hat{c})^2}{320a^*\bar{t}\mu_0^2}},$$

for all $0 < \mu_0 < \mu_1$. Here a_*, a^* are defined by (9.2), (9.3). In particular as $\mu \to 0$, $u_t^\mu(x) \to 0$ uniformly in $\underline{t} \leq t \leq \bar{t}, x \in R^r$ P-a.s..

It is very interesting to reconsider Example 9.1 by varying the coefficient of the random noise. It is impossible to compare the values $3\sin\frac{t}{\mu^2}$ and $k = 1$ which is from Theorem 8.1 (here we take $\hat{c} = 1$). Nevertheless we can compare the values $\lim_{\sigma\to+\infty} \frac{1}{2\sigma}\int_0^\sigma 9\sin^2 s ds = \frac{9}{4}$ and $\lim_{\sigma\to+\infty} \frac{1}{2\sigma}\int_0^\sigma 1^2 ds = \frac{1}{2}$. The former mean value is bigger than the latter one. In Example 9.2 the noise forces the solution to zero while the later noise does not by Theorem 8.1. This shows that $a = \lim_{\sigma\to+\infty} \frac{1}{2\sigma}\int_0^\sigma k^2(s)ds$ is the appropriate quantity to classify the multiplicative random noise perturbations $k(\cdot)dw$..

Example 9.2. Consider

$$du_t^\mu(x) = [\frac{\mu^2}{2}\Delta u_t^\mu(x) + \frac{1}{\mu^2}(1 - u_t^\mu(x))u_t^\mu(x)]dt + 3\sin\frac{t}{\mu^2}u_t^\mu(x)dw_{\frac{t}{\mu^2}}, \tag{9.30}$$

with an initial point source. Calculation tells us that $a = a^* = a_* = \lim_{\sigma\to\infty} \frac{1}{2\sigma}\int_0^\sigma k^2(s)ds = \frac{9}{4} > 1 = \hat{c}$. Then by Theorem 9.3, the solution $u_t^\mu(x) \to 0$ as $\mu \to 0$ $P-a.s.$ for any $(t,x) \in (0,+\infty) \times R^r$.

E. If $a^* = a_* = +\infty$ suppose there exist $\alpha > 0$, $0 < a_{*\alpha} \leq a_\alpha^* < +\infty$ such that

$$a_{*\alpha} = \varliminf_{\sigma\to+\infty} \frac{1}{2\sigma^{1+\alpha}}\int_0^\sigma k^2(s)ds, \tag{9.31}$$

$$a^*_\alpha = \overline{\lim_{\sigma\to+\infty}} \frac{1}{2\sigma^{1+\alpha}} \int_0^t k^2(\sigma)d\sigma. \tag{9.32}$$

For any $\bar{t} > \underline{t} > 0$ by (9.31), (9.32) we have $\mu_1(\bar{t}, \underline{t}) > 0$ such that for $0 < \mu \le \mu_1$, $\underline{t} \le t \le \bar{t}$

$$\frac{3}{4}a_{*\alpha} \le \frac{1}{2\left(\frac{t}{\mu^2}\right)^{1+\alpha}} \int_0^{\frac{t}{\mu^2}} k^2(\sigma)d\sigma \le \frac{5}{4}a^*_\alpha, \tag{9.33}$$

and

$$\mu^{2(1+\alpha)} \log ||T_0||_\infty + \mu^{2\alpha}\hat{c}\bar{t} \le \frac{1}{4}a_{*\alpha}\underline{t}^{1+\alpha}. \tag{9.34}$$

For $0 < \mu_0 \le \mu_1$, let

$$\Omega_0^{\mu_0} = \{\omega \in \Omega : -\mu^{2(1+\alpha)} \int_0^t k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}} < \frac{1}{4}a_{*\alpha}\underline{t}^{1+\alpha}, \text{ for all } 0 < \mu < \mu_0, \underline{t} \le t \le \bar{t}\}. \tag{9.35}$$

Then $P(\Omega_0^{\mu_0}) > 1 - e^{-\frac{(a_{*\alpha}\underline{t}^{1+\alpha})^2}{32\int_0^{\frac{t}{\mu_0^2}} k^2(\sigma)d\sigma\mu_0^{4(1+\alpha)}}} > 1 - e^{-\frac{(a_{*\alpha}\underline{t}^{1+\alpha})^2}{80a^*_\alpha\bar{t}^{1+\alpha}\mu_0^{2(1+\alpha)}}}$. Therefore for $0 < \mu_0 \le \mu_1$ and $\omega \in \Omega_0^{\mu_0}$, $\underline{t} \le t \le \bar{t}$, $0 < \mu < \mu_0$, taking the logarithm on both sides of (9.4) and multiplying $\mu^{2(1+\alpha)}$

$$\begin{aligned}
&\mu^{2(1+\alpha)} \log u_t^\mu(x) \\
\le&\mu^{2(1+\alpha)} \log ||T_0|| + \mu^{2\alpha}\hat{c}t - \frac{1}{2}\mu^{2(1+\alpha)} \int_0^t k^2(\frac{t-s}{\mu^2})d\frac{s}{\mu^2} \\
&- \mu^{2(1+\alpha)} \int_0^t k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}} \\
\le&\frac{1}{4}a_{*\alpha}t^{1+\alpha} - \frac{3}{4}a_{*\alpha}t^{1+\alpha} + \frac{1}{4}a_{*\alpha}t^{1+\alpha} \\
=&-\frac{1}{4}a_{*\alpha}t^{1+\alpha}.
\end{aligned} \tag{9.36}$$

From this we obtain the following theorem:

Theorem 9.4. *Assume condition (I), and there exists $\alpha > 0, 0 < a_{*\alpha} \le a^*_\alpha < \infty$ with (9.31) and (9.32) and T_0 is bounded, S_0 is nonnegative. Then for any $\bar{t} > \underline{t} > 0$, there exists $\mu_1(\bar{t}, \underline{t}) > 0$ such that*

$$\begin{aligned}
&P\{\omega \in \Omega : u_t^\mu(x) > e^{-\frac{a_{*\alpha}t^{1+\alpha}}{4\mu^{2(1+\alpha)}}},\ \textit{some } \underline{t} \le t \le \bar{t}, x \in R^r, 0 < \mu < \mu_0\} \\
&< e^{-\frac{(a_{*\alpha}\underline{t}^{1+\alpha})^2}{80a^*_\alpha\bar{t}^{1+\alpha}\mu_0^{2(1+\alpha)}}},
\end{aligned} \tag{9.37}$$

for all $0 < \mu_0 < \mu_1$. In particular as $\mu \to 0$, $u_t^\mu(x) \to 0$ uniformly in $\mathcal{K}$ P-a.s..

Remark 9.4. *Theorem 9.4 can be generalised in the following case: if there exist* $\alpha(t) > 0$ *with* $\frac{\alpha(t)}{t} \to +\infty$ *as* $t \to +\infty$, *and* $0 < \tilde{a}_* \leq \tilde{a}^* < \infty$ *such that*

$$\tilde{a}_* = \frac{1}{2} \varliminf_{\sigma \to +\infty} \frac{1}{\alpha(\sigma)} \int_0^\sigma k^2(s)ds, \tag{9.38}$$

$$\tilde{a}^* = \frac{1}{2} \varlimsup_{\sigma \to +\infty} \frac{1}{\alpha(\sigma)} \int_0^\sigma k^2(s)ds. \tag{9.39}$$

We leave this case as an exercise for the reader.

F. If $k \in L^2(0, +\infty)$, then $a = 0$. Therefore Theorem 9.1 and Theorem 9.2 are valid. However in the following we will prove that the perturbation is very weak. The phenomenon is quite different from the case when $a > 0$. We modify the proof in §5 to prove if $k \in L^2(0, \infty)$, the solution to equation (9.1) converges to the deterministic approximate travelling wave almost surely. For this we consider condition (I), (I'), (II), (N^{**}) (DZ) the same as in §5 and condition

(L^2). The function $k \in L^2(0, +\infty)$.

The following theorem is a corollary of Theorem 9.1, if we note $\tilde{V}_t^k(x) = V_t(x)$ (where $V_t(x)$ is defined by (5.3)). However the large deviation probability estimate is far better than that in Theorem 9.1.

Theorem 9.5. *Assume the condition (I), (L^2), that* $\bar{c}, S_0$ *are* C^2 *with* S_0 *bounded below, and that* T_0 *is bounded and measurable. Then for any compact subset* $\mathcal{K}$ *of* $\{(t,x) : V_t(x) < 0, t > 0\}$, *there exists* $\mu_1(\mathcal{K}) > 0$ *such that*

$$P\{\omega \in \Omega : \sup_{(t,x)\in\mathcal{K}} u_t^\mu(x,\omega) > e^{-\frac{\delta}{\mu^2}}, \text{ for some } \quad 0 < \mu \leq \mu_0\} < e^{-\frac{\delta^2}{8\int_0^{+\infty} k^2(\sigma)d\sigma\mu_0^4}}, \tag{9.40}$$

for any $0 < \mu_0 < \mu_1$, *where* $\delta = -\frac{1}{2}\sup\{V_t(x) : (t,x) \in \mathcal{K}\}$. *In particular, as* $\mu \to 0$, $u_t^\mu(x,\omega) \to 0$, *uniformly in* $\mathcal{K}$ *P-a.s..*

Proof. We only need to note that for $k \in L^2(0,\infty)$, $P(\Omega_0^{\mu_0}) > 1 - e^{-\frac{\delta^2}{8\mu_0^4\int_0^{\frac{t}{\mu_0}} k^2(s)ds}} > 1 - e^{-\frac{\delta^2}{8\int_0^\infty k^2(s)ds\mu_0^4}}$ in the proof of Theorem 9.1. ‡‡

Lemma 9.2. *Suppose* $T_0(x)$ *is a nonnegative bounded function,* $S_0(x)$ *is a nonnegative continuous function, and conditions (II), (L^2). Then for any* $0 < \epsilon < 1$ *and* $\bar{t} > \underline{t} > 0$, *there exists* $\mu_1(\bar{t}, \underline{t}, \epsilon) > 0$, *such that*

$$P\{\omega \in \Omega : \sup_{0<\mu\leq\mu_0} \sup_{\underline{t}\leq t\leq\bar{t}, x\in R^r} u_t^\mu(x,\omega) > 1 + \epsilon\} < e^{-I_1(\mu_0,\epsilon,\underline{t},\bar{t})}, \tag{9.41}$$

for all $0 < \mu_0 < \mu_1$ *with* $I_1(\mu_0, \epsilon, \underline{t}, \bar{t}) = \frac{1}{2}\left(\inf_{\underline{t}\le t\le \bar{t}} min\left\{\frac{\epsilon^2}{18\int_{\frac{t}{2\mu_0^2}}^{\frac{t}{\mu_0^2}} k^2(s)ds}, \frac{\alpha^2\underline{t}^2}{8\mu_0^4\int_0^\infty k^2(s)ds}\right\}\right).$

Proof. For any $\mu_0 > 0$ set

$$\Omega_0^{(1)} = \{\omega \in \Omega : -\int_0^s k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}}(\omega) \le \frac{1}{3}\epsilon, \text{ all } 0 \le s \le \frac{1}{2}t, 0 < \mu < \mu_0\}, \quad (9.42)$$

$$\Omega_0^{(2)} = \{\omega \in \Omega : -\mu^2\int_0^s k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}}(\omega) \le \frac{1}{2}\alpha\underline{t}, \text{ all } \frac{1}{2}t \le s \le t, 0 < \mu < \mu_0\}. \quad (9.43)$$

Then $P(\Omega_0^{(1)}) > 1 - e^{-\frac{\epsilon^2}{18\int_{\frac{t}{2\mu_0^2}}^{\frac{t}{\mu_0^2}} k^2(s)ds}}$ and $P(\Omega_0^{(2)}) > 1 - e^{-\frac{\alpha^2 t^2}{8\mu_0^4\int_0^\infty k^2(s)ds}}$. Let $\Omega_0 = \Omega_0^{(1)} \cap \Omega_0^{(2)}$. From (L^2), we know $I_1 \to +\infty$ as $\mu_0 \to 0$. Taking μ_1 smaller if necessary we assume $I_1 \ge \log 2$. So $P(\Omega_0) > 1 - e^{-I_1}$. Then the proof proceeds as the proof of Lemma 5.1 if we use

$$\hat{\Omega} = \{\hat{\omega} \in \hat{\Omega} : \tau < \frac{1}{2}t\} + \{\hat{\omega} \in \hat{\Omega} : \frac{1}{2}t \le \tau < t\} + \{\hat{\omega} \in \hat{\Omega} : \tau \ge t\}, \quad (9.44)$$

where τ is defined on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ for fixed $\omega \in \Omega_0$ by

$$\tau = \tau^\mu = \inf\{s : u^\mu(t-s, x+\mu B_s, \omega) \le 1 + \frac{1}{3}\epsilon\}. \quad (9.45)$$

‡‡

Example 9.3. For $k(s) = e^{-\frac{1}{2}s}$, $\mu_0^4 I_1(\mu_0, \epsilon, \mathcal{K}) = \bigcirc(1)$ as $\mu_0 \to 0$.

For a compact subset $\mathcal{K}$ of D define $\mathcal{N}_{\mathcal{K}}$ as in the proof of Theorem 5.1 with η the first exit time. Let $\tilde{X}_s^{x,\mu} = X_{s\wedge\eta}^{x,\mu}$. By the same argument of Lemma 2.2 and Theorem 9.5 we have the following Lemma:

Lemma 9.3. *Assume all conditions of Theorem 9.5. Then if* $0 \le \theta_i \le \frac{1}{2}t$, $i = 1, 2$, *for any compact subset* $\mathcal{K}$ *of* $\{(t,x) : V_{t-s}(z_s^t(x)) < 0, \theta_1 \le s \le t - \theta_2\}$ *and* $\epsilon^* > 0$, *there exist* $\mu_0 > 0$, $\delta > 0$ *and* $\Omega_1 \subset \Omega$ *with* $P(\Omega_1) > 1 - e^{-\frac{t^2}{8\int_0^{+\infty} k^2(s)ds\mu_0^4}}$ *such that if* $\omega \in \Omega_1$,

$$\hat{P}\{\hat{\omega} \in \hat{\Omega} : \mu^2 \sup_{\theta_1\le s\le t-\theta_2,(t,x)\in\mathcal{K}} \log u_{t-s}^\mu(\tilde{X}_s^{x,\mu}(\hat{\omega}), \omega) < -\delta, \text{ for all } \quad 0 < \mu \le \mu_0\}$$
$$> 1 - \epsilon^*. \quad (9.46)$$

In particular, for almost all $\omega \in \Omega$, and $\theta_1 \le s \le t - \theta_2$, as $\mu \to 0$,

$$\frac{u^{\mu}_{t-s}(\tilde{X}^{x,\mu}_s, \omega)}{\mu^2} \to 0, \text{ in } \hat{P} \text{ probability}. \tag{9.47}$$

Lemma 9.4. *Assume conditions (I'), (II), (N^{**}), (L^2) and that $\bar{c}$ and S_0 are C^2 with S_0 nonnegative and that T_0 is positive and continuous. Then for any $\gamma > 0$ and any compact subset $\mathcal{K}$ of $\{(t,x) : V_t(x) = 0\}$, there exist $\mu_1(\mathcal{K}, \gamma) > 0$, $\delta(\mathcal{K}, \gamma) > 0$ such that*

$$P\{\omega \in \Omega : \mu^2 \log u^{\mu}_t(x,\omega) \ge -\gamma, \text{ all } 0 < \mu \le \mu_0, (t,x) \in \mathcal{K}\} > 1 - e^{-\frac{\delta}{\mu_0^4}}, \tag{9.48}$$

for all $0 < \mu_0 < \mu_1$.

Proof. The lemma follows Lemma 9.4 and similar argument of Lemma 5.3. ‡‡

Theorem 9.6. *Assume conditions (I'), (II), (N^{**}), (DZ), (L^2) and that $\bar{c}$ and S_0 are C^2 with S_0 nonnegative, and that T_0 is positive, bounded, and continuous. Then for any $0 < \varepsilon < 1$ and any compact subset $\mathcal{K}$ of $\{(t,x) : V_t(x) > 0\}$, there exist $\mu_1(\mathcal{K}, \epsilon) > 0$, $\delta(\mathcal{K}, \varepsilon) > 0$, such that*

$$P\{\omega \in \Omega : \sup_{0<\mu\le\mu_0} \sup_{(t,x)\in\mathcal{K}} |u^{\mu}_t(x,\omega) - 1| > \epsilon\} < e^{-I(\mu_0,\varepsilon,\mathcal{K})}, \tag{9.49}$$

for all $0 < \mu_0 < \mu_1$ with $I(\mu_0, \epsilon, \mathcal{K}) = \frac{1}{2} \inf_{\underline{t}\le t\le\bar{t}} \min\left\{ \frac{\varepsilon^2}{128 \int_{\frac{t}{2\mu_0^2}}^{\frac{t}{\mu_0^2}} k^2(s)ds}, \frac{\alpha^2(h \wedge \frac{1}{2}t)^2}{32\mu_0^4 \int_0^{\infty} k^2(s)ds}, \frac{\delta}{\mu_0^4} \right\}$ *where* $\underline{t} = \inf\{t : (t,x) \in \mathcal{K}\}$, $\bar{t} = \sup\{t : (t,x) \in \mathcal{K}\}$. *In particular $u^{\mu}_t(x,\omega) \to 1$, uniformly in $\mathcal{K}$, P-a.s..*

Proof. Define α, h the same as in the proof of Theorem 5.4. Simple analysis implies that there exists $\mu_1^{(1)} > 0$ such that if $0 < \mu \le \mu_1^{(1)}$,

$$\frac{1}{2} \int_0^{\frac{t}{2}} k^2(\frac{t-s}{\mu^2}) d\frac{s}{\mu^2} \le \frac{1}{8}\varepsilon, \tag{9.50}$$

$$\frac{1}{2}\mu^2 \int_0^t k^2(\frac{t-s}{\mu^2}) d\frac{s}{\mu^2} \le \frac{1}{4}\alpha(\frac{1}{2}t \wedge h). \tag{9.51}$$

For $0 < \mu_0 < \mu_1^{(1)}$ let

$$\Omega_1^{\mu_0} = \{\omega \in \Omega : -\frac{1}{2}\int_0^s k^2(\frac{t-s}{\mu^2})d\frac{s}{\mu^2} - \int_0^s k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}}(\omega) \geq -\frac{1}{4}\epsilon, \quad \text{for all } 0 < s < \frac{1}{2}t, 0 < \mu \leq \mu_0\}, \tag{9.52}$$

$$\Omega_2^{\mu_0} = \{\omega \in \Omega : -\frac{1}{2}\mu^2\int_0^s k^2(\frac{t-s}{\mu^2})d\frac{s}{\mu^2} - \mu^2\int_0^s k(\frac{t-s}{\mu^2})dw_{\frac{t-s}{\mu^2}}(\omega) \geq -\frac{1}{2}\alpha(h \wedge \frac{1}{2}t), \quad \text{for all } h \wedge \frac{1}{2}t < s < t, 0 < \mu \leq \mu_0\}, \tag{9.53}$$

then $P(\Omega_1) > 1 - e^{-\frac{\epsilon^2}{128\int_{\frac{t}{2\mu_0^2}}^{\frac{t}{\mu_0^2}} k^2(s)ds}}$ and $P(\Omega_2) > 1 - e^{-\frac{\alpha^2(h\wedge\frac{1}{2}t)^2}{32\mu_0^4\int_0^{\infty} k^2(s)ds}}$. The rest of the proof proceeds as Theorem 5.4 by Lemma 9.5. ‡‡

Example 9.4. For $k(s) = se^{-\frac{1}{2}s}$, $\mu_0^4 I(\mu_0, \epsilon, \mathcal{K}) = \bigcirc(1)$ as $\mu_0 \to 0$.

Remark 9.6. *The estimate of I depends very much on the estimate of* $\int_{\frac{t}{2\mu_0^2}}^{\frac{t}{\mu_0^2}} k^2(s)ds$. *If* $\frac{1}{\mu_0^4}\int_{\frac{t}{2\mu_0^2}}^{\frac{t}{\mu_0^2}} k^2(s)ds = \bigcirc(1)$ *as* $\mu_0 \to 0$, *e.g. in Example 9.4, then* $\mu_0^4 I(\mu_0, \epsilon, \mathcal{K}) = \bigcirc(1)$ *as* $\mu_0 \to 0$. *Otherwise* $I(\mu_0) \sim \frac{1}{\int_{\frac{t}{2\mu_0^2}}^{\frac{t}{\mu_0^2}} k^2(s)ds}$. *For example when* $k(s) = \frac{1}{1+s}$, $\int_{\frac{t}{2\mu_0^2}}^{\frac{t}{\mu_0^2}} k^2(s)ds = \frac{\mu_0^2 t}{(t+\mu_0^2)(t+2\mu_0^2)}$. *Therefore* $\mu_0^2 I(\mu_0, \epsilon, \mathcal{K}) = \bigcirc(1)$ *as* $\mu_0 \to 0$.

Remark 9.7. *If* $k \notin L^2(0, +\infty)$, *generally speaking* $I(\mu_0)$ *does not converge to* $+\infty$ *as* $\mu_0 \to 0$, *i.e.* e^{-I} *does not converge to* 0 *as* $\mu_0 \to 0$. *For example when* $k(s) = \frac{1}{(1+s)^{\frac{1}{4}}} \to 0$ *as* $s \to +\infty$, $\int_{\frac{t}{2\mu_0^2}}^{\frac{t}{\mu_0^2}} k^2(s)ds = 2\frac{\frac{t}{\mu_0^2}}{\sqrt{1+\frac{t}{\mu_0^2}} + \sqrt{1+\frac{t}{2\mu_0}}}$. *Therefore* $I \to 0$ *as* $\mu_0 \to 0$, *i.e.* $e^{-I} \to 1$ *as* $\mu_0 \to 0$. *That is to say that we can not get a good set in* Ω *which is almost all* Ω *in which the solution* $u_t^\mu(x)$ *converges to a deterministic travelling wave if k does not belong to* $L^2(0, +\infty)$ *even if* $k(s) \to 0$ *as* $s \to +\infty$. *We have seen quite different phenomena already.*

Remark 9.8. *From the results of this section we can classify the random perturbations in three levels:*

(i). $k \in L^2(0, +\infty)$, *the perturbation to travelling wave is very weak. In fact we proved that the solution of the stochastic p.d.e. converges to the deterministic travelling wave solution P-a.s..*

(ii). $a_* > \sup_{(t,x)\in D} \frac{1}{t}\int_0^t \overline{c}(\Phi_s(\Phi_t^{-1}(x)))ds$, *the perturbation is very strong. In fact we proved in* §9D *and* §9E *that the solution of the stochastic equation converges to zero* P*-a.s., i.e. the perturbation forces the solution to die.*

(iii). $a = a_* = a^* \leq \sup_{(t,x)\in D} \frac{1}{t}\int_0^t \overline{c}(\Phi_s(\Phi_t^{-1}(x)))ds$, *we get the wave front formula* $\tilde{V}_t^k(x) = 0$ *where* $\tilde{V}_t^k(x)$ *is defined by (9.9), and the estimate for the crest. The situation is similar to mild noise in* §$2-5$.

Remark 9.9. *For the KPP equation* $\overline{c}(x) = \hat{c}$, *then the criteria in (ii) and (iii) of Remark 9.8 are just* $a_* > \hat{c}$ *and* $a = a_* = a^* \leq \hat{c}$ *and* $\frac{1}{2}k^2 > \hat{c}$, *or* $\frac{1}{2}k^2 < \hat{c}$ *if we suppose* k *is a constant too. The same remark is valid for Theorems 9.2, 9.3 and 9.4.*

Chapter IV. The Travelling Wave Fronts of Nonlinear Reaction-Diffusion Systems via Freidlin's Stochastic Approach

The travelling wave fronts of reaction-diffusion systems were also discussed by Fife (1979), Smoller (1983) and Murray (1989). Because of the difficulties of the algebraic computation and the qualitative theory of high dimensional ordinary differential equations (the ordinary differential equations are usually of dimension more than 2), the discussion of travelling wave fronts of the reaction-diffusion systems becomes very difficult. It is clear that the spatial wave phenomenon in reaction diffusion systems is richer than in single species models. Except for wave fronts, wave trains and spiral waves are also possible in systems. Many questions about systems remain open, in particular, important questions with regard to the existence of travelling waves (Volpert and Volpert (1990)). In the case of a multi-reactant system, where the diffusionless system has several steady states, travelling wave fronts, which joined steady states, were studied by Smoller (1983), Murray (1989) and Volpert and Volpert (1990). Freidlin (1992) started to study nonlinear reaction-diffusion systems. In his literature, he only considered the case where the nonlinear reaction terms have the following forms

$$f_k(u) = c_{kk}(u)u_k + \sum_{j=1,j\neq k}^{n} c_{kj}u_j, \quad k = 1, 2, \cdots, n \tag{0.1}$$

where $c_{kj}(k \neq j)$ are positive constants.

Because of the importance and interest of the problem both from mathematics or mathematical physics point of review, and from other applications, we study the travelling wave solution of nonlinear reaction diffusion systems with very general reaction terms

$$f_k(x, u) = \sum_{j=1}^{n} c_{kj}(x, u)u_j, \quad k = 1, 2, \cdots, n. \tag{0.2}$$

basically following Freidlin's method in this chapter. Suppose the diffusionless system has an unstable steady points at point $O = (0, 0, \cdots, 0)$ and asymptotically stable one at $a = (a_1, a_2, \cdots, a_n) > 0$ and $c_{kj}(x,u) \le c_{kj}(x,0) = \hat{c}_{kj}(x)$. We apply some ideas of ergodic theory (e.g. Frobenius Theorem) to study the the travelling wave front of the reaction diffusion system with nonlinear reaction terms

$$f_k(u) = \sum_{j=1}^{n} c_{kj}(u)u_j, \quad k = 1, 2, \cdots, n. \tag{0.3}$$

We suppose the matrix $\hat{C} = (\hat{c}_{ij}) = (c_{ij}(0))$ is a nonnegative irreducible matrix (we call this kind of system the reaction diffusion systems with nonlinear ergodic interactions) satisfying stability condition (2.II) at the point a, and we obtain the existence and the speed of travelling wave. In physics, the results mean that in a system, if each two states have direct or indirect interaction, then the system may have a travelling wave front with a common speed. Generally speaking, the result is not true if the matrix C is reducible. But under the condition

$$\min_{1\le k\le n} \sum_{j=1}^{n} \hat{c}_{kj}(x) = \max_{1\le k\le n} \sum_{j=1}^{n} \hat{c}_{kj}(x), \tag{0.4}$$

and stability condition (3.II) on the point a, the result is still true for the system with nonlinear reaction terms (0.3). In fact we can treat nonlinear reaction terms such as (0.2) under this condition. We study this case in §3.

The main contributions of this chapter are that not only the reaction-diffusion system with nonlinear term (0.1), but also more general systems with nonlinear reaction terms (0.2) or (0.3) are considered, and the interaction of each two states in the system is not required to interact directly, but indirectly, or as (0.4).

§1. The Generalised Solution of the n-Dimensional Nonlinear Cauchy Problem

In order to define the generalised solution of an n-dimensional nonlinear Cauchy problem, we need the n-dimensional Feynman-Kac Formula, which was given by Stroock (1970) and Babbit (1970) first independently and was discussed by Pinsky (1972, 1974). This formula was much improved and widely used on manifolds by Elworthy (1982). We give the n-dimensional Feynman-Kac formula first in a suitable way. Consider the n-dimensional linear Cauchy problem first

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + C(t,x)u, \\ u(0,x) = g(x). \end{cases} \tag{1.1}$$

Here $t \in [0,+\infty)$, $x \in R^r$, $u : [0,+\infty) \times R^r \to R^n$, $g : R^r \to R^n$, $C(t,x) = (c_{ij}(t,x))$ is an $n \times n$ matrix function of t and x,

$$L = \frac{1}{2}\sum_{i,j=1}^{r} a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x)\frac{\partial}{\partial x^i},$$

and

$$Lu = \begin{pmatrix} Lu_1 \\ Lu_2 \\ \cdot \\ \cdot \\ \cdot \\ Lu_n \end{pmatrix}.$$

We suppose a^{ij}, b^i $(i,j = 1,\cdots,r)$ are bounded and Lipschitz continuous, g_j $(j = 1,\cdots,n)$ are bounded and continuous, c_{ij} $(i,j = 1,2,\cdots,n)$ are bounded and continuous with respect to t and Lipschitz continuous with respect to x. The matrix $(a^{ij}(x))$ is assumed to be nonnegative definite. We shall suppose that there exists a real matrix $\sigma(x) = (\sigma_{ij}(x))$ whose elements satisfy a Lipschitz condition such that

$$\sigma(x) \cdot \sigma^*(x) = (a^{ij}(x)).$$

Let us denote by (X_t^x, P) the Markov family and by (X_t, P^x) the Markov process corresponding to the following stochastic differential equations

$$\begin{cases} dX_t^x = \sigma(X_t^x)dw_t + b(X_t^x)dt, \\ X_0^x = x, \end{cases} \tag{1.2}$$

where w_t is an r-dimensional Wiener process. Let D_s be the matrix satisfying the following equations (C^* stands for the transpose of the matrix C)

$$\begin{cases} \dfrac{dD_s}{ds} = C^*(t-s, X_s^x)D_s, \\ D_0 = I. \end{cases}$$

Let $u(t,x)$ be the $C^{1,2}$ solution of Cauchy problem (1.1). Then applying Itô's Formula to $D_s^* u(t-s, X_s^x)$, we have

$$\begin{aligned} & D_t^* u(0, X_t^x) - u(t,x) \\ = & \int_o^t D_s^* \left[-\frac{\partial u(t-s, X_s^x)}{\partial t} + Lu(t-s, X_s^x) + C(t-s, X_s^x)u(t-s, X_s^x) \right] ds \\ & + \int_0^t (\nabla_x D_s^* u(t-s, X_s^x), \sigma(X_s^x)dw_s) \end{aligned}$$

Notice that the first integral on the right hand side of the above equality is zero since $u(t,x)$ is the solution of problem (1.1), and the second integral has zero expectation according to the martingale property of stochastic integrals. So we have

$$u(t,x) = ED_t^* u(0, X_t^x) = ED_t^* g(X_t^x). \tag{1.3}$$

We call (1.3) the n-dimensional Feynman-Kac Formula.

Now we consider an n-dimensional nonlinear Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + C(x,u)u, \\ u(0,x) = g(x). \end{cases} \tag{1.4}$$

Here $t \in [0, +\infty)$, $x \in R^r$, $u : [0, +\infty) \times R^r \to R^n$, $g : R^r \to R^n$, $C(x,u) = (c_{ij}(x,u))$ is an $n \times n$ matrix function of x and u,

$$L = \frac{1}{2} \sum_{i,j=1}^{r} a^{ij}(x,u) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x,u) \frac{\partial}{\partial x^i},$$

and

$$Lu = \begin{pmatrix} Lu_1 \\ Lu_2 \\ \cdot \\ \cdot \\ \cdot \\ Lu_n \end{pmatrix}.$$

We suppose a^{ij} $(i,j = 1,2,\cdots,r)$ are bounded and Lipschitz continuous, b^i $(i = 1,2,\cdots,r)$ and $g_j(x)$ $(j = 1,2,\cdots,n)$ are bounded and have bounded first-order derivatives in x and u. c_{ij} are assumed to be bounded and have bounded derivatives with respect to x and u. The matrix $(a^{ij}(x,u))$ $(i,j = 1,2,\cdots n)$ is assumed to be nonnegative definite. We shall suppose that there exists a real matrix $\sigma(x,u) = (\sigma_{ij}(x,u))$ whose elements are bounded and Lipschitz continuous with respect to x and u such that

$$\sigma(x,u) \cdot \sigma^*(x,u) = (a^{ij}(x,u)).$$

Let us introduce the generalised solution of problem (1.4). We assume at first that a classical solution $u(t,x)$ of problem (1.4) exists. If this vector function is substituted into the coefficients, then to the operator

$$\mathcal{L} = -\frac{\partial}{\partial t} + L = -\frac{\partial}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{r} a^{ij}(x,u) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x,u) \frac{\partial}{\partial x^i},$$

there corresponds a Markov family and Markov process (homogeneous in time) in the state space $(-\infty, T] \times R^r, T > 0$. They are defined by the stochastic differential equations

$$\begin{cases} X_s^{t,x} = x + \int_0^s \sigma(X_{s_1}^{t,x}, u(t-s_1, X_{s_1}^{t,x}))dw_{s_1} + \int_0^s b(X_{s_1}^{t,x}, u(t-s_1, X_{s_1}^{t,x}))ds_1, \\ t-s = t-s. \end{cases} \tag{1.5}$$

For negative t, we define $u(t,x)$ by putting $u(t,x) = g(x)$ for $t \leq 0$. The solution of problem (1.4) may be written in the form as (1.3)

$$u(t,x) = ED_t^*(X_\cdot^{t,x}, u(t_\cdot, X_\cdot^{t,x})) \cdot g(X_t^{t,x}), \tag{1.6}$$

where $D_s(X_\cdot^{t,x}, u(t_\cdot, X_\cdot^{t,x}))$ is the matrix satisfying the following equation

$$\begin{cases} \dfrac{dD_s(X_\cdot^{t,x}, u(t_\cdot, X_\cdot^{t,x}))}{ds} = C^*(X_s^{t,x}, u(t-s, X_s^{t,x})) \cdot D_s(X_\cdot^{t,x}, u(t_\cdot, X_\cdot^{t,x})), \\ D_0(X_\cdot^{t,x}, u(t_\cdot, X_\cdot^{t,x})) = I. \end{cases} \tag{1.7}$$

Thus the classical solution of the Cauchy problem (1.4), if it is exists, together with the function $X_s^{t,x}$ and $D_s(X_\cdot^{t,x}, u(t_\cdot, X_\cdot^{t,x}))$, satisfies the systems (1.5), (1.6) and (1.7). In general, we introduce the generalised solution of the Cauchy problem (1.4) as follows:

Definition 1.1. *The n-dimensional vector function* $u : [0, +\infty) \times R^r \to R^n$ *is called a generalised solution of problem (1.4), provided it, together with some* $X_s^{t,x}$ *and* $D_s(X_\cdot^{t,x}, u(t_\cdot, X_\cdot^{t,x}))$*, satisfies systems (1.5), (1.6) and (1.7).*

From the definition it is obvious that

A). A classical solution is a generalised one.

B). If the generalised solution has bounded first- and second-order derivatives in x and first-order derivatives in t, and if these derivatives are uniformly continuous, then the generalised solution is a classical one.

§2. The Wave Front in the System with Nonlinear Ergodic Interactions

In this section, we consider an n-dimensional reaction diffusion system with a small parameter ϵ,

$$\begin{cases} \dfrac{\partial u^\epsilon(t,x)}{\partial t} = L^\epsilon u^\epsilon(t,x) + \dfrac{1}{\epsilon} C(u^\epsilon(t,x)) u^\epsilon(t,x), \\ u^\epsilon(0,x) = g(x), \end{cases} \tag{2.1}$$

where $t \in [0,+\infty)$, $x \in R^r$, $u^\epsilon : [0,+\infty) \times R^r \to R^n$, $C : R^r \times R^n \to L(R^n, R^n)$, $g : R^r \to R^n$, and

$$L^\epsilon = \epsilon L = \frac{\epsilon}{2} \sum_{i,j=1}^{r} \frac{\partial}{\partial x^i} \left(a^{ij}(x) \frac{\partial}{\partial x^j} \right),$$

$$L^\epsilon u^\epsilon = \begin{pmatrix} L^\epsilon u_1^\epsilon \\ L^\epsilon u_2^\epsilon \\ \cdot \\ \cdot \\ \cdot \\ L^\epsilon u_n^\epsilon \end{pmatrix}.$$

We suppose g is nonnegative bounded vector valued function (A vector is called nonnegative if all its components are nonnegative), and $c_{ij}(u^\epsilon)$ are bounded and Lipschitz continuous. We suppose that $a^{ij}(x)$ are smooth and bounded and have bounded derivatives of each order, and the matrix $(a^{ij}(x))$ is nonnegative definite. We shall suppose that there exists a real matrix $\sigma(x) = (\sigma_{ij}(x))$ whose elements are smooth and bounded and have bounded derivatives of each order such that

$$\sigma(x) \cdot \sigma^*(x) = (a^{ij}(x)).$$

We assume that the diffusionless system has in $R_+^n = \{u \in R^n, u_1 \geq 0, u_2 \geq 0, \cdots, u_n \geq 0\}$ two equilibrium points: an unstable one at point $O = (0, 0, \cdots, 0)$ and an asymptotically stable one at a point $a = (a_1, a_2, \cdots, a_n)$ (suppose $a_i > 0$, $i = 1, 2, \cdots, n$) and that all the integral curves in the region $R_+^n - \{0\}$ do not leave R_+^n and are attracted to the point a. We consider conditions:

(2.I). $c_{ij}(u) \leq \hat{c}_{ij}$, $\hat{c}_{ij}$ *are nonnegative constants,* $i, j = 1, 2, \cdots, n$.

(2.I'). $c_{ij}(u) \leq c_{ij}(0) = \hat{c}_{ij}$, $\hat{c}_{ij}$ *are nonnegative constants,* $i, j = 1, 2, \cdots, n$.

(2.II). *If* $0 \leq u_p < a_p, p = 1, 2, \cdots, n$, *then*

$$c_{ii}(u) > 0, \quad c_{ij}(u) \geq 0, \quad i, j = 1, 2, \cdots, n,$$

and there exists a constant α *(*$0 < \alpha \leq \min\{a_i : 1 \leq i \leq n\}$*) such that if* $\sum_{p=1}^{n} u_p \geq \frac{1}{2}\alpha$, $u_p \geq 0$, $p = 1, 2, \cdots, n$, *then the matrix* $\bar{D}_s(v)$ *defined by*

$$\begin{cases} \dfrac{d}{ds} \bar{D}_s(v) = \dfrac{1}{\epsilon} \bar{C}^*(v) \bar{D}_s(v), \\ \bar{D}_0(v) = I, \end{cases}$$

satisfies $\bar{D}_{ijt}(v) \geq 0,\ i,j = 1,2,\cdots,n,$ *for* $\inf_{1\leq j\leq n} v_j \leq 0$ *and*

$$|\bar{D}_{ijt}(v)| \to 0, \ \text{as } \epsilon \to 0, \ i,j = 1,2,\cdots,n,$$

where $\bar{C}(v) = (\bar{c}_{ij}(v))$ *is a matrix with*

$$\sum_{j=1}^{n} \bar{c}_{ij}(v)v_j = -\sum_{j=1}^{n} c_{ij}(u)u_j$$

and

$$v_j = a_j - u_j.$$

(2.III). $\max_{1\leq j\leq n} \sup_{x\in R^r} g_j(x) \leq \min_{1\leq j\leq n} a_j.$

By $(X_t^\epsilon, P_x^\epsilon)$ we denote the diffusion process in R^r corresponding to the operator L^ϵ. We choose the action functional $\lambda(\epsilon)S_{0T}(\phi)$ in the space $C_{0T}(R^r)$ associated to the family of the process $(X_t^\epsilon, P_x^\epsilon)$, i.e.

$$S_{0T}(\phi) = \begin{cases} \frac{1}{2}\int_0^T \sum_{i,j=1}^{r} a_{ij}(\phi_s)\dot{\phi}_s^i\dot{\phi}_s^j ds & \text{for absolutely continuous } \phi \in C_{0T}(R^r), \\ +\infty & \text{for other } \phi \in C_{0T}(R^r), \end{cases} \tag{2.2}$$

$$\lambda(\epsilon) = \frac{1}{\epsilon},$$

where $(a_{ij}(x)) = (a^{ij}(x))^{-1}$. That $\lambda(\epsilon)S_{0t}(\phi)$ defined by (2.2) is the action functional corresponding to $(X_t^\epsilon, P_x^\epsilon)$ was proved by Varadhan [3] while the special case for $a_{ij} = 1(i = j = 1)$ was first introduced and proved by Schilder (1966). Elworthy and Truman (1981) used Schilder's results on manifolds which was proved by Molchanov (1975). Wentzell and Freidlin (1970) generalised Varadhan's results.

According to the large deviation theory (c.f. Stroock (1984), Deuschel and Stroock (1989), R. Leandre (1989)), if B is a closed subset in $C_{0t}(R^r)$, then

$$\overline{\lim_{\epsilon\downarrow 0}}\, \epsilon \ln P_x\{X_\cdot^\epsilon \in B\} \leq -\inf\{S_{0t}(\phi) : \phi \in B\}, \tag{2.3}$$

uniformly in any compact subset of $\{(t,x) \in [0,\infty) \times R^r\}$. Let $\rho_{0t}(\phi_1, \phi_2)$ be the distance of ϕ_1 and ϕ_2 in the topological space $C_{0t}(R^r)$. In the following, we will also use the following conclusion (see Freidlin (1985), R. Leandre (1989)), for any function

$\phi \in C_{0t}(R^r), \phi_0 = x$, and arbitrary $\delta_1, \delta_2 > 0$, there is an $\epsilon_0(\delta_1, \delta_2) > 0$ such that for $0 < \epsilon \leq \epsilon_0$,

$$P_x^\epsilon\{\rho_{0t}(X^\epsilon, \phi) < \delta_1\} \geq exp\{-\frac{1}{\epsilon}(S_{0t}(\phi) + \delta_2)\}. \tag{2.4}$$

In order to give the main results of this section, we quote the well-known Frobenius Theorem.

Definition 2.1. *(c.f. Gantmacher 1974) An $n \times n$ matrix $\hat{C} = (\hat{c}_{ij})$ is called reducible if the index $1, 2, \cdots, n$ can be split into two complementary sets (without common indices) $i_1, i_2, \cdots, i_\mu, k_1, k_2, \cdots, k_\nu$ $(\mu + \nu = n)$ such that*

$$\hat{c}_{i_\alpha k_\beta} = 0, \quad \alpha = 1, 2, \cdots, \mu, \beta = 1, 2, \cdots, \nu.$$

Otherwise the matrix is called irreducible.

We will say a matrix is nonnegative if each entry is nonnegative.

Frobenius Theorem. *(Frobenius (1908, 1909, 1912), c.f. Gantmacher (1974)) An irreducible nonnegative $n \times n$ matrix $\hat{C}$ always has a positive eigenvalue γ that is a simple root of its characteristic equation, the moduli of all the other eigenvalues do not exceed γ. To the "maximal" eigenvalue γ there corresponds an eigenvector with positive coordinates.*

The following proposition is very important for us.

Proposition 2.1. *A nonnegative irreducible matrix $\hat{C}$ cannot have two linearly independent nonnegative eigenvectors.*

Define condition

(2.IV). *The matrix $\hat{C} = (\hat{c}_{ij})$ defined by condition (2.I) or (2.I') is nonnegative irreducible.*

Denote by γ the eigenvalue of the matrix $\hat{C}$ with largest real part. According to the Frobenius Theorem such an eigenvalue is real, simple and corresponds to an eigenvector with positive coordinates. According to Proposition 2.1, there is no other eigenvector with positive coordinates which is linearly independent of the above eigenvector. Denote by $e = (e_1, e_2, \cdots, e_n)^T$ the positive eigenvector such that $\sum_{i=1}^{n} e_i = 1$.

Let us denote by $\hat{D}_s$ the matrix satisfying

$$\begin{cases} \dfrac{d}{ds}\hat{D}_s = \dfrac{1}{\epsilon}\hat{C}^*\hat{D}_s, \\ \hat{D}_0 = I. \end{cases} \tag{2.5}$$

Lemma 2.1. *Let $\hat{D}_t = (\hat{D}_{ijt})$ be defined by (2.5). If $\hat{C}$ is a nonnegative irreducible matrix, then*

$$\lim_{\epsilon\downarrow 0}\epsilon \cdot \ln \sum_{j=1}^{n} \hat{D}_{jit} = \gamma t, \ i = 1, 2, \cdots, n, \tag{2.6}$$

uniformly in $t \in [0, +\infty)$.

Proof. From (2.5) we know

$$\frac{d}{ds}\sum_{j=1}^{n} e_j \hat{D}_{jis} = \frac{1}{\epsilon}\gamma \sum_{j=1}^{n} e_j \hat{D}_{jis}.$$

So

$$\sum_{j=1}^{n} e_j \hat{D}_{jit} = e_i \cdot e^{\frac{1}{\epsilon}\gamma t}.$$

Hence

$$\epsilon \cdot \ln \min_{1\le j\le n} e_j + \epsilon \cdot \ln \sum_{j=1}^{n} \hat{D}_{jit} \le \epsilon \ln e_i + \gamma t \le \epsilon \cdot \ln \sum_{j=1}^{n} \hat{D}_{jit}.$$

Therefore we have (2.6). ‡‡

Now define

$$R_{0t}(\phi) = \gamma t - S_{0t}(\phi). \tag{2.7}$$

Lemma 2.2. *Suppose g is a nonnegative bounded n-dimensional vector valued function and its support $G_0 = \bigcup_{j=1}^{n}\{x \in R^r, g_j(x) > 0\}$ satisfies $[G_0] = [(G_0)]$ and it is continuous inside G_0 and outside G_0. The matrix $\hat{C}$ is assumed to be nonnegative irreducible. Then*

$$\overline{\lim_{\epsilon\downarrow 0}}\, \epsilon \ln \sum_{j=1}^{n} E_x^\epsilon \hat{D}_{jit} g_j(X_t^\epsilon) \le \sup\{R_{0t}(\phi) : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0\}, \tag{2.8}$$

uniformly in any compact subset of $\{(t, x) \in [0, +\infty) \times R^r\}$.

Proof. Let $\mathcal{K}$ be a compact subset of $\{(t,x) \in [0,+\infty) \times R^r\}$. From (2.3), (2.6) and the boundedness of g_j we know that for any $\delta > 0$, there is a $\epsilon_0(\delta) > 0$ such that for $0 < \epsilon < \epsilon_0$, $(t,x) \in \mathcal{K}$,

$$\epsilon \ln P_x^\epsilon\{X_t^\epsilon \in G_o\} \leq -\inf\{S_{0t}(\phi) : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0\} + \frac{1}{3}\delta, \tag{2.9}$$

and

$$\epsilon \ln \sum_{j=1}^{n} \hat{D}_{jit} \leq \gamma t + \frac{1}{3}\delta, \tag{2.10}$$

and

$$\epsilon \ln \left(\max_{1 \leq j \leq n} \sup_{x \in R^r} |g_j(x)| \right) \leq \frac{1}{3}\delta. \tag{2.11}$$

So from (2.9), (2.10), (2.11), we have for $0 < \epsilon < \epsilon_0, (t,x) \in \mathcal{K}$,

$$\begin{aligned} &\epsilon \ln \sum_{j=1}^{n} E_x^\epsilon \hat{D}_{jit} g_j(X_t^\epsilon) \\ \leq &\epsilon \ln P_x^\epsilon\{X_t^\epsilon \in G_0\} + \epsilon \ln \sum_{j=1}^{n} \hat{D}_{jit} + \epsilon \ln \left(\max_{1 \leq j \leq n} \sup_{x \in R^r} |g_j(x)| \right) \\ \leq &\sup\{R_{0t}(\phi) : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0\} + \delta. \end{aligned}$$

That proves the lemma. ‡‡

Let by $D_s(u^\epsilon(t_., X_.^{x,\epsilon}))$ denote the matrix satisfying

$$\begin{cases} \dfrac{d}{ds} D_s(u^\epsilon(t_., X_.^{x,\epsilon})) = \dfrac{1}{\epsilon} C^*(u^\epsilon(t-s, X_s^{x,\epsilon})) \cdot D_s(u^\epsilon(t_., X_.^{x,\epsilon})), \\ D_0(u^\epsilon(t_., X_.^{x,\epsilon})) = I, \end{cases} \tag{2.12}$$

and write

$$V(t,x) = sup\{R_{0t}(\phi) : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0\}, \tag{2.13}$$

$$B = \{(t,x) : V(t,x) < 0\}.$$

Theorem 2.1. *Suppose the general conditions about the operator L and matrix C are true. Suppose g is a nonnegative bounded n-dimensional vector valued function whose support* $G_0 = \cup_{j=1}^n \{x : g_j(x) > 0\}$ *satisfying* $[G_0] = [(G_0)]$ *and g is continuous in* G_0 *and outside* G_0. *If (2.I) and (2.IV) are true, then for* $(t,x) \in B$, *we have*

$$\lim_{\epsilon \downarrow 0} u_i^\epsilon(t,x) = 0, \quad i = 1, 2, \cdots, n, \tag{2.14}$$

the convergence is uniform in any compact subset of $\{(t,x) \in [0,+\infty) \times R^r, V(t,x) < 0\}$.

Proof. According to the n-dimensional Feynman-Kac formula, we have

$$u^\epsilon(t,x) = E_x^\epsilon D_t^*(u^\epsilon(t_\cdot, X_\cdot^\epsilon))g(X_t^\epsilon).$$

From condition (2.I) and the comparison theorem of ordinary differential equations (see Lakshmikantham and Leela (1969)), it is easy to know

$$\begin{aligned} u_i^\epsilon(t,x) =& E_x^\epsilon \sum_{j=1}^n D_{jit}(u^\epsilon(t_\cdot, X_\cdot^\epsilon))g_j(X_t^\epsilon) \\ \leq & E_x^\epsilon \sum_{j=1}^n \hat{D}_{jit} g_j(X_t^\epsilon). \end{aligned} \tag{2.15}$$

From Lemma 2.2 we have

$$\overline{\lim_{\epsilon \downarrow 0}} \, \epsilon \ln u_i^\epsilon(t,x) \leq V(t,x),$$

uniformly in any compact subset of $\{(t,x) \in [0,+\infty) \times R^r\}$. So if $V(t,x) < 0$, $\lim\limits_{\epsilon \downarrow 0} u_i^\epsilon(t,x) = 0$. This convergence is uniform in any compact subset of $\{(t,x) \in [0,+\infty) \times R^r, V(t,x) < 0\}$. ‡‡

Lemma 2.3. *If the general conditions on* C *and* g *and the operator* L *are true, then condition (2.II) on* C *implies*

$$\overline{\lim_{\epsilon \downarrow 0}} \, u_i^\epsilon(t,x) \leq a_i, \quad i = 1, 2, \cdots, n,$$

uniformly in any compact subset of $\{t \in (0,+\infty)\}$ *and in* $x \in R^r$.

Proof. We take the transformation

$$v_i^\epsilon = a_i - u_i^\epsilon, \tag{2.16}$$

then the Cauchy problem (2.1) becomes

$$\begin{cases} \dfrac{\partial v_i^\epsilon}{\partial t} = \epsilon L v_i^\epsilon + \dfrac{1}{\epsilon} \sum\limits_{j=1}^n \overline{c}_{ij}(v^\epsilon) v_j^\epsilon, \\ v_i^\epsilon(0,x) = a_i - g_i(x). \end{cases} \tag{2.17}$$

Here

$$\sum_{j=1}^{n} \bar{c}_{ij}(v^{\epsilon})v_j^{\epsilon} = -\sum_{j=1}^{n} c_{ij}(u^{\epsilon})u_j^{\epsilon}.$$

Let $\lambda > 0$ be a small positive number and define a Markov time

$$\tau = \tau^{\lambda,\epsilon} = \inf\{s : \sum_{j=1}^{n} |v_j^{\epsilon}(t-s, X_s^{x,\epsilon})| \leq \lambda\}.$$

By $\bar{D}_s(v^{\epsilon}(t_{\cdot}, X_{\cdot}^{x,\epsilon}))$ we denote a matrix satisfying

$$\begin{cases} \frac{d}{ds}\bar{D}_s(v^{\epsilon}(t_{\cdot}, X_{\cdot}^{x,\epsilon})) = \frac{1}{\epsilon}\bar{C}^{*}(v^{\epsilon}(t-s, X_s^{x,\epsilon})) \cdot \bar{D}_s(v^{\epsilon}(t_{\cdot}, X_{\cdot}^{x,\epsilon})), \\ \bar{D}_0(v^{\epsilon}(t_{\cdot}, X_{\cdot}^{x,\epsilon})) = I. \end{cases} \tag{2.18}$$

According to the Feynman-Kac formula we have that for t in a compact subset of $(0, +\infty)$ and $x \in R^r$, there is an $\epsilon_0(\lambda) > 0$ such that for $0 < \epsilon < \epsilon_0$,

$$\begin{aligned} & v_i^{\epsilon}(t,x) \\ =& E_x^{\epsilon} \sum_{j=1}^{n} v_j^{\epsilon}(t - \tau \wedge t, X_{\tau\wedge t}^{\epsilon})\bar{D}_{ji\tau\wedge t}(v^{\epsilon}(t_{\cdot}, X_{\cdot}^{\epsilon})) \\ =& E_x^{\epsilon} \sum_{j=1}^{n} v_j^{\epsilon}(t-\tau, X_{\tau}^{\epsilon})\bar{D}_{ji\tau}(v^{\epsilon}(t_{\cdot}, X_{\cdot}^{\epsilon}))\chi_{\tau<t} \\ &+ E_x^{\epsilon} \sum_{j=1}^{n} v_j^{\epsilon}(0, X_t^{\epsilon})\bar{D}_{jit}(v^{\epsilon}(t_{\cdot}, X_{\cdot}^{\epsilon}))\chi_{\tau\geq t} \\ \geq& -\lambda[P_x^{\epsilon}\{\tau < t\} + P_x^{\epsilon}\{\tau \geq t\}] \\ \geq& -\lambda. \end{aligned}$$

From (2.16) we know

$$u_i^{\epsilon}(t,x) \leq a_i + \lambda.$$

From this we get the lemma. ‡‡

From the proof it is not difficult to get

Corollary 2.3. *Under conditions of Lemma 2.3 we have*

$$u_i(t,x) \leq a_i \vee \max_{1\leq j\leq n} \sup_{x\in R^r} g_j(x).$$

Lemma 2.4. *Assume the general condition on C, g and operator L are satisfied and g_j $(j = 1, 2, \cdots, n)$ have common support G_0 such that $[G_0] = [(G_0)]$, and g is continuous in G_0 and outside G_0. If (2.I'), (2.II), (2.III) and (2.IV) are true, then for any compact subset $\mathcal{K}$ of $\{(t,x) \in (0,+\infty) \times R^r, V(t,x) = 0\}$, and any $\delta > 0$, there is an $\epsilon_0(\mathcal{K}, \delta) > 0$ such that for $0 < \epsilon < \epsilon_0$ and all $(t,x) \in \mathcal{K}$,*

$$u_p^\epsilon(t,x) \geq e^{-\frac{\delta}{\epsilon}}, \quad p = 1, 2, \cdots, n. \tag{2.19}$$

Proof. Let $\mathcal{K}$ be a compact subset of $\{(t,x) \in (0,+\infty) \times R^r, V(t,x) = 0\}$ and $(t,x) \in \mathcal{K}$. By the definition of $V(t,x)$ Freidlin (1992) shown that $\forall \delta > 0$, there is a function ϕ in $C_{0t}(R^r)$ with $\phi_0 = x$, $\phi_t \in G_0$ such that

$$R_{0t}(\phi) = \gamma t - S_{0t}(\phi) > -\frac{\delta}{12},$$

and $V(t-s, \phi_s) < 0$ for $0 < s < t$. Now for small $\theta > 0$, we can alter ϕ_s near $s = t$ to find a function $\overline{\phi} \in C_{0t}$ with $\overline{\phi}_0 = x, \overline{\phi}_t \in (G_0)$, $\rho_{0t}(\phi, \overline{\phi}) < \delta$ and $R_{0t}(\overline{\phi}) > -\dfrac{\delta}{6}$ such that

$$V(t-s, \overline{\phi}_s) < 0, \text{for } \theta \leq s \leq t - \theta.$$

Define

$$K_\theta = distance[\{(t-s, \overline{\phi}_s) : \theta \leq s \leq t-\theta\}, \{(s,y) \in [0,+\infty) \times R^r, V(s,y) = 0\}].$$

By our construction of $\overline{\phi}$, the distance k_θ will be positive. From the result of Theorem 2.1 we know that as $\epsilon \downarrow 0$, $u_j^\epsilon(t-s, y)$ $(j = 1, 2, \cdots, n)$ tend to zero for all (s,y) such that $|y - \overline{\phi}_s| < \frac{1}{2}K_\theta$ and $\theta \leq s \leq t - \theta$ uniformly in $(t,x) \in \mathcal{K}$. So $\exists\ \theta_0(\delta), \epsilon_0^{(1)}(\delta^*) > 0$ such that for $0 < \theta < \theta_0, 0 < \epsilon < \epsilon_0^{(1)}$ and for $|\underline{\phi}_s - \overline{\phi}_s| \leq \dfrac{1}{2}K_\theta$, $\theta \leq s \leq t - \theta$,

$$\max_{1 \leq i,j \leq n} \sup_{\theta \leq s \leq t-\theta} [\hat{c}_{ij} - c_{ij}(u^\epsilon(t-s, \underline{\phi}_s))] \leq \delta^*, \tag{2.20}$$

where $\delta^* \leq \dfrac{1}{2}\min\{\hat{c}_{ij} : (i,j) \in Q\}$ can be very small and

$$\theta \cdot \max_{1 \leq i \leq n} \sum_{j=1}^{n} \hat{c}_{ij} \leq \frac{1}{12}\delta, \tag{2.21}$$

where $Q = \{(i,j) \in \{1,2,\cdots,n\} \times \{1,2,\cdots,n\} : c_{ij}(u) \not\equiv 0\}$. Define $K_0 = \frac{1}{2}\min\{K_\theta, \rho(\overline{\phi}, R^r - G_0)\}$. From the boundedness of g and (2.4), we know there is an $\epsilon_0^{(2)}(\mathcal{K},\delta) > 0$ such that for $0 < \epsilon < \epsilon_0^{(2)}$, $(t,x) \in \mathcal{K}$,

$$\min_{1\leq j\leq n} \inf_{\rho(x,\overline{\phi})<K_0} g_j(x) \geq e^{-\frac{\delta}{6\epsilon}}, \tag{2.22}$$

and

$$P_x^\epsilon\{\rho_{0t}(\overline{\phi}, X_\cdot^\epsilon) < K_0\} \geq e^{-\frac{1}{\epsilon}\left(S_{0t}(\overline{\phi})+\frac{1}{6}\delta\right)}. \tag{2.23}$$

Denote by $D_{[\theta,s]}(u^\epsilon(t_\cdot, X_\cdot^{x,\epsilon}))$ a matrix satisfying

$$\begin{cases} \dfrac{d}{ds}D_{[\theta,s]}(u^\epsilon(t_\cdot, X_\cdot^{x,\epsilon})) = \dfrac{1}{\epsilon}C^*(u^\epsilon(t-s, X_s^{x,\epsilon})) \cdot D_{[\theta,s]}(u^\epsilon(t_\cdot, X_\cdot^{x,\epsilon})), \\ D_{[\theta,\theta]}(u^\epsilon(t_\cdot, X_\cdot^{x,\epsilon})) = I, \end{cases} \tag{2.24}$$

$$\theta \leq s \leq t-\theta.$$

It is evident that for $\rho(\underline{\phi}, \bar{\phi}) < K_0$,

$$\sum_{j=1}^n D_{jpt}(u^\epsilon(t_\cdot, \underline{\phi}_\cdot)) \geq \sum_{j=1}^n D_{jp[\theta,t-\theta]}(u^\epsilon(t_\cdot, \underline{\phi}_\cdot)), \tag{2.25}$$

and, using (2.20),

$$\frac{d}{ds}D_{[\theta,s]}(u^\epsilon(t_\cdot, \underline{\phi}_\cdot)) \geq \frac{1}{\epsilon}\underline{C}^* \cdot D_{[\theta,s]}(u^\epsilon(t_\cdot, \underline{\phi}_\cdot)), \tag{2.26}$$

where $\underline{C} = (\underline{c}_{ij})$, $\underline{c}_{ij}$ are defined by

$$\underline{c}_{ij} = \begin{cases} \hat{c}_{ij} - \delta^*, & \text{for } (i,j) \in Q, \\ 0, & \text{for } (i,j) \in \{1,2,\cdots,n\} \times \{1,2,\cdots,n\} - Q. \end{cases} \tag{2.27}$$

It is evident that $\underline{C}$ is also a nonnegative irreducible matrix. Let $\underline{D}_{[\theta,s]}$ $(\theta \leq s \leq t-\theta)$ be a matrix satisfying

$$\begin{cases} \dfrac{d}{ds}\underline{D}_{[\theta,s]} = \dfrac{1}{\epsilon}\underline{C}^* \cdot \underline{D}_{[\theta,s]}, \\ \underline{D}_{[\theta,\theta]} = I. \end{cases} \tag{2.28}$$

According to the comparison theorem of ordinary differential equations (c.f. Lakshmikantham and Leela (1969)), we have that for $\rho(\underline{\phi}, \bar{\phi}) < K_0$,

$$\sum_{j=1}^n D_{jp[\theta,t-\theta]}(u^\epsilon(t_\cdot, \underline{\phi}_\cdot)) \geq \sum_{j=1}^n \underline{D}_{jp[\theta,t-\theta]}. \tag{2.29}$$

Let $\{\nu_s^\epsilon\}$ be the Markov chain in the phase space $\{1, 2, \cdots, n\}$ defined by:

$$\begin{cases} P\{\nu_{s+\Delta}^{p,\epsilon} = j | \nu_s^{p,\epsilon} = i\} = \underline{c}_{ij}\Delta + o(\Delta),\ \Delta \downarrow 0,\ i \neq j, \\ P\{\nu_{s+\Delta}^{p,\epsilon} = i | \nu_s^{p,\epsilon} = i\} = 1 - \sum\limits_{j=1, j\neq i}^{n} \underline{c}_{ij}\Delta + o(\Delta),\ \Delta \downarrow 0, \\ \nu_0^{p,\epsilon} = p. \end{cases} \tag{2.30}$$

According to Freidlin (1983, 1985, 1991, 1992),

$$u_p^\epsilon(t_1, x) = E_p e^{\frac{1}{\epsilon}\int_\theta^{t_1} \underline{c}_{\nu_{\frac{s}{\epsilon}}} ds}, \quad k = 1, 2, \cdots, n, \tag{2.31}$$

is the solution of Cauchy problem

$$\begin{cases} \dfrac{\partial u_p^\epsilon(t_1, x)}{\partial t_1} = L^\epsilon u_p^\epsilon(t_1, x) + \dfrac{1}{\epsilon}\sum\limits_{j=1}^{n} \underline{c}_{pj} u_j^\epsilon(t_1, x), \\ u_p^\epsilon(\theta, x) = 1, \\ p = 1, 2, \cdots, n. \end{cases} \tag{2.32}$$

Here

$$\underline{c}_p = \sum_{j=1}^{n} \underline{c}_{pj}.$$

According to n-dimensional Feynman-Kac Formula in §1, we know

$$u_p^\epsilon(t_1, x) = \sum_{j=1}^{n} \underline{D}_{jp[\theta, t_1]}, \quad k = 1, 2, \cdots, n, \tag{2.33}$$

is also the solution of Cauchy problem (2.32). From the uniqueness, we know

$$\sum_{j=1}^{n} \underline{D}_{jp[\theta, t-\theta]} = E_p e^{\frac{1}{\epsilon}\int_\theta^{t-\theta} \underline{c}_{\nu_{\frac{s}{\epsilon}}} ds}. \tag{2.34}$$

Let denote by $\underline{D}_s$ $(0 \leq s \leq t)$ a matrix satisfying

$$\begin{cases} \dfrac{d}{ds}\underline{D}_s = \dfrac{1}{\epsilon}\underline{C}^* \cdot \underline{D}_s, \\ \underline{D}_0 = I. \end{cases} \tag{2.35}$$

As (2.34) we have

$$\sum_{j=1}^{n} \underline{D}_{jpt} = E_p e^{\frac{1}{\epsilon}\int_0^{t} \underline{c}_{\nu_{\frac{s}{\epsilon}}} ds}. \tag{2.36}$$

From (2.21) we have

$$\int_\theta^{t-\theta} \underline{c}_{\nu_{\frac{s}{\epsilon}}} ds \geq \int_0^t \underline{c}_{\nu_{\frac{s}{\epsilon}}} ds - \frac{\delta}{6}. \tag{2.37}$$

Let $\underline{\gamma}$ be the maximal eigenvalue of the matrix $\underline{C}$. According to Lemma 2.1 we have

$$\lim_{\epsilon\downarrow 0} \epsilon \ln \sum_{j=1}^{n} \underline{D}_{jit} = \underline{\gamma} t,$$

uniformly in $t \in [0, +\infty)$. So there is an $\epsilon_0^{(3)}(\delta) > 0$ such that for $0 < \epsilon \leq \epsilon_0^{(3)}$ and $t \in [0, +\infty)$,

$$\sum_{j=1}^{n} \underline{D}_{jit} \geq e^{\frac{1}{\epsilon}(\underline{\gamma} t - \frac{1}{6}\delta)}. \tag{2.38}$$

From the definition of $\underline{C}$ and (2.20) there is an $\epsilon_0^{(4)}(\delta) = \epsilon_0^{(1)}(\delta_0^*(\delta)) > 0$ such that if $0 < \epsilon \leq \epsilon_0^{(4)}(\delta)$ then $0 < \delta^* < \delta_0^*$ then

$$\underline{\gamma} \geq \gamma - \frac{1}{6t}\delta. \tag{2.39}$$

Let $\epsilon_0 = \min\{\epsilon_0^{(2)}, \epsilon_0^{(3)}, \epsilon_0^{(4)}\}$. From the n-dimensional Feynman-Kac formula and (2.22), (2.23), (2.25), (2.29), (2.34), (2.36)-(2.39), Corollary 2.3, condition (2.III) and the positivity of $C(u)$ for $0 \leq u_q \leq a_q, q = 1, 2, \cdots n$, we know for $0 < \epsilon < \epsilon_0$ and $(t, x) \in \mathcal{K}$,

$$\begin{aligned}
u_p^\epsilon(t,x) &= E_x^\epsilon \sum_{j=1}^{n} D_{jpt}(u^\epsilon(t,, X_\cdot^\epsilon)) g_j(X_t^\epsilon) \\
&\geq \min_{1\leq j\leq n} \inf_{\rho_{0t}(x,\overline{\phi})<K_0} g_j(x) \cdot E_x^\epsilon \left\{ \chi_{\rho_{0t}(X_\cdot^\epsilon,\overline{\phi})<K_0} e^{\frac{1}{\epsilon}\left(\epsilon \ln \sum_{j=1}^{n} D_{jpt}(u^\epsilon(t,, X_\cdot^\epsilon))\right)} \right\} \\
&\geq e^{-\frac{\delta}{6\epsilon}} E_x^\epsilon \left\{ \chi_{\rho_{0t}(X_\cdot^\epsilon,\overline{\phi})<K_0} \cdot e^{\frac{1}{\epsilon}\left(\epsilon \ln \sum_{j=1}^{n} D_{jp[\theta,t-\theta]}(u^\epsilon(t,, X_\cdot^\epsilon))\right)} \right\} \\
&\geq e^{-\frac{\delta}{6\epsilon}} P_x^\epsilon\{\rho_{0t}(X_\cdot^\epsilon, \overline{\phi}) < K_0\} \cdot e^{\frac{1}{\epsilon}\left(\epsilon \ln \sum_{j=1}^{n} \underline{D}_{jp[\theta,t-\theta]}\right)} \\
&\geq e^{-\frac{\delta}{6\epsilon}} \cdot e^{-\frac{1}{\epsilon}\left(S_{0t}(\overline{\phi}) + \frac{1}{6}\delta\right)} \cdot E_p e^{\frac{1}{\epsilon}\left(\int_\theta^{t-\theta} \underline{c}_{\nu_{\frac{s}{\epsilon}}} ds\right)} \\
&\geq e^{-\frac{1}{\epsilon}\left(S_{0t}(\overline{\phi}) + \frac{2}{6}\delta\right)} \cdot E_p e^{\frac{1}{\epsilon}\left(\int_0^t \underline{c}_{\nu_{\frac{s}{\epsilon}}} ds - \frac{1}{6}\delta\right)} \\
&\geq e^{-\frac{1}{\epsilon}\left(S_{0t}(\overline{\phi}) + \frac{3}{6}\delta\right)} \cdot \sum_{j=1}^{n} \underline{D}_{jpt} \\
&\geq e^{-\frac{1}{\epsilon}\left(S_{0t}(\overline{\phi}) + \frac{3}{6}\delta\right)} \cdot e^{\frac{1}{\epsilon}\left(\underline{\gamma} t - \frac{1}{6}\delta\right)}
\end{aligned}$$

$$\ge e^{\frac{1}{\epsilon}\left[-\frac{5}{6}\delta+\gamma t-S_{0t}(\overline{\phi})\right]}$$
$$\ge e^{-\frac{\delta}{\epsilon}}.$$

‡‡

Theorem 2.2. *Assume the general conditions about C, g and operator L are satisfied and g_j have the common support G_0 such that $[G_0] = [(G_0)]$, g is continuous in G_0 and outside G_0. If hypotheses (2.I'), (2.II), (2.III) and (2.IV) are true, then for $V(t,x) > 0$, we have*

$$\lim_{\epsilon\downarrow 0} u_i^\epsilon(t,x) = a_i, \quad i = 1, 2, \cdots, n,$$

the convergence is uniform in any compact subset of $\{(t,x) \in (0,+\infty)\times R^r : V(t,x) > 0\}$.

Proof. From Corollary 2.3 and condition (2.III), we know

$$u_i^\epsilon(t,x) \le a_i.$$

Let $\mathcal{K}$ be a compact subset of $\{(t,x) \in (0,+\infty)\times R^r : V(t,x) > 0\}$. In the following we will show

$$\underline{\lim}_{\epsilon\downarrow 0} u_i^\epsilon(t,x) \ge a_i,$$

uniformly in $\mathcal{K}$. For this, first we will show that there is an $\epsilon_0^{(1)} > 0$ such that $0 < \epsilon < \epsilon_0^{(1)}$ and any $(t,x) \in \mathcal{K}$,

$$\sum_{j=1}^n u_j^\epsilon(t,x) \ge \frac{1}{2}\alpha. \tag{2.40}$$

We define two Markov times:

$$\tau_1 = \tau_1^{\epsilon,\lambda} = \inf\{s : \sum_{j=1}^n u_j^\epsilon(t-s, X_s^{x,\epsilon}) \ge \alpha\}, \tag{2.41}$$

$$\tau_2 = \tau_2^\epsilon = \inf\{s : V(t-s, X_s^{x,\epsilon}) = 0\}, \tag{2.42}$$

and write

$$\tau = \tau_1 \wedge \tau_2.$$

So we have

$$
\begin{aligned}
&\sum_{p=1}^{n} u_p^\epsilon(t,x) \\
=&\sum_{p=1}^{n} E_x^\epsilon \sum_{j=1}^{n} D_{jp\tau}(u^\epsilon(t_., X_.^\epsilon))u_j^\epsilon(t-\tau, X_\tau^\epsilon) \\
=&E_x^\epsilon \sum_{p,j=1}^{n} D_{jp\tau_1}(u^\epsilon(t_., X_.^\epsilon))u_j^\epsilon(t-\tau_1, X_{\tau_1}^\epsilon)\chi_{\tau=\tau_1} \\
&+E_x^\epsilon \sum_{p,j=1}^{n} D_{jp\tau_2}(u^\epsilon(t_., X_.^\epsilon))u_j^\epsilon(t-\tau_2, X_{\tau_2}^\epsilon)\chi_{\tau=\tau_2}
\end{aligned}
\tag{2.43}
$$

From the definition of τ_1 and the condition (2.II) we know

$$
\begin{aligned}
&E_x^\epsilon \sum_{p,j=1}^{n} D_{jp\tau_1}(u^\epsilon(t_., X_.^\epsilon))u_j^\epsilon(t-\tau_1, X_{\tau_1}^\epsilon)\chi_{\tau=\tau_1} \\
\geq &E_x^\epsilon \sum_{j=1}^{n} u_j^\epsilon(t-\tau_1, X_{\tau_1}^\epsilon)\chi_{\tau=\tau_1} \\
\geq &\alpha P_x^\epsilon\{\tau=\tau_1\}.
\end{aligned}
\tag{2.44}
$$

Write $V_0 = V(t,x) > 0$ and choose $h > 0$ such that

$$\inf\{V(s,y) : |s-t| < h, |x-y| < h\} > \frac{1}{2}V_0.$$

Write

$$\alpha^* = \inf_{0 \leq \sum_{p=1}^{n} u_p \leq \frac{1}{2}\alpha, u_p \geq 0, \text{ for } p \in \{1,2,\cdots,n\}} \varliminf_{\epsilon \downarrow 0} \epsilon \ln \sum_{j,p=1}^{n} D_{jqh}(u).$$

From condition (2.II) it is evident that $\alpha^* > 0$. Let us select $\delta \in (0, \frac{1}{4}\alpha^*)$. By the definition of τ_2, we have

$$(t-\tau_2, X_{\tau_2}^{x,\epsilon}) \in Z = \{(t,x) \in (0,+\infty) \times R^r, V(t,x) = 0\}, a.s.$$

Therefore for any $\beta \in (0, \frac{1}{2})$, there is a compact subset $\mathcal{K}^*$ in Z with $P(\Omega_0) > 1-\beta$ for

$$\Omega_0 = \{\omega \in \Omega : (t-\tau_2, X_{\tau_2}^{x,\mu} \in \mathcal{K}^*\}.$$

From Lemma 2.4 we know that there is an $\epsilon_0^{(2)} > 0$ such that for $0 < \epsilon < \epsilon_0^{(2)}$,

$$u_j^\epsilon(t-\tau_2, X_{\tau_2}^{x,\epsilon}) > e^{-\frac{\delta}{\epsilon}}, \quad j = 1, 2, \cdots n, \text{ on } \Omega_0,$$

and

$$e^{\frac{\alpha^*}{4\epsilon}} \geq \alpha.$$

Let

$$\tau_3 = \tau_3^\epsilon = \inf\{s : |X_s^{x,\epsilon} - x| = h\}.$$

It is clear that $P_x^\epsilon\{\tau_3 < b\} \to 0$ as $\epsilon \downarrow 0$ for any $b > 0$. So for $0 < \epsilon < \epsilon_0^{(2)}$,

$$\begin{aligned}
&E_x^\epsilon \sum_{p,j=1}^{n} D_{jp\tau_2}(u^\epsilon(t_\cdot, X_\cdot^\epsilon))u_j^\epsilon(t-\tau_2, X_{\tau_2}^\epsilon)\chi_{\tau=\tau_2} \\
\geq& e^{-\frac{\delta}{\epsilon}} \cdot E_x^\epsilon e^{\epsilon \ln \sum\limits_{p,j=1}^{n} D_{jp\tau_2}(u^\epsilon(t_\cdot, X_\cdot^\epsilon))} \chi_{\tau=\tau_2}\chi_{\Omega_0} \\
\geq& e^{\frac{1}{\epsilon}\left(\frac{1}{2}\alpha^* - \delta\right)} \left[P_x^\epsilon\{\tau = \tau_2 < \tau_3\} - \beta\right] \\
\geq& \alpha \left[P_x^\epsilon\{\tau = \tau_2\} - P_x^\epsilon\{\tau_3 < \tau_2\} - \beta\right].
\end{aligned} \tag{2.45}$$

Noting that there is an $\epsilon_0^{(3)} > 0$ such that for $0 < \epsilon < \epsilon_0^{(3)}$ we have $P_x^\epsilon\{\tau_3 \leq \tau_2\} < \frac{1}{2} - \beta$, so (2.45), together with (2.43) and (2.44), implies (2.40) for $0 < \epsilon < \epsilon_0^{(1)} = \min\{\epsilon_0^{(2)}, \epsilon_0^{(3)}\}$.

Now taking the transformation (2.16) we get the system (2.17). Let λ be a sufficiently small number and define a Markov time

$$\tau_4 = \tau_4^{\epsilon,\lambda} = \inf\{s : \sum_{j=1}^{n} v_j^\epsilon(t-s, X_s^{x,\epsilon}) \leq \lambda\}. \tag{2.46}$$

By $\bar{D}_s(v^\epsilon(t_\cdot, X_\cdot^{x,\epsilon}))$ we denote a matrix satisfying (2.18). Let $\epsilon_0^{(4)}(\lambda) > 0$ such that for $0 < \epsilon < \epsilon_0^{(4)}$ and any $(t,x) \in \mathcal{K}$,

$$\max_{1\leq j\leq n} \sup_{x\in R^r} |a_j - g_j(x)| \cdot \sup_{0\leq \sum\limits_{j=1}^{n} v_j \leq \lambda} \sum_{j=1}^{n} \overline{D}_{jit}(v) < \lambda, i = 1, 2, \cdots, n. \tag{2.47}$$

Write $\epsilon_0(\lambda) = \min\{\epsilon_0^{(1)}, \epsilon_0^{(4)}(\lambda)\}$. According to the n-dimensional Feynman-Kac Formula and strong Markov property, (2.40), (2.47) and condition (2.II) we have for

$0 < \epsilon < \epsilon_0$ and any $(t, x) \in \mathcal{K}$,

$$
\begin{aligned}
&|v_i^\epsilon(t,x)| \\
=&|E_x^\epsilon \sum_{j=1}^n v_j^\epsilon(t-\tau_4 \wedge t, X_{\tau_4\wedge t}^\epsilon)\bar{D}_{ji\tau_4\wedge t}(v^\epsilon(t_\cdot, X_\cdot^\epsilon)| \\
\leq&|E_x^\epsilon \sum_{j=1}^n v_j^\epsilon(t-\tau_4, X_{\tau_4}^\epsilon)\bar{D}_{ji\tau_4}(v^\epsilon(t_\cdot, X_\cdot^\epsilon))\chi_{\tau_4\leq t}| \\
&+|E_x^\epsilon \sum_{j=1}^n v_j^\epsilon(0, X_t^\epsilon)\bar{D}_{jit}(v^\epsilon(t_\cdot, X_\cdot^\epsilon))\chi_{\tau_4>t}| \\
\leq&\lambda\left[P_x^\epsilon\{\tau_4 \leq t\} + P_x^\epsilon\{\tau_4 > t\}\right] \\
=&\lambda.
\end{aligned}
$$

This implies

$$u_i^\epsilon(t,x) \geq a_i - \lambda.$$

So we prove the theorem. ‡‡

Example 2.1. Consider the nonlinear 2-dimensional Cauchy problem

$$
\begin{cases}
\dfrac{\partial u_1}{\partial t} = \dfrac{\epsilon}{2}\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{2}{\epsilon}(1-u_1)u_1 + \dfrac{1}{\epsilon}(1-u_1)u_2, \\
\dfrac{\partial u_2}{\partial t} = \dfrac{\epsilon}{2}\dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{3}{\epsilon}(1-u_2)u_1 + \dfrac{4}{\epsilon}(1-u_2)u_2, \\
u_1(0,x) = \chi_{x\leq 0}, \quad u_2(0,x) = \chi_{x\leq 0}.
\end{cases}
\tag{2.48}
$$

According to Theorem 2.1 and Theorem 2.2, we get as $\epsilon \downarrow 0$,

$$
(u_1^\epsilon(t,x), u_2^\epsilon(t,x)) \to \begin{cases} (0,0), & \text{for } x > \sqrt{10}t, \\ (1,1), & \text{for } x < \sqrt{10}t. \end{cases}
$$

Remark 2.1. *In physics, condition (2.IV) means that a system has nonlinear ergodic interactions. The results in this section means these kinds of system may have a travelling wave with a common speed.*

Remark 2.2. *Generally speaking if the matrix $\hat{C}$ is not irreducible, the results of this section are not true.*

Example 2.2. Consider nonlinear 2-dimensional Cauchy problem

$$
\begin{cases}
\dfrac{\partial u_1}{\partial t} = \dfrac{\epsilon}{2}\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{1}{\epsilon}(1-u_1)u_1, \\
\dfrac{\partial u_2}{\partial t} = \dfrac{\epsilon}{2}\dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{2}{\epsilon}(1-u_2)u_2, \\
u_1(0,x) = \chi_{x\leq 0}, \quad u_2(0,x) = \chi_{x\leq 0}.
\end{cases}
\tag{2.49}
$$

It is easy to see that as $\epsilon \downarrow 0$,

$$u_1^\epsilon(t,x) \to \begin{cases} 0, & \text{for } x > \sqrt{2}t, \\ 1, & \text{for } x < \sqrt{2}t. \end{cases}$$

and

$$u_2^\epsilon(t,x) \to \begin{cases} 0, & \text{for } x > 2t, \\ 1, & \text{for } x < 2t. \end{cases}$$

We don't have a common speed for u_1 *and* u_2 *here. Note that the largest eigenvalue* γ *of a nonnegative irreducible matrix* $\hat{C}$ *satisfies*

$$\min_{1\le i\le n} \sum_{j=1}^{n} \hat{c}_{ij} \le \gamma \le \max_{1\le i\le n} \sum_{j=1}^{n} \hat{c}_{ij}.$$

If a nonnegative matrix $\hat{C}$ *is not irreducible, but*

$$\min_{1\le i\le n} \sum_{j=1}^{n} \hat{c}_{ij} = \max_{1\le i\le n} \sum_{j=1}^{n} \hat{c}_{ij},$$

we will prove in the next section that the results of this section are still true, i.e., there exists a travelling wave front with speed as $\gamma = \sum_{j=1}^{n} \hat{c}_{ij}$ *in this section. In fact we can investigate more general systems under this condition.*

Remark 2.3. *If we define the Riemannian metric* $d(x,y)$ *in the space* R^r *as*

$$d(x,y) = \inf\left\{ \int_0^1 \sqrt{\sum_{i,j=1}^{n} a_{ij}(\phi_s)\dot{\phi}_s^i \dot{\phi}_s^j} ds : \phi \in C_{01}(R^r), \phi_0 = x, \phi_1 = y \right\}, \tag{2.50}$$

then

$$\inf\left\{ \int_0^t \sum_{i,j=1}^{n} a_{ij}(\phi_s)\dot{\phi}_s^i \dot{\phi}_s^j ds : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0 \right\} = \frac{1}{2t} d^2(x, G_0).$$

Hence

$$V(t,x) = \gamma t - \frac{1}{2t} d^2(x, G_0). \tag{2.51}$$

Therefore the conclusions in this section can be expressed by the Huygens principle as

$$\lim_{\epsilon\downarrow 0} u_i^\epsilon(t,x) = 0, \;\; i = 1, 2\cdots, n, \;\; \textit{if } d(x, G_0) > \sqrt{2\gamma} t,$$

$$\lim_{\epsilon\downarrow 0} u_i^\epsilon(t,x) = a_i, \ i = 1, 2 \cdots, n, \ \textit{if } d(x, G_0) < \sqrt{2\gamma}t.$$

§3. The Wave Front of the System with Nonlinear Reducible Interactions

In this section, we consider an n-dimensional generalised KPP equation with small parameter ϵ,

$$\begin{cases} \dfrac{\partial u^\epsilon(t,x)}{\partial t} = L^\epsilon u^\epsilon(t,x) + \dfrac{1}{\epsilon} C(x, u^\epsilon(t,x)) u^\epsilon(t,x), \\ u^\epsilon(0,x) = g(x), \end{cases} \tag{3.1}$$

where $t \in [0, +\infty)$, $x \in R^r$, $u^\epsilon : [0, +\infty) \times R^r \to R^n$, $g : R^r \to R^n$, $C : R^r \times R^n \to L(R^n, R^n)$, and L^ϵ is the same as those in the last section. The conditions on L^ϵ and g are the same as last section and $c_{ij}(x, u^\epsilon)$ are bounded and Lipschitz continuous in u.

As in §2, by $(X_t^\epsilon, P_x^\epsilon)$ we denote the diffusion process in R^r corresponding to the operator L^ϵ, the action functional $\lambda(\epsilon) S_{0T}(\phi)$ in the space $C_{0T}(R^r)$ associated to the family of the process $(X_t^\epsilon, P_x^\epsilon)$. According to Varadhan's large deviation theory, we have if $F : C_{0t}(R^r) \to R^1$ is bounded and continuous, and B is a closed subset in $C_{0t}(R^r)$, then

$$\lim_{\epsilon\downarrow 0} \lambda^{-1}(\epsilon) \ln E_x e^{\lambda(\epsilon) F(X_\cdot^\epsilon)} \cdot \chi_B(X_\cdot^\epsilon) = \sup_{\xi\in B} [F(\xi) - S(\xi)], \tag{3.2}$$

uniformly in any compact subset of $\{(t,x) \in [0, +\infty) \times R^r\}$.

We consider hypotheses

(3.I). $c_{ij}(x,u) \le \hat{c}_{ij}(x) \le c$, *$\hat{c}_{ij}$ is uniformly continuous and c is a constant,* $i, j = 1, 2, \cdots, n$.

(3.I'). $c_{ij}(x,u) \le c_{ij}(x,0) = \hat{c}_{ij}(x) \le c$, *$\hat{c}_{ij}$ is uniformly continuous and c is a constant,* $i, j = 1, 2, \cdots, n$.

(3.II). *If* $0 \le u_p < a_p, p = 1, 2, \cdots, n$, *then*

$$c_{ii}(x,u) > 0, \quad c_{ij}(x,u) \ge 0, \quad i, j = 1, 2, \cdots, n,$$

and there exists a constant α $(0 < \alpha \le \min\{a_i : 1 \le i \le n\})$ *such that if* $\sum_{p=1}^{n} u_p \ge \frac{1}{2}\alpha,\ \ u_p \ge 0,\ p = 1, 2, \cdots, n$, *the matrix* $\bar{D}_s(x, v)$ *defined by*

$$\begin{cases} \dfrac{d}{ds}\bar{D}_s(x,v) = \dfrac{1}{\epsilon}\bar{C}^*(x,v)\bar{D}_s(x,v), \\ \bar{D}_0(x,v) = I, \end{cases} \tag{3.3}$$

satisfies $\bar{D}_{ijt}(x, v) \ge 0,\ i, j = 1, 2, \cdots, n$, *for* $\inf_{1\le j\le n} v_j \le 0$ *and*

$$|\bar{D}_{ijt}(x,v)| \to 0,\ \ as\ \epsilon \to 0,\ \ i, j = 1, 2, \cdots, n, \tag{3.4}$$

where $\bar{C}(x, v) = (\bar{c}_{ij}(x, v))$ *is a matrix with*

$$\sum_{j=1}^{n} \bar{c}_{lj}(x,v)v_j = -\sum_{j=1}^{n} c_{lj}(x,u)u_j, \tag{3.5}$$

and

$$v_j = a_j - u_j,\ \ j = 1, 2, \cdots, n.$$

(3.III). $\max_{1\le j\le n} \sup_{x\in R^r} g_j(x) \le \min_{1\le j\le n} a_j.$

(3.IV). $\min_{1\le i\le n} \sum_{j=1}^{n} \hat{c}_{ij}(x) = \max_{1\le i\le n} \sum_{j=1}^{n} \hat{c}_{ij}(x), \quad x \in R^r.$

Let $\hat{D}_t(\phi)$ be the matrix satisfying

$$\begin{cases} \dfrac{d\hat{D}_t(\phi)}{dt} = \dfrac{1}{\epsilon}\hat{C}^*(\phi_t)\hat{D}_t(\phi), \\ \hat{D}_0(\phi) = I, \end{cases} \tag{3.6}$$

and $D_s(X_{\cdot}^{x,\epsilon}, u^\epsilon(t_{\cdot}, X_{\cdot}^{x,\epsilon}))$ be a matrix satisfying

$$\begin{cases} \dfrac{d}{ds}D_s(X_{\cdot}^{x,\epsilon}, u^\epsilon(t_{\cdot}, X_{\cdot}^{x,\epsilon})) = \dfrac{1}{\epsilon}C^*(X_s^{x,\epsilon}, u^\epsilon(t-s, X_s^{x,\epsilon})) \cdot D_s(X_{\cdot}^{x,\epsilon}, u^\epsilon(t_{\cdot}, X_{\cdot}^{x,\epsilon})), \\ D_0(X_{\cdot}^{x,\epsilon}, u^\epsilon(t_{\cdot}, X_{\cdot}^{x,\epsilon})) = I. \end{cases} \tag{3.7}$$

Next define

$$R_{0t}(\phi) = \int_0^t \sum_{j=1}^{n} \hat{c}_{ij}(\phi_s)ds - S_{0t}(\phi). \tag{3.8}$$

Lemma 3.1. *Suppose g is a nonnegative bounded n-dimensional vector valued function . We shall put $G_0 = \bigcup_{j=1}^{n}\{x \in R^r, g_j(x) > 0\}$ and assume that $[G_0] = [(G_0)]$, g is continuous in G_0 and outside G_0. Then if condition (3.IV) is satisfied, we have*

$$\overline{\lim_{\epsilon\downarrow 0}}\, \epsilon \ln \sum_{j=1}^{n} E_x^\epsilon \hat{D}_{jit}(X_\cdot^\epsilon) g_j(X_t^\epsilon) \le \sup\{R_{0t}(\phi) : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0\}, \quad (3.9)$$

uniformly in any compact subset of $\{(t,x) \in [0,+\infty) \times R^r\}$.

Proof. Let $\mathcal{K}$ be a compact subset of $\{(t,x) \in [0,+\infty) \times R^r\}$. From (3.2) and the boundedness of $g_j(x)$ we know that $\forall \delta > 0$, there is an $\epsilon_0(\delta) > 0$ such that for $0 < \epsilon < \epsilon_0$ and any $(t,x) \in \mathcal{K}$,

$$\epsilon \ln E_x^\epsilon e^{\frac{1}{\epsilon}\int_0^t \sum_{j=1}^{n} \hat{c}_{ij}(X_s^\epsilon)ds} \chi_{G_0}(X_t) \le \sup\{R_{0t}(\phi) : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0\} + \frac{1}{2}\delta,$$

and

$$\epsilon \ln \left(\max_{1\le j\le n} \sup_{x\in R^r} |g_j(x)| \right) \le \frac{1}{2}\delta.$$

So for $0 < \epsilon < \epsilon_0$ and $(t,x) \in \mathcal{K}$, from condition (3.IV) we have

$$\begin{aligned}
&\epsilon \ln \sum_{j=1}^{n} E_x^\epsilon \hat{D}_{jit}(X_\cdot^\epsilon) g_j(X_t^\epsilon) \\
\le& \epsilon \ln E_x^\epsilon e^{\frac{1}{\epsilon}\left(\epsilon \ln \sum_{j=1}^{n} \hat{D}_{jit}(X_\cdot^\epsilon)\right)} \chi_{G_0} + \epsilon \ln \left(\max_{1\le j\le q} \sup_{x\in R^r} |g_j(x)| \right) \\
\le& \epsilon \ln E_x^\epsilon e^{\frac{1}{\epsilon}\sum_{j=1}^{n}\int_0^t \hat{c}_{ij}(X_s^\epsilon)ds} \chi_{G_0} + \epsilon \ln \left(\max_{1\le j\le q} \sup_{x\in R^r} |g_j(x)| \right) \\
\le& \sup\{R_{0t}(\phi) : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0\} + \delta.
\end{aligned}$$

That proves the Lemma. ‡‡

Next define

$$V(t,x) = \sup\{R_{0t}(\phi) : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0\},$$

$$B = \{(t,x) : V(t,x) < 0\}.$$

Theorem 3.1. *If the hypotheses of Lemma 3.1 are satisfied and the hypotheses (3.I) and (3.IV) are true, then for $(t,x) \in B$, we have*

$$\lim_{\epsilon\downarrow 0} u_i^\epsilon(t,x) = 0, \quad i = 1, 2, \cdots, n, \quad (3.10)$$

the convergence is uniform in any compact subset of $\{(t,x) \in [0,+\infty) \times R^r : V(t,x) < 0\}$.

Proof. According to n-dimensional Feynman-Kac formula (1.6) we have

$$u^\epsilon(t,x) = E_x^\epsilon D_t^*(X_\cdot^\epsilon, u^\epsilon(t_\cdot, X_\cdot^\epsilon))g(X_t^\epsilon).$$

From condition (3.I) and the comparison theorem of ordinary differential equations (see Lakshmikantham and Leela (1969)), it is easy to know that for ϵ sufficiently small,

$$\begin{aligned} u_i^\epsilon(t,x) =& E_x^\epsilon \sum_{j=1}^n D_{jit}(X_\cdot^\epsilon, u^\epsilon(t_\cdot, X_\cdot^\epsilon))g_j(X_t^\epsilon) \\ \leq& E_x^\epsilon \sum_{j=1}^n \hat{D}_{jit}(X_\cdot^\epsilon)g_j(X_t^\epsilon) \end{aligned} \tag{3.11}$$

So from Lemma 3.2 we have

$$\overline{\lim_{\epsilon\downarrow 0}}\, \epsilon \ln u_i^\epsilon(t,x) \leq V(t,x),$$

uniformly in any compact subset of $\{(t,x) \in [0,+\infty) \times R^r\}$. So if $V(t,x) < 0$, $\lim_{\epsilon\downarrow 0} u_i^\epsilon(t,x) = 0$ uniformly in any compact subset of $\{(t,x) \in [0,+\infty) \times R^r : V(t,x) < 0\}$. ‡‡

Lemma 3.2. *Suppose the general conditions on* C, g *and the operator* L *are true. If condition (3.II) on* C *is satisfied, then for any* $q = 1, 2, \cdots, n$,

$$u_q^\epsilon(t,x) \leq a_q \vee \sup_{x \in R^r} g_q(x).$$

Proof. Define a Markov time

$$\tau = \tau^\epsilon = \inf\{s : u_q^\epsilon(t-s, X_s^{x,\epsilon}) \leq a_q \vee \sup_{x\in R^r} g_q(x)\}.$$

By the same argument in the proof of Lemma 2.3, we get the lemma ‡‡

We say condition (N) is fulfilled if

$$\begin{aligned} &V(t,x) = \\ &\sup\{R_{0t}(\phi) : \phi \in C_{0t}(R^r), \phi_0 = x, \phi_t \in G_0, V(t-s,\phi_s) < 0, \text{for } 0 < s < t\} \end{aligned} \tag{3.12}$$

for (t,x) such that $V(t,x) \le 0$.

Lemma 3.3. *Assume the general condition on C, g and operator L are satisfied and g_j $(j = 1, 2, \cdots, n)$ have the common support G_0 such that $[G_0] = [(G_0)]$, and g is continuous inside G_0 and outside G_0. If condition (3.I'), (3.II), (3.III), (3.IV) and (N) are true, then for any compact subset $\mathcal{K}$ of $\{(t,x) \in (0,+\infty) \times R^r : V(t,x) = 0\}$, for any $\delta > 0$, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and all $(t,x) \in \mathcal{K}$,*

$$u_p^\epsilon(t,x) \ge e^{-\frac{\delta}{\epsilon}}, p = 1, 2 \cdots, n.$$

Proof. Let $\mathcal{K}$ be a compact subset of $\{(t,x) \in (0,+\infty) \times R^r : V(t,x) = 0\}$ and $(t,x) \in \mathcal{K}$. By the definition of $V(t,x)$ and condition (N), $\forall \delta > 0$, there is a function ϕ in $C_{0t}(R^r)$ with $\phi_0 = x$, $\phi_t \in G_0$ such that

$$R_{0t}(\phi) = \int_0^t \sum_{j=1}^n \hat{c}_{ij}(\phi_s)ds - S_{0t}(\phi) > -\frac{\delta}{12},$$

and $V(t-s, \phi_s) < 0$ for $0 < s < t$. Now for small $\theta > 0$, we can alter ϕ_s near $s = t$ to find a function $\overline{\phi} \in C_{0t}$ with $\overline{\phi}_0 = x, \overline{\phi}_t \in (G_0)$, $\rho_{0t}(\phi, \overline{\phi}) < \delta$ and $R_{0t}(\overline{\phi}) > -\frac{\delta}{6}$ such that

$$V(t-s, \overline{\phi}_s) < 0, \text{for } \theta \le s \le t - \theta.$$

Define

$$K_\theta = distance[\{(t-s, \overline{\phi}_s) : \theta \le s \le t-\theta\}, \{(s,y) \in [0,+\infty) \times R^r, V(s,y) = 0\}].$$

By our construction of $\overline{\phi}$, the distance k_θ will be positive. From the result of Theorem 3.1 we know $u_j^\epsilon(t-s,y)$ $(j = 1, 2, \cdots, n)$ tend to zero for all (s,y) such that $|y - \overline{\phi}_s| < \frac{1}{2}K_\theta$ and $\theta < s < t - \theta$ uniformly in $(t,x) \in \mathcal{K}$. Write

$$\delta^* = \min\left\{\frac{1}{2} \min_{(i,j)\in Q} \inf_{\rho(x,\overline{\phi})<\frac{1}{2}K_\theta} \hat{c}_{ij}(x), \frac{\delta}{6nt}\right\},$$

where

$$Q = \{(i,j) \in \{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\} : c_{ij}(x,u) \not\equiv 0\}.$$

We can choose $\theta_0(\delta), \epsilon_0^{(1)}(\delta) > 0$ such that for $0 < \theta < \theta_0$ and $0 < \epsilon < \epsilon_0^{(1)}$, $|\underline{\phi}_s - \overline{\phi}_s| \leq \frac{1}{2}K_\theta$, $\theta \leq s \leq t-\theta$,

$$\max_{1\leq i,j\leq n} \sup_{\theta\leq s\leq t-\theta} \left[\hat{c}_{ij}(\underline{\phi}_s) - c_{ij}(\underline{\phi}_s, u^\epsilon(t-s, \underline{\phi}_s))\right] < \delta^*, \tag{3.13}$$

and

$$\int_\theta^{t-\theta} \sum_{j=1}^n \hat{c}_{ij}(\underline{\phi}_s)ds > \int_\theta^{t-\theta} \sum_{j=1}^n \hat{c}_{ij}(\bar{\phi}_s)ds - \frac{1}{6}\delta, \tag{3.14}$$

and

$$\int_\theta^{t-\theta} \sum_{j=1}^n \hat{c}_{ij}(\bar{\phi}_s)ds > \int_0^{t} \sum_{j=1}^n \hat{c}_{ij}(\bar{\phi}_s)ds - \frac{1}{6}\delta. \tag{3.15}$$

Let $K_0 = \frac{1}{2}\min\{K_\theta, \rho(\overline{\phi}, R^r - G_0)\}$. From the boundedness of g and (2.4) we know there is an $\epsilon_0^{(2)} > 0$ such that for $0 < \epsilon < \epsilon_0^{(2)}$,

$$\min_{1\leq j\leq n} \inf_{\rho(x,\overline{\phi})<K_0} g_j(x) \geq e^{-\frac{\delta}{6\epsilon}}, \tag{3.16}$$

$$P_x^\epsilon\{\rho_{0t}(\overline{\phi}, X_\cdot^\epsilon) < K_0\} \geq e^{-\frac{1}{\epsilon}\left(S_{0t}(\overline{\phi})+\frac{1}{6}\delta\right)}. \tag{3.17}$$

Denote by $D_{[\theta,s]}(X_\cdot^{x,\epsilon}, u^\epsilon(t_\cdot, X_\cdot^{x,\epsilon}))$ a matrix satisfying

$$\begin{cases} \dfrac{d}{ds}D_{[\theta,s]}(X_\cdot^{x,\epsilon}, u^\epsilon(t_\cdot, X_\cdot^{x,\epsilon})) = \dfrac{1}{\epsilon}C^*(X_s^{x,\epsilon}, u^\epsilon(t-s, X_s^{x,\epsilon})) \cdot D_{[\theta,s]}(X_\cdot^{x,\epsilon}, u^\epsilon(t_\cdot, X_\cdot^{x,\epsilon})), \\ D_{[\theta,\theta]}(X_\cdot^{x,\epsilon}, u^\epsilon(t_\cdot, X_\cdot^{x,\epsilon})) = I, \end{cases}$$

$$\theta \leq s \leq t-\theta.$$

It is evident that for $|\underline{\phi}_s - \bar{\phi}_s| \leq K_0, \theta \leq s \leq t-\theta$,

$$\sum_{j=1}^n D_{jpt}(\underline{\phi}_\cdot, u^\epsilon(t_\cdot, \underline{\phi}_\cdot)) \geq \sum_{j=1}^n D_{jp[\theta,t-\theta]}(\underline{\phi}_\cdot, u^\epsilon(t_\cdot, \underline{\phi}_\cdot)). \tag{3.18}$$

and, using (3.13) we know

$$\begin{aligned} &\frac{d}{ds}\sum_{j=1}^n D_{jp[\theta,s]}(\underline{\phi}_\cdot, u^\epsilon(t_\cdot, \underline{\phi}_\cdot)) \\ =&\frac{1}{\epsilon}\sum_{i,j=1}^n c_{ij}(\underline{\phi}_s, u^\epsilon(t-s, \underline{\phi}_s)) \cdot D_{ip[\theta,s]}(\underline{\phi}_\cdot, u^\epsilon(t_\cdot, \underline{\phi}_\cdot)) \\ \geq&\frac{1}{\epsilon}\sum_{j=1}^n \left(\hat{c}_{ij}(\underline{\phi}_s) - \delta^*\right) \cdot \sum_{i=1}^n D_{ip[\theta,s]}(\underline{\phi}_\cdot, u^\epsilon(t_\cdot, \underline{\phi}_\cdot)). \end{aligned}$$

So we have

$$\sum_{j=1}^{n} D_{jp[\theta,t-\theta]}(\underset{.}{\phi}, u^{\epsilon}(t,,\underset{.}{\phi})) \geq e^{\frac{1}{\epsilon}\left(\int_{\theta}^{t-\theta}\sum_{j=1}^{n}\hat{c}_{ij}(\underset{.}{\phi}_s)ds-\frac{\delta}{6}\right)}. \tag{3.19}$$

Let $\epsilon_0 = \min\{\epsilon_0^{(1)}, \epsilon_0^{(2)}\}$. So according to n-dimensional Feynman-Kac formula, (3.14)-(3.19), Lemma 3.1, condition (3.III) and the positivity of $C(x,u)$ for $0 \leq u_q \leq a_q, q = 1, 2, \cdots, n$, we have for $0 < \epsilon < \epsilon_0$ and $(t,x) \in \mathcal{K}$,

$$\begin{aligned}
u_p^{\epsilon}(t,x) &= E_x^{\epsilon}\sum_{j=1}^{n} D_{jpt}(X_{.}^{\epsilon}, u^{\epsilon}(t,, X_{.}^{\epsilon}))g_j(X_t^{\epsilon}) \\
&\geq \min_{1\leq j\leq n}\inf_{\rho_{0t}(x,\overline{\phi})<K_0} g_j(x)\cdot E_x^{\epsilon}\{\chi_{\rho_{0t}(X_{.}^{\epsilon},\overline{\phi})<K_0}\}e^{\frac{1}{\epsilon}\left(\epsilon\ln\sum_{j=1}^{n}D_{jpt}(X_{.}^{\epsilon},u^{\epsilon}(t,,X_{.}^{\epsilon}))\right)} \\
&\geq e^{-\frac{\delta}{6\epsilon}}E_x^{\epsilon}\left\{\chi_{\rho_{0t}(X_{.}^{\epsilon},\overline{\phi})<K_0}e^{\frac{1}{\epsilon}\left(\sum_{j=1}^{n}D_{jp[\theta,t-\theta]}(X_{.}^{\epsilon},u^{\epsilon}(t,,X_{.}^{\epsilon}))\right)}\right\} \\
&\geq e^{-\frac{\delta}{6\epsilon}}E_x^{\epsilon}\left\{\chi_{\rho_{0t}(X_{.}^{\epsilon},\overline{\phi})<K_0}e^{\frac{1}{\epsilon}\left(\int_{\theta}^{t-\theta}\sum_{j=1}^{n}\hat{c}_{ij}(X_s^{\epsilon})ds-\frac{1}{6}\delta\right)}\right\} \\
&\geq e^{-\frac{\delta}{3\epsilon}}P_x^{\epsilon}\{\rho_{0t}(X_{.}^{\epsilon},\overline{\phi}) < K_0\}e^{\frac{1}{\epsilon}\left(\int_{\theta}^{t-\theta}\sum_{j=1}^{n}\hat{c}_{ij}(\overline{\phi}_s)ds-\frac{1}{6}\delta\right)} \\
&\geq e^{-\frac{\delta}{2\epsilon}}\cdot e^{-\frac{1}{\epsilon}\left(S_{0t}(\overline{\phi})+\frac{1}{6}\delta\right)}\cdot e^{\frac{1}{\epsilon}\left(\int_0^t\sum_{j=1}^{n}\hat{c}_{ij}(\overline{\phi}_s)ds-\frac{1}{6}\delta\right)} \\
&\geq e^{\frac{1}{\epsilon}\left[-\frac{5}{6}\delta+\int_0^t\sum_{j=1}^{n}\hat{c}_{ij}(\overline{\phi}_s)ds-S_{0t}(\overline{\phi})\right]} \\
&\geq e^{-\frac{\delta}{\epsilon}}.
\end{aligned}$$

‡‡

By Lemma 3.2 and 3.3, we can prove the following theorem with the same argument as in the proof of Theorem 2.2.

Theorem 3.2. *Assume the general conditions about C, g and operator L are satisfied and g_j have the common support G_0 such that $[G_0] = [(G_0)]$ and g is continuous in G_0 and outside G_0. If hypotheses (3.I'), (3.II), (3.III), (3.IV) and condition (N) are true, then for $V(t,x) > 0$, we have*

$$\lim_{\epsilon\downarrow 0} u_q^{\epsilon}(t,x) = a_q, \quad q = 1, 2, \cdots n,$$

the convergence is uniform in any compact subset of $\{(t,x) \in (0,+\infty) \times R^r : V(t,x) > 0\}$.

Example 3.1. Consider the nonlinear 2-dimensional Cauchy problem

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{\epsilon}{2}\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{2}{\epsilon}(1-u_1)u_1, \\ \dfrac{\partial u_2}{\partial t} = \dfrac{\epsilon}{2}\dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{1}{\epsilon}(1-u_2)u_1 + \dfrac{1}{\epsilon}(1-u_2)u_2, \\ u_1(0,x) = \chi_{x\le 0}, \quad u_2(0,x) = \chi_{x\le 0}. \end{cases} \tag{3.20}$$

It is evident that the irreducible condition is not satisfied. So Theorem 2.1, 2.2 are not valid here. But according to Theorem 3.1 and Theorem 3.2, we get as $\epsilon \downarrow 0$,

$$(u_1^\epsilon(t,x), u_2^\epsilon(t,x)) \to \begin{cases} (0,0), & \text{for } x > 2t, \\ (1,1), & \text{for } x < 2t. \end{cases}$$

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