A NEW CLASS OF THETA FUNCTION IDENTITIES IN TWO VARIABLES

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Abstract. We describe a new series of identities, which hold for certain general theta series, in two completely independent variables. We provide explicit examples of these identities involving the Dedekind eta function, Jacobi theta functions, and various theta functions of Ramanujan.

Introduction

Let $z \in \mathcal{H} = \{x + yi : x, y \in \mathbb{R}, y > 0\}$ and for each $x \in \mathbb{R}$ set $q^x = \exp(2\pi ixz)$ and $e(x) = \exp(2\pi ix)$. The Dedekind eta-function is defined by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

In [1] the following identity was proved.

$$27\eta^3(3z)\eta^3(3w) = \eta^3\left(\frac{z}{3}\right)\eta^3\left(\frac{w}{3}\right) + i\eta^3\left(\frac{z+1}{3}\right)\eta^3\left(\frac{w+1}{3}\right) - \eta^3\left(\frac{z+2}{3}\right)\eta^3\left(\frac{w+2}{3}\right),$$

for all $z, w \in \mathcal{H}$.

In this paper we will generalize this identity for theta functions of the following kind:

$$\Theta_{\nu,a,b,m,\psi}(z) = \sum_{n=-\infty}^{\infty} \psi(n)(an+b)^{\nu} q^{\frac{(an+b)^2}{m}},$$

where $\nu, a, m, b, \in \mathbb{Z}$, $(a, b) = 1$ and $0 \le |b| < a$ and ψ is an additive character modulo a.

1. Generalized Identities

Fix two sets ν, a, b, m, ψ and ν', a', b', m', ψ', satisfying the above conditions and, for simplicity, denote

$$f(z) = \Theta_{\nu,a,b,m,\psi}(z), \quad g(z) = \Theta_{\nu',a',b',m',\psi'}(z).$$

The main result that we prove in this paper is the following identity.

Theorem 1.1. *Let p be an odd prime such that $a|(p-1)$ and $a'|(p-1)$, then*

$$\sum_{j=0}^{p-1} f\left(\frac{z+mj}{p}\right) g\left(\frac{w-m'jk}{p}\right) = \psi(c)\psi(c')\, p^{\nu+\nu'+1}\, f(pz)g(pw),$$

for all $z, w \in \mathcal{H}$, where k is any quadratic non-residue modulo p, $c = b(p-1)/a$ and $c' = b'(p-1)/a'$.

Proof: First let

$$f_j(z) = f\left(\frac{z+mj}{p}\right),$$

and note that

$$f_j(z) = \sum_{n=-\infty}^{\infty} \psi(n)(an+b)^\nu e((an+b)^2 j/p) q^{\frac{(an+b)^2}{mp}}.$$

Thus f_j depends only on the congruence class of j modulo p. Thus the vector

$$\alpha(z) = (f_0(z), f_1(z), \ldots, f_{p-1}(z)) \in \mathbb{C}^p,$$

lies in the linear subspace V_p of $\mathbb{C}^p$ spanned by the vectors

$$v_d = \left(1, e(d^2/p), e(2d^2/p), \ldots, e((p-1)d^2/p)\right),$$

where d runs through the integers.

There are $(p+1)/2$ distinct squares modulo p, so there are $(p+1)/2$ distinct v_d which are linearly independent. Hence V_p has dimension $(p+1)/2$.

Now let

$$f_\infty(z) = \psi(c)p^{\nu+1/2} f(pz) = \psi(c)p^{\nu+1/2} \sum_{n=-\infty}^{\infty} \psi(n)(an+b)^\nu q^{\frac{(an+b)^2 p}{m}},$$

where $c = b(p-1)/a$.

We express f_∞ in terms of the f_j, i.e. we derive an identity involving only one variable z, for these functions. Consider

$$\begin{aligned} \sum_{j=0}^{p-1} f_j(z) &= \sum_{n=-\infty}^{\infty} \psi(n)(an+b)^\nu q^{\frac{(an+b)^2}{mp}} \sum_{j=0}^{p-1} e((an+b)^2 j/p) \\ &= p \sum_{\substack{n=-\infty \\ p|(an+b)}}^{\infty} \psi(n)(an+b)^\nu q^{\frac{(an+b)^2}{mp}}. \end{aligned}$$

But $p|(an+b)$ if and only if $n = ps + c$ where $s \in \mathbb{Z}$. Thus

$$\begin{aligned} \sum_{j=0}^{p-1} f_j(z) &= p \sum_{s=-\infty}^{\infty} \psi(ps+c)(aps+bp)^\nu q^{\frac{(aps+bp)^2}{mp}} \\ &= p^{\nu+1}\psi(c) \sum_{s=-\infty}^{\infty} \psi(s)(as+b)^\nu q^{\frac{(as+b)^2 p}{m}} \\ &= \sqrt{p}\, f_\infty(z). \end{aligned}$$

Now we see that

$$F(z) = \left(f_0(z), f_1(z), \ldots, f_{p-1}(z), f_\infty(z)\right)$$

lives in the vector space W_p spanned by the vectors

$$w_0 = (1, 1, 1, \ldots, 1, \sqrt{p})$$

and

$$w_d = \left(1, e(d^2/p), e(2d^2/p), \ldots, e((p-1)d^2/p), 0\right),$$

where d runs through the integers prime to p. Again W_p has dimension $(p+1)/2$.

Now, if we let

$$g_j(z) = g\left(\frac{z - m'jk}{p}\right) \qquad \text{and} \qquad g_\infty(z) = \psi'(c')p^{\nu'+1/2} g(pz),$$

then

$$G(w) = \big(g_0(w), g_1(w), \ldots, g_{p-1}(w), g_\infty(w)\big) \in W_p$$

and is spanned by

$$w_0' = w_0 = (1, 1, 1, \ldots, 1, \sqrt{p})$$

and

$$w_d' = \big(1, e(-kd^2/p), e(-2kd^2/p), \ldots, e(-(p-1)kd^2/p), 0\big).$$

Next we define

$$B(\mathbf{z}, \mathbf{u}) = \sum_{j=0}^{p-1} z_j u_j - z_\infty u_\infty,$$

a bilinear form on $\mathbb{C}^{p+1}$.

Clearly for a, b not both zero modulo p, $B(w_a, w_b') = \sum_{j=0}^{p-1} e\left(\frac{j(a^2-kb^2)}{p}\right) = 0$, since k is a quadratic non-residue of p.

On the other hand,

$$B(w_0, w_0') = \sum_{j=0}^{p-1} 1 - (\sqrt{p})^2 = 0.$$

Thus $B(\mathbf{z}, \mathbf{u}) = 0$ for all $\mathbf{z}, \mathbf{u} \in W_p$. In particular, $B(F(z), G(w)) = 0$, i.e.

$$\sum_{j=0}^{p-1} f_j(z) g_j(w) = \psi(c)\psi'(c')\, p^{\nu+\nu'+1}\, f(pz)g(pw),$$

as was to be shown. □

Now, the above argument required that $a \,|\, (p-1)$ and $a' \,|\, (p-1)$. We can make a small modification to deal with the case where $a \,|\, (p+1)$. Indeed we simply note that for the original theta function we defined,

$$\Theta_{\nu,a,b,m,\psi}(z) = (-1)^\nu \sum_{n=-\infty}^{\infty} \psi(n)(an-b)^\nu q^{\frac{(an-b)^2}{m}}.$$

Now our argument goes through much the same as before, except that we now require $\psi(n) = \psi(-n)$, etc., i.e. ψ and ψ' must now be real characters. We thus have the following.

Theorem 1.1.1. *Let p be an odd prime such that $a|(p+1)$ and $a'|(p+1)$, then*

$$\sum_{j=0}^{p-1} f\left(\frac{z+mj}{p}\right) g\left(\frac{w-m'jk}{p}\right) = -\psi(c)\psi(c')\,(-p)^{\nu+\nu'+1}\, f(pz)g(pw),$$

for all $z, w \in \mathcal{H}$, where k is any quadratic non-residue modulo p, $c = b(p+1)/a$ and $c' = b'(p+1)/a'$.

Theorem 1.1.2. *Let p be an odd prime such that $a|(p-1)$ and $a'|(p+1)$, then*

$$\sum_{j=0}^{p-1} f\left(\frac{z+mj}{p}\right) g\left(\frac{w-m'jk}{p}\right) = (-1)^{\nu'}\psi(c)\psi(c')\, p^{\nu+\nu'+1}\, f(pz)g(pw),$$

for all $z, w \in \mathcal{H}$, where k is any quadratic non-residue modulo p, $c = b(p-1)/a$ and $c' = b'(p+1)/a'$.

We note that the space V_p is essentially the same as $\mathcal{Q}$ in [2].

2. Examples

Example 1:

Let $f(z) = \eta^3(z)$ and $g(w) = \eta^3(w)$. By Jacobi's formula,

$$\eta(z) = q^{1/8}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$$

and can be rewritten as $\Theta_{\nu,a,b,m,\psi}(z)$ with $\nu = 1$, $a = 4$, $b = 1$, $m = 8$ and ψ the trivial character.

Our main result then yields:

$$p^3\eta(pz)^3\eta(pw)^3 = \sum_{j=0}^{p-1}\eta\left(\frac{z+8j}{p}\right)^3\eta\left(\frac{w-8jk}{p}\right)^3,$$

for any odd prime p.

The case $p = 3$ and $k = -1$ is precisely the identity of [1] cited at the beginning of this paper.

Example 2:

Let $f(z) = \eta(z)$. Then by Euler's pentagonal number formula

$$\eta(z) = q^{1/24}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2} = \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24},$$

corresponding to one of our theta functions for $\nu = 0$, $a = 6$, $b = 1$, $m = 24$ and $\psi(n) = (-1)^n$.

It follows that for $p > 3$,

$$p\eta(pz)\eta(pw) = \sum_{j=0}^{p-1}\eta\left(\frac{z+24j}{p}\right)\eta\left(\frac{w-24jk}{p}\right),$$

where k is a quadratic non-residue modulo p.

Example 3:

Let $g(z) = q^{1/24}\phi^2(-q)f(-q)$ where $f(q)$ and $\phi(q)$ are theta series defined by Ramanujan, with $q = e(z)$. This function has the expansion

$$g(z) = \sum_{n=-\infty}^{\infty}(6n+1)q^{\frac{(6n+1)^2}{24}},$$

and so it corresponds to one of our theta functions with $\nu = 1$, $a = 6$, $b = 1$, $m = 24$, and ψ the trivial character.

Our results yields the following.

$$p^3 g(pz)g(pw) = \sum_{j=0}^{p-1} g\left(\frac{z+mj}{p}\right) g\left(\frac{w+mj}{p}\right),$$

for all $p \equiv 7, 11 \pmod{12}$.

A simple variation of this result exists for $p \equiv 1, 5 \pmod{12}$.

Example 4:

Let $g(z) = q^{1/6}\psi(q^2)f(-q)$ where $f(q)$ and $\psi(q)$ are theta series defined by Ramanujan, with $q = e(z)$. This function has the expansion

$$g(z) = \sum_{n=-\infty}^{\infty}(3n+1)q^{\frac{(3n+1)^2}{6}},$$

and so it corresponds to one of our theta functions with $\nu = 1$, $a = 3$, $b = 1$, $m = 6$, and ψ the trivial character.

Our theorems yields the following.

$$p^3 g(pz)g(pw) = \sum_{j=0}^{p-1} g\left(\frac{z+mj}{p}\right) g\left(\frac{w+mj}{p}\right),$$

for all $p \equiv 7, 11 \pmod{12}$.

There is a simple variation of this for $p \equiv 1, 5 \pmod{12}$.

Example 5:

Our theorem clearly induces identities for each of the Jacobi theta null-values:

$$\begin{aligned}
\theta_1'(q) &= \sum_{n=-\infty}^{\infty} i^{(2n-1)}(2n+1)q^{(2n+1)^2/4} \\
\theta_2(q) &= \sum_{n=-\infty}^{\infty} q^{(2n+1)^2/4} \\
\theta_3(q) &= \sum_{n=-\infty}^{\infty} q^{n^2} \\
\theta_4(q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}
\end{aligned}$$

For example, $\theta_1'(q) = -i\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2/4} = -i\Theta_{1,2,1,4,(-1)^n}$.

Our main result yields:

$$p^3\theta_1'(pz)\theta_1'(pw) = \sum_{j=0}^{p-1} \theta_1'\left(\frac{z+8j}{p}\right) \theta_1'\left(\frac{w-8jk}{p}\right),$$

for any odd prime p.

Example 6 (mixed identity):

On page 369 of Ramanujan's Lost Notebook [3], he defines

$$F_\alpha(q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^\alpha q^{\frac{n^2+n}{2}},$$

$$G_\beta(q) = \sum_{n=-\infty}^{\infty} (-1)^n(6n+1)^\beta q^{\frac{3n^2+n}{2}}.$$

Clearly $(1-(-1)^\alpha)q^{1/8}F_\alpha(q) = \sum_{n=-infty}^{\infty}(-1)^n(2n+1)^\alpha q^{\frac{(2n+1)^2}{8}} = \Theta_{\alpha,2,1,8,(-1)^n}(q)$.

Similarly $q_\beta^{1/12}(q) = \Theta_{\beta,6,1,12,(-1)^n}(q)$.

Our final theorem then yields:

$$\sum_{j=0}^{p-1} F_\alpha\left(\frac{z+8j}{p}\right) G_\beta\left(\frac{w+24j}{p}\right) = p^{\alpha+\beta+1} e\left(\frac{p^2-1}{8p}z\right) e\left(\frac{p^2-1}{24p}w\right) F_\alpha(pz)G_\beta(pw),$$

for all primes $p \equiv 11 \pmod{12}$.

References

[1] Bruce C. Berndt and William B. Hart, *An identity for the Dedekind eta-function involving two independent complex variables*, preprint.
[2] Robin Chapman, *Determinants of Legendre symbol matrices*, Acta Arith. 115, (2004), pp. 231–244.
[3] S. Ramanujan, *The lost notebook and other unpublished papers*, Narosa, New Delhi, 1988.

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