

**Original citation:**

Chapman, Robin, Hart, William B. and Choon Toh, Pee. (2010) A new class of theta function identities in two variables. *Journal of Combinatorics and Number Theory*, Volume 2 (Number 3). pp. 201-208. ISSN 1942-5600

**Permanent WRAP url:**

<http://wrap.warwick.ac.uk/43605/>

**Copyright and reuse:**

The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions. Copyright © and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable the material made available in WRAP has been checked for eligibility before being made available.

Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

**A note on versions:**

The version presented here may differ from the published version or, version of record, if you wish to cite this item you are advised to consult the publisher's version. Please see the 'permanent WRAP url' above for details on accessing the published version and note that access may require a subscription.

For more information, please contact the WRAP Team at: [publications@warwick.ac.uk](mailto:publications@warwick.ac.uk)

warwick**publications**wrap

highlight your research

<http://wrap.warwick.ac.uk>

# A NEW CLASS OF THETA FUNCTION IDENTITIES IN TWO VARIABLES

ROBIN CHAPMAN, WILLIAM B. HART, PEE CHOON TOH

ABSTRACT. We describe a new series of identities, which hold for certain general theta series, in two completely independent variables. We provide explicit examples of these identities involving the Dedekind eta function, Jacobi theta functions, and various theta functions of Ramanujan.

## INTRODUCTION

Let  $z \in \mathcal{H} = \{x + yi : x, y \in \mathbb{R}, y > 0\}$  and for each  $x \in \mathbb{R}$  set  $q^x = \exp(2\pi i x z)$  and  $e(x) = \exp(2\pi i x)$ . The Dedekind eta-function is defined by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

In [1] the following identity was proved.

$$27\eta^3(3z)\eta^3(3w) = \eta^3\left(\frac{z}{3}\right)\eta^3\left(\frac{w}{3}\right) + i\eta^3\left(\frac{z+1}{3}\right)\eta^3\left(\frac{w+1}{3}\right) - \eta^3\left(\frac{z+2}{3}\right)\eta^3\left(\frac{w+2}{3}\right),$$

for all  $z, w \in \mathcal{H}$ .

In this paper we will generalize this identity for theta functions of the following kind:

$$\Theta_{\nu, a, b, m, \psi}(z) = \sum_{n=-\infty}^{\infty} \psi(n)(an + b)^{\nu} q^{\frac{(an+b)^2}{m}},$$

where  $\nu, a, m, b \in \mathbb{Z}$ ,  $(a, b) = 1$  and  $0 \leq |b| < a$  and  $\psi$  is an additive character modulo  $a$ .

## 1. GENERALIZED IDENTITIES

Fix two sets  $\nu, a, b, m, \psi$  and  $\nu', a', b', m', \psi'$ , satisfying the above conditions and, for simplicity, denote

$$f(z) = \Theta_{\nu, a, b, m, \psi}(z), \quad g(z) = \Theta_{\nu', a', b', m', \psi'}(z).$$

The main result that we prove in this paper is the following identity.

**Theorem 1.1.** *Let  $p$  be an odd prime such that  $a|(p-1)$  and  $a'|(p-1)$ , then*

$$\sum_{j=0}^{p-1} f\left(\frac{z + mj}{p}\right) g\left(\frac{w - m'jk}{p}\right) = \psi(c)\psi(c') p^{\nu+\nu'+1} f(pz)g(pw),$$

for all  $z, w \in \mathcal{H}$ , where  $k$  is any quadratic non-residue modulo  $p$ ,  $c = b(p-1)/a$  and  $c' = b'(p-1)/a'$ .

Proof: First let

$$f_j(z) = f\left(\frac{z + mj}{p}\right),$$

and note that

$$f_j(z) = \sum_{n=-\infty}^{\infty} \psi(n)(an + b)^\nu e((an + b)^2 j/p) q^{\frac{(an+b)^2}{mp}}.$$

Thus  $f_j$  depends only on the congruence class of  $j$  modulo  $p$ . Thus the vector

$$\alpha(z) = (f_0(z), f_1(z), \dots, f_{p-1}(z)) \in \mathbb{C}^p,$$

lies in the linear subspace  $V_p$  of  $\mathbb{C}^p$  spanned by the vectors

$$v_d = (1, e(d^2/p), e(2d^2/p), \dots, e((p-1)d^2/p)),$$

where  $d$  runs through the integers.

There are  $(p+1)/2$  distinct squares modulo  $p$ , so there are  $(p+1)/2$  distinct  $v_d$  which are linearly independent. Hence  $V_p$  has dimension  $(p+1)/2$ .

Now let

$$f_\infty(z) = \psi(c)p^{\nu+1/2} f(pz) = \psi(c)p^{\nu+1/2} \sum_{n=-\infty}^{\infty} \psi(n)(an + b)^\nu q^{\frac{(an+b)^2 p}{m}},$$

where  $c = b(p-1)/a$ .

We express  $f_\infty$  in terms of the  $f_j$ , i.e. we derive an identity involving only one variable  $z$ , for these functions. Consider

$$\begin{aligned} \sum_{j=0}^{p-1} f_j(z) &= \sum_{n=-\infty}^{\infty} \psi(n)(an + b)^\nu q^{\frac{(an+b)^2}{mp}} \sum_{j=0}^{p-1} e((an + b)^2 j/p) \\ &= p \sum_{\substack{n=-\infty \\ p|(an+b)}}^{\infty} \psi(n)(an + b)^\nu q^{\frac{(an+b)^2}{mp}}. \end{aligned}$$

But  $p|(an + b)$  if and only if  $n = ps + c$  where  $s \in \mathbb{Z}$ . Thus

$$\begin{aligned} \sum_{j=0}^{p-1} f_j(z) &= p \sum_{s=-\infty}^{\infty} \psi(ps + c)(aps + bp)^\nu q^{\frac{(aps+bp)^2}{mp}} \\ &= p^{\nu+1} \psi(c) \sum_{s=-\infty}^{\infty} \psi(s)(as + b)^\nu q^{\frac{(as+b)^2 p}{m}} \\ &= \sqrt{p} f_\infty(z). \end{aligned}$$

Now we see that

$$F(z) = (f_0(z), f_1(z), \dots, f_{p-1}(z), f_\infty(z))$$

lives in the vector space  $W_p$  spanned by the vectors

$$w_0 = (1, 1, 1, \dots, 1, \sqrt{p})$$

and

$$w_d = (1, e(d^2/p), e(2d^2/p), \dots, e((p-1)d^2/p), 0),$$

where  $d$  runs through the integers prime to  $p$ . Again  $W_p$  has dimension  $(p+1)/2$ .

Now, if we let

$$g_j(z) = g\left(\frac{z - m'jk}{p}\right) \quad \text{and} \quad g_\infty(z) = \psi'(c')p^{\nu'+1/2} g(pz),$$

then

$$G(w) = (g_0(w), g_1(w), \dots, g_{p-1}(w), g_\infty(w)) \in W_p$$

and is spanned by

$$w'_0 = w_0 = (1, 1, 1, \dots, 1, \sqrt{p})$$

and

$$w'_d = (1, e(-kd^2/p), e(-2kd^2/p), \dots, e(-(p-1)kd^2/p), 0).$$

Next we define

$$B(\mathbf{z}, \mathbf{u}) = \sum_{j=0}^{p-1} z_j u_j - z_\infty u_\infty,$$

a bilinear form on  $\mathbb{C}^{p+1}$ .

Clearly for  $a, b$  not both zero modulo  $p$ ,  $B(w_a, w'_b) = \sum_{j=0}^{p-1} e\left(\frac{j(a^2 - kb^2)}{p}\right) = 0$ , since  $k$  is a quadratic non-residue of  $p$ .

On the other hand,

$$B(w_0, w'_0) = \sum_{j=0}^{p-1} 1 - (\sqrt{p})^2 = 0.$$

Thus  $B(\mathbf{z}, \mathbf{u}) = 0$  for all  $\mathbf{z}, \mathbf{u} \in W_p$ . In particular,  $B(F(z), G(w)) = 0$ , i.e.

$$\sum_{j=0}^{p-1} f_j(z) g_j(w) = \psi(c) \psi'(c') p^{\nu+\nu'+1} f(pz) g(pw),$$

as was to be shown.  $\square$

Now, the above argument required that  $a \mid (p-1)$  and  $a' \mid (p-1)$ . We can make a small modification to deal with the case where  $a \mid (p+1)$ . Indeed we simply note that for the original theta function we defined,

$$\Theta_{\nu, a, b, m, \psi}(z) = (-1)^\nu \sum_{n=-\infty}^{\infty} \psi(n) (an - b)^\nu q^{\frac{(an-b)^2}{m}}.$$

Now our argument goes through much the same as before, except that we now require  $\psi(n) = \psi(-n)$ , etc., i.e.  $\psi$  and  $\psi'$  must now be real characters. We thus have the following.

**Theorem 1.1.1.** *Let  $p$  be an odd prime such that  $a \mid (p+1)$  and  $a' \mid (p+1)$ , then*

$$\sum_{j=0}^{p-1} f\left(\frac{z + mj}{p}\right) g\left(\frac{w - m'jk}{p}\right) = -\psi(c) \psi'(c') (-p)^{\nu+\nu'+1} f(pz) g(pw),$$

for all  $z, w \in \mathcal{H}$ , where  $k$  is any quadratic non-residue modulo  $p$ ,  $c = b(p+1)/a$  and  $c' = b'(p+1)/a'$ .

**Theorem 1.1.2.** *Let  $p$  be an odd prime such that  $a \mid (p-1)$  and  $a' \mid (p+1)$ , then*

$$\sum_{j=0}^{p-1} f\left(\frac{z + mj}{p}\right) g\left(\frac{w - m'jk}{p}\right) = (-1)^{\nu'} \psi(c) \psi'(c') p^{\nu+\nu'+1} f(pz) g(pw),$$

for all  $z, w \in \mathcal{H}$ , where  $k$  is any quadratic non-residue modulo  $p$ ,  $c = b(p-1)/a$  and  $c' = b'(p+1)/a'$ .

We note that the space  $V_p$  is essentially the same as  $\mathcal{Q}$  in [2].

## 2. EXAMPLES

**Example 1:**

Let  $f(z) = \eta^3(z)$  and  $g(w) = \eta^3(w)$ . By Jacobi's formula,

$$\eta(z) = q^{1/8} \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}$$

and can be rewritten as  $\Theta_{\nu, a, b, m, \psi}(z)$  with  $\nu = 1$ ,  $a = 4$ ,  $b = 1$ ,  $m = 8$  and  $\psi$  the trivial character.

Our main result then yields:

$$p^3 \eta(pz)^3 \eta(pw)^3 = \sum_{j=0}^{p-1} \eta\left(\frac{z+8j}{p}\right)^3 \eta\left(\frac{w-8jk}{p}\right)^3,$$

for any odd prime  $p$ .

The case  $p = 3$  and  $k = -1$  is precisely the identity of [1] cited at the beginning of this paper.

**Example 2:**

Let  $f(z) = \eta(z)$ . Then by Euler's pentagonal number formula

$$\eta(z) = q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24},$$

corresponding to one of our theta functions for  $\nu = 0$ ,  $a = 6$ ,  $b = 1$ ,  $m = 24$  and  $\psi(n) = (-1)^n$ .

It follows that for  $p > 3$ ,

$$p\eta(pz)\eta(pw) = \sum_{j=0}^{p-1} \eta\left(\frac{z+24j}{p}\right) \eta\left(\frac{w-24jk}{p}\right),$$

where  $k$  is a quadratic non-residue modulo  $p$ .

**Example 3:**

Let  $g(z) = q^{1/24} \phi^2(-q) f(-q)$  where  $f(q)$  and  $\phi(q)$  are theta series defined by Ramanujan, with  $q = e(z)$ . This function has the expansion

$$g(z) = \sum_{n=-\infty}^{\infty} (6n+1) q^{\frac{(6n+1)^2}{24}},$$

and so it corresponds to one of our theta functions with  $\nu = 1$ ,  $a = 6$ ,  $b = 1$ ,  $m = 24$ , and  $\psi$  the trivial character.

Our results yields the following.

$$p^3 g(pz)g(pw) = \sum_{j=0}^{p-1} g\left(\frac{z+mj}{p}\right) g\left(\frac{w+mj}{p}\right),$$

for all  $p \equiv 7, 11 \pmod{12}$ .

A simple variation of this result exists for  $p \equiv 1, 5 \pmod{12}$ .

**Example 4:**

Let  $g(z) = q^{1/6} \psi(q^2) f(-q)$  where  $f(q)$  and  $\psi(q)$  are theta series defined by Ramanujan, with  $q = e(z)$ . This function has the expansion

$$g(z) = \sum_{n=-\infty}^{\infty} (3n+1) q^{\frac{(3n+1)^2}{6}},$$

and so it corresponds to one of our theta functions with  $\nu = 1$ ,  $a = 3$ ,  $b = 1$ ,  $m = 6$ , and  $\psi$  the trivial character.

Our theorems yields the following.

$$p^3 g(pz)g(pw) = \sum_{j=0}^{p-1} g\left(\frac{z+mj}{p}\right) g\left(\frac{w+mj}{p}\right),$$

for all  $p \equiv 7, 11 \pmod{12}$ .

There is a simple variation of this for  $p \equiv 1, 5 \pmod{12}$ .

**Example 5:**

Our theorem clearly induces identities for each of the Jacobi theta null-values:

$$\begin{aligned} \theta'_1(q) &= \sum_{n=-\infty}^{\infty} i^{(2n-1)}(2n+1)q^{(2n+1)^2/4} \\ \theta_2(q) &= \sum_{n=-\infty}^{\infty} q^{(2n+1)^2/4} \\ \theta_3(q) &= \sum_{n=-\infty}^{\infty} q^{n^2} \\ \theta_4(q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \end{aligned}$$

For example,  $\theta'_1(q) = -i \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)q^{(2n+1)^2/4} = -i\Theta_{1,2,1,4,(-1)^n}$ .

Our main result yields:

$$p^3 \theta'_1(pz)\theta'_1(pw) = \sum_{j=0}^{p-1} \theta'_1\left(\frac{z+8j}{p}\right) \theta'_1\left(\frac{w-8jk}{p}\right),$$

for any odd prime  $p$ .

**Example 6 (mixed identity):**

On page 369 of Ramanujan's Lost Notebook [3], he defines

$$\begin{aligned} F_\alpha(q) &= \sum_{n=0}^{\infty} (-1)^n (2n+1)^\alpha q^{\frac{n^2+n}{2}}, \\ G_\beta(q) &= \sum_{n=-\infty}^{\infty} (-1)^n (6n+1)^\beta q^{\frac{3n^2+n}{2}}. \end{aligned}$$

Clearly  $(1-(-1)^\alpha)q^{1/8}F_\alpha(q) = \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^\alpha q^{\frac{(2n+1)^2}{8}} = \Theta_{\alpha,2,1,8,(-1)^n}(q)$ .

Similarly  $q_\beta^{1/12}(q) = \Theta_{\beta,6,1,12,(-1)^n}(q)$ .

Our final theorem then yields:

$$\sum_{j=0}^{p-1} F_\alpha\left(\frac{z+8j}{p}\right) G_\beta\left(\frac{w+24j}{p}\right) = p^{\alpha+\beta+1} e\left(\frac{p^2-1}{8p}z\right) e\left(\frac{p^2-1}{24p}w\right) F_\alpha(pz)G_\beta(pw),$$

for all primes  $p \equiv 11 \pmod{12}$ .

## REFERENCES

- [1] Bruce C. Berndt and William B. Hart, *An identity for the Dedekind eta-function involving two independent complex variables*, preprint.
- [2] Robin Chapman, *Determinants of Legendre symbol matrices*, Acta Arith. 115, (2004), pp. 231–244.
- [3] S. Ramanujan, *The lost notebook and other unpublished papers*, Narosa, New Delhi, 1988.

*E-mail address:* `rjc@maths.ex.ac.uk`

*E-mail address:* `wbhart@math.uiuc.edu`

*E-mail address:* `mattpc@nus.edu.sg`