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The Y-combinator in Scott's Lambda-Calculus Models

(Revised Version)

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Assume the notation and terminology of Dana Scott's paper "Models for the Lambda-Calculus". In this note I want to exhibit the relationship between the lambda-calculus "paradoxical operator"

$$Y = \lambda x((\lambda y.x(yy))\lambda y.x(yy))$$

and the minimal fixpoint operator

$$Y^* = \lambda x. \bigsqcup_{n=o}^{\infty} x^n \ \Omega$$

obtained by regarding the lambda-calculus model as a lattice. Intuitively, one expects that

$$Y = Y^*$$

should hold in all Scott's models; and this is indeed the case in the models constructed as in his paper; however there is an (unexpected?) complication, in that a slight alteration in the construction obtains another class of models in which $Y \neq Y^*$. This anomaly lacks (so far) any complete rationalization; one looks for grounds on which to reject such "pathological" models, but so far I know of no completely convincing ones.

Since Y^*, from lattice theory, obtains the <u>minimal</u> fixpoint, and Y, by beta-reductions, is certainly another fixpoint operator, it must be the case that

$$Y^* \sqsubseteq Y \quad .$$

The difficulties arise over the converse question, whether $Y \sqsubseteq Y^*$, i.e. whether $Yx \sqsubseteq \bigsqcup_{n=o}^{\infty} x^n \, \Omega$ for all x.

Abbreviate Yx by writing

$$X = \lambda y . x(yy)$$

then $Yx = XX = \bigsqcup_{n=1}^{\infty} X_n \, X_{n-1}$, by Scott.

Now note the following:

(a) Using Scott's methods, it is straightforward that

$$X = \bigsqcup_{n=o}^{\infty} \lambda y{:}D_n . x_{n+1}(\phi_n yy)$$

We need something stronger, viz.

$$X_{n+1} = \lambda y{:}D_n . \; x_{n+1}(\phi_n yy) \quad , n \geqslant 0 \quad .$$

(i.e. that the right hand side is a "best approximation" in D_{n+1} to X). To show this, we need that

$$\psi_n \; (\lambda y{:}D_n . \; x_{n+1}(\phi_n yy)) = \lambda y{:}D_{n-1} . \; x_n(\phi_{n-1} yy), \; n > 0 \; .$$

Now remember the following identities:

(i) $\psi_n(u(\phi_n v)) = \psi_{n+1} \; u \; v$ (by defn. of ψ_{n+1})

(ii) $\phi_n \; (uv) = \phi_{n+1} \; u \; (\phi_n v)$ (by defn. of ϕ_{n+1} & since $\psi_n(\phi_n v) = v$)

Then, for n > 0

$$\psi_n(\lambda y{:}D_n . \; x_{n+1}(\phi_n yy)) = \lambda y{:}D_{n-1} . \psi_{n-1}(x_{n+1}(\phi_n(\phi_{n-1} y)(\phi_{n-1} y))$$ (by defn. of ψ_n)

$$= \lambda y{:}D_{n-1} . \psi_{n-1}(x_{n+1}(\phi_{n-1}(\phi_{n-1} yy)))$$

(from (ii))

$$= \lambda y{:}D_{n-1} . \; \psi_n \; x_{n+1}(\phi_{n-1} yy)$$

(from (i))

$$= \lambda y{:}D_{n-1} . \; x_n(\phi_{n-1} yy)$$

which is the expression we wanted.

(b) Now for $n > 0$

$$X_{n+1}\ X_n = (\lambda y{:}D_n.\ x_{n+1}(\phi_n yy))X_n$$
$$= x_{n+1}(\phi_n\ X_n\ X_n)$$
$$= x_{n+1}(\phi_{n-1}(X_n(\psi_{n-1}X_n))) \quad \text{(from defn. of } \phi_n)$$
$$= x_{n+1}(\phi_{n-1}(X_n\ X_{n-1})) \quad .$$

But this provides a simple recurrence relation, so that

$$X_{n+1}\ X_n = x_{n+1}(\phi_{n-1}(x_n(\phi_{n-2}(\cdots\cdots x_2(\phi_o(X_1\ X_o))\cdots\cdots))))$$

Hence $Yx = \bigsqcup_{n=o}^{\infty} X_{n+1}\ X_n$

$$= \bigsqcup_{n=1}^{\infty} x_{n+1}(x_n(\cdots\cdots x_2(X_1\ X_o)\cdots\cdots))$$

dropping the ϕ's.

(c) Everything now depends on the initial value $X_1\ X_o$, which is determined by the choice of ϕ_o, ψ_o .

In Scott's case

$$\phi_o = \lambda x{:}D_o.\ \lambda y{:}D_o.\ x$$
$$\psi_o = \lambda x{:}D_1.\ x\Omega$$

so that
$$X_1 = \lambda y{:}D_o.\ x_1(\phi_o yy)$$
$$= \lambda y{:}D_o.\ x_1 y$$
$$= x_1$$
$$X_o = \psi_o\ X_1 = x_1\ \Omega$$

and
$$X_1\ X_o = x_1(x_1\ \Omega) \ .$$

Therefore, in this case,

$$Yx = \bigsqcup_{n=1}^{\infty} x_{n+1}(x_n(\cdots\cdots x_2(x_1(x_1\ \Omega))\cdots\cdots)) \ .$$

But $x_n \sqsubseteq x$, $n \geqslant 1$

therefore $Yx \sqsubseteq \bigsqcup_{n=o}^{\infty} x^n \, \Omega = Y^* x$

so that in this case $Y = Y^*$.

(d) An alternative choice of ϕ_o, ψ_o provides the anomaly; viz. suppose D_o has a <u>compact</u> element $a \neq \Omega$, and consider the following possible ϕ_o, ψ_o :

$$\phi_o = \lambda x{:}D_o .\ \lambda y{:}D_o .(y \sqsupseteq a \rightarrow x,\ \Omega)$$

$$\psi_o = \lambda x{:}D_1 .xa$$

(The compactness condition is necessary just for ϕ_o to be continuous, and holds e.g. for all elements of a finite D_o, or of Scott's lattice N). Notice that this choice of ϕ_o, ψ_o is O.K., i.e. that ϕ_o, ψ_o are continuous and

$$\psi_o(\phi_o\ x) = x$$

$$\phi_o(\psi_o\ x) \sqsubseteq x$$

are satisfied; so Scott's construction is repeatable on this basis, and obtains a respectable model D_∞ of the lambda-calculus.

But now what is YI in such a model? With x = I we have

$$X_1 = \lambda y{:}D_o .I_1(\phi_o yy)$$

$$= \lambda y{:}D_o .\phi_o yy \qquad \text{since } I_1 = \lambda x{:}D_o .x$$

$$= \lambda y{:}D_o (y \sqsupseteq a \rightarrow y,\ \Omega)$$

$$X_o = X_1\ a = a$$

and $$X_1\ X_o = a$$

$$\text{Therefore } YI = \bigsqcup_{n=1}^{\infty} I_{n+1}(I_n \cdot\cdot\cdot\cdot\cdot\cdot I_2(a))$$

$$= a \neq \Omega \;! \quad \text{since } I_n = \lambda x{:}D_{n-1}.x$$

But $Y^*I = \Omega$

Therefore $Y \neq Y^*$ in this version.

(Actually, it turns out in such models that

$$Y = \lambda x. \bigsqcup_{n=o}^{\infty} x^n(xa \sqsupseteq a \rightarrow a, \Omega) \;;$$

this produces the minimal fixpoint of x which contains a, if $xa \sqsupseteq a$, and the "correct" minimal fixpoint otherwise.)

Additional Remarks:

1. For ϕ_o of the form

$$\lambda x.\ \lambda y.\ (y \sqsupseteq a \rightarrow x, \Omega)$$

any $x \in D_\infty$, Yx is the minimal fixpoint of x which is λ-definable from x.

(Note that, in D_∞

$$a = \lambda x.\ x \sqsupseteq a \rightarrow a, \Omega .$$

Hence

$$x \sqsupseteq a \iff ax \sqsupseteq a \iff xa \sqsupseteq a .$$

and $x, y \sqsupseteq a \Rightarrow xy \sqsupseteq a$.

All combinators $\sqsupseteq a$, since

$$Saaa = (aa)(aa) = a \Rightarrow Saa \sqsupseteq a \Rightarrow \cdots S \sqsupseteq a$$

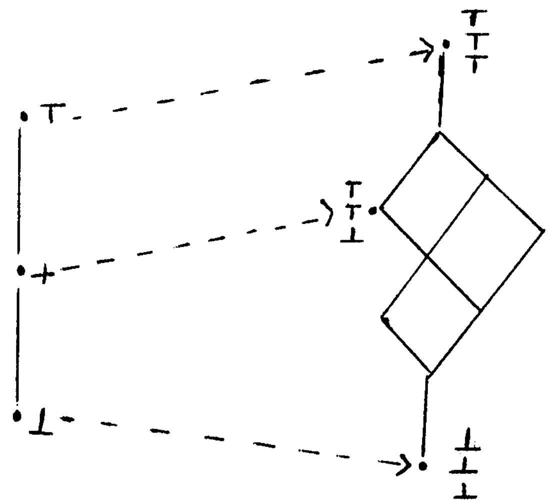
& $Kaa = a \Rightarrow Ka \sqsupseteq a \Rightarrow K \sqsupseteq a$)

2. The obvious generalization of (1) fails, since

with $D_o = \begin{smallmatrix}\top\\ +\\ \bot\end{smallmatrix}$, ϕ_o as below, we get

$$YI = \top \text{ in } D_\infty$$

which is certainly not the minimal λ-definable element of D_∞ .



In this case:

$$X_1 = \begin{smallmatrix}\top\\ \top\\ \bot\end{smallmatrix} \qquad X_o = +$$

so $X_1\, X_o = \top \in D_o$

$= \top \in D_\infty$