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The Y-combinator in Scott's Lambda-Calculus Models

(Revised Version)

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The Y-combinator in Scott's Lambda-Calculus Models.

(Revised Version)

David Park

Assume the notation and terminology of Dana Scott's paper "Models for the Lambda-Calculus". In this note I want to exhibit the relationship between the lambda-calculus "paradoxical operator"

$$Y = \lambda x((\lambda y.x(yy))\lambda y.x(yy))$$

and the minimal fixpoint operator

$$Y^* = \lambda x. \bigsqcup_{n=0}^{\infty} x^n \Omega$$

obtained by regarding the lambda-calculus model as a lattice. Intuitively, one expects that

$$Y = Y^*$$

should hold in all Scott's models; and this is indeed the case in the models constructed as in his paper; however there is an (unexpected?) complication, in that a slight alteration in the construction obtains another class of models in which $Y \neq Y^*$. This anomaly lacks (so far) any complete rationalization; one looks for grounds on which to reject such "pathological" models, but so far I know of no completely convincing ones.

Since Y^* , from lattice theory, obtains the minimal fixpoint, and Y , by beta-reductions, is certainly another fixpoint operator, it must be the case that

$$Y^* \sqsubseteq Y \quad .$$

The difficulties arise over the converse question, whether $Y \subseteq Y^*$,
i.e. whether $Yx \subseteq \bigsqcup_{n=0}^{\infty} x^n \Omega$ for all x .

Abbreviate Yx by writing

$$X = \lambda y. x(yy)$$

then $Yx = XX = \bigsqcup_{n=1}^{\infty} X_n X_{n-1}$, by Scott.

Now note the following:

(a) Using Scott's methods, it is straightforward that

$$X = \bigsqcup_{n=0}^{\infty} \lambda y: D_n. x_{n+1}(\phi_n yy)$$

We need something stronger, viz.

$$X_{n+1} = \lambda y: D_n. x_{n+1}(\phi_n yy), \quad n \geq 0.$$

(i.e. that the right hand side is a "best approximation" in D_{n+1} to X).

To show this, we need that

$$\psi_n(\lambda y: D_n. x_{n+1}(\phi_n yy)) = \lambda y: D_{n-1}. x_n(\phi_{n-1} yy), \quad n > 0.$$

Now remember the following identities:

$$\begin{aligned} \text{(i)} \quad \psi_n(u(\phi_n v)) &= \psi_{n+1} u v && \text{(by defn. of } \psi_{n+1}) \\ \text{(ii)} \quad \phi_n(uv) &= \phi_{n+1} u(\phi_n v) && \left(\begin{array}{l} \text{by defn. of } \phi_{n+1} \\ \text{\& since } \psi_n(\phi_n v) = v \end{array} \right) \end{aligned}$$

Then, for $n > 0$

$$\begin{aligned} \psi_n(\lambda y: D_n. x_{n+1}(\phi_n yy)) &= \lambda y: D_{n-1}. \psi_{n-1}(x_{n+1}(\phi_n(\phi_{n-1} y)(\phi_{n-1} y))) && \text{(by defn. of } \psi_n) \\ &= \lambda y: D_{n-1}. \psi_{n-1}(x_{n+1}(\phi_{n-1}(\phi_{n-1} yy))) \\ &&& \text{(from (ii))} \\ &= \lambda y: D_{n-1}. \psi_n x_{n+1}(\phi_{n-1} yy) \\ &&& \text{(from (i))} \\ &= \lambda y: D_{n-1}. x_n(\phi_{n-1} yy) \end{aligned}$$

which is the expression we wanted.

(b) Now for $n > 0$

$$\begin{aligned}
 X_{n+1} X_n &= (\lambda y : D_n \cdot x_{n+1} (\phi_n y y)) X_n \\
 &= x_{n+1} (\phi_n X_n X_n) \\
 &= x_{n+1} (\phi_{n-1} (X_n (\psi_{n-1} X_n))) \quad (\text{from defn. of } \phi_n) \\
 &= x_{n+1} (\phi_{n-1} (X_n X_{n-1})) \quad .
 \end{aligned}$$

But this provides a simple recurrence relation, so that

$$X_{n+1} X_n = x_{n+1} (\phi_{n-1} (x_n (\phi_{n-2} (\dots x_2 (\phi_0 (X_1 X_0)) \dots))))$$

$$\begin{aligned}
 \text{Hence } Yx &= \bigsqcup_{n=0}^{\infty} X_{n+1} X_n \\
 &= \bigsqcup_{n=1}^{\infty} x_{n+1} (x_n (\dots x_2 (X_1 X_0) \dots))
 \end{aligned}$$

dropping the ϕ 's.

(c) Everything now depends on the initial value $X_1 X_0$, which is determined by the choice of ϕ_0, ψ_0 .

In Scott's case

$$\begin{aligned}
 \phi_0 &= \lambda x : D_0 \cdot \lambda y : D_0 \cdot x \\
 \psi_0 &= \lambda x : D_1 \cdot x \Omega
 \end{aligned}$$

so that

$$\begin{aligned}
 X_1 &= \lambda y : D_0 \cdot x_1 (\phi_0 y y) \\
 &= \lambda y : D_0 \cdot x_1 y \\
 &= x_1
 \end{aligned}$$

$$X_0 = \psi_0 X_1 = x_1 \Omega$$

and

$$X_1 X_0 = x_1 (x_1 \Omega) .$$

Therefore, in this case,

$$Yx = \bigsqcup_{n=1}^{\infty} x_{n+1} (x_n (\dots x_2 (x_1 (x_1 \Omega)) \dots)) .$$

But $x_n \sqsubseteq x$, $n \geq 1$

therefore $Yx \sqsubseteq \bigsqcup_{n=0}^{\infty} x^n \Omega = Y^*x$

so that in this case $Y = Y^*$.

(d) An alternative choice of ϕ_0, ψ_0 provides the anomaly; viz. suppose D_0 has a compact element $a \neq \Omega$, and consider the following possible ϕ_0, ψ_0 :

$$\phi_0 = \lambda x:D_0. \lambda y:D_0.(y \sqsupseteq a \rightarrow x, \Omega)$$

$$\psi_0 = \lambda x:D_1.xa$$

(The compactness condition is necessary just for ϕ_0 to be continuous, and holds e.g. for all elements of a finite D_0 , or of Scott's lattice N). Notice that this choice of ϕ_0, ψ_0 is O.K., i.e. that ϕ_0, ψ_0 are continuous and

$$\psi_0(\phi_0 x) = x$$

$$\phi_0(\psi_0 x) \sqsubseteq x$$

are satisfied; so Scott's construction is repeatable on this basis, and obtains a respectable model D_{∞} of the lambda-calculus.

But now what is YI in such a model? With $x = I$ we have

$$\begin{aligned} X_1 &= \lambda y:D_0.I_1(\phi_0 yy) \\ &= \lambda y:D_0.\phi_0 yy \quad \text{since } I_1 = \lambda x:D_0.x \\ &= \lambda y:D_0.(y \sqsupseteq a \rightarrow y, \Omega) \end{aligned}$$

$$X_0 = X_1 a = a$$

and $X_1 X_0 = a$

$$\begin{aligned} \text{Therefore } YI &= \bigsqcup_{n=1}^{\infty} I_{n+1}(I_n \dots \dots I_2(a)) \\ &= a \dagger \Omega ! \quad \text{since } I_n = \lambda x : D_{n-1} . x \end{aligned}$$

But $Y^*I = \Omega$

Therefore $Y \neq Y^*$ in this version.

(Actually, it turns out in such models that

$$Y = \lambda x . \bigsqcup_{n=0}^{\infty} x^n (xa \sqsupseteq a \rightarrow a, \Omega) ;$$

this produces the minimal fixpoint of x which contains a , if $xa \sqsupseteq a$,
and the "correct" minimal fixpoint otherwise.)

Additional Remarks:

1. For ϕ_0 of the form

$$\lambda x. \lambda y. (y \sqsupseteq a \rightarrow x, \Omega)$$

any $x \in D_\infty$, Yx is the minimal fixpoint of x which is λ -definable from x .

(Note that, in D_∞

$$a = \lambda x. x \sqsupseteq a \rightarrow a, \Omega .$$

Hence

$$x \sqsupseteq a \Leftrightarrow ax \sqsupseteq a \Leftrightarrow xa \sqsupseteq a .$$

$$\text{and } x, y \sqsupseteq a \Rightarrow xy \sqsupseteq a .$$

All combinators $\sqsupseteq a$, since

$$Saaa = (aa)(aa) = a \Rightarrow Saa \sqsupseteq a \Rightarrow \dots S \sqsupseteq a$$

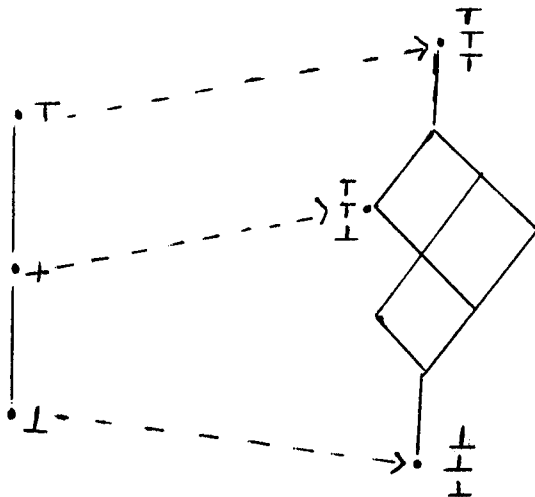
$$\& Kaa = a \Rightarrow Ka \sqsupseteq a \Rightarrow K \sqsupseteq a \quad)$$

2. The obvious generalization of (1) fails, since

with $D_0 = \begin{Bmatrix} \top \\ + \\ \perp \end{Bmatrix}$, ϕ_0 as below, we get

$$YI = \tau \text{ in } D_\infty$$

which is certainly not the minimal λ -definable element of D_∞ .



In this case:

$$X_1 = \begin{Bmatrix} \top \\ \top \\ \perp \end{Bmatrix} \quad X_0 = +$$

$$\text{so } X_1 X_0 = \tau \in D_0$$

$$= \tau \in D_\infty$$