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CATEGORY-THEORETIC SOLUTION
OF
RECURSIVE DOMAIN EQUATIONS

by

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- CATEGORY-THEORETIC SOLUTION

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1. Introduction. The solution of a recursive domain equation, of the form

\[ D \overset{\cong}{=} F(D) \]  

may be viewed as the finding of a fixpoint (up to isomorphism) of the functor \( F \). This has led to the idea of formulating a category-theoretic analogue of Tarski's fixpoint theorem for lattices, as a basis for a general method of solution for this kind of equation; see especially Reynolds [1], Wand [2], Plotkin [3].

In seeking to adapt Tarski's results to the category situation, one would expect the role of continuous functions to be taken by colimit-preserving functors. In all the versions developed up till now, however, the major role is taken by a quite different notion of continuity of functors ("continuity on morphism-sets"), having to do with orderings of the hom-sets. Apparently it was thought that requirement of preservation of (w-)colimits was too difficult to handle ([2]).

The purpose of the present note is to show that, on the contrary, a much better organization of the theory can be achieved by taking preservation of \( w \)-colimits (or \( w \)-continuity, as we shall call it) as the basic notion. The main theorem (Theorem 1) now takes the form of an exact generalization of the lattice-theoretic fixpoint theorem, rather than only an analogue of it. More substantially, since the theorem applies to arbitrary categories admitting \( w \)-colimits (no ordering of hom-sets is needed) the range of application is wider: see Sec. 2 below. Furthermore, the application of the theorem to a given class \( C \) of domains takes on a simpler form: instead of the manipulations with three distinct categories (\( K, K_P, K_R \) in Wand's notation) characteristic of the "continuity on morphism
sets" approach, we consider just the category of "embeddings" in C (Definition 2).

In Section 2 we discuss the existence of \( \omega \)-colimits in certain relevant categories. The material here is fairly standard. One point, however, is worth noting. We have not found the abstract approach of Wand [2] to be worth the effort it involves. In all the relevant categories, the objects are sets (with structure); and it involves little more than a routine verification to show that the set-theoretic inverse limit is also the colimit. The abstract method, however, brings with it a double complication when we try to apply it to a concrete category. We must first show that the set-theoretic inverse limit is the (category-theoretic) limit; and then, with the aid of special conditions (cf. Wand's "Condition A"), that the limit is also the colimit.

Section 3 concerns the \( \omega \)-continuity of various useful functors. In addition to the ones usually discussed, we have included a brief treatment of Plotkin's powerdomain construction.

2. The fixed point theorem. The "lattice-theoretic" fixedpoint theorem for continuous functions may be stated in a strong form as follows. Let \( (P, \preceq) \) be a poset in which every increasing \( \omega \)-chain has a lub, let \( a \in P \) and let \( f : P \to P \) be an \( \omega \)-continuous function (that is, \( f \) preserves lubs of \( \omega \)-chains) such that \( a \leq fa \). Then the set of post-fixedpoints of \( f \) greater than \( a \) (that is, \( \{ x \mid f(x) \preceq a \land x \neq a \} \) ) has a least element, namely \( \lim_n f^n(a) \) - which is, moreover a fixedpoint of \( f \). We seek the appropriate generalization of this result to categories.

Definition 1. An \( \omega \)-chain is a functor from \( \omega \) (the natural numbers with the standard linear ordering) into \( K \). \( K \) is said to admit \( \omega \)-colimits if every
w-chain in $K$ has a colimit. A functor $F: K \to K'$ is $w$-continuous if $F$ transforms any colimit diagram for an $w$-chain $\Gamma$ in $K$ into a colimit diagram (for $F\Gamma$) in $K'$.

An $w$-chain in $K$ may be pictured like this:

$$\Gamma: \cdots \xrightarrow{a} \cdots$$

We adopt the viewpoint that a colimit (diagram) for $\Gamma$ is an initial object in the category of cones from $\Gamma$. Corresponding to the sequence $<f^n(a)>$ in the above formulation of the fixedpoint theorem, we will have an $w$-chain

$$\Delta = a \xrightarrow{\theta} Fa \xrightarrow{F\theta} F^2a \cdots$$

where $F: K \to K'$ is now an $w$-continuous functor. Our program is to construct the category $\mathcal{P}(K,F,\theta)$ of "post-fixed-objects along $\theta$" and "post-fixed-arrows" of $F$, and to show that - roughly speaking - $\text{colim} \Delta$ is initial in this category.

Let $F$ be an endofunctor of a category $K$. We say that an object $x$ of $K$ is a post-fixedpoint of $F$ via $y$ if $yFx = x$, and an arrow $\pi: A \to B$ is a post-fixed-arrow of $F$ via $y, \delta$ if

$$\begin{array}{ccc}
B & F & B \\
\delta & \downarrow & \delta \\
\pi & \downarrow & \pi \\
A & F & A
\end{array}$$

commutes. If now $\theta: a \to Fa$, define the category $\mathcal{P}(K,F,\theta)$ as follows: the objects are the triples $<\alpha, M, \gamma>$, where $M$ is a post-fixedpoint of $F$ via $\gamma$, and $\alpha = \gamma \text{ id } \theta$ (equivalently, the objects are the commuting squares

$$\begin{array}{ccc}
a & \xrightarrow{\theta} & Fa \\
\alpha & \downarrow & \pi \downarrow \\
M & \gamma & FM
\end{array}$$

while $\text{Hom}(<\alpha, M, \gamma>, <\alpha', M', \gamma'>)$ is the set of arrows (in $K$) $\pi: M \to M'$ such that (1) $\pi \alpha = \alpha'$, and (2) $\pi$ is a post-fixed arrow of $F$ via $\gamma, \gamma'$.

Suppose now that $F$ is $w$-continuous, and that $\lambda$ is a colimit cone for
Let $K$ admit $w$-colimits and $F$ be $w$-continuous. Then $L$ is a fixedpoint of $F$ via $\psi^{-1}$, and $<\lambda_0, L, \psi^{-1}>$ is initial in $PF(K, F, \theta)$.

Proof. That $L$ is a fixedpoint of $F$ via $\psi^{-1}$ is contained in the preceding remarks. Turning to the initiality of $<\lambda_0, L, \psi^{-1}>$, we note first that any object $<\alpha, \lambda, \gamma>$ of $PF(K, F, \theta)$ determines a cone $\mu$ from $\Delta$ to $\lambda_2$ as follows:

$$\mu_0 = \alpha$$
$$\mu_{n+1} = \gamma F \mu_n$$

(In this sense, $<\lambda_0, L, \psi^{-1}>$ determines $\lambda$.) Moreover, any arrow

$$\pi: <\alpha, \lambda, \gamma> \rightarrow <\alpha', \lambda', \gamma'>$$

is a morphism of the corresponding cones $\mu, \mu'$ (That is, $\mu_n' = \pi \mu_n$, $n=0, 1, \ldots$ ). For $\mu_0' = \alpha' = \pi \alpha = \pi \mu_0$ ; while (induction step) $\mu_{n+1}' = \gamma' F \mu_n' = \gamma' F (\pi \mu_n) = \pi \gamma F \mu_n = \pi \mu_{n+1}'$. Hence, the only possible arrow from $<\lambda_0, L, \psi^{-1}>$ to $<\alpha', \lambda, \gamma'>$ is the unique arrow $\sigma: \lambda \rightarrow \mu$.

It remains to show that $\sigma$ is indeed an arrow of $PF(K, F, \theta)$, that is, that $\sigma$ is a post-fixed-arrow of $F$ via $\psi^{-1}, \gamma$. By the preceding remarks, the (unique) arrow from $\lambda_0$ to $\mu$ is $\sigma \psi^{-1}$. But we can also show that it is $\gamma F \sigma$:

$$\mu_0 = \alpha = \gamma F \alpha = \gamma F (\sigma \lambda_0) = \gamma F \lambda_0$$
$$\mu_{n+1} = \gamma F \mu_n = \gamma F (\sigma \lambda_n) = \gamma F \lambda_{n+1}$$

Hence $\sigma \psi^{-1} = \gamma F \sigma$ ; the proof is complete.

This theorem is formally similar to \textit{Wend}'s Theorem 3. The content (and proof) is different, due to the changed notion of continuity of functors.
Also, we have brought in the terminology of post-fixedpoints (and arrows), in the hope of making the result more intuitive.

In most applications of the theorem, a is initial in $K$. (The important exception is the construction of models of the $\lambda$-calculus via the domain equation $D = [D \to D]$.) In this case, the conclusion of the theorem can be simplified:

**Corollary 1.** Under the hypotheses of Theorem 1, suppose that $a$ is initial in $K$. For any post-fixedpoint $M$ of $F$ via $\gamma$, there is a unique $\sigma : L \to M$ such that $\sigma$ is a post-fixed-arrow of $F$ via $\gamma^{-1}, \gamma$.

The following lemma will be useful (via Corollary 2) in establishing $w$-continuity of functors:

**Lemma 1.** Let $K'$ admit $w$-colimits, and $F : K \to K'$. Then $F$ is $w$-continuous iff the following condition holds for every $w$-chain $A$ in $K$ with colimit cone $\lambda$: if $\mu$ is a colimit cone for $F\lambda$ in $K'$, the arrow $\psi$ from $\mu$ to $F\lambda$ is an isomorphism.

**Proof.** The necessity of the condition is obvious. For sufficiency, assume the condition satisfied. Then there is an arrow from $F\lambda$ to any cone $\nu$ from $F\lambda$, viz. $\sigma\psi^{-1}$, where $\sigma : \mu \to \nu$; and there is at most one such arrow, for if $\psi_1, \psi_2 : F\lambda \to \nu$ are distinct, then so are $\psi_1\psi_1^{-1}, \psi_2\psi_2^{-1} : \mu \to \nu$. (Alternative proof: verify that $\psi$ is an isomorphism in the category of cones from $F\lambda$.)

### 3. Existence of $w$-colimits

**Definition 2.** An $w$-cpo is a poset in which every (ascending) $w$-chain has a lub. A map $f : D \to D'$, where $D, D'$ are $w$-cpo's, is $w$-continuous if $f(\langle a_i \rangle) = \bigcup f(a_i)$ for every $w$-chain $\langle a_i \rangle$ in $D$. An $w$-continuous map $f : D \to D'$ is an embedding if $f$ possesses an $w$-continuous right adjoint $f' : D' \to D$. $w$-CPO is the category of $w$-cpo's with embeddings as arrows.
Some of these concepts were already introduced informally at the beginning of Sec. 2. A more familiar characterization of embeddings is as follows: \( f : D \to D' \) is an embedding iff there is an (\( \omega \)-continuous) \( f' : D \to D \) such that \( f' f = I_D \) and \( f f' \leq I_{D'} \) (so that \( f, f' \) form a projection pair).

The prefix \( \omega \)- will usually be omitted (from the terms introduced in Definition 2). In fact, nothing hinges on our use of \( \omega \)-cpo's rather than the more restricted class of cpo's (in which all directed sets have lubs). The point simply that the \( \omega \)-notions generalize more readily to categories: see remarks at the end of this section.

Usually, the categories in which we need to solve recursive domain equations are full subcategories of \( \text{CPC}^\mathcal{N} \). There follows a concise treatment of the construction of \( \omega \)-colimits in \( \text{CPC}^\mathcal{N} \). This is little more than a summary of (parts of) the existing proofs for complete lattices (Reynolds, Wand).

**Theorem 2.** Let \( \Delta = D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} \ldots \) be an \( \omega \)-chain in \( \text{CPC}^\mathcal{N} \). Let \( D \) be the inverse limit \( \{ \langle x \rangle_{n \in \omega} \mid x_n \in D_n \text{ and } f'_n(x_{n+1}) = x_n \} \) with the induced (componentwise) partial ordering. Then \( D \), together with the maps \( i_n : D_n \to D \) given by \( i_n(x_n) = \langle f_m(x_n) \rangle_{m \in \omega} \), is a colimit for \( \Delta \). (Here we have used the standard notation

\[
\begin{pmatrix}
  f'_m & \cdots & f'_{n-1} & m < n \\
  I_{D_n} & & & m = n \\
  f_{m-1} & \cdots & f_n & m > n
\end{pmatrix}
\]

**Proof.** This falls into a series of lemmas and sublemmas. For the easiest cases we just quote the main relevant fact(s), rather than giving a detailed proof. Certain elementary facts are used implicitly, for example: for \( n \geq m \), \( f_{mn} \) is an embedding, with \( f'_{mn} = f_{mn} \).
(1) $D$ is a cpo. (Continuity of the $f_n'$)

(2) Each $i_n$ is an embedding, with adjoint $i_n'$ given by: $i_n'(<x_{m\in\omega}>)_n = x_n$.

(i) $<f_m(x_n)>_{m\in\omega} \in D$ (since $f_{m\leq n+1} f_m = f_m$)

(ii) $i_n' i_n = 1_D$

(iii) $i_n'(<x_m>_{m\in\omega}) < x_m$ (since $i_n' f_{m\leq n} (x_{m\leq n}) (n \leq p) f_{m\leq p} (x_{m\leq p}) = x_p$).

(3) Given a cone $\lambda$ from $\Delta$ to $L$, we have the mediating map $\psi:D \rightarrow L$:

$<x_m>_{m\in\omega} \rightarrow \bigsqcup_{m\in\omega} \lambda_m (x_m)$, with adjoint $\psi':L \rightarrow D : y \rightarrow <\lambda'_n(y)>_{n\in\omega}$.

(i) $\psi$ is well-defined, i.e. $<\lambda_m (x_m)>_{m\in\omega}$ is increasing. (Since $\lambda$ is a cone, $\lambda_m (x_m) = \lambda_{m+1} f_m (x_m) = \lambda_{m+1} f_m f_m (x_{m+1}) \subseteq \lambda_{m+1} (x_{m+1})$).

(ii) $\psi'$ is well-defined, i.e. $\psi'(y) \in D$. This is equivalent to $f_n' \lambda_{n+1}'(y) = \lambda_n'(y)$. But this follows, by taking adjoints, from $\lambda_{n+1} f_n = \lambda_n$.

(iii) $\psi' \psi = 1_D$. For $(\psi' \psi(<x_m>_{m\in\omega}))_n = \lambda'_n (\bigsqcup_{m\in\omega} \lambda_m (x_m)) = \bigsqcup_{m\in\omega} \lambda_m (x_m) = \bigsqcup_{m\in\omega} \lambda'_m (x_m) = \lambda'_n (x_n) = x_n$.

(iv) $\psi ' \subseteq 1_L$. For $\psi ' \subseteq \bigsqcup_{m\in\omega} \lambda'_m$, and each $\lambda_n' \subseteq I_L$.

(v) $\lambda_n = \psi i_n$. For $\psi i_n (x_n) = \bigsqcup_{m\in\omega} \lambda_m f_m (x_m) = \lambda_n (x_n)$.

(4) There is at most one mediating map $\varphi:D \rightarrow L$. For if $\lambda_n = \varphi i_n$, then $i_n \varphi (y) = \lambda_n'(y)$, so that $\varphi'(y)$ must be $<\lambda_n'(y)>_{n\in\omega}$. Hence $\varphi'$, and so $\varphi$, is uniquely determined.

Corollary 2. $\lambda$ is a colimiting cone for $\Delta$ iff $\bigsqcup_{n \in \omega} \lambda_n \circ \lambda'_n = 1_L$.

Proof. By Lemma 1, $\lambda$ is colimiting iff $\psi$ is an isomorphism. This holds iff $\psi \psi' = 1$. Now $\psi \psi' = \bigsqcup_{n \in \omega} \lambda_n \circ \lambda'_n$.

If $K$ is any full subcategory of $\text{CPO}^\omega$, it follows from Theorem 2 that $K$ admits colimits provided that the inverse limit of any $\omega$-chain in $K$ is an object of $K$. In particular, this holds for the algebraic cpo's, which may be introduced - using sequences instead of directed sets, as in Definition 2 - as follows:
Definition 3. A cpo $D$ is **countably algebraic** if there is a countable subset $B$ of $D$ such that (1) every $x \in D$ is the lub of an increasing sequence of elements of $B$, and (2) for any increasing sequence $\langle e_i \rangle_i$ in $B$ and any $a \in B$, if $a \subseteq \bigcup e_i$, then $a \subseteq e_i$ for some $i$.

The qualifier "countably" will usually be omitted. One readily shows that the elements of $B$ are exactly the **finite** (=isolated) elements of $D$, and that the definition agrees with that given by Plotkin (for "algebraic cpo") except for one (minor) point: we do not require algebraic cpo's to have least elements.

For the proof that $\mathbf{ACP}_C^E$ (the category of algebraic cpo's and embeddings) is closed under the inverse limit construction, see [3]. One other elementary fact which we shall need is the following: if $(\mathbb{Q}, \leq)$ is a countable preordered set, then the collection $\mathbb{Q}$ of all directed subsets of $\mathbb{Q}$, ordered by inclusion, is an algebraic cpo; the finite elements of $\mathbb{Q}$ are the sets of the form $[a] = \{b \mid b \leq a\}$, for $a \in \mathbb{Q}$.

The basic notion of this section is that of an $\omega$-complete poset. As has already been hinted, however, the definitions and results can be formulated more generally, in terms of ($\omega$-complete) **categories**. A generalization of this kind has been worked out by Lehmann [4], with a view to applications to the semantics of non-deterministic programs; in this approach the semantic domains are themselves categories. In a slightly different direction, [5] introduces a notion of "algebraic category", got by generalizing Definition 3; we return to this point in Sec.4. It should be emphasized that Theorem 1 applies without modification in the more general situation; this is an important advantage which the formulation in terms of $\omega$-continuity has over that in terms of local continuity (continuity on morphism sets).
4. Continuity of special functors. The functor \( F \) in the domain equation (1) will typically be composed out of the following basic functors: \(+, \times, \to\) and (in case we follow Plotkin's "powerdomain" approach [3] to non-deterministic semantics) the definition, and proof of \( w \)-continuity, of \(+\) and are entirely straightforward. In the present section we consider \( \to \) and .

The functor \( \to: (\text{CPO}^E)^2 \to \text{CPO}^E \) is defined on objects by
\[
\to(<D,D'>) = [D \to D']
\]
(the cpo of continuous functions from \( D \) to \( D' \)), and on arrows by
\[
\to(<p,q>) = \lambda f.q'f.p'
\]
\( \to(<p,q>) \) is an embedding, with adjoint \( \lambda g.q'g.p \).

**Theorem 3.** \( \to \) is \( w \)-continuous.

**Proof.** Suppose that \( \Delta \) is a cone in \( (\text{CPO}^E)^2 \); that is, \( \Delta \) is in effect a pair of cones \( D_0 \to D_1 \to \ldots, E_0 \to E_1 \to \ldots \) in \( \text{CPO}^E \). Let these cones have colimits \( D,E \) via the embeddings \( d_n:D_n \to D, e_n:E_n \to E \). \( \to(\Delta) \) is a cone with vertex \( [D \to E] \) and embeddings \( \varphi_n:[D_n \to E_n] \to [D \to E]: f \to e_n f d_n \); we have to show that this cone is colimiting. But this follows by Corollary 2, since
\[
\bigcup \varphi_n \varphi' = \lambda g. \bigcup e_n e'_n g d_n d'_n
\]
\[
= \\lambda g. \bigcup e_n e'_n (\bigcup d_n d'_n) \quad \text{(by continuity of \( \cup \))}
\]
\[
= \lambda g g \quad \text{(by Corollary 2)}
\]
\[
= \text{I}.
\]

Turning to the powerdomain, let \( D \) be an algebraic cpo, and let \( M(D) \) be the set of non-empty finite sets of finite elements of \( D \). \( M(D) \) is given the preorder \( \sqsubseteq_M \) (the "Hilber ordering") defined as follows:
\[
A \sqsubseteq_M B \iff \forall a \in A \exists b \in B. a \leq b \quad \text{and} \quad \forall b \in B \exists a \in A. a \leq b.
\]
It was shown in [5] that the powerdomain of $D$, as defined by Plotkin, is isomorphic to $\overline{M(D)}$; we will define $(\overline{D})$ as $\overline{M(D)}$. To describe the action of $\rho$ on arrows of $ACPO^E$, we proceed as follows. Given any continuous $f : D \to D'$ ($D,D'$ algebraic), we define the "extension" $\hat{f}$ of $f$ to $\overline{(D)}$ by:

$$\hat{f}(X) = \bigcup_{A \in X} \{ f(A) \}$$

(of course, the operation $[\cdot]$ is here taken w.r.t. the preorder $\leq_M$ of $M(D')$)

The following properties are immediate: $f$ is a continuous function;

$$I_D = I_{\overline{\rho}(D)}; \ f \mapsto \hat{f}; \ \hat{\cdot} \text{ is monotone (that is, } f \leq g \implies \hat{f} \leq \hat{g}). \text{ It follows that if } f \text{ is an embedding, then } \hat{f} \text{ is an embedding (with adjoint } \hat{f}'\text{).}

Thus, if we define $\rho$ on arrows by: $\rho(f) = \hat{f}$, then $\rho : ACPO^E \to ACPO^E$ is indeed a functor.

It will be useful to have the property of local continuity for this functor:

**Lemma 2.** If $f_n : D \to D'$ is an increasing sequence of continuous functions ($D,D'$ algebraic), then $\{ \bigcup_n f_n \}^\ddagger = \bigcup_n \hat{f}_n^\ddagger$.

**Proof.** The conclusion of the lemma is equivalent to the following statement: for any $A \in M(D)$, $[(\bigcup_n f_n)(a)] = \bigcup_n [\hat{f}_n(a)]$. The right-to-left inclusion here is trivial. For the left-to-right inclusion, suppose that $B \subseteq_M (\bigcup f_n)(A)$. Since the elements of $B$ are finite, we can choose (1) $k$ such that $\forall a \in f_k(A) \exists b \in B. b \subseteq a$, and (2) $l$ such that $\forall b \in B (a \in f_l(A), b \subseteq a$. Setting $m = \max(k,l)$, we have $B \subseteq_M f_m(A)$, and so $B \subseteq \bigcup f_n(\cdot)$.

**Theorem 4.** is $w$-continuous.

**Proof.** Again using Corollary 2, it suffices to show that if $\bigcup_n f_n : D \to D$ (where each $f_n : D \to D$ is an embedding), then $\bigcup_n \hat{f}_n f_n^\ddagger = 1_D$.

Now

$$\bigcup_n \hat{f}_n f_n^\ddagger = \bigcup_n (f_n f_n^\ddagger) = \bigcup_n \hat{f}_n f_n^\ddagger = 1_D (\text{by Lemma 2})$$

$$= I_D = 1_D$$
The functors \( \rightarrow \) and \( \rho \) have been defined for different categories. If we wish to solve a recursive domain equation which involves both functors, we need to have a single category which is closed under both of them. For this purpose Plotkin introduces the **SFP objects** - these are the \( \text{cpo}'s \) which are colimits of \( \omega \)-chains of \( \text{finite} \) \( \text{cpo}'s \). Every SFP object is algebraic, by the remarks following Definition 3 (closure under inverse limit construction of \( \text{ACPO}^E \)). Thus \( \text{SFP}^E \) (category of SFP objects and embeddings) is a full subcategory of \( \text{ACPO}^E \). If \( D, E \) are finite, then \( \rho(D) \) and \( D \rightarrow E \) are (trivially) finite. It follows by Theorems 3, 4 that \( \text{SFP}^E \) is closed under both constructions.

It remains only to show that \( \text{SFP}^E \) admits \( \omega \)-colimits. A proof of this result is given in [3]. An alternative proof - which proceeds by way of showing that \( \text{SFP}^E \) is an "algebraic category" - may be found in [5].

**REFERENCES**


[2] Wand, M., Fixed-point constructions in order-enriched categories, TR No. 23, Indiana University Computer Science Department, April 1975

