ON THE STRUCTURE OF FREE FINITE STATE MACHINES

BY

W. M. BEYNON

Department of Computer Science
University of Warwick
COVENTRY CV4 7AL
ENGLAND.

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1. Introduction

As explained by Birkhoff and Lipson in [1], a finite state machine $M$ (without outputs) can be considered as an algebra with two "phyla":

$$S = \text{set of states, } I = \text{input alphabet}$$

and a single operator: $T : S \times I \rightarrow S$, the transition function of $M$.

Given $M = (S, I)$ and a pair of integers $(m, n)$ there is an associated machine $U_{m, n}(M)$ freely generated as an algebra by states $t_1, \ldots, t_m$ and input symbols $e_1, \ldots, e_n$ subject to the relations which hold within $M$. Explicitly $U_{m, n}(M) = (\mathcal{A}, \mathcal{J})$ where

$$\mathcal{J} = \{e_1, \ldots, e_n\}$$

and each state in $\mathcal{A}$ consists of an equivalence class of expressions of the form

$$t_i w(e_1, \ldots, e_n) \text{ where } 1 \leq i \leq m, w \in \mathcal{J}^*$$

and $t_i w(e_1, \ldots, e_n)$ and $t_j v(e_1, \ldots, e_n)$ are equivalent if for all pairs of maps $(t_1, \ldots, t_m) \not\rightarrow S$ and $(e_1, \ldots, e_n) \not\rightarrow I$ the relation:

$$f(t_i) w(g(e_1), \ldots, g(e_n)) = f(t_j) v(g(e_1), \ldots, g(e_n))$$

holds in $M$. The transition function then maps $(t_i w(e_1, \ldots, e_n), e_i)$ to $t_i (w(e_1, \ldots, e_n) e_i)$.

Definition Using the notation introduced above, it will be convenient to refer to a pair of maps $f : \{t_1, \ldots, t_m\} \rightarrow S$ and $g : \{e_1, \ldots, e_n\} \rightarrow I$ as a phyla-preserving mapping from $\{t_1, \ldots, t_m, e_1, \ldots, e_n\}$ to $M$ or an interpretation of $\{t_1, \ldots, t_m, e_1, \ldots, e_n\}$ in $M$. 

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The proof of the following theorem is to be found in [1].

**Theorem**  
(i) \( U_{m,n}(M) \) is a finite state machine.  
(ii) \( U_{m,n}(M) \) is generated by the \( m \) states \( t_1, \ldots, t_m \) and \( n \) input symbols \( e_1, \ldots, e_n \).  
(iii) If \( \pi \) denotes the canonical phyla-preserving map from the set \( \{ t_1, \ldots, t_m, e_1, \ldots, e_n \} \) to \( U_{m,n}(M) \), and \( \theta \) is any phyla-preserving map from \( \{ t_1, \ldots, t_m, e_1, \ldots, e_n \} \) to \( M \), then there is an unique algebra homomorphism \( \phi : U_{m,n}(M) \to M \) such that \( \theta = \phi \pi \).  
(iv) \( U_{m,n}(M) \) is an epimorphic image of any other finite state machine having property (iii).

**Definition** If \( t_i w(e_1, \ldots, e_n) \) and \( t_j v(e_1, \ldots, e_n) \) are equivalent in \( U_{m,n}(M) \), then  
\[ t_i w(e_1, \ldots, e_n) = t_j v(e_1, \ldots, e_n) \]  
is a universal relation in \( M \).

2. The case \( m > 1 \)

**Theorem 1:** For a machine \( M = (S,I) \) to have a universal relation of the form  
\[ t_i w(e_1, \ldots, e_n) = t_j v(e_1, \ldots, e_n) \text{ with } i \neq j \]  
it is necessary and sufficient that for each input \( \alpha \) in \( I \) there should exist a state \( t(\alpha) \) such that \( (a) \) \( t(\alpha) \cdot \alpha = t(\alpha) \)  
and \( (b) \) for each \( s \) in \( S \) there is a non-negative integer \( r(s) \) such that \( \alpha^r(s) = t(\alpha) \)

**Proof:** Suppose that \( M \) has a universal relation \( U \) of the form  
\[ t_i w(e_1, \ldots, e_n) = t_j v(e_1, \ldots, e_n) \text{ for } i \neq j \]  
If \( \alpha \in I \), there is a submachine \( M_\alpha = (S,\alpha^k) \) of \( M \), which is a disjoint union of \( k \) machines of the following type:
Since \( U \) holds under all interpretations \((f,g)\) for which \(g(e_i) = a\) for \(1 \leq i \leq n\), it is clear that \(k = 1\). Moreover, taking interpretations \((f,g)\) such that \(f(t_i) = p_o, f(t_j) = p_o \cdot a^c\) for some non-negative integer \(c\) and \(g(e_i) = a\) for \(1 \leq i \leq n\), it follows that

\[
p_o \cdot \alpha(w) = p_o \cdot \alpha(v) + c
\]

in \(M\) for \(c = 0, 1, 2, \ldots\). This establishes that \(\overline{p \alpha} = \overline{p}\), so that conditions (a) and (b) are satisfied with \(t(\alpha) = \overline{p}\).

For the converse, suppose that given input \(a\) in \(I\), there is a \(t(\alpha)\) for which conditions (a) and (b) hold. Then let \(r(\alpha) = \max r(s), s \in S\) and \(r = \max r(\alpha)\). It is clear that the relation \(s \cdot a^r = t \cdot a^r\) holds for all \(s, t \in S\) and all \(a \in I\); that is, the relation \(t_1 \cdot x_1^r = t_2 \cdot x_1^r\) holds universally in \(M\).

**Corollary to Theorem 1:**

Unless a relation of the form \(t_1 \cdot x_1^r = t_2 \cdot x_1^r\) holds universally in the machine \(M\), the finite state machine \(U_{m,n}(M)\) is (up to isomorphism) \(m\) disjoint copies of \(U_{1,n}(M)\).

3. **Structure of \(U_{1,n}(M)\)**

**Definition** Let \(K\) be a finite monoid generated by elements \(x_1, \ldots, x_n\).

The machine \(m(K,X)\) associated with the monoid \(K\) generated by \(X\) has a set of states \(K\), input alphabet \(X = \{x_1, \ldots, x_n\}\) and transition function \(K \times X \to K\) defined by multiplication in \(K\). The machine \(m(K,X)\) will be called a **monoid machine**. If \(K\) is a group, then \(m(K,X)\) is a **group machine** or **Cayley diagram**.


Theorem 2: (i) If $M$ is a finite state machine, then, for $n \geq 1$, $U_{1,n}(M)$ is isomorphic to the monoid machine $\mathcal{M}(K,X)$, where $K$ is the monoid freely generated by $X = \{x_1, \ldots, x_n\}$ subject to the relations:

$$w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$$

where $tw(e_1, \ldots, e_n) = tv(e_1, \ldots, e_n)$

is a universal relation in $M$.

(ii) Let $K$ be a finite monoid generated by $X = \{x_1, \ldots, x_n\}$

For $\mathcal{M}(K,X)$ to be isomorphic with $U_{1,n}(M)$ for some finite state machine $M$, it is necessary and sufficient that for each relation $w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ in $K$ and each map $f : \{1, 2, \ldots, n\}$, the relation $w(x_{f(1)}, \ldots, x_{f(n)}) = v(x_{f(1)}, \ldots, x_{f(n)})$ also holds in $K$. If this condition is satisfied then $U_{1,n}(\mathcal{M}(K,X)) = \mathcal{M}(K,X)$.

(iii) For $U_{1,n}(M)$ to be a group machine ($n \geq 1$) it is necessary and sufficient that for some non-trivial $w$ in $G^n$, a relation of the form:

$$tw(e_1, \ldots, e_n) = t$$

holds universally in $M$.

Definition. When the necessary and sufficient conditions (stated in (ii) above) for $\mathcal{M}(K,X)$ to be isomorphic with $U_{1,n}(M)$ for a finite state machine $M$ are satisfied, $X$ is said to generate $K$ universally or to generate a universal presentation of $K$.

Proof: (i) The elements of $U_{1,n}$ are equivalence classes of expressions of the form:

$$tw(e_1, \ldots, e_n)$$

where $tw(e_1, \ldots, e_n)$ and $tv(e_1, \ldots, e_n)$ are equivalent if $tw(e_1, \ldots, e_n) = tv(e_1, \ldots, e_n)$ is a universal relation in $M$, with transition function defined by

$$(tw(e_1, \ldots, e_n), e_i) \mapsto t(w(e_1, \ldots, e_n)e_i)$$
The map \( tw(e_1, \ldots, e_n) = w(x_1, \ldots, x_n) \) then clearly induces an isomorphism \( U_{1,n}(M) = K \).

(iii) Suppose \( M(K,X) = U_{1,n}(M) \). Then if the relation
\[
w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)
\]
holds in \( K \) then \( tw(e_1, \ldots, e_n) = tv(e_1, \ldots, e_n) \)
is a universal relation in \( M \). Thus given any map \( f : \{1,2,\ldots,n\} \), the relation
\[
tw(e_{f(1)}, \ldots, e_{f(n)}) = tv(e_{f(1)}, \ldots, e_{f(n)})
\]
holds universally in \( M \), whence \( w(x_{f(1)}, \ldots, x_{f(n)}) = v(x_{f(1)}, \ldots, x_{f(n)}) \)
in \( K \).

Conversely, suppose that if \( w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \) in \( K \) and \( f \) is a map \( \{1,2,\ldots,n\} \), then \( w(x_{f(1)}, \ldots, x_{f(n)}) = v(x_{f(1)}, \ldots, x_{f(n)}) \).
It follows that the relation \( tw(e_1, \ldots, e_n) = tv(e_1, \ldots, e_n) \) holds universally in \( M(K,X) \). Conversely if \( tw(e_1, \ldots, e_n) = tv(e_1, \ldots, e_n) \) is a universal relation in \( M(K,X) \) then certainly \( w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \)
in \( K \) (interpreting \( t \) as 1, and \( e_i \) as \( x_i \) for \( i = 1,2,\ldots,n \)). The isomorphism
\[
U_{1,n}(M(K,X)) = M(K,X)
\]
follows from (i).

(iii) Let \( x_1, \ldots, x_n \) generate \( U_{1,n}(M) \) freely subject to the relations:
\[
w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)
\]
where \( tw(e_1, \ldots, e_n) = tv(e_1, \ldots, e_n) \)
is a universal relation in \( M \). Since \( x_1^r = 1 \) for some \( r \gg 1 \), the relation \( te_i^r = t \) must hold universally in \( M \) for some \( r \).

Conversely, suppose \( tw(e_1, \ldots, e_n) = t \) holds universally in \( M \), with \( w \) non-trivial. Then given \( f : \{1,2,\ldots,n\} \) the relation
\[
w(x_{f(1)}, \ldots, x_{f(n)}) = 1 \]
holds in \( U_{1,n}(M) \). In particular, \( w(x_1, \ldots, x_1) = 1 \)
for each \( i \), which proves the existence of \( x_i^{-1} \) for each \( i \), as \( w \) is non-trivial.
Definition Let $M = (S, I)$ be a finite state machine, and let $F(S)$ denote the semigroup of mappings $S \rightarrow S$ under composition. For each $a$ in $I$, let $T(a)$ be the map $S \rightarrow S$ in $F(S)$. The map $T$ extends naturally to a semigroup homomorphism $I^* \rightarrow F(S)$. The image of this homomorphism is the **syntactic monoid** $\mathcal{J}(M)$ of $M$.

**Lemma:** For each $n > 1$, $U_{1,n}(M)$ and $U_{1,n}(\mathcal{M}(\mathcal{J}(M), T(I)))$ are isomorphic.

**Proof:** Suppose that $sw(a_1, \ldots, a_n) = sv(a_1, \ldots, a_n)$ for all $s$ in $S$ and all $a_i$ in $I$. Then $w(T(a_1), \ldots, T(a_n))$ and $v(T(a_1), \ldots, T(a_n))$ represent the same element of $\mathcal{J}(M)$, so that $fw(T(a_1), \ldots, T(a_n)) = fv(T(a_1), \ldots, T(a_n))$ for all $f$ in $\mathcal{J}(M)$ and all $a_i$ in $I$.

Conversely, if $fw(T(a_1), \ldots, T(a_n)) = fv(T(a_1), \ldots, T(a_n))$ for all $f$ in $\mathcal{J}(M)$ and all $a_i$ in $I$, then $w(T(a_1), \ldots, T(a_n)) = v(T(a_1), \ldots, T(a_n))$ in $\mathcal{J}(M)$. Thus $sw(a_1, \ldots, a_n) = sv(a_1, \ldots, a_n)$ in $M$ for all $s$ in $S$ and all $a_i$ in $I$.

This proves the required isomorphism.

It is evident that a universal relation of the form $tw(e_1, \ldots, e_m) = tv(e_1, \ldots, e_m)$ holds in a monoid machine $\mathcal{M}(K, X)$ if and only if $w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ for all $x_i$ in $X$. This result will be used in the proof of the next theorem, which describes a simple method for constructing $U_{1,n}(M)$ when $M$ is a monoid machine.

**Theorem 3:** Let $K$ be a finite monoid generated by $X = \{x_1, \ldots, x_n\}$.

Let $X^k$ be the set of rows of the $n$ by $m^n$ array whose columns are the elements of $X^n$. Then $X^k$ has $n$ elements $x_1, \ldots, x_n$, which generate a submonoid $K^k$ of $K^{m^n}$, and $U_{1,n}(\mathcal{M}(K, X))$ and $\mathcal{M}(K^k, X^k)$ are isomorphic.

**Proof:** Suppose that $w(y_1, \ldots, y_n) = v(y_1, \ldots, y_n)$ in $K$ for all interpretations of $y_1, \ldots, y_n$ in $X$. Then the identity $w(y_1, \ldots, y_n) = v(y_1, \ldots, y_n)$ necessarily holds in $K^k$. 
Conversely \( w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \) in \( K^* \) entails \( w(y_1, \ldots, y_n) = v(y_1, \ldots, y_n) \) for all interpretations of \( y_1, \ldots, y_n \) in \( X \), each interpretation corresponding to a projection of the identity \( w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \) onto a single component.

4. Illustrative Examples

Example 1:

Let \( M \) be the machine having three states, and input alphabet \( \{a, b\} \), as indicated below:

![Diagram of a machine with three states and input alphabet \{a, b\}]

(This machine is considered by Birkhoff and Lipson in [1]).

For this machine \( M \), \( \mathcal{A}(M) \) is the subsemigroup of maps \( \{1, 2, 3\} \) generated by \( a, b \) where

\[
\overline{a}(1) = 2, \quad \overline{a}(2) = 3, \quad \overline{a}(3) = 2
\]

and \( \overline{b}(1) = 1, \quad \overline{b}(2) = 2, \quad \overline{b}(3) = 1 \)

The syntactic monoid then consists of five maps viz. 1, \( \overline{a}, \overline{b}, \overline{a}^2, \overline{ab} \), and the additional relations \( \overline{a}^3 = \overline{a}, \overline{b}^2 = \overline{b}, \overline{b.a} = \overline{a}, \overline{a^2b} = \overline{b}, \overline{aba} = \overline{a^2} \) hold.

The machine \( m = m(\mathcal{A}(M), \{\overline{a}, \overline{b}\}) \) is:
The free machine \( U_{1,2}(\mathcal{M}) = U_{1,2}(M) \) is now the semigroup machine associated with the subsemigroup of \( \mathcal{M}^4 \) generated by

\[
A = (\overline{a}, \overline{a}, \overline{b}, \overline{b}) \quad \text{and} \quad B = (\overline{a}, \overline{b}, \overline{a}, \overline{b})
\]

It has 9 elements viz.:

1, A, B, \( A^2 = (\overline{a^2}, \overline{a^2}, \overline{b}, \overline{b}) \), \( B^2 = (\overline{a^2}, \overline{b}, \overline{a^2}, \overline{b}) \),

\[
AB = (\overline{a^2}, \overline{ab}, \overline{a}, \overline{b}), \quad BAB = (\overline{a}, \overline{ab}, \overline{a^2}, \overline{b})
\]

\[
BA = (\overline{a^2}, \overline{a}, \overline{ab}, \overline{b}) \quad \text{and} \quad BA^2 = (\overline{a}, \overline{a^2}, \overline{ab}, \overline{b})
\]

The resulting semigroup machine is:

(Note that there is an error in the representation of \( U_{1,n}(M) \) given by Birkhoff and Lipson in [1], and that a similar error occurs in [2]. The relation \( AB^2 = BA^2 \) does not hold universally in \( M \) as the diagrams in [1] and [2] suggest).

Example 2:

Let \( N \) be the machine with 3 states, and input alphabet \( \{a, b\} \), as represented below:
In this case, $A(N)$ is the subgroup of the semigroup of maps $\{1,2,3\}$ consisting of all permutations, with generators $\overline{a} = (23)$, $\overline{b} = (123)$. The machine $\mathcal{M}(S(N), \{a, b\})$ is then a Cayley diagram for the symmetric group $S_3$ viz:

![Diagram]

By the previous results, $U_{1,2}(M) \cong U_{1,2}(\mathcal{M})$ is the group machine associated with the subgroup of $S_3^4$ generated by $A = (\overline{a}, \overline{a}, \overline{b}, \overline{b})$ and $B = (\overline{a}, \overline{b}, \overline{a}, \overline{b})$ generated by $A = (\overline{a}, \overline{a}, \overline{b}, \overline{b})$ and $B = (\overline{a}, \overline{b}, \overline{a}, \overline{b})$. Since $(AB)^2 = (1,1,1,1)$, and it can be shown that $(\overline{a}, \overline{a}, \overline{b})$ and $(\overline{a}, \overline{b}, \overline{a})$ generate the subgroup of $C_1 \times S_3^2$ consisting of triples $(c, p, q)$ such that $pq$ and $c$ are permutations of the same parity, it follows that $U_{1,2}(M)$ is (up to isomorphism) the group machine $\mathcal{M}(G, X)$ where $G = S_3 \times S_3 \times C_3$ and $X = \{(\overline{a}, \overline{b}, \overline{a}), (\overline{b}, \overline{a}, \overline{b})\}$. This result would be difficult to obtain by the direct method for computing $U_{1,2}(M)$ described in [1].

5. **Monoids with a universal presentation**

It has been shown in the previous section that every free finite state machine is of the form $\mathcal{M}(S, X)$ where $S$ is a finite monoid, and $X = \{x_1, \ldots, x_n\}$ is a set of generators for $S$. In this section, necessary conditions on $S$ and $X$ for $\mathcal{M}(S, X)$ to be a free finite state machine are described. For instance, it is evident that if $X$ generates $S$ universally then the set of relations between $x_1, \ldots, x_n$ holding in $S$ is invariant under any permutation of $\{1,2,\ldots,n\}$. This fact is relevant for the interpretation of the results and proofs of this section.
Lemma 3: If the elements $x_1, \ldots, x_n$ generate the finite monoid $S$ universally, then either $x_1 = x_2 = \ldots = x_n$ or $x_1, x_2, \ldots, x_n$ are pairwise distinct and generate $S$ irredundantly.

Proof: Suppose that $x_1 = w(x_2, \ldots, x_n)$. Let $f$ be the map $\{1, 2, \ldots, n\}$ such that $f(1) = 2$ and $f(i) = i$ for $i > 2$. Then $x_{f(1)} = x_2 = w(x_{f(2)}, \ldots, x_{f(n)}) = w(x_2, \ldots, x_n) = x_1$, whence $x_1 = x_2 = \ldots = x_n$.

Notation: Let $P$ be a partition of $\{1, 2, \ldots, n\}$. The rank of $P$ (the number of blocks in the partition $P$) will be denoted by $\rho(P)$.

Theorem 4: Let $S$ be a finite monoid generated universally by distinct generators $x_1, \ldots, x_n$. For each partition $P$ let $E(P)$ be the smallest congruence on $S$ such that $x_i$ and $x_j$ are congruent for all $(i, j)$ in $P$.

Then

(i) the map $E$ is a join-preserving bijection from the lattice of partitions of $\{1, 2, \ldots, n\}$ (ordered by $P \leq Q$ if $P$ is a refinement of $Q$), to the congruence lattice of $S$.

(ii) the quotient $S/E(P)$ is isomorphic with the subsemigroup $S_{\rho(P)}$ of $S$ generated by $x_1, \ldots, x_{\rho(P)}$.

Proof: (ii) Let $F$ be the monoid freely generated by $e_1, \ldots, e_n$.

Define $p : \{1, 2, \ldots, n\}$ by setting $p(i) = \text{smallest integer in the block of } F$ which contains $i$. There is a unique monoid homomorphism $\phi : F \rightarrow G$ such that $\phi(e_i) = x_{p(i)}$ for $i = 1, 2, \ldots, n$. Since $w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ implies $w(x_{p(1)}, \ldots, x_{p(n)}) = v(x_{p(1)}, \ldots, x_{p(n)})$, there is a monoid homomorphism $\phi' : G \rightarrow G$, induced by $\phi$, such that $\phi'(w(x_1, \ldots, x_n)) = w(x_{p(1)}, \ldots, x_{p(n)})$. Consider the equivalence relation $E$ on $S$ defined by $x \equiv y$ if and only if there exist $w$ and $v$ in $F$ such that $w(x_1, \ldots, x_n) = x$, $v(x_1, \ldots, x_n) = y$ and $w(x_{p(1)}, \ldots, x_{p(n)}) = v(x_{p(1)}, \ldots, x_{p(n)})$. Clearly $(i, j) \in P$ implies $(x_i, x_j) \in E$, whilst it is easy to show that $\text{Ker } \phi' = E \subseteq E(P)$. Since $E(P)$ is the smallest congruence in which $x_i$ and $x_j$ are equivalent whenever $(i, j) \in P$, it
follows that \( \text{Ker } \phi' = E(P) \). Thus \( S/E(P) \) is isomorphic to \( \text{Im } \phi' \), the submonoid of \( S \) generated by \( \{x_{p(1)}, \ldots, x_{p(n)}\} \), and this set comprises \( \rho(P) \) distinct elements.

(i) If \( P \) is a refinement of \( Q \), then certainly \( E(P) \subseteq E(Q) \).

Moreover \( P < Q \) ensures \( \rho(P) > \rho(Q) \), so \( S/E(P) \) and \( S/E(Q) \) are non-isomorphic by (ii) and the previous lemma. Thus \( P < Q \) entails \( E(P) \subseteq E(Q) \).

Now \( E(P \vee Q) \) is the congruence generated by the relations \( x_i = x_j \) for \( (i,j) \in P \vee Q \). Since \( P \vee Q \) is the smallest equivalence relation which contains both \( P \) and \( Q \), it follows that \( E(P \vee Q) = E(P \cup Q) = E(P) \vee E(Q) \), showing that \( E \) is a join-preserving map.

Suppose that \( E(P) = E(Q) \). Then \( E(P) = E(P) \vee E(Q) = E(P \vee Q) \). Since \( P \vee Q \geq P \), this implies \( Q \leq P \). Similarly \( P \leq Q \), so that \( E \) is a bijective map.

**Note:** The map \( E \) is not in general a lattice homomorphism.

Let \( S \) be universally generated by \( x_1, \ldots, x_n \), and suppose that \( x_1 \) (and thus each generator) has stem of length \( c \) and period \( t \).

Suppose that \( w(e_1, \ldots, e_n) \) and \( v(e_1, \ldots, e_n) \) are elements of length \( l(w) \) and \( l(v) \) respectively in \( F \), the monoid freely generated by \( e_1, \ldots, e_n \). If \( w(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) \) in \( S \), then \( x_1 \equiv w(x_1, \ldots, x_1) = v(x_1, \ldots, x_1) = x_1 \) \( l(v) \) whence either

(i) \( l(w) = l(v) < c \)

or (ii) \( \min (l(w), l(v)) > c \)

and \( l(w) \equiv l(v) \) (mod \( t \))

Given an element \( x \) in \( S \), it is then consistent to define the length of \( x \) as the unique number \( l(x) \) such that if \( w(x_1, \ldots, x_n) = x \) then \( l(x) \equiv l(w) \) (mod \( t \)) and \( l(x) < c + t \).
Corollary: Let $U$ be the partition of $\{1,2,\ldots,n\}$ consisting of a single block. Then $(x,y) \in E(U)$ if and only if $f(x) = f(y)$.

Proof: Let $w(x_1,\ldots,x_n) = x$ and $v(x_1,\ldots,x_n) = y$. Then $f(x) = f(y)$ if and only if $w(x_1,\ldots,x_1) = v(x_1,\ldots,x_1)$, and this is equivalent to $(x,y) \in E(U)$ as observed in the proof of part (ii) of the theorem.

6. Algebras with a universal presentation

Necessary conditions for a finite monoid to possess a universal presentation have already been described. In this section, stronger conditions are derived for special varieties of monoid.

Theorem 5: Let $S$ be an upper semilattice with least element $0$ (i.e. a monoid $(S,\vee)$ in which the binary operation $\vee$ is commutative and idempotent and $0$ is the identity element).

The generators $x_1,\ldots,x_n$ of $S$ generate a universal presentation of $S$ if and only if either $x_1 = x_2 = \ldots = x_n$ or $x_1,\ldots,x_n$ freely generate $S$ as an upper semilattice with zero element.

Proof: The sufficiency of the stated conditions is clear. Accordingly, it suffices to show that if a relation of the form

$$(U) \bigvee_{i \in A} x_i = \bigvee_{i \in B} x_i \quad A,B \subseteq \{1,2,\ldots,n\}$$

holds in $S$ then either $x_1 = x_2 = \ldots = x_n$ or $A = B$.

Assume without loss of generality that $A \not= \phi$. Then if $B = \phi$ the relation $(U)$ is of the form

$$\bigvee_{i \in A} x_i = 0$$

whence $x_i = 0$ for all $i$ in $A$, and $x_1 = x_2 = \ldots = x_n = 0$. 

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Suppose \( A, B \) both non-empty, and let \( I = A \cap B, \overline{A} = A \setminus I \) and \( \overline{B} = B \setminus I \). If \( \overline{A} = \overline{B} = \emptyset \) then \( A = B \). Otherwise assume without loss of generality that \( \overline{A} \neq \emptyset \) and let \( f : \{1,2,\ldots,n\} \) be such that

\[
\bigvee_{i \in A} x_f(i) = \bigvee_{i \in B} x_f(i)
\]

Since \((U)\) is a universal relation,

whence \( x_1 = x_2 = \ldots = x_n \). If \( I \neq \emptyset \), then \( x_2 \vee x_1 = x_2 \) whence (by universality and commutativity)

\[
x_1 = x_1 \vee x_2 = x_2 \vee x_1 = x_2.
\]

**Theorem 6:** Let \( G \) be a finite Abelian group. The elements \( g_1',\ldots,g_n' \) of \( G \) are the generators of a universal presentation if and only if for some \( t \) and some \( d \) dividing \( t \), the group \( G \) is freely generated by \( g_1',\ldots,g_n' \) subject to the relations:

\[
\begin{align*}
\forall i & \quad g_i^t = 1 \\
\forall i,j & \quad g_i g_j = g_j g_i \\
\end{align*}
\]

\[
\forall i,j \quad g_i^d = g_j^d
\]

\[
(*)
\]

**Proof:** The group \( G \) with free presentation on generators \( g_1',\ldots,g_n' \) subject to the relations \((*)\) is universally presented on generators \( g_1',\ldots,g_n' \), since the set of relations \((*)\) is closed under the application of any function \( f : \{1,2,\ldots,n\} \) to the indices of the \( g_i' \)’s.

Conversely, suppose that \( g_1',\ldots,g_n' \) universally generate \( G \) and have common order \( t \). Then \( G \) has a free presentation on \( g_1',\ldots,g_n' \) with relations

\[
\begin{align*}
\forall i & \quad g_i^t = 1 \\
\forall i,j & \quad g_i g_j = g_j g_i \\
\end{align*}
\]

and other relations of the form \( \prod_{i=1}^{n} g_i^{r_i} = 1 \). Since \( g_i^t = g_j^t = 1 \), there is a least number \( d \) such that \( g_i^d = g_j^d \) (for some, whence all
pairs of indices \((i,j)\). As \(g_i^d = g_j^d\) and \(g_i^d = g_j^d\) ensure

\[ g_i^{HCF(d,t)} = g_j^{HCF(d,t)} \]

it must be that \(d\) divides \(t\). If the

relation \(\prod_{i=1}^{n} g_i^{r_i} = 1\) holds in \(G\), then \(\prod_{i=1}^{n} g_{f(i)}^{r_i} = 1\) for all

maps \(f: \{1,2,\ldots,n\}\). In particular, if \(r = \sum_{i=1}^{n} r_i\) then \(g_1^{r} = 1\),

whence \(t\) divides \(r\). Moreover, \(g_1^{r_1} g_2^{r_1} = 1\), showing that \(g_1^{r_1} = g_2^{r_1}\)

and thus that \(d\) divides \(r_1\). By symmetry, \(d\) divides \(r_i\) for each \(i\), so

that the relation \(\prod_{i=1}^{n} g_i^{r_i} = 1\) is a consequence of the set of relations

\[ g_i^d = g_j^d \quad \text{for all} \quad i \text{ and } j. \]

**Cor.1:**

\(G\) is an Abelian group universally generated by elements \(g_1, \ldots, g_n\)
of order \(t\) if and only if \(G\) and \(C_t \times C_d^{n-1} \cong \langle \alpha \rangle \times \langle \beta \rangle^{n-1}\) are isomorphic via

the mapping \(\phi\) such that \(\phi(g_i) = (\alpha,1,\ldots,1)\) and \(\phi(g_i) = (\alpha,1,\ldots,1,\beta,1,\ldots,1)\)

for \(i = 2,3,\ldots,n\).

**Proof:** It is not difficult to show that the group freely generated by \(g_1, \ldots, g_n\)

subject to the relations (\#) is indeed isomorphic to \(C_t \times C_d^{n-1}\) via the

mapping \(\phi\).

7. **Groups with a universal presentation**

Necessary and sufficient conditions for a finite Abelian group to have a universal presentation are given in Theorem 6. The results and examples in this section relate to the harder (and unresolved) problem of determining which finite non-Abelian groups admit a universal presentation.

The following result is a corollary to Theorem 6:
Cor. 2 to Theorem 6:

Suppose that $g_1, \ldots, g_n$ generate a universal presentation for the finite group $G$. Let $G'$ be the commutator subgroup of $G$. The images $\bar{g}_1, \ldots, \bar{g}_n$ of $g_1, \ldots, g_n$ generate a universal presentation of $G/G'$. In particular, $G/G'$ is isomorphic with $C_t \times C_d^{n-1}$ for some positive integers $k$ and $d$ where $d$ divides $k$.

Proof: The elements of $G'$ are products of commutators. Thus if $w(g_1, \ldots, g_n) \in G'$ and $f : \{1, 2, \ldots, n\}$ is any map, then $w(g_{f(1)}, \ldots, g_{f(n)}) \in G'$. That is, the relations imposed upon $g_1, \ldots, g_n$ by taking the quotient by $G'$ hold universally in $G/G'$. By Theorem 6, $G/G'$ (being a finite Abelian group) is isomorphic with some $C_t \times C_d^{n-1}$.

The next result is the analogue for groups of Theorem 4.

Theorem 7: Let $G$ be a finite group generated universally by distinct generators $g_1, \ldots, g_n$. For each partition $P$, let $N(P)$ be the normal subgroup of $G$ generated by all elements of the form $g_i g_j^{-1}$ such that $(i, j) \in P$.

Then

(i) the map $N$ is a join-preserving bijection from the lattice of partitions of $\{1, 2, \ldots, n\}$ (ordered by refinement) to the lattice of normal subgroups of $G$.

(ii) the quotient $G/N(P)$ is isomorphic with the subgroup $G_{\rho(P)}$ of $G$ generated by $g_1, \ldots, g_{\rho(P)}$, and $G$ is isomorphic to a semi-direct product of $G_{\rho(P)}$ and $N(P)$.

Proof: It suffices to show that $G = G_{\rho(P)} * N(P)$; the other results are interpretations of Theorem 4.

For $i = 1, 2, \ldots, n$ let $q(i)$ be the least integer such that $(i, q(i)) \in P$. Let $\theta : G \to G$ be the group homomorphism such that $\theta(g_i) = g_{q(i)}$ (c.f. proof of Theorem 4 (ii)). Then $\text{Ker} \theta = N(P)$, and $\text{Im} \theta = <g_{q(1)}, \ldots, g_{q(n)}>$. Note that $g_i = g_{q(i)} (g_{q(i)}^{-1} g_i) \in \text{Im} \theta . \text{Ker} \theta$. 

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so that $G = \text{Im} \theta \ast \text{Ker} \theta$. Moreover, if $g \in \text{Ker} \theta \ast \text{Im} \theta$, then $\theta(g) = g = 1$.

Thus $G = \text{Im} \theta \ast \text{Ker} \theta = C_p(R) \ast N(P)$

**Corollary 1:** If $G$ has a universal presentation by generators $g_1, \ldots, g_n$ of common order $t$ then the elements of $G$ of length 0 form a normal subgroup $N$ of $G$, and $G = C_t \ast N$.

**Proof:** See the corollary to Theorem 4, and apply Theorem 7 (ii).

**Corollary 2:** If $k > 4$, then the symmetric group $S_k$ has no universal presentation.

**Proof:** Suppose $g_1, \ldots, g_n$ are permutations generating $S_k$ universally, and let $g_1, \ldots, g_n$ have common order $t$. Since $S_k = C_t \ast N$ for some normal subgroup $N$, it must be that $N = A_k$ and $t = 2$. On the other hand, in view of Theorem 7 (i), $n \leq 2$. But, if a group is generated by two elements of order 2 it is dihedral (see [4] p.49 Ex.1).

8. **Examples of groups with universal presentations**

**Example 1:**

As suggested by the proof of the previous corollary, the dihedral group $D_n$ of order $2n$ has a universal presentation by two generators of order 2, viz. $< x, y \mid x^2 = y^2 = (xy)^n = 1 >$. In particular $S_3(\approx D_3)$ is universally generated by a pair of transpositions.

**Example 2:**

Every finite Burnside group $B(t,n)$ (which is generated by $n$ elements $x_1, \ldots, x_n$ subject to relations $g^t = 1$ for every $g$ in $B(t,n)$) is universally generated by its canonical generating set.

The Burnside group $B(3,3)$ of order 2187 illustrates that the map $N$ in Theorem 7 (and likewise the map $E$ in Theorem 4) is not in general a lattice homomorphism. As described in [3], every element of $B(3,3)$ has a unique representation of the form:
where $0 \leq a_1, b_1, c \leq 2$. Let $P$ be the partition $(12)(3)$ and $Q$ the partition $(1)(23)$. The partition $P \wedge Q$ is $(1)(2)(3)$ whence $N(P \wedge Q) = \{1\}$.

But $(x_1, x_3)^2 (x_2, x_3)^2 \in N(P) \cap N(Q)$ (it reduces to 1 under adjunction of the relation $x_1 = x_2$ or $x_2 = x_3$) whence $N(P) \cap N(Q) \neq N(P \wedge Q)$.

**Example 3:**

The group $A_4$ is universally generated by $x$ and $y$ subject to the relations:

$$x^3 = y^3 = (xy)^3 = (xy^2)^2 = 1$$

The Cayley diagram associated with this presentation is:

![Cayley Diagram](image)

**Example 4:**

The group $G$ of order 56 generated by $x$ and $y$ subject to the relations

$$x^2 y x y^3 = y^2 x y x^3 = 1$$

is universally generated by $x$ and $y$, which are elements of order 7 (see [4] p.60). The semi-direct product decomposition of $G$ referred to in Corollary 2 to Theorem 7 exhibits $G$ as $C_7 \rtimes C_2^3$. 
References


