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Orbifold Riemann–Roch and Hilbert Series

by

Shengtian Zhou

Thesis

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Contents

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Declarations

Chapter 2 is mostly an introduction to known background information about stacks that we need. Chapter 3 follows the ideas in [Nir]. Chapter 4 is joint work with A. Buckley and M. Reid. Chapter 5 is original, to the best of my knowledge.

Abstract

A general Riemann–Roch formula for smooth Deligne–Mumford stacks was obtained by Toen [Toë99]. Using this formula, we obtain an explicit Riemann–Roch formula for quasismooth substacks of weighted projective space, following the ideas in [Nir]. The Riemann-Roch formula enables us to study polarized orbifolds in terms of the associated Hilbert series. Given a polarized projectively Gorenstein quasismooth pair $(\mathcal{X}, \bigoplus_{d\in\mathbb{Z}} \mathcal{O}(d))$, we want to parse the Hilbert series $P(t) = \sum_{d\geq 0} h^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(d))t^d$ according to the orbifold loci. For $\mathcal{X}$ with only isolated orbifold points, we give a parsing such that each orbifold point corresponds to a closed term, which only depends on the orbifold type of the point and has Gorenstein symmetry property and integral coefficients. Similarly, for the case when $\mathcal{X}$ has dimension ≤ 1 orbifold loci, we can also parse the Hilbert series into closed terms corresponding to orbifold curves and dissident points as well as isolated orbifold points. Our parsing of Hilbert series reflects the global symmetry property of the Gorenstein ring $\bigoplus_{d\geq 0} H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(d))t^d$ in terms of its local data.

Chapter 1

Introduction

The aim of this thesis is to rewrite the Riemann–Roch formula for stacks of Toën [Toë99], in the case of projective Gorenstein orbifolds, and using this formula to parse Hilbert series in such a way that:

1. The Hilbert series is written as a sum of terms, which are in one-to-one correspondence with the orbifold strata. Each term depends only on the local property of the stratum and the global canonical weight of the orbifold.

2. Gorenstein symmetry property and integrality are manifest in each term.

Our motivation are applications to explicit problems in algebraic geometry, such as, constructions of 3-folds and 4-folds with small invariants or given invariants, classification of 3-folds with small invariants, classification and constructions of Gorenstein rings of small codimension.

More precisely, given a polarized orbifold $(\mathcal{X}, \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_{\mathcal{X}}(d))$, the associated graded ring is given by $R = \bigoplus_{d\geq 0} H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(d))$ and its Hilbert series is defined by $P(t) = \sum_{d\geq 0} h^0(\mathcal{X}, \mathcal{O}(d))t^d$. The main goal of this thesis is to write the Hilbert series into different terms according to the orbifold loci of $\mathcal{X}$ as described above.

By *orbifolds*, we mean quasismooth substacks inside a weighted projective stack as we will introduce in Chapter 2. The reason we introduce stacks is that considering orbifolds as schemes cannot keep the information of the codimension 0 or 1 orbifold behavior. For example, $\mathbb{P}(1, 2)$, $\mathbb{P}(2, 4)$ and $\mathbb{P}^1$ are isomorphic as schemes. Under this setup, we will be able to study Hilbert series in terms of the orbifold strata of $\mathcal{X}$.

To study the Hilbert series, we need to calculate the dimension of Riemann–Roch spaces $h^0(\mathcal{O}_{\mathcal{X}}(d))$. Even though Riemann–Roch for orbifolds has been developed in various places (see, for example, [BFM75]), only a few formulas are explicit and calculable, namely, the formulas developed in [Rei87] for orbifolds with only isolated orbifold points and in [BS05] for 3-folds with orbifold curves. Moreover, none of these formulas deal with orbifolds with codimension 0 or codimension 1 orbifold loci. In Chapter 3 of this thesis, following the idea of [Nir], we obtain an explicit Riemann–Roch formula for $\mathcal{O}_{\mathcal{X}}(d)$ for quasismooth projective orbifolds in all dimensions by applying the general Riemann–Roch formula of stacks in [Toë99]. The Riemann–Roch formula for $\chi(\mathcal{O}_{\mathcal{X}}(d))$ we obtain is given as a sum over all the components of the inertia stack, that is, a sum over all the orbifold loci of $\mathcal{X}$. We call the part that corresponds to each orbifold loci the *contribution* from this orbifold loci.

Having the Riemann–Roch formula, we apply it to $\mathcal{O}_{\mathcal{X}}(d)$ in the case when $\mathcal{X}$ is a quasismooth projective orbifold and the orbifold loci on $\mathcal{X}$ only consist of isolated points. Then naturally we can also express $P(t)$ as a sum over the orbifold points of $\mathcal{X}$ by applying the Riemann–Roch formula directly. However, this parsing of the Hilbert series is not easy to control, in the sense that each of the summands is a rational function in t whose numerator is a polynomial with rational coefficients. Our main theorem 4.2.1 in Chapter 4 says that, for a quasismooth projective Gorenstein orbifold with isolated orbifold points, we can parse the Hilbert series into different parts according to the orbifold points, each of which can be determined by the type of the orbifold point and can be calculated using easy computer algebra. The parts in our parsing corresponding to the orbifold points are given by the InverseMod function, or *ice cream functions*. In Chapter 5 we extend our parsing theorem to quasismooth projectively Gorenstein orbifolds with dimension ≤ 1 orbifold loci. Studying Hilbert series helps us to construct examples of orbifolds as in [IF00] or in the graded ring database of G. Brown. Our theorems for parsing Hilbert series according to orbifold loci can be used to construct orbifolds with required invariants in an effective way. We give some examples of this applications in Section 4.7 and Section 5.5.

Chapter 2

Introduction to Stacks

In this chapter, we introduce the language of stacks to study weighted projective spaces and their subspaces. In the traditional language, weighted projective spaces and their subspaces are considered as Proj of graded rings, or as quotients of the $\mathbb{C}^*$ action on their affine cones. However, this description is in some sense coarse. Here we first give an introduction to stacks. Then we give some definitions and terminology that we use later in this thesis. Note that we assume throughout this thesis G *is abelian* as it is enough for our purpose.

2.1 Introduction to stacks

In this section, we give an introduction to stacks. In particular, we are interested in quotient stacks, as weighted projective spaces and their quasismooth subvarieties are naturally associated with quotient stack structures. Given a smooth variety X and a smooth affine group scheme G (e.g., a finite group or $\mathbb{C}^*$), we want to consider the quotient stack $[X/G]$ and find the relation between $[X/G]$ and its moduli space X/G.

Note that throughout this thesis when we talk about quotient stacks we always assume the group G is abelian, as it is enough for our purpose.

We start with the definition of stacks. In the following definition, to simplify notation, we denote by $X|_i$ the pullback f_i^*X where $f_i : U_i \to U$ and X is an object of $\mathcal{X}(U)$, and by $X_i|_{ij}$ the pullback $f_{ij,i}^*X_i$ where $f_{ij,i} : U_i \times_U U_j \to U_i$ and X_i is an object of $\mathcal{X}(U)$.

Definition 2.1.1. [Góm01] Let S be a base scheme and (Sch/S) the category of schemes over S with a Grothendieck topology. A stack is a sheaf $\mathcal{X} : (\mathrm{Sch}/S) \to$ (Groupoids) of groupoids, i.e., a pseudo-functor that satisfies the following axioms. Let $\{U_i \to U\}_{i \in I}$ be a covering of U in the site (Sch/S). Then

1. (Gluing of morphisms) If X and Y are two objects of $\mathcal{X}(U)$, and $\varphi_i\colon X|_i \to Y|_i$ are morphisms for all i such that $\varphi_i|_{ij} = \varphi_j|_{ij}$ for all i, j, then there exists a morphism $\eta\colon X \to Y$ such that $\eta|_i = \varphi_i$ for all i.

2. (Monopresheaf) If X and Y are two objects of $\mathcal{X}(U)$, and $\varphi\colon X \to Y$, $\psi\colon X \to Y$ are morphisms such that $\varphi|_i = \psi|_i$ for all i, then $\varphi = \psi$.

3. (Gluing of objects) If X_i are objects of $\mathcal{X}(U_i)$ and $\varphi_{ij}\colon X_j|_{ij} \to X_i|_{ij}$ are morphisms satisfying the cocycle condition $\varphi_{ij}|_{ijk} \circ \varphi_{jk}|_{ijk} = \varphi_{ik}|_{ijk}$ for all i, j, k, then there exists an object X of $\mathcal{X}(U)$ and $\varphi_i\colon X|_i \cong X_i$ such that $\varphi_{ji} \circ \varphi_i|_{ij} = \varphi_j|_{ij}$ for all i, j.

Definition 2.1.2. A morphism from $\mathcal{X}$ to $\mathcal{Y}$ is a natural transformation between these two functors. Two stacks $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic if and only if they are naturally isomorphic as functors.

Remark 2.1.1. A scheme can be seen naturally as a stack as follows. Given a scheme X, define the following functor:

$$\begin{aligned} X : (\mathrm{Sch}/S) &\longrightarrow (\mathrm{Sets}) \\ Y &\mapsto \mathrm{Hom}\,(Y, X). \end{aligned}$$

One can check that this functor satisfies the above conditions, and thus defines a stack. A stack is representable if it is isomorphic to a scheme. Therefore, if a stack has an object with an automorphism other than identity, then this stack cannot be represented by a scheme.

Remark 2.1.2. Given a scheme X and a stack $\mathcal{X}$, there is a categorical equivalence from $\mathrm{Hom}(X, \mathcal{X})$ to $\mathcal{X}(X)$ which sends every morphism $f\colon X \to \mathcal{X}$ to $f(\mathrm{id}_X)$ (Yoneda Lemma). Therefore, we sometimes specify a morphism from X to $\mathcal{X}$ by its corresponding element in $\mathcal{X}(X)$.

Remark 2.1.3. For any x, y in $\mathcal{X}(U)$, define $\mathrm{Isom}_U(x,y)$ to be the functor

$$\begin{array}{rcl} \mathrm{Isom}_U(x,y)\colon (\mathrm{Sch}/U) & \longrightarrow & (\mathrm{Sets}) \\ V \to U & \mapsto & \mathrm{Hom}\,(x|_V, y|_V). \end{array}$$

By conditions 1 and 2 above, we know that for a stack $\mathcal{X}$ the functor $\mathrm{Isom}_U(x,y)$ is a sheaf on the site (Sch/U).

Example 2.1.1. *Let $S = \mathrm{Spec}\,\mathbb{C}$ and let X be a scheme over S with a G action. Then $[X/G]$ is defined as a pseudo-functor*

$$[X/G]\colon (Sch/S) \to (\mathrm{Groupoids}),$$

which sends each scheme Y over S to $[X/G](Y)$ or $[X/G]_Y$, where $[X/G]_Y$ is a category whose objects are G-torsors E on Y with equivariant maps $E \to X$ and whose arrows are the morphisms between the G-torsors compatible with the map to X. Note that all morphisms between G-torsors are isomorphisms and that therefore $[X/G]_Y$ is a groupoid. Also, for each covering Y_i of Y, we can check that the above three conditions hold by the definition of the G-torsors and descent theory. Therefore $[X/G]$ is a stack.

For two morphisms of stacks $f\colon \mathcal{X} \to \mathcal{Z}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$, the fiber product $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ (or simply $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$) is defined to be the functor

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}\colon (\mathrm{Sch}/S) \to (\mathrm{Groupoids}),$$

which sends each scheme U to the category whose objects consist of triples (x, y, α), where $x \in \mathcal{X}(U)$, $y \in \mathcal{Y}(U)$ and α is a morphism $\alpha : f(x) \to g(y)$ in $\mathcal{Z}(U)$, and whose arrows between two objects (x, y, α) and (x', y', β) are given by $\phi\colon x \to x'$ and $\varphi\colon y \to y'$ such that

$$\begin{array}{ccc} f(x) & \xrightarrow{\alpha} & g(y) \\ {\scriptstyle f(\phi)}\downarrow & & \downarrow{\scriptstyle g(\varphi)} \\ f(x') & \xrightarrow[\beta]{} & g(y') \end{array}$$

commutes. One can check that this category is a groupoid and $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a stack when $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are all stacks.

A morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is called representable if for any U in

(Sch/S) and any morphism $U \to \mathcal{Y}$, the fiber product $U \times_{\mathcal{Y}} \mathcal{X}$ is representable (isomorphic to a scheme). Let "P" be a property of morphisms of schemes which is local in nature on the target for the topology chosen on (Sch/S) (e.g., étale), and which is stable under arbitrary base change, for instance, separated, quasi-compact, unramified, flat, smooth, étale, surjective, finite type, locally of finite type and so on. Then we say that a morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ satisfies "P" if it is representable, and for every $U \to \mathcal{Y}$, the pullback $U \times_{\mathcal{Y}} \mathcal{X} \to U$ satisfies "P".

For a stack to be a Deligne-Mumford stack, it must satisfy further:

Definition 2.1.3. (Deligne-Mumford stacks) Let (Sch/S) be a category of S-schemes with the étale topology. Let $\mathcal{X}$ be a stack. Assume

1. The diagonal $\Delta_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, quasi-compact and separated.
2. There exists a scheme U and an étale surjective morphism $u\colon U \to \mathcal{X}$. Here U is called an étale atlas.

Then $\mathcal{X}$ is a Deligne-Mumford stack (or simply DM stack).

Often, the most natural presentation of a DM stack has a smooth, rather than étale atlas. Thus the following criterion for a stack to be a DM stack is useful.

Proposition 2.1.1. *(Theorem* 4.21 *[DM69]) Let $\mathcal{X}$ be a stack over the étale site (Sch/S). Assume*

1. *the diagonal $\Delta_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, quasi-compact, separated and* ***unramified****.*
2. *there exists a scheme U of finite type over S and a* ***smooth*** *surjective morphism $u\colon U \to \mathcal{X}$.*

Then $\mathcal{X}$ is a Deligne-Mumford stack.

Lemma 2.1.2. *(See Proposition* 4.3.1 *[LMB00] and Example* 7.17 *[Vis89]) Let G be a smooth group scheme. Suppose G acts on X such that the stabilizer group of each of the geometric points is finite and reduced. Then $\Delta\colon [X/G] \to [X/G] \times [X/G]$ is representable, quasi-compact, separated and unramified.*

Proof First, we need to show that the diagonal map $\Delta\colon [X/G] \to [X/G] \times [X/G]$ is representable, quasi-compact, and separated. In fact, by Proposition 5.15 in

[BCE+06], we only need to show that for trivial G-torsors x and y in $[X/G]$ over a scheme U, the sheaf $\mathrm{Isom}_U(x, y)$ is representable by a scheme, quasi-affine over U.

Suppose x is equal to a trivial G-torsor $U \times G$ with an equivariant map $f : U \times G \to X$ and y is equal to $U \times G$ with an equivariant map $g : U \times G \to X$. Note that any equivariant map from $U \times G$ to X is determined by a morphism from $U \times e_G$ (where e_G is the identity of the group G) to X. Let us denote by a, b respectively the maps that determine f, g. Given a morphism $\phi : V \to U$, the category $\mathrm{Isom}_U(x, y)(V)$ consists of the isomorphisms between $\phi^* x$ and $\phi^* y$. Note that $\phi^* x$ (respectively, $\phi^* y$) is given by the trivial G-torsor $V \times G$ with equivariant map $V \times G \to X$ which is given by $f \circ (\phi, \mathrm{id})$ (respectively, $g \circ (\phi, \mathrm{id})$). An isomorphism between $\phi^* x$ and $\phi^* y$ is given by a φ in the following:

$$\begin{array}{ccc} V \times G & \xrightarrow{f \circ (\phi,\, \mathrm{id})} & X \\ \downarrow & \overset{\varphi}{\searrow} & \uparrow {\scriptstyle g \circ (\phi,\, \mathrm{id})} \\ V & \longleftarrow & V \times G \end{array}$$

where $\varphi : V \times G \to V \times G$ sending (v, g) to $(v, \alpha(v) g)$ and $\alpha : V \to G$ such that the diagram commutes, i.e. $a(\phi(v)) g = b(\phi(v)) \alpha(v) g$. Hence we see that the category $\mathrm{Isom}_U(x, y)(V)$ is equivalent to the category with objects formed of maps $\alpha : V \to G$ satisfying $a(\phi(v)) = b(\phi(v)) \alpha(v)$. On the other hand, consider the fiber product $U \times G \times_{X \times X} X$ of the morphism $U \times G \to X \times X$ sending (u, g) to $(a(u), b(u) g)$ and the diagonal map $X \to X \times X$. One can check that this scheme represents $\mathrm{Isom}_U(x, y)$ and $U \times G \times_{X \times X} X$ is quasi-affine over U.

Next, we need to show that $\Delta : [X/G] \to [X/G] \times [X/G]$ is unramified. In fact, let $P = \mathrm{Spec}\, k$ where k is an algebraically closed field. Any morphism from P to $[X/G]$ corresponds to an orbit of a geometric point $P \to X$. Given two morphisms η_1, η_2 from P to $[X/G] \times [X/G]$, we know $\mathrm{Isom}_P(\eta_1, \eta_2)$ is empty unless η_1, η_2 correspond to the same orbit, in which case $\mathrm{Isom}_U(\eta_1, \eta_2)$ is isomorphic to the scheme-theoretic stabilizer of a point in the orbit. Moreover, the following diagram is cartesian:

$$\begin{array}{ccc} \mathrm{Isom}_P(\eta_1, \eta_2) & \longrightarrow & [X/G] \\ \downarrow & & \downarrow {\scriptstyle \Delta} \\ P & \xrightarrow{(\eta_1, \eta_2)} & [X/G] \times [X/G] \end{array}$$

Thus Δ is unramified since the stabilizers of the geometric points of X are finite and reduced. □

Example 2.1.2. *(Continued) When each geometric point of X has finite reduced stabilizers, the stack $[X/G]$ is actually a Deligne-Mumford stack using the lemma and the fact that the map $p\colon X \to [X/G]$ corresponding to the trivial principal G-bundle $X \times G \to X$ is a smooth surjective morphism. In fact, for any scheme U in (Sch/S) and any map from U to $[X/G]$ corresponding to a G-torsor E over U with equivariant map to X in $[X/G]_U$, the fiber product $U \times_{[X/G]} X$ can be represented by E, that is, we have the following cartesian diagram:*

$$\begin{array}{ccc} E & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ U & \xrightarrow{E} & [X/G] \end{array}$$

Thus p is smooth and surjective.

The existence of a scheme U such that there is a smooth surjective morphism $u\colon U \to \mathcal{X}$ is important and useful since the following proposition gives us another way to describe the same stack.

Proposition 2.1.3. *([LMB00]) Let $\mathcal{X}$ be a stack, U a scheme and $u\colon U \to \mathcal{X}$ a morphism of stacks. Consider the groupoid*

$$X_1 := U \times_{\mathcal{X}} U \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} U$$

deduced canonically from the stack $\mathcal{X}$. If u is surjective, then the morphism

$$\Phi\colon [X_1 \rightrightarrows U] \to \mathcal{X}$$

is an isomorphism of stacks.

Remark 2.1.4. Here $[X_1 \rightrightarrows U]$ is the stack associated with the groupoid scheme $X_1 \rightrightarrows U$ (Section 2.4.3 and Section 3.4.2 [LMB00]). In the case of $[X/G]$, if u is the map $u\colon X \to [X/G]$ corresponding to the trivial bundle $X \times G \to X$, then the fiber product $X \times_{[X/G]} X$ is canonically isomorphic to $X \times G$. Then Proposition 2.1.3 tells us that the stack $[X \times G \rightrightarrows X]$ associated to this groupoid scheme $X \times G \rightrightarrows X$ is isomorphic to $[X/G]$. Thus knowing the group action on X or giving the presentation $X \times G \rightrightarrows X$ will determine the stack $[X/G]$ following the steps of constructing the associated stack.

Next we will introduce the inertia stack of a stack.

Definition 2.1.4. Given a stack $\mathcal{X}$, the inertia stack $I_{\mathcal{X}}$ of $\mathcal{X}$ is defined to be the fiber product $\mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X},\Delta} \times \mathcal{X}$ of the diagonal map $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$.

Remark 2.1.5. By the definition of the fiber product of stacks, the inertia stack is equivalent to the 2-functor

$$\begin{array}{rcl} I_{\mathcal{X}}\colon (\mathrm{Sch}/S) & \longrightarrow & (\mathrm{Groupoids}) \\ U & \mapsto & \{(u,\alpha) \mid u \in \mathcal{X}(U)\}, \end{array}$$

where α is an automorphism of $u \in \mathcal{X}(U)$.

Lemma 2.1.4. *Suppose an affine abelian group scheme G acts on the variety X with finite and reduced stabilizers. Then the inertia stack $I_{[X/G]}$ of the stack $[X/G]$ is isomorphic to $\coprod_{g\in G}[X^g/G]$, where $X^g = \{x \in X \mid gx = x\}$.*

PROOF The inertia stack of $[X/G]$ maps each scheme U to the category of triples (E, φ, α) where E is a G-torsor over U, φ is the equivariant map $\varphi\colon E \to X$ and α is an automorphism of the G-torsor compatible with maps to X, i.e., $\varphi = \varphi \circ \alpha$. First, suppose E is a trivial G-torsor over G, i.e., $E \cong U \times G$. Then the automorphism α is given by $(\mathrm{id}, a(u))$ where $a : U \to G$ is a morphism such that $\varphi(u, g) = \varphi(u, a(u)g)$, which implies that $\varphi(u, g) = \varphi(u, g)a(u)$. Therefore $a(u)$ lies in the stabilizer group of every image of φ. But G acts on X with only finite and reduced stabilizers, so $a(u)$ is a constant, say g_0. Hence φ factors through X^{g_0}. This gives a map from the inertia stack $I_{[X/G]}$ to $\coprod_{g\in G}[X^g/G]$ and one can check that it is an isomorphism. □

Definition 2.1.5. Let $\mathcal{X}$ be a stack. A sheaf $\mathcal{F}$ on $\mathcal{X}$ is the following data:

1. For each morphism $X \to \mathcal{X}$ where X is a scheme, a sheaf $\mathcal{F}_X$ on X.
2. For each commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \downarrow \\ & & \mathcal{X} \end{array}$$

an isomorphism $\varphi_f \colon \mathcal{F}_X \xrightarrow{\cong} f^*\mathcal{F}_Y$, satisfying the cocycle condition, i.e., for any commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{X} & & \end{array}$$

we have $\varphi_{g \circ f} = f^*\varphi_g \circ \varphi_f$.

We say that $\mathcal{F}$ is coherent if all the $\mathcal{F}_X$ are coherent. Similarly for quasi-coherent, locally free and so on. A morphism of sheaves $s \colon \mathcal{F} \to \mathcal{G}$ is defined to be a collection of morphisms $h_X \colon \mathcal{F}_X \to \mathcal{G}_X$ compatible with the isomorphisms φ.

For the stack $[X/G]$, the above definition of sheaves implies that a sheaf on $[X/G]$ determines a G-equivariant sheaf on X, and vice versa. In fact, given any $g \in G$, there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha_g} & X \\ & {\scriptstyle f'}\searrow & \downarrow{\scriptstyle f} \\ & & [X/G] \end{array}$$

where $f : X \to [X/G]$ corresponds to the trivial torsor $X \times G$ with the canonical equivariant map (the group action of X) to X, the morphism α_g is an automorphism of X induced by multiplication of $g \in G$, and $f' : X \to [X/G]$ corresponds to the trivial bundle $X \times G$ with the equivariant map given by sending $(x, a) \in X \times G$ to gax. One can check that for such morphisms f, α_g and f' this diagram commutes. Thus, by definition of sheaves on $[X/G]$, for any sheaf $\mathcal{F}$, we have an isomorphism of sheaves $\alpha_g^*\mathcal{F}_X \cong \mathcal{F}_X$ on X for any $g \in G$ satisfying the cocycle condition, which determines an equivariant structure of $\mathcal{F}$ on X. Conversely, given an equivariant sheaf $\mathcal{F}_X$ on X, for any $U \to [X/G]$ corresponding to a G-torsor E together with an equivariant morphism α to X, we can assign a sheaf on U through $\alpha^*\mathcal{F}$ by descent since $P \to U$ is smooth and surjective and having local sections.

Definition 2.1.6. A substack of $\mathcal{X}$ is a morphism of stacks $\mathcal{Y} \to \mathcal{X}$ which is represented by embedding of schemes. A substack is open, or closed, or dense, if the representing embeddings are open, or close, or dense.

Let $[X/G]$ be a quotient stack with G as a smooth affine group scheme. Let

$X \times G \rightrightarrows X$ be the canonical presentation of $[X/G]$, i.e., the first arrow represents the projection, denoted by p, and the second arrow represents the action, denoted by a. A substack of $[X/G]$ is given by a stack determined by a presentation $R \rightrightarrows U$ with induced structure from $X \times G \rightrightarrows X$, where U satisfies that $p^{-1}(U) = a^{-1}(U)$ and $R = p^{-1}(U)$. A substack is closed, or open, or dense if U is closed, or open, or dense in X.

We refer to [Vis89] for further definitions about Chow groups of an algebraic stack.

2.2 Weighted projective spaces as stacks

Now that we have introduced the language of stacks, we will use it here to study weighted projective spaces as quotient stacks. We first give some basic definitions, and then we study line bundles and tangent bundles on weighted projective stacks.

2.2.1 Basic definitions

First we define the weighted projective stack $\underline{\mathbb{P}}(a_0, \dots, a_n)$ or simply $\underline{\mathbb{P}}$ to be the quotient stack $[\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*]$ with $\mathbb{C}^*$ action

$$\lambda : (x_0, \dots, x_n) \mapsto (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n), \text{ for all } \lambda \in \mathbb{C}^*,$$

where the a_i are positive integers. It has a smooth presentation as follows:

$$\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^* \rightrightarrows \mathbb{C}^{n+1} \setminus \{0\},$$

where the upper arrow represents the first projection and the second arrow represents the action. By comparing these two arrows, we can study the stabilizer groups of the points in $\mathbb{C}^{n+1} \setminus \{0\}$. Observe that $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ with only finite reduced stabilizers. In fact, the stabilizer groups are subgroups of $\mathbb{C}^*$, which can only be μ_r, i.e., rth roots of unity. Following from the argument in Example 2.1.2, the stack $\underline{\mathbb{P}}(a_0, \dots, a_n)$ is a smooth DM stack. Here we use the notation $\underline{\mathbb{P}}(a_0, \dots, a_n)$ to distinguish it from the usual scheme version of the weighted projective space $\mathbb{P}(a_0, \dots, a_n)$.

Definition 2.2.1. A stack is projective if it is a closed substack of a weighted projective stack $\underline{\mathbb{P}}$. A projective substack of $\underline{\mathbb{P}}$ is given by $[C \setminus \{0\}/\mathbb{C}^*]$, where

C is an affine cone invariant under the $\mathbb{C}^*$ action and is defined by Spec R for some $R = \mathbb{C}[x_0, \ldots, x_n]/I$ with a weighted homogeneous ideal I in the graded ring $\mathbb{C}[x_0, \ldots, x_n]$ whose gradings are given by the action.

Remark 2.2.1. Our definition of projective stack is one of the several definitions in literature, and it is a special case, which is enough to serve our purpose.

To distinguish this from the notion $\operatorname{Proj} R$, which is equal to $C \setminus \{0\}/\mathbb{C}^*$, we denote $[C \setminus \{0\}/\mathbb{C}^*]$ by $\operatorname{Stac} R$. We say that $\operatorname{Stac} R$ is integral if Spec R is integral. Unless otherwise mentioned, the stacks considered in this thesis are always integral. We say $\operatorname{Stac} R$ is a hypersurface if I is a principal ideal, and $\operatorname{Stac} R$ is a complete intersection if I is a complete intersection. By $\mathcal{X}_d$, we mean that $\mathcal{X}_d$ is a general hypersurface of degree d, that is, I is generated by a polynomial involving all degree d monomials. Similarly, for $\mathcal{X}_{d_1,d_2}$ and so on.

As an analog of [IF00], we define quasismoothness of $\operatorname{Stac} R$ as follows.

Definition 2.2.2. Let $\operatorname{Stac} R$ be the quotient stack $[\operatorname{Spec} R \setminus \{0\}/\mathbb{C}^*]$ defined by the graded ring R. Then $\operatorname{Stac} R$ is called *quasismooth* if its affine cone $\operatorname{Spec} R$ is smooth outside the origin, or equivalently $\operatorname{Stac} R$ is a smooth DM stack.

By analogy with the definition of singularity type of cyclic quotient singularities in [Rei87], we give the definition of orbifold type.

Definition 2.2.3. A geometric point P in $\operatorname{Stac} R$, i.e., a $\mathbb{C}^*$ bundle over Spec $\mathbb{C}$ with an equivariant map to $C \setminus \{0\}$ with respect to the $\mathbb{C}^*$ action

$$\begin{array}{ccc} \operatorname{Spec} \mathbb{C} \times \mathbb{C}^* & \xrightarrow{\phi} & C \setminus \{0\} \\ \downarrow & & \\ \operatorname{Spec} \mathbb{C} & & \end{array}$$

is said to be an *orbifold point* (or an *orbipoint*) *of type* $\frac{1}{r}(b_1, \ldots, b_n)$ if the automorphism group of P is μ_r and the image of ϕ has local parameters $y_1, \ldots, y_n$ such that $\mathbb{C}^*$ acts on $y_1, \ldots, y_n$ with weights $b_1, \ldots, b_n$ respectively. We can also talk about higher dimensional orbifold loci. For example, a curve $\mathcal{C}$ is said to be an *orbicurve of type* $\frac{1}{s}(c_1, \ldots, c_{n-1})$ if every generic point on $\mathcal{C}$ is of type $\frac{1}{s}(0, c_1, \ldots, c_{n-1})$.

Definition 2.2.4. A point P lying on a curve $\mathcal{C}$ is called a dissident point if it has a bigger automorphism group than a generic point on $\mathcal{C}$. Similarly for higher dimensional cases.

Remark 2.2.2. Equivalently, we can also think about the corresponding point on the affine cone $C = \operatorname{Spec} R$. A point P on $\operatorname{Stac} R$ corresponds to an orbit of $\mathbb{C}^*$ in C. A point P is said to be an orbipoint of type $\frac{1}{r}(b_1, \ldots, b_n)$ if its orbit has local coordinates $y_1, \ldots, y_n$, and the stabilizer group of this orbit acts on $y_1, \ldots, y_n$ with weights $b_1, \ldots, b_n$. Therefore, for simplicity, when we refer to a point in $\operatorname{Stac} R$, we just specify the corresponding orbit in C (or a point on the orbit).

Remark 2.2.3. If $\operatorname{Stac} R$ is quasismooth, then the orbifold behavior can only happen on the intersection of the affine cone $\operatorname{Spec} R \setminus \{0\}$ with a coordinate plane.

Remark 2.2.4. If there are only orbifold loci of codimension ≥ 2 on $\operatorname{Stac} R$, then every orbifold point of type $\frac{1}{r}(b_1, \ldots, b_n)$ gives rise to a cyclic quotient singularity on $\operatorname{Proj} R$ of type $\frac{1}{r}(b_1, \ldots, b_n)$, and vice versa.

For the criterion of quasismoothness of $\operatorname{Stac} R$ as well as for how to find the orbifold type of the orbifold loci, we refer to [IF00].

Example 2.2.1. *Consider the weighted projective stack $\underline{\mathbb{P}}(1,3,7)$. Then the point $(0,0,1)$ is an orbipoint of type $\frac{1}{7}(1,3)$, and the point $(0,1,0)$ is an orbipoint of type $\frac{1}{3}(1,7)$. Now let f be the polynomial $x^{10} + yz$, where x, y, z are coordinates on $\mathbb{C}^3$. Then f is a homogeneous polynomial with respect to the $\mathbb{C}^*$ action. We can check that $\operatorname{Stac} \mathbb{C}[x,y,z]/(f)$ is quasismooth and has two orbipoints $(0,1,0)$ and $(0,0,1)$ of type $\frac{1}{3}(1)$ and $\frac{1}{7}(1)$ respectively.*

2.2.2 Line bundles and tangent bundles

Recall that a sheaf on $\underline{\mathbb{P}}(a_0, \ldots, a_n)$ is a $\mathbb{C}^*$-equivariant sheaf on $\mathbb{C}^{n+1} \setminus \{0\}$. In this way we can give a complete description of all line bundles on $\underline{\mathbb{P}}(a_0, \ldots, a_n)$ up to isomorphism. The only line bundle up to isomorphism on $\mathbb{C}^{n+1} \setminus \{0\}$ is of the form $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}$, and to make it equivariant, the only possible action is the following:

$$\begin{array}{rcl} \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C} \times \mathbb{C}^* & \to & \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C} \\ (x_0, \ldots, x_n, y) \times \lambda & \mapsto & (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n, \lambda^d y). \end{array}$$

One can check that this is equivariant and thus gives a line bundle on $\underline{\mathbb{P}}(a_0, \ldots, a_n)$, denoted by $\mathcal{O}_{\underline{\mathbb{P}}}(d)$. The global sections of such a line bundle are given by equivariant sections under the same action, which are the maps

$$\begin{aligned} s : \mathbb{C}^{n+1} \setminus \{0\} &\rightarrow \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C} \\ (x_0, \ldots, x_n) &\mapsto (x_0, \ldots, x_n, f(x_0, \ldots, x_n)), \end{aligned}$$

where $f(x_0, \ldots, x_n)$ is a polynomial of degree d with $\deg x_i = a_i$. Thus the dimension of the global sections $h^0(\underline{\mathbb{P}}(a_0, \ldots, a_n), \mathcal{O}_{\underline{\mathbb{P}}}(d))$ is given by the number of degree d monomials in $x_0, \ldots, x_n$, as for global sections of $\mathcal{O}_{\mathbb{P}}(d)$. Thus the graded ring $\bigoplus_{d \geq 0} H^0(\underline{\mathbb{P}}, \mathcal{O}(d))$ equals $k[x_0, \ldots, x_n]$ with $\text{weight}(x_i) = a_i$.

Next question is how one can associate the line bundle with its corresponding divisor. Let s be a section in $\Gamma(\underline{\mathbb{P}}, \mathcal{O}(d))$. Then s is an invariant function on $\mathbb{C}^{n+1}$ and thus it defines a subscheme D which is invariant under the $\mathbb{C}^*$ action. Then D determines a substack $\underline{D}$ of $\underline{\mathbb{P}}$ with the presentation $D \times \mathbb{C}^* \rightrightarrows D$. We say $\underline{D}$ is the associated divisor of $\mathcal{O}(d)$ on $\underline{D}$.

As we saw above, the line bundle $\mathcal{O}(d)$ on $\underline{\mathbb{P}}$ is locally free of rank one by definition (see Definition 2.1.5). However, it is not always associated with a Cartier divisor but associated with a $\mathbb{Q}$-divisor. For example, on the stack $\mathbb{P}(2, 4)$, the line bundle $\mathcal{O}(1)$ does not have any global sections but $\mathcal{O}(2)$ has a global section, which has an associated divisor $\underline{D}$. Therefore, $\mathcal{O}(1)$ is associated with $\frac{1}{2}\underline{D}$.

Similarly to line bundles, locally free sheaves (respectively, coherent sheaves) on $\underline{\mathbb{P}}(a_0, \ldots, a_n)$ are also given as $\mathbb{C}^*$-equivariant locally free sheaves (respectively, coherent sheaves) on $\mathbb{C}^{n+1} \setminus \{0\}$. In particular, the tangent sheaf of $\underline{\mathbb{P}}(a_0, \ldots, a_n)$ is given by the equivariant $\mathbb{C}^n$-bundle as a subbundle of the following:

$$\begin{aligned} \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{n+1} \times \mathbb{C}^* &\rightarrow \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{n+1} \\ (x_0, \ldots, x_n) \times (y_0, \ldots, y_n) \times \lambda &\mapsto (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n) \times (\lambda^{a_0} y_0, \ldots, \lambda^{a_n} y_n), \end{aligned}$$

defined by $\sum y_i \frac{\partial}{\partial x_i} = 0$ where $(\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_n})$ is the basis for $\mathbb{C}^{n+1}$ at each point. Therefore we have the following Euler exact sequence:

$$0 \rightarrow \mathcal{O}_{\underline{\mathbb{P}}} \rightarrow \mathcal{O}_{\underline{\mathbb{P}}}(a_0) \bigoplus \cdots \bigoplus \mathcal{O}_{\underline{\mathbb{P}}}(a_n) \rightarrow \mathcal{T}_{\underline{\mathbb{P}}} \rightarrow 0.$$

For the substack $\mathcal{X} = [C \setminus \{0\}/\mathbb{C}^*]$ of some weighted projective stack $\underline{\mathbb{P}}$, we can define line bundles and tangent bundles as equivariant vector bundles on the affine cone $C \setminus \{0\}$ with respect to the $\mathbb{C}^*$ action. In particular, we define $\mathcal{O}_{\mathcal{X}}(d)$ to be the restriction of $\mathcal{O}_{\underline{\mathbb{P}}}(d)$ on $\mathcal{X}$. For a quasismooth substack $\mathcal{X}$ of $\underline{\mathbb{P}}$, we also have the following exact sequence

$$0 \to \mathcal{T}_{\mathcal{X}} \to \mathcal{T}_{\underline{\mathbb{P}}}|_{\mathcal{X}} \to \mathcal{N}_{\mathcal{X}|\underline{\mathbb{P}}} \to 0.$$

Consequently, if $\mathcal{X}$ is a quasismooth codimension r substack of $\underline{\mathbb{P}}$, then the canonical sheaf of $\mathcal{X}$ can be given by the adjunction formula $\omega_{\mathcal{X}} = \omega_{\underline{\mathbb{P}}} \bigotimes \bigwedge^r \mathcal{N}_{\mathcal{X}|\underline{\mathbb{P}}}$.

2.2.3 The relation between $\operatorname{Proj} R$ and $\operatorname{Stac} R$

In this section, we want to see some relations between a quasismooth projective stack and its coarse moduli space. First let us take a look at an example.

Example 2.2.2. *Consider the simplest example $\mathbb{P}^1$. Let P be a point on $\mathbb{P}^1$. We know that the graded ring $R_1 = \bigoplus_{d\geq 0} H^0(\mathcal{O}_{\mathbb{P}^1}(\lfloor \frac{dP}{2} \rfloor))$ is given by $k[x,y]$ with x, y in degree $1, 2$ respectively. Therefore $\operatorname{Stac}\ R_1 = \underline{\mathbb{P}}(1,2)$. Since $\frac{P}{2}$ is a $\mathbb{Q}$-ample divisor on $\mathbb{P}^1$, we also know that $\operatorname{Proj} R_1 \cong \mathbb{P}^1$. Similarly, the graded ring $R_2 = \bigoplus_{d\geq 0} H^0(\mathcal{O}_{\mathbb{P}^1}(\lfloor \frac{dP}{4} \rfloor))$ is given by $k[x,y]$ with x, y in degree $1, 4$. Therefore $\operatorname{Stac} R_2 = \underline{\mathbb{P}}(1,4)$. Let P' be another point on $\mathbb{P}^1$ different from P. The graded ring $R_3 = \bigoplus_{d\geq 0} H^0(\mathcal{O}_{\mathbb{P}^1}(\lfloor \frac{dP}{3} \rfloor + \lfloor \frac{dP'}{5} \rfloor) = k[x,y,z]/(f_8)$, where x, y, z are of degree $1, 3, 5$ and f_8 is a degree 8 homogeneous polynomial. In this case, we get $\operatorname{Stac} k[x,y,z]/(f_8)$ with two orbipoints of type $\frac{1}{3}(1)$ and $\frac{1}{5}(1)$ respectively. Now we have three graded rings R_1, R_2, R_3 who have the same Proj, but $\operatorname{Stac} R_1 \neq \operatorname{Stac} R_2 \neq \operatorname{Stac} R_3$.*

From this example, we see that different polarizations of $\mathbb{P}^1$ give different graded rings. Even though they are isomorphic to each other as Proj, they are different as stacks. In fact, Theorem 3.5 in [Dem88] states this more precisely. Here we translate the theorem to our language.

Theorem 2.2.1. *(M. Demazure) If a quasismooth projective stack, given by $\operatorname{Stac} R$, has no codimension 0 orbifold loci, then there exists a $\mathbb{Q}$-Weil divisor H on $X = \operatorname{Proj} R$ such that the ring $\bigoplus_{d\geq 0} H^0(\mathcal{O}_X(\lfloor dH \rfloor))$ is isomorphic to R as graded rings.*

This theorem tells us that, if $\operatorname{Stac} R_1$ and $\operatorname{Stac} R_2$ are different, but $\operatorname{Proj} R_1$ and $\operatorname{Proj} R_2$ are equal, then R_1, R_2 are given by the graded rings associated to different $\mathbb{Q}$-Weil divisors. Analyzing the proof of this theorem, we notice that when $\operatorname{Stac} R$ has no codimension 0 or 1 orbifold loci, we can choose the divisor H to be a genuine Weil divisor and $R = \bigoplus_{d\geq 0} H^0(X, \mathcal{O}_X(dH))$.

Chapter 3

Orbifold Riemann–Roch via Stacks

B. Toën obtained a general Riemann–Roch theorem for all smooth DM stacks in [Toë99]. However, the generality of the theory makes the formula difficult to use in calculations. Nironi in [Nir] gave an explicit Riemann–Roch theorem for weighted projective stacks via this general Riemann–Roch theorem. Following the same ideas, we here obtain an explicit Riemann–Roch formula for all quasismooth substacks of weighted projective stacks.

3.1 Riemann–Roch theorem

In [Toë99], Theorem 4.10, Toën gave a Riemann–Roch formula for sheaves on smooth Deligne–Mumford stacks. In this section we are going to first recall the ideas of the proof of this general Riemann–Roch theorem, and then translate it to our case where the stacks concerned are quasismooth substacks of weighted projective stacks.

3.1.1 Idea of the Riemann–Roch formula

Here we will go through the argument working with Deligne–Mumford quotient stacks $[X/G]$ for simplicity (recall that we assume throughout G is abelian) and also because we are mainly concerned with this type of stack.

Given a Deligne–Mumford stack $\mathcal{X}$, let $\mathrm{Vect}(\mathcal{X})$ (repectively, $\mathrm{Coh}(\mathcal{X})$) be the category of vector bundles (respectively, coherent sheaves) on $\mathcal{X}$. In [Toë99], Toën uses Quillen's higher K-theory [Qui73], which defines $K_*(\mathcal{X})$ to be the homotopy groups of the classifying space BQVect and $G_*(\mathcal{X})$ to be the homotopy goups of BQCoh, see [Qui73] for details. Theroem 1 of [Qui73] says $K_0(\text{Vect}(\mathcal{X}))$ is canonically isomorphic to the Grothendieck group K_0, i.e. the free group genetated by vector bundles on $\mathcal{X}$ modulo the relation induced by exact sequences. For orbifolds, we know that every coherent sheaf is a quotient of a vector bundle and therefore the natural morphism $K_0(\mathcal{X}) \to G_0(\mathcal{X})$ is an isomorphism.

Next, we need to set up the link between vector bundles on $\mathcal{X}$ and vector bundles on its inertia stack $I_{\mathcal{X}}$. Theorem 3.15 in [Toë99] defines a map

$$\phi\colon K_0(\mathcal{X}) \to K_0(I_{\mathcal{X}}) \otimes \Lambda,$$

where $\Lambda = \mathbb{Q}(\mu_\infty)$ and μ_∞ is the group of all the roots of unity. This map is the composition of two maps. The first is $\pi^*\colon K(\mathcal{X}) \to K(I_{\mathcal{X}})$, where π is the natural map $\pi : I_{\mathcal{X}} \to \mathcal{X}$. Recall that a vector bundle $\mathcal{V}$ on $I_{\mathcal{X}}$ is given by the following data:

- To every section $s : U \to \mathcal{X}$ and every automorphism $\alpha \in \text{Aut}(s)$, where $U \in \text{Sch/S}$, one associates a vector bundle $\mathcal{V}_{s,\alpha}$ over U.

- For every pair (s, α) in $I_{\mathcal{X}}(U)$ and (s', α') in $I_{\mathcal{X}}(V)$, every morphism $f : V \to U$ of S-schemes, and every isomorphism $H : f^*(s, \alpha) \cong (s', \alpha')$, there is an isomorphism of vector bundles:

  $$\varphi_{f,H} : f^*\mathcal{V}_{s,\alpha} \cong \mathcal{V}_{s',\alpha'}.$$

- For all pair of morphisms of S-schemes

  $$W \xrightarrow{g} V \xrightarrow{f} U,$$

  all objects (s, α) in $I_{\mathcal{X}}(U)$, $(s^{'}, \alpha^{'})$ in $I_{\mathcal{X}}(V)$, $(s^{''}, \alpha^{''})$ in $I_{\mathcal{X}}(W)$ and all isomorphisms $H_1 : f^*(s, \alpha) \cong (s^{'}, \alpha^{'})$ and $H_2 : g^*(s^{'}, \alpha^{'}) \cong (s^{''}, \alpha^{''})$, there is an equality:

  $$g^*\varphi_{f,H_1} \circ \varphi_{g,H_2} \cong \varphi_{f\circ g, g^*H_1 \circ H_2}.$$

Then $\pi^* : K(\mathcal{X}) \to K(I_{\mathcal{X}})$ can be given as follows: for any vector bundle $\mathcal{V}$ on $\mathcal{X}$ (see Definition 2.1.5), $(\pi^*\mathcal{V})_{s,\alpha}$ on all pairs (s, α), with $s : U \to \mathcal{X}$ and $\alpha \in \text{Aut}(s)$, are all given by the sheaf $\mathcal{V}_U$ of $\mathcal{V}$ on the section $s : U \to \mathcal{X}$.

The second map of the composition ϕ is the map $\mathrm{dec}\colon K(I_{\mathcal{X}}) \to K(I_{\mathcal{X}}) \otimes \Lambda$ which decomposes sheaves into their eigensheaves. In fact, for all objects $(s, \alpha) \in I_{\mathcal{X}}(U)$ and automorphisms α of s in $\mathcal{X}(U)$, α defines an isomorphism $H\colon (s, \alpha) \to (s, \alpha)$ in $I_{\mathcal{X}}(U)$. Therefore by the above description, a vector bundle $\mathcal{V}_{(s,\alpha)}$ on U comes naturally with an action of the cyclic group $\langle \alpha \rangle$. Since α is of finite order r, the action can be diagonalized canonically as $\mathcal{V}_{(s,\alpha)} \cong \mathcal{V}^{(\varepsilon)}_{(s,\alpha)} \bigoplus W_{(s,\alpha)}$, where α acts on $\mathcal{V}^{(\varepsilon)}_{(s,\alpha)}$ by multiplication of ε, and ε is in the r-th roots of unity. In this way, one can define a subbundle $\mathcal{V}^{(\varepsilon)}$ of $\mathcal{V}$ on $I_{\mathcal{X}}$. The map dec sends every vector bundle $\mathcal{V}$ to the sum of eigen subbundles $\bigoplus_{\varepsilon \in \mu_\infty} \varepsilon \mathcal{V}^{(\varepsilon)}$.

Combining these two maps π^* and dec, we get $\phi = \mathrm{dec} \circ \pi^* : K_0(\mathcal{X}) \to K_0(I_{\mathcal{X}}) \otimes \Lambda$ which sets up the link between the K_0-theory of the stack and K_0-theory of its inertia stack.

These maps can be given explicitly for quotient stacks. Recall that for a quotient stack $[X/G]$, a sheaf on $[X/G]$ is equivalent to a G-equivariant sheaf on X, and $I_{[X/G]}$ is isomorphic to $\bigsqcup_{g \in G} [X^g/G]$, where X^g is the fixed locus of g for every $g \in G$. Given an G-equivariant vector bundle $\mathcal{V}$ on X, then $\mathcal{V}$ restricted to the fixed locus X^g is still G-equivariant on X^g for any g since X^g is invariant under the G action. Therefore $\mathcal{V}$ is mapped to a sheaf on $I_{[X/G]}$ by restricting to each component of the inertia stack and one can check that this is the same as $\pi^*\mathcal{V}$. Given an equivariant vector bundle $\mathcal{V}$ on X^g for some $g \in G$, g acts on X^g trivially and thus g acts on the fibers of the vector bundle. Thus $\mathcal{V}$ can be decomposed into eigensheaves $\mathcal{V} = \bigoplus_{\varepsilon \in \mu_r} \mathcal{V}^{(\varepsilon)}$, where g acts on the subsheaf $\mathcal{V}^{(\varepsilon)}$ through multiplication by ε. In this case, dec sends each $\mathcal{V}$ to the direct sum $\bigoplus_{\varepsilon \in \mu_r} \varepsilon \mathcal{V}^{(\varepsilon)} \in K([X^g/G]) \bigotimes \Lambda$.

One more concept we need to set up is the conormal bundle of the inertia stack $\mathcal{N}^*$. In the case of a quotient stack $[X/G]$, this notion is straightforward since each component of the inertia stack $\bigsqcup_{g \in G} [X^g/G]$ is naturally embedded in the originally stack $[X/G]$ and therefore the conormal bundle of each component in $[X/G]$ is well defined. In fact, the tangent sheaf $\mathcal{T}_{[X/G]}$ comes from a equivariant sheaf of X and it is naturally equivariant when restricted on X^g. The tangent sheaf of $[X^g/G]$ also results from an equivariant sheaf on X^g. Therefore the quotient of these two tangent sheaves is still equivariant on X^g, which defines the normal bundle of $[X^g/G]$ in $[X/G]$. In this way we obtain the normal bundle of the inertia stack $I_{[X/G]}$ in $[X/G]$.

Now let $\alpha_{\mathcal{X}} = \mathrm{dec}\ (\lambda_{-1}(\mathcal{N}^*))$, where $\lambda_{-1}(\mathcal{N}^*) = \sum (-1)^i \wedge^i \mathcal{N}^*$ as in [FL85]. Then Riemann–Roch can be obtained by combining the following two diagrams.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two smooth stacks. For every proper morphism $f : \mathcal{X} \to \mathcal{Y}$ the following diagram given in Lemma 4.11 in [Toë99] commutes:

$$\begin{array}{ccc} K_0(\mathcal{X}) & \xrightarrow{\alpha_{\mathcal{X}}^{-1}\phi} & K_0(I_{\mathcal{X}}) \bigotimes \Lambda \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle If_*} \\ K_0(\mathcal{Y}) & \xrightarrow[\alpha_{\mathcal{Y}}^{-1}\phi]{} & K_0(I_{\mathcal{Y}}) \bigotimes \Lambda \end{array}$$

where f_* is given by $\sum_i (-1)^i R^i f_*(-)$ and If is induced by f. Another commutative diagram given in Lemma 4.12 in [Toë99] is the following:

$$\begin{array}{ccc} K_0(I_{\mathcal{X}}) \otimes \Lambda & \xrightarrow{\mathrm{Ch}(-)\,\mathrm{Td}_{I_{\mathcal{X}}}} & A(I_{\mathcal{X}}) \otimes \Lambda \\ {\scriptstyle If_*}\downarrow & & \downarrow{\scriptstyle If_*} \\ K_0(I_{\mathcal{Y}}) \otimes \Lambda & \xrightarrow[\mathrm{Ch}(-)\,\mathrm{Td}_{I_{\mathcal{Y}}}]{} & A(I_{\mathcal{Y}}) \otimes \Lambda \end{array}$$

where Ch and Td are the Chern character and the Todd character which can be defined in the usual way. Here one can take $A(I_{\mathcal{X}})$ or $A(I_{\mathcal{Y}})$ to be the rational Chow group of $I_{\mathcal{X}}$ or $I_{\mathcal{Y}}$ defined in [Vis89], Definition 3.4. Combining these two commutative diagrams, we arrive at the Grothendieck Riemann–Roch theorem obtained by Toën.

Theorem 3.1.1. *(B.Toën) Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth stacks satisfying the following conditions:*

- *the moduli space of $\mathcal{X}$ is a quasiprojective scheme;*
- *all the coherent sheaves over $\mathcal{X}$ are quotients of locally free sheaves.*

Define the representation Todd class $\mathrm{Td}_{\mathcal{X}}^{rep}$ to be $\mathrm{Ch}(\alpha_{\mathcal{X}}^{-1})\,\mathrm{Td}_{I\mathcal{X}}$. Then for any $\mathcal{F} \in K_0(\mathcal{X})$ and any proper morphism $f : \mathcal{X} \to \mathcal{Y}$, one has:

$$If_*(\mathrm{Ch}(\phi(\mathcal{F}))\,\mathrm{Td}_{\mathcal{X}}^{rep}) = \mathrm{Ch}(\phi(f_*(\mathcal{F}))\,\mathrm{Td}_{\mathcal{Y}}^{rep}\,.$$

Remark 3.1.1. It might be possible to prove the above formula directly, without using the functorial properties of Quillen's higher K-theory.

3.1.2 Riemann–Roch formula for quasismooth projective stacks

To write down the Riemann–Roch formula for quasismooth stacks, we need to introduce some more notation (see [Nir]).

Let $\mathcal{X}$ be a quasismooth projective substack $\operatorname{Stac} R$ inside $\underline{\mathbb{P}}(a_0, \ldots, a_n)$, where $R = k[x_0, \ldots, x_n]/J$ with J a weighted homogeneous ideal. Let $I_{\underline{\mathbb{P}}}$ (resp. $I_{\mathcal{X}}$) be the inertia stack of $\underline{\mathbb{P}}$ (resp. $\mathcal{X}$). Then there is a natural embedding $I_{\mathcal{X}} \hookrightarrow I_{\underline{\mathbb{P}}}$, which is given in each component of $I_{\underline{\mathbb{P}}}$, say $\underline{\mathbb{P}}(a_{i_0}, \ldots, a_{i_m})$, by the substack $\mathcal{Y} = \operatorname{Stac} R'$, where $R' = k[x_{i_0}, \ldots, x_{i_m}]/J \cap k[x_{i_0}, \ldots, x_{i_m}]$.

Let $S = \{\text{all subsets of } \{a_0, \ldots, a_n\}\}$. The subset S_0 of S is defined as follows:

$$S_0 = \left\{ \{a_{i_0}, \ldots, a_{i_m}\} \in S \;\middle|\; \begin{array}{l} \nexists a_{j_0}, \ldots, a_{j_l} \; s.t. \\ \gcd(a_{j_0}, \ldots, a_{j_l}) = \gcd(a_{i_0}, \ldots, a_{j_m}) \text{ and} \\ \{a_{i_0}, \ldots, a_{i_m}\} \subset \{a_{i_0}, \ldots, a_{i_l}\} \end{array} \right\}.$$

In other words, it contains the subsets of $\{a_0, \ldots, a_n\}$ which are the largest among these who have the same greatest common divisors. For instance, let $S = \{1, 3, 4, 6\}$ then $S_0 = \{\{1, 3, 4, 6\}, \{4, 6\}, \{3, 6\}, \{4\}, \{6\}\}$. Moreover, for each of the subsets $s = \{a_{i_0}, \ldots, a_{i_m}\} \in S_0$ with $r = \gcd\,(a_{i_0}, \ldots, a_{i_m})$, we associate to it a set τ_s, which is defined by

$$\tau_s = \left\{ \varepsilon \in \mu_r \;\middle|\; \begin{array}{l} \varepsilon \notin \mu_q, \text{ if there exists } \{a_{j_0}, \ldots, a_{j_l}\} \in S_0 \; s.t. \\ q = \gcd\,(a_{j_0}, \ldots, a_{j_l}) \text{ and } q|r \end{array} \right\}.$$

Take the above example. To $s = \{6\} \in S_0$, we associate the set $\{\varepsilon \in \mu_6 \,|\, \varepsilon^2 \neq 1 \text{ and } \varepsilon^3 \neq 1\}$. Using these notation, the inertia stack $I_{\underline{\mathbb{P}}}$ is given by $\sqcup_{s \in S_0} (\underline{\mathbb{P}}(s) \times \tau_s)$, where $\underline{\mathbb{P}}(s) = \underline{\mathbb{P}}(a_{i_1}, \ldots, a_{i_m})$ and $x_{i_j} \in s$ with weight a_{i_j}. If we let $\mathcal{Y}_s$ be the substack of $\underline{\mathbb{P}}(s)$ defined by the ideal J, then the inertia stack $I_{\mathcal{X}}$ of $\mathcal{X}$ is given by $\sqcup_{s \in S_0} (\mathcal{Y}_s \times \tau_s)$.

Remark 3.1.2. There is another way to describe the inertia stack. Given the set of weights $S = \{a_0, \ldots, a_n\}$, let F be the set $\{\frac{l}{a_i} \,|\, 0 \leq l < a_i\}$. For each $f \in F$, there exists a subset S_f of S given by

$$S_f = \{a_i \in S \,|\, f a_i \in \mathbb{Z}\}.$$

For instance, let $S = \{1, 3, 4, 6\}$. Then $F = \{0, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}\}$. Take $f = \frac{1}{3}$

for example. We have $S_f = \{3, 6\}$. Using these notation, the inertia stack $I_{\underline{\mathbb{P}}}$ is given by $\sqcup_{f\in F}\underline{\mathbb{P}}(S_f)$, where $\underline{\mathbb{P}}(S_f) = \underline{\mathbb{P}}(a_{i_1}, \ldots, a_{i_m})$ and $a_{i_j} \in S_f$. If let $\mathcal{Y}_f$ be the substack of $\underline{\mathbb{P}}(S_f)$ defined by the ideal J, then the inertia stack $I_{\mathcal{X}}$ of $\mathcal{X}$ is given by $\sqcup_{f\in F}\mathcal{Y}_f$. This is an easier way to describe the inertia stack, but the first way is more convenient for our purpose and we will use the first in the following statements.

Having set up all the notation we need, we can state the Riemann–Roch formula for quasismooth stacks.

Proposition 3.1.2. *Let $\mathcal{X}$ be a quasismooth substack in a weighted projective stack $\underline{\mathbb{P}}(a_0, \ldots, a_n)$, and let $\mathcal{V}$ be a vector bundle on $\mathcal{X}$. Using the above notation, one has*

$$\chi(\mathcal{V}) = \sum_{s\in S_0}\sum_{\varepsilon\in\tau_s}\Big[\frac{\mathrm{Ch}(\phi(\mathcal{V}))\,\mathrm{Td}\,\mathcal{Y}_s}{\mathrm{Ch}(\lambda_{-1}(\mathrm{dec}\,(\mathcal{N}_s^*)))}\Big]_{\dim\mathcal{Y}_s},$$

where $\mathcal{N}_s^*$ is the conormal bundle of Y_s inside X and $[-]_{\dim Y_s}$ represents the codimension $\dim Y_s$ part in the Chow group. In particular, when $\mathcal{V} = \mathcal{O}(d)$, then

$$\chi(\mathcal{O}(d)) = \sum_{s\in S_0}\sum_{\varepsilon\in\tau_s}\Big[\frac{\varepsilon^d\,\mathrm{Ch}(\mathcal{O}(d))\,\mathrm{Td}\,\mathcal{Y}_s}{\mathrm{Ch}(\lambda_{-1}(\mathrm{dec}\,(\mathcal{N}_s^*)))}\Big]_{\dim\mathcal{Y}_s}.$$

Proof In Theorem 3.1.1, if we take $\mathcal{Y}$ to be a point, we will get the Hizebruch–Riemann–Roch formula for a vector bundle $\mathcal{V} \in K_0(X)$. In the first diagram above Theorem 3.1.1, the map $\alpha_{\mathcal{X}}^{-1}\phi$ sends $\mathcal{V}$ to a direct sum of sheaves on $I_{\mathcal{X}}$, and in the second diagram, we can calculate Ch and Td componentwise on $I_{\mathcal{X}}$. Then we obtain the Riemann–Roch formula for vector bundles on $\mathcal{X}$. In particular, if $\mathcal{V} = \mathcal{O}(d)$, then for each $\mathcal{Y}_s$ and each element $\varepsilon \in \tau_s$, one has $\phi(\mathcal{V}) = \varepsilon^d\mathcal{V}|_{\mathcal{Y}_s}$. □

Remark 3.1.3. Let $\mathcal{Y}_s$ be one of the components of the inertia stack of $I_{\mathcal{X}}$ and τ_s the set associated to it. For each $\varepsilon \in \tau_s$, ε acts on the normal bundle $\mathcal{N}_s$ of $\mathcal{Y}_s$ and decomposes the normal bundle into eigen-subbundles $\bigoplus_{i=1}^{l}\mathcal{N}_i$, where $\mathcal{N}_i$ are not necessarily line bundles. Therefore, we have to work with Chern roots of $\mathcal{N}_i$ when it has rank ≥ 2. Note that every Chern root of $\mathcal{N}_s$ comes with an eigen value. We will use this implicitly in our statements in the following.

Suppose the normal bundle $\mathcal{N}_s$ of rank r in $\mathcal{X}$ of $\mathcal{Y}_s$ can be decomposed into the direct sum $\bigoplus_{i=1}^{l}\mathcal{N}_i$ under the group $\langle\varepsilon\rangle$ action for each $\varepsilon \in \tau_s$, and each $\mathcal{N}_i$ has eigenvalue ε^{a_i}, then the denominator of the formula in the proposition can be written as

$$\begin{aligned}\mathrm{Ch}(\lambda_{-1}(\mathrm{dec}(\mathcal{N}^*))) &= \mathrm{Ch}(\lambda_{-1}(\bigoplus_{i=1}^{l} \varepsilon^{-a_i}\mathcal{N}_i^*)) \\ &= \mathrm{Ch}(\prod_{i=1}^{l}(1-\varepsilon^{-a_i}\mathcal{N}_i^*)) = \prod_{i,j}(1-\varepsilon^{-a_i}e^{-v_{i_j}}),\end{aligned}$$

where v_{i_j} are the Chern roots of $\mathcal{N}_i$. Moreover, we can express the inverse

$$\frac{1}{(1-\varepsilon^{-a_i}e^{-v_{i_j}})} = \frac{1}{1-\varepsilon^{-a_i}} - \frac{\varepsilon^{-a_i}}{(1-\varepsilon^{-a_i})^2}v_{i_j} + \\ (\frac{\varepsilon^{-a_i}}{(1-\varepsilon^{-a_i})^3} - \frac{\varepsilon^{-a_i}}{2(1-\varepsilon^{-a_i})^2})v_{i_j}^2 + \text{higher order terms} .$$

This expression is very useful, as we will see in the concrete cases below.

Using the formula in Proposition 3.1.2 and the above remark, for quasismooth stacks with concrete orbifold loci one can express this formula in terms of Dedekind sums. Suppose a quasismooth stack $\mathcal{X}$ of dimension n has only isolated orbipoints. Let $\mathcal{B} = \{P \text{ of type } \frac{1}{r}(b_1, \ldots, b_n)\}$ be the collection of all the isolated orbipoints. Then the formula in Proposition 3.1.2 can be written as:

Corollary 3.1.3. *Given $\mathcal{X}$ as above, the Riemann–Roch formula for $\mathcal{O}_\mathcal{X}(d)$ is given by*

$$\chi(\mathcal{O}_\mathcal{X}(d)) = [\mathrm{Ch}(\mathcal{O}_\mathcal{X}(d))\,\mathrm{Td}_\mathcal{X}]_n + \sum_{P\in\mathcal{B}} \frac{1}{r} \sum_{\varepsilon\in\mu_r,\varepsilon\neq 1} \frac{\varepsilon^d}{\prod_i(1-\varepsilon^{-b_i})}. \tag{3.1}$$

Proof In this case, the only components for the inertia stack are the stack itself and the orbipoints. Each of the orbipoints of type $\frac{1}{r}(b_1, \cdots, b_n)$ is associated with $r-1$ components of the inertia stack, namely $\sqcup_{\varepsilon\in\mu_r,\varepsilon\neq 1}[C(P)/\mathbb{C}^*]\times\varepsilon$, where $C(P)$ is the orbit of P. Now consider one of the components $[C(P)/\mathbb{C}^*]\times\varepsilon$ corresponding to a singular point of type $\frac{1}{r}(b_1, \ldots, b_n)$ with normal bundle $\mathcal{N}$. Then one has

$$\mathrm{Ch}(\lambda_{-1}(\mathrm{dec}\ (\mathcal{N}^*))) = \prod_i(1-\varepsilon^{-b_i}e^{-v_i}),$$

where v_i are the Chern roots of $\mathcal{N}$. Since each component $[C(P)/\mathbb{C}^*]\times\varepsilon$ is of dimension 0, we have that $\mathrm{Ch}(\phi(\mathcal{O}(d))) = \varepsilon^d$ and $\mathrm{Td} = 1$. Therefore,

$$\Big[\frac{\varepsilon^d \,\mathrm{Ch}(\mathcal{O}(d))\,\mathrm{Td}_{\mathcal{Y}_s}}{\mathrm{Ch}(\lambda_{-1}(\mathrm{dec}(\mathcal{N}_s^*)))}\Big]_0 = \frac{1}{r}\frac{\varepsilon^d}{\prod_i (1-\varepsilon^{-b_i})},$$

where $\frac{1}{r}$ is the degree of the point. Summing over all the components we get the formula. □

Remark 3.1.4. Note for $d = 0$, one obtains

$$\chi(\mathcal{O}_{\mathcal{X}}) = \mathrm{Td}_n + \sum_{P\in\mathcal{B}} \frac{1}{r} \sum_{\varepsilon\in\mu_r,\varepsilon\neq 1} \frac{1}{(1-\varepsilon^{-b_1})\cdots(1-\varepsilon^{-b_n})},$$

where Td_n represents the top Todd class of $\mathcal{X}$. Thus replacing the Td_n via the above equality (3.1) gives the same formula as in [Rei87].

Now suppose $\mathcal{X}$ has orbifold loci of dimension ≤ 1, and it has

- a set of orbicurves $\mathcal{B}_C = \{$ orbicurves of type $\frac{1}{r}(a_1, \ldots, a_{n-1})\}$, and
- a set of orbipoints $\mathcal{B}_P = \{$orbipoints of type $\frac{1}{s}(b_1, \ldots, b_n)\}$.

In this case, we have $I_{\mathcal{X}} = \mathcal{X} \sqcup_{\mathcal{B}_C} (\sqcup_{\varepsilon\in\mu_r,\varepsilon\neq 1}\mathcal{C}\times\varepsilon) \sqcup_{\mathcal{B}_P} (\sqcup_{\varepsilon\in\mu_s,\varepsilon^{b_i}\neq 1} P\times\varepsilon)$ and the Riemann–Roch formula is given by

Corollary 3.1.4. *Given such an $\mathcal{X}$ with only orbifold loci of dimension ≤ 1, one has*

$$\chi(\mathcal{O}_{\mathcal{X}}(d)) = [\mathrm{Ch}(\mathcal{O}_{\mathcal{X}}(d))\,\mathrm{Td}_{\mathcal{X}}]_n + \sum_{P\in\mathcal{B}_P} M_P + \sum_{\mathcal{C}\in\mathcal{B}_C} M_{\mathcal{C}},$$

where M_P for a point P of type $\frac{1}{s}(b_1, \ldots, b_n)$ is given by

$$\frac{1}{s}\sum_{\varepsilon\in\mu_r,\varepsilon^{-b_i}\neq 1} \frac{\varepsilon^d}{\prod_i(1-\varepsilon^{-b_i})},$$

while $M_{\mathcal{C}}$ for a curve $\mathcal{C}$ of type $\frac{1}{r}(a_1, \ldots, a_{n-1})$ is given by

$$\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1} \frac{\varepsilon^d}{\prod(1-\varepsilon^{-a_i})} d \deg H|_{\mathcal{C}} - \frac{1}{2r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1} \frac{\varepsilon^d}{\prod(1-\varepsilon^{-a_i})} \deg K_{\mathcal{C}}$$
$$-\sum_{i=1}^{n-1}\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1} \frac{\varepsilon^{d-a_i}}{(1-\varepsilon^{-a_i})^2\prod_{j\neq i}(1-\varepsilon^{-a_j})} \deg\gamma_i,$$

where H is the divisor (possibly $\mathbb{Q}$-divisor) corresponding to the sheaf $\mathcal{O}_{\mathcal{X}}(1)$, and γ_i's are the Chern roots of the normal bundle $\mathcal{N}$.

Proof As in the proof of Corollary 3.1.3 we obtain the part coming from orbifold points M_P. An orbicurve of type $\frac{1}{r}(a_1, \ldots, a_{n-1})$ will give rise to $r-1$ components in the inertia stack of $\mathcal{X}$, namely, $\sqcup_{\varepsilon \in \mu_r, \varepsilon \neq 1} \mathcal{C} \times \varepsilon$. Then for the component $\mathcal{C} \times \varepsilon$ we will have

$$\begin{aligned}
& [\frac{\varepsilon^d \operatorname{Ch}(\mathcal{O}(d)) \operatorname{Td}_{\mathcal{C}}}{\operatorname{Ch}(\lambda_{-1}(\mathcal{N}^*))}]_1 \\
= \; & [(1 + dH|_{\mathcal{C}})(1 + \frac{1}{2} c_1(\mathcal{T}_C)) \prod_{i=1}^{n-1} (\frac{1}{1-\varepsilon^{-a_i}} - \frac{\varepsilon^{-a_i}}{(1-\varepsilon^{-a_i})^2} \gamma_i)]_1 \\
= \; & \frac{\varepsilon^d (dH|_{\mathcal{C}} + \frac{1}{2} c_1(\mathcal{T}_C))}{\prod_{i=1}^{n-1} (1-\varepsilon^{-a_i})} - \sum_{i=1}^{n-1} \frac{\varepsilon^{d-a_i}}{(1-\varepsilon^{-a_i})^2 \prod_{j \neq i} (1-\varepsilon^{-a_j})} \deg \gamma_i,
\end{aligned}$$

where H is the $\mathbb{Q}$-divisor corresponding to $\mathcal{O}(1)$, $\mathcal{T}_{\mathcal{C}}$ is the tangent sheaf of $\mathcal{C}$, and γ_i are the Chern roots of $\mathcal{N}$. Summing these over the $r-1$ components in the inertia stack, we get the above formula. □

Remark 3.1.5. In the above formula, by abuse of notation, we write $\deg H|_{\mathcal{C}}$ for the number given by the intersection number of rH with $\mathcal{C}$, because in this way the coefficients can be given in the form of Dedekind sums as in Section 3.2. Similarly for $\deg K_{\mathcal{C}}$, the $\deg K_{\mathcal{C}}$ here is given by r times degree of the divisor $K_{\mathcal{C}}$, where $K_{\mathcal{C}}$ is the canonical divisor of $\mathcal{C}$ as a stack. For example, $\mathcal{C} = \mathbb{P}(2,4)$ has $\deg K_{\mathcal{C}} = 2 \times (-\frac{6}{8}) = -\frac{3}{2}$. We will also use the same convention in the following.

Remark 3.1.6. Here if let $d = 0$, then one obtains again the relation between $\chi(\mathcal{O}_{\mathcal{X}})$ and Td_n of $\mathcal{X}$ as follows:

$$\begin{aligned}
\chi(\mathcal{O}_{\mathcal{X}}) \; = \; & \operatorname{Td}_n + \sum_{P \in \mathcal{B}_P} \frac{1}{s} \sum_{\varepsilon \in \mu_r, \varepsilon^{-b_i} \neq 1} \frac{1}{\prod_i (1-\varepsilon^{-b_i})} + \sum_{C \in \mathcal{B}_C} (-\frac{1}{2r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{1}{\prod (1-\varepsilon^{-a_i})} \\
& \deg K_C - \sum_{i=1}^{n-1} \frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^{-a_i}}{(1-\varepsilon^{-a_i})^2 \prod_{j \neq i} (1-\varepsilon^{-a_j})} \deg \gamma_i),
\end{aligned}$$

where γ_i are as in Corollary 3.1.4. Also note that for each $1 \le i \le n-1$ we have

$$\begin{aligned}
& \frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^{d-a_i} - \varepsilon^{-a_i}}{(1-\varepsilon^{-a_i})^2 \prod_{j \neq i}(1-\varepsilon^{-a_j})} \\
= & \frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{(\varepsilon^{d-a_i} - \varepsilon^d + \varepsilon^d - 1 - (\varepsilon^{-a_i} - 1))}{(1-\varepsilon^{-a_i})^2 \prod_{j \neq i}(1-\varepsilon^{-a_j})} \\
= & -\frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^d - 1}{\prod_i (1-\varepsilon^{-a_i})} + \frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^d - 1}{(1-\varepsilon^{-a_i})^2 \prod_{j \neq i}(1-\varepsilon^{-a_j})}.
\end{aligned}$$

By the adjunction formula we also have $K_{\mathcal{C}} = K_{\mathcal{X}} \otimes \wedge^{n-1}\mathcal{N}$. Then putting the above equality into the formula in Corollary 3.1.4 we obtain the following:

$$\chi(\mathcal{O}_X(d)) = \chi(\mathcal{O}_X) + RR + \sum_{P \in \mathcal{B}_P} M'_P + \sum_{C \in \mathcal{B}_C} M'_C$$

where $RR = ([\mathrm{Ch}(\mathcal{O}_X(d))\,\mathrm{Td}_X]_n - \mathrm{Td}_n)$, for a point of singularity type $\frac{1}{r}(b_1, \dots, b_n)$ the term M'_P is given by

$$\frac{1}{s} \sum_{\varepsilon \in \mu_r, \varepsilon^{-b_i} \neq 1} \frac{\varepsilon^d - 1}{\prod_i (1-\varepsilon^{-b_i})},$$

and for the curve C of type $\frac{1}{r}(a_1, \dots, a_{n-1})$ the term M'_C is given by

$$\begin{aligned}
& \frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^d}{\prod(1-\varepsilon^{-a_i})} \deg dH|_{\mathcal{C}} - \frac{1}{2r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^d - 1}{\prod(1-\varepsilon^{-a_i})} \deg K_{\mathcal{X}}|_{\mathcal{C}} \\
& - \sum_{i=1}^{n-1} \frac{1}{2r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{(\varepsilon^d - 1)(1+\varepsilon^{-a_i})}{(1-\varepsilon^{-a_i})^2 \prod_{j \neq i}(1-\varepsilon^{-a_j})} \deg \gamma_i.
\end{aligned}$$

Of course, we can also write out the formula for quasismooth stacks with orbifold loci of dimension ≥ 2 in the same way. Maybe for later reference, we just write out the formula for varieties with dimension ≤ 2 orbifold loci without much explanation.

Corollary 3.1.5. *Let $\mathcal{X}$ be a quasismooth stack of dimension n. Suppose that $\mathcal{X}$ has the following orbifold loci:*

- *the orbipoints $\mathcal{B}_P = \{P$ of type $\frac{1}{s}(b_1, \dots, b_n)\}$,*

- *the orbicurves* $\mathcal{B}_{\mathcal{C}} = \{\mathcal{C}$ *of type* $\frac{1}{r}(a_1, \dots, a_{n-1})\}$*,*
- *the orbisurfaces (orbifold loci of dimension* 2*)* $\mathcal{B}_{\mathcal{S}} = \{\mathcal{S}$ *of type* $\frac{1}{l}(h_1, \dots, h_{n-2})\}$*.*

Then one has

$$\chi(\mathcal{O}(d)) = [\mathrm{Ch}(\mathcal{O}(d))\,\mathrm{Td}_X]_n + \sum_{P\in\mathcal{B}_P} M_P + \sum_{\mathcal{C}\in\mathcal{B}_{\mathcal{C}}} M_{\mathcal{C}} + \sum_{\mathcal{S}\in\mathcal{B}_{\mathcal{S}}} M_{\mathcal{S}},$$

where M_P *and* $M_{\mathcal{C}}$ *are given as in Corollary 3.1.4, and* $M_{\mathcal{S}}$ *for a surface of singularity type* $\frac{1}{l}(h_1, \dots, h_{n-2})$ *is given by summing the following over* $\varepsilon \in \mu_l, \varepsilon \neq 1$*:*

$$\begin{aligned}
&[\frac{\mathrm{Ch}\,(\mathcal{O}(d))\,\mathrm{Td}_{\mathcal{S}}}{\prod(1-\varepsilon^{-h_i}e^{-\gamma_i})}]_2 = \\
&\frac{\varepsilon^d}{\prod_{i=1}^{n-2}(1-\varepsilon^{-h_i})}\big(\frac{1}{12}(c_1(\mathcal{S})^2 + c_2(\mathcal{S})) + \frac{1}{2}dH(dH + c_1(\mathcal{S}))\big) \\
&+\sum_{i=1}^{n-2}\frac{\varepsilon^{d-a_i}}{(1-\varepsilon^{-a_i})^2\prod_{j\neq i}(1-\varepsilon^{-a_j})}(\frac{1}{2}c_1(\mathcal{S}) + dH)\gamma_i \\
&+\sum_{1\le i<j\le n-2}\frac{\varepsilon^{d-a_i-a_j}}{(1-\varepsilon^{-a_i})^2(1-\varepsilon^{-a_j})^2\prod_{k\neq i,k\neq j}(1-\varepsilon^{-a_k})}\gamma_i\gamma_j \\
&+\sum_{i=1}^{n-2}(\frac{\varepsilon^{d-a_i}}{(1-\varepsilon^{-a_i})^3\prod_{j\neq i}(1-\varepsilon^{-a_j})} - \frac{\varepsilon^{d-a_i}}{2(1-\varepsilon^{-a_i})^2\prod_{j\neq i}(1-\varepsilon^{-a_j})})\gamma_i^2,
\end{aligned}$$

where H *is the* $\mathbb{Q}$*-divisor corresponding to* $\mathcal{O}(1)$ *and* γ_i *are the Chern roots of* $\mathcal{N}$*.*

Remark 3.1.7. The general formula as in Proposition 3.1.2 should be possible to obtain for “quasismooth” substacks of toric stacks since a toric stack can also be seen as a global quotient stack (see for example [FMN07]).

3.2 Calculating Dedekind sums

Before going any further, we would like to study the Dedekind sums appeared in the formulas so that we will be able to characterize and calculate them. Here by

Dedekind sum, we mean a sum of the form:

$$\begin{aligned}\sigma_i(\frac{1}{r}(a_1,\ldots,a_n)) &= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon^{a_i}\neq 1}\frac{\varepsilon^i}{(1-\varepsilon^{-a_i})\ldots(1-\varepsilon^{-a_n})}\\ &= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon^{a_i}\neq 1}\frac{\varepsilon^{-i}}{(1-\varepsilon^{a_i})\cdots(1-\varepsilon^{a_n})},\end{aligned}$$

where $(a_1,\ldots,a_n)$ is a sequence of positive integers such that $a_i \bmod r \neq 0$ for all i. Such sums are closely related to traditional Dedekind sums, thus we still refer it as the ith Dedekind sum, denoted by $\sigma_i(\frac{1}{r}(a_1,\ldots,a_n))$ or simply σ_i. We write δ_i for $\sigma_i-\sigma_0$, that is,

$$\delta_i = \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon^{a_i}\neq 1}\frac{\varepsilon^{-i}-1}{(1-\varepsilon^{a_i})\cdots(1-\varepsilon^{a_n})}.$$

When $n=1$ and $(a,r)=1$, there is a compact expression for $\delta_i(\frac{1}{r}(a))$.

Lemma 3.2.1. *When $(a,r)=1$,*

$$\delta_i(\frac{1}{r}(a)) = \sigma_i(\frac{1}{r}(a)) - \sigma_0(\frac{1}{r}(a)) = \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{-i}-1}{1-\varepsilon^a} = -\frac{\overline{bi}}{r},$$

where b is the inverse of a modulo r, i.e., $ab=1$ mod r. In particular, this gives

$$\sigma_0(\frac{1}{r}(a)) = \frac{r-1}{2r}.$$

Proof Let $ab = 1$ mod r, then $(\varepsilon^a)^{\overline{bi}} = \varepsilon^i$, where $\overline{bi}$ represents the smallest nonnegative residue of bi modulo r (similarly in what follows). Thus

$$\begin{aligned}\varepsilon^{r-i}-1 &= (\varepsilon^a)^{r-\overline{bi}}-1\\ &= ((\varepsilon^a)^{r-\overline{bi}-1}+\cdots+1)(\varepsilon^a-1).\end{aligned}$$

Note that $\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\varepsilon^m = -1$ for all $m\neq 0$. Then

$$\begin{aligned}\delta_i = \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{-i}-1}{1-\varepsilon^a} &= -\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}((\varepsilon^a)^{r-\overline{bi}-1}+\cdots+1)\\ &= -\frac{1}{r}(\underbrace{(-1+\cdots+(-1))}_{r-\overline{bi}-1}+r-1) = -\frac{\overline{bi}}{r}.\end{aligned}$$

Moreover since $\sum_{i=0}^{r-1} \sigma_i(\frac{1}{r}a) = 0$, one has

$$\sigma_0 = \frac{\sum_{i=0}^{r-1} \overline{bi}}{r^2} = \frac{r-1}{2r},$$

because b is coprime to r and thus $\overline{bi}$ will run over $1, \ldots, r-1$ for $0 \leq i \leq r-1$.

Example 3.2.1. *Take $r = 5$, $a = 3$, and one has $b = 2$. Thus for $i = 1, \ldots, 4$, the $\delta_i(\frac{1}{5}(a))$ are: $-2/5, -4/5, -1/5, -3/5$.*

The following proposition allows us to calculate all σ_i in general (see also [Buc07] for a different proof).

Proposition 3.2.2. *Given positive integers r and $a_1, \ldots, a_n$ such that a_i are not divisible by r, let $h = \gcd(\prod_{j=1}^{n}(1 - t^{a_j}), \frac{1-t^r}{1-t})$. Then $\sum_{i=0}^{r-1} \sigma_i t^i$ is the inverse of $\prod_{j=1}^{n}(1 - t^{a_j})$ modulo $\frac{1-t^r}{h(1-t)}$, that is,*

$$(\sum_{i=0}^{r-1} \sigma_i t^i) \prod_{j=1}^{n}(1 - t^{a_j}) = 1 \bmod \frac{1 - t^r}{h(1-t)}.$$

Proof Observe that

$$\frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} (1 + \varepsilon^{-1}\zeta + \cdots + \varepsilon^{-(r-1)}\zeta^{r-1}) \frac{(1 - \zeta^{a_1}) \cdots (1 - \zeta^{a_n})}{(1 - \varepsilon^{a_1}) \cdots (1 - \varepsilon^{a_n})} = 1$$

for all $\zeta \in \mu_r$, $\zeta^{a_i} \neq 1$, $\zeta \neq 1$. In fact, when $\varepsilon \neq \zeta$ we have $\sum_{i=0}^{r-1}(\zeta^{-1}\varepsilon)^i = 0$ as $\zeta^{-1}\varepsilon$ is still a rth roots of unity, and when $\varepsilon = \zeta$ we have $\sum_{i=0}^{r-1}(\zeta^{-1}\varepsilon)^i = r$. Thus we have shown that for all the roots of $\frac{1-t^r}{h(1-t)}$ the left hand side of the equality equals 1, which is equivalent to:

$$(\sum_{i=0}^{r-1} \sigma_i t^i)(1 - t^{a_1}) \cdots (1 - t^{a_n}) = 1 \bmod \frac{1 - t^r}{h(1-t)}.$$

We are done. □

Using this proposition, we can calculate $\sigma_i(\frac{1}{r}(a_1, \ldots, a_n))$ by a computer program. In fact, since $h = \gcd(\frac{1-t^r}{1-t}, \prod_{j=1}^{n}(1 - t^{a_j}))$, by the Euclidean algorithm there exists a unique $\alpha(t)$ of degree $\leq r - \deg h - 2$ and $\beta(t) \in \mathbb{C}[t]$ (in fact, $\alpha(t)$

and $\beta(t)$ are in $\mathbb{Q}[t]$) such that

$$\alpha(t)\prod_{i=1}^{n}(1-t^{a_i})+\beta(t)\frac{1-t^r}{h(1-t)}=1.$$

This implies that $\alpha(t)$ is also the inverse of $\prod_{i=1}^{n}(1-t^{a_i})$ modulo $\frac{1-t^r}{h(1-t)}$, and therefore $\alpha(t)=\sum_{i=1}^{n}\sigma_i t^i \mod \frac{1-t^r}{h(1-t)}$, i.e.,

$$\sum_{i=0}^{r-1}\sigma_i t^i=\alpha(t)+f(t)\frac{1-t^r}{h(1-t)},$$

where $f(t)$ is a polynomial of degree $\deg h$. In particular, $f(t)$ is a constant when $h=1$. If $h\neq 1$, then $f(t)$ will have $\deg h+1$ undetermined coefficients. Thus we need $\deg h+1$ relations among the coefficients of the right hand side to determine $f(t)$ and hence σ_i. Note that for each $w_i=(a_i,r)\neq 1$ and any $\varepsilon\in\mu_r$, one has $1+\varepsilon^{w_i}+\cdots+\varepsilon^{w_i(r/w_i-1)}=0$. Thus

$$\sum_{l=0}^{r/w_i-1}\sigma_{w_i l+k}=\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon^{a_i}\neq 1}\frac{(1+\varepsilon^{w_i}+\cdots+\varepsilon^{r-w_i})\varepsilon^k}{(1-\varepsilon^{a_i})\cdots(1-\varepsilon^{a_n})}=0.$$

for $k=0,1,\ldots,w_i-1$. Then for every such w_i there are w_i-1 independent relations. Suppose w_{i_j} $(j=1,\ldots,l)$ are all such w_i, then we have $\sum_{j=1}^{l}(w_{i_j}-1)=\deg h$ relations between the σ_i's. One more relation comes from the fact that $\sum_{i=0}^{r-1}\sigma_i=0$. Therefore we have in total $\deg h+1$ independent relations among σ_i, which gives us enough linear equations to determine $f(t)$ and hence σ_i. This in particular implies σ_i's are rational numbers. The following MAGMA program uses above ideas and output $\sigma_0,\ldots,\sigma_{r-1}$ if we input r and the sequence $LL=[a_1,\ldots,a_n]$.

Program 3.2.3.
```
function  Contribution(r, LL)
QQ:=Rationals();
Poly<t>:=PolynomialRing(QQ);
L:=[Integers()|i: i in LL]; n:=#LL;
pi:=&*[(1-t^i):i in L]; A:=Poly!((1-t^r)/(1-t));
G:=GCD(pi, A); dG:=Degree(G);
B:=Poly!(A/G); dB:=Degree(B);
a,be,c:=XGCD(pi, B); dbe:=Degree(be);
R<[v]>:=PolynomialRing(QQ,dG+2);
```

```
va:=Name(R,dG+2);
bnew:=&+[Coefficient(be,i)*va^i: i in [0..dbe]];
RR:=&+[v[i]*va^(i-1):  i in [1..dG+1]];
Bnew:=&+[Coefficient(B,i)*va^i: i in [0..dB]];
AA:=bnew-RR*Bnew;
S:=[Coefficient(AA,va, 0)] cat [Coefficient(AA, va, r-i): i in [1..r-1]];
empty:=[];
for a in L do
dd:=GCD(a,r); tt:=r/dd;
relations:=empty cat [&+[S[dd*l+i]: l in [0..tt-1]]: i in [1..dd]];
empty:=relations;
end for;
Mat:=Matrix(QQ,[[Coefficient(empty[i],v[j],1):j in [1..dG+1]]:i in
[1..#empty]]);
zero:=[0: i in [1..dG+2]];
V:=-Vector(QQ,[Evaluate(empty[i],zero): i in [1..#empty]]);
MF:=Transpose(Mat); x,y,z:=IsConsistent(MF,V);
yy:=&+[y[i+1]*va^(i):i in [0..dG]];
sigma:=bnew-yy*Bnew;
Sigma:=[QQ!Coefficient(sigma, va, 0)] cat [QQ!Coefficient(sigma, va,
i): i in [1..r-1]];
return  Sigma;
end function;
```

Now we can do calculations on concrete examples.

Example 3.2.2. *Consider the substack $\mathcal{X}_{11}$ of $\mathbb{P}(1,2,3,5)$, where $\mathcal{X}_{11}$ is defined by $f = x_0^{11} + x_1^4 x_2 + x_1 x_2^3 + x_0 x_3^2$. We can check that it is quasismooth and has 3 orbipoints $P_1 = (0,1,0,0)$, $P_2 = (0,0,1,0)$ and $P_3 = (0,0,0,1)$ of type $\frac{1}{2}(1,1)$, $\frac{1}{3}(1,2)$, and $\frac{1}{5}(2,3)$ respectively. Hence the formula for the sheaves $\mathcal{O}_{\mathcal{X}}(d)$ is given by*

$$\chi(\mathcal{O}_{\mathcal{X}}(d)) = [\mathrm{Ch}(\mathcal{O}_{\mathcal{X}}(d))\,\mathrm{Td}_{\mathcal{X}}]_2 + M_{P_1} + M_{P_2} + M_{P_3},$$

where M_{P_1}, M_{P_2} and M_{P_3} are given by Dedekind sums as in (3.1), and

$$[\mathrm{Ch}(\mathcal{O}_{\mathcal{X}}(d))\,\mathrm{Td}_{\mathcal{X}}]_2 = \mathrm{Td}_2 + \frac{1}{2} dH(dH - K_{\mathcal{X}}),$$

where H *is* $c_1(\mathcal{O}_{\mathcal{X}}(1))$*. By the exact sequence*

$$0 \to \mathcal{T}_{\mathcal{X}} \to \mathcal{T}_{\underline{\mathbb{P}}}|_{\mathcal{X}} \to \mathcal{N}_{\mathcal{X}|\underline{\mathbb{P}}} \to 0,$$

we know that $c_t(\mathcal{T}_{\mathcal{X}})c_t(\mathcal{N}_{\mathcal{X}|\underline{\mathbb{P}}}) = c_t(\mathcal{T}_{\underline{\mathbb{P}}})|_{\mathcal{X}}$*, and thus we have* $c_1(\mathcal{T}_{\mathcal{X}}) = -K_{\mathcal{X}} = 0$ *and*

$$c_2(\mathcal{T}_{\mathcal{X}}) = c_2(\mathcal{T}_{\underline{\mathbb{P}}}|_{\mathcal{X}}) - c_1(\mathcal{T}_{\mathcal{X}})c_1(\mathcal{N}_{\mathcal{X}|\underline{\mathbb{P}}}) = \frac{451}{30}.$$

Hence $\mathrm{Td}_2 = \frac{1}{12}(c_1(\mathcal{T}_{\mathcal{X}})^2 + c_2(\mathcal{T}_{\mathcal{X}})) = \frac{451}{360}$*. Now we use our Program 3.2.3 to compute* $M_{P_i}(d)$ *for* $0 \le i \le 3$*.*

```
>f:=func<d|451/360+1/2*d^2*11/30>;
>MP1:=Contribution(2,[1,1]);
>MP2:=Contribution(3,[1,2]);
>MP3:=Contribution(5,[2,3]);
>[f(d)+MP1[d mod 2 +1]+MP2[d mod 3+1]+MP3[d mod 5+1]: d in [1..10]];
[ 1, 2, 3, 4, 6, 8, 10, 13, 16, 20 ]
```

The last output gives us $\chi(\mathcal{O}(d))$ *for* $1 \le d \le 10$*.*

Next, we give an example with curve orbifold loci and dissident points.

Example 3.2.3. *Let* $\mathcal{X}$ *be a quasi-smooth Calabi-Yau 3-fold given by* $\mathcal{X}_{80} \subset \mathbb{P}^4(3, 5, 7, 25, 40)$*. It is of degree* $2/2625$ *and has an orbifold curve* $\mathcal{C}_{80} \subset \mathbb{P}(5, 25, 40)$ *of type* $\frac{1}{5}(2, 3)$ *and a point basket* $\mathcal{B} = \{\frac{1}{3}(1, 1, 1), \frac{1}{7}(4, 5, 5), \frac{1}{25}(3, 7, 15)\}$*, among which the point of type* $\frac{1}{25}(3, 7, 15)$ *is a dissident point. Then according to the Riemann–Roch formula in Corollary 3.1.4, we have several parts in the formula, which correspond to the connected components of the associated inertia stack. The first part is given by:*

$$r_1 = [\mathrm{Ch}(\mathcal{O}_X(d))\,\mathrm{Td}_{\mathcal{X}}]_3,$$

where the Chern character is given by $\mathrm{Ch}(\mathcal{O}_{\mathcal{X}}(d)) = 1 + dH + d^2H^2/2 + d^3H^3/6$*. To calculate the Todd class, we use the exact sequence:*

$$0 \to \mathcal{T}_{\mathcal{X}} \to \mathcal{T}_{\mathbb{P}}|_{\mathcal{X}} \to \mathcal{N}_{\mathcal{X}|\mathbb{P}} \to 0.$$

Since X is a hypersurface, we have $\mathcal{N}_{X|\mathbb{P}} = \mathcal{O}_{\mathcal{X}}(80)$*. It follows that*

$$\begin{aligned} c_t(\mathcal{T}_{\mathcal{X}}) &= c_t(\mathcal{T}_{\mathbb{P}}|_{\mathcal{X}})c_t^{-1}(\mathcal{N}) \\ &= (1+3t)(1+5t)(1+7t)(1+35t)(1+40t)(1+80Ht)^{-1} \\ &= 1 + 2046H^2u^2 - 143960H^3t^3 + \text{higher order terms.} \end{aligned}$$

That is, $c_1(X) = 0$, $c_2(X) = 2046H^2$, $c_3(X) = -143960H^3$. *Thus*

$$\begin{aligned} r_1 &= [(1 + dH + d^2H^2/2 + d^3H^3/6)(1 + 1/2c_1 + 1/12(c_1 + c_2^2) + 1/24c_1c_2)]_3 \\ &= 1/6d^3H^3 + 341/2dH^3, \end{aligned}$$

where $H^3 = \frac{80}{3\cdot 5\cdot 7\cdot 25\cdot 40} = 2/2625$.

The second part comes from the orbifold curve $\mathcal{C}_{80} \subset \mathbb{P}(5, 25, 40)$*, whose normal bundle is given by* $\mathcal{N} = \mathcal{O}_{\mathcal{C}}(3) \bigoplus \mathcal{O}_{\mathcal{C}}(7)$*. Thus the second part* r_2 *is given as follows:*

$$\begin{aligned} &\frac{1}{5}\sum_{\varepsilon\in\mu_5,\varepsilon\neq 1}\frac{\varepsilon^d}{(1-\varepsilon^{-2})(1-\varepsilon^{-3})}d\deg H|_{\mathcal{C}} - \frac{1}{2\cdot 5}\sum_{\varepsilon\in\mu_5,\varepsilon\neq 1}\frac{\varepsilon^d}{(1-\varepsilon^{-3})(1-\varepsilon^{-5})}\deg K_{\mathcal{C}} \\ &-\frac{1}{5}\sum_{\varepsilon\in\mu_5,\varepsilon\neq 1}\frac{\varepsilon^{d-3}}{(1-\varepsilon^{-3})^2(1-\varepsilon^{-2})}\deg\gamma_1 - \frac{1}{5}\sum_{\varepsilon\in\mu_5,\varepsilon\neq 1}\frac{\varepsilon^{d-7}}{(1-\varepsilon^{-7})^2(1-\varepsilon^{-3})}\deg\gamma_2, \end{aligned}$$

where $d\deg H|_{\mathcal{C}}$ *is given by* $c_1(\mathcal{O}_{\mathcal{X}}(d)|_{\mathcal{C}})$*, and* γ_1, γ_2 *are the first Chern classes of* $\mathcal{O}_{\mathcal{C}}(3)$, $\mathcal{O}_{\mathcal{C}}(7)$ *respectively. Moreover, we know that the canonical class of* $\mathcal{C}$ *is given by* $c_1(\mathcal{O}_{\mathcal{C}}(10))$ *and* $\deg H|_C = 5 \cdot \frac{2}{125}$.

Then the remaining parts come from these 3 *singular points, and hence they are given by:*

$$\begin{aligned} r_3 = &\frac{1}{3}\sum_{\varepsilon\in\mu_3,\varepsilon\neq 1}\frac{\varepsilon^d}{(1-\varepsilon^{-1})^3} + \frac{1}{7}\sum_{\varepsilon\in\mu_7}\frac{\varepsilon^d}{(1-\varepsilon^{-4})(1-\varepsilon^{-5})^2} + \\ &\frac{1}{25}\sum_{\varepsilon\in\mu_{25},\varepsilon^5\neq 1}\frac{\varepsilon^d}{(1-\varepsilon^{-3})(1-\varepsilon^{-7})(1-\varepsilon^{-15})}. \end{aligned}$$

Using the Program 3.2.3, we can calculate the Dedekind sums in the formula. The following are codes in MAGMA program.

```
>h:=2/2625;
>r1:=func<d|(1/6*d^3+341/2*d)*h>;
```

```
>s1:=Contribution(5,[2,3]);
>s2:=Contribution(5,[3,3,2]);
>s3:=Contribution(5,[3,2,2]);
>kc:=10; ga1:=3; ga2:=7;
>r2:=func<d|(s1[d mod 5+1]*d-1/2*kc*s1[d mod 5+1]-ga1*s2[(d-3) mod 5+1]
 -ga2*s3[(d-7) mod 5+1])*2/25>;
>c1:=Contribution(3,[1,1,1]);
>c2:=Contribution(7,[4,5,5]);
>c3:=Contribution(25,[3,7,15]);
>r3:=func<d|c1[d mod 3+1]+c2[d mod 7+1]+c3[d mod 25+1]>;
>rr:=[r1(d)+r2(d)+r3(d): d in [2..10]];
>rr;
[ 0, 1, 0, 1, 1, 1, 1, 1, 2 ]
```

The last output gives the plurigenera for degree $2, \dots, 10$.

3.3 Riemann–Roch on the moduli space

In the last section, we obtained the Riemann-Roch formula for line bundles $\mathcal{O}_{\mathcal{X}}(d)$ on $\mathcal{X} = \operatorname{Stac} R$. Now we want to deduce the Riemann-Roch formula for $\mathcal{O}_X(d)$ on its moduli space $X = \operatorname{Proj} R$. For this we just need to set up the link between $\mathcal{X}$ and X.

Let $\pi : \mathcal{X} \to X$ be the map induced by the quotient map $\hat{\pi} : \mathrm{Spec} R \backslash \{0\} \to X$. Then π is the natural map from $\mathcal{X}$ to X inducing a bijection between the geometric points of $\mathcal{X}$ and X. Recall that we define $\mathcal{O}_{\mathcal{X}}(d)$ to be the line bundle descended from an equivariant line bundle on the affine cone, but we can also define it on an étale cover of $\mathcal{X}$, in which case we can see clearly that $\pi_*(\mathcal{O}_{\mathcal{X}}(d)) = \mathcal{O}_X(d)$. Calculating the Čech cohomology on $\mathcal{X}$ and X gives us $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(d)) = H^i(X, \mathcal{O}_X(d))$ for all i, and therefore $\chi(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(d)) = \chi(X, \mathcal{O}_X(d))$.

In this way, we can transfer the formula for $\chi(\mathcal{O}_{\mathcal{X}}(d))$ to the coarse moduli space X to get a formula for $\mathcal{O}_X(d)$. Recall that the formula for $\chi(\mathcal{O}_{\mathcal{X}}(d))$ is given by a sum over all the components of the inertia stack $I_{\mathcal{X}}$, which implies that the formula on X will sum over all the singular strata of X. We also know that the morphism $\pi_* : A(\mathcal{X}) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}$ between Chow groups given in [Vis89] or [BCE$^+$06] is an isomorphism. For an integral closed substack of $\mathcal{Y}$, the map π_* sends $[\mathcal{Y}]$ to $[\frac{1}{g_{\mathcal{Y}}}\pi(\mathcal{Y})]$, where $g_{\mathcal{Y}}$ is the order of the generic stabilizer group of $\mathcal{Y}$.

When the quasismooth stack $\mathcal{X} = \operatorname{Stac} R$ has only codimension ≥ 2 orbifold loci, then the coarse moduli space given by $X = \operatorname{Proj} R$ has cyclic quotient singularities in one to one correspondence with the orbifold loci on $\operatorname{Stac} R$. Take the case when there are only curve singularities as an example.

Proposition 3.3.1. *Let X be a quasismooth variety of dimension ≥ 3 in weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$. Let $\mathcal{B} = \{C$ of singular type $\frac{1}{r}(a_1, \ldots, a_{n-1})\}$ be all the singular loci on X. Then*

$$\chi(\mathcal{O}(d)) = [\operatorname{Ch}(\mathcal{O}(d)) \operatorname{Td}_X]_n + \sum_{C \in \mathcal{B}} M_C$$

where M_C is given by

$$\frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^d}{\prod(1-\varepsilon^{-a_i})} d \deg H|_C - \frac{1}{2r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^d - 1}{\prod(1-\varepsilon^{-a_i})} \deg K_X|_C$$
$$- \sum_{i=1}^{n-1} \frac{1}{2r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{(\varepsilon^d - 1)(1+\varepsilon^{-a_i})}{(1-\varepsilon^{-a_i})^2 \prod_{j \neq i}(1-\varepsilon^{-a_j})} \deg \gamma_i.$$

where H is the Weil divisor associated to $\mathcal{O}_X(1)$, and γ_i are the the Chern roots of the orbibundle $\mathcal{N}$.

Proof Here we just need to point out that intersection number $\deg K_X|_C$ is defined as follows: Let $\hat{C} = \hat{\pi}^{-1}(C)$. Then $\mathcal{C} = [\hat{C}/\mathbb{C}^*]$ is a substack of $\mathcal{X}$, which maps to C by π. Since $\pi^* K_X = K_{\mathcal{X}}$, by projection formula, we have

$$K_{\mathcal{X}} \cdot \mathcal{C} = \pi_*(K_{\mathcal{X}} \cdot \mathcal{C}) = \pi_*(\pi^* K_X \cdot \mathcal{C}) = K_X \cdot \frac{1}{r} C.$$

Similarly for $H|_C$. □

Remark 3.3.1. Here the definition of $\deg H|_C$ coincides with the one given in [BS05]. Therefore as a special case we can recover the formula in [BS05].

3.3.1 Orbifolds with codimension 1 orbifold loci

In this section, we consider the case when there are codimension 1 orbifold loci. If codimension 1 orbifold loci are the only orbifold loci, then the problem is easy and the whole calculation can be reduced to Riemann–Roch for $\mathbb{Q}$-Weil divisors on smooth varieties.

We first consider the curve case. When $\mathcal{X}$ is quasismooth of dimension 1 with orbipoints $\mathcal{B} = \{P \text{ of type } \frac{1}{r}(a)\}$, the Riemann–Roch formula can be written as:

$$\chi(\mathcal{O}_{\mathcal{X}}(d)) = -\frac{1}{2}\deg K_{\mathcal{X}} + \deg dH + \sum_P \frac{1}{r} \sum_{\varepsilon\in\mu_r,\varepsilon\neq 1} \frac{\varepsilon^d}{1-\varepsilon^{-a}},$$

where $H = c_1(\mathcal{O}_{\mathcal{X}}(1))$. Note by Lemma 3.2.1, one can reduce this to

$$\begin{aligned}
\chi(\mathcal{O}_{\mathcal{X}}(d)) &= -\frac{1}{2}\deg K_{\mathcal{X}} + \sum_P \frac{1}{r} \sum_{\varepsilon\in\mu_r,\varepsilon\neq 1} \frac{1}{1-\varepsilon^{-a}} + \deg dH + \sum_P \frac{1}{r} \sum_{\varepsilon\in\mu_r,\varepsilon\neq 1} \frac{\varepsilon^d - 1}{1-\varepsilon^{-a}} \\
&= -\frac{1}{2}(\deg K_{\mathcal{X}} - \sum_P \frac{r-1}{r}) + \deg dH - \sum_P \frac{\overline{bd}}{r}, \quad \text{or in another form:} \\
\chi(\mathcal{O}_{\mathcal{X}}(d)) &= \chi(\mathcal{O}_{\mathcal{X}}) + \deg dH - \sum_P \frac{\overline{bd}}{r}.
\end{aligned}$$

Recall here b is the inverse of a mod r. Since there is only codimension 1 orbifold loci, we deduce that the coarse moduli space of $\mathcal{X}$ is a smooth curve. Therefore, the second equality above is just the Riemann–Roch formula for smooth curves and $\deg K_{X,\text{ sch}} = \deg\,(K_{\mathcal{X}} - \sum_{P\in\mathcal{B}} \frac{r-1}{r} P)$ (see [Rei]). That is,

$$\chi(\mathcal{O}_{\mathcal{X}}(d)) = \chi(\mathcal{O}_X(d)) = -\frac{1}{2}\deg K_{X,sch} + \deg\lfloor dH' \rfloor,$$

where H' is given by the $\mathbb{Q}$ divisor $\sum_{P\in\mathcal{B}} \frac{b}{r} P'$ and P' is the image of P under the map $\pi : \mathcal{X} \to X$.

Example 3.3.1. *Take the substack $\mathcal{X}_8$ of $\mathbb{P}(1,3,5)$. There are two orbifold points on it, namely $(0,1,0)$ of type $\frac{1}{3}(1)$ and $(0,0,1)$ of type $\frac{1}{5}(1)$. Also note the canonical class of $\mathcal{X}$ is given by $\mathcal{O}_{\mathcal{X}}(-1)$. Then one can calculate*

$$\chi(\mathcal{O}(d)) = 1 + \frac{8d}{15} - \frac{\overline{d}}{3} - \frac{\overline{d}}{5}.$$

On the other hand, we have seen in Example 2.2.2 that R can be obtained by $R = \bigoplus_{d\geq 0} \mathcal{O}_X(\lfloor dH \rfloor)$, where H is given by $\frac{P_1}{3} + \frac{P_2}{5}$ for two different points P_1, P_2 on $\mathbb{P}^1$. Thus the above formula is just $\chi(\mathcal{O}(\lfloor dH \rfloor)) = 1 + \deg\lfloor dH \rfloor$.

More generally, for the higher dimensional case, we can obtain similar results but there will be complication: for example, it can happen that one has a higher codimension orbifold locus lying on a codimension 1 orbifold locus.

Chapter 4

Hilbert Series Parsing for Isolated Orbifold Points

In this chapter, for a polarized quasismooth projectively Gorenstein orbifold (as a stack) $(\mathcal{X}, \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_{\mathcal{X}}(d))$ with only isolated orbifold points, we parse the associated Hilbert series into simple pieces, each of which is integral and Gorenstein symmetric. We start with introducing some definitions and basic properties of Hilbert series. Then we state the main theorem, which was a conjecture in [Rei], and introduce the so-called *ice cream function*. Afterwards we give a detailed proof of the theorem and some applications. This chapter is part of the joint paper [BRZ] with A. Buckley and M. Reid.

4.1 Definitions and notations

Let $\mathcal{X}$ be a projective stack and H a $\mathbb{Q}$-ample divisor on $\mathcal{X}$ (Here H is taken to be the associated divisor of $\mathcal{O}_{\mathcal{X}}(1)$). The graded ring associated to this pair is given by

$$R(\mathcal{X}, H) = \bigoplus_{d\geq 0} H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(dH)),$$

and its Hilbert series is defined to be

$$P(t) = \sum_{d\geq 0} h^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(dH))t^d.$$

Here we call $h^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(dH))$ the d-th plurigenus of this pair.

Note that $R(\mathcal{X}, H)$ is a finitely generated $\mathbb{C}$-algebra, that is, there exists a weighted homogeneous ideal I of $\mathbb{C}[x_0, \ldots, x_m]$ such that $R = \mathbb{C}[x_0, \ldots, x_m]/I$. Therefore $R(\mathcal{X}, H)$ is a finitely generated graded $\mathbb{C}[x_0, \ldots, x_m]$-module. By the Hilbert syzygy theorem, there exists a finite free resolution

$$0 \leftarrow R(\mathcal{X}, H) \leftarrow A_0 = \mathbb{C}[x_0, \ldots, x_m] \leftarrow A_1 \leftarrow \cdots \leftarrow A_\gamma \leftarrow 0, \tag{4.1}$$

where $A_i = \bigoplus_{j=1}^{k_i} A_0(-b_{i,j})$, $i \geq 1$. This resolution gives us information about the structure of the ring R, namely, it has relations in degree $b_{1,j}$, first syzygies in degree $b_{2,j}$, second syzygies in degree $b_{3,j}$, and so on. From this resolution, we can also read out the associated Hilbert series

$$P(t) = \frac{1 - \sum_j t^{b_{1,j}} + \sum_j t^{b_{2,j}} + \cdots + (-1)^\gamma \sum_j t^{b_{\gamma,j}}}{(1 - t^{a_0}) \cdots (1 - t^{a_m})}, \tag{4.2}$$

where a_i is the weight of x_i for each i.

When the ring $R(\mathcal{X}, H)$ (or simply R) is Gorenstein, the resolution (4.1) has length $\gamma = \operatorname{codim} X$, and it also satisfies the so-called Gorenstein symmetry, i.e., $A_\gamma = A_0^\vee \otimes A_0(-a)$ and $A_{\gamma-i} = A_i^\vee \otimes A_0(-a)$ for some integer a, where $A_i^\vee$ is the dual $\operatorname{Hom}(A_i, R)$ of A_i. Such a is defined to be *the adjunction number*, because we have

$$\omega_{\mathcal{X}} = \omega_{\mathbb{P}(a_0, \ldots, a_m)} \otimes \mathcal{O}_{\mathcal{X}}(a) = \mathcal{O}_{\mathcal{X}}\big(a - \sum_{i=0}^{m} a_i\big).$$

Definition 4.1.1. When $R(\mathcal{X}, H)$ is a Gorenstein graded ring, the canonical sheaf $\omega_{\mathcal{X}}$ of X is given by a line bundle $\mathcal{O}_{\mathcal{X}}(k_{\mathcal{X}})$. We call $k_{\mathcal{X}}$ *the canonical weight* of the pair $(\mathcal{X}, H)$.

The Gorenstein symmetry of the resolution induces the following property of the Hilbert series associated to a Gorenstein graded ring.

Lemma 4.1.1. *Suppose R is a Gorenstein graded ring of dimension $n+1$, and let a be the adjunction number of $\mathcal{X} = \operatorname{Stac} R$. Then*

$$P(t) = (-1)^{n+1} t^{k_{\mathcal{X}}} P(1/t), \tag{4.3}$$

where $k_{\mathcal{X}} = a - \sum_{i=0}^m a_i$ is the canonical weight, i.e., $k_{\mathcal{X}} H \sim K_{\mathcal{X}}$.

Definition 4.1.2. We refer to property (4.3) for a rational function in t as *Gorenstein symmetry of degree $k_{\mathcal{X}}$ in dimension n*. Often, when the dimension is clear, we will just say *Gorenstein symmetry of degree $k_{\mathcal{X}}$*.

PROOF We calculate $P(1/t)$:

$$\begin{aligned} P(1/t) &= \frac{1-\sum t^{-b_{1,j}}+\sum t^{-b_{2,j}}+\cdots+(-1)^{\gamma}t^{-a}}{(1-t^{-a_0})\cdots(1-t^{-a_m})} \\ &= \frac{(-1)^{\gamma}t^{-a}(t^a-\sum t^{a-b_{1,j}}+\cdots+1)}{(-1)^{m+1}t^{-(a_1+\cdots+a_m)}(1-t^{a_0})\cdots(1-t^{a_m})} \\ &= (-1)^{n+1}t^{-(a-\sum_{i=0}^m a_i)}P(t). \end{aligned}$$

Putting $a-\sum_{i=0}^m a_i = k_{\mathcal{X}}$, we are done. □

Because we will talk about Gorenstein symmetric property of a rational function intensively later, we want to point out some obvious facts.

Definition 4.1.3. Given a Laurent polynomial $f(t)=\sum_{i=l_1}^{l_2} b_it^i$, if $b_{l_1}\neq 0$ and $b_{l_2}\neq 0$, we say it has support $[l_1,l_s]$. If $[l_1,l_2]\subset[L_1,L_2]$, then we can say $f(t)$ is supported in the interval $[L_1,L_2]$.

Suppose $f(t)$ has support $[l_1,l_2]$. Then it is *palindromic* if $b_{l_1+j}=b_{l_2-j}$ for all $j\in\mathbb{Z}$, and we say $f(t)$ is *palindromic of degree l_1+l_2*.

Remark 4.1.1.

1. One fact is about the relation between Gorenstein symmetry and palindromic polynomials. By definition, a rational function $\frac{B(t)}{(1-t^{a_0})\cdots(1-t^{a_n})}$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$ if and only if $B(t)=\sum_{i=l_1}^{l_s} b_it^i$ is palindromic and $l_1+l_2-\sum_{i=0}^n a_i = k_{\mathcal{X}}$ (or equivalently $B(t)$ is palindromic of degree $k_{\mathcal{X}}+\sum_{i=0}^n a_i$).
2. The second fact is that the sum or the difference of two Gorenstein symmetric polynomial of degree $k_{\mathcal{X}}$ in the same dimension is also Gorenstein symmetric of the same degree in the same dimension.

Definition 4.1.4. The pair $(\mathcal{X},H)$ is called *projectively Gorenstein* if the ring $R(\mathcal{X},H)=\bigoplus_{d\geq 0}H^0(\mathcal{X},\mathcal{O}(dH))$ is Gorenstein.

Here are some examples for projective stacks without orbifold loci whose canonical rings are Gorenstein (quasismooth stacks without orbifold loci can be represented by their moduli spaces, and therefore we can refer to them as schemes) and whose associated Hilbert series satisfy Gorenstein symmetry as in (4.3).

Example 4.1.1. *Given a smooth curve C of general type with genus $g = h^0(\mathcal{O}_C(K_C))$, the Hilbert series associated to (C, K_C) is given by*

$$P(t) = \frac{1 + (g-2)t + (g-2)t^2 + t^3}{(1-t)^2}.$$

Let X be a smooth surface of general type with invariants $p_g = h^0(K_{\mathcal{X}})$, $q = h^1(\mathcal{O}_X)$ and K^2. Assume $q = 0$. Then using Kodaira vanishing, this implies that $H^1(X, dK_X) = 0$ for all d, and so the graded ring $R(X, K_X)$ is Gorenstein by [GW78]. The Hilbert series $P(t) = \sum_{d\geq 0} h^0(X, \mathcal{O}_X(dK_X))t^d$ is given by

$$P(t) = \frac{1 + (p_g - 3)t + (K^2 - 2p_g + 4)t^2 + (p_g - 3)t^3 + t^4}{(1-t)^3}.$$

Notice that the Hilbert series itself encodes invariants of the variety.

Given a projectively Gorenstein pair $(\mathcal{X}, H)$, we have seen that the associated Hilbert series $P(t)$ is Gorenstein symmetric. Actually there are more good properties coming with the Gorenstein condition. The properties introduced in [Wat81] can be translated into our language.

Proposition 4.1.2. *Let $\mathcal{X}$ be a quasismooth projective stack, and H a $\mathbb{Q}$-ample divisor. Then $R(\mathcal{X}, H)$ is a Cohen–Macaulay ring if and only if $H^p(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(dH)) = 0$ for $1 \leq p < n$ and for every $d \in \mathbb{Z}$. Moreover, if $R(\mathcal{X}, H)$ is a Cohen–Macaulay ring, then $R(\mathcal{X}, H)$ is Gorenstein if and only if $K_{\mathcal{X}} \sim_{\text{linear}} k_{\mathcal{X}} H$ for some $k_{\mathcal{X}} \in \mathbb{Z}$.*

Furthermore, requiring R to be Gorenstein also restricts the types of orbifold points that can occur on $\operatorname{Stac} R$.

Proposition 4.1.3. *Let $(\mathcal{X}, H)$ be a quasismooth projectively Gorenstein pair. Suppose P is an orbifold point on $\mathcal{X} = \operatorname{Stac} R(\mathcal{X}, H)$ of type $\frac{1}{r}(a_1, \ldots, a_n)$. Then $\sum_{i=1}^{n} a_i + k_{\mathcal{X}} = 0 \bmod r$.*

Proof First when $\operatorname{Stac} R$ is just $\mathbb{P}(a_0, a_1, \ldots, a_m)$, then the orbifold points are of the form $\frac{1}{a_j}(a_0, \ldots, \widehat{a_j}, \ldots, a_n)$ for some j. In this case, $k_{\mathcal{X}} = -\sum_{i=0}^{m} a_i$ and therefore $\sum_{i=0, i\neq j}^{m} a_i + k_{\mathcal{X}} = 0 \bmod a_j$.

If $R = \mathbb{C}[x_0, \ldots, x_m]/(f_1, \ldots, f_c)$ is a complete intersection, then $k_{\mathcal{X}} = \sum_{i=1}^{c} d_i - \sum_{j=0}^{m} a_j$ where $d_i = \deg f_i$ and weight x_j are given by a_j. Since $\mathcal{X}$ is quasismooth, a point P on $\operatorname{Stac} R$ is always of type $\frac{1}{r}(a_1, \ldots, a_n)$ (reorder the weight for simplicity) with r a divisor of a_0. If its corresponding orbit has local coordinates $y_1, \dots, y_n$ with $\mathbb{C}^*$ action of weights $a_1, \dots, a_n$ respectively, then after reordering the f_is we can have $d_i = \deg f_i = a_{n+i} \bmod r$. Hence, we also have $\sum_{i=1}^n a_i + k_{\mathcal{X}} = 0 \bmod r$.

In the general case, we can analyze locally using Gorenstein condition. It follows from a similar argument as in the complete intersection case. □

To prepare for the statement of the theorem, we introduce two concepts here.

Definition 4.1.5. The number $k_{\mathcal{X}} + n + 1$ is defined to be the coindex of the pair (X, H), denoted by c (first introduced in [Muk89]). By the adjunction formula, the coindex is invariant under passing to a hyperplane section of degree 1.

Definition 4.1.6. For coprime polynomials $A, F \in \mathbb{Q}[t]$ and $\gamma \in ZZ$, we set

$$\mathrm{InvMod}(A, F, \gamma) = B.$$

That is, $B \in \mathbb{Q}[t, t^{-1}]$ is the uniquely determined Laurent polynomial supported in $[t^\gamma, \dots, t^{\gamma+d-1}]$ such that $AB \equiv 1 \bmod F$. For different $\gamma \in \mathbb{Z}$, these inverses are congruent modulo F, but different polynomials in general. We also write $\mathrm{InvMod}(A, F)$ with unspecified support for any inverse of A modulo F in $\mathbb{Q}[t]$.

4.2 The main result

Let $(\mathcal{X}, H)$ be a polarized quasismooth projectively Gorenstein orbifold with only isolated orbifold points. We give a parsing of the Hilbert series associated to $(\mathcal{X}, H)$ according to its orbifold loci. Each part of the parsing has integral coefficients and Gorenstein symmetric properties of the same degree $k_{\mathcal{X}}$ (where $k_{\mathcal{X}}$ is the canonical weight of $\mathcal{X}$ with polarization H). In this parsing, we call the parts corresponding to orbifold loci *ice cream functions*, given by InverseMod functions. The result expresses $P_{\mathcal{X}}(t)$ in a closed form that can be calculated readily with some simple computer algebra.

Theorem 4.2.1. *Let $(\mathcal{X}, H)$ be a polarized n-dimensional orbifold which is projectively Gorenstein. Let $k_{\mathcal{X}} \in \mathbb{Z}$ be the canonical weight of $(\mathcal{X}, H)$. Assume that the orbifold loci of $(\mathcal{X}, H)$ consist of isolated points $\mathcal{B} = \{Q$ of type $\frac{1}{r}(a_1, \dots, a_n)\}$.*

Then the Hilbert series of $\mathcal{X}$ can be parsed into

$$P_{\mathcal{X}}(t) = P_I(t) + \sum_{Q \in \mathcal{B}} P_{\mathrm{orb},Q}(t)$$

where

- *the initial term is of the form* $P_I(t) = \frac{I(t)}{(1-t)^{n+1}}$, *where* $I(t)$ *is the unique integral palindromic polynomial of degree* $c = k_{\mathcal{X}} + n + 1$ *(the coindex) such that* $P_I(t)$ *as power series equals* $P_{\mathcal{X}}(t)$, *up to and including degree* $\lfloor \frac{c}{2} \rfloor$. *If* $c < 0$, *then* $P_I(t) = 0$.

- *the orbifold term for* $Q \in \mathcal{B}$ *of type* $\frac{1}{r}(a_1, \dots, a_n)$ *is of the form* $P_{\mathrm{orb},Q}(t) = \frac{C(t)}{(1-t)^n(1-t^r)}$, *with*

$$C(t) = \mathrm{InvMod}\,(\prod \frac{1-t^{a_i}}{1-t}, \frac{1-t^r}{1-t}, \left\lfloor \frac{c}{2} \right\rfloor + 1)$$

the unique Laurent polynomial supported in $\left[\lfloor \frac{c}{2} \rfloor + 1,\ \lfloor \frac{c}{2} \rfloor + r - 1\right]$ *equal to the inverse of* $\prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t}$ *modulo* $\frac{1-t^r}{1-t}$. *Also* $C(t)$ *has integral coefficients, and* $P_{\mathrm{orb},Q}(t)$ *is Gorenstein symmetric of deg* $k_{\mathcal{X}}$.

We first see an example to explain the statements in the theorem.

Example 4.2.1. *Consider the hypersurface* $(\mathcal{X}_{10} = \mathrm{Stac}\, R, \mathcal{O}_{\mathcal{X}}(1))$ *in* $\underline{\mathbb{P}}(1,1,2,2,3)$, *where* $R = k[x_0, x_1, x_2, x_3, x_4]/(f_{10})$ *and* f_{10} *is a general homogeneous polynomial of degree* 10. *Then* $\mathcal{X}_{10}$ *is a* 3*-fold with* 5 *orbifold points* $Q_1, \dots, Q_5$ *of type* $\frac{1}{2}(1,1,1)$ *along* $\mathbb{P}^1\langle y_1, y_2\rangle$ *and an orbifold point* $Q = (0,0,0,0,1)$ *of type* $\frac{1}{3}(1,2,2)$. *It has canonical weight* $k_{\mathcal{X}} = 1$ *and coindex* $c = k_{\mathcal{X}} + n + 1 = 5$. *The Hilbert series is as follows: The initial term*

$$P_I(t) = \frac{1 - 2t + 3t^2 + 3t^3 - 2t^2 + t^5}{(1-t)^4}$$

handles $P_0 = 1$, $P_1 = 2$, $P_2 = 5$. *The orbifold terms*

$$P_{\mathrm{orb},Q}(t) = \frac{-t^3 - t^4}{(1-t)^3(1-t^3)},\ P_{\mathrm{orb},Q_i}(t) = \frac{-t^3}{(1-t)^3(1-t^2)},\ \textit{for all } i = 1, \dots, 5,$$

take care of the periodicity. One can check that the numerators are given by InverseMod as stated in the theorem. In fact, take the point Q *of type* $\frac{1}{3}(1,2,2)$ *as an* *example. We see that*

$$\begin{aligned}(-t^3-t^4)\frac{(1-t)(1-t^2)^2}{(1-t)^3} &= (-t^3-t^4)(1+t)^2 \\ &= (1-t^3)(1+t)^3-(1+t+t^2)(1+t)-(1+t+t^2)+1 \\ &= 1 \bmod \frac{1-t^3}{1-t}.\end{aligned}$$

Hence we have

$$P_I(t)+\sum_{i=1}^{5} P_{\mathrm{orb},Q_i}(t)+P_{\mathrm{orb},Q}(t)=\frac{1-t^{10}}{(1-t)^2(1-t^2)^2(1-t^3)}.$$

Here the numerator of $P_I(t)$ is palindromic of degree $c=5$, so that $P_I(t)$ is Gorenstein symmetric of degree 1. The $P_{\mathrm{orb}}(t)$ terms are also integral and Gorenstein symmetric of degree 1, they start with t^3, and so do not affect the first three plurigenera P_0, P_1 and P_2.

4.3 Fun calculations with the ice cream function

We now give some observations and fun calculations (an explanation of the name ice cream function). The classic example we have is "Income $\frac{3}{7}$ per day means ice cream on Wednesdays, Fridays and Sundays". Consider the step function $d \mapsto \lfloor \frac{3d}{7} \rfloor$, where $\lfloor\ \rfloor$ denotes the rounddown, or integral part. As Hilbert series, it gives

$$P(t)=\sum_{d\geq 0}\lfloor\frac{3d}{7}\rfloor t^d = 0+0t+0t^2+t^3+t^4+2t^5+2t^6+3t^7+\cdots,$$

which has the rational form

$$P(t)=\frac{t^3+t^5+t^7}{(1-t)(1-t^7)}. \tag{4.4}$$

In fact, notice that the function $\lfloor \frac{3d}{7} \rfloor$ increases by 1 when $d=0,3,5$ modulo 7, so that

$$(1-t)P(t)=t^3+t^5+t^7+t^{10}+t^{12}+t^{14}+\cdots$$

gives us all the days when one gets an extra ice cream, which repeats weekly. Thus, multiplying with $(1-t^7)$ gives

$$(1-t^7)(1-t)P(t) = t^3 + t^5 + t^7.$$

Note that $P(t)$ can also be written as

$$P(t) = \frac{3}{7}\frac{t}{(1-t)^2} + \frac{-\frac{3}{7}t - \frac{6}{7}t^2 - \frac{2}{7}t^3 - \frac{5}{7}t^4 - \frac{1}{7}t^5 - \frac{4}{7}t^6}{1-t^7}, \tag{4.5}$$

which can naturally appear in the Hilbert series associated to some orbifold curve with an orbifold point of type $\frac{1}{7}(5)$. In fact, in Section 3.3.1, the Riemann–Roch formula for $\chi(\mathcal{O}_{\mathcal{C}}(dH))$, where $(\mathcal{C}, H)$ is a polarized quasismooth curve with an orbifold point of type $\frac{1}{7}(5)$ and $\mathcal{O}_{\mathcal{X}}(H)$ is of type $\frac{1}{7}(5)$, can be given as

$$\chi(\mathcal{O}_C(dD)) = \chi(\mathcal{O}_{\mathcal{X}}) + \deg dD - \{\frac{3d}{7}\},$$

since $3\cdot 5 = 1 \bmod 7$. Here the fractional part $-\{\frac{3d}{7}\}$ appears as the contribution from the orbifold point, that is, the Dedekind sums.

Even though the forms in (4.4) and (4.5) are equivalent, we prefer (4.4) since (4.5) has integral coefficients in the numerator and it is Gorenstein symmetric of some degree in the sense of Lemma 4.1.1. One can also change the degree of Gorenstein symmetry by shifting up and down the exponent of t in the numerator, i.e.,

$$\frac{t^{-4}+t^{-2}+1}{(1-t)(1-t^7)} = \frac{t^{-4}+t^{-2}+1}{(1-t)^2} + \frac{t^3+t^5+t^7}{(1-t)(1-t^7)},$$

or

$$\frac{-t^{-1}-t-t^2-t^4}{(1-t)(1-t^7)} = \frac{-t^{-1}+1-t}{(1-t)^2} + \frac{t^3+t^5+t^7}{(1-t)(1-t^7)}.$$

Among these possible shifts as Laurent polynomials with length less than or equal to $6 = 7-1$ in the numerator, $\frac{t^{7i}(t^3+t^5+t^7)}{(1-t)(1-t^7)}$ is Gorenstein symmetric of degree $2+14i$ and $\frac{t^{7i}(-t^{-1}-t-t^2-t^4)}{(1-t)(1-t^7)}$ is Gorenstein symmetric of degree $-5+14i$, and no others. We will give a proof of this statement in a general context in later sections.

Our aim is to see the relation of (4.4) and (4.5) and transform the periodic contribution to Hilbert series from each isolated orbifold point into a closed form as in (4.4), which has integral coefficients and Gorenstein symmetric property.

4.4 First parsing

To parse the Hilbert series according to the orbifold points, we use the Riemann–Roch formula in Corollary 3.1.1 or Theorem 8.5 in [Rei87], which says that for a quasismooth stack $(\mathcal{X}, H)$ with only isolated orbifold points $\mathcal{B} = \{Q \text{ of type } \frac{1}{r}(a_1, \dots, a_n)\}$, the Euler number $\chi(\mathcal{O}_{\mathcal{X}}(dH))$ can be given by

$$\chi(\mathcal{O}_{\mathcal{X}}(dH)) = [\mathrm{Ch}(\mathcal{O}_{\mathcal{X}}(dH))\,\mathrm{Td}_{\mathcal{X}}]_n + \sum_{Q\in\mathcal{B}} \sum_{\varepsilon\in\mu_r, \varepsilon\neq 1} \frac{\varepsilon^d}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})},$$

where

$$\begin{aligned}
\mathrm{Ch}(\mathcal{O}_{\mathcal{X}}(dH)) &= 1 + dH + \frac{1}{2}d^2H^2 + \frac{1}{6}d^3H^3 + \frac{1}{24}d^4H^4 + \cdots, \\
\mathrm{Td}_{\mathcal{X}} &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2^2) + \frac{1}{24}c_1c_2 - \\
&\quad \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4) + \cdots,
\end{aligned}$$

where the c_i are the Chern classes of the tangent sheaf $\mathcal{T}_{\mathcal{X}}$, and in particular, $c_1 = -k_{\mathcal{X}}H$. Therefore $[\mathrm{Ch}(\mathcal{O}_{\mathcal{X}}(dH))\,\mathrm{Td}_{\mathcal{X}}]_n$ is a polynomial in d of degree n, denoted by $p(d)$. First suppose $k_{\mathcal{X}} \geq 0$. Then the associated Hilbert series can be given by

$$\begin{aligned}
P(t) &= \sum_{d\geq 0} h^0(\mathcal{O}(dh))t^d \\
&= \sum_{d\geq 0} (\chi(\mathcal{O}(dh)) + (-1)^{n+1}h^0(\mathcal{O}(k_{\mathcal{X}} - d)))t^d \\
&= \sum_{d=0}^{k_{\mathcal{X}}} (p(d) + (-1)^{n+1}h^0(\mathcal{O}(k_{\mathcal{X}} - d)))t^d + \\
&\quad \sum_{d>k_{\mathcal{X}}} p(d)t^d + \sum_{d\geq 0}\sum_{P\in\mathcal{B}} \frac{1}{r} \sum_{\varepsilon\neq 1} \frac{\varepsilon^d}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})} t^d.
\end{aligned}$$

The second equality above is due to vanishing of the middle cohomologies (see Proposition 4.1.2) and Serre duality. Notice that when $d > k_{\mathcal{X}}$, the coefficient of t^d is the polynomial $p(d)$. Now we can reduce the first and second part of the Hilbert series by differencing. Here by "differencing" we mean that given a series $\sum_{i\geq 0} h(i)t^i$

with $h(t)$ a polynomial, we multiply by $(1-t)$:

$$\begin{aligned}(1-t)\sum_{i\geq 0} h(i)t^i = \quad & h(0) \quad +h(1)t+h(2)t^2+\cdots \\ & \qquad\quad -h(0)t-h(1)t^2-h(2)t^3-\cdots \\ = \quad & h(0) \quad +(h(1)-h(0))t+\cdots+(h(i)-h(i-1))t^i+\cdots ,\end{aligned}$$

and the resulting series has coefficients $h(i)-h(i-1)$ for $i \geq 1$, which is a polynomial of degree reduced by 1. Therefore by "differencing" $n+1$ times we have that

$$(1-t)^{n+1}\left(\sum_{d=0}^{k_{\mathcal{X}}}(p(d)+(-1)^{n+1}h^0(\mathcal{O}(k_{\mathcal{X}}-d)))t^d+\sum_{d>k_{\mathcal{X}}}p(d)t^d\right)=A(t)$$

is a polynomial of degree $k_{\mathcal{X}}+n+1$. Similarly, when $k_{\mathcal{X}}<0$, the Hilbert series will just be

$$P(t)=\sum_{d\geq 0}p(d)t^d+\sum_{d\geq 0}\sum_{P\in\mathcal{B}}\frac{1}{r}\sum_{\varepsilon\neq 1}\frac{\varepsilon^d}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})}t^d.$$

Then by "differencing" $\sum_{d\geq 0}p(d)t^d$ a total of $n+1$ times, for the first part of the above equality, we obtain

$$\sum_{d\geq 0}p(d)t^d=\frac{b_0+\cdots+b_nt^n}{(1-t)^{n+1}}.$$

Therefore, we arrive at the following proposition:

Proposition 4.4.1. *Let $(\mathcal{X},H)$ be a projectively Gorenstein pair. Suppose $\mathcal{X}$ is quasismooth and has only isolated orbifold points $\mathcal{B}=\{Q \text{ of type } \frac{1}{r}(a_1,\ldots,a_n)\}$. Then the associated Hilbert series can be written as follows:*

$$P(t)=\frac{A(t)}{(1-t)^{n+1}}+\sum_{Q\in\mathcal{B}}P_{\mathrm{per},Q}(t),$$

where $A(t)$ is a polynomial of degree $k_{\mathcal{X}}+n+1$ when $k_{\mathcal{X}}\geq 0$, and is of degree n otherwise. The periodic term corresponding to an orbifold point Q of type $\frac{1}{r}(a_1,\ldots,a_n)$ is given by $P_{\mathrm{per},Q}(t)=\frac{N_{\mathrm{per},Q}(t)}{1-t^r}$ with $N_{\mathrm{per},Q}=\sum_{d=0}^{r-1}\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^dt^d}{(1-\varepsilon^{-a_i})\cdots(1-\varepsilon^{-a_n})}$.

Remark 4.4.1. Note that $A(t)$ necessarily has fraction coefficients due to the appearance of singularities. If $\mathcal{X}$ has no orbifold loci, and $(\mathcal{X}, H)$ is projectively Gorenstein, the same argument will give

$$P(t) = \frac{A(t)}{(1-t)^{n+1}},$$

where $A(t)$ is a palindromic polynomial of degree $k_{\mathcal{X}}+n+1$ with integral coefficients. Thus, $P(t)$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$ (see [Rei]). Our aim for parsing the Hilbert series is in some sense to achieve integral conditions and Gorenstein symmetry in the case with orbifold loci.

4.5 From $P_{\text{per}}(t)$ to $P_{\text{orb}}(t)$

As the second step of our parsing of $P(t)$, we want to adjust each part in Proposition 4.4.1 to be Gorenstein symmetric and with integral coefficients. The idea is to move some part of $\frac{A(t)}{(1-t)^{n+1}}$ to each of the periodic parts $\frac{P_{\text{per},Q}(t)}{(1-t^r)}$ to make the coefficients of both parts integral. We start by studying the periodic term.

Using the notion in Section 3.2, for a point Q of singular type $\frac{1}{r}(a_1, \dots, a_n)$, we can write $N_{\text{per},Q}(t) = \sum_{d=0}^{r-1} \sigma_d(\frac{1}{r}(a_1, \dots, a_n))t^d$, or simply $N_{\text{per},Q}(t) = \sum_{d=0}^{r-1} \sigma_d t^d$. By Proposition 3.2.2, $P_{\text{per},Q}(t)$ satisfies

$$N_{\text{per},Q}(t) \prod_{i=1}^{n} (1 - t^{a_i}) = 1 \bmod \frac{1-t^r}{1-t}. \tag{4.6}$$

4.5.1 Moving the support

The following algebraic lemma tells us how we can move some part of $\frac{A(t)}{(1-t)^{n+1}}$ to the periodic term.

Lemma 4.5.1. *Consider a polynomial $B = \sum_{i=1}^{r-1} b_i t^i \in \mathbb{Q}[t]$, and suppose we have r, $n \in \mathbb{N}$ and an interval $[\gamma+1, \gamma+r-1]$. Then there exists a unique Laurent polynomial*

$$C = \sum_{j \in [\gamma+1, \gamma+r-1]} \alpha_j t^j \in \mathbb{Q}[t, t^{-1}]$$

supported in $[\gamma+1, \gamma+r-1]$ such that

$$C - (1-t)^n B = \frac{1-t^r}{1-t} L,$$

with L a Laurent polynomial. That is,

$$\frac{L}{(1-t)^{n+1}} + \frac{B}{1-t^r} = \frac{C}{(1-t)^n(1-t^r)}.$$

PROOF The quotient ring

$$V = \mathbb{Q}[t]/(1+t+\cdots+t^{r-1})$$

is an $(r-1)$-dimensional vector space over $\mathbb{Q}$ based by $1, t, \ldots, t^{r-2}$. Note that t maps to an invertible element of V, so that also

$$V = \mathbb{Q}[t, t^{-1}]/(1+t+\cdots+t^{r-1})$$

and the $r-1$ elements t^j for $j \in [\gamma+1, \gamma+r-1]$ form another basis of V. Therefore, the class of $(1-t)^n B$ modulo the ideal of $\mathbb{Q}[t, t^{-1}]$ generated by $1+t+\cdots+t^{r-1}$ can be written in a unique way as a linear combination of t^j for $j \in [\gamma+1, \gamma+r-1]$. □

Following from the above lemma, we can rewrite our periodic term into the form

$$\frac{C(t)}{(1-t)^n(1-t^r)} = \frac{L(t)}{(1-t)^{n+1}} + \frac{N_{\mathrm{per},Q}(t)}{(1-t^r)}, \tag{4.7}$$

where $C(t)$ is in a chosen support of length $r-1$. Actually, if we choose the right support, $C(t)$ will have integral coefficients and palindromic property.

Proposition 4.5.2. *Let $(\mathcal{X}, H)$ be a projectively Gorensten pair and $c = k_{\mathcal{X}} + n + 1$ be the coindex. For each isolated orbifold point Q of type $\frac{1}{r}(a_1, \ldots, a_n)$, there exists a unique Laurent polynomial $C(t)$ as in (4.7), determined as the InverseMod function*

$$\mathrm{InverseMod}\,(\prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t}, \frac{1-t^r}{1-t}, \lfloor \frac{c}{2} \rfloor + 1),$$

i.e., the inverse of $\prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t}$ modulo $\frac{1-t^r}{1-t}$ supported in $[\lfloor \frac{c}{2} \rfloor + 1, \lfloor \frac{c}{2} \rfloor + r - 1]$. Moreover, with the chosen support, $C(t)$ has integral coefficients and $\frac{C(t)}{(1-t)^n(1-t^r)}$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$.

PROOF Rewrite (4.7) as follows:

$$C(t) = P_{\mathrm{per},Q}(t)(1-t)^n + L(t)\frac{1-t^r}{1-t}, \tag{4.8}$$

and use the property of $P_{\text{per},Q}(t)$ as in (4.6). One can deduce that

$$C(t)\prod_{i=1}^{n}\frac{1-t^{a_i}}{1-t}=1 \bmod \frac{1-t^r}{1-t},$$

i.e., $C(t)$ is the inverse of $\prod_{i=1}^{n}\frac{1-t^{a_i}}{1-t}$ modulo $\frac{1-t^r}{1-t}$. We prove the Gorenstein symmetry property in the following.

Definition 4.5.1. In (4.7), $\frac{C(t)}{(1-t)^n(1-t^r)}$ with $C(t)$ supported in $[\lfloor\frac{c}{2}\rfloor+1,\lfloor\frac{c}{2}\rfloor+r-1]$ is defined to be the *orbifold term*, denoted by $P_{\text{orb},Q}(t)$. The term $\frac{L(t)}{(1-t)^{n+1}}$ such that $P_{\text{orb},Q}(t)=P_{\text{per},Q}(t)+\frac{L(t)}{(1-t)^{n+1}}$ is called the *growing term*, denoted by $P_{\text{grow},Q}(t)$.

To prove that with the chosen support $C(t)$ has integral coefficients and Gorenstein symmetry, there are two different ways. The first way is through explicit calculations. Let $\sigma_i=\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^i}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})}$ as before.

Proposition 4.5.3. *$C(t)$ with chosen support in $[\gamma+1,\cdots,\gamma+r-1]$ is equal to*

$$t^{\gamma+1}\sum_{j=0}^{r-2}\Theta_j t^j,$$

where $\Theta_j=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left(\sigma_{\gamma+1+j-i}-\sigma_{\gamma-i}\right)$ and all $\Theta_j\in\mathbb{Z}$.

Proof By (4.8), one can see that $C(t)$ equals $(1-t)^n(\sum_{i=0}^{r-1}\sigma_{r-i}t^i)$ modulo $\frac{1-t^r}{1-t}$. For any integers k,m, we have

$$t^k\sum_{i=0}^{r-1}\sigma_i\,t^i\equiv\sum_{i=0}^{r-1}\sigma_i\,t^{i+k}\equiv\sum_{j=k}^{k+r-1}\sigma_{j-k}\,t^j\equiv$$
$$\sum_{j=0}^{r-1}\sigma_{j-k}\,t^j\equiv\sum_{j=0}^{r-2}\left(\sigma_{j-k}-\sigma_{r-k-1}\right)\,t^j,$$

where $\equiv$ denotes congruence modulo $1+t+\cdots+t^{r-1}$. This implies that

$$t^{-\gamma-1}(1-t)^n\sum_{j=0}^{r-1}\sigma_j\,t^j=\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}\,t^{i-\gamma-1}\right)\sum_{j=0}^{r-1}\sigma_j\,t^j\equiv$$
$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{j=0}^{r-2}\left(\sigma_{\gamma+1+j-i}-\sigma_{\gamma-i}\right)\,t^j=\sum_{j=0}^{r-2}\Theta_j\,t^j.$$

To prove that the Θ_j are all integers, we can rewrite their expressions as

$$\begin{aligned}
\Theta_j &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{i-\gamma-1-j}-\varepsilon^{i-\gamma}}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})}\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{-\gamma-1-j}-\varepsilon^{-\gamma}}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\varepsilon^i\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{(\varepsilon^{-\gamma-1-j}-\varepsilon^{-\gamma})(1-\varepsilon)^n}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})}.
\end{aligned}$$

Note that $\gcd(a_i, r) = 1$, for $i = 1, \ldots, n$, as the orbifold points are isolated, and therefore there exist some integers k_i such that $a_i k_i = 1 \bmod r$. Then the above equality can be written as

$$\Theta_j = \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}(\varepsilon^{-1-j}-1)\varepsilon^{-\gamma}\frac{1-\varepsilon^{a_1k_1}}{1-\varepsilon^{a_1}}\cdots\frac{1-\varepsilon^{a_nk_n}}{1-\varepsilon^{a_n}}.$$

For any $\alpha \in \mathbb{Z}$, one has $\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\varepsilon^{\alpha} = -1$ when $\alpha \neq 0 \bmod r$, and it is $r-1$ otherwise. Thus one has $\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}(\varepsilon^{\alpha}-\varepsilon^{\beta})f(\varepsilon) = 0 \bmod r$ for any polynomial f in ε. This proves that Θ_j are all integers. □

Now if we let $\gamma = \lfloor \frac{c}{2} \rfloor$, the symmetry can be seen in the following proposition.

Proposition 4.5.4. *Let Θ_j be as in Proposition 4.5.3. We then have the following:*

- *If the coindex $c = k_{\mathcal{X}} + n + 1$ is even, then $\sigma_{\lfloor \frac{c}{2}\rfloor - l} = (-1)^n\sigma_{\lfloor \frac{c}{2}\rfloor + l - n - 1}$ for all $l \in \mathbb{Z}$. In particular, $\Theta_{r-2} = 0$ and $\Theta_j = \Theta_{r-3-j}$ for $j = 0, 1, \ldots, r-3$.*
- *If the coindex c is odd, then $\sigma_{\lfloor \frac{c}{2}\rfloor - l} = (-1)^n\sigma_{\lfloor \frac{c}{2}\rfloor + l - n}$ for all $l \in \mathbb{Z}$. In particular, $\Theta_j = \Theta_{r-2-j}$ for $j = 0, 1, \ldots, r-2$.*

Proof When $c = k_{\mathcal{X}} + n + 1$ is even, then $\lfloor \frac{c}{2} \rfloor = \frac{c}{2}$. Taking into consideration that $k_{\mathcal{X}}+\sum_{i=1}^n a_i = 0 \mod r$, then

$$\begin{aligned}
\sigma_{\frac{c}{2}-l} &= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{-\frac{c}{2}+l}}{(1-\varepsilon^{a_i})\cdots(1-\varepsilon^{a_n})}\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{-\frac{c}{2}+l}}{\varepsilon^{a_1}(\varepsilon^{-a_1}-1)\cdots\varepsilon^{a_n}(\varepsilon^{-a_n}-1)}\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}(-1)^n\frac{\varepsilon^{-(a_1+\cdots+a_n)-\frac{c}{2}+l}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})}\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}(-1)^n\frac{\varepsilon^{\frac{c}{2}+l-n-1}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})} = (-1)^n\sigma_{\frac{c}{2}+l-n-1}.
\end{aligned}$$

Applying the above equality to $\sigma_{r-2-i+\frac{c}{2}+1}$, we have

$$\sigma_{r-2-i+\frac{c}{2}+1} = \sigma_{\frac{c}{2}-i+r-1} = \sigma_{\frac{c}{2}-(i-r+1)} = (-1)^n\sigma_{\frac{c}{2}+i-r+1-n-1} = (-1)^n\sigma_{\frac{c}{2}+i-n}$$

Thus,

$$\begin{aligned}
\Theta_{r-2} &= \sum_{i=0}^n(-1)^i\binom{n}{i}(\sigma_{r-2-i+\frac{c}{2}+1}-\sigma_{\frac{c}{2}-i})\\
&= \sum_{i=0}^n(-1)^i\binom{n}{i}\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{(-1)^n\varepsilon^{\frac{c}{2}+i-n}-\varepsilon^{\frac{c}{2}-i}}{(1-\varepsilon^{-a_1}\cdots(1-\varepsilon^{-a_n}))}\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{\frac{c}{2}}((-1)^n\sum_{i=0}^n(-1)^i\binom{n}{i}\varepsilon^{i-n}-\sum_{i=0}^n(-1)^i\binom{n}{i}\varepsilon^{-i})}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})}\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{-\frac{c}{2}}}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})}((-1)^n(\varepsilon-1)^n-(1-\varepsilon)^n) = 0,
\end{aligned}$$

and

$$\begin{aligned}
\Theta_{r-3-j}-\Theta_j &= \sum_{i=0}^n\binom{n}{i}(\sigma_{r-3-j-i+\frac{c}{2}+1}-\sigma_{j-i+\frac{c}{2}+1})\\
&= \sum_{i=0}^n\binom{n}{i}((-1)^n\sigma_{\frac{c}{2}+j+i+1-n}-\sigma_{j+\frac{c}{2}+1-i})\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{\frac{c}{2}+j+1}\sum_{i=0}^n\binom{n}{i}((-1)^n\varepsilon^{i-n}-\varepsilon^{-i})}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})} = 0.
\end{aligned}$$

Similarly for c odd. $\square$

Combining Proposition 4.5.3 and Proposition 4.5.4, we get the integral and Gorenstein properties for $P_{\mathrm{orb},Q}(t)$, which finishes the proof of Proposition 4.5.2.

4.5.2 Another proof of the Gorenstein symmetry of $P_{\mathrm{orb},Q}(t)$

Here is another way to see the integral and symmetric properties of $P_{\mathrm{orb},Q}(t)$.

Proposition 4.5.5. *Let Q be an isolated orbifold point of type $\frac{1}{r}(a_1,\dots,a_n)$, and $P_{\mathrm{orb},Q}(t)=\frac{C(t)}{(1-t)^n(1-t^r)}$ given as in Proposition 4.5.2. Then $C(t)$ can be determined by $\prod_{i=1}^n \frac{1-t^{a_ib_i}}{1-t^{a_i}}$ modulo $\frac{1-t^r}{1-t}$ with support in $[\lfloor\frac{c}{2}\rfloor+1,\lfloor\frac{c}{2}\rfloor+r-1]$, where $0<b_i<r$ satisfy $a_ib_i=1$ mod r. Moreover, $C(t)$ has naturally integral coefficients and satisfies Gorenstein symmetry.*

Proof Note that the inverse of $\prod_{i=1}^n\frac{1-t^{a_i}}{1-t}$ modulo $\frac{1-t^r}{1-t}$ can be given by $\prod_{i=1}^n\frac{1-t^{a_ib_i}}{1-t^{a_i}}$ with $0<b_i<r$, and $a_ib_i=1$ mod r. In fact,

$$\prod_{i=1}^n\frac{1-t^{a_i}}{1-t}\prod_{i=1}^n\frac{1-t^{a_ib_i}}{1-t^{a_i}}-1=\prod_{i=1}^n\frac{1-t^{a_ib_i}}{1-t}-1=\frac{t(1-t^{a_ib_i-1})}{1-t}=0 \bmod \frac{1-t^r}{1-t}.$$

Therefore

$$C(t)=\prod_{i=1}^n\frac{1-t^{a_ib_i}}{1-t^{a_i}}=\prod_{i=1}^n(1+t^{a_i}+\cdots+t^{(b_i-1)a_i}) \bmod \frac{1-t^r}{1-t}.$$

Thus, $C(t)$ will always have integral coefficients no matter which support of length $\le r-1$ we choose.

Using this characterization of $C(t)$, we can also see the symmetry quite easily. The idea is to find some natural palindromic polynomial first and then move it back to the support required. We will use the following lemma:

Lemma 4.5.6. 1. *Let $f(t)\in\mathbb{Q}[t]$ be a palindromic polynomial supported in $[\gamma+1,\gamma+l]$ with $0\le l\le r-1$. Then given $m\in\mathbb{Z}$, there is a unique polynomial $g(t)=f(t)$ mod $\frac{1-t^r}{1-t}$ supported in $[\gamma+mr+1,\gamma+(m+1)r-1]$, and obviously $g(t)$ is also palindromic.*

2. *If $f(t)\in\mathbb{Q}[t]$ is palindromic, supported in $[\gamma+1,\gamma+l-1]$, then there exists a palindromic polynomial $g(t)=f(t)$ mod $\frac{1-t^r}{1-t}$ with support in $[\gamma+\lfloor\frac{l}{2}\rfloor+2,\gamma+$*

$\lfloor \frac{l}{2} \rfloor + r]$ *when* l *is odd, and with support in* $[\gamma + \frac{l}{2} + 2, \gamma + \frac{l}{2} + r - 1]$ *when* l *is even.*

Proof For the first part, it is easy to see that we only need to shift the degree of each term up or down by $|mr|$. For the second part, just note that subtracting $a_1 t^{\gamma+1} \frac{1-t^r}{1-t}$ from $f(t)$ will cancel out two terms, namely $a_1 t^{\gamma+1} + a_1 t^{\gamma+l}$, and do the similar process to the resulting polynomial. We will finally obtain a palindromic polynomial with support as stated. $\square$

Now trim the polynomial $\prod_{i=1}^{n}(1 + t^{a_i} + \cdots + t^{(b_i-1)a_i})$ from its two ends by subtracting multiples of $(t^i + t^{\sum a_i(b_i-1)-i}) \frac{1-t^r}{1-t}$ until we reach a polynomial of length $\leq r$ around its symmetric center. If the length equals r, then we can still reduce it further to length $\leq r-2$ by deducting some multiple of $\frac{1-t^r}{1-t}$. The polynomial obtained this way, say $f(t)$, is equal to $\prod_{i=1}^{n}(1 + t^{a_i} + \cdots + t^{(b_i-1)a_i})$ modulo $\frac{1-t^r}{1-t}$, and it is obviously palindromic since $\prod_{i=1}^{n}(1 + t^{a_i} + \cdots + t^{(b_i-1)a_i})$ and $\frac{1-t^r}{1-t}$ are all palindromic.

To show that we can move $f(t)$ to the right support, we now analyze the situation case by case.

- When $\sum_{i=1}^{n} a_i(b_i-1)+1$ and r are both odd, then $\prod_{i=1}^{n}(1+t^{a_i}+\cdots+t^{(b_i-1)a_i})$ can be reduced by trimming from two ends into a palindromic polynomial supported in $[\frac{\sum a_i(b_i-1)-r+1}{2} + 1, \frac{\sum a_i(b_i-1)-r+1}{2} + r - 2]$.
- When $\sum_{i=1}^{n} a_i(b_i-1)+1$ is odd and r is even, then $\prod_{i=1}^{n}(1+t^{a_i}+\cdots+t^{(b_i-1)a_i})$ can be reduced into a polynomial of length $r-1$ supported in $[\frac{\sum a_i(b_i-1)-r}{2} + 1, \frac{\sum a_i(b_i-1)-r}{2} + r - 1]$.
- When $\sum_{i=1}^{n} a_i(b_i-1)+1$ and r are both even, then $\prod_{i=1}^{n}(1+t^{a_i}+\cdots+t^{(b_i-1)a_i})$ can be reduced into a polynomial of length $r-2$ supported in $[\frac{\sum a_i(b_i-1)-r+1}{2} + 1, \frac{\sum a_i(b_i-1)-r+1}{2} + r - 2]$.
- When $\sum_{i=1}^{n} a_i(b_i-1)+1$ is even and r is odd, then $\prod_{i=1}^{n}(1+t^{a_i}+\cdots+t^{(b_i-1)a_i})$ can be reduced into a polynomial of length $r-1$ supported in $[\frac{\sum a_i(b_i-1)-r+1}{2} + 1, \frac{\sum a_i(b_i-1)-r+1}{2} + r - 1]$.

Take the first case, for example. Note that using the fact that $\sum_{i=1}^{n} a_i + k_{\mathcal{X}} = 0 \bmod r$ (Proposition 4.1.3) and $a_i b_i = 1 \bmod r$, one has $\sum_{i=1}^{n} a_i(b_i - 1) - r + 1 = k_{\mathcal{X}} + n + 1 \bmod r$, i.e., $k_{\mathcal{X}} + n + 1 = \sum a_i(b_i - 1) - r + 1 + mr$ for some $m \in \mathbb{Z}$. If m is an even number, then $\frac{\sum a_i(b_i-1)-r+1}{2} - \frac{k_{\mathcal{X}}+n+1}{2}$ is a multiple of r. By Lemma 4.5.6

(1) above, we can move $f(t)$ to the support $[\frac{k_{\mathcal{X}}+n+1}{2}+1, \frac{k_{\mathcal{X}}+n+1}{2}+r-2]$, which gives $C(t)$. On the other hand, if m is an odd number, then using Lemma 4.5.6 (2), we know $g(t) = f(t) \bmod \frac{1-t^r}{1-t}$ supported in $[\frac{\sum a_i(b_i-1)-r+1}{2} + \frac{r-1}{2} + 1, \frac{\sum a_i(b_i-1)-r+1}{2} + \frac{r-1}{2} + r - 1]$ is also palindromic and $\frac{\sum a_i(b_i-1)-r+1}{2} + \frac{r-1}{2} - \lfloor \frac{k_{\mathcal{X}}+n+1}{2} \rfloor$ is a multiple of r, and therefore again using Lemma 4.5.6 (1) we can move the support to $g(t)$ to $[\lfloor \frac{k_{\mathcal{X}}+n+1}{2} \rfloor + 1, \lfloor \frac{k_{\mathcal{X}}+n+1}{2} \rfloor + r - 1]$ and get $C(t)$. Thus $C(t)$ is palindromic and $P_{\mathrm{orb,Q}}(t)$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$.

Using similar arguments for the remaining three cases, we are done. □

Remark 4.5.1. We will use the analysis in this proof several times in the next chapter. The idea is to prove that some fractional function is Gorenstein symmetric of certain degree, we can prove the numerator of the fraction is palindromic in certain support, see Remark 4.1.1.

Remark 4.5.2. Compare this with Proposition 4.5.4. From the analysis, we can also conclude that when the coindex $c = k+n+1$ is even, the numerator $C(t)$ of $P_{\mathrm{orb}(t)}$ is supported in $[\lfloor \frac{c}{2} \rfloor + 1, \lfloor \frac{c}{2} \rfloor + r - 2]$; and when the coindex c is odd, $C(t)$ is supported in $[\lfloor \frac{c}{2} \rfloor + 1, \lfloor \frac{c}{2} \rfloor + r - 1]$.

4.5.3 Program for calculating $P_{\mathrm{orb},Q}(t)$

In Proposition 4.5.3, the characterization of the numerator of $P_{\mathrm{orb}}(t)$ as inverse of $\prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t}$ enables us to write a computer program to calculate it. In fact, since the orbifold points are isolated, we know that gcd $(a_i, r) = 1$ and gcd $(\frac{1-t^{a_i}}{1-t}, \frac{1-t^r}{1-t}) = 1$. Therefore, there exist $\alpha(t)$ and $\beta(t)$ in $\mathbb{Q}[t]$ such that

$$\alpha(t) \prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t} + \beta(t) \frac{1-t^r}{1-t} = 1,$$

and thus $\alpha(t)$ is the inverse of $\prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t} \bmod \frac{1-t^r}{1-t}$. Thus $C(t)$ can be obtained by moving $\alpha(t)$ to the right support modulo $\frac{1-t^r}{1-t}$.

The following is a so-called *Qorb* program in MAGMA, which was first given by Reid in [Rei]. For each orbifold point Q of type $\frac{1}{r}(a_1, \ldots, a_n)$, we have $P_{\mathrm{orb},Q}(t) = \mathrm{Qorb}(r, [a_1, \ldots, a_n], k_{\mathcal{X}})$, where $k_{\mathcal{X}}$ is the canonical weight.

Program 4.5.7. `function Qorb(r,LL,k)`

```
L := [ Integers() | i : i in LL ];
if (k + &+L) mod r ne 0
```

```
   then error "Error: Canonical weight not compatible";
end if;
n := #LL;
Pi := &*[ R | 1-t^i : i in LL];
h := Degree(GCD(1-t^r, Pi));
l := Floor((k+n+1)/2+h);
de := Maximum(0,Ceiling(-l/r));
m := l + de*r;
A := (1-t^r) div (1-t);
B := Pi div (1-t)^n;
H,al_throwaway,be:=XGCD(A,t^m*B);
return t^m*be/(H*(1-t)^n*(1-t^r)*t^(de*r));
end function;
```

4.6 The initial term

The previous section established $P_{\mathrm{orb},Q}(t)$ for an orbifold point Q of type $\frac{1}{r}(a_1, \cdots, a_n)$, which is obtained by moving some growing part from $\frac{A(t)}{(1-t)^{n+1}}$ in Proposition 4.4.1 to the periodic term $P_{\mathrm{per},Q}(t)$. In this section, we will study the remaining part, i.e., the initial term

$$\frac{I(t)}{(1-t)^{n+1}} = P(t) - \sum_{Q \in \mathcal{B}} P_{\mathrm{orb},Q}(t),$$

which is denoted by $P_I(t)$. Note that $P_I(t)$ is also Gorenstein symmetric of weight $k_{\mathcal{X}}$. The following lemma tells us how to determine it.

Lemma 4.6.1. *Let $P(t) = \sum_{m \geq 0} P_m t^m$ be the Hilbert series of $(\mathcal{X}, H)$. The initial term of the form $\frac{I(t)}{(1-t)^{n+1}}$ is uniquely determined by the first $\lfloor \frac{c}{2} \rfloor$ coefficients of $P(t)$. Moreover,*

$$I(t) = \sum_{j=0}^{c} I_j t^j, \quad \textit{where } I_j = \begin{cases} \sum_{l=0}^{j} (-1)^{j-l} P_l \binom{n+1}{j-l}, & j \leq \lfloor \frac{c}{2} \rfloor \\ \sum_{l=0}^{c-j} (-1)^{c-j-l} P_j \binom{n+1}{c-j-l}, & \lfloor \frac{c}{2} \rfloor < j \leq c \end{cases}.$$

Proof For $c = k_{\mathcal{X}} + n + 1 < 0$, we have $P_I(t) = 0$ due to Gorenstein symmetry. In fact, if $I(t) \neq 0$ and we let $I(t) = \sum_{j=l_1}^{l_2} I_j t^j$, then because $P_I(t)$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$, we have $l_1 + l_2 - (n+1) = k_{\mathcal{X}}$ with l_1, $l_2 < 0$, which implies that

$l_1 \leq \frac{k_{\mathcal{X}}+n+1}{2}$. However, our $P_{\mathrm{orb},Q}(t)$ are designed to start from degree $\lfloor \frac{k+n+1}{2} \rfloor + 1$ as power series, which implies that $I(t)$ has to start from degree $\lfloor \frac{k+n+1}{2} \rfloor + 1$, that is, one has $l_1 \geq \lfloor \frac{k+n+1}{2} \rfloor + 1$, which is a contradiction. Therefore, $I(t) = 0$.

For $c > 0$, since all $P_{\mathrm{orb}}(t)$ will start from degree $\lfloor \frac{c}{2} \rfloor + 1$ as series, then the initial part $P_I(t)$ will have same coefficients as $P(t)$ up to and including degree $\lfloor \frac{c}{2} \rfloor$. By the Gorenstein symmetry of $P_I(t)$, we can deduce that $I(t)$ is palindromic of degree $c = k_{\mathcal{X}} + n + 1$. It follows that we can calculate $P_I(t)$ as follows: Let $A_0 = \sum_{i=0}^{\lfloor \frac{c}{2} \rfloor} P_i t^i$ and $A_1 = A_0(1-t)^{n+1}$. Then let a_i be the coefficient of t^i in A_1 for $i \in [0, \lfloor \frac{c}{2} \rfloor]$. Therefore, $A(t)$ is given by $\sum_{i=0}^{\lfloor \frac{c}{2} \rfloor} a_i(t^i + t^{c-i})$. Formally, it can be stated as in the lemma. $\square$

In particular, we can write out some special cases. For a dimension 2 orbifold with $k_{\mathcal{X}} = 0$ with coindex $c = 0 + 2 + 1 = 3$, if we let $P_1 = h^0(\mathcal{O}(H)) = g + 1$, then

$$P_I(t) = \frac{1 + (g-2)t + (g-2)t^2 + t^3}{(1-t)^3};$$

for a dimension 3 orbifold with $k_{\mathcal{X}} = -1$, if given that $P_1 = h^0(\mathcal{O}(H)) = g + 2$, one also has

$$P_I(t) = \frac{1 + (g-2)t + (g-2)t^2 + t^3}{(1-t)^4};$$

similarly, for a dimension 3 orbifold with $k_{\mathcal{X}} = 0$, one has

$$P_I(t) = \frac{1 + (P_1 - 4)t + (P_2 - 4P_1 + 6)t^2 + (P_1 - 4)t^3 + t^4}{(1-t)^4};$$

for a dimension 4 orbifold with $k_{\mathcal{X}} = -1$, one has

$$P_I(t) = \frac{1 + (P_1 - 5)t + (P_2 - 5P_1 + 10)t^2 + (P_1 - 5)t^2 + t^4}{(1-t)^5}.$$

Comparing this to the case for nonsingular varieties (see Example 4.1.1 or Remark 4.4.1), we see that the initial term is analogous to the Hilbert series of a smooth pair (X, H). However, we have to be careful with this. Even though $P_I(t)$ is designed to handle the first few plurigenera $P_1, \ldots, P_{\lfloor \frac{c}{2} \rfloor}$, it is definitely not the leading term of the Hilbert function controlling the order of growth of the plurigenera.

4.7 Examples and applications

4.7.1 Ideas of the application

Now given a quasismooth projectively Gorenstein orbifold pair $(\mathcal{X}, H)$ with only isolated orbifold points, Theorem 4.2.1 parses the Hilbert series according to orbifold points, in which the initial term is determined by the first $\lfloor \frac{c}{2} \rfloor$ plurigenera and the orbifold term is determined by the orbifold type and the canonical weight of the polarized pair. In some sense, we break the Hilbert series into pieces of "building blocks". Now we can put together different "building blocks" to build up Hilbert series.

The idea of using our parsing to construct orbifolds with certain invariants is the following: Given a required canonical weight $k_{\mathcal{X}}$ and the first $\lfloor \frac{c}{2} \rfloor$ plurigenera, we can write out an initial term as in Section 4.6. Given the number of orbifold points and their orbifold types, we can calculate the corresponding orbifold terms using Program 4.5.7. Then summing them together we get a Hilbert series $P(t)$. To find the graded ring. we then write $P(t)$ into a rational form as in (4.2) and guess the possible resolution of the graded ring and hence the structure of the ring. In the end, we need to check whether $\operatorname{Stac} R$ is an orbifold with required properties.

However, note that not every Hilbert series we obtain will give rise to a sensible ring that we can understand. Moreover, the rational form of the Hilbert series as in (4.2) cannot determine the resolution uniquely, since terms of the same degree but of different signs may cancel each other. Therefore, Hilbert series just suggest a possible resolution of the graded ring.

Since our rings are always Gorenstein, to get the ring from its Hilbert series, we can use the following structure theorem of Gorenstein rings.

Theorem 4.7.1. *(Serre, Buchsbaum–Eisenbud) Suppose R is of the form $\mathbb{C}[x_1, \dots, x_n]/I$ for some weighted homogeneous ideal I. Then R is Gorenstein if and only if I is a Gorenstein ideal. I is Gorenstein of codimension 1 if and only if I is principal; I is Gorenstein of codimension 2 if and only if I is a complete intersection; I is Gorenstein of codimension 3 if and only if I is defined by the $2m \times 2m$ Pfaffians of a $2m+1 \times 2m+1$ skew symmetric matrix of rank $2m$.*

For codimension 4 Gorenstein rings, unprojection techniques have been developed in [PR04a], [PR04b].

4.7.2 Examples of constructing orbifolds

In this section, we give some examples of our parsing theorem and show how to use it to construct orbifolds. Recall that $\mathcal{X}_d$ represents a general hypersurface defined by a homogeneous polynomial of degree d and $\mathcal{X}_{d_1,d_2}$ the complete intersection of two general hypersurfaces of degree d_1, d_2.

Example 4.7.1. *Consider the following three orbifolds who have trivial canonical sheaves. We choose the polarization to be $\mathcal{O}(1)$ for all of them.*

- $\mathcal{S}_5 = \operatorname{Stac} k[x,y,z,t]/(f_5) \subset \underline{\mathbb{P}}(1,1,1,2)$ *with an orbifold point* $Q = (0,0,0,1)$ *of type* $\frac{1}{2}(1,1)$*;*
- $\mathcal{S}_7 = \operatorname{Stac} k[x,y,z,t]/(f_7) \subset \underline{\mathbb{P}}(1,1,2,3)$ *with orbifold points* $Q_1 = (0,0,1,0)$ *of type* $\frac{1}{2}(1,1)$ *and* $Q_2 = (0,0,0,1)$ *of type* $\frac{1}{3}(1,2)$*;*
- $\mathcal{S}_{11} = \operatorname{Stac} k[x,y,z,t]/(f_{11}) \subset \underline{\mathbb{P}}(1,2,3,5)$ *with orbifold points* $Q_1 = (0,1,0,0)$ *of type* $\frac{1}{2}(1,1)$*,* $Q_2 = (0,0,1,0)$ *of type* $\frac{1}{3}(1,2)$ *and* $Q_3 = (0,0,0,1)$ *of type* $\frac{1}{5}(2,3)$*;*

where f_i is a general polynomial of degree i in the corresponding graded ring. All three surfaces have $k_{S_i} = 0$ and $c = 3$. The Hilbert series can be parsed into $P_{S_i}(t) = P_I + \sum_{Q \in \mathcal{B}_i} P_{\mathrm{orb},Q}(t)$, in which $P_I(t)$ can be calculated as in Lemma 4.6.1 and P_{orb} can be calculated using Program 4.5.7. This gives

$$
\begin{aligned}
P_{\mathcal{S}_5}(t) &= P_I(t) + P_{\mathrm{orb},Q}(t) \\
&= \frac{1+t^3}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^2)}; \\
P_{\mathcal{S}_7}(t) &= P_I(t) + P_{\mathrm{orb},Q_1}(t) + P_{\mathrm{orb},Q_2}(t) \\
&= \frac{1-t-t^2+t^3}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^2)} + \frac{t^2+t^3}{(1-t)^2(1-t^3)}; \\
P_{\mathcal{S}_{11}}(t) &= P_I(t) + P_{\mathrm{orb},Q_1}(t) + P_{\mathrm{orb},Q_2}(t) + P_{\mathrm{orb},Q_3}(t) \\
&= \frac{1-2t-2t^2+t^3}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^2)} + \frac{t^2+t^3}{(1-t)^2(1-t^3)} + \frac{2t^2+t^3+t^4+2t^5}{(1-t)^2(1-t^5)}.
\end{aligned}
$$

One notices the same contribution from orbifold points of the same type when the orbifolds have the same canonical weight. Next we want to use the above data to find more orbifolds with certain orbifold points. Since the orbifold term only depends on the orbifold type of the point and the canonical weight, in the rest of this section we will use notation $P_{\text{orb}}(\frac{1}{r}(a_1,\dots,a_n))$ to denote the orbifold term for the point of type $\frac{1}{r}(a_1,\dots,a_n)$ when the canonical weight is clear.

Example 4.7.2. *Based on the first example with $\mathcal{S}_5 \subset \underline{\mathbb{P}}(1,1,1,2)$, if we want one more orbifold point of the same type and keep the other invariants unchanged, then we have the following data:*

- $P_0 = 1,\ P_1 = 3$*;*
- 2 *points of type* $\frac{1}{2}(1,1)$.

We have $P_I = (1+t^3)/(1-t)^3$ *and* $P_{\text{orb}} = t^2/(1-t)^2(1-t^2)$ *as in the last example. Therefore we have*

$$P(t) = P_I + 2P_{\text{orb}}(\frac{1}{2}(1,1)) = \frac{1-t^3-t^4+t^7}{(1-t)^3(1-t^2)^2},$$

which suggests that such an orbifold can be a complete intersection $\mathcal{X}_{f,g}$ with f,g two general homogeneous polynomial of degree 3 and 4 respectively inside $\underline{\mathbb{P}}(1,1,1,2,2)$. We can easily analyze that it has two orbifold points at the intersection of $\{g=0\}$ with $\underline{\mathbb{P}}(2,2)$.

Next, based on $\mathcal{S}_{11} \subset \underline{\mathbb{P}}(1,2,3,5)$, we want to construct an orbifold with

- $P_0 = 1,\ P_1 = 1$*;*
- 2 *points of type* $\frac{1}{3}(1,2)$*;*
- 1 *point of type* $\frac{1}{5}(2,3)$.

Then similarly one can calculate

$$P_I = \frac{1-2t-2t^2+t^3}{(1-t)^3},$$

and then

$$P(t) = P_I(t) + 2P_{\text{orb}}(\frac{1}{3}(1,2)) + P_{\text{orb}}(\frac{1}{5}(2,3)) = \frac{1-t^6-t^8+t^{14}}{(1-t)(1-t^2)(1-t^3)^2(1-t^5)}.$$

Thus, this orbifold with the required property can be given by a complete intersection $\mathcal{X}_{6,8}$ in $\underline{\mathbb{P}}(1,2,3,3,5)$. With the same $P_I(t)$, if one wants 2 points of type $\frac{1}{5}(2,3)$

with no other orbifold points, then one has

$$P(t) = P_I(t) + 2P_{\mathrm{orb}}(\frac{1}{5}(2,3)) = \frac{1-t^6-t^{10}+t^{16}}{(1-t)(1-t^2)(1-t^3)(1-t^5)^2},$$

which gives an orbifold with trivial canonical sheaf and two orbifold points as required.

The above example just gives us some idea about how the combination of certain P_I and P_{orb} may give some reasonable Hilbert series $P(t)$ which helps us to find the required orbifolds. It is also very efficient to construct some higher dimensional canonical orbifolds. In the following, we give some examples, where we construct canonical 3-folds (canonical weight equals 1).

To do experiments, we give a program for the initial term. Suppose we want an orbifold of dimension n with canonical weight $k_{\mathcal{X}}$ with $P_0, P_1, \ldots, P_{\lfloor \frac{c}{2} \rfloor}$ as the first $\lfloor \frac{c}{2} \rfloor$ plurigenera. The following program gives us the initial term.

Program 4.7.2. `function initial(L,k,n)`

```
f:=&+[L[i]*t^(i-1): i in [1..#L]];
pp:=R!(f*(1-t)^(n+1));
c:=k+n+1;
if IsEven(c) eq true then
return (&+[Coefficient(pp, i )*(t^i+t^(c-i)):i in [0..c div 2-1]]+
Coefficient(pp,c div 2)*t^(Floor(c/2)))/(1-t)^(n+1);
else
return &+[Coefficient(pp,i)*(t^i+t^(c-i)):i in [0..Floor(c/2)]]
/(1-t)^(n+1);
end if;
end function;
```

Example 4.7.3. *Suppose we want to construct a canonical* 3 *fold with* $P_0 = 1$, $P_1 = 2$, $P_2 = 4$. *Suppose we can have* 0, 1 *or* 2 *of the orbifold points of types* $\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,2,2)$ *and* $\frac{1}{5}(1,1,2)$. *Using Program 4.5.7, we can calculate* $P_{\mathrm{orb}}(t)$ *for each of these points. Then we can search using the following MAGMA program:*

```
q1:= Qorb (5,[1,1,2],1);
q2:= Qorb (3,[1,2,2],1);
q3:= Qorb (2,[1,1,1],1);
```

```
for i,j, k in [0..2] do
p:= initial ([1,2,4],1,3)+i*q1+j*q2+k*q3;
p* &*[(1-t^i): i in [1,1,2,3,5]]; [i,j,k];
end for;
```

Analyzing the output of the above program, we have the following canonical 3-folds:

- $\mathcal{X}_{13} \subset \mathbb{P}(1,1,2,3,5)$ *has one orbifold point for each type above.*
- $\mathcal{X}_{16} \subset \mathbb{P}(1,1,2,3,8)$ *has* 1 *point of orbifold type* $\frac{1}{3}(1,2,2)$ *and* 2 *points of type* $\frac{1}{2}(1,1,1)$.
- $\mathcal{X}_{20} \subset \mathbb{P}(1,1,2,5,10)$ *has* 2 *points of orbifold type* $\frac{1}{5}(1,2,2)$ *and* 2 *points of type* $\frac{1}{2}(1,1,1)$.
- $\mathcal{X}_{6,10} \subset \mathbb{P}(1,1,2,3,3,5)$ *has* 2 *points of orbifold type* $\frac{1}{3}(1,2,2)$.
- $\mathcal{X}_{8,9} \subset \mathbb{P}(1,1,2,3,4,5)$ *has* 1 *point of type* $\frac{1}{5}(1,1,2)$ *and* 2 *points of type* $\frac{1}{2}(1,1,1)$.
- $\mathcal{X}_{8,10} \subset \mathbb{P}(1,1,2,3,5,5)$ *has* 1 *point of type* 2 *points of type* $\frac{1}{5}(1,2,2)$ *and* $\frac{1}{3}(1,2,2)$.

These are all the complete intersections and hypersurfaces under this condition. There are also codimension 3 *outputs. For example, one of the outputs tells us that a canonical* 3*-fold with* 2 *points of type* $\frac{1}{5}(1,1,2)$ *and one point of type* $\frac{1}{2}(1,1,1)$ *gives rise to the Hilbert series*

$$P(t) = \frac{1 - 2t^8 - 2t^9 - t^{10} + t^{12} + 2t^{13} + 2t^{14} - t^{22}}{(1-t)^2(1-t^2)(1-t^3)(1-t^4)(1-t^5)^2},$$

which suggests that the orbifold we want can be given as a codimension 3 *substack in* $\mathbb{P}(1,1,2,3,4,5,5)$ *defined by the* 4×4 *Pfaffians of a* 5×5 *skew symmetric matrix with the following degree in each entry:*

$$\begin{pmatrix} 6 & 5 & 5 & 4 \\ & 5 & 5 & 4 \\ & & 4 & 3 \\ & & & 3 \end{pmatrix}.$$

Let $x_1, x_2, y, z, t, s_1, s_2$ be the coordinates of $\mathbb{P}(1, 1, 2, 3, 4, 5, 5)$. Then we can fill out each entry of the matrix with a general polynomial of the indicated degrees to make sure that we get the right orbifold points. For example, we fill in as follows:

$$\begin{pmatrix} z^2 & s_1 & -s_2 & t \\ & x_1^5 + s_2 & -x_2^5 - s_1 & y^2 + t \\ & & y^2 - t & z + x_2^3 \\ & & & z + x_1^3 \end{pmatrix},$$

which gives 5 equations, that is, the 4×4 Pfaffians

$$\begin{aligned} pf_1 &= x_1^8 + x_2^8 + zs_1 + zs_2 + y^4 - t^2 + \cdots \\ pf_2 &= zs_1 + zs_2 + ty^2 - t^2 + s_1x_1^3 + s_2x_2^3 \\ pf_3 &= z^3 - s_1t + s_2t - x_2^5t + z + x_1^3 - tx_2^5 \\ pf_4 &= z^3 + s_1t - s_2t + x_1^5t + s_1y^2 + \cdots \\ pf_5 &= s_1^2 - s_2^2 - z^2t + \cdots \end{aligned}$$

We can check that this gives a quasismooth orbifold with 2 orbifold points of type $\frac{1}{5}(1, 1, 2)$ along $\mathbb{P}(5, 5)$ and 1 point of type $\frac{1}{2}(1, 1, 1)$ along $\mathbb{P}(2, 4)$. We can also vary the coefficients of the polynomials in the matrix, and in this way, we find a family of codimension 3 canonical 3-folds with required properties.

Similarly there are other outputs giving us candidates for codimension 3 canonical 3-folds. We donot list them one by one, but just remark that with help of computer programs, this method of finding varieties with certain invariants is quite efficient (compare with [ABR02] and [Rei00]). For the codimension 4 guys we need to use the unprojection method developed in [PR04a], [PR04b].

Chapter 5

Hilbert Series Parsing for Orbifold Loci of Dimension ≤ 1

As we have seen in Chapter 4, to parse Hilbert series according to orbifold loci is helpful for constructing orbifolds and is also interesting in its own right. In this chapter, we want to find an analogue parsing for the case when not only isolated singularities but also dimension 1 orbifold loci and dissident points appear. The difficulty here is how to separate the contributions from a dissident point and the curves it lies on. We can only give a tentative answer to this question, and we hope to give a complete description of the part concerning orbicurves in the future.

5.1 Statement of the theorem and some examples

We first recall from Chapter 3 the Riemann–Roch formula for a quasismooth projective stack with orbifold loci of dimension ≤ 1. Let $\mathcal{X}$ be a quasismooth projective stack of dimension n. Suppose $\mathcal{X}$ has a basket of orbifold curves $\mathcal{B}_{\mathcal{C}} = \{\text{curves of type } \frac{1}{r}(a_1, \dots, a_{n-1})\}$ and a basket of orbifold points $\mathcal{B}_P = \{\text{points of type } \frac{1}{s}(b_1, \dots, b_n)\}$. Then one has

$$\chi(\mathcal{O}_{\mathcal{X}}(d)) = \left[\mathrm{Ch}(\mathcal{O}_{\mathcal{X}}(d))\,\mathrm{Td}_{\mathcal{X}}\right]_n + \sum_{\mathcal{C} \in \mathcal{B}_{\mathcal{C}}} M_{\mathcal{C}} + \sum_{P \in \mathcal{B}_P} M_P,$$

where for each point of type $\frac{1}{s}(b_1, \ldots, b_n)$, the term M_P is given by

$$M_P = \frac{1}{s} \sum_{\varepsilon \in \mu_s, \varepsilon^{b_i} \neq 1} \frac{\varepsilon^d}{\prod_{i=1}^n (1 - \varepsilon^{-b_i})},$$

and for a curve of type $\frac{1}{r}(a_1, \ldots, a_{n-1})$, the term $M_{\mathcal{C}}$ is given by

$$\frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^d}{\prod (1 - \varepsilon^{-a_i})} d \deg H|_{\mathcal{C}} - \frac{1}{2r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^d}{\prod (1 - \varepsilon^{-a_i})} \deg K_{\mathcal{X}}|_{\mathcal{C}}$$
$$- \sum_{i=1}^{n-1} \frac{1}{2r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^d (1 + \varepsilon^{-a_i})}{(1 - \varepsilon^{-a_i})^2 \prod_{j \neq i} (1 - \varepsilon^{-a_j})} \deg \gamma_i,$$

where H is the associated divisor of $\mathcal{O}(1)$ and γ_i are the Chern roots of the normal bundle $\mathcal{N}$ of $\mathcal{C}$.

Using this formula, we can write the Hilbert series associated to $(\mathcal{X}, H)$ into the following form as we did for the isolated case in Section 4.4.

Proposition 5.1.1. *Let $\mathcal{X}$ be a quasismooth projective orbifold with polarization $\mathcal{O}(1)$. Let $\mathcal{B}_p$ and $\mathcal{B}_{\mathcal{C}}$ be the orbifold loci given above. Then the Hilbert series $P(t) = \sum_{d \geq 0} h^0(\mathcal{O}(d)) t^d$ can be written as*

$$P(t) = \frac{A(t)}{(1-t)^{n+1}} + \sum_{Q \in \mathcal{B}_P} P_{\mathrm{per},Q}(t) + \sum_{\mathcal{C} \in \mathcal{B}_{\mathcal{C}}} P_{\mathrm{per},\mathcal{C}}(t),$$

where $A(t)$ is a polynomial of degree $k_{\mathcal{X}} + n + 1$ if $k_{\mathcal{X}} \geq 0$; otherwise $A(t)$ is of degree n. The term $P_{\mathrm{per}}(t)$ for a point Q of type $\frac{1}{s}(b_1, \ldots, b_n)$ is given by

$$P_{\mathrm{per},Q}(t) = \frac{\sum_{i=1}^{s-1} \frac{1}{s} \sum_{\varepsilon \in \mu_s, \varepsilon^{b_i} \neq 1} \frac{\varepsilon^i}{(1-\varepsilon^{-b_1}) \cdots (1-\varepsilon^{-b_n})} t^i}{1 - t^s},$$

and the term $P_{\mathrm{per},\mathcal{C}}(t)$ for a curve $\mathcal{C}$ of type $\frac{1}{r}(a_1, \ldots, a_{n-1})$ is given by

$$\begin{aligned} P_{\mathrm{per},\mathcal{C}}(t) \quad = \quad & \frac{\sum_{i=1}^r i \sigma_i t^i}{1 - t^r} \deg H|_{\mathcal{C}} + \frac{(\sum_{i=1}^r \sigma_i t^i) t^r}{(1 - t^r)^2} r \deg H|_{\mathcal{C}} - \\ & \frac{\sum_{i=0}^{r-1} \sigma_i t^i}{1 - t^r} \frac{1}{2} \deg K_{\mathcal{X}}|_{\mathcal{C}} - \sum_{j=1}^{n-1} \frac{\sum_{i=0}^{r-1} \delta_{i,j} t^i}{1 - t^r} \frac{1}{2} \deg \gamma_j, \end{aligned}$$

where $\sigma_i = \sigma_i(\frac{1}{r}(a_1, \ldots, a_{n-1}))$ is given by $\frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^i}{(1-\varepsilon^{-a_1}) \cdots (1-\varepsilon^{-a_{n-1}})}$ and $\delta_{i,j} =$

$\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^i(1+\varepsilon^{-a_j})}{(1-\varepsilon^{-a_j})^2\prod_{i\neq j}(1-t^{a_i})}$. *The γ_i are given as before.*

PROOF To see this, just note that the first term in the Riemann–Roch formula is a polynomial in d of degree n and the contributions from points are periodic. Also note that

$$\frac{a_1t+2a_2t^2+\cdots+ra_rt^r}{1-t^r}+\frac{(a_1t+a_2t^2+\cdots+a_rt^r)rt^r}{(1-t^r)^2}$$
$$=a_1t+2a_2t^2+\cdots+(r-1)a_{r-1}t^{r-1}+ra_rt^r+(r+1)a_1t^{r+1}+$$
$$(r+2)a_2t^{r+2}+\cdots.$$

For more details, see the proof of Proposition 4.4. □

The above parsing roughly gives us how each orbifold locus appears in the Hilbert series, but we want a parsing with each of the parts corresponding to orbifold loci characterized in a closed form, analogue to Theorem 4.2.1. The following theorem parses the Hilbert series in such a way.

Theorem 5.1.2. *Let $\mathcal{X}$ be a quasismooth projective stack of dimension n with a polarization $\mathcal{O}(1)$. Suppose $(\mathcal{X},\mathcal{O}(1))$ is projectively Gorenstein, and $\mathcal{X}$ has a basket of orbifold curves $\mathcal{B}_\mathcal{C}=\{$curve $\mathcal{C}$ of type $\frac{1}{r}(a_1,\ldots,a_{n-1})\}$ and a basket of orbifold points $\mathcal{B}_Q=\{$point Q of type $\frac{1}{s}(b_1,\ldots,b_n)\}$. Then the Hilbert series associated to $(\mathcal{X},\mathcal{O}(1))$ can be uniquely parsed into the form*

$$P(t)=P_I(t)+\sum_{Q\in\mathcal{B}_Q}P_{\mathrm{orb},Q}(t)+\sum_{\mathcal{C}\in\mathcal{B}_\mathcal{C}}P_{\mathrm{orb},\mathcal{C}}(t),$$

where

1. *the initial term $P_I(t)$ is of the form $\frac{I(t)}{(1-t)^{n+1}}$, where $I(t)$ is a polynomial of degree $c=k_\mathcal{X}+n+1$ and palindromic. $P_I(t)$ has the same coefficients as $P(t)$ as power series up to and including degree $\lfloor\frac{c}{2}\rfloor$.*

2. *the orbifold term $P_{\mathrm{orb},Q}(t)$ for a point Q of type $\frac{1}{s}(b_1,\ldots,b_n)$ is given by $\frac{Q(t)}{(1-t)^nh(1-t^s)}$, where $h=\gcd\left((1-t^{b_1})\cdots(1-t^{b_n}),\frac{1-t^s}{1-t}\right)$ and $Q(t)$ is the inverse of $\prod\frac{1-t^{b_i}}{1-t}$ modulo $\frac{1-t^s}{(1-t)h}$ supported in $[\lfloor\frac{c}{2}\rfloor+1+\deg h,\lfloor\frac{c}{2}\rfloor+s-1]$. For each Q, the numerator $Q(t)$ has integral coefficients and $P_{\mathrm{orb},Q}(t)$ is Gorenstein symmetric of degree $k_\mathcal{X}$.*

3. *the orbifold term* $P_{\mathrm{orb},\mathcal{C}}(t)$ *for a curve* $\mathcal{C}$ *of type* $\frac{1}{r}(a_1,\ldots,a_{n-1})$ *can be given in two parts, that is,*

$$g_{\mathcal{C}}(t)\frac{S_{\mathcal{C},1}(t)}{(1-t)^{n-1}(1-t^r)^2}+\frac{S_{\mathcal{C},2}(t)}{(1-t)^n(1-t^r)},\tag{5.1}$$

where

- $S_{\mathcal{C},1}(t)$ *is given by the inverse of* $\prod_{i=1}^{n-1}\frac{1-t^{a_i}}{1-t}$ *mod* $\frac{1-t^r}{1-t}$, *supported in the integral* $[\lfloor\frac{c+r}{2}\rfloor+1,\lfloor\frac{c+r}{2}\rfloor+r-1]$. *Then* $S_{\mathcal{C},1}(t)$ *has integral coefficients and* $\frac{S_{\mathcal{C},1}(t)}{(1-t)^{n-1}(1-t^r)^2}$ *is Gorenstein symmetric of degree* $k_{\mathcal{X}}$.
- $g_{\mathcal{C}}(t)$ *is a Laurent polynomial with integral coefficients, which is supported in* $[-\lfloor\frac{r}{2}\rfloor+1,-\lfloor\frac{r}{2}\rfloor+r-1]$, *and* $g_{\mathcal{C}}(t)$ *is palindromic centered at degree* 0. *Moreover,* $g_{\mathcal{C}}(t)$ *is determined by the degree of the curve and the dissident points it passes through, as described in Section 5.3.1. In particular, when there are no dissident points on* $\mathcal{C}$, $g(t)=r\deg H|_{\mathcal{C}}$ *is an integer.*
- $\frac{S_{\mathcal{C},2}(t)}{(1-t)^n(1-t^r)}$ *has integral coefficients and is Gorenstein symmetric of degree* $k_{\mathcal{X}}$.

Remark 5.1.1. The point of this theorem is to state explicitly how each term is constructed from orbipoints, orbicurves, their normal bundle and the global canonical weight. However, to give a complete description of $S_{\mathcal{C},2}(t)$ in terms of the normal bundle of the curve is still work in progress.

We will prove this theorem in the later sections step by step. Now we want to give some examples to verify (or clarify) the statements in the theorem.

Example 5.1.1. *Let* $\mathcal{X}_{12}$ *be a general degree* 12 *hypersurface inside* $\mathbb{P}^4(1,2,2,3,4)$ *with polarization* $\mathcal{O}(1)$. *Then* $k_{\mathcal{X}}=0$ *and* $c=0+3+1=4$. *Note that it has an orbicurve* $C_{12}\subset\mathbb{P}(2,2,4)$ *of degree* $3/2$ *of type* $\frac{1}{2}(1,1)$. *The Hilbert series associated to* $(\mathcal{X},\mathcal{O}_{\mathcal{X}}(1))$ *can be parsed into*

$$P(t)=\frac{1-t^{12}}{(1-t)(1-t^2)^2(1-t^3)(1-t^4)}=P_I(t)+P_{\mathcal{C}}(t)$$

where

- $P_I(t)=\frac{1-3t+5t^2-3t^3+t^4}{(1-t)^4}$ *is the initial term. Written as power series,* $P_I(t)=1+t+3t^2+7t^3+\cdots$ *while* $P(t)=1+t+3t^2+4t^3+\cdots$.

- $P_{\mathcal{C}}(t) = 3\frac{-t^3}{(1-t)^2(1-t^2)^2}$. *Here we do not have the second part in (5.1)of the orbifold curve term (see a general statement in Proposition 5.3.8). The coefficient* 3 *is given by* $2\deg H|_{\mathcal{C}}$ *because there is no dissident points on the curve.*

Example 5.1.2. *Take a general hypersurface* $\mathcal{X}$ *of degree* 36 *inside* $\underline{\mathbb{P}}^5(1,4,5,6,9,10)$. *We can analyze the orbifold loci on* $\mathcal{X}$. *It has two types of orbifold points, namely the point* $P_1 = (0,\dots,0,1)$ *of type* $\frac{1}{10}(1,4,5,9)$ *and* 2 *points* P_2, P_3 *on the coordinate axis* $x_0 = x_1 = x_2 = x_5 = 0$ *of type* $\frac{1}{3}(1,1,1,2)$. *The* P_1 *is a dissident point, and it lives on the curve* $\mathcal{C} = \mathcal{C}_{36} \subset \underline{\mathbb{P}}(4,6,10)$ *of type* $\frac{1}{2}(1,1,1)$ *as well as the curve* $L = \underline{\mathbb{P}}(5,10)$ *of type* $\frac{1}{5}(1,4,4)$. *Given the polarization* $\mathcal{O}(1)$ *on* $\mathcal{X}$, *the associated Hilbert series* $P(t)$ *can be parsed into*

$$\begin{aligned} P(t) &= \frac{1-t^{36}}{(1-t)(1-t^4)(1-t^5)(1-t^6)(1-t^9)(1-t^{10})} \\ &= P_I(t) + P_{\mathrm{orb},\,P_1}(t) + P_{\mathrm{orb},\,P_2}(t) + P_{\mathrm{orb},\,P_3}(t) + P_{\mathrm{orb},\mathcal{C}}(t) + P_{\mathrm{orb},L}(t), \end{aligned}$$

where

- *the intial term* $P_I(t) = \frac{1-4t+6t^2-4t^3+6t^4-4t^5+t^6}{(1-t)^5}$.
- *the orbifold point terms are given by* $P_{\mathrm{orb},\,P_1}(t) = \frac{-t^9+t^{10}-t^{11}}{(1-t)^2(1-t^2)(1-t^5)(1-t^{10})}$, *and* $P_{\mathrm{orb},\,P_2}(t) = P_{\mathrm{orb},\,P_3}(t) = \frac{-t^4}{(1-t)^4(1-t^3)}$.
- *the orbifold curve term* $P_{\mathrm{orb},\mathcal{C}}(t) = 0\frac{-t^4}{(1-t)^3(1-t^2)^2}$ *and the orbifold curve term* $P_{\mathrm{orb},L}(t) = (t+1/t)\frac{t^7}{(1-t)^3(1-t^5)^2} + \frac{-2t^4-3t^5-2t^6}{(1-t)^4(1-t^5)}$.

Note that the degree of the curve $\mathcal{C} = \mathcal{C}_{36} \subset \underline{\mathbb{P}}(4,6,10)$ *is* $3/10$, *but gives no contribution in this parsing. This is because the dissident point* P_1 *"bites off" its contribution* $3/5\frac{-t^4}{(1-t)^3(1-t^2)^2}$. *Similarly, for the curve* $L = \underline{\mathbb{P}}(5,10)$, *which is of degree* $1/10$, *the dissident point* P_1 *"bites off"* $(-t+1/2-1/t)\frac{t^7}{(1-t)^3(1-t^5)^2}$ *from this curve contribution, and* $g_L(t)$ *is given by* $5\deg H|_{\mathcal{C}} - (-t+1/2-1/t) = (t+1/t)$. *We will explain what "bite off" means in Section 5.3.1.*

5.2 Contributions from dissident points

Now we start a proof of Theorem 5.1.2. We consider the formula in Proposition 5.1.1 piece by piece and try to adjust each of them to be of the form described in our theorem. Note that the parts corresponding to isolated orbifold points can be treated in the same way as in Chapter 4, so we only need to consider the remaining parts, namely the parts corresponding to orbifold curves and dissident points. This section deals with the contribution from dissident points.

For an orbifold point of type $\frac{1}{s}(b_1, \dots, b_n)$ (see Definition 5.2.1), *dissident* means that there exists some b_i such that $(s, b_i) \neq 1$. Furthermore, if we assume that the orbifolds we consider here only have orbifold loci of dimension ≤ 1, then there do not exist i, j such that $\gcd(s, b_i, b_j) \neq 1$. In this case, for each of the $w_i = \gcd(s, b_i) \neq 1$, there is a curve of type $\frac{1}{w_i}(\overline{b_1}, \dots, \widehat{b_i}, \dots, \overline{b_n})$ passing through this point, where $\widehat{b_i}$ means that b_i is omitted, and $\overline{b_j}$ gives the smallest nonnegative residue of b_j mod w_i.

Recall that the periodic term from a dissident point Q of type $\frac{1}{s}(b_1, \dots, b_n)$ in the Hilbert series is given by

$$P_{\mathrm{per},Q}(t) = \frac{\sum_{i=0}^{s-1} \frac{1}{s} \sum_{\varepsilon \in \mu_s, \varepsilon^{b_i} \neq 1} \frac{\varepsilon^i}{(1-\varepsilon^{-b_1}) \cdots (1-\varepsilon^{-b_n})} t^i}{1 - t^s}.$$

By Proposition 3.2.2, the numerator of $P_{\mathrm{per},Q}(t)$, denoted by $N_{\mathrm{per},Q}(t)$, satisfies

$$N_{\mathrm{per}}(t) \prod_{i=0}^{n} (1 - t^{b_i}) = 1 \mod \frac{1 - t^s}{(1-t)h}, \tag{5.2}$$

where $h = \gcd\left(\prod_{i=1}^{n}(1 - t^{b_i}), \frac{1-t^s}{1-t}\right)$. As in Chapter 4, we want to move some other parts in the Hilbert series to $P_{\mathrm{per},Q}(t)$ so that we obtain $P_{\mathrm{orb},Q}(t)$ with integral coefficients and satisfying the Gorenstein symmetric property.

Proposition 5.2.1. *Let $w_i = \gcd(s, b_i)$. There exists a unique $Q(t)$ supported in $[\lfloor \frac{c}{2} \rfloor + 1 + \deg h, \lfloor \frac{c}{2} \rfloor + s - 1]$ given by the equation*

$$\frac{Q(t)}{\prod_{i=1}^{n}(1 - t^{w_i})(1 - t^s)} = P_{\mathrm{per},Q}(t) + \frac{A(t)}{(1-t)^{n+1}} + \sum_{1 \leq i \leq n,\, w_i \neq 1} \frac{B_i(t)}{(1 - t^{w_i})^2}$$

where $A(t)$, $B_i(t)$ are some Laurent polynomials, and $Q(t)$ can be determined by

$$Q(t) \prod_{i=1}^{n} \frac{1 - t^{b_i}}{1 - t^{w_i}} = 1 \bmod \frac{1 - t^s}{(1-t)h},$$

that is, $Q(t)$ is the inverse of $\prod_{i=1}^{n} \frac{1-t^{b_i}}{1-t^{w_i}} \bmod \frac{1-t^s}{(1-t)h}$. Furthermore, $Q(t)$ has integral coefficients, and $\frac{Q(t)}{(1-t)^n h (1-t^s)}$, denoted by $P_{\mathrm{orb},Q}(t)$, is Gorenstein symmetric of *degree $k_{\mathcal{X}}$.*

Proof Note that the equality can be rewritten as

$$Q(t) = N_{\mathrm{per},Q}(t)(1-t)^n h + A(t)\frac{(1-t^s)h}{1-t} + \sum_{1\le i\le n,\, w_i\ne 1} B_i(t)(1-t)^n \frac{(1-t^s)h}{(1-t^{w_i})^2},$$

and in our case $\prod_{i=1}^n \frac{1-t^{w_i}}{1-t} = h$. Therefore, one can write the above equality as

$$Q(t) = N_{\mathrm{per},Q}(t)\prod_{i=1}^n (1-t^{w_i}) + \frac{1-t^s}{(1-t)h}(A(t)h^2 + \sum_{1\le i\le n,\, w_i\ne 1} B_i(t)\frac{(1-t)^{n+1}h^2}{(1-t^{w_i})^2}).$$

By the above equality and (5.2), we deduce that $Q(t)$ is the inverse of $\prod \frac{1-t^{b_i}}{1-t^{w_i}}$ mod $\frac{1-t^s}{(1-t)h}$.

Moreover, suppose $w_{i_1}, \dots, w_{i_k}$ are all the w_i that are not equal to 1. Then $h = \gcd\,(\prod_{i=1}^n (1-t^{b_i}), \frac{1-t^s}{1-t}) = \prod_{j=1}^k \frac{1-t^{w_{i_j}}}{1-t}$ and $\gcd\,(h^2, \frac{(1-t)^{n+1}h^2}{(1-t^{w_{i_1}})^2}, \dots, \frac{(1-t)^{n+1}h^2}{(1-t^{w_{i_k}})^2}) = 1$. Then by the same idea as in Lemma 4.5.1, there is a unique $Q(t)$ supported in $[\lfloor \frac{c}{2} \rfloor + 1 + \deg h, \lfloor \frac{c}{2} \rfloor + s - 1]$.

To see that $Q(t)$ has integral coefficients, note that the inverse of $\prod_{i=1}^n \frac{1-t^{b_i}}{1-t^{w_i}}$ can be given by $\prod_{i=1}^n \frac{1-t^{\alpha_i b_i}}{1-t^{b_i}}$ mod $\frac{1-t^s}{(1-t)h}$, where α_i is the smallest positive integer such that $\alpha_i b_i = w_i$ mod s. Since $Q(t)$ with length $\le s - \deg h - 1$ can be obtained by moving $\prod_{i=1}^n \frac{1-t^{\alpha_i b_i}}{1-t^{b_i}}$ modulo $\frac{1-t^s}{(1-t)h}$, we conclude that $Q(t)$ has integral coefficients.

To prove the Gorenstein symmetry of $P_{\mathrm{orb},Q}(t)$, we analyze analogously to the proof in Section 4.5.2. We reduce the support of the polynomial $\prod_{i=1}^n \frac{1-t^{\alpha_i b_i}}{1-t^{b_i}}$ modulo $\frac{1-t^s}{1-t}$ and then modulo $\frac{1-t^s}{(1-t)h}$. Since $\prod_{i=1}^n \frac{1-t^{\alpha_i b_i}}{1-t^{b_i}}$ and $\frac{1-t^r}{(1-t)h}$ as polynomials are both palindromic, we can prove that for the chosen support of $Q(t)$, the orbifold term $P_{\mathrm{orb},Q}(t)$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$. For a complete proof we need to analyze all cases as we did in Section 4.5.2. Here we show one of the cases, and the rest are similar.

Note that $\prod_{i=1}^n \frac{1-t^{\alpha_i b_i}}{1-t^{b_i}}$ is a polynomial of degree $\sum_{i=1}^n (\alpha_i - 1)b_i$ and $\frac{1-t^s}{1-t}$ is a polynomial of degree $s-1$. Suppose $\sum_{i=1}^n (\alpha_i - 1)b_i + 1$ and s are both even. Then by trimming $\prod_{i=1}^n \frac{1-t^{\alpha_i b_i}}{1-t^{b_i}}$ modulo $\frac{1-t^s}{1-t}$ from both ends, we obtain a palindromic polynomial of length $s-2$ supported in

$$[\frac{\sum_{i=1}^n (\alpha_i - 1)b_i - 1}{2} - \frac{s-2}{2} + 1, \frac{\sum_{i=1}^n (\alpha_i - 1)b_i - 1}{2} + \frac{s-2}{2}], \tag{5.3}$$

and by moving a bit forward (see Lemma 4.5.6, 2) we can also get another palindromic polynomial supported in

$$[\frac{\sum_{i=1}^n(\alpha_i-1)b_i+1}{2}+1, \frac{\sum_{i=1}^n(\alpha_i+1)b_i-1}{2}+s-2]. \tag{5.4}$$

Then we trim them further modulo $\frac{1-t^s}{(1-t)h}$. If $\deg h$ is even, then we obtain from (5.3) a palindromic polynomial supported in

$$[\frac{\sum_{i=1}^n(\alpha_i-1)b_i-1}{2}-\frac{s-2-\deg h}{2}+1, \frac{\sum_{i=1}^n(\alpha_i-1)b_i-1}{2}+\frac{s-2-\deg h}{2}],$$

and we obtain from (5.4) a palindromic polynomial supported in

$$[\frac{\sum_{i=1}^n(\alpha_i-1)b_i+1}{2}+\frac{\deg h}{2}+1, \frac{\sum_{i=1}^n(\alpha_i+1)b_i-1}{2}+\frac{\deg h}{2}+(s-\deg h-1)-1].$$

Notice that

$$\begin{aligned}
&\sum_{i=1}^n(\alpha_i-1)b_i-1-(s-2-\deg h)\\
=\ &\sum_{i=1}^n\alpha_ib_i-\sum_{i=1}^n b_i-1-s+\deg h+2\\
=\ &\sum_{i=1}^n(w_i-1)+n-\sum_{i=1}^n b_i-1-s+\deg h+2 \bmod s\\
=\ &2\deg h+n+k_{\mathcal{X}}+1 \bmod s,
\end{aligned}$$

since we know that $\sum_{i=1}^n b_i+k_{\mathcal{X}}=0 \bmod r$ and $\deg h=\sum_{i=1}^n(w_i-1)$. Therefore we can finally use Lemma 4.5.6, 1 to move the support to

$$[\frac{c}{2}+\deg h+1, \frac{c}{2}+s-2].$$

If $\deg h$ is odd, we just need to replace $\frac{\deg h}{2}$ by $\lfloor\frac{\deg h}{2}\rfloor$ and replace $\frac{c}{2}$ by $\lfloor\frac{c}{2}\rfloor$, and the rest of arguments are similar.

Thus we obtain in the end a palindromic polynomial with integral coefficients supported in $[\lfloor\frac{c}{2}\rfloor+\deg h+1, \lfloor\frac{c}{2}\rfloor+s-1]$ which is the inverse of $\prod_{i=1}^n\frac{1-t^{b_i}}{1-t^{w_i}}$ mod $\frac{1-t^s}{(1-t)h}$. □

Remark 5.2.1. We should remark here that we made a choice of the form for the dissident point contribution in our Hilbert series parsing. This choice gives us integral coefficients for the numerator of $P_{\rm orb}(t)$, but it also gives us the denominator of $P_{\rm orb}$ in the form $(1-t^{w_1})\cdots(1-t^{w_n})(1-t^s)$ for a dissident point of type $\frac{1}{s}(b_1,\ldots,b_n)$, where $w_i = \gcd\,(b_i, s)$. Using this choice, to obtain $P_{\rm orb}(t)$ we have to move some parts of the terms to $P_{\rm per}(t)$ from curves that pass through this point as well as some growing part (see Section 5.3.1).

Remark 5.2.2. As in Remark 4.5.2, we have a more precise description of the support of the palindromic polynomial $Q(t)$, that is, when the coindex $c = k_{\mathcal{X}}+n+1$ is even, the support of $Q(t)$ is in $[\lfloor\frac{c}{2}\rfloor+\deg h+1, \lfloor\frac{c}{2}\rfloor+s-2]$; when the coindex $c = k_{\mathcal{X}}+n+1$ is odd, the support of $Q(t)$ is in $[\lfloor\frac{c}{2}\rfloor+1, \lfloor\frac{c}{2}\rfloor+s-1]$.

Remark 5.2.3. Notice that $\prod\frac{1-t^{b_i}}{1-t}/h$ and $\frac{1-t^s}{1-t}/h$ have no common factors. Hence, we can calculate $Q(t)$ using the XGCD in the MAGMA program, i.e., the inverse of $\prod\frac{1-t^{b_i}}{1-t}/h$ mod $\frac{1-t^s}{1-t}/h$ is given by $\alpha(t)$ in the following equality:

$$\alpha(t)\prod\frac{1-t^{b_i}}{1-t}/h+\beta(t)\frac{1-t^s}{1-t}/h = 1,$$

and one can shift the support of $\alpha(t)$ to get $Q(t)$. The following program is analogue to Program 4.5.7, but it applies to a wider range of types of orbifold points (including the isolated case), that is, it applies to dissident points on curves or dissident points on a higher dimensional orbifold locus. The following program is obtained with help of M. Reid.

Program 5.2.2.
```
function Qorb(r,LL,k)
L := [Integers() | i : i in LL];
if (k + &+L) mod r ne 0
   then error "Error: Canonical weight not compatible";
end if;
n := #L; Pi := Denom(L);
A := (1-t^r) div (1-t); B := Pi div (1-t)^n;
H := GCD(A, B); M := &* [GCD(A, 1-t^i) : i in L];
shift := Floor(Degree(M*H)/2);
l := Floor((k+n+1)/2+shift+1);
de := Maximum(0,Ceiling(-l/r));
m := l + de*r;
G, al_throwaway, be := XGCD(A div H, t^m*B div M);
```

```
return t^m*be/(M*(1-t)^n*(1-t^r)*t^(de*r));
end function;
```

5.3 Contributions from curves

This section deals with the parts that correspond to orbicurves in our parsing. Recall from Proposition 5.1.1 that for an orbicurve of type $\frac{1}{r}(a_1, \ldots, a_{n-1})$, the original shape of its contribution to the Hilbert series is given by the following:

$$P_{\mathcal{C}}(t) = \frac{\sum_{i=1}^{r} i\sigma_i t^i}{1-t^r} \deg H|_{\mathcal{C}} + \frac{(\sum_{i=1}^{r} \sigma_i t^i) t^r}{(1-t^r)^2} r \deg H|_{\mathcal{C}} - \quad (5.5)$$

$$\frac{\sum_{i=0}^{r-1} \sigma_i t^i}{1-t^r} \frac{1}{2} k_{\mathcal{X}} \deg H|_{\mathcal{C}} - \sum_{j=1}^{n-1} \frac{\sum_{i=0}^{r-1} \delta_{i,j} t^i}{1-t^r} \frac{1}{2} \deg \gamma_j, \quad (5.6)$$

where $\sigma_i = \sigma_i(\frac{1}{r}(a_1, \ldots, a_{n-1}))$ is given by $\frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^i}{(1-\varepsilon^{-a_1}) \cdots (1-\varepsilon^{-a_{n-1}})}$ and $\delta_{i,j} = \frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^i (1+\varepsilon^{-a_j})}{(1-\varepsilon^{-a_j})^2 \prod_{i \neq j} (1-\varepsilon^{-a_i})}$.

We want to show that the above expression can be adjusted to the form $\frac{M(t)}{(1-t)^{n-1}(1-t^r)^2}$, which is Gorenstein symmetric of degree $k_{\mathcal{X}}$. We first deal with the parts related to the normal bundle, namely, $\frac{\sum_{i=0}^{r-1} \delta_{i,j} t^i}{1-t^r} \frac{1}{2} \deg \gamma_j$, for $1 \leq j \leq n-1$.

Lemma 5.3.1. *There exists a unique $N_j(t)$ supported in $[\lfloor \frac{c}{2} \rfloor + 1, \lfloor \frac{c}{2} \rfloor + r - 1]$ in the following:*

$$\frac{N_j(t)}{(1-t)^n (1-t^r)} = \frac{\sum_{i=0}^{r-1} \delta_{i,j} t^i}{1-t^r} + \frac{A_j(t)}{(1-t)^{n+1}},$$

for each $1 \leq j \leq n-1$. Moreover, $N_j(t)$ satisfies

$$N_j(t) \frac{1-t^{a_1}}{1-t} \cdots (\frac{1-t^{a_j}}{1-t})^2 \cdots \frac{1-t^{a_{n-1}}}{1-t} = 1 + t^{a_j} \bmod \frac{1-t^r}{1-t}$$

for all j. Consequently, $N_j(t)$ has integral coefficients and $\frac{N_j(t)}{(1-t)^n(1-t^r)}$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$.

Proof Observe that

$$(\sum_{i=0}^{r-1} \delta_{i,j} t^i)(1-t_1^a) \cdots (1-t^{a_j})^2 \cdots (1-t^{a_{n-1}}) = 1 + t^{a_j} \bmod \frac{1-t^r}{1-t}.$$

Then the rest follows as we did before. □

Now we are going to study the first three terms in (5.5). Putting these three terms together we have

$$\frac{\sum_{i=0}^{r-1}(i-\frac{k_{\mathcal{X}}}{2})\sigma_i t^i+\sum_{i=0}^{r-1}(r-i+\frac{k_{\mathcal{X}}}{2})\sigma_i t^{r+i}}{(1-t^r)^2}\deg H_{\mathcal{C}}.$$

By adding some growing term we can write this into the following form

$$\frac{N(t)}{(1-t)^{n-1}(1-t^r)^2}=\frac{\sum_{i=0}^{r-1}(i-\frac{k_{\mathcal{X}}}{2})\sigma_i t^i+\sum_{i=0}^{r-1}(r-i+\frac{k_{\mathcal{X}}}{2})\sigma_i t^{r+i}}{(1-t^r)^2}+\frac{V(t)}{(1-t)^{n+1}},$$

where $N(t)$ is supported in $[\lfloor\frac{c}{2}\rfloor+1,\lfloor\frac{c}{2}\rfloor+2r-2]$. Therefore $N(t)$ is given by

$$(\sum_{i=0}^{r-1}(i-\frac{k_{\mathcal{X}}}{2})\sigma_i t^i+\sum_{i=0}^{r-1}(r-i+\frac{k_{\mathcal{X}}}{2})\sigma_i t^{r+i})(1-t)^{n-1} \tag{5.7}$$

moved to the right support modulo $(\frac{1-t^r}{1-t})^2$.

Lemma 5.3.2. *$N(t)$ is a palindromic polynomial and $\frac{N(t)}{(1-t)^{n-1}(1-t^r)^2}$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$.*

Proof To prove that $N(t)$ is palindromic, the idea is that we first move the support of the polynomial (5.7) to $[\lfloor\frac{c}{2}\rfloor,\lfloor\frac{c}{2}\rfloor+2r-1]$ modulo $(1-t^r)^2$ and then move the support to $[\lfloor\frac{c}{2}\rfloor+1,\lfloor\frac{c}{2}\rfloor+2r-2]$ modulo $(\frac{1-t^r}{1-t})^2$. Note that for any integer b we have

$$\begin{aligned}&t^b(\sum_{i=0}^{r-1}(i-\frac{k_{\mathcal{X}}}{2})\sigma_i t^i+\sum_{i=0}^{r-1}(r-i+\frac{k_{\mathcal{X}}}{2})\sigma_i t^{r+i})\\ =\;&\sum_{i=0}^{r-1}(-b+i-\frac{k_{\mathcal{X}}}{2})\sigma_{-b+i}t^i+\sum_{i=0}^{r-1}(r+b-i+\frac{k_{\mathcal{X}}}{2})\sigma_{-b+i}t^{r+i}\bmod(1-t^r)^2.\end{aligned}$$

Now using this equality, for any integer γ we obtain

$$
\begin{aligned}
& t^{\gamma}(1-t)^{n-1}(\sum_{i=0}^{r-1}(i-\frac{k_{\mathcal{X}}}{2})\sigma_i t^i + \sum_{i=0}^{r-1}(r-i+\frac{k_{\mathcal{X}}}{2})\sigma_i t^{r+i}) \\
\equiv\ & \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}t^{\gamma+j}(\sum_{i=0}^{r-1}(i-\frac{k_{\mathcal{X}}}{2})\sigma_i t^i + \sum_{i=0}^{r-1}(r-i+\frac{k_{\mathcal{X}}}{2})\sigma_i t^{r+i}) \\
\equiv\ & \sum_{i=0}^{r-1}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(-(\gamma+j)+i-\frac{k_{\mathcal{X}}}{2})\sigma_{-(\gamma+j)+i}t^i + \\
& \sum_{i=0}^{r-1}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(r+(\gamma+j)-i+\frac{k_{\mathcal{X}}}{2})\sigma_{-(\gamma+j)+i}t^{r+i},
\end{aligned}
$$

where $\equiv$ means equality modulo $(1-t^r)^2$. Here we want to show that if we choose $\gamma = -\lfloor\frac{c}{2}\rfloor$, the last polynomial above is palindromic, and we denoted it by $L_\gamma(t)$.

Now let ρ_i be the coefficient of degree i in $L_\gamma(t)$. We show that when c is even, $\rho_{2r-1} = 0$ and $\rho_i = \rho_{2r-2-i}$; when c is odd, $\rho_i = \rho_{2r-1-i}$. Therefore $L_\gamma(t)$ is palindromic in the support $[0, \lfloor\frac{c}{2}\rfloor + 2r - 1]$. Here we only show it for the case when c is even; the other case is similar. When c is even, we have

$$
\begin{aligned}
\rho_{2r-1} &= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(r+(-\frac{c}{2}+j)-(r-1)+\frac{k_{\mathcal{X}}}{2})\sigma_{-(-\frac{c}{2}+j)+r-1} \\
&= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}j\sigma_{-\frac{c}{2}-j-1} + (\frac{1-n}{2})\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\sigma_{\frac{c}{2}-j-1} \\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq1}\frac{\varepsilon^{\frac{c}{2}-1}(\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}j\varepsilon^{-j})}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})} + \\
&\quad \frac{1-n}{2}\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq1}\frac{\varepsilon^{\frac{c}{2}-1}(\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\varepsilon^{-j})}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})} \\
&= (1-n)\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq1}\frac{\varepsilon^{\frac{c}{2}-2}(1-\varepsilon^{-1})^{n-2}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})} + \\
&\quad \frac{1-n}{2}\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq1}\frac{\varepsilon^{\frac{c}{2}-1}(1-\varepsilon^{-1})^{n-1}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})} \\
&= \frac{1-n}{2}\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq1}\frac{\varepsilon^{\frac{c}{2}-1}(1+\varepsilon^{-1})(1-\varepsilon^{-1})^{n-2}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})}.
\end{aligned}
$$

By the above expression, we can see that $\rho_{2r-1} = 0$ by the following fact:

$$\begin{aligned}
\rho_{2r-1} &= \frac{1-n}{2}\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{\varepsilon^{-\frac{c}{2}+1}(1+\varepsilon)(1-\varepsilon)^{n-2}}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_{n-1}})}\\
&= \frac{1-n}{2}\frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{(-1)^{n-2}\varepsilon^{-\frac{c}{2}+1+n-1}(1+\varepsilon^{-1})(1-\varepsilon^{-1})^{n-2}}{(-1)^{n-1}\varepsilon^{a_1+\cdots+a_{n-1}}(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})}\\
&= -\rho_{2r-1},
\end{aligned}$$

where the last equality is due to the fact that $\sum_{i=1}^{n-1} a_i = -k_{\mathcal{X}} \mod r$.

To prove $\rho_i = \rho_{2r-2-i}$, we first simplify the expression of ρ_i and ρ_{2r-2-i}. First, we have

$$\begin{aligned}
\rho_i &= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(-(-\frac{c}{2}+j)+i-\frac{k_{\mathcal{X}}}{2})\sigma_{-(-\frac{c}{2}+j)+i}\\
&= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(-j+\frac{n+1}{2}+i)\sigma_{-j+\frac{c}{2}+i}\\
&= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(-j)\sigma_{-j+\frac{c}{2}+i}+\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\sigma_{-j+\frac{c}{2}+i}(\frac{n+1}{2}+i)\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{(n-1)(1-\varepsilon^{-1})^{n-2}\varepsilon^{\frac{c}{2}+i-1}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})}+\\
&\quad \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{(1-\varepsilon^{-1})^{n-1}\varepsilon^{\frac{c}{2}+i}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})}(\frac{n+1}{2}+i),
\end{aligned}$$

where the last equality uses the binomial expansion $(1-t)^{n-1} = \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}t^j$ and its derivative $(n-1)t(1-t)^{n-2} = \sum_{j=0}^{n-1}(-1)^j j\binom{n-1}{j}t^j$.

Recall that in Proposition 4.5.4 we have $\sigma_{\frac{c}{2}-l} = (-1)^{n-1}\sigma_{\frac{c}{2}+l-n-1}$. Therefore we can simplify ρ_{2r-2-i} as follows:

$$\begin{aligned}
\rho_{2r-2-i} &= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(r+(-\frac{c}{2}+j)-(r-2-i)+\frac{k_{\mathcal{X}}}{2})\sigma_{-(-\frac{c}{2}+j)+r-2-i}\\
&= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(j+\frac{-n+3}{2}+i)\sigma_{\frac{c}{2}-i-j-2}\\
&= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(-(n-1-j)+\frac{n+1}{2}+i)(-1)^{n-1}\sigma_{\frac{c}{2}+i+j+2-n-1}\\
&= (-1)^n\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(n-1-j)\sigma_{\frac{c}{2}+i-((n-1)-j)}+\\
&\quad (-1)^{n-1}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\sigma_{\frac{c}{2}+i-((n-1)-j)}(\frac{n+1}{2}+i)\\
&= \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{(n-1)(1-\varepsilon^{-1})^{n-2}\varepsilon^{\frac{c}{2}+i-1}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})}+\\
&\quad \frac{1}{r}\sum_{\varepsilon\in\mu_r,\varepsilon\neq 1}\frac{(1-\varepsilon^{-1})^{n-1}\varepsilon^{\frac{c}{2}+i}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})}(\frac{n+1}{2}+i),
\end{aligned}$$

where the last equality uses the fact that $\sum_{j=0}^{n-1}(-1)^j(n-1-j)\varepsilon^{-(n-1-j)}=(n-1)\varepsilon^{-1}(\varepsilon^{-1}-1)^{n-2}$.

Therefore, $\rho_i=\rho_{2r-2-i}$ for all $0\leq i\leq r-1$. Since $N(t)$ supported in $[\lfloor\frac{c}{2}\rfloor+1,\lfloor\frac{c}{2}\rfloor+2r-2]$ is given by $L_{-\lfloor\frac{c}{2}\rfloor}$ modulo the palindromic polynomial $(\frac{1-t^r}{1-t})^2$, we obtain that $N(t)$ is also palindromic. □

Combining Lemma 5.3.1 and Lemma 5.3.2, we can get the following conclusion, which gives a global view of the curve contribution in our Hilbert series parsing.

Proposition 5.3.3. *The total contribution from a curve $\mathcal{C}$ of type $\frac{1}{r}(a_1,\ldots,a_{n-1})$ is given by*

$$\frac{M(t)}{(1-t)^{n-1}(1-t^r)}=\frac{N(t)}{(1-t)^{n-1}(1-t^r)^2}\deg H|_{\mathcal{C}}+\sum_{j=1}^{n-1}\frac{N_j(t)}{(1-t)^n(1-t^r)}\frac{\deg\gamma_i}{2},$$

and we this denote by $P_{\mathcal{C}}(t)$. Moreover, $P_{\mathcal{C}}(t)$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$.

Even though we can prove that the curve contribution $P_{\mathcal{C}}(t)$ has the Gorenstein symmetry property, we cannot characterize it as a whole using some ice cream function (or InverseMod function) as we did for point contributions. However, we can give a characterization as an ice cream function for the "order 2" part, more precisely, we can put the part

$$P_{\text{per},\mathcal{C}}(t) = \frac{t^r \sum_{i=1}^{r} \frac{1}{r} \sum_{\varepsilon \in \mu_r, \varepsilon \neq 1} \frac{\varepsilon^i}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_{n-1}})} t^i}{(1-t^r)^2}$$

into an ice cream function by the following lemma.

Lemma 5.3.4. *There exists a unique $S_1(t)$ supported in $[\lfloor \frac{c+r-1}{2} \rfloor + 1, \lfloor \frac{c+r-1}{2} \rfloor + r - 1]$, satisfying*

$$\frac{S_1(t)}{(1-t)^{n-1}(1-t^r)^2} = P_{\text{per},\mathcal{C}}(t) + \frac{B(t)}{1-t^r} + \frac{A(t)}{(1-t)^{n+1}}.$$

Consequently, $S_1(t)$ can be determined by the inverse of $\prod \frac{1-t^{a_i}}{1-t} \bmod \frac{1-t^r}{1-t}$ with the chosen support. Moreover, $S_1(t)$ has integral coefficients and $\frac{S_1(t)}{(1-t)^{n-1}(1-t^r)^2}$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$.

Proof See proof of Proposition 4.5.2. □

Taking into consideration of the coefficient of $P_{\text{per},\mathcal{C}}(t)$, we know that

$$P_{\mathcal{C}}(t) - r \deg H|_{\mathcal{C}} \frac{S_1(t)}{(1-t)^{n-1}(1-t^r)^2}$$

is of the form $\frac{S_2(t)}{(1-t)^n(1-t^r)}$, which is also Gorenstein symmetric of degree $k_{\mathcal{X}}$. That is,

$$P_{\mathcal{C}}(t) = r \deg H|_{\mathcal{C}} \frac{S_1(t)}{(1-t)^{n-1}(1-t^r)^2} + \frac{S_2(t)}{(1-t)^n(1-t^r)},$$

and we denote the first part by $P_{\mathcal{C},1}(t)$ and the second part by $P_{\mathcal{C},2}(t)$.

For curves without dissident points this gives a nice form, since $r \deg H|_{\mathcal{C}}$ is an integer (see proof of Proposition 5.3.6). When there are no dissident points on the curve $\mathcal{C}$, then $P_{\mathcal{C}}(t)$ gives us the $P_{\text{orb},\mathcal{C}}(t)$ in our theorem 5.1.2. However, when there are dissident points on the curve, the number $r \deg H|_{\mathcal{C}}$ is possibly fractional. We will see in the next section how orbifold terms we chose for the dissident points affect the number $r \deg H|_{\mathcal{C}}$.

5.3.1 Orbicurves with dissident points

Recall that in Proposition 5.2.1, choosing $P_{\mathrm{orb},Q}(t)$ in our parsing, we need to move some parts from the curve terms in the Hilbert series to $P_{\mathrm{per},\,Q}(t)$. We will see that after subtracting all the parts which the dissident points "bite off", the remaining curve contributions have integral coefficients. Here we can only measure how much the dissident point "bites off" from the first part of the curve contribution $P_{\mathcal{C},1}(t)$. We cannot control precisely how the dissident points affect the second part $P_{\mathcal{C},2}(t)$, but we can prove what each dissident point "bites off" from the second part is Gorenstein symmetric of degree $k_{\mathcal{X}}$.

Proposition 5.3.5. *Let Q be a dissident point of type $\frac{1}{s}(b_1,\ldots,b_n)$. Let $w_i=(s,b_i)$ and $P_{\mathrm{orb},Q}(t)=\frac{Q(t)}{(1-t^{w_1})\cdots(1-t^{w_n})(1-t^s)}$ be the term given in Proposition 5.2.1. Then when $w_i\neq 1$, there is a curve $\mathcal{C}_i$ of type $\frac{1}{w_i}(\overline{b_1},\ldots,\widehat{b_i},\ldots,\overline{b_n})$ passing through this point. Then the point Q bites off the following contribution from $P_{\mathcal{C}_i,1}(t)$:*

$$\mathrm{bit}_{Q,w_i}(t)\frac{S_{1,w_i}(t)}{(1-t)^{n-1}(1-t^{w_i})^2},$$

where $S_{1,w_i}(t)$ is given as in Lemma 5.3.4. The coefficient $\mathrm{bit}_{Q,w_i}(t)$ is a Laurent polynomial supported in $[-\lfloor\frac{w_i}{2}\rfloor+1,\lfloor\frac{w_i}{2}\rfloor-1]$ and is Gorenstein symmetric of degree 0 (in the sense that $\mathrm{bit}_{Q,w_i}(t)=(t)^0\mathrm{bit}_{Q,w_i}(1/t)$), determined uniquely by

$$\mathrm{bit}_{Q,w_i}(t)=\frac{w_i}{s}Q(t)\prod_{j\neq i}\frac{1-t^{b_j}}{1-t^{w_j}}\mod\frac{1-t^{w_i}}{1-t},$$

Moreover, $\mathrm{bit}_{Q,w_i}(t)$ has integral coefficients except for the constant term.

Proof Recall from Proposition 5.2.1 that the orbifold term $P_{\mathrm{orb}}(t)$ for a dissident point Q of type $\frac{1}{s}(b_1,\ldots,b_n)$ is given by

$$\frac{Q(t)}{\prod_{i=1}^n(1-t^{w_i})(1-t^s)}=\frac{N_{\mathrm{per},Q}(t)}{1-t^s}+\frac{A(t)}{(1-t)^{n+1}}+\sum_{1\le i\le n,w_i\neq 1}\frac{B_i(t)}{(1-t^{w_i})^2}.$$

We can rewrite this in the form

$$\frac{Q(t)}{\prod_{i=1}^n(1-t^{w_i})(1-t^s)}=\frac{N_{\mathrm{per},Q}(t)}{1-t^s}+\frac{A'(t)}{(1-t)^{n+1}}$$
$$+\sum_{1\le i\le n,w_i\neq 1}(\mathrm{bit}_{Q,w_i}(t)\frac{S_{1,w_i}(t)}{(1-t)^{n-1}(1-t^{w_i})^2}+\frac{D_i(t)}{(1-t)^n(1-t^{w_i})}),$$

which gives

$$
\begin{aligned}
Q(t) &= N_{\mathrm{per},Q}(t)\prod_{i=1}^{n}(1-t^{w_i}) + \frac{1-t^s}{(1-t)h}(A'(t)h^2 + \\
&\sum_{1\le i\le n, w_i\ne 1}(\mathrm{bit}_{Q,w_i}S_{1,w_i}(t) + D_i(t)\frac{1-t^{w_i}}{1-t})(\prod_{j\ne i}\frac{1-t^{w_j}}{1-t})^2),
\end{aligned}
$$

where $h = \gcd\,(\prod_{i=1}^{n}(1-t^{b_i}), \frac{1-t^s}{1-t}) = \prod_{i=1}^{n}\frac{1-t^{w_i}}{1-t}$ and each $S_{1,w_i}(t)$ is the inverse of $\prod_{j\ne i}\frac{1-t^{b_j}}{1-t} \mod \frac{1-t^{w_i}}{1-t}$. Note that h^2 and $(\prod_{j\ne i}\frac{1-t^{w_j}}{1-t})^2$, for $i = 1, \ldots, n$, are coprime, which ensures that we can move $Q(t)$ to the right support; $S_{1,w_i}(t)$ and $\frac{1-t^{w_i}}{1-t}$ are coprime, which enables us to choose $\mathrm{bit}_{Q,w_i}(t)$ modulo $\frac{1-t^{w_i}}{1-t}$. We claim that we can choose $\mathrm{bit}_{Q,w_i}(t)$ to be Gorenstein symmetric of degree 0. In fact, by the above equality, we know that bit_{Q,w_i} satisfies

$$
\begin{aligned}
\mathrm{bit}_{Q,w_i}(t) &\equiv Q(t)\frac{(1-t)h}{1-t^s}(\prod_{j\ne i}\frac{1-t}{1-t^{w_j}})^2\prod_{j\ne i}\frac{1-t^{b_j}}{1-t} \\
&\equiv Q(t)\frac{1}{1+t^{w_i}+\cdots+t^{\frac{s}{w_i}-1}}\prod_{j\ne i}\frac{1-t^{b_j}}{1-t^{w_j}} \\
&\equiv \frac{w_i}{s}Q(t)\prod_{j\ne i}\frac{1-t^{b_j}}{1-t^{w_j}} \mod \frac{1-t^{w_i}}{1-t}.
\end{aligned}
$$

Since $Q(t)$ and $\prod_{j\ne i}\frac{1-t^{b_j}}{1-t^{w_j}}$ are symmetric, we deduce that $\mathrm{bit}_{Q,w_i}(t)$ can be reduced to be Gorenstein symmetric of degree 0 modulo $\frac{1-t^{w_i}}{1-t}$ (This can be done as in Section 4.5.2). Moreover, we know that the constant part of bit_{Q,w_i} is given by $\frac{\alpha_i w_i}{s}$ plus an integer, where α_i is the smallest positive integer such that $\alpha_i b_i = w_i \mod s$, and apart from the constant term, bit_{Q,w_i} has integral coefficients. In fact, recall that $Q(t) = \prod_{i=1}^{n}\frac{1-t^{\alpha_i b_i}}{1-t^{b_i}} + \beta(t)\frac{1-t^s}{(1-t)h}$. We plug this into the above equality and get

$$
\begin{aligned}
\mathrm{bit}_{Q,w_i}(t) &\equiv \frac{w_i}{s}(\prod_{i=1}^{n}\frac{1-t^{\alpha_i b_i}}{1-t^{b_i}} + \beta(t)\frac{1-t^s}{(1-t)h})\prod_{j\ne i}\frac{1-t^{b_j}}{1-t^{w_j}} \\
&\equiv (\frac{w_i}{s}\frac{1-t^{\alpha_i b_i}}{1-t^{b_i}}\prod_{j\ne i}\frac{1-t^{\alpha_i b_j}}{1-t^{w_j}} + \frac{w_i}{s}\beta(t)\frac{1-t^s}{1-t^{w_i}}\prod_{j\ne i}\frac{1-t^{\beta_j w_j}}{1-t^{w_j}}\prod_{j\ne i}\frac{1-t^{b_j}}{1-t^{w_j}}) \\
&\equiv (\frac{\alpha_i w_i}{s} + \beta(t)\prod_{j\ne i}\frac{1-t^{\beta_j w_j}}{1-t^{w_j}}\prod_{j\ne i}\frac{1-t^{b_j}}{1-t^{w_j}}) \mod \frac{1-t^{w_i}}{1-t},
\end{aligned}
$$

where β_j satisfies $\beta_j w_j = 1 \bmod w_i$. Such β_j exist because w_j and w_i are coprime for $j \neq i$. Note that the second part of the last equality, $\beta(t)\prod_{j\neq i}\frac{1-t^{\beta_j w_j}}{1-t^{w_j}}\prod_{j\neq i}\frac{1-t^{b_j}}{1-t^{w_j}}$, is a polynomial with integral coefficients. Since $\mathrm{bit}_{Q,w_i}(t)$ is uniquely determined with chosen support, then the constant term of $\mathrm{bit}_{Q,w_i}(t)$ is given by $\frac{\alpha_i w_i}{s}$ plus some integer, and apart from the constant term, it has only integral coefficients. □

Here we give one example to explain the last proposition.

Example 5.3.1. *Given a point Q of type $\frac{1}{10}(1,4,5,9)$, we have $w_1 = w_4 = 1$, $w_2 = 2$ and $w_3 = 5$. Then Q lies on both a curve of type $\frac{1}{2}(1,1,1)$ and a curve of type $\frac{1}{5}(1,4,4)$. By the last proposition, it bites off bit_{Q,w_2} from the curve of type $\frac{1}{2}(1,1,1)$, which is given by $3/5\frac{S_{1,w_2}(t)}{(1-t)^3(1-t^2)^2}$. In fact, $P_{\mathrm{orb},Q}(t)$ can be calculated using Program 5.2.2, which gives us*

$$P_{\mathrm{orb},Q}(t) = \frac{-t^9+t^{10}-t^{11}}{(1-t)^2(1-t^2)(1-t^5)(1-t^{10})}.$$

By the above proposition, we know that

$$\mathrm{bit}_{Q,w_2}(t) = \frac{2}{10}(-t^9+t^{10}-t^{11})\frac{1-t^9}{1-t} \bmod \frac{1-t^2}{1-t} = 3/5.$$

Similarly, for $\mathrm{bit}_{Q,w_3}(t)$ we have

$$\mathrm{bit}_{Q,w_3}(t) = \frac{5}{10}(-t^9+t^{10}-t^{11})\frac{1-t^4}{1-t^2}\frac{1-t^9}{1-t} \bmod \frac{1-t^5}{1-t} = -t+1/2-1/t.$$

Remark 5.3.1. Note that the parts, which a dissident point bites off from each of the curves it lies on, are determined by its orbifold type and do not depend on the ambient orbifold it lives in.

Now we know how each dissident point affects the curves it lies on. Given a curve $\mathcal{C}$ of type $\frac{1}{r}(a_1,\dots,a_{n-1})$ with a set $\mathcal{T}$ of dissident points on it, we have the following:

Proposition 5.3.6. *$r\deg H|_{\mathcal{C}} - \sum_{Q\in\mathcal{T}}\mathrm{bit}_{Q,r}(t)$ has integral coefficients and is Gorenstein symmetric of degree 0.*

Proof The only thing we need to prove is that its constant term is an integer. Since we only consider the case when there are only orbifold loci of dimension ≤ 1, for each dissident point $Q \in \mathcal{T}$ of type $\frac{1}{s_Q}(b_{Q,1},\dots,b_{Q,n})$, there exists exactly one

$b_{Q,i}$ such that gcd $(b_{Q,i}, s_Q) = r$ and gcd $(b_{Q,j}, r) = 1$ for all $j \neq i$. For convenience, we denote this $b_{Q,i}$ by b_Q. Recall that the constant term each dissident point bites off from the curve contribution is given by $\frac{\alpha_Q r}{s_Q}$ plus some integer, where $\alpha_Q b_Q = r$ mod s_Q, which is also equivalent to $\alpha_Q \frac{b_Q}{r} = 1$ mod $\frac{s_Q}{r}$.

Since this only concerns the curve $\mathcal{C}$, we can restrict the problem to $\mathcal{C}$. Suppose $\mathcal{C}$ is defined by I in $\underline{\mathbb{P}}(c_1, \ldots, c_l)$, where the c_i are divisible by r. Consider the curve $\mathcal{C}'$ defined by the same ideal I in $\underline{\mathbb{P}}(\frac{c_1}{r}, \ldots, \frac{c_l}{r})$ with shifted weights for each of the variables. Then the degree $\deg H'|_{\mathcal{C}'}$ of the curve $\mathcal{C}'$ is given by $r \deg H|_{\mathcal{C}}$. The dissident point Q restricted to the curve $\mathcal{C}'$ is an orbifold point of type $\frac{1}{s_Q/r}(\frac{b_Q}{r})$. Recall in Section 3.3.1 that the Euler characteristic of $\mathcal{O}_{\mathcal{C}'}(1)$ is given by $\chi(\mathcal{O}_{\mathcal{C}'}) + \deg H'|_{\mathcal{C}'} - \sum_Q \frac{\alpha_Q}{s_Q/r}$ if the curve has orbifold points of type $\frac{1}{s_Q/r}(b_Q/r)$. Thus, we see that $r \deg H|_{\mathcal{C}} - \sum_Q \frac{\alpha_Q r}{s_Q} = \deg H'|_{\mathcal{C}'} - \sum_Q \frac{\alpha_Q}{s_Q/r} = \chi(\mathcal{O}_{\mathcal{C}'}(1)) - \chi(\mathcal{O}_{\mathcal{C}'})$ is an integer. We are done. □

Example 5.3.2. *Now we can return to Example 5.1.2 to work out the coefficients for the $P_{C,1}(t)$ for each of the curves.*

So far, we have only considered how dissident points on the curve affects the first part $P_{\mathcal{C},1}(t)$ of the curve term. Now we want to see how the second term $P_{\mathcal{C},2}(t)$ is affected by the dissident points. Note that even though we cannot control precisely the parts that the dissident points bite off from the second piece $P_{\mathcal{C},2}(t)$, we can assert the following:

Proposition 5.3.7. *Let Q be an orbifold point of type $\frac{1}{s}(b_1, \ldots, b_n)$ on $\mathcal{X}$. Suppose $w_i = \gcd(s, b_i) \neq 1$ for $1 \leq i \leq l$ (possibly after reordering the b_i), and let $\mathcal{C}_i$ be the orbifold curve of type $\frac{1}{w_i}(b_1, \ldots, \widehat{b_i}, \ldots, b_n)$ that passes through Q. Then with the $P_{\mathrm{orb},Q}(t)$ given in Proposition 5.2, $P_{\mathrm{orb},Q}(t)$ "bites off" from the second part $P_{\mathcal{C}_i,2}$ a rational function that is Gorenstein symmetric of degree $k_{\mathcal{X}}$.*

Proof Note that the numerator $N_{\mathrm{per},Q}(t)$ of the periodic term $P_{\mathrm{per},Q}(t)$ is divisible by $h(t) = \prod_{i=1}^{n} \frac{1-t^{w_i}}{1-t}$, where $w_i =$ gcd (b_i, s) for all i. Then there exists a unique $n(t)$ supported in $[\lfloor \frac{c}{2} \rfloor + 1 + \lfloor \frac{\deg h}{2} \rfloor, \lfloor \frac{c}{2} \rfloor + r - 1 - \lfloor \frac{\deg h}{2} \rfloor]$ in the following equality:

$$\frac{n(t)}{(1-t)^n m(t)} = \frac{N_{\mathrm{per},Q}(t)}{h(t)m(t)} + \frac{A(t)}{(1-t)^{n+1}},$$

where $m(t) = \frac{1-t^s}{h(t)}$ and $A(t)$ is some Laurent polynomial. Equivalently, we have

$$n(t) = \frac{N_{\mathrm{per},Q}(t)}{h(t)}(1-t)^n + A(t)\frac{m(t)}{1-t}.$$

Hence $n(t)$ is the inverse of $\prod_{i=1}^n \frac{1-t^{b_i}}{1-t}$ mod $\frac{1-t^s}{(1-t)h(t)}$ by Proposition 3.2.2. One can prove that $\frac{n(t)}{(1-t)^n m(t)}$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$ as before. Therefore,

$$P_{\mathrm{orb},Q}(t) - \frac{n(t)}{(1-t)^n m(t)} - \sum_{i=1}^{l} \mathrm{bit}_{Q,\mathcal{C}_i}(t) \frac{S_{1,w_i}(t)}{(1-t)^{n-1}(1-t^{w_i})^2}$$

is Gorenstein symmetric of degree $k_{\mathcal{X}}$, which is of the form $\frac{S(t)}{(1-t)^{n+1-l}(1-t^{w_1})\dots(1-t^{w_l})}$. This is the sum of what $P_{\mathrm{orb},Q}(t)$ bites off from the second part $P_{\mathcal{C}_i,2}(t)$ of each curve $\mathcal{C}_i$, that is

$$\frac{S(t)}{(1-t)^{n+1-l}(1-t^{w_1})\dots(1-t^{w_l})} = \frac{s_1(t)}{(1-t)^n(1-t^{w_1})} + \dots + \frac{s_l(t)}{(1-t)^n(1-t^{w_l})},$$

where $\frac{s_i(t)}{(1-t)^n(1-t^{w_i})}$ represents the bite from the second part of the curve $\mathcal{C}_i$, and $s_i(t)$ is supported in $[\lfloor \frac{c}{2} \rfloor + 1, \lfloor \frac{c}{2} \rfloor + w_i - 1]$. Now we need to prove that the Gorenstein symmetry of the sum implies the Gorenstein symmetry of $\frac{s_i(t)}{(1-t)^n(1-t^{w_1})}$ for all $0 \leq i \leq l$. In fact, the above equality can be rewritten as

$$S(t) = \sum_{i=1}^{l} s_i(t) \prod_{j=1, j\neq i}^{l} \frac{1-t^{w_j}}{1-t}.$$

Now if we take the last equality modulo $\frac{1-t^{w_i}}{1-t}$, then

$$S(t) = s_i(t) \prod_{j=1, j\neq i}^{l} \frac{1-t^{w_j}}{1-t} \mod \frac{1-t^{w_i}}{1-t},$$

which implies that

$$s_i(t) = S(t) \prod_{j=1, j\neq i}^{l} \frac{1-t^{w_j v_{i_j}}}{1-t^{w_j}} \mod \frac{1-t^{w_i}}{1-t},$$

where $w_j v_{i_j} = 1 \mod w_i$. In this way we can prove as before that $s_i(t)$ is Gorenstein symmetric with the support $[\lfloor \frac{c}{2} \rfloor + 1, \lfloor \frac{c}{2} \rfloor + w_i - 1]$. □

Combining Propositions 5.3.6 and 5.3.7, we know that after subtracting what each of the dissident points bites off from the curve, the remaining contribution from the curve $\mathcal{C}$ in the Hilbert series is given in the following form:

$$(r \deg H|_{\mathcal{C}} - \sum_{Q \in \mathcal{T}} \mathrm{bit}_{Q,r}(t)) \frac{S_1(t)}{(1-t)^{n-1}(1-t^r)^2} + \frac{S_2(t)}{(1-t)^n(1-t^r)}, \tag{5.8}$$

where each part is Gorenstein symmetric of degree $k_{\mathcal{X}}$. We denote the above expression by $P_{\mathrm{orb},\mathcal{C}}(t)$ for a curve with dissident points.

5.3.2 A special case

For an orbifold curve, we have seen that in general its contribution in our Hilbert series parsing consists of two parts as in (5.8). The following proposition says that for an orbifold curve of type $\frac{1}{2}(1, \ldots, 1)$, we only have the first part of the contribution.

Proposition 5.3.8. *Let $(\mathcal{X}, H)$ be a projectively Gorenstein pair. Suppose there is an orbifold curve of singularity type $\frac{1}{2}(1, \ldots, 1)$, and that there are dissident points of type $\mathcal{T} = \{Q \text{ of type } \frac{1}{2s}(2b_{Q,1}, \ldots, b_{Q,n})\}$ living on $\mathcal{C}$ (by assumption we have $\gcd(b_{Q,i}, 2s) = 1$ for all i). Then the orbifold term for this curve $\mathcal{C}$ can be given by*

$$P_{\mathrm{orb},\mathcal{C}}(t) = \alpha \frac{t^{\lfloor \frac{c+1}{2} \rfloor + 1}}{(1-t)^{m-1}(1-t^2)^2}, \tag{5.9}$$

where $\alpha = 2 \deg H|_{\mathcal{C}} - \sum_{Q \in \mathcal{T}} \mathrm{bit}_Q(t)$ and $\mathrm{bit}_Q(t)$ are determined as in Proposition 5.3.5.

Proof Note that since $(\mathcal{X}, H)$ is projectively Gorenstein, then by Proposition 4.1.3 we know that $n - 1 + k = 0 \bmod 2$. Therefore the coindex $c = k + n + 1$ is always even. The second part from the curve contribution is of the form $\frac{t^{\lfloor \frac{c}{2} \rfloor + 1}}{(1-t)^n(1-t^2)}$. When c is even, this part cannot be Gorenstein symmetric of degree $k_{\mathcal{X}}$. Then it has to be zero, and so in this case the curve contribution term in our parsing only consists of the first part. □

5.4 Initial term and the end of the proof

Now to finish the parsing of our Hilbert series, we are left with the initial term. Recall that our orbifold $\mathcal{X}$ has orbifold curves $\mathcal{B}_{\mathcal{C}}$ and orbifold points $\mathcal{B}_Q$. We have given an orbifold term for each orbifold locus in our parsing of the Hilbert series, namely, $P_{\mathrm{orb},\mathcal{C}}(t)$ and $P_{\mathrm{orb},Q}(t)$. Then the remaining part is

$$P(t) - \sum_{\mathcal{C}\in\mathcal{B}_{\mathcal{C}}} P_{\mathrm{orb},\mathcal{C}}(t) - \sum_{Q\in\mathcal{B}_Q} P_{\mathrm{orb},Q}(t),$$

which we define to be the initial term $P_I(t)$. Since each term in the above expression is Gorenstein symmetric of degree $k_{\mathcal{X}}$, then $P_I(t)$ is also Gorenstein symmetric of degree $k_{\mathcal{X}}$.

Recall that we required the orbifold term in the Hilbert series for points and curves to have numerators with support starting from $\lfloor \frac{c}{2} \rfloor + 1$, and therefore the initial term needs to take care of the first $\lfloor \frac{c}{2} \rfloor + 1$ terms, namely, $P_0, \dots, P_{\lfloor \frac{c}{2} \rfloor}$, in the Hilbert series. Since $P_I(t)$ is Gorenstein symmetric of degree $k_{\mathcal{X}}$, it can be given as in Lemma 4.6.1. In particular, $P_I(t)$ has a numerator with integral coefficients by construction.

Now we have our parsing as follows:

$$P(t) = P_I(t) + \sum_{Q\in\mathcal{B}_Q} P_{\mathrm{orb},Q}(t) + \sum_{\mathcal{C}\in\mathcal{B}_{\mathcal{C}}} (P_{\mathcal{C},1}(t) + P_{\mathcal{C},2}(t)).$$

There is one more point we need to prove, that is, the integral condition for the second part of the curve contribution, namely, $P_{\mathcal{C},2}(t)$ for each orbifold curve $\mathcal{C}$. However, we know that the sum $\sum_{\mathcal{C}\in\mathcal{B}_{\mathcal{C}}} P_{\mathcal{C},2}(t)$ has integral coefficients, which is of the form

$$\frac{S(t)}{(1-t)^{n+1}\prod_{\mathcal{C}\in\mathcal{B}_{\mathcal{C}}} \frac{1-t^{r_{\mathcal{C}}}}{1-t}}.$$

Recall that $P_{\mathcal{C},2}(t)$ is of the form $\frac{S_{\mathcal{C},2}(t)}{(1-t)^n(1-t^{r_{\mathcal{C}}})}$. Then

$$\sum_{\mathcal{C}\in\mathcal{B}_{\mathcal{C}}} P_{\mathcal{C},2}(t) = \sum_{\mathcal{C}\in\mathcal{B}_{\mathcal{C}}} \frac{S_{\mathcal{C},2}(t)}{(1-t)^n(1-t^{r_{\mathcal{C}}})}.$$

Therefore, $S_{\mathcal{C},2}(t)$ is given by $S(t)(\prod_{\mathcal{C}'\neq\mathcal{C}} \frac{1-t^{r_{\mathcal{C}'}}}{1-t})^{-1} \bmod \frac{1-t^{r_{\mathcal{C}}}}{1-t}$ (see proof of Proposition 5.3.7), which proves that $S_{\mathcal{C},2}(t)$ has integral coefficients as usual. This finishes the proof of our Theorem 5.1.2.

5.5 Examples and applications

In this section, we give some examples of our Hilbert series parsing formula. Then we apply this to construct orbifolds with certain invariants and orbifold loci. First, let us see some examples of our parsing formula with pure orbicurves (that is, orbicurves without dissident points).

Example 5.5.1. *Let $\mathcal{X}_{10}$ be a degree 10 hypersurface in $\mathbb{P}(1,1,1,2,2,2)$ and $\mathcal{O}(1)$ be the polarization. This is a canonical 4-fold with an orbicurve of type $\frac{1}{2}(1,1,1)$. We know $k_{\mathcal{X}} = 1$ and $c = 1+4+1 = 6$. Also we can calculate the degree of the curve*

$$\deg H|_{\mathcal{C}} = \frac{10 \cdot 2}{2 \cdot 2 \cdot 2} = \frac{5}{2}.$$

Thus the parsing of Hilbert series is given by

$$P(t) = P_I(t) + 5P_{\mathcal{C}}(t),$$

where P_I can be calculated using Program 4.7.2, which gives

$$P_I(t) = \text{initial}\left([1,3,9,19],1,4\right) = \frac{1-2t+4t^2-6t^3+4t^4-2t^5+t^6}{(1-t)^5}.$$

and $P_{\mathcal{C}}$ can be calculated using Program 4.5.7, and it gives

$$P_{\mathcal{C}}(t) = \text{Qorb}\left(2,[1,1,1],3\right)/(1-t^2) = \frac{t^4}{(1-t)^3(1-t^2)^2}.$$

Example 5.5.2. *Consider the following two 4-folds:*

- *let $(\mathcal{X}_1, \mathcal{O}(1))$ be a general hypersurface of degree 16 in $\mathbb{P}(1,1,1,3,3,8)$. Then it has an orbicurve $\mathcal{C} = \mathbb{P}(3,3)$ of type $\frac{1}{3}(1,1,2)$.*
- *let $(\mathcal{X}_2, \mathcal{O}(1))$ be a general hypersurface of degree 13 in $\mathbb{P}(1,1,1,3,3,5)$. Then it has an orbicurve $\mathcal{C}' = \mathbb{P}(3,3)$ of type $\frac{1}{3}(1,1,2)$ and an orbipoint of type $\frac{1}{5}(1,1,1,3)$.*

Note that these 4-folds both have canonical weight -1 and coindex $c = -1+4+1 = 4$. They all have the same plurigenera $1,3,6$ in degree $0,1,2$ respectively. Therefore, *they have the same initial term, which can be calculated by Program 4.7.2. This gives*

$$P_I(t) = \mathrm{initial}\,([1,3,6],-1,4) = \frac{1-2t+t^2-2t^3+t^4}{(1-t)^5}.$$

They also both have an orbicurve of type $\frac{1}{3}(1,1,2)$ *of the same degree* $\frac{1}{3}$*, for which we can calculate the first part of the curve contribution by Program 4.5.7 or Program 5.2.2, that is,*

$$P_{\mathcal{C},1} = 3\deg H|_{\mathcal{C}}\,\mathrm{Qorb}(3,[1,1,2],-1+3)/(1-t^3) = \frac{-t^4}{(1-t)^3(1-t^3)^2}.$$

where the second part of the curve parsing can be calculated by its Gorenstein property and an extra information of the third plurigenus. Now for $(\mathcal{X}_1, \mathcal{O}(1))$ *we write out our parsing*

$$\begin{aligned} P_1(t) &= P_I(t) + P_{\mathcal{C},1}(t) + P_{\mathcal{C},2}(t) \\ &= \frac{1-2t+t^2-2t^3+t^4}{(1-t)^5} + \frac{-t^4}{(1-t)^3(1-t^3)^2} + \frac{4t^3}{(1-t)^4(1-t^3)}. \end{aligned}$$

For $(\mathcal{X}_2, \mathcal{O}(1))$*, our parsing is*

$$\begin{aligned} P_1(t) &= P_I(t) + P_{\mathrm{orb},Q}(t) + P_{\mathcal{C}',1}(t) + P_{\mathcal{C}',2}(t) \\ &= \frac{1-2t+t^2-2t^3+t^4}{(1-t)^5} + \frac{t^3+t^5}{(1-t)^4(1-t^5)} + \\ &\quad \frac{-t^4}{(1-t)^3(1-t^3)^2} + \frac{3t^3}{(1-t)^4(1-t^3)}. \end{aligned}$$

where $P_{\mathrm{orb},Q}(t)$ *is calculated by* $\mathrm{Qorb},(5,[1,1,1,3],-1)$ *and the second part of the curve contribution is calculated as above.*

As one may notice that even though the orbifold types of the two orbicurves $\mathcal{C}$ *and* $\mathcal{C}'$ *are the same, the second parts of the curve contributions are different. This is because the second part of the curve contribution is related to the normal bundle of the curve.*

Now we have seen some examples of our Hilbert series parsing formula. We want to construct orbifolds with this parsing as in Section 5.5. Here we have a simple example.

Example 5.5.3. *Suppose we want to construct an orbifold of dimension* 3 *with* *trivial canonical sheaf with the following data:*

- *the first three plurigenera:* $P_0 = 1,\ P_1 = 1,\ P_2 = 2$;
- *an orbicurve* $\mathcal{C}_1$ *of type* $\frac{1}{2}(1,1)$ *and an orbicurve* $\mathcal{C}_2$ *of type* $\frac{1}{3}(1,2)$;
- *a dissident point* Q_1 *of type* $\frac{1}{9}(1,2,6)$ *and a dissident point* Q_2 *of type* $\frac{1}{6}(1,2,3)$.

Suppose such an orbifold exist, then in our Hilbert series parsing we should have

$$P_I(t) = \text{initial}\,([1,1,2],0,3) = \frac{1-3t+4t^2-3t^3+t^4}{(1-t)^4}.$$

We should also have a term related to the curve of type $\frac{1}{2}(1,1)$*, that is,*

$$P_{\text{orb},\mathcal{C}_1}(t) = \text{Qorb}\,(2,[1,1],2)/(1-t^2) = \frac{-t^3}{(1-t)^2(1-t^2)^2},$$

and a term related to the curve of type $\frac{1}{3}(1,2)$*, which is given by*

$$\begin{aligned} P_{\text{orb},\mathcal{C}_2}(t) &= P_{\mathcal{C}_2,1}(t) + P_{\mathcal{C}_2,2}(t) = \text{Qorb}\,(3,[1,2],3)/(1-t^3) + P_{\mathcal{C}_2,2}(t) \\ &= \frac{-t^4}{(1-t)^2(1-t^3)^2} + \frac{S(t)}{(1-t)^3(1-t^3)}, \end{aligned}$$

where $S(t)$ *should be given by* t^3 *multiplied with some integer due to its Gorenstein symmetry property. Moreover, for these two dissident points we should also have orbifold terms*

$$\begin{aligned} P_{\text{orb},Q_1}(t) &= \text{Qorb}\,(9,[1,2,6],0) = \frac{t^6-t^7+t^8}{(1-t)^2(1-t^3)(1-t^9)}; \\ P_{\text{orb},Q_1}(t) &= \text{Qorb}\,(6,[1,2,3],0) = \frac{t^6}{(1-t)(1-t^2)(1-t^3)(1-t^6)}. \end{aligned}$$

To find such an orbifold, we can do the following search:

```
pi:=initial([1,1,2],0,3);
q1:=Qorb(2,[1,1],2)/(1-t^2);
q2:=Qorb(3,[1,2],3)/(1-t^3);
q3:=Qorb(9,[1,2,6],0);
q4:=Qorb(6,[1,2,3],0);
for  i,j,k in [0..3] do
```

```
p:=pi+i*q1+j*q2 + k*t^3/Denom([1,1,1,3])+q3+q4;
p*Denom([1,2,3,6,9]);[i,j,k];
end for;
```

Among the outputs (here for simplicity we do not consider the candidates that are codimension ≥ 4), we have two candidates that possibly gives us such orbifolds, namely, when $i = 0$, $j = 2$, $k = 1$, we have a Hilbert series

$$P_1(t) = \frac{1 - t^9 - 3t^{12} + 3t^{18} + t^{21} - t^{30}}{(1-t)(1-t^2)(1-t^3)^2(1-t^6)^2(1-t^9)},$$

and when $i = 1$, $j = 0$, $k = 1$, we have a Hilbert series

$$P_2(t) = \frac{1 - t^{10} - 2t^{12} - t^{13} - t^{15} + t^{16} + t^{18} + 2t^{19} + t^{21} - t^{31}}{(1-t)(1-t^2)(1-t^3)(1-t^4)(1-t^6)^2(1-t^9)}.$$

Now we analyze these two Hilbert series one by one. In the first case, $P_1(t)$ suggests a codimension 3 orbifold in $\mathbb{P}(1,2,3,3,6,6,9)$. Denote the variables of $\mathbb{P}(1,2,3,3,6,6,9)$ by $x, y, z_1, z_2, t_1, t_2, w$. Then it can be given by 4×4 Pfaffians in the following 5×5 skew symmetric matrix

$$\begin{pmatrix} w & a_9 & b_9 & c_6 \\ & t_1 & d_6 & e_3 \\ & & t_2 & z_1 \\ & & & z_2 \end{pmatrix}$$

where a_9, b_9, c_6, d_6, e_3 represent general homogeneous polynomials of degrees $9, 9, 6, 6, 3$ respectively. Then the Pfaffians are given by the following equations

$$\begin{aligned} pf_1 &= t_1 z_2 - t_2 e_3 + z_1 d_6, \\ pf_2 &= z_2 a_9 - b_9 z_1 + c_6 t_2, \\ pf_3 &= w z_2 - b_9 e_3 + d_6 c_6, \\ pf_4 &= w z_1 - a_9 e_3 + t_1 c_6, \\ pf_5 &= w t_2 - a_9 d_6 + b_9 t_1. \end{aligned}$$

we can check that the orbifold defined by these equations has the property we required. For example, we see that the point $(0, \ldots, 0, 1)$ has local parameters x, y, t_2, and its orbifold type is given by $\frac{1}{9}(1,2,6)$. Similarly, we can check for other orbifold loci.

Now in the sencond case, the Hilbert series $P_2(t)$ suggests an orbifold own-

ing these properties can be given by a codimension 3 *orbifold in* $\mathbb{P}(1,2,3,4,6,6,9)$. *Denote its coordinates by* $(x, y, z, t, w_1, w_2, v)$. *Then this orbifold can be defined by Pfaffians in the following matrix*

$$\begin{pmatrix} v & a_9 & t^2 + b_8 & c_6 \\ & d_7 & w_2 & y^2 + t + e_4 \\ & & w_1 & t \\ & & & z \end{pmatrix}$$

and we can check that general choices of these homogeneous polynomials will give us an orbifold with the required properties.

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