The Harmonic Map Heat Flow from Surfaces

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Summary

We present a study of the harmonic map heat flow of Eells and Sampson in the case that the domain manifold is a surface. Particular emphasis has been placed on the singularities which may occur, as described by Struwe, and the analysis of the flow despite these.

In Chapter 1 we give a brief introduction to the theory of harmonic maps and their flow. Further details are to be found in [9] and [10]. In the case that the domain manifold is a surface we describe the existence theory for the heat flow and the theory of bubbling.

In Chapter 2 we investigate the question of the uniformity in time of the convergence of the heat flow to a bubble tree at infinite time. In Section (2.1) (page 28) we give the first example of a non-uniform flow. In contrast, Theorem (2.2) (page 30) provides conditions under which the convergence is uniform and any bubbles which form are rigid.

In Chapter 3 we give the first example of a nontrivial bubble tree - in other words we give a flow in which more than one bubble develops at the same point at infinite time.

In Chapter 4 we discuss in what sense two flows are close when their initial maps are close. We formulate this question in various ways, providing examples of instability and an 'infinite time' stability result (Theorem (4.2), page 56) using techniques developed in Chapter 2.

From the theory of bubbling as described in Chapter 1, if an initial map has less energy than is required for a bubble, then the subsequent flow cannot blow up. In Chapter 5 we ask conversely whether given enough energy for a bubble, we can find an initial map leading to blow-up.

In the appendix we outline a plausible construction of a flow which can be analysed at two different sequences of times to give convergence to two different bubble trees, with different numbers of bubbles.

A warning

We note here that many of the theorems in this thesis assume implicitly that the heat flow is from a surface, and we always assume compactness of the domain and target.

Chapter 1

Introduction

1.1 Basic definitions and notions

Let $(\mathcal{M},\gamma)$ and $(\mathcal{N},g)$ be compact Riemannian manifolds, the latter without boundary. For a suitably differentiable map $v:\mathcal{M}\to\mathcal{N}$ we may assign an energy

$$E(v)=\int_{\mathcal{M}} e(v)\,d\mathcal{M},$$

where the energy density $e(v)$ is defined by

$$e(v)=\frac{1}{2}|dv|^2=\frac{1}{2}\gamma^{\alpha\beta}g_{ij}(v)\frac{\partial v^i}{\partial x^\alpha}\frac{\partial v^j}{\partial x^\beta},$$

using local coordinates $x=(x^1,\ldots,x^m)$ on $\mathcal{M}$ and $v=(v^1,\ldots,v^n)$ on $\mathcal{N}$. We will often see v as a map to $\mathcal{N}$ embedded isometrically inside $\mathbb{R}^N$ for some N. Such an embedding is always possible by the Nash embedding theorem, and will be used implicitly to consider Sobolev spaces of maps from $\mathcal{M}$ to $\mathcal{N}$. In this case, the energy density may be written

$$e(v)=\frac{1}{2}|\nabla v|^2=\frac{1}{2}\gamma^{\alpha\beta}\frac{\partial v^i}{\partial x^\alpha}\frac{\partial v^i}{\partial x^\beta},$$

with $v=(v^1,\ldots,v^N)$.

A regular map $v:\mathcal{M}\to\mathcal{N}$ is said to be 'harmonic' if it is a critical point of the energy functional E. Harmonic maps generalise several common notions from the Calculus of Variations. When $\mathcal{M}=S^1$, they represent closed geodesics in $\mathcal{N}$. Alternatively, when $\mathcal{N}=\mathbb{R}$, they are the ordinary harmonic functions from $\mathcal{M}$.

We define the 'tension' $\mathcal{T}(v)$ of a map $v : \mathcal{M} \to \mathcal{N}$ to be the negation of the L^2-gradient of E. Precisely, the tension is the vector field along the map v (ie. $\mathcal{T}(v) \in \Gamma(v^*(T\mathcal{N}))$) which satisfies

$$dE_v(\phi) = -\int_{\mathcal{M}} \langle \mathcal{T}(v), \phi \rangle d\mathcal{M},$$

for any $\phi \in \Gamma(v^*(T\mathcal{N}))$. Explicitly, the tension is

$$\mathcal{T}(v) = \left(\Delta_{\mathcal{M}} v^l + \gamma^{\alpha\beta} {}_{\mathcal{N}}\Gamma^l_{ij}(v) \frac{\partial v^i}{\partial x^\alpha} \frac{\partial v^j}{\partial x^\beta} \right) \frac{\partial}{\partial v^l},$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator

$$\Delta_{\mathcal{M}} = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \right),$$

in which $\gamma = det(\gamma_{\alpha\beta})$, and ${}_{\mathcal{N}}\Gamma^l_{ij}$ denotes the Christoffel symbols of the target $\mathcal{N}$. Alternatively, we may write

$$\mathcal{T}(v) = \gamma^{\alpha\beta} \left(\frac{\partial^2 v^l}{\partial x^\alpha \partial x^\beta} - {}_{\mathcal{M}}\Gamma^\sigma_{\alpha\beta} \frac{\partial v^l}{\partial x^\sigma} + {}_{\mathcal{N}}\Gamma^l_{ij}(v) \frac{\partial v^i}{\partial x^\alpha} \frac{\partial v^j}{\partial x^\beta} \right) \frac{\partial}{\partial v^l},$$

from which we see, by taking normal coordinates, that the identity map is always harmonic. If we are considering the target to be isometrically embedded in $\mathbf{R}^N$, then we write the tension as the vector in $\mathbf{R}^N$

$$\mathcal{T}(v) = (\Delta_{\mathcal{M}} v)^T = \Delta_{\mathcal{M}} v + A(v)(\nabla v, \nabla v) \equiv \Delta_{\mathcal{M}} v + \gamma^{\alpha\beta} A^l_{ij}(v) \frac{\partial v^i}{\partial x^\alpha} \frac{\partial v^j}{\partial x^\beta}, \tag{1.1}$$

where $(\Delta_{\mathcal{M}} v)^T(x)$ is the component of $(\Delta_{\mathcal{M}} v)(x)$ in $T_{v(x)}\mathcal{N}$, and A^l_{ij} is the second fundamental form of the embedding of $\mathcal{N}$ in $\mathbf{R}^N$.

As an example, we consider the case that the domain $\mathcal{M}$ is a subset of $\mathbf{R}^m$, and the target $\mathcal{N}$ is a sphere S^n embedded in $\mathbf{R}^{n+1}$. The tension is then given simply as

$$\mathcal{T}(v) = \Delta v + v|\nabla v|^2,$$

where Δ is the standard Laplacian.

From the definition of the tension, we see that the harmonic maps are precisely the regular solutions of the elliptic equation

$$\mathcal{T}(v) = 0, \tag{1.2}$$

the Euler-Lagrange equation for E. We call any weak solution $v \in W^{1,2}$ of (1.2) a 'weakly harmonic' map.

The 'harmonic map heat flow' is defined to be L^2-gradient flow on the energy. Precisely, the flow is a solution $u : \mathcal{M} \times [0, \infty) \to \mathcal{N}$ to the parabolic equation

$$\frac{\partial u}{\partial t} = \mathcal{T}(u(\cdot, t)), \qquad u(\cdot, 0) = u_0, \qquad u(\cdot, t)|_{\partial\mathcal{M}} = u_0|_{\partial\mathcal{M}}. \tag{1.3}$$

We refer to this equation as the 'heat equation,' to the map u_0 as the 'initial map,' and to the map $u_0|_{\partial\mathcal{M}}$ as the boundary values. We will often abbreviate $u(\cdot, t)$ to $u(t)$.

For smooth flows, we find that the energy of the flow decays in time according to

$$\frac{d}{dt}E(u(t)) = -\|\mathcal{T}(u(t))\|^2_{L^2(\mathcal{M})}. \tag{1.4}$$

In particular, we see that

$$E(u(T)) - \lim_{t\to\infty} E(u(t)) = \int_T^\infty \|\mathcal{T}(u(t))\|^2_{L^2(\mathcal{M})} dt, \tag{1.5}$$

which we will use periodically.

Most of this thesis, and indeed everything beyond the first chapter, will concern only the case in which the domain $\mathcal{M}$ is a surface, which is special for a variety of reasons. The first reason is that the energy is conformally invariant. So if we have a conformal diffeomorphism $h : \mathcal{P} \to \mathcal{M}$ (with $\mathcal{P}$ another surface) and any map $v : \mathcal{M} \to \mathcal{N}$ then $E(v \circ h) = E(v)$. Thus harmonicity is preserved by precomposition by a conformal map. Moreover, as the identity map is always harmonic, we see that conformal maps between surfaces are harmonic. An important example of this is that the Möbius transformations between 2-spheres are harmonic. Alternatively, we find that the energy is invariant under conformal changes of domain metric. We record however that the tension is definitely not conformally invariant, and is scaled inversely to any scaling of the domain metric.

Following the above observations, we may establish a relationship between the energy of a map v and the area of its image $A(v)$ (which we measure here with multiplicity).

Proposition 1.1 *Suppose $v : \mathcal{M} \to \mathcal{N}$ is sufficiently regular, and $\mathcal{M}$ is a surface. Then we have $E(v) \geq A(v)$, with equality precisely when v is conformal.*

Proof. It is sufficient to prove the result locally. By taking local isothermal coordinates, and using the invariance of energy discussed above, we may assume that $\mathcal{M}$ is the 2-disc

D, say. The result then follows immediately:

$$E(v) = \frac{1}{2}\int_D \{|v_x|^2 + |v_y|^2\}dxdy \geq \int_D \{|v_x|^2|v_y|^2 - \langle v_x, v_y\rangle^2\}^{\frac{1}{2}}dxdy = A(v).$$

∎

In the case that the domain is a 2-sphere, we can exploit the extra global restriction with Liouville's theorem to find that the harmonic maps enjoy a further property:

Proposition 1.2 *If $\mathcal{M}$ is a 2-sphere, and $v : \mathcal{M} \to \mathcal{N}$ is harmonic, then v is conformal.*

Before offering a proof, we recall some complex notation. After taking isothermal coordinates x and y on the domain, we call $z = x + iy$ a local complex coordinate. Moreover, we write $dz = dx + idy$ and $d\bar{z} = dx - idy$. The metric of $\mathcal{M}$ may be written as $\sigma^2(z)dzd\bar{z}$. For a fixed domain chart, we may consider

$$w_z = \frac{1}{2}(w_x - iw_y) \qquad \text{and} \qquad w_{\bar{z}} = \frac{1}{2}(w_x + iw_y)$$

where w will normally, but not always, be complex valued. We have associated geometric objects (independent of the chart) $w_z dz$ and $w_{\bar{z}} d\bar{z}$.

Proof. (Proposition (1.2).)

Let $z = x + iy$ be a complex coordinate on $\mathcal{M}$ and consider the quadratic differential $\phi(v)dz^2$ where

$$\phi(v) = (|v_x|^2 - |v_y|^2) - (2v_x \cdot v_y)i.$$

This is just the $(2,0)$ part of v^*g. As the domain $\mathcal{M}$ is a 2-sphere, we may take z to be a global complex coordinate (ie. we conformally chart $\mathcal{M}$ minus a point with $\mathbf{C}$, which is permitted by the Uniformisation Theorem [1]). For example when $\mathcal{M}$ is the round 2-sphere, we may take a stereographic chart. Let us take such a chart and see v, and ϕ, as a map from $\mathbf{C}$. Calculation shows that

$$\phi_{\bar{z}} = 2v_z \cdot \mathcal{T}(v),$$

for any sufficiently regular map v, and therefore as v is harmonic, we must have that $\phi_{\bar{z}} = 0$. As $\phi(v)dz^2$ is in fact invariant of the chart we took for the domain, we must have that $[\phi(v)](z) \to 0$ as $z \to \infty$. We may then appeal to Liouville's theorem to find that $\phi(v)$ must be identically zero. This is clearly equivalent to v being conformal. ∎

1.2 Classical problems and results

The fundamental existence problem of harmonic maps is the following:

Problem 1.3 *Given a map $v_0 : \mathcal{M} \to \mathcal{N}$, can we smoothly deform it to a harmonic map $v : \mathcal{M} \to \mathcal{N}$, whilst preserving the boundary values in the case that $\mathcal{M}$ is a manifold with boundary?*

This problem is in some sense an extension of the ordinary Dirichlet problem for harmonic functions. The most significant progress ever made in addressing the question was published in 1964 by Eells and Sampson [11]. The work introduced the harmonic map heat flow for the first time, and used it to provide the deformation required, subject to curvature restrictions on the target. We will be stating the results of Eells and Sampson in the case in which the domain is without boundary, as they were originally. Domains with boundary were dealt with by Hamilton in [18].

The first result of theirs which we give establishes short time existence of the flow, whatever the dimension of the domain.

Theorem 1.4 *Suppose we have $u_0 \in C^1(\mathcal{M},\mathcal{N})$. Then there exists $t_1 > 0$ depending only on $e(u_0)$ and $\mathcal{N}$ such that* (1.3) *has a unique solution*

$$u \in C^1(\mathcal{M} \times [0,t_1),\mathcal{N}) \bigcap C^\infty(\mathcal{M} \times (0,t_1),\mathcal{N}).$$

The main result of Eells and Sampson declared that the flow was 'global' (ie. it existed for all time) subject to curvature restrictions on the target, and led to a positive partial answer to Problem (1.3) for domains of any dimension. We give their result incorporating an addition of Hartman [19] which we will discuss later.

Theorem 1.5 *Suppose that $\mathcal{N}$ has nonpositive sectional curvature. Then given any $u_0 \in C^1(\mathcal{M},\mathcal{N})$,* (1.3) *has a unique solution $u \in C^1(\mathcal{M}\times[0,\infty),\mathcal{N}) \bigcap C^\infty(\mathcal{M}\times(0,\infty),\mathcal{N})$. Moreover,*

$$u_\infty = \lim_{t\to\infty} u(t) \tag{1.6}$$

exists C^k-uniformly (for all $k \geq 0$) and defines a harmonic map homotopic to u_0.

We also mention a much more recent extension of the theorem of Eells and Sampson, which is due to Ding and Lin [7] and holds for domains of any dimension.

Theorem 1.6 *Let $\mathcal{M}$ be without boundary, and let $(\tilde{\mathcal{N}}, \tilde{g})$ be the universal covering of $(\mathcal{N}, g)$. Suppose that $\tilde{\mathcal{N}}$ admits a strictly convex function $\rho \in C^2(\tilde{\mathcal{N}})$ with quadratic growth - in other words*

(i) $\nabla^2 \rho \geq c_0 \tilde{g}$ *on* $\tilde{\mathcal{N}}$,

(ii) $0 \leq \rho(y) \leq c_1 d^2_{\tilde{\mathcal{N}}}(y, y_0) + c_2$ *for all* $y \in \tilde{\mathcal{N}}$,

where $c_0, c_1, c_2 > 0$ are constants, $y_0 \in \tilde{\mathcal{N}}$ and $d_{\tilde{\mathcal{N}}}$ is the distance on $\tilde{\mathcal{N}}$. Then there exists a global solution u of (1.3) and $u(t_i)$ converges to a harmonic map for some sequence $t_i \to \infty$. Moreover, we have that

$$\|u(t)\|_{C^1(\mathcal{M})} \leq C(E(u_0)) \text{ for } t > 1.$$

This theorem is an extension of the theorem of Eells and Sampson as if the sectional curvature of $\mathcal{N}$ is nonpositive, then $\rho \equiv d^2_{\tilde{\mathcal{N}}}(\cdot, y_0)$ is a strictly convex function with quadratic growth.

In contrast to the existence results we have given to partially answer Problem (1.3), there are situations in which the answer to the problem is negative. In the case that the domain is without boundary, the first example was given by Eells and Wood [12].

Theorem 1.7 *There is no degree one harmonic map from (T^2, γ) to (S^2, g) whatever the metrics γ and g.*

In the case that the domain has boundary, we have the following result of Lemaire [22, Theorem 3.2] which provides an endless supply of examples.

Theorem 1.8 *Any harmonic map from a contractible surface with boundary with constant boundary values, is itself constant.*

In particular, when $\mathcal{M}$ is a flat 2-disc, if we choose a target $\mathcal{N}$ with $\pi_2(\mathcal{N}) \neq 0$ and take v_0 to have constant boundary values but lie in a nontrivial homotopy class, then we cannot deform it to a harmonic map.

1.3 The nature of the energy functional for different domain dimensions

The properties of the energy functional are dependent on the dimension of the domain. For domains of dimension one, the functional is very well behaved, and satisfies the Palais-Smale condition. Moreover, in this case, we find the following description of the heat flow. (See also [11] and [24].)

Theorem 1.9 *Suppose that $\mathcal{M}$ is one dimensional. Then for any $u_0 \in C^1(\mathcal{M},\mathcal{N})$ there exists a unique solution u to* (1.3) *which is smooth on $\mathcal{M} \times (0,\infty)$, and a sequence of times $t_i \to \infty$ such that*

$$u(t_i) \to u_\infty$$

in $C^{1,\alpha}$ (for any $\alpha < \frac{1}{2}$) as $i \to \infty$, where u_∞ is a constant speed geodesic.

In Section (2.1) we shall see that the convergence is *not* uniform in time in general. Of course, if we assumed that the target had nonpositive sectional curvature, then the result would be automatic, with uniform convergence, by Theorem (1.5).

Proof. (Theorem (1.9).)

By Theorem (1.4) we may find a unique flow over a time interval $[0,t_1)$ for some $t_1 > 0$. Without loss of generality we may assume $t_1 \in (0,\infty]$ is the largest possible time for which there exists a flow $u \in C^\infty(\mathcal{M} \times (0,t_1),\mathcal{N})$.

A simple calculation (especially simple if we see $\mathcal{N}$ embedded in Euclidean space) reveals that the energy density evolves according to

$$\frac{\partial e(u(\theta,t))}{\partial t} - \frac{\partial^2 e(u(\theta,t))}{\partial \theta^2} = -|\mathcal{T}(u(\theta,t))|^2 \leq 0.$$

An application of the parabolic maximum principle to this inequality immediately provides the estimate

$$\sup e(u(\cdot,t)) \leq \sup e(u_0), \tag{1.7}$$

for all $t \in (0, t_1)$.

Suppose that $t_1 < \infty$. Together with (1.7), the heat equation (1.3) and (1.1) tell us that $|\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial \theta^2}| \in L^\infty(\mathcal{M} \times (0,t_1))$ and we may apply linear parabolic theory - see [21, IV, Theorem 9.1] and [21, II, Lemma 3.3] - to establish that $|\frac{\partial u}{\partial t}|, |\frac{\partial^2 u}{\partial \theta^2}| \in L^q(\mathcal{M} \times (0,t_1))$ for all $q < \infty$, and subsequently that $\frac{\partial u}{\partial \theta} \in C^{0,\alpha}(\mathcal{M} \times (0,t_1])$ for any $\alpha \in (0,1)$. In particular we have that $u(t_1) \in C^1(\mathcal{M},\mathcal{N})$ and we can re-apply Theorem (1.4) to continue the flow to give a weak solution to the heat equation on $\mathcal{M} \times (0, t_1 + \varepsilon)$ for some $\varepsilon > 0$ which is smooth except possibly at time $t = t_1$. However, the extended flow is smooth at $t = t_1$ as like above, we have $\frac{\partial u}{\partial \theta} \in C^{0,\alpha}(\mathcal{M} \times (0, t_1 + \varepsilon))$ and thus $|\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial \theta^2}| \in C^{0,\alpha}(\mathcal{M} \times (0, t_1 + \varepsilon))$ and consequently we can commence a Schauder iteration argument on the heat equation to obtain smoothness. This contradicts the maximality of t_1.

So we have a solution $u \in C^\infty(\mathcal{M} \times (0,\infty), \mathcal{N})$. It remains to find a sequence of times $t_i \to \infty$ at which the flow converges to a harmonic map, as the harmonic maps are all constant speed geodesics when $\mathcal{M}$ is one dimensional. In the light of (1.5) we may find $t_i \to \infty$ such that $\mathcal{T}(u(t_i)) \to 0$ in $L^2(\mathcal{M})$. Together with (1.7) the form of the tension (1.1) then tells us that $\|\frac{\partial^2 u}{\partial \theta^2}(t_i)\|_{L^2} \leq C$ for all i, and hence that

$$\|u(t_i)\|_{W^{2,2}} \leq C.$$

The compact embedding $W^{2,2} \hookrightarrow C^{1,p}$ for $p < \frac{1}{2}$ then leads to the strong convergence

$$u(t_i) \to u_\infty$$

in $C^{1,p}$ (after passing to a subsequence) for some map u_∞, and by passing to the limit in the heat equation we find that u_∞ is harmonic. ∎

When the domain has dimension at least three, the situation is much more difficult. Loosely speaking, a map which is restricted to a nontrivial homotopy class may find that to reduce its energy to the minimum possible, it must become infinitely concentrated. This is best demonstrated by the simple example of the homotopy class of maps between m-spheres, for $m \geq 3$, containing the identity map, in which the infimum of energy is zero (see [11]). We remark that a characterisation of the situations in which the minimum energy in a homotopy class is zero, and the corresponding references, are given in [10, (2.5)].

As the energy is reduced by concentration in some cases, and the heat flow seeks to reduce its energy as quickly as possible, we would expect the heat flow to have singularities for domains of dimension at least three. The first examples of finite time singularities in the heat flow were given for such domains by Coron and Ghidaglia [6].

When the domain has dimension two, we are in a borderline case, and this is the case we concentrate on in this work. The Palais-Smale condition fails. The key to understanding variational problems is often to isolate the natural space of mappings associated to the problem, and to understand this. From the definition of the energy, we expect the Sobolev space $W^{1,2}$ to be the natural space in which to work. The structure of the space $W^{1,p}$ undergoes a significant transition as p increases beyond the dimension of the domain. In particular, discontinuous mappings are suddenly introduced. The fact that $W^{1,2}$ does not quite embed continuously into C^0 is the central reason for many of the phenomena we will observe in this thesis.

Useful intuition is gained in the two dimensional case, by considering the degree one map between round 2-spheres given in complex coordinates as

$$z \longrightarrow \lambda z, \tag{1.8}$$

for $\lambda > 0$. This map is always harmonic, and always has energy 4π, as we mentioned in the discussion of conformal invariance in Section (1.1). As λ decreases to zero, or increases to infinity, the map becomes infinitely concentrated. In language that we will introduce shortly, the map converges to a constant map and a bubble is formed at the blow-up point.

1.4 Heat flow from surfaces

1.4.1 Existence and basic properties

We now consider the case when the dimension of the domain is two. (This will be assumed in theorems unless otherwise stated.) The fundamental work in this special case was published in 1985 by Michael Struwe [31]. He proved the existence of a flow which is regular except at finitely many points in space-time, and gave the initial description of the behaviour of the flow at these singularities. We begin with his existence theorem, including the extension to domains with boundary given by Chang [2], and incorporating a uniqueness result of Freire [14].

Theorem 1.10 *For each initial map $u_0 \in W^{1,2}(\mathcal{M},\mathcal{N})$, with the boundary value condition $u_0|_{\partial\mathcal{M}} \in C^{2,\alpha}(\partial\mathcal{M},\mathcal{N})$ in the case that $\partial\mathcal{M} \neq \emptyset$, there exists precisely one solution u of the heat equation* (1.3) *amongst those maps in $W^{1,2}(\mathcal{M}\times[0,\infty),\mathcal{N})$ for which $E(u(t))$ is decreasing. Moreover, u is smooth in $\mathcal{M}\times(0,\infty)$ away from at most a finite number of points.*

In the sequel, when we talk of the flow corresponding to or subsequent to an initial map u_0, we will by default be referring to the solution u to (1.3) constructed in the above theorem.

Although existence will be the crucial point for us, we briefly discuss uniqueness. Freire's result (which extended earlier work of Freire [13] and Rivière [27]) is a parabolic version of a theorem of Frédéric Hélein which declares that weakly harmonic maps from surfaces are regular [20]. Given that many of the techniques used by Freire come from the work of Hélein, we note that Hélein's result follows immediately from Theorem (1.10). To see this, we take a weakly harmonic map from a surface as an initial map for the flow. Then the stationary flow is the unique solution to the heat equation and is singular at at most finitely many points. As the flow is stationary, it must be globally regular, and in particular, the initial map must be regular. It is not known if the condition that $E(u(t))$ is decreasing is necessary to obtain uniqueness of the flow. We speculate, without justification, that this could at least be weakened to

$$\limsup_{t \downarrow T} E(u(t)) \leq E(u(T)) + \inf\{E(v) \mid v : S^2 \to \mathcal{N} \text{ is nonconstant and harmonic }\},$$

for all $T > 0$. With higher dimensional domains, the flow need not be unique, and an example has been given by Coron [5].

Given the existence element of Theorem (1.10) we can establish the following description of the flow at infinite time. To avoid technical complications we will only be detailing a proof in the case that $\mathcal{M}$ is without boundary.

Theorem 1.11 *Let u be the solution of the heat equation* (1.3) *introduced in Theorem* (1.10). *Then there exists a sequence of times $t_i \to \infty$, a harmonic map $u_\infty : \mathcal{M} \to \mathcal{N}$ and a finite set of points $\{x^1, \ldots x^m\} \subset \mathcal{M}$ such that*

(i) $u(t_i) \rightharpoonup u_\infty$, weakly in $W^{1,2}(\mathcal{M})$ (and hence strongly in L^p for any $p \geq 1$) as $i \to \infty$,

(ii) $u(t_i) \to u_\infty$, strongly in $W^{2,2}_{loc}(\mathcal{M}\backslash\{x^1, \ldots x^m\})$ as $i \to \infty$.

We will refer to the map u_∞ as the 'body map' (also refered to as the 'limit map' or the 'principal map') and to the points $x^1, \ldots x^m$ as 'bubble points.' The latter terminology will be justified at a later stage. If the set of bubble points is non-empty, we say that the flow has singularities at 'infinite time' at the points $x^1, \ldots x^m$. Of course, we are assuming that $\{x^1, \ldots x^m\}$ is minimal in the sense that the convergence $u(t_i) \to u_\infty$ is not strong in $W^{2,2}$ in any neighbourhood of any point x^i. When $u(t_i)$ converges for some sequence of times $t_i \to \infty$ as above, we are liable to say that u 'subconverges' as $t \to \infty$.

Remark 1.12 The $W^{2,2}$ convergence in Theorem (1.11) will be improved later to smooth convergence. (See Remark (2.15) on Page 40.) For now, $W^{2,2}$ convergence, which in particular implies C^0 convergence, will suffice.

In order to prove Theorem (1.11), we will have to control the second spatial derivatives of the flow. We will do this with the following estimate in which we use the notation $S_r = \{(x, y) \in \mathbb{R}^2 \mid x, y \in [-\frac{r}{2}, \frac{r}{2})\}$ for squares, and the abbreviation $S = S_1$.

Lemma 1.13 *For any $p \in [1,2)$, there exists $\varepsilon > 0$ such that for any $v \in W^{2,p}(S,\mathcal{N})$ with $E(v) < \varepsilon$ there holds the estimate*

$$\|v\|_{W^{2,p}(S_{\frac{1}{2}})} \leq C(1 + \|\nabla v\|_{L^2(S)} + \|\mathcal{T}(v)\|_{L^2(S)}).$$

Although the lemma as stated suffices for our purposes, it is still true with $p = 2$. We give a proof in the vein of [28].

Proof. From (1.1) in the introduction we recall that

$$\Delta v = \mathcal{T}(v) - A(v)(\nabla v, \nabla v).$$

Taking a smooth cut-off function $\varphi : S \to [0,1]$ such that $\varphi(S_{\frac{1}{2}}) = \{1\}$ but with compact support in S, we may calculate that

$$|\varphi \Delta v| \leq C(|\mathcal{T}(v)| + |\nabla(\varphi v)|.|\nabla v| + |\nabla v|),$$

and consequently that

$$|\Delta(\varphi v)| \leq C(1 + |\nabla v| + |\mathcal{T}(v)| + |\nabla(\varphi v)|.|\nabla v|).$$

Taking L^p norms over S, standard regularity theory then gives us

$$\|\varphi v\|_{W^{2,p}} \leq C(1 + \|\nabla v\|_{L^p} + \|\mathcal{T}(v)\|_{L^p} + \||\nabla(\varphi v)|.|\nabla v|\|_{L^p}).$$

We may estimate the final term firstly with Hölder's inequality and then with Sobolev's inequality, to get

$$\||\nabla(\varphi v)|.|\nabla v|\|_{L^p} \leq \|\nabla(\varphi v)\|_{L^{\frac{2p}{2-p}}} \|\nabla v\|_{L^2} \leq C\|\varphi v\|_{W^{2,p}} E(v)^{\frac{1}{2}}.$$

Consequently, for $E(v)$ sufficiently small, we can absorb the final term, to get

$$\|\varphi v\|_{W^{2,p}} \leq C(1 + \|\nabla v\|_{L^p} + \|\mathcal{T}(v)\|_{L^p}),$$

and the lemma follows. ∎

Proof. (Theorem (1.11).)

We begin by selecting a sequence of times $t_i \to \infty$ such that $\mathcal{T}(u(t_i)) \to 0$ in $L^2(\mathcal{M})$ as $i \to \infty$. This is possible from Equation (1.5). We immediately use the boundedness of the energies $E(u(t_i)) \leq E(u_0)$ and the Banach-Alaoglu Theorem to pass to a subsequence which converges weakly in $W^{1,2}$ to a map u_∞ say.

By choosing a finite number of conformal charts $\varphi_i : S \to \mathcal{M}$ for the domain such that $\bigcup_i \varphi_i(S_{\frac{1}{2}}) = \mathcal{M}$, and treating each individually, we may assume that the domain is in fact a 2-square S, and prove the claimed convergence from part (ii) on $S_{\frac{1}{2}}$. Let us cover the domain by the $(2n-1)^2$ squares $\mathcal{S}_j$ of side length $\frac{1}{n}$ with centres $(\frac{i}{2n}, \frac{j}{2n})$ for $|i|, |j| < n$. Then no point in S is covered more than 4 times, but $S_{\frac{1}{2}}$ is covered by the $(2n-1)^2$ squares with the same centres, but of side length $\frac{1}{2n}$. Clearly we cannot have more than $\frac{4E(u_0)}{\varepsilon}$ of the squares $\mathcal{S}_j$ on which the energy of $u(t_i)$ is at least ε. By passing to a subsequence of $\{t_i\}$, we may restrict such squares to a set (of size no more than $\frac{4E(u_0)}{\varepsilon}$) independent of i. We denote the union of the squares in this set by A. For each of the other squares $\mathcal{S}_j$ we may apply Lemma (1.13) after dilation, to find that

$$\|u(t_i)\|_{W^{2,p}(S\backslash A)} \leq C,$$

for some C independent of i. We may therefore pass to a further subsequence so that the sequence $\{u(t_i)\}$ is weakly convergent in $W^{2,p}(S\backslash A)$, with limit u_∞ as before. Taking $p \in (\frac{4}{3}, 2)$ the embedding $W^{2,p} \hookrightarrow W^{1,4}$ is compact. Passing to yet another subsequence, we obtain strong convergence in $W^{1,4}(S\backslash A)$ and in C^0 as a byproduct. Given these strengths of convergence, we may pass to the limit in the equation

$$\mathcal{T}(u_i) = \Delta u_i + A(u_i)(\nabla u_i, \nabla u_i)$$

where $u_i = u(t_i)$, to find that u_∞ is weakly harmonic, and indeed harmonic. Applying standard regularity theory to

$$\Delta(u_i - u_\infty) = \mathcal{T}(u_i) + A(u_\infty)(\nabla u_\infty, \nabla u_\infty) - A(u_i)(\nabla u_i, \nabla u_i),$$

we find that $u(t_i)$ is strongly convergent to u_∞ in $W^{2,2}(S\backslash A)$. Finally, by choosing n sufficiently large, we may shrink A so that $S\backslash A$ encloses any given compact subset of $\mathcal{M}\backslash\{x^1, \ldots x^m\}$. The final sequence of times is selected via a diagonal argument as A is shrunk, to ensure convergence on any compact subset of $\mathcal{M}\backslash\{x^1, \ldots x^m\}$ independent of the sequence. We remark that the map u_∞ is harmonic everywhere by virtue of the removable singularity theorem of Sacks and Uhlenbeck [28]. ∎

So in general, when the domain is a surface, we have some sort of convergence to a harmonic map at infinite time. By considering the examples given at the end of Section (1.2) of homotopy classes containing no harmonic map, we may reverse the strategy of Eells and Sampson to limit the smoothness of the flow, or the smoothness of the convergence to a harmonic map at infinite time. If we take an initial map which we cannot deform continuously to a harmonic map, and we consider the subsequent heat flow, then the flow cannot be both continuous and converge in C^0 to a harmonic map. The two possibilities are that the homotopy class of the flow switches at a finite time singularity, or at one of the points $x^1, \ldots x^m$ from Theorem (1.11) at infinite time. In fact, we will shortly give examples of flows which show each of these phenomena occuring. We remark that the strong convergence in L^p for $p < \infty$ mentioned in Theorem (1.11) is therefore optimal in the sense that C^0 convergence certainly is not true in general.

1.4.2 Blow-up and bubbling

Struwe's fundamental work [31] also described the structure of the singularities mentioned above. It was known from the work of Eells and Sampson that at any singularity, the gradient of the map could not be uniformly bounded, but rather that it had to 'blow up.'

Struwe showed that the flow could be rescaled appropriately to extract a 'harmonic sphere' (in other words a nonconstant harmonic map from S^2 to $\mathcal{N}$) in the limit. In other words, he showed that the 'bubbling' phenomenon of Sacks and Uhlenbeck occured. Precisely, his result is the following, which he gave in the case of domains without boundary - similar results for the case with boundary were given by Chang [2], who demonstrated that bubbling can only occur in the interior of $\mathcal{M}$.

Theorem 1.14 *For an initial map $u_0 \in W^{1,2}(\mathcal{M},\mathcal{N})$, let u be the solution of (1.3) introduced in Theorem (1.10). Let $(x_0, T) \in \mathcal{M} \times (0,\infty]$ be a singular point of the flow - in other words either one of the singular points with $T < \infty$ mentioned in Theorem (1.10) or a point (x_0, ∞) where x_0 is one of the points mentioned in Theorem (1.11). Then there exist sequences $a_i \to x_0$, $t_i \uparrow T$, $R_i \downarrow 0$ and a nonconstant harmonic map $\overline{u_0} : \mathbb{R}^2 \to \mathcal{N}$ such that as $i \to \infty$,*

$$u_i(x) \equiv u(\exp_{a_i}(R_i x), t_i) \to \overline{u_0} \text{ in } W^{2,2}_{loc}(\mathbb{R}^2, \mathcal{N}).$$

Moreover, $\overline{u_0}$ extends to a smooth harmonic map $S^2 \to \mathcal{N}$ which we refer to as a 'bubble'.

After Struwe's theorem there were several immediate questions. The first was whether energy is conserved during the process of bubbling. In other words, if we have a singular point (x_0, T) and an open set $U \subset \mathcal{M}$ such that $x_0 \in U$ but that $\overline{U} \times \{T\}$ contains no other singular point but (x_0, T), then do we have

$$\lim_{i \to \infty} E_U(u(t_i)) = E_U(u(T)) + E(\overline{u_0}) \tag{1.9}$$

(where we write $u(\infty) = u_\infty$ if necessary) for the sequence $\{t_i\}$ from Theorem (1.14)? We have the used the notation

$$E_U(v) = \int_U e(v) d\mathcal{M}$$

for $v : \mathcal{M} \to \mathcal{N}$. Note that from Struwe's work, we automatically have that the left-hand-side is greater than or equal to the right-hand-side in (1.9).

In Chapter (3) we shall give the first example in which we do not have equality in (1.9). The example only works in the case $T = \infty$. The loss of energy arises because more than one bubble develops at the same point. In fact, if we are prepared to rescale to find more than one bubble at each singularity, then we do have conservation of energy, as described in the following theorem due to Qing [26] in the case in which the target is a sphere, and independently to Ding and Tian [8] and to Wang [34] in the general case.

Theorem 1.15 *Let us pick bounded open sets $V \subset \mathbb{R}^2$ and $U \subset \mathcal{M}$ with $x_0 \in U$ and with a conformal diffeomorphism $\phi : V \to U$. We can always do this by taking isothermal coordinates. Moreover, by making the sets smaller if necessary, we may assume that (x_0, T) is the only singular point in $U \times \{T\}$, and by translating V we may assume that $\phi(0) = x_0$.*

Then there exists finitely many nonconstant harmonic maps $\{\omega_k\}_{k=1}^m$ from S^2 to $\mathcal{N}$ which we see as maps from $\mathbb{R}^2$ by stereographic projection, together with sequences

(i) $\{t_i\}$ with $t_i \uparrow T$,

(ii) $\{\{a_i^k\}\}_{k=1}^m$ in $\mathbb{R}^2$ with $\lim_{i\to\infty} a_i^k = 0$ for $1 \le k \le m$, and

(iii) $\{\{\lambda_i^k\}\}_{k=1}^m$ with $\lambda_i^k > 0$ for $1 \le k \le m$ and any i, and $\lim_{i\to\infty} \lambda_i^k = 0$ for $1 \le k \le m$,

such that

$$\frac{\lambda_i^k}{\lambda_i^j} + \frac{\lambda_i^j}{\lambda_i^k} + \frac{|a_i^k - a_i^j|^2}{\lambda_i^k \lambda_i^j} \to \infty \qquad as\ i \to \infty, \tag{1.10}$$

and

$$\lim_{i\to\infty} E_V(u(t_i)) = E_V(u(T)) + \sum_{k=1}^{m} E(\omega_k), \tag{1.11}$$

and moreover,

$$u(\phi(x), t_i) - \sum_{k=1}^{m} \left(\omega_k \left(\frac{x - a_i^k}{\lambda_i^k} \right) - \omega_k(\infty) \right) \to u(\phi(x), T)$$

strongly in $W^{1,2}(V, \mathbb{R}^N)$ as $i \to \infty$, where in the case that $T = \infty$, we read u_∞ for $u(T)$.

We remark that the fact that limit in (1.11) exists is not immediately clear, and could be considered part of the theorem. We remark further that the theorem could be restated using the exponential map as a projection rather than ϕ.

Theorem (1.15) tells us that the loss of energy in the flow may be exactly compensated by subtracting a set of bubbles. The condition (1.10) forces bubbles which develop close together to develep at different rates. In this case, we say that one bubble develops on top of another, to form a 'bubble tree.' We note that the theorem does not force equality between $u(x_0, T)$ and $\omega_k(\infty)$ for any k. In the case of inequality, we say that the bubble ω_k is attached via a 'neck.'

Returning to Theorem (1.14) in the case that $\partial\mathcal{M} \neq \emptyset$, we remark that a heuristic reason why bubbles cannot develop at the boundary - which was proved by Chang in [2] - is that the boundary conditions appear approximately constant upon dilation of the flow about a boundary point, and so any bubble map is forced to be constant on an equator. However, any harmonic map from S^2 which is constant on an equator must itself be constant, as we see by applying Theorem (1.8) to the hemispheres on either side of the equator.

1.4.3 Regularity by ruling out bubbles

The description of the structure of bubbling given in the previous section means that we can rule out finite time singularities and establish subconvergence of the flow at infinite time in $W^{2,2}$ if we can show that there do not exist any appropriate harmonic maps from

S^2 to $\mathcal{N}$ which could serve as bubble maps. (In fact the $W^{2,2}$ subconvergence can be improved to smooth subconvergence - refer to Remark (2.15).)

One way of doing this is to show that the flow has less energy than is required to produce a bubble. This will be discussed in Chapter (5). Here we use restrictions on the geometry of the target.

Theorem 1.16 *Suppose $\Omega \subset \mathcal{N}$ is open and admits a strictly convex function $\psi : \overline{\Omega} \to \mathbb{R}$ with $\psi(y) = \psi_0$ for all $y \in \partial\Omega$ (so $\psi(x) < \psi_0$ for all $x \in \Omega$). Then if $u_0(\mathcal{M}) \subset \Omega$, the subsequent flow u satisfies $u(\mathcal{M} \times [0, \infty)) \subset \Omega$ and is globally smooth. Moreover, if $\mathcal{M}$ is boundaryless then $u(t)$ converges to a constant map uniformly as $t \to \infty$ in C^0 and $W^{1,2}$.*

By using the function $\psi = d^2_{\mathcal{N}}(\cdot, x_0)$ we may specialise to two useful corollaries.

Corollary 1.17 *If $\mathcal{N}$ is a round sphere of any dimension and $\Omega \subset \mathcal{N}$ is an open hemisphere, then whenever $u_0(\mathcal{M}) \subset \Omega$, the subsequent flow u satisfies $u(\mathcal{M} \times [0, \infty)) \subset \Omega$ and is globally smooth. Moreover, if $\mathcal{M}$ is boundaryless then $u(t)$ converges to a constant map uniformly as $t \to \infty$ in C^0 and $W^{1,2}$.*

Corollary 1.18 *Whenever $u_0(\mathcal{M}) \subset \mathbf{B}_r(x)$, a geodesic ball of radius r centred at x, for sufficiently small r, the subsequent flow u satisfies $u(\mathcal{M} \times [0, \infty)) \subset \mathbf{B}_r(x)$ and is globally smooth. Moreover, if $\mathcal{M}$ is boundaryless then $u(t)$ converges to a constant map uniformly as $t \to \infty$ in C^0 and $W^{1,2}$.*

Remark 1.19 Of course, if we have a flow which converges without bubbles to a constant map, then we may apply Corollary (1.18) to find that the convergence is uniform in time.

Remark 1.20 In fact, as we will mention in Remark (2.15) the convergence in Theorem (1.16) and its two corollaries is smooth.

Before proving Theorem (1.16) we must recall a theorem of Gordon [16].

Theorem 1.21 *Dropping the restriction that $\mathcal{M}$ is a surface but imposing that $\mathcal{M}$ is boundaryless, if $\Omega \subset \mathcal{N}$ is open and admits a strictly convex function then any harmonic map $v : \mathcal{M} \to \mathcal{N}$ with $v(\mathcal{M}) \subset \Omega$ is necessarily constant.*

Proof. Calculation shows that

$$\Delta_{\mathcal{M}}(\psi \circ v) = \nabla^2 \psi(\nabla v, \nabla v). \tag{1.12}$$

Consequently, as $\psi : \Omega \to \mathbf{R}$ is convex, $\psi \circ v : \mathcal{M} \to \mathbf{R}$ is subharmonic and hence constant. Returning to (1.12) and using the strict convexity of ψ, we find that ∇v is zero, and hence that v is constant. ∎

Proof. (Theorem (1.16).)

Let us choose $\varepsilon > 0$ sufficiently small so that $u_0(\mathcal{M}) \subset \Omega_\varepsilon \equiv \{x \in \Omega \mid \psi(x) \leq \psi_0 - \varepsilon\}$. The parabolic equivalent of (1.12) is

$$\left(\frac{\partial}{\partial t} - \Delta_{\mathcal{M}}\right)(\psi \circ v) = -\nabla^2 \psi(\nabla u, \nabla u) \leq 0.$$

Applying the parabolic maximum principle, we find that $|\psi \circ u(t)| \leq \psi_0 - \varepsilon$ for all $t > 0$ (ie. $u(t)(\mathcal{M}) \subset \Omega_\varepsilon$). Moreover the flow cannot blow up as there are no nonconstant harmonic maps $S^2 \to \mathcal{N}$ which could serve as bubble maps, by Theorem (1.21) - any such bubble map would have its image enclosed in $\Omega_\varepsilon \subset \Omega$.

It remains to prove uniform convergence to a constant map in the case that $\mathcal{M}$ is boundaryless. By Theorems (1.11) and (1.14) we must have that $u(t_i) \to u_\infty$ in $W^{2,2}$ for some sequence $t_i \to \infty$ and some harmonic map u_∞ with $u_\infty(\mathcal{M}) \subset \Omega$. By Theorem (1.21) again, u_∞ must be constant - say $u_\infty(x) = l$ for all $x \in \mathcal{M}$. The convergence to u_∞ is uniform in C^0 as $t \to \infty$ because we may repeat the argument above with $\psi = d^2_{\mathcal{N}}(\cdot, l)$, which is always convex over some neighbourhood of l, to establish that $\sup_{x \in \mathcal{M}} d_{\mathcal{N}}(u(x,t), l)$ is decreasing in t. Convergence in $W^{1,2}$ rapidly follows from the fact that the energy is decreasing, and therefore must decrease monotonically to zero. ∎

Before leaving this section, we note that the theory we have seen can be used to improve Theorem (1.6) of Ding and Lin, in the case that the domain is a surface. We see that if the universal cover $\tilde{\mathcal{N}}$ of $\mathcal{N}$ supports a strictly convex function (with no growth condition necessary) then the flow cannot blow up either at finite or infinite time. This is because any bubble map $S^2 \to \mathcal{N}$ could be lifted to a nonconstant (harmonic) map $S^2 \to \tilde{\mathcal{N}}$ which is impossible by Theorem (1.21). This argument applies also in the case that the domain has boundary, unlike Theorem (1.6).

1.4.4 Examples of blow-up

As we mentioned earlier, by choosing an initial map in a homotopy class devoid of harmonic maps, we can be sure that a singularity develops in the subsequent flow. However, the question of whether singularities occur at finite time or infinite time is not so easy. In fact, singularities may indeed occur at finite time as well as at infinite time, though it was after the work of Struwe [31] describing the form of any singularities that Chang, Ding and Ye [4] gave the first example.

Let us consider maps between the 2-disc D and S^2. In this scenario, the flow evolves according to

$$\frac{\partial u}{\partial t} = \Delta u + u|\nabla u|^2.$$

In the examples which follow, we will take initial maps with a form of rotational symmetry, which is preserved under the flow, and it will therefore be beneficial to take polar coordinates (r, ϕ) on the domain and spherical polar coordinates (h, φ) on the target. A point (h, φ) then corresponds to the point $(\sin h \cos\varphi, \sin h \sin\varphi, \cos h) \in S^2 \hookrightarrow \mathbb{R}^3$. We consider initial maps of the form

$$(r, \phi) \to (h_0(r), \phi) \tag{1.13}$$

for $h_0 : [0, 1] \to \mathbb{R}$ with $h_0(0) = 0$, and these have corresponding flows

$$(r, \phi, t) \to (h(r, t), \phi).$$

In these coordinates, the heat equation (1.3) becomes

$$\begin{cases} \dfrac{\partial h}{\partial t} = \dfrac{\partial^2 h}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial h}{\partial r} - \dfrac{\sin h \cos h}{r^2}, \\ h(\cdot, 0) = h_0, \\ h(0, t) = 0, \ h(1, t) = b, \end{cases} \tag{1.14}$$

for some fixed b. By studying (1.14) for different values of b, and different initial maps h_0 we may arrive at the following two complementary results from [3] and [4] respectively.

Theorem 1.22 *Suppose that $u_0 : D \to S^2$ has the form (1.13) with $h_0(0) = 0$ and $\|h_0\|_\infty \leq \pi$. Then the flow u with initial map u_0 has no finite time singularities.*

Theorem 1.23 *Suppose that $u_0 : D \to S^2$ has the form (1.13) with $h_0(0) = 0$ and $|h_0(1)| > \pi$. Then the flow u with initial map u_0 blows up in finite time. In particular, there is no solution $u \in C(D \times (0,\infty), S^2)$.*

By taking $b = \pi$ in Theorem (1.22) the initial map u_0 has constant boundary values, but lies in a nontrivial homotopy class. By applying Theorem (1.8) of Lemaire as in the discussion at the end of Section (1.4.1), this then provides an example of infinite time blow-up.

The proof of Theorem (1.23) involves constructing a subsolution to (1.14) which blows up in finite time.

Both of the above theorems have analogues for the case of heat flow between 2-spheres. The basic idea is to consider heat flow from a hemisphere rather than a disc, and then glue two solutions along the equator. As for the details, observe that the upper closed hemisphere S^2_+ is isometric to D with the metric $\sigma^2(x)ds^2$ where ds^2 is the Euclidean metric of $\mathbf{R}^2$ and

$$\sigma(x) = \frac{2}{1+|x|^2}.$$

The heat equation with the new metric is then

$$\frac{\partial u}{\partial t} = \sigma^{-2}(x)(\Delta u + u|\nabla u|^2)$$

and with the symmetry as above, (1.3) becomes

$$\begin{cases} \dfrac{\partial h}{\partial t} = \dfrac{1}{\sigma^2}\left(\dfrac{\partial^2 h}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial h}{\partial r} - \dfrac{\sin h \cos h}{r^2}\right), \\ h(\cdot,0) = h_0, \\ h(0,t) = 0,\ h(1,t) = b, \end{cases} \tag{1.15}$$

The addition of the σ^{-2} factor does not affect the analysis of Chang, Ding and Ye, and we have equivalents of Theorems (1.22) and (1.23) from S^2_+ rather than D. Now, to create a flow between 2-spheres which blows up, we set $b = \frac{3}{2}\pi$ and take a corresponding, symmetric $u_0 : S^2_+ \to S^2$ which leads to blow-up, and extend the map to the whole of S^2 via the relation

$$u_0 \circ R_1 = R_1 \circ u_0,$$

where $R_1 : S^2 \to S^2$ is defined by $R_1(x, y, z) = (x, y, -z)$. This gives a degree one map (with $u_0|_{\partial S^2_+} = id|_{\partial S^2_+}$) which leads to blow-up. Alternatively, set $b = 2\pi$, this time extending the map u_0 to the whole of S^2 via

$$u_0 \circ R_1 = R_2 \circ u_0, \tag{1.16}$$

where $R_2 : S^2 \to S^2$ is defined by $R_2(x, y, z) = (-x, -y, z)$ and R_1 is as before. This gives a degree zero map (with $u_0(\partial S^2_+)$ =the north pole).

Alternatively, to create a flow between 2-spheres which blows up at infinite time, take $b = \pi$ and a corresponding, symmetric $u_0 : S^2_+ \to S^2$ with $h_0(1) = \pi$ and $\|h_0\|_\infty \leq \pi$. Then extend via (1.16) again to give a suitable degree zero initial map.

We record that all these examples from S^2 blow up simultaneously at the north and south pole, and that the degree of the flow remains the same after the time of the singularity. It is, however, possible to modify the examples of finite time blow-up so that the degree of the flow actually increases at the singularity.

1.5 Uniformity of the flow at infinite time

Let us return to the main result of Eells and Sampson, namely Theorem (1.5). In the original form of the theorem (as found in [11]) the convergence of the flow was given as

$$u_\infty = \lim_{i \to \infty} u(t_i)$$

for some sequence of times $t_i \to \infty$. Although this sufficed to establish a positive partial answer to Problem (1.3) the question of uniformity of the convergence in time was left open at that stage. The question was settled, in the case of targets with nonpositive sectional curvature by Hartman in [19] to give Theorem (1.5) as we stated it.

Other uniformity results are obtainable via application of the very general work of Leon Simon [29]. His theory was originally applicable in the case that the target was real analytic, but a result of Gulliver and White [17] allowed alternative application, essentially to the case of flow between 2-spheres (with arbitrary metrics).

Simon's result [29, Corollary 2] specialised to the case of the harmonic map heat flow is the following:

Theorem 1.24 *Consider the case that $\mathcal{N}$ is real analytic. Suppose that u is a smooth solution of* (1.3) *and that there exists a sequence of times $t_i \to \infty$ and a harmonic map u_∞ such that*

$$\lim_{i\to\infty} \|u(t_i) - u_\infty\|_{C^2} = 0.$$

Then we have the uniform convergence

$$\lim_{t\to\infty} \|u(t) - u_\infty\|_{C^2} = 0.$$

Of course, this theorem is never applicable to flows which blow up. It will become apparent later in the thesis (see Remark (2.15)) that in the case of heat flow from surfaces, the condition that there are no bubbles at infinite time is sufficient to obtain the C^2 convergence required for Theorem (1.24). Of course we still require that the target is real analytic (or that the flow is between 2-spheres) to apply Simon's theory. However, it is possible to argue directly to establish uniformity results for flows between 2-spheres without bubbles, using adaptations of the arguments of Leon Simon. We do not give details as all such results will be special cases of more general work in Chapter (2).

Chapter 2

Uniformity of the flow

In this chapter we will be considering the question of uniformity of the flow at infinite time. Here, and in all subsequent work in this thesis, we will be considering only the case that the domain is a surface. Moreover, in this chapter we will only be considering domain manifolds without boundary. The starting point for our discussion is Theorem (1.11). For the convenience of the reader, we recall that we found a sequence of times $t_i \to \infty$, a harmonic map $u_\infty : \mathcal{M} \to \mathcal{N}$ and a finite set of points $\{x^1, \ldots x^m\} \subset \mathcal{M}$ such that

(i) $u(t_i) \rightharpoonup u_\infty$, weakly in $W^{1,2}(\mathcal{M})$ (and hence strongly in L^p for any $p \geq 1$) as $i \to \infty$,

(ii) $u(t_i) \to u_\infty$, strongly in $W^{2,2}_{loc}(\mathcal{M}\backslash\{x^1, \ldots x^m\})$ as $i \to \infty$.

In this chapter we will be examining whether this convergence depends on the sequence of times $\{t_i\}$ or whether it holds uniformly as $t \to \infty$. For reference, we pose the following specific problems.

Problem 2.1 *With the notation above, do we have convergence of the forms*

(i) $u(t) \rightharpoonup u_\infty$, *weakly in* $W^{1,2}(\mathcal{M})$ *as* $t \to \infty$,

(ii) $u(t) \to u_\infty$, *strongly in* $W^{2,2}_{loc}(\mathcal{M}\backslash\{x^1, \ldots x^m\})$ *as* $t \to \infty$,

(iii) $E_B(u(t)) \to \overline{E_B}$ as $t \to \infty$, for some $\overline{E_B}$, where $B \subset \mathcal{M}$ is any small closed ball containing exactly one of the points $x^1, \dots x^m$?

As we shall see, in general the convergence may be far from uniform, though we will give hypotheses under which we establish various forms of uniform convergence, many of them being exponential in rate.

2.1 Examples of non-uniform flows

We give an example of a flow which demonstrates that in general, the uniform convergence of part (i) of Problem (2.1) does not hold. No bubbling occurs. The flow has a 'winding' behaviour and has a circle of accumulation points. In particular, we could find two sequences $t_i \to \infty$ and $s_i \to \infty$ and two distinct harmonic maps u_1 and u_2 such that

$$u(t_i) \to u_1 \qquad \text{and} \qquad u(s_i) \to u_2$$

in $W^{2,2}$ say, as $i \to \infty$. The example will also show that a perturbation of a locally energy minimising harmonic map may move far away under the heat flow, and we will refer back to this property in Chapter (4).

Let the domain be S^2 and the target $\mathbf{R}^2 \times S^2$. It is not important that $\mathbf{R}^2$ is noncompact - as we shall see, we are only concerned with a bounded region, so we could change it to a flat 2-torus. We give the domain the standard metric, but give the target a warped metric - if g and h are the standard metrics on $\mathbf{R}^2$ and S^2 respectively, then at a point $(z,x) \in \mathbf{R}^2 \times S^2$, we define the metric to be $g(z) + f(z)h(x)$ where $f : \mathbf{R}^2 \to \mathbf{R}_+$ is to be determined. In other words, the target is $\mathbf{R}^2 \times_f S^2$. We consider initial maps of the form $u_0(x) = (z_0, x)$ where z_0 is independent of x. Such maps give solutions of the heat equation (1.3) of the form $u(x,t) = (z(t), x)$, where $z : \mathbf{R}_+ \to \mathbf{R}^2$, and z evolves according to

$$\frac{dz}{dt} = -\nabla f(z(t)). \tag{2.1}$$

In other words, we have reduced the heat flow to finite-dimensional gradient flow for z on the function f. It remains to choose the function f so that z may not have a unique limit, and so that moving z an arbitrarily small amount from a point z_0 with $\nabla f(z_0) = 0$ (which corresponds to a harmonic map) will make the solution of (2.1) move away from

z_0. To achieve this we take a 'downwardly spiralling gramophone record'

$$f(r,\theta) = \begin{cases} 1 & \text{if } r \leq 1 \\ 1 + e^{-\frac{1}{r-1}} \left(\sin(\frac{1}{r-1} + \theta) + 2 \right) & \text{if } r > 1, \end{cases} \tag{2.2}$$

where (r, θ) are polar coordinates. Taking initial conditions with $r = 2$ say, the solution for z will spiral in to give the circle $r = 1$ as the accumulation set. Moreover, any point with $r = 1$ is a stationary point, but perturbing r to be slightly larger will make the solution of (2.1) spiral around, and move at least a distance 2 away.

Of course, we are not restricted to using 2-spheres in the example. In particular, the same idea will work for domains of any dimension by considering flow from S^n to $\mathbb{R}^2 \times S^n$ for $n \geq 1$. We remark that we do not know if a flow can be non-uniform if $\mathcal{M}$ has boundary.

Although we have provided a counterexample to part (i) of Problem (2.1) and also to part (ii) as a byproduct, there remains the question of whether energy may flow into and out of a region for all time to provide a counterexample to part (iii). In particular, we have Leon Simon's problem of 'the uniqueness of the positions of bubbles,' which asks whether we may pick two sequences of times $t_i \to \infty$ and $s_i \to \infty$ such that the sequences $\{u(t_i)\}$ and $\{u(s_i)\}$ both have convergence as in Theorem (1.11) but with bubbles developing at a different set of points. In fact, we believe that not only is this possible, but that we could pick two sequences of times so that for one sequence we have bubbles forming, but that for another sequence we have no bubbles whatsoever. We discuss this further in an appendix, where we sketch a counterexample.

2.2 A uniformity result

Let us consider the case of heat flow between round 2-spheres.

After defining, for maps $v : \mathcal{M} \to \mathcal{N}$, a local energy

$$E_{(x,r)}(v) = \int_{\mathbb{B}_r(x)} e(v),$$

where $\mathbb{B}_r(x)$ is the geodesic ball of radius r centred at x in S^2, we may state our main uniformity result.

Theorem 2.2 *Suppose we have a solution u of the heat equation* (1.3) *as found in Theorem* (1.10) *with both the domain $\mathcal{M}$ and target $\mathcal{N}$ being round 2-spheres. Suppose moreover that at infinite time, the body map and any bubbles all share a common orientation. Then with the definition of u_∞ and $\{x^k\}_{k=1}^m$ as in Theorem* (1.15) *we have that*

(i) $u(t) \rightharpoonup u_\infty$ *uniformly as* $t \to \infty$ *weakly in* $W^{1,2}(S^2, \mathbb{R}^3)$ *- and hence strongly in* $L^p(S^2, \mathbb{R}^3)$ *for any* $p \in [1, \infty)$,

(ii) $u(t) \to u_\infty$ *uniformly as* $t \to \infty$ *in* $C^k_{loc}(S^2 \backslash \{x^1, \ldots x^m\}, \mathbb{R}^3)$ *for any* $k \in \mathbb{N}$,

(iii) for any $r > 0$ *sufficiently small and* $k \in \{1, \ldots m\}$, *we have that* $E_{(x^k, r)}(u(t))$ *converges to a limit* $F_{k,r}$ *uniformly as* $t \to \infty$.

In addition, given $\Omega \subset\subset S^2 \backslash \{x^1, \ldots x^m\}$ *there exists* $C = C(u_0, \Omega) > 0$ *and* $\gamma > 0$ *a universal constant such that for all* $t \geq 0$

(iv) $\|u(t) - u_\infty\|_{L^2(S^2)} \leq Ce^{-\gamma t}$,

(v) $\|u(t) - u_\infty\|_{W^{1,2}(\Omega)} \leq Ce^{-\gamma t}$,

(vi) $|E_{(x^k, r)}(u(t)) - F_{k,r}| \leq Ce^{-\gamma t}$.

This theorem not only gives us uniqueness of the limit map u_∞, but also solves (in our situation) the problem of 'the uniqueness of the positions of the bubbles.' As mentioned earlier, this means that we cannot take the flow at another sequence of times $t_i \to \infty$ to get a different set of blow-up points $\{x^k\}$.

Before we proceed to the proof of the theorem, we give a condition on the initial map under which the hypotheses of Theorem (2.2) will always be satisfied. A proof will be offered towards the end of this chapter, after we have developed the necessary technology (Page 45).

Proposition 2.3 *The condition that the body map and bubbles at infinite time share a common orientation will certainly be satisfied providing*

$$E(u_0) \leq 8\pi + 4\pi |deg(u_0)|$$

We need barely mention that if no bubbles develop in a flow at a sequence of times $t_i \to \infty$ from Theorem (1.14) then the hypotheses are again satisfied.

2.2.1 Complex notation

To prove Theorem (2.2) we will have to introduce some complex analytic techniques. The principal problem to be overcome in the proof is that many of the tools of analysis which we would like to apply are rendered inappropriate by the formation of bubbles. Loosely speaking, abstract functional analytic techniques fail because within the well-known spaces of maps with strong enough topology to control the flow, any bubbles will scupper all hopes of convergence. As we shall see, the complex analysis will be used to filter out the singularity of the flow which prevented analysis, whilst preserving enough information to control the movement of the flow.

Let us consider maps $u : S^2 \to S^2$. We use $z = x + iy$ as a complex coordinate on the domain, obtained by stereographic projection, and write the metric as $\sigma^2 dz d\bar{z}$, where

$$\sigma(z) = \frac{2}{1+|z|^2},$$

as we assume the domain to be a round 2-sphere. Similarly we have a complex coordinate on the target which we denote also by u, and a metric $\rho^2 du d\bar{u}$.

To the map u we associate the energy densities

$$e_{\partial}(u) = \frac{\rho^2(u)}{\sigma^2}|u_z|^2, \qquad e_{\bar{\partial}}(u) = \frac{\rho^2(u)}{\sigma^2}|u_{\bar{z}}|^2,$$

and

$$e(u) = e_{\partial}(u) + e_{\bar{\partial}}(u).$$

The corresponding energies are

$$E_{\partial}(u) = \int_{S^2} e_{\partial}(u) = \frac{i}{2}\int_{\mathbf{C}} \rho^2 |u_z|^2 dz \wedge d\bar{z},$$

$$E_{\bar{\partial}}(u) = \int_{S^2} e_{\bar{\partial}}(u) = \frac{i}{2}\int_{\mathbf{C}} \rho^2 |u_{\bar{z}}|^2 dz \wedge d\bar{z},$$

and

$$E(u) = E_{\partial}(u) + E_{\bar{\partial}}(u). \tag{2.3}$$

Of course, holomorphicity and anti-holomorphicity of u correspond precisely to the conditions $E_{\bar{\partial}}(u) = 0$ and $E_{\partial}(u) = 0$ respectively. The Jacobian of u is given by

$$J(u) = e_{\partial}(u) - e_{\bar{\partial}}(u),$$

and consequently we see that

$$E_{\partial}(u) - E_{\bar{\partial}}(u) = 4\pi deg(u). \tag{2.4}$$

For a fixed target chart, we may form

$$\tau = \frac{4}{\sigma^2}\left(u_{z\bar{z}} + \frac{2\rho_u}{\rho}u_z u_{\bar{z}}\right) \tag{2.5}$$

The associated geometric object is $\tau\frac{\partial}{\partial u}$, and the tension is related through the equality

$$\mathcal{T}(u) = \tau\frac{\partial}{\partial u} + \bar{\tau}\frac{\partial}{\partial \bar{u}}.$$

Many of the definitions and statements above can be adapted for other orientable domain and target surfaces.

To avoid any possible confusion, we record that seeing u as the heat flow (so u maps into the manifold $\mathcal{N}$) the heat equation is $\frac{\partial u}{\partial t} = \mathcal{T}$, whilst writing the value of u in terms of a complex coordinate on the target, the heat equation may be written $\frac{\partial u}{\partial t} = \tau$.

Before leaving this section, we note that for maps between a 2-sphere and a surface, Proposition (1.2) may be translated to give

Proposition 2.4 *Any harmonic map from S^2 to an orientable surface is either holomorphic or anti-holomorphic.*

In fact, the calculation is marginally simpler in this case as we may write the value of the map in terms of a local complex coordinate u on the target to find that $\phi(u) = \rho^2(u)u_z u_{\bar{z}}$. We then rapidly arrive at $\phi_{\bar{z}} = \rho^2(\bar{u}_z\tau + u_z\bar{\tau}) = 0$ and may proceed as before.

With the hindsight of Proposition (2.4) the hypothesis that all bubbles and the body map share a common orientation is precisely that they are all holomorphic, or all anti-holomorphic.

For the case of maps between 2-spheres (with the standard metric on the target for simplicity) we have the following extension.

Proposition 2.5 *Let $v : S^2 \to S^2$ be harmonic. Then v is an absolute minimum of energy in its homotopy class, and*

$$E(v) = 4\pi|deg(v)|.$$

Proof. From (2.3) and (2.4) we find that for any $v : S^2 \to S^2$

$$E(v) = 2E_{\partial}(v) + 4\pi deg(v) = 2E_{\bar{\partial}}(v) - 4\pi deg(v) \geq 4\pi|deg(v)|$$

with equality precisely when either $E_{\partial}(v) = 0$ or $E_{\bar{\partial}}(v) = 0$. ∎

Remark 2.6 Consequences of this proposition include that all degree zero harmonic maps between 2-spheres are constant, and that the minimum energy of a nonconstant harmonic map between 2-spheres is 4π. Note that we would have to replace 4π by the area of the target sphere here and in Proposition (2.5) if we chose not to use the standard metric on the target.

In fact, as we shall now see, the case that the target is any oriented surface other than a 2-sphere is even simpler.

Proposition 2.7 *Any harmonic map from S^2 to an oriented surface of positive genus is constant.*

In particular, the flow in this situation cannot have any singularities (as no bubble maps are possible) and must converge to a constant (uniformly in time by Remark (1.19)).

Proof. (Proposition (2.7).)

Recall that as $\mathcal{N}$ has positive genus, we have that $\pi_2(\mathcal{N}) = 0$ and hence all maps between S^2 and $\mathcal{N}$ lie in the trivial homotopy class.

We could construct a proof by generalising the method of the previous lemma, to show that the holomorphicity (or anti-holomorphicity) of the harmonic maps being considered force them to be absolute minimisers of energy and hence constant.

As an alternative, we lift the harmonic map to the universal cover of $\mathcal{N}$, which is either the 2-disc or the plane. By Proposition (2.4) and Liouville's Theorem, this lifted harmonic map, and hence the original map, must be constant. ∎

We mention that in practice, orientability of the target only serves to eliminate the case $\mathcal{N} = \mathbf{R}P^2$. For other targets we may lift the harmonic map to the orientable cover of $\mathcal{N}$ before applying the proposition.

The preceeding three propositions are well known.

2.2.2 The key estimate

In this section we derive a key estimate controlling the $\bar{\partial}$-energy in terms of the tension. The estimate is very similar to the key estimate of Leon Simon in his important paper [29]. However, in the special case of maps between 2-spheres, and with the harmonic map energy functional E, we are able to reduce Simon's hypothesis of $W^{2,2}$ closeness to a harmonic map, to just smallness of the $\bar{\partial}$-energy. This makes it applicable to maps with bubbles, assuming the bubbles and the body map are all anti-holomorphic (or all holomorphic).

Very loosely speaking, the harmonic map heat flow can only keep moving energy about for all time if the total energy is dissipated very slowly. The point of the estimate will be to show that this cannot happen, and thus that the heat flow becomes 'rigid' for large times. We will return to make some concluding remarks in this vein following the proof of Theorem (2.2).

Lemma 2.8 *There exists $\varepsilon_0 > 0$ and $\kappa > 0$ such that providing $u : S^2 \to S^2$ satisfies $E_{\bar{\partial}}(u) < \varepsilon_0$, we have the estimate*

$$E_{\bar{\partial}}(u) \leq \kappa \|\mathcal{T}(u)\|^2_{L^2(S^2)} \tag{2.6}$$

This lemma is very much a global result. Any such local result would imply the conformality of any harmonic map from a surface to a 2-sphere, which does certainly not hold. The necessity of a smallness of energy condition as in the lemma is evident by considering the identity map, for which the left-hand-side is equal to 4π whilst the right-hand-side is zero. Curiosity begs the question of how large ε_0 may be, and in particular whether it could take the value 4π. In addition, for a given, perhaps optimal value of ε_0 it would be interesting to know the value of κ. We stress that these questions are not prompted by potential applications, however.

Before proving Lemma (2.8) we recall the following lemma from [35, Theorem 2.8.4] or [30, Chapter V, Theorem 1, page 119].

Lemma 2.9 *Suppose we have an operator I on functions $f : \mathbf{C} \to \mathbf{R}$ given by*

$$(If)(w) = \int_{\mathbf{C}} \frac{f(z)}{|z-w|} \frac{i}{2} dz \wedge d\bar{z},$$

then for $q \in (1,2)$ we have the estimate

$$\|If\|_{L^{\frac{2q}{2-q}}(\mathbf{C})} \leq C(q)\|f\|_{L^q(\mathbf{C})}.$$

Proof. (Lemma (2.8).)

Fix global complex coordinates z and u on the domain and target respectively. With these coordinates, we consider the quantity $\rho^2 u_z$. To begin with, we calculate

$$\begin{aligned} (\rho^2 u_z)_{\bar{z}} &= \rho^2 u_{z\bar{z}} + 2\rho\rho_u u_{\bar{z}} u_z + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} u_z & (2.7) \\ &= \tfrac{1}{4}\sigma^2\rho^2\tau + 2\rho\rho_{\bar{u}}|u_z|^2, & (2.8) \end{aligned}$$

by (2.5). In particular, as $\rho_{\bar{u}} = -\frac{1}{2}u\rho^2$, we see that

$$|(\rho^2 u_z)_{\bar{z}}| \leq |\sigma^2\rho\tau| + \rho^2|u_z|^2. \tag{2.9}$$

We now apply Cauchy's theorem for C^∞ functions to the function $\rho^2 u_z$ to get, for $|w| < r$

$$\rho^2 u_z(w) = \frac{1}{2\pi i}\int_{\partial D_r} \frac{\rho^2 u_z}{z-w} dz + \frac{1}{2\pi i}\int_{D_r} \frac{(\rho^2 u_z)_{\bar{z}}}{z-w} dz \wedge d\bar{z},$$

where $D_r = \{z \in \mathbf{C} \ : \ |z| < r\}$.

Observe that as $u_z\frac{\partial}{\partial u}\otimes dz$ is independent of the coordinates z and u, its norm $|\frac{\rho}{\sigma}u_z|$ in our fixed coordinates must be bounded and therefore $|\rho^2 u_z| \to 0$ as $|z| \to \infty$. Consequently, we may let r tend to infinity to find that

$$\rho^2 u_z(w) = \frac{1}{2\pi i}\int_{\mathbf{C}} \frac{(\rho^2 u_z)_{\bar{z}}}{z-w} dz \wedge d\bar{z}.$$

Combining this with (2.9), we have

$$|\rho^2 u_z(w)| \leq \frac{1}{\pi}\int_{\mathbf{C}} \frac{1}{|z-w|}(|\sigma^2\rho\tau| + \rho^2|u_z|^2)\frac{i}{2}dz \wedge d\bar{z}.$$

Now we observe that the right hand side is independent of which stereographic chart we took for the target (note for example that $|\rho\tau|$ is the length of $\tau\frac{\partial}{\partial u}$) so by taking a chart for which $0 \in \mathbf{C}$ corresponds to $u(w)$, we obtain the estimate

$$|\rho u_z(w)| \leq \frac{1}{2\pi}\int_{\mathbf{C}} \frac{1}{|z-w|}(|\sigma^2\rho\tau| + \rho^2|u_z|^2)\frac{i}{2}dz \wedge d\bar{z}. \tag{2.10}$$

But now, all the terms are independent of the target chart, allowing us to change target chart, or equivalently to move w with a fixed chart.

To develop estimate (2.10) we need to appeal to the theory of Riesz potentials, and in particular to Lemma (2.9). This produces, for $q \in (1,2)$, the first of the estimates

$$\begin{aligned}
\|\rho u_z\|_{L^{\frac{2q}{2-q}}(\mathbf{C})} &\leq C\left(\|\sigma^2\rho\tau\|_{L^q(\mathbf{C})} + \|\rho^2|u_z|^2\|_{L^q(\mathbf{C})}\right) && (2.11)\\
&\leq C\left(\|\rho\tau\|_{L^q(S^2)} + \|\rho u_z\|^2_{L^{2q}(\mathbf{C})}\right) && (2.12)\\
&\leq C\left(\|\rho\tau\|_{L^2(S^2)} + \|\rho u_z\|_{L^2(\mathbf{C})}\|\rho u_z\|_{L^{\frac{2q}{2-q}}(\mathbf{C})}\right), && (2.13)
\end{aligned}$$

whilst the last follows from applying Hölder's inequality to both terms. The constant C is changing, of course, but remains independent of u. Now, as $E_{\bar{\partial}}(u) = \|\rho u_z\|^2_{L^2(\mathbf{C})}$, we see that there exists $\varepsilon_0 > 0$ such that providing $E_{\bar{\partial}}(u) \leq \varepsilon_0$, we may absorb the second term on the right hand side into the left hand side to give

$$\|\rho u_z\|_{L^{\frac{2q}{2-q}}(\mathbf{C})} \leq C\|\mathcal{T}\|_{L^2(S^2)}.$$

We want to reduce the exponent $\frac{2q}{2-q}$ down to 2, but to do so we must operate on a bounded domain. We again exploit the geometry by retreating to

$$\left(\int_{S^2_+} e_{\bar{\partial}}(u)\right)^{\frac{1}{2}} = \|\rho u_z\|_{L^2(D_1)} \leq C\|\rho u_z\|_{L^{\frac{2q}{2-q}}(D_1)} \leq C\|\mathcal{T}\|_{L^2(S^2)},$$

(where S^2_+ is the hemisphere corresponding to the points in the domain with $|z| < 1$) repeating the estimate with the 'opposite' domain chart to give a $\bar{\partial}$-energy estimate over the remaining hemisphere, and combining the two to find

$$E_{\bar{\partial}}(u) \leq C\|\mathcal{T}\|^2_{L^2(S^2)}.$$

■

Remark 2.10 We remark that we have in fact proved more than stated in that we can control the L^p norm of $e_\partial(u)$ for any $p \in [1, \infty)$ not just $p = 1$. We speculate, ahead of time, that this will be enough to rule out necks (see Section (1.4.2)) in flows as considered in (2.2).

Remark 2.11 Our key lemma is the part of this theory which requires the hypotheses on the domain, target and flow. As we shall see, the ∂-energy $E_\partial(u(t))$ is exactly half the energy still left to be dissipated during the flow, but in fact whenever we have an estimate of the form

$$\text{(energy left to dissipate)} \leq \|\mathcal{T}\|^p_{L^2}$$

for some $p > 1$, the forthcoming proofs will be valid. Of course, such an estimate will not be true in general as the conclusions of our theorems are not true in general.

Although we only give details of the case of round 2-spheres, we mention that the key lemma as stated implies the same lemma with a deformed domain metric, and that the proof can be modified to imply the same lemma with a deformed target metric.

2.2.3 Basic energy control and regularity theory

To prove the smooth convergence of part (ii) of Theorem (2.2) we will need some regularity theory controlling the C^k norms of the flow away from any points where the energy density concentrates, We will take the opportunity to give a more comprehensive theory than appears to be necessary, to justify some claims made elsewhere in this thesis.

We begin with a result of Struwe - refer to [31, Lemma 3.10′] - which applies for any domain surface $\mathcal{M}$. To state it, we need the measure of concentration of a flow over a set $\Omega \subset\subset \mathcal{M}$ defined by

$$\mathcal{E}(R, \Omega, a, b) = \sup_{(x,t)\in\Omega\times[a,b)} E_{(x,R)}(u(t)).$$

Later we will use the abbreviation

$$\mathcal{E}(R, \Omega) = \mathcal{E}(R, \Omega, 0, \infty).$$

Lemma 2.12 *There exists $\varepsilon_1 > 0$ such that the following is true. For all $s \geq \tau > 0$ and $T > 0$, whenever we have a flow u with initial map u_0 satisfying*

$$\mathcal{E}(R, \Omega, s - \tau, s + T) < \varepsilon_1$$

for some $R > 0$ and $\Omega \subset\subset \mathcal{M}$, then the C^k norms of u are bounded uniformly on $\Omega \times [s, s+T)$ in terms of $E(u_0)$, τ, T, k and $\mathcal{M}$, but independently of s.

Successive iterations of the lemma above give us the regularity lemma we shall require in this chapter.

Lemma 2.13 *There exists $\varepsilon_1 > 0$ such that if a flow u satisfies $\mathcal{E}(R, \Omega) < \varepsilon_1$ for some $R > 0$ and $\Omega \subset\subset \mathcal{M}$, then the C^k norms of u are bounded uniformly on $\Omega \times [1, \infty)$.*

Let us digress to study the movement of energy into and out of local regions. We take such local measurements with a 'cut energy' Θ_v of a map $v : \mathcal{M} \to \mathcal{N} \hookrightarrow \mathbb{R}^N$ defined by

$$\Theta_w = \Theta_v^{(r,s)} = \frac{1}{2} \int_{\mathcal{M}} \varphi^2 |\nabla v|^2 d\mathcal{M},$$

where $\varphi : \mathcal{M} \to [0, 1]$ is the cut-off function about $y \in \mathcal{M}$

$$\varphi(x) = \begin{cases} 1 & \text{if } d(x, y) \leq r \\ 1 + \frac{1}{s}(r - d(x, y)) & \text{if } r < d(x, y) < r + s \\ 0 & \text{if } d(x, y) \geq r + s \end{cases}$$

in which $d(\cdot, \cdot)$ represents geodesic distance on $\mathcal{M}$, and $r, s > 0$ are sufficiently small. We abbreviate

$$\Theta(t) = \Theta^{(r,s)}(t) = \Theta^{(r,s)}_{u(t)}.$$

Lemma 2.14 *For a flow u with initial map u_0, the evolution of the cut-energy $\Theta = \Theta(t)$ is constrained in terms of the tension $\mathcal{T} = \mathcal{T}(u(t))$ according to*

$$-\|\mathcal{T}\|^2_{L^2(\mathcal{M})} - \frac{C}{s} \Theta^{\frac{1}{2}} \|\mathcal{T}\|_{L^2(\mathcal{M})} \leq \frac{d\Theta}{dt} \leq -\|\mathcal{T}\|^2_{L^2(\mathcal{M})} + \frac{C}{s} \Theta^{\frac{1}{2}} \|\mathcal{T}\|_{L^2(\mathcal{M})}, \qquad (2.14)$$

where C depends only on $\mathcal{M}$. Consequently we find that

(i)

$$\Theta(t) - \Theta(t_0) \leq \frac{C}{s^2} E(u_0)(t - t_0)$$

for $t \geq t_0 \geq 0$, and in particular

$$E_{(y,R)}(u(t)) \leq E_{(y,2R)}(u(t_0)) + \frac{C}{R^2}E(u_0)(t-t_0),$$

(ii) for any $\varepsilon > 0$, there exists a $T \geq 0$ (dependent on ε, s and the flow u) such that for $t > t_0 > T$ we have

$$|\Theta(t) - \Theta(t_0)| \leq \varepsilon(t - t_0 + 1)$$

and in particular for $t, t_0 > T$ we find that

$$E_{(y,R)}(u(t)) \leq E_{(y,2R)}(u(t_0)) + \varepsilon(|t-t_0| + 1).$$

We record that part (i) has been given in [31, Lemma 3.6] already.

Proof. The cut energy Θ evolves according to

$$\frac{d\Theta}{dt} = \int_{\mathcal{M}} \varphi^2 \nabla u^i . \nabla \mathcal{T}^i d\mathcal{M} = -\int_{\mathcal{M}} \varphi^2 |\mathcal{T}|^2 d\mathcal{M} - 2\int_{\mathcal{M}} \varphi(\nabla\varphi . \nabla u^i)\mathcal{T}^i d\mathcal{M}.$$

Using the fact that the first term on the right hand side is negative, and applying Hölder's inequality to the second term, we arrive immediately at the principle estimate (2.14) of Lemma (2.14).

By applying Young's inequality to the second of the inequalities in (2.14) we find that

$$\frac{d\Theta}{dt} \leq \frac{C}{s^2}\Theta \leq \frac{C}{s^2}E(u_0),$$

and part (i) of the lemma follows easily.

An alternative combination of Young's inequality and (2.14) leads rapidly to

$$\left|\frac{d\Theta}{dt}\right| \leq \frac{\varepsilon}{E(u_0)}\Theta + \left(1 + \frac{C\,E(u_0)}{\varepsilon s^2}\right)\|\mathcal{T}\|^2. \tag{2.15}$$

Choosing T sufficiently large so that

$$\int_T^\infty \|\mathcal{T}(u(\xi)\|^2 d\xi \leq \frac{\varepsilon}{\left(1 + \frac{C\,E(u_0)}{\varepsilon s^2}\right)}$$

(which we can do by (1.5)) and integrating (2.15) we arrive at part (ii) of the lemma. ∎

Remark 2.15 Our digression into the study of the cut energy may be used to establish the improved regularity promised in Remark (1.12) and elsewhere in this thesis. Recall that in Theorem (1.11) for a flow u we found a sequence of times $t_i \to \infty$, a set of points $\{x^1, \ldots x^m\} \subset \mathcal{M}$ and a harmonic map $u_\infty : \mathcal{M} \to \mathcal{N}$ such that $u(t_i) \to u_\infty$ strongly in $W^{2,2}_{loc}(\mathcal{M}\backslash\{x^1, \ldots x^m\})$ as $i \to \infty$. We will now argue that this convergence is in C^k_{loc}.

Let us fix $\Omega \subset\subset \mathcal{M}\backslash\{x^1, \ldots x^m\}$, and choose $R > 0$ sufficiently small so that for all $x \in \Omega$ we have $\mathbb{B}_{2R}(x) \subset \mathcal{M}\backslash\{x^1, \ldots x^m\}$ and $E_{(x,2R)}(u_\infty) < \frac{\varepsilon_1}{4}$. As $u(t_i) \to u_\infty$ in $W^{1,2}_{loc}(\mathcal{M}\backslash\{x^1, \ldots x^m\})$ as $i \to \infty$, we can find $N \in \mathbb{N}$ such that for all $i > N$ we have

$$E_{(x,2R)}(u(t_i)) < \frac{\varepsilon_1}{2}$$

where N is independent of x. Applying part (ii) of Lemma (2.14) we see that for all $i > N$, for possibly larger N (still independent of x) we have

$$\sup_{t \in [t_i - 1, t_i + 1)} E_{(x,R)}(u(t)) < \varepsilon_1$$

and therefore

$$\mathcal{E}(R, \Omega, t_i - 1, t_i + 1) < \varepsilon_1.$$

We may now bring Lemma (2.12) into play to find that

$$\|u(t_i)\|_{C^k(\Omega)} < C(k)$$

for all i, so using the Ascoli-Arzéla Theorem we have, by passing to a subsequence,

$$u(t_i) \to u_\infty \text{ in } C^{k-1}(\Omega) \text{ as } i \to \infty,$$

for any k. Passing to a further subsequence using a diagonal argument, we may make the convergence valid for any $\Omega \subset\subset \mathcal{M}\backslash\{x^1, \ldots x^m\}$ (and indeed any k) for a fixed sequence of times $\{t_i\}$.

2.2.4 Proof of the main uniformity result

Proof. (Theorem (2.2).)

Without loss of generality, we will assume that the body map and all the bubbles are anti-holomorphic, rather than holomorphic.

We begin by recalling from (1.4) that

$$\frac{d}{dt}E(u(t)) = -\|\mathcal{T}(u(t))\|^2_{L^2(S^2)}. \tag{2.16}$$

Moreover, as a combination of (2.3) and (2.4) gives

$$E_{\partial}(u) = \frac{1}{2}\left(E(u) + 4\pi deg(u)\right), \tag{2.17}$$

we see that

$$\frac{d}{dt}E_{\partial}(u(t)) = -\frac{1}{2}\|\mathcal{T}(u(t))\|^2_{L^2(S^2)}. \tag{2.18}$$

The conservation of energy established in Theorem (1.15), due to Qing in this case, together with the fact that the maps $\{\omega_k\}$ are all anti-holomorphic tells us that at the sequence of times $\{t_i\}$, the ∂-energy is converging to zero. As the ∂-energy is decreasing (equation (2.18)) we then see that

$$E_{\partial}(u(t)) \to 0 \qquad \text{as} \qquad t \to \infty. \tag{2.19}$$

In particular, there exists a time T such that if $t \geq T$, then $E_{\partial}(u(t)) < \varepsilon_0$, where ε_0 is defined in Lemma (2.8). As we are concerned only with the asymptotics of the heat flow, we may suppose for simplicity that $T = 0$. (This will not affect parts (iv) to (vi) either.) Moreover, we may assume that no finite time blow-up occurs.

The combination of Lemma (2.8) and equation (2.18) now yields

$$\frac{d}{dt}E_{\partial}(u(t)) \leq -\frac{1}{2\kappa}E_{\partial}(u(t))$$

and hence, writing $\gamma = \frac{1}{4\kappa}$, we find that

$$E_{\partial}(u(t)) \leq E_{\partial}(u(t_0))e^{-2\gamma(t-t_0)},$$

which will be necessary to establish the exponential convergence of parts (iv) to (vi) of Theorem (2.2). An alternative application of Lemma (2.8) gives

$$-\frac{d}{dt}\left(E_{\partial}(u(t))\right)^{\frac{1}{2}} = \frac{1}{4}\left(E_{\partial}(u(t))\right)^{-\frac{1}{2}}\|\mathcal{T}(u(t))\|^2_{L^2(S^2)} \geq \frac{1}{4\sqrt{\kappa}}\|\mathcal{T}(u(t))\|_{L^2(S^2)},$$

and thus, for $t_0 \in [0,\infty)$

$$\int_{t_0}^{\infty}\|\mathcal{T}(u(t))\|_{L^2(S^2)}dt \leq 4\sqrt{\kappa}\left[E_{\partial}(u(t_0))\right]^{\frac{1}{2}}. \tag{2.20}$$

The first application of (2.20) is the calculation

$$\sup_{t\in[t_0,\infty)}\|u(t)-u_\infty\|_{L^2(S^2)} \leq \int_{t_0}^{\infty}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(S^2)}dt = \int_{t_0}^{\infty}\|\mathcal{T}(u(t))\|_{L^2(S^2)}dt \leq 4\sqrt{\kappa}\left(E_{\partial}(u(t_0))\right)^{\frac{1}{2}}. \tag{2.21}$$

The exponential decay of $E_\partial(u(t_0))$ in t_0 then gives us the L^2 exponential convergence of part (iv) of Theorem (2.2). In pursuit of the other parts, however, we are satisfied with the weaker statement

$$u(t) \to u_\infty \text{ in } L^2(S^2, \mathbb{R}^3) \text{ as } t \to \infty. \tag{2.22}$$

A consequence of this is that

$$u(t) \rightharpoonup u_\infty \text{ weakly in } W^{1,2}(S^2, \mathbb{R}^3) \text{ as } t \to \infty. \tag{2.23}$$

This is because otherwise we could pick a sequence of times $\{t_i\}$ to give

$$|\langle u(t_i), \alpha\rangle - \langle u_\infty, \alpha\rangle| > \delta, \tag{2.24}$$

where α is some test function, $\delta > 0$, and $\langle\cdot,\cdot\rangle$ is the inner product of $W^{1,2}(S^2, \mathbb{R}^3)$. Then, as the total energy E is bounded ($E(u(t)) \leq E(u_0)$) we could pass to a subsequence of times (also called $\{t_i\}$) such that

$$u(t_i) \rightharpoonup \beta \text{ weakly in } W^{1,2}(S^2, \mathbb{R}^3) \text{ as } i \to \infty, \tag{2.25}$$

for some β. The convergence would then be strong in $L^2(S^2, \mathbb{R}^3)$, and so by (2.22) we would have $\beta = u_\infty$. There would then be a contradiction between (2.24) and (2.25). So (2.23) holds, which is part (i) of Theorem (2.2).

Of course, as we have already used above in the case $p = 2$, the compactness of the inclusion of $W^{1,2}$ in L^p for any $p \in [1, \infty)$ tells us that weak convergence in $W^{1,2}$ implies strong convergence in L^p and thus

$$u(t) \to u_\infty \text{ in } L^p(S^2, \mathbb{R}^3) \text{ as } t \to \infty.$$

We now proceed to consider the evolution of the cut energy Θ. Recall that by Theorem (1.15) there exists a number $l^{(r,s)}$ such that

$$\Theta(t_i) \to l^{(r,s)}$$

as $i \to \infty$. Moreover, providing the point $y \in \mathcal{M}$ around which we have taken the cut energy is not a bubble point, we can reduce r and s to make $l^{(r,s)}$ as small as desired. Indeed, if there is no bubble point in $\mathbb{B}_{r+s}(y)$ then $l^{(r,s)} = \Theta^{(r,s)}_{u_\infty}$. Our objective is to control $\Theta(t)$ for $t \in (t_i, t_{i+1})$. This will enable us, among other things, to apply Lemma (2.13) to get the higher regularity of part (ii) of the theorem.

Let us return to the principal estimate (2.14) of Lemma (2.14). Taking the second inequality and abandoning the first term on the right hand side, we see that

$$\frac{d\Theta}{dt} \leq \frac{C}{s}\Theta^{\frac{1}{2}}\|\mathcal{T}\|_{L^2(S^2)} \leq \frac{C}{s}E(u_0)^{\frac{1}{2}}\|\mathcal{T}\|_{L^2(S^2)}.$$

Integrating this, we see that the cut energy can only vary within the restriction

$$\Theta(t) - \Theta(t_0) \leq \frac{C}{s}E(u_0)^{\frac{1}{2}}\int_{t_0}^{t}\|\mathcal{T}(u(\xi))\|_{L^2(S^2)}d\xi,$$

and thus, by (2.20)

$$\Theta(t) - \Theta(t_0) \leq \frac{C}{s}E(u_0)^{\frac{1}{2}}\left(E_{\partial}(u(t_0))\right)^{\frac{1}{2}}, \tag{2.26}$$

where $t > t_0$, of course.

Estimate (2.26) is exactly what we were looking for as it controls $\Theta(t)$ for $t \in [t_i, t_i + 1)$, enabling us to deduce that

$$\Theta^{(r,s)}(t) \to l^{(r,s)} \qquad \text{as} \qquad t \to \infty. \tag{2.27}$$

For any $\Omega \subset\subset S^2 \backslash \{x^1, \ldots x^m\}$ this provides us with the uniform control of concentration

$$\mathcal{E}(R, \Omega) < \varepsilon_1,$$

for some R, enabling us to apply Lemma (2.13) to get

$$\|u(t)\|_{C^k(\Omega)} \leq C(k) \text{ uniformly for } t \in [1, \infty),$$

for all k. We can deduce the convergence of a subsequence of any sequence $u(t_i)$ in $C^k(\Omega)$ for any k, and hence, as we know that $u(t) \to u_\infty$ uniformly as $t \to \infty$ in L^2 say, we can establish part (ii) of Theorem (2.2) via the obvious contradiction argument.

Having established part (ii), part (iii) of Theorem (2.2) then follows from a further application of (2.27).

To prove part (v) we can no longer get away with controling just the rate of *increase* of Θ as in (2.26). We now need to control $\Theta(t_0) - \Theta(t)$ for $t > t_0$. To do so, we take the first inequality in the principal estimate (2.14) of Lemma (2.14) which tells us that

$$-\frac{d\Theta}{dt} \leq \|\mathcal{T}\|^2_{L^2(S^2)} + \frac{C}{s}\Theta^{\frac{1}{2}}\|\mathcal{T}\|_{L^2(S^2)} \leq \|\mathcal{T}\|^2_{L^2(S^2)} + \frac{C}{s}E(u_0)^{\frac{1}{2}}\|\mathcal{T}\|_{L^2(S^2)}.$$

Integrating this from t_0 to $t > t_0$ gives us

$$\Theta(t_0) - \Theta(t) \leq \int_{t_0}^{\infty}\|\mathcal{T}(u(\xi))\|^2_{L^2(S^2)}d\xi + \frac{C}{s}E(u_0)^{\frac{1}{2}}\int_{t_0}^{\infty}\|\mathcal{T}(u(\xi))\|_{L^2(S^2)}d\xi,$$

allowing application of (2.20) and the integral of (2.18) to get

$$\Theta(t_0) - \Theta(t) \le 2E_\partial(u(t_0)) + \frac{C}{s}E(u_0)^{\frac{1}{2}}[E_\partial(u(t_0))]^{\frac{1}{2}}. \tag{2.28}$$

Let us now calculate

$$\frac{d}{dt}\left(\frac{1}{2}\int_{S^2}\varphi^2|\nabla(u(t)-u_\infty)|^2\right) = \frac{d}{dt}\Theta(t) - \frac{d}{dt}\left(\int_{S^2}\varphi^2\nabla u(t).\nabla u_\infty\right). \tag{2.29}$$

The second term on the right hand side is controlled by

$$\left|\frac{d}{dt}\left(\int_{S^2}\varphi^2\nabla u(t).\nabla u_\infty\right)\right| = \left|\int_{S^2}\varphi^2\nabla\mathcal{T}(u(t)).\nabla u_\infty\right| \le C(u_\infty,r,s)\|\mathcal{T}(u(t))\|_{L^2(S^2)},$$

where we have integrated by parts and used Hölder's inequality. Together with (2.28) and (2.20) we now have enough information to integrate (2.29). Assuming that there are no bubble points inside $\mathbf{B}_{r+s}(y)$, the support of φ, we obtain

$$\begin{aligned}\frac{1}{2}\int_{S^2}\varphi^2|\nabla(u(t)-u_\infty)|^2 &\le \Theta(t) - \Theta_{u_\infty} - \int_t^\infty C(u_\infty,r,s)\|\mathcal{T}(u(\xi))\|_{L^2(S^2)}d\xi \\ &\le C(E_\partial(u(t)) + [E_\partial(u(t))]^{\frac{1}{2}}) \\ &\le Ce^{-\gamma t},\end{aligned}$$

where the constant C is at least independent of t. We observe that this is enough to establish part (v) of the theorem.

Part (vi) of the theorem may be proved with an adaptation of the last calculation, or by combining part (v) with the exponential convergence

$$|\Theta(t) - l^{(r,s)}| \le 2E_\partial(u(t)) + \frac{C}{s}E(u_0)^{\frac{1}{2}}[E_\partial(u(t))]^{\frac{1}{2}} \le Ce^{-\gamma t}$$

which follows from (2.26) and (2.28). ∎

For future reference, we collect some of the more striking or useful estimates from the preceeding proof in the following separate theorem. Part (vi) is a slightly sharper version of part (iv) which is used in [33].

Theorem 2.16 *There exists universal constants $\varepsilon_0,\kappa,C > 0$ such that the following is true. Let $u_0 : S^2 \to S^2$ be an initial map satisfying $E_\partial(u_0) < \varepsilon_0$, and let u be the corresponding heat flow. Then for any $t \ge t_0 \ge 0$ we have*

(i) $E_\partial(u(t)) \le E_\partial(u(t_0))e^{-\frac{(t-t_0)}{2\kappa}}$,

(ii) $\int_t^\infty \|\mathcal{T}(u(\xi))\|_{L^2(S^2)}d\xi \le 4\sqrt{\kappa}\,[E_\partial(u(t))]^{\frac{1}{2}}$,

(iii) $\|u(t) - u_\infty\|_{L^2(S^2)} \le 4\sqrt{\kappa}[E_\partial(u(t))]^{\frac{1}{2}}$,

(iv) $\Theta(t) - \Theta(t_0) \le \frac{C}{s}E(u_0)^{\frac{1}{2}}[E_\partial(u(t_0))]^{\frac{1}{2}}$,

(v) $\Theta(t_0) - \Theta(t) \le 2E_\partial(u(t_0)) + \frac{C}{s}E(u_0)^{\frac{1}{2}}[E_\partial(u(t_0))]^{\frac{1}{2}}$,

(vi) $\Theta(t)^{\frac{1}{2}} - \Theta(t_0)^{\frac{1}{2}} \le \frac{C}{s}[E_\partial(u(t_0))]^{\frac{1}{2}}$.

With the hindsight of Theorem (2.2) we can see that each part of the theorem relied on establishing the finiteness of

$$\int_{t_0}^\infty \|\mathcal{T}(u(\xi))\|_{L^2(S^2)}d\xi,$$

or its exponential decay as t_0 increases. It can be noted that by integrating (1.4) we always have the finiteness of

$$\int_0^\infty \|\mathcal{T}(u(\xi))\|^2_{L^2(S^2)}d\xi,$$

but what could go wrong, and must in examples of nonuniqueness as in Section (2.1), is that $\|\mathcal{T}(u(t))\|_{L^2(S^2)}$ could decay too slowly. From (1.4) we see that this is equivalent to the energy left to dissipate decaying too slowly. The point of the key lemma (Lemma (2.8)) is to ensure that the energy is dissipated sufficiently fast. Using (1.4) again, and the fact that $E_\partial(u(t))$ is exactly half the energy left to dissipate (see the beginning of the proof of Theorem (2.2) for example) the lemma tells us that

$$(\text{Rate of dissipation of energy}) \ge C(\text{Energy left to dissipate})$$

and so the energy left to dissipate decays exponentially which we find is sufficient.

Finally, an earlier proposition still awaits a proof.

Proof. (Proposition (2.3).)

If u_0 is harmonic then there is nothing to prove, so suppose otherwise. Without loss of generality, we may assume that

$$E(u_0) \le 8\pi + 4\pi deg(u_0).$$

By combining (2.3) and (2.4) once more, and recalling that $E_\partial(u(t))$ decreases in t, we find that

$$E_\partial(u(t)) < E_\partial(u_0) = \frac{1}{2}(E(u_0) - 4\pi deg(u_0)) \le 4\pi$$

for all $t > 0$; and as the bubbles and the body map (if nonconstant) each carry energy of at least 4π, we may deduce that none of them are holomorphic, so each must be anti-holomorphic. ∎

Chapter 3

A nontrivial example of a bubble tree

In this chapter we will construct what we believe to be the first example of the formation of a nontrivial bubble tree in the harmonic map heat flow. In other words, we give a flow in which more than one bubble develops at the same point. Alternatively, we may see the flow as an example in which one bubble, as captured by Struwe in Theorem (1.14), cannot account for the loss in energy or the change in homotopy class as we pass to the limit of $u(t_i)$. The bubbles occur at infinite time and develop at different scales. It is not known whether this phenomenon can occur at finite time.

Theorem 3.1 *There exists a target manifold $\mathcal{N}$, and an initial map $u_0 : D \to \mathcal{N}$ where D is a 2-disc, such that the subsequent flow u blows up at infinite time at precisely one point, but so that upon analysing the blow-up with Theorem (1.15), we must have two bubbles developing at that point (in other words $m = 2$). In fact we have $a_i^1 = a_i^2 = 0$ for all $i \in \mathbb{N}$ and consequently the bubbles develop at different scales in that (by swapping the bubbles if necessary) we have $\frac{\lambda_i^1}{\lambda_i^2} \to \infty$ as $i \to \infty$.*

The example relies on the construction of a target with a warped metric. A target like this has already been considered in the context of the harmonic map heat flow by Jie Qing, to provide an (unpublished) example of the development of a neck (as defined in Section (1.4.2)).

The remainder of this chapter is devoted to the proof of Theorem (3.1).

Let the domain $\mathcal{M}$ be the flat 2-disc of radius π with polar coordinates (r, ϕ), and the target $\mathcal{N}$ be $S^2 \times S^2$ with a metric to be described shortly.

We parameterise each S^2 considered in this proof with spherical polar coordinates as used in Section (1.4.4). Let $h(\theta, \phi)$ be the standard metric on S^2 at the point (θ, ϕ).

In $\mathcal{N}$ we parameterise the first S^2 by (α, A) and the second by (β, B). At the point $(\alpha, A; \beta, B)$ in $\mathcal{N}$, we set the metric to be $h(\alpha, A) + f(\alpha)h(\beta, B)$, where $f(\alpha) \equiv 1$ would give the standard metric on $S^2 \times S^2$. We choose $f : [0, \pi] \to \mathbf{R}$ to be any smooth function satisfying

(i) $f(\alpha) = 1$ for $0 \leq \alpha \leq \frac{\pi}{2}$,

(ii) $f'(\alpha) > 0$ for $\frac{\pi}{2} < \alpha < \pi$,

(iii) $v : S^2 \to \mathbf{R}$, defined by $v(\theta, \phi) = f(\theta)$, is smooth

So with v as above, we have $\mathcal{N} = S^2 \times_v S^2$. We refer to f as well as v as a warping function.

Let us consider symmetric maps of the form

$$(r, \phi) \to (\alpha(r), \phi, \beta(r), \phi).$$

This symmetry is preserved under the heat flow. We will be using the fixed, constant boundary conditions

$$u(\pi, \phi, t) = (\pi, \phi, \pi, \phi),$$

and will start with the initial map

$$u_0(r, \phi) = (r, \phi, r, \phi). \tag{3.1}$$

Writing the heat flow as

$$u(r, \phi, t) = (\alpha(r, t), \phi, \beta(r, t), \phi),$$

the evolution equations for α and β are

$$\frac{\partial \alpha}{\partial t} = \frac{\partial^2 \alpha}{\partial r^2} + \frac{1}{r}\frac{\partial \alpha}{\partial r} - \frac{\sin\alpha\cos\alpha}{r^2} - \frac{f'(\alpha)}{2}\left(\frac{\partial \beta}{\partial r}\right)^2 - \frac{f'(\alpha)}{2}\frac{\sin^2\beta}{r^2}, \tag{3.2}$$

$$\frac{\partial \beta}{\partial t} = \frac{\partial^2 \beta}{\partial r^2} + \frac{1}{r}\frac{\partial \beta}{\partial r} - \frac{\sin\beta\cos\beta}{r^2} + \frac{f'(\alpha)}{f(\alpha)}\frac{\partial \alpha}{\partial r}\frac{\partial \beta}{\partial r}. \tag{3.3}$$

By the symmetry imposed, and the finiteness of any blow-up points, blow-up may only occur at $r = 0$. We will argue that both α and β blow up at infinite time, and that rescaling to capture a bubble will not account for the blow-up of both α and β - they must blow up at different rates. The first step is to argue that both α and β blow up at some stage. This is evidently true as any harmonic map from $\mathcal{M}$ with constant boundary values, is constant by Theorem (1.8), and the initial map (3.1) is homotopically nontrivial, because the projections onto either S^2 of the target $\mathcal{N}$ are nontrivial.

The next step is to argue that α and β do not blow up at finite time. For this, we will employ the following result from [3] which Chang and Ding used to establish the global existence result Theorem (1.22).

Lemma 3.2 *Suppose that $d > 0$ and we have a smooth function $h_0 : [0, d] \to [0, \pi]$ with $h_0(0) = 0$ and $h_0(d) = \pi$. Then we may find a (unique) smooth solution $h : [0, d] \times [0, \infty) \to [0, \pi]$ of*

$$\begin{cases} \dfrac{\partial h}{\partial t} = \dfrac{\partial^2 h}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial h}{\partial r} - \dfrac{\sin h \cos h}{r^2}, \\ h(\cdot, 0) = h_0, \\ h(0, t) = 0, \;\; h(d, t) = \pi. \end{cases} \tag{3.4}$$

We will use Lemma (3.2) to generate supersolutions to the solutions of (3.2) and (3.3). Finite-time singularities in the heat flow will then be ruled out by what we learnt about blow-up in Section (1.4.2). To begin with, we set $d = \pi$, and choose h_0 to be any function as in Lemma (3.2) with $h_0(r) > r = \alpha(r, 0)$ when $r \in (0, \pi)$. Applying Lemma (3.2) to get a function h, and comparing this function to the solution α of (3.2) using the maximum principle (as in [3] and [25]) we see that

$$h(r, t) > \alpha(r, t) \qquad \text{for all } (r, t) \in (0, \pi) \times [0, \infty).$$

(Note that we are using the fact that the final two terms in (3.2) are negative.) Consequently, we see that α does not blow up in finite time.

Next we turn to β, the solution of (3.3). Suppose that β blows up in finite time - at time $t = T$ say. As α exists for all time, we know that $\frac{\partial \alpha}{\partial r}$ is bounded for $t \in [0, T]$. In particular, as $\alpha(0, t) = 0$ for all t, there exists $d \in (0, \pi)$ such that if $(r, t) \in [0, d] \times [0, T]$ then $\alpha(r, t) < \frac{\pi}{2}$, and hence $f'(\alpha(r, t)) = 0$ from (i) on Page 48. Consequently, β evolves for $(r, t) \in [0, d] \times [0, T]$ under the same equation as h in Lemma (3.2). Set $h_0 : [0, d] \to [0, \pi]$ to be $h_0(r) = \frac{\pi}{d} r$, so that $h_0(r) > r = \beta(r, 0)$, and apply Lemma (3.2) to get the corresponding function h. Comparing h with β using the maximum principle as before, we see that

$$h(r, t) > \beta(r, t) \qquad \text{for all } (r, t) \in (0, d) \times [0, T],$$

contradicting the fact that β blows up at time $t = T$. Hence β does not blow up in finite time.

So we have established that α and β blow up at infinite time, and not at finite time. It remains to prove that there does not exist one bubble which can account for the blow-up of both α and β.

We will use the following consequence of Theorem (1.8) of Lemaire several times.

Lemma 3.3 *For any $0 < d < \pi$, there does not exist a smooth solution $g : [0, d] \to [0, \pi]$ to*

$$\begin{cases} 0 = \dfrac{d^2 g}{d\theta^2} + \dfrac{1}{\tan\theta} \dfrac{dg}{d\theta} - \dfrac{\sin g \cos g}{\sin^2 \theta}, \\ g(0) = 0, \; g(d) = \pi. \end{cases} \tag{3.5}$$

Proof. The reason for this is simply because if g was a solution to (3.5) then we would have a nonconstant harmonic map with constant boundary values

$$(\theta, \phi) \to (g(\theta), \phi)$$

from the part of S^2 with $0 \leq \theta \leq d$ to S^2, which would contradict Theorem (1.8). ∎

Suppose one bubble accounts for the blow-up of both α and β. Then the bubble would be a harmonic map from S^2 to $\mathcal{N}$ which we could write as

$$(\theta, \phi) \to (\alpha(\theta), \phi, \beta(\theta), \phi), \tag{3.6}$$

where α and β are smooth functions with

$$\alpha(0) = \beta(0) = 0, \qquad \alpha(\pi) = \beta(\pi) = \pi \qquad 0 \leq \alpha, \beta \leq \pi, \tag{3.7}$$

satisfying the system

$$0 = \frac{d^2\alpha}{d\theta^2} + \frac{1}{\tan\theta}\frac{d\alpha}{d\theta} - \frac{\sin\alpha\cos\alpha}{\sin^2\theta} - \frac{f'(\alpha)}{2}\left(\frac{d\beta}{d\theta}\right)^2 - \frac{f'(\alpha)}{2}\frac{\sin^2\beta}{\sin^2\theta}, \tag{3.8}$$

$$0 = \frac{d^2\beta}{d\theta^2} + \frac{1}{\tan\theta}\frac{d\beta}{d\theta} - \frac{\sin\beta\cos\beta}{\sin^2\theta} + \frac{f'(\alpha)}{f(\alpha)}\frac{d\alpha}{d\theta}\frac{d\beta}{d\theta}. \tag{3.9}$$

However, such a bubble cannot exist:

Lemma 3.4 *There do not exist any harmonic maps $S^2 \to \mathcal{N}$ of the form* (3.6) *satisfying* (3.7), (3.8) *and* (3.9).

Proof. Suppose such a map exists. We will find that (3.8) contains a contradiction. Let us multiply (3.8) by $2\sin^2\theta\frac{d\alpha}{d\theta}$, and integrate over the region $(0,\theta)$. We find that

$$\begin{aligned} \sin^2\theta\left(\frac{d\alpha}{d\theta}\right)^2 &= \sin^2\alpha + \int_0^\theta f'(\alpha(\zeta))\sin^2\zeta\left(\frac{d\beta}{d\theta}(\zeta)\right)^2\frac{d\alpha}{d\theta}(\zeta)d\zeta \\ &\quad + \int_0^\theta f'(\alpha(\zeta))\sin^2\beta(\zeta)\frac{d\alpha}{d\theta}(\zeta)d\zeta. \end{aligned} \tag{3.10}$$

Define

$$\theta_0 = \inf\{\theta \in (0,\pi) \mid \alpha(\theta) = \frac{\pi}{2}\} \in (0,\pi).$$

So for $\theta \in (0,\theta_0)$ we have $f'(\theta) = 0$ by (i) on Page 48.

Of course $\alpha(\theta_0) = \frac{\pi}{2}$, but we also know that $\beta(\theta_0) < \pi$. This is because otherwise we would have $\beta(\theta_0) = \pi$ and by setting $d = \theta_0$ and $g = \beta|_{[0,\theta_0]}$ and applying Lemma (3.3), we would have a contradiction.

Next, by setting $\theta = \theta_0$ in (3.10) we see that $\frac{d\alpha}{d\theta}(\theta_0) \neq 0$, and hence that

$$\frac{d\alpha}{d\theta}(\theta_0) > 0.$$

Suppose that $\frac{d\alpha}{d\theta}(\theta) > 0$ for all $\theta \in (\theta_0,\pi)$. Then by setting $\theta = \pi$ in (3.10) we have a contradiction - for $\theta \in (\theta_0,\pi)$ we would have $f'(\theta) > 0$, $\frac{d\alpha}{d\theta}(\theta) > 0$ and $\sin^2\beta$ not identically equal to zero, so the final term of (3.10) and thus the whole right-hand-side, would be strictly positive.

So we must have $\theta \in (\theta_0, \pi)$ with $\frac{d\alpha}{d\theta}(\theta) = 0$. Set

$$\theta_1 = \inf\{\theta \in (\theta_0, \pi) \mid \frac{d\alpha}{d\theta}(\theta) = 0\} \in (\theta_0, \pi).$$

Putting $\theta = \theta_1$ into (3.10), the left hand side is zero, and all the terms on the right are nonnegative, so they must all be zero. In particular, we must have $\alpha(\theta_1) = \pi$, and $\left(\frac{d\beta}{d\theta}\right)^2 \equiv \sin^2 \beta \equiv 0$ on $\theta \in (\theta_0, \theta_1)$. But then setting $d = \theta_1$, and $g = \alpha|_{[0,\theta_1]}$ and applying Lemma (3.3), we have a contradiction. ∎

So one bubble cannot account for the blow-up of both α and β, and we have finished the argument of the chapter.

Remark 3.5 It would be interesting to extend the fact that there is no one suitable bubble which can account for the change in homotopy - as stated in Lemma (3.4) - to prove that there are no harmonic spheres homotopic to the diagonal embedding

$$(\theta, \phi) \rightarrow (\theta, \phi, \theta, \phi).$$

Remark 3.6 The above construction would also work if the domain was the upper hemisphere of S^2 rather than the 2-disc. By connecting two such examples together, we could construct an example in which the domain was S^2 itself.

Chapter 4

Stability of the flow

In this chapter we will be discussing the stability of the heat flow under perturbations of the initial map. In other words, given a solution to the heat equation (1.3), if we alter the initial map u_0 slightly and find the subsequent heat flow, is it in any sense close to the original one? Of course, the answers to this question are extremely dependent on the norms with which we measure, and in particular on whether they are strong enough to detect bubbles (as is the $W^{1,2}$ norm for instance). A special case of this question is when the original flow is stationary - in other words when the initial map of the original flow is harmonic - and we will be able to give many of our examples of instability in this case.

As in almost all of this thesis, we will only be considering the case that the domain is a surface. Moreover, as usual there is an emphasis on our favourite case of flow between 2-spheres, or more generally flow from a 2-sphere.

4.1 Examples of instability

We begin with a simple example to show that initial maps which are close in C^∞ may lead to divergent flows. Take the domain and target to be T^2 and S^2 respectively. Consider a harmonic map which wraps the torus around an equator of S^2. Taking coordinates $(\theta,\phi) \in S^1 \times S^1$ on the domain and spherical polar coordinates (α,φ) as used in Section

(1.4.4) on the target, we could write such a map as

$$(\theta,\phi) \to (\frac{\pi}{2},\phi)$$

if we use appropriate metrics. Taking this map as an initial map, the subsequent flow is, of course, stationary in time. However, if we take the perturbed initial map

$$(\theta,\phi) \to (\frac{\pi}{2}-\varepsilon,\phi)$$

for some small ε, then it maps entirely into a hemisphere of the target, and by Corollary (1.17) the subsequent flow must exist for all time and converge to a constant map. The same principle can be used to give an example where the domain is S^2. In this case, we map the domain isometrically into an equator of S^3 to give an initial map whose flow is stationary. As before, a perturbation takes the image of the map strictly into a hemisphere (of S^3 now) and the corresponding flow will exist for all time and converge to a constant.

Although the examples above are simpler, we have already observed instability in Section (2.1). As above, we could pick a stationary original flow. That situation had the further interesting property that the harmonic maps

$$x \to (z_0,x) \text{ for } |z_0| = 1$$

from S^2 to $\mathbf{R}^2 \times S^2$ which were unstable, were energy minimising. They therefore contrast with the work of Leon Simon - see Theorem (4.5) below - in which such instability is ruled out under the hypothesis that the target is real analytic.

The discussion above provides examples only of instability over infinite time intervals, and leaves open the possibility of finite time perturbation results. We remark that if we have a flow which blows up within a finite time interval $[0,T)$ then the flow is likely, in general, to be unstable in some sense. To justify this, consider a heat flow u between 2-spheres with initial map u_0 which blows up in finite time. (Recall that examples are given in Section (1.4.4).) Now let $\varphi_\varepsilon : S^2 \to S^2$ be a rotation of the 2-sphere defined by $\varphi_\varepsilon(\theta,\phi) = (\theta,\phi+\varepsilon)$ in spherical polar coordinates, for which $\varphi_\varepsilon(x)$ is not a blow-up point, for some blow-up point $x \in S^2$. Then the flow from the perturbed initial map $u_0 \circ \varphi_\varepsilon$ will be precisely $u(\varphi_\varepsilon(\cdot),\cdot)$. Only the original flow will blow up at the point x, and therefore it is clear that the two flows will diverge in $W^{1,2}$ as we approach the time of blow-up. In Section (4.3) we will give a further simple example of a flow with finite time blow-up for which a perturbation removes all singularities. However in this example we must raise the dimension of the target.

4.2 Stability results

In contrast to the discussion in the previous section, we may prove a stability result over finite time intervals providing the original flow does not blow up during the time interval considered. As usual, the domain is required to be a surface, and moreover we will only consider domains without boundary.

Lemma 4.1 *Suppose we have two solutions of the heat equation u and v with initial maps u_0 and v_0. Suppose moreover that $T > 0$ and that the flow u has no bubbles up to time $t = T$ (in other words u is smooth for $t \in (0, T]$).*

Then for all $\varepsilon > 0$, there exists $\delta > 0$ independent of v such that if

$$\|u_0 - v_0\|_{W^{1,2}(\mathcal{M})} < \delta,$$

then

$$\|u(t) - v(t)\|_{W^{1,2}(\mathcal{M})} < \varepsilon \textit{ for all } t \in [0, T].$$

Proof. (Lemma (4.1).)

We sketch the proof, which essentially follows from the work of Struwe.

For $\varepsilon_1 > 0$ we may choose R sufficiently small so that

$$\sup_{(x,t)\in\mathcal{M}\times[0,T]} E_{(x,2R)}(u(t)) < \frac{\varepsilon_1}{4}.$$

This is possible as u is regular up to time T. Then for $\|u_0 - v_0\|_{W^{1,2}(\mathcal{M})}$ sufficiently small, we may ensure that

$$\sup_{x\in\mathcal{M}} E_{(x,2R)}(v_0) < \frac{\varepsilon_1}{2}.$$

Then by part (i) of Lemma (2.14), for $\eta > 0$ sufficiently small we have that

$$\sup_{(x,t)\in\mathcal{M}\times[0,\eta]} E_{(x,R)}(v(t)) < \varepsilon_1.$$

Here η is dependent on u only in terms of R, and essentially independent of v (totally independent if we assume $\|u_0 - v_0\|_{W^{1,2}} < 1$ say). The lemma then follows for $T \leq \eta$ by

[31, Remark 3.9]. By dividing up the interval $[0, T]$ into intervals of length no more than η and applying the lemma for $T \leq \eta$ iteratively, we establish the lemma for general, finite, T. ∎

Although, as we have already discussed, we cannot in general extend this result to cover infinite time intervals (even if there is no blow-up in the original flow) we will be able to exploit some of the theory developed in Chapter (2) to prove the following result.

Theorem 4.2 *Suppose we have two solutions of the heat equation u and v between round 2-spheres, with initial maps u_0 and v_0. Suppose moreover that for the flow u there are no bubbles at finite time and that the bubbles and the body map at infinite time all share a common orientation (so all are holomorphic or all anti-holomorphic).*

Then with $\{x^k\}$ the blow-up points of the flow u as in Theorem (1.11), we have that for all $\varepsilon > 0$, $\Omega \subset\subset S^2 \backslash \{x^1, \ldots x^m\}$ and $r > 0$ sufficiently small, there exists $\delta > 0$ independent of v such that if

$$\|u_0 - v_0\|_{W^{1,2}(S^2)} < \delta,$$

then

(i) $\|u(t) - v(t)\|_{L^2(S^2)} < \varepsilon$ *for all* $t > 0$,

(ii) $\|u(t) - v(t)\|_{W^{1,2}(\Omega)} < \varepsilon$ *for all* $t > 0$,

In other words, if we start a new flow close to the original one, then it will stay close in L^2 for all time, and close in $W^{1,2}$ away from the blow-up points. We remark that the perturbed flow may blow up in finite time if the original flow blew up at infinite time. Examples from Section (4.1) justify our hypotheses on the domain and the target. We evidently have the following corollary.

Corollary 4.3 *The set $\mathcal{I}$ of initial maps u_0 for which the subsequent flow u is nonsingular for all time and converges without bubbles at some sequence of times $t_i \to \infty$ as in Theorem (1.15), is open in the $W^{1,2}(S^2, S^2)$ topology.*

Moreover, denoting by $\mathcal{R}$ the subset of $W^{1,2}(S^2 \times [0,\infty), S^2)$ with finite norm

$$\|u\| = \sup_{t \in [0,\infty)} \|u(t)\|_{W^{1,2}(S^2)},$$

the map $\mathcal{F} : \mathcal{I} \to \mathcal{R}$ which associates to each u_0 its flow u, is continuous if we use the $W^{1,2}(S^2, S^2)$ topology on the domain.

The second part of the corollary fails without hypotheses on the domain and the target, as is shown by examples in Section (4.1). In fact, an example to be given in Section (4.3) will show that even the first part of the corollary fails if we drop the condition on the target.

Proof. (Theorem (4.2).)

As in the proof of Theorem (2.2) we will assume without loss of generality that the body map and all the bubbles are anti-holomorphic, rather than holomorphic.

The basic idea of the proof is to use Lemma (4.1) to show that the flows stay close until the ∂-energy is small, and then use the techniques we developed in the proof of Theorem (2.2). We set ε_0 as in Lemma (2.8) in anticipation.

Let u_∞ be the body map of the flow u and v_∞ a body map for v.

Part (i) of Theorem (4.2) will follow from part (iii) of Theorem (2.16) (cf. (2.21)). For any $\eta_1, \eta_2 > 0$, we may choose T sufficiently large so that

$$\|u(t) - u_\infty\|_{L^2} < \eta_1 \qquad \text{for} \qquad t \geq T,$$

from part (i) of Theorem (2.2) and

$$E_\partial(u(t)) < \min(\eta_2, \frac{\varepsilon_0}{2}) \qquad \text{for} \qquad t \geq T,$$

as $E_\partial(u(t)) \to 0$ as $t \to \infty$ (cf.(2.19)) and

$$\|u(t) - u_\infty\|_{W^{1,2}(\Omega)} < \eta_3 \qquad \text{for} \qquad t \geq T,$$

from part (ii) of Theorem (2.2). Then for any $\eta_4 > 0$, we may apply Lemma (4.1) to find $\delta > 0$ such that providing $\|u_0 - v_0\|_{W^{1,2}(S^2)} < \delta$, we have

$$\|u(t) - v(t)\|_{W^{1,2}(S^2)} < \min(\eta_4, \frac{\varepsilon_0}{2}, \varepsilon) \qquad \text{for all} \qquad t \in [0,T]. \tag{4.1}$$

Then we must have

$$E_\partial(v(t)) \leq E_\partial(v(T)) < \min(\eta_2 + \eta_4, \varepsilon_0) \qquad \text{for} \qquad t \geq T,$$

and so from part (iii) of Theorem (2.16) (cf. (2.21)) we may estimate

$$\|v(t) - v_\infty\|_{L^2} < C(\eta_2 + \eta_4)^{\frac{1}{2}} \qquad \text{for} \qquad t \geq T.$$

Combining the above, we find that

$$\begin{aligned}
\|u(t) - v(t)\|_{L^2} &\leq \|u(t) - u_\infty\|_{L^2} + \|u_\infty - u(T)\|_{L^2} + \|u(T) - v(T)\|_{L^2} \\
&\quad + \|v(T) - v_\infty\|_{L^2} + \|v_\infty - v(t)\|_{L^2} \\
&< 2\eta_1 + \eta_4 + 2C(\eta_2 + \eta_4)^{\frac{1}{2}},
\end{aligned}$$

for $t \geq T$, and thus by taking η_1, η_2, η_4 sufficiently small and using (4.1) again, we establish part (i) of Theorem (4.2).

To establish part (ii) we must control locally the oscillation of the first order part of the $W^{1,2}$ norm, and we do so in a manner similar to that of the proof of part (v) of Theorem (2.2).

With φ and Θ as in Chapter (2) we calculate

$$\frac{d}{dt}\left(\frac{1}{2}\int_{S^2} \varphi^2 |\nabla(v(t) - u_\infty)|^2\right) = \frac{d}{dt}\Theta_{v(t)} - \frac{d}{dt}\left(\int_{S^2} \varphi^2 \nabla v(t).\nabla u_\infty\right). \tag{4.2}$$

The second term on the right hand side is controlled by

$$\left|\frac{d}{dt}\left(\int_{S^2} \varphi^2 \nabla v(t).\nabla u_\infty\right)\right| = \left|\int_{S^2} \varphi^2 \nabla \mathcal{T}(v(t)).\nabla u_\infty\right| \leq C(u_\infty, r, s)\|\mathcal{T}(v(t))\|_{L^2(S^2)},$$

where we have integrated by parts and used Hölder's inequality.

Together with parts (ii) and (iv) of Theorem (2.16) we now have enough information to integrate (4.2) giving, for $t \geq T$,

$$\begin{aligned}
&\frac{1}{2}\int_{S^2} \varphi^2 |\nabla(v(t) - u_\infty)|^2 - \frac{1}{2}\int_{S^2} \varphi^2 |\nabla(v(T) - u_\infty)|^2 \\
&\leq \Theta_{v(t)} - \Theta_{v(T)} + \int_T^t C(u_\infty, r, s)\|\mathcal{T}(v(\xi))\|_{L^2(S^2)} d\xi \\
&\leq \frac{C}{s} E(v(T))^{\frac{1}{2}}[E_\partial(v(T))]^{\frac{1}{2}} + C(u_\infty, r, s)\,(E_\partial(v(T)))^{\frac{1}{2}} \\
&\leq C\,(E_\partial(v(T)))^{\frac{1}{2}} \leq C(\eta_2 + \eta_4),
\end{aligned}$$

where the constant C on the final line is independent of the flow v assuming we insist that $\|u_0 - v_0\|_{W^{1,2}(S^2)} < 1$ (for example) so that we have a bound on the energy of $v(T)$. Taking η_2 and η_4 sufficiently small, we may make the right hand side as small as we desire. Consequently, for any $\eta_5 > 0$ we can ensure that

$$\|v(t) - u_\infty\|_{W^{1,2}(\Omega)} - \|v(T) - u_\infty\|_{W^{1,2}(\Omega)} \leq \eta_5.$$

Combining everything again, we see that

$$\|v(t) - u(t)\|_{W^{1,2}(\Omega)} \leq \|v(t) - u_\infty\|_{W^{1,2}(\Omega)} + \|u_\infty - u(t)\|_{W^{1,2}(\Omega)} \leq \eta_5 + (\eta_3 + \eta_4) + \eta_3,$$

for $t \geq T$, so by taking η_3, η_5 sufficiently small and making η_4 smaller if necessary (and using (4.1) again) we establish part (ii) of Theorem (4.2). ∎

We complete this section by giving some of the perturbation results which were either known prior to this thesis, or could be derived without difficulty. All of these will concern the stability of harmonic maps - in other words they will assume that the original flow to be perturbed is stationary in time. However, when we can prove the stability of a harmonic map to which a flow subconverges, we can establish the stability of the flow - all we must do is use a finite time perturbation result until the perturbed flow has entered the domain of attraction of the the limit of the original flow. Finite time perturbation results are always available - even in the case that the domain is not a surface (though maybe not of the strength of Lemma (4.1) in that the initial maps may have to be close in a stronger norm than that used to compare flows) by linearising the heat equation about the original flow and applying the Implicit Function Theorem for Banach spaces.

In the case that $\mathcal{N}$ has nonpositive sectional curvature, whatever the dimension of the domain, we can apply the work of Hartman [19] to find that if $h : \mathcal{M} \to \mathcal{N}$ is harmonic and $u_0 : \mathcal{M} \to \mathcal{N}$ is sufficiently close to h in C^0, then the subsequent flow u (as found in Theorem (1.5)) is such that

$$\sup_x d_{\mathcal{N}}(h(x), u(x,t))$$

is decreasing in time. In particular, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever u_0 satisfies $\|u_0 - h\|_{C^0} < \delta$, the subsequent flow satisfies $\|u(t) - h\|_{C^0} < \varepsilon$ for all $t \geq 0$.

A related, though easier situation is when the original flow converges to a constant map. Simple arguments along the lines of those in Section (1.4.3) (see Corollary (1.18)) show that a flow starting at a C^0 perturbation of a constant map converges to a nearby constant map.

Although we know that energy minimising harmonic maps are not necessarily stable from the example in Section (2.1), Naito [23] has proved the following theorem, for domains of any dimension.

Theorem 4.4 *Suppose $h : \mathcal{M} \to \mathcal{N}$ is harmonic with strongly positive Jacobi operator. Then there exists $\varepsilon > 0$ such that if $\|u_0 - h\|_{W^{m,2}} < \varepsilon$ for some $m > \frac{1}{2}\dim \mathcal{M} + 2$ then there exists a global solution u to (1.3) in $C^{\frac{1}{2}}([0,\infty); W^{m-1,2}(\mathcal{M},\mathcal{N})) \bigcap C^0([0,\infty); W^{m,2}(\mathcal{M},\mathcal{N}))$ such that $u(t) \to h$ in $W^{m,2}(\mathcal{M},\mathcal{N})$ as $t \to \infty$.*

The work of Leon Simon [29] gives us the following result, which we state in the case that the domain is a surface, though it holds for general domains if we strengthen the norms used.

Theorem 4.5 *Consider the case that $\mathcal{N}$ is real analytic. Suppose $h \in C^\infty(\mathcal{M},\mathcal{N})$ is locally energy minimising in the sense that*

$$E(h) \leq E(\tilde{h}) \text{ whenever } \|h - \tilde{h}\|_{C^5} < \varepsilon,$$

where $\varepsilon > 0$ is given. Then there are constants $\delta = \delta(\varepsilon, \mathcal{M}, \mathcal{N}, h)$ and $\alpha = \alpha(\mathcal{N}, h) \in (0,1)$ such that if $\|u_0 - h\|_{C^7} < \delta$ then there is a smooth solution u of (1.3) on $\mathcal{M} \times [0,\infty)$ with $\lim_{t\to\infty} u(t) = u_\infty$ for some harmonic map $u_\infty : \mathcal{M} \to \mathcal{N}$ such that $E(u_\infty) = E(h)$ and $\|u_\infty - h\|_{C^5} < \min\{\delta^\alpha, \frac{\varepsilon}{2}\}$.

4.3 A further example of instability

The principal example of this section will be a harmonic map from S^2 which has initial maps arbitrarily close in C^∞ whose flows blow up (at infinite time). The example could be altered to make the harmonic map energy minimising.

Our starting point is the target used in the example of a nontrivial bubble tree which we constructed in the previous chapter. Let us define

$$\mathcal{P}_f = S^2 \times_v S^2, \tag{4.3}$$

where $v : S^2 \to (0,\infty)$ is of the form $v(\theta,\phi) = f(\theta)$. Unlike in the previous chapter let us consider flow from S^2 to $\mathcal{P}_f$ rather than from D to $\mathcal{P}_f$. The crucial property of $\mathcal{P}_f$ which we will be exploiting is that for a constant warping function f, the diagonal embedding is harmonic; and yet when f is given an arbitrarily small slope, the harmonicity is destroyed and the map must separate out into two bubbles under the heat flow. In particular, this flow will blow up. Although this appears to be no good as we have to perturb a harmonic map to give blow-up rather than take a harmonic map and perturb the metric, all that is required is to add a system to the target (increasing the dimension) on whose state the function f depends. We can then perturb this additional system to effectuate a perturbation of the metric. At this point we switch to the details.

We begin by giving the equations of heat flow from S^2 to $\mathcal{P}_f$. Let us parameterise $\mathcal{P}_f$ by (α, A, β, B) - two sets of spherical polar coordinates as in Chapter (3) - and the domain S^2 by (θ,ϕ). For initial maps

$$(\theta,\phi) \to (\alpha_0(\theta), \phi, \beta_0(\theta), \phi),$$

with $\alpha_0, \beta_0 : [0,\pi] \to [0,\pi]$ satisfying

$$\alpha_0(0) = \beta_0(0) = 0, \qquad \alpha_0(\pi) = \beta_0(\pi) = \pi, \qquad \alpha_0(\theta), \beta_0(\theta) \in (0,\pi) \text{ for } \theta \in (0,\pi),$$

the subsequent heat flow takes the form

$$(\theta,\phi,t) \to (\alpha(\theta,t), \phi, \beta(\theta,t), \phi),$$

where α and β satisfy $\alpha(\theta,t), \beta(\theta,t) \in (0,\pi)$ for $\theta \in (0,\pi)$ and evolve according to

$$\frac{\partial \alpha}{\partial t} = \frac{\partial^2 \alpha}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial \alpha}{\partial \theta} - \frac{\sin\alpha\cos\alpha}{\sin^2\theta} - \frac{1}{2}f'(\alpha)\left(\left(\frac{\partial \beta}{\partial \theta}\right)^2 + \frac{\sin^2\beta}{\sin^2\theta}\right), \tag{4.4}$$

$$\frac{\partial \beta}{\partial t} = \frac{\partial^2 \beta}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial \beta}{\partial \theta} - \frac{\sin\beta\cos\beta}{\sin^2\theta} + \frac{f'(\alpha)}{f(\alpha)}\frac{\partial \alpha}{\partial \theta}\frac{\partial \beta}{\partial \theta}. \tag{4.5}$$

We first observe that if $\alpha_0(\theta) = \beta_0(\theta) = \theta$ and f is constant, then the initial map is harmonic. However, if f is perturbed to be as in (i) - (iii) on Page 48 (in particular nonconstant and increasing) then we may apply Lemma (3.4) to find that the flow must blow up - the lemma tells us that there is no appropriate body map to converge to in the case that there is no blow-up. This is all that we require in this section, though we remark that in the case that f is increasing, there seems to be a tendency for α to blow up at $\theta = \pi$ and for β to blow up at $\theta = 0$ - ie. the flow seems to separate the initial map into two bubbles. If we insist that f is constant on closed neighbourhoods of $\theta = 0$ and $\theta = \pi$

(but is strictly increasing in between), we can argue rigorously as in Chapter (3) that α cannot blow up at $\theta = 0$, β cannot blow up at $\theta = 0$ at finite time, α blows up at $\theta = \pi$ (possibly in finite time) and, β cannot blow up at $\theta = \pi$ before α has blown up.

Before giving the example, we admit that in many senses it is not the simplest possible. However, we justify our choice as it serves as a gentle introduction to the ideas required for our sketched example of 'disappearing bubbles' to be given in the appendix.

Theorem 4.6 *Setting $\mathcal{M} = S^2$ with the standard metric and $\mathcal{N} = S^2 \times S^2 \times S^2 \times S^1$ for some appropriate warped metric, there exists a harmonic map $\Phi : \mathcal{M} \to \mathcal{N}$ and an initial map u_0 arbitrarily close to Φ in C^∞ for which the subsequent flow blows up.*

Proof. The reader is advised to be already familiar with the example of the flow into a warped product that we saw in Section (2.1) as well as the behaviour of flows from S^2 to $\mathcal{P}_f$ as described above.

Let us parameterise the three copies of S^2 in the target with the spherical polar coordinates (α, A), (β, B) and (γ, C), and the S^1 by $\rho \in (-\pi, \pi]$. With $h(\theta, \phi)$ and $k(\rho)$ the standard metrics on S^2 and S^1 respectively, we equip $\mathcal{N}$ with the metric

$$h(\alpha, A) + p(\alpha, \rho)h(\beta, B) + q(\rho)h(\gamma, C) + k(\rho), \tag{4.6}$$

where p and q are positive functions to be determined. It will shortly be possible to interpret this metric in terms of our previous examples of warped products. Imagine an initial map of the form

$$(\theta, \phi) \to (\theta, \phi, \theta, \phi, \theta, \phi, \rho_0), \tag{4.7}$$

for some constant ρ_0. The heat flow then takes the form

$$(\theta, \phi, t) \to (\alpha(\theta, t), \phi, \beta(\theta, t), \phi, \theta, \phi, \rho(\theta, t)) \tag{4.8}$$

where α, β and ρ solve

$$\begin{aligned}
\frac{\partial \alpha}{\partial t} &= \frac{\partial^2 \alpha}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial \alpha}{\partial \theta} - \frac{\sin\alpha\cos\alpha}{\sin^2\theta} - \frac{1}{2}\frac{\partial p}{\partial \alpha}(\alpha, \rho)\left(\left(\frac{\partial \beta}{\partial \theta}\right)^2 + \frac{\sin^2\beta}{\sin^2\theta}\right), &(4.9)\\
\frac{\partial \beta}{\partial t} &= \frac{\partial^2 \beta}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial \beta}{\partial \theta} - \frac{\sin\beta\cos\beta}{\sin^2\theta} + \frac{1}{p(\alpha,\rho)}\frac{\partial p}{\partial \alpha}(\alpha, \rho)\frac{\partial \alpha}{\partial \theta}\frac{\partial \beta}{\partial \theta}, &(4.10)\\
\frac{\partial \rho}{\partial t} &= \frac{\partial^2 \rho}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial \rho}{\partial \theta} - q'(\rho) - \frac{1}{2}\frac{\partial p}{\partial \rho}(\alpha, \rho)\left(\left(\frac{\partial \beta}{\partial \theta}\right)^2 + \frac{\sin^2\beta}{\sin^2\theta}\right). &(4.11)
\end{aligned}$$

We observe that for an initial map as given, if we drop the ρ-dependence of p, then the flow decouples to give a flow into the first two copies of S^2 (compare this with (4.4) - (4.5)) and a flow into the $S^2 \times S^1$ which remains (compare this with Section (2.1) - ρ then evolves under gradient flow on q).

Let us choose the warping function q to be

$$q(\rho) = 2 + \cos \rho.$$

The basic properties of q which this ensures are that $q(\rho)$ has a strict maximum when $\rho = 0$, a strict minimum when $\rho = \pi$, and that $q'(\rho) < 0$ for $\rho \in (0, \pi)$. We choose p to be constant in α when $\rho = 0$, and increasing in α for $\rho \neq 0$. More precisely, we choose p to be smooth, and satisfy

$$\begin{gathered}
\frac{\partial p}{\partial \alpha} = 0, \text{ for } \rho = 0, \ \alpha \in [0, \pi], \\
p(\alpha, \rho) = f(\alpha), \text{ for some } f \text{ as in (i)-(iii) on Page 48, when } \rho \in [\frac{\pi}{2}, \pi] \text{ and } \alpha \in [0, \pi], \\
\frac{\partial p}{\partial \rho} \leq 0, \ \frac{\partial p}{\partial \alpha} \geq 0, \text{ for all } \rho, \alpha \in [0, \pi], \\
p(\cdot, \rho) = p(\cdot, -\rho) > 0.
\end{gathered} \tag{4.12}$$

Then we may set

$$\Phi(\theta, \phi) = (\theta, \phi, \theta, \phi, \theta, \phi, 0)$$

to be the harmonic map required for the theorem. Harmonicity can be confirmed geometrically or analytically. We then take our initial map to be as in (4.7) with ρ_0 small, and positive, say. As we shall see, the final two terms (ie. the q and the p terms) of (4.11) then push ρ towards π. If it were not for the final term, this would be simply gradient descent. In fact, owing to this final term, ρ does not remain independent of θ, however by the maximum principle we have

$$\xi(t) \leq \rho(\theta, t) \leq \pi$$

for any $t > 0$ and $\theta \in [0, \pi]$ where $\xi : [0, \infty) \to [0, \pi]$ is the solution to

$$\frac{\partial \xi}{\partial t} = -q'(\xi), \qquad \xi(0) = \rho_0.$$

In particular, there exists a $T > 0$ such that $\rho(\cdot, t) \in (\frac{\pi}{2}, \pi]$ for $t > T$. By (4.12) we see that from time T onwards, α and β decouple from ρ and evolve exactly under the equations (4.4) and (4.5) hence leading to blow-up. ∎

As promised in the previous section the example above shows that in contrast with Corollary (4.3) the set of initial maps with flows without any finite or infinite time blow-up, is not open in general. That the set of initial maps with flows which blow up in finite time is not open in general is evident by taking a known example of finite time blow-up between 2-spheres and considering it as a flow from S^2 into an equator of S^3. By perturbing the initial map so that the image lies entirely in a hemisphere of S^3, we may argue with Corollary (1.17) (as in Section (4.1)) that the subsequent flow converges smoothly to a constant. Similarly, the set of initial maps with smooth flows which blow up in infinite time is not open in general.

Returning again to the case of heat flow between 2-spheres, we remark that we do not know if the set of initial maps with flows which blow up in finite time is open or not. A further example in the next chapter (see Section (5.3)) will show that the set of initial maps with flows which blow up at infinite time but not at finite time, is not open.

It would be interesting to establish results proving that, under hypotheses on the domain and target (for instance that both are 2-spheres) that any map may be perturbed to give an initial map whose subsequent flow is regular and converges smoothly at infinite time. Of course, for some combinations of domain and target, we may pick an initial map in a homotopy class devoid of harmonic maps, and the subsequent flow will always blow up, however we perturb it within its homotopy class.

Chapter 5

The energy and concentration required for singularities

5.1 The minimum energy required for blow-up

In this chapter we discuss the extent to which low energy corresponds to globally regular solutions in the harmonic map heat flow. As usual, we shall always assume that the domain $\mathcal{M}$ is a surface.

Our starting point is the simple consequence of the theory of Struwe described in Section (1.4.2) that if we do not have enough energy to produce a bubble, then we cannot have any singularities. This was remarked by Struwe [31, Remark 4.4] and is central to our discussion:

Theorem 5.1 *Suppose that we have an initial map $u_0 : \mathcal{M} \to \mathcal{N}$ which satisfies*

$$E(u_0) < \inf\{E(v) \mid v : S^2 \to \mathcal{N} \text{ is nonconstant and harmonic }\} > 0, \tag{5.1}$$

then the heat flow does not have any finite or infinite time singularities.

Remark 5.2 In the case of heat flow between 2-spheres (using the standard target sphere metric) the condition (5.1) is that

$$E(u_0) < 4\pi,$$

by Remark (2.6).

The statement that the infimum in (5.1) is strictly positive whatever the target, follows from a more general lemma.

Lemma 5.3 *Suppose $\mathcal{M}$ is (a surface) without boundary. Then there exists $\varepsilon = \varepsilon(\mathcal{M}, \mathcal{N})$ such that any harmonic map $v : \mathcal{M} \to \mathcal{N}$ with $E(v) < \varepsilon$ is necessarily constant.*

Proof. We consider $\mathcal{N}$ to be embedded in Euclidean space, and without loss of generality we may assume that

$$\int_{\mathcal{M}} v \, d\mathcal{M} = 0.$$

By (1.1) we have, for fixed $p \in (1,2)$

$$\|\Delta v\|_{L^p} \leq C(\mathcal{N}) \|\nabla v\|_{L^{2p}}.$$

Applying standard regularity theory to the left hand side and Hölder's inequality to the right, we find that

$$\|v\|_{W^{2,p}} \leq C \left(\|\nabla v\|_{L^2} \|\nabla v\|_{L^{\frac{2p}{2-p}}} + \|v\|_{L^p} \right).$$

The estimation continues with the inequalities of Sobolev, Hölder and Poincaré to give

$$\|v\|_{W^{2,p}} \leq C E(v)^{\frac{1}{2}} \left(\|\nabla v\|_{L^p} + \|v\|_{W^{2,p}} \right) + C \|\nabla v\|_{L^2},$$

at which point, for ε sufficiently small, we may absorb the second derivatives (and apply Hölder again) to find

$$\|v\|_{W^{2,p}} \leq C E(v)^{\frac{1}{2}} \|\nabla v\|_{L^2} + C E(v)^{\frac{1}{2}}.$$

As $W^{2,p}$ embeds continuously in L^∞ (note that $p > 1$) we may conclude the estimations with

$$\|v\|_{L^\infty} \leq C \left(E(v) + E(v)^{\frac{1}{2}} \right).$$

By reducing ε if necessary, we may force the image of v into a small ball, and applying Theorem (1.21) with the hindsight of Section (1.4.3) we find that v must be constant. ■

We wish to know whether Theorem (5.1) is optimal, or whether more energy than the energy of one harmonic sphere is required for blow-up. We restrict our discussion, as usual, mainly to the case of heat flow between 2-spheres. We remark that the questions we address are of a very different nature in the case when the domain has boundary. This will be apparent from the work of Chang, Ding and Ye (see Section (1.4.4)) following this chapter. We begin with a simple result we proved originally in [32] which shows that given extra topological restrictions, more energy than 4π is required to produce a bubble.

Theorem 5.4 *Suppose $u_0 \in C(S^2, S^2) \bigcap W^{1,2}(S^2, S^2)$ is of degree zero, and satisfies $E(u_0) \leq 8\pi$. Then the subsequent flow u is globally regular and converges in C^∞ to a constant map as $t \to \infty$.*

Proof. We will use the decomposition of the energy into ∂-energy and $\bar{\partial}$-energy that we described in Section (2.2.1). For convenience we recall that for $u : S^2 \to S^2$ we have

$$E(u) = E_{\partial}(u) + E_{\bar{\partial}}(u), \tag{5.2}$$

and

$$E_{\partial}(u) - E_{\bar{\partial}}(u) = 4\pi deg(u), \tag{5.3}$$

whilst $E_{\bar{\partial}}(u) = 0$ if and only if u is holomorphic and $E_{\partial}(u) = 0$ if and only if u is anti-holomorphic. We also recall that both the ∂-energy and the $\bar{\partial}$-energy decrease during the heat flow (at the same rate) and that both are positive quantities.

Suppose that blow-up occurs. Then the bubble map must carry energy of at least 4π, from Remark (2.6). Moreover, the bubble map, as it is a harmonic map from S^2 to a surface, must be either holomorphic or anti-holomorphic, so it must carry ∂ or $\bar{\partial}$-energy of at least 4π by (5.2). Therefore we must have either that $E_{\partial}(u_0) > 4\pi$ or that $E_{\bar{\partial}}(u_0) > 4\pi$. But by using the fact that $deg(u_0) = 0$ in (5.3), we then have that both $E_{\partial}(u_0) > 4\pi$ and $E_{\bar{\partial}}(u_0) > 4\pi$ which tells us that $E(u_0) > 8\pi$, a contradiction.

So no blow-up occurs either at finite or infinite time, and hence by the work in Section (2.2) we know that the flow converges smoothly to a harmonic map, which must be of degree zero, and therefore constant by Remark (2.6). We remark that we need not rely on the results in Section (2.2) as we could first prove C^0 subconvergence to a constant map and then use more elementary arguments - see Remarks (1.19) and (1.20). ∎

This result is optimal in the sense that if the energy is allowed above 8π then blow-up may occur. To see this, consider an example of infinite time blow-up between 2-spheres mentioned in Section (1.4.4). Such an example blows up at the north and south pole at infinite time, and converges to a constant elsewhere. By analysing with Theorem (1.15) we find that

$$\lim_{t\to\infty} E(u(t)) = 8\pi,$$

and hence we may take $u(t)$ as an initial map with energy arbitrarily close to 8π (for sufficiently large t) which leads to blow-up.

We are left wondering whether the result remains true if we drop the condition on the degree. As any map of degree at least two has energy larger than 8π (see (5.2) and (5.3)) we need only look at the case in which the degree is one. In fact, we discover that in general, Theorem (5.1) is optimal:

Theorem 5.5 *For any $\varepsilon > 0$ there exists an initial map $u_0 : S^2 \to S^2$ with*

$$E(u_0) < 4\pi + \varepsilon$$

such that the subsequent heat flow blows up in finite time.

We will give examples with the same form of rotational symmetry as was used in Section (1.4.4) to demonstrate the theorem. As in previous examples we adopt spherical polar coordinates on the domain and target, and write a symmetric map as

$$U_\alpha(\theta, \phi) = (\alpha(\theta), \phi),$$

where $\alpha : [0, \pi] \to \mathbb{R}$. This symmetry is preserved under the heat flow.

If we allow U_α to evolve according to the heat equation, then α evolves according to

$$\frac{\partial \alpha}{\partial t} = \frac{\partial^2 \alpha}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial \alpha}{\partial \theta} - \frac{\sin\alpha \cos\alpha}{\sin^2\theta} \tag{5.4}$$

When the heat flow is from a sphere, with degree one, we use the boundary conditions $\alpha(0,t) = 0$ and $\alpha(\pi, t) = \pi$, and (5.4) holds for $0 < \theta < \pi$. However, we will have cause to consider the heat flow from the upper hemisphere S^2_+, in which case we still use the equation (5.4) but for $0 < \theta < \frac{\pi}{2}$ and with boundary conditions $\alpha(0,t) = 0$ and $\alpha(\frac{\pi}{2}, t) = c$ for some c. In the latter case, we still use the notation U_α to refer to the map from S^2_+.

In order to prove Theorem (5.5) we will utilise the examples of finite time blow-up given by Chang, Ding and Ye [4], which were described in Theorem (1.23). The proof of that theorem, as mentioned in Section (1.4.4), carries over to the case that the domain is S^2_+:

Theorem 5.6 *Suppose that $u_0 \in C^1(S^2_+, S^2)$ is of the form $u_0 = U_{\alpha_0}$ with $\alpha_0(0) = 0$ and $\alpha_0(\frac{\pi}{2}) > \pi$. Then the subsequent flow $u = U_\alpha$ blows up in finite time. In particular, $u \notin C(S^2_+ \times [0,\infty), S^2)$. Assuming that $\alpha_0(\theta) > 0$ for all $\theta \in (0, \frac{\pi}{2}]$, at the time of the singularity T, the boundary condition $\alpha(0,t) = 0$ which held for $t < T$, switches to $\alpha(0,T) = \pi$.*

Of course, the boundary conditions of u never change.

We are now in a position to prove Theorem (5.5). The initial map resembles the identity map which has been conformally concentrated at a point so that a bubble has nearly formed there. Of course we cannot take exactly this map as it is harmonic. We compare the subsequent flow to one of the examples of Chang, Ding and Ye with the maximum principle. The difficulty in the proof is to apply the maximum principle despite the boundary conditions of Chang, Ding and Ye's examples. As we shall see later, this is more than just a technicality.

Proof. (Theorem (5.5).)

We consider the heat flow in terms only of α. When we pick a map α_0, say, in the proof, we assume that it is smooth and moreover that U_{α_0} is smooth.

It is important to consider what happens to create the blow-up of Theorem (5.6). As the blow-up time is approached, the gradient of α gets very large at $\theta = 0$ and a small interval $[0, \varepsilon]$ is mapped onto all, or almost all, of $[0, \pi]$ by α.

We begin by picking a map $\beta_0 : [0, \frac{\pi}{2}] \to [0, \pi + \delta]$ with $\beta_0(0) = 0$ and $\beta_0(\frac{\pi}{2}) = \pi + \delta$ for $\delta > 0$ small, such that $E(U_{\beta_0}) < 4\pi + \frac{\varepsilon}{2}$.

Next we pick a map $\gamma_0 : [0, \pi] \to [0, \pi + 2\delta]$ say, with $\gamma_0(\frac{\pi}{2}) > \pi + \delta$ and boundary conditions $\gamma_0(0) = \gamma_0(\pi) = 0$.

Now imagine the heat flow U_γ with initial map U_{γ_0}. From Theorem (1.4) the heat flow evolves smoothly for some small time interval, so γ evolves smoothly initially and we may find $T > 0$ such that for $t \in [0,T]$ we have no blow-up and we have the condition $\gamma(\frac{\pi}{2}, t) > \pi + \delta$ still.

Now take the heat flow U_β with initial map U_{β_0}. By Theorem (5.6) this must eventually blow up - say at time $t = S$. We are interested in the function $\sigma_0 = \beta(\cdot, S - \frac{T}{2})$ because we know that U_{σ_0} is an initial map with energy less than $4\pi + \frac{\varepsilon}{2}$ for which the heat flow U_σ blows up in time $t = \frac{T}{2}$. Note that by making T smaller if necessary we may assume that $S > \frac{T}{2}$.

We are now in a position to choose our initial map to satisfy the theorem. We take a map $\alpha_0 : [0,\pi] \to [0, \pi + 3\delta]$ say, with boundary conditions $\alpha_0(0) = 0$ and $\alpha_0(\pi) = \pi$ which is above both σ_0 and γ_0. In other words, $\alpha_0(\theta) > \sigma_0(\theta)$ for $\theta \in (0, \frac{\pi}{2}]$ where σ_0 is defined, and $\alpha_0(\theta) > \gamma_0(\theta)$ for $\theta \in (0,\pi]$. Assuming we chose γ_0 appropriately (and δ was sufficiently small) we may choose this so that U_{α_0} has energy less than $4\pi + \varepsilon$.

Now consider the heat flow U_α with initial map U_{α_0} and suppose that it does not blow up in finite time. By the maximum principle (as used in [3] and [4]) applied to α and σ, and the fact that $\alpha_0 = \alpha(\theta,0) > \sigma(\theta,0) = \sigma_0$ for $\theta \in (0, \frac{\pi}{2}]$, we must have that

$$\alpha(\theta,t) > \sigma(\theta,t) \tag{5.5}$$

for $\theta \in (0,\frac{\pi}{2}]$ and for $t \in [0,\eta)$ where $\eta \in (0, \frac{T}{2}]$ is chosen so that $\alpha(\frac{\pi}{2},t) > \sigma(\frac{\pi}{2},t) = \pi + \delta$. However by the maximum principle applied to α and γ and the fact that $\alpha_0 = \alpha(\theta,0) > \gamma(\theta,0) = \gamma_0$ for $\theta \in (0,\pi]$, we must have that $\alpha(\theta,t) > \gamma(\theta,t)$ for $t \in [0,T]$ and $\theta \in (0,\pi]$ and in particular that

$$\alpha(\frac{\pi}{2},t) > \gamma(\frac{\pi}{2},t) > \pi + \delta$$

for $t \in [0,T]$. Therefore we must be able to take $\eta = \frac{T}{2}$. But then as σ blows up in time $\frac{T}{2}$, by (5.5) α must blow up within time $\frac{T}{2}$ which is a contradiction.

So the idea was to take a map which would be pushed into blowing up by the example of Chang, Ding and Ye, by the maximum principle providing the map did not foul Chang, Ding and Ye's boundary values - so we protected the boundary values with another map (γ) for enough time for the blow-up to occur. ∎

To summarise the situation, if the initial energy is less than 4π then blow-up cannot occur and if it is between 4π and 8π then it can occur precisely when the degree of the initial map is one.

We remark that in the example constructed to prove Theorem (5.5) the initial map comes equipped with a concentration that is well on the way to forming a bubble. This still does not answer the question of whether the heat flow can 'move through the conformal group'. For example, suppose we start with a map between 2-spheres which is close in $W^{1,2}$ to the identity, and thus has energy slightly larger than 4π - then the possibility that the heat flow could concentrate the map at one point is certainly not ruled out on energy grounds. The flow at some larger time could look like a concentrated Möbius transformation.

In fact this is not possible, as we will demonstrate shortly. We will use techniques from Chapter (2) to control the flow of energy into a given region of the domain in terms of the amount of ∂-energy (or $\bar{\partial}$-energy) there is left to dissipate.

Of course, from the properties (5.2) and (5.3) of the $\bar{\partial}$-energy and ∂-energy, for a degree one map the condition $E(u_0) < 4\pi + \varepsilon_0$ implies that either $E_\partial(u_0) < \frac{\varepsilon_0}{2}$ or $E_{\bar{\partial}}(u_0) < \frac{\varepsilon_0}{2}$.

Before stating the result ruling out the possibility of moving through the conformal group, we must define

$$\mathcal{F}(u_0, R) = \sup_{x \in S^2} \int_{\mathbb{B}_R(x)} e(u_0),$$

where $\mathbb{B}_R(x)$ is a geodesic disc of radius R centred at x as usual.

Theorem 5.7 *For all $R > 0$ and $k \in (0,1)$, there exists $\varepsilon > 0$ such that whenever we have an initial map $u_0 : S^2 \to S^2$ satisfying $\mathcal{F}(u_0, R) < 4\pi k$ and $E(u_0) < 4\pi + \varepsilon$, then the subsequent heat flow u does not blow up.*

We remark that it is elementary to prove that restricting the energy to be only a little over 4π implies that the heat flow cannot move very quickly, but it is far from clear that the flow cannot blow up given as much time as it wants.

Proof. All the work has already been done in establishing Theorem (2.16). Suppose that a bubble does develop - at a point $y \in S^2$. Recalling the cut energy from Chapter (2), we take $r = s = \frac{R}{2}$ so that φ is supported in $\mathbb{B}_R(y)$ and is equal to one in a neighbourhood of y. Then as $\mathcal{F}(u_0, R) < 4\pi k$, we must have that $\Theta(0) = \Theta_{u_0} < 4\pi k$. For a bubble to develop, at time $T \in (0, \infty]$ say, we must have $\limsup_{t\uparrow T} \Theta(t) \geq 4\pi$, the minimum energy for a bubble.

So we simply must take $\varepsilon < \varepsilon_0$, and ε sufficiently small to control $E_\partial(u)$ and hence, by part (vi) of Theorem (2.16), to control the amount by which Θ may fluctuate, giving a contradiction. In detail, taking C to be the constant of Theorem (2.16), we can take

$$\varepsilon = \frac{8\pi s^2(1-\sqrt{k})^2}{C^2}$$

and then calculate

$$2\sqrt{\pi}(1-\sqrt{k}) < \limsup_{t\uparrow T} \Theta(u(t))^{\frac{1}{2}} - \Theta(u_0)^{\frac{1}{2}} \leq \frac{C}{s}(E_\partial(u_0))^{\frac{1}{2}} \leq \frac{C}{s}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{2}} \leq 2\sqrt{\pi}(1-\sqrt{k}),$$

which clearly is not possible. ∎

5.2 The effect on the flow of domain metric scaling

Here we give an application of the theory given above. Given a harmonic map from a surface, we may change the metric on the domain conformally and still have a harmonic map. Therefore, if we intend to use the harmonic map heat flow to give information about harmonic maps, we may scale the domain metric before using the flow. Scaling the domain metric does not give exactly the same flow - as mentioned in Section (1.1) the velocity with which the map evolves at each point is scaled by a factor inversely proportional to the factor used to scale the domain metric. So we have a hope of choosing a conformally equivalent domain metric which gives a more useful flow. Of course, this technique would turn out to be pointless if any two such flows were conjugate in some sense, or had the same 'topology'. However, we are given hope with the following result.

Theorem 5.8 *There exists an initial map $u_0 : S^2 \to S^2$ such that the corresponding heat flow with the standard metric on the domain gives blow-up in finite time, but the corresponding heat flow with a certain other fixed, conformally equivalent metric on the domain exists for all time and converges smoothly at infinite time.*

Proof. (Sketch.)

We describe the proof rather than let things become too technical. The initial map u_0 will be a map such as in the examples constructed in Theorem (5.5). The construction in that theorem is sufficiently flexible to allow us to assume that there is a homothety $\phi : S^2 \to S^2$ such that $u_0 \circ \phi^{-1}$ is sufficiently close in $W^{1,2}$ to the identity map to be an initial map whose heat flow does not blow up, by Theorem (5.7). We then change the metric on the sphere in order to expand the region of concentration, and shrink the rest. The effect of this is to get rid of the concentration. In fact, we take the pullback under ϕ of the standard metric g. This ensures that the domain is still a round 2-sphere. Moreover, as $\phi : (S^2, \phi^* g) \to (S^2, g)$ is an isometry, the heat flow with initial map $u_0 : (S^2, \phi^* g) \to (S^2, g)$ will not blow up, as it is basically the same flow as the flow with initial map $u_0 \circ \phi^{-1} : (S^2, g) \to (S^2, g)$. Precisely, if u is the flow starting from u_0 (with domain metric $\phi^* g$) then the flow starting from $u_0 \circ \phi^{-1}$ (with domain metric g) will be $u \circ \phi^{-1}$. Smooth convergence follows from Theorem (2.2). ∎

So we deformed the metric to slow down the flow where it previously wanted to blow up.

5.3 A flow with a nontrivial holomorphic limit

As promised in [33] we now give an example of a flow which satisfies the hypotheses of Theorem (2.2) and in which bubbles develop. In other words, it is a flow between 2-spheres in which bubbling occurs at infinite time, with all the bubbles and the body map sharing the same orientation. In fact, the body map is constant, and there is one bubble. Of course Theorem (2.2) applies to any flow between 2-spheres without bubbles at infinite time.

In fact, no example is known of infinite time blow-up between 2-spheres in which the body map is nonconstant. In the case that all bubbles share the same orientation as the body map, examples with nonconstant body map (and a nonzero number of bubbles) cannot have the rotational symmetry on which we normally rely for analysis. In the case that bubbles have opposing orientation to a nonconstant body map, there seems to be a tendency for the bubble to develop at finite time.

We turn to the construction. The idea is to take a homotopy of initial maps starting with one whose subsequent flow does not blow up, and ending with one with a flow that does. The first of these flows which blows up, in this construction, will do so at infinite time and have the desired properties.

Let us choose a homotopy $u_0^s = U_{\alpha_0^s}$ (with $s \in [0,1]$) of maps between the identity map u_0^0 and an example of an initial map leading to finite time blow-up u_0^1 from Theorem (5.5). We have $\alpha_0^0(\theta) = \theta$, and we require also that $\alpha_0^s(\theta)$ is increasing in s for fixed θ, and that for each $s \in (0,1]$ there is precisely one value $\theta \in (0,\pi)$ solving $\alpha_0^s(\theta) = \pi$.

Now let us consider the heat flow $u^s = U_{\alpha^s}$ starting at each of the maps $u_0^s = U_{\alpha_0^s}$. Then by the maximum principle, we find that $\alpha^s(\theta,t)$ is increasing in s for fixed θ and t, and that for each $s \in (0,1]$ there is precisely one value $\theta \in (0,\pi)$ solving $\alpha^s(\theta) = \pi$, prior to any blow-up of α^s.

Let us define

$$\xi = \inf\{s \in [0,1] \mid u^s \text{ blows up either at infinite or finite time }\}.$$

Then we claim that u^ξ is smooth for all finite time, but blows up at infinite time. First of all, by Corollary (4.3) we see that u^ξ must blow up. Otherwise, we could rapidly contradict the definition of ξ. So it remains to rule out finite time blow-up.

Let us suppose u^ξ blows up at a finite time T. Then by the properties of the functions α^s mentioned above, and the maximum principle, we may argue that $\alpha^\xi(\theta,T) > \pi$ for all $\theta \in (0,\pi)$. All that we must do to establish a contradiction is to find an initial map U_{β_0} with $\beta_0(0) = 0$, $\beta_0(\pi) = \pi$ and $\beta_0(\theta) \in (0, \alpha^\xi(\theta,T))$ for all $\theta \in (0,\pi)$ whose subsequent flow leads to blow-up. This would then allow us to find an s slightly smaller than ξ such that $\alpha^s(\theta,T) > \beta_0(\theta)$ for all $\theta \in (0,\pi)$ and hence, by the maximum principle as usual, α^s would blow up for some $t > T$ contradicting the definition of ξ. The construction of β_0 may be made in the same way as we constructed the examples of initial maps with energy slightly larger than 4π which led to blow-up, in Theorem (5.5). All we must do is choose δ sufficiently small so that the comparison maps (denoted by γ and β in Theorem (5.5)) and then the constructed map (denoted by α in the theorem) all lie below $\alpha^\xi(\cdot,T)$.

We remark that u_0^ξ may be perturbed to $u_0^{\xi-\varepsilon}$ for some small ε, to give an initial map with a regular flow (no finite or infinite time blow-up). If we had defined

$$\xi = \inf\{s \in [0,1] \mid u^s \text{ blows up at finite time }\},$$

we could again argue that u^ξ blows up at infinite time, and then a perturbation of u_0^ξ to $u_0^{\xi+\varepsilon}$ for some small ε, would give an initial map with a flow which blows up in finite time.

Appendix - An example of 'disappearing bubbles'

In this appendix we indicate that the harmonic map heat flow can, in general, have extremely complicated convergence properties at infinite time. In contrast to the uniformity result Theorem (2.2) given in Section (2.2) we sketch an example of the 'nonuniqueness of bubbles.' Unlike in the rest of this thesis, we do not give sufficient details to claim total rigour.

Recall the description of bubbling at infinite time as given in Theorem (1.15). In our example, a bubble which exists at one sequence of times $t_i \to \infty$ does not exist at another sequence $s_i \to \infty$ - so the value of m is dependent on the sequence of times. The energy of the bubble at the original sequence of times is absorbed into a higher energy body map at the new sequence of times.

The basic idea is as in Section (4.3) - to consider flow from S^2 to $\mathcal{P}_f$ (see (4.3)) and make the slope of the warping function f depend on the state of a different heat flow. Unlike in Section (4.3) we now require the slope of the warping function to oscillate very slowly between being positive and being negative, rather than just switching once from being zero to being positive. The functions α and β - see Section (4.3) - then oscillate between blowing up at $\theta = 0$ and $\theta = \pi$ (α blows up at 0 when β blows up at π and vice versa). We then choose the new sequence of times $\{s_i\}$ to catch α in the transition between these two concentrated states when approximately equal to the identity, say.

We will let the slope of the warping function f depend on the state of the spiralling system described in Section (2.1). As that system spirals round, for one half of the spiral the warping function f will have positive gradient, and for the other half, negative gradient.

A problem is that the coupling between the spiralling system and the $\mathcal{P}_f$ system makes the spiralling system lose its simple form $(x,t) \to (z(t),x)$, as the z gains an x dependence (see Section (2.1)). Once we have given some details, we hope that this will appear irrelevant.

As described above, our target is $\mathcal{N} = S^2 \times S^2 \times S^2 \times \mathbb{R}^2$. We parameterise this by $(\alpha, A, \beta, B, \gamma, C, \rho)$ as in Section (4.3) though with $\rho = (\rho_1, \rho_2)$ taking values in $\mathbb{R}^2$ now. We use a metric (4.6) where k is now the standard metric on $\mathbb{R}^2$ rather than S^1. Taking an initial map of the form (4.7), the flow is of the form (4.8), where again we now have $\rho_0 \in \mathbb{R}^2$ and $\rho = (\rho_1, \rho_2)$ mapping into $\mathbb{R}^2$. The evolution of α and β remains as in (4.9) and (4.10), though ρ now satisfies

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial \rho}{\partial \theta} - \nabla q(\rho) - \frac{1}{2}\nabla_\rho p(\alpha,\rho)\left(\left(\frac{\partial \beta}{\partial \theta}\right)^2 + \frac{\sin^2\beta}{\sin^2\theta}\right), \tag{5.6}$$

where $\nabla_\rho p = (\frac{\partial p}{\partial \rho_1}, \frac{\partial p}{\partial \rho_2})$.

Let us choose q to be the function (2.2) - at least for the moment. Of course, if p was independent of ρ, the final term in (5.6) would vanish, and with initial conditions $\rho(\theta,0) = \rho_0$, we would have a solution $\rho(\theta,t) = \rho(t)$ evolving under gradient flow on q. In fact, we will give p a ρ-dependence so that $\frac{\partial p}{\partial \alpha} < 0$ when $\rho_1 < 0$ and $\frac{\partial p}{\partial \alpha} > 0$ when $\rho_1 > 0$.

Let us consider a sequence of linear functions $f_n : [0,\pi] \to \mathbb{R}$ such that

$$f_n(0) = 1 + \frac{1}{n}, \qquad f_n(\pi) = 1 + \frac{1}{n+1}, \qquad \text{for } n \text{ odd},$$
$$f_n(0) = 1 + \frac{1}{n+1}, \qquad f_n(\pi) = 1 + \frac{1}{n}, \qquad \text{for } n \text{ even},$$

so that $f_n(x)$ is decreasing in n for fixed x, and the slope of f_n alternates between positive and negative.

Let us provisionally define p for ρ on the trajectory of gradient descent for q. Observe that if ρ follows this gradient descent then ρ_1 changes sign twice each time ρ winds around the origin. After ρ_1 has changed sign exactly n times, we ask that $p(\cdot,\rho) = f_n$ (for ρ on this arc of the gradient descent trajectory). We will shortly address the discontinuities of p at the points where $\rho_1 = 0$, and the fact that p is only defined for values of ρ on the gradient descent trajectory.

As ρ_1 changes sign, the flow alternates between concentrating α at 0 and β at π, and concentrating α at π and β at 0. Once n has grown large, and the energy of the flow has decreased towards 12π (at least 4π of energy is taken by each sphere in the target) the flow restricted to either of the first two spheres in the target will begin to resemble a homothety of the identity as in (1.8) at each time. By modifying q to change the spiralling speed of gradient descent, and hence the time given for α and β to concentrate in each direction, we can ensure that the flow becomes arbitrarily concentrated in alternating senses (and thus that the value of λ, where the flow restricted to the first sphere in the target resembles (1.8), oscillates almost between zero and infinity). Our new sequence of times is to be chosen between the extremes of oscillation so that α resembles the identity ($\lambda = 1$).

Unfortunately the discontinuity in p is not permitted, and we still have to extend the definition of p to $\mathbf{R}^2$. Firstly we smooth out the discontinuity of p. We do this so that the final term in (5.6) points in the same direction as the second to last term - the effect of this is to make this final term push ρ around the old trajectory. Then we extend p to $\mathbf{R}^2$ smoothly and so that p increases as we move away from the old gradient descent trajectory. Unfortunately the smoothing of p causes ρ to 'smear' - ie. gain a θ dependence. However, this effect is small, and occurs only during brief periods of transition whereas the first two terms on the right hand side of (5.6) - ie. the laplacian terms - continually counteract this, pulling the image of $\rho(\cdot, t)$ back together. Though the action of the laplacian will push ρ slightly off the original trajectory of the gradient descent, this does not seem important, assuming we extended p to $\mathbf{R}^2$ appropriately.

We end with the remark that it seems possible to adapt the example above to enable the selection of a sequence of times $s_i \to \infty$ at which $u(s_i) \to u_\infty$ smoothly for some u_∞ - ie. no bubbles occur whatsoever, despite the occurence of bubbles at another sequence of times.

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