A Metalogical Analysis of Vagueness: An Exploratory Study into the Geometry of Logic

FELIX HOVSEPIAN

October 1992

A Thesis Submitted to The University of Warwick for the Degree of Doctor of Philosophy in the Faculty of Science

Department of Computer Science

"Several authors have recently suggested that more exotic logics, especially 'fuzzy logic', are necessary in order to capture the essentially imprecise nature of human deduction. While agreeing that we have to look beyond first–order logic, I find the usual arguments advanced for the use of fuzzy logic most unconvincing"

Patrick J. Hayes

Dedicated to my mother and the memory of my father.

Acknowledgements

To my mother and my family: your patience and understanding throughout the entire process is unsurpassed.

Research into the foundations of any subject is a solitary undertaking, one that requires much determination and is made possible only with the kindness of others who deserve a special mention.

I would like to to express my heartfelt thanks to Dr David Randell, who provided many long and valuable discussions on philosophical and foundational issues of logic and AI. Without David's friendship this thesis would not have come to fruition. Special thanks also go to Rolf Howarth and Dr John Vaudin, who have always showed kindness and provided support during some very difficult times. I thank Dr Nick Francis, Ian Gent, Dr Guy Saward and Jeff Smith for their friendship and continual encouragement. And Dr Steve Russ, for his guidance on Bolzano's work and other issues concerning the history of mathematics.

My thanks are extended to my supervisors. Dr Tony Cohn for providing me with the opportunity to do this thesis. In addition I would like to thank him for his guidance and his unending tolerance. I would also like to thank Dr Iain Alexander–Craig for taking over from Tony and providing interesting discussions on perception and theories of truth.

During the final writing–up stage of this thesis I was employed by Parallax Solutions Ltd, whose support is gratefully acknowledged.

I would also like to thank Professors Ian N. Stewart (Mathematics Institute, University of Warwick), and Patrick J. Hayes (Stanford University) for their support and continued interest in this project.

Declaration

The material presented for this thesis is (unless explicitly stated) my own work.

Abstract

As early as 1958 John McCarthy stressed the importance of formulating common sense knowledge, and common sense reasoning, in a rigourous manner. Today, this is considered to be the central problem in Artificial Intelligence (AI). A strong advocate of this view is Patrick Hayes, who in 1974 argued that fuzzy logic was not a useful mechanism for representing vague terms, and suggested a better formalism could be developed using Zeeman's Tolerance Geometry.

Five years later, Hayes complained about AI's emphasis on toy world's and suggested that a suitable project would be to formalise our common sense knowledge of the (everyday) physical world. A project now known as Naïve Physics (NP). In this project, Hayes discussed his attempts at describing the intuitive notion of objects touching using topological techniques, and indicated that Tolerance Geometry would be a better framework for capturing this notion.

This thesis investigates Hayes' suggestion of developing Tolerance Geometry into a formal framework in which one can capture such intuitive terms as bodies touching, and characterising such vague terms as being tall.

The analysis in this thesis begins with a (formal) investigation of the Sorites paradox. This puzzle is singled out because it clearly illustrates the problems raised by any formal analysis of vagueness in any language.

The analyses of vagueness indicate that vague predicates possess *continuous interpretations*, and thence demonstrate the need for a spatial structure to be incorporated into the formalised metalanguage. This metalanguage then provides the framework for the proof that the Sorites is insoluble in a logic with a truth–set given by {0,1}, but consistent in a logic with truth–set given by {0,u,1}. Furthermore, this investigation reveals that Zadeh has confused the notions of *continuity* and the *continuum*, and therefore his theory of fuzzy sets rest on a mistaken assumption.

Contents

3 Mathematical Structures

4 Development of Tolerance Relations

5 The Sorites

6 Analyses of Vagueness : A Review

7 The Continuum

8 Spatial Structures in Metalogic

9 Further work and Conclusions

References

Appendices

1 Introduction

"All of this intricacy came from taking the idea of 'touching' seriously, and illustrates again the way in which trying to capture one concept with some breadth of application can force major changes to large parts of the growing theory."

Patrick J. Hayes

1·1 ORIGINS OF THE THESIS

This thesis develops Patrick Hayes' suggestions of using Tolerance Geometry: firstly, as a formal mechanism to represent *common sense* notions (such as two objects touching one another), and secondly, to formalise common sense notions that are considered to be *vague*. At first sight this thesis topic might appear to be somewhat esoteric, but in fact has its roots with the very beginnings of Artificial Intelligence.

In 1950 Alan Turing (Turing[1950]) first put forward the idea that a computer could in principle be programmed to exhibit intelligent behaviour. However, it wasn't until the Dartmouth Conference held in 1956 – six years later – that the subject ***Artificial Intelligence (AI)*** was truly conceived under that title. The Dartmouth Conference was organised by John McCarthy (with Marvin Minsky), and was the first time that the four most influential men in AI (together with Herbert Simon and Alan Newell) were to meet.

Recently (Hoffman[1990]), McCarthy re–affirmed his early vision of AI, stating that the formulation of common sense knowledge is of crucial importance for the development of intelligent robots. McCarthy's approach has also been championed by Patrick Hayes who in a series of articles Hayes[1970,1974a,1979a,1985] argued for the need to produce formal representation languages which could be used to represent knowledge associated with the everyday world.

In his famous 'Naïve Physics Manifesto' (Hayes[1979b]), Hayes complained about AI's preoccupation with toy worlds, and suggested that a suitable project would be to create a formalisation of our common sense knowledge of the (everyday) physical world. This led to the approach now known as ***Naïve Physics*** (***NP***).

While common sense knowledge, and common sense concepts may appear to be easily understood and open to analysis, this apparent simplicity will soon be swept aside to reveal the underlying complexity. Hayes aptly illustrates this in his early attempts to formalise the simple notion of two objects touching one another.

Touching is a clear intuitive concept. We use touching to describe situations in which there are two objects with no discernible space between them. Despite this apparent simplicity, Hayes claimed that this was one concept which he found "especially vexing" (Hayes[1985,p.21]). In order to capture this concept, he posited the notion of an infinitesimally thin layer of space between the objects. But this seemed

> *"... unintuitive and in any case does not address the basic issue, which is that our intuitive local space is, indeed, probably not a topological space". Hayes[1985,p.21].*

Hayes has consequently argued that Zeeman's work on Tolerance Geometry (Zeeman[1962]) is better placed to provide a formalisation for this concept, than the traditional approach based on topology. Following Hayes, I not only show how a tolerance geometry can be used to formally represent two objects touching, but also to represent a large class of vague terms.

Zadeh[1965,1975a,1978,1983] has continually argued that humans successfully deal with concepts that are essentially vague. He has criticised classical logic for being inadequate as a representational mechanism for these concepts. Zadeh's solution is to abandon classical set theory and replace it with *fuzzy sets*, which Zadeh[1965,1983] developed specifically to deal with such imprecise *common sense* concepts[1]. Moreover, he proposes that we should also abandon classical logic in favour of his new *fuzzy logic*.

While the use of fuzzy sets as a formal extension of conventional set theory (such as that presented by Goguen[1969]) is an interesting exercise in formal systems, my objections begin with Zadeh's claim that we should use fuzzy sets in preference to classical sets to cope with imprecise terms. Zadeh simply fails to present an adequate analysis of the subject matter he is attempting to capture within this formalism. For example, it is all very well stating that 'most Frenchman are not very tall' cannot be represented in classical logic because the sets constituting the denotations of the predicate 'tall' are fuzzy rather than crisp (Zadeh[1983,p.61]), however, Zadeh provides little analysis of why this is true. He simply asserts that it is better to re–express the whole situation in a different language.

Zadeh is right in asserting that humans cope well with imprecise concepts. However, following Hayes, I too believe that Zadeh is mistaken in his proposed solution. Firstly, one *can* use conventional set theory to represent such concepts, and secondly fuzzy sets are *not* the correct solution to this problem. I therefore claim, that Davis'[1990,p.25] view that Zadeh's theory of fuzzy sets is the only one that adequately deals with the problem posed by vagueness is at best a mistaken one, and at worst, simply wrong. In fact, as I show in chapter 8, we do not need an uncountable number of truth values (as assumed by the fuzzy set approach), but only three.

Criticizing someone else's work for having failed in certain respects is all very well, so long as one can present an alternative solution which addresses the criticisms. Vagueness

[1]"The fuzzy logic underlying approximate reasoning can provide what traditional logical systems cannot – an appropriate computational framework for dealing with common sense knowledge", Zadeh[1983].

is not an easy topic to grapple with – as has been shown by two thousand years of philosophical debate – so one must therefore exercise great care in one's analysis and formalisation of such concepts.

In this thesis I present a full analysis of an old puzzle (the Sorites), together with a formalisation of the puzzle that demonstrates *why* the puzzle is a paradox. I also *prove* that such a puzzle is inconsistent if expressed in a language whose semantics only allows a truth–set with two values represented by {0,1}. The Sorites is one of the oldest and best known puzzles, which clearly demonstrates the problems associated with attempting to formalise vague concepts[1].

1·2 METHODOLOGY AND SCOPE

Given the formidable task of representing common sense knowledge (see for example, Hobbs&Moore[1985]), some restrictions have to be acknowledged in a thesis of this nature. Even considering well defined subject of Tolerance Geometry, the formal treatment of objects touching and the analysis of vague terms, the subject matter opens up into a labyrinth of complex notions. On vagueness alone, the researcher already faces a considerable body of philosophical literature to familiarise themselves with.

The main approach in the thesis has been heavily influenced by Hilbert's work in metamathematics (Hilbert[1927]) and by Tarski's formalisation of the metalanguage of formal systems (Tarski[1933]). Both authors set out to formalise the metalanguage which is then used to discuss the structure of the object language. Tarski and Hilbert both assume that one is allowed to use conventional and accepted mathematics in the metalanguage – which includes amongst other notions that of naïve set theory.

Clearly, a formalised metalanguage is not necessary in every situation, but it an important step to take if one wishes to *prove* any results concerning the structure of the object

[1]This puzzle is invariably discussed in articles which discuss vagueness (e.g. Wright[1976]).

language. For instance, proofs of completeness are conducted in the metalanguage and could not be accomplished if one did not have any formal mechanisms.

Formalised metalanguages are also important from the point of view of AI, as for example in theorem proving where one may need to *prove* some result concerning the inference rules employed in the formal system. Alternatively, one may wish to automate some kind of formal 'control' mechanism for the inference rules as in the GOLUX project (see Hayes[1974b]). Furthermore, a formalised metalanguage provides the mechanism to ensure that any semantic notions used in the object language are clearly specified.

In this thesis, I *prove* that there are no object languages which will allow one to state the Sorites in a consistent manner, while satisfying certain metalogical conditions. It is impossible to prove such a result without the aid of a formalised metalanguage.

1·3 STRUCTURE OF THE THESIS

The thesis proper begins with Chapter 2 on the basic principles of logic and metalogic. Here I discuss such issues as: what Frege meant by a thought, and what is meant by an argument. In this chapter I also describe the kinds of assumptions which are commonly made regarding the structure of the metalanguage for a logic.

In chapter 3, I introduce the concept of *nearness* as it is formalised in mathematics, first using the conventional technique of topological spaces, then by using tolerance spaces. *Continuous* functions are introduced as those functions which map one space into another in a manner which preserves the essential characteristics of that space. Continuous functions are one instance of what mathematicians call *structure preserving functions*, and are discussed here for both topological and tolerance spaces. In this chapter I also introduce the notions of connectedness and density, which will be used later to formalise the notion of a (mathematical) continuum.

The approach adopted here is somewhat different to that taken in conventional textbooks on geometry and topology. In my discussion I include historical developments as well as a number of related topics which are not readily available in the literature (for example, a description of Frechet's theory of V–spaces).

Tolerance relations initially give the impression that they are a direct generalisation of equivalence relations, until one attempts to understand their structural properties that is. The effect a tolerance relation has when defined on a set is to 'glue' the elements of this set together. This is clearly the opposite effect to what one would expect form an equivalence relation which partitions the set it is defined on. As one would expect, the most useful techniques in modern mathematics namely equivalence relations, quotient sets and projection mappings cannot be defined once we forfeit transitivity. This leads naturally to the question: do analogous correspondences or constructions exist for tolerance relations? The answer to this question is presented in chapter 4.

Chapter 4 first develops the theory of tolerance relations in a manner which emulates the theory of equivalence relations. I then demonstrate that the fundamental correspondence which exists between *equivalence relations* and *partitions*, can be shown to exist between *tolerance relations* and *covers*. I also describe what happens to the canonical projections in such cases.

In chapter 3, I consider the route Klein took when he used a group action to describe the notion of equivalence in a geometry, and thereby to define the geometry itself. In chapter 4, I follow Klein in defining the notion of *indistinguishability* as a development of a certain kind of external operation, which is almost a group action on the underlying set. I call such an external operation a *t–subset action*, which differs from a group by failing to satisfy the axiom of closure. The formulation presented here enables me to describe a framework in which the notions of *identity*, *equality* and *similarity* are unified.

In the final section of chapter 4, I describe how tolerance spaces may be used to formalise the intuitive notion of touching. This approach (as Hayes predicted) is more intuitive than that presented by the topological version. Furthermore, it does not suffer from the same problems of attachment as in the topological case (details of how a topology can be used to attach objects together can be found in Eisenberg[1974], under the section on *Quotient Spaces*).

It was clear from my search through the literature on vagueness, that the Sorites was undoubtedly the most important paradigm in vagueness. I knew that Sorites predicates were vague, and that (according to Wright) vague predicates have an associated notion, small changes of which are required to be tolerated by the vague predicates. I then linked Wright's notion of tolerance with that of Zeeman (Tolerance Geometry), and conjectured that if one could construct a logic with tolerance geometry in its metalanguage, then one could control the inference by using the tolerance structure, and thereby block the paradox. In §5·2, I present the history and my analysis for the Sorites. In this analysis I point out that the Sorites may only occur when two different though related conditions to be satisfied. Namely, the existence of a spatial structure, which provide the notion of 'nearness', and the requirement that the predicate should preserve this structure.

Chapter 6 is essentially a review of the currently accepted philosophical views on vagueness. In particular I pay special attention to the classic papers by Russell[1923], Black[1937] and those by Wright[1975,1976,1987].

Russell's analysis provides the emphasis that vagueness is a metalinguistic notion. Black disagreed with Russell's analysis, and he accused Russell of having confused vagueness and ambiguity, and offered an alternate analysis in which he viewed vagueness as a statistical phenomenon. Black's analysis is adopted by Zadeh who (allbeit implicitly) used it in the foundations of fuzzy sets.

Wright's articles are generally considered to be the state of the art in philosophical analysis of vagueness, and are the ones I shall utilise most in this thesis. In his articles Wright asserts that vague predicates often involve a notion of tolerance, but does not formalise this concept as I do here.

The notion of a continuum is discussed in a number of articles which analyse vagueness. Typically, they point out the difficulties encountered with 2–valued logic if one assumes the continuum as a basic structure into which predicates are mapped. These articles invariably fail to describe what one may formally understand by a continuum. Moreover, given the general lack of analysis of this concept, it is difficult to assess the validity of these approaches. This is particularly apparent where the authors claim to deal with vague predicates using fuzzy sets.

The first section of Chapter 7 shows how the notion of a continuum has evolved through the centuries, culminating with the formal definition of a (mathematical) continuum offered by Cantor.

The subtle distinction between the notions of a *continuum* (a property of a set) and that of *continuity* (a property of a function) was not made until the works of Bolzano and Cantor towards the end of the last century. This may appear a long time ago, however, these topics have been studied since the time of the Ancient Greeks, therefore in relative terms it is a recent discovery.

In Chapter 7, I also discuss the distinction between mathematical points and the points of our perception. This distinction has been emphasized by many mathematicians from Parmenides to Cantor. I discuss it again here because it is important to decide the exact nature of the subject matter that we wish to capture formally. I also argue that a perceptual continuum (which I call an impression continuum) should be represented by a *connected tolerance space*, rather than the mathematical definition presented by Cantor. I then use

this framework to demonstrate how the comment "... nearby places on the skin are wired to nearby places in the brain" (Minsky[1987]), may be formalised.

Chapter 8 is really the heart of this thesis. This chapter begins with a section discussing how one may use geometry in the metalanguage. In the following section I present my analysis of vague predicates, and propose that we should not require vague predicates to take values in a *continuum*, but rather, that we should require them to be *continuous*. Using descriptions of vague predicates from chapter 6, I demonstrate that these descriptions are simply another way of requiring that vague predicates to be structure preserving (i.e. continuous). This justifies my claim that Zadeh confused the two notions of continuity and the continuum. Therefore, Hayes was correct in his criticism that fuzzy sets are not necessary to capture vague notions. In fact as it turns out, all that we need is a logic which has a truth–set given by {0,u,1} – where u stands for 'unknown' – rather than that given by the uncountable set [0,1].

Having shown that vague predicates should be *structure preserving*, it remains to give a description of the structures which they are required to *preserve*. I argue that there are only two structures which enable one to describe the notions of continuity. These are the two spatial structures of topology, and tolerance geometry. The consequence of such structures on the underlying logic are subsequently analysed.

The notion of a continuum is frequently discussed in articles on vagueness, and therefore it is an important issue to address. Since a continuum is formally defined in terms of a spatial structure, one must therefore introduce such a structure into the metalanguage (clearly at this stage the metalanguage must already be a formal one). Given the existence of such a structure, I can then investigate what the affect will be on the underlying logic if we required the predicates extensions to be continuous.

In having presented the formal mechanisms, and the analysis of the Sorites puzzle in chapter 5, I describe the puzzle in a formal manner and explain why the paradox arises, by

translating the analysis (given in chapter 5), into the following conditions:

(i) Soritical predicate should be structure preserving,

(ii) the domain (or at least that part which the predicate is defined) must be a continuum,

(iii) the truth–set is given by {0,1};

I then prove that these statements are mathematically inconsistent.

One question which naturally arises is: does there exist a logic with a different truth–set which can satisfy the other two conditions? If so, what is its structure? The answer to the first question is yes, there is a logic which allows one to keep conditions (i) and (ii), and still have a consistent set of statements. There is a partial answer to the second question; the logics which satisfies the metalogical conditions and the conditions of the Sorites, are Kleene's (*Strong* and *Weak*) three valued logics. Therefore, the Sorites can be resolved in a logic which is different to the classical one.

In the final section of chapter 8, I discuss what I consider to be an important point about Kleene's logic, namely, Kleene's own epistemological interpretation of the third value ('u') in the truth–set used. According to Kleene, this value does not represent a truth value as such. Epistomologically, it represents the value 'unknown whether true or false'. I conjecture that the value 'u' may be used in the case of an individual for which it is *intrinsically uncertain* whether the chosen predicate applies or not (using Peirce's definition of vague predicates).

Despite the number of articles produced since Hayes first published his NP programme, almost all of them have attempted to solve the problems raised by Hayes using topological techniques. To the best of my knowledge the only other article which deals with tolerance techniques (in any context) is that by Kaufmann[1991], who uses the theory presented in Dodson[1975] to describe the motion of a rachet. Kaufmann develops Poston's intuitive description of motion, together with a formalism which is non–monotonic. This is an

interesting article but one that suffers form the use of an opaque notation and is unintuitive.

1·4 CONTRIBUTIONS

The following summarises my contribution in this thesis.

(1) Having extended the works of Zeeman[1962] and Poston[1971] on tolerance geometry, I demonstrate that there is a correspondence between tolerance relations and covers. This correspondence emulates that existing between equivalence relations and partitions. I also show how the notions of identity, equality and similarity can be unified into one framework. (Chapter 4).

(2) Also in chapter 4, I demonstrate how the notion of touching may be expressed in an intuitive yet rigorous manner using a tolerance space (as required by Hayes).

(3) In chapter 8, I analyse the literature on the Sorites and vagueness (as presented in chapter 6), and argue that vague predicates are required to be *structure preserving* (or *continuous*). This is different to Zadeh's demands that they should take values in a *continuum*. I claim that Zadeh has confused the two notions, and that fuzzy sets do not possess the right properties as a framework for representing vague concepts.

(4) The analysis presented in the first few sections of chapter 8, required me to incorporate spatial structures into the metalanguage of a logic, which is then be used as a constraint on the semantics for the object language. With this kind of structure, I demonstrate that the Sorites is *insoluble* in logic with a truth–set given by {0,1} – assuming the analysis given in chapter 5.

(5) Using the spatial structures on the set {0,u,1}, I show how the Sorites can be stated in a consistent manner in a 3–valued logic. Furthermore, I show that there are only two such logics which satisfy my metalogical conditions.

1·5 PRELIMINARIES (TO THE REST OF THE THESIS)

As this is a thesis in (formal) AI, I shall assume the reader has a knowledge of AI literature together with specific knowledge of McCarthy's[1958] and Hayes'[1970,1971,1974a,1974b,1979b,1985] articles as well as Welham&Hayes[1985]. Knowledge of NP beyond that covered by the above articles is not required, neither is the reader required to to have any other specialist knowledge in AI in order to be able to read this thesis. The only other background knowledge assumed is a working knowledge of naïve set theory comparable to that presented in Halmos[1960], elementary first order logic and Tarskian semantics. A basic understanding of topology (see for example Mendelson[1973] or Eisenberg[1974]) and Tarski's formalisation of the metalanguage in (Tarski[1933]) is useful, but not essential. In this respect I have included background material in Chapters 2 and 3 and in appendices A and B which should be sufficient for reading this thesis.

2 Axiomatics and Metalogic

"All sciences have truth as their goal; but logic is also concerned with it in a quite different way: logic has much the same relation to truth as physics has to weight or heat. To discover truths is the task of all sciences; it falls to logic to discern the laws of truth."
G. Frege

This chapter is of an introductory nature. Its aim is to present the *intuitive* meaning for such terms as *thought, argument, metalogic* and so on, as they will be used later.

The first section deals with Frege's notion of thought and argument, which includes two extensive citations from his work. There are two reasons for this, firstly, the sections of work (to be cited) are controversial, and are best presented as direct citations. And secondly, I believe that paraphrasing this material would not have helped to clarify its content.

The second section describes the evolution of the *axiomatic* method and the introduction of Hilbert's *metamathematics*, a topic which Hilbert invented to describe such notions as a *proof* in a formal manner. Metamathematics is nowadays called *proof theory*.

The addition of semantic notions to metamathematics, takes one from metamathematics to *metalogic* proper. The rest of the chapter is concerned with the sort of structures that are required to be in the metalanguage, so that certain properties of the object language can be expressed in a formal manner.

2·1 WHAT IS LOGIC?

The word 'logic' is derived from the ambiguous Greek word 'λογική', which can be interpreted in a number of ways. Some consider logic as a study of words, while others (like myself) prefer to interpret it as a study of *reasoning*.

Reasoning is a complex mental process, that should be distinguished from the more rudimentary process of 'thinking'. What distinguishes reasoning from other forms of thinking is the use of *reasons*, where we assume certain pieces of information to be true and use them to arrive at new ones.

In this respect logic is concerned with the nature of *thinking*. However, in logic we deal with the *products* of thinking and reasoning as opposed to the *actual* process of thinking and reasoning as we encounter them everyday (which fall under the realm of psychology). In other words we are interested not in the *arguings* which we perform as humans, but in the *arguments* themselves.

I shall use the word *'logic'* to mean the study of *reasoning*, or more precisely: the analytical theory of the art of reasoning, whose goals are to systemtize and codify principles of valid argument.

2·1·1 Frege's concept of thought

In order to clarify the meaning of 'principles of valid argument' scholars separated out the notions of 'thought' and 'truth'. Frege distinguishes the psychological principles governing the process of thought from the structural properties of thought which reflect properties of the real world.

Frege[1977] describes a *thought* as the sense of a sentence, which is imperceptible by the senses but "gets clothed in the perceptible garb of a sentence, and thereby we are able to grasp it". In other words, Frege sanctions the thought as being expressed by a sentence, while carefully denying the converse[1].

[1]Namely, that every sentence expresses a thought.

It is interesting to note that Frege made a distinction between expressing a thought and an assertion, Frege[1977,p7]:

> *"... two things must be distinguished in an assertoric sentence: the content, which it has in common with the corresponding propositional question: and the assertion. the former is the thought or at least contains the thought. So it is possible to express the thought without laying it down as true. The two things are so closely joined in an assertoric sentence that it is easy to overlook their separability. Consequently we distinguish:*
>
> *(1) the grasp of a thought – thinking,*
> *(2) the acknowledgement of the truth of a thought – the act of judgement*[1]
> *(3) the manifestation of this judgement – assertion."*

According to Frege[1977] scientific advances are made in three steps. The first step is the grasping of a thought, next comes the appropriate investigation which results in the acceptance of the truth for this thought, and finally, we express acknowledgement of the truth in the form of an assertoric sentence.

The most profound consequence of Frege's work is clearly laid out in the preface of the translation, where Geach (Frege[1977]) states that in 'Der Gedanke' Frege affirms:

> *"(1) that any thought is by its nature communicable, (2) that thoughts about private sensations and sense–qualities, and about the Cartesian I, are by their nature incommunicable. It is an immediate consequence that there can be no such thoughts. .... But though he [Frege] never drew this conclusion, Wittgenstein was to draw it." [my emphasis].*

Thus Frege claims that we have a thought which we express as a sentence, and communicate it to someone else, who *grasps* it.

Frege is drawing a distinction between the sense of a sentence, namely a 'thought' which he regards as an abstract object, and an 'idea' which is a purely mental entity. Haack[1978] describes this clearly saying: *"Since ideas are mental, they are, Frege argues, essentially private; so you can no more have my idea than you can have my toothache"*.

---

[1]It seems to me that thought and judgement have not hitherto been adequately distinguished.

The most important part of these statement made by Frege is that: private sensations and sense-qualities are *not* thoughts. The consequence of which is that such notions should be excluded from the bounds of logic. Finally, Frege[1977, or 1956] asserts that 'a thought' is something for which the question of truth can arise at all. This indicates that he is taking 'thought' to be the 'truth-bearers' in his theory. The notion of 'truth-bearers' is an important issue in the philosophy of logic and is discussed in Haack[1978], and at some length by Lemmon[1966].

2·1·2 Arguments

I shall describe what I mean by an 'argument' in terms of the intuitive notion of a thought. An ***argument*** can be understood as a sequence of ***thoughts***, the final one of which is usually signalled by the use of such terms[1] as ***'hence'*** or ***'therefore'***. The sequence of thoughts which precede the 'therefore' are called ***premises***, and the final thought is called a ***conclusion***.

An argument is said to be:

(i) ***valid*** if it consists of premises linked in such a way that, if they are true, then the conclusion drawn for them must also be true;

(ii) ***sound*** only if all the premises are true and it is valid as well. The soundness of an argument ensures the truth of the conclusion.

When we perform an ***act of reasoning*** we establish the conclusion based on the evidence presented by the premises. If we arrived at the conclusion by thinking alone, that is to say without the independent investigation of its truth, then we say that the conclusion was ***inferred*** from the premises. An ***inference*** is a special sort of reasoning, where we come to know the truth of the conclusion without recourse directly to sense perceptions or immediate experience.

[1]These terms are often called, ***inference markers.***

2·1·3 Reasoning as movement of thought

An ***inference*** is defined as a rule which links a set of statements (the premises), with another (the conclusion). In a recent article, Walton[1990] prefers to view an inference not just as a relationship between a collection of statements and another, but as a 'movement' from the premises to the conclusion which has a direction associated with it.

One can generalize this notion of 'movement' to reasoning as: *a process, which given an argument 'moves' from the premises to the conclusion.* Given that the premises and the conclusion are expressions of thoughts, we may view reasoning as 'the movement of thought'.

2·2 AXIOMATICS

Twentieth century mathematical research is being increasingly influenced by the axiomatic approach which has a long and distinguished ancestry. This method began to flourish from about the middle of the last century when mathematicians like Frege, Russell, Peirce and Hilbert began to study axioms and apply the axiomatic technique to logic.

2·2·1 Evolution Of The Axiomatic Theory

Euclidean Geometry is a development of the method used by Euclid in his *Elements* (*c.* 300 B.C.). The Elements start with a list of definitions, postulates and axioms, from which Euclid proceeds to derive a large number of theorems. It seems that the *axioms* were intended to convey the principles of reasoning which would be valid in any science, while the *postulates* are intended to be assertions about the subject matter to be studied. It is important to appreciate the difference between these concepts, since they indicate the structure of the axiomatic method which was destined to become a significant step in the development of mathematical logic. Euclidean geometry has often been used as a prototype for the *axiomatic method.* A technique which can be considered to be the forerunner of the present-day concept of an *axiomatic theory.*

In general the word ***theory*** is used to describe a collection of statements which explain a given subject matter. A subset of this collection is chosen to denote the 'true' statements, and is called the ***distinguished subset.*** In an axiomatic theory, the distinguished subset is no longer the collection of true statements but the collection of 'theorems' or 'provable' statements. The manner in which we classify these 'provable' statements is by initially choosing a subset from the collection of all statements, and declaring them to be the 'axioms'. 'Theorems' are then defined to be those statements which can be deduced from the axioms of the logic.

In any situation which we accept the system of logic, and the truth of the axioms, we must also accept the truth of the *theorems*. Thus, in an axiomatic theory the notion of *truth* is relegated to potential applications of the theory. It is the user of the axiomatic theory who, having accepted such a system must concern themselves with the truth of the axioms.

2·2·2 Informal theories

In mathematics it is common practice initially to formulate axiomatic theories as *informal theories*. An informal theory is one which presupposes set theory and a theory of inference, and where the 'theory of deduction' is the intuitive one commonly used in mathematics. It should not, however be assumed that the intuitive theory of inference used by mathematicians is somehow not rigorous, quite the contrary, the underlying logic can be clearly spelled out if necessary[1].

Structures

A ***structure*** $\mathcal{U}$ for a theory **T**, is a pair $\langle |\mathcal{U}|, \sigma \rangle$, where $|\mathcal{U}|$ is a non-empty set called the ***domain*** of the interpretation over which the individual variables range. The function σ (called an ***interpretation function***) assigns appropriate operations on $|\mathcal{U}|^n$ to symbols of **T** .

[1]This is precisely the sort of ideas Hilbert was concerned with in his metamathematics.

If $\mathcal{U}$ is a structure for T and S is a sentence of T (a T-sentence), then the sentence which results on assignment of *meaning* to each primitive term of T that occurring in S will be called an ***interpretation*** of S in $\mathcal{U}$. If a sentence of S is interpreted as a true statement in $\mathcal{U}$, then we shall say that S is ***true in*** $\mathcal{U}$, or that $\mathcal{U}$ is a ***model of S***. $\mathcal{U}$ is called a ***model of*** T if it is a model for each axiom in T.

The manner in which these axiomatic theories are defined uses what may be termed as *set–theoretical predicates*. For example, ***group theory*** is defined as follows: **G** is a ***group*** if and only if there is a set X, a binary operation • in X, and an element e of X such that $\mathbf{G} = \langle X, \bullet, e \rangle$ and

G1. for all $a,b,c \in X$, $a \bullet (b \bullet c) = (a \bullet b) \bullet c$,

G2. for all $a \in X$, $a \bullet e = e \bullet a = a$,

G3. for each $a \in X$ there exists an $a' \in X$ such that $a \bullet a' = a' \bullet a = e$.

2·2·3 Metamathematics

In the late nineteenth century the subject of metamathematics emerged from the need to deal with the notions of completeness and consistency in a precise manner. Hilbert recognized that such an endeavour could only succeed if theories were *formalized* so that the definition of 'proof' could be made entirely explicit. This endeavour saw the emergence of a new field of study called ***metamathematics***.

In order to formalise a theory, we need the apparatus of a *formal theory*, which can be described as: a completely symbolic language constructed according to certain rules from the alphabet of primitive symbols, together with an inference mechanism. A formal theory itself can be the subject of study, in which case it is called an ***object language***. To study such a language we need to describe its properties in a second language, which we call the ***metalanguage***. A good example of the object language/metalanguage distinction is given by an English speaking person learning to speak French, in which case English is the metalanguage and French the object language.

Amongst the components of the object language which are studied in the metalanguage are:

(1) the *syntax* – which is simply the formal grammar of the theory, and

(2) the ***axioms and rules of inference***, which supply the analogues of 'arguments' in the formal theory.

In the metalanguage we wish to study the formal properties of a formal theory, therefore we need to define such notions as *theorem* and *proof* for the metalanguage. Hilbert[1] proposed that the metalanguage chosen for such a study should belong to *intuitive and informal mathematics*, and therefore any constructions should be expressible in ordinary language with the addition of mathematical symbols. Furthermore, the theorems about the formal theory expressed in the metalanguage must be clearly understood, and the principles of deduction universally accepted.

Consistency and Completeness

Among the properties which an informal theory may possess, the most important are those of consistency and completeness.

Suppose that the logic we are using for our informal theory **T** includes the rule of inference: from A and (A implies B), infer B.

We shall call such a theory ***inconsistent*** if, for some formula $A \in \mathbf{T}$, both A and $\neg A$ are provable in this system[2]. Inconsistency is an extremely serious matter. In an inconsistent theory, whose logic includes the above rule of inference together with the tautology $(A \rightarrow (\neg A \rightarrow B))$, one is able to prove any formula $B \in \mathbf{T}$; in other words every formula in **T** is a theorem. Such a theory is worthless, because it excludes nothing. Thus it is

[1]David Hilbert[1927], who is recognized as the founder of proof theory, starts his seminal paper by stating that no science (or mathematics) can be founded by logic alone. He argues that certain entities must already be "... given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought."

[2]A theory is ***consistent*** iff it is not inconsistent.

important to show that a theory is consistent, the way in which we normally show this is by finding a model for it.

If $\mathcal{U}$ is a model of some consistent theory T, then every theorem of T is true in $\mathcal{U}$. Conversely, we say that T is ***complete for*** $\mathcal{U}$ if every statement $A \in T$ whose interpretation is a true statement of $\mathcal{U}$, is a theorem of T. A ***logical truth*** of a theory T is a statement of T which is true in every model of T. We call a theory T ***deductively complete*** iff every logical truth of T is provable (i.e. a theorem).

In the 1920's Hilbert and his team started work on the programme which proposed to prove the consistency of elementary number theory. However, in 1931 Gödel proved the impossibility of such an endeavour for any formal theory – including elementary number theory – by using constructive methods. This effectively put an end to the Hilbert programme, though it did not spell the doom of metamathematics, but merely served to indicate that the subject had its limitations.

2·3 METALOGIC

2·3·1 What is metalogic?

In 1930 Gödel[1930] demonstrated the completeness of predicate logic. This was followed a year later by his incompleteness proof (Gödel[1931]). In just two years Gödel had managed to show the importance of semantic notions[1] (informally, ***semantics*** can be considered as a method of assigning *meaning* to strings of symbols in a language).

Gödel's theorems not only had a profound effect on the Hilbert programme, they also influenced Tarski's and Carnap's views on *truth* and *proof* in a logical system. According

[1]Although to be fair, Gödel was not the first to give a 'completeness' result. The first completeness proof was by Emil Post (Post [1921]), who investigated the *formal completeness* of the propositional calculus. An axiomatic theory T is said to be ***formally complete*** provided that any theory T', which is the result of adding to the axioms of T, a statement of T which is not already a theorem of T, is inconsistent.

to Coffa[1987,p.548], the difficulties concerning Carnap[1937] in his 'Logical Syntax of Language' were the definitions for mathematical truth and logical consequence.

In recognising the difference between the notions of truth and theoremhood, Carnap decided to change his approach. Instead of considering a generalization of the concept of inference, he introduced the novel notion of a *valuation*. Carnap however, was not alone in realising this. Tarski had also become aware of the philosophical implications regarding Gödel's investigations. Up until 1933 Tarski had a purely proof-theoretic analysis for the notion of consequence, however in the light of Gödel's results he was forced to question their adequacy.

Tarski's investigations into the notion of truth resulted in one of the most significant contributions to metalogic. In his paper Tarski[1933][1] investigates whether the notion of truth for a formal language could be defined in an appropriate metalanguage. Tarski[1933,p.153] concerned himself with *grasping the intentions that are contained in a classical conception of truth, a truth corresponding with reality*, which he captured by his 'semantic definition of truth'. Tarski attributes this notion to Aristotle[*Metaphysics*]:

> *"By this I mean a definition which we can express in the following words: 'a true sentence is one which says that the state of affairs is so and so, and the state of affairs indeed is so and so'."*

It is important to note that Tarski's and Hilbert's notion of formalized languages were distinct. As Tarski[1933,p.166] himself explained:

> *"It remains perhaps to add that we are not interested here in 'formal' languages and sciences in one special sense of the word 'formal', namely sciences to the signs and expressions of which no meaning is attached. For such sciences the problem here discussed has no relevance, it is not even meaningful. We shall always ascribe concrete and, for us, intelligible meanings to the signs which occur in the languages we shall consider."*

[1]Coffa [1987, p.568] has an interesting story about a discussion Carnap had with Tarski about the definition of *truth*: *"They were at a coffee-house and Carnap challenged Tarski to explain how truth was defined for an empirical sentence such as 'this table is black'. Tarski answered that 'this table is black' is true if and only if this table is black; and then – Carnap explained – 'the scales fell from my eyes' "*.

It is important to emphasize the value of semantics. According to Van Benthem[1981], those who only work with pure syntax are

> *"... groping around in a self-imposed dark. For, he need only realize that his language refers to a reality outside — and the helping hand of semantics will open a window through which the fresh air of real life rushes in."*

The addition of semantics (of the object language) takes us from metamathematics into metalogic proper. Following Wang[1974], I define ***Metalogic*** to be the study of syntax and semantics, of formal languages and formal systems.

2·3·2 The metalanguage

The metalanguage I shall use in this thesis will be the one proposed by Hilbert, namely the language of intuitive and informal mathematics. However, the mathematics which was universally accepted by mathematicians in Hilbert's day is clearly different to the mathematics accepted today.

Quotation marks, use and mention

Tarski[1933] indicates the importance of making a clear distinction between entities which exist in the *object language* and those which exist in the *metalanguage*. For example, in the metalanguage one can talk about: (a) Ann the human being, an *object* in the domain; and (b) 'Ann' the *name* of the entity.

Technically, the word 'Ann' is ***used*** in (a), and ***mentioned*** in (b). A difficulty arises in that both uses of 'Ann' above are words in the metalanguage. The technical distinction between the two is achieved via ***quotation marks***. According to Quine[1990], quotation marks make all the difference between talking about the words and talking about snow.

Concatenation and Structural descriptions

There are two further concepts, not belonging to set theory but which are needed in the metalanguage (Tarski[1933]), and therefore have to be stated separately.

Concatenation is a two-place function defined on pairs of symbols (x and y say), which produces the compound symbol x immediately followed by y.

Structural descriptive names is a procedure which Tarski uses to produce a correlate of every quotation–name which is free from quotation marks, but possesses the same extension. This can be achieved if we have a name for every symbol in the object language in our metalanguage (for details see Tarski[1933, p156]).

Definition: An ***expression*** is defined to be the result of concatenating together a finite number of symbols.

I should point out that a rigorous treatment of an object language requires both notions of concatenation and structural descriptive names. However, for my purposes, these strictly technical notions would unnecessarily complicate matters, and therefore I have mentioned them here for completeness only.

Set theory in the metalanguage

There are some constructs which belong to the metalanguage which are neither in the *domain* nor in the *object language*, for example the notion of a *pair*. According to Quine[1990, p.35], when we say that the pair ⟨3,5⟩ satisfies the sentence 'x<y', we are only assuming that the domain of objects of the object language includes the numbers 3 and 5. However, we do not need to assume that this domain includes the pair ⟨3,5⟩ which belongs to the apparatus of my study of the object language.

2·3·3 Metatheoretical issues

In his seminal paper of 1933[1], Tarski set out to formalise the notion of truth and how an expression in the formal language relates to certain constructs in the domain of interpretation.

In describing Tarski's theory of truth, Quine[1990,p.11] asserts that :

[1]Tarski's paper was the first attempt to demonstrate the possibility of a formalized metalanguage.

"... the truth predicate serves, as it were, to point through the sentence to the reality; it serves as a reminder that though sentences are mentioned, reality is still the whole point."

In what follows I shall describe some of the issues concerned with this semantics, commencing with the sort of properties our system should possess.

Theory of the syntax

The following principle is often used to restrict the formal language.

Principle 1: The collection of well-formed formulas of the object language must be an inductively defined set.

Formal semantics : Theory of meaning

Metalogicians use formal semantics to explicate such notions as *truth* and *validity*. Thus ***formal semantics*** can be described as the study of the relationship between the expressions of a formal system and the objects in a domain of interpretation.

The presentation of a semantics for a language presupposes a *theory of meaning*. There are a number of these, the one I shall now describe is the one relevant to the semantics proposed by Tarski[1933].

As mentioned earlier, Tarski was interested in capturing the notion of truth in a formal manner. A notion which can be informally described as a kind of *correspondence* between asserted sentences and 'the way the world is', or sentences and 'state of affairs'. The notion of *correspondence* is the key, according to which expressions pick out entities in the domain, in much the way we as human beings perform ostensive acts every day where by picking things out we *point* to things in the real world. This relation of picking out is particularly important and is consequently taken as a basic semantic relation. Nowadays, we would say that we are ***referring to*** or ***denoting*** an entity in the domain.

What an expression denotes or refers to is certainly part of its meaning, and the theory of meaning restricted to this concept is called the ***referential theory of meaning*** (***RTM***). There are essentially two parts to this theory:

The ontological category: In this part of the theory we need to stipulate a collection of entities in the domain which the basic or atomic expressions of the language refer to.

The semantic rules: We also need rules to determine the referent of complex expressions in terms of the referent of their parts. Such rules are called ***semantic rules***. Often these rules mirror the formation rules in the syntax, hence there will be as many semantic rules as there are formation rules in the language.

We need to ensure that the semantic rules possess the properties we need, therefore we require the next principle to be satisfied.

Principle 2: Given a domain, the referent of a complex expression is determined in a rule-like manner from the referents of its atomic parts.

Finally, I shall mention that in Tarski's theory he takes *sentences as his truth bearers*, therefore sentences have truth-values as their referents. Thus applying principle 2 to sentences we have:

Principle 3: The truth-value of a complex sentence is determined in a rule-like manner from the referents of its immediate parts.

2·4 FORMALIZED THEORIES

I have already discused informal theories earlier in this chapter. I now wish to discuss *formalized theories*. Recall from §2·2·2 that, abstractly any group of sentences concerning a given subject matter may be regarded as a theory. In the case of a *deductive theory* the distinguished subset constitutes the *assertions* or *theses* of the deductive theory.

<u>Definition</u>: A ***(first order)theory*** **T** is defined by specifying a formal system for first order predicate logic, $\mathfrak{L}$, together with the following:

(1) a collection Δ of non-logical constants;

(2) a collection Σ of closed formulas of $\mathfrak{T}$.

A first order theory **T**, is said to be ***closed*** iff it is closed under the relation of *syntactic consequence* ('$\vdash$'). A closed theory is also called a ***deductive theory.***

If the language which is used to encapsulate the theory is complete, then the closure condition is equivalent to the theory T being closed under the relation of *semantic consequence* ('$\models$'). This connection is important because many authors define a theory to be a set of sentences closed under *semantical* consequence, rather than *syntactical* consequence (see Enderton[1972] and Chang & Keisler[1973]).

2·4·1 Models of formal theories

The process of formalizing a theory regarding a particular subject matter is carried out in a number of steps. The first step is to lay down the domain and the appropriate relations, this is sometimes called the *ontology* of the theory. On this level, the first mathematical concepts are obtained, by *abstraction from real situations*; for example, the fundamental geometrical concepts such as point and line arose by abstracting from reality. The second step involves the concept of a ***proposition***, which needs to made precise, and an *interpretation of the propositions on the domain in question* is defined. Finally, an *axiom system* and a relationship of *derivability* are stipulated.

2·5 SUMMARY

In this chapter I have outlined some of the important notions in logic, in particular the notion of *thought* which was in common use at the beginning of the century, but has since been dropped. I have re–introduced this notion here because it helps one to identify what could in principle be expressed by the formalism.

In the rest of this chapter I discussed the mechanisms which are essential for describing the structure of an object language in a formal manner. The metalanguage used by both

Hilbert (in his metamathematics) and Tarski (in his article which dealt with formal theories of truth and formalised the metalanguages for first order logic) contain naïve set theory, in addition to other more specialised mechanisms.

The manner in which Tarski proved theorems concerning object language expressions, was to translate the whole entity into the metalanguage and use the naïve set theory as a formal tool. These details are often overlooked when describing the semantics of a language using the notion of satisfaction, which was constructed by Tarski to describe the notion of truth without the aid of any semantic primitives.

This chapter is intended to provide some familiarity with the subject matter of metalogic, and should be contrasted with the kind of things which are the concerns of logic. Familiarity with the general concepts outlined here will be assumed in chapters 5, 6 and 8.

3 Mathematical Structures

"The true philosophy is written in that great book of nature which lies ever open before our eyes but no one can read unless he has first learned to understand the language and to know the characters in which it is written. It is written in mathematical language, and the characters are triangles, circles, and other geometrical figures."

Galileo Galilei

This chapter is concerned with geometrical concepts in mathematics. The first part of the chapter deals with the geometric structures as proposed by Klein in his Erlanger Programme, in which he views an *equivalence* in terms of a *group acting on a set.* The second part is concerned with the important notion of an *abstract space*, which was first contemplated by Bolzano in the early nineteenth century.

The approach adopted in this chapter is unusual in many respects, for example abstract spaces are developed from the intuitive concept of *nearness*, instead of the familiar presentation involving open-sets (described in abstract set-theoretical axioms) commonly found in textbooks on topology. Nonetheless, the familiar axiomatic approach is useful for defining some of the more technical material and is therefore introduced in §3·3·3 and used thereafter.

The final section is an introduction to another kind of abstract space, called a tolerance space. In this section the reader is introduced to some technical definitions and theorems from tolerance geometry.

A proof from Poston's Thesis (Poston[1971]) is also included here, since the theorem is utilised later in this thesis and the original proof contains a number of typographical errors. The section is concluded with an interesting example illustrating how tolerance geometry may be used to describe motion.

3·1 GEOMETRY

Geometry is often considered as a deductive science, namely that it is a body of theorems deduced from a set of axioms. However, in his inaugural address at the University of Erlanger(1872), Klein[1903] introduced a new unifying principle for geometry, which is both elegant and intuitive.

3·1·1 Klein's characterisation of geometry

Klein's Erlanger programme captures an important class of geometries, each of which can be considered as the *invariant* theory of a *transformation group*. In order to describe the meaning and evolution of this definition, we need to define the terms 'transformation' and 'invariance'.

Definition: Let S be a set. Then a ***transformation*** of S is a bijective function of S onto itself. A collection $\Gamma(S) \subseteq \mathcal{F}(S,S)$ of transformations, which under the operation of function composition forms a group, is called a ***transformation group***[1].

Wiener[1948] describes a *transformation* of a system as some alteration in which each element is sent into another. As an example he considers two configurations of the solar system, one at time t_1 and the other at time t_2. The change in the configuration can be considered as a transformation.

[1]Also called a ***symmetry group***.

Associated with the notion of a transformation is the concept of an *invariant*. Let T be a transformation of the set S, then any relation $P \subseteq S^n$ on S is called an ***invariant*** of the transformation T if

$$P(x_1, \ldots, x_n) = P(T(x_1), \ldots, T(x_n)).$$

If P is invariant under all the transformations of a group $\Gamma(S)$, then we call it a $\Gamma(S)$–*invariant* of S.

Klein adopted a viewpoint which allowed a geometry to be characterised by a group of transformations and the primary subjects of study were the *invariants* under this group. In other words, for each transformation group $\Gamma(S)$ of S, the $\Gamma(S)$–***geometry*** of S is the study of the $\Gamma(S)$–invariants of S.

> *Definition (Klein, Erlanger Programme): **Geometry** is the science which studies the properties of figures preserved under the transformations of a certain group of transformations, or as one also says, the science which studies the invariants of a group of transformations.*

In the application to geometry, the set S in the above definition is often a *space*. What distinguishes a *space* as opposed to a mere collection of points is some concept that *binds* the points together. The notion which is commonly chosen to bind the elements of a set to form a space is one which embodies the intuitive notion of *position* in that set.

Let $\langle S; \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n \rangle$ be a space, the elements of which we call the ***points***; and any property or relation definable from the structure $\langle \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n \rangle$ we call ***geometric***. Each relation, $\mathcal{R}_i \in \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n\}$, is required to be a $\Gamma(S)$–***invariant*** of any transformation group $\Gamma(S)$ of that space. In other words, the transformation group is required to preserve the structure of the underlying space.

In geometry one studies the properties of ***figures***, defined as arbitrary point subsets of the space[1] S. Such subsets, however, are too general and we must restrict the sort of point

[1]According to Johnson[1977] this definition first appeared in Bolzano[1817] who called them ***spatial objects***.

sets we allow so that the term 'figure' may have a *geometric* meaning. Therefore, we shall define any subset which is definable in terms of the geometric properties to be a ***geometric figure*** (also called a ***configuration***).

Definition: Let $\langle S; \mathcal{R}_1, \ldots \mathcal{R}_n \rangle$ be a space. A transformation T of S is called an ***automorphism*** if it preserves the structure, $\langle \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n \rangle$. In other words if the relations $\mathcal{R}_1, \ldots \mathcal{R}_n$ are all *invariant* under T.

One of the simplest examples of a *space* is a set equipped with a notion of a *distance.*

Definition: Let S be a set. A function $d: S \times S \rightarrow \mathbb{R}$ is called a ***metric*** (or ***distance function***), if for all $p,q \in S$, the function d satisfies the following axioms:

(M1) $d(p,q)$ is zero if and only if $p = q$,

(M2) $d(p,q) = d(q,p)$

(M3) $d(p,r) \leq d(p,q) + d(q,r)$, called the ***triangle inequality.***

A set equipped with a metric is called a ***metric space***, and is denoted by $\langle S,d \rangle$.

A transformation T of a metric space $\langle S,d \rangle$ which preserves the distance between points is called an ***isometry***[1]. The set of all isometries for a space form a group called the ***isometry group***, denoted by '$\mathcal{I}(S)$'.

The isometry group for the Euclidean space, is sometimes called the *group of motions.* The reasoning behind this is as follows: Imagine a rigid body in space, which moves from position A to position A′. The portions of space occupied by the body in two of its positions are *congruent.* The notion of congruence, as well as the physical applications of this notion, are discussed at some length by Weyl[1949].

If a geometric figure F is invariant under a subgroup $\mathcal{S}(F)$ of the isometry group $\mathcal{I}(S)$, then $\mathcal{S}(F)$ is called the ***symmetry group*** of F. The size of the symmetry group for a geometric figure is a measure of the degree of symmetry of that figure.

---

[1]In other words for all points x and y in S we must have, $d(x,y) = d(T(x),T(y))$.

Examples 1:

(a) In ***Euclidean geometry***, a figure and its image under an *isometry* are considered to be geometrically equivalent. They are said to be ***congruent***. Therefore by Klein's definition Euclidean geometry can be considered as the ***geometry of isometries*** (or congruence). The ***geometry of similarity*** is obtained by extending the isometry group, to include dilations (change of scale) resulting in a group called the ***similarity group***. The invariants in this geometry are angles, and length ratios, though the notion of length is absent.

(b) If the transformation group $\mathcal{G}(\mathbb{R}^2)$ is taken to be the group of all *continuous* bijective functions of the plane to itself (called the topological transformations), the associated geometry is the ***topology*** of the plane. Thus, for Klein, *topology* was the study of the group of transformations which preserved the notion of nearness. Therefore, this geometry only recognised the continuity properties of figures.

The evolution of Klein's Ideas

Klein was aware that any *science* could be specified by naming the objects, and the properties of these objects, which the science studied. Klein viewed Euclidean geometry in exactly this manner. The 'objects' were all possible plane figures and solids, which could be described as 'point sets' in the plane or space[1]. The properties of figures considered in geometry are entirely specified by an indication of what figures are to be considered as *equivalent*. Thus, given a figure F, geometry studies only those properties of F which are shared by all the figures G considered equivalent to F. It can be clearly seen that the key to this approach is the definition of 'equivalent'.

Klein first of all considered the following as a definition of 'equivalent':

Let S be a set and $\mathcal{H}(S,S) \subseteq \mathcal{F}(S,S)$ a collection of transformations on S. Then two figures F and G are to be considered *equivalent* if they can be sent into one another by a

[1]An arbitrary collection of points does *not* constitute a *figure*. *Geometric figures* are obtained by placing a restriction on the family of admissible point sets.

transformation in $\mathcal{H}(S,S)$. Klein was aware that any such definition for the relation of equality could only be meaningful, if it satisfied the axioms for an *equivalence* relation. Klein's true genius was in demonstrating the connection between the axioms for an equivalence relation and those for a group:

(1) *Reflexivity* is translated as the existence of an identity function in $\mathcal{H}(S,S)$;

(2) *Symmetry* is translated as the existence of inverse transformations for each member of $\mathcal{H}(S,S)$;

(3) *Transitivity* is analogous to the condition that the collection $\mathcal{H}(S,S)$ is closed under composition of transformations.

Definition: Let S be a set and $\mathcal{G}(S)$ a group of transformations of S. Then two figures F and G in S are said to be ***equivalent*** if and only if $\exists \varphi \in \mathcal{G}(S)$ such that $\varphi(F) = G$.

We can now see the notion of a group action emerging. A geometry is determined by a set M and a group G acting on M. A different geometry is obtained by varying the group G, as we saw in the examples above.

The definition of geometry given above is the one usually extrapolated from Klein's Erlanger Programme, however, this description is incomplete. A complete specification of geometry requires:

(a) a group G acting on a set M, which consists of all those transformations of M which preserve all the properties of the geometric figures in M, that are of interest;

(b) a 'generating' element. Namely the simplest element in the set M, which is the building block of all geometric figures to be considered.

Details of this completion can be found in Yaglom[1988,p118–122].

Invariants

Klein's *Erlanger Programme* of 1872, in which Klein conceived of a geometry to be the *invariant* theory of some transformation[1] group, had an enormous impact on mathematics.

[1]Klein placed such emphasis on invariants, that he even called the geometries, 'the invariant theory of ... transformations'.

Even though his definition for a 'geometry' does not capture all of geometry[1], it nonetheless guided geometric thinking for a long time. These ideas have been carried beyond mathematics into physics, where one is interested in 'physical laws' that need to be expressed in a coordinate invariant manner. The importance of invariants are discussed by Weyl[1949].

3·2 ABSTRACT SPACES

As mentioned earlier, a space is distinguished form a mere collection of points, by virtue of some concept that binds the points together. For example, a Euclidean space is formed by adding a distance function, '$\|\langle\ ,\ \rangle\|$', to the set $S = \{\langle x,y\rangle \mid x,y \in \mathbb{R}\}$. Then, we may consider the point $q \in S$ to be *nearer* to the point $p \in S$ than $r \in S$, if $\|\langle p,q\rangle\| \leq \|\langle p,r\rangle\|$. Thus the distance between points tells us how *close* or *near* the points are to each other. This is tantamount to assigning position to points of S, *relative* to one another, (the notion of position was so important that Leibniz named the subject, ***Analysis Situs*** – meaning *position analysis*[2]).

3·2·1 Bolzano's Contribution to Geometry and Topology

Bernard Bolzano may be considered as the most underrated mathematician to date, the most likely reason for this being that his ideas were simply to far ahead of their time.

Bolzano was a prolific writer, and was primarily interested in issues concerning the foundations of mathematics[3]. The particular article which is of concern to us here is one written by Bolzano in 1817 concerning the definitions of what he called *spatial objects*[4]. What is remarkable about this article is that it predates the commonly cited references on

[1]'algebraic geometry' for example does not come under this scheme.

[2]It is common to find Gauss being given the credit for the discovery of the name, *Analysis Situs*.

[3]There are only a few works which have been translated into English. For a readable account of his work during the period 1804 to 1817, see Russ[1980].

[4]Where a ***spatial object*** is to be considered as a system or collection of points (finite or infinite).

abstract spaces, and topological concepts such as neigbourhoods[1], by some *fifty* years.

It is evident from this paper that Bolzano must have been one of the first mathematicians to have thought about an *abstract space* as we know it today. The reasons for this conjecture are as follows. The first article that was ever published by Bolzano was his 1804 work on elementary geometry, which according to Johnson[1977] contained the "seeds of Bolzano's later geometrical ideas and some of the principles of his mathematical philosophy". In this short article Bolzano is already very cautious about the composition of the straight line, by expressing the fact that there must be something "*qualitatively* different from a mere system of points"[2]. Which he stresses again in his 1817 paper:

> *"... we must look not only at the set of points but also at the way they are put together", Russ[1980,p.154] citing Bolzano[1817].*

There is no direct statement by Bolzano on what constitutes an abstract space, however this is not altogether unreasonable, since the works mentioned predate that of Cantor by at least sixty years, and those of Fréchet by some ninety years.

3·2·2 The early topologists

In view of the discussion on Bolzano's contributions, it is difficult to establish the exact beginnings of topology. However, it is known that Gauss played a significant role in its development. Gauss' enthusiasm for the topic was enough to generate an interest in others. For example, Johann Listing[3] (a fellow professor) was encouraged by Gauss to publish his 'Vorstudien' in 1847 (Listing[1847]), and by the time Riemann arrived at Göttingen, topology was already an established branch of mathematics.

---

[1]According to Russ[1980], the topological concept of a neighbourhood, appears here for the first time.

[2]Russ[1980,p.154]. The language which Bolzano uses to describe his ideas involve such notions as sets, and relationship between members of the sets, indicate he must have developed some of these notions.

[3]Although Listing was one of the earliest workers in the field, he is now almost completely forgotten.

There is a body of evidence to suggest that Gauss had a considerable influence on the young Riemann, and by all accounts it was Gauss who guided Riemann into topology. Nevertheless, what is less well known is that Riemann was a member of Listing's seminar in 1850, who undoubtedly also encouraged Riemann's interest in the field.

Listing not only influenced Riemann, but also had a profound effect on Peirce[CP], who wrote: *"There is no possible question that the most important topical investigation that has ever yet been conducted, – it is probably the most important that ever will be undertaken, – is that of Johann Benedict Listing ..."*.

3·2·1 Klein's view of topology

Topology may be described informally, as a subject which is concerned with those properties of geometric figures that remain invariant when the figures are deformed in any way that does not create new points, or fuse existing ones. The important characteristic which these transformations are required to possess is the ability to carry *nearby* points into *nearby* points; a property called *continuity*. A continuous bijective transformation whose inverse is continuous is called a ***topological transformation***.

In 1872 Klein defined a topology as a *geometry* which recognizes only the *continuity* properties of figures. At that time he took for granted the notion of a *space*, which although justifiable then, soon became a problem. As topology progressed, so did the need for a definition of a *space*, which needed to be specified in a manner that retained its essential characteristics.

As mentioned earlier a space is distinguished from a mere collection of points by virtue of some concept that binds the points together. For example, in Euclidean space the distance between the points tells us how close the points are to each other. Whilst geometers consider such notions as *nearness* to be the essential characteristics for a space[1], analysts

[1]The notion of 'limit point' was introduced into topology by Riesz in 1906, and taken up again by Kuratowski, who found it convenient to state his axioms in terms of the *closure* function defined on the power set of the space. See Eisenberg[1974].

consider the notion of *limit points* for the sets in the space as the essential.

3·2·2 Fréchet's notion of an abstract space

In 1837 Dirichlet gave a definition of a function, in which he separated the 'rule of association' from the range and the domain, implicitly assumed to be subsets of the reals. It was Fréchet[1906][1] who made the conceptual jump from the subsets of the reals to an arbitrary set of elements. According to Manheim[1964], it was Fréchet's investigations which resulted in the notion of an *abstract space* considered as a set of objects (the *points* of the space) together with a set of relations among these points.

3·2·3 Metric spaces

The axioms for a *metric space,* which capture the intuitive notion of a distance were introduced into mathematics for the first time by Fréchet. In this section I shall describe some properties of such spaces.

Definition: Let $\langle S,d\rangle$ be a metric space, and p a point of S. Then an ***open ε-ball*** of p is defined by: $N_p(\varepsilon) = \{q \mid d(p,q) < \varepsilon\}$.

A subset N of S is called a ***neighbourhood*** of p if there exists an $\varepsilon > 0$ such that $N_p(\varepsilon) \subseteq N$.

Intuitively we can think of a neighbourhood N of a point $p \in \langle S,d\rangle$, as the collection of points in S that are sufficiently 'near' to p.

Connected sets

Connectedness is an important property of a domain and can be formally defined as follows:

Definition: Given a metric space $\langle S,d\rangle$ a finite collection of points $\{a_1, \ldots, a_n\}$ is called an ***ε-chain*** if $d(a_1,a_2) < \varepsilon$, $d(a_2,a_3) < \varepsilon$, ..., $d(a_{n-1},a_n) < \varepsilon$. Two points c and d in S are said to be

[1]Manheim[1964, p.116].

ε–connected in S if there exists an ε–chain $(a_1, \ldots, a_n)$ contained in S and such that $c = a_1$ and $d = a_n$. If c and d are ε–connected for every $\varepsilon > 0$, then we say that they are ***well–chained.*** A metric space is said to be ***connected*** if every pair of its points in a metric space are well–chained.

Theorem 1: If X is a connected space having at least two distinct points, then X contains an uncountable number of points (i.e. the cardinality of X is at least $\aleph_1$).

Proof: See Eisenberg[1974].

3·3 TOPOLOGICAL SPACES

At the beginning of this century a new sort of topology (called ***general topology*** or ***point set topology***) began to emerge, considered as a generalization of geometry based entirely on set theory. Lefschetz[1949,p3] states that topology

> *"... begins where sets are implemented with some cohesive properties enabling one to define continuity. The sets are then called topological spaces. Thus topology is a branch of general set theory, the creation of Georg Cantor. That his ideas have led to a profound influence in all mathematics is well known. In topology this influence has been decisive."*

3·3·1 Neighbourhood structures

According to Manheim[1964], Fréchet, Riesz and Weyl were instrumental in *advancing* the notion of an abstract space. However it was Felix Hausdorff who saw how to generalize the intuitive notion of *nearness* into the mathematically more precise concept of a neighbourhood for these abstract spaces[1], thereby providing the generality and precision necessary to establish general topology as a separate discipline. In 1918 Fréchet published his investigations into abstract spaces, which by now included the remarkable *V-spaces.*

[1]I disagree with Manheim[1964], who claims that Hermann Weyl was the first person to mention the concept of a *neighbourhood*, (in 1913). As I have already mentioned, the evidence indicates that Bolzano had already used the concept of a neighbourhood in his 1817 article on geometry.

Definition: A ***V-space*** (or ***Fréchet V-space***) is a set X together with a function

$$\mathcal{N}: X \to \mathbb{P}(\mathbb{P}(X)) \quad \text{defined} \quad \text{by } x \to \mathcal{N}_x$$

such that[1] $\mathcal{N}_x \neq \varnothing$ for *any* $x \in X$. The elements $\mathcal{N}_x$ are called the ***neighbourhoods*** of x, and the V-space is denoted by '$\langle X, \mathcal{N} \rangle$'.

V-spaces are the most general *topological structures* that have been studied. In many respects it is surprising that one actually obtains non-trivial results from such a simple structure.

Definition: Let $\langle X, \mathcal{N} \rangle$ be a V-space, such that $\mathcal{N}$ satisfies the following

(N1) $x \in N_x$ for all $N_x \in \mathcal{N}_x$;

(N2) the intersection of any two neighbourhoods of a point belongs to $\mathcal{N}_x$; and furthermore the intersection contains a neighbourhood of that point. Formally: given $N_x, N'_x \in \mathcal{N}_x$, there exists $N^*_x \in \mathcal{N}_x$ such that $N^*_x \subseteq N_x \cap N'_x \in \mathcal{N}_x$.

Then $\mathcal{N}$ is called a ***(topological) neighbourhood system***[2] for X, and the V–space a ***neighbourhood structure.***

Definition: A neighbourhood structure $\langle X, \mathcal{N} \rangle$ is said to be ***Hausdorff***, if there exists a pair of disjoint neighbourhoods for every distinct pair of points, p and q in X.

3·3·2 Formalisation of the concept of nearness

The fundamental notion of *nearness* is itself encouched in terms of the mathematically more precise notion of a neighbourhood.

Let A be a subset of the neighbourhood structure $\langle X, \mathcal{N} \rangle$. Then a point p in X is said to be:

(1) ***near*** A (denoted by '$p\delta A$'), if every neighbourhood of p contains a point of A.

---

[1]This restriction on $\mathcal{N}$ implies that $\mathcal{N}_x$ is not empty, but this does not exclude the empty set from belonging to $\mathcal{N}_x$.

[2]These are not Hausdorff's original axioms, for these see Hausdorff[1914].

(2) ***near a sequence*** $\langle p_n \rangle$ if either $p = p_n$ infinitely often or p is near the set of other values of the sequence.

(3) a ***limit point*** of A if every neighbourhood of p contains a point of A other than p itself. The set of all limit points of a set A is called the ***derived set*** of A.

These ideas were first introduced by Frigyes Riesz in 1908, and are described mainly in terms of metrics in Cameron[1974].

Definition: Let $f: \langle X, \mathcal{N} \rangle \rightarrow \langle Y, \mathcal{M} \rangle$ be a function between two neighbourhood structures. Then f is said to be ***continuous at c*** in X, provided that for all subsets $A \subseteq X$, if c is near A then f(c) is near $f^{\rightarrow}(A)$; symbolically[1]: $\forall_{A \subseteq X} (c \delta A \rightarrow f(c) \delta f^{\rightarrow}(A))$. We say that the function f is ***continuous*** on X if it is continuous at every point of X.

A continuous invertible function between two neighbourhood structures whose inverse is also continuous is called a ***homeomorphism***[2].

Definition: Let A and B be subsets of a neighbourhood structure $\langle X, \mathcal{N} \rangle$. Then:

(1) the subset A is called ***open*** if every point in A is not near the complement of A. Formally, the subset A is open if for each $x \in A$ there exists a neighbourhood $N_x \in \mathcal{N}$ such that $N_x \subseteq A$.

(2) the subset B is called ***closed*** if B contains all its limit points.

(3) the ***interior*** of A consists of all those points in A which are not near the complement of A (denoted by 'A°').

(4) the ***boundary*** of A consists of all those points near to both A and its complement (denoted by '∂A').

(5) the ***closure*** of A consists of A plus all those points near A (denoted by '$A^{\bullet}$').

The definition of a neighbourhood system given in the previous section differs from those presented in Mendelson[1973] and Eisenberg[1974]. However, the two systems can be

[1]See Appendix A for notation.

[2]Homeomorphisms are important because they induce a one-to-one correspondence between the open sets in the domain with those in the codomain.

made to correspond as follows: given a neighbourhood structure $\langle X,\mathcal{N}\rangle$ we can then define a collection of new sets: $n_x = \{V \mid \text{for some } N\in\mathcal{N}, x\in N\subseteq V\}$, which correspond to the neighbourhoods in Mendelson[1973] and Eisenberg[1974].

Definition: Given a neighbourhood structure $\langle X,\mathcal{N}\rangle$, the collection of all open sets denoted by 'τ' is called a ***topology*** on X generated by $\mathcal{N}$, denoted by '$\langle X,\tau\rangle$'. The structure $\langle X,\tau\rangle$ is called a ***topological space.***

Examples 2: of topological spaces

1. The ***discrete*** topology: A set X becomes a topological space if we allow *all* nonempty sets to be neighbourhoods.
2. The ***indiscrete*** topology: A set X becomes a topological space by stipulating that the only neighbourhood is the set X itself.
3. The ***standard*** topology on the real line. We obtain this topology by taking the collection of all open intervals to be the neighbourhoods.

3·3·3 Further topological concepts

Topological spaces defined by the family of open sets

In order to lay the foundations for a systematic study of topology, it is convenient to use the *invariant* concept of an *open set* as a starting point.

Definition: A set X, together with a family of subsets τ, is called a ***topological space*** if:

(O1) $\emptyset\in\tau$, and $X\in\tau$;

(O2) if $A_i\in\tau$ for all $i\in I$, then $\bigcup_{i\in I} A_i\in\tau$;

(O3) if $A\in\tau$ and $B\in\tau$, then $A\cap B\in\tau$.

The elements of τ are called the ***open sets*** of X, and the elements of X are called the ***points*** of the topological space $\langle X,\tau\rangle$.

A function, $f:\langle X,\tau\rangle \to \langle Y,\tau'\rangle$ between two topological spaces is said to be ***continuous*** (with respect to the topologies τ, τ') if $U\in\tau$ then $f^{\leftarrow}(U)\in\tau'$.

Bases & Subbases

Definition: Let $\langle X,\tau\rangle$ be a topological space. A collection $\mathcal{B}\subseteq\mathbb{P}(X)$ is called a ***base*** for $\langle X,\tau\rangle$ if it satisfies the following conditions:

(1) if $B\in\mathcal{B}$ then $B\in\tau$;

(2) if $C\in\tau$ then there exists some collection $\{B_i \mid B_i\in\mathcal{B}\}$ such that $C=\bigcup_i B_i$.

Definition: Let $\langle X,\tau\rangle$ be a topological space. A collection $\mathcal{C}\subseteq\mathbb{P}(X)$ is called a ***subbase*** for $\langle X,\tau\rangle$ if and only if finite intersection of members of $\mathcal{C}$ form a base for τ.

The entities which I have chosen to call topological neighbourhoods are identical to what others call a base, as is shown by the theorem below.

Theorem 2 : Let X be a set. Let $\mathcal{B}$ be a collection of subsets of X having the properties (N1) and (N2). Then there exists a unique topology τ on X of which $\mathcal{B}$ is a base.

Proof: See Eisenberg[1974].

Generated topologies

One way of generating a topology from a family $\mathcal{A}$ of subsets of a set X is by treating $\mathcal{A}$ as a subbase, thereby obtaining a unique topology on X called the topology ***generated*** by $\mathcal{A}$.

Alternatively, given a function $f:X\rightarrow\langle Y,\tau'\rangle$ from a set X to a topological space $\langle Y,\tau'\rangle$, we may wish to define a topology on X such that the function f will be continuous. The simplest way to achieve the construction of such a topology is as follows.

Definition: Let $f:X\rightarrow\langle Y,\tau'\rangle$ be a function from a set X to a topological space $\langle Y,\tau'\rangle$, then the family of subsets $\tau=\{f^{\leftarrow}(U) \mid U\in\tau'\}$ generates a topology on X called the topology ***induced*** by f. Similarly, for a function $f:\langle X,\tau\rangle\rightarrow Y$, the induced topology on Y is given by the family of subsets $\tau=\{U\in\mathbb{P}(Y) \mid f^{\leftarrow}(U)\in\tau\}$.

Example 3: Consider any subset Y of a topological space $\langle X,\tau\rangle$. Then the topology induced by the insertion function **I**, namely the family of subsets given by:

$\tau_Y = \{I^{\leftarrow}(U) = U \cap Y \mid U \in \tau\}$, is called the ***relative topology*** on Y, and the topological space $\langle Y, \tau_Y \rangle$ is called a ***subspace*** of $\langle X, \tau \rangle$.

Isolated points and density

It is not difficult to show that in a metric space a *limit point* p of A (for $A \subseteq \langle X,d \rangle$ and $p \in X$) actually contains infinitely many distinct points of A in each of its neighbourhoods. Intuitively, points of A different from p can be considered to *accumulate* or *cluster* about p. *Isolated points* for a subset A of a topological space $\langle X, \tau \rangle$, can be viewed as the antithesis of limit points, having a neighbourhood with no other points of A than itself.

<u>Definition</u>: Let A be a subset of the topological space $\langle X, \tau \rangle$, and $p \in A$. Then p is called an ***isolated point*** of A if p is not a limit point of the rest of the set A, in other words $p \notin (A - \{p\})^c$.

We can see from the definition above that a point $p \in A$ is an isolated point of A if and only if the singleton set $\{p\}$ is *open* in the subspace A.

<u>Definition</u>: A subset D of a topological space $\langle X, \tau \rangle$ is said to be ***dense*** in $\langle X, \tau \rangle$ when D intersects each non-empty open subset of $\langle X, \tau \rangle$.

Informally, if a subset D of a topological space $\langle X, \tau \rangle$ is dense in $\langle X, \tau \rangle$, then there are points of D arbitrarily close to each point of $\langle X, \tau \rangle$. In other words

D is dense in $\langle X, \tau \rangle$ if and only if $D^\circ = X$.

<u>Definition</u>: A subset D of a space $\langle X, \tau \rangle$ is said to be ***perfect***[1] if D is closed and every point of D is a limit point of D.

Note how a perfect set does not contain any isolated points.

[1]This property is sometimes called –misleadingly– ***dense-in-itself***.

Compactness

Compactness is one of the two most important properties that a topological space can possess. It can be thought of as the topological analogy of 'bounded' or 'finiteness', and it is this property which enables one to replace problems involving infinite sets with those involving only finite ones.

The notion of a compact space can only be defined formally using the notion of a *cover*, which I shall now define.

Definition: A family of sets $\mathcal{A} = \{A_i\}$, is said to ***cover*** the set A (or be a ***cover for*** A), if A is contained in the union of the members of $\mathcal{A}$, i.e. $\mathcal{A} \subseteq \bigcup_i A_i$. In other words each $a \in A$ belongs to some $A_i \in \mathcal{A}$.

Definition: A subset A of a topological space $\langle X,\tau \rangle$ is ***compact*** if every cover $\mathcal{A}$ for A consisting of open sets is reducible to a finite cover of open sets.

3·3·4 Connectedness

Connectedness is the second important topological property. Intuitively, one thinks of a space as being *connected* if it is in one piece, in other words the space can not be divided into two pieces unless the pieces are *near* each other.

We call two subsets A and B of a topological space $\langle X,\tau \rangle$ ***separated*** if

(1) A and B are disjoint (i.e. $A \cap B = \varnothing$);

(2) A and B do not contain any points which are near the other (i.e. $A \cap B^c = \varnothing$ and $A^c \cap B = \varnothing$).

Definition: Let C be a subset of the topological space $\langle X,\tau \rangle$. Then C is called ***connected*** if there *cannot* be found two subsets $A,B \in \tau$ such that $C \cap A$ and $C \cap B$ are disjoint non–empty sets whose union is C.

For example, the natural numbers $\mathbb{N}$ form a connected topological space when given a topology consisting of the open sets, $\{x \in \mathbb{N} \mid 1 \leq x \leq n\}$.

Observe that separation of sets is a much stronger requirement than simple disjointness. For example, consider the usual topology on the real line $\mathbb{R}$, then the two sets (a,b] and (b,c] are disjoint but are not separated. Moreover, any subset C of an arbitrary topological space $\langle X,\tau\rangle$, which is not equal to $\varnothing$ or X and is both open and closed, will disconnect that space.

Theorem 3: Let ι denote the discrete topology on $\{0,1\}$. Then a topological space $\langle X,\tau\rangle$ is connected if and only if the only continuous functions $f:\langle X,\tau\rangle \rightarrow \langle\{0,1\},\iota\rangle$ are the constant functions.

Proof: See Mendelson[1973].

Theorem 4: A topological space is connected if and only if:

(a) it cannot be represented as the union of two non-empty separated sets.

(b) each subset A of $\langle X,\tau\rangle$ which is not equal to $\varnothing$ or X has a non-empty boundary.

Proof: See Eisenberg[1974].

Theorem 5: Let X be a countable connected space. Then each continuous real valued function on X is constant.

Proof: Eisenberg[1974, p.347].

Components

If a topological space is not connected, then intuitively it must decompose into a number of pieces each of which will be connected. These pieces are formally called *components*.

Definition: Let $\langle X,\tau\rangle$ be a topological space. Suppose *Comp* is the relation on X, in which two points of X are related if they belong to a connected subset of X. This is defined formally as: $x\ \mathit{Comp}\ y \leftrightarrow \exists$ a connected subset A of $\langle X,\tau\rangle$ with $\{x,y\}\subseteq A$.

If $a\in X$, then the *Comp*-class of a (denoted by '*Comp*(a)') is called the ***component*** of a in X.

Theorem 6: The binary relation *Comp* on X is an equivalence relation.

Proof: See Mendelson[1973].

An element of the quotient set X/*Comp* (denoted by '*Comp*(X)') is called a ***component*** of X.

Definition: A topological space $\langle X,\tau\rangle$ is said to be ***totally disconnected*** if for each pair of points $p,q\in X$, there exist a pair of subsets $A,B\in\tau$ such that $p\in A$, $q\in B$, $A\cap B=\varnothing$ and $A\cup B=X$.

The components in a totally disconnected space consist of the singletons $\{x\}$.

Examples 4:

(1) The real line $\mathbb{R}$ with the topology generated by the collection of open-closed intervals (a,b] is totally disconnected. This example shows that the phrase 'the reals are connected' is inaccurate because it does not mention the topology – in reality of course it is implicitly assumed that we are dealing with the *standard* topology for the real line, in which case it is *connected.*

(2) The set $\mathbb{Q}$ of rational numbers as a subspace of the real line with the standard topology is totally disconnected.

An application of connectedness in analysis

The *Fixed-point theorem* (a theorem in analysis), states that every continuous function $f:[0,1]\rightarrow[0,1]$ has a point $c\in$ **Dom**(f) such that $f(c)=c$. The proof of this uses the following theorem.

Theorem 7 (***Intermediate value theorem***): Let $f:\langle X,\tau\rangle\rightarrow\mathbb{R}$ be a continuous function on a connected topological space $\langle X,\tau\rangle$. Let $a,b\in X$ with $f(a)<f(b)$, and suppose $c\in\mathbb{R}$ with $f(a)<c<f(b)$. Then there exists some $x\in X$ with $f(x)=c$.

Proof: See Eisenberg[1974] or Mendelson[1973].

From this one can show that if real valued non-constant continuous function exists on a connected topological space $\langle X,\tau\rangle$, then X is *uncountable.*

Metrizability

Any metric space $\langle X,d\rangle$ will ***induce*** a topological space $\langle X,\tau\rangle$ by taking the neighbourhoods to be all the open ε-balls in $\langle X,d\rangle$.

Definition: A topological space $\langle X,\tau\rangle$ is said to be ***metrizable*** if τ is the topology induced by some metric on X.

Theorem 8: If $\langle X,\tau\rangle$ is a countable connected topological space which contains more than one point, then $\langle X,\tau\rangle$ is *not* metrizable.

Proof: See Eisenberg[1974] or Mendelson[1973].

Path connectedness

The concept of connectedness is made more intuitive by the 'geometric' notion of a path in a topological space. As an intuitive example, if we consider the space to be Britain then two places might be considered as path connected if one can walk from one to the other.

Definition: A ***path*** from a point c to a point d in a topological space $\langle X,\tau\rangle$ is a continuous function $\sigma:[0,1] \rightarrow X$ with $\sigma(0)=c$ and $\sigma(1)=d$. Here c is called the ***initial point*** and d the ***terminal point*** of the path σ, and σ is said to be a path ***from c to d*** in X. The subset $\mathrm{Im}(\sigma)$ of X is called a ***curve*** in X.

A subset B of the topological space $\langle X,\tau\rangle$ is called ***path connected*** if for every pair of points $c,d\in X$ there exists a path from c to d which is wholly contained in B. The maximal path connected subsets of the space $\langle X,\tau\rangle$ are called the ***path connected components.***

3·3·5 Localisation

Informally, the concept of localisation of a topological property such as connectedness is the requirement that the property holds for *small* enough open sets, even though it may not hold for the space as a whole.

We say that a property 'P' is *locally* true at a point x in a topological space $\langle X,\tau \rangle$ (or $\langle X,\tau \rangle$ is *locally* P at x), if there is an open set containing x and having property P, or whose closure has the property.

3·4 TOLERANCE GEOMETRY

Tolerance spaces were discovered independently by Christopher Zeeman and Henri Poincaré (who used the term 'physical continuum'). The ideas were placed on a firm mathematical footing by Zeeman[1962, 1965, 1968] and by a graduate student of his called Tim Poston, who used the unfortunate[1] term 'fuzzy geometry' to describe the same ideas.

Poincaré is without doubt one of the greatest mathematicians of the twentieth century. Amongst his many contributions include the development of modern-day topology. Zeeman, a geometer, is famous for his contributions to topology and in particular to dynamical systems and catastrophe theory. Both mathematicians had a similar reason for developing the notion of a tolerance, namely that visual perception is not adequately modelled by mathematical tools of geometry and topology. This is discussed at length in Poincaré[1905,1913], and in Zeeman[1965].

The central idea in tolerance geometry is to capture mathematically the notion of 'indistinguishability' in visual perception, and thereby describe *continuity* as the concept of 'preserving indistinguishability'. Poston[1971] argues indistinguishability is *"a better formalization of the intuitive idea of continuity than is the topological one"*.

[1]Unfortunate, because it sounds as if it is related to Zadeh's fuzzy sets, which it is not.

I shall introduce the following definitions and theorems from Poston and Zeeman which will be needed later[1].

Definition: A ***tolerance space***, $\langle X,\xi\rangle$ is a set X with a symmetric, reflexive relation $\xi \subseteq X \times X$, the ***tolerance*** on X. If $\langle x,y\rangle \in \xi$ (denoted by '$x\xi y$'), then x is said to be ***within ξ–tolerance*** of y, or x and y are ***indistinguishable***.

The collection $\{\langle x,x\rangle \mid x \in X\}$ is denoted by 'δ' and called the ***discrete tolerance***, and the space $\langle X,\delta\rangle$ is called the ***discrete tolerance space***. The set X is the ***underlying set*** of the tolerance space $\langle X,\xi\rangle$ and a ***set-theoretic*** map to or from $\langle X,\xi\rangle$ is a map to or from X.

Definition: Let $f:\langle X,\xi\rangle \rightarrow \langle Y,\eta\rangle$ be a map between two tolerance spaces $\langle X,\xi\rangle$ and $\langle Y,\eta\rangle$. We say that f is a ***tolerance map***, if $x\xi y$ in $\langle X,\xi\rangle$ implies that $f(x)\eta f(y)$ in $\langle Y,\eta\rangle$.
If the converse also holds we call f a ***tolerance embedding***. We call f a ***tolerance homeomorphism*** if the further condition also holds that every point of Y is within tolerance of **Im**(f); in other words, given $y \in Y$ there exists $x \in X$ such that $f(x)\eta y$.

Example 5 (Poston[1971]): Consider the points in your field of vision, and the tolerance of being unable to distinguish them.

Example 6 (Zeeman[1965]): Let $\langle X,\xi\rangle$ be the tolerance space where X is the set of atoms in the page with tolerance one millimetre. Let $\langle Y,\eta\rangle$ be the tolerance space where Y is the set of points on this page with the tolerance one millimetre. Let $f:X \rightarrow Y$ be the function that assigns to each atom the nearest point on the surface of the paper. Then f is a tolerance homeomorphism. This example illustrates the very powerful idea that a finite set X can be homeomorphic to an infinite set Y, to within tolerance[2].

---

[1]In this section a 'map' is a synonym for a 'function'.

[2]This has important for representing information in our brains, which are finite in capacity, but have the power to handle continuous information, which is infinite in quality, to *within some tolerance.*

Definition: Let $f:X \to \langle Y,\eta\rangle$ be a map from a set X to a tolerance space $\langle Y,\eta\rangle$. Define a tolerance ξ on X by defining: $x\xi y$ if $f(x)\eta f(y)$. We call ξ the ***pull-back*** of η and write $\xi = f^{-1}(\eta)$. We describe the process as ***pulling η back across f.*** It follows immediately that f is then a tolerance embedding of $\langle X,\xi\rangle$ in $\langle Y,\eta\rangle$.

Observe that if $f:X \to \langle Y,\eta\rangle$ is *injective*, then $f:\langle X,f^{-1}(\eta)\rangle \to \langle Y,\eta\rangle$ is an *embedding*. If f is an *inclusion*, then $f^{-1}(\eta)$ is the ***subspace tolerance*** on X, and $\langle X,f^{-1}(\eta)\rangle$ is a ***subspace*** of $\langle Y,\eta\rangle$. A tolerance space $\langle X,\xi\rangle$ together with a distinguished subspace Y, is denoted by (X,Y) (or (X,Y,ξ)), and is called a ***tolerance-pair.***

Definition (Zeeman[1962]): We define two tolerance spaces $\langle X,\xi\rangle$ and $\langle Y,\eta\rangle$ to be ***related*** if there exists a relation $\alpha \subseteq X \times Y$, such that $\xi = \alpha \circ \alpha^{-1}$, and $\eta = \alpha^{-1} \circ \alpha$. In other words, $\langle x,x'\rangle \in \xi$ if and only if there is a $y \in Y$ such that both $\langle x,y\rangle$ and $\langle x',y\rangle$ are in α. Similarly for η.

Definition: Suppose $\langle X,\xi\rangle$ and $\langle Y,\eta\rangle$ are tolerance spaces, then $\xi \times \eta$ gives the ***tolerance product*** on the cartesian product $X \times Y$.

Suppose $\langle X,\xi\rangle$ and $\langle Y,\eta\rangle$ are tolerance spaces. Then we may construct a tolerance from ξ^{η} on $\mathcal{F}(X,Y)$ by defining $f\,\xi^{\eta}\,g$ if their graphs in $X \times Y$ are within tolerance under $\xi \times \eta$. Formally we have:

Definition: Two maps $f,g:\langle X,\xi\rangle \to \langle Y,\eta\rangle$ between two tolerance spaces are said to be ***indistinguishable*** if $x\xi y$ implies $f(x)\eta g(y)$. This gives a tolerance on the set $\mathcal{F}(X,Y)$ of tolerance maps from X to Y, and the resulting tolerance space is the ***function space.*** Notice that a map is a tolerance map if it is indistinguishable from itself.

For a tolerance space $\langle X,\xi\rangle$, the set $N(x) = \{y \mid y\xi x\}$, is called a ***t–neighbourhood*** of x in X. The t–neighbourhood of a subset A of X is the union of the t–neighbourhoods of all its points, namely

$$N(A) = \bigcup_{x \in A} N(a).$$

Definition: A tolerance space $\langle X,\xi\rangle$ is said to be ***compact*** if there exists a finite subset $Y \subseteq X$ such that $N(Y) = X$.

Given a tolerance space $\langle X,\xi\rangle$, we can construct a new tolerance Ξ on the power set $\mathbb{P}(X)$ by defining $A \Xi B$ if every point of A is within ξ–tolerance of some point of B and vice versa.

Definition (Zeeman[1962]): A tolerance ξ on X induces a tolerance Ξ on the lattice $\mathbb{P}(X)$ of subsets of X as follows: given $A, A' \subseteq X$, write $A \Xi A'$, pronounced A is ***indistinguishable*** from A′, if $A \subseteq N(A')$ and $A' \subseteq N(A)$. Then the relation Ξ is a tolerance relation on $\mathbb{P}(X)$.

Definition: Let (X,Y) be a tolerance-pair, then define:

(a) the ***interior boundary*** $\partial_X^-(Y)$ of Y in X to be the set $Y \cap N(X-Y)$.

(b) the ***exterior boundary*** $\partial_X^+(Y)$ of Y in X to be the set $N(Y) \cap (X-Y)$.

(c) the ***border*** $\partial_X(Y)$ of Y in X to be the subset: $\{(a,b) \mid a \in Y, b \in X-Y \text{ and } (a,b) \in \xi\}$.

(d) the ***interior*** Y° of Y with respect to X is the subset $Y - \partial_X^-(Y)$.

In the adjacent figure, the symbols '*' and '–' denote the points, and the relationships between the points, respectively.

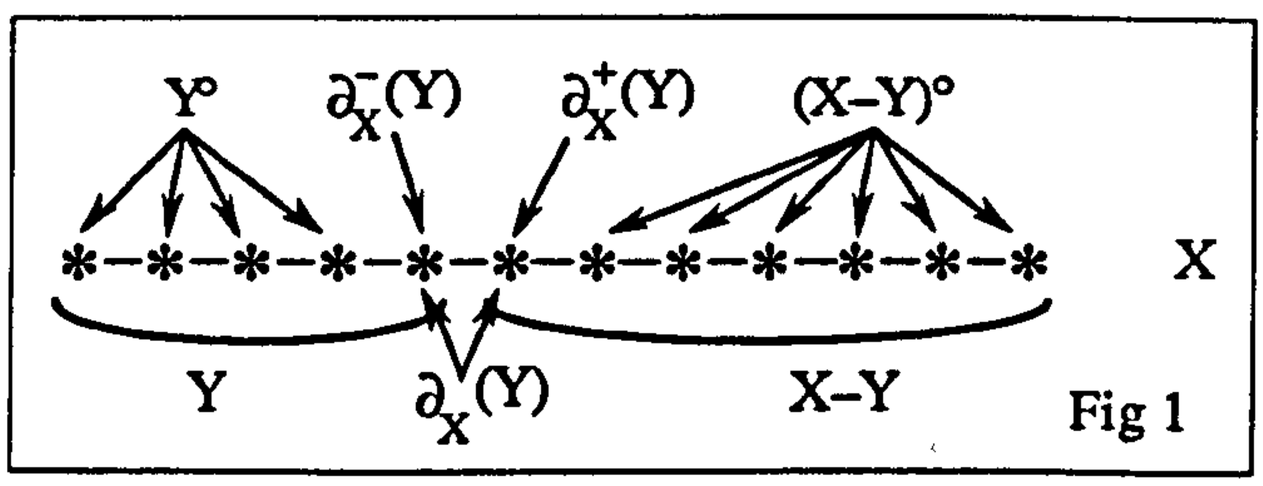


Fig 1

In following Bolzano, I shall use the phrase '***spatial structure***' to mean a V-space together with any suitable axioms[1].

3·4·1 Paths & connectedness in tolerance geometry

Definition: The ***standard tolerance*** $\approx$ on the set $\mathbb{N}_0$ of non-negative integers is the relation $\{\langle m,n\rangle \mid |m-n| \leq 1\}$. Any subset of $\mathbb{N}_0$ will have the subspace tolerance induced

[1]Therefore both topological spaces and tolerance spaces are spatial structures.

from $\approx$ unless otherwise stated. The tolerance space $\langle\{m, m+1, \ldots, n\}, \approx\rangle$ will be denoted by '$[\![m,n]\!]$'.

A ***path of length*** m in a tolerance space $\langle X,\xi\rangle$ is a tolerance map $\phi:[\![0,m]\!] \to \langle X,\xi\rangle$. We say the path ϕ ***begins*** at $\phi(0)$ and ***ends*** at $\phi(m)$, or that ϕ is a path from $\phi(0)$ to $\phi(m)$. The points $\phi(0)$ and $\phi(m)$ are connected by ϕ, and we may write: $\phi(0) \to \phi(m)$.

The strict finiteness of this definition is in marked contrast to the non-denumerability involved in topological paths.

Definition: An ***m-path*** in a tolerance space $\langle X,\xi\rangle$ is a tolerance map $\phi:\langle \mathbb{N}_0,\approx\rangle \to \langle X,\xi\rangle$ in association with a number $m\in \mathbb{N}_0$ such that if $n\geq m$ then $\phi(n)=\phi(m)$. If $m'>m$, m' is *a* bound for ϕ, while m is ***the*** bound for ϕ. A ***path*** in $\langle X,\xi\rangle$ is a tolerance map $\phi:\langle \mathbb{N}_0,\approx\rangle \to \langle X,\xi\rangle$ such that a bound exists, without a specific choice of bound.

Definition: A tolerance space $\langle X,\xi\rangle$ is ***connected*** if for all pairs $(x,x')\in X\times X$, there exists a path $x \to x'$. Given $x_0\in X$, the ***component*** of x_0 in X is the set:

$$\{x\in X \mid \exists \text{ a path } x_0 \to x\}.$$

Evidently, ρ = 'connected by a path of some length' is an equivalence relation on X, and the component of x_0 is its ρ-equivalence class. A non-empty ρ-equivalence class is called a ***component*** of X.

Theorem 9 (Poston[1971], p.27): A tolerance space $\langle X,\xi\rangle$ is connected if and only if there is no tolerance epimorphism (i.e. surjective tolerance mapping),

$$\langle X,\xi\rangle \to \langle\{0,2\},\delta\rangle.$$

Proof:

$\leftarrow$: Suppose $\langle X,\xi\rangle$ is connected, and consider a set-theoretic epimorphism

$$\eta:\langle X,\xi\rangle \to \langle\{0,2\},\delta\rangle.$$

As η is a surjection, we may take $x_0\in\eta^{-1}\{0\}$, $x_2\in\eta^{-1}\{2\}$. Since $\langle X,\xi\rangle$ is connected, there exists a tolerance map $\phi:[\![0,m]\!] \to \langle X,\xi\rangle$, for some m, such that $\phi(0)=x_0$, $\phi(m)=x_2$.

Then if $K = \{n \in [\![0,m]\!] \mid \eta \circ \phi(n) = 0\}$, we have $K \neq \varnothing$, and $m \notin K$.

Thus if $\kappa = \max(K)$, then $\{\kappa, \kappa+1\} \subseteq [\![0,m]\!]$.

Therefore, since ϕ is a tolerance map and $\kappa \approx \kappa+1$, implies $\langle \phi(\kappa), \phi(\kappa+1) \rangle \in \xi$. But,

$\eta \times \eta \langle \phi(\kappa), \phi(\kappa+1) \rangle = \langle \eta \circ \phi(\kappa), \eta \circ \phi(\kappa+1) \rangle = \langle 0,2 \rangle \notin \delta$.

Therefore η is not a tolerance map. Thence, if $\langle X,\xi \rangle$ is connected, there does not exists a tolerance epimorphism: $\langle X,\xi \rangle \to \langle \{0,2\},\delta \rangle$.

$\rightarrow$: Suppose $\langle X,\xi \rangle$ *not* connected. Then $\exists x_0, x_2 \in \langle X,\xi \rangle$ with no path connecting them. Let P_0 be the component in $\langle X,\xi \rangle$ of x_0. Then

$$(x \in P_0) \;\&\; (\langle x,x' \rangle \in \xi) \rightarrow (x' \in P_0),$$

for $[\![0,1]\!] \to \langle X,\xi \rangle$, is a path from x to x′, so we have $x_0 \rho x$, $x \rho x'$, hence $x' \in P_0$.

Conversely, $(x \in P_0) \;\&\; (\langle x,x' \rangle \in \xi) \rightarrow (x' \in P_0)$.

Now define:

$$\eta : \langle X,\xi \rangle \to \langle \{0,2\},\delta \rangle$$
$$x \to 0 \text{ if } x \in P_0$$
$$x \to 2 \text{ if } x \notin P_0.$$

Then η is a tolerance map, for

$$\begin{aligned} \langle x,x' \rangle \in \xi \quad &\rightarrow \quad (x \in P_0 \;\&\; x' \in P_0) \text{ or } (x \notin P_0 \;\&\; x' \notin P_0) \\ &\rightarrow \quad \eta(x) = \eta(x') \\ &\rightarrow \quad \langle \eta(x), \eta(x') \rangle \in \delta, \end{aligned}$$

and η is a surjection, for $\eta(x_0,x_2) = \{0,2\}$.

Hence if $\langle X,\xi \rangle$ is not connected, there exists a tolerance epimorphism $\langle X,\xi \rangle \to \langle \{0,2\},\delta \rangle$.

Otherwise, if there does not exist a tolerance epimorphism $\langle X,\xi \rangle \to \langle \{0,2\},\delta \rangle$, then $\langle X,\xi \rangle$ is connected.

QED.

3·4·2 Metric properties of tolerance spaces

Definition (Poston[1971]): For any tolerance space $\langle X,\xi\rangle$ define the ***hop–metric***

$$d: X\times X \to \mathbb{N}_0 \cup \{\omega\}$$

$$\langle x,x'\rangle \to \min\{m \mid \exists \text{ an m-path } x \to x'\}$$

$$\omega \text{ if there is no path } x \to x'.$$

We call $d(x,x')$ the ***hop–distance*** from x to x′.

The reason why $d(x,y)$ is called a *hop–metric*, is because the function d satisfies the axioms **(M1)** – **(M3)** of §3·1·1.

Any tolerance map $f:\langle X,\xi\rangle \to \langle Y,\eta\rangle$ preserves m-paths for all m, hence for all $x,x'\in X$ we have $d(f(x),f(x')) \le d(x,x')$. Furthermore, if f is a tolerance homeomorphism its inverse also has this property, so that $d(f(x),f(x')) = d(x,x')$.

3·4·3 Intermediate value theorem

Definition: A ***totally ordered tolerance space*** $\langle X,\xi,>\rangle$ is a tolerance space $\langle X,\xi\rangle$ with a total ordering > on it such that:

$$a>b\,,\ b>c\,,\ a\xi c \ \to\ a\xi b\,,\ b\xi c\,.$$

Definition: A ***semiordering*** on a set X is a pair of binary relations P, I on X such that for all $a,b,c,d\in X$:

S1. Exactly one of aPb , bPa , aIb obtains

S2. aIa

S3. aPb , bIc , cPd → aPd

S4. aPb , bPc , bId → not both aId and cId

Observe that I is a tolerance relation.

Definition: A ***weak semiordering*** on a set X is a pair of binary relations P,I on X such that for all $a,b,c\in X$:

WS1. Exactly one of aPb, bPa, aIb obtains

WS2. aIa

WS3. aPb , bPc → aPc

Theorem 10: If $\langle X,\xi\rangle$ is a connected tolerance space, $\langle Y,\eta,>\rangle$ is a totally ordered tolerance space, and (Z,I,P) is a weakly semiordered set, then if

$f:X \to Y$ and $g:X \to Z$ are tolerance maps, we have for any $x,x' \in X$:

(i) if $f(x)>y_0>f(x')$ then $\exists x_0 \in X$ such that $f(x_0)\eta y_0$.

(ii) if $g(x)Pz_0Pg(x')$ then $\exists x_0' \in X$ such that $g(x_0')Iz_0$.

Proof: See Poston[1971, p.34].

3·4·4 Tolerable topology

A topological space $\langle X,\tau\rangle$ is *tolerable* if there exists a tolerance ξ on X such that the set $\{N(x) \mid x \in X\}$ of t-neighbourhoods serves as a *sub-basis* for the set τ of open sets of X. If a space is not tolerable, it is said to be ***intolerable***.

As an example, consider the distance ($\|x-y\|$) between two points in the Euclidean plane $\mathbb{R}^2$, defined by $\sqrt{\{(x_1-y_1)^2 + (x_2-y_2)^2\}}$. Then the ***natural tolerance*** on $\mathbb{R}^2$ is given by $x\xi y \leftrightarrow \|x-y\|<1$. The set of t-neighbourhoods of points of $\mathbb{R}^2$ form a sub-basis for the usual topology: every open set in $\mathbb{R}^2$ is a union of finite intersections of open intervals of length 2, and every such union is open. Moreover, the hop distance between two points is precisely their Euclidean distance rounded up to the next integer.

3·4·5 Poston's description of motion

I would like to conclude this chapter with an example of Newtonian motion, as presented by Poston in his thesis. This *intuitive* example demonstrates the simple elegance of this theory.

In Newtonian mechanics we often describe motion of a particle by giving its position at each moment of time, with the requirement that this be continuous. This is formalised by a continuous map:

$$p: \text{TIME} \rightarrow \text{SPACE}$$

$$t \rightarrow p(t) = \text{position at time } t.$$

We may convert this into tolerance geometry, by defining a tolerance on SPACE and a tolerance on TIME, with the proviso that the particle occupies indistinguishable positions at indistinguishable times.

Extend this notion to the motion of a *body* (call it 'B') – an *object* – rather than a point mass in space. For the sake of simplicity assume that the body is one-dimensional, consisting of points in a line, where each point is indistinguishable from its two nearest neighbours on either side. The position of the body B in space S consists of a map from B to S, therefore the collection of all its positions in space will be given by the function space $\mathcal{F}(B,S)$. The *motion* of B will consist of a map:

$$m: \text{TIME} \rightarrow \mathcal{F}(B,S).$$

If the motion is to be perceived as continuous we must define a tolerance both on TIME and on $\mathcal{F}(B,S)$, with the requirement that m is a tolerance map. The tolerance structure on TIME and SPACE will be homeomorphic images of the integers, where $x\xi y \leftrightarrow |x-y| \leq 1$.

Suppose the position of B at time τ is represented by Fig 2.

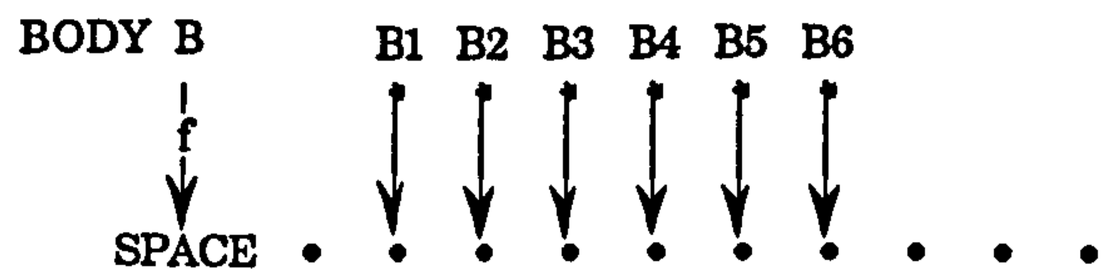


Fig 2

Then since motion is described as a map for each moment such that maps specifying positions at indistinguishable moments are indistinguishable, as *maps*, means that the position at time $\tau+1$ *cannot* be that represented by Fig 3.

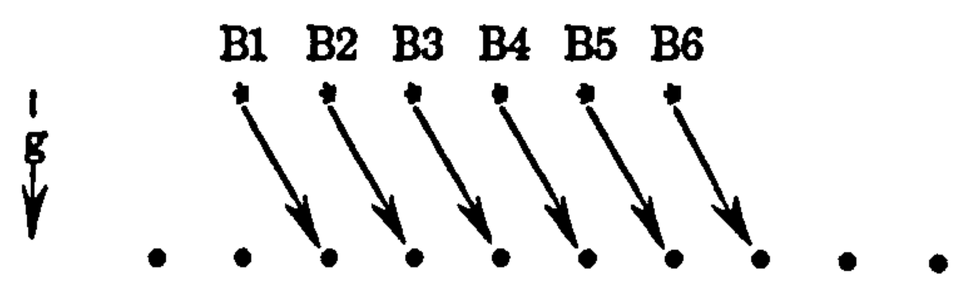


Fig 3

However,

$\langle f,g\rangle \notin \xi^{\xi}$ since $f(B5)\xi g(B6)$.

The position g is connected to f, but only via a path shown in Fig 4.

Thus the body B takes six hops to travel a distance that a particle could go in one, and so moves at only a fraction of the speed.

Notice while the body is in motion, the space it occupies is one hop shorter.

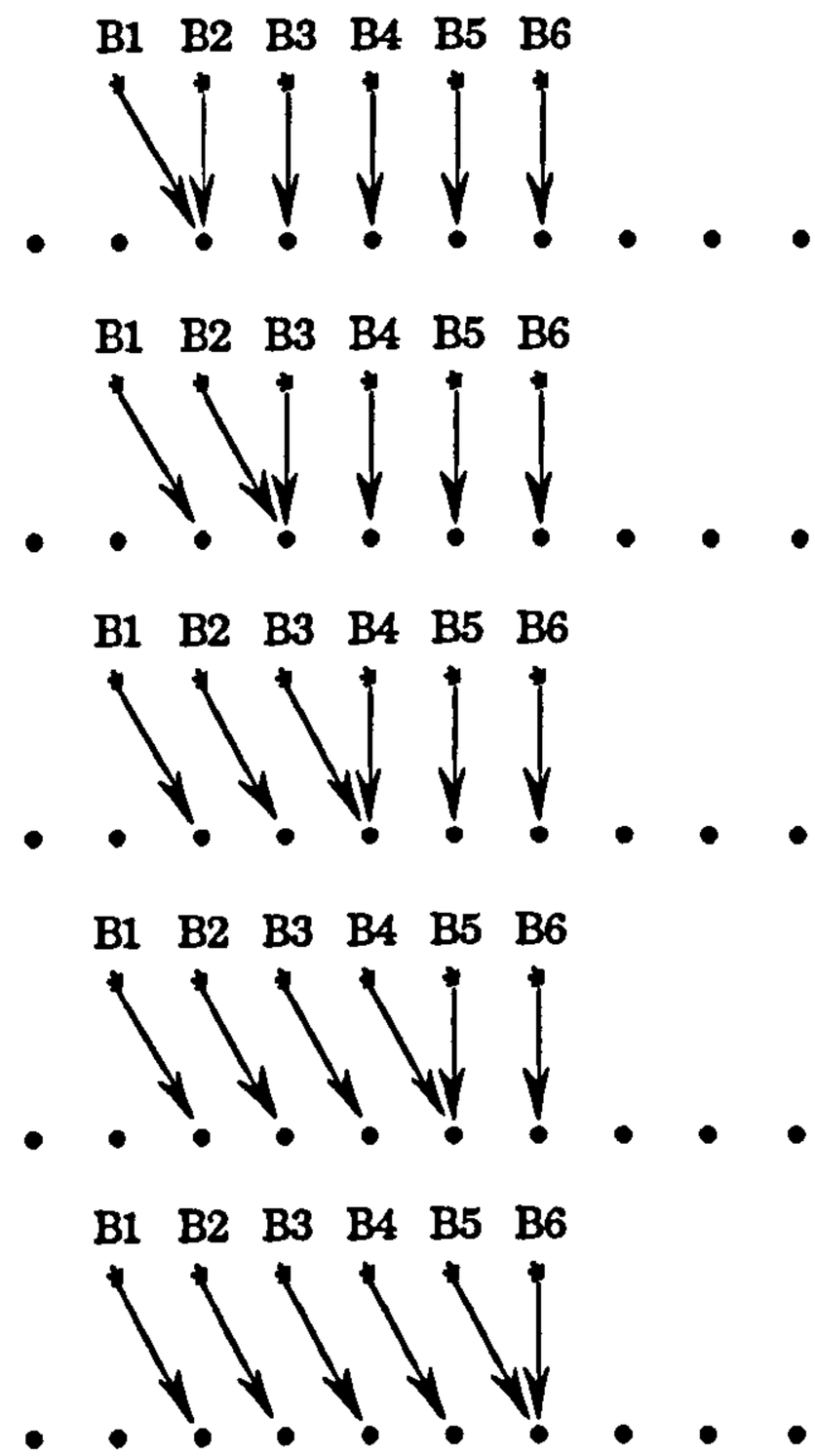


Fig 4

3·5 SUMMARY

In this chapter I have presented the notions of geometry as adopted by Klein. He used the idea of a group acting on a set to define the notion of 'equivalence' for two subsets of a set (namely, that the subsets A and B of S are 'equivalent for this geometry if there exists an element φ of the group such that $\varphi(A) = B$). This led Klein to provide the definition of 'geometry' in his Erlanger Programme[1].

[1]Where Klein's view of geometry differs from the classical one taken by Hilbert is in the axioms concerning congruence, which Klein replaces with the appropriate group axioms.

Naturally, an arbitrary set together with a subgroup of transformations will not result in a very worthwhile geometry. Usually the set S will have some structure, (namely it will be a space) and the subgroup will be required to preserve this structure, thus the subgroup will be a subgroup of *automorphisms* rather than just transformations of S.

The important notion of a *space*, considered as a set together with some concept which provides the bond between the elements was also introduced. This leads naturally to the idea that a continuous function was nothing more than a structure preserving function, where the structure preserved is a *spatial structure*. Topology and tolerance geometry were then introduced as two examples of such structures, and discussed at some length.

4 Development of Tolerance Relations

"Thus logic and intuition have each their necessary rôle. Each is indispensable. Logic, which alone can give certainty, is the instrument of demonstration; intuition is the instrument of invention."

Henri Poincaré

Having introduced the concept of a tolerance space in the previous chapter, I shall develop this theory in §4·1 in a manner which parallels the theory of equivalence relations and quotient sets. In this section I shall describe how a tolerance relation leads to a cover, in much the same way that an equivalence relation leads to a partition of the underlying set. Thereby demonstrating the one–to–one correspondence between the collection of tolerance relations and the collection of covers for a given set. This result is similar to the relationship which exists between the collection of equivalence relations and the collection of partitions for a given set.

It is an easy task to characterise a tolerance space, but a much more difficult one to comprehend it intuitively. The similarity between the axioms for a tolerance relation and that of an equivalence relation is misleading[1], the two structures have an opposite effect on the set upon which they are defined. Equivalence relations are used as a classification technique – they partition a set. Tolerance relations, on the other hand, have the effect of 'gluing' the elements together to form a space akin to a topological space.

[1]The connection between equivalence relations and tolerance relations are not discussed by Zeeman or Poincaré.

In the following section (§4·2) I shall use the similarity between a tolerance relation and an equivalence relation to introduce the notion of *t–subset* action on a set, which is a generalisation of a group action on a set. Following Klein, I shall then use this newly developed t–subset action to capture the notion of similarity in geometry. The approach I have adopted here is similar to that taken by Klein, who used a group action to capture the notion of *equivalence.*

Next, I shall discuss the various descriptions for the notion of similarity which has been stated over the years by geometers. However, the definitions formulated to date have often failed to capture the intuitive concept correctly. In this section I shall describe a formulation which overcomes some of these difficulties.

In the penultimate section I discuss how the theory of tolerance relations is used to capture an outstanding problem in AI, namely the problem of formally describing the notion of two objects *touching*. This notion is formulated here in a manner which is intuitive, yet does not lead to philosophical or formal difficulties[1].

The approach in this chapter is *not* in the same spirit as that which may be found in Poston[1971] or Dodson[1974, 1975], both of whom insist in encapsulating their work in terms of category theory. The use of which not only makes the mathematics very abstract by removing all intuition from the subject, it also makes the work inaccessible to some who may have otherwise benefited from it.

4·1 TOLERANCE STRUCTURES REVEALED

<u>Notation</u>: Consider a relation $\mathcal{R} \subseteq X \times Y$ between X and Y. Then $\mathcal{R}$ can be replaced by $\mathcal{R}: X \times Y \rightarrow \{0,1\}$, which has a dual $R: X \rightarrow \{0,1\}^Y = \mathbb{P}(Y)$. The subset R(a) of Y is denoted by 'R^a', and is considered to be identical to the $\mathcal{R}$-class[2] of a. The subset 'R_b' is

[1]See Randell[1991] for details.

[2]See appendix A for definition of $\mathcal{R}$-class.

similarly defined using the other dual[1], $R:Y \rightarrow \mathbb{P}(X)$.

In this section we will assume that whenever a cover Ω is given for a set X, that $\varnothing \notin \Omega$.

4·1·1 Covers and partitions

It is well known[2] that a given partition Π of a set X will induce an equivalence relation $\mathcal{E}_\Pi$ on X. The induced relation being defined by:

$$\langle x,y\rangle \in \mathcal{E}_\Pi \text{ iff } \exists T \in \Pi \text{ such that } x,y \in T.$$

If a partition Π induces an equivalence relation, what sort of relation will a cover[3] Ω induce?

<u>Lemma 1</u>: Let Ω be a cover for a set X. Then the relation $\mathcal{R}_\Omega$ is defined by

$$\langle x,y\rangle \in \mathcal{R}_\Omega \text{ iff } \exists S \in \Omega \text{ such that } x,y \in S,$$

and is a tolerance relation, which I shall call the relation ***induced*** by Ω.

<u>Proof</u>: (1) The relation $\mathcal{R}_\Omega$ is reflexive. The fact that Ω is a cover implies the existence of at least one set $S \in \Omega$ such that $x \in X$ belongs to S, and by the definition reflexivity of $\mathcal{R}_\Omega$ follows.

(2) The relation $\mathcal{R}_\Omega$ is also symmetric for similar reasons.

QED.

There are instances where one may wish to define a tolerance on a cover, without having to define one on the underlying set. For example, we may have a situation where the extensions of all the predicates in our formal language forms a cover for the domain. Then, by the above, we may define two predicates to be indistinguishable without having a prior need for to define indistinguishability between objects.

---

[1]Notice that $R_b = (R^{-1})^b$. This may explain why R_b is sometimes called the inverse-image of $b \in Y$.

[2]See appendix A.

[3]The difference between a partition Π and a cover Ω is that members of a cover are allowed to overlap one another.

Example: Let Ω be a cover for X. Define a tolerance λ on Ω by

$$\text{for } S,T \in \Omega, \qquad S\lambda T \text{ if } S \cap T \neq \varnothing.$$

The only tolerance relation which is defined by Zeeman on a cover is for the special case of the power set, namely $\Omega = \mathbb{P}(X)$. In this case he uses the tolerance ξ on the underlying set X to induce[1] a tolerance on the power set.

So far we have seen how a cover induces a tolerance relation on the domain. We now need to show the converse.

Lemma 2: Let $\langle X,\xi \rangle$ be a tolerance space. Then the set of t–neighbourhoods,

$$N(X) = \{N(x) \mid x \in X\} \text{ forms a cover for X.}$$

Proof: Since the relation ξ is reflexive, there exists at least one (neighbourhood) $M \in N(X)$ for every $x \in X$. Notice that, for every $M \in N(X)$ M is not empty, since M is a t–neighbourhood for some $x \in X$. Therefore N(X) forms a cover for X.

QED.

Theorem 1: To every cover of a set there corresponds a tolerance relation, and vice versa.

Proof: By Lemmas 1 and 2 above.

QED.

4·1·2 Induced relations

The previous section demonstrated that *covers* have the same relationship to tolerance relations, as *partitions* have to equivalence relations[2]. Formally we have:

Definition 1: Let $R:X \rightarrow \mathbb{P}(Y)$ be a relation between two sets X and Y. Then the relation, $\mathcal{K}_R$ is defined on $X \times X$ by

$$\langle x,\bar{x} \rangle \in \mathcal{K}_R \text{ iff } \exists y \in Y \text{ such that } \langle x,y \rangle \in R \text{ and } \langle \bar{x},y \rangle \in R$$

[1]See §3·4.

[2]See the second theorem in §A·5.

and is called the relation ***induced*** on X by R. Two points $x,\bar{x}$ in X, are said to be ***R–associates*** if $\langle x,\bar{x}\rangle \in \mathcal{K}_R$.

Lemma 3: For any $x \in X$, if $R^x \neq \varnothing$, then x is R–related to itself. Moreover, $\mathcal{K}_R$ is reflexive on **Supp**(R).

Proof: If $R^x \neq \varnothing$ then $\exists y \in Y$ such that $\langle x,y\rangle \in R$. By definition of $\mathcal{K}_R$ it follows that $\langle x,x\rangle \in \mathcal{K}_R$.

QED.

Theorem 2: Let $R:X \rightarrow \mathbb{P}(Y)$ be a relation between two sets X and Y, such that **Dom**(R) = **Supp**(R). Then the binary relation $\mathcal{K}_R$ defined by

$$\langle x,\bar{x}\rangle \in \mathcal{K}_R \text{ iff } \exists y \in Y \text{ such that } \langle x,y\rangle \in R \text{ and } \langle \bar{x},y\rangle \in R,$$

is a tolerance relation on **Dom**(R) = X.

Proof: (1): Since **Dom**(R) = **Supp**(R), we have that $R^x \neq \varnothing$ for any $x \in X$. Thus, by Lemma 3, $\mathcal{K}_R$ is reflexive.

(2): The relation $\mathcal{K}_R$ is symmetric, since it is defined in a symmetric manner.

QED.

The condition **Dom**(R) = **Supp**(R) rules out all 'partial' relations, in other words those relations which are not defined everywhere on the domain.

Consider the situation where the relation R is required to satisfy the conditions for a function, namely:

(1) **Dom**(R) = **Supp**(R);

(2) for all $x \in X$, R^x is a singleton set.

Then as we have already seen, condition (1) will generate a tolerance relation $\mathcal{K}_R$ on X. Condition (2) alters the definition for $\mathcal{K}_R$

from: $\langle x,\bar{x}\rangle \in \mathcal{K}_R$ iff $\exists y \in Y$ such that $\langle x,y\rangle \in R$ and $\langle \bar{x},y\rangle \in R$,

to: $\langle x,\bar{x}\rangle \in \mathcal{K}_R$ iff $R(x) = R(\bar{x})$.

The extra condition required for a function thus introduces transitivity into the construction and thus forces the induced relation into an equivalence relation. Therefore the induced relation may be viewed as a direct generalisation of the equivalence kernel.

4·1·3 Projections and V–space structures

Given an equivalence relation $\mathcal{E}$ on a set X, then one may form the quotient set $X/\mathcal{E}$ and construct a canonical projection $p:X \to X/\mathcal{E}$, where every point $x \in X$ is mapped to its equivalence class. It is important to note that to each point $x \in X$ belongs to *one* and only *one* equivalence class. In the case of a tolerance relation (which I shall denote by ξ), one may proceed in a similar manner. Intuitively the set N(X) of t–neighbourhoods corresponds to the quotient set in the case of an equivalence relation. However, in this case each point $x \in X$ may belong to more than one t–neighbourhood.

One may be inclined to mimic the canonical projection (for an equivalence relation) and construct a projection map where each point in X is mapped to its t–neighbourhood,

$$p:X \to N(X) \qquad \text{defined by} \qquad p(x) = N(x).$$

In other words, p(x) is the t–neighbourhoods of $x \in X$. As it turns out this approach is not very helpful, since each point of X may belong to several t–neighbourhoods in N(X). Which means the relationship between the sets X and N(X) is not that of a *function*, as in the case of the equivalence relation, but a *relation*. This relation I shall denote by: $\pi \subseteq X \times N(X)$, and define as

$$\pi:X \to \mathbb{P}(N(X)) \qquad \text{defined by} \qquad \pi(x) = \{M \in N(X) \mid x \in M\}.$$

Informally, the set $\pi(x)$ consists of all neighbourhoods in N(X) which contain x, and may therefore contain many neighbourhoods which are different[1]. This clearly demonstrates that the tolerance space has a spatial structure, that is to say, it is a V–space[2]. In order to

[1]Formally, $\pi(x)$ could be thought of as: $\{N(y) \mid y \in X \text{ and } x \in N(y)\}$.

[2]For the definition of a V–space see §3·3·1 of chapter 3. Observe that in constructing this function, we have only used the property that N(X) forms a cover for the set X.

discuss the nature of the relationship between two neighbourhoods contained in $\pi(x)$ it is necessary to define the notion of second and higher order neighbourhoods.

For a tolerance space $\langle X,\xi\rangle$, the t–neighbourhood of a subset $A\subseteq X$ is defined to be the union of t–neighbourhoods for all its points, namely: $N(A) = \bigcup_{x\in A} N(a)$. Apart from the natural restriction on A (namely that $A \neq \varnothing$), there are no other restrictions. The case where $A = N(a)$ – for some $a\in A$ – I shall denote by '$N^2(a)$', and call the ***second neighbourhood*** of a. Clearly, one may now repeat the process to obtain $N^3(a)$ and higher order neighbourhoods. Second order neighbourhoods for subsets are defined similarly.

One may view the operation of taking neighbourhoods of a subset in a tolerance space as analogous to taking the *closure* of a subset in a topological space. However, this analogy is rather tenuous since the closure operation for a topological space satisfies the property: $A^{cc} = A^c$; which is not generally true in a tolerance space, i.e. $N(N(A)) \neq N(A)$.

Let us now extend our list of axioms by adding transitivity, and suppose that N(y) and N(w) are two different neighbourhoods which belong to $\pi(x)$. Since N(y) and N(w) are different there must be at least one point, z (say), which belongs to one neighbourhood but not the other. For the sake of argument let $z\in N(y)$ and $z\notin N(w)$.

Using the definition of t–neighbourhood together with the following facts,

(i) $N(y), N(w) \in \pi(x)$, and

(ii) $z\in N(y)$ and $z\notin N(w)$;

enables one to derive the following expressions,

(iii) $x\xi y$ and $x\xi w$;

(iv) $z\xi y$ and $z\not\xi w$.

Applying the axiom of symmetry for the relation ξ to the expressions in (iii) gives $y\xi x$, and $x\xi w$. The axiom of transitivity for ξ allows one to derive $y\xi w$. Applying transitivity to $z\xi y$ and $y\xi w$, yields $z\xi w$ which contradicts the second expression in (iv). This

demonstrates that adding the axiom of transitivity to our list of axioms reduces the number of neighbourhoods in $\pi(x)$ to one. This explains why the canonical projection is so useful in the case of an equivalence relation[1]. Let us now consider what happens in the case of a tolerance space.

Theorem: Let $\langle X,\xi\rangle$ be a tolerance space, and $\pi:X \rightarrow \mathbb{P}(N(X))$ the projection as defined above. A neighbourhood $N(y)$ of a point $y\in X$ which belongs to $\pi(x)$ is indistinguishable from the neighbourhood $N(x)$ of the point $x\in X$, when $N(y)$ and $N(x)$ are considered as members of the tolerance space $\langle \mathbb{P}(X),\Xi\rangle$.

Proof: The condition $N(y)\in \pi(x)$ implies $x\xi y$, which by symmetry also implies $y\xi x$. Given an arbitrary point $z\in N(x)$ we have $x\xi z$, from which we may deduce that $z\in N^2(y)$. Hence, $N(x)\subseteq N^2(y)$. A similar argument gives the result $N(y)\subseteq N^2(x)$. Therefore, $N(x)$ and $N(y)$ are indistinguishable in the tolerance space $\langle \mathbb{P}(X),\Xi\rangle$.

QED.

In the case of an equivalence relation we know that if $N(y)\in p(x)$ then $N(y) = N(x)$. In the case of a tolerance relation, if $N(y)\in \pi(x)$ then $N(y)\Xi N(x)$. One may state this in everyday terms by saying that if two neighbourhoods are not the same then they may still be similar. Thus the notion of similarity (which is not in general transitive) is formally expressible as a tolerance relation.

The collection $N(X)$ used above may be replaced by an arbitrary cover Ω, of X. In which case the projection function will become:

$$\pi:X \rightarrow \Omega, \qquad \text{where} \qquad \pi(x) = \{M\in \Omega \mid x\in M\}.$$

Following a similar argument to the one above, one can define a relation $\pi\subseteq X\times\Omega$ where,

$$\pi:X \rightarrow \mathbb{P}(\Omega) \qquad \text{is defined by} \quad \pi(x) = \{M\in \Omega \mid x\in M\}.$$

This clearly demonstrates that a cover can generate a V–space.

[1]Since $\pi(x)$ only contains one element, namely $N(x)$, enables one to identify $\pi(x) = \{N(x)\}$ with $N(x)$, hence reducing $\mathbb{P}(\mathbb{P}(X))$ to $\mathbb{P}(X)$, in this special case.

One may summarise this result by observing that the set of neighbourhoods for a tolerance relation defined on a set X forms a cover for X. One may contrast this with the set of equivalence classes which partition the underlying set.

4·2 TOLERANCE RELATIONS AND GROUP ACTIONS

In the previous section I described how a tolerance relation may be viewed as a *qualitative* geometry, akin to a topology. In the same section I also demonstrated the connection between an equivalence and a tolerance relation. As a result of these discussions one is naturally lead to the following question: is there a connection between tolerance spaces and the notion of geometry, as defined by Klein in his Erlanger Programme? In order to investigate such a conjecture one must return to Klein's original observations concerning equivalence relations and group actions.

The difficulty one faces in such an endeavour is a conceptual one, namely, what is the notion we are attempting to capture? Klein, for example, was attempting to capture the notion of *equivalence* (in geometry), therefore Klein knew that his characterisation would have to satisfy the axioms for an equivalence relation.

For a given set S, Klein conjectured that in a geometrical setting the notion of equivalence would need to be described using some set of transformations $\mathcal{H}(S,S)$ of the underlying set S. What Klein needed to do was to find a structure on $\mathcal{H}(S,S)$ which would satisfy the axioms for an equivalence relation. Klein discovered that a group structure could in principle satisfy all these axioms. An observation which turned out to be instrumental in his definition of a *geometry*[1].

In following Klein, I shall hypothesize that the concept we seek to capture is that of *indistinguishability*, which is characterised by the axioms of reflexivity and symmetry.

[1]Klein expressed the essential characteristics of a geometry in terms of a group acting on a set S. This approach was discussed in chapter 3.

I shall begin by considering a set S and a collection of functions $\mathfrak{S}(S) \subseteq \mathcal{F}(S,S)$. Following the arguments presented in chapter 3, one can see that it is necessary to have the following properties:

(i) $\mathbb{1}_S \in \mathfrak{S}(S)$, (consequence of reflexivity);

(ii) if $\psi \in \mathfrak{S}(S)$ then $\psi^{-1} \in \mathfrak{S}(S)$, (consequence of symmetry).

The third condition which Klein required, namely that of transitivity, is the one axiom which I shall omit – since this axiom represents the closure of $\mathfrak{S}(S)$ under functional composition.

Any subset $\mathfrak{S}(S)$ of the transformation group $\Gamma(S)$ for the set S which satisfies the conditions (i) and (ii) above, I shall call a ***t–subset*** of $\Gamma(S)$. Note however, that if $\mathfrak{S}(S)$ satisfies the closure axiom, then $\mathfrak{S}(S)$ satisfies all the axioms for a group, and thereby will have the affect of partitioning the domain.

<u>Definition</u>: Let S be a set and $\mathfrak{S}(S)$ a t–subset of $\Gamma(S)$, the transformation group of S. Then two figures F and G in S are said to be ***indistinguishable*** if and only if $\exists \psi \in \mathfrak{S}(S)$ such that $\psi(F) = G$.

As mentioned earlier[1] a transformation group $\mathcal{G}(S)$ has an *action* on S, which one may describe as the existence of a function

$$\mathcal{G}(S) \times S \to S \qquad \text{defined by} \qquad (\psi, x) \to \psi \cdot x.$$

Similarly, one may define an external binary operation

$$\mathfrak{S}(S) \times S \to S \qquad \text{defined by} \qquad (\theta, x) \to \theta \cdot x,$$

which satisfies:

(i) $\mathbb{1}_S \cdot x = x$; since $\mathbb{1}_S \in \mathfrak{S}(S)$.

(ii) If $\varphi, \psi \in \mathfrak{S}(S)$ such that $\varphi \cdot \psi \in \mathfrak{S}(S)$ then $\varphi(\psi(x)) = (\varphi \cdot \psi) \cdot x$.

Condition (ii) states that if the composition of two members of $\mathfrak{S}(S)$ belongs to $\mathfrak{S}(S)$, then transforming a point of S by the first transformation followed by the next has the

[1]See chapter 3 and appendix A.

same effect as transforming the same point using the composite of the two transformations. I shall call the external operation, $\mathcal{E}(S)\times S \rightarrow S$ (which is *almost* a group action on S), a ***t–subset action*** on S.

If we now replace the action of a group on a set with a t–subset action, then instead of generating an equivalence relation we end up with a tolerance relation, and hence a cover for the set. To illustrate how this can be achieved we first of all need to construct the *orbits* for each of the points in the underlying set.

The ***orbit***[1] of a point $x \in S$ under the action of a transformation group $\mathcal{G}(S)$ is given by: $\mathcal{G}(S)\cdot x = \{y \mid y = \varphi \cdot x \text{ for some } \varphi \in \mathcal{G}(S)\}$. In a similar fashion I shall define the ***t–neighbourhood*** of a point $x \in S$ to be $\mathcal{E}(S)\cdot x$. It should be clear that the collection of all t–neighbourhoods, $\{\mathcal{E}(S)\cdot x \mid x \in S\}$, will form a cover for S.

One obvious difficulty arises when both *equivalence* and *indistinguishability* are required in a particular theory. In this situation one will need to express the relationship between the transformation group $\mathcal{G}(S)$ and the t–subset $\mathcal{E}(S)$, for the set S. I therefore propose that the t–subset $\mathcal{E}(S)$ should be a *superset* of the transformation group $\mathcal{G}(S)$ for the set S. The reasoning underlying this decision may be stated as follows: if I consider two objects to be equivalent then I certainly consider them to be indistinguishable, however, I am able to think of instances where I would not wish the converse to to be true[2]. Stating this formally we have: $\mathcal{G}(S) \subseteq \mathcal{E}(S) \subseteq \Gamma(S)$. The reason for not having the strict subset relationship is to cover the case when $\mathcal{G}(S)$ is chosen to be equal to $\mathcal{E}(S)$, in which case we forfeit the notion of indistinguishability.

To summarise, one may say that a transformation group $\mathcal{G}(S)$ acting on a set S enables one to define the notion of *equivalence*, whereas the t–subset $\mathcal{E}(S)$ of the transformation group may similarly be used to capture the notion of *indistinguishability*.

---

[1]See appendix A.

[2]As an informal example one may consider two twins, who are indistinguishable but are not equivalent.

4·2·1 An Example of a t–subset action

Let $\mathfrak{S}(S) \subseteq \Gamma(S)$ have a t–subset action on the set S. The t–neighbourhoods of a point $a \in S$ is defined by: $N^a = \{y \mid y = \psi_i(a) \text{ for some } \psi_i \in \mathfrak{S}(S)\}$. One may now define the tolerance space $\langle S,\xi \rangle$, where two points $x,y \in S$ may be defined as tolerant if $y \in N^x$ or by symmetry $x \in N^y$. Formally stated we have: $\exists \psi_i \in \mathfrak{S}(S)$ such that $y=\psi_i(x)$. Since the t–neighbourhoods of the points in S are elements of the power set $\mathbb{P}(S)$, one may consider two t–neighbourhoods N^x and N^y to be indistinguishable if $N^x \Xi N^y$ in the induced[1] tolerance space $\langle \mathbb{P}(S),\Xi \rangle$.

Let Ω denote the collection of all t–neighbourhoods for the set S, (i.e. $\Omega = \{N^x \mid x \in S\}$). Clearly $\Omega \subseteq \mathbb{P}(S)$ is a cover for S. Moreover, we may restrict the induced tolerance relation Ξ to Ω, producing the tolerance space $\langle \Omega,\Xi \rangle$ which I shall call the ***$\mathfrak{S}$–induced space*** on S. The purpose of constructing the $\mathfrak{S}$–induced space $\langle \Omega,\Xi \rangle$ is to obtain a tolerance space whose points may not only be considered indistinguishable, but may also overlap one another. This overlapping is not sanctioned in the tolerance space $\langle S,\xi \rangle$, or in fact in any tolerance space, since the underlying *set* requires its members to be distinct. However, it is possible to achieve this result in the case of $\langle \Omega,\Xi \rangle$, since I use a tolerance space $\langle S,\xi \rangle$ rather than a set for my underlying structure.

4·3 IDENTITY, EQUALITY AND SIMILARITY

The notions of equality and identity are philosophically very complex, and I do not intend to enter into the subtle philosophical arguments[2] which discuss whether identity should be treated in the object language or in the metalanguage. I do, however, wish to discuss the subtle distinction between *equality* and *identity*.

I myself have found often these concepts very confusing, and the clearest exposition I

---

[1]Zeeman defines $N^x \Xi N^y$ if and only if $N^y \subseteq N(N^x)$ and $N^x \subseteq N(N^y)$). See chapter 3 for details.

[2]See Williams[1989] or Morris[1984] for such discussions.

have managed to discover is by Bolzano[1820–30 and 1804][1], who not only warns the novice of the confusion but also provides the clearest explanation of the difference. According to Bolzano[1820–30], identity is

> *"... the concept that arises in our minds when we observe one and the same object several times";*

which he contrasts with equality, a concept

> *"... that arises whenever we observe various objects and find that they are subsumable under the same objective ideas".*

In order to follow Bolzano's intuition consider the statement 'an entity A is the same as an entity B, or it is different'; Bolzano[1804,II§1] then differentiates the different ones into 'equal' and 'unequal'. The point Bolzano is making here is that identity is a concept which applies to *one* thing at a time, whereas equality is a concept which applies to *two* different entities. Thus, according to Bolzano[1804,II§1], when one states: 'the thing A is identical to the thing B', one means: A and B were assumed to be different, but proved to be the same. Bolzano makes a similar comment about *properties* of things[2].

As an example of the distinction between equality and identity, consider an illustration which has been copied. The two illustrations may not be considered as *identical* (there are two different *things*), but may be considered as *equal.*

The final concept I shall consider in this section is that of *similarity*. Similarity is often considered to be a weaker form of *equality*, and is commonly used in geometry to identify properties of shape. However, it is evident from Bolzano's writings that he was dissatisfied with the manner in which this concept was deployed in mathematics[3].

The first definition of similarity (for geometrical objects) comes from Euclid[4]:

---

[1]See part II§1 of Bolzano[1804] translation in appendix A of Russ[1980].

[2]"*Properties* of things can be identical or different. But, insofar as they are hypothesized and considered as things themselves, they are *eo ipso* different and can now be called equal or unequal." See Russ[1980].

[3]See Russ[1980, 3.2.4]; see also Bolzano[1804,I§24] as translated by Russ[1980].

[4]From Book 6 of the Elements.

> *Similar rectilinear figures are such as have their angles severally equal and the sides about the equal angles proportional.*

Bolzano[1804I§16] modifies this definition to:

> *Two spatial objects are called similar if all characteristics which arise from the comparison of the parts of each one among themselves of the parts of each one of them, are equal in both, or if by every possible comparison among themselves of the parts of each, no unequal characteristics can be observed.*

According to Russ[1980,p.128] Bolzano's definition is a kind of compromise between Euclid's and the one defined by Leibniz:

> *Those things are similar in which it is not possible to discover, by consideration of themselves alone, whether they are to be distinguished*

which is reflected in the definition[1] by Wolff, who defines *similarity* as the:

> *... the correspondence by which things are distinguished from one another by the mind, and further, similar things cannot be distinguished from one another without, for example, the help of a measure*

Despite the fact that Bolzano does not explicitly mention Wolff's definition, it is quite probable that he was aware of it since he mentions that:

> *... there may be ideas that have so many common attributes that they are often confounded. These ideas shall be called 'similar ideas', Bolzano[1820–30].*

As one can see from the various definitions of similarity above, the concept of similarity was stated quite simply in Euclid but has subsequently became complex. In my opinion the definition Bolzano had in mind is very close to that of Leibniz in spirit. For example, George[1] asserts that similar objects can be distinguished only because of their relations to other objects. This observation fits in well with Leibniz' definition.

[1]From Wolff's *Anfangsgründe aller mathematischen Wissenschaften*, cited in Russ[1980,p130].

4·3·1 A unified definition

In the previous chapter I described how Klein characterised the *geometry* on a set S by using a transformation group $\mathcal{G}(S)$ acting on S. Moerover, Klein defined the notion of *equivalence* for two spatial objects (F_1 and F_2 say) as follows:

F_1 is ***equivalent*** to F_2 iff $\exists \varphi \in \mathcal{G}(S)$ such that $\varphi(F_1) = F_2$.

It is important to note how the equivalence for two spatial objects is *relative* to the chosen transformation group $\mathcal{G}(S)$ acting on S. This means if we choose the group $\mathcal{G}(S) \subseteq \Gamma(S)$ such that $\mathcal{G}(S) \neq \Gamma(S)$ then there are bijections of S ($\eta \in \Gamma(S)$ say) which do not belong to $\mathcal{G}(S)$. Informally this means that the two subsets F_1 and F_2 are equivalent as *subsets* of S (i.e. $\exists\, \eta \in \Gamma(S)$ such that $\eta(F_1) = F_2$), but *not* as *spatial objects* in the $\mathcal{G}(S)$–geometry of S – since $\nexists \zeta \in \mathcal{G}(S)$ such that $\zeta(F_1) = F_2$.

One may extend this and define two spatial objects G_1 and G_2 to be ***identical*** if and only if $\mathbb{1}_{\mathcal{G}(S)}(G_1) = G_2$.

Just as the notion of equivalence of two spatial objects is relative to transformation group $\mathcal{G}(S)$, one may define the notion of similarity in a manner which makes it relative to a t–subset $\mathcal{E}(S)$ acting on S. In keeping with Klein I shall define the concept of similarity for two spatial objects H_1 and H_2 as follows: select a t–subset $\mathcal{E}(S)$ of $\Gamma(S)$ such that $\mathcal{G}(S) \subseteq \mathcal{E}(S) \subseteq \Gamma(S)$, and define

H_1 to be ***similar***[1] to H_2 iff $\exists \varphi \in \mathcal{E}(S)$ such that $\varphi(H_1) = H_2$.

One may now also use this formalism to describe the scenario where two spatial objects F_1 and F_2 are similar but not equivalent. In this situation one must choose a t–subset $\mathcal{E}(S)$ and a transformation group $\mathcal{G}(S)$ such that $\mathcal{G}(S) \subseteq \mathcal{E}(S) \subseteq \Gamma(S)$.

[1]This is in keeping with ideas of similarity possessed by Bolzano[1804,I§10] who commented that "... the word 'equal' says rather more than the word 'similar' so that if two objects are called equal they must already be similar".

One way to formalise the above concepts is to extend the metalogic of a first order object language as follows: Let $\mathfrak{L}_2$ denote a 2–sorted first order language[1] with sorts τ_1 and τ_2. Now consider the individuals of sort τ_1 as elements of the domain $\mathcal{U}$, and the individuals of sort τ_2 as elements of a group $\mathcal{G}(\mathcal{U})$ acting on the domain $\mathcal{U}$. We will need to formalise the group action by a symbol '•' of sort (τ_2, τ_1, τ_1) such that '•(g,x,y)' is true if $g \bullet x = y$, and shall also need some way of indicating the t–subset action $\mathcal{E}(\mathcal{U})$ on the domain $\mathcal{U}$. For a domain $\mathcal{U}$, interpretation function σ, group $\mathcal{G}(\mathcal{U})$ acting on the domain $\mathcal{U}$, and a t–subset action $\mathcal{E}(\mathcal{U})$ on the domain $\mathcal{U}$, the semantics of *identity*, *equivalence* and *indistinguishability* represented by $\mathbb{1}$, = and ~ respectively are as follows.

$x \mathbb{1} y$	is true iff	$[\sigma(x) = \mathbb{1} \bullet \sigma(y)]$;
$x = y$	is true iff	$(\exists g \in \mathcal{G}(\mathcal{U}))[\sigma(x) = g \bullet \sigma(y)]$;
$x \sim y$	is true iff	$(\exists \alpha \in \mathcal{E}(\mathcal{U}))[\sigma(x) = \alpha \bullet \sigma(y)]$.

The conceptual difference between the relations of *tolerance* and *equivalence* can now be stated as follows. The expression 'a = b' is often taken to mean that the object denoted by 'a' is the same as the object denoted by 'b'; in other words one has *two* names for the *same* object. Whereas in the expression 'a~b' one has *two* names and *two* objects. Therefore in the inference from 'a = b' and 'P(a)' to 'P(b)' one is making a statement about the same object; whereas in the inference form 'a~b' and 'P(a)' one is stating something about a *different* object.

4·4 Tolerance spaces in AI: an example

So far in this chapter I have dealt with theoretical issues regarding tolerance relations, in this section I wish to present an example of this theory, namely the problem of representing the common sense notion of two objects *touching*.

The question of how to represent this concept was first investigated by Hayes[1979]. Hayes was primarily interested in using first order logic as a mechanism to represent

[1]For an example of a sorted logic see Cohn[1983 or 1987].

common sense notions, and that of of touching was just one which he studied in detail. This work is now regarded as part of a subject called ***Naïve geometry***, which is itself a branch of Naïve Physics. 'Naïve geometry' as a term first appeared in a memo authored by Welham&Hayes[1985] which extended some of the ideas discussed by Hayes in his earlier paper (Hayes[1979]).

In his paper Hayes[1979] discusses how one may use topological ideas to formalise intuitive common sense notions such as, *inside*, *outside* and *touching*. However, the branch of topology which Hayes is drawing his intuition from is that of *homology theory*, and Hayes himself admits that this approach

> *"... works up to a point, but seems **unintuitive** and in any case does not address the basic issues, which is that our intuitive local space is, indeed, probably not a topological space", [my emphasis], Hayes[1985,p.22].*

Moreover, in Hayes' opinion intuitive space is a tolerance space rather than a topological one[1].

4·4·1 Hayes' formalisation of touching

I shall begin by briefly describing how Hayes[1979] captures the common sense notion of two objects touching by using the idea of a *face* (borrowed from *homology theory* a branch of *algebraic topology*, see Hocking&Young[1961]). Intuitively, one may think of the face of an object as a part of its boundary (in the case of a 3D object it will be part of its surface); for example the face of a cube is exactly what one would expect it to be. The face 'f' of an object 'O' is denoted by 'Face(f,O)'.

Hayes now argues that it is intuitively clear that the surface of a solid object is a part of that object. In other words, a ***solid object*** (considered as a piece of space) is defined to contain its boundaries (faces) (Hayes[1979,p.80 & 1985,p.21]). Recall from §3·3·2 that a set which contains its boundary is by definition a *closed* set, therefore, solid objects are closed sets.

[1]Hayes[1985,p.22].

Following Hayes[1985,p.21] one can immediately see the difficulty here: intuitively, two closed sets cannot touch without having some points shared between them, so solid objects can *never* touch. Hayes escapes this difficulty by introducing an infinitesimally thin layer of space[1] between any two solid objects that touch one another. Formally, he states that two objects are defined to be *touching* one another if they share an outer directed face:

$$\text{Touching}(O_1,O_2) \quad \text{if and only if} \quad \exists d(\text{Outer}(d,O_1) \wedge \text{Outer}(d,O_2)).$$

Hayes points out that this is not the same as two objects sharing a face, which is used to define the concept of two objects being *joined* together.

4·4·2 Formalisation of touching using tolerance geometry

The notion of touching is a simple intuitive concept which is very difficult to formalise. In particular one must exercise great care not to confuse the formalisation of this concept with that of *fusing* or *joining* two objects together. The manner in which topologists formalise the concept of *fusing* two objects is by *identifiying*[2] one or more points from the two objects, a process which is known as a *quotient space*[3] construction.

In this section we are not interested in formalising the notion of fusing but the intuitive notion of touching, as expressed in such intuitive statements as: 'the body A *touches* the body B'. As a first attempt one may say two objects *touch* if they share a boundary point. However, if two objects share a boundary point then we have in effect fused the two objects together at their respective boundaries, and we have already agreed that this is not the concept we seek to formalise. In asserting that two objects A and B touch one another I am not stating that they be fastened to one another in some fashion, merely that they are *near* one another. We need a closer analysis of the situation.

---

[1]which he calls a 'directed surface', see Hayes[1979,p.83] for details.

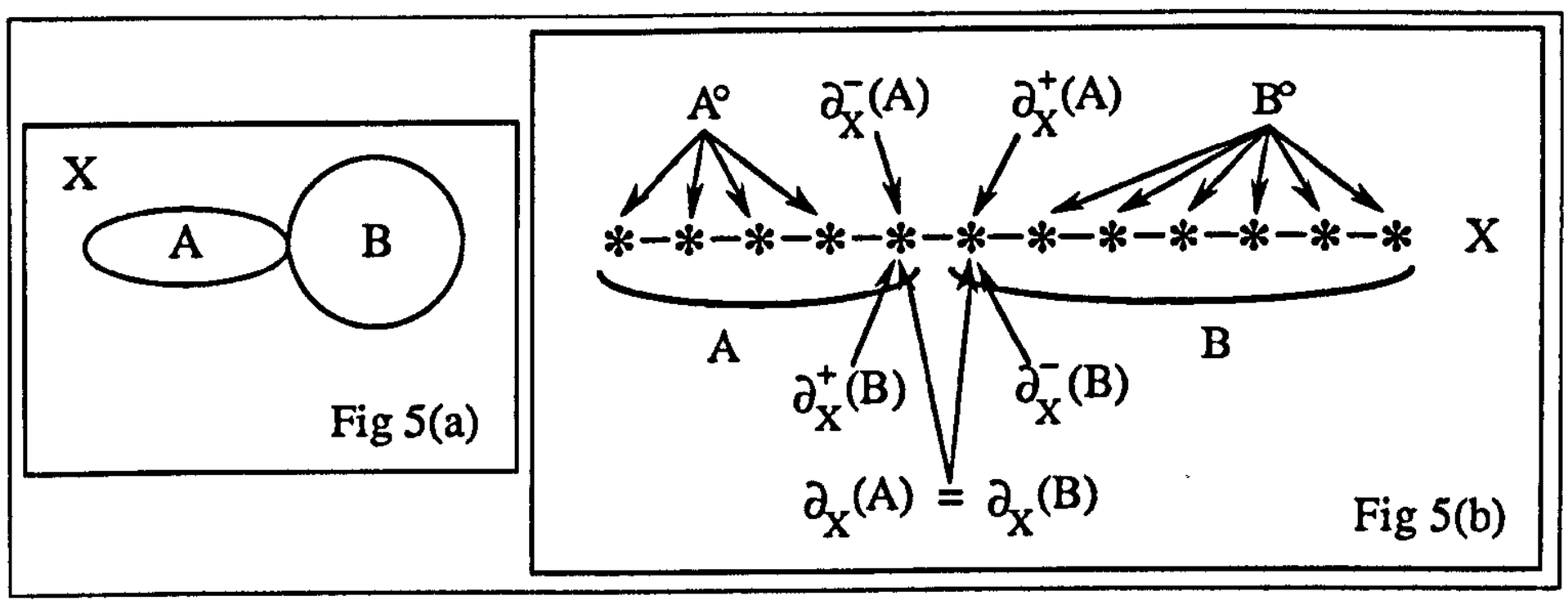
[2]By using an equivalence relation.

[3]Also called an *identification space*. See Eisenberg[1974] or Mendelson[1973] for details.

Before one can begin to describe this notion, one must define what one means by an object in such a space.

Definition: Let $\langle X,\xi \rangle$ be a tolerance space. A *t–object*, in $\langle X,\xi \rangle$, is denoted by 'Φ' and is defined to be a non–empty subspace[1] of $\langle X,\xi \rangle$.

Consider two objects touching as illustrated in Fig 5(a).



The diagram in Fig 5(b) illustrates two objects A and B, which are depicted as the collection of symbols '*', together with the relationship between the elements of the objects depicted by the symbols '–'. This illustration also demonstrates the *interior boundary* ($\partial_X^-(A)$), the *exterior boundary* ($\partial_X^+(A)$), and the *border* ($\partial_X(A)$) of the object A. Recall from §3·4 that the *interior boundary*, *exterior boundary* and *border* are automatically defined for any tolerance pair (Φ,X), moreover, the subspace Φ always contains its *interior* boundary.

One may consider the exterior boundary ($\partial_X^+(\Phi)$) on a t–object to be analogous to Hayes' outer directed face. Thence one may reformulate Hayes' definition of touching as:

$$\text{Touching}_2(\Phi_1,\Phi_2) \qquad \text{if and only if} \qquad \exists x[\,(x \in \partial_X^+(\Phi_1)) \wedge (x \in \partial_X^+(\Phi_2))\,].$$

Nonetheless this formalisation is still not as intuitive as one would like. Common sense tells us that two objects (A and B say) touch one another if there exists a point in A which is *indistinguishable* from a point in B. It is clear from the aforesaid, that the relation of

[1]Typically, we shall also want this subspace to be connected.

touching involves the points which belong to the borders and boundaries of the objects concerned.

By considering the diagram in Fig 5(b) one is able to capture the notion of touching by stating, that there exists an interior boundary point of A which is an exterior boundary point of B, and vice versa[1]. However, all this is best captured in terms of border points.

Definition: Let Φ_1 and Φ_2 be two t–objects in a tolerance space $\langle X,\xi\rangle$. Then, for $Z=\langle x,y\rangle$ $(x,y\in X)$, we have

$$\Phi_1 \textit{ touches } \Phi_2 \quad \text{if and only if} \quad \exists Z[(Z\in \partial_X(\Phi_1)) \wedge (Z\in \partial_X(\Phi_2))].$$

In this case the relation of 'touches' becomes: two objects *touch* if they share a border point (note that the border points in a tolerance spaces are collection of *pairs*, and not of individual points[2]. Informally, this definition captures the notion that two objects touch if there are two points x in one object and y in the other, such that x is *indistinguishable* form y.

The definition for *touches* above has some pleasing properties. For example, two t–objects Φ_1 and Φ_2 may touch without actually sharing any points. This is often a stumbling point for topologists since in their case the boundary points are either the same, or different, since they have no concept of *indistinguishability*.

Another property is that a subset A of a connected tolerance space $\langle X,\xi\rangle$, touches its complement X–A. One may visualise this by setting B equal to the complement of A (i.e. B=X–A), in Fig.5(b) above. $\exists y[\ (y\in \partial_X^-(\Phi_2)) \wedge (y\in \partial_X^+(\Phi_1))\ ]$.

This description of touching appears to be simpler and more intuitive than that presented in Hayes[1979][3], who characterises the same concept using such entities as *directed surfaces* and *faces*; concepts which are derived from algebraic topology.

---

[1] Φ_1 *touches* Φ_2 if and only if $\exists x[\ (x\in \partial_X^-(\Phi_1)) \wedge (x\in \partial_X^+(\Phi_2))\ ]$. Note that we do not need the clause $\exists y[\ (y\in \partial_X^-(\Phi_2)) \wedge (y\in \partial_X^+(\Phi_1))\ ]$ since tolerance relations are symmetric.

[2] See chapter 3 for details.

[3] Or the modified version in Welham&Hayes[1985].

4·5 SUMMARY

In the first section of this chapter I demonstrated how the theory of tolerance relations may be developed in a manner which parallels the theory of equivalence relations and equivalence kernels. In an attempt to clarify this connection I shall summarise the salient points of each.

An equivalence relation $\mathcal{E}$ defined on a set X enables one to form a set of equivalence classes $X/\mathcal{E}$ together with the projection, $p:X \rightarrow X/\mathcal{E}$. There are two important points about this construction. Firstly, the quotient set $X/\mathcal{E}$ partitions the underlying set X. Secondly, the converse is also true, that is to say, any partition Π of the set X corresponds to an equivalence relation on X.

By analogous constructions, one may assert that given any tolerance relation ξ on a set X, one is able to form the set of t–neighbourhoods N(X) together with the projection $\pi:X \rightarrow \mathbb{P}(N(X))$. Moreover, this construction demonstrates that N(X) forms a cover for the underlying set X. The converse also holds, namely, given any cover Ω for the set X one may construct a tolerance relation from it. Thus,

$\mathcal{E}$ corresponds to Π, and

ξ corresponds to Ω.

In other words: '*an equivalence relation is to a partition as a tolerance relation is to a cover*'.

The next point involves the connection between functions and relations defined between two sets. It is well known that any function $f : X \rightarrow Y$, from a set X to a set Y, may be used to generate a partition of the domain. This operation involves the construction of an *equivalence kernel* on the domain, whereupon the quotient set $X/\mathcal{K}_f$ provides the required partition. This construction can be emulated in the case of a relation, $R \subseteq S \times T$, that satisfies the condition: **Dom**(R) = **Supp**(R). In a similar manner we my construct an *induced relation* $\mathcal{K}_R$ on the domain, except in this case the collection of neighbourhoods

form a cover for the domain.This may be summarised as: '*equivalence kernels and partitions are to a function, what induced relations and covers are to a relation*'.

It is interesting to note that a *cover* has the *intuitive* effect of 'gluing' the elements from the underlying domain, which results in a space like a topological space. Since any tolerance relation may be used to *induce* such a cover, it too may be thought of as having a similar effect on the domain. However, the introduction of a transitivity axiom converts a tolerance relation into an equivalence relation, which *intuitively* has the effect of separating the domain into equivalence classes, or in other words, *partitioning* the set. One should appreciate the significant difference between an equivalence relation and a tolerance relation, despite the feeling that they should have somehow been similar.

Klein used the action of a group on the domain to generate the *equivalence* relation, which would enable him to describe equivalence between geometric configurations. In a similar way I defined a *t–subset action* on a set, which enabled me to generate a *tolerance* relation. The t–subset action on a set may be used to describe the notion of similarity much in the same way as Klein used the notion of a group action to define the equivalence of geometric figures.

In the next section of this chapter I turn my attention to the problem of formalising the notion of similarity, a concept which has been around since the time of Euclid but has ever since evaded all attempts to capture the notion in a formal manner. Finally, I describe how it is possible to unify the concepts of *identity*, *equality* and *similarity* by using a group action on a set.

In the penultimate section, I discussed how the theory of tolerance relations may be applied to the problem of describing the common sense notion of *touching*. This is an outstanding problem in AI, and many attempts have used topological techniques to capture this notion, the first of which appeared in an article by Hayes[1979] and was subsequently modified by Welham&Hayes[1985].

A different manner of using topological tools is described in Randell[1991]. Randell takes a nominalist view in his work, but is forced to introduce the notion of a point in order to formulate the notion of touching. While solving one issue the introduction of points presents others. For example, if two objects touch if they share a boundary point (essentially the definition of touching in Randell[1991]), how do we then define the situation when two points touch? I am *not* claiming that topology can not be used to represent finite entities, merely that the topic becomes unwieldy in such situations, particularly when one is attempting to deal with finite objects which appear in *real-world* common-sense settings.

Finally, I have demonstrated (§4·4) how the addition of a tolerance structure on the domain allows one to represent the notion of touching in a rigorous yet intuitive manner. The introduction of a tolerance relation into the metalanguage, and hence into the semantics obviously needs treatment. In the following chapters I shall investigate the affect of adding such a structure to the metalanguage. This endeavour was partially motivated by the statement made by Welham&Hayes[1985,p.1] who state that in their axiomatisation "*... no coordinate system or metric is introduced and the concepts of the intended interpretation are topological*". This statement can only be understood in an intuitive framework, since the metalanguage for first order logic (as defined by Tarski[1933]) does not support any spatial structures.

5 The Sorites

"What is involved in treating these examples as genuinely paradoxical is a certain tolerance in the concepts which they respectively involve, a notion of a degree of change too small to make any difference, as it were."

Crispin Wright

An ancient paradox which is frequently quoted in the literature as a paradigm involving a *vague predicate* is the *Sorites*. This paradox demonstrates the difficulties one is likely to face in an attempt to capture such vague notions in first order logic. Since the Sorites is of such central importance to vagueness it is prudent to starts one's analysis with this puzzle.

This chapter is divided into three sections plus a summary. The first section describes the historical development of the Sorites. In the second section I present a formal analysis of the puzzle together with some associated formal discussions.

5·1 SORITICAL ARGUMENTS

Traditionally, the Sorites is expressed as an argument about heaps. Haack[1967,p.112] for example presents the following argument: given that one grain of sand does not constitute a heap, and adding one grain to something less than a heap does not make it into a heap, it follows that no amount of sand added in this manner constitutes a heap.

This we know cannot be true, yet the original premisses are clearly true, and given that the reasoning from the premisses to the conclusion seems perfectly valid.

An alternative version of the Sorites appears as the *Phalakros* (or the bald man) puzzle[1], where grains of sand are replaced with hairs on a persons head.

The basic structure of these arguments are the same. Essentially, a Sorites argument is one in which any number of premisses and terms appear which can satisfy the criteria for syllogisms, and where each conclusion forms the premiss of the succeeding syllogism[2].

5·1·1 Origin of the Sorites

According to Moline[1969] the Sorites was first[3] formulated by Aristotle's contemporary, Eubulides of the Megarian school. The evidence for this suggestion comes from the writings of Diogenes Laërtius. In his *Philosophorum (II, 108)* Diogenes proclaims that Eubulides proposed many arguments in logic, including the *Sorites*, the *Phalakros* and the *Liar*. Some authors (including Moline) believe that the statement in Diogenes is evidence that Eubulides was the inventor of the paradoxes, however, others like Barnes[1982], point out that Diogenes does not actually say that Eubulides invented these arguments.

Moline[1969] dates the origin of the Sorites to the earlier collection of paradoxes attributed to Zeno (of Elea). In particular Moline argues that Eubulides seems to have been influenced by Zeno's *Millet seed paradox*, which Flew[1979] citing Aristotle[Physics, 250a 19–25] describes as follows. Given that one millet seed falling makes no sound, but one thousand seeds do, this would seem to suggest that one thousand nothings are something. However, Barnes[1982] dismisses this claim, pointing out that the millet seed paradox is not formulated in the classical form as a Sorites argument. Barnes agrees with Moline, in that as far as one can tell, Eubulides was the first person to state the Sorites in the present form.

[1]See Davis[1990].

[2]Flew[1979].

[3]Barnes[1982] quotes Abraham[Genesis] as the first to formulate a soritical type argument.

Several suggestions have been made regarding the aim of the Sorites. Beth[1959] claims that the Sorites was an attack on Aristotle's theory of potential infinity, while Professor Weinberg[1] suggests it was an aimed at the *Law of the Excluded Middle.* In contrast to these Moline[1969] using historical evidence, maintains that it was an attack on Aristotle's doctrine of the *Mean.* As far as we know, Eubulides' attack on Aristotle was of a personal nature, for according to Diogenes "Eubulides actually was at odds with Aristotle and slandered him a great deal" (see Barnes[1982]).

What is clear is that Eubulides' attack was aimed at Aristotle's philosophy; although there is some evidence to support the idea that it was aimed at Aristotle's doctrine of the mean[2], there is no direct evidence of this. In fact, Barnes goes so far as to say that Aristotle was not aware of the Sorites.

5·2 ANALYSIS OF THE SORITES

5·2·1 Sorites: Wang's paradox

In 1975 Dummett presented a paper on 'Wang's Paradox', which Dummett claims is a version of the Sorites. The paradox is formulated as an inductive argument as follows:

0 is small;

If n is small, n+1 is small:

Therefore, every number is small.

Dummett's options for resolving the paradox are as follows:

1. The first premise (the basis step) is false[3];

2. The rule of universal instantiation is invalid[4];

---

[1]See Moline[1969].

[2]See Barnes[1982].

[3] '0 is small' is false.

[4]for each particular m, we may not derive, 'If m is small, then m+1 is small' from 'For every n, if n is small, then n+1 is small'.

3. Modus ponens is invalid[1];

4. The induction step is not true[2];

Dummett's analysis leads him to conclude that the problem is with step 4:

> *"... if the statement 'If m is small, then m+1 is small' and 'm is small' is not true, it must be false, i.e., its negation must be true. But the negation of this statement is equivalent to: 'For some n, n is small and n+1 is not small', whereas it seems to us a priori that it would be absurd to specify any number as being small, but such that its successor is not small".*

According to Dummett, the above argument assumes at least three principles of classical two-valued logic, which are questionable when dealing with vague terms. The three principles are (Dummett[1975, p.308]):

1. that any statement must either be true or false;

2. that from the negation of 'For every n, A(n)' we can infer 'For some n, not A(n)'; and

3. that from the negation of 'If A, then B' we can infer the truth of 'A'.

Note that among the above principles, the rule of mathematical induction is *absent*. As Dummett points out, the reason for this appears to be that one can generate the paradox from a finite number of statements of the form, 'If n is small, then n+1 is small'. Therefore the rule of mathematical induction is not the culprit of the paradox – since the paradox can equally well be generated *without* it.

Barnes[1982] also demonstrates that the Sorites argument (form) is not a problem regarding mathematical induction, but something much more serious. Therefore any analysis of the Sorites argument should not just address this particular case.

---

[1]at least for some values of m, we may not derive, 'm+1 is small' from the premisses 'If m is small, then m+1 is small' and 'm is small'.

[2] 'For every n, if n is small, then n+1 is small'.

5·2·2 Formal analysis of the Sorites

One of the earliest formal accounts of the Sorites can be found at the end of Diogenes Laërtius' survey of Stoicism, which translates as:

> *"It is not the case that two are few and three are not also; it is not the case that these are and four are not also (and so on up to ten thousand). But two are few: therefore ten thousand are also". Barnes[1982,p.27].*

This can be analysed formally as an argument which has a large number of premisses, the first one of which is categorical, while the rest are of the form 'not both (P and not–Q)' which is equivalent to '$P \rightarrow Q$'. The logical structure of this argument takes the form:

$$
\begin{array}{c}
Fa_1 \\
Fa_1 \rightarrow Fa_2 \\
\cdot \\
\cdot \\
Fa_{n-1} \rightarrow Fa_n \\
\hline
Fa_n
\end{array}
$$

This analysis is similar to that which appears in Barnes[1982]. Barnes calls the predicate 'F' 'soritical with respect to the sequence $\langle a_1, \ldots, a_n \rangle$' if:

(1) The subjects (the a_i's) are always groups or sets of things. The subjects are ordered according to the subscripts.

(2) The predicate 'F' is required to satisfy the following three conditions:

(i) '$F(a_1)$' is true of a_1;

(ii) '$F(a_n)$' is false of a_n;

(iii) each adjacent pair of subjects in the sequence must, to all appearances[1], be

[1]Barnes uses the phrase 'to all appearances', by which he simply means that "not every argument of the logical form set out above is a Sorites" – not every predicate will produce puzzles if it is substituted for 'F' in the schema above.

indistinguishable with respect to 'F'. In other words, given any two adjacent subjects, a_i and a_{i+1}, either 'F' is true of both or 'F' is false of both.

In the above formalisation of the Sorites it is plain that the argument produces problems for one of the most fundamental principles of logic, namely the inference rule modus ponens.

5·2·3 Sorites: A puzzle involving tolerant predicates

In his 1987 article, Wright re-evaluates his earlier work (Wright[1975,1976]), in particular he reviews what he calls the 'governing view'. The governing view consists of two claims. Firstly, that anyone who is to be considered as a master of a language must at some level have internalized a definite set of semantics and syntactic rules, definitive of the particular language in question, and secondly, that such rules may be characterized and understood.

The governing view itself is not particularly important here, but Wright's analysis of it is. He describes the following example of the Sorites. Consider a series of homogeneously coloured patches, ranging from a red to orange, such that each patch is *just* discriminable in colour from its immediate neighbours. Thus every transition from a patch to its successor (moving in the direction red to orange), involves a marginal change of shade. The sense of a colour predicate, like 'red', is such that it preserves the applicability of the predicate through very small changes in shade. This incites Wright to state that what is involved:

> *"... in treating these examples as genuinely paradoxical is a certain tolerance in the concepts which they respectively involve, a notion of a degree of change too small to make any difference, as it were." Wright[1976, p.229], [my emphasis].*

Wright[1976] expresses the notion of a tolerant predicate more formally as follows. Suppose the concept Φ is related to a predicate F, such that any object which falls under F may be changed to one which does not, simply by a sufficient change in respect of Φ.

Then F is ***tolerant*** with respect to Φ if there is also some change in respect of Φ insufficient to affect the way in which F correctly applies to particular cases.

The notion of tolerance is central to Wright's view, especially so in the case of the Sorites. The Sorites is a puzzle involving certain vague predicates, which Wright claims should be treated as tolerant. Furthermore, he points out that our inability to 'draw the line' should be considered as a consequence of the tolerance of the predicates in question.

5·2·4 Sorites and mathematical induction

The Sorites is sometimes connected with the inference pattern known as 'mathematical induction'[1], for example Davis[1990] states "... by *induction*, you can take away all the sand, and still have a heap, which is absurd". This view is also shared by Black[1963,p.4] who begins his examination of the Sorites by conceding its validity, then claiming that the "... pattern of reasoning is the familiar and unimpeachable one known as mathematical induction."

Smith[1984] assumes Black's analyses and makes the bold claim that the Sorites *should* be formulated in a form which requires the use of mathematical induction. Moreover, Smith claims that any argument which can not be expressed in this form is not a genuine Sorites type argument. However, having seen three different analysis concerning the Sorites, all of which demonstrate that the puzzle can be formulated in a manner which does not display the form of a mathematical induction, together with the fact that mathematical induction was not formulated until the middle ages, it would seem that on this point Smith was mistaken in his analysis.

Another author influenced by Black's analysis is Roy Sorenson. Sorenson's four articles (Sorenson[1985,1988a,1988b,1990]) are good examples of the point made by Wright[1987,p.282] who explains that "... in comparison with what is needed, it is not unduly harsh to say that much of the literature on this topic which has mushroomed over

[1]Mathematical induction is formalised as: Given (i) 'F(0)', and (ii) 'For any n, if F(n) then F(n+1)', then infer 'For any n, F(n)'.

the last decade or so has amounted to little more than tinkering". Sorenson[1988b,p.3] states that Sorites arguments are most compactly formulated as mathematical inductions, a statement which though true will not help to resolve the issues, since the paradox may be generated in other forms. Moreover, Sorenson[1988b,p.51] asserts that "... since Eubulides' mathematical induction has only two premises ..."; as we have seen from the analysis presented, Eubulides' paradox was not formulated in a form which we would call mathematical induction. Therefore, it is possible that Sorenson also has an erroneous view of the Sorites.

Some aspects of Sorenson's analysis appear to be new. For example, in Sorenson[1985] he produces an ingenious argument to demonstrate that the predicate 'vague' is itself vague. But, Deas[1989] provides a sound argument which shows that Sorenson's approach does not succeed. Furthermore, Deas[1989,p.26] claims that Sorenson has

> *"... merely disguised an already familiar Sorites argument, and the vagueness he attributes to 'vague' is rightfully that of quite another predicate".*

The problem of Sorites is not that a particular puzzle can be formulated in a manner which displays the form of a mathematical induction, but that it strikes at the very heart of logic, namely modus ponens.

5·3 SUMMARY

In this chapter I have discussed the origins of the Sorites and discussed some of the philosophical analyses available, together with a formal analysis of my own.

Very little is known about Eubulides apart from his puzzles and paradoxes. However, one may conclude from the available evidence that Eubulides did discover the Sorites in its original form. We also know that Aristotle was aware of the Zenoean *Millet Seed Paradox*, since he commented upon this puzzle in his writings. However, there is no such evidence to suggest that Aristotle was aware of the Sorites.

The formal analysis presented in §5·2 describes how the Sorites is not only associated with a predicate, but also with a particular sequence of objects chosen from the domain. This means one is not simply able to state that a predicate is vague, since such a predicate may be vague when considered with one sequence, but perfectly normal when considered with another. The other two issues addressed in §5·2 are:

(i) The fallacious assertion that the Sorites indicates a problem with the rule of inference commonly known as 'mathematical induction'. In §5·2·4 I discuss how the paradox may be generated without the use of this inference rule, therefore one may conclude that the puzzle is independent of this rule.

(ii) Wright's analysis of this paradox which led Wright to assert that what is involved in these examples is a "certain tolerance in the concepts". That is to say, Wright associates a concept Φ with each predicate F, such that any object which falls under F may be changed to one which does not, with a sufficient change in Φ. Wright defines the predicate F to be *tolerant* with respect to the concept Φ if there is also some change in respect of Φ insufficient to effect the way in which F correctly applies to a particular case.

This chapter serves as an introduction to the rest of the thesis, where I will investigate the nature of vague terms, and develop the formal mechanisms necessary to deal with these notions.

6 Analyses of Vagueness: A Review

"Apart from representation, whether cognitive or mechanical, there can be no such thing as vagueness or precision; things are what they are, and there is an end of it."

Bertrand Russell

One puzzle often discussed in articles on vagueness is the Sorites, and it was for this reason that it was treated separately in the previous chapter. In this chapter I shall consider the nature of vagueness from a general point of view.

I shall begin my discussion with a section briefly describing what is meant by the various laws of logic. Obviously there are many descriptions of these laws, some clearer than others. The set presented here closely follows that found in Haack[1967]. The reason for this lies with articles in the literature which invariably mention how 'vagueness violates the laws of logic', or how 'the logic which deals with vagueness can not be two-valued' (for example Negoita[1985,p.49]). These statements are only useful if one has a clear understanding of the terms used.

In §6·2, I discuss the salient features of the classical papers by Russell and Black. I believe it is important to cover these articles here since they considered to be the pioneering articles in this field of study.

Other topics covered in this chapter include *observational predicates* which may be considered to be the richest source of vague predicates, and *borderline* cases for vague predicates.

Finally, I discuss the 'fuzzy' approach to vagueness. This is currently the most popular approach adopted by psychologists, engineers and those working in AI. The criticisms of fuzzy logic I shall discuss here are mainly those offered by Haack[1979], since my own concerns regarding this topic are criticisms of the underlying set theory, let alone the logic which is constructed using this set theory.

6·1 THE LAWS OF LOGIC

In order to argue which laws (or principles) of logic are preserved or violated under certain situations, one must have a common understanding of what constitutes these laws.

The principles of (classical) logic are defined to be:

(1) **The Principle of Bivalence (*PB*)**, which states that every statement is true or false.

(2) **The Principle of Excluded Middle (*EM*)**[1], which commits one to holding that for any statement p, the statement 'p or not-p' is true.

(3) **The Principle of Biexclusion (*PBX*)**, which states no statement is both true and false.

(4) **The Principle of Non-Contradiction (*NC*)**, which commits one to holding that for any statement p, the statement 'p and not-p' is false.

And finally one of the most important conditions for classical logic:

(5) **Tarski's Material Adequacy Condition** for definitions of truth (*T*), is the principle that for any statement p, 'p' is true if and only if p.

[1]Also called the **Law of Excluded Middle (*LEM*)**. The Law of the Excluded Middle is the syntactic counterpart to the semantic principle of Bivalence, in just the same way as the Principle of Non-contradiction is the syntactic counterpart of the semantic Biexclusion.

These principles can be generalised for use in other systems of logic. For example, Sanford[1976] is interested in many-valued systems for which he presents the following generalisation;

(1) **The Principle of Valence,** as the commitment that every statement has a truth-value;

(2) **The Principle of Exclusion,** as the commitment that no statement has more than one truth-value;

(3) **The Principle of N-value Semantics,** that there are just n truth-values.

It is of interest to note that according to Kneale&Kneale[1984, p.46] Tarski's T condition may have been anticipated by Aristotle. Taking the principle of non-contradiction and the principle of the excluded middle, Kneale&Kneale[1984] show that together with Aristotle's assertion 'it is false to say of that which is that it is not, or of that which is not that it is and it is true to say of that which is that it is or that which is not that it is not', combine to yield the equivalence of 'it is true that p' and 'p'. Similarly one may obtain the equivalence of 'it is false that p' and 'not–p'. In this definition of truth, PB and EM are considered to be equivalent.

The manner in which the principles T and PB interact, presents serious difficulties for the design of a metalanguage. Haack[1967] maintains that if the object language is chosen to be three-valued, then the manner in which we interpret the intermediate truth values, and whether we wish the principles T and PB to hold, will determine whether the metalanguage is also three-valued.

Haack[1967,p.66] also emphasizes the importance of placing a distinction between the principles EM and PB, since they are not equivalent in all systems of logic. Moreover, Haack states that many philosophers use the expression 'law of the excluded middle' and 'principle of bivalence' to mean the same thing, and gives as an example the work by

Taylor[1962], who speaks of the principle that 'any proposition is, either true, or if not true, false, i.e. '$p \vee \neg p$''.

In following Haack I shall distinguish between the principles of PB and EM, as follows:

(a) in a system the principle *PB holds* if every formula is either true or false, and

(b) in a system the principle *EM holds* if the formula '$p \vee \neg p$' is a theorem of the system.

As regards to a logical system being two-valued, Haack argues that it is relatively straightforward to say whether a system has a two-valued or many-valued characteristic matrix. In the case of a many-valued system, the problem is much more difficult, since one needs to decide whether the intermediate values are to count as truth-values, and if so, whether PB should be dropped. In other words, it is *not* a simple matter to say whether or not these principles of logic are true for a particular system. As an example Haack mentions Kleene's system, in which EM is *not* a theorem, but PB *holds*.

It should be clear from the aforesaid that any claims made regarding the logic of vagueness not being two-valued must be viewed with some scepticism, since such statements are often made without the necessary clarification of what would constitute a two-valued system. One way around this would be is to state that a particular system did not satisfy the conditions laid down by Tarski for his semantics, since this is a rigourous statement which can be formally verified or falsified.

6·2 THE CLASSICAL PAPERS ON VAGUENESS

Two papers on vagueness by Russell[1923] and Black[1937] are classics in the philosophical analysis of vagueness. These articles are commonly referred to by other authors who use them as a starting point for their own analyses. It is for this reason that these articles receive special attention here.

6·2·1 Bertrand Russell

Russell[1923][1] begins by pointing out that many who have realised that words can be vague, make the jump and infer that things are vague also. Russell accuses them of "... the fallacy of verbalism", by which the properties of words are mistaken for properties of things. He states the problem as follows:

> *"Vagueness and precision alike are characteristics which can only belong to a representation, of which language is an example. They have to do with the relation between a representation and that which it represents. Apart from representation whether cognitive or mechanical, there can be no such thing as vagueness or precision; things are what they are, and there is the end of it."*

Russell maintains that while we normally understand vagueness as it applies to cognitive matters, the concept of vagueness appears in every kind of representation.

For Russell, a representation is vague when the relation between the representing system (a symbolic system, or some language) and the represented system (i.e. some world) is not one-one but one-many[2]. As an example he states that a small-scale map is more vague than a large-scale one, because the former is deficient in the minor details which are present in the large-scale one.

In his commentary on Russell's paper, Kohl[1969] points out that Russell is not saying that 'red' is vague because it lacks a meaning in the sense of lacking an intension. However, (as Kohl points out) Russell takes a word as being vague if the word can always *"cause a state of unsettled judgement concerning the extent of its application"*.

When Russell says that the relationship between the representing and the represented is one-many, he does not intend us to interpret that as meaning an entity (say α) in the representing system *actually* represents different entities (say a and b) in the represented system, but rather, that it is *possible* that α might equally well represent a or b. In other

[1]All the quotations in this section are from this paper.

[2]Russell is changing the relationship represented by the interpretation function, from a function to a relation. A change which can be seen as analogous to the change form an equivalence relation to a tolerance relation.

words the entity α will only represent *one* entity at any one time, however a collection of *possible* ones exist. Thus in the case of the smudged photograph, when Russell states that the picture "might equally represent Brown or Jones", he does not mean that the picture *actually* represents *both* Brown and Jones, but one or the other.

An important point made by Russell is that "the law of the excluded middle is true when precise symbols are employed, but it is not true when symbols are vague, as in fact all symbols are." Here Russell claims that *all* symbols are vague, with the consequence that the law of the excluded middle is *never* satisfied. This somewhat disturbing result leads one to conclude that some qualification is needed, namely that we are dealing with words whose definition involves a "sensible element".

In summary, Russell defines the relationship between a representation and that which it represents as vague, if the relationship is one-many. Therefore Russell's definition should not be treated as an attempt at an explanation of the ordinary usage of the word 'vague', but rather an attempt to produce a definition which should be used in a metaphysical description of vagueness.

6·2·2 Max Black

Fourteen years after Russell's paper another classic appeared, authored by Black[1937].

In his paper Black[1] makes a distinction between generality, ambiguity and vagueness. Black criticizes Russell's definition of vagueness which he claims confuses notions of vagueness and generality.

For Black, a symbol is defined to be ***general*** if the symbol can be applied to a multiplicity of objects in the field of reference, and is ***ambiguous*** if the symbol can be associated with a finite number of alternative meanings which have the same phonetic form. However, a symbol's ***vagueness*** is characterised as a "*...feature of the boundary of its extension and is not constituted by the extension itself*". Thus, a term can be shown

[1]For which he is often quoted.

to be vague by producing what Black calls ***'borderline cases'***, namely, those individuals to which it is impossible either to apply or not apply the term.

Black attributes vagueness to a symbol when there exist objects for which it is intrinsically impossible to say whether or not the symbol applies. He identifies what he calls the 'fringe of the symbol's field of application', being the set of all such objects where the symbol's application is not determinate. This indeterminacy is of central importance to Black's concept of vagueness. He illustrates this by using the word 'chair' as an example. The word 'chair' is vague, because we can conceive of objects whose membership in the class of chairs is incurably 'uncertain' or 'doubtful'. He uses this example to illustrate the notion of vagueness for this word by the lack of specification of its boundary.

A simple way of dealing with the problem of borderline cases is to assign them to a region of 'doubtful application'. This region would then be a sort of no man's land, lying between two regions, one in which the term definitely applied and the other where it definitely did not. According to Black this view is mistaken. The assumption that the set of objects to which the application of the vague symbol is doubtful is well-defined is actually inconsistent with the usual meaning of negation. In order to demonstrate this inconsistency, I shall follow Black by introducing a set of symbols to facilitate the illustration.

Let 'L' denote a vague symbol, the vagueness of which consists in the impossibility of applying 'L' to certain members of a series. Let us call this series 'S', which is composed of a finite number (ten say), of terms x (i.e. $S = \{x_1, \ldots, x_{10}\}$[1]). The rank of each term in the series will be used as its name. Finally the region of 'doubtful application' or 'fringe' will consist of the terms x_5 and x_6. As usual the notation 'L(x)' will mean *L applies to x*, and '¬L(x)' will mean *L does not apply to* x, or L(x) is false.

Suppose that $L(x_i)$ is true for i = 1, 2, 3, 4, false for i = 7, 8, 9, 10, and $L(x_5)$ and $L(x_6)$ are 'doubtful'. If L(x) is asserted for any x, then we are positively excluding it only from

[1]There is, of course, no special significance in the choice of the numbers.

the region $x_7, \ldots, x_{10}$, since we cannot be sure, when L(x) is asserted, that x does not perhaps occur in the range x_5, x_6. It can only follow that to assert L(x) is tantamount to confining x to the region $x_1, \ldots, x_6$. We can construct a similar argument with respect to ¬L(x), the assertion of which is to positively exclude x from the region, $x_1, \ldots, x_4$, confining it to the region, $x_5, \ldots, x_{10}$.

The above analysis forces Black to conclude that our inability to find a logical interpretation of the doubtful assertions, using the standard two truth values, forces one to admit that the range of application of L(x), and that of ¬L(x), *overlap* in the fringe. This incites Black to infer that the

> *"... formal properties of logical negation are incompatible with an interpretation which allows the domain and the complementary domain of the propositional function to overlap" ; [my emphasis].*

From this Black concludes that such a point of view will be implausible unless it is possible to define a new sense of negation. The solution he offers is to replace the distinction between fringe and the region of certain application by a quantitative mechanism which allows degrees of change. Black proposes to correlate this with "the indeterminacy in the divisions made by a group of observers"[1].

Black starts his analysis by defining a '*discrimination* of x with respect to L, or a DxL for short', to be a situation in which a user of a language makes a decision whether to apply L or ¬L to an object x'. Moreover, he makes the assumption that while vague terms involve variations in use by different persons, these variations must be systematic and obey statistical laws if one symbol is to be distinguished from another.

The '*consistency* of application of L to x' is defined by taking any number of DxL's involving the same object x but with different observers[2], and considering the limit of the

---

[1]Note that Black's definition involves the following three notions: *"a language, a situation in which a user of a language is trying to apply a symbol L to an object x, and the consistency of application of L to x".*

[2]What is interesting is that Black claims that the result of a DxL is almost independent of the observer.

ratio of those who judge that L should apply to x to those who judge ¬L is to apply to x. The consistency of the application, C, is clearly a function of both L and x, and is therefore written as 'C(L,x)'.

Next, Black considers the case where the symbol L is applied to a linear series, called S. Black claims that as we pass along the series, the corresponding values of C(L,x) will have larger values at the outset, corresponding to the region of 'certain application of L'. The values of C(L,x) decrease until values near to zero are reached, which corresponds to the region of 'certain application of ¬L'. Moreover, he claims that a list of exact values of C(L,x) corresponding to each member x of S will be an exact description of L's vagueness. The curve obtained by plotting the sequence of values, C(L,x), against the series, S, is called a ***consistency profile*** for the application of L to the series S. The vagueness and ambiguity of L is then deduced from the 'geometry' of the consistency profile.

According to Black, the analysis of L is a specific consistency profile, which he denotes by L′. He further suggests that we may view this consistency distribution as an indication of the *degree* to which L′ is applicable to the corresponding terms of the series S. In effect, what Black has achieved in this analysis of vagueness, is to replace the propositional function L(x) of a single variable, by a function of two variables, L(x,c), which is read as 'L applies to x with consistency c'.

The *statistical* analysis provides a distribution for the group of language users, rather than an individual, and it is based upon the assumption that users can *decide* one of, applying L or applying ¬L. This analysis is fine if the symbol 'L' is not vague. However, if the symbol is vague, then by the very definition of vagueness presented by Black, there will exist objects "for which it is intrinsically impossible to say whether or not the symbol applies". What Black is proposing one should do in these cases is to take a survey of all those who can decide whether or not to apply the symbol, and take the ratio of those

answering 'apply', to those who answer 'don't apply'. This situation seems to make the rather bold assumption that objects which are borderline cases for one person, are not borderline cases for everyone else.

Criticism of Black's solution

Black's analysis for a borderline case is very much in line with Peirce's, however having defined these troublesome boundary cases, Black offers a solution which does not appear to conform to his analysis. Black argues that if we accept some reasonable premisses, then the range of application of L(x), and ¬L(x) overlap in the fringe, from which he concludes that the formal properties of logical negation are incompatible with an interpretation that allows this to happen. Black's solution is via his notion of *discrimination*. This he defines as a situation in which a user of a language makes the decision whether or not to apply L to an individual.

The difficulty here is not Black's analysis, but the manner in which he suggests that we should capture it. For example, he suggests that we replace the 'intrinsic uncertainty' displayed by one language user for a vague predicate, by a distribution which is obtained from a group of language users. This approach does not appear to address the relevant issues.

Suppose I took a coloured ball into the street and asked a number of people whether the ball was coloured red or not. And suppose I came across someone for whom it was 'intrinsically uncertain' whether the term 'red' applied to this object, namely the ball. In other words someone replies 'I can't decide' to my question. According to Black, it is acceptable to assign a value obtained by the ratio of all those who answered 'red' applied to those who answered the contrary (denote this value by α, say).

Black is taking a statistical approach here, and statistically speaking the value α is interpreted as follows; if I were to ask the question n times I should expect $n \times \alpha$ of the answers to be 'yes' the ball is red, and the rest to be 'no' it is not red. This is not at all in

the spirit of what is meant by *intrinsically uncertain*. By the analysis presented above, intrinsically uncertain means the question being asked is essentially undecidable. This means, no matter how many times one was to repeat the question, the person being questioned will never be able to decide whether to apply a certain term or not. This is an example of a situation where statistical methods are being used inappropriately.

Vagueness is not a phenomenon concerned with the lack of information, but one concerned with indecision. The distinction being that in the first case one does not have all the information to decide the outcome of a situation; whereas in the second case, it is the inability to make a decision irrespective of how much information becomes available.

6·3 VAGUENESS IS A METALINGUISTIC NOTION

It is easy to agree with Schefe[1980,p.190] that, "... vagueness has to be dealt with metalinguistically". Nevertheless one still needs to present an argument to support this position.

In his article Russell[1923] presents a *general* analysis of vagueness, and does not relate it to any particular system used for representation. In particular, he does not mention anything regarding the formal representational techniques such as logic. However, since what Russell has to say is relevant to all representational systems, it is clearly also relevant to formal logic.

Let us apply Russell's definition to a standard formal logic (i.e. a formal system together with its semantical machinery). There is clearly only *one* relationship between a representation and that which it represents, namely the interpretation function.

The interpretation function forms a part of the logical structure of a language, which by definition is an entity in the metalanguage. Therefore, any characteristic of such an entity must also be discussed and described in the metalanguage. It is difficult – if not impossible – to describe such notions in the object language, unless one decides to adopt a

non-standard logic. Thus, by adhering to Russell's analysis in the formal case, one has to accept that vagueness is a metalinguistic notion.

Russell's analysis is not the only one which indicates that vagueness must be treated metalinguistically. In his analysis, Black claims that the vagueness of a symbol is described *exactly* by its consistency profile. The manner in which Black defines the consistency profile for a symbol makes it analogous to providing an interpretation for that symbol. A procedure which is without doubt a metalinguistic operation.

Black also states that vagueness is characterised as a feature of the boundary of an extension, and is not constituted by the extension itself. A symbol can be shown to be vague by producing borderline cases, but how can one produce such cases in the object language since no extensions have been assigned at this stage?

One possible solution to this situation might be to have a mechanism in the object language that states whether a particular term is a boundary term for a particular predicate. This solution actually exacerbates the situation, since one now has to give a semantics for such a term and there is no machinery available in any standard logic to cope with this situation. I agree with the fact that vagueness is a metalinguistic notion, I shall describe my approach in chapter 8.

The discussion above indicates that if one accepts Russell's analysis then vagueness is a metalinguistic entity. Alternatively, one can accept Black's analysis, in which case one needs to take a closer look at borderline cases.

6·4 BORDERLINE CASES

Borderline cases were discussed by Black[1937], who attributed the symbol's vagueness to the existence of objects for which "*... it is intrinsically impossible to say either that the symbol in question does, or does not, apply*".

However, Black was *not* the first to define vagueness in this manner. According to Brock[1979], Peirce developed the notion of a 'borderline case' in providing the strictest sense for the term 'vague'. The evidence for this claim can be found in Peirce's entry on 'vagueness' in Baldwin's dictionary:

> *"A proposition is vague when there are possible states of things concerning which it is intrinsically uncertain whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition. By intrinsically uncertain we mean not uncertain in consequence of any ignorance of the interpreter, but because the speaker's habits of language were indeterminate". Baldwin[1902].*

The notion of a borderline case is obviously closely connected to that of vagueness, and is therefore discussed in a number of articles, two of which are of particular interest here, namely Machina[1972] and Sanford[1976].

In his article, Machina describes a semantics which he claims to have adopted from Goguen[1969] and Zadeh[1972]. The semantics is very close to the classical semantics for first order logic, with the essential difference being the manner in which predicates are interpreted. Predicates are assigned what Machina calls, 'a predicate interpretation function', which takes as argument atomic formulae of the form 'R(a,b)' and returns values in a set I (called the 'index' of the interpretation).

Machina[1972,p.231] makes an interesting choice for the elements of the index set I, namely, **'In'**, **'Out'**, and **'Borderline'**. The meaning of the values **'In'** and **'Out'**, as assigned by the interpretation function, are self evident. However, if a given interpretation function assigns the value **'Borderline'** to the pair ⟨R, ⟨a,c⟩⟩, then Machina interprets this as the interpretation function being unable to decide whether the pair ⟨a,c⟩ is in the extension of 'R' (or not, as the case may be).

Machina demonstrates his awareness of the fact that the set I may require a more complicated structure, and opts for the unit interval of all real numbers[1] between 0 and 1.

[1]Machina[1972,p.233]. Machina's characterization of 'degrees of truth' as 'degrees of correspondence', is the most intuitively pleasing I have come across.

In a sequel article, Machina[1976] firmly moves in the direction towards infinite-valued logics.

Machina[1976] describes the logic which he began to construct in his previous article. The two assumptions he makes about the metatheory are,

(a) the logic should be normal, meaning that the connectives shall be defined in a manner such that when they operate on propositions with classical values, they return propositions with classical values.

(b) the desired logical system should be truth-functional with regard to its connectives.

Machina[1976,p.61] explains that the problem is not to devise a new system, but rather to lay down well-motivated constraints on the system we are seeking. In Machina's case, he finds that the Łukasiewicz system (known as $Ł_\infty$) satisfies all of his requirements, and he therefore proposes that $Ł_\infty$ is well-suited to serve as a logic of vagueness[1].

6·4·1 Vagueness and uncertainty

As mentioned earlier, Peirce describes a proposition as vague:

> *"... when there are possible states of things concerning which it is intrinsically uncertain ..." [Baldwin[1902], my emphasis].*

He goes on to explain that the word 'uncertain' does not mean 'uncertain' in the usual sense of the word. This 'uncertainty', is different from the uncertainty which is a consequence of ignorance and can therefore be modelled formally by the use of probability theory.

The uncertainty which can be modelled by probability concerns the lack of information to make a decision; once the information is available the difficulty can be overcome. Moreover, this type of uncertainty conforms to the rules of standard logic and thus excludes borderline cases.

[1]Not the interpretation Łukasiewicz himself placed on it.

One way to consider the probability of an event is as a measure of the ratio of the number of successes to the number of failures, or if one prefers, the number of times a situation can be true, to the number of times it turns out to be false. But in each experiment it must either be true or false.

The confusion between vagueness and probability is quite common. For example, Sorenson[1988b,p.62] states that vagueness is a type of uncertainty, and like all other types of uncertainty falls under the scope of probability theory. Furthermore, Sorenson argues that if an entity x is a borderline case of a predicate F, then it is impossible for anyone to know that x is an F. Thus, no amount of inquiry, be it empirical or conceptual, will resolve the uncertainty.

However, as mentioned earlier, (formal) probability theory requires that the *events* be clear cases of either occurring or not, there is no room for borderline cases. Thus Sorenson[1990,p.9] is correct in placing a distinction between the inquiry-resistent cases which are the borderline entities, and the clear cases. However, he must have overlooked something if he believes that the borderline cases as described by him, can be captured by probability theory.

6·4·2 The 'Definitely' operator

In his short article, Chandler[1967] presents a remarkable analysis of 'borderline cases' of a predicate. Chandler carefully identifies that some of the confusion often attributed to the notion of predicates possessing borderline cases is due to a misunderstanding of the concept of *negation*. He argues that there are a number of forms of negation (partial, total and bare – see Chandler[1967,pp.807 &808] for details).

Chandler[1967,p.812] suggests that one intuitive way of capturing these concepts is by introducing a notion of determinacy, where an individual x has the property P ***determinately*** (written: ***DP(x)***) iff an omniscient being would know that x has the property.

Having defined the notion of *determinately*, Chandler explains that by 'x is a borderline case of P' he means "it is not determinate that x has the property P and it is not determinate that x does not have the property P"; namely

$$BP(x) \leftrightarrow [\,[\neg DP(x)] \wedge [\neg D\neg P(x)]\,].$$

The introduction of the 'D' operator into the object language is clearly a modification of standard logic, and will obviously necessitate an alteration of the semantics. This view is explicitly stated by Dummett[1975], who questions the bivalency of logic and states that the semantics of a language which deals with determinate statements will be different from the semantics of standard logic.

Dummett, like Chandler, proposes the introduction of a new operator ***'Definitely'*** into the logic. This operator restricts the conditions concerning the degree of definiteness about the truth of a statement, or its application.

Dummett is quick to point out that vagueness is not eliminated by the introduction of this operator. In fact, he argues that vagueness is an indispensable feature of a language, and that we could not get along with a language in which all terms were definite. This is to a certain extent agreeing with Wittgenstein who also held a similar view[1].

6·5 OBSERVATIONAL PREDICATES

In his discussion of vagueness, Dummett[1975] maintains that "... vagueness is an essential feature of a language, at least of any language which is to contain *observational predicates*." Here, the term ***observational predicate*** is used by him to mean a predicate whose application can be decided merely by employing our sense-organs.

Dummett is not the only author who discusses vagueness in terms of observational predicates. Wright[1976,p.233] also presents an argument for supposing that colour predicates are tolerant with respect to marginal changes in shade, an argument which is

[1]A view which is opposite to that held by Frege, namely that the presence of vague expressions cause a language to become incoherent.

based upon the ostensive manner in which we learn our basic vocabulary. Wright views colour predicates as purely observational, in other words if it is possible to tell what colour the object is, it can be determined just by looking, for the look of an object decides its colour. Information from the senses determines the applicability of an observational predicate. Thus any distinction between objects derived from sensory information, cannot be captured solely by means of such predicates – what Wright calls "sensorily–equivalent items".

Wright[1987,p.292] refers to Peacocke[1981] for a refinement of an observational predicate, which Peacocke defines as "... one whose application can be determined from the kind of *experience* produced by that object in standard conditions".

The connection between vagueness and our sensory mechanisms is also mentioned by Russell, who argues that we cannot distinguish between two glasses of water, one of which is wholesome and the other infected by bacteria with our naked eyes. We could distinguish between the two glasses if we had a microscope, but without the aid of such an instrument

> *"... the difference is only inferred from the differing effects of things which are sensibly indistinguishable. It is this fact that things which our senses do not distinguish produce different effects ... that has led us to regard the knowledge derived from the senses as vague. And the vagueness of the knowledge derived form the senses infects all words in the definition of which there is a sensible element."*

According to Russell, our physiological make–up is responsible for the fact that different stimuli can produce indistinguishable sensations. This gives rise to the question as to whether or not the sensations are really different, like their stimuli, and only our power to discriminate between them is deficient, or whether the sensations themselves are sometimes identical in relevant respects even when the stimuli differ in irrelevant respects.

Recognition of our physiological ability to produce indistinguishable sensations from different stimuli is by far the most important step towards a better understanding of this phenomenon.

Dummett[1975] takes this one step further, and states that the phenomenon of vagueness is connected with the indistinguishability relation between the sensations being non–transitive. Wright[1975] also arrives at the same conclusion, though according to him the non–transitivity of our discriminations may be viewed as a pervasive structural feature of our sense–experience.

As an example of an indistinguishability relation, Wright[1976,p.233] describes a series of colour patches starting with red and ending with orange, with the proviso that neighbouring patches are indistinguishable in colour (c.f. §5·2·3). As Wright correctly points out, the possibility of constructing such a series depends upon the non-transitivity of colour discriminations[1]. Wright considers the outcome of allowing the indiscriminability of the colour patches to behave transitively, and concludes that "... we shall have lost what was distinctive of that series: the appearance of ***continuous*** change" from red to orange. Furthermore Wright explains that if matching behaved in a transitive manner among the shades of colour, then no such series of colour patches could give the impression of continuous transformation of colour. Therefore, if we require matching to function non–transitively among a finite set of colour patches, it is sufficient that the colour patches be arranged so as to form a phenomenal ***continuum***.

Wright and Dummett are but two amongst many who use the words 'continuous' and 'continuum' in their analyses. For example, Sanford[1976,p.197] describes 'alive' as a vague predicate, since it too allows for borderline cases. In order to argue this point, he considers the following example: on Tuesday the patient is gravely ill, but definitely alive; by Friday morning there is no reasonable doubt that he is dead. Sanford argues that the

[1]It should be clear from the construction of this series that we can generate a Sorites paradox from this set-up.

principle of bivalence requires any statement to be either true or false, when applied to the statement: 'the patient is alive' this has the consequence that death was instantaneous.

Sanford then argues that since dying is a process involving

> *"... a continuum of momentary states no one of which is the last at which the person is alive or the first which the person is no longer alive. If the transition from being alive to not being alive is continuous, it seems appropriate to have a continuum of values to assign to statements..." Sanford[1976,p.199]. [my emphasis].*

Sanford clarifies this by stating that every discernible difference of semantic status between statements about borderline cases be reflected by a difference in assignable truth–values. This statement appears reasonable enough, however, Sanford's claim[1] that "... if there is a continuum of values, there must be infinitely many" is somewhat contentious. For example, we do not need more than 10,000 truth values to manage the borderline cases of 'short'[2], since we cannot measure a person's height with an accuracy greater than a tenth of a millimetre. This situation appears to be a little unrealistic, and Sanford[1976] himself admits to the artificial nature of this situation by admitting that there is "... something ironic about responding to the imprecision of natural language by adopting a semantics which allows infinitely precise discrimination of truth-values".

Sanford's previous article (Sanford[1975]) was concerned with a logic of vagueness which he calls 'Borderline Logic'. In this article, Sanford argued the case for an infinite–valued semantics, where the set of real numbers between 0 and 1 were to be considered as the set of truth values. What is interesting about this article is Sanford's introduction of a determinacy operator ('D') into his object language, as suggested by Chandler[1967].

Sanford at least indicates what he means by a continuum – namely the real numbers between 0 and 1. There are many authors who don't; for example, Margalit[1976, p.211]

---

[1]Sanford[1976,p.198].

[2]We don't even need this many.

mentions that *"indefiniteness has to do with things which are on a continuum"*, but makes no attempt to describe what a 'continuum' is. If one considers some of the statements made by Sanford, then one can assume that he is using the word 'continuum' in a manner which indicates he is thinking of a mathematical continuum. A mathematical continuum is by its very definition, infinite. However, this is not a necessary feature of a perceptual continuum, which I shall describe in the following chapters.

6·6 TWO FORMAL APPROACHES TO VAGUENESS

I now shall describe two formal approaches to vagueness; the first of which is currently the one favoured by the philosophers, and the second is the most popular approach as adopted by the AI community.

6·6·1 The approach using supervaluations

The theory of *supervaluations* was originally developed by van Fraassen[1969] to represent failures of presupposition. Supervaluations are based on the idea that valuations in which truth value gaps[1] occur should agree as much as possible with classical valuations. Briefly, a supervaluation assigns to a compound formula a value which all classical valuations would assign, if there was such a unique value, otherwise it would assign no value at all[2]. The appeal of such an approach appears to be that it will assign 'true' to all classical tautologies and 'false' to all classical contradictories.

The concept of supervaluations is a generalisation of the standard semantics, in which we generalise the notion of a valuation from being a function to a partial function. The truth value of sentence is determined by considering all the classical completions of this partial function.

A *supervaluation* is thus defined in three steps: First a partial valuation ρ is given, then its set of classical completions $[\rho]$ is determined, and finally a supervaluation R is defined

[1]For an exposition of truth value gaps see van Fraassen[1966].

[2]Thus in many respects the semantics is three valued.

as the function recording the unanimous assignments of [ρ]. To sentences for which there is no unanimity, the supervaluation does not assign any value.

For example, all classical valuations which either assign 'true' or 'false' to 'p', assign 'true' to 'p∨¬p', thus the supervaluation would also assign 'true' to 'p∨¬p'.

As far as I am aware the only scholar to use the supervaluation approach is Fine[1975]. Fine's work is based on a number of assumptions, one of which is that the manner in which we consider a sentence to be true. According to Fine[1975, p.278]:

> *"... a [vague] sentence is true if it is true for all for all ways of making it completely precise"*

Firstly, the way in which Fine constructs his theory to enable him to define '... for all ways of making it completely precise' involves the assumption that the valuation function is initially partial. Thus some sentences are not assigned a truth value, and I agree with Peirce that vagueness is a feature which is characterised by a 'glut' of values and not by a 'gap'. I propose that a truth 'gap' is a characteristic feature of a predicate which applied to an object results in a 'meaningless' sentence.

Secondly, supervaluations are *not* truth-functional. Intimately related to the notion of truth–functionality is the concept of *substitutability salva veritate.* This is the property of valid substitutability of one sentence for another, of like truth-value, in a larger sentence. Mathematically, the relation 'P↔Q' defined by $\upsilon(P) = \upsilon(Q)$, is a congruence relation which admits substitutability of co–referential parts without altering the reference of the whole. This kind of substitutability is important, because it forms part of the what is meant when we describe classical logic as extensional.

The approach based on supervaluations has been briefly covered here because of its importance to philosophers. A readable account of the topic can be found in Sainsbury[1988][1].

[1]Sanford[1976] presents a different argument for rejecting supervaluational approaches in favour of many valued ones.

6·6·2 Vagueness in AI : the 'fuzzy' approach

Fuzzy set theory was invented by Lotfi Zadeh in the mid 1960's, and has flourished ever since. Having developed the original set theory, Zadeh took the natural route and started to work on what he called *Fuzzy Logic*. This was intended as a logic for dealing with *fuzzy* or *imprecise* information, such as that used everyday by human beings.

It is not clear however, who first made the suggestion that fuzzy sets were the proper modification of classical sets for dealing with *vagueness*. Zadeh himself is very careful not to mention 'vagueness', instead, he uses the term *fuzziness*. For example in Zadeh[1972, p.4] he mentions how, despite its fundamental importance, 'fuzziness' has not attracted much attention in the scientific literature. Nevertheless, Dubois & Prade[1980,p.1] point out that *"'fuzziness' is what Black[1937] calls 'vagueness'..."*.

Many authors[1] discussing fuzzy sets who mention the Sorites paradox invariably adopt Black's philosophical analysis of vagueness. For example, Goguen[1969,p.334] asserts that Black distinguishes three kinds of inexactness (vagueness, generality and ambiguity), and all three can be represented by fuzzy sets. In particular, Goguen claims that *vagueness* occurs when the function takes values other than just 0 and 1.

Vagueness in AI

There is a vast body of literature dealing with uncertainty in AI, however, the number of articles on vagueness is relatively meagre, and what there is deals with vagueness using some kind of 'fuzzy' technique. For instance, Davis[1990] briefly mentions vagueness and asserts that fuzzy sets/logic is the proper *formal* setting for describing such notions. Another recent article is that by Fehrer[1990], who despite describing an interesting calculus provides little or no analysis for the complex notion of vagueness, which makes it particularly difficult to discern whether he has actually managed to capture the concept.

[1]Amongst the many one can include, Lefaivre[1974] and Negoita[1985].

Fuzzy logic

The rationale for developing fuzzy logic appears to be guided by the observation that informal arguments suffer from vagueness and indeterminacy. Those engaged in developing such logics argue that the classical logic should be 'fuzzified' to obtain a logic which is applicable to such informal arguments. However, there is a great deal of confusion regarding the term 'fuzzy logic'. According to Haack[1979,p.437], it appear to be used in the literature to refer to two related but nevertheless distinct enterprises:

(I) the interpretation of familiar infinitely many-valued logics in terms of fuzzy set theory,

(II) the development, on the basis of (I), of a family of new logical systems in which the truth-values are themselves fuzzy sets.

I follow Zadeh[1975a] and use the term '*fuzzy logic*' to mean the systems characterized by (II) above, and follow Haack[1979], in which many–valued logics interpreted in (I) are called the '*base logics*'.

According to Haack[1979], there are two stages of 'fuzzification' from classical to fuzzy logics:

(i) a move from 2-valued to indenumerably many-valued logic as a result of treating object language predicates as denoting fuzzy rather than classical sets, which yield the base logics; and

(ii) a move to countably many fuzzy truth-values as a result of treating the metalanguage predicates 'true' and 'false', as denoting fuzzy subsets of the set of values of the base logic, thereby yielding 'fuzzy logic' proper.

The first stage is nothing more than a 'conventional' deviant logic, and it is not until the second stage that the radical departure from classical logic takes place, and a true 'fuzzy logic' is generated. Many of the applications of 'fuzzy logic' have been applications of the base logic, and in some cases such logics have been *misunderstood* as constituting fuzzy logic proper.

In the second stage of the fuzzification process, the set of points in the interval [0,1] are replaced by fuzzy subsets of [0,1] and called ***fuzzy truth-values.*** Acknowledging the fact that the collection of fuzzy subsets of [0,1] are unmanageable[1], Zadeh introduces the notion of a (countable) set consisting of '*linguistic truth-values*'. In order to understand the this stage of fuzzification, we have to look at the notion of a '*linguistic variable*', a concept which Zadeh introduces in a series of three articles[2]. According to Haack[1979,p.439], a linguistic variable is a non-fuzzy variable ranging over a collection, the members of which are fuzzy variables, and each one of which indicates a fuzzy restriction.

One can conceive of object language predicates as determining a fuzzy set in place of a classical one, then apply the same idea in the metalanguage treating the predicate 'true' in a similar manner.

Haack[1979] argues the above at some length in her article, and concludes by rejecting the arguments offered in favour of fuzzy logic. She refuses to acknowledge the need for the more radical fuzzy logics. However, she also stresses that her criticisms do not apply to base logics.

Professor Haack is not the only logician to repudiate Zadeh's contentions. According to Schefe[1980] Zadeh's fuzzy-algorithmic approach cannot cope with the complex nature of vagueness, and states that he can

> *"... neither accept Zadeh's modelling of truth nor his modelling of 'linguistic' reasoning."*

Furthermore, Schefe claims that Zadeh's model of 'fuzzy' reasoning has some formal deficiencies.

---

[1]Fuzzy subsets of [0,1] are functions of the form [0,1] → [0,1]. The cardinality of the collection of all such fuzzy subsets, is strictly greater than the cardinality of the set of real numbers.

[2]Zadeh[1975b, 1975c, 1975d].

6·7 SUMMARY

In this chapter I have attempted to discuss many of the pertinent points selected from a number of famous, and some not so well known articles in the literature concerning the notion of vagueness. In my discussions I have highlighted the issues which I regard as essential characteristics of this concept. The characteristic points may be highlighted as follows.

According to Russell, vagueness is a characteristic feature of the relationship between a representation and that which it represents. If one accepts this, then one also accept that vagueness can only be described in the metalanguage for a representation.

Black takes a similar view to Peirce, and defines a term to be vague if one can show the existence of borderline cases for it, that is to say, individuals for which it is *intrinsically impossible* either to apply or not apply the term. Black criticises Russell for having confused the notion of vagueness with that of ambiguity, and instead he offers a different analysis which leads him to suggest a statistical solution to the problem of vagueness. I argue the case against this approach in the chapter.

Black's definition of a borderline case is very similar to that expressed by Peirce some twenty years before. Peirce defines a symbol to be vague if there exist individuals for which it is *intrinsically uncertain* whether the predicate applies to that individual or not. However, Peirce is quick to point out that by intrinsically uncertain he does not mean uncertain as a consequence of any ignorance, but a notion which is closer to that of indeterminacy.

Chandler offers a formalisation for the notion of borderline cases by introducing a new operator into the object language, which he denotes by the letter 'D' and calls it 'determinately'. This operator enables Chandler to define a borderline case for a predicate P as: $BP(x) \leftrightarrow [\,[\neg DP(x)] \wedge [\neg D\neg P(x)]\,]$. This operator is also discussed by Dummett who calls it a 'definitely' operator, but the two operators apparently differ in name only.

One of the most important popular examples of vague predicates are what are commonly described as *observational predicates*. These are predicates whose application is decided merely by the use of our sense organs. Dummett argues that vagueness is an essential feature of any language which contains observational predicates, and is not alone in asserting this. Russell also states that knowledge which is derived from the senses should be considered as vague.

Finally (and briefly) I discuss the approach which is commonly used in the literature to discuss vagueness, namely the fuzzy approach. This approach is commonly adopted by those authors who are keen to cite examples of concepts which cannot be captured by classical logic. I discuss the basics of this approach and present the arguments and criticisms for dismissing fuzzy logic as a viable approach for dealing with vagueness. I do not offer any criticisms of my own as far as fuzzy logic is concerned for two reasons. Firstly, I believe that Haack's criticisms are already strong enough to cast a shadow over fuzzy logic. Secondly, my concerns are at a lower level, that is to say I disagree with the use of fuzzy sets in this context, let alone the use of fuzzy logic. My criticism of fuzzy sets is presented in some detail in chapter 8.

7 The Continuum

"... nothing but geometry can furnish a thread for the labyrinth of the composition of the continuum, maxima and minima, and of the unassignable and the infinite, and no one will arrive at a truly solid metaphysic who has not passed through that labyrinth."

Gottfried Wilhelm Leibniz

There is a close connection between the concept of vagueness and the notion of the *continuum*, which has been discussed in chapter 6 – and will be treated fully in the next chapter once I have established what is meant by saying something is a continuum.

The difficulty as far as the logic is concerned arises when one attempts to define a predicate on a continuum which is required to preserve the concept of *nearness*. For example, if one considers the continuum represented by the colour spectrum, then as Wright[1976] asserts, we would expect colour predicates to be tolerant of marginal changes in shade. Various similar examples were mentioned in the previous chapter.

The conventional way to deal with predicates defined on a continuum, that are required to be structure preserving is by asserting that the predicate itself should take values in a continuum. This approach is described in Zadeh[1965], Sanford[1976], Lakoff[1973] and others. What is missing in many of these articles, though, is a formal description of what constitutes a continuum.

In this chapter, I describe how the notion of the continuum has evolved over the centuries, and discuss the important distinction between a *mathematical* continuum and a *perceptual* (or *phenomenological*) continuum. It is necessary to make this distinction because here lies the root cause of paradoxes which are essentially concerned with a perceptual continuum, but are formalised using a mathematical one.

The distinction between the two types of continua were made as far back as Zeno (who in his paradoxes tried to demonstrate this very fact), but it wasn't until Leibniz wrote his treatise on the continuum that the two notions were explicitly distinguished. Nevertheless, the confusion between the two continua is still exists today.

Another confusion which arises frequently, and which has lasted even longer than that mentioned above is that, between the notions of *continua* and *continuity*. This confusion was not clarified until the works of Bolzano and Cantor last century.

Finally, I shall present a formal definition for an *impression continuum* and use it to formalise Minsky's statement, which describes how nearby points on our skins are wired to nearby places in our brains.

7·1 EVOLUTION OF THE CONCEPT OF A CONTINUUM

The nature of the continuum lies at the very heart of metaphysics, and presents one of the most important and difficult subjects in philosophical analysis[1]. This may explain why mankind has been intrigued by this concept for many years, and has experienced great difficulties in grasping it.

7·1·1 The Ancient Greeks

In their efforts to describe the structure of space, the Ancient Greeks faced problems which we now associate with the notion of a continuum. According to Weyl[1949], it is to the Greeks that we owe the concept of space as defined by relations holding between

[1]See Russell[1900].

sets of objects. The structure of space presented a true obstacle for the Ancient Greeks, and in trying to understand the underlying nature of *space*[1] led philosophers such as Parmenides (of Elea) and Aristotle to identify what we now call *metaphysics*.

The Ancient Greeks were originally interested in such questions to do with motion, in particular, how does one describe motion? or, how does one describe continuous motion? and is motion cinematographic (i.e. composed of discrete frames or instances)? which naturally lead them to consider such problems as, is space a continuum? or is it discrete? does it contain points? and so on. The Greek philosophers were thus introduced to the problems of space through a need to describe motion. The problems which they encountered are best illustrated by a series of well–known paradoxes constructed by the philosopher/mathematician known as Zeno of Elea[2].

The Greeks already had a framework in which to describe the structure of space, namely that of geometry. However, as already mentioned they were the first to define the notion of space by the relations which existed between sets of objects. Using geometry these objects were described as collections of points, which were treated as primitives. One may define a *space* as a collection of undefined entities known as *points*, together with the relations which hold between these *points*. The objects in this space may then be considered as *configurations* of points.

One of the most important issues raised by mathematicians and some philosophers, concerns the (undefined) notion of a *point*. It has consistently been stressed that mathematical points, such as those found in geometry, must *not* be confused with the points of our perception, which are considered to be considered the smallest distinguishable units of our perception[3].

---

[1]According to Weyl[1949]: "Nowhere do mathematics, natural science, and philosophy permeate one another so intimately as in the problem of space.

[2]See Grünbaum[1968] for a detailed study of Zeno's paradoxes.

[3]The distinction between mathematical and perceptual points was first made by Parmenides.

According to Sweeney[1972], Parmenides accepted geometry per se, he simply denied the leap from the intelligible geometric figure to the physical world. Parmenides considered geometrical objects to be objects of *thought*, something used to describe *reality*. The physical manifestations of these geometrical objects were then considered to be objects of *sense*. Thus, Parmenides was the first to separate geometrical objects from their physical manifestation.

The confusion between the points in mathematics and the points of our perceptual space can be dated back to the Pythagoreans, who believed that the universe was somehow composed of units. Zeno (*c.* 490 B.C.) accused the Pythagoreans of having confounded these units with the points of geometry, and attempted to demonstrate the absurd consequences that followed from their beliefs by devising four arguments which demonstrated the impossibility of motion.

Zeno grouped his arguments into two types. The first assumes that space and time to be infinitely divisible, in which case motion is continuous[1], while the second assumes that space and time possess indivisible minima, in which case motion is fundamentally cinematographic[2]. He argued that taking either assumption, motion was impossible. Zeno also produced about 40 paradoxes concerned essentially with the concept of plurality[3].

I do not discuss Zeno's paradoxes in detail since they are readily available to the reader (see Grünbaum[1968]). I should like to stress that Zeno devised these paradoxes to illustrate the distinction between the points of perception and those of mathematics. This issue is also addressed by Grünbaum[1968,p.44], who emphatically states that

> *"... the structure which scientific theory attributes to an interval of physical space cannot be understood as isomorphic with the elements, properties, and relations encountered in the perceptual space furnished by the sensory*

[1]The paradoxes of the Stadium and Achilles attack this alternative.

[2]The flying arrow and the moving rows were designed to attack this alternative.

[3]These included the famous Millet Seed Paradox which according to Flew[1979], was designed to demonstrate the scope and validity of sense experience and the importance of differences of degrees.

organs of the human body when we sense a line segment." [emphasis in the original].

In discussing the distinction between perceptual and mathematical spaces, Grünbaum[1968] explains how a chalk line on a blackboard may qualify as *continuous* in the sensory sense, on the grounds that it does not exhibit any gaps which are noticeable by means of our unaided senses. Nevertheless, this line does *not* constitute a mathematical continuum.

Plato (*c.* 428 B.C.) also distinguished between the objects of sense–perception and those of pure mathematics[1]. According to Taylor[1919], Plato took a scientific truth to be exact and definite, something which was true once and for all. In other words, scientific truths were not contingent. Plato distinguished this type of truth from the judgements which we base on our sense-perception of the visible and tangible world.

In a similar vein, Aristotle (*c.* 385 B.C.) also admitted that there was a disparity between sense–perception and scientific knowledge. But, according to Taylor[1919], sense–perception by itself can never provide us with a scientific truth, since it only supplies us with a fact, never the *reason* for it.

The ancient Greeks[2] clearly understood the difference between the notions of *continuity* and *discreteness*. Nevertheless, they often confused the notions of *continuity* (a property of a function) and *continuum* (a property of a set). There appears to be no evidence to suggest that any attempt was made to distinguish these concepts, until Leibniz (1646–1716) commenced his investigations into the nature of the continuum.

7·1·2 Leibniz

Leibniz's theory of the continuum is very complex, and as mentioned earlier even Bertrand Russell admitted experiencing considerable difficulty with this topic.

---

[1]See Sorabji[1983,p.358].

[2]Aristotle studied the notions of continuity in some detail, his treatment can be found in Physics (VI). See Sorabji[1983].

Nevertheless, Leibniz made a significant contribution to our understanding of the continuum, and no study of the topic would be complete without mentioning his approach.

According to Russell[1900], Leibniz was motivated by the fact that matter (which he carefully distinguished from space) is extended[1]. As far as Leibniz was concerned, extension just like duration is a property of a body, therefore a body has its own extension and duration, but not its own space and time. This is just one example of the kind of subtle distinction made by Leibniz in his work, which can only be fully appreciated nowadays because of modern developments in topology and the theory of relativity.

Like Plato and Aristotle before him, Leibniz distinguished between different kinds of points. He classified these into: *metaphysical points*, *mathematical points* and *physical points*. Physical points are what we see, they have an extension and are not truly indivisible. Mathematical points on the other hand are purely abstract, have no extension and are not real. According to Russell[1900,p.104], Leibniz viewed mathematical points are *"mere modalities, being a mere aspect or quality of the actual terms, which are metaphysical points or monads"*. Thus Leibniz concluded that a body consisting of mathematical points is not real but something mental.

Leibniz was trying to differentiate between the the *actual* and the *ideal*, and this naturally lead him to consider notions of perceivers and perception. As MacDonald Ross[1989] explains, we commonly understand the world as consisting of objects of perception, which are separate from, and common to, different perceivers. These objects of perception were considered to be mental constructs by Leibniz. Moreover, Leibniz held that if one presented a cube to a number of people, then each would have their own perceptual image of the cube, none of which would correspond to the standard geometrical description of a cube. Leibniz maintained that the only channel of

[1] *"A body can change space, but cannot leave its extension"*, Russell[1900,p.101].

communication between perceivers is through their perceived bodies[1], thus the perceived body is of considerable importance and it is what we conceive as immediate reality.

Leibniz also distinguished between sensation and perception. Suppose we touch a hot object, this gives us a sensation of heat and hence we *perceive* the object to be hot[2]. Thus the heat of the object causes us to have a sensation (i.e we experience the heat), but learn to identify the fact that the object is hot by having this sensation. This point is pertinent to AI since it describes *hot* as something which we learn to identify based on the *sensory experience* of *heat*. Thus our understanding of the concept of *hot* is based on sensory input, and therefore cannot be construed as some arbitrary mental construct. In spite of the fact that we can construct a theory for *heat*, we cannot construct a theory of *hot*, as for example Fehrer[1990] intends.

Leibniz's investigations into the continuum involve notions which are associated with perception and the perceiving entities. He does however discuss mathematical continua employing terminology which intuitively mean some kind of collection or aggregate, but they are not defined in a manner which we may recognise as a set[3]. The concept of a set is much clearer in the articles published by Bernard Bolzano.

7·1·3 Bolzano

While Bolzano's contributions to the study of continua is unquestionable, his contribution to our understanding of the notion of a *set* is rarely mentioned. Bolzano was the first to attempt a definitive theory of sets, evidence for which can be found in Bolzano[1820–30] and the famous *Paradoxien* (Bolzano[1851]). His work on the infinite was more philosophical than mathematical, but one can still find words which appear later in

[1]MacDonald Ross[1989,p.89].

[2]Further discussion of this point can be found in Ishiguro[1990,p.65].

[3]Russell[1900,p.245]. It is significant that Leibniz believed the very notion of a mathematical continuum necessitated the use of some kind of infinity, but expressed no such opinion about the other forms of continua.

Cantor's formalisations. For example, Bolzano[1] explains the meaning of *'class'* ['Inbegriff'] by providing a number of near synonyms like 'collection' or 'aggregate'. The notion of a *set* ['Menge'] is then introduced as a class where the manner of connection between the elements is not specified.

Having presented a definition for a set, Bolzano then defines what is meant by the assertion that two sets are to be considered as *equivalent*, a concept we use to this day. Building on these concepts Bolzano introduces the concept of a *unit* and that of a *manifold*. A *unit of kind A* is defined to be an object which has a certain property A; a *manifold* (*of kind A*) is a class whose parts are units[2] of kind A. Having developed a rudimentary set theory Bolzano then presents the following definition for a (mathematical) continuum

> *"... A **continuum** is present when, and only when, we have an aggregate of simple entities (instances, points or substances) so arranged that each individual member of the aggregate has, at least one member of the aggregate for a neighbour." [my emphasis], Bolzano[1851,p.129].*

Bolzano[1851] then describes an ***isolated point*** as one which fails to be so thickly surrounded by neighbours as to have at least one at each individual and sufficiently small distance from it. Thus any collection of points containing an isolated point will fail to constitute a continuum – by definition.

Despite making some frank criticism of Bolzano's work, Georg Cantor (1845–1918)[3] still recognised Bolzano as *"... an extremely acute philosopher and mathematician"*. It is Cantor's work which I shall turn my attention to next.

7·1·4 Cantor

Cantor's pioneering work on set theory was brought about by the need to express certain ideas in the theory of real numbers. Cantor's detailed criticism of Bolzano's work

[1]According to George (Bolzano[1820–30,p.126]).

[2]Bolzano[1820–30,p.129].

[3]Bolzano[1852,p.38].

demonstrates that Cantor had studied the material at some length and was familiar with this work, in which Bolzano had already laid the foundations of naïve set theory. The benefit which Cantor must have received from Bolzano's work must therefore be acknowledged.

Cantor's work was closely related to that of Richard Dedekind who's theory of irrational numbers involved a notion which Dedekind called a 'cut', details of which can be found in his book (Dedekind[1901]). Cantor criticized Dedekind's approach on the basis that the 'cuts' defined by him do not appear naturally in the subject matter of analysis. Instead, Cantor approached the problem by assuming the existence of rational numbers and then constructing[1] the class of 'real numbers'. The class of real numbers include the irrational numbers, each one of which is an *infinite* collection of symbols, in marked contrast to the pair of symbols required to represent a rational. Constructing the set of real numbers was the first step towards Cantor's dream of discovering the nature of the continuum.

Cantor[1872] (and Dedekind[1901]) had already criticised the traditionally assumed *continuity* of space, in particular, Cantor had already stressed that the connection between the real numbers (the *arithmetical* domain) and the points of the real line in geometry (the *geometrical* domain) was arbitrary, since it was axiomatic. This led Cantor to conclude that continuity of space was little more than a free construction of the mind, with no guarantee that it conformed to the reality of phenomenological space. Ten years later (in 1882) Cantor was able to use his set theory to demonstrate that continuous motion was possible in a discontinuous space[2].

Cantor[1883a] draws upon the theory of real numbers to develop a purely arithmetic analysis of the continuum[3]. Starting from an n-dimensional arithmetical space G_n, that is to say the totality of n-tuples $\langle x_1, \ldots, x_n \rangle$, in which each x can take any real value, together

[1]Details of this construction can be found in Stewart&Tall[1977].

[2]See Cantor[1882] (Cantor[1932,p.156]), or Dauben[1979,p.85] for details.

[3]According to Dauben[1979,p.107] Cantor edited the Grundlagen (Cantor[1883b]) and placed the original section 10 at the very end, to emphasize the important role of the continuum in directing and motivating his research.

with the normal Euclidean distance function defined on G_n. Cantor posed the question: under what conditions would a set of points P from the space G_n constitute a continuum?

According to Cantor, a set (such as P) constitutes a ***continuum*** in G_n if it possesses two basic properties, namely, that of being *connected* and of being *perfect*[1]. By connected Cantor meant the following: a point set P is called *connected* if for every two of its points t and s, and an arbitrary given positive number ε, there always exist a *finite* number of points $t_1, \ldots, t_n$ of P exist such that the distances $tt_1, t_1t_2, \ldots, t_ns$ are all less than ε. Cantor then states:

> *"All the geometric point-continua known to us are, as is easy to see, connected; and I believe, now, that I recognize in these two predicates 'perfect' and 'connected' the necessary and sufficient characteristics of a point-continuum." Cantor[1932,p.194] and Cantor[1915,p.72].*

Cantor criticized Bolzano's definition for only capturing one property of the continuum, namely that of being connected[2]. Similarly, he criticized Dedekind's work, for only emphasizing the other property, namely, that which it shared with all other perfect sets.

At about the same time that Cantor and Dedekind were constructing their set theory in Europe, Charles Peirce(1839–1914) was constructing one of his own in the United States. Peirce made many major contributions in a number of different fields, including a system of first order logic similar to that developed by Frege.

7·1·5 Peirce

Peirce starts his analysis by posing the question[3]: "What is continuity?". He first considers Kant's definition of a continuous series, namely, that between any two points of the series a third can always be found. Peirce accused Kant of confounding *infinite divisibility* with *continuity*, arguing that such series possessed certain 'gaps'. For example, the rational

[1]For a definition of this property see chapter 3.

[2]In a footnote Cantor[1932,p.208] defines a ***semicontinuum*** as a connected but imperfect set.

[3]Peirce[1930–58,§6.164–6.213].

numbers satisfy the infinite divisibility criterion but also contain 'gaps', where the irrational numbers such as $\sqrt{2}$ should be.

Peirce then turns his attention to the work done by Bolzano, and praises him for having

> *"... put Human Reason under an eternal debt by laying the foundations of this science, ...", Peirce[CP,§6.175].*

Despite making complimentary comments towards Cantor, Peirce actually attacks Cantor's definition on the grounds that: *"... it turns upon metrical considerations; while the distinction between a continuous and discontinuous series is manifestly non-metrical"*, Peirce[1930–58,§6.120]. Peirce is of course correct. The notion of continuity or a continuum should not be based on the notion of a metric. However, the conditions stated by Cantor may be re–stated in topological terms, which would avoid these difficulties.

7·1·6 The modern definition of a continuum

The definition of a continuum which is in use today may be stated as follows:

<u>Definition</u> ([Hocking&Young[1961,p.43]): A compact connected space is called a ***continuum***.

Often the definition of a continuum also requires the space to be Hausdorff (see for example, Eisenberg[1974,p.348]).

This definition is essentially that formulated by Cantor[1932] and *incorrectly*[1] translated by Russell[1903]. This definition for a continuum has come about because Russell[1903,p.100] translated Cantor's notion of a space being *dense–in–itself* [überall dicht] as *compactness*. The difficulty here is that the real numbers, which everyone agrees are a paradigm for a continuum, fail to satisfy this definition on the grounds that they do *not* form a compact space.

[1]Here Russell describes compactness as the condition *"that between any two terms of a series there are others"*. See Huntington[1905,p.176].

Cantor considered such entities as curves and surfaces to be continua (in the mathematical sense). These entities are still studied by mathematicians today, except they are now called *manifolds*[1].

It is rather intriguing that Cantor used the term 'manifold' [Mannigfaltigkeit] in his work. For example, in the series of five articles which deal with the real numbers, Cantor gives the title: "Über unendliche lineare Punktmannigfaltigkeiten" which literally translates to: 'concerning infinite linear point–manifolds'. However, Cantor uses the word 'set' interchangeably with the word 'manifold', sometimes even defining them to mean the same thing. This situation is somewhat puzzeling, since by the time Cantor was publishing his work, the word 'Mannigfaltigkeit' had already been given an intuitive meaning by Gauss and Kant, and a formal meaning by Bolzano. Cantor was well versed with the works of Bolzano, and would therefore have been aware of the careful manner in which Bolzano distinguished these words. Why Cantor chose to ignore Bolzano's definitions is not clear.

The term 'manifold' has its roots in a famous article by Riemann[1854] (which is an extract form his doctoral dissertation, translated in Clifford[1873]). In this article, Riemann explains that there are essentially two different types of manifolds, the *discrete* and the *continuous*. Riemann states that discrete manifolds are very common, unlike continuous ones which are rare and the only simple example of one are the "... positions of perceived objects and colours". This may explain why mathematicians today use the notion of manifold in place of the notion of continuum as defined by Cantor.

7·1·7 Poincaré

The property of being connected, that is to say it should be in one piece, is very much at the heart for the notion of a continuum and therefore must also be a feature of the perceptual continua. Density, however, is another matter.

[1]Manifolds are mathematical structures used in such areas as differential geometry and topology. See Choquet–Bruhat[1982] for details.

Density is the property which allows one to reason that there is no smallest number between 0 and an arbitrary small number, or that every interval contains a sub–interval. Thus *density* is the property which one can most easily describe as *infinite divisibility*.

Recall the example of the continuous chalk line drawn on a blackboard[1], described in §7·1·1, which one can analyse using conventional mathematical techniques involving such entities as dimensionless points. However, this chalk line is *perceptually* continuous, that is to say it is an indivisible whole, whose components can only be individuated by our intellect. Therefore, objects of our perception do not contain points, or any other entity which may be considered as infinitely divisible, but is made up of *quanta* of sensation which are only divisible by our imagination.

Poincaré was one of the first mathematicians to acknowledge that the perceptual continuum (or 'physical continuum' as he called it) *did* have a structure of its own which could be analysed. However, since this view of the continuum was radically different to the conventional (mathematical) one, anyone considering such an analysis would need to be careful about the tools and techniques used in such an endeavour.

In his 'Science and Hypothesis', Poincaré[1905] demonstrated in a precise manner the paradoxical nature of a perceptual continuum that arises if conventional mathematics was used to express it. He argues that the perceptual continuum is derived from the immediate evidence of our senses, and proceeds to describe the following experiment. A weight E of 10g and a weight F of 11g produce indistinguishable sensations. However another weight G of 12g, while producing an indistinguishable sensation to F, is readily distinguishable from E. Poincaré[1905,p.22] then concludes that the results of the experiment may be *"... expressed by the following relations: $A = B$, $B = C$, $A < C$; which may be regarded as the formula of the physical continuum"*. Here, Poincaré is demonstrating how equality on the perceptual domain fails to be transitive.

[1]From Grünbaum[1968].

In a later article Poincaré[1913, p.241] he states that: "Each of the elements of this continuum consists of a manifold of impressions ...", two elements of which are either distinguishable or are not. These elements are analogous to the points in geometry, except that one cannot say that these have a no size.

7·2 CONTINUITY AND THE CONTINUA

7·2·1 Phenomenological continua

In the previous section I discussed how the notion of a continuum has evolved over the years, and how the scholars who investigated the characteristics of this concept have emphasized the difference between the (formal) notion of a mathematical continuum and that of our perception. Clearly the two notions share certain properties, but the distinctions are also important.

As defined by Cantor, a *mathematical* continua consists of a *set* together with a spatial structure which has the following properties:

(i) connectedness and

(ii) perfectness.

Typically, a mathematical continua is identified with the idea of a geometrical surface, in which case the underlying set consists of points which have *no* extension (i.e. they have *no* size), and in this case perfectness is identified with infinite divisibility.

A *phenomenological* (or *perceptual*) continua on the other hand is best described by the example of the chalk line drawn on the blackboard. It too can be described by a set together with some relation between its elements. In this case however, the elements do have a finite size, and therefore the property of infinite divisibility is not suitable. Following Poincaré I shall now define a *phenomenological* continua.

<u>Definition</u>: An ***phenomenological*** (or ***impression***)***continua*** is a connected tolerance space.

A phenomenological continuum shares a number of feature with a mathematical one. For example, they are both connected, and remarkably enough a phenomenological continua is also perfect (although one needs to consider the condition of being perfect in a different light to infinite divisibility). Recall from §3·3·3 that a perfect (topological) space contained no isolated points, which also makes sense in a tolerance space, therefore define a tolerance space to be perfect when it contains no isolated points. It is clear form this definition that any space that was connected would also be perfect – since there would be a finite path between every point in the space.

It is an interesting point to consider whether the converse of the above is also true, which seems to depend largely on how we define an isolated point in this context. Clearly, the notion of an isolated point has to be related to the idea of a path between points in a tolerance space, but do we wish to define an isolated point to be one which just fails to have a *finite* path to it from any other point in the space, or should we disallow *countable* paths also? If one defined an isolated point to be one which did not have any finite paths connecting it to any other point then we are encroaching on the topic of connectedness. I believe the way forward is to allow countable paths for the case of connectedness, but I have not investigated this further.

7·2·1 The distinction between continuity and continua

The distinction between the concepts of continuity and the continuum was not made until the end of the last century, by mathematicians such as Bolzano and Cantor. It is a particularly important distinction for the purposes of this thesis, and one which I shall now emphasize.

Continuity is a property of a function.

Thus a continuous function is a structure preserving function, one which preserves the *nearness* structure defined on its domain.

A continuum is a characteristic property of a space.

Thus a set may be deemed to be a continuum only after a spatial structure has been assigned to it which is perfect and connected.

In the next chapter I shall use this distinction to discuss why I believe Zadeh's analysis concerning vague predicates as those which take values in a continuum, to be an inaccurate one.

7·3 IMPRESSION SPACES

In the first section of this chapter I discussed how the notion of a *mathematical* continuum was distinct from that of a *perceptual* continuum. This distinction has been made by mathematicians and philosophers alike since the time of Parmenides and Zeno. Nonetheless, very few articles have devoted any time to the rigourous development of the perceptual continuum, much of the scholarly effort being concentrated on capturing the mathematical notion instead.

Poincaré was the first mathematician to study the nature of the perceptual continuum, and reports that conventional mathematics (as it stood at the time) was unsuitable as a mechanism for representing this notion. In fact, he went on to claim that the structures available in conventional mathematics could not in principle capture this notion, and that a new type of structure had to be constructed.

Much of Poincaré's work is philosophical in nature, and a rigourous study of the kind of structures Poincaré had in mind were not undertaken until Zeeman[1] constructed his *Tolerance Geometry* in 1962. In a series of three articles (Zeeman[1962,1965,1968]) Zeeman describes how his theory may be used to formulate a mathematical model of human visual perception.

[1]Ironically, Zeeman developed his ideas independently of Poincaré's studies.

7·3·1 Sensors and perception

Human visual perception is a topic which has been at the centre of much discussion both in philosophy and psychology. However, I do not wish to discuss the intricacies of this subject here, other than to define a very simple model which will be more than adequate for my purposes.

Our bodies are genetically supplied with millions of sensors[1]. These continually detect objects and changes in our external and internal environment. For example, Minsky[1987,p.111] explains that our skins are constructed with a myriad of nerves that run from every skin spot to the brain and enables our skin to *feel* its environment.

Every sensory experience[2] involves the activity of many different sensors. For example, an object touching the skin will undoubtedly activate many sensors there, producing a collection of neural stimuli in the brain. This collection I shall call an ***impression*** of the object. An ***impression space*** is the collection of all possible neural stimuli produced in the brain by the same type of sensors.

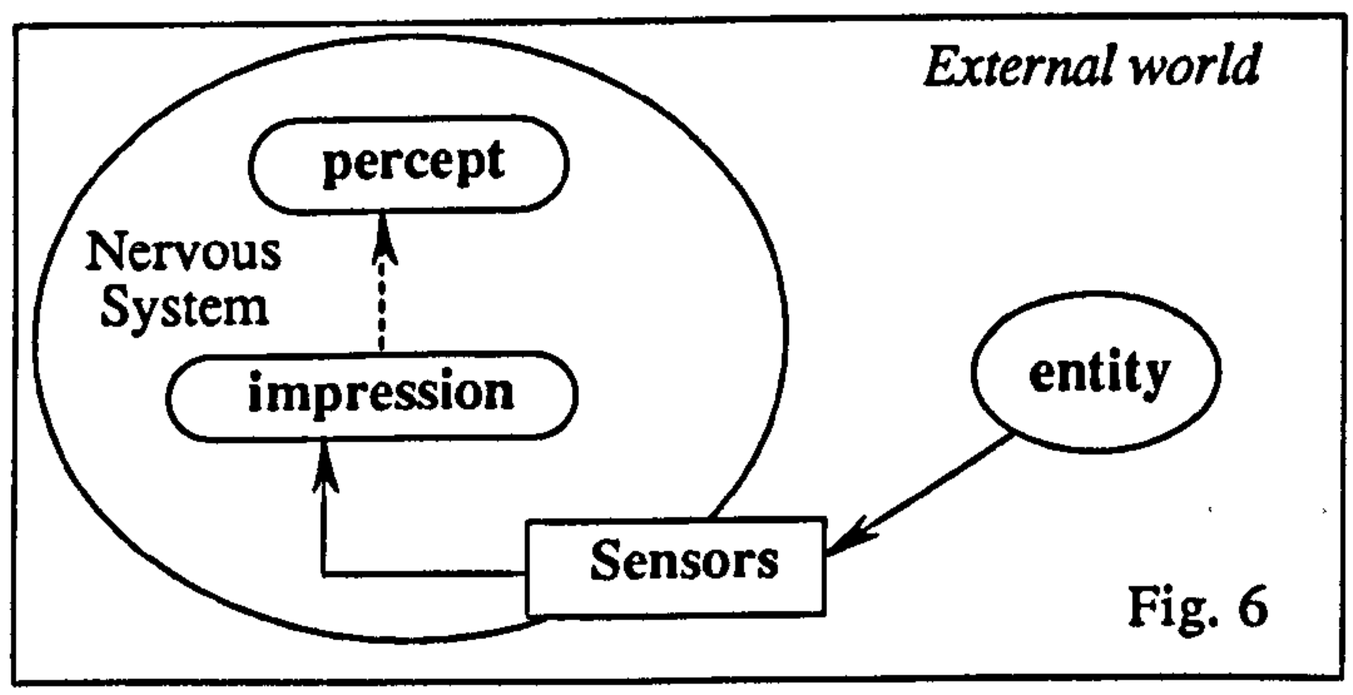


Fig. 6

The sensory mechanism works by converting external energy to a neural impression. This is then processed by the brain to produce a percept (see Fig. 6). We are aware from psychological experiments that a single object, which gives rise to a single impression, can generate two different percepts (for example, the Necker cube. See Attneave[1971] or Poston&Stewart1978]).

[1]"External event detecting agents" as Minsky[1987,p.113] describes them. Also see Arbib[1972].
[2]Terminology used by Minsky[1987,p.111].

The impression of an object is not directly known – one can never describe one's nerve impulses to another human. We do, however, know of their existence from experiments in physiology and neurology. What *is* known to the conscious mind is the *percept*, an entity which is constructed by the mind from the information contained in the impression of the object. This percept may be considered as a 'mental' representation of the physical object and it is this which is stored by the brain rather than the impression[1].

To clarify some points concerning the ideas just described, I shall assume that there is an external world which contains, us together with other objects. These objects interact with our sensors to produce neural impressions in our brains. These impressions (or stimuli) are not the same as the objects which caused them, neither are the percepts constructed by the mind from the impressions, the same as the objects, or the impressions of them.

There are a number of relationships here, including that between the objects, that between the collection of objects and the impressions they induce in the brain (via the sensory mechanisms), and the relationship between the impressions and the percepts.

In order to discuss these relationships in a formal manner, let

$\mathcal{O}$ represent the set of *objects*;

$\mathcal{S}$ represent the set of *sensors*,

$\mathbb{I}$ represent the *impression space*.

Minsky's statement regarding the relationship between the configuration of sensors in the skin and the configuration of respective neural stimuli in the brain, can be formalised, provided one accepts the following postulate.

<u>Postulate 1</u>: The impression space $\mathbb{I}$ is a tolerance space $\langle \mathbb{I}, \xi \rangle$.

This postulate basically asserts that nearness for neural stimuli is hardwired into our brains.

---

[1]I am aware that the hypothesis I have just presented is in the oldest area of philosophy, namely our knowledge of the external world. However, I do not wish to discuss the philosophical implications of them in this thesis. Interesting articles in the literature include Russell[1972] and Peirce[1930–58,§2.141].

There are two ways of interpreting Minsky's statement that:

nearby places on the skin are wired to nearby places in the brain.

The first interpretation is to assume that we only have the concept of nearness for neural stimuli, together with the fact that sensors are *wired–up* to the brain. In other words, there is a function, $\varphi : \mathcal{S} \rightarrow \langle \mathbb{I}, \xi \rangle$, from the set of sensors to the impression space.

The second interpretation is that we actually have a notion of nearness for the sensors as well as for the neural stimuli, in which case we need a postulate that states the following:

Postulate 2: The collection of all sensors $\mathcal{S}$ constitute a tolerance space $\langle \mathcal{S}, \delta \rangle$.

In this case, Minsky's statement can be formalised as a requirement that the function, $\varphi : \langle \mathcal{S}, \delta \rangle \rightarrow \langle \mathbb{I}, \xi \rangle$ be a tolerance function. Of these alternatives I prefer the first, since it makes fewer assumptions about our physiological make–up.

Formalisation of Minsky's statement

Given the function $\varphi : \mathcal{S} \rightarrow \langle \mathbb{I}, \xi \rangle$, one can form a tolerance space on $\mathcal{S}$, simply by pulling ξ back across φ, to give:

$$\varphi : \langle \mathcal{S}, \varphi^{-1}(\xi) \rangle \rightarrow \langle \mathbb{I}, \xi \rangle$$

This construction guarantees that the function φ is a tolerance function. For each object $O \in \mathcal{O}$, there exists a function, $\eta_O : O \rightarrow \langle \mathcal{S}, \varphi^{-1}(\xi) \rangle$ which essentially describes the interac of the object with our sensors, and hence provides a formal description of how we obtain an impression of the object $O \in \mathcal{O}$.

We may now obtain a tolerance for the points on the object by pulling $\varphi^{-1}(\xi)$ back across η_O,

$$\eta_O : \langle O, \eta_i^{-1}(\varphi^{-1}(\xi)) \rangle \rightarrow \langle \mathcal{S}, \varphi^{-1}(\xi) \rangle.$$

This function defines two points to be indistinguishable on the object if they are indistinguishable in the space $\langle \mathcal{S}, \varphi^{-1}(\xi) \rangle$. Since the indistinguishability of points in $\langle \mathcal{S}, \varphi^{-1}(\xi) \rangle$ is determined by the indistinguishability of their impressions, this means that

two points in the object are indistinguishable if their impressions are indistinguishable. In other words, the concept of nearness for points on an object is dictated by the concept of nearness in the impression space.

We obtain an impression of an object as follows. An object $O \in \mathcal{O}$, is detected by a number of sensors $\eta_O(O) \subseteq \mathcal{S}$. These in turn produce an impression $\varphi(\eta_O(O)) \subseteq \mathbb{I}$. If $\varphi(\eta_O(O))$ is connected then we have an impression of the object O which is unbroken, something we would intuitively describe as *continuous*.

Our intuition of an object being connected is very much dependent on the impression of that object, and is clearly related to the sensory mechanism used to detect it (it may differ depending on the mechanism). For example, a table top is considered to be connected at the macroscopic level, while at the microscopic level it consists of molecules which by their very nature are discrete entities. In this scenario the table top is physically *discrete* but is detected in such a way as to cause a connected impression, and is therefore considered to be *continuous*.

Describing the impression of an object in this manner has the advantage that objects in $\mathcal{O}$ may change without their impressions having to change, since the change in the object may be *too* small to be detected by the sensory mechanism being used.

It also gives us the apparatus to describe notions such as the *indistinguishability* of objects. To do this we need to construct the induced tolerance space, $\langle \mathbb{P}(\mathcal{S}), \Delta \rangle$, where Δ represented the tolerance induced by $\varphi^{-1}(\xi)$. Given two objects, $O_1, O_2 \in \mathcal{O}$, there exist associated functions,

$$\eta_{O_1} : \langle O_1, \eta_i^{-1}(\varphi^{-1}(\xi)) \rangle \rightarrow \langle \mathcal{S}, \varphi^{-1}(\xi) \rangle \quad \text{and} \quad \eta_{O_2} : \langle O_2, \eta_i^{-1}(\varphi^{-1}(\xi)) \rangle \rightarrow \langle \mathcal{S}, \varphi^{-1}(\xi) \rangle$$

which map the objects into the space $\mathcal{S}$.

The images, $\mathrm{Im}(\eta_{O_1})$ and $\mathrm{Im}(\eta_{O_2})$, are both subsets of the space $\mathcal{S}$, in other words, $\mathrm{Im}(\eta_{O_1}), \mathrm{Im}(\eta_{O_2}) \in \mathbb{P}(\mathcal{S})$, thus we can use the tolerance Δ, on $\mathbb{P}(\mathcal{S})$ to define the indistinguishability of the two subsets, $\mathrm{Im}(\eta_{O_1})$ and $\mathrm{Im}(\eta_{O_2})$.

We can make use of the above definition of indisitinguishability and invoke Poston's description of motion (§3·4·5) to describe our impression of continuous motion formally. In keeping with Poston, I shall describe the position of the object O, by a function $\eta_O \in \mathcal{F}(O,\mathcal{S})$, which consists of all the positions of the moving object[1] (clearly, as the object O moves there will be a number of these functions). This approach now ties in with Poston's example in §3·4·5, where he described the motion of a body by the function, $\mathbf{m}: \text{TIME} \rightarrow \mathcal{F}(O,\mathcal{S})$.

The conventional approach used to describe continuous involves the use of real numbers as a model of time, and has all the drawbacks of attempting to unite the two notions of continua which have clearly been distinguished in the earlier sections of this chapter.

The material discussed in this section justifies the use of the term 'impression continua' as defined in §7·2·1, since it is our *impression* of the object which determines whether it is a continuum or not, and not the *physical* make–up of the object itself.

7·4 SUMMARY

In this chapter we have seen how the notion of a continuum evolved throughout history. In particular we should note the important distinction between a mathematical and perceptual continuum; a distinction which was first made by Parmenides and Zeno, who constructed paradoxes to demonstrate that such a distinction was necessary.

The mathematical continuum was finally formalised by Cantor at the end of the last century, when he defined a (mathematical) continuum to be any set which possessed the properties of *perfectness* and *connectedness*. One can clearly observe that the very definition of a continuum requires a spatial structure to be given.

In §7·2·1, I discussed the distinction between a mathematical and phenomenological continua. Here, I stressed that a *mathematical* continua had the property of being perfect,

[1]Since the configuration involving the space, $\mathcal{S}$ and $\mathbf{I}$, are fixed, we may safely ignore the extra level of mapping from the sensor space to the impression space by considering two points to be indistinguishable if they belong to $\varphi^{-1}(\xi)$.

which in this context equated to being infinitely divisible. I claim that this is an unnecessary property for our perception of a continua, and hence define an *impression* continuum to be a connected tolerance space. As it turns out this continuum may also be defined to be perfect if we take the requirement of perfectness to mean the absence of isolated points.

In §7·2·2, I emphasized the important distinction between the concepts of a *continuum* and that of *continuity*. One which is the property of a domain, whilst the other is a property of a function. The confusion between these two concepts lasted from the Ancient Greeks to the end of the last century, when Bolzano and Cantor clarified the situation.

When defining a continuous function we must declare a spatial structure, so that the property of continuity may be verified with respect to this particular structure. It makes no sense to call a function continuous without first declaring a spatial structure for it to preserve. The final section (§7·3), I described how a body appears to be *continuous* (to us as perceivers), despite the fact that it is actually physically discrete. One may use this theory to describe continuous motion.

The description of a continuous manner adopted here is that which is commonly described as cinematographic, in other words the body moves in imperceptible jerks. Formally, this is represented by the body having indistinguishable impressions at indistinguishable positions in space, and for us to perceive continuous motion the body must be in indistinguishable positions at indistinguishable times.

Philosophically, this position is very interesting. Since we have a threshold built into our sensory mechanism such that movement which is below the threshold will not be detected, this means that objects in the physical world may be deformed in certain ways without being detected. For example, a body may change its position in the physical world without our sensory mechanism detecting such a move, but since the body has *actually* moved, we may *not* describe it as being in the same position. We must therefore

acknowledge the existence of the concept of *indistinguishable* states, and hence the existence of motion which appears *continuous* in spite being physically *discrete.*

Russell[1972,p.120] was therefore correct when he stated that there was no reason

> *"...to suppose that the space of our immediate experience possesses mathematical continuity."*

8 Spatial Structures in Metalogic

"I believe that we lack another analysis properly geometric or linear which expresses location directly as algebra expresses magnitude."
Gottfried Wilhelm Leibniz

In this chapter I shall take a formal approach to the various mathematical issues and philosophical viewpoints, regarding the phenomenon of vagueness raised in earlier chapters. I shall start the chapter with a brief look at the article by Tarski which motivated this work.

Some metalogical conditions are then presented, which I shall require any logic to satisfy. I then turn my attention to the geometrical issues of predicate interpretations, presenting ideas which I shall use in my analysis of vague concepts.

The fuzzy set theorists assert that vague predicates should not possess a sharp boundary, but should allow for a gradual transition from membership to non–membership. They claim this can be achieved by requiring the predicates to take values in a *continuum*. Unfortunately, words such as, 'continuum' or 'boundary' (which are used in these articles) can only be defined formally in the context of a *geometry*, which is not given.

In this chapter I shall argue that the notion of a geometry is essential to a correct analysis of vagueness. In the analysis I shall shortly present, demonstrates that the difficulty with

vague predicates is not that they do not take a value in a *continuum*, but that we require them to be *continuous*.

If we accept that vague predicates are to be considered as continuous, and define them on some continuum, then the mathematics of spatial structures dictates that the image of the characteristic function for this predicate will also be a continuum. However, note that this is a consequence of the predicate being defined on a continuum, and not a part of the definition for a vague predicate, as is the case in fuzzy set theory. As a result, I argue that attempts to use fuzzy set theory, or fuzzy logic to capture vague concepts, are at best inadequate.

Having argued that interpretations of vague predicates are required to be continuous, I investigate the kinds of spatial structure that may be defined on the domain/codomain of these interpretation functions. This topic is covered in §8·7 – 8·9. In section 8·10, I demonstrate that the Sorites (as analysed in §5·2·2), is insoluble in a logic with a truth–set $\{0,1\}$.

In section 8·11, I increase the cardinality of the truth–set to three (namely, $\{0,u,1\}$), and describe the kind of spatial structures which may be defined on this set, as well as the domain of interpretation. I demonstrate that, given my metalogical conditions described in section 2, then there can only be *one* tolerance structure on this set, which I call ε. As a result of my findings, I describe (§8·12) the structure of a logic which is constrained by this tolerance space. I shall also discuss the reasons, why other well–known 3–valued logics fail to satisfy the conditions presented in §8·11.

In §8·14, I apply the 3–valued logic described in the earlier sections. In particular, I demonstrate that the Sorites can be resolved in a logic structured by the tolerance space $\langle\{0,u,1\},\varepsilon\rangle$. In the final two sections I discuss the interpretations of this logic, and state how one may define a vague predicates in a formal manner.

8·1 GEOMETRICAL TECHNIQUES IN LOGIC

The motivation for using geometrical techniques in a logic comes from Frege, who advocates the view that conceptual aids such as pictures can provide an invaluable aid to our understanding of the subject matter[1]. In particular, Frege[1988,p.139] describes how one can represent "... *concepts in extension by areas on a plane*". Developing this viewpoint, I shall reason that terminology used in examples concerned with vagueness actually involve terms which are *geometrical* in nature. Therefore, if we wish to interpret these terms in a *formal* manner, then we must introduce some kind of *geometry* into the metalanguage.

The relationship between logic and topology dates back to 1937, when Tarski[1937] first carried out his investigation into the formalisation of intuitionistic *sentential* calculus using topological techniques. I shall briefly describe this article here for two reasons. Firstly, because it is a landmark in the relationship between logic and topology. And secondly, this was the article which provided the initial impetus for this chapter. In this section I also draw crucial distinctions between this work and that to be presented in the rest of the thesis.

8·1·1 Tarski's work

The connection between sentential calculus and the calculus of classes (set theory) arises when sentential variables are interpreted as set variables ranging over the subsets of some (arbitrary) set $\mathbb{V}$. The logical connectives are interpreted as operations on these sets. The classical identities are those expressions which are assigned the value $\mathbb{V}$. This value is assigned to these identities irrespective of the subsets that are assigned to their constituent variables (see Mostowski[1966]).

In 1937, Tarski discovered a similar kind of connection between the *intuitionistic calculus* and the class $\mathbb{U}$, of subsets of $\mathbb{V}$ (with the exception that the sentential variables are only

[1]Clearly, one must exercise care in such cases.

allowed to range over $\mathbb{U}$, and the operations are the appropriate ones over $\mathbb{U}$). In particular, he was able to show that we obtain a polynomial identically equal to $\mathbb{V}$ if and only if the formula we started with was intuitionistically provable.

Tarski's major achievement was to show that if we considered $\mathbb{V}$ to be a suitable topological space[1], then $\mathbb{U}$ could be considered as the collection of all the *open* subsets of $\mathbb{V}$ (in other words, $\mathbb{U}$ was a topology on $\mathbb{V}$, in which the empty set $\varnothing$ would correspond to any false proposition). The connectives in this situation would correspond to the following operations on $\mathbb{U}$:

$$(p \wedge q) \longleftrightarrow (P \cap Q)$$

$$(p \vee q) \longleftrightarrow (P \cup Q)$$

$$(p \rightarrow q) \longleftrightarrow [(\mathbb{V} - P) \cup Q]^{\circ}$$

clearly, $(\neg p)$ corresponds to $(\mathbb{V} - P)^{\circ}$.

Tarski managed to prove that one could also use topological spaces in this manner to describe classical sentential logic, provided the topological space satisfied certain conditions such as being *totally disconnected*, (see Tarski[1937] for details).

There are a number of important differences between Tarski's discoveries and the work presented in this thesis. Firstly, Tarski worked with *sentential logic*, in particular matrix style propositional languages. Secondly, he was interested in replacing the truth-set $\mathbb{V}$ with a topological space $\langle D,\tau \rangle$, in order to find the connection between the topological space $\langle D,\tau \rangle$, and the intuitionistic calculus. The connection which Tarski sought, corresponded to the relationship that existed between the theory of classes and sentential calculus.

In contrast, I am interested in *first order logic*, in which the predicates are required to have structure preserving interpretations. My primary aim is not construct a logic which

[1]Tarski had the Euclidean plane, $\mathbb{E}^2$ in mind, however his results are more general.

parallels other languages, but one which will allow me to state the Sorites without introducing an inconsistency.

8·2 METALOGICAL ISSUES

As asserted by Machina[1976], it is important to describe the conditions which the metatheory will be required to satisfy. The specific conditions I shall require are as follows:

(1) the truth values, 'truth' and 'falsity', must be distinguishable;

(2) the logical system must be truth–functional;

(3) the logic must be normal[1];

(4) the connectives must be structure preserving.

Conditions (1), (2) and (3) are implicitly required by many formal logics, in particular by classical first order logic. Condition (4) is the only new one and will be discussed in detail later in §8·6.

8·3 GEOMETRICAL VIEWS OF PREDICATE EXTENSIONS

Following Frege, I shall present some possible geometrical views for the extension/ antiextension[2] of a predicate. The diagrams below indicate which objects fall under the extension of a predicate, and those that fall under its antiextension.

Diagram in Fig 7(a), depicts how the extension/antiextension of a predicate 'P(x)' is traditionally depicted as a Venn diagram.

Diagram in Fig 7(b), depicts the same information, except here the domain of interpretation is shown explicitly as the striped region. The extension/antiextension is shown as a *surface* suspended over the domain. However, there is no reason to assume that the pair should be depicted as a single surface. One can consider the extension/

[1]i.e. when the connectives operate on propositions with classical values, they should return classical values.

[2]The set of things which it is correct to deny the predicate.

antiextension as separate *surfaces* which cover the domain in some manner, such as that shown in Fig 7(c).

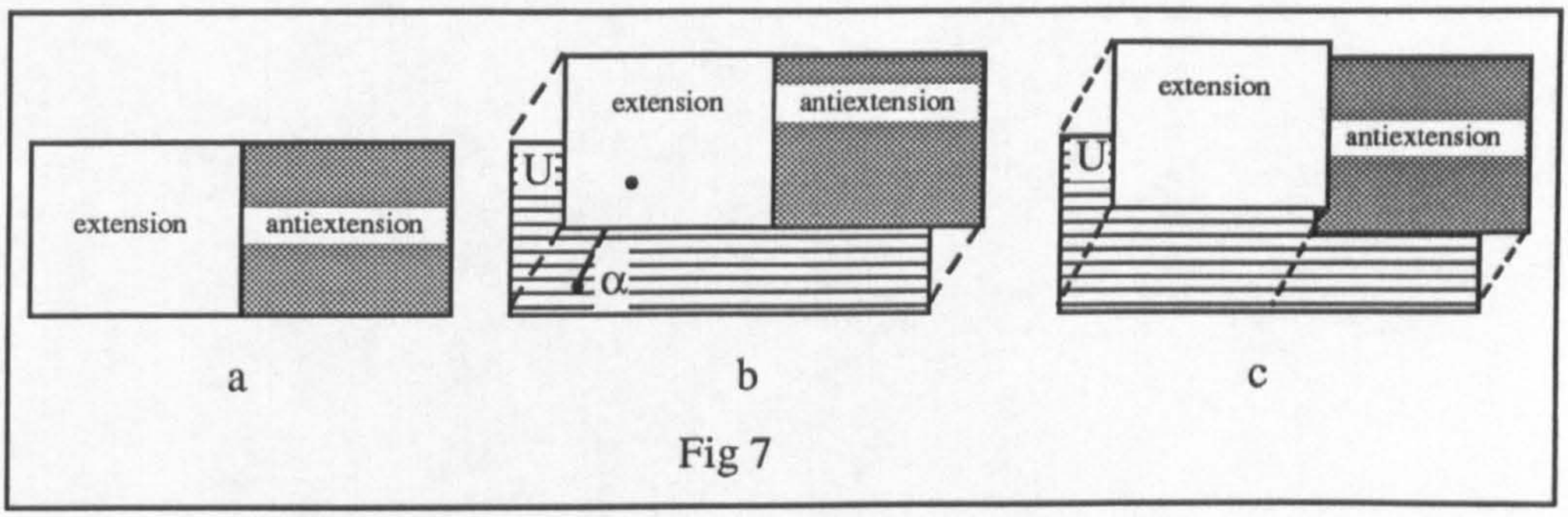


Fig 7

Recognising the change between the two views of the extension/antiextension as depicted in diagrams (b) and (c) is an important step towards an understanding the relationship between the two surfaces and the domain. Diagram (c) depicts the graph of a function which assigns a truth value to the sentence 'P(α)' for every object α in the domain U. In other words it is a *graph* of the interpretation for the symbol 'P'. This diagram is intended to convey the idea that an object α in the domain U satisfies the predicate 'P' (i.e. 'P(α)' is true).

8·4 THE INTERPRETATION FUNCTION

Many articles in the literature that discuss the concept of vagueness often restrict their presentations to monadic predicates, therefore I too shall begin my analysis of the interpretation function for the special case of monadic predicates.

8·4·1 Analysis of monadic predicates

Given a formal language $\mathfrak{L}$, the supporting logical structure $\langle |\mathcal{U}|,\sigma\rangle$ provides an interpretation of every symbol in $\mathfrak{L}$. The interpretation of a predicate symbol 'P' in the language $\mathfrak{L}$, is commonly described as a subset of the domain, $P^{\mathcal{U}} \subseteq |\mathcal{U}|$ which is mathematically equivalent to the function:

$$\chi_{PU} : |\mathcal{U}| \rightarrow \{0,1\}.$$

Notation: In the following, the symbol '$|\mathcal{U}|$' will be replaced with the symbol '$\mathcal{U}$' where no confusion is likely to arise. I shall also use the notation $P^{\mathcal{U}}$ instead of χ_{PU}.

Let us study the nature of the function $P^{\mathcal{U}}$ as a *fibration*[1] with fibres $P^{\mathcal{U}}{}_1$ and $P^{\mathcal{U}}{}_0$, (Fig 8).

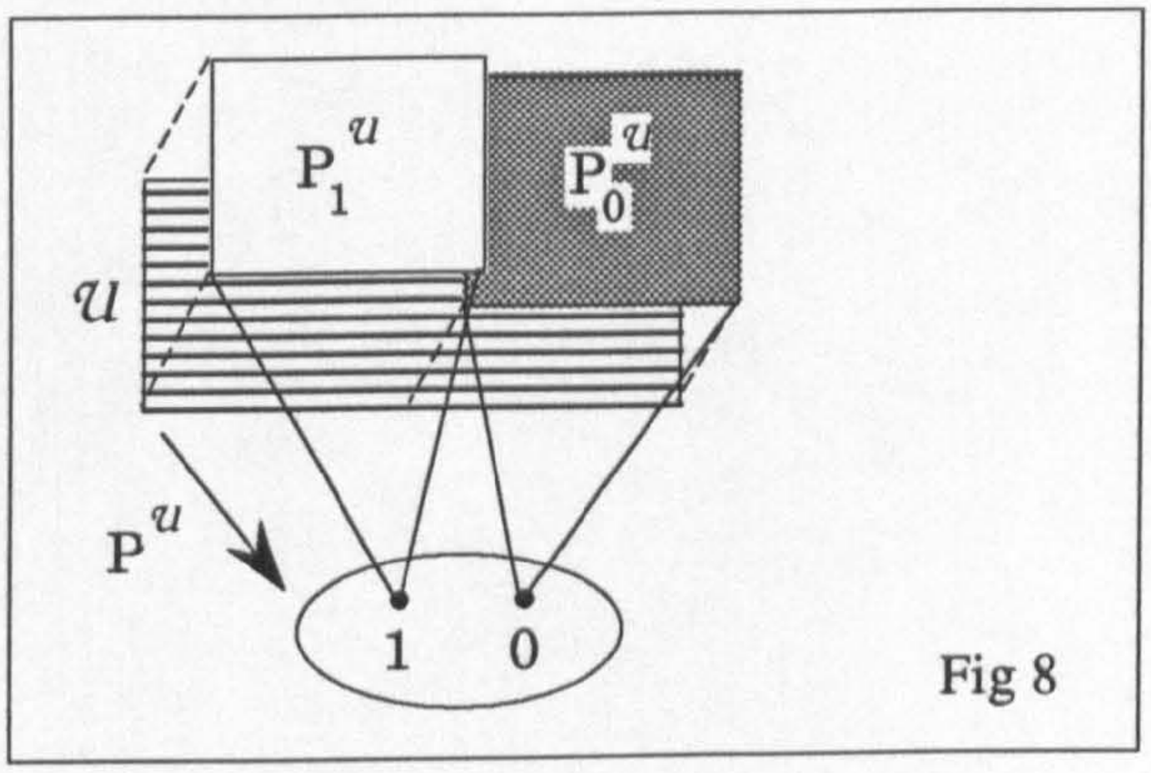


Fig 8

Since $P^{\mathcal{U}}$ is required to be a function, it must satisfy the following conditions:

(a) that every member of the domain must be mapped onto some member of the codomain;

(b) no member of the domain shall be mapped onto more than one member of the codomain.

One may translate these conditions into relationships between the fibres $P^{\mathcal{U}}{}_0$ and $P^{\mathcal{U}}{}_1$:

(†) $$P^{\mathcal{U}}{}_0 \cup P^{\mathcal{U}}{}_1 = \mathcal{U} \quad \text{and} \quad P^{\mathcal{U}}{}_0 \cap P^{\mathcal{U}}{}_1 = \varnothing.$$

Mathematically, these conditions on the fibres of a relation, express the necessary conditions to determine whether a *relation* qualifies as a *function* or not. One can now clearly see the connection between the equations in (†), and the logical principles[2] PV and PE:

$P^{\mathcal{U}}{}_0 \cup P^{\mathcal{U}}{}_1 = \mathcal{U}$ is a formalisation of PV; and

$P^{\mathcal{U}}{}_0 \cap P^{\mathcal{U}}{}_1 = \varnothing$ is a formalisation of PE.

---

[1]This concept is discussed briefly in Appendix A.

[2]The explanation of the terms PV and PE can be found in chapter 6.

Therefore we can assert that, from a mathematical point of view, the logical principles PV and PE are conditions which ensure the relationship between a domain and the set of truth–values (namely the interpretation) is always *functional* and never *relational*. In other words, the logical principles PV and PE ensure that the *interpretation function* is precisely that, a *function* and not just a *relation*.

Let us now concentrate on the fibres shown in Fig 9, and consider what happens when we increase the cardinality of the codomain. Fig 9 depicts how the fibres change when we change the codomain from {0,1} to {0,1,2}.

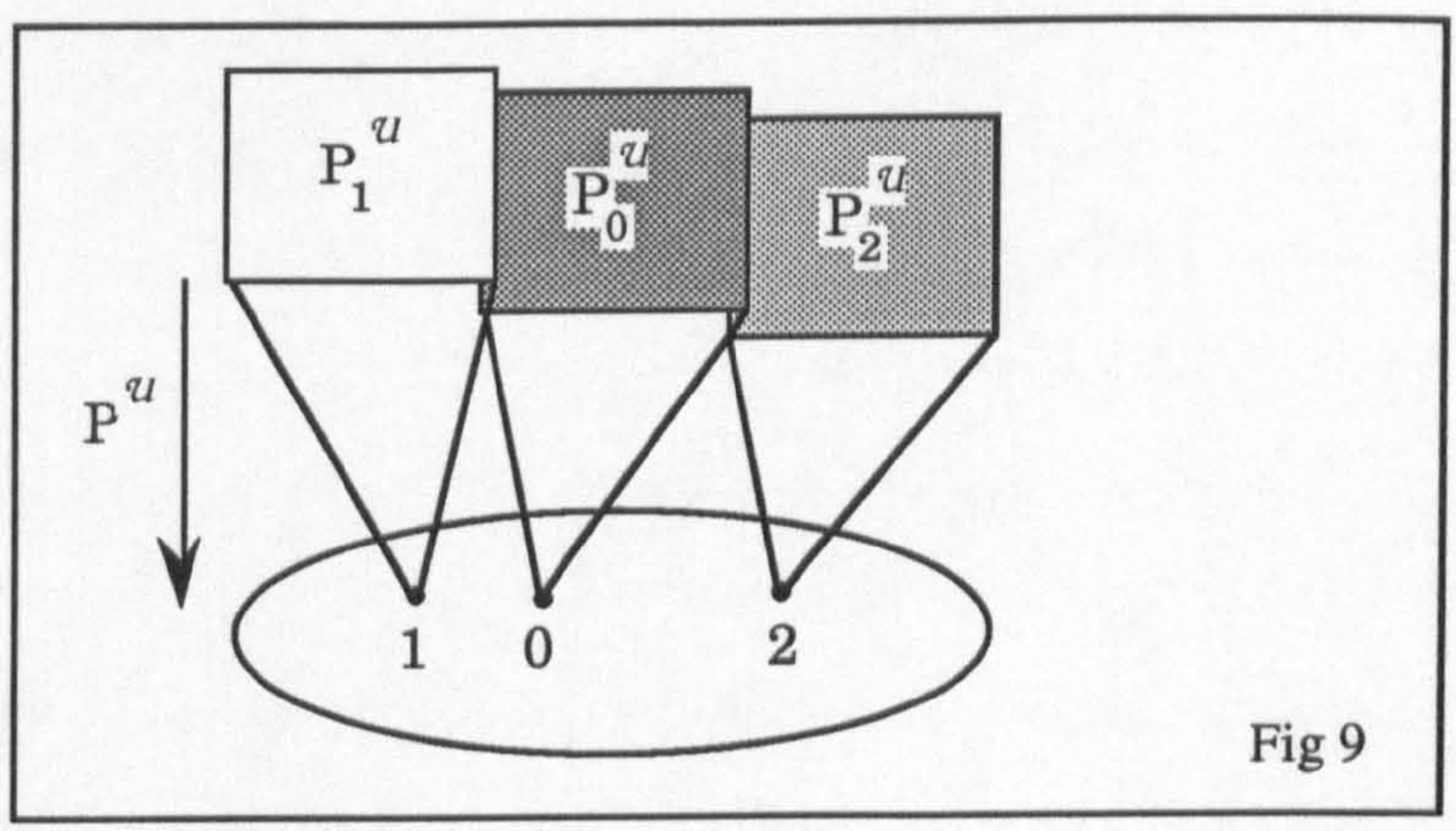


Fig 9

Note how the cardinality of the set of fibres increases as we increase the cardinality of the codomain. it is possible to prove that given a function there is a direct relationship between the cardinality of the set of fibres and the cardinality of the codomain.

It should be clear from the illustration that no matter how many truth–values we allow (that is to say no matter what the cardinality of the codomain happens to be) we shall always have a *sharp* cut-off point between the different fibres. Thus, if one accepts this particular analysis, one must also accept that simply by increasing the cardinality of the truth–values, we will not have resolved the problem of sharp boundaries. As we shall shortly see a fuzzy set is defined to have a truth–set which consists of the real numbers between 0 and 1. By our analysis, this will have the effect of dividing the domain into an

uncountable number of pieces (there is no reason to assume that we have any kind of cohesion between the different members of the set, which means we shall still have sharp cut–off points, only now we have an uncountable number of them).

8·5 FUZZY SETS

In order to understand the fundamentals of fuzzy set theory, and the connection between fuzzy sets and vague predicates, one needs to have a clear view of how set theory and logic are linked together.

There is an axiom in naïve set theory (called *the axiom of abstraction*) which associates a set with every predicate in a formal language. This axiom (or one of its derivatives) basically states that any open sentence '$\phi(x)$' defines a set A, where the members of A are precisely those objects for which the statement '$\phi(x)$' is *true*. Thus, the open sentence '$\phi(x)$' in x, acts like a condition which a given object must satisfy in order for it to be considered as a member of the set. Therefore, given a predicate and a domain, one may use the predicate as an open sentence and invoke the axiom of abstraction to generate a subset of that domain.

8·5·1 Zadeh's fuzzy set theory

In both his original and in subsequent articles, Zadeh[1965, 1983] states that many of the classes which we encounter in everyday life do not have a precisely defined *criteria of membership*. Zadeh uses these examples to to argue the case in favour of open sentences which do not have to have a 'yes'/'no' answer. While Zadeh states that such imprecisely defined classes do not constitute a set in the mathematical sense of the word, he argues that they play an important role in human thinking.

It was to deal with these imprecise classes that which induced Zadeh into setting –up his theory of fuzzy sets. Zadeh defines a ***fuzzy set*** as a class of objects with a *continuum* of grades of membership. A fuzzy set is characterised by its ***membership*** (or characteristic)

function, which assigns to each object a *grade* of membership ranging between zero and one[1].

Having defined his theory of fuzzy sets, Zadeh then takes the following route in defining an ***imprecise predicate***.

Step one, he acknowledges that every subset of a domain is mathematically equivalent to its characteristic function.

Step two, he generalises the characteristic function,

$$\chi_P : \mathcal{U} \rightarrow \{0,1\}$$

to,

$$\mu_P : \mathcal{U} \rightarrow [0,1];$$

and calls it a ***membership function***.

Step three, Zadeh defines a fuzzy set to be a set which corresponds to this characteristic (membership) function.

Step four, he invokes the axiom of abstraction (backwards) and defines an ***imprecise predicate*** as that predicate which gives rise to the membership function, μ_P.

According to Zadeh[1972,p.5], every fuzzy subset A of the domain $\mathcal{U}$ has a membership function $\mu_A(x)$, which associates with every point x in $\mathcal{U}$ its grade of membership in A. Note how this approach does not produce *sets* as such, but *subsets* of a given domain[2] (which is by definition a classical set).

Criticisms of fuzzy set theory

Despite its recent popularity, fuzzy set theory still faces a number of outstanding criticisms, the best known being: "where do the membership functions come from?", or as Hayes[1974a] citing McCarthy states: "... where do all those numbers come from?". Other criticisms include, the difficulty in choosing the membership function for a

[1]Zadeh[1965,p.338].

[2]See Kaufmann[1975].

combination of concepts (see Smith&Medin[1981]), and what does the membership function mean? (see French[1989]).

Initially, attempts were made to answer these criticisms by appealing to certain characteristics of human thinking. For example, Bellman&Zadeh[1977] argue that fuzzy logic is an appropriate conceptual framework for *approximate reasoning*, since it captures the basic premise that human perception involves classes of objects which do not have an abrupt change from membership to non–membership. Moreover, Bellman&Zadeh[1977] argue that the model of reasoning embodied in fuzzy logic aims to accommodate the pervasive imprecision of human thinking. This viewpoint is corroborated by some psychologists. Oden[1979], for example, deals with psycholinguistic issues in human cognition. He concludes that the competency for processing fuzzy semantic information appears to be a fundamental characteristic of human cognition. Hersh&Caramazza[1976] have conducted experiments, to verify the conjecture that people comprehend vague concepts as if the concepts were represented as fuzzy sets. They conclude that natural language concepts can be described more precisely using this framework. Nevertheless, this view is *not* shared by all psychologists.

Oherson&Smith[1981] not only express their concerns regarding the use of fuzzy set theory by the prototype theorists in psychology; they also state their concern about infinite–valued logics in general, which they claim are unsuitable techniques for modelling psychological phenomena.

One interesting viewpoint is outlined by Brownell&Caramazza[1978], who reason that a subject shown a pictorial stimulus which falls in the fringe area of a category may find it difficult to decide whether the object belongs to a category or not. They claim that this difficulty may be due to the "perceptual confusion" suffered by the subject, in other words, *the category boundaries might be precise and well defined, but the subjects have trouble judging exactly where the pictorial stimulus falls.*

The psychological evidence suggesting that human information processing is based on the type of information claimed by Zadeh is by no means conclusive. Currently the psychologists do not seem to have a consistent view of the matter.

I have two criticisms of fuzzy sets. Firstly, I am concerned not so much with 'where do these membership functions come form', but the fact that the membership function takes values in the set [0,1]. The difficulty here is best described by an example. Consider the predicate 'Red', which we shall consider as being represented by some fuzzy set, $\mu_{Red} : \mathcal{U} \rightarrow [0,1]$. Some object, $a \in \mathcal{U}$, will then have a membership value of $\mu_{Red}(a)$, which represents the grade of 'redness' for this object. However, $\mu_{Red}(a)$ is a real number, and (by definition of real numbers) it is uniquely defined only by an infinite sequence of natural numbers. Informally, this state of affairs describes a situation in which I am unable to state whether the object a is Red or not, yet I am able to express *how* Red this object is with an *infinite* degree of accuracy. This does *not* seem plausible.

Secondly, Zadeh observed that these *imprecise classes* of objects that are pervasive in human reasoning possess ill–defined boundaries. He clearly states that the way to approach this problem is to have sets which have a *continuum* for grades of membership. In other words, Zadeh proposes to change the truth–value set to a *continuum*. I shall shortly argue that the difficulty we have with these classes is not that we do not have a continuum for grades of membership, but the fact that informally we require the characteristic function to be *continuous*. For some domains this does not cause a problem, but for those which are *connected* in some manner, it does. The problem we face has to do with the *connectedness* of the domain and *not* the *truth–value set*, as claimed by Zadeh. Therefore, even if we alter the codomain of the characteristic function to be a *continuum*, we may still not achieve the results which Zadeh wishes for.

8·5·2 Fuzzy sets and the concept of nearness

One of the key issues concerning fuzzy set is that they are

> *"... classes of objects in which the transition from membership to non–membership is gradual rather than abrupt",*
> *Bellman&Zadeh[1977,p.106];*

I shall not argue whether Zadeh's analysis is correct or not, what I am interested in here is whether his solution can achieve what he claims. According to Zadeh[1965], the way to deal with these classes is to alter their characteristic function, such that the codomain {0,1} is replaced with a *continuum*. In other words a change from a two–valued truth–set {0,1}, to an uncountable number of truth–values[1] such as [0,1].

In changing the codomain from {0,1} to {0, ..., n} then to $\mathbb{Q}$, or even to $\mathbb{R}$, only changes the cardinality of the codomain from two, through the finite and countable to the uncountable[2]. Clearly, changing the number of truth–values will allow one to describe the notion of a gradual change, however, it will not allow one to describe a *smooth* or *continuous* change. From a purely set theoretic point of view, changing the codomain from {0,1} to [0,1] has allowed us to partition the domain into a larger number of pieces.

The real numbers by themselves do *not* constitute a continuum. They represent the best paradigm for the concepts of continua, but without a spatial structure the real numbers are nothing more than an uncountable collection of points[3].

My criticism of this approach is twofold. Firstly, if Zadeh wishes to discuss continua, then he should first of all define a spatial structure. Secondly, what Zadeh wishes to describe is a continuous transition from membership to non–membership, not just a gradual one.

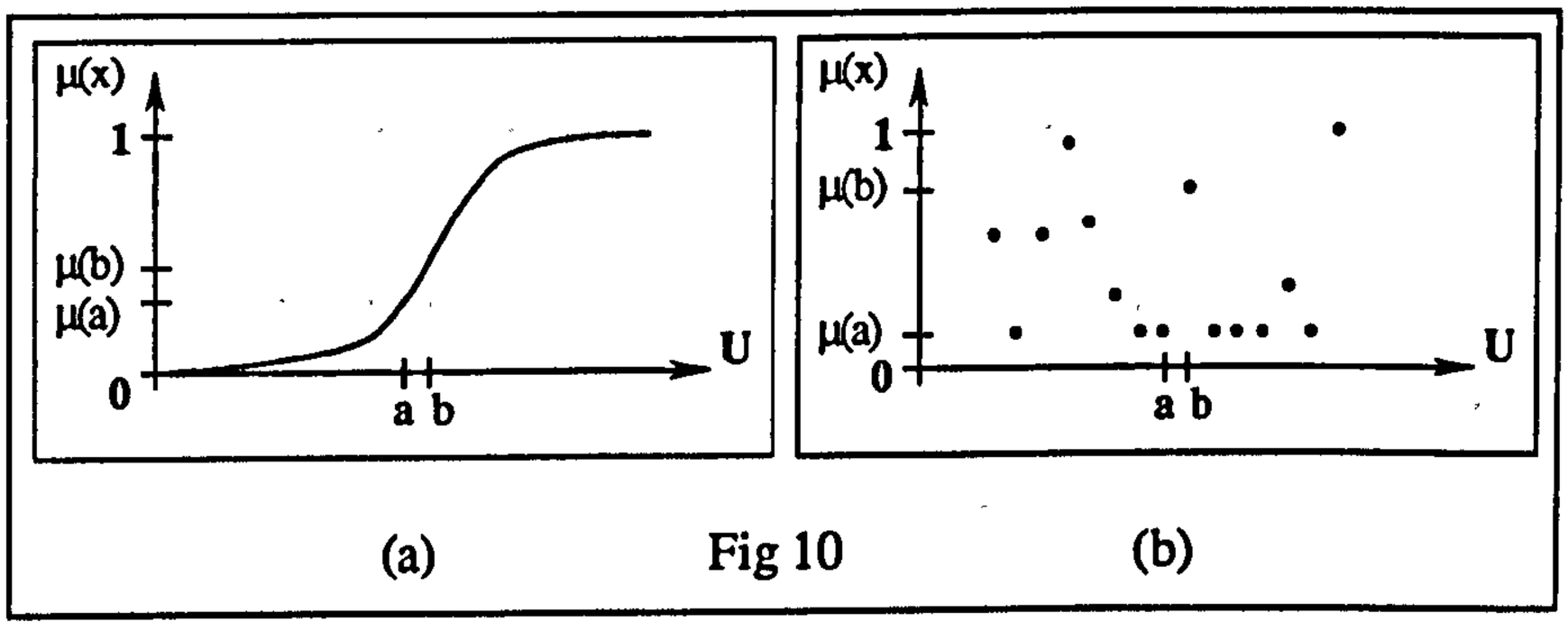
---

[1]Clearly, there is nothing unusual about logics being base on uncountable number of truth–values, but as mentioned in chapter 6 this is only the first stage of the fuzzification, and fuzzy logic proper involves a second stage.

[2]From a set theoretic point of view, the cardinality of $\mathbb{R}$ is the same as the cardinality of [0,1].

[3]With certain other structures defined upon them; for example an ordering structure, see Stewart&Tall[1977] for more details.

My first criticism might perhaps be a little pedantic; after all Zadeh may be considering the real numbers, together with its standard topology. The evidence for my second criticism comes from the illustration used by Zadeh (amongst many others) to depict membership functions. These are typically[1] like the one illustrated in Fig 10(a).



(a) Fig 10 (b)

In this illustration we have a situation where the two points 'a' and 'b' are near to each other in the domain U, and their membership values μ(a) and μ(b) are also near one another in the codomain. This allures one into believing that the membership function is *continuous*. However, there are no formal requirements in the foundations of fuzzy sets which demands that this should be the case, and there is nothing stopping us from defining a membership function like that depicted in Fig 10(b).

The notion of continuity I am using here is the formal notion, which is not the same as that commonly mentioned in the literature on fuzzy sets. For example, when Ng&Abramson[1990,p.38] use the term 'continuous' to assert that a fuzzy set is a class of objects with a "continuous grade of membership", they are using the term to mean the set of truth–values constitute a continuum (Gaines[1976] uses the term in a similar fashion).

Spatial structures are not only needed to capture the notion of continuity accurately, but also to capture the notions of ambiguity and vagueness. In describing the theory of fuzzy sets, Kandel[1982,p.25] asserts that the notions of ambiguity and vagueness

[1]Zadeh[1975b], or Lakoff[1973].

> *"... depend upon there being some notion of nearness or continuity in the universe." [My emphasis].*

One may conclude from the arguments presented above that if fuzzy sets are to achieve their aim, then it is imperative that some kind of spatial structure is incorporated into its foundations.

8·6 STRUCTURE PRESERVING PREDICATES

Clearly, the notion of *continuity* (or *structure preservation*) is not applicable to all predicates. For example, the predicate which represents the set of all even integers is not required to be continuous. Therefore the kind of predicates to which such a notion may be applicable are *vague predicates*[1].

8·6·1 Continuity of vague predicates

Firstly, we should note that a *symbol* cannot be continuous, since continuity is a property of a function. Therefore, when one is talking about vague predicates being continuous, what we really should be doing is discussing the continuity of their *interpretation* (or characteristic function). Secondly, as mentioned earlier in §3·3·2, in order to discuss the continuity of a function we require two spatial structures to be present: one on the domain and another on the codomain. If we do not assume that such structures exist then there is nothing further to discuss; the given function *cannot* be continuous.

We can decide whether to consider structure preservation by looking for situations in which there is some notion of *nearness* on the domain, such that *nearby* points are required to have *nearby* truth–values by a chosen predicate. For example, consider the following statement:

> *"... if colour predicates are observational, any pair of patches indistinguishable in colour must satisfy the condition that any colour predicate applicable to either is applicable to both." Wright[1976,p.233].*

[1]I shall present a definition of a vague predicate towards the end of this chapter, for the time being the reader is asked to use the intuitive meaning of vagueness.

If we interpret 'indistinguishable' as 'near', then what Wright is asserting here is that if two colour patches are *near* to one another, then they are both the same colour. In other words, the interpretation of a given colour predicate will assign truth–values to these colour patches which are *near* to one another in the codomain[1]. Therefore what Wright is demanding here is that his colour predicates should be assigned interpretations which are continuous. Clearly, we would need to discuss the spatial structure on the domain before this statement could be accepted.

Colour predicates are a special case of something much more important described by Wright, namely the notion of a *tolerant predicate*. Wright[1976,p.229] begins by associating a *concept* (denoted by Φ) with a predicate F, and assumes that F applies to some particular case. The predicate F is then defined to be ***tolerant*** with respect to Φ if it is possible to introduce a positive change (of some degree) in respect of Φ, which is insufficient to change the application of F.

Let us consider Wright's definition of tolerance further. Suppose that the predicate F applies to the object a (i.e. F(a) is true), and also suppose that the predicate F is tolerant with respect to the concept Φ. Since the predicate F applies to objects and according to Wright a small positive change in the concept Φ (which I shall denote by 'δ_Φ'), will be insufficient to change the application of F, can only mean that the small change in Φ has resulted in a different object (denoted by: '$\delta_\Phi(a)$'), such that $F(\delta_\Phi(a))$ is also true.

One can view the concept Φ as follows. According to Wright, a predicate F can then be considered to be tolerant:

> if the predicate applies to an object in the domain, then it applies to any object which is produced by a small change in the concept Φ.

We may restate this condition as follows:

---

[1]For the time being, let us assume that the structure on the codomain, {0,1}, is the following. The points 0 and 1 are *not* near to each other; which means the only point in the codomain which is near to 0, is 0 itself. Similarly for the point 1.

(†) If 'F(a)' is true for some object a in the domain, and the object 'Φ(a)' denotes another object in the domain which is produced by a small change in the concept Φ. Then '$F(\delta_\Phi(a))$' is also true.

The condition stated in (†) expresses the requirement that the predicate F should have a structure preserving interpretation, where the structure to be preserved is dictated by the concept Φ.

Similar requirements of continuity are made by other authors. For example, Peacocke[1981] considers the case where two objects produce experiences in us that are not qualitatively discriminable from one another. Peacock then requires, that any predicate should definitely apply to both, or to neither, of these two objects. This is a condition which can be considered to be a continuity condition. For instance, when two objects are stated as being qualitatively indistinguishable from one another, we are in fact asserting that these objects are qualitatively *near* one another. One can restate this as:

> if two objects produce indistinguishable experiences in us, then any predicate applying to one applies to both.

Which one can further analyse to,

> if two objects produce indistinguishable experiences in us, then any predicate applied to these two objects will produce truth values which are near one another.

So far in this section I have been discussing the continuity of vague predicates in general. Let us now be more specific and consider predicates which occur in Sorites type arguments.

8·6·2 Are Sorites predicates continuous?

The Sorites was discussed and analysed in chapter 4. Here, I shall discuss whether Sorites type predicates are structure preserving (or continuous).

Let us begin with the analysis of the Sorites as presented by Wright[1976,p.229], who claims that there is a certain concept of tolerance involved in these examples. Wright's notion of a tolerant predicate was analysed in the previous section (paragraph (†)). This is an alternative way of stating that the interpretation for the predicate is required to be structure preserving. By this analysis, Wright requires predicates involved in Sorites type arguments to be *continuous*.

Consider the analysis of the Sorites offered by Sainsbury. Sainsbury[1989,p.33] defines a predicate, F, to be a ***Sorites predicate*** if and only if F is associated with a dimension of comparison, Φ, such that:

(S1) very small Φ–differences do not affect the applicability of F;

(S2) large Φ–differences do affect the applicability of F;

(S3) Φ–differences are cumulative: large Φ–differences can be attained by a series of small Φ–differences.

Sainsbury's dimension of comparison acts very much like a metric on the domain. Given a point (b, say) in the domain for which the predicate F applies, then condition (S1) requires that all points which are near[1] to b should also be F's. Thus, condition (S1) states the requirement that the predicate F should be interpreted as a structure preserving function.

In §5·2·2 of chapter 5, I presented a formal analysis of the Sorites, in which I defined a predicate F was defined to be 'soritical with respect to the sequence $\langle a_1, \ldots, a_n \rangle$' if it satisfied a number of conditions. One of these conditions was that:

Each adjacent pair of subjects in the sequence must be indistinguishable with respect to 'F'. In other words, given any two adjacent subjects, a_i and a_{i+1}, either 'F' is true of both or that 'F' is false of both[2].

[1]i.e. the point b and the other point have a small Φ–difference.

[2]Condition 2(iii) in §5·2·2.

The condition that the two subjects should be adjacent is another way of stating that the chosen subjects are *near* one another. It was assumed in §8·4·1 that if we kept the set of truth–values to {0,1}, then the only value near 0, was 0 itself – and similarly for 1. Therefore, the condition that the predicate is either true of both or false of both is just another way of asserting that the predicate assigns nearby truth–values to nearby arguments. This condition is of course just a special case of that described in paragraph (†) above.

In this section I have discussed how some of the statements made in the philosophical literature can be translated into the terminology of structure preserving functions. One can summarise this section with the following postulate:

Postulate 1: Interpretations of vague predicates are required to be structure preserving.

What we need to do now is to consider the structure on the domain which these predicates are required to preserve.

8·7 SPATIAL STRUCTURES ON THE DOMAIN/CODOMAIN

In the literature on vagueness there are a number of articles (shortly to be mentioned) which point to problems associated with predicates being defined on a continuum of some kind. In this section I shall focus my attention to these specific kinds of spatial structures.

8·7·1 Does the domain need to be a continuum?

There are many instances of continua mentioned in the literature concerning vagueness. For example, Russell[1923] asserts that

> *"It is perfectly obvious, since colours form a* ***continuum****, that there are shades of colour concerning which we shall be in doubt whether to call them red or not ..." [my emphasis].*

The continuum which Russell is describing here is a structure on the domain of the predicate, and should not be confused with the structure on the truth–values. Another example is given by Sanford[1976] who presumes that dying is sometimes a

> *"... process involving a continuum of momentary states no one of which is the last at which the person is alive ...".*

Once again the continuum mentioned above is on the domain and *not* on the truth–values.

Apart from the obvious cases of vague predicates already mentioned in this thesis, one can construct many more from the concepts we use in our everyday lives. For example, take the predicates, 'Young', 'Old', 'Tall', 'Fat', et cetera , all of which are defined on some aspect of the human body.

Upon closer examination of the predicates which we use in our everyday language, one finds that they are typically defined on some kind of continuum. For example, 'Old' and 'Young' are defined on the continuum of age, which is it self based on the continuum of time (as measured by a clock). The predicate 'Tall' is defined on the continuum of height, which is itself defined on the continuum of space (as measured by distance). There are many such examples, and in fact it is much harder to think of *intuitive* examples of predicates which are *not* continuous. Typically, the discontinuous predicates are designed to serve a specific (often artificial) purpose, such as parking restrictions which suddenly come into effect at some specified time.

The difficult task is not finding such continuous predicates[1], but to identify a structure which faithfully represents these predicates and their domains.

8·7·2 Should the truth–value set form a continuum?

As I mentioned in chapter 7, Cantor defined a continuum as a spatial structure which was connected and was free from isolated points[2]. Then by definition any continuous function

[1]Or ones defined on some kind of continuum.

[2]I am not going to use the currently accepted form of the definition, namely a topological space which is connected, compact and Hausdorff. For the reasons I explained in chapter 7.

must preserve both these properties. That is to say, any structure preserving function defined on a continuum has an image which must also be a continuum (otherwise it would fail to be structure preserving). This is true whether we choose the definition given by Cantor or the modern definition.

Therefore, in metalogical terms, this states that any predicate which is non–constant must have an image which is also a continuum. If the predicate is defined in such a manner that the image equals the codomain (i.e. it is surjective) then the codomain is obviously required to be a continuum. I therefore conclude that it is quite natural to require the domain of definition of a vague predicate to have the structure of a continuum. Moreover, the truth–value set is also required to be a continuum for all those predicates which are assumed to be non–trivial[1].

Note, however, that I am not requiring vague predicates to take values in a continuum (as, for example in the case of fuzzy sets), but, in the case above, the truth–value set is required to form a continuum as a *consequence* of a vague predicate being defined on a continuum. Clearly, we may define a vague predicate on other types of spaces, in which case no such consequence follows.

In the next section I shall formalise some of the concepts discussed in this section using topological structures as an example of a spatial structure.

8·8 TOPOLOGICAL STRUCTURES ON THE DOMAIN/CODOMAIN

Let $\langle \mathcal{U},\sigma \rangle$ be a structure for a language $\mathfrak{L}$, and $P^{\mathcal{U}} : \mathcal{U} \rightarrow \{0,1\}$ an interpretation for a monadic predicate symbol 'P'. If we require the function $P^{\mathcal{U}}$ to be continuous, we must declare a spatial structure on both $\mathcal{U}$ and on $\{0,1\}$. Prior to this discussion I shall define what is meant by a non–trivial predicate.

<u>Definition</u>: The interpretation $P^{\mathcal{U}}$ of a ***non–trivial*** predicate is required to be a function such that $0,1 \in \mathrm{Im}(P^{\mathcal{U}})$.

[1]A notion which is defined in §8·8.

8·8·1 The topology on the set {0,1} of truth–values

The only possible topologies on {0,1} are the following:

(i) $\{\varnothing, \{0\}, \{1\}, \{0,1\}\}$ (*discrete* topology);

(ii) $\{\varnothing, \{0,1\}\}$ (*indiscrete* topology);

(iii) $\{\varnothing, \{1\}, \{0,1\}\}=\tau^*$ (*Sierpinski* topology);

(iv) $\{\varnothing, \{0\}, \{0,1\}\}$ which is homeomorphic to the Sierpinski.

Assuming that we are interested in *non–trivial predicates*, then the topologies defined on the codomain, together with the requirement for the function P^u to be continuous, restricts our choice of topologies on the domain. These restrictions will now be dealt with case by case.

Case (i): Given that {0,1} has the discrete topology restricts the topological structures on the domain to those which are *not* connected (by Theorem 3, §3·3·4).

Case (ii): If {0,1} has the indiscrete topology then one can have *any* topology on the domain, and the function P^u will always be continuous.

Case (iii) & (iv): The continuity of P^u will depend on the different topologies defined on the domain.

Case (ii) has an unpleasant philosophical property, in that it identifies the members of the codomain, namely 0 and 1. This means we no longer are able to distinguish between the two truth values. This topology will no longer be considered since it fails to conform to condition (1) of §8·2.

Cases (iii) and (iv), have the displeasing feature that the topologies in these cases give rise to an asymmetry in the truth–values. Informally, this means I can only distinguish between one truth–value and the other, but *not* vice versa. These topologies are also non–Hausdorff, which makes it awkward to construct any visual[1] examples for them.

[1]Topological spaces which are not Hausdorff are often difficult to deal with, sometimes for the simple reason that we cannot find an intuitive example for them.

8·8·2 The topology on the domain

Let us now turn to the topology on the domain of interpretation for the predicate, and prove the following theorem.

Theorem 1: Let $\langle \mathcal{U},\sigma \rangle$ be a structure for a language $\mathfrak{L}$, and $P^{\mathcal{U}}$ an interpretation for a monadic predicate symbol, 'P'. If we require that

(a) the predicate is to be a vague, non–trivial, predicate

(b) the topological space $\langle \mathcal{U},\tau \rangle$ has the structure of a continuum,

(c) conditions (1) – (4) of §8·2 are to be satisfied,

then there are no topological structures on the truth–value set $\{0,1\}$ which satisfies these conditions.

Proof: Since $P^{\mathcal{U}}$ is considered to be structure preserving, then it must preserve the topological structure on the domain. Since the topological space $\langle \mathcal{U},\tau \rangle$ has the structure of a continuum, the image of $P^{\mathcal{U}}$ (which in this case is the set $\{0,1\}$, since the predicate is non–trivial) must be both *connected* and free from any *isolated* points. There are only two topologies on the set $\{0,1\}$ which are connected, namely:

(i) the indiscrete topology, and

(ii) the Sierpinski topologies.

We have already ruled out the indiscrete topology on the grounds that it does not satisfy certain metalogical conditions in §8·2. The Sierpinski topologies are ruled out since they contain an isolated point (for example the point 1 in case (iii)). Therefore, we may conclude that there are no topologies on the set $\{0,1\}$ which satisfy both the conditions (a) and (b) of the theorem.

QED.

One can clearly have a continuum on the domain with a classical logical structure, so long as we are willing to accept that the function $P^{\mathcal{U}}$ will be discontinuous. Of course, intuitively this means that there will be two objects which are for all intents and purposes

indistinguishable, but which can be distinguished using the predicate P. For example, two objects may apparently be indistinguishable in colour, but one will be considered to be red while the other is not.

Let us now consider what apply the analysis above to a tolerance structure instead of a topological structure.

8·9 TOLERANCE STRUCTURES ON THE DOMAIN/CODOMAIN

In this section, I shall use the following theorems regarding the structure preserving maps on connected tolerance spaces.

Lemma 1: The composition of two tolerance functions is a tolerance function.

Proof: Let $f : \langle X,\xi\rangle \rightarrow \langle Y,\eta\rangle$ and $g : \langle Y,\eta\rangle \rightarrow \langle Z, \zeta\rangle$ be two tolerance functions. Then the composition, $g\bullet f : \langle X,\xi\rangle \rightarrow \langle Z, \zeta\rangle$ is a tolerance function if: $x\xi y$ then $g\bullet f(x)\zeta g\bullet f(y)$. Since f is a tolerance function, we have: $x\xi v \rightarrow f(x)\eta f(v)$; and since g is a tolerance function we have: $y\eta w \rightarrow g(y)\zeta g(w)$. By letting $y=f(x)$ and $w=f(v)$, it is clear that: $x\xi v \rightarrow g\bullet f(x)\zeta g\bullet f(v)$. Therefore, the composition $g\bullet f$ of two tolerance functions is a tolerance function.

QED.

Theorem 2: The tolerance image of a connected set is connected.

Proof: Let $E\subseteq X$ be a connected set, and let $f:\langle X,\xi\rangle \rightarrow \langle Y,\eta\rangle$ be a tolerance function. Then I claim f(E) is connected[1]. Let $p,q\in f(E)$, then there exist p^* and q^* in E such that $f(p^*)=p$ and $f(q^*)=q$. But E is connected, therefore there exists a path, $g:[\![0,m]\!] \rightarrow \langle X,\xi\rangle$ such that $g(0)=p^*$ and $g(m)=q^*$ and $g([0,m])\subseteq E$. Since the composition of two tolerance functions is a tolerance function (by Lemma 1, above), we have that the composition $f\bullet g:[\![0,m]\!] \rightarrow \langle Y,\eta\rangle$ is a tolerance function. Furthermore, $f\bullet g(0)=f(p^*)=p$, $f\bullet g(m)=f(q^*)=q$, and $f\bullet g([0,m])=f(g([0,m]))\subseteq f(E)$.

[1]I am abusing the notation here, since I should really be using $f^{\rightarrow}(E)$ instead of f(E).

Therefore f(E) is connected.

QED.

8·9·1 Tolerance structures on the set {0,1} of truth–values

In this section I shall mirror the topological analysis presented in §8·8·1, using a tolerance structure instead.

Let $\langle \mathcal{U},\sigma \rangle$, be a structure for a language $\mathfrak{L}$, and $P^{\mathcal{U}}$ an interpretation for a monadic predicate symbol 'P'. The possible tolerance structure on {0,1} are:

(i) $\varnothing$;

(i) $\{\{0\},\{1\}, \{0,1\}\}$;

(iii) $\{\{0\},\{1\}\}$.

The first structure is a formality, and is of little practical use. The second structure has the effect of *gluing* the two points '0' and '1' together, in this space we cannot distinguish between the values 'truth' and 'falsity'. Therefore, this structure fails to satisfy condition (1) of §8·2, and will not be considered.

The tolerance space $\{\{0\},\{1\}\}$ behaves very much like the discrete topology, it consists of two distinguishable points and is clearly not connected.

8·9·2 Tolerance structures on the domain

Before one can present results similar to the previous section, we have to agree upon a definition of a continuum in the case of a tolerance space. In chapter 7, I defined an impression continuum to be a connected tolerance space and I believe that this is an adequate definition for our purposes here – since it conforms to Cantor's definition for a continuum in that it is connected and free from isolated points.

<u>Theorem 3</u>: Let $\langle \mathcal{U},\sigma \rangle$ be a structure for a language $\mathfrak{L}$, and $P^{\mathcal{U}}$ an interpretation for a monadic predicate symbol, 'P'. If we require that

(a) the predicate to be a vague non–trivial predicate

(b) the tolerance space $\langle \mathcal{U}, \xi \rangle$ to have the structure of a continuum, and

(c) conditions (1) – (4) of §8·2 are to be satisfied,

then there are no tolerance structures on the truth–value set $\{0,1\}$ which satisfies these conditions.

Proof: Since the predicate is non–trivial, we have $\mathrm{Im}(P^{\mathcal{U}}) = \{0,1\}$. Since the predicate P is vague, by postulate 1 (§8·6·2), it is structure preserving. Therefore, since the domain $\langle \mathcal{U}, \xi \rangle$ is a continuum it is by definition connected. By Theorem 2 above, the tolerance image of a connected set must be connected, in other words $\mathrm{Im}(P^{\mathcal{U}}) = \{0,1\}$ must have a connected tolerance structure. There are no such possible structures which satisfy the above conditions as well as those stated in §8·2.

QED.

I shall now use this, and previous, results in my analysis of the Sorites in two–valued logic.

8·10 THE SORITES IS INSOLUBLE IF EXPRESSED IN A LOGIC WITH {0,1} AS A TRUTH–SET

In §5·2·2 predicates were considered to be soritical with respect to a particular sequence $\langle a_0, \ldots, a_n \rangle$, if they satisfied certain conditions. In the next subsection I shall discuss the relevant tolerance issues concerning such sequences.

8·10·1 Tolerance relations and sequences

Formally, a sequence of objects $\langle a_0, \ldots, a_n \rangle$ from a set A is considered to be the image of a function $s : [\![0, n]\!] \to A$. I shall define a tolerance relation ξ on $\mathrm{Im}(s) \times \mathrm{Im}(s)$ by[1]:

(‡) $\quad s(k)\ \xi\ s(m) \quad$ if and only if $\quad k \approx m.$

We already know that $k \approx m$ is true if and only if $|k-m| \leq 1$. Therefore, we may restate condition (‡) as:

[1]The standard tolerance relation '≈' was defined in §3·4·1.

(‡‡) $s(k)\ \xi\ s(m)$ if and only if $|k-m|\leq 1$.

Given $s(k), s(m) \in s(\mathbb{N})$, there exists a path $s(k), s(k+1), \ldots, s(m)$, such that each element is tolerant to its neighbours, therefore one may view the function, $s:[\![0, n]\!] \rightarrow A$, as a tolerance path in the set A. This has the following consequences.

(i) The function s is a tolerance function.

(ii) The space $\langle Im(s),\xi\rangle$ is a connected tolerance space.

Thus, any sequence may be viewed as a means of connecting the elements of its image in the codomain. Let the set A denote the collection of all non–negative numbers; then we can prove that the set A is a connected tolerance space.

Theorem 4: The tolerance space $\langle \mathbb{N}_0, \approx\rangle$ is connected.

Proof: Let $m, n \in \mathbb{N}_0$, and assume without loss of generality that $m \leq n$. Then consider the path $\phi:[\![0,n-m]\!] \rightarrow \mathbb{N}_0$, defined by $\phi(0)=m$ and $\phi(n-m)=n$. Clearly, ϕ is a tolerance function which connects the points $m,n \in \mathbb{N}_0$. Therefore, the tolerance space $\langle \mathbb{N}_0, \approx\rangle$ is connected. A similar argument shows that the tolerance space $\langle \mathbb{N}, \approx\rangle$ is also connected. *QED.*

8·10·2 Why the Sorites Is Insoluble

In this sub–section I shall demonstrate that the Sorites, as stated in Chapter 5, cannot be resolved in two–valued logic.

In §5·2·2, I demonstrated that there were a number of important components associated with the Sorites. These can be summarised as follows:

The predicate 'F' is 'soritical with respect to the sequence $\langle a_1, \ldots, a_n\rangle$' if:

(1) the a_i's are ordered according to their subscripts,

(2) 'F' is true of a_1, and false of a_n, and

(3) given any two adjacent subjects, a_i and a_{i+1}, either 'F' is true of both or 'F' is false of both.

I shall now demonstrate that these components taken together are inconsistent.

(a) Firstly, the requirement that the subjects (the a_i's) are ordered according to their subscripts, may be translated as follows. The sequence $\langle a_1, \ldots, a_n \rangle$ is the image of a function $s:\mathbb{N} \rightarrow A$. According to the theory presented in §8·10·1, we may define a tolerance relation ξ on $\mathbf{Im}(s) \times \mathbf{Im}(s)$, such that the tolerance space $\langle \mathbf{Im}(s), \xi \rangle$ is connected.

The space $\langle \mathbf{Im}(s), \xi \rangle$ is a *connected* tolerance space. *Thus the sequence has the effect of connecting a subset of the codomain.*

(b) Secondly, I have already argued that condition (3) is simply another way of expressing the requirement that *the predicate 'F' is to have a structure preserving interpretation* (§8·6·2).

(c) Thirdly, the condition that there are cases for which the predicate 'F' takes a value 'true', and cases for which it takes the value 'false', demonstrates that *the interpretation of the predicate 'F' is not a constant function.*

In the above analysis the predicate 'F' was defined to be Soritical for a *particular sequence* $\langle a_1, \ldots, a_n \rangle$ of objects in the domain. A predicate can therefore be soritical for some sequences in a given domain and not for others. For example, if we know that 'F' is true for the object b, in the domain, then clearly 'F' is not soritical for the sequence $\langle b, \ldots, b \rangle$. Given that a *sequence* is formally defined as the image of some function ($s:\mathbb{N} \rightarrow A$, say), one can then restate the above conditions as: a predicate 'F' is deemed to be Soritical for a given subset $\mathbf{Im}(s)$ of the domain if the conditions (1)–(3) above hold.

The logic which is commonly assumed in discussions regarding the Sorites is often classical two–valued logic; i.e. a logic which is constructed with a metalanguage that only possesses the entities described in chapter 2. In this metalanguage there are no spatial structures defined, therefore I shall consider the classical truth–set $\{0,1\}$ as having the discrete tolerance space structure, namely $\langle \{0,1\}, \delta \rangle$, (where $\delta = \{\langle 0,0 \rangle, \langle 1,1 \rangle\}$).

Theorem 5: The Sorites, as analysed above, cannot be resolved in a logic whose truth–set is given by {0,1}.

Proof: The conditions above state the following:

(a) the domain is a connected tolerance space, namely a tolerance continuum;

that is to say, the subset **Im**(s) can be thought of as a *connected* tolerance space, ⟨**Im**(s),ξ⟩.

(b) the soritical predicate is structure preserving;

since the predicate is required to take *nearby* truth–values for *nearby* elements in the set **Im**(s).

(c) the soritical predicate is non–trivial;

since the predicate is true at some point of **Im**(s), while false at another.

By theorem 3 above there are no tolerance structures on the truth–set {0,1} which satisfy conditions (a)–(c) above, together with those stated in §8·2.

QED.

The reason for this result is easy to see. The only structure preserving functions from a connected space to the discrete space ⟨{0,1},δ⟩ are the constant functions, which we explicitly disallow.

This theorem demonstrates that, from a spatial structure point of view, there is very little one can do to a logic with {0,1} as its truth–set to solve the above paradox. One is therefore forced to consider logics with truth–sets of higher cardinalities.

8·11 TOLERANCE STRUCTURES ON THE TRUTH–SET {0,u,1}

I am interested in investigating the structure of a logic which has a truth–set of minimum cardinality, yet satisfies conditions stated 8·11(a) – (d) below.

8·11(a) the tolerance space on the domain will be a continuum[1],

[1]These are important spaces, because they appear to be the kind which can capture such entities as colour spectra, and other impression continua.

8·11(b) the interpretations (P^u) of the predicate symbols ('P') will be structure preserving functions. That is to say, the predicates we consider will be vague predicates.

8·11(c) the predicate P is non–trivial, (i.e. $0,1 \in \mathrm{Im}(P^u)$).

8·11(d) the logic will satisfy the metalogical conditions stated in §8·2.

Since we require the functions P^u to be structure preserving, and the domain to be a continuum, then Theorem 2, tells us that the codomain of P^u (namely the truth–set) must also be connected. We already know that it is not possible to construct a non–trivial connected tolerance space on the set {0,1}, therefore one must investigate whether it is possible to construct a continuum on sets of higher cardinality. As I am seeking a minimum change to the truth–set, I shall begin with a truth–set of cardinality three (which I shall denote by: {0,u,1}). The third value denoted by the symbol 'u', is used intuitively to indicate *unknown* truth value. The reason for this will become apparent towards the end of this chapter. Given the set {0,u,1} the next step is to calculate all the possible tolerance relations on this set.

The number of possible tolerance relations on the set {0,u,1} are easy to calculate with the aid of the table in Fig 11. Since tolerance relations are required to be reflexive, means the diagonal must all be 1's.

	0	u	1
0	1	#	#
u		1	#
1			1

Fig 11

The symmetry condition enables us to ignore the values in the shaded triangle, since these will be determined once the values denoted by the #'s have been chosen. Each of the values denoted by a # can either be a 1 or a 0, and since there are only three such values, means that there can only be *eight* possible tolerance structures on this set. They are:

(i) $\{\langle 0,0\rangle, \langle u,u\rangle, \langle 1,1\rangle\} = \delta$;

(ii) $\{\langle 0,0\rangle, \langle u,u\rangle, \langle 1,1\rangle, \langle 0,u\rangle, \langle u,0\rangle\}$;

(iii) $\{\langle 0,0\rangle, \langle u,u\rangle, \langle 1,1\rangle, \langle 1,u\rangle, \langle u,1\rangle\}$;

(iv) $\{\langle 0,0\rangle, \langle u,u\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle\}$;

(v) $\{\langle 0,0\rangle, \langle u,u\rangle, \langle 1,1\rangle, \langle 0,u\rangle, \langle u,0\rangle, \langle 1,u\rangle, \langle u,1\rangle\} = \varepsilon$;

(vi) $\{\langle 0,0\rangle, \langle u,u\rangle, \langle 1,1\rangle, \langle 0,u\rangle, \langle u,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle\}$;

(vii) $\{\langle 0,0\rangle, \langle u,u\rangle, \langle 1,1\rangle, \langle 1,u\rangle, \langle u,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle\}$;

(viii) $\{\langle 0,0\rangle, \langle u,u\rangle, \langle 1,1\rangle, \langle 1,u\rangle, \langle u,1\rangle, \langle 1,u\rangle, \langle u,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle\} = \iota$.

Case (i): The relation δ, is an equivalence relation, and will therefore partition our domain, thus its behaviour in a metalogical situation will resemble a conventional (discrete) three valued system.

Cases (ii) & (iii): The relation given in case (ii) is depicted in Fig.12.

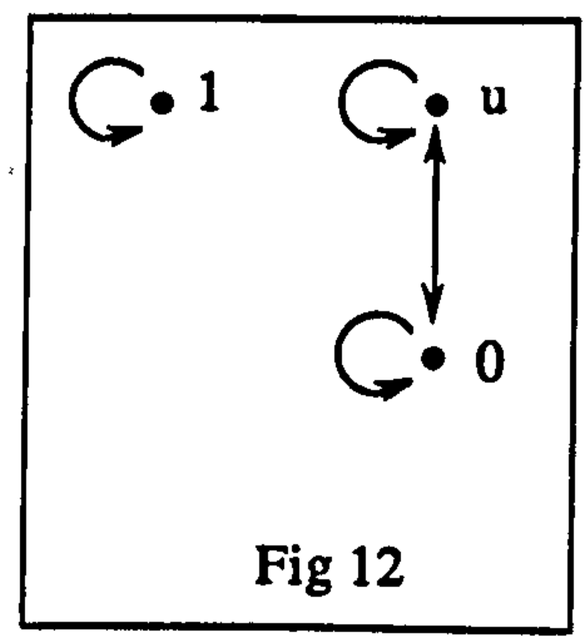


Fig 12

In this structure, the values 0 and u are considered to be indistinguishable, and are separated from the value 1. Therefore, we have the ability to distinguish between truth and falsity, however, we have no mechanism which will allow us to get from the value 0 to the value 1. This structure also fails condition 8·11(c) above. Thus, this structure and the one represented by case (iii) can be considered as unsuitable.

Cases (iv), (vi), (vii) & (viii): One can immediately discard these structures, since in these relations, the points 0 and 1 are considered to be indistinguishable. In a metalogical setting this means we cannot distinguish between truth and falsity. This structure fails condition (1) of §8·2.

Case (v): This relation is depicted in Fig.13.

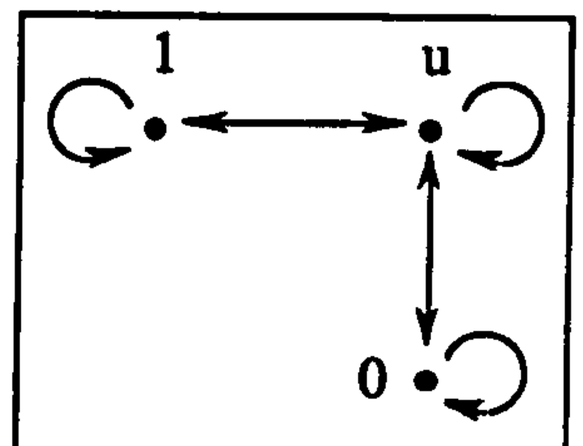


Fig 13

The illustration is largely self explanatory. The values 0 and 1 are *distinguishable*, while the value u is *indistinguishable* from both, 0 and 1. Distinguishability of truth and falsity is a necessary requirement from a metalogical point of view. The value u connects the other values together, thereby producing a connected tolerance space. It also provides one with a mechanism for moving (in a continuous manner) from the the value 0 to 1, and vice versa.

In this analysis I have actually proved the following theorem, that the only tolerance relation which provides us with a possible structure that satisfies my requirements is the one given by ε (see Fig 13).

Theorem 6: There is only *one* tolerance structure which satisfies the conditions 8·11(a)–(d), namely the one represented by ε.

In the next section I shall investigate the structure of a logic satisfying conditions 8·11(a)–(d), using the tolerance relation ε.

8·12 LOGICS CONSTRAINED BY THE TOLERANCE SPACE ⟨{0,u,1},ε⟩

In this section I shall describe the truth tables for a logic structured by the tolerance relation, ε.

Theorem 7: There are only two logics with truth-sets {0,u,1} that satisfy conditions 8·11(a) – (d), and are constrained by the tolerance structure ε.

Proof: Let us consider the truth–tables for disjunction, ∨:

The #'s in Fig14 indicate truth–values which have yet to be determined. In order to determine these values I shall need to use the tolerance properties of a product space, namely that ⟨x,y⟩ is indistinguishable from ⟨v,w⟩, if and only if x is indistinguishable from v, and y is indistinguishable from w.

∨	0	u	1
0	0	#	1
u	#	#	#
1	1	#	1

Fig 14

(i) The truth–value of ∨(u,u), has to be indistinguishable from each one of the other values; this can only be the case if ∨(u,u)=u.

(ii) The truth–value of ∨(u,0)=∨(0,u), has to be indistinguishable from each one of the following: ∨(0,0)=0; ∨(u,u)=u; ∨(0,1)=1; ∨(u,1)=#.

Since ∨(u,0) has to be indistinguishable from both 0 and 1, it can only take the value u, that is to say, ∨(u,0)=∨(0,u)=u.

(iii) The truth–value of ∨(u,1)=∨(1,u), has to be indistinguishable from each one of the following: ∨(0,1)=1; ∨(0,u)=u; ∨(u,u)=u; ∨(1,1)=1.

Therefore the value of ∨(u,1) is not uniquely determined, it can be either u, or 1.

If we choose ∨(u,1)=1, then we shall get the truth–table shown in Fig.15

Since the value of ¬u needs to be tolerant to both 0 and 1, and there is only one value which satisfies this namely u. Therefore the truth–table for negation can *only* be that shown in Fig 16.

∨	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

Fig 15

¬	
0	1
u	u
1	0

Fig 16

The same process applied to the truth–table for conjunction, shown in Fig.17. In classical logic the truth value of $(A \wedge B)$ is the same as that of $\neg(\neg A \vee \neg B)$, and since we wish this logic to be normal we may insist that the same holds for this logic. In which case we have:

$$(u \wedge 0) = \neg(\neg u \vee \neg 0) = \neg(u \vee 1) = \neg(1) = 0.$$

$\wedge$	0	u	1
0	0	#	0
u	#	u	u
1	0	u	1

Fig 17

Thus, the truth–table for conjunction must be the one shown in Fig.18.

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

Fig 18

$\rightarrow$	0	u	1
0	1	#	1
u	#	#	#
1	0	#	1

Fig 19

$\rightarrow$	0	u	1
0	1	1	1
u	u	u	1
1	0	u	1

Fig 20

If we now focus on the truth–table for the conditional (shown in Fig 19), then one can begin to replace the unknown values as follows. The truth–value of $\rightarrow(u,u)$ has to be indistinguishable from both 0 and 1, therefore $\rightarrow(u,u)=u$. Similarly for $\rightarrow(1,u)=\rightarrow(u,1)$. The truth–value of $\rightarrow(u,0)=\rightarrow(0,u)$ has to be indistinguishable from both u and 1, therefore at this stage it is indeterminate, it can be either. However, if we require that the truth–value of $\rightarrow(A,B)$ be the same as $\neg A \vee B$, then the truth–value of $\rightarrow(u,0)$ is given by: $\rightarrow(u,0)=\neg u \vee 0 = 1 \vee 0 = 1$. The truth–table for this is shown in Fig 20.

These truth–tables are the well known tables for the ***strong*** connectives in Kleene's 3–valued logic. The choice for the truth–value of $\vee(u,1)=1$, was completely arbitrary.

With equal validity I may have chosen it to be $\vee(u,1)=u$, then the same arguments would have lead to the following truth–tables for the various connectives.

$\vee$	0	u	1
0	0	u	1
u	u	u	u
1	1	u	1

Fig 21

$\neg$	
0	1
u	u
1	0

Fig 22

$\rightarrow$	0	u	1
0	1	u	1
u	u	u	u
1	0	u	1

Fig 23

$\wedge$	0	u	1
0	0	u	0
u	u	u	u
1	0	u	1

Fig 24

These truth–tables are the well known tables for the ***weak*** connectives in Kleene's 3–valued logic.

Therefore, there are only two logics which have truth–sets {0,u,1} and which satisfy the conditions 8·11(a)–(d), and they are the Kleene Strong and Weak 3–valued logics.

QED.

It interesting to note, that the truth tables for each connective in the different Kleene logics are indistinguishable as far as the spatial structure is concerned.

As a consequence of the way the truth tables for Kleene's strong three valued logic is defined, one may state them in an elegant manner which makes no explicit mention of the undefined value ('u'):

$\neg$A	is True	iff	A is False;
$\neg$A	is False	iff	A is True;
A$\wedge$B	is True	iff	A is True and B is True;
A$\wedge$B	is False	iff	A is False or B is False;

$A \vee B$ is True	iff	A is True or B is True;
$A \vee B$ is False	iff	A is False and B is False;
$A \rightarrow B$ is True	iff	A is False or B is True;
$A \rightarrow B$ is False	iff	A is True and B is False.

8·13 OTHER WELL-KNOWN 3-VALUED LOGICS

There are a large number of 3–valued logics[1], and it is not possible to consider them all here to discover which satisfy and which fail these conditions. Here, I have considered many of the well–known ones described by Haack[1967], and I shall indicate where they fail to satisfy the conditions 8·11(a) – (d) specified above. For further details of the connectives see Haack[1967].

Łukasiewicz's 3–valued logic

Łukasiewicz[1920] defined a 3–valued logic to deal with the problem of future contingents. In this logic those sentences whose truth values are not determined, receive the third value 'u'. The truth–table for the conditional in this logic is given in Fig 25.

The connective $\rightarrow_{Ł}$ fails to be a tolerance function, because $\rightarrow_{Ł}$ (u,u) and $\rightarrow_{Ł}$ (1,0) are *intolerant*, when ⟨u,u⟩ and ⟨1,0⟩ are *tolerant* in the product space.

Therefore Łukasiewicz's 3–valued logic has to be rejected on this point.

$\rightarrow_{Ł}$	0	u	1
0	1	u	1
u	u	1	u
1	0	u	1

Fig 25

[1]See Hayes[1975] or Dienes[1949].

Bochvar's 3–valued logics

Bochvar's 3–valued logics are characterised by two different sets of truth–tables for his connectives. The ***internal connectives*** are the same as Kleene's *weak* connectives, and are therefore acceptable by my conditions.

Bochvar's ***external connectives*** are defined by a set of truth–tables which includes that for negation shown in Fig 26. This negation operator is not a tolerant function, since $\neg_B(1)$ is intolerant to $\neg_B(u)$, despite the fact that 1 is tolerant to u; this consequently rules out this set of connectives.

$\neg_B$	
0	1
u	1
1	0

Fig 26

Reichenbach's 3–valued logic

Reichenbach's logic fails on the alternative implication connective as defined in Fig 27.

For example, $\rightarrow_R(u,u)$ is intolerant to $\rightarrow_R(1,u)$, despite the fact that $\langle u,u\rangle$ and $\langle 1,u\rangle$ are tolerant in the product space.

$\rightarrow_R$	0	u	1
0	1	1	1
u	1	1	1
1	0	0	1

Fig 27

Post's Many–valued logic (for the case of three values)

Clearly, this logic also fails to satisfy the specified conditions; $\neg_P(u)$ is intolerant to $\neg_P(0)$, despite the fact that u is tolerant to 0.

$\neg_P$	
0	1
u	0
1	u

Fig 28

8·14 APPLICATIONS OF A LOGIC WITH AN ε-STRUCTURING

I have already argued that examples commonly discussed in the literature on vagueness involve the notion of a spatial structure. Firstly, I shall demonstrate how a non–trivial predicate defined on some connected space is forced to take the value $u \in \{0,u,1\}$ for some member of that domain. Secondly, I shall demonstrate how a logic structured by ε can be used to resolve the Sorites in 3–valued logic. Finally, I shall describe how mathematical induction may be viewed from a geometrical point of view.

8·14·1 Non–trivial predicates with codomain $\langle\{0,u,1\},\varepsilon\rangle$

Let 'P(x)' be a predicate in a language $\mathfrak{L}$, and let $\langle\mathcal{U},\xi\rangle$ be a connected tolerance space. Let the interpretation of the predicate 'P(x)' be given by the function $P^{u}:\langle\mathcal{U},\xi\rangle \rightarrow \langle\{0,u,1\},\varepsilon\rangle$. If this predicate is assumed to be non–trivial, then P^{u} will *necessarily* take the value 'u' for some member of the domain $\mathcal{U}$.

<u>Theorem 8</u>: Let $\langle\mathcal{U},\xi\rangle$ be a connected tolerance space and $f:\langle\mathcal{U},\xi\rangle \rightarrow \langle\{0,u,1\},\varepsilon\rangle$ be a tolerance function. If $0,1 \in \mathbf{Im}(f)$, then the function f is surjective.

<u>Proof</u>: Suppose that $f:\langle\mathcal{U},\xi\rangle \rightarrow \langle\{0,u,1\},\varepsilon\rangle$ is a tolerance function from a connected tolerance space to $\langle\{0,u,1\},\varepsilon\rangle$, where $0,1 \in \mathbf{Im}(f)$, but $u \notin \mathbf{Im}(f)$, since $u \notin \mathbf{Im}(f)$ implies $\mathbf{Im}(f) = \{0,1\}$ which is a disconnected set under ε. Theorem 2 (§8·9) states that the tolerance image of a connected set is connected, which contradicts the fact that $\mathbf{Im}(f) = \{0,1\}$ forms a disconnected set. Therefore $u \in \mathbf{Im}(f)$, and the function f is surjective.

QED.

Note that this theorem has been proved without the aid of any ordering relations on the domain $\langle\mathcal{U},\xi\rangle$. This is a special result which works for the particular codomain above. If we wished to generalise this result then we will certainly need some kind of ordering structure on the domain, which would enable us to invoke the *intermediate value theorem*

(Theorem 12, §3·4·3), and obtain a similar result. Theorem 2 (§8·9) above provides the mechanism which allows us to assert the existence of at least *one* point, which will be assigned the value 'u' by *every* non–trivial predicate.

8·14·2 Why the Sorites can be resolved in a 3–valued logic

If we now reconsider the conditions stated in §8·10·2 for the Sorites, we can see a reason why the paradox was generated, namely that the truth–set did not form a continuum. Moreover, it did not contain enough elements to do so. This difficulty is resolved in the case of a 3–valued truth–set such as {0,u,1}, which allows one to state the following theorem.

Theorem 9: The conditions for the Sorites which lead to a contradiction in the case of a logic with a truth–set {0,1}, are avoided in a logic with a truth–set {0,u,1}, that is structured by the tolerance relation ε.

Proof: The conditions for the Sorites, as stated in §8·10·2, are the following:

(a) the domain is a connected tolerance space, namely a tolerance continuum.

(b) the soritical predicate is structure preserving.

(c) the soritical predicate is non–trivial.

However, this time no inconsistency results from the above conditions (or in fact the ones in §8·11(a)–(d)), since we do have a tolerance structure which is a continuum, namely that given by ⟨{0,u,1},ε⟩. This tolerance space avoids the earlier difficulties caused by Theorem 2, which required the image of every non–trivial structure preserving predicate on the continuum to have an image which was also a continuum. In providing an example of a tolerance space which satisfies all the conditions above, I have demonstrated that these conditions are not inconsistent in a logic structured by ε.

QED.

What is unusual about the results of Theorems 5 and 8 is the manner in which they were derived. *Never* before has there been a *formal proof* that the Sorites was inconsistent

when expressed in a logic with a truth–set given by {0,1}. Theorem 8, demonstrates that the same puzzle can be expressed in consistent manner in a logic with a truth–set given by {0,u,1}. There are many articles in the literature (e.g. Negoita[1985,p.47]), which *describe* how the Sorites occurs, and why this indicates the inadequacy of classical logic to capture the complex everyday arguments performed by human beings. However, Negoita, never actually proves this point he simply *discusses* the difficulties associated with it.

As a corollary, applying Theorem 8 to this analysis of the Sorites shows that there will always be some individual in the domain for which it will never be known whether the chosen predicate applies or not.

I have now discussed how conventional Sorites arguments may be analysed in a geometric manner. Since there are some (e.g. Smith[1984]), who argue that the Sorites is a puzzle concerning the rule of mathematical induction, it is important to consider this rule of inference in a similar vein.

8·14·3 Mathematical induction : a geometric analysis

In §8·10·1, I demonstrated how a tolerance relation ξ could be defined for the sequence of objects $\langle a_1, \ldots, a_n \rangle$ chosen from a set A, by considering the sequence (in the proper formal manner) as the image of a function $s : [\![0,n]\!] \to A$. The tolerance relation ξ on $\mathrm{Im}(s) \times \mathrm{Im}(s)$ was defined by:

$$(\ddagger) \quad s(k)\, \xi\, s(m) \text{ if and only if } k \approx m.$$

In the same section I demonstrated how one could prove that the tolerance space $\langle \mathbb{N}_0, \approx \rangle$ was connected, by simply taking A to be $\mathbb{N}_0$.

This result can be generalised to the following. Given a connected tolerance space $\langle X, \eta \rangle$ and a function $s : X \to A$, one may define a tolerance ξ on the image $\mathrm{Im}(s) \times \mathrm{Im}(s)$ by:

$$s(k)\, \xi\, s(m) \text{ if and only if } k\, \eta\, m,$$

and has the following consequences

(i) the function s, becomes a tolerance function, and

(ii) $\langle Im(s),\xi\rangle$ becomes a connected tolerance space.

Note, how in this generalisation there is nothing in the theory which dictates that the domain of the sequence, $s:[\![0,n]\!] \rightarrow A$, should be a finite set (i.e. $[\![0,n]\!]$). We may therefore extend this to cover infinite sequences simply by extending the domain of the sequences to be $\mathbb{N}_0$. Clearly, this is only interesting if the set A is also infinite.

The function s has the effect of tracing a path in the set A, irrespective of whether we choose the set $[\![0,n]\!]$, or $\mathbb{N}_0$ for its domain. The only difference between the two paths are that one is finite, while the other is infinite.

In having to express mathematical induction in a geometric manner, one needs to consider abstractions of the components involved in the proof. This has two affects, firstly, one needs to identify the various components, and how they interact with one another. Secondly, in having to describe this schema in a *geometric* setting, one needs to have a clearer understanding of the characteristic nature of each component, since conceptual misunderstandings can easily be highlighted by poor illustrations.

<u>Notation</u>: In the following I shall use the notion '+(n)' to denote 'n+1'.

Mathematical induction is commonly described as follows:

(P1) 'F(0)' and

(P2) 'For any n, if F(n) then F(+(n))'

(C) 'For any n, F(n)'.

There are a number of points to note about this proof schema. Firstly, the proof itself involves the following three components:

(1) consider the first premise, P1, in which we have a predicate that is true at the point 0.

(2) the second premise (P2) states that, if the predicate 'F' is true at some point n then it is also true at the point n+1.

(3) the conclusion then states that if (1) and (2) hold then 'F' is true at all points in $\mathbb{N}_0$.

One can separate the proof by mathematical induction into a number of distinct steps. Firstly, the function '+' is simply a mechanism for describing the intuitive notion of *movement* in the set, which satisfies:

(a) injective;

(b) but *not* surjective, since $\nexists\, p \in \mathbb{N}_0$ such that $0 = +(p)$.

The function '+' is shown geometrically in Fig.29.

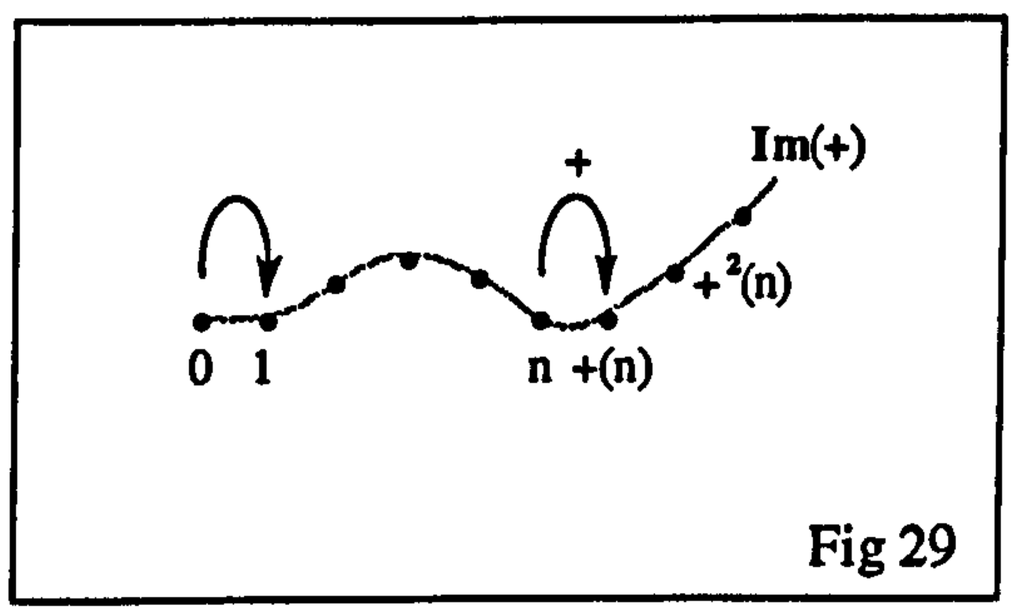


Fig 29

It is important to note that the function '+' is automatically given by the definition of the set $\mathbb{N}_0$, therefore in this case $\mathrm{Im}(+) \cup \{0\} = \mathbb{N}_0$. If we choose another function $s: \mathbb{N}_0 \rightarrow \mathbb{N}_0$, we shall still have the function '+' but now we shall have another function from $\mathbb{N}_0$ to itself. Note that term 'F(n+1)' is now replaced by 'F(s(n))'.

The elements of the proof can now be analysed geometrically as follows.

(A) The set of numbers $\mathbb{N}_0$, is defined using a function '+', which provides the structure for $\mathbb{N}_0$. Thus, one may define a tolerance ξ on the set $\mathbb{N}_0$, such that $\langle \mathbb{N}_0, \xi \rangle$ is a *connected* tolerance space.

(B) Any predicate 'F' is defined not just on the the *set* $\mathbb{N}_0$ but on the *space* $\langle \mathbb{N}_0, \xi \rangle$.

Furthermore, condition (2) above requires the predicate 'F' to be structure preserving (i.e. if 'F(n)' is true then 'F(+(n))' must also true).

(C) Geometrically, any structure preserving function from a connected space into the space $\langle\{0,1\}, \delta\rangle$, can only be a constant function. Since the predicate 'F' is defined to be true at a point $0 \in \mathbb{N}_0$, then it must take the same value for all points of $\mathbb{N}_0$. Hence, 'F' must be true for *all* points of $\mathbb{N}_0$.

One may summarise the above as follows. From a geometric point of view, the collection $\mathbb{N}_0$ is not just a *set* but a connected tolerance *space* $\langle\mathbb{N}_0,\zeta\rangle$, where the tolerance relation ζ is defined using the successor function s. Mathematical induction may then be restated as the requirement that any structure preserving function on the space $\langle\mathbb{N}_0, \zeta\rangle$ should be a constant function.

8·15 INTERPRETATION OF THE THIRD VALUE IN KLEENE'S THREE-VALUED LOGICS

According to Haack[1967] Kleene interprets his connectives as follows.

1 is interpreted as true (certainly)

u is interpreted as true or false (but undecidable which)

0 is interpreted as false (certainly)

Thus expressions which are assigned the value, u, are considered to be undecidable, but are nevertheless either true or false, though it is not possible to tell which.

Kleene[1952] constructed this theory in order to capture the logic of those expressions in mathematics that can in principle be decided by an algorithm. Kleene's idea was to divide the expressions into three classes:

(i) those expressions for which there is an algorithm[1] showing that it is true;

(ii) those which are undecided, where no algorithm exists which will establish the

[1] *"Decidable by the algorithms (i.e. by the use of only such information about the predicates as can be obtained by the algorithms) to be true"*, Kleene[1952,p.336].

truth or falsity of the expression[1].

(iii) those for which there is an algorithm showing that it is false[2];

Kleene[1952,p.336] also provides an epistemic reading of these values as:

(i) *known to be true*;

(ii) *unknown whether true or false*;

(iii) *known to be false.*

What makes Kleene's three–valued logic so special is that the third truth–value is not a third value at all, as Kleene[1952,p.333] himself asserts:

> *"The third 'truth value' u is thus not on a par with the other two t and f in our theory".*

The third truth value is commonly interpreted as a truth–value gap (for example see Turner[1984,p.32]). I disagree with this interpretation for the following reasons. A truth–value gap occurs when the extension and antiextension of a predicate fail to cover the domain, that is to say there is literally a gap between the extension and antiextension. This interpretation is not the one offered by Kleene.

Kleene interprets his third value epistemically as *unknown whether true or false*, this means Kleene interprets the third value as a *glut*, where a glut can be considered as the region in which the extension and antiextension *overlap*. The overlap is the region in the domain where the expression has *a* truth–value, but we are unable to discover which one.

I now conjecture that this region where this overlap, is what what Peirce meant when he stated that a borderline case for a predicate was an individual, for which the application or non–application of the predicate symbol was considered to be 'intrinsically uncertain'.

---

[1] *"Undecidable by the algorithms whether true or false"*, Kleene[1952,p.336].

[2] *"Decidable by the algorithms to be false"*, Kleene[1952,p.336].

8·16 SUMMARY

This chapter brings the geometric vein running through this thesis to a conclusion. Here, all the topics and ideas which have been discussed and argued are focused into the theorems, and it is therefore without doubt the most important in the thesis. The geometrical approach adopted in this chapter is completely novel, and has not appeared before in the literature.

The chapter begins by describing how predicate extensions (and the interpretation function) may be viewed in a geometric manner, which provides the mechanism for arguing the case against Zadeh's fuzzy sets.

Zadeh (amongst others see Negoita[1985]) asserts that human information processing is imprecise – in which classes do not have an abrupt change of membership, but a gradual one, from which Zadeh concludes that the interpretation of a predicate must take values in a continuum. I argue here, that Zadeh is in actual fact mistaken in drawing this conclusion, because he has confused the notion of a *continuum* with that of *continuity*.

In §8·6, I discuss the analyses from previous chapters and demonstrating that they are simply another way of requiring the interpretation of vague predicates to be continuous. Clearly, by requiring any function to be continuous one must immediately declare some kind of spatial structure on the domain/codomain. Such an endeavour is only possible if one is using a formalised metalanguage which possesses a geoemtrical structure.

In the following sections (§8·7, 8·8 and 8·9) I discuss various spatial structures on the domain and the codomain, and describe why some are not possible given the metalogical constraints placed on the logic. In particular, I prove that there are no spatial structure on the truth–set given by {0,1} which will be consistent with the requirements that the domain is a continuum and the predicate extension is continuous. The results of §8·7, §8·8 and §8·9 can now be utilised to show that the Sorites *cannot* be resolved in a logic with a truth–set {0,1} – Theorem 5 §8·10·2.

In §8·11, I increase the cardinality of the truth–set form {0,1} to {0,u,1} and demonstrate the various tolerance structures which are possible on a such a set (given certain metalogical constraints). These constraints are of a general nature and are used in reducing the number of such structures to *one* (which I call ε).

In §8·12, I investigate the logical structure of a language constrained by this tolerance space in its metalanguage, and prove in Theorem 9, that there are only *two* such logics, namely, the Kleene Strong and Weak, 3–valued logics[1]. These 3–valued logics are in fact indistinguishable with respect to the tolerance space ε.

The results of §8·11, §8·12 and §8·13 enable me to prove that the Sorites *can* be resolved in a logic with ⟨{0,u,1},ε⟩ for a truth–set.

[1]Details of how this kind of logic may be automated, see Gent[1992] or Schmitt[1986,1990].

9 Further work and Conclusions

"... we must look not only at the set of points but also at the way they are put together."

Bernard Bolzano

9·1 FURTHER WORK

In this thesis I have developed and applied Tolerance Geometry to a number of different areas in AI, metalogic and mathematics. However, there still remains a great deal to be developed. In this section I shall indicate the areas which I believe can benefit from further investigation.

9·1·1 Necessary work relating to chapter 8

In chapter 8, I discussed, argued and demonstrated various issues concerning the semantics of vague predicates. It is now necessary to formalise the syntax of the language (e.g. give formal definition for well formed expressions of the language), and extend the semantics to cover n–place predicates and functions. The syntax of these languages I intend to be as similar to classical first order logic as possible (see for example Kleene[1952]).

The semantical issues concerning n–place predicates should not cause difficulties, since the interpretation of an n–place predicate is defined to be, $\chi_{PU}: \mathcal{U}^n \rightarrow \{0,1\}$; and the

tolerance on $\mathcal{U}^n$ is defined as the product of the tolerances on $\mathcal{U}$. However, I do not believe that we should require all predicates to be continuous, since there are occasions when we specifically design a predicate to be discontinuous (e.g. eligibility to vote at the age of majority). One way to accomodate discontinuous predicates into the formal language is to exploit the formal properties of a sorted logic.

As for function symbols, it is not clear that we should require all function symbols to be continuous, the interpretation of a function symbol 'f' in a structure $\langle \mathcal{U},\sigma \rangle$ is defined as: $f^{\mathcal{U}} : \mathcal{U}^n \rightarrow \mathcal{U}$, which we may require to be a tolerance map. As I explained above, we probably do not want *all* function symbols to be interpreted as tolerance functions on the domain.

It may seem rather artificial to assign *sorts* to those predicates and functions which are required to be continuous, since there does not appear to be such structure in our everyday natural languages. Nevertheless, we are attempting to model our common sense notions of the everyday world, and it is therefore necessary to use whatever means are at our disposal to achieve the 'correct' model. Simple reflection is sufficient to recognize that many everyday predicates are continuous, indeed very few are actually discontinuous. But these are issues which still need to be resolved.

9·1·2 Similarity of objects induced by the relation χ

Intuitively, we consider objects to be *similar* if they share some property. The more properties they share the greater the similarity between the two objects. I shall now indicate how the development of induced relations presented in §4·1·2, might be used to capture this common sense notion of *similarity* between two objects of the real world.

The interpretation function for all *monadic* predicates, and all objects, in a domain $\mathcal{U}$ is defined by the function: $\chi : \mathbf{Prd} \times \mathcal{U} \rightarrow \{0,1\}$,

which has as dual the function: $\chi_{PU} : \mathcal{U} \rightarrow \{0,1\}$.

which represents the interpretation for an sole predicate 'P'.

The function χ can be considered as a relation,

$$\chi \subseteq \mathcal{U} \times \mathbf{Prd}^1,$$

which assigns to each object in $\mathcal{U}$ *all* those properties possessed by that object. There are two extensions of the relation χ, the first is the deviant extension considered by the fuzzy set community (discussed in §8·5). The second is a conventional development of this relation using the theory developed in §4·1·2, which I shall now consider.

Let us make the assumption that **Dom**(χ) = **Supp**(χ), which in this context is means that every object must possess at least one property. Define the (tolerance) relation ***induced*** by χ as:

$$a \sim b \text{ if and only if } \chi(a) \cap \chi(b) \neq \varnothing.$$

This is a formal way of stating that two objects in $\mathcal{U}$ are considered to be related if there exists at least one property which is common to both. Clearly, one can view this relation as a kind of similarity between objects. Furthermore, one can grade this relationship by making the following definition:

the object a bears a greater resemblance to the object b than to c, if the following is true:

$$\chi(a) \cap \chi(c) \subseteq \chi(a) \cap \chi(b),$$

which states that the objects a and b share a greater number of properties than the objects a and c. Note the use of the subset relation in this definition[2].

The development here is clearly at it most elementary stages, but it would be interesting to see how this can be applied to common sense knowledge representation scenarios. Also to discover the kind of reason we as humans perform on similar entities.

---

[1]**Prd** is the collection of all predicate symbols. See Appendix B for details.

[2]I would like to thank Ian Gent for his comments regarding this relation.

9·1·3 Semantics of the 'determinately' operator

In §6·4·2 I described the work of Chandler[1967], who introduced the notion of an individual x *determinately* having the property P (denoted by: 'DP(x)'). What remains to be shown is a semantics for this operator.

Let $\mathfrak{L}$ be a first order language with a structure $\langle \mathcal{U},\sigma \rangle$, and suppose that the domain $\mathcal{U}$ is enhanced with a tolerance space structure $\langle \mathcal{U},\xi \rangle$. Then the operator 'D' can be given the following semantics,

$$\sigma[DP(x)] \text{ is true if and only if } x \in P^{\circ},$$

where P° denotes the interior of the set P (see §3·4 for details). This is a very intuitive notion. I believe that we – as humans – operate on a kind of *prototype theory*[1], in which we define certain core members of the class to be true (or false) and expect to derive the rest by assuming that predicates are continuous.

Realistically, this approach can only succeed after we have the formal definition of the language together with a calculus to enable us to perform inferences. However, it should eventually provide a better way of dealing with such categories than that using fuzzy sets.

9·2 SUMMARY

The problem of dealing with vagueness in a formal framework has been the main aim of this thesis. I have discussed many other issues along the way in order to indicate that vagueness is not a simple notion, but integrates an apparently wide range of topics which are central to AI.

The main thrust of this thesis comes late – in chapter 8. This is mainly due to the very difficult and disparate topics that were in need of integration before a proper analysis of vagueness could be attempted.

In this thesis I have developed the work of Zeeman[1962] and Poston[1971] on tolerance geometry (chapter 4). In particular, I have demonstrated how tolerance relations and

[1]See Osherson&Smith[1981], and Rosch[1973] for details of prototype theories of cognition.

covers have a correspondence which emulates that between equivalence relations and partitions. This is a new result, and one which should be invaluable to those wishing to formalise taxonomies or (class structures) whose classes overlap. I developed this result primarily to allow one to express the notion of indistinguishability in an object language, which I had failed to achieve in my first attempt to resolve the Sorites.

In chapter 4, I also demonstrated how the outstanding problem of formally representing the common sense notion of touching, as expressed in Hayes[1979a,1985], could be achieved in an intuitive manner using tolerance spaces. The comment by Hayes[1985] which stated that he believed that this could be done was my introduction to tolerance geometry. However, the task of actually representing this solution was made difficult by the fact that not only did Hayes not develop this comment any further, neither did anyone else. Very little work has been done on tolerance relations since Poston's[1971] PhD thesis. There are probably less than twenty articles on the topic, theoretical as well as those dealing with applications of the theory.

In chapter 8 (§8·3), I discuss the geometrical issues concerning the interpretation of a monadic predicate, which I use to argue that fuzzy set theory, as described by Zadeh, has precarious foundations. In fact in §8·5 and §8·6, I argue that we should not require predicates to take values in a *continuum* (as Zadeh suggested), but rather that their interpretations should be *continuous*. It appears that Zadeh has confused the notion of continuity (a property of a *function*) with that of the continuum (a property of a *domain*), when analysing common sense predicates. This subtle distinction makes all the difference between having a logic with an uncountable number of truth values, and one which has three.

Having discussed the analyses of vagueness presented earlier in chapter 6, I argue the case for introducing a spatial structure into the formal metalanguage (§8·6). Having a formal metalanguage enables me to translate the formal analysis of the Sorites (presented

in §5·2·2) into the metalanguage, where I am able to *prove* that the puzzle, as stated, is insoluble in a logic with a truth–set {0,1}.

I also investigate the possibility of resolving the Sorites in a logic with a truth–set given by {0,u,1}. I *prove* that there can only be *one* tolerance relation which satisfies the quite general metalogical conditions expressed in §8·2, and those required by the Sorites (§8·11). This tolerance relation I call ε. I prove that there can only be *two* logics which satisfy these conditions, and have semantics which are constrained by the tolerance space ε. I therefore conclude that the Sorites can be expressed in a consistent manner, in a logic with a truth–set given by {0,u,1}, and a formal metalanguage structured by a tolerance relation, ε.

9·3 CONCLUSIONS

McCarthy and Hayes have consistently stated, that the central issue in AI is – and has always been – how to formally represent our common sense knowledge of the everyday physical world. Intuitive statements such as, 'the man is *tall*', 'the cup is *hot*', and 'Bertrand's jumper is *red*', clearly involve terms which are vague. Thus, if we wish to represent this kind of knowledge, then sooner or later be we will be forced to address the issues concerning vagueness. This point is clearly recognised by Davis[1990], who states that "in some respect the concepts of common sense knowledge are vague, in a sense that goes beyond uncertainty and incompleteness".

According to Davis, vagueness is a notion which is theoretically *extremely* difficult to deal with, and claims that the *only* formal mechanism available for dealing with vague concepts is Zadeh's theory of fuzzy sets. Whilst accepting Davis' statement that vagueness is a difficult topic to deal with in a formal framework, I reject his claim regarding Zadeh's fuzzy sets.

If one is to achieve the goals described by McCarthy and in particular by Hayes, then we must find ways of dealing with vague predicates in a formal manner. My analyses of vagueness have shown that vague predicates possess *continuous* extensions, which we can only restate in a formal framework if we use a formalised metalanguage that carries a spatial structure. Furthermore, the same analyses can be used to show why Zadeh's assertion that vague predicates take values in a *continuum* (as suggested by Zadeh) is inadequate to capture this complex notion.

There are advantages in having a configuration of two formalised languages with one which incorporates a spatial structure and acts as the metalanguage for the other, such a configuration not only keeps our semantics clean, but also enables us to represent entities at two distinct levels. Moreover, we can prove results about the language which we use to represent our everyday world (as for example whether it is powerful enough to cope with vague predicates).

As Davis describes, the difficulties concerning the representation of vague information are made vivid by the ancient paradox called the Sorites, and it is therefore important to deal with this puzzle first. By using the various analyses of the Sorites paradox, I am able to present an analysis of my own and thence *prove* that such a puzzle *cannot* be solved in a 2–valued logic, but can be resolved in a 3–valued logic. This significant result can only be proved with the aid of a formalised metalanguage which supports a spatial structure.

One can only conclude by stating that Hayes was correct. Tolerance Geometry can be used in the metalanguage to produce a cleaner and more powerful formal solution to the problem of representing vague predicates, than that offered by the fuzzy set theorists. Moreover, we don't need a logic with an uncountable number of truth–values but only three.

A Concepts in Mathematics

The material in this appendix is intended to be used as a guide to the notation, and the various constructions used in this thesis.

A·1 INTRODUCTION

Modern mathematics studies the pattern of relationships which is exhibited by collections of objects. The 'collection of objects' is captured by the mathematical concept of a set, and is considered to be the basic raw material of modern mathematical thought. The 'pattern of relationships' is called a *structure*, and represents the mathematicians analogy of the fundamental notion of form. A set endowed with a structure is usually called a *mathematical structure.*

There are basically two kinds of structures in mathematics: *algebraic* and *analytic*. A structure which permits the composition of elements, resulting in an another element of the set, is often considered to be an algebraic structure. Analytic strucutres can be further subdivided into, *topological* and *measure* structures. Topological spaces are those structures where the relationship between the elements of the underlying set can be characterized by the concept of *neighbourhoods.* Whereas measure spaces are commonly used to formulate notions of extent.

A·2 CANTORIAN SET THEORY

The mathematicians which were instrumental in the development of the mathematical discipline we call set theory include: Bolzano[1820–30], Schröder[1890], Peirce[CP],

Dedekind[1872] and of course Cantor[1895], who is still regarded by many to be the founding figure of this mathematical discipline.

A·2·1 The Concept of a Set

The concept of a set has emerge as the most fundamental notion of modern mathematics. However, because of its generality, it is extremely difficult to give a logically unassailable definition of a set. Cantor[1895, p.282][1] defined a *set* or (Menge) as:

> *"Definition: By a 'set' we mean any collection M into a whole of definite, distinct objects m (which are called the 'elements' of M) of our perception [Anschaung] or of our thought."*

I believe that Cantor was trying to capture two fundamental points in his definition. Firstly, that a collection of objects should be regarded as a single entity. And secondly, that a set is completely determined by its members. Cantor achieves the second point by using the words 'distinct' and 'definite' to describe the sort of objects qualified to appear as members of a set, and can be taken to have the following interpretation: Consider two objects qualified to appear as elements of a particular set, then we must be able to determine whether the two objects are

(i) the same or different;

(ii) whether each object is, or is not, an element of the set.

These points actually appear in Cantor's own writings. The first point comes from Cantor[1883,p.204][1]:

> *"By an 'aggregate' or 'set' I mean generally any multitude which can be thought of as a whole, i.e. any collection of definite elements which can be united by a law into a whole."*

The second point can be found in Cantor[1882,p.150][2]:

> *"I call an aggregate (a collection, a set) of elements which belong to any domain of concepts **well-defined**, if it must be regarded as **internally determined** on the basis of its definition and in consequence of the logical principle of the excluded middle [ausgeschlossenen Dritten]. It must also be*

[1]Dauben[1979], p.170, also in Kneale & Kneale[1962], p.439.

[2]Dauben[1979], p.83. See also Cantor[1915], p.46.

> *internally determined whether any object belonging to the same domain of concepts belongs to the aggregate in question as an element or not, and whether two objects belonging to the set, despite formal differences, are equal to one another or not." [my emphasis].*

The Membership Relation

It is plain from the definition of a set, that there is an intrinsic relation between objects and sets, namely that of belonging or membership. It is this notion of membership which is often considered to be the principle concept of set theory, and not the concept of a set.

A·2·2 The Axioms of Cantors Set Theory

The Axiom of Extensionality[1]. Two sets are equal iff they have the same members.

A notion which is of some importance for describing a set, is that of an 'open sentence'. Intuitively, an ***open sentence in x*** is defined to mean a finite sequence made up from words and the symbol x, such that when each occurrence of x is replaced by the same name of an object a true or false statement will result. Quine[1968] discusses this construct, and gives 'x has fins' as an example. According to Quine[1968,p.486][2] this open sentence

> *"... is supposed to determine both an attribute, that of finnedness, and a class, that of fin bearers (past, present and future). The class may be called the extension of the open sentence"*

The Axiom of Abstraction. An open sentence '$\phi(x)$' defines a set A by the convention that the members of A are precisely those objects a such that '$\phi(a)$' is a true statement.

In moden notation the set determined by an open sentence '$\phi(x)$' is symbolized by: '$\{x \mid \phi(x)\}$'. However, traditionally circumflex accents were used to describe a set (or class). As Quine[1968, p.486] points out there were two ways

[1]I shall not consider the *axiom of choice* (see Johnstone[1987]), which together with the axioms I shall mention form the basis of Cantor's set theory.

[2]In this section I use property and attribute interchangeably.

> *"... of using circumflex accents in connexion with open sentences, one for the abstraction of attributes and the other for the abstraction of classes; thus '$\hat{x}$ has fins' names the attribute and '$\hat{x}$ (x has fins)' the class."*

A rather intuitive description of the principle of abstraction is provided by Frege in his original system. Frege claimed that every attribute ϕ had a class $\hat{x}(\phi x)$ as its extension, or in other words *every property determined a set.* Formally then, the axiom of abstraction is:

$$\exists y \forall x\, [x \in y \leftrightarrow \phi(x)].$$

It is with thanks to the axiom of abstraction that we can say that for a given set

$$A=\{x \mid \phi(x)\},\ a \in A \text{ if and only if '}\phi(a)\text{' is true.}$$

An open sentence '$\phi(x)$' in x then acts like a condition or property, which a given object must satisfy to be a member of A. Classically, as Quine[1968] mentions it, the members of $\hat{x}(\phi x)$ are such that: $\forall y\, [y \in \hat{x}(\phi x) \leftrightarrow \phi(y)]$.

A·3 ZERMELO-FRAENKEL SET THEORY

In 1901 Bertrand Russell[1] showed how the indiscriminate use of the abstraction axiom in Cantor's set theory could lead to contradictions. It wasn't until 1908 when the mathematician Zermelo used an axiomatic approach to demostrate how Cantor's set theory could be rescued. Zermelo's own theory was modified by Skolem and Fraenkel to produce what is now known as ***Zermeol-Fraenkel set theory*** (***ZF-set theory***).

The most significant difference between the two set theories, is the replacement of the axiom of abstraction with the following axiom of specification[2].

The Axiom of Specification. To every set A and every open sentence '$\phi(x)$' there corresponds a set B whose elements are exactly those elements of A for which '$\phi(x)$' is true. Formally,

$$\forall z \exists y \forall x\, [x \in y \leftrightarrow x \in z \wedge \phi(x)].$$

---

[1]There appears to be reasonable evidence to suggest that Zermelo discovered this paradox first. See Zermelo[1908], footenote 9 p.191.

[2]See Halmos [1960]. This axiom is sometimes also called the *axiom schema of Separation.*

Intuitively, the axiom of specification is not very different from Cantor's axiom of abstraction. The essential difference is not that the axiom of specification requires us to choose our members from some pre-determined collection, but the implicit condition that, the open sentence should apply to every member of this pre-determined discretionary collection (z).

A·4 CONCEPTS IN SET THEORY

The set of all subsets of a given set A is called the ***power set*** of A, and is symbolized by '$\mathbb{P}(A)$'. I use the phrase ***family of sets*** to mean a collection of sets, which are assumed to be subsets of some unspecified set.

Definition: A family of sets $\mathcal{A} = \{A_i\}$, is said to ***cover*** a set A (or be a ***cover for*** A), if A is contained in the union of the members of $\mathcal{A}$, i.e. $A \subseteq \bigcup_i A_i$. In other words each $a \in A$ belongs to some $A_i \in \mathcal{A}$.

Definition: A collection Π of subsets of a set X, is called a ***partition*** of X, if:

(i) Π is a cover for X

(ii) If $A, B \in \Pi$ & $A \neq B$ then $A \cap B = \varnothing$

A·4·1 The Concept of a Relation: Definitions and Analysis

In mathematics the notion of a binary relation is used as a means of describing the 'bond' or 'link' between the objects in a set A and those in a set B. The existence (or nonexistence) of the 'bond' is determined by an open sentence 'P(x,y)', in which 'P(a,b)' is either true or false for the elements $a \in A$ and $b \in B$.

Definition: Let A and B be two sets. Then a ***binary relation $\mathcal{R}$ from A to B*** (or ***between A and B***) is defined by a pair ($A \times B$, P(x,y)) whose first component is the cartesian product $A \times B$, and whose second component is an open sentence 'P(x,y)'. The set A is called the ***domain*** and is denoted by '**Dom**($\mathcal{R}$)'. The set B is called the ***codomain*** and is denoted by '**Codom**($\mathcal{R}$)'.

Associated with every binary relation $R=(A\times B, P(x,y))$ is a subset $\mathcal{G}(\mathcal{R})$ of $A\times B$ consisting of those pairs $\langle a,b\rangle \in A\times B$ for which $P(a,b)$ is true. This subset is conventionally written:

$\mathcal{G}(\mathcal{R}) = \{\langle a,b\rangle \in A\times B \mid P(a,b)\}$ and called the ***graph*** of the binary relation $\mathcal{R}$.

If $P(a,b)$ is true I shall write $a\mathcal{R}b$ or $\langle a,b\rangle \in \mathcal{R}$, which reads 'a is $\mathcal{R}$-***related*** to b'. And if $P(a,b)$ is false I shall write $a\not\mathcal{R}b$ or $\langle a,b\rangle \notin \mathcal{R}$ which reads 'a is not $\mathcal{R}$-related to b'.

<u>Definition</u>: Every binary relation $\mathcal{R}$ from A to B has an ***inverse relation*** $\mathcal{R}^{-1}$ from B to A, which is defined by: $\mathcal{R}^{-1} = \{\langle b,a\rangle \in B\times A \mid \langle a,b\rangle \in \mathcal{R}\}$

<u>Definition</u>: Let $\mathcal{R}$ be a binary relation from A to B. The collection of all those points in A which have some $b\in B$ $\mathcal{R}$-related to them, is called the ***support*** of $\mathcal{R}$ and is symbolized by: $\mathbf{Supp}(\mathcal{R}) = \{a\in A \mid \text{for some } b\in B, \langle a,b\rangle \in \mathcal{R}\}$.

<u>Definition</u>: Let $\mathcal{R}$ be a binary relation from A to B. For an element $a\in A$ the collection of all those elements in B which are $\mathcal{R}$-related to a, is called the ***image of a under*** $\mathcal{R}$ or the $\mathcal{R}$-***class*** of a; symbolized by: $\mathcal{R}^{a} = \{b\in B \mid \langle a,b\rangle \in \mathcal{R}\}$.

The image of the whole domain is called ***the image of*** $\mathcal{R}$, and is symbolized by:

$$\mathbf{Im}(\mathcal{R}) = \{b\in B \mid \text{for some } a\in A, \langle a,b\rangle \in \mathcal{R}\}.$$

Obviously, one can use the above definition to define the image of a subset $Y\subseteq A$ under $\mathcal{R}$ as follows: $\mathcal{R}^{Y} = \bigcup_{y\in Y} \mathcal{R}^{y}$.

Let $\mathcal{R}$ be a binary relation from A to B. If $\mathbf{Dom}(\mathcal{R}) = \mathbf{Codom}(\mathcal{R})$ (i.e. $A=B$), then $\mathcal{R}$ is said to be a ***relation in*** (or ***on***) A.

Let $\mathcal{R}$ be a relation from A to B and $\mathcal{S}$ a relation from B to C. Then the relation from A to C which consists of all ordered pairs $\langle a,c\rangle \in A\times C$ such that, for some $b\in B$, $\langle a,b\rangle \in \mathcal{R}$ and $\langle b,c\rangle \in \mathcal{S}$, is called the ***composition of*** $\mathcal{R}$ and $\mathcal{S}$. Symbolized by '$\mathcal{S}\circ\mathcal{R}$' and its graph is given by

$$\mathcal{G}(\mathcal{S}\circ\mathcal{R}) = \{\langle x,y\rangle \mid x\in A, y\in C; \text{ for some } b\in B, \langle x,b\rangle \in \mathcal{R} \text{ and } \langle b,y\rangle \in \mathcal{S}\}$$

Definition: Let $\mathcal{R}$ be a relation in A. Then $\mathcal{R}$ is said to be:

(a) ***reflexive*** iff: for all $x \in A$, $x\mathcal{R}x$;

(b) ***symmetric*** iff: for all $x,y \in A$ if $x\mathcal{R}y$ then $y\mathcal{R}x$;

(c) ***transitive*** iff: for all $x,y,z \in A$, if $x\mathcal{R}y$ & $y\mathcal{R}z$ then $x\mathcal{R}z$;

One important property of a binary relation on a set which is used frequently in mathematics and in logic, but is rarely formally defined can be stated as follows:

Definition: Let $\mathcal{R}$ a binary relation on a set A. A subset $X \subseteq A$ is said to be ***closed*** with respect to $\mathcal{R}$ if and only if: for all $x,y \in A$ if $x \in X$ and $x\mathcal{R}y$, then $y \in X$.

A·4·2 The Concept of a Function

Preliminary Notions

A rather good intuitive definition for a function is given by Church[1941]:

> *"A function is a rule of correspondence by which when anything is given (as argument) another thing (the value of the function for that argument) may be obtained."*

This definition is very similar to the one given originally by Dirichlet in the last century, who defined a ***numerical variable*** to be a symbol which can stand for any one of a collection of numbers. Then formulated the following definition for a function:

> *"We shall say that y is a function of x if x and y are variables and so related by a rule that the assignment of a number to x determines a unique assignment of a number to y."*

Let $\mathcal{R}=(A \times B, P(x,y))$ be a binary relation from A to B, and place a restriction on the relation such that $\mathcal{R}$ assigns a unique element of the set B to each element of the set A. The difficulty appears when I try and convert the open sentence 'P(x,y)' into a rule which given $a \in A$, return $b \in B$. The open sentence 'P(x,y)' will only return a *true* or *false* answer for a given pair $\langle a,b \rangle \in A \times B$, but will not necessarily allow me to derive the unique $b \in B$ which relates to a given $a \in A$. Given Dirichlet's definition, it is a trivial matter to convert it into the current terminology. Given a rule **F**, define 'F(a)' to mean

'the result of the rule **F** when applied to a', and thereby, define the open sentence 'P(x,y)' to be true for all pairs $\langle a,b\rangle \in A \times B$ where $b = F(a)$.

<u>Definition</u>: Let $\mathcal{R}$ be a binary relation from A to B. $\mathcal{R}$ is said to be a *function* from A to B if:

(i) $\mathbf{Dom}(\mathcal{R}) = \mathbf{Supp}(\mathcal{R})$

(ii) for each $a \in \mathbf{Dom}(\mathcal{R})$, the image of a under $\mathcal{R}$ (i.e. $\mathcal{R}^{a}$) is a singleton set.

Two functions f and g, are said to be *equivalent* if

(i) $\mathbf{Dom}(f) = \mathbf{Dom}(g)$

(ii) $\mathcal{G}(f) = \mathcal{G}(g)$.

Church[1941, p2], explicitly calls this approach 'functions in extension':

> *"... we regard the operation or rule of correspondence, which constitutes the function, as being first given, and the range of arguments then determined as consisting of the things to which the operation is applicable. This is a departure from the point of view usual in mathematics ..."*

If the notion of a function I have been using so far is what Church calls 'functions in extension', the obvious question is what does he mean by a 'function in intension'? I shall quote Church's own exposition:

> *"It is possible, however, to allow two functions to be different on th ground that the rule of correspondence is different in meaning in two cases although always yielding the same result when applied to any particular argument."*

It is interesting to note that even though mathematicians have a formal definition for a function in terms of its extension, they often prefer to use the more intuitive intensional notion as expressed by Dirichlet & Church, for their everyday use. The notions of intensionality and extensionality is studied in detail by Sloman[1965], who calls the intensional functions ***rogators***, and provides an insight into their characteristics.

In the following I shall follow the common trend of identifying the relation/function with its graph.

Further Notions

The function $f: A \to B$ induces a pair of functions, $f^{\rightarrow}$ and $f^{\leftarrow}$, on the power sets, which are called the ***associated set functions*** induced by f. They are defined as follows:

$$f^{\rightarrow}: \mathbb{P}(A) \to \mathbb{P}(B) \quad \text{defined by} \quad f^{\rightarrow}(X) = \{f(x) \mid x \in X\}$$

and, $$f^{\leftarrow}: \mathbb{P}(B) \to \mathbb{P}(A) \quad \text{defined by} \quad f^{\leftarrow}(Y) = \{x \in A \mid f(x) \in Y\}.$$

MacLane&Birkoff[1979] cover this in more detail, however their treatment is in terms of categories. I should point out that they use the notation $f^{*}(Y)$ and $f_{*}(X)$, in place of my $f^{\leftarrow}(Y)$ and $f^{\rightarrow}(X)$, respectively.

<u>Definition</u>: Let $f: A \to B$ and $g: C \to D$ be two function. The ***cross product function*** $f \times g$ is defined as the function

$$f \times g: A \times B \to C \times D \quad \text{defined by} \quad (a,b) \to (f(a), g(b)).$$

Sequences

Let X be an arbitrary set. A ***sequence*** s in X is then defined to be the function,

$$s: \mathbb{N} \to X.$$

Since s is a function, image of $k \in \mathbb{N}$ would normally denoted by s(k), however, it is customary to write 's_k' for s(k) and call it the 'kth term of the sequence'. The notation '$\langle s_k \rangle$' is also used to denote the sequence s.

A·4·3 Dual Functions

The set of all functions from A to B is called a ***function set***, and is denoted by $\mathcal{F}(A,B)$. Formally, $\mathcal{F}(A,B) = \{f \mid f: A \to B\}$.

Let X, Y, C be sets and $F: X \times Y \to C$ an arbitrary map. By fixing an element $x \in X$, we have the function H_x defined by

$$H_x: \{x\} \times Y \to C \qquad \text{by } H_x = F \circ I$$

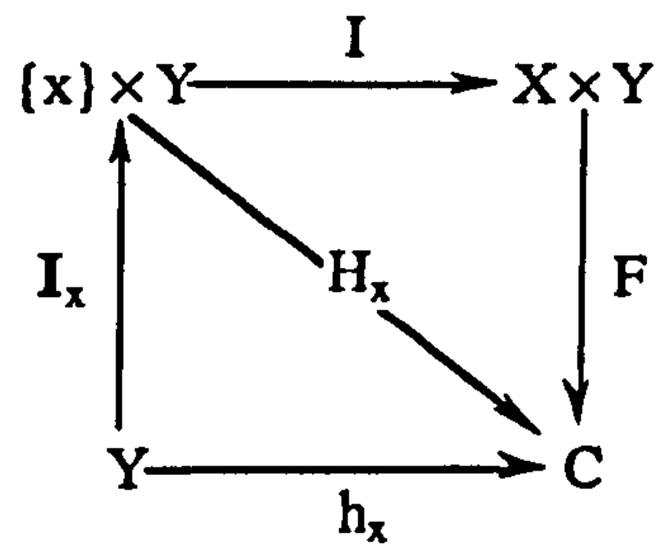
$$(x,y) \to H_x(x,y) = F \circ I(x,y) = F(x,y)$$

By composing H_x with the insertion[1] I_x, I shall construct a function h_x, where

[1]The function, $I = \mathbb{1}|_X : X \to A$, is called an ***insertion***, since it 'inserts' elements of X back into A.

$$h_x : Y \to C \qquad \text{defined by } h_x = H_x \circ I_x$$
$$y \to h_x(y),$$

and, $$h_x(y) = H_x \circ I_x\,(y) = H_x(x,y) = F(x,y).$$



Therefore, for each $x \in X$ we have a function $h_x : Y \to C$, which is used to define the function Φ:

$$\Phi : X \to C^Y \qquad \text{defined by} \qquad x \to h_x$$

Consider the converse; suppose we are given a function $\Phi : X \to C^Y$, then for each $x \in X$, $\Phi(x)$ is a map from Y into C. Replace $\Phi(x)$ it by Φ_x for ease of use, then

$$\Phi_x : X \to C^Y \qquad \text{defined by} \qquad y \to \Phi_x(y)$$

Finally, using the function Φ_x, one can define the function F

$$F : X \times Y \to C \qquad \text{defined by} \qquad (x,y) \to \Phi_x(y).$$

The functions F and Φ are called ***dual functions.***

MacLane&Birkoff[1979], approach the same idea in a different manner, by using a function called an ***evaluation***:

$$e : C^Y \times Y \to C \qquad \text{defined by} \qquad e(f,y) = f(y),$$

which returns a value f(y) when applied to a function f and an element $y \in Y$. Notice by fixing $y \in Y$, we obtain a function

$$e_y : C^Y \to C \qquad \text{defined by} \qquad e_y(f) = f(y),$$

which acts like a projection function from C^Y onto C. The idea that a function of two variables can be decomposed into a function of one variable which returns a function of one variable as a value, is clearly presented in Schönfinkel[1924]. According to

Church[1941], Schönfinkel was the originator of this idea.

A·4·4 Characteristic functions

The introduction of the characteristic function offers a convenient formalism to describe sets. Let $\mathbb{P}(X)$ denote the power set of X, and $\{0,1\}^X$ denote the set of all functions from X into $\{0,1\}$.

Theorem: There is a bijection between the sets $\{0,1\}^X$ and $\mathbb{P}(X)$[1].

Proof: Suppose X and $\mathbf{Y}$ are two sets, such that we can construct a function

$$\chi : \mathbf{Y} \times X \rightarrow \{0,1\}$$

between X and $\mathbf{Y}$. For $\mathbf{Y} = \mathbb{P}(X)$, we can construct a function

$$\chi : \mathbb{P}(X) \times X \rightarrow \{0,1\}$$

such that for any subset S of X

$$\chi(S,x) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ if } x \notin S \end{cases}$$

For the subset S of X, the dual function χ_S to χ, is called the ***characteristic function*** of the subset S. To every member S of $\mathbb{P}(X)$ there is a function $\chi_S \in \{0,1\}^X$ such that $\chi(S,x) = \chi_S$. Conversely, given any function $\psi \in \{0,1\}^X$, we can define a subset S_ψ of X by:

$$S_\psi = \{x \mid \psi(x) = 1\}.$$

QED.

If in the function $\chi : \mathbf{Y} \times X \rightarrow \{0,1\}$, we let $\mathbf{Y} = \mathbf{Prd}$ (i.e. the set of all monadic predicate symbols), we obtain an interpretation function

$$\chi : \mathbf{Prd} \times X \rightarrow \{0,1\},$$

which for a fixed symbol P, gives the associated dual function to χ, namely

$$\chi_P : X \rightarrow \{0,1\}$$

[1]We can construct a Boolean algebra out of the set $\{0,1\}^X$, by defining the following operations: $(\chi \wedge \psi)(x) = \max\{\chi(x), \psi(x))$; $(\chi \vee \psi)(x) = \min\{\chi(x), \psi(x))$; $\bar{\chi}(x) = 1-\chi(x)$. Since $\mathbb{P}(X)$ is also a Boolean algebra under the operations, $\cap, \cup, -$. The function $\Phi:\langle \mathbb{P}(X), \cap, \cup, -\rangle \rightarrow \langle \{0,1\}^X, \wedge, \vee, ^-\rangle$, defined by $\Phi(S) = \chi_S$, is an isomorphism of Boolean algebras.

which is the standard definition for the interpretation of, the predicate symbol P.

Intuitively the function $\chi : \mathbf{Prd} \times X \rightarrow \{0,1\}$ operating on a pair $\langle P,a \rangle$, determines whether the predicate symbol P holds true for the object a.

A·5 EQUIVALENCE RELATIONS AND THE EQUIVALENCE KERNEL

If we seek the foundations of almost any aspect of modern mathematics, we shall dicover the notion of an equivalence relation being used. In conjunction with this notion we find a construction called, the 'equivalence kernel'. It is important to emphasize the significance of this construction, since it is used in so many disparate areas of mathematics.

Definition: A relation $\mathcal{E}$ on X which is reflexive, symmetric and transitive is called an ***equivalence relation.***

Definition: Given an equivalence relation $\mathcal{E}$ on A, the $\mathcal{E}$-class of $a \in A$ is called the ***equivalence class*** of $a \in A$. The collection of all equivalence classes for the equivalence relation $\mathcal{E}$ is called the ***quotient set*** of A by $\mathcal{E}$, and is written as: $A/\mathcal{E} = \{\mathcal{E}^a \mid a \in A\}$. The function which takes every element onto its equivalence class, is called the ***canonical projection***, and defined by

$$p : A \rightarrow A/\mathcal{E} \qquad \text{defined by} \qquad a \rightarrow \mathcal{E}^a$$

The Equivalence Kernel

Definition: Let $f : A \rightarrow B$ be an arbitrary function between two sets. Define the ***equivalence kernel*** $\mathcal{K}_f$, to be the relation:

$$a\mathcal{K}_f b \qquad \text{iff} \qquad f(a) = f(b)$$

The equivalence class $\mathcal{K}_f^a$ is called a ***fibre*** over B, and consists of elements which have the same image under f. The canonical projection onto the quotient set $A/\mathcal{K}_f$ is given by:

$$p : A \rightarrow A/\mathcal{K}_f \qquad \text{defined by} \qquad a \rightarrow \mathcal{K}_f^a$$

Theorem: Any function $f: A \rightarrow B$ can be decomposed into $f = f^* \circ p$, where p is the surjective canonical projection, $p: A \rightarrow A/\mathcal{K}_f$, and f^* is the injective function,

$$f^*: A/\mathcal{K}_f \rightarrow B \text{ induced by } f.$$

Proof: See MacLane&Birkoff[1979].

Theorem: Let A be a set.

(1) If $\mathcal{E}$ is an equivalence relation on A, then the quotient set $A/\mathcal{E}$ is a partition of A.

(2) If Π is a partition of A, then the relation $\mathcal{E}_\Pi$ on A given by

$$x\ \mathcal{E}_\Pi\ y \quad \text{iff} \quad \text{there exists some } T \in \Pi \text{ such that } x,y \in T$$

is an equivalence relation on A, and is said to be ***induced*** by Π.

Proof: See Eisenberg[1974].

A·6 MATHEMATICAL STRUCTURE

Definition (Gandy[1973]): A ***mathematical structure*** is specified by a domain (or set) of objects and a collection of relations over that domain.

An ***abstract mathematical structure***[1], is a structure where the nature of the individuals in the domain are of no importance, nor is the nature of the specified relations.

Definition: An ***(internal) binary operation*** $\cdot$ on a set A is a function from the cartesian product $A \times A$ into A, i.e.

$$\cdot : A \times A \rightarrow A.$$

An ***(external) binary operation*** $*$ on a set A by a set X, is a function

$$* : X \times A \rightarrow A$$

where the sets X and A are distinct.

Obviously, one can extend this definition to an ***n-ary operation*** as:

$$\cdot : X^n A \rightarrow A.$$

[1]It was Boubaki [1939] who first unified the various new notions to appear in mathematics since the begining of this century into a general concept of 'mathematical structure'.

Notation: A mathematical structure is written as a list $\langle S; \mathcal{R}_1, \mathcal{R}_2, \ldots; f_1, f_2, \ldots \rangle$, where S denotes the set or domain, the $\mathcal{R}_i$'s denote relations on S, and the f_i's operations on S.

Definition (MacLane&Birkoff[1979]): An ***algebraic system*** is a set S together with one or more n-ary operations on S which satisfy specified axioms (identities or other conditions). A ***morphism*** $\varphi : \langle S;\oplus \rangle \rightarrow \langle T;\otimes \rangle$ between two algebraic systems $\langle S;\oplus \rangle$ and $\langle T;\otimes \rangle$ is defined to be a function from S to T such that[1]

$$\varphi(x \oplus y) = \varphi(x) \otimes \varphi(y) \qquad \text{for all } x,y \in X.$$

A bijective morphism is called an ***isomorphism***.

A·7 FIBRATIONS AND GROUP ACTIONS

A·7·1 Fibrations

Given a map $f : A \rightarrow B$ between two sets A and B, the normal (*extensional*) intuition is to consider the domain A as the primary object, upon which the function f acts, by 'mapping' each element a of A onto the element f(a) in the codomain B. However, there is a second way of viewing this set up, namely by invoking the equivalence kernel of f. In this case we consider the codomain B to be the primary object, and the think of the function $f : A \rightarrow B$ as the family $\{f^{\leftarrow}(b) \mid b \in B\}$ of subsets of A. Each member $f^{\leftarrow}(b)$ of the family is called a ***fibre*** over B. In §A·5 I illustrated how such a family could be constructed, except there I used the notation, '$A / \mathcal{K}_f$' to denote the family '$\{f^{\leftarrow}(b) \mid b \in B\}$'.

Notation: If we decide to view the function $f : A \rightarrow B$, as a family of fibers over B, then it is common to use the notation 'f_b' to denote the fiber '$f^{\leftarrow}(b)$'.

A·7·2 Group Actions

Informally speaking, the basic idea is to use the group structure to move elements of the set, to other elements of the same set.

---

[1]Alternatively, $(\varphi \bullet \oplus = \otimes \bullet (\varphi \times \varphi) : S \times S \rightarrow T)$.

I have already mentioned that an external binary operation φ on a set M by a set G, is a function, $\varphi : G \times M \rightarrow M$. Now suppose that the set G posesses a group structure, in which case we say that the group G has an ***action*** on the set M.

Definition: A group G is said to have an ***action on a set M*** (and M is called a ***G-set***) if there exists a map

$$\varphi : G \times M \rightarrow M$$

$$(g,x) \rightarrow \varphi(g,x)$$

such that the following conditions are satisfied:

(i) $\varphi(e,x) = x$ for all x in M;

(ii) $\varphi(g, \varphi(h,x)) = \varphi(gh,x)$.

This action of G on M is called a ***left action.*** A right action is similarly defined.

By using the duality theorem we can rephrase the function $\varphi : G \times M \rightarrow M$, as $\varphi : G \rightarrow M^M$. Thus for each fixed $g \in G$ we have a function $\varphi_g : M \rightarrow M$, using this notation the conditions (i) and (ii) can be written as: $\varphi_e(x) = x$; and $\varphi_g \bullet \varphi_h = \varphi_{gh}$ respectively.

Definition: The action of a group G on a set M is said to be, ***transitive*** if for every $x,y \in M$ there exists $g \in G$ such that $\varphi_g(x) = y$. In other words given two points in M, there always exists an element in G which will map one point onto the other.

Orbits

Let,

$$\varphi : G \times M \rightarrow M$$

$$(g,x) \rightarrow g \cdot x$$

be an action of the group G on the set M. Select an element $x \in M$, and consider the set which consists of all points which are obtained from x by applying an element of the group G. This set is called the orbit of x under the action of G, and it is easy to see that the set of orbits under the action of G will partition the set M.

Definition: Let the group G have an action on the set M. Define the relation $\mathcal{R}$ in M by:

$x \mathcal{R} y$ if and only if $y = g \cdot x$,

where x and y are points in M and g is some element of the group G.

The equivalence class of the point x is given by:

$$G \cdot x = \{y \mid y = g \cdot x \text{ for some } g \in G\} = \{g \cdot x \mid g \in G\}$$

and is called the ***orbit*** of x under the action of G. Note that G has a transitive action on each of the orbits $G \cdot x$.

A·8 INDUCTIVE STRUCTURES

The concept of *mathemtical induction* is one of the most interesting techniques in mathematics, and certainly the most important for the metalogician.

A ***recursive*** definition of S contains three essential components:

(a) the ***base clause***: the specification of the ***initial objects***,

(b) the ***inductive clause***: the specification of the ***operations***,

(c) the ***closure clause***: which states that nothing else is a member of S.

Thus S is defined as the set of all the elements in the set **I** which are used as the building blocks, together with any new elements which can be obtained from the objects in **I** by using any elements from **O**. The ideas of inductive definitions are akin to the notion of a subset closed under a binary relation.

In the light of the above definition, we can state the example of the natural numbers formally, as (Stewart&Tall[1977, p.146]):

Suppose there exists a set $\mathbb{N}_0$ and a function $s : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that

(N1) s is not surjective; there exists $0 \in \mathbb{N}$ such that $s(n) \neq 0$ for any $n \in \mathbb{N}_0$.

(N2) s is injective; if $s(m) = s(n)$ then $m = n$.

(N3) If $S \subseteq \mathbb{N}_0$ is such that $0 \in S$; and if $n \in S$ then $s(n) \in S$ for all $n \in \mathbb{N}_0$, then $S = \mathbb{N}_0$.

B Formal Languages, Systems and Semantics

B·1 Formal Languages and Systems

The basic objects of metalogic are *formal systems.*

Definition: A ***formal language***, $\mathfrak{L}$, is identified by the set of its ***well-formed formulas*** (***formulas*** or ***wffs*** for short). The ***well-formed formulas*** for $\mathfrak{L}$, are defined by:

(a) a set A, of ***symbols*** called the ***alphabet*** of the language, which is devided into ***logical*** and ***non-logical symbols***;

(b) a set Fr, of ***formation rules*** determining which expressions are to be considered as well-formed formulas.

Both part (a) and (b) must be defined without any recourse to an interpretation for the language. Non-logical symbols stand for something in the world (domain), whereas logical symbols do not stand for anything in the world, but serve to combine smaller expressions of the language into larger ones.

By taking a formal language we can add to it a *calculus*, thereby obtaining a formal system.

Definition: A ***formal system*** $\mathfrak{L}$ is a formal language $\mathfrak{L}$ together with a ***calculus*** (or ***deductive apparatus***) or given by:

(a) selecting a subset Ax of $\mathfrak{L}$, called the ***axioms*** of $\mathfrak{L}$,

and/or (b) a set Tr, of ***transformation rules*** .

The transformation rules warrant the derivation of some formulas from others.

Definition: The ***transformation rules, Tr,*** for a formal system $\mathfrak{T}$ is defined to be a collection of *finite* relations over $\mathfrak{T}$, that is a subset of $\mathfrak{T}^n$. An element of *Tr* is called an ***rule of inference*** of $\mathfrak{T}$.

Definition: Given a formal system $\mathfrak{T}$ and a set Γ of formulas of $\mathfrak{T}$, we say that a formula y is ***immediately inferred*** from the set Γ if there exists a rule of inference $R \in Tr$, and a finite sequence $a_1,\ldots,a_{n-1}$ of elements of Γ such that $\langle a_1,\ldots,a_{n-1}, y\rangle \in R$.

Definition: Given a subset Γ of formulas of a formal system $\mathfrak{T}$, we say that a formula $y \in \Gamma$ is ***deducible*** from the hypotheses Γ, if there exists a finite sequence $a_1,\ldots,a_n$ of formulas such that y is a_n, and such that every member of the sequence is either

(a) an element of Γ,

(b) an axiom of $\mathfrak{T}$,

(c) immediately inferred from a set of prior members of the sequence by some rule of inference.

The finite sequence $a_1,\ldots,a_n$ itself is called a ***formal deduction*** from the hypotheses Γ. And if Γ is *empty* the sequence $a_1,\ldots,a_n$ is called a ***formal proof***[1] in $\mathfrak{T}$, and in this case the formula y is said to be ***provable*** or a ***theorem*** of $\mathfrak{T}$.

The set of all formulas which can be deduced from a given set of hypotheses Γ is denoted by $Cn(\Gamma)$, the set of ***syntactic consequences*** of Γ. The theorems are thus the set $Cn(Ax)=Cn(\varnothing)$. I shall use the notation '$\Gamma \vdash y$' instead of $y \in Cn(\Gamma)$, and '$\vdash y$' instead of '$\varnothing \vdash y$'. The notation '$\vdash y$' means y is a theorem or y is provable. It is very important to note that the sign '$\vdash$' is *not* a symbol of the object language, it is a sign indicating an argument, and therefore a symbol of the metalanguage proper. I am abusing the notation

[1]This definition is somewhat different to the one Hilbert gave in his 1927 paper: *"A proof is an array that must be given as such to our perceptual intuition; it consists of inferences according to the schema " $C, C \rightarrow L \vdash L$ ",where each of the premisses, that is, the formulas C and $C \rightarrow L$ is an array, either is an axiom or results directly from an axiom by substitution, or else coincides with the end formula of an inference occuring earlier in the proof or results from it by substitution".*

here, because I should really be indexing the symbol '$\vdash$' with the symbol '$\mathfrak{L}$' to indicate the formal system which I am using.

B·2 SENTENTIAL LOGIC

B·2·1 The Formal Language

Definition: The set $\overline{\mathcal{S}}$ of well-formed formulas for the language of ***sentential logic*** is defined by induction, as follows:

Alphabet:

non-logical symbols: $\mathcal{S}=\{P_1, P_2, \ldots, P_n, \ldots\}$ called ***sentence symbols***

logical symbols: '$\neg$', '$\vee$', '$\wedge$', '$\rightarrow$' and '$\leftrightarrow$'; called ***sentence connectives***, together with the ***punctuation symbols*** '(' and ')'.

Formation rules:

A set $O_{\mathcal{S}}=\{\mathcal{E}_{\neg},\mathcal{E}_{\vee},\mathcal{E}_{\wedge},\mathcal{E}_{\rightarrow},\mathcal{E}_{\leftrightarrow}\}$ defined as follows:

$\mathcal{E}_{\neg}(\alpha) = (\neg\alpha)$: called the ***negation*** of α,

$\mathcal{E}_{\vee}(\alpha,\beta) = (\alpha\vee\beta)$: called the ***disjunction*** of α and β,

$\mathcal{E}_{\wedge}(\alpha,\beta) = (\alpha\wedge\beta)$: called the ***conjunction*** of α and β,

$\mathcal{E}_{\rightarrow}(\alpha,\beta) = (\alpha\rightarrow\beta)$: called the ***conditional*** of α and β, where α is called the ***antecedent***, and β the ***consequent***,

$\mathcal{E}_{\leftrightarrow}(\alpha,\beta) = (\alpha\leftrightarrow\beta)$: called the ***biconditional*** of α and β.

Well-formed formulas of $\overline{\mathcal{S}}$:

(i) *atomic formulas* (*initial objects*, or *atomic sentences*): $\mathcal{S}$,

(ii) *sentences* (formulas of sentential logic): any α and β which belong to $\mathcal{S}$ or can be generated by applying one or more of : $\mathcal{E}_{\neg},\mathcal{E}_{\vee},\mathcal{E}_{\wedge},\mathcal{E}_{\rightarrow}$ and $\mathcal{E}_{\leftrightarrow}$.

(iii) nothing else is in $\overline{\mathcal{S}}$.

B·2·2 The Formal System

There are a number of different calculi available for $\mathcal{S}$, examples include the Hilbert style *axiomatic* approach and Gentzen style *natural deduction* approach.

Historically, logicians characterized logical truths of the sentential logic in terms of axiom systems (for example see Hilbert[1927]). The logical truths were defined inductively, by taking a collection of sentences as the *initial objects* (called ***axioms***), together with a collection of inference rules as the *operations*. The inductive set of theorems, was defined as the closure of the axioms under the rules of inference.

If the collection of axioms was defined as a finite set then the collection was simply enumerated. It was possible to give infinite sets by using a finte set of what are commonly called ***axioms schema***, which were essentially sentential 'forms', from which axioms could be constructed by replacing the letters in the axiom schema by sentences.

<u>Definition</u>: The formal system $\mathcal{S}$ is defined as the structure $\langle Th, Ax, \mathcal{I} \rangle$ such that:

(a) Ax is the set of all sentences having the forms defined by the following axiom schemas:

(1) $(\alpha \rightarrow (\beta \rightarrow \alpha))$,

(2) $((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))$,

(3) $((\neg\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \alpha))$.

(b) The operation $\mathcal{I}$ is defined as the function which maps any pair of sentences having the form '$(\alpha \rightarrow \beta)$' and '$\alpha$' to $\mathcal{I}(\alpha, \alpha \rightarrow \beta) = \beta$.

(c) Th is the smallest inductively defined set which is the closure of Ax under $\mathcal{I}$.

As the main theme of the thesis is metalogic and *not* logic, it is unimportant which calculus is chosen, so long as I have a way of denoting provable formulas.

I have illustrated a Hilbert style axiom system as an example of how the proof theory can be defined inductively. For details of this approach see Hunter[1971], Lyndon[1966], or Enderton[1972].

B·2·3The Formal Semantics

Definition: The set $\mathbb{V}=\{\mathbb{T}, \mathbb{F}\}$, is called the set of ***truth-values***. The symbol '$\mathbb{T}$' is called ***truth***, and the symbol '$\mathbb{F}$' ***falsity***.

Definition: A ***truth assignment*** υ for the set $\mathcal{S}$ of sentence symbols is a function

$$\upsilon : \mathcal{S} \rightarrow \mathbb{V}$$

assigning either $\mathbb{T}$ or $\mathbb{F}$ to each sentence symbol in $\mathcal{S}$.

In order to be able to use the recursion theorem we must first of all define a collection of operations $\mathbf{O}_{\mathbb{V}}=\{\mathcal{F}_{\neg},\mathcal{F}_{\vee},\mathcal{F}_{\wedge},\mathcal{F}_{\rightarrow},\mathcal{F}_{\leftrightarrow}\}$ on $\mathbb{V}$, one for each one of the formation rules in $\mathbf{O}_{\mathcal{S}}$. The operations $\mathbf{O}_{\mathbb{V}}$ are often called ***truth functions*** and are the formal counterpart of the *semantic rules*.

Definition: An ***interpretation*** for the language of sentential logic ($\overline{\mathcal{S}}$), is defined to be the unique extension $\upsilon' : \overline{\mathcal{S}} \rightarrow \mathbb{V}$ of a truth assignment

$\upsilon : \mathcal{S} \rightarrow \mathbb{V}$ which satisfies the conditions:

(0) $\upsilon'(P_i) = \upsilon(P_i)$, for each $P_i \in \mathcal{S}$.

For any $\alpha, \beta \in \overline{\mathcal{S}}$:

(1) $\upsilon'(\mathcal{E}_{\neg}(\alpha)) = \mathcal{F}_{\neg}(\upsilon'(\alpha))$

(2) for each $\pi \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$: $\upsilon'(\mathcal{E}_{\pi}(\alpha,\beta)) = \mathcal{F}_{\pi}(\upsilon'(\alpha), \upsilon'(\beta))$.

The function υ' is guaranteed to exist and be unique by the *recursion theorem*[1].

Definition: A well-formed formula $\alpha \in \overline{\mathcal{S}}$ of the sentential logic, is said to be ***true in the interpretation*** υ' if $\upsilon'(\alpha) = \mathbb{T}$.

Notation: In the following I shall denote the formal language of the sentential logic, $\overline{\mathcal{S}}$ by $\mathcal{S}$, and an interpretation υ' of $\overline{\mathcal{S}}$ by υ.

Definition: An interpretation υ, is said to be a ***model*** of a formula $\alpha \in \mathcal{S}$ of the sentential logic, if $\upsilon(\alpha) = \mathbb{T}$.

---

[1]See Enderton[1972].

Definition: A formula $\alpha \in \mathcal{S}$ of the sentential logic is said to be a ***logical truth***, if $\upsilon(\alpha)=\mathbb{T}$ for every interpretation υ of $\mathcal{S}$. A logical truth α of $\mathcal{S}$ is denoted by: $\models \alpha$.

The symbol '$\models$' belongs to the metalanguage, and is an abbreviation for: 'α' is a logical truth of $\mathcal{S}$.

Definition: A $\alpha \in \mathcal{S}$ is said to be a ***semantic consequence*** of the set of formulas Γ, iff there is no interpretation of $\mathcal{S}$ for which every member of Γ is true and α is false. This is denoted by: $\Gamma \models \alpha$.

B·3 FIRST ORDER PREDICATE LOGIC

B·3·1 The Formal Language

Definition: The alphabet for the language $\mathcal{P}$ of ***First order predicate logic*** (abbreviated to ***FOL***) is defined as follows:

Variables (a) a set **Var** = $\{x_1, x_2, x_3, \ldots\}$[1] of ***individual variables***

Constants

logical:

(b) ***sentence connectives***: '$\neg$', '$\vee$', '$\wedge$', '$\rightarrow$' and '$\leftrightarrow$';

(c) ***punctuation symbols:*** '(' and ')';

(d) ***quantifier symbols:*** '$\forall$' and '$\exists$';

non-logical:

(e) a set **Con** = $\{a_1, a_2, a_3, \ldots\}$ of ***individual constant symbols.***

(f) a set **Prd** = $\{P_1, P_2, P_3, \ldots\}$ of ***predicate symbols***, (arity greater than zero);

(g) a set **Fun** = $\{f_1, f_2, f_3, \ldots\}$ of ***function symbols***, (arity greater than zero);

[1]It is very important that the variables $x_1, x_2, x_3, \ldots$ are viewed to be in a particular sequence given by the ordering of the subscripts.

Definition: Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ be expressions consisting of symbols from the alphabet of $\overline{\mathcal{P}}$, then for each n-arity function symbol f, we define an n-place ***term building operation*** $\mathcal{E}_f$ on expressions by: $\mathcal{E}_f(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n) = f\varepsilon_1\ \varepsilon_2\ \ldots\ \varepsilon_n$

Definition: The set **Tm** of ***terms*** is defined inductively as follows:

(a) an individual variable is a term, and an individual constant is a term,

(b) for each function symbol f, of arity n, and terms $t_1, t_2, \ldots, t_n$, then

$\mathcal{E}_f(t_1, t_2, \ldots, t_n) = ft_1\ t_2 \ldots t_n$, is a term,

(c) nothing else is a term.

A term in which no variables occur is called a ***closed term*** (or ***ground term***).

The terms in our formal language are analogous to the nouns in a colloquial language, and are used to define the atomic formulas, which are in turn used as the building blocks for defining the well-formed formulas.

Definition: Let P be a predicate of arity n, and $t_1, t_2, \ldots, t_n$ be n terms. Then the expression defined by the formation rule: $\mathcal{E}_{AF}(P, t_1, t_2, \ldots, t_n) = P(t_1, t_2, \ldots, t_n)$, is called an ***atomic formula***.The set of atomic formulas is denoted by **Af**.

Definition: The set $\mathcal{P}$ of well-formed formulas for the language of ***first order predicate logic*** is defined by induction as follows:

(a) atomic formulas are well-fromed formulas,

(b) if α and β are well-fromed formulas then so are:

$\mathcal{E}_{\neg}(\alpha)$, $\mathcal{E}_{\pi}(\alpha,\beta)$ (for each $\pi \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$),

$\mathcal{E}_{\forall}(x, \alpha) = \forall_x(\alpha)$ and $\mathcal{E}_{\exists}(x, \alpha) = \exists_x(\alpha)$.

(c) nothing else is a well-formed formula.

Next we need to define the notion of a 'free variable'.

Definition: A variable y is said to be ***free*** in a formula α, if:

(a) α is atomic and y occurs in α,

(b) y is free in $\mathcal{E}_{\neg}(\alpha)$, if y is free in α,

(c) y is free in $\mathcal{E}_{\pi}(\alpha,\beta)$ (for each $\pi \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$), if y is free in α or β,

(d) y is free in $\mathcal{E}_{\forall}(x, \alpha) = \forall_x(\alpha)$ or $\mathcal{E}_{\exists}(x, \alpha) = \exists_x(\alpha)$, if $x \neq y$.

A variable y which is not free in a formula α, is said to be ***bound*** in α. If the formula α contains the variables '$x_1, x_2, \ldots, x_n$', then this will be indicated by the notation '$\alpha(x_1, x_2, \ldots, x_n)$', and the entire set of free variables in α is denoted by, '$Fv(\alpha)$'.

A well-formed formula α, is said to be ***closed*** (or called a ***sentence***), if it contains no free occurrence of any variable.

For examples of formal systems see Enderton[1972, p.69] or Mates[1965].

B·3·2 The Formal Semantics

In sentential logic we had truth assignments which provided us with a tool for interpretating the sentential symbols. In first order predicate logic we the concept of a 'structure' to help us interpret the symbols in the formal language.

<u>Definition</u>: A ***structure*** $\mathcal{U}$ for a first order predicate logic, $\mathcal{P}$, is defined to be a pair $\langle |\mathcal{U}|, \sigma \rangle$, where '$|\mathcal{U}|$' is a set called the ***domain*** (or ***universe***) and 'σ' is a function called the ***interpretation function***, such that $\mathcal{U}$ assigns[1]:

(1) to each predicate symbol P of arity n, an n-ary relation $P^{\mathcal{U}} \subseteq |\mathcal{U}|^n$;

(2) to each constant symbol c a member $c^{\mathcal{U}}$ of $|\mathcal{U}|$;

(3) to each function symbol f of arity n, an n-ary operation $f^{\mathcal{U}} : |\mathcal{U}|^n \rightarrow |\mathcal{U}|$.

Given a formula $\alpha \in \mathcal{P}$ we want to formally define 'α is true in $\mathcal{U}$'. The main difficulty is copping with the free variables in a formula, for example how do we know whether 'P(x)' is 'true' in some structure $\mathcal{U}$, or even whether it even makes sense to talk about 'truth' of an open formula. Tarski solved the problem by introducing the notion of 'satisfaction'.

[1]It is convenient to assume that the domain $|\mathcal{U}|$ is non-empty.

Definition: Let $\mathcal{U}$ be a structure for a language $\mathcal{P}$, a ***valuation*** (or ***environment***) for **Var** in $\mathcal{U}$, is defined to be a function: $g : \mathbf{Var} \rightarrow |\mathcal{U}|$

which by using recursion we extend to the set terms **Tm** as follows: $\overline{g} : \mathbf{Tm} \rightarrow |\mathcal{U}|$

(1) for each $x \in \mathbf{Var}$, $\overline{g}(x)=g(x)$;

(2) for each $c \in \mathbf{Con}$, $\overline{g}(c)=c^{\mathcal{U}}$;

(3) for each $f \in \mathbf{Fun}$ and $t_i \in \mathbf{Tm}$, $\overline{g}(\mathcal{E}_f(t_1, t_2, \ldots, t_n))= f^{\mathcal{U}}(\overline{g}(t_1), \overline{g}(t_2), \ldots, \overline{g}(t_n))$.

Notation: Once again I shall use 'g' instead of '$\overline{g}$', since it helps with the clasrity of the concepts being described.

If $\alpha \in \mathcal{P}$, $x \in \mathbf{Fv}(\alpha)$ and $t \in \mathbf{Tm}$, then the formula denoted by '$\alpha[t/x]$' is the formula resulting from substituting t for every free occurrence of x in α. Suppose we have two functions $g, h : \mathbf{Tm} \rightarrow \mathcal{U}$, where h assumes the same values as g except for $y \in \mathbf{Tm}$, where $h(y)=a$. Then the function h is denoted by $g(a/y)(x)$.

Definition: Let $\mathcal{U}$ be a structure for a language $\mathcal{P}$, α a well-formed formula of $\mathcal{P}$, and g a valuation for the the set of variables in $\mathcal{U}$. Then g is said to ***satisfy*** α in $\mathcal{U}$ (denoted by:'$\mathcal{U} \models \alpha[g]$'), if:

(1) $P(t_1, t_2, \ldots, t_n) \in \mathbf{Af}$ then $\mathcal{U} \models P(t_1, t_2, \ldots, t_n)[g]$ iff the ordered n-tuple $\langle g(t_1), g(t_2), \ldots, g(t_n)\rangle \in P^{\mathcal{U}}$.

(2) $\mathcal{U} \models \neg\alpha[g]$ iff it is not the case that $\mathcal{U} \models \alpha[g]$.

(3) $\mathcal{U} \models (\alpha \wedge \beta)[g]$ iff $\mathcal{U} \models \alpha[g]$ and $\mathcal{U} \models \beta[g]$.

(4) $\mathcal{U} \models (\alpha \vee \beta)[g]$ iff $\mathcal{U} \models \alpha[g]$ or $\mathcal{U} \models \beta[g]$.

(5) $\mathcal{U} \models (\alpha \rightarrow \beta)[g]$ iff either it is not the case that $\mathcal{U} \models \alpha[g]$ or $\mathcal{U} \models \beta[g]$ or both.

(6) $\mathcal{U} \models (\alpha \leftrightarrow \beta)[g]$ iff

(a) both $\mathcal{U} \models \alpha[g]$ and $\mathcal{U} \models \beta[g]$,

(b) neither $\mathcal{U} \models \alpha[g]$ nor $\mathcal{U} \models \beta[g]$.

(7) for each $x_i \in \mathbf{Var}$:

(a) $\mathcal{U} \models \forall_{x_i}\alpha[g]$ iff for every element $d \in |\mathcal{U}|$, $\mathcal{U} \models \alpha[g(d/x_i)]$;

(b) $\mathcal{U} \models \exists_{x_i}\alpha[g]$ iff for some element $d \in |\mathcal{U}|$, $\mathcal{U} \models \alpha[g(d/x_i)]$.

If ϕ is a well-formed formula which has no free variables, then $\mathcal{U} \models \phi[g]$ for any g, in this case we shall use the notation $\mathcal{U} \models \phi$, and say 'ϕ is ***true*** in $\mathcal{U}$' or '$\mathcal{U}$ is a ***model*** of ϕ'.

I have deliberately not used the notation '$\mathcal{U} \not\models \alpha[g]$' to stand for 'it is not the case that $\mathcal{U} \models \alpha[g]$', since this can cause difficulties when dealing with *partial logics.*

Tarski did not use assignments in his 1933 paper, instead he used *sequences.* Recall that an infinite sequence $\langle g(1), g(2), \ldots \rangle$ in $|\mathcal{U}|$, is nothing more than a function $g : \mathbb{N} \rightarrow |\mathcal{U}|$. This together with the fact that the variables were strictly ordered, meant that each element $g(i)$ could implicitely be assigned to the variable x_i, so that $g(x_i)$ is defined to be $g(i)$.

<u>Definition</u>: A formula $\alpha \in \mathcal{P}$ is said to be a ***semantic consequence*** of the set of formulas Γ (denoted $\Gamma \models \alpha$), iff for *every* structure $\mathcal{U}$ and valuation g, such that g satisfies every member of Γ in $\mathcal{U}$, also satisfies α in $\mathcal{U}$. If the set $\Gamma = \varnothing$, then '$\varnothing \models \alpha$' is denoted by $\models \alpha$', and called a ***logical truth.***

The truth value of a formula $\alpha \in \mathfrak{L}$ in a structure $\mathcal{U}$ is denoted by $\|\alpha\|_{\mathcal{U}}$ and is defined as:

$\|\alpha\|_{\mathcal{U}} = \mathbb{T}$ if $\mathcal{U} \models \alpha$ and $\mathbb{F}$ otherwise.

However, some authors (see Lyndon[1966] for example), define $\| \ \|_{\mathcal{U}}$ directly, and then define $\models$ in terms of $\| \ \|_{\mathcal{U}}$. If we replace $\mathbb{T}$ by 1 and $\mathbb{F}$ by 0, then the inductive definition of $\| \ \|_{\mathcal{U}}$ includes such formulas as:

$\|\neg\alpha\|_{\mathcal{U}} = 1 - \|\alpha\|_{\mathcal{U}}$,

$\|(\alpha \wedge \beta)\|_{\mathcal{U}} = \min\{\|\alpha\|_{\mathcal{U}}, \|\beta\|_{\mathcal{U}}\}$,

$\|(\alpha \vee \beta)\|_{\mathcal{U}} = \max\{\|\alpha\|_{\mathcal{U}}, \|\beta\|_{\mathcal{U}}\}$.

This notation with a valuation g, becomes: $\|\alpha\|_{\mathcal{U},g}$.

This notation is particularly useful when discussing many-valued logics.

Other notations are also possible, for example Turner[1984] calls my valuation an 'assignment function' and uses the notation: $M \models_{\mathfrak{s}} \alpha$.

B·3·3 The Formal System

I shall not define a particular formal system, since all that I shall require in the remainder of the thesis is the notion of a well-formed formula α, being a *theorem* (denoted as usual by, '$\vdash \alpha$'), or being a *syntactical consequence* of a set of formulas Γ (denoted by '$\Gamma \vdash \alpha$'). Details of particular axiomatisatioms can be found in Hunter[1971].

B·4 MATRIX LANGUAGES

The syntax of a language is described by an inductive structure consisting of atomic formulas and formation rules. In contrast the semantics is described by a structure consisting of a set of truth-values, used to define semantical consequence, together with a collection of truth-functions (semantical rules), which organize the truth values and serve as counterparts to the formation rules. In defining the syntax and semantics of sentential logic in the manner above, makes it particularly easy to generalize to a ***matix language.***

<u>Definition 28</u>: A ***logical matrix*** $\mathbb{M}$ is defined as a structure $\langle \mathbb{V}, \mathbb{D}, O_{\mathbb{V}} \rangle$ where:

(a) $\mathbb{V}$ is a set whose members are called ***truth-values,***

(b) $\mathbb{D}$ is a subset of $\mathbb{V}$, whose members are called the ***designated truth-values,***

(c) $O_{\mathbb{V}}$ is a set of operations on $\mathbb{V}$.

The *syntax* $\mathcal{S}^*$, of a ***sentential matrix language*** is defined in exactly the same manner as the syntax, $\mathcal{S}$ of the standard sentential language, namely by defining an alphabet $\mathcal{S}$, together with a set, $O_{\mathcal{S}}$ of formation rules. Where the two can differ, is in their semantics.

<u>Definition</u>: The *semantics* for $\mathcal{S}^*$, is given by

(1) a logical matrix $\mathbb{M}=\langle \mathbb{V}, \mathbb{D}, O_{\mathbb{V}} \rangle$, such that for each $\mathcal{E}_n \in O_{\mathcal{S}}$ there exists $\mathcal{F}_n \in O_{\mathbb{V}}$ of the same arity.

(2) a truth assignment, $\upsilon : \mathcal{S} \rightarrow \mathbb{V}$, which is uniquely extended to an interpretation υ, such that: $\upsilon(\mathfrak{C}_n(\alpha_1, \alpha_2, \ldots \alpha_n)) = \mathfrak{F}_n(\upsilon(\alpha_1), \upsilon(\alpha_2), \ldots, \upsilon(\alpha_n))$.

There is just one thing left to be defined, the notion of semantic consequence.

<u>Definition</u>: Let $\mathcal{S}^*$, be a sentential matrix language with a matrix $\mathbb{M}$. Then a formula $\alpha \in \mathcal{S}^*$, is said to be a ***semantic consequence*** of a set Γ of formulas ($\Gamma \models_{\mathbb{M}} \alpha$), iff: for every interpretation υ and every $\beta \in \Gamma$, $\upsilon(\beta) \in \mathbb{D}$ only if $\upsilon(\alpha) \in \mathbb{D}$.

For further details of matrix based languages see Hunter[1971, p.121], or Tarski[1937].

References

Arbib, M. A. [1972]: *The Metaphorical Brain*, Wiley-Interscience, New York.

Aristotle, *Metaphysics*, 1061a28–1061b3; 1061b21–25.

Attneave, F.[1971]: 'Multistability in perception', *Scientific American*, **226**(6), pp.62–71.

Baldwin. [1902]: *Dictionary of Philosophy and Psychology*, **2**.

Barnes, J. [1982]: 'Medicine, experience and logic', in *Science & Speculation, Studies in Hellenistic Theory & Practice*, ed by. J. Branes, J. Brunschwig, M. Burnyeat & M. Schofield. Cambridge U.P., Cambridge.

Bellman, R. E. & Zadeh, L. A. [1977]: 'Local and Fuzzy Logics', in Dunn & Epstein[1977], pp.103–165.

Beth, E. W. [1959]: *The Foundations of Mathematics*, North Holland, Amsterdam.

Black, M. [1937]: 'Vagueness: An exercise in logical analysis', *Philosophy of Science*, **4**, pp.427.

Black, M. [1963]: 'Reasoning with loose concepts', *Dialogue*, **2**, pp.1–12. Also in *Margins of Precision, Essays in Logic and Language*, by Max Black, Cornell U.P., Ithaca, 1970.

Bolzano, B. [1804]: *Betrachtungen über einige Gegenstände der Elementargeometrie*, Prage.

Bolzano, B. [1817]: *Die drie Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen, ohne die Annahmen des Archimedes, und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst; zugleich als Probe einer gänzlichen Umstaltung der Raumwissenshaft, allen Mathematikern zur Prüfung vorgelegt.* Leipzig.

Bolzano, B. [1820–30]: *Theory of Science*, transl. by R. George, Basil Blackwell, Oxford, 1972.

Bolzano, B. [1851]: *Paradoxien (Paradoxes of the Infinite)*, transl. by F. Prihonsky, RKP, London, 1950.

Bourbaki, N. [1939]: *Elements of Mathematical Theory of Sets*, Addison Wesley, Cambridge, Mass. (Translated into English in 1968).

Brock, J. E. [1979]: 'Principal Themes in Peirce's Logic of Vagueness' in *Studies in Peirce's Semiotic*, ed by J. E. Brock, Institute for Studies in Pragmatics, Lubbock Texas.

Brownell, H. H. & Carmazza, A. [1978]: 'Categorizing with overlapping categories', *Memory & Cognition*, **6** (5), pp.481–490.

Cameron, P., Hocking, J. G. & Naimpally, S. A. [1974]: 'Nearness – a better approach to continuity and limits', *American Mathematical Monthly*, **81**, pp.739–745.

Cantor, G. [1872]: 'Ueber die Ausdenung eine Satzes aus der Theorie der trigonometrischen Reihen. *Mathematiche Annalen*, **5**, pp.123–132. In Cantor[1932], pp.92–101.

Cantor, G. [1882]: 'Ueber unendliche lineare Punktmannigfaltigkeiten, Part 3,', *Math. Ann.*, **20**, pp.113–121. In Cantor[1932], pp.149–157.

Cantor, G. [1883a]: 'Ueber unendliche lineare Punktmannigfaltigkeiten, Part 5', *Math. Ann.*, **21**, pp.545–591. In Cantor[1932], pp.165–208.

Cantor, G. [1883b]: *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematische-philosophischer Versuch in der Lehre des Unendlichen*, Teubner, Leipzig.

Cantor, G. [1895]: 'Beiträge zur Begründung der transfiniten Mengenlehre', Part I, *Mathematische Annalen*, **46**, pp. 481–512.

Cantor, G. [1915]: *Contributions to the Founding of the theory of Transfinite Numbers*, trans. by P. E. B. Jourdain, Dover, NY.

Cantor, G. [1932]: *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ed by E. Zermelo. Springer, Berlin.

Carnap, R. [1937]: *The Logical Syntax of Language*, transl. by A. Smeaton, RKP, London.

Chandler, H. S. [1967]: 'Excluded middle', *Journal of Philosophy*, **64** (24), pp.807–814.

Chang, C. C. & Keisler, H. J. [1973]: *Model Theory*, North-Holland, Amsterdam.

Choquet-Bruhat, Y., DeWitt-Morette, C. & Dillard-Bleick, M. [1982]: *Analysis, Manifolds and Physics*, North-Holland, Amsterdam.

Church, A. [1941]: *The Calculus of Lambda-conversion*, Princeton U. P., London.

Clifford, W. K. [1873]: "On the hypothesis which lie at the bases of geometry" (Translation of Riemann[1854]), *Nature*, **183** (14). Reprinted in: *General Theory of Relativity*, by C. W. Kilmimster, Pergammon Press, 1973.

Coffa, A. [1987]: 'Carnap, Tarski and the search for truth', *Noûs*, **21**, pp.547–572.

Cohn, A. G. [1983]: *Mechanising a Particularly Expressive Many Sorted Logic*, PhD Thesis, Department of Computer Science, University of Essex.

Cohn, A. G. [1987]: 'A More Expressive Formulation of Many Sorted Logic', *Journal of Automated Reasoning*, **3** (2), pp.113–200.

Dauben, J. W. [1979]: *GEORG CANTOR, His Mathematics and Philosophy of the Infinite*, Harvard U.P., Cambridge, Mass.

Davis, E. [1990]: *Representations of Commonsense Knowledge*, Morgan Kaufmann, San Mateo, California.

Deas, R. [1989]: 'Sorenson's sorites', *Analysis*, **49**, pp.26–31.

Dedekind, J. W. R. [1901]: *Essays on The Theory of Numbers*, (English translation of: *Stetigkeit und irrationale Zahlen (1872)*, and*Was sind und was sollen die Zahlen? (1888)*). transl. by W. W. Beman, Dover, NY, 1963.

Dienes, P. [1949]: 'On ternary logic', *The Journal of Symbolic Logic*, **14** (2), pp.85–94.

Dodson, C. T. J. [1974]: 'Hazzy Spaces & Fuzzy Spaces', *Bulletin of the London Mathematical Society*, **6**, pp.191–197.

Dodson, C. T. J. [1975]: 'Tangent structures for Hazzy Spaces', *Journal of the London Mathematical Society*, **11** (2), pp.465–473.

Dubois, D. & Prade, H. [1980]: *Fuzzy Set Theory, Academic Press*, New York.

Dunn, J. M. & Epstein, G. (eds)[1977]: *Modern Uses of Multiple–Valued logic*, Reidel, Dordrecht–Holland.

Dummett, M. [1975]: 'Wangs Paradox', *Synthese*, **30**, pp.301–325.

Eisenberg, M. [1974]: *Topology*, Holt Rinehart & Winston, New York.

Enderton, H. [1972]: *A Mathematical Introduction to Logic*, Academic Press, New York.

Fehrer, D. [1990]: 'A resolution calculus for a logic based on vaguely defined predicates, *Proceedings of the 9th European Conference on Artificial Intelligence*, Stockholm, Sweden.

Fine, K. [1975]: 'Vagueness, truth and logic', *Synthese*, **30**, pp.265–300.

Flew, A. [1979]: *A Dictionary of Philosophy*, Pan Books, London.

Frege, G. [1879]: *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, transl. in van Heijenoort[1967].

Frege, G. [1956]: 'The thought: a logical inquiry', transl. by A. M. & Marcelle Quinton, Mind, 65, pp. 289–311.

Frege, G. [1977]: *Logical Investigations*, transl. by P.T. Geach & R. H. Stoothoff, Blackwell, Oxford.

Frege, G. [1988]: *Philosophical Writings of Gottlob Frege*, 3rd edition, transl. by P. Geach & M. Black, Blackwell, Oxford.

French,S. [1989]: 'Fuzzy Sets: The Unanswered Questions", University of Leeds, Department of Computer Studies, Research Report No. 89.6, Leeds, UK.

Frétchet, M. [1906]: 'Sur quelques points du calcul fonctionnel', *Rendiconti del Circolo Matematico di Palermo*, **22**, Palermo, Sede Della Societa.

Gabbay, D. & Guenthner, F. (eds) [1983]: *Handbook of Philosophical Logic*, 4 vols., Reidel. Pub. Co., Amsterdam.

Gaines, B. R. [1976]: 'Foundations of fuzzy reasoning', *International Journal of Man Machine Studies*, **8** (6), pp.623–668.

Gandy, R. [1973]: 'Structure in mathematics' in *Structuralism: An introduction*, ed by D.Robey, Clarendon Press, Oxford.

Gent, I. P. [1992]: 'Analytic proof systems for classical and modal logics of restricted quantification', PhD Thesis (pending), Department of Computer Science, University of Warwick.

Gödel, K. [1930]: 'Die Vollständigkeit der Axiome des logischen Funktionenkalküls, *Monatshefte für Mathematik und Physik*, **37**, pp.349–360; transl. in van Heijenoort[1967], pp.582–591.

Gödel, K. [1931]: 'Über formal unentsheidbare Sätze der Principia Mathematica und verwandter Systeme I', *Monatshefte für Mathematik und Physik*, **38**, pp.173–198; transl. in van Heijenoort[1967], pp.596–616.

Goguen, J. A. [1969]: 'The logic of inexact concepts', *Synthese*, **19**, pp.325–378.

Grünbaum, A. [1968]: *Modern Science and Zeno's Paradoxes*, George Allen and Unwin, London.

Haack, S. [1967]: *Deviant Logics*, Cambridge U.P., Cambridge.

Haack, S. [1978]: *Philosophy of Logics*, Cambridge U.P., Cambridge.

Haack, S. [1979]: 'Do we need Fuzzy Logic?', *International Journal of Man-Machine Studies*, **1**, pp.437–445.

Halmos, P. R. [1960]: *Naive Set Theory*, Springer-Verlag, New York.

Hausdorff, F. [1914]: *Grundzüge der Mengenlehre*, Von Weit, Leipzig.

Hayes, P. J. [1970]: 'Robotologic', in *Machine Intelligence*, **5**, pp.533–554.

Hayes, P. J. [1971]: 'A logic of actions', *Machine Intelligence*, **6**, Edinburgh U.P., Edinburgh. pp.495–520.

Hayes, P. J. [1974a]: 'Some problems and non-problems in representation theory', *Proc. AISB Summer Conf.*, University of Sussex. pp.63–79.

Hayes, P. J. [1974b]: 'The language GOLUX', University of Essex, unpublished technical report.

Hayes, P. J. [1975]: *Three Valued Logic & Computer Science*, University of Essex, CSM–6.

Hayes, P.J. [1979a]: 'Naive Physics 1: Ontology for Liquids'. University of Essex, unpublished technical report. Published in Hobbs&Moore[1985], pp.71–107.

Hayes, P. J. [1979b]: 'The naive physics manifesto', in *Expert Systems in the Micro-Electronic Age*,

Hayes, P. J. [1985]: 'The second naive physics manifesto', in Hobbs&Moore[1985], pp.1–36.

Hersh, H, M. & Caramazza, A. [1976]: 'A Fuzzy Set Approach to Modifiers and Vagueness in Natural Language', *Journal of Experimental Psychology – General*, **105** (3), pp.254–276.

Hilbert, D. [1927]: 'Die Grundlagen der Mathematik', *Abhandlungen aus dem mathematischen Seminar ser Hamburgischen Universität 6 (1928)*, p.65–85. transl. in van Heijenoort[1967], pp.464–479.

Hocking, J. G. & Young, G. S. [1961]: *Topology*, Dover, New York, 1988.

Hobbs, J. R. & Moore, R. C. (eds) [1985]: *Formal Theories of the Commonsense World*, Ablex, Norwood, NJ.

Hoffman, R. G. [1990]: 'John McCarthy: Approaches to Artificial Intelligence', *IEEE Expert*, June 1990, pp.87–89.

Hunter, G. [1971]: *Metalogic*, Macmillan, London.

Huntington, E. V. [1905]: 'The Continuum as a type of order: An exposition of the modern theory', *Annals of Mathematics*, **6**, pp.151–184; and **7**, pp.15–43.

Ishiguro, H. [1990]: *Leibniz's Philosophy of Logic and Language*, 2nd Ed, Cambridge U.P., Cambridge.

Johnson, D. M. [1977]: 'Prelude to dimension theory: The geometrical investigations of Bernard Bolzano', *Archive for the History of the Exact Sciences*, **17**, p.275.

Johnstone, P. T. [1987]: *Notes on Logic and Set Theory*, Cambridge U.P., Cambridge.

Kandel, A. [1982]: *Fuzzy Techniques in Pattern Recognition*, John Wiley, New York.

Kaufmann, A. [1975]: *Introduction to the Theory of Fuzzy Subsets*, Cambridge U.P., Cambridge.

Kaufman, S. G. [1991]: 'A formal theory of spatial reasoning', in *Proceedings of the 2nd International Conference on Prionciples of Knowledge Representation and Reasoning*.

Klein, F. [1903]: 'Vergleichende Betrachtungen über neure geometrische Forschungen', *Mathematische Annalen*, **43**. Initially published by University of Erlangen in 1872.

Kleene, S. C. [1952]: *Introduction to Metamathematics*, North Holland, Amsterdam.

Klemke, E. D. (ed)[1968]: *Essays on Frege*, Illinois U.P., Illinois.

Kneale, W. & Kneale, M. [1984]: *The Development of Logic*, Clarendon Press, Oxford.

Kohl, M. [1969]: 'Bertrand Russell on Vagueness', *Australasian J. of Philosophy*, **47** (1), pp.31–41.

Lakoff, G. [1973]: 'Hedges: A study in meaning criteria and the logic of fuzzy concepts', *J. Philo. Logic*, **2**, pp458–508.

Lambert, K. (ed)[1969]: *The Logical Way of Doing Things*, Yale U.P., New Haven.

Lefaivre, R. A. [1974]: 'The representation of fuzzy knowledge', *Journal of Cybernetics*, **4** (2), pp.57–66.

Lefschetz, S. [1949]: *Introduction to Topology*, Princeton U. P., Princeton, NJ.

Listing, J. B. [1847]: *Vorstudien zur Topologie*, Göttingen.

Lemmon, E. J. [1966]: 'Sentences, statements and propositions', in *British Analytical Philosophy*, eds. B. Williams & A. Montefiore, RKP, London.

Łukasiewicz, J. [1920]: 'On 3–valued logic', in McCall[1967].

Lyndon, R. C. [1966]: *Notes on Logic*, Van Nostrand, Princeton NJ.

Machina, K. F. [1972]: 'Vague predicates', *American Philosophical Quareterly*, **9** (3), pp.225–233.

Machina, K. F. [1976]: 'Truth, belief, and vagueness', *Journal of Philosophical Logic*, **5**, pp.47–78.

Mac Lane, S. & Birkoff, G. [1979]: *Algebra*, 2nd Ed., MacMillan, NY.

MacDonald Ross, G. [1989]: *Leibniz*, Oxford U.P., Oxford.

Mamdani, E. H. & Gaines, B. R. (eds) [1981]: *Fuzzy Reasoning and its Applications*, Academic Press, London.

Manheim, J. H. [1964]: *The Genesis of Point Set Topology*, Pergamon Press, London.

Margalit, A. [1976]: 'Vagueness in vogue', *Synthese*, **33**, pp.211–221.

Mates, B. [1965]: *Elementary Logic*, Oxford U.P., Oxford.

McCall, S. (ed.)[1967]: *Polish Logic, 1920–1939*, Oxford U.P., Oxford.

McCarthy, J. [1958]: 'Programs with common sense', in *Mechanisation of Thought Processes. Proc Symp. Nat. Phys. Lab.*, **1**, Her Majesty's Stationary Office, London. pp.77–84.

McCarthy, J. & Hayes, P. J. [1969]: 'Some philosophical problems from the standpoint of artificial intelligence', in *Machine Intelligence*, **4**, pp.463–502.

McCulloch, W. S. [1954]: 'Through the den of the metaphysician', *British Journal of Philosophy of Science*,**5**, pp.18–31.

McDermott, D. [1983]: 'Under what conditions can a machine attribute meanings to symbols?', in *Proceedings of the 8th International Joint Conference on Artificial Intelligence*, pp.45–46.

Mendelson, B. [1973]: *Introduction to Topology*, 2nd Ed, Allyn & Bacon, Boston.

Metlzer, B. & Michie, D. (eds)[1969]: *Macine Intelligence*, **4**, Edinburgh U.P., Edinburgh.

Metlzer, B. & Michie, D. (eds)[1970]: *Machine Intelligence*, **5**, Edinburgh U.P., Edinburgh.

Minsky, M. L. [1987]: *The Society of Mind*, Picador, London.

Moline, J. [1969]: 'Aristotle, Eubulides and the Sorites', *Mind*, **78**, pp.393–406.

Morris, T. V.[1984]: *Understanding Identity Statements*, Aberdeen U.P., Aberdeen.

Mostowski, A. [1966]: *Thisrty Years of Foundational Studies*, Blakwell, Oxford.

Negoita, C. J. [1985]: *Expert Systems and Fuzzy Systems*, Benjamin/Cummings Pub., Menlo Park, California.

Ng, K. & Abramson, B. [1990]: 'Uncertainty management in expert systems', *IEEE Expert*, April 1990, pp.29–47.

Oden, G. C. [1979]: 'Fuzzy propositional approach to pscholinguistic problems: An application of fuzzy set theory in Cognitive Science", in *Advances in Fuzzy Sets: Theory and Applications*, ed by, M. M. Gupta, R. K. Ragade and R. R. Yager; North–Holland.

Osherson, D. N. & Smith, E. E. [1981]: 'On the adequacy of prototype theory as a theory of concepts', *Cognition*, **9**, pp.35–58.

Peirce, C. S. [CP, (1930–58)]: *Collected Papers*, eds. by C. Hartshorne, P. Weiss & Burks, Harvard U.P., Boston, Mass.

Peacocke, C. [1981]: 'Are vague predicates incoherent?', *Synthese*, **46**, pp.121–141.

Poincaré, H. [1905]: *Science and Hypothesis*, Walter Scott Pub. Co. Dover, New York, 1952.

Poincaré, H. [1913]: *The Value of Science*, Dover, New York, 1958.

Post, E. L. [1921]: 'Introduction to the general theory of elementary propositions', *American J. of Mathematics*, **43**, pp.163–185. Also in van Heijenoort[1967], pp.264–283.

Poston, T. [1971]: *Fuzzy Geometry*, PhD Thesis, Mathematics Institute, University of Warwick, Coventry.

Poston, T. & Stewart, I. N. [1978]: 'Nonlinear modelling of multistable perception', *Behavioural Science*, **23**, pp.318–334.

Quine, W. V. [1968]: 'On Frege's way out', in Klemke[1968], pp.485–501.

Quine, W. V. [1990]: *Pursuit of Truth*, Harvard U.P., Cambridge, Mass.

Randell, D. A. [1991]: *Analysing the Familiar: Reasoning About Space and Time in the Everyday World*, PhD Thesis, Department of Computer Science, University of Warwick.

Riemann, B. [1854]: 'Über die Hypothesen, welche der Geometrie zu Grunde liegen', in *The Collected Works of B. Riemann*, ed by H. Weber, Dover, New York.

Rosch, E. [1973]: 'On the internal structure of perceptual and semantic categories'. In T. E. Moore (ed), *Cognitive Development and Acquisition of Language*, New York, Academic Press.

Russ, S. B. [1980]: *The mathematical works of Beranrd Bolzano published between 1804 and 1817*, PhD Thesis, Department of Mathematics, Open University, Milton Keynes.

Russell, B. [1900]: *A Critical Exposition of the Philosophy of Leibniz*, Cambridge U.P., Cambridge.

Russell, B. [1903]: *Principles of Mathematics*, Cambridge U.P., Cambridge.

Russell, B. [1923]: 'Vagueness', *Australasian Journal of Philosophy*, **1**, pp.84–92.

Russell, B.[1972]: *Our Knowledge of the External World*, 6th Ed, Allen and Unwin, London.

Sainsbury, R. M. [1988]: *Paradoxes*, Cambridge U.P., Cambridge.

Sainsbury, R. M. [1989]: 'Tolerating Vagueness', *Proceedings of the Aristotlean Society*, **89**, pp.33–48.

Sainsbury, R. M. [1991]: 'Is there higher–order vagueness', *Philosophical Quarterly*, **41** (163), pp.167–182.

Sanford, D. H. [1975]: 'Borderline Logic', *American Philosophical Quarterly*, **12** (1), pp.29.

Sanford, D. H. [1976]: 'Competing semantics of vaguenes: many values versus super-truth', *Synthese*, **33**, pp.195–210.

Self, K. [1990]: 'Designing with fuzzy logic', *IEEE Spectrum*, November 1990, pp.42–105.

Schefe, P. [1980]: 'On the foundations of reasoning with uncertain facts and vague concepts', *International Journal of Man Machine Studies*, **12**, pp.35–62. Also in Mamdani & Gaines[1981], pp.189–216.

Schmitt, P. H. [1986]: 'Computational aspects of three–valued logic', in *Proceedings of the 8th Conference on Automated Deduction* (ed by J. H. Siekmann), Oxford, UK.

Schmitt, P. H. [1990]: 'Perspectives in multiple–valued logic', in Natural Language and Logic, ed R. Studer, Lecture Notes in AI (No. 459), Springer Verlag.

Schröder, E. [1890]: *Voresungen über die Algebra der Logik*, 3 volumes, Leipzig, 1890–1905.

Schönfinkel, M. [1924]: 'Über die Bausteine der mathematischen Logik', *Mathematische Annalen*, **92**, pp.305–316. transl. in van Heijenoort [1967].

Sloman, A. [1965]: 'Functions and Rogators' in *Formal Systems and Recursive Functions*, II, ed. by N. Crossley & M. Dummett, North-Holland, Amsterdam. pp.156–175.

Smith, E. E. & Medin, D. L. [1981]: *Catergories and Concepts*, Harvard U.P., Cambridge, Mass.

Smith, J. W. [1984]: 'The surprise examination on the paradox of the heap', *Philosophical Papers*, **13**, pp.43–56.

Sorabji, R. [1983]: *Time, Creation and the Continuum: Theories in Antiquity and the early middle ages*, Duckworth, London.

Sorenson, R. A. [1985]: 'An argument for the vagueness of 'vague'', *Analysis*, **45**, pp.134–137.

Sorenson, R. A. [1988a]: 'Precification by means of vague predicates', *Notra Dame Journal of Formal Logic*, **29** (2), pp.267–275.

Sorenson, R. A. [1988b]: 'Vagueness, measurement, and blurriness', *Synthese*, **75**, pp.45–82.

Sorenson, R. A. [1990]: 'Vagueness implies cognitivism', *American Philosophical Quarterly*, **27** (1), pp.1–14.

Stewart, I. N. & Tall, D. O. [1977]: The Foundations of Mathematics, Oxford U.P., Oxford.

Sweeney, L. [1972]: *Infinity in the presocrates: A bibliographical and philosophical study*. Martinus Nijhoff, Amsterdam.

Tarski, A. [1937]: 'Sentential calculus and topology', in *Logic, Semantics and Metamathematics*, in Tarski[1956], pp.421–454.

Tarski, A. [1933]: 'The concept of truth in formalised languages', (Origionally appeared in Polish (1933)) in Tarski[1956], pp.152–278.

Tarski, A. [1956]: *Logic, Semantics and Metamathematics*, (papers 1923 – 1938, ed and transl by J. H. Woodger), Oxford U.P., Oxford.

Taylor, A. E. [1919]: *Aristotle*, Dover, NewYork, 1955.

Taylor, R. [1962]: 'Fatalism', *Philosophical Review*, **71**, pp.56–66

Turing, A. M. [1950]: 'Computing machinery and intelligence', *Mind*, **59**, pp.433–460.

Turner, R. [1984]: *Logics for Artificial Intelligence*, John Wiley, New York.

van Benthem, J. F. A. K. [1981]: 'Why is semantics what?', in *Formal Methods in the study of Language*, (part 1), ed by, J. A. G. Groenendijk, T. M. V. Janssen & M. B. J. Stokhof, Mathematisch Centrum, Amsterdam.

van Fraassen, B. C. [1966]: 'Singular terms, truth value gaps, and free logic', *Journal of Philosophy*, **63**, 481–495.

van Fraassen, B. C. [1969]: 'Presuppositions, supervaluations, and free logic. In Lambert[1969], pp.67–91.

van Heijenoort, J. (ed.)[1967]: *From Frege to Gödel, A Source Book in Mathematical Logic, 1879–1931*, Harvard U.P., Cambridge, Mass.

Walton, D. N. [1990]: 'What is reasoning? what is an argument?', *Journal of Philosophy*, **87** (8), pp.399–419.

Wang, H. [1974]: From Mathematics to Philosophy, RKP, London.

Welham, B. & Hayes, P. J. [1985]: 'Notes on Naïve Geometry', Hewlett Packard Lab., technical report, Bristol, UK.

Weyl, H. [1949]: *Philosophy of Mathematics and Natural Sciences*, Princeton U.P., Princeton, NJ.

Wiener, N. [1948]: *Cybernetics*, The Technology Press, Cambrisge, Mass.

Williams, C. J. F.[1989]: *What is Identity*, Oxford U.P., Oxford.

Wright, C. [1975]: 'On the Coherence of Vague Predicates', *Synthese*, **30**, pp.325–365.

Wright, C. [1976]: 'Language mastery and the sorites paradox', in *Truth and Meaning: Essays in Semantics*, ed by G. Evans & J. McDowell, Oxford U.P., Oxford, pp.223–247.

Wright, C. [1987]: 'Further reflections on the sorites paradox', *Philosophical Topics*, **15** (1), pp.227–290.

Yaglom, I. M. [1988]: *Felix Klein and Sophus Lie, Evolution of the Idea of Symmetry in the Nineteenth Century*, transl. by. S. Sossinsky, Birkhäuser, Boston.

Zadeh, L. A. [1965]: 'Fuzzy sets', *Information and Control*, **8**, pp.338–353.

Zadeh, L. A. [1972]: 'A fuzzy-set-theoretic interpretation of linguistic hedges', *Journal of Cybernetics*, **2** (3), pp.4–34.

Zadeh, L. A. [1975a]: 'Fuzzy logic and approximate reasoning', *Synthese*, **30**, pp.407–428.

Zadeh, L. A. [1975b]: 'The concept of a linguistic variable and its applications to approximate reasoning – I', *Information Sciences*, **8**, pp.199–249.

Zadeh, L. A. [1975c]: 'The concept of a linguistic variable and its applications to approximate reasoning – II', *Information Sciences*, **8**, pp.301–357.

Zadeh, L. A. [1975d]: 'The concept of a linguistic variable and its applications to approximate reasoning – III', *Information Sciences*, **9**, pp.43–80.

Zadeh, L. A. [1978]: 'Fuzzy sets as a basis for a theory of possibility', *Fuzzy Sets and Systems*, **1**, pp.3–28.

Zadeh, L. A. [1983]: 'Commonsense Knowledge Representation Based on Fuzzy Logic', *IEEE Computer*, October 1983, pp.61–65.

Zeeman, E. C. [1962]: 'The topology of the brain and visual perception', in *Topology of 3-manifolds*, ed by K. Fort, Prentice Hall, Englewood Cliffs, NJ. pp.240–256.

Zeeman, E. C. [1965]: 'Topology of the Brain', in *Mathematics and Computer Science in Biology and Medicine*, MRC, London, pp.277–292.

Zeeman, E. C. & Buneman, O. P. [1968]: Tolerance Spaces and the Brain', in *Towards a Theoretical Biology*, **1**, ed. by C. H. Waddington, Edinburgh U.P., Edinburgh, pp.140–151.

Zermelo, E. [1908]: 'A new proof of the possibility of a well-ordering', in van Heijenoort[1967], pp.183–198.