A SIMPLE PROOF OF KRAMKOV'S RESULT ON UNIFORM SUPERMARTINGALE DECOMPOSITIONS

SAUL JACKA[1]
University of Warwick

Abstract: we give a simple proof of Kramkov's uniform optional decomposition in the case where the class of density processes satisfies a suitable closure property. In this case the decomposition is previsible.

Keywords: UNIFORM SUPERMARTINGALE; UNIFORM OPTIONAL DECOMPOSITION; UNIFORM PREDICTABLE DECOMPOSITION

AMS subject classification: 60G15

§1 Introduction

In [4], Kramkov showed that for a suitable class of probability measures, $\mathcal{P}$, on a filtered measure space $(\Omega, \mathcal{F}, \mathcal{F}_t; t \geq 0)$, if S is a non-negative supermartingale under all $\mathbb{Q} \in \mathcal{P}$, then there is a uniform optional decomposition of S into the difference between a $\mathcal{P}$-uniform local martingale and an increasing optional process. In this note we give (in Theorem 2.2) a simple proof of this result in the case where the martingale logarithms of the density processes of the p.m.s in $\mathcal{P}$ (taken with respect to a suitable reference p.m.) are closed under scalar multiplication (and hence continuous).

The applications in [4] refer to the financial set-up, where $\mathcal{P}$ is the collection of Equivalent Martingale Measures for a collection of discounted securities $\mathcal{X}$, and S is the payoff to a superhedging problem for an American option, so that

$$S_t = \operatorname*{ess\ sup}_{\mathbb{Q} \in \mathcal{P}} \ \operatorname*{ess\ sup}_{\text{stopping times } \tau \geq t} \mathbb{E}[X_\tau | \mathcal{F}_t],$$

where X is the claims process for the option.

Other examples are a multi-period coherent risk-measure where the risk measure ρ_t is given by

$$\rho_t(X) = \operatorname*{ess\ sup}_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}[X | \mathcal{F}_t]$$

(see [4]) and the Girsanov approach to a control set-up, where S is given by the same formula, but $\mathcal{P}$ corresponds to a collection of costless controls on X (see, for example, [1]).

§2 Uniform supermartingale decomposition

We assume that we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, satisfying the usual conditions, and a collection, $\mathcal{P}$, of probability measures on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} \sim \mathbb{P}$, for all $\mathbb{Q} \in \mathcal{P}$.

We note that, since $\mathbb{Q} \sim \mathbb{P}$, $\Lambda_t^{\mathbb{Q}} \stackrel{def}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$ is a positive $\mathbb{P}$-martingale, with $\Lambda_0^{\mathbb{Q}} = 1$.

Lemma 2.1. We may write $\Lambda_t^{\mathbb{Q}} = \mathcal{E}(\lambda^{\mathbb{Q}})_t$, where $\mathcal{E}$ is the Doleans-Dade exponential and $\lambda_t^{\mathbb{Q}} = \int_0^t \frac{d\Lambda^{\mathbb{Q}}{}_s}{\Lambda^{\mathbb{Q}}_{s-}}$, so that $\lambda^{\mathbb{Q}}$ is a $\mathbb{P}$-local martingale with jumps strictly bounded below by -1.

[1] I am most grateful to an anonymous referee for pointing out an error in an earlier draft of this paper and for very helpful comments on the presentation. I also thank Sigurd Assing for helpful suggestions and comments.

Proof From Theorem II.8.3 of [4], if neither $\Lambda_t^{\mathbb{Q}}$ nor $\Lambda_{t-}^{\mathbb{Q}}$ vanishes then $\lambda^{\mathbb{Q}}$ exists. The fact that $\Lambda^{\mathbb{Q}}$ does not vanish follows from the stronger statement that, since $\mathbb{Q} \sim \mathbb{P}$, $\Lambda^{\mathbb{Q}}$ is strictly positive. This also implies, once we have established its existence, the condition on the jumps of $\lambda^{\mathbb{Q}}$.

The fact that $\Lambda_{t-}^{\mathbb{Q}}$ does not vanish follows via the following argument. First note that $\frac{d\mathbb{P}}{d\mathbb{Q}}\big|_{\mathcal{F}_t} = \big(\Lambda_t^{\mathbb{Q}}\big)^{-1}$, so $\big(\Lambda_t^{\mathbb{Q}}\big)^{-1}$ is a $\mathbb{Q}$-martingale. Now $\mathcal{M}^{loc}$, the class of local martingales, is equal to H_1^{loc}, the localisation of $H_1 = \{\text{martingales } M : \mathbb{E}[\sup_{0\le t<\infty} |M_t|] < \infty\}$ [see [2]). So, it follows that there is a localising sequence (T_n) of stopping times increasing ($\mathbb{Q}$ and hence) $\mathbb{P}$-a.s. to ∞ such that $\mathbb{E}\big[\sup_{t\le T_n}\big(\Lambda_t^{\mathbb{Q}}\big)^{-1}\big] < \infty$. It follows from the integrability that $\sup_{t\le T_n}\big(\Lambda_t^{\mathbb{Q}}\big)^{-1}$ is $\mathbb{P}$-a.s. finite and hence $\inf_{t\le T_n}\Lambda_t^{\mathbb{Q}}$ is $\mathbb{P}$-a.s. strictly positive. Thus $\Lambda_{t-}^{\mathbb{Q}}$ is $\mathbb{P}$-almost surely positive on the stochastic interval $[[0, T_n]]$. Now letting $n \uparrow \infty$ we see that the second requirement is satisfied. ▫

We denote by $\mathcal{L}$ the collection $\{\lambda^{\mathbb{Q}};\, \mathbb{Q} \in \mathcal{P}\}$ and by $\mathcal{L}^{loc}$ the usual localisation of $\mathcal{L}$.

Theorem 2.2 *Suppose that*

i) $\mathbb{P} \in \mathcal{P}$; *and*

ii) $\mathcal{L}^{loc}$ *is closed under scalar multiplication;*

then any $\mathcal{P}$-uniform non-negative supermartingale, S, possesses a class-uniform Doob-Meyer predictable decomposition, i.e. we may write S uniquely as

$$S = M - A,$$

where M is a $\mathcal{P}$-uniform local martingale and A is a locally integrable predictable increasing process with $A_0 = 0$.

Remark: Notice that condition (ii) implies that every element of $\mathcal{L}^{loc}$ is continuous. This follows since: any element of $\mathcal{L}^{loc}$ has jumps bounded below by -1; then if $\delta\lambda \in \mathcal{L}^{loc}$ for all $\delta \in \mathbb{R}$, by taking appropriately large positive and negative values of δ, we see that the jumps of λ must be of size zero.

Proof of Theorem 2.2: take $\mathbb{Q} \in \mathcal{P}$, with $\Lambda^{\mathbb{Q}} = \mathcal{E}(\lambda^{\mathbb{Q}})$. Now S is a non-negative $\mathbb{P}$-supermartingale. It follows that S is locally of class D and so we can take the (unique) Doob-Meyer decomposition of S with respect to $\mathbb{P}$: $S = M - A$, where M is a $\mathbb{P}$-local martingale and A is a predictable increasing process.

Now S is a non-negative $\mathbb{Q}$-supermartingale if and only if $S\Lambda^{\mathbb{Q}}$ is a non-negative $\mathbb{P}$-supermartingale. Thus, since it follows from the remark that $\Lambda^{\mathbb{Q}}$ is continuous,

$$\begin{aligned} S\Lambda^{\mathbb{Q}}_{\cdot} &= S_0 + \int_0^{\cdot} S_{t-}d\Lambda_t^{\mathbb{Q}} + \int_0^{\cdot} \Lambda_t^{\mathbb{Q}} dS_t + < S, \Lambda^{\mathbb{Q}} >. \\ &= S_0 + \int_0^{\cdot} S_{t-}d\Lambda_t^{\mathbb{Q}} + \int_0^{\cdot} \Lambda_t^{\mathbb{Q}} dM_t + \int_0^{\cdot} \Lambda_t^{\mathbb{Q}}(d < \lambda^{\mathbb{Q}}, M >_t - dA_t) \end{aligned} \tag{2.2}$$

is a (non-negative) $\mathbb{P}$-supermartingale. Now since the first two terms in the last line of (2.2) are local martingales, whilst the last is a predictable process of finite variation, it follows that the last term must be decreasing.

Now we claim that we must then have

$$< \lambda^{\mathbb{Q}}, M >^+ << A, \text{ with } \frac{d < \lambda^{\mathbb{Q}}, M >^+}{dA} \leq 1, \tag{2.2}$$

where $< \lambda^{\mathbb{Q}}, M >^+$ and $< \lambda^{\mathbb{Q}}, M >^-$ are, respectively, the increasing processes corresponding to the positive and negative components in the Hahn decomposition of the signed measure induced by $< \lambda^{\mathbb{Q}}, M >$.

This follows from the more general statement: if μ, m^+ and m^- are three σ-finite measures on a measurable space $(\Omega, \mathcal{F})$ and

(i) m^+ and m^- are mutually singular

and

(ii) $\nu \stackrel{def}{=} \mu - m^+ + m^-$ is also a measure, then

$$m^+ << \mu, \text{ with } \frac{dm^+}{d\mu} \leq 1.$$

To see this, suppose that m^+ is supported by B and $m^-(B) = 0$ then take $A \in \mathcal{F}$ with $m^+(A) > 0$. It follows that $m^+(A) = m^+(A \cap B)$ and $m^-(A \cap B) = 0$ so

$$\nu(A \cap B)) = \mu(A \cap B) - m^+(A \cap B) \geq 0.$$

Thus $\mu(A) \geq \mu(A \cap B) > 0$ and

$$m^+(A) = m^+(A \cap B) \leq \mu(A \cap B) \leq \mu(A).$$

Now $\mathcal{L}^{loc}$ is closed under scalar multiplication so that, again localising if necessary, we may assume that $\delta\lambda \in \mathcal{L}$ and so, defining $\mathbb{Q}^\delta$ by $\Lambda^{\mathbb{Q}^\delta} \stackrel{def}{=} \mathcal{E}(\delta\lambda^{\mathbb{Q}})$, we see that (2.2) holds with $\lambda^{\mathbb{Q}}$ replaced by $\delta\lambda^{\mathbb{Q}}$ for any $\delta \in \mathbb{R}$. Letting $\delta \to \infty$ we see that $\frac{d<\lambda^{\mathbb{Q}},M>^+}{dA} = 0$, whilst letting $\delta \to -\infty$ we see that $\frac{d<\lambda^{\mathbb{Q}},M>^-}{dA} = 0$. It follows immediately that

$$< \lambda^{\mathbb{Q}}, M > \equiv 0.$$

To complete the proof we need simply observe that

$$M\Lambda^{\mathbb{Q}} = M_0 + \int M_{t-} d\Lambda_t^{\mathbb{Q}} + \int \Lambda_t^{\mathbb{Q}} dM_t + \int \Lambda_t^{\mathbb{Q}} d < \lambda^{\mathbb{Q}}, M >_t,$$

and hence M is a $\mathbb{Q}$-local martingale and since $\mathbb{Q}$ is arbitrary, the result follows □

Remark: We note that if $\mathcal{P}$ consists of the EMMs for a vector-valued martingale M and the underlying filtration supports only continuous martingales (for example if it is the filtration of a multi-dimensional Wiener process), then the conditions of Theorem 2.2 are satisfied. This follows since, under these conditions, if λ is a $\mathbb{P}$-local martingale then

$$\lambda \in \mathcal{L}^{loc} \Leftrightarrow < \lambda, M >= 0,$$

and the same then holds for any multiple of λ.

REFERENCES

[1] Benes, V: Existence of Optimal Stochastic Control Laws (1971), *SIAM J. of Control and Optim.* **9**, 446-472.

[2] Jacod, J: Calcul Stochastique et Problèmes de Martingales, LNM 714, Springer, Berlin (1979) .

[3] Jacod, J and Shiryaev, A: Limit Theorems for Stochastic Processes *2nd Edition*, Springer, Berlin, (2003).

[4] Kramkov, D: Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets (1996), *Prob. Th & Rel. Fields* **105**, 459-479.

[5] Riedel, F: Dynamic Coherent Risk Measures (2004), *Stoch. Proc. and Appl.* **112**, 185-200.

DEPARTMENT OF STATISTICS
UNIVERSITY OF WARWICK
COVENTRY
CV4 7AL
UK
s.d.jacka@warwick.ac.uk