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Declarations

I declare that, to the best of my knowledge and unless otherwise stated, all the work in this thesis is original. I confirm that this thesis has not been submitted for a degree at another university.

Abstract

A reduced divisor $D = V(f) \subset \mathbb{C}^n$ is *free* if the sheaf $\mathrm{Der}(-\log D) := \{\delta \in \mathrm{Der}_{\mathbb{C}^n} \mid \delta(f) \in (f)\mathcal{O}_{\mathbb{C}^n}\}$ of logarithmic vector fields is a locally free $\mathcal{O}_{\mathbb{C}^n}$-module. It is *linear* if, furthermore, $\mathrm{Der}(-\log D)$ is globally generated by a basis consisting of vector fields all of whose coefficients, with respect to the standard basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$ of the space $\mathrm{Der}_{\mathbb{C}^n}$ of vector fields on $\mathbb{C}^n$, are linear functions.

In principle, linear free divisors, like other kinds of singularities, might be expected to appear in non-trivial parameterised families. As part of this thesis, however, we prove that for *reductive linear free divisors*, there are no formally non-trivial families, where a linear free divisor is reductive if its associated Lie algebra is reductive, thus reductive linear free divisors are *formally rigid*.

To prove this and to understand better the class of free divisors, we introduce a rigorous deformation theory for germs of free and linear free divisors. A *(linearly) admissible deformation* is a deformation in which we deform a germ of a (linear) free divisor $(D, 0) \subset (\mathbb{C}^n, 0)$ in such a way that each fiber of the deformation is still a (linear) free divisor and that the singular locus of $(D, 0)$ is deformed flatly. Moreover, we explain how to use the de Rham logarithmic complex to compute the space of first order infinitesimal admissible deformations and the Lie algebra cohomology complex to compute the space of first order infinitesimal linearly admissible deformations.

Introduction

The main idea of this thesis is to introduce a deformation theory for germs of free and linear free divisors.

The idea of using the approach described in this thesis comes from the work of C. Sevenheck and D. van Straten [53], [54], where they deform a germ of a Lagrangian singularity in such way that each fiber of the deformation is still Lagrangian.

Free divisors were introduced by K. Saito in [48], and linear free divisors by R.-O. Buchweitz and D. Mond in [7]. Free divisors are ubiquitous in singularity theory. For example, the discriminants of versal deformations of isolated complete intersection singularities, of space curve singularities, and of singularities of functions on space curves, are always free divisors. However, not much is known on the behavior of free and linear free divisors under deformations.

A reduced divisor $D = V(f) \subset \mathbb{C}^n$ is *free* if the sheaf $\mathrm{Der}(-\log D) := \{\delta \in \mathrm{Der}_{\mathbb{C}^n} \mid \delta(f) \in (f)\mathcal{O}_{\mathbb{C}^n}\}$ of logarithmic vector fields is a locally free $\mathcal{O}_{\mathbb{C}^n}$-module. It is *linear* if, furthermore, $\mathrm{Der}(-\log D)$ is globally generated by a basis consisting of vector fields all of whose coefficients, with respect to the standard basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$ of the space $\mathrm{Der}_{\mathbb{C}^n}$ of vector fields on $\mathbb{C}^n$, are linear functions. The simplest example is the normal crossing divisor, but the main source of examples, motivating Saito's definition, is deformation theory, where discriminants and bifurcation sets are frequently free divisors.

The problem of deciding if a hypersurface $D \subset \mathbb{C}^n$ is free was solved by K. Saito in [48]. In his paper, he proved that it is enough to find n logarithmic vector fields such that the module they generate is closed under Lie brackets and such that the determinant of the matrix of their coefficients is a reduced defining equation for D.

To find deformations of a germ of a (linear) free divisor we use a relative version of Saito's criterion in order to construct a deformation where each fiber is still a (linear) free divisor and we deform the singular locus in a flat way.

Let us now fix $(D, 0) \subset (\mathbb{C}^n, 0)$ a germ of a free divisor and (S, s) a germ of a complex space. An *admissible deformation* of $(D, 0)$ over (S, s) consists of a flat morphism $\phi\colon (X,x) \longrightarrow (S,s)$ with an isomorphism $(D,0) \longrightarrow (X_s,x) := (\phi^{-1}(s),x)$ such that

$$\operatorname{Der}(-\log X/S)/\mathfrak{m}_{S,s}\operatorname{Der}(-\log X/S) = \operatorname{Der}(-\log D)$$

where $\mathfrak{m}_{S,s}$ is the maximal ideal of $\mathcal{O}_{S,s}$ and $\operatorname{Der}(-\log X/S) := \{\delta \in \operatorname{Der}(-\log X) \mid \delta \in \operatorname{Der}_{\mathbb{C}^n\times S/S}\}$. If, in addition, we suppose that $(D,0)$ is linear, then we define a *linearly admissible deformation* of $(D,0)$ over (S,s) as an admissible deformation of $(D,0)$ over (S,s) such that there exists a basis of $\operatorname{Der}(-\log X/S)$ as $\mathcal{O}_{\mathbb{C}^n\times S,(0,s)}$-module consisting of vector fields all of whose coefficients are linear in $x_1,\dots,x_n$.

As in any other deformation theory, the first type of deformations that we study are the *infinitesimal (linearly) admissible deformations*, i.e. the (linearly) admissible deformations over $T_\epsilon := \operatorname{Spec}(\mathbb{C}[t]/(t^2))$. To do so, for a germ of a free divisor $(D,0)$, we introduce the space of isomorphism classes of infinitesimal admissible deformations modulo the trivial deformations, which we will denote by $\mathcal{FT}^1(D)$. Furthermore, for a germ of a linear free divisor $(D,0)$, we introduce the space of isomorphism classes of infinitesimal linearly admissible deformations modulo the trivial ones, and denote it by $\mathcal{LFT}^1(D)$.

In order to study the previous two spaces, we use two complexes from $\mathcal{D}$-module theory (see for example [8]) and Lie algebras cohomology (see [33]), respectively. The first is the complex $\mathcal{C}^\bullet$ with modules

$$\mathcal{C}^p := \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}\big(\bigwedge^p \operatorname{Der}(-\log D), \operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D)\big)$$

and differentials

$$(d^p(\psi))(\delta_1\wedge\cdots\wedge\delta_{p+1}) := \sum_{i=1}^{p+1}(-1)^i[\delta_i,\psi(\delta_1\wedge\cdots\wedge\widehat{\delta_i}\wedge\cdots\wedge\delta_{p+1})]+$$

$$+\sum_{1\le i<j\le p+1}(-1)^{i+j-1}\psi([\delta_i,\delta_j]\wedge\delta_1\wedge\cdots\wedge\widehat{\delta_i}\wedge\cdots\wedge\widehat{\delta_j}\wedge\cdots\wedge\delta_{p+1}).$$

When the divisor D is linear, similarly, we introduce the complex $\mathcal{C}_0^\bullet$ defined by

$$\mathcal{C}_0^p := Hom_{\mathbb{C}}\big(\bigwedge^p \operatorname{Der}(-\log D)_0, (\operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D))_0\big)$$

and the differentials

$$(d_0^p(\psi))(\delta_1\wedge\cdots\wedge\delta_{p+1}) := \sum_{i=1}^{p+1}(-1)^i[\delta_i,\psi(\delta_1\wedge\cdots\wedge\hat{\delta_i}\wedge\cdots\wedge\delta_{p+1})]+$$

$$+\sum_{1\le i<j\le p+1}(-1)^{i+j-1}\psi([\delta_i,\delta_j]\wedge\delta_1\wedge\cdots\wedge\hat{\delta_i}\wedge\cdots\wedge\hat{\delta_j}\wedge\cdots\wedge\delta_{p+1})$$

where $\mathrm{Der}(-\log D)_0$ and $(\mathrm{Der}_{\mathbb{C}^n}/\mathrm{Der}(-\log D))_0$ are the weight zero part of the graded modules $\mathrm{Der}(-\log D)$ and $\mathrm{Der}_{\mathbb{C}^n}/\mathrm{Der}(-\log D)$, respectively.

Introducing the above two complexes allows us to compute the two spaces that we have just defined. In fact, the germ at the origin of the first cohomology sheaf of the complex $\mathcal{C}^\bullet$ is isomorphic to $\mathcal{FT}^1(D)$, i.e. $\mathcal{H}^1(\mathcal{C}^\bullet)_0\cong\mathcal{FT}^1(D)$ and the germ at the origin of the first cohomology sheaf of the complex $\mathcal{C}_0^\bullet$ is isomorphic to $\mathcal{LFT}^1(D)$, i.e. $H^1(\mathcal{C}_0^\bullet)_0\cong\mathcal{LFT}^1(D)$.

Among linear free divisors, there is an important subclass: the *reductive* linear free divisors, where a linear free divisor $D\subset\mathbb{C}^n$ is called reductive if its associated Lie algebra $\mathfrak{g}_D$ is reductive, see [25]. Thanks to the theory of representations of Lie algebras, see [19] and [33], we can say more in the case that $(D,0)$ is reductive: $\mathcal{LFT}^1(D)=0$ and hence $(D,0)$ is formally rigid, where a germ of a (linear) free divisor is called formally rigid if all its (linearly) admissible deformations are *trivial*, i.e. formally isomorphic to the product deformation.

Another important class of free divisors are the *Koszul free divisors*, where a free divisor D is Koszul if the principal symbols of a basis of $\mathrm{Der}(-\log D)$ form a regular sequence, see [45]. For this class of free divisors, F.J. Calderón Moreno and L. Narvaéz Macarro, in [8] and in [12], were able to develop more deeply the $\mathcal{D}$-module theory. Thanks to their work, we are able to say something more in this case. In fact, under some technical assumption, that are equivalent to the possibility to put a logarithmic connection on $\mathrm{Der}_{\mathbb{C}^n}$ and $\mathrm{Der}(-\log D)$, we can say that all $\mathcal{H}^i(\mathcal{C}^\bullet)$ are constructible sheaves of finite dimensional complex vector spaces.

Another interesting fact is that if $(D,0)\subset(\mathbb{C}^n,0)$ is a germ of a free divisor such that there exists a germ of a free divisor $(D',0)\subset(\mathbb{C}^{n-1},0)$ with $(D,0)=(D'\times\mathbb{C},0)$, then to compute $\mathcal{FT}^1(D)$ it is enough to compute $\mathcal{FT}^1(D')$, i.e. $\mathcal{FT}^1(D)\cong\pi^{-1}\mathcal{FT}^1(D')$, where $\pi\colon(D,0)\longrightarrow(D',0)$ is the projection. This is in contrast with the classical deformation theory of singularities, where if $(X',0)$ is a germ of a non-rigid singularity, then the space $T^1(X)$ of infinitesimal deformations of $(X,0)=(X'\times\mathbb{C},0)$ is necessarily infinite dimensional.

If $(D,0)$ is the germ of a *weighted homogenous free divisor*, i.e. a free divisor defined by weighted homogenous equation, then $\mathcal{FT}^1(D)$ is always finite dimensional and if the divisor is a reduced plane curve of weighted degree k, then $\mathcal{FT}^1(D)\cong\mathbb{C}[x,y]_k/J(D)\cap\mathbb{C}[x,y]_k$, where $\mathbb{C}[x,y]_k$ is the space of homogeneous polynomials of degree k and $J(D)$ is the Jacobian ideal of the divisor D.

In [16] and [18], T. de Jong and D. van Straten developed a deformation theory for germs of non-isolated singularities where the singular locus is deformed in a flat way. Because a free divisor $D \subset \mathbb{C}^n$, for $n \geq 3$, is a non-isolated singularity where the singular locus is Cohen-Macaulay of codimension 2 in $\mathbb{C}^n$, so of codimension 1 in D, see [2], we can apply their theory to the germs of free divisors in order to obtain another description of their deformations. It turns out that the two approaches are equivalent.

In the following paragraphs we give an outline of the thesis. In chapter 1, we recall the notions of free divisor introduced by K. Saito and of linear free divisor introduced by R.-O. Buchweitz and D. Mond. We describe the main properties of the module of logarithmic vector fields and its relations with the modules of logarithmic forms. We describe the relation between linear free divisors and subgroups of $\mathrm{GL}_n(\mathbb{C})$ and we point out that each linear free divisor can be seen as a discriminant in a prehomogeneous vector space. We expand the description of a basis of the module of logarithmic vector fields given in [25]. Finally, we make use of the theory developed in [27] to study the ring associated to the radical of the Jacobian of a free divisor, in the case that the latter ideal is Cohen-Macaulay of codimension 2.

In chapter 2, we recall the basic notions of $\mathcal{D}$-module theory for free divisors. We recall the notion of logarithmic connection on a $\mathcal{O}_{\mathbb{C}^n}$-module. We show under which assumptions we can define a logarithmic connection on $\mathrm{Der}_{\mathbb{C}^n}$ and $\mathrm{Der}(-\log D)$. We recall the definitions of the logarithmic Spencer complex and of the logarithmic de Rham complex and we show the connection between the two. Finally, we recall the notion of Koszul free divisors and we show that for them, the logarithmic de Rham complex is a perverse sheaf.

The third chapter contains the main results of this thesis. We introduce the notions of admissible and linearly admissible deformations using the language of functors, and we show that the functors defined are *deformation functors* in the sense of M. Schlessinger, see [50]. We show how to use the Lie algebra cohomology complex to compute the space of infinitesimal linearly admissible deformations and the logarithmic de Rham complex to compute the space of infinitesimal admissible deformations. We describe the properties of these spaces, with emphasis on the cases of reductive linear free divisors and Koszul free divisors. We analyse in more detail the case of weighted homogeneous free divisors. We describe how to use the theory developed by T. de Jong and D. van Straten in [16] to introduce another way to deform a free divisor and we show that the two approaches are equivalent.

Finally, there are three appendices. The first one concerning the classic functorial deformation theory studied by M. Schlessinger, M. Artin and others. The second one concerning Lie algebras, their representations and their cohomology. The third one contains two examples: one is the computation of $\mathcal{LFT}^1(D)$ for a non-reductive linear free divisor in $\mathbb{C}^5$ and the other of $\mathcal{FT}^1(D)$ for a weighted homogeneous free divisor in $\mathbb{C}^2$.

Chapter 1

Free and linear free divisors

This chapter is an exposition of the main objects used in this thesis. We recall the notion of free and linear free divisors that were respectively introduced by K. Saito in [48] and by R.O. Buchweitz and D. Mond in [7]. We give characterisations of freeness and we describe the relations between linear free divisors and subgroups of $\mathrm{GL}_n(\mathbb{C})$. We show that each linear free divisor can be seen as a discriminant in a prehomogeneous vector space. Furthermore, we describe the structure of logarithmic vector fields for a linear free divisor. Finally, we study the rank and ring conditions for free divisors, looking with particular interest at the radical of the Jacobian.

The material of this chapter is mainly taken from: [48], [25], [26] and [7].

1.1 Basic notions

In this section we recall the notions of logarithmic forms and logarithmic vector fields, and the notions of free and linear free divisors. We show properties and characterisations of freeness. Finally, we recall some important subclasses of free divisors, such as Euler-homogeneous and locally quasi-homogeneous free divisors.

The notion of logarithmic differential forms and logarithmic vector fields were first used by K. Saito in order to study the Gauss-Manin connection of the singularity A_3. He introduced the analytic sheaves $\Omega^1_S(\log D)$ and $\mathrm{Der}_S(-\log D)$ of a reduced divisor D in a smooth complex manifold S. The hypersurfaces D for which the sheaves of $\mathcal{O}_S$-modules $\Omega^q_S(\log D)$ and $\mathrm{Der}_S(-\log D)$ are locally free are called *free divisors*. Saito gave a criterion, see Proposition 1.1.16, to decide whether given logarithmic differential forms (resp. logarithmic vector fields) form a basis of $\Omega^1_S(\log D)$ (resp. $\mathrm{Der}_S(-\log D)$).

The material of this section is mainly taken from [48] and [46].

Proposition 1.1.1. ([48], (1.1)) *Let U be a domain of $\mathbb{C}^n$ and let $D \subset U$ be a hypersurface of U defined by an equation $f = 0$, where f is holomorphic on U. Let ω be a meromorphic q-form on U, which may have poles only along D. Then the following four conditions for ω are equivalent*

1. *$f\omega$ and $fd\omega$ are holomorphic on U;*
2. *$f\omega$ and $df \wedge \omega$ are holomorphic on U;*
3. *There exist a holomorphic function g, a holomorphic $(q-1)$-form ξ and a holomorphic q-form η on U such that*

   a) $\dim_{\mathbb{C}}(D \cap \{x \in U \mid g(x) = 0\}) \leq n-2$,
   b) $g\omega = \frac{df}{f} \wedge \xi + \eta$;
4. *There exists an $(n-2)$-dimensional analytic set $A \subset D$ such that the germ of ω at any point $p \in D \setminus A$ belongs to $\frac{df}{f} \wedge \Omega^{q-1}_{U,p} + \Omega^q_{U,p}$, where $\Omega^q_{U,p}$ denotes the module of germs of holomorphic q-form on U at p.*

Definition 1.1.2. *A meromorphic q-form on U is called a* q-form with logarithmic poles along D *or* logarithmic q-form, *if it satisfies the equivalent conditions of Proposition 1.1.1.*

Definition 1.1.3. *Let $D \subset \mathbb{C}^n$ be a hypersurface and let $f_p = 0$ be a reduced equation for D, locally at $p \in \mathbb{C}^n$. A meromorphic q-form is logarithmic along D at p, if $f_p\omega$ and $f_p d\omega$ are holomorphic. We write*

$$\Omega^q_p(\log D) := \{\textit{germs of logarithmic } q\textit{-form at } p\}$$

and denote the corresponding sheaf on $\mathbb{C}^n$ by $\Omega^q(\log D)$.

Notice that for $q = 1$ we have the following inclusions

$$\Omega^1_{\mathbb{C}^n,p} \subset \Omega^1_p(\log D) \subset \frac{1}{f_p}\Omega^1_{\mathbb{C}^n,p},$$

where f_p is a reduced equation for D, locally at $p \in \mathbb{C}^n$.

Remark 1.1.4. *By Definition and Proposition 1.1.1:*

1. *$\Omega^q(\log D)$, $q = 0, \ldots, n$ are coherent $\mathcal{O}_{\mathbb{C}^n}$-modules;*
2. *$\bigoplus_{q=0}^n \Omega^q(\log D)$ is an $\mathcal{O}_{\mathbb{C}^n}$-exterior algebra;*

3. $\bigoplus_{q=0}^{n} \Omega^q(\log D)$ *is closed under exterior differentiation;*

4. $\Omega_p^0(\log D) = \Omega_{\mathbb{C}^n,p}^0$;

5. $\Omega_p^n(\log D) = \dfrac{1}{f_p}\Omega_{\mathbb{C}^n,p}^n$.

Let us fix now a coordinate system $x_1, \dots, x_n$ on $\mathbb{C}^n$.

Remark 1.1.5. *Let $\omega_1, \dots, \omega_n$ be n elements of $\Omega_p^1(\log D)$. Then the wedge product $\omega_1 \wedge \cdots \wedge \omega_n$ has a local presentation $a(x)\dfrac{dx_1 \wedge \cdots \wedge dx_n}{f_p}$ for a certain holomorphic function $a(x)$.*

We can also study logarithmic vector fields along a divisor in a complex manifold. These vector fields δ appear naturally as tangent vectors $\delta(p)$, $p \in D$, to the divisor D in its smooth points. Later in Lemma 1.1.8, it is shown that the module of logarithmic vector fields at a point is dual to the module of logarithmic differential 1-forms.

Definition 1.1.6. ([48], Definition 1.4) *Let $D \subset \mathbb{C}^n$ be a hypersurface and let $f_p = 0$ be a reduced equation for D, locally at $p \in \mathbb{C}^n$. A holomorphic vector field δ on $\mathbb{C}^n$ is called* logarithmic *if it satisfies the following equivalent conditions*

1. *For any smooth point p of D, the tangent vector $\delta(p)$ of p is tangent to D;*

2. *For any point p of D, the derivative δf_p of the local equation for D belongs to the ideal $(f_p)\mathcal{O}_{\mathbb{C}^n,p}$.*

We write

$$\mathrm{Der}_p(-\log D) := \{\delta \in \mathrm{Der}_{\mathbb{C}^n,p} \mid \delta(f_p) \in (f_p)\mathcal{O}_{\mathbb{C}^n,p}\}$$

where $\mathrm{Der}_{\mathbb{C}^n,p}$ is the set of germs of holomorphic vector fields on $\mathbb{C}^n$ at p and

$$\mathrm{Der}(-\log D) \text{ is the sheaf of germs of } \bigcup_{p\in\mathbb{C}^n} \mathrm{Der}_p(-\log D).$$

Notice that $\mathrm{Der}(-\log D)$ is not empty. In fact if f_p is a reduced equation for $D \subset \mathbb{C}^n$ locally at $p \in \mathbb{C}^n$, then

$$f_p \cdot \mathrm{Der}_{\mathbb{C}^n,p} \subset \mathrm{Der}_p(-\log D) \subset \mathrm{Der}_{\mathbb{C}^n,p}\,.$$

Remark 1.1.7. *By Definition 1.1.6 we get*

1. $\mathrm{Der}(-\log D)$ *is a coherent $\mathcal{O}_{\mathbb{C}^n}$-submodule of* $\mathrm{Der}_{\mathbb{C}^n}$*, where* $\mathrm{Der}_{\mathbb{C}^n}$ *is the sheaf of holomorphic vector fields on $\mathbb{C}^n$;*

2. $\mathrm{Der}(-\log D)$ *is closed under the Lie bracket product* $[\cdot,\cdot]$;

3. *Let* $\delta_1, \ldots, \delta_n$ *be any* n *elements of* $\mathrm{Der}_p(-\log D)$. *Write* $\delta_1 \wedge \cdots \wedge \delta_n = g(x)\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_n$. *Then* $g(x)$ *belongs to the ideal* $(f_p)\mathcal{O}_{\mathbb{C}^n,p}$, *i.e. at any smooth point* $p \in D, \delta_1(p), \ldots, \delta_n(p)$ *are tangent to the* $(n-1)$*-dimensional space* D *and therefore they are linear dependent.*

Let us now consider the maps

$$\mathrm{Der}_{\mathbb{C}^n,p} \times \Omega^q_{\mathbb{C}^n,p} \longrightarrow \Omega^{q-1}_{\mathbb{C}^n,p}$$

defined by

$$(\delta, \omega) \mapsto \delta \cdot \omega, q = 1, \ldots, n,$$

and

$$\mathrm{Der}_{\mathbb{C}^n,p} \times \Omega^q_{\mathbb{C}^n,p} \longrightarrow \Omega^{q}_{\mathbb{C}^n,p}$$

defined by

$$(\delta, \omega) \mapsto L_\delta(\omega), q = 1, \ldots, n,$$

they are the pairing between forms and vector fields, and the Lie derivative.

Notice that both notions can be defined for logarithmic vector fields and differential forms. A priori they may only be defined as meromorphic forms. The following Lemma shows that they are actually holomorphic.

Lemma 1.1.8. ([48], Lemma 1.6) *The above notions of the inner product and the Lie derivative are extended to*

$$\mathrm{Der}_p(-\log D) \times \Omega^q_p(\log D) \longrightarrow \Omega^{q-1}_p(\log D) \quad \textit{defined by} \quad (\delta, \omega) \mapsto \delta \cdot \omega,$$

$$\mathrm{Der}_p(-\log D) \times \Omega^q_p(\log D) \longrightarrow \Omega^{q}_p(\log D) \quad \textit{defined by} \quad (\delta, \omega) \mapsto L_\delta(\omega).$$

In addition, by the pairing

$$\mathrm{Der}_p(-\log D) \times \Omega^1_p(\log D) \longrightarrow \mathcal{O}_{\mathbb{C}^n,p}$$

each module is the dual $\mathcal{O}_{\mathbb{C}^n,p}$*-module of the other.*

Corollary 1.1.9. $\Omega^1_p(\log D)$ *and* $\mathrm{Der}_p(-\log D)$ *are reflexive* $\mathcal{O}_{\mathbb{C}^n,p}$*-modules. In particular, when* $n = 2$, *then* $\Omega^1(\log D)$ *and* $\mathrm{Der}(-\log D)$ *are locally free* $\mathcal{O}_{\mathbb{C}^2}$*-modules.*

All the previous material is devoted to give the following:

Definition 1.1.10. *A reduced divisor $D \subset \mathbb{C}^n$ is called a* free divisor *if the sheaf* $\mathrm{Der}(-\log D)$ *of logarithmic vector fields is a locally free $\mathcal{O}_{\mathbb{C}^n}$-module, or equivalently if the sheaf $\Omega^1(\log D)$ is a locally free $\mathcal{O}_{\mathbb{C}^n}$-module.*

Notice that any reduced plane curve is a free divisor by Corollary 1.1.9. For example, if we consider the cusp singularity $D \subset \mathbb{C}^2$ given by $f = x^3 - y^2$, then a basis of $\mathrm{Der}(-\log D)$ is formed by $\delta_1 = 2x\partial/\partial x + 3y\partial/\partial y$ and $\delta_2 = 2y\partial/\partial x + 3x^2\partial/\partial y$, and a basis of $\Omega^1(\log D)$ is obtained by forming the dual basis to (δ_1, δ_2).

Definition 1.1.11. *A non-zero element $\delta \in \mathrm{Der}_{\mathbb{C}^n}$ is* homogeneous of polynomial degree p *if $\delta = \sum_{i=1}^n f_i\partial/\partial x_i$ and $f_i \in \mathcal{O}_{\mathbb{C}^n}$ is an homogeneous element of degree p, for $1 \le i \le n$. In this case we write* $\mathrm{pdeg}(\delta) = p$.

Between all free divisors, an important role is played by the linear ones, i.e. the free divisors $D \subset \mathbb{C}^n$ which have the property that $\mathrm{Der}(-\log D)$ has a basis consisting of vector fields that are homogeneous of degree one with respect to the natural grading.

Definition 1.1.12. *A reduced divisor $D \subset \mathbb{C}^n$ is called a* linear free divisor *if it is free and there is a basis for $\mathrm{Der}(-\log D)$ as $\mathcal{O}_{\mathbb{C}^n}$-module consisting of vector fields all of whose coefficients, with respect to the standard basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$ of the space $\mathrm{Der}_{\mathbb{C}^n}$, are linear functions, i.e. they are all homogeneous polynomials of degree* 1.

Remark 1.1.13. *With respect to the standard grading of $\mathrm{Der}_{\mathbb{C}^n}$, i.e., $\deg x_i = 1$ and $\deg \partial/\partial x_i = -1$ for every $i = 1, \ldots, n$, such vector fields have weight zero.*

Remark 1.1.14. *Let $\delta \in \mathrm{Der}_{\mathbb{C}^n}$ a weight zero vector field. Then there exists a $n \times n$ matrix A with coefficients in $\mathbb{C}$ such that we can write $\delta = xA\partial^t$, where $\partial = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$.*

Definition 1.1.15. *We denote by $\mathrm{Der}(-\log D)_0$ the finite dimensional Lie subalgebra of $\mathrm{Der}(-\log D)$ consisting of the weight zero logarithmic vector fields.*

Checking if a divisor is free or not directly from the definition is not very practical. However, there is a nice criterion to understand it easily:

Proposition 1.1.16. (SAITO'S CRITERION) ([48], Theorem 1.8) *i) The hypersurface $D \subset \mathbb{C}^n$ is a free divisor in the neighbourhood of a point p if and only if $\bigwedge^n \Omega^1_p(\log D) = \Omega^n_p(\log D)$, i.e. if there exist n elements $\omega_1, \ldots, \omega_n \in \Omega^1_p(\log D)$ such that*

$$\omega_1 \wedge \cdots \wedge \omega_n = \alpha \frac{dx_1 \wedge \cdots \wedge dx_n}{f}$$

where α is a unit. Then the set of forms $\{\omega_1, \ldots, \omega_n\}$ form a basis for $\Omega^1_p(\log D)$. Moreover, we have

$$\Omega^q_p(\log D) = \bigoplus_{i_1,\ldots,i_q} \mathcal{O}_{\mathbb{C}^n,p}\omega_{i_1} \wedge \cdots \wedge \omega_{i_q}$$

for $q = 1, \ldots, n$.

ii) The hypersurface $D \subset \mathbb{C}^n$ is a free divisor in the neighbourhood of a point p if and only if there exist n germs of vector fields $\delta_1, \ldots, \delta_n \in \mathrm{Der}_p(-\log D)$ such that the determinant of the matrix of coefficients $[\delta_1, \ldots, \delta_n]$, with respect to some, or any, $\mathcal{O}_{\mathbb{C}^n,p}$-basis of $\mathrm{Der}_{\mathbb{C}^n,p}$, is a reduced equation for D at p, i.e. it is a unit multiple of f_p. In this case, $\delta_1, \ldots, \delta_n$ form a basis for $\mathrm{Der}_p(-\log D)$.

Definition 1.1.17. *In the notation of the Proposition 1.1.16, the matrix $[\delta_1, \ldots, \delta_n]$ is called a* Saito matrix *of D.*

Lemma 1.1.18. ([48], Lemma 1.9) *Let $\delta_i = \sum_{j=1}^n a_i^j(x)\partial/\partial x_j$, $i = 1, \ldots, n$, be a system of holomorphic vector fields at $p \in \mathbb{C}^n$ such that*

1. *$[\delta_i, \delta_j] \in \sum_{k=1}^n \mathcal{O}_{\mathbb{C}^n,p}\delta_k$ for $i, j = 1, \ldots, n$;*
2. *$\det(a_i^j) = f$ defines a reduced hypersurface D.*

Then for $D = V(f)$, $\delta_1, \ldots, \delta_n$ belongs to $\mathrm{Der}_p(-\log D)$, and hence $\delta_1, \ldots, \delta_n$ forms a basis of $\mathrm{Der}_p(-\log D)$.

There is also an algebraic version of Saito's criterion that does not refer to vector fields directly but characterizes the Taylor series of the function f defining a free divisor:

Proposition 1.1.19. *A formal power series $f \in R = \mathbb{C}[[x_1, \ldots, x_n]]$ defines a free divisor if it is reduced, i.e. squarefree, and there is an $n \times n$ matrix A with entries in R such that*

$$\det A = f$$

and

$$(\nabla f)A \equiv (0, \ldots, 0) \bmod f,$$

where $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is the gradient of f, and the last condition just expresses that each entry of the vector $(\nabla f)A$ is divisible by f in R. The columns of A can then be viewed as the coefficients of a basis, with respect to the partial derivatives $\partial/\partial x_i$, of the logarithmic vector fields along the divisor $f = 0$.

Example 1.1.20. *The normal crossing divisor* $D = V(x_1 \cdots x_n) \subset \mathbb{C}^n$ *is a linear free divisor. In fact* $\mathrm{Der}(-\log D)$ *has basis* $x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n$. *Up to isomorphism it is the only linear free divisor among hyperplane arrangements. See Chapter 4 of [46].*

We now describe another way to understand if a divisor is free or not by looking at its singular locus. We prove the result only in the case of weighted homogeneous divisors, and give references for the general case, because this proof contains material useful for the remaining of the thesis.

Theorem 1.1.21. *A reduced divisor* $D \subset \mathbb{C}^n$ *defined by an equation* $f \in \mathcal{O}_{\mathbb{C}^n}$ *is free at the origin if and only if* $\mathcal{O}_{\mathbb{C}^n}/J(D)$ *is* 0 *or* $(n-2)$*-dimensional Cohen-Macaulay, where* $J(D)$ *is the Jacobian ideal of* D*, i.e. the ideal of* $\mathcal{O}_{\mathbb{C}^n}$ *generated by* f *and all its partial derivatives.*

Proof. Suppose that f is weighted homogeneous with respect to strictly positive weights $(a_1, \ldots, a_n)$. Consider then the Euler vector field $\delta_0 = \sum_{i=1}^n a_i x_i \partial/\partial x_i$. Then $\delta_0 \in \mathrm{Der}(-\log D)$ because $\delta_0(f) = \deg(f)f$. Moreover, we define the submodule

$$\mathrm{Ann}(D) := \{\delta \in \mathrm{Der}(-\log D) \mid \delta(f) = 0\}$$

of $\mathrm{Der}(-\log D)$. Then we have a decomposition

$$\mathrm{Der}(-\log D) = \mathcal{O}_{\mathbb{C}^n}\delta_0 \oplus \mathrm{Ann}(D)$$

because $\delta - \dfrac{\delta(f)}{f}\dfrac{1}{\deg(f)}\delta_0 \in \mathrm{Ann}(D)$ for any $\delta \in \mathrm{Der}(-\log D)$. Since any projective module over a local ring is free, $\mathrm{Der}(-\log D)$ is free if and only if $\mathrm{Ann}(D)$ is free.

We have an exact sequence

$$0 \longrightarrow \mathrm{Ann}(D) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{C}^n}^n \xrightarrow{\beta} \mathcal{O}_{\mathbb{C}^n} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{C}^n}/J(D) \longrightarrow 0\ ,$$

where

$$\alpha(\sum_{i=1}^n f_i\partial/\partial x_i) = (f_1, \ldots, f_n)^t \quad \text{for} \quad \sum_{i=1}^n f_i\partial/\partial x_i \in \mathrm{Ann}(D),$$

$$\beta((g_1, \ldots, g_n)^t) = \sum_{i=1}^n g_i\partial f/\partial x_i \quad \text{for} \quad (g_1, \ldots, g_n)^t \in \mathcal{O}_{\mathbb{C}^n}^n$$

and γ is the natural projection. Thus $\mathrm{Ann}(D)$ is free if and only if the homological dimension of $\mathcal{O}_{\mathbb{C}^n}/J(D)$ is less than three.

Recall the Auslander-Buchsbaum equality (cf. [20], Theorem 19.9)

$$\mathrm{depth}(\mathcal{O}_{\mathbb{C}^n}/J(D)) + \mathrm{hdim}(\mathcal{O}_{\mathbb{C}^n}/J(D)) = \dim(\mathcal{O}_{\mathbb{C}^n}) = n,$$

if $\mathcal{O}_{\mathbb{C}^n}/J(D) \neq 0$. On the other hand, we have $\dim(\mathcal{O}_{\mathbb{C}^n}/J(D)) \leq n-2$, then

$$\mathrm{hdim}(\mathcal{O}_{\mathbb{C}^n}/J(D)) \leq 2$$

$$\iff \mathrm{depth}(\mathcal{O}_{\mathbb{C}^n}/J(D)) \geq n-2 \geq \dim(\mathcal{O}_{\mathbb{C}^n}/J(D))$$

$$\iff \mathrm{depth}(\mathcal{O}_{\mathbb{C}^n}/J(D)) = n-2 = \dim(\mathcal{O}_{\mathbb{C}^n}/J(D))$$

$$\mathcal{O}_{\mathbb{C}^n}/J(D) \text{ is Cohen-Macaulay of dimension } n-2,$$

if $\mathcal{O}_{\mathbb{C}^n}/J(D) \neq 0$. When $\mathcal{O}_{\mathbb{C}^n}/J(D) = 0$, $\mathrm{Ann}(D)$ is obviously free. Thus we have proved the theorem in the case that f is weighted homogeneous.

For the proof in the general case see [2], §2. □

Proposition 1.1.22. *Let D be a divisor in $\mathbb{C}^n$ defined by a weighted homogeneous equation. Then D is free at the origin of $\mathbb{C}^n$ if and only if it is free at every point of $\mathbb{C}^n$.*

Proof. If D is a free divisor, then $\mathrm{Der}(-\log D)$ is free at every point, in particular it is free at the origin.

Assume that $\mathrm{Der}_0(-\log D)$ is free. Then there exists an open neighborhood U of the origin such that $D \cap U$ is free at every point of U, because $\mathrm{Der}(-\log D)$ is coherent. Recall that D has a good $\mathbb{C}^*$-action in $\mathbb{C}^n$ because D is defined by a weighted homogeneous polynomial. Thus we know that $\mathrm{Der}_y(-\log D)$ is a free $\mathcal{O}_{\mathbb{C}^n,y}$-module for any $y \in \mathbb{C}^n$. □

Notice that the previous result is false if we drop the hypothesis that D be weighted homogeneous. It is enough to consider a surface in $\mathbb{C}^3$ with an isolated singularity away from the origin. For example, if we take $D = V(x^2+y^2+z^2+2z+4) \subset \mathbb{C}^3$, then it is smooth at the origin and hence it is free there but at the singular point it is not free by Theorem 1.1.21.

Proposition 1.1.23. *Let $D \subset \mathbb{C}^n$ be a divisor. Suppose that $D = \bigcup_{i=1}^k D_i$ is the irreducible decomposition of D and that D is defined by the reduced equation $f = \Pi_{i=1}^k f_i$, where f_i corresponds to the irreducible component D_i for $i = 1, \ldots, k$. Then*

$$\mathrm{Der}(-\log D) = \bigcap_{i=1}^{k} \mathrm{Der}(-\log D_i).$$

Proof. It is sufficient to prove the statement for $k = 2$. Consider then coprime functions $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^n}$. If $\delta \in \mathrm{Der}_{\mathbb{C}^n}$ then

$$\begin{aligned}\delta \in \mathrm{Der}(-\log D) &\iff \delta(f_1 f_2) \in (f_1 f_2)\mathcal{O}_{\mathbb{C}^n} \\ &\iff f_1\delta(f_2) + f_2\delta(f_1) \in (f_1 f_2)\mathcal{O}_{\mathbb{C}^n} \\ &\iff \delta(f_1) \in (f_1)\mathcal{O}_{\mathbb{C}^n} \text{ and } \delta(f_2) \in (f_2)\mathcal{O}_{\mathbb{C}^n} \\ &\iff \delta \in \mathrm{Der}(-\log D_1) \cap \mathrm{Der}(-\log D_2)\end{aligned}$$

as required. □

Proposition 1.1.24. *Let $D \subset \mathbb{C}^n$ be a free divisor defined by a weighted homogeneous equation. Then* $\mathrm{Der}(-\log D)$ *has a basis consisting of weighted homogeneous vector fields.*

Proof. If D is defined by a weighted homogeneous equation, then its Jacobian ideal is weighted homogeneous and hence its syzygy module has a weighted homogeneous basis consisting of $n-1$ elements by Theorem 1.1.21. Now to obtain a basis of $\mathrm{Der}(-\log D)$ consisting of weighted homogeneous elements, it is enough to add the Euler vector field to the $n-1$ just obtained as basis of the syzygy module. □

Theorem 1.1.25. *Let $D = V(f)$ be a reduced divisor in $\mathbb{C}^n$ defined by a homogeneous equation and let $\delta_1, \ldots, \delta_n \in \mathrm{Der}(-\log D)$ be homogeneous and linearly independent over $\mathcal{O}_{\mathbb{C}^n}$. Then D is free with basis $\delta_1, \ldots, \delta_n$ if and only if $\sum_{i=1}^n \mathrm{pdeg}(\delta_i) = \deg(f)$.*

Proof. If D is a free divisor with basis $\delta_1, \ldots, \delta_n$, it follows from Proposition 1.1.16 that $\sum_{i=1}^n \mathrm{pdeg}(\delta_i) = \deg(f)$.

Suppose now that $\sum_{i=1}^n \mathrm{pdeg}(\delta_i) = \deg(f)$. Since by hypothesis, $\delta_1, \ldots, \delta_n$ are linearly independent, then $\det[\delta_1, \ldots, \delta_n] \neq 0$. Moreover, we can write $\det[\delta_1, \ldots, \delta_n] = gf$ with $g \in \mathcal{O}_{\mathbb{C}^n}$ a non-zero homogeneous polynomial. Since $\deg(\det[\delta_1, \ldots, \delta_n]) = \sum_{i=1}^n \mathrm{pdeg}(\delta_i) = \deg(f)$, we see that $g \in \mathbb{C}^*$. The conclusion follows from Proposition 1.1.16. □

We now recall some particular types of divisors. These are of special interest, since they are a generalization of weighted homogeneous divisors. Often results about free divisors are much simpler to prove for these types of divisors and can then be generalized.

Definition 1.1.26. *A divisor $D \subset \mathbb{C}^n$ is* strongly Euler-homogeneous *at x if there is a local equation f for D around x and a logarithmic vector field $\delta \in \mathrm{Der}_x(-\log D)$ vanishing at x such that $\delta(f) = f$. If D is strongly Euler-homogeneous at each point, we simply say that it is strongly Euler-homogeneous.*

Definition 1.1.27. *A divisor $D \subset \mathbb{C}^n$ is* Euler-homogeneous *at x if there is a local equation f for D around x and a logarithmic vector field $\delta \in \mathrm{Der}_x(-\log D)$ such that $\delta(f) = f$. If D is Euler-homogeneous at each point, we simply say that it is Euler-homogeneous.*

Notice that the set of points where a divisor is Euler-homogeneous is open. In addition, it is clear that strongly Euler-homogeneous divisors are Euler-homogeneous. In stead, not all Euler-homogeneous divisors are strongly Euler-homogeneous. If we consider $D = V(z(x^4 + xy^4 + y^5)) \subset \mathbb{C}^3$, then D is Euler-homogeneous but is not strongly Euler-homogeneous at points $(0, 0, z_0)$, where $z_0 \neq 0$. This example also shows that in general the set of points where a divisor is strongly Euler-homogeneous is not open. See [14], Section 1.

Definition 1.1.28. *A divisor D in a n-dimensional complex manifold M is* locally quasi-homogeneous *if at each point $x \in D$, there are local coordinates $(U; x_1, \ldots, x_n)$ centered at x with respect to which $D \cap U$ has a weighted homogeneous defining equation with strictly positive weights.*

Examples of locally quasi-homogeneous free divisors are free hyperplane arrangements and discriminants of stable maps in Mather's "nice dimensions".

Definition 1.1.29. *A divisor D in a n-dimensional complex manifold M is called* weakly locally quasi-homogeneous *if at each point $x \in D$, there are local coordinates $(U; x_1, \ldots, x_n)$ centered at x with respect to which $D \cap U$ has a weighted homogeneous defining equation with all weights ≥ 0 and at least one > 0.*

It is obvious that a locally quasi-homogeneous divisor is also weakly locally quasi-homogeneous and that a locally quasi-homogeneous divisor is Euler-homogeneous at every point. However, the notions of weakly locally quasi-homogeneous and locally quasi-homogeneous are not equivalent. In fact if we consider $D = V(f) = V(xy(x + y)(xz + y)) \subset \mathbb{C}^3$ then, at any point, f is weighted homogeneous with weights $(1, 1, 0)$ and so D is weakly locally quasi-homogeneous. However, it is not locally quasi-homogeneous because at the origin is not possible to find all strictly positive weight such that f is weighted homogeneous. See [9], Section 1.

We now give a characterisation of free Euler-homogeneous divisors in terms of logarithmic differential forms.

Proposition 1.1.30. ([22], Proposition 1.29) *Let $D \subset \mathbb{C}^n$ be a free divisor defined locally at $p \in \mathbb{C}^n$ by a reduced $f_p \in \mathcal{O}_{\mathbb{C}^n,p}$. Then D is Euler-homogeneous at p if and only if there exists a basis $\omega_1, \ldots, \omega_n$ of $\Omega^1_p(\log D)$ such that $\frac{df_p}{f_p}$ can be chosen as ω_1.*

We now introduce a relative version of logarithmic vector fields. They will have a key role in Chapter 3.

Definition 1.1.31. *Let S be a complex space. Then $\mathrm{Der}_{\mathbb{C}^n \times S/S}$ is the set of vector fields on $\mathbb{C}^n \times S$ without components in the S direction, i.e.*

$$\mathrm{Der}_{\mathbb{C}^n \times S/S} := \{\chi \in \mathrm{Der}_{\mathbb{C}^n \times S} \mid \pi_*\chi = 0\},$$

where $\pi \colon \mathbb{C}^n \times S \longrightarrow S$ is the second projection.

Definition 1.1.32. *Let S be a complex space and let $D \subset \mathbb{C}^n \times S$ be a divisor. Then*

$$\mathrm{Der}(-\log D/S) := \mathrm{Der}(-\log D) \cap \mathrm{Der}_{\mathbb{C}^n \times S/S}.$$

Remark 1.1.33. $\mathrm{Der}_{\mathbb{C}^n \times S/S}$ *and* $\mathrm{Der}(-\log D/S)$ *are both coherent sheaves of $\mathcal{O}_{\mathbb{C}^n \times S}$-modules.*

1.2 Linear free divisors and subgroups of $\mathrm{GL}_n(\mathbb{C})$

In this section, we recall how to associate to a linear free divisor in $\mathbb{C}^n$ a subgroup G_D° of $\mathrm{GL}_n(\mathbb{C})$ and also under which assumptions a subgroup of $\mathrm{GL}_n(\mathbb{C})$ is isomorphic to G_D° for some linear free divisor $D \subset \mathbb{C}^n$.

The material of this section is mainly taken from: [25], [26], [7] and [24].

Definition 1.2.1. *Let $D \subset \mathbb{C}^n$ be a divisor defined by a homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots x_n]$ of degree n. Then we denote by L_D the Lie algebra*

$$\{xA\partial^t \mid xA\partial^t(f) \in \mathbb{C} \cdot f\} \subset \Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))$$

of weight 0 global logarithmic vector fields.

Remark 1.2.2. *$D \subset \mathbb{C}^n$ is a linear free divisor if and only if L_D contains a basis of* $\mathrm{Der}(-\log D)$ *as $\mathcal{O}_{\mathbb{C}^n}$-module.*

Definition 1.2.3. *Let $D = V(f) \subset \mathbb{C}^n$ be a linear free divisor. We denote by G_D the subgroup*

$$\{A \in GL_n(\mathbb{C}) \mid A(D) = D\} = \{A \in GL_n(\mathbb{C}) \mid f \circ A \in \mathbb{C} \cdot f\}$$

of $\mathrm{GL}_n(\mathbb{C})$ *with identity component* G_D° *and Lie algebra* $\mathfrak{g}_D$*.*

Definition 1.2.4. ([52]) *A Lie subgroup* G *of* $\mathrm{GL}_n(\mathbb{C})$ *is* reductive *if it has finitely many connected components, its Lie algebra* $\mathfrak{g}$ *is reductive (see Definition B.1.28) and the centre* Z_{G° *of the identity component of* G *consists of semisimple transformations.*

For connected algebraic groups the definition of reductiveness, cf. [47] Definition 11.30, is simpler than the definition for Lie groups.

Definition 1.2.5. *A connected algebraic group* G *is* reductive *if it has no normal connected abelian subgroups except tori, or, equivalently, if its unipotent radical is trivial.*

Lemma 1.2.6. *Any reductive algebraic group is reductive as Lie group.*

Proof. It follows from [47], Aside 8.26, Theorem 9.13 and Theorem 15.1. □

Definition 1.2.7. *Let* $D \subset \mathbb{C}^n$ *be a linear free divisor. We call* D reductive *if* $\mathfrak{g}_D$ *is a reductive Lie algebra, see Definition B.1.28.*

Lemma 1.2.8. ([25], Lemma 2.2) G_D° *is an algebraic subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *and* $\mathfrak{g}_D = \{A \mid xA^t\partial^t \in L_D\}$*.*

The following Lemma appeared in the draft version of [25].

Lemma 1.2.9. *Let* $D \subset \mathbb{C}^n$ *be a linear free divisor. If* $\mathfrak{g}_D$ *is reductive then* G_D° *is reductive as algebraic group.*

Proof. By definition, we have to show that G_D° has no connected normal commutative subgroups except tori. Let H be such a subgroup. Then its Lie algebra $\mathfrak{h}$ is a commutative ideal in $\mathfrak{g}_D$ by [47], Proposition 13.18 (b). In particular $\mathfrak{h}$ is solvable and hence contained in the radical of $\mathfrak{g}_D$. The assumption that $\mathfrak{g}_D$ is reductive means that this radical has no nilpotent part and it is therefore contained in the semisimple part of $\mathrm{Der}(-\log D)$. Thus, H is contained in the diagonal subgroup $(\mathbb{C}^*)^n \subset GL_n(\mathbb{C})$ and is then a torus by [25], Lemma 3.6 (1). □

Lemma 1.2.10. ([25], Lemma 2.3) *The complement* $\mathbb{C}^n \setminus D$ *of a linear free divisor is an orbit of* $G_D^\circ \subset \mathrm{GL}_n(\mathbb{C})$ *with finite isotropy group.*

Proof. For $p \in \mathbb{C}^n$, the orbit $G_D^\circ \cdot p$ is a smooth locally closed subset of $\mathbb{C}^n$ whose boundary is a union of strictly lower dimensional orbits. The orbit map $G_D^\circ \longrightarrow G_D^\circ \cdot p$ sends $I_n + \epsilon \cdot A$ to $p + \epsilon \cdot pA$ and induces a tangent map

$$\mathfrak{g}_D \twoheadrightarrow T_p(G_D^\circ \cdot p),\ A \mapsto pA^t. \tag{1.1}$$

For $p \notin D$, $\mathrm{Der}_p(-\log D)$ and hence also L_D are n-dimensional. Then by Lemma 1.2.8 and (1.1), $T_p G_D^\circ \cdot p$ and hence $G_D^\circ \cdot p$ are n-dimensional, which implies the finiteness of the isotropy group of p in G_D°. As this holds for all $p \notin D$, the boundary of $G_D^\circ \cdot p$ must be D and then $G_D^\circ \cdot p = \mathbb{C}^n \setminus D$. □

Reversing our point of view, we might try to find algebraic subgroups $G \subset \mathrm{GL}_n(\mathbb{C})$ that define linear free divisors. This requires, by definition, that G is n-dimensional, connected and that by Lemma 1.2.10, there is an open orbit. The complement of the open orbit D is then a candidate for a free divisor. Indeed D is a divisor: comparing with (1.1), D is defined by the discriminant determinant

$$f := \det(A_1 x^t \cdots A_n x^t)$$

where $A_1, \ldots, A_n$ is a basis of the Lie algebra $\mathfrak{g}$ of G. As the entries of the defining polynomial are linear, f is a homogeneous polynomial of degree n. Thus, if f is not reduced, D can not be linear free. On the other hand, Lemma 1.1.18 shows the following:

Lemma 1.2.11. ([25], Lemma 2.4) *Let the n-dimensional algebraic group G acts linearly on $\mathbb{C}^n$ with an open orbit. If f is reduced, then D is a linear free divisor.*

Definition 1.2.12. *We call a linear free divisor $D = V(f) \subset \mathbb{C}^n$* special *if* $\mathrm{Aut}(f) \subset \mathrm{SL}_n(\mathbb{C})$. *This means that if $\chi \in \mathrm{Ann}(D) = \{\delta \in \mathrm{Der}(-\log D) \mid \delta(f) = 0\}$ then* $\mathrm{trace}(\chi) = 0$.

Lemma 1.2.13. ([15], Corollary 2.9) *Let $D = V(f) \subset \mathbb{C}^n$ be a reductive linear free divisor. Then D is special.*

Example 1.2.14. *i) The normal crossing divisor of Example 1.1.20 is a reductive linear free divisor because $\mathfrak{g}_D = \mathbb{C}^n$.*

ii) Consider the divisor $D = V((y^2 + xz)z) \subset \mathbb{C}^3$. This is a linear free divisor because we can take the matrix

$$A = \begin{bmatrix} x & 4x & -2y \\ y & y & z \\ z & -2z & 0 \end{bmatrix}$$

as its Saito matrix. Moreover, if we consider σ the vector field represented by the second column of A, i.e. $\sigma = 4x\partial/\partial x + y\partial/\partial y - 2z\partial/\partial z$, we have that $\sigma \in \mathrm{Ann}(D)$ and $\mathrm{trace}(\sigma) = 3$ *and hence by Lemma 1.2.13, D is a non-reductive linear free divisor.*

The following is an example of a series of locally quasi-homogeneous non-reductive linear free divisors. See [25] for more details.

Example 1.2.15. ([25], Example 5.1) *For $n \geq 2$, consider the non-reductive group B_n of $n \times n$ invertible upper triangular matrices. It acts on the space $\mathrm{Sym}_n(\mathbb{C})$ of symmetric $n \times n$ matrices by transpose conjugation*

$$B \cdot S = B^t S B$$

where $B \in B_n$ and $S \in \mathrm{Sym}_n(\mathbb{C})$. The dimensions of B_n and $\mathrm{Sym}_n(\mathbb{C})$ are both equal to $(n+1)n/2$. Moreover, the discriminant determinant f is reduced and defines a linear free divisor $D = V(f) \subset \mathrm{Sym}_n(\mathbb{C})$ and $\mathrm{Der}(-\log D)_0$ can be identified with the Lie algebra of B_n. Let S be a $n \times n$ symmetric matrix of indeterminates and S_j be the $j \times j$ matrix obtained by deleting the last $n - j$ rows and columns of S, then $f = \prod_{j=1}^{n} \det(S_j)$.

1.3 Linear free divisors and prehomogeneous vector spaces

In this section, we recall the notion of prehomogeneous vector spaces, as described in [37], and describe the connection between them and linear free divisors. Prehomogeneous vector spaces have been partially classified in [49], [36], [38], [40] and [39]. In the last part of the section, we describe which linear free divisors appear in these classifications.

Basic notions and classification of irreducible linear free divisors

Definition 1.3.1. *Let G be a connected linear algebraic group and ρ a rational representation of G, i.e. it is a rational map of algebraic varieties, on a finite dimensional vector space V, all defined over $\mathbb{C}$. We call such a triplet (G, ρ, V) a* prehomogeneous vector space *if V has a Zariski dense G-orbit.*

Definition 1.3.2. *Let G be a connected linear algebraic group. A rational homomorphism $\chi \colon G \longrightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is called a* rational character *of G. The group of all rational character of G will be denoted by $X(G)$.*

Definition 1.3.3. *Rational characters $\chi_1, \dots, \chi_l$ are called* multiplicatively independent *if they generate a free abelian group of rank l in $X(G)$.*

Definition 1.3.4. *Let (G, ρ, V) be a triplet where ρ is not necessarily irreducible. A non-constant function $f \in \mathcal{O}_V$ is called a* semi-invariant *or* relative invariant *of* *the triplet (G, ρ, V), if there is a rational character $\chi \in X(G)$ satisfying $f(\rho(g)x) = \chi(g)f(x)$ for any $g \in G$ and $x \in V$. The ring spanned by the semi-invariants of the triplet (G, ρ, V) is denoted by $SI(G, V)$.*

Proposition 1.3.5. ([49], §4 Proposition 3) *A semi-invariant is, up to a constant multiple, uniquely determined by its corresponding character. In particular, any semi-invariant is a homogeneous function.*

Definition 1.3.6. *Let $\chi \in X(G)$ then the set of semi-invariants with character χ is denoted by $SI(G, V)_\chi$.*

It is possible to describe easily the space of semi-invariants in the case that the action of G has an open orbit.

Theorem 1.3.7. ([49]) *If the action of G on the vector space V has an open orbit, then the ring $SI(G, V)$ is a polynomial ring*

$$SI(G, V) = \mathbb{C}[f_1, \ldots, f_k]$$

for some collection of algebraically independent and irreducible semi-invariants $f_1, \ldots, f_k$. Moreover, if $f_i \in SI(G, V)_{\chi_i}$, then $\chi_1, \ldots, \chi_k$ are independent in the space of characters $X(G)$.

Corollary 1.3.8. *Under the assumptions of Theorem 1.3.7, the set of characters χ such that $SI(G, V)_\chi \neq 0$ forms a free abelian semigroup isomorphic to $\mathbb{N}^k$. In particular, if f is any semi-invariant with character χ, then $f = uf_1^{a_1} \cdots f_k^{a_k}$, where u is a unit in $\mathbb{C}$ and the $a_i \geq 0$ are the unique integers such that $\chi = \sum_{i=1}^k a_i \chi_i$ in the space $X(G)$.*

If we consider $D = V(f) \subset \mathbb{C}^n$ a linear free divisor and $\rho\colon G_D^\circ \longrightarrow \mathrm{GL}_n(\mathbb{C})$ the inclusion map, then $(G_D^\circ, \rho, \mathbb{C}^n)$ is a prehomogeneous vector space and the equation f is a semi-invariant associated to a non-trivial character. Moreover, if h is a non-zero semi-invariant and $x \notin D$, then $h(x)$ cannot vanish. If it did, then it would vanish everywhere on the orbit of x, which is dense. In other words, the zero locus of any semi-invariant must be contained in D.

Putting together these facts and Theorem 1.3.7, we have the following:

Proposition 1.3.9. *Let $D \subset \mathbb{C}^n$ be a linear free divisor. Then $SI(G_D^\circ, \mathbb{C}^n) = \mathbb{C}[f_1, \ldots, f_k]$, for some collection of algebraically independent and irreducible semi-invariants $f_1, \ldots, f_k$, and $f = f_1 \cdots f_k$ is a reduced equation for D.*

Definition 1.3.10. *Let $D = V(f) \subset \mathbb{C}^n$ be a linear free divisor and let $\chi \in X(G_D^\circ)$ be the non-trivial character associated to f. We define $H = \mathrm{GL}_n(\mathbb{C})_f^\circ \subset G_D^\circ$, where $\mathrm{GL}_n(\mathbb{C})_f = \ker \chi$ is the isotropy group of f.*

Definition 1.3.11. *Let $D \subset \mathbb{C}^n$ be a linear free divisor. We call D* semisimple *if H is semisimple. We call D* abelian *if H is abelian.*

Proposition 1.3.12. ([26], Lemma 2.6) *For a reductive linear free divisor $D \subset \mathbb{C}^n$, the number of irreducible components of D equals the dimension of the center of G_D°. In particular, D is irreducible if and only if H is semisimple.*

We now recall some basic concepts from theory of prehomogeneous vector spaces.

Definition 1.3.13. *Two triplets (G, ρ, V) and (G', ρ', V') are called* equivalent *if there exist a rational isomorphism $\sigma\colon \rho(G) \longrightarrow \rho'(G')$ and an isomorphism $\tau\colon V \longrightarrow V'$, both defined over $\mathbb{C}$, such that the following diagram is commutative for all $g \in G$*

$$\begin{array}{ccc} V & \xrightarrow{\tau} & V' \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \sigma\rho(g)} \\ V & \xrightarrow{\tau} & V' \end{array}$$

This equivalence relation is denoted by $(G, \rho, V) \cong (G', \rho', V')$.

Proposition 1.3.14. ([37], Proposition 7.40) *Let G be a reductive algebraic group. Then a triplet (G, ρ, V) is equivalent to its dual (G, ρ^*, V^*), where ρ^* is the contragredient representation of ρ on the dual vector space V^* of V.*

Definition 1.3.15. *Let $(\mathrm{SL}_n(\mathbb{C}), \Lambda_1, V(n))$ denote the standard n-dimensional representation of $\mathrm{SL}_n(\mathbb{C})$. We say that two triplets (G, ρ, V) and (G', ρ', V') are* castling transforms *of each other, when there exist a triplet $(\tilde{G}, \tilde{\rho}, V(m))$ and a positive number n with $m > n \geq 1$ such that*

$$(G, \rho, V) \cong (\tilde{G} \times SL_n(\mathbb{C}), \tilde{\rho} \otimes \Lambda_1, V(m) \otimes V(n))$$

and

$$(G', \rho', V') \cong (\tilde{G} \times SL_{m-n}(\mathbb{C}), \tilde{\rho}^* \otimes \Lambda_1, V(m)^* \otimes V(m-n)),$$

where $\tilde{\rho}^*$ is the contragredient representation of $\tilde{\rho}$ on the dual vector space $V(m)^*$ of $V(m)$.

Definition 1.3.16. *A triplet (G, ρ, V) is called* reduced *if there is no castling transform (G', ρ', V') of (G, ρ, V) with $\dim V' < \dim V$.*

Definition 1.3.17. *We say that two triplets* (G, ρ, V) *and* (G', ρ', V') *belong to the same* castling class *when one is obtained from the other by a finite number of castling transforms and in this case we write* $(G, \rho, V) \sim (G', \rho', V')$.

Proposition 1.3.18. ([49], §2 Proposition 12) *Each castling class contains one and, up to equivalence relation, only one reduced triplet.*

In the case that two triplets are in the same castling class there are some interesting consequences. In particular:

Proposition 1.3.19. ([26], Proposition 2.10) *Suppose that* $(G, \rho, V) \sim (G', \rho', V')$. *Then the following statements hold true*

1. *The generic isotropy subgroups of* (G, ρ, V) *and* (G', ρ', V') *are isomorphic;*
2. $\dim G - \dim V = \dim G' - \dim V'$;
3. *If* (G, ρ, V) *is a prehomogeneous vector space then so is* (G', ρ', V');
4. *If* (G, ρ, V) *is a prehomogeneous vector space and the complement of the open orbit is a linear free divisor with group* G*, then the same goes for* (G', ρ', V')*, mutatis mutandis. The number of irreducible components of these divisors are the same.*

Remark 1.3.20. ([36]) *Let* (G, ρ, V) *be a triplet with* G *reductive and let* $l \geq 1$ *be an integer. Then we may assume that* $G = \mathrm{GL}_1(\mathbb{C})^t \times G_1 \times \cdots \times G_k$ *with* $t \leq l$*,* ρ *is the composition of scalar multiplications of* $\mathrm{GL}_1(\mathbb{C})^t$ *on* $V = V_1 \oplus \cdots \oplus V_l$ *and* $\rho_1 \oplus \cdots \oplus \rho_l$*, where each* G_i *is a simple algebraic group and* ρ_j *is an irreducible representation of* $G_1 \times \cdots \times G_k$ *on* V_j.

In [49], M. Sato and T. Kimura classified irreducible prehomogeneous vector spaces up to castling transformations. From Proposition 1.3.12, we know that irreducible reductive linear free divisors live in irreducible representations of their group. Thus, up to castling transformations, every irreducible reductive linear free divisor appears as the complement of the orbit in one of the irreducible prehomogeneous vector spaces classified by Sato and Kimura. Hence, we have:

Theorem 1.3.21. ([26], Theorem 2.11) *Up to castling transformations, there are only four irreducible reductive linear free divisors*

1. $D = \{0\} \subset \mathbb{C}$ *with* $H = \{e\}$;
2. $D = V(y^2z^2 - 4xz^3 - 4y^3w = 18xyzw - 27w^2x^2) \subset \mathbb{C}^4$ *with* $G = \mathrm{GL}_2(\mathbb{C})$ *and* $H = \mathrm{SL}_2(\mathbb{C})$;

3. $D \subset \mathbb{C}^{12}$ *with* $G = \mathrm{SL}_3(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ *and* $H = \mathrm{SL}_3(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$*;*

4. $D \subset \mathbb{C}^{40}$ *with* $G = \mathrm{SL}_5(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C})$ *and* $H = \mathrm{SL}_5(\mathbb{C}) \times \mathrm{SL}_4(\mathbb{C})$*.*

Theorem 1.3.22. ([26], Theorem 2.12) *The normal crossing divisor is the only abelian linear free divisor.*

New classifications

In [36], T. Kimura classified all the prehomogeneous vector spaces with $k = 1$ up to castling transformations. Thus, up to castling transformations, every linear free divisor with such a group appears as the complement of the open orbit in one of the prehomogeneous vector spaces classified in the article. Hence we obtain:

Theorem 1.3.23. *Up to castling transformations, there are just two linear free divisors with* $G_D^\circ = \mathrm{GL}_1^t \times G$

1. $D = V((ae + bf)(ce + df)(ad - bc)) \subset \mathbb{C}^6$ *with* $G_D^\circ = \mathrm{GL}_1(\mathbb{C})^3 \times \mathrm{SL}_2(\mathbb{C})$*;*

2. $D = V((ac - b^2)(cd^2 - 2bde + ae^2)) \subset \mathbb{C}^5$ *with* $G_D^\circ = \mathrm{GL}_1(\mathbb{C})^2 \times \mathrm{SL}_2(\mathbb{C})$*;*

Definition 1.3.24. *A prehomogeneous vector space* (G, ρ, V) *is called* 2-simple *when*

1. $G = \mathrm{GL}_1(\mathbb{C})^l \times G_1 \times G_2$*, with simple algebraic groups* G_1 *and* G_2*;*

2. ρ *is the composition of a rational representation* ρ' *of* $G_1 \times G_2$ *of the form* $\rho' = \rho_1 \otimes \rho'_1 + \cdots + \rho_k \otimes \rho'_k + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t)$ *with* $k+s+t = l$*, where* ρ_i, σ_i *(resp.* ρ'_j, τ_j*) are non-trivial irreducible representations of* G_1 *(resp.* G_2*) and the scalar multiplications* $\mathrm{GL}_1(\mathbb{C})^l$ *on each irreducible component* V_i*, where* $V = V_1 \oplus \cdots \oplus V_l$*.*

Definition 1.3.25. *We say that a 2-simple prehomogeneous vector space* (G, ρ, V) *is of* type I *if* $k \geq 1$ *and at least one of* $(\mathrm{GL}_1(\mathbb{C}) \times G_1 \times G_2, \rho_i \otimes \rho'_i)$ *for* $i = 1. \ldots, k$*, is a non-trivial prehomogeneous vector space.*

In [38], T. Kimura and others classified all 2-simple prehomogeneous vector spaces of type I up to castling transformations and in [40] and [39] they listed all the semi-invariants of the prehomogeneous vector space classified. Thus, up to castling transformations, every linear free divisor with such a group appears as the complement of the open orbit in one of the prehomogeneous vector spaces classified in the article. Hence we obtain:

Theorem 1.3.26. *Up to castling transformations, there are four linear free divisors with such a* G_D°

1. $D = V((b^2c^2-2abcd+a^2d^2+b^2e^2+d^2e^2-2abef-2cdef+a^2f^2+c^2f^2)(-b^2g^2-d^2g^2-f^2g^2+2abgh+2cdgh+2efgh-a^2h^2-c^2h^2-e^2h^2)) \subset \mathbb{C}^8$ *with* $G_D^\circ = \mathrm{GL}_1(\mathbb{C})^2 \times \mathrm{SO}_3 \times \mathrm{SL}_2(\mathbb{C})$*;*

2. $D = V((bc-ad)(hi-gj)(deg-bfg-ceh+afh)(dei-bfi-cej+afj)) \subset \mathbb{C}^{10}$ *with* $G_D^\circ = \mathrm{GL}_1(\mathbb{C})^3 \times \mathrm{Sp}_1 \times \mathrm{GL}_2(\mathbb{C})$*;*

3. $D = V((ad-bc)(gi-h^2)(-d^2e^2g+2bdefg-b^2f^2g+2cde^2h-2bcefh-2adefh-c^2e^2i+2abf^2h+2acefi-a^2f^2i)) \subset \mathbb{C}^9$ *with* $G_D^\circ = \mathrm{GL}_1(\mathbb{C})^2 \times \mathrm{Sp}_1 \times \mathrm{GL}_2(\mathbb{C})$*;*

4. $D \subset \mathbb{C}^{15}$ *with* $G_D^\circ = \mathrm{GL}_1(\mathbb{C})^4 \times \mathrm{Sp}_1 \times \mathrm{SL}_3(\mathbb{C})$.

1.4 Structure of logarithmic vector fields

In this section we describe the structure of the module of logarithmic vector fields for a linear free divisor, see [28] and [25], and we give a bound on the number of semisimple vector fields in a basis of such module.

Let $D \subset \mathbb{C}^n$ be a linear free divisor defined by the homogeneous polynomial $f = \det((\delta_i(x_j))_{i,j}) \in \mathbb{C}[x_1, \dots, x_n]$ of degree n, where $\delta_1, \dots, \delta_n$ is a basis of weight zero vector fields of $\mathrm{Der}(-\log D)$. Then $\delta_i(f) \in \mathbb{C} \cdot f$ and there is the standard Euler vector field $\chi = \sum_{i=1}^n x_i \partial/\partial x_i \in \langle \delta_1, \dots, \delta_n \rangle_{\mathbb{C}}$.

Since $\chi(f)/f = n \neq 0$, we can assume that $\delta_1 = \chi$ and $\delta_i(f) = 0$ for $i = 2, \dots, n$. So $\delta_2, \dots, \delta_n$ is a degree zero basis of the annihilator $\mathrm{Ann}(D)$ of D. Since χ vanishes only at the origin, the origin of the affine coordinate system $x_1, \dots, x_n$ is uniquely determined. A coordinate change between two degree zero bases of $\mathrm{Der}(-\log D)$ can always be chosen linear. Among all possible linear coordinate changes, let $s+1$ be the maximal number of linearly independent diagonal weight zero logarithmic vector fields.

For $\delta \in \mathrm{Der}_{\mathbb{C}^n}$ a weight zero vector field, we write δ_S for its semisimple part and δ_N for its nilpotent part. Then we have the following:

Theorem 1.4.1. ([25], Theorem 6.1) *Let* $D = V(f) \subset \mathbb{C}^n$ *be a linear free divisor. Then there exists a global degree zero basis* $\chi, \sigma_1, \dots, \sigma_s, \nu_1, \dots, \nu_{n-s-1}$ *such that*

1. $[\chi, \sigma_i] = 0$ *and* $[\chi, \nu_j] = 0$ *for all* $i = 1, \dots, s$ *and* $j = 1, \dots, n-s-1$*;*

2. *the* σ_i *are simultaneously diagonalizable with eigenvalues in* $\mathbb{Q}$ *and* $\sigma_i(f) = 0$*;*

3. *the* ν_j *are nilpotent and* $\nu_j(f) = 0$*;*

4. $[\sigma_i, \nu_j] \in \mathbb{Q} \cdot \nu_j$ *and* $\sum_j [\sigma_i, \nu_j]/\nu_j + \mathrm{trace}(\sigma_i) = 0$*;*

5. *if* $\delta \in \Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))$ *is a weight zero vector field such that* $[\chi, \delta] = 0$ *and* $[\sigma_i, \delta] = 0$ *for* $i = 1, \ldots, s$*, then* $\delta_S \in \langle \chi, \sigma_1, \ldots, \sigma_s \rangle_{\mathbb{C}}$.

Moreover, $s \geq 1$ *and if* $s = n - 1$ *then* $f = x_1 \cdots x_n$ *defines a normal crossing divisor.*

In Theorem 1.4.1, one can perform the Gauss algorithm on the diagonals $\sigma_1, \ldots, \sigma_s$. Then $\sigma_i \equiv x_i \partial/\partial x_i \mod \sum_{j=s+1}^{n} \mathbb{C} \cdot x_j \partial/\partial x_j$.

The following Lemma is useful in many examples:

Lemma 1.4.2. ([25], Lemma 6.3) *Let* $\sigma = \sum_{i=1}^{n} w_i x_i \partial/\partial x_i$*. Then* $x_i \partial/\partial x_j$ *is an eigenvector of* ad_σ *for the eigenvalue* $w_i - w_j$.

We now improve this description of $\mathrm{Der}(-\log D)$, putting a lower bound on s and noticing that each logarithmic vector field that is nilpotent annihilates each irreducible component of D.

Proposition 1.4.3. *Let* $D \subset \mathbb{C}^n$ *be a linear free divisor, let* $f = \prod_{i=1}^{k} f_i$ *be a reduced defining equation for* D *written as a product of irreducible polynomials and let* $s+1$ *be the number of semisimple vector fields in the basis of* $\mathrm{Der}(-\log D)$ *given by Theorem 1.4.1. Then* $s + 1 \geq k$.

Proof. We can suppose that $f_1, \ldots, f_k$ are the semi-invariant polynomials given by Proposition 1.3.9 and let $\chi_1, \ldots, \chi_k$ be the corresponding independent characters.

We want to show that there exist $\sigma_1, \ldots, \sigma_k \in \mathrm{Der}(-\log D)$ such that $df_j(\sigma_i) = \delta_{ij} f_j$. This will conclude the proof because if they are not semisimple, we can take their semisimple part and if they exist, they are automatically linearly independent.

Because $\chi_1, \ldots, \chi_k$ are independent, so are $d_e\chi_1, \ldots, d_e\chi_k$ as elements of the dual space of $\mathfrak{g}_D$. Hence there exist $\nu_1, \ldots, \nu_k \in \mathfrak{g}_D$ such that $d_e\chi_i(\nu_j) = \delta_{ij}$. Let now $\sigma_1, \ldots, \sigma_k$ be the corresponding vector fields on $\mathbb{C}^n$. Because f_i is a semi-invariant with character χ_i, we have that $f_i(gx) = \chi_i(g) f_i(x)$ for all $x \in \mathbb{C}^n$ and $g \in G_D^\circ$. Differentiating the previous expression with respect to g, we have that $d_x f_i(\sigma_j) = d_e\chi_i(\nu_j) f_i = \delta_{ij} f_i$. □

Notice that if in the previous Proposition we consider the case $k = 1$, then the inequality is strict because we always have $s \geq 1$, as proved in Theorem 1.4.1.

Example 1.4.4. 1. *Consider the linear free divisor* $D = V((yz + xw)zw) \subset \mathbb{C}^4$ *with Saito matrix*

$$\begin{bmatrix} x & 3x & x & z \\ y & 0 & 2y & -w \\ z & z & -z & 0 \\ w & -2w & 0 & 0 \end{bmatrix}$$

This linear free divisor has 3 *components and* 3 *semisimple vector fields in the basis from Theorem 1.4.1.*

2. *Consider the linear free divisor* $D = V(y^2z^2 - 4xz^3 - 4y^3w + 18xyzw - 27x^2w^2) \subset \mathbb{C}^4$ *with Saito matrix*

$$\begin{bmatrix} x & 0 & x & y \\ y & 3x & y & 2z \\ z & 2y & -z & 3w \\ w & z & -3w & 0 \end{bmatrix}$$

This linear free divisor has just 1 *component but* 2 *semisimple vector fields in the basis from Theorem 1.4.1.*

Proposition 1.4.5. ([6], Proposition 11.8) *Let* G *be a connected algebraic group with Lie algebra* $\mathfrak{g}$ *and* $\nu \in \mathfrak{g}$*. Then* ν *is semisimple if and only if it is tangent to a torus in* G*.*

Proposition 1.4.6. *Let* $D \subset \mathbb{C}^n$ *be a linear free divisor, let* $f = \prod_{i=1}^{k} f_i$ *be a reduced defining equation for* D *written as a product of irreducible polynomials and let* $\nu \in \mathrm{Der}(-\log D)$ *be a nilpotent vector field. Then* $\nu \in \mathrm{Ann}(V(f_i))$ *for all* $i = 1, \ldots, k$*.*

Proof. To prove the statement it is enough to show that for every $g \in SI(G_D^\circ, \mathbb{C}^n)$ we have $dg(\nu) = 0$.

Consider $g \in SI(G_D^\circ, \mathbb{C}^n)$ with character χ and let $v \in \mathfrak{g}_D$ be the corresponding nilpotent element to ν. Then we have that $dg(\nu) = d_e\chi(v)g$, so it is enough to prove that $d_e\chi(v) = 0$.

We can assume that the character $\chi\colon G_D^\circ \longrightarrow \mathbb{C}^*$ is non-trivial, then χ induces an isomorphism of a one dimensional quotient of G_D°, a torus, onto the image. The corresponding one dimensional quotient of the Lie algebra $\mathfrak{g}_D$ is then isomorphic to the Lie algebra of the 1-torus and thus consists of semisimple elements by Proposition 1.4.5. In particular, all nilpotent elements of $\mathfrak{g}_D$ must lie in the kernel of $d_e\chi$. □

Remark 1.4.7. *The conclusion of Proposition 1.4.6 does not hold if* ν *is semisimple, also if* $\nu \in \mathrm{Ann}(D)$*.*

Proof. Consider the linear free divisor $D = V(f) = V((yz + xw)zw) \subset \mathbb{C}^4$ and $\sigma = x\partial/\partial x + 2y\partial/\partial y - z\partial/\partial z \in \mathrm{Der}(-\log D)$. Then $\sigma(f) = 0$ but $\sigma(yz + xw) = yz + xw \neq 0$ and hence $\sigma \notin \mathrm{Ann}(yz + xw)$. □

1.5 Ring and rank conditions for free divisors

In this section we consider R a n-dimensional local Cohen-Macaulay ring with maximal ideal $\mathfrak{m}$ and A a $n \times n$ matrix over R with transpose $\Lambda := A^t$. We assume that $f := \det(A)$ is a reduced non-zero divisor in R and set $D := V(f)$. Notice that by Cramer's rule f annihilates $M := \mathrm{coker}(A)$ which is hence a module over $R_D := R/(f)R$.

The general theory

Definition 1.5.1. *The k-th* Fitting ideal *of M is the ideal of R generated by the $(n-k) \times (n-k)$ minors of A and is denoted by $F^k(M)$.*

Notice that $F^k(M)$ is an invariant of M and is independent of the presentation A. For more properties of Fitting ideals see Chapter 20 of [20].

We denote by m^i_j the generator of $F^1(M)$ obtained from A by deleting row i and column j and by $g_1, \ldots, g_n$ the images in M of the standard basis of R^n.

Remark 1.5.2. *By Cramer's rule $m^i_j g_k = m^k_j g_i$, for all $i, j, k = 1, \ldots, n$.*

Definition 1.5.3. *We say that the* ring condition (RC) *holds for M (or A) if $R/F^1(M)$ is Cohen-Macaulay of codimension 2 and the map $End_R(F^1(M)R_D) \hookrightarrow Hom_R(F^1(M)R_D, R_D)$ is surjective.*

Notice that the previous property is a property of the pair $(F^0(M), F^1(M))$ and hence of M.

Definition 1.5.4. *We say that the* rank condition (rc) *holds for A if, possibly after left multiplication by some invertible matrix over R, the ideal $F^1(M)$ is generated by the maximal minors of the matrix obtained from A by deleting one of its rows and* $\mathrm{grade}(F^1(M)) \geq 2$.

Remark 1.5.5. *By the Hilbert-Burch Theorem (see [20], Theorem 20.15), (rc) implies that $F^1(M)R_D$ is a maximal Cohen-Macaulay R_D-module.*

The next few Propositions and Lemmas appeared in a draft version of [27].

Proposition 1.5.6. *(rc) is a property of M.*

Proof. Let B be an invertible $n \times n$ matrix with entries in R. Then

$$(\mathrm{ad}(AB))^t = (\mathrm{ad}(A))^t (\mathrm{ad}(B))^t,$$

so if all the entries in $(\mathrm{ad}(A))^t$ are linear combinations of the entries in the last row then the same is true for $(\mathrm{ad}(AB))^t$. Thus if (rc) holds for A, then it holds also for AB.

By linear algebra on $R/\mathfrak{m}$, left and right multiplication by invertible matrices brings A to the form

$$\begin{bmatrix} A_0 & 0 \\ 0 & I_r \end{bmatrix}$$

where I_r is the $r \times r$ identity matrix and all entries of A_0 lie in $\mathfrak{m}$. Evidently (rc) holds for A if and only if holds for A_0. So given a second square presentation matrix A' of M, we may assume that both A and A' have entries in $\mathfrak{m}$ and hence are minimal presentations. Then by Nakayama's Lemma, A and A' have the same size and there are invertible $n \times n$ matrices B and C such that $A' = BAC$. It follows that (rc) holds for A if and only if holds for A'. □

Definition 1.5.7. *Let S be a ring. A* fractional ideal *of S is a finitely generated S-submodule of the ring of fraction $Q(S)$ which contains a non-zero divisor.*

We now prove that if A satisfies (rc), then M is a ring, i.e. it has a ring structure with respect to which R_D, embedded via $r \mapsto r \cdot 1_M$, is a subring.

Lemma 1.5.8. *An element $r \in R$ is zero in R_D if $r \in fR_{\mathfrak{p}}$ for all minimal primes $\mathfrak{p}$ over f. In particular, r is non-zero divisor if $r \notin \mathfrak{p}R_{\mathfrak{p}}$ for all such primes.*

Proof. With $\mathrm{Spec}(R)$ also D is Cohen-Macaulay and hence R_D is unmixed, that is, all associated primes of R_D are minimal. □

Lemma 1.5.9. *If D is reduced then $F^1(M)R_D$ contains a non-zero divisor. In case M is a ring and $g_n = 1_M$, some linear combination $U_n = \sum_j u_j m_j^n$ is a non-zero divisor in R_D.*

Proof. Let $\mathfrak{p}$ be a prime divisor of D. By assumption, $R_{\mathfrak{p}}$ is smooth and D is reduced. So by Cohen's structure Theorem (see [20], Theorem 7.7), the completion of $R_{\mathfrak{p}}$ is a formal power series ring $\widehat{R_{\mathfrak{p}}} = K[[t]]$, with $K = R/\mathfrak{m}$ and $t = f$. Considering A as a map from $\mathrm{Spec}(R)$ to a matrix space $\mathrm{Spec}(K[X_j^i])$, we can write $f = \det \circ A$, from which we obtain by the chain rule

$$1 = \frac{\partial f}{\partial t} = \sum_{i,j} \frac{\partial \det}{\partial X_j^i} \circ A \cdot \frac{\partial A_j^i}{\partial t} = \sum_{i,j} m_j^i \frac{\partial A_j^i}{\partial t}$$

So there is an m_j^i which is not in $tK[[t]] = \widehat{fR_{\mathfrak{p}}}$ and hence not in $fR_{\mathfrak{p}}$. Then the claim follows from Lemma 1.5.8 by taking suitable linear combination of the m_j^i. If

M is a ring and $g_n = 1_M$, the latter can be written using Remark 1.5.2 as

$$\sum_{i,j} u_{i,j} m^i_j = \sum_{i,j} u_{i,j} g_i m^n_j = \sum_j u_j m^n_j = U_n,$$

where $u_j = \sum_i u_{i,j} g_i$. $\square$

Lemma 1.5.10. *If there exist $u_1, \dots, u_n \in R_D$ such that for some $k \in \{1, \dots, n\}$, $U_k = \sum_j u_j m^k_j$ is a non-zero divisor in R_D, then $g_1, \dots, g_n$ have the same relations over R_D as $U_1, \dots, U_n$ and as the columns $(m^k_1, \dots, m^k_n)$ of $\mathrm{ad}(A)$. In case M is a ring and $g_n = 1_M$, U_n is a non-zero divisor in R_D.*

Proof. The first claim follows from the exactness of the 2-periodic sequence

$$\cdots \xrightarrow{\mathrm{ad}(A)} R^n_D \xrightarrow{A} R^n_D \xrightarrow{\mathrm{ad}(A)} R^n_D \xrightarrow{A} \cdots \tag{1.2}$$

Generically on D, A has rank $n-1$ by Lemma 1.5.8 and 1.5.9 and hence $\mathrm{ad}(A)$ has rank 1. It follows, using Lemma 1.5.8, that $\sum_k \alpha_k U_k = 0$ in R_D is equivalent to $\sum_k \alpha_k m^k_j = 0$ for all j. By the exactness of the 2-periodic sequence (1.2), this is equivalent to $(\alpha_1, \dots, \alpha_n)^t$ being the column space of A and thus $\sum_k \alpha_k g_k = 0$. This proves the first claim.

Now, if M is a ring and $g_n = 1_M$, then by Remark 1.5.2, $U_k = g_k U_n$ and the second claim follows. $\square$

Proposition 1.5.11. *If D is reduced, then M is isomorphic to a fractional ideal between R_D and $Q(R_D)$.*

Proof. By left and right multiplication of A by invertible matrices, i.e. choosing a new set of generators of M and a new set of generators for the relations among these generators, one can arrange that $m^n_n \notin \mathfrak{p}_k$, for any k. By Lemma 1.5.8, this element is then a non-zero divisor in $F^1(M)R_D$. To see this, fix k and j and consider the set

$$\{(\alpha_1, \dots, \alpha_n) \in K^n \mid \sum_i \alpha_i m^i_j \in \mathfrak{p}_k\}.$$

By Lemma 1.5.9, this set is algebraic and not equal to K^n for some $j(k)$. As $K = R/\mathfrak{m}$ is infinite, there exists $(\alpha_1, \dots, \alpha_n) \in K^n$ such that for each k, there is a $j(k)$ such that

$$\sum_i \alpha_i m^i_{j(k)} \notin \mathfrak{p}_k.$$

Left multiply $\mathrm{ad}(\Lambda)$ by some $P \in \mathrm{GL}_n(K)$ with last row $(\alpha_1, \dots, \alpha_n)$, this corresponds to left multiplying A by a unit times P^{-1}. Now $m^n_{j(k)} \notin \mathfrak{p}_k R_{\mathfrak{p}_k}$ for any k.

The sets

$$\{(\beta^1, \dots, \beta^n) \in K^n \mid \sum_j \beta^j m_j^n \in \mathfrak{p}_k\}$$

depending on k are then once again algebraic and not equal to K^n. Thus there exists $(\beta^1, \dots, \beta^n)$ such that $\sum_j \beta^j m_j^n \notin \mathfrak{p}_k$ for all k. As before, right multiplying $\mathrm{ad}(\Lambda)$ by $Q \in \mathrm{GL}_n(K)$ with last column $(\beta^1, \dots, \beta^n)$. Now m_n^n is a non-zero divisor in $F^1(M)R_D$.

By Lemma 1.5.10, M embeds into $Q(R_D)$ by sending g_k to m_n^k/m_n^n for $k = 1, \dots, n$. Since g_n is sent to 1_{R_D}, the image of M contains a non-zero divisor and thus M is a fractional ideal as claimed. □

Evidently, many different embeddings of M into $Q(R_D)$ are possible and, in case M has a multiplicative structure making it into a ring, there is no reason why the embedding $M \hookrightarrow Q(R_D)$ in Proposition 1.5.11 should be a multiplicative homomorphism. Nevertheless if A satisfies (rc), there is an embedding which achieves just this. We shall need the following:

Lemma 1.5.12. ([17], Proposition 1.10) *Let A' be the matrix obtained from A by deleting its last row and let $I \subset R$ be the ideal generated by the maximal minors of A'. If I has codimension 2, then the ideal $I_D := IR_D$ has free resolution of the form*

$$0 \longrightarrow R^n \xrightarrow{\Lambda} R^n \xrightarrow{(m_1^n, \dots, m_n^n)} I_D \longrightarrow 0\,.$$

Theorem 1.5.13. (cf. [27], Theorem 3.3) *(rc) implies (RC) and that M is a ring isomorphic to the ring $End_R(F^1(M)R_D)$, which is canonically embedded between R_D and $Q(R_D)$. Moreover, M is generated over R_D by $\psi_1, \dots, \psi_n \in Q(R_D)$ where*

$$\psi_i m_j^n = m_j^i$$

for all $i, j = 1, \dots, n$.

Proof. Suppose that $F^1(M)$ is generated by the maximal minors of A', where A' is the matrix obtained from A by deleting its last row, then Lemma 1.5.12 yields a presentation

$$0 \longrightarrow R^n \xrightarrow{\Lambda} R^n \xrightarrow{(m_1^n, \dots, m_n^n)} F^1(M)R_D \longrightarrow 0\,.$$

Dualising this presentation with respect to R_D gives the exact sequence

$$0 \longrightarrow Hom_R(F^1(M)R_D, R_D) \longrightarrow R_D^n \xrightarrow{A} R_D^n$$

so $Hom_R(F^1(M)R_D, R_D)$ is isomorphic to $\ker_{R_D}(A) \cong \mathrm{coker}_{R_D}(A) = M$ by exactness of the 2-periodic sequence (1.2). However $\ker_{R_D}(A) = \mathrm{image}_{R_D}(\mathrm{ad}(A))$, again by exactness of (1.2). Thus $Hom_R(F^1(M)R_D, R_D)$ is generated by the homomorphisms $\psi_1, \dots, \psi_n$, where

$$\psi_i(m_j^n) = m_j^i$$

for all $i, j = 1, \dots, n$. Clearly all homomorphisms $F^1(M)R_D \longrightarrow R_D$ map into $F^1(M)R_D$, so (RC) holds and $M \cong End_R(F^1(M)R_D)$.

Now, $End_R(F^1(M)R_D)$ is a ring with multiplicative structure given by composition. In fact, it embeds as a fractional ideal in $Q(R_D)$ containing R_D: pick a non-zero divisor $m \in F^1(M)R_D$ and map $\psi \in End_R(F^1(M)R_D)$ to $\psi(m)/m \in Q(R_D)$. This embedding is independent of choice of m: if m_1 and m_2 are both non-zero divisor in $F^1(M)R_D$, then

$$m_1\psi(m_2) = \psi(m_1 m_2) = m_2\psi(m_1).$$

Under this embedding, composition in $End_R(F^1(M)R_D)$ becomes multiplication in $Q(R_D)$. □

Theorem 1.5.14. ([44], Theorem 3.4) *If M is a ring and D is reduced, then (rc) holds.*

Combining Proposition 1.5.6 and Theorems 1.5.13 and 1.5.14, we have

Theorem 1.5.15. ([44], Theorem 3.4, Proposition 3.14) *If D is reduced, then (rc) holds for M if and only if M is a ring.*

Remark 1.5.16. *Let $I \subset R$ be an ideal such that R/I is Cohen Macaulay of codimension 2 and suppose $f \in I$ is reduced. By Hilbert-Burch Theorem, we can consider Λ' the $n \times (n-1)$ syzygy matrix of I. See [3], Theorem 5.1. Because $f \in I$, then we can add an extra column to Λ', obtaining a matrix Λ whose determinant is f.*

Definition 1.5.17. *In the notation of the Remark 1.5.16, we call the matrix Λ a* HB matrix factorisation *of $f \in I$.*

Proposition 1.5.18. *Let Λ be an HB matrix factorisation of $f \in I$ with transpose $A = \Lambda^t$. If I is reduced, then*

$$A \text{ satisfies (rc)} \iff F^1 = I \iff J(f) \subset I \iff (\langle f \rangle, I) \text{ satisfies (RC)},$$

where $J(f)$ is the Jacobian ideal of f. If these conditions hold, then $M = \mathrm{coker}(A)$ is a ring.

Proof. This comes from putting together Theorem 1.12 from [17] with the fact that $F^1_R(\mathrm{coker}(A)) = I$ if and only if $f + J(f) \subset I$ proved in Proposition 3.1 of [16]. $\square$

The case of the radical of the Jacobian

Suppose now that $D = V(f) \subset \mathbb{C}^n$ is a free divisor. By Theorem 1.1.21, the Jacobian ideal $J(D)$ is Cohen-Macaulay of codimension 2 but in general, it is not reduced. Consider then the ideal $\sqrt{J(D)}$. If it is Cohen-Macaulay of codimension 2, then by Hilbert-Burch Theorem it has a resolution

$$0 \longrightarrow \mathcal{O}^{n-1}_{\mathbb{C}^n} \xrightarrow{\Lambda'} \mathcal{O}^n_{\mathbb{C}^n} \longrightarrow \sqrt{J(D)} \longrightarrow 0\,.$$

Because $f \in J(D) \subset \sqrt{J(D)}$, we can add an extra column to Λ', obtaining a HB matrix factorisation Λ for f. We can then apply Proposition 1.5.18 and study the ring $M = \mathrm{coker}(A)$, where $A = \Lambda^t$. We will write $\tilde{D} := \mathrm{Spec}(M)$.

If we consider the normal crossing divisor $D = V(x_1 \cdots x_n) \subset \mathbb{C}^n$, then $\sqrt{J(D)} = J(D) = (x_2 \cdots x_n, \ldots, x_1 \cdots \widehat{x_i} \cdots x_n, \ldots, x_1 \cdots x_{n-1})$ and so it is Cohen-Macaulay of codimension 2. In this case, it is easy to describe $\tilde{D}$. In fact, we have the following:

Proposition 1.5.19. *Let $D \subset \mathbb{C}^n$ be the normal crossing divisor. Then $\tilde{D}$ is the normalisation of D, i.e. the disjoint union of the components of D.*

Proof. Because $\sqrt{J(D)} = J(D) = (x_2 \cdots x_n, \ldots, x_1 \cdots \widehat{x_i} \cdots x_n, \ldots, x_1 \cdots x_{n-1})$, we can take

$$\Lambda = \begin{bmatrix} x_1 & 0 & \ldots & 0 & 0 \\ -x_2 & x_2 & \ldots & 0 & 0 \\ 0 & -x_3 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & x_{n-1} & 0 \\ 0 & 0 & \ldots & -x_n & x_n \end{bmatrix}$$

Applying rows and columns operations to Λ will give isomorphic modules. Hence we can transform Λ is in the diagonal form

$$\Lambda = \begin{bmatrix} x_1 & 0 & \ldots & 0 & 0 \\ 0 & x_2 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & x_{n-1} & 0 \\ 0 & 0 & \ldots & 0 & x_n \end{bmatrix}$$

It is now obvious that $A = \Lambda^t$ is the presentation matrix of the normalisation of D. $\square$

Proposition 1.5.20. *Let $D = V(f) \subset \mathbb{C}^n$ be a free divisor and $x \in \mathrm{Reg}(D)$. Then $M_x \cong \mathcal{O}_{D,x}$.*

Proof. Because we have the exact 2-periodic sequence (1.2) with $\det(A) = f$, then M is a rank 1 maximal Cohen-Macaulay $\mathcal{O}_D$-module. Consider now $x \in \mathrm{Reg}(D)$, then M_x is a rank 1 Cohen-Macaulay module over the regular ring $\mathcal{O}_{D,x}$ and hence it is free of rank 1. This then implies that $M_x \cong \mathcal{O}_{D,x}$. $\square$

Lemma 1.5.21. ([43], Exercise 1.6) *Let S be a ring, $I, P_1, \ldots, P_r$ ideals of S such that $P_3, \ldots, P_r$ are prime and suppose that I is not contained in any of the P_i. Then there exists an element $l \in I$ not contained in any P_i.*

Proposition 1.5.22. *Let $D \subset \mathbb{C}^n$ be a free divisor. Then $\tilde{D}$ is reduced.*

Proof. By Proposition 1.5.20, M_x is reduced if $x \in \mathrm{Reg}(D)$, hence if $g \in M$ such that $g^k = 0$ for some k, then $\mathrm{Supp}(g) \subset \mathrm{Sing}(D)$ and so $g \in H^0_{\mathrm{Sing}(D)}(M)$.

By Theorem 3.8 from [32], $H^0_{\mathrm{Sing}(D)}(M) = 0$ if and only if there exists $l \in \sqrt{J(D)}$ a non-zero divisor of M. On the other hand, by Theorem 3.1 from [20], the set of zero-divisors of M is equal to $\bigcup_{P \in \mathrm{Ass}(M)} P$. Because each associated prime of M describes a component of $\tilde{D}$ and because M is Cohen-Macaulay, then $P \subsetneq \sqrt{J(D)}$ for all $P \in \mathrm{Ass}(M)$. Then by Lemma 1.5.21, there exists $l \in \sqrt{J(D)}$ a non-zero divisor of M and so $H^0_{\mathrm{Sing}(D)}(M) = 0$. As a consequence $g = 0$ and so M is reduced. $\square$

Definition 1.5.23. *A space X is* weakly normal *if every continuous function $X \longrightarrow \mathbb{C}$ which is holomorphic on the smooth part of X is in fact holomorphic on all of X.*

Notice that any smooth space and any normal space is weakly normal, see [41] for more details.

Example 1.5.24. *Consider the free divisor $D = V(f) = V(x(xz-y^2)) \subset \mathbb{C}^3$. Then $\sqrt{J(D)} = (x, y)$ and so we can take*

$$A = \begin{bmatrix} y & -x \\ xz & -xy \end{bmatrix}.$$

By Theorem 1.5.13, we know that we need to introduce a new generator, $w := \psi_1$, and that M is given by $\mathcal{O}_{\mathbb{C}^4}/I$, where $I = (f, xw - xy, yw - xz, w^2 - xz) = (x, w) \cap (xz - y^2, y - w)$. Notice that $\tilde{D}$ is weakly normal, in fact if we consider $\psi(y, z) \in$

$\mathcal{O}_{\mathbb{C}^4}/(x,w)$ *and* $\phi(x,z,w) \in \mathcal{O}_{\mathbb{C}^4}/(xz-y^2, y-w)$ *such that* $\psi(0,z) = \phi(0,z,0)$*, then we can define* $\Phi(x,y,z,w) = \psi(y,z) + \phi(x,z,w) - \phi(0,z)$*.*

Proposition 1.5.25. *Let* $D = V(f) \subset \mathbb{C}^n$ *be a linear free divisor. If* $n \leq 4$*, then* $\sqrt{J(D)}$ *is Cohen-Macaulay of codimension* 2*.*

Proof. Such linear free divisors are classified in [25]. The statement follows by a direct computation. □

Remark 1.5.26. *There exist free divisors* $D = V(f) \subset \mathbb{C}^n$ *such that* $\sqrt{J(D)}$ *is not Cohen-Macaulay.*

Proof. Consider A a 3×4 matrix of indeterminates and f the product of all the 3×3 minors of A. Then $D = V(f) \subset \mathbb{C}^{12}$ is a reductive linear free divisor, see [7]. Then $\sqrt{J(D)}$ is the intersection of 7 ideals: the ideal of the 3×3 minors of A and the 6 ideals $I_{i,j}$, where for $1 \leq i < j \leq 4$, $I_{i,j}$ is the ideal of 2×2 minors of the columns i and j of A. A computation with the computer algebra system Macaulay 2, see [29], shows that the projective dimension is 4 and not 2. □

Lemma 1.5.27. ([2], Proposition, §1) *Let* $D = V(f) \subset \mathbb{C}^n$ *be a free divisor and let* $\delta_1, \ldots, \delta_n$ *be a basis of* $\mathrm{Der}(-\log D)$ *such that* $\delta_i(f) = g_i f$ *for some* $g_i \in \mathcal{O}_{\mathbb{C}^n}$*. Then the Jacobian ideal of* D *is equal to the ideal of maximal minors of the* $n \times (n+1)$ *matrix obtained by using the column* $(g_1, \ldots, g_n)^t$ *to augment the transpose of the Saito matrix relative to* $\delta_1, \ldots, \delta_n$*.*

Proposition 1.5.28. *Let* $D \subset \mathbb{C}^n$ *be a free divisor, let* S *be a Saito matrix and let* I *be the ideal of submaximal minors of* S*. Then* $V(J(D)) = V(I)$ *and hence,* $\sqrt{J(D)} = \sqrt{I}$*.*

Proof. By Lemma 1.5.27, we have the inclusion $J(D) \subset I$ and so $V(I) \subset V(J(D))$.

On the other hand, at a point on $V(J(D))$ the rank of S is smaller that $n-1$ and so all the submaximal minors of S vanish. Hence, $V(J(D)) \subset V(I)$. □

Definition 1.5.29. *An element* $g \in \mathrm{GL}_n(\mathbb{C})$ *is a* reflection *if it has finite order and its fixed point set is a hyperplane* H_g*. We call* H_g *the* reflecting hyperplane *of* g*. A finite subgroup* $G \subset \mathrm{GL}_n(\mathbb{C})$ *is called a* reflection group *if it is generated by reflections. The set* $\mathcal{A}(G)$ *of reflecting hyperplanes of a reflection group* G *is called the* reflection arrangement *of* G*.*

Definition 1.5.30. *A* Coxeter group $G \subset \mathrm{GL}_n(\mathbb{C})$ *is a reflection group with presentation*

$$\langle r_1, \ldots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ for $i \neq j$. In case there are no relations between r_i and r_j, we make the convention that $m_{ij} = \infty$.

Definition 1.5.31. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a Coxeter group. Then the reflection arrangement $\mathcal{A}(G)$ is called a* Coxeter arrangement*.*

For more details on Coxeter groups, we refer to the book of J.E. Humphreys [34].

Example 1.5.32. *For any integer $l \geq 1$, the Coxeter arrangement A_l is defined by $V(\prod_{1\leq i<j\leq l+1}(x_i - x_j)) \subset \{\sum_{i=1}^{l+1} x_i = 0\}$.*

Example 1.5.33. *For any integer $m \geq 3$, the Coxeter arrangement $I_2(m)$ is the line arrangement consisting of the diagonals of a regular m-sided polygon centred at the origin. Here diagonal means a line bisecting the polygon, joining two vertices or the midpoints of opposite sides if m is even, or joining a vertex to the midpoint of the opposite side if m is odd. For example, if $m = 4$ we have $V(xy(x+y)(x-y)) \subset \mathbb{C}^2$.*

Remark 1.5.34. ([46]) *Let $\mathcal{A}$ be a Coxeter arrangement. Then $\mathcal{A}$ is free.*

Example 1.5.35. *Consider A_2. We can describe it as $V((x-y)(2x+y)(x+2y)) \subset \mathbb{C}^2$ with Saito matrix*

$$S_2 = \begin{bmatrix} x & 2y^2 \\ y & 2x^2 + 3xy - 3y^2 \end{bmatrix}$$

Notice that the radical of the Jacobian ideal and the ideal of the submaximal minors of S_2 coincide and they are both equal to the maximal ideal (x, y). Notice also that S_2^t satisfies (rc).

More in general, we have the following:

Proposition 1.5.36. ([27], 4.1) *Let $\mathcal{A}$ be a Coxeter arrangement and let S be a Saito matrix for $\mathcal{A}$. Then S^t satisfies (rc).*

Proposition 1.5.37. *Consider a Coxeter arrangement $\mathcal{A}$. Then $\sqrt{J(\mathcal{A})}$ is Cohen-Macaulay of codimension 2.*

Proof. Let S be a Saito matrix for $\mathcal{A}$ and let I be the ideal of submaximal minors of S. By Proposition 1.5.28, $\sqrt{J(\mathcal{A})} = \sqrt{I}$.

At a generic point of the singular locus of $\mathcal{A}$, the arrangement is of type A_2, $A_1 + A_1$ or $I_2(m)$. In the first case I is radical by Example 1.5.35. In the second case, we have a normal crossing and so I is radical. In the third case, $I = (x, y)$ because $I_2(m)$ is a central line arrangement. Hence I is radical at generic points. By Proposition 1.5.36, S satisfies (rc) and so I is Cohen-Macaulay of codimension 2. This then implies that I is a radical ideal.

Because $\sqrt{J(\mathcal{A})} = \sqrt{I} = I$, then also $\sqrt{J(\mathcal{A})}$ is Cohen-Macaulay of codimension 2. $\square$

Proposition 1.5.38. (cf. [27], Proposition 4.25) *Consider the Coxeter arrangement A_l. Then $\tilde{A}_l$ is Cohen-Macaulay, weakly normal and it is isomorphic to the union L_l of the coordinate $(l-1)$ planes $L_{i,j} := \{x_i = x_j = 0\} \subset \mathbb{C}^{l+1}$.*

Proof. By Proposition 1.5.28, Lemma 1.5.12 and the fact that the ideal of submaximal minors of a Saito matrix is radical, we can take a Saito matrix as presentation matrix for the radical of the Jacobian ideal. The result follows from Lemma 4.24 and Proposition 4.25 of [27]. $\square$

More in general, we have the following:

Proposition 1.5.39. *Consider a Coxeter arrangement $\mathcal{A}$ of ADE-type. Then $\tilde{\mathcal{A}}$ is a Cohen-Macaulay and weakly normal space.*

Proof. By Theorem 2.7 of [27], $\tilde{\mathcal{A}}$ is Cohen-Macaulay.

Consider now $x \in \operatorname{Reg}(\mathcal{A})$. Then by Proposition 1.5.20, $\tilde{\mathcal{A}}_x$ is smooth and hence weakly normal. Furthermore, because we consider only ADE-type, at a generic point p of the singular locus of $\mathcal{A}$ the arrangement is of type A_2 or $A_1 + A_1$, and in the first case $\tilde{\mathcal{A}}_p$ is weakly normal by Proposition 1.5.38 while in the second case, it is weakly normal by Proposition 1.5.19.

We need now only to check the set X of points of codimension 2 in $\mathcal{A}$. However, because $\mathcal{A}$ is Cohen-Macaulay, by Hartogs Theorem, each continuos function on $\mathcal{A}$ that is holomorphic on $\mathcal{A} \setminus X$ is actually holomorphic on all of $\mathcal{A}$, and hence $\tilde{\mathcal{A}}$ is weakly normal by definition. $\square$

The previous result is false for other types of Coxeter arrangement. In fact we have the following:

Example 1.5.40. *Consider the reflection arrangement for $I_2(4)$ given by $D = V(xy(x-y)(x+y)) \subset \mathbb{C}^2$ with the transpose of the Saito matrix equal to*

$$A = \begin{bmatrix} x & y \\ 2y^2 & 2x^2 + 3xy - 3y^2 \end{bmatrix}$$

By Theorem 1.5.13, we know that we need to introduce a new generator, $z := \psi_1$. Moreover, a direct Macaulay 2 computation shows that M is isomorphic to $\mathcal{O}_{\mathbb{C}^3}/I$, where $I = (y^2 + z, x) \cap (z, y) \cap (y^2 + 4z, x + y) \cap (y^2 - 2z, x - y)$. Notice that $\tilde{D}$ is the union of 4 smooth branches and that M is not weakly normal because the four tangent spaces at the origin to the four components of M all lie in the xy-plane.

The author does not know whether there exists a subclass of free divisors for which the radical of the Jacobian ideal is Cohen-Macaulay of codimension 2.

Chapter 2

$\mathcal{D}$-module theory for free divisors

The aim of this chapter is to recall the basic concepts of $\mathcal{D}$-module theory for free divisors. This material will have a central role in the third chapter.

We recall the notion of logarithmic connection on a $\mathcal{O}_{\mathbb{C}^n}$-module and show how the existence of such a logarithmic connection is equivalent to the existence of a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module structure. We show how to put the $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module structure induced by the adjoint representation on $\mathrm{Der}_{\mathbb{C}^n}$ and $\mathrm{Der}(-\log D)$. We introduce the logarithmic Spencer complex and the logarithmic de Rham complex and we show the relations between the two complexes. Finally, we recall the notion of Koszul free divisors and show that for them the de Rham logarithmic complex is perverse.

The material of this chapter is essentially taken from [8] and [12].

2.1 $\mathcal{V}$-filtration

In this section we introduce the notion of the $\mathcal{V}$-filtration for a free divisor $D \subset \mathbb{C}^n$ and we show the connection between this filtration and the module of logarithmic vector fields of D.

Definition 2.1.1. *We denote by $\mathcal{D}_{\mathbb{C}^n}$ the sheaf of differential operators over $\mathbb{C}^n$ defined by*

$$\mathcal{D}_{\mathbb{C}^n} := \{ \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n} p_\alpha \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n} \mid p_\alpha \in \mathcal{O}_{\mathbb{C}^n} \}$$

and by $F^\bullet$ the filtration of $\mathcal{D}_{\mathbb{C}^n}$ defined by the order of the operators, i.e.

$$F^k(\mathcal{D}_{\mathbb{C}^n}) := \{\sum_{|\alpha|\leq k} p_\alpha \frac{\partial^{\alpha_1}}{\partial x_1}\cdots\frac{\partial^{\alpha_n}}{\partial x_n} \mid p_\alpha \in \mathcal{O}_{\mathbb{C}^n}\},$$

where for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$.

Notice that by definition if $x \in \mathbb{C}^n$, then $\mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x}) \cong \mathcal{O}_{\mathbb{C}^n,x}[\xi_1, \dots, \xi_n]$ for some indeterminates $\xi_1, \dots, \xi_n$.

We denote by $\mathcal{D}_{\mathbb{C}^n}[\star D]$ the sheaf of meromorphic differential operators with poles along D and by $\Omega^\bullet_{\mathbb{C}^n}[\star D]$ the meromorphic de Rham complex with poles along D.

Definition 2.1.2. *Let $D \subset \mathbb{C}^n$ be a divisor defined by the ideal I. We define the $\mathcal{V}$-filtration relative to D on $\mathcal{D}_{\mathbb{C}^n}$ by*

$$\mathcal{V}_k^D(\mathcal{D}_{\mathbb{C}^n}) := \{P \in \mathcal{D}_{\mathbb{C}^n} \mid P(I^j) \subset I^{j-k}\ \forall j \in \mathbb{Z}\}$$

for all $k \in \mathbb{Z}$, where $I^j = \mathcal{O}_{\mathbb{C}^n}$ when j is negative. Similarly, we define

$$\mathcal{V}_k^D(\mathcal{D}_{\mathbb{C}^n,x}) := \{P \in \mathcal{D}_{\mathbb{C}^n,x} \mid P(f^j) \subset f^{j-k}\ \forall j \in \mathbb{Z}\},$$

where f is a local equation for D at x. If there is no confusion, we denote $\mathcal{V}_k^D(\mathcal{D}_{\mathbb{C}^n})$ and $\mathcal{V}_k^D(\mathcal{D}_{\mathbb{C}^n,x})$ simply by $\mathcal{V}_k(\mathcal{D}_{\mathbb{C}^n})$ and $\mathcal{V}_k(\mathcal{D}_{\mathbb{C}^n,x})$, respectively.

Definition 2.1.3. *A* logarithmic differential operator *is a differential operator of degree zero with respect to the $\mathcal{V}$-filtration.*

Notice that by definition

$$\mathrm{Der}(-\log D) = \mathrm{Der}_{\mathbb{C}^n} \cap \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) = \mathcal{G}r^1_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})).$$

Furthermore, because

$$F^1(\mathcal{D}_{\mathbb{C}^n}) = \mathcal{O}_{\mathbb{C}^n} \oplus \mathrm{Der}_{\mathbb{C}^n},$$

we also have that

$$F^1(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) = \mathcal{O}_{\mathbb{C}^n} \oplus \mathrm{Der}(-\log D).$$

Remark 2.1.4. *The inclusion* $\mathrm{Der}(-\log D) \subset \mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))$ *gives rise to a canonical graded morphism of graded algebras*

$$\kappa\colon \mathcal{S}ym_{\mathcal{O}_{\mathbb{C}^n}}(\mathrm{Der}(-\log D)) \longrightarrow \mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})).$$

Similarly, for $x \in D$, we have a canonical graded morphism of graded $\mathcal{O}_{\mathbb{C}^n,x}$-algebras

$$\kappa_x\colon \operatorname{Sym}_{\mathcal{O}_{\mathbb{C}^n,x}}(\operatorname{Der}_x(-\log D)) \longrightarrow \operatorname{Gr}_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})),$$

which is the stalk of κ at x.

Definition 2.1.5. *Consider $\delta = \sum_{i=1}^n a_i\partial/\partial x_i \in \operatorname{Der}_{\mathbb{C}^n,x}$. Then the* principal symbol *of δ is $\sigma(\delta) := \sum_{i=1}^n a_i\xi_i \in \operatorname{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x}) = \mathcal{O}_{\mathbb{C}^n,x}[\xi_1,\dots,\xi_n]$.*

It turns out that in order to describe $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})$, we need only to know a basis of $\operatorname{Der}(-\log D)$. In fact, we have the following:

Theorem 2.1.6. ([8], Theorem 2.1.4) *Let $D \subset \mathbb{C}^n$ be a free divisor and $x \in D$. Consider a basis $\delta_1,\dots,\delta_n$ of $\operatorname{Der}_x(-\log D)$. Each logarithmic operator P can be written, in a unique way, as a polynomial*

$$P = \sum_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n} \beta_\alpha \delta_1^{\alpha_1}\cdots\delta_n^{\alpha_n},$$

where $\beta_\alpha \in \mathcal{O}_{\mathbb{C}^n,x}$. In other words, the ring of logarithmic operators is the $\mathcal{O}_{\mathbb{C}^n,x}$-subalgebra of $\mathcal{D}_{\mathbb{C}^n,x}$ generated by logarithmic derivations

$$\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x}) = \mathcal{O}_{\mathbb{C}^n,x}[\delta_1,\dots,\delta_n] = \mathcal{O}_{\mathbb{C}^n,x}[\operatorname{Der}_x(-\log D)].$$

Remark 2.1.7. *As a immediate consequence of the previous Theorem, we obtain an isomorphism*

$$\alpha\colon \operatorname{Gr}_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})) \longrightarrow \mathcal{O}_{\mathbb{C}^n,x}[\sigma(\delta_1),\dots,\sigma(\delta_n)].$$

Corollary 2.1.8. ([8], Corollary 2.1.6) *If D is free at x, the morphism κ_x from the symmetric algebra $\operatorname{Sym}_{\mathcal{O}_{\mathbb{C}^n,x}}(\operatorname{Der}_x(-\log D))$ to $\operatorname{Gr}_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x}))$ of Remark 2.1.4 is an isomorphism of graded $\mathcal{O}_{\mathbb{C}^n,x}$-algebras. As a consequence, if D is a free divisor, the canonical morphism*

$$\kappa\colon \mathcal{S}ym_{\mathcal{O}_{\mathbb{C}^n}}(\operatorname{Der}(-\log D)) \longrightarrow \mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))$$

is an isomorphism.

Corollary 2.1.9. ([8], Corollary 2.1.7) *$\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$ is a coherent sheaf of rings.*

Proof. By Theorem 9.16 of [5], we have only to prove that $\mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))$ is coherent, but this sheaf is locally isomorphic to the polynomial ring $\mathcal{O}_{\mathbb{C}^n}[T_1\dots,T_n]$, which is coherent by Lemma 3.2, VI of [4]. □

2.2 Equivalence between $\mathcal{O}_{\mathbb{C}^n}$-modules with a logarithmic connection and left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules

In this section we recall the notion of logarithmic connection for a $\mathcal{O}_{\mathbb{C}^n}$-module and we show how a logarithmic connection allows us to introduce the logarithmic de Rham complex. Finally, we describe the equivalence between the structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module and the existence of a logarithmic connection. See [8] for more details.

Definition 2.2.1. *Let $\mathcal{M}$ be a $\mathcal{O}_{\mathbb{C}^n}$-module. A* connection with logarithmic poles along *D on $\mathcal{M}$ or a* logarithmic connection *on $\mathcal{M}$ is a homomorphism over $\mathbb{C}$*

$$\nabla\colon \mathcal{M} \longrightarrow \Omega^1(\log D) \otimes \mathcal{M},$$

that verifies Leibniz's identity

$$\nabla(hm) = dh \otimes m + h\nabla(m)$$

for any $h \in \mathcal{O}_{\mathbb{C}^n}$ and $m \in \mathcal{M}$, where d is the exterior derivative over $\mathcal{O}_{\mathbb{C}^n}$. We denote $\Omega^q(\log D) \otimes \mathcal{M}$ by $\Omega^q(\log D)(\mathcal{M})$ for any q.

Let $\mathcal{M}$ be a $\mathcal{O}_{\mathbb{C}^n}$-module with ∇ a logarithmic connection. We can define the following left $\mathcal{O}_{\mathbb{C}^n}$-linear morphism

$$\nabla'\colon\ \mathrm{Der}(-\log D) \longrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$$

$$\delta \mapsto \nabla_\delta$$

where $\nabla_\delta(m) = \langle \delta, \nabla(m)\rangle$.

Notice that the morphism ∇' verifies Leibniz's condition

$$\nabla_\delta(hm) = \delta(h)m + h\nabla_\delta(m) \tag{2.1}$$

for any $\delta \in \mathrm{Der}(-\log D)$, $h \in \mathcal{O}_{\mathbb{C}^n}$ and $m \in \mathcal{M}$.

On the other hand, given a left $\mathcal{O}_{\mathbb{C}^n}$-linear morphism

$$\nabla'\colon\ \mathrm{Der}(-\log D) \longrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$$

verifying Leibniz's condition (2.1), we can define

$$\nabla\colon \mathcal{M} \longrightarrow \Omega^1(\log D)(\mathcal{M})$$

with $\nabla(m)$ the element of $\Omega^1(\log D)(\mathcal{M}) = \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathrm{Der}(-\log D), \mathcal{M})$ such that $\nabla(m)(\delta) = \nabla'(\delta)(m)$.

This shows us the following:

Proposition 2.2.2. *Let $\mathcal{M}$ be a $\mathcal{O}_{\mathbb{C}^n}$-module. A logarithmic connection ∇ on $\mathcal{M}$ is equivalent to a $\mathcal{O}_{\mathbb{C}^n}$-linear map $\nabla' \colon \mathrm{Der}(-\log D) \longrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ that verifies Leibniz's condition* (2.1)*.*

Definition 2.2.3. *A logarithmic connection ∇ is* integrable *if, for each $\delta, \delta' \in \mathrm{Der}(-\log D)$, it verifies*

$$\nabla_{[\delta,\delta']} = [\nabla_\delta, \nabla_{\delta'}], \tag{2.2}$$

where $[\ ,\ ]$ represents the Lie bracket in $\mathrm{Der}(-\log D)$ and the commutator in $\mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$.

The following Example will have a crucial role in Chapter 3.

Example 2.2.4. *Consider $\mathcal{M} = \mathrm{Der}_{\mathbb{C}^n} / \mathrm{Der}(-\log D)$. Then we can introduce on $\mathcal{M}$ the map defined by*

$$\nabla'(\delta) = \nabla_\delta = [\delta, -].$$

Notice that ∇' is $\mathcal{O}_{\mathbb{C}^n}$-linear. To see that, consider $h \in \mathcal{O}_{\mathbb{C}^n}, \delta \in \mathrm{Der}(-\log D)$ and $\nu \in \mathcal{M}$. Then $\nabla_{h\delta}(\nu) = [h\delta, \nu] = h[\delta, \nu] - \nu(h)\delta = h[\delta, \nu] = h\nabla_{\delta(\nu)}$, because $\nu(h)\delta \in \mathrm{Der}(-\log D)$ and so it is zero in $\mathcal{M}$. It is clear that it satisfies Leibniz's condition (2.1) *and, by Jacoby identity, it satisfies also condition* (2.2)*. Hence by Proposition 2.2.2, ∇_δ defines an integrable logarithmic connection on $\mathcal{M}$.*

Notice that if we take $\mathcal{M} = \mathrm{Der}(-\log D)$ or $\mathrm{Der}_{\mathbb{C}^n}$, then $\nabla'(\delta) = [\delta, -]$ does not in general define a logarithmic connection on $\mathcal{M}$ because ∇' is not $\mathcal{O}_{\mathbb{C}^n}$-linear.

Definition 2.2.5. *Given a logarithmic connection ∇ on $\mathcal{M}$ and the exterior derivative d, we can construct a morphism*

$$\nabla^q \colon \Omega^q(\log D)(\mathcal{M}) \longrightarrow \Omega^{q+1}(\log D)(\mathcal{M})$$

for each $q = 1, \ldots, n$, defined by

$$\nabla^q(\omega \otimes m) = d\omega \otimes m + (-1)^q \omega \wedge \nabla(m),$$

where ω and m are sections of the sheaves $\Omega^q(\log D)(\mathcal{M})$ and $\mathcal{M}$, respectively.

Remark 2.2.6. *Let ∇ be a logarithmic connection. Then ∇ is integrable if and only if $\nabla^q \circ \nabla^{q-1} = 0$ for every q.*

Definition 2.2.7. *Let $\mathcal{M}$ be a $\mathcal{O}_{\mathbb{C}^n}$-module with ∇ an integrable logarithmic connection on $\mathcal{M}$. We call the* logarithmic de Rham complex *of $\mathcal{M}$ and we denote by $\Omega^\bullet(\log D)(\mathcal{M})$, the complex of sheaves of complex vector spaces*

$$0 \longrightarrow \mathcal{M} \xrightarrow{\nabla} \Omega^1(\log D)(\mathcal{M}) \xrightarrow{\nabla^1} \cdots \xrightarrow{\nabla^{n-1}} \Omega^n(\log D)(\mathcal{M}) \longrightarrow 0\,.$$

Example 2.2.8. *Suppose that $\mathcal{M} = \mathcal{O}_{\mathbb{C}^n}$ and that the logarithmic connection ∇ is equal to the exterior derivative $d\colon \mathcal{O}_{\mathbb{C}^n} \longrightarrow \Omega^1(\log D)$, then the morphisms $\nabla^q\colon \Omega^q(\log D) \longrightarrow \Omega^{q+1}(\log D)$ define the logarithmic de Rham complex of Saito. See [48].*

The following will have an important role in the third chapter.

Definition 2.2.9. *Let $\mathcal{M}$ be a $\mathcal{O}_{\mathbb{C}^n}$-module and ∇ an integrable logarithmic connection on $\mathcal{M}$. We define the following complex*

$$0 \longrightarrow \mathcal{M} \xrightarrow{d^0} \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\textstyle\bigwedge^1 \operatorname{Der}(-\log D), \mathcal{M}) \xrightarrow{d^1} \cdots$$

$$\cdots \xrightarrow{d^{n-1}} \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\textstyle\bigwedge^n \operatorname{Der}(-\log D), \mathcal{M}) \longrightarrow 0$$

where the differentials are defined by

$$(d^p(\psi))(\delta_1 \wedge \cdots \wedge \delta_{p+1}) := \sum_{i=1}^{p+1} (-1)^i \nabla_{\delta_i}(\psi(\delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \delta_{p+1}))+$$

$$+ \sum_{1 \le i < j \le p+1} (-1)^{i+j-1} \psi([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \widehat{\delta_j} \wedge \cdots \wedge \delta_{p+1}),$$

for $p = 1, \ldots, n-1$ and for $p = 0$, by

$$d^0(m) := (\delta \mapsto \nabla_\delta(m)).$$

Consider now the ring $R := \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) = \cup_{k \ge 0} R_k$, where $R_0 := \mathcal{O}_{\mathbb{C}^n} \subset R_1$ and $R_k := F^k(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))$. Then the ring $\mathcal{G}r(R)$ is commutative and the canonical morphism

$$\alpha\colon Sym_{R_0}(\mathcal{G}r^1(R)) \longrightarrow \mathcal{G}r(R),$$

defined by

$$\alpha(s_1 \otimes \cdots \otimes s_t) := s_1 \cdots s_t,$$

is an isomorphism. In this situation, R_1 is an (R_0, R_0)-bimodule and a Lie algebra, with $[x, y] = xy - yx \in R_1$ because $\mathcal{G}r(R)$ is commutative. In addition, R_0 is a (R_0, R_0)-subbimodule of R_1 such that the two induced structures of R_0-module over the quotient R_1/R_0 are the same.

Let $T_{R_0}(R_1) := R_0 \oplus R_1 \oplus (R_1 \otimes_{R_0} R_1) \oplus \cdots$ be the tensor algebra of R_1 and let

$$\psi \colon T_{R_0}(R_1) \longrightarrow R = \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$$

be the canonical morphism defined by the inclusion $R_1 \subset R$. Then we have the following:

Proposition 2.2.10. ([8], Proposition 2.2.5) *The morphism ψ induces an isomorphism*

$$\phi \colon \frac{T_{R_0}(R_1)}{J} \longrightarrow \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$$

defined by

$$\phi(i(y_1) \otimes \cdots \otimes i(y_t) + J) := y_1 y_2 \cdots y_t,$$

where i is the inclusion of R_1 in the tensor algebra and J is the two sided ideal generated by the elements

1. *$a - i(a)$, for $a \in R_0 \subset R_1$;*
2. *$i(x) \otimes i(y) - i(y) \otimes i(x) - i([x, y])$, for $x, y \in R_1$.*

We now present the main result of the section:

Corollary 2.2.11. ([8], Corollary 2.2.6) *Let $D \subset \mathbb{C}^n$ be a free divisor and let $\mathcal{M}$ be a $\mathcal{O}_{\mathbb{C}^n}$-module. An integrable logarithmic connection on $\mathcal{M}$ gives rise to a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module structure on $\mathcal{M}$ and vice versa.*

Proof. A $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{M}$ with an integrable logarithmic connection ∇ has a natural structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module defined by its structure as $\mathcal{O}_{\mathbb{C}^n}$-module. Let μ be the morphism of $(\mathcal{O}_{\mathbb{C}^n}, \mathcal{O}_{\mathbb{C}^n})$-bimodules

$$\mu \colon R_1 = \mathcal{O}_{\mathbb{C}^n} \oplus \operatorname{Der}(-\log D) \longrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$$

defined by

$$\mu(a)(m) := am, \ \mu(\delta)(m) := \nabla_\delta(m),$$

for every $a \in \mathcal{O}_{\mathbb{C}^n}$, $\delta \in \operatorname{Der}(-\log D)$ and $m \in \mathcal{M}$. Moreover, μ induces a morphism $\nu \colon T_{R_0}(R_1) \longrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ and, as $\nu(J) = 0$, we have a morphism

$$\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \cong \frac{T_{R_0}(R_1)}{J} \longrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M}),$$

which defines a structure of $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module on $\mathcal{M}$.

On the other hand, a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module structure on $\mathcal{M}$ defines an integrable logarithmic connection ∇ on the $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{M}$ by

$$\nabla' \colon \operatorname{Der}(-\log D) \longrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$$

$$\nabla_\delta(m) := \delta \cdot m.$$

□

A left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module structure on $\mathcal{M}$ defines a logarithmic de Rham complex. This complex coincides with the one introduced in Definition 2.2.7. In local coordinates $(U; x_1, \ldots, x_n)$, with $\delta_1, \ldots, \delta_n$ a local basis of $\operatorname{Der}(-\log D)$ and $\omega_1, \ldots, \omega_n$ its dual basis, the differential of the complex is defined by

$$\nabla^p(U)(\omega \otimes m) = d\omega \otimes m + \sum_{i=1}^{n} ((\omega_i \wedge \omega) \otimes \delta_i \cdot m),$$

for any section $\omega \in \Omega^1_{\mathbb{C}^n}(\log D)$ and $m \in \mathcal{M}$.

Example 2.2.12. *In the case of the left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module $\mathcal{O}_{\mathbb{C}^n}$, defined as $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module in a natural way, $P \cdot g = P(g)$ for any holomorphic function g and any logarithmic operator P. This canonical structure is obviously equivalent to the integrable logarithmic connection over $\mathcal{O}_{\mathbb{C}^n}$ defined naturally by the exterior derivative*

$$\nabla_\delta(g) = \delta \cdot dg = \delta(g).$$

2.3 $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules

The aim of this section is to recall the basic properties of $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules. For more details see [12].

Let now $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules. Then we can introduce a natural structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module on the $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathcal{M}, \mathcal{N})$. The structure is defined by

$$(\delta h)(m) := -h(\delta m) + \delta h(m),$$

where δ is a logarithmic derivation and h and m are local sections respectively of $\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathcal{M}, \mathcal{N})$ and $\mathcal{M}$.

In a similar way, we can introduce a natural structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module on the $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{N}$. The structure is defined by

$$\delta(m \otimes n) := (\delta m) \otimes n + m \otimes (\delta n),$$

where δ is a logarithmic derivation and m, n are local sections respectively of $\mathcal{M}$ and $\mathcal{N}$.

Remark 2.3.1. *Consider the invertible sheaf $\mathcal{O}_{\mathbb{C}^n}[mD] \subset \mathcal{O}_{\mathbb{C}^n}[\star D]$. If f is a reduced local equation of D at $x \in D$, then f^{-m} is a local basis of $\mathcal{O}_{\mathbb{C}^n,x}[mD]$ over $\mathcal{O}_{\mathbb{C}^n,x}$.*

Proposition 2.3.2. ([12],§2) *The natural isomorphisms of $\mathcal{O}_{\mathbb{C}^n}$-modules*

$$\mathcal{O}_{\mathbb{C}^n}[(m+m')D] \cong \mathcal{O}_{\mathbb{C}^n}[mD] \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{O}_{\mathbb{C}^n}[m'D]$$

and

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathcal{O}_{\mathbb{C}^n}[mD], \mathcal{O}_{\mathbb{C}^n}) \cong \mathcal{O}_{\mathbb{C}^n}[-mD]$$

are isomorphisms also of $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules.

Definition 2.3.3. *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. Then we denote by $\mathcal{M}[mD]$ the locally free $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{O}_{\mathbb{C}^n}[mD]$ and by $\mathcal{M}^*$ the locally free $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathcal{M}, \mathcal{O}_{\mathbb{C}^n})$.*

Notice that by Definition, $\mathcal{M}[mD]$ and $\mathcal{M}^*$ are both endowed with a natural structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module.

Proposition 2.3.4. ([12],§2) *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$ and let $\mathcal{N}$ be a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. Then the isomorphism of $\mathcal{O}_{\mathbb{C}^n}$-modules $\mathcal{M}^* \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{N} \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathcal{M}, \mathcal{N})$ is also an isomorphism of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules.*

Let us consider now $\mathcal{P}$ and $\mathcal{Q}$ two right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules. Then we can introduce a natural structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module on the $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathcal{P}, \mathcal{Q})$. The structure is defined by

$$(\delta h)(p) := h(p\delta) - h(p)\delta,$$

where δ is a logarithmic derivation and h and p are local sections respectively of $\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathcal{P}, \mathcal{Q})$ and $\mathcal{P}$.

Similarly, if we consider $\mathcal{P}$ a right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module and $\mathcal{N}$ a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module, then we can introduce a natural structure of right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module on the $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{N}$. The structure is defined by

$$(p \otimes n)\delta := (p\delta) \otimes n - p \otimes (\delta n),$$

where δ is a logarithmic derivation and p, n are local sections respectively of $\mathcal{P}$ and $\mathcal{N}$.

The following two Propositions have a proof similar to assertions A.4 and A.6 of [21], respectively. See [12], Lemma 2.1.

Proposition 2.3.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules. Then one has the isomorphism*

$$\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathcal{M}, \mathcal{N})) \cong \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{N})$$

given by $\psi \mapsto \psi(1)$.

Proposition 2.3.6. *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$ and let $\mathcal{N}$ be a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. Then there is a natural isomorphism in the derived category*

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{N}) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{M}^* \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{N}).$$

Proposition 2.3.7. ([12],§2) *Let $\mathcal{P}$, $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules. Then the canonical isomorphisms of $\mathcal{O}_{\mathbb{C}^n}$-modules*

$$\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{N} \cong \mathcal{N} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$$

and

$$(\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{N} \cong \mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} (\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{N})$$

are also isomorphisms of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules. If $\mathcal{P}$ is a right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module, then the second isomorphism is an isomorphism of right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules.

Remark 2.3.8. *By [48], $\Omega^n(\log D) = \Omega^n \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{O}_{\mathbb{C}^n}[D]$.*

Proposition 2.3.9. ([12], Proposition 2.2.1) *The natural structure of right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module of $\Omega^n(\log D)$ coincides with the one of $\Omega^n \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{O}_{\mathbb{C}^n}[D]$ coming from the natural structure of right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module on Ω^n and the natural structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module on $\mathcal{O}_{\mathbb{C}^n}[D]$.*

Remark 2.3.10. *Let $\mathcal{M}$ be a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. Then the $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$ is also a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-bimodule. The left structure is given by the left structure on $\mathcal{M}$ and $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$, the right structure is given only by the right structure on $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$.*

Lemma 2.3.11. ([12], Lemma 2.2.2) *Let $\mathcal{M}$ be a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. Then there exists a unique natural isomorphism of $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-bimodules*

$$\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \cong \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$$

that sends $1 \otimes m$ to $m \otimes 1$.

Corollary 2.3.12. ([12], Corollary 2.2.3) *Let $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules. Then there is a natural morphism of right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules*

$$\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{N} \longrightarrow \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))$$

that is an isomorphism if $\mathcal{N}$ is also locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$.

Proposition 2.3.13. ([12], Proposition 2.2.4) *Let $\mathcal{P}$ be a right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module and let $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules. Then one has the following isomorphism*

$$(\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}) \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{N} \cong \mathcal{P} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} (\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{N})$$

defined by $(p \otimes m) \otimes n \mapsto p \otimes (m \otimes n)$.

Remark 2.3.14. *Consider $\mathcal{P}$ a right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. Then the $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$ is a right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module in two ways. The first one is given by the right structure on $\mathcal{P}$ and the left structure on $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$, the second one is given only by the right structure on $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$.*

Lemma 2.3.15. ([12], Lemma 2.3.1) *Let $\mathcal{P}$ be a right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. Then there is a $\mathcal{O}_{\mathbb{C}^n}$-linear involution of $\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$ that interchange the two previous structures of right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module.*

Corollary 2.3.16. ([12], Corollary 2.3.2) *Let $\mathcal{P}$ be a right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. Then $\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$ is locally free of finite rank as right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module with the first structure from Remark 2.3.14.*

Theorem 2.3.17. ([12], Theorem 2.3.3) *Let $\mathcal{P}$ be a right $\mathcal{D}_{\mathbb{C}^n}$-module and let $\mathcal{M}$ be left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. Then the natural morphism*

$$\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \longrightarrow \mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} (\mathcal{D}_{\mathbb{C}^n} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M})$$

defined by

$$p \otimes m \mapsto p \otimes (1 \otimes m)$$

is right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-linear and the induced morphism

$$(\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}) \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{D}_{\mathbb{C}^n} \longrightarrow \mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} (\mathcal{D}_{\mathbb{C}^n} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M})$$

is an isomorphism of right $\mathcal{D}_{\mathbb{C}^n}$-modules.

Corollary 2.3.18. ([12], Corollary 2.3.4) *Let $\mathcal{P}$ be a right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$ and let $\mathcal{M}$ be a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that admits a locally free resolution. Then there is a natural isomorphism in the derived category of right $\mathcal{D}_{\mathbb{C}^n}$-modules*

$$(\mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}) \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{D}_{\mathbb{C}^n} \cong \mathcal{P} \otimes_{\mathcal{O}_{\mathbb{C}^n}} (\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}).$$

2.4 An integrable logarithmic connection on $\mathrm{Der}_{\mathbb{C}^n}$ and $\mathrm{Der}(-\log D)$

This section is devoted to the construction of a structure of $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules for $\mathrm{Der}_{\mathbb{C}^n}$ and $\mathrm{Der}(-\log D)$. This new construction will have an important role in the Chapter 3 and in particular in the proof of Theorem 3.4.8.

From Example 2.2.4, we know that the adjoint representation does not give rise to a logarithmic connection either on $\mathrm{Der}_{\mathbb{C}^n}$ or on $\mathrm{Der}(-\log D)$. In particular, if we consider $\mathcal{M} = \mathrm{Der}_{\mathbb{C}^n}$ or $\mathrm{Der}(-\log D)$, then the map $\nabla'\colon \mathrm{Der}(-\log D) \longrightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ defined by $\nabla'(\delta) = [\delta, -]$ is not $\mathcal{O}_{\mathbb{C}^n}$-linear but satisfies the integrability condition (2.2). Furthermore, in general if we force $\mathcal{O}_{\mathbb{C}^n}$-linearity from $\nabla'(\delta_i)$ on a chosen basis of $\mathrm{Der}(-\log D)$, then we lose the integrability condition. In this section we describe conditions which ensure us that we keep integrability. This implies that under these conditions we are able to put on $\mathrm{Der}_{\mathbb{C}^n}$ and on $\mathrm{Der}(-\log D)$ a structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules.

Let us fix $D \subset \mathbb{C}^n$ a free divisor and $\delta_i = \sum_{j=1}^n a_{ij}\partial/\partial x_j$, $i = 1, \ldots, n$ a basis of $\mathrm{Der}(-\log D)$, where $a_{ij} \in \mathcal{O}_{\mathbb{C}^n}$ for all $i, j = 1, \ldots, n$. We know that $\mathrm{Der}(-\log D)$ forms a Lie subalgebra of $\mathrm{Der}_{\mathbb{C}^n}$, hence we can write

$$[\delta_i, \delta_j] = \sum_{k=1}^n b^i_{jk}\delta_k,$$

where $b^i_{jk} \in \mathcal{O}_{\mathbb{C}^n}$ for all $i, j, k = 1, \ldots, n$ and similarly we can write

$$[\delta_i, \frac{\partial}{\partial x_j}] = \sum_{k=1}^n c^i_{jk}\frac{\partial}{\partial x_k},$$

where $c^i_{jk} \in \mathcal{O}_{\mathbb{C}^n}$ for all $i, j, k = 1, \ldots, n$. In this way we obtain the data of $2n$ matrices $B_i := (b^i_{jk})$ and $C_i := (c^i_{jk})$ of holomorphic functions on $\mathbb{C}^n$. Let us call $\delta \cdot \partial := [\delta, \partial]$ for any derivation ∂ and any logarithmic derivation δ. Then

$$\delta_i \cdot \underline{\delta}^t = B_i \underline{\delta}^t, \quad 1 \leq i \leq n$$

and

$$\delta_i \cdot \underline{\partial}^t = C_i \underline{\partial}^t, \quad 1 \leq i \leq n$$

where $\underline{\delta} = (\delta_1, \ldots, \delta_n)$ and $\underline{\partial} = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$.

Lemma 2.4.1. *For* $i, j = 1, \ldots, n$

$$\delta_i(C_j) - \delta_j(C_i) + [C_j, C_i] = \sum_{k=1}^n b^i_{jk} C_k$$

if and only if

$$\sum_{k=1}^n a_{kl} \frac{\partial(b^i_{jk})}{\partial x_r} = 0, \quad \forall\, l, r = 1, \ldots, n.$$

Proof. We first notice that by definition $c^i_{jk} = -\partial(a_{ik})/\partial x_j$. The first equality is an equality between matrices, hence we can check it entry by entry. Let $1 \geq l, r \geq 0$. We now check the entry (l, r). In this case the expression becomes

$$-\delta_i(\frac{\partial(a_{jr})}{\partial x_l}) + \delta_j(\frac{\partial(a_{ir})}{\partial x_l}) + \sum_{k=1}^n \frac{\partial(a_{jk})}{\partial x_l}\frac{\partial(a_{ir})}{\partial x_k} +$$

$$-\sum_{k=1}^n \frac{\partial(a_{ik})}{\partial x_l}\frac{\partial(a_{jr})}{\partial x_k} = -\sum_{k=1}^n b^i_{jk}\frac{\partial(a_{kr})}{\partial x_l}.$$

Consider now the Jacobi identity

$$[[\delta_i, \delta_j], \frac{\partial}{\partial x_l}] + [[\delta_j, \frac{\partial}{\partial x_l}], \delta_i] + [[\frac{\partial}{\partial x_l}, \delta_i], \delta_j] = 0.$$

The coefficient of $\partial/\partial x_r$ of the previous expression is

$$\delta_i(\frac{\partial(a_{jr})}{\partial x_l}) - \delta_j(\frac{\partial(a_{ir})}{\partial x_l}) - \sum_{k=1}^n \frac{\partial(a_{jk})}{\partial x_l}\frac{\partial(a_{ir})}{\partial x_k} +$$

$$+\sum_{k=1}^n \frac{\partial(a_{ik})}{\partial x_l}\frac{\partial(a_{jr})}{\partial x_k} - \sum_{k=1}^n b^i_{jk}\frac{\partial(a_{kr})}{\partial x_l} - \sum_{k=1}^n a_{kr}\frac{\partial(b^i_{jk})}{\partial x_l} = 0.$$

Hence, the first equality is satisfied if and only if

$$\sum_{k=1}^{n} a_{kr} \frac{\partial(b^i_{jk})}{\partial x_l} = 0.$$

□

Proposition 2.4.2. *We can define a structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module on* $\mathrm{Der}_{\mathbb{C}^n}$ *if*

$$\sum_{k=1}^{n} a_{kr} \frac{\partial(b^i_{jk})}{\partial x_l} = 0, \quad \forall\ i, j, l, r = 1, \ldots, n.$$

Proof. To define a structure of left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module on $\mathrm{Der}_{\mathbb{C}^n}$, we define the action of δ_i on any derivation ∂ by

$$\delta_i \bullet \partial := [\delta_i, \partial],$$

or in other words

$$\delta_i \bullet \underline{\partial}^t := C_i \underline{\partial}^t, \quad 1 \leq i \leq n.$$

The structure just introduced is well defined if and only if

$$(\delta_i\delta_j - \delta_j\delta_i) \bullet \underline{\partial}^t = (\sum_{k=1}^{n} b^i_{jk}\delta_k) \bullet \underline{\partial}^t.$$

An easy computation shows us that this is true if and only if

$$\delta_i(C_j) - \delta_j(C_i) + [C_j, C_i] = \sum_{k=1}^{n} b^i_{jk} C_k$$

hence we can conclude by Lemma 2.4.1. □

Notice that the action on $\mathrm{Der}_{\mathbb{C}^n}$ of any logarithmic derivation $\delta = \sum_{k=1}^{n} \beta_k \delta_k$ is given by

$$\delta \bullet \underline{\partial}^t = \sum_{k=1}^{n} \beta_k C_k \underline{\partial}^t, \tag{2.3}$$

hence this is not the adjoint action.

Lemma 2.4.3. *For* $i, j = 1, \ldots, n$

$$\sum_{k=1}^{n} a_{lk} \frac{\partial(b^i_{jr})}{\partial x_k} = 0, \quad \forall\ l, r = 1, \ldots, n$$

if and only if

$$\delta_i(B_j) - \delta_j(B_i) + [B_j, B_i] = \sum_{k=1}^{n} b^i_{jk} B_k.$$

Proof. It is similar to the proof of Lemma 2.4.1. □

Proposition 2.4.4. *We can define a structure of left* $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$*-module on* $\mathrm{Der}(-\log D)$ *if*

$$\sum_{k=1}^{n} a_{lk} \frac{\partial(b^i_{jr})}{\partial x_k} = 0, \quad \forall\ i, j, l, r = 1, \ldots, n.$$

Proof. As the proof of Proposition 2.4.2. □

Notice that, as in the case of $\mathrm{Der}_{\mathbb{C}^n}$, the action on $\mathrm{Der}(-\log D)$ of any logarithmic derivation $\delta = \sum_{k=1}^{n} \beta_k \delta_k$ is given by

$$\delta \bullet \underline{\partial}^t = \sum_{k=1}^{n} \beta_k B_k \underline{\partial}^t, \tag{2.4}$$

hence also this action is not the adjoint one.

Corollary 2.4.5. *Let* $D \subset \mathbb{C}^n$ *be a linear free divisor. Then* $\mathrm{Der}_{\mathbb{C}^n}$ *and* $\mathrm{Der}(-\log D)$ *are left* $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$*-modules, where the action is given by* (2.3) *and* (2.4)*, respectively.*

Proof. In this case $b^i_{jk} \in \mathbb{C}$ and so the two previous conditions are trivially fulfilled. □

Corollary 2.4.6. *Let* $D \subset \mathbb{C}^2$ *be a free divisor defined by a weighted homogeneous equation. Then* $\mathrm{Der}_{\mathbb{C}^2}$ *and* $\mathrm{Der}(-\log D)$ *are left* $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^2})$*-modules.*

Proof. Because D is defined by f a weighted homogenous equation and because $\mathrm{Der}(-\log D)$ is a free $\mathcal{O}_{\mathbb{C}^2}$-module of rank 2, then we can choose χ, δ as a basis of $\mathrm{Der}(-\log D)$, where χ is an Euler vector field and $\delta(f) = 0$. Then $[\chi, \delta] = \alpha\delta$, where $\alpha \in \mathbb{C}$ and so all the $b^i_{jk} \in \mathbb{C}$. Hence the two previous conditions are trivially fulfilled. □

The author thinks that the approach used to put a logarithmic connection on $\mathrm{Der}_{\mathbb{C}^n}$ and $\mathrm{Der}(-\log D)$ is a particular case of the notion of integrability up to homotopy, see [1].

2.5 The logarithmic Spencer complex

In this section, we recall the definition of the logarithmic Spencer complex and we describe its basic properties.

Definition 2.5.1. *Let $D \subset \mathbb{C}^n$ be a free divisor. We call the* logarithmic Spencer complex *and we denote by $\mathcal{S}p^\bullet(\log D)$, the complex*

$$0 \longrightarrow \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^n \operatorname{Der}(-\log D) \xrightarrow{\varepsilon_{-n}} \cdots$$

$$\cdots \xrightarrow{\varepsilon_{-2}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^1 \operatorname{Der}(-\log D) \xrightarrow{\varepsilon_{-1}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$$

where the differentials are defined by

$$\varepsilon_{-p}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) := \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \delta_p) +$$

$$+ \sum_{1 \le i < j \le p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \widehat{\delta_j} \wedge \cdots \wedge \delta_p),$$

for all $2 \le p \le n$ and for $p = 1$ we have

$$\varepsilon_{-1}(P \otimes \delta) := P\delta.$$

Definition 2.5.2. *We can augment the complex $\mathcal{S}p^\bullet(\log D)$ by another morphism*

$$\varepsilon_0 \colon \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \longrightarrow \mathcal{O}_{\mathbb{C}^n}$$

defined by

$$\varepsilon_0(P) := P(1).$$

We call the new complex $\tilde{\mathcal{S}p}^\bullet(\log D)$.

We denote by $\mathcal{S}p^\bullet[\star D] = \mathcal{D}_{\mathbb{C}^n}[\star D] \otimes_{\mathcal{D}_{\mathbb{C}^n}} \mathcal{S}p^\bullet$ the *meromorphic Spencer complex* of $\mathcal{O}_{\mathbb{C}^n}[\star D]$, where $\mathcal{S}p^\bullet$ is the usual Spencer complex of $\mathcal{O}_{\mathbb{C}^n}$.

Theorem 2.5.3. ([8], Theorem 3.1.2) *The complex $\mathcal{S}p^\bullet(\log D)$ is a locally free resolution of $\mathcal{O}_{\mathbb{C}^n}$ as a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module.*

Proof. To see the exactness of $\tilde{\mathcal{S}p}^\bullet(\log D)$ we define a discrete filtration $G^\bullet$ such that it induces an exact graded complex

$$G^k(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^p \operatorname{Der}(-\log D)) := F^{k-p}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^p \operatorname{Der}(-\log D),$$

$$G^k(\mathcal{O}_{\mathbb{C}^n}) := \mathcal{O}_{\mathbb{C}^n}.$$

We have that

$$\mathcal{G}r_{G^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^p \mathrm{Der}(-\log D)) = \mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))[-p] \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^p \mathrm{Der}(-\log D),$$

$$\mathcal{G}r_{G^\bullet}(\mathcal{O}_{\mathbb{C}^n}) = \mathcal{O}_{\mathbb{C}^n}.$$

As the above filtration are compatible with the differential of the complex $\tilde{\mathcal{S}p}^\bullet(\log D)$, we can consider the complex $\mathcal{G}r_{G^\bullet}(\tilde{\mathcal{S}p}^\bullet(\log D))$

$$0 \longrightarrow \mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))[-n] \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^n \mathrm{Der}(-\log D) \xrightarrow{\psi_{-n}} \cdots$$

$$\cdots \xrightarrow{\psi_{-2}} \mathcal{G}r_{F^\bullet}\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))[-1] \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^1 \mathrm{Der}(-\log D) \xrightarrow{\psi_{-1}}$$

$$\xrightarrow{\psi_{-1}} \mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \xrightarrow{\psi_0} \mathcal{O}_{\mathbb{C}^n} \longrightarrow 0\ ,$$

where the local expression of the differential is defined by

$$\psi_{-p}(G \otimes (\delta_{j_1} \wedge \cdots \wedge \delta_{j_p})) = \sum_{i=1}^{p} (-1)^{i-1} G\sigma(\delta_{j_i}) \otimes (\delta_{j_1} \wedge \cdots \wedge \widehat{\delta_{j_i}} \wedge \cdots \wedge \delta_{j_p}),$$

for all $2 \le p \le p$ and for $p = 1, 0$ we have

$$\psi_{-1}(G \otimes \delta_i) = G\sigma(\delta_i), \quad \psi_0(G) = G_0,$$

with $\delta_1, \ldots, \delta_n$ a basis of $\mathrm{Der}(-\log D)$. This complex is the Koszul complex of the ring

$$\mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \cong \mathcal{S}ym_{\mathcal{O}_{\mathbb{C}^n}}(\mathrm{Der}(-\log D))$$

with respect to the $\mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))$ regular sequence $\sigma(\delta_1), \ldots \sigma(\delta_n)$ in the ring $\mathcal{G}r_{F^\bullet}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))$ and consequently, it is exact. □

More generally, we can introduce the following:

Definition 2.5.4. *Let $D \subset \mathbb{C}^n$ be a free divisor and let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. We denote by $\mathcal{S}p^\bullet(\log D)(\mathcal{M})$, the complex*

$$0 \longrightarrow \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^n \mathrm{Der}(-\log D) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \xrightarrow{\varepsilon_{-n}} \cdots$$

$$\cdots \xrightarrow{\varepsilon_{-2}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^1 \mathrm{Der}(-\log D) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \xrightarrow{\varepsilon_{-1}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}$$

where the differentials are defined by

$$\varepsilon_{-p}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p) \otimes m) := \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \delta_p) \otimes m +$$

$$- \sum_{i=1}^{p} (-1)^{i-1} P \otimes (\delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \delta_p) \otimes (\delta_i m) +$$

$$+ \sum_{1 \leq i < j \leq p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \widehat{\delta_j} \wedge \cdots \wedge \delta_p) \otimes m,$$

for all $2 \leq p \leq n$ *and for* $p = 1$ *we have*

$$\varepsilon_{-1}(P \otimes \delta \otimes m) := P\delta \otimes m - P \otimes \delta m.$$

Definition 2.5.5. *We can augment the complex* $\mathcal{S}p^{\bullet}(\log D)(\mathcal{M})$ *by another morphism*

$$\varepsilon_0 \colon \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M} \longrightarrow \mathcal{M},$$

defined by

$$\varepsilon_0(P \otimes m) := Pm.$$

We call the new complex $\tilde{\mathcal{S}p}^{\bullet}(\log D)(\mathcal{M})$.

Theorem 2.5.6. ([12], (1.2)) *Let* $\mathcal{M}$ *be a free* $\mathcal{O}_{\mathbb{C}^n}$*-module with an integrable logarithmic connection. The complex* $\mathcal{S}p^{\bullet}(\log D)(\mathcal{M})$ *is a locally free resolution of* $\mathcal{M}$ *as a left* $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$*-module.*

Proof. The proof is similar to the one of Theorem 2.5.3, see [12], (1.2). □

Lemma 2.5.7. ([8], Lemma 3.1.3) *Let* f *be a local equation of the free divisor* $D \subset \mathbb{C}^n$ *at* $x \in D$*. For every logarithmic operator* $P \in \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})$*, there exist, for each integer* p*, a logarithmic operator* $Q \in \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})$ *and an integer* k *such that* $f^{-p}P = Qf^{-k}$.

Remark 2.5.8. *For every operator* $Q \in \mathcal{D}_{\mathbb{C}^n,x}[\star D]$*, we can always find a strictly positive integer* m *such that* $f^m Q \in \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})$*. Equivalently, for each meromorphic differential operator* Q*, there exist a positive integer* p *and a logarithmic operator* Q' *such that we can write* $Q = f^{-p}Q'$.

We now introduce several morphisms that we will use later.

Lemma 2.5.9. ([8], Lemma 3.1.5) *We have the following isomorphisms*

1. $\mathcal{O}_{\mathbb{C}^n}[\star D] \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \tilde{\hookrightarrow} \mathcal{D}_{\mathbb{C}^n}[\star D] \tilde{\hookleftarrow} \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{O}_{\mathbb{C}^n}[\star D]$;

2. $\alpha\colon \mathcal{D}_{\mathbb{C}^n}[\star D] \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{O}_{\mathbb{C}^n} \cong \mathcal{O}_{\mathbb{C}^n}[\star D]$, $\alpha(P \otimes g) := P(g)$;

3. $\rho\colon \mathcal{D}_{\mathbb{C}^n}[\star D] \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{D}_{\mathbb{C}^n}[\star D] \cong \mathcal{D}_{\mathbb{C}^n}[\star D]$, $\rho(P \otimes Q) := PQ$.

Proposition 2.5.10. ([8], Proposition 3.1.6) *We have the following isomorphisms of complexes of $\mathcal{D}_{\mathbb{C}^n}[\star D]$-modules*

1. $\mathcal{D}_{\mathbb{C}^n}[\star D] \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{S}p^\bullet \cong \mathcal{S}p^\bullet[\star D]$;

2. $\mathcal{D}_{\mathbb{C}^n}[\star D] \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{S}p^\bullet(\log D) \cong \mathcal{S}p^\bullet[\star D]$.

2.6 The logarithmic de Rham complex

The aim of this section is to recall that the logarithmic de Rham complex and the dual logarithmic Spencer complex are isomorphic complexes.

Consider $D \subset \mathbb{C}^n$ a free divisor. By the equality $\Omega^p(\log D) = \bigwedge^p \Omega^1(\log D)$ and by the fact that $\Omega^1(\log D) \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathrm{Der}(-\log D), \mathcal{O}_{\mathbb{C}^n})$, we can construct a natural isomorphism

$$\gamma^p\colon \Omega^p(\log D) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\bigwedge^p \mathrm{Der}(-\log D), \mathcal{O}_{\mathbb{C}^n}),$$

defined by

$$\gamma^p(\omega_1 \wedge \cdots \wedge \omega_p)(\delta_1 \wedge \cdots \wedge \delta_p) := \det(\omega_i \cdot \delta_j)_{1 \le i,j \le p}.$$

Similarly, for every divisor $D \subset \mathbb{C}^n$, we have a standard canonical isomorphism

$$\lambda^p\colon \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\bigwedge^p \mathrm{Der}(-\log D), \mathcal{O}_{\mathbb{C}^n}) \longrightarrow \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^p \mathrm{Der}(-\log D), \mathcal{O}_{\mathbb{C}^n}),$$

defined by

$$\lambda^p(\alpha)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) := P(\alpha(\delta_1 \wedge \cdots \wedge \delta_p)).$$

Composing the two isomorphism γ^p and λ^p, we can construct a natural isomorphism

$$\psi^p = \lambda^p \circ \gamma^p\colon \Omega^p(\log D) \longrightarrow \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^p \mathrm{Der}(-\log D), \mathcal{O}_{\mathbb{C}^n}),$$

for $p = 0, \ldots, n$, defined locally by

$$\psi^p(\omega_1 \wedge \cdots \wedge \omega_p)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) := P(\det(\omega_i \cdot \delta_j)_{1 \le i,j \le p}).$$

The previous definitions can be given in a more general setting:

Definition 2.6.1. *If $\mathcal{M}$ is a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module, given an integer $p \in \{1, \ldots, n\}$, there exist the following canonical isomorphisms*

$$\gamma^p_{\mathcal{M}} \colon \Omega^p(\log D)(\mathcal{M}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\bigwedge^p \mathrm{Der}(-\log D), \mathcal{M}),$$

defined by

$$\gamma^p((\omega_1 \wedge \cdots \wedge \omega_p) \otimes m)(\delta_1 \wedge \cdots \wedge \delta_p) := \det(\omega_i \cdot \delta_j)_{1\le i,j\le p} \cdot m,$$

$$\lambda^p_{\mathcal{M}} \colon \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\bigwedge^p \mathrm{Der}(-\log D), \mathcal{M}) \longrightarrow \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^p \mathrm{Der}(-\log D), \mathcal{M}),$$

defined by

$$\lambda^p(\alpha)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) := P(\alpha(\delta_1 \wedge \cdots \wedge \delta_p))$$

$$\psi^p_{\mathcal{M}} = \lambda^p_{\mathcal{M}} \circ \gamma^p_{\mathcal{M}} \colon \Omega^p(\log D)(\mathcal{M}) \longrightarrow \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^p \mathrm{Der}(-\log D), \mathcal{M}),$$

defined locally by

$$\psi^p_{\mathcal{M}}((\omega_1 \wedge \cdots \wedge \omega_p) \otimes m)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) := P \cdot \det(\omega_i \cdot \delta_j)_{1\le i,j\le p} \cdot m.$$

Proposition 2.6.2. *The isomorphisms $\lambda^\bullet_{\mathcal{M}}$ induce an isomorphism of complexes between $\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\bigwedge^\bullet \mathrm{Der}(-\log D), \mathcal{M})$ and $\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \bigwedge^\bullet \mathrm{Der}(-\log D), \mathcal{M})$.*

Proof. The isomorphisms $\lambda^\bullet_{\mathcal{M}}$ commute with the differentials on the two complexes also because they are one the dual of the other. □

Theorem 2.6.3. ([8], Theorem 3.2.1) *If $\mathcal{M}$ is a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module, the complexes of sheaves of $\mathbb{C}$-vector spaces $\Omega^\bullet(\log D)(\mathcal{M})$ and $\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{S}p^\bullet(\log D), \mathcal{M})$ are canonically isomorphic.*

Proof. The general case is solved if we prove the case $\mathcal{M} = \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$, using the isomorphisms

$$\Omega^\bullet(\log D)(\mathcal{M}) \cong \Omega^\bullet(\log D)(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M},$$

$$\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{S}p^\bullet(\log D), \mathcal{M}) \cong \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{S}p^\bullet(\log D), \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}.$$

For $\mathcal{M} = \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$, we obtain the right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-isomorphisms

$$\phi^p = \psi^p_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \colon \Omega^p(\log D)(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \longrightarrow \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{S}p^{-p}(\log D), \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})),$$

whose local expression are

$$\phi^p((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) := P \cdot \det(\omega_i \cdot \delta_j)_{1 \le i,j \le p} \cdot Q.$$

To prove that these isomorphisms produce an isomorphism of complexes we have to check that they commute with the differential of the complex. Thanks to the second isomorphism of Proposition 2.5.10

$$\mathcal{D}_{\mathbb{C}^n}[\star D] \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{S}p^\bullet(\log D) \cong \mathcal{S}p^\bullet[\star D],$$

we obtain a natural morphism of complexes of sheaves of right $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules

$$\tau^\bullet \colon \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{S}p^\bullet(\log D), \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \longrightarrow \mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}[\star D]}(\mathcal{S}p^\bullet[\star D], \mathcal{D}_{\mathbb{C}^n}[\star D]),$$

locally defined by

$$\tau^p(\alpha)(R \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) := f^{-k}\alpha(P \otimes (f\delta_1 \wedge \cdots \wedge f\delta_p)),$$

where P is a local section of $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$ such that $Rf^{-p} = f^{-k}P$. See Lemma 2.5.7. The morphism τ^p are injective, because

$$\alpha(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = \tau^p(\alpha)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)).$$

Let us see the following diagram is commutative

$$\begin{array}{ccc}
\Omega^p(\log D)(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) & \xrightarrow{\quad j^p \quad} & \Omega^p[\star D](\mathcal{D}_{\mathbb{C}^n}[\star D]) \\
\Big\downarrow{\scriptstyle \phi^p} & & \Big\downarrow{\scriptstyle \Phi^p} \\
\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{S}p^p(\log D), \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) & \xrightarrow{\tau^p} & \mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}[\star D]}(\mathcal{S}p^p[\star D], \mathcal{D}_{\mathbb{C}^n}[\star D])
\end{array}$$

for each $p \geq 0$, where the Φ^p are the isomorphisms

$$\Phi^p \colon \Omega^p[\star D](\mathcal{D}_{\mathbb{C}^n}[\star D]) \longrightarrow \mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}[\star D]}(\mathcal{D}_{\mathbb{C}^n}[\star D] \otimes \bigwedge^p \mathrm{Der}_{\mathbb{C}^n}, \mathcal{D}_{\mathbb{C}^n}[\star D]),$$

defined by

$$\Phi^p((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) := P \cdot \det(\omega_i \cdot \delta_j)_{1 \le i,j \le p} \cdot Q.$$

Given $\omega_1, \ldots, \omega_p$ local sections of $\Omega^1(\log D)$, Q and P local sections of $\mathcal{D}_{\mathbb{C}^n}[\star D]$ and

$\delta_1, \ldots, \delta_p$ local sections of $\mathrm{Der}_{\mathbb{C}^n}$, we have that

$$(\tau^p \circ \phi^p)((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)(R \otimes (\delta_1 \wedge \cdots \wedge \delta_p))$$

$$= f^{-k} \phi^p((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)(P \otimes (f\delta_1 \wedge \cdots \wedge f\delta_p))$$

$$= f^{-k} P \cdot \det(\omega_i \cdot f\delta_j) \cdot Q = R \cdot f^{-p} \det(\omega_i \cdot f\delta_j) \cdot Q = R \cdot \det(\omega_i \cdot \delta_j) \cdot Q$$

$$= (\Phi^p \circ j^p)((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)(R \otimes (\delta_1 \wedge \cdots \wedge \delta_p)),$$

where P is a local section of $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$ such that $Rf^{-p} = f^{-k}P$. But $\Phi^\bullet$, $j^\bullet$ and $\tau^\bullet$ are morphisms of complexes and $\tau^\bullet$ is injective, hence we deduce that the ϕ^p commute with the differential and so define an isomorphism of complexes

$$\phi^\bullet \colon \Omega^\bullet(\log D)(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \longrightarrow \mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{S}p^\bullet(\log D), \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})),$$

as we wanted to prove. □

Corollary 2.6.4. *If $\mathcal{M}$ is a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module, the complexes of sheaves of $\mathbb{C}$-vector spaces $\Omega^\bullet(\log D)(\mathcal{M})$ and $\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\bigwedge^\bullet \mathrm{Der}(-\log D), \mathcal{M})$ are isomorphic.*

Proof. It is a direct consequence of the Proposition 2.6.2 and of the Theorem 2.6.3. □

Corollary 2.6.5. *There exists a canonical isomorphism in the derived category*

$$\Omega^\bullet(\log D)(\mathcal{M}) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{M}).$$

Proof. By Theorem 2.5.3, the complex $\mathcal{S}p^\bullet(\log D)$ is a locally free resolution of $\mathcal{O}_{\mathbb{C}^n}$ as a left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. So, we have only to apply the Theorem 2.6.3. □

Notice that in the special case that $\mathcal{M} = \mathcal{O}_{\mathbb{C}^n}$, we have that the complexes $\Omega^\bullet(\log D)$ and $\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{S}p^\bullet(\log D), \mathcal{O}_{\mathbb{C}^n})$ are canonically isomorphic and so, there exists a canonical isomorphism

$$\Omega^\bullet(\log D) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{O}_{\mathbb{C}^n}).$$

2.7 Koszul free divisors

In this section, we recall the notion of Koszul free divisors and their basic properties. Furthermore, we describe the theory of the logarithmic Spencer complex in the case of Koszul free divisors.

Definition 2.7.1. ([8], Definition 4.1.1) *Let $D \subset \mathbb{C}^n$ be a divisor. We say that D is a* Koszul free divisor *at $x \in \mathbb{C}^n$ if it is free at x and there exists a basis $\delta_1, \ldots, \delta_n$ of* $\mathrm{Der}_x(-\log D)$ *such that the sequence of symbols $\sigma(\delta_1), \ldots, \sigma(\delta_n)$ is regular in* $\mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})$. *If D is a Koszul free divisor at every point, we simply say that it is a Koszul free divisor.*

Notice that for a free divisor D, to be Koszul is equivalent to being holonomic in the sense of Definition 3.8 from [48], i.e. the logarithmic stratification of D is locally finite. See [25], Theorem 7.4.

Remark 2.7.2. *If a basis of* $\mathrm{Der}_x(-\log D)$ *satisfies the condition of Definition 2.7.1, then every basis does.*

Example 2.7.3.

1. *The normal crossing divisor of Example 1.1.20 is Koszul free.*
2. ([11], Example 2.8) *Consider the free divisor $D = V(2^8z^3 - 2^7x^2z^2 + 2^4x^4z + 2^43^2xy^2z - 2^2x^3y^2 - 3^3y^4) \subset \mathbb{C}^3$ with Saito matrix*
$$[\delta_1, \delta_2, \delta_3] = \begin{bmatrix} 6y & 4x^2 - 48z & 2x \\ 8z - 2x^2 & 12xy & 3y \\ -xy & 9y^2 - 16xz & 4z \end{bmatrix}.$$
Then the sequence of symbols $\sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3)$ is regular in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^3,x})$ for any $x \in \mathbb{C}^3$ and so D is Koszul free.
3. ([11], Example 4.2) *Consider the free divisor $D = V(xy(x+y)(y+xz)) \subset \mathbb{C}^3$ with Saito matrix*
$$\begin{bmatrix} x & x^2 & 0 \\ y & -y^2 & 0 \\ 0 & -z(x+y) & xz+y \end{bmatrix}.$$
Then D is Euler-homogeneous but is not Koszul free.

Remark 2.7.4. ([11], Remark 2.4) *Let $D \subset \mathbb{C}^n$ be a free divisor. Then D is Koszul at x if and only if* $\mathrm{depth}((\sigma(\delta_1), \ldots, \sigma(\delta_n)), \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})) = n$, *where $\delta_1, \ldots, \delta_n$ is a basis for* $\mathrm{Der}_x(-\log D)$. *Furthermore, by coherence, if a divisor is Koszul free at a point, then it is a Koszul free divisor near that point.*

Proposition 2.7.5. ([10], Example 1.11) *Let $D \subset \mathbb{C}^2$ be a reduced divisor. Then D is a Koszul free divisor.*

Proof. Suppose that f is a local reduced equation of D at $x \in \mathbb{C}^2$. $\mathrm{Der}_x(-\log D)$ is a reflexive $\mathcal{O}_{\mathbb{C}^2,x}$-module and hence, it is free. So, we have only to check that the symbols σ_1, σ_2 of a basis δ_1, δ_2 of $\mathrm{Der}_x(-\log D)$ form a $\mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})$-regular sequence. Let us suppose they are not. Then they have a common factor $g \in \mathcal{O}_{\mathbb{C}^2,x}$, because they are symbols of order 1. If g is a unit, we divide one of them by g and eliminate the common factor. If g is not a unit, it would be in contradiction with Proposition 1.1.16, because the determinant of the Saito matrix of the basis δ_1, δ_2 would have as factor g^2, with g not invertible, while this determinant has to be equal to f multiplied by a unit. □

The Koszul free divisor behave well under products. In fact, we have the following:

Proposition 2.7.6. ([10], Proposition 1.10 1)) *Let $D \subset \mathbb{C}^n$ be a divisor such that there exists a divisor $D' \subset \mathbb{C}^{n-1}$ and $D = D' \times \mathbb{C}$. Then D is a Koszul free divisor if and only if D' is a Koszul free divisor.*

Proposition 2.7.7. ([10], Proposition 1.10 2)) *Let $D \subset \mathbb{C}^n$ and $D' \subset \mathbb{C}^r$ be two divisors. Then*

1. *the divisor $(D \times \mathbb{C}^r) \bigcup (\mathbb{C}^n \times D') \subset \mathbb{C}^{n+r}$ is free if D and D' are both free;*
2. *the divisor $(D \times \mathbb{C}^r) \bigcup (\mathbb{C}^n \times D') \subset \mathbb{C}^{n+r}$ is Koszul free if D and D' are both Koszul free.*

Proposition 2.7.8. ([10], Corollary 4.2) *Let $D \subset \mathbb{C}^n$ be a free divisor and let $\Sigma \subset D$ be a discrete set of points. If D is Koszul free at all $y \in D \setminus \Sigma$, then D is Koszul free.*

Proposition 2.7.9. ([10], Theorem 4.3) *Every locally quasi-homogeneous free divisor is Koszul free.*

Corollary 2.7.10. *Every free divisor that is locally quasi-homogeneous at the complement of a discrete set is Koszul free.*

Proposition 2.7.11. ([8], Proposition 4.1.2) *Let $D \subset \mathbb{C}^n$ be a Koszul free divisor at $x \in \mathbb{C}^n$ and consider $\delta_1, \ldots, \delta_n$ a basis of* $\mathrm{Der}_x(-\log D)$. *Then we have that*

$$\sigma(\mathcal{D}_{\mathbb{C}^n,x}(\delta_1, \ldots, \delta_n)) = \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})(\sigma(\delta_1), \ldots, \sigma(\delta_n)).$$

The following Theorem is a generalisation of Proposition 4.1.3 from [8]. The statement, without a proof, is already present in Section 1.2 of [12].

Theorem 2.7.12. *Let $D \subset \mathbb{C}^n$ be a Koszul free divisor and let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. Then the complex $\mathcal{D}_{\mathbb{C}^n} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{S}p^\bullet(\log D)(\mathcal{M})$ is concentrated in degree 0.*

Proof. We can work locally. Fix a point $x \in D$ and a reduced equation f for D at x. To prove that the complex $\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{S}p^\bullet(\log D)(\mathcal{M})_x$ is concentrated in degree zero, we define a filtration $G^\bullet$ such that the graded complex has the same property. Consider

$$G^k(\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \bigwedge^p \mathrm{Der}_x(-\log D) \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \mathcal{M}_x)$$

$$:= F^{k-p}(\mathcal{D}_{\mathbb{C}^n,x}) \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \bigwedge^p \mathrm{Der}_x(-\log D) \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \mathcal{M}_x.$$

Clearly, this filtration is compatible with the differentials of the complex. Consider now the complex $\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{S}p^\bullet(\log D)_x$ with the filtration

$$G^k(\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \bigwedge^p \mathrm{Der}_x(-\log D)) := F^{k-p}(\mathcal{D}_{\mathbb{C}^n,x}) \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \bigwedge^p \mathrm{Der}_x(-\log D),$$

also in this case, this filtration is compatible with the differentials of the complex. Hence, we have that

$$\mathrm{Gr}_{G^\bullet}(\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{S}p^\bullet(\log D)(\mathcal{M})_x)$$

$$= \mathrm{Gr}_{G^\bullet}(\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{S}p^\bullet(\log D)_x) \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \mathcal{M}_x.$$

Because $\mathcal{M}$ is free and hence flat over $\mathcal{O}_{\mathbb{C}^n,x}$, to conclude is enough to show that $\mathrm{Gr}_{G^\bullet}(\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{S}p^\bullet(\log D)_x)$ is concentrated in degree zero. Because

$$\mathrm{Gr}_{G^\bullet}(\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \bigwedge^p \mathrm{Der}_x(-\log D)) = \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})[-p] \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \bigwedge^p \mathrm{Der}_x(-\log D),$$

the complex $\mathrm{Gr}_{G^\bullet}(\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{S}p^\bullet(\log D)_x)$ looks like

$$0 \longrightarrow \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})[-n] \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \bigwedge^n \mathrm{Der}_x(-\log D) \xrightarrow{\psi_{-n}} \cdots$$

$$\xrightarrow{\psi_{-2}} \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})[-1] \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \bigwedge^1 \mathrm{Der}_x(-\log D) \xrightarrow{\psi_{-1}} \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x}) \longrightarrow 0\ ,$$

where the local expression of the differential is defined by

$$\psi_{-p}(G \otimes (\delta_{j_1} \wedge \cdots \wedge \delta_{j_p})) := \sum_{i=1}^{p} (-1)^{i-1} G\sigma(\delta_{j_i}) \otimes (\delta_{j_1} \wedge \cdots \wedge \widehat{\delta_{j_i}} \wedge \cdot \wedge \delta_{j_p}),$$

for $p = 2, \ldots, n$ and for $p = 1$

$$\psi_{-1}(g \otimes \delta_i) := G\sigma(\delta_i),$$

where $\delta_i, \ldots, \delta_n$ is a basis of $\mathrm{Der}_x(-\log D)$. This complex is the Koszul complex of the ring $\mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})$ with respect to the sequence $\sigma(\delta_1), \ldots, \sigma(\delta_n)$. By hypothesis, D is Koszul free and hence, this sequence is regular and so, the complex $\mathrm{Gr}_{G^\bullet}(\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{S}p^\bullet(\log D)_x)$ is concentrated in degree zero. □

We now present Proposition 4.1.2 from [8] as a corollary of the previous Theorem.

Corollary 2.7.13. *Let $D \subset \mathbb{C}^n$ be a Koszul free divisor and consider $\delta_1, \ldots, \delta_n$ a basis of $\mathrm{Der}_x(-\log D)$. Then the complex $\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{S}p^\bullet(\log D)_x$ is a resolution of the quotient module $\frac{\mathcal{D}_{\mathbb{C}^n,x}}{\mathcal{D}_{\mathbb{C}^n,x}(\delta_1,\ldots,\delta_n)}$.*

Proof. It is enough to apply Theorem 2.7.12 to $\mathcal{M} = \mathcal{O}_{\mathbb{C}^n}$. □

2.8 Perversity of the logarithmic de Rham complex

In this section we recall the notion of Spencer free divisors and show that every Koszul divisor is Spencer. We recall the notion of admissible $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules and prove, in Theorem 2.8.8, that if D is Koszul, every free $\mathcal{O}_{\mathbb{C}^n}$-module is admissible. Notice that the last property was stated without a proof as Proposition 1.2.3 in [12]. Finally, we show that the logarithmic the Rham complex is perverse if D is Koszul. The last fact will have an important role in the third chapter and in particular in the proof of Theorem 3.4.8.

Definition 2.8.1. *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. We say that $\mathcal{M}$ is* admissible *if the complex $\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}$ is concentrated in degree 0 and if $\mathcal{D}_{\mathbb{C}^n} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}$ is a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module.*

Notice that asking that the complex $\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}$ is concentrated in degree 0 is equivalent to ask that the complex $\mathcal{D}_{\mathbb{C}^n} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{S}p^\bullet(\log D)(\mathcal{M})$ is concentrated in degree 0.

Definition 2.8.2. *A divisor $D \subset \mathbb{C}^n$ is said to be* Spencer*, if the module $\mathcal{O}_{\mathbb{C}^n}$ is admissible.*

Definition 2.8.3. *A divisor $D \subset \mathbb{C}^n$ is said to be* pre-Spencer*, if the complex $\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{O}_{\mathbb{C}^n}$ is concentrated in degree* 0*.*

Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. Suppose that $x \in D$ and that $\mathcal{M}_x$ is generated by $m_1, \ldots, m_r$ as $\mathcal{O}_{\mathbb{C}^n,x}$-module. Then the left $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})$-module structure on $\mathcal{M}_x$ is determined by the action of $\delta_1, \ldots, \delta_n$, a basis of $\mathrm{Der}_x(-\log D)$

$$\delta_i m_j := \sum_{k=1}^{r} a_k^{ij} m_k,$$

for all $i = 1, \ldots, n, j = 1, \ldots, r$ and where $a_k^{ij} \in \mathcal{O}_{\mathbb{C}^n,x}$.

Notice that in this situation, we have that $m_1, \ldots, m_r$ also generate $\mathcal{M}_x$ over $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})$. Hence, we have a surjection

$$\pi \colon \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})^r \longrightarrow \mathcal{M}_x.$$

Lemma 2.8.4. *The kernel of π is generated by*

$$\Delta_{ij} := (-a_1^{ij}, \ldots, -a_{j-1}^{ij}, \delta_i - a_j^{ij}, -a_{j+1}^{ij} \ldots, -a_r^{ij}),$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, r$.

Proof. By Theorem 2.1.6, we have that $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x}) = \mathcal{O}_{\mathbb{C}^n,x}[\delta_1, \ldots, \delta_n]$. Consider now $P = (P_1, \ldots, P_r) \in \ker(\pi)$. We will prove the statement by induction on $p_0 = \max\{\deg P_j\}$.

If $p_0 = 0$, then each P_i is in $\mathcal{O}_{\mathbb{C}^n,x}$ and the statement is obvious. We suppose now that the result holds when $p_0 < d$. Let $p_0 = d$. Note that if $Q \in \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x}) = \mathcal{O}_{\mathbb{C}^n,x}[\delta_1, \ldots, \delta_n]$, then we can rewrite it as

$$Q = Q_1(\delta_1 - a_j^{1j}) + \cdots + Q_n(\delta_n - a_j^{nj}) + R,$$

for all $j = 1, \ldots, r$, where $\deg(Q_i) < \deg(Q)$ and $\deg(R) = 0$. Without lost of generality we can suppose that $\deg(P_1) = p_0$, by the previous note, we can rewrite

$$P_1 = Q_1(\delta_1 - a_1^{11}) + \cdots + Q_n(\delta_n - a_1^{n1}) + R$$

and hence

$$P = (P_1, \ldots, P_r) = \sum_{k=1}^{n} Q_k \Delta_{k1} + (R, P_2', \ldots, P_r'),$$

where $P_i' \in \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})$ and $\deg(P_i') \leq \deg(P_i)$ for $i = 2, \ldots, r$. Now there are two possible cases

1. $\max\{\deg P'_j\} < d$;

2. $\max\{\deg P'_j\} = d$.

In the first case, we can conclude applying the induction hypothesis to $(R, P'_2, \ldots, P'_r)$. In the second case, we can apply the method before again and eventually, we will be in the first case. □

Corollary 2.8.5. *In the situation before, the module* $\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{M}_x$ *is presented as* $\mathcal{D}^r_{\mathbb{C}^n,x}/L$*, where* L *is the submodule generated by* Δ_{ij} *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, r$.

Proposition 2.8.6. *Let* $D \subset \mathbb{C}^n$ *be a Koszul free divisor at* x *and consider* $\mathcal{M}$ *a* $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$*-module that is locally free of finite rank over* $\mathcal{O}_{\mathbb{C}^n}$*. Then we have that*

$$\sigma(\mathcal{D}_{\mathbb{C}^n,x}(\Delta_{11}, \ldots, \Delta_{nr})) = \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})(\sigma(\Delta_{11}), \ldots, \sigma(\Delta_{nr})).$$

Proof. Notice now that $\sigma(\Delta_{ij}) = (0, \ldots, 0, \sigma(\delta_i), 0, \ldots, 0)$ with δ_i in the j-th place. Similarly to Proposition 2.7.11, we have then that

$$\sigma(\mathcal{D}_{\mathbb{C}^n,x}(\Delta_{11}, \ldots, \Delta_{nr})) = (\sigma(\mathcal{D}_{\mathbb{C}^n,x}(\delta_1, \ldots, \delta_n)))^r =$$

$$= (\mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})(\sigma(\delta_1), \ldots, \sigma(\delta_n)))^r = \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})(\sigma(\Delta_{11}), \ldots, \sigma(\Delta_{nr})).$$

□

Corollary 2.8.7. *If* $D \subset \mathbb{C}^n$ *is a Koszul free divisor and* $\mathcal{M}$ *is a* $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$*-module that is locally free of finite rank over* $\mathcal{O}_{\mathbb{C}^n}$*, then* $\mathcal{D}_{\mathbb{C}^n} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}$ *is a holonomic* $\mathcal{D}_{\mathbb{C}^n}$*-module.*

Proof. For each x we have

$$\mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n,x})} \mathcal{M}_x) = \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})^r / \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})(\sigma(\Delta_{11}), \ldots, \sigma(\Delta_{nr})),$$

but $\sigma(\Delta_{ij}) = (0, \ldots, 0, \sigma(\delta_i), 0, \ldots, 0)$ with δ_i in the j-th place. Hence

$$\mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})^r / \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})(\sigma(\Delta_{11}), \ldots, \sigma(\Delta_{nr}))$$

$$\cong (\mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x}) / \mathrm{Gr}_{F^\bullet}(\mathcal{D}_{\mathbb{C}^n,x})(\sigma(\delta_1), \ldots, \sigma(\delta_n)))^r$$

and so it has dimension n. □

Theorem 2.8.8. (cf. [12], Proposition 1.2.3) *Let $D \subset \mathbb{C}^n$ be a Koszul free divisor and let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. Then $\mathcal{M}$ is admissible.*

Proof. It is a consequence of Theorem 2.7.12 and Corollary 2.8.7. □

Corollary 2.8.9. *Any Koszul free divisor is Spencer.*

Proof. Apply Theorem 2.8.8 to $\mathcal{M} = \mathcal{O}_{\mathbb{C}^n}$. □

Remark 2.8.10. ([45], Example 3.1) *The reciprocal of the previous Corollary is false. It is enough to consider $D = V(xy(x+y)(x+yz)) \subset \mathbb{C}^3$ of Example 2.7.3, in fact D is Spencer but not Koszul.*

We can introduce the following functor in the derived category

$$\mathbb{D}_{\mathcal{D}_{\mathbb{C}^n}} : D^b_{coh}(\mathcal{D}_{\mathbb{C}^n}) \longrightarrow D^b_{coh}(\mathcal{D}_{\mathbb{C}^n})$$

defined by

$$\mathbb{D}_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{M}) := \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\Omega^n, \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{M}, \mathcal{D}_{\mathbb{C}^n}))[n].$$

Similarly, we can consider the following functor in the derived category

$$\mathbb{D}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} : D^b_{coh}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \longrightarrow D^b_{coh}(\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))$$

defined by

$$\mathbb{D}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}) := \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\Omega^n(\log D), \boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})))[n].$$

Notice that by Proposition 3.1 of [12], if $\mathcal{M}$ is a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$, then $\mathbb{D}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}) = \mathcal{M}^*$.

Proposition 2.8.11. ([12], Theorem 3.1.1) *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. Then there is a natural isomorphism in the derived category*

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}, \mathcal{D}_{\mathbb{C}^n}) \cong \Omega^n \otimes_{\mathcal{O}^n_{\mathbb{C}}} (\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}^*[D])[-n].$$

Proof. There is a canonical isomorphism

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}, \mathcal{D}_{\mathbb{C}^n}) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{D}_{\mathbb{C}^n}).$$

Moreover, because $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$ and $\mathcal{M}$ are coherent module, we have

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{D}_{\mathbb{C}^n}) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{D}_{\mathbb{C}^n}.$$

By Proposition 3.1 of [12], $\boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n}))$ is concentrated in degree n and we have a natural isomorphism of $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules

$$Ext^n_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}, \mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})) \cong \Omega^n(\log D) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}^*.$$

Hence we have

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}, \mathcal{D}_{\mathbb{C}^n}) \cong (\Omega^n(\log D) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}^*) \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{D}_{\mathbb{C}^n}[-n].$$

But we have that

$$\begin{aligned}
&(\Omega^n(\log D) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}^*) \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{D}_{\mathbb{C}^n}[-n] \\
&\cong ((\Omega^n \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{O}_{\mathbb{C}^n}[D]) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}^*) \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{D}_{\mathbb{C}^n}[-n] \\
&\cong (\Omega^n \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{M}^*[D]) \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{D}_{\mathbb{C}^n}[-n] \\
&\cong \Omega^n \otimes_{\mathcal{O}^n_{\mathbb{C}}} (\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}^*[D])[-n].
\end{aligned}$$

□

The following is a generalisation of Corollary 3.1.2 of [12], where the property is stated only for $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules that are locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$.

Corollary 2.8.12. *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. Then there is a natural isomorphism in the derived category*

$$\mathbb{D}_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}) \cong \mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}^*[D].$$

Proof. It is explained in [13], Theorem 4.5 in the language of Lie-Rienhart algebras. □

Corollary 2.8.13. *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. If $\mathcal{M}$ is admissible, then also $\mathcal{M}^*[D]$ is admissible.*

Proof. By definition of admissible module, $\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}$ is concentrated in degree zero and it is holonomic, but then its $\mathcal{D}_{\mathbb{C}^n}$ dual has the same properties and hence, by the previous Corollary, $\mathcal{M}^*[D]$ is admissible. □

The following is a generalisation of Corollary 3.1.5 of [12], where the property is stated only for $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-modules that are locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$.

Corollary 2.8.14. *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module. Then there exists a natural isomorphism in the derived category*

$$\Omega^\bullet(\log D)(\mathcal{M}) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}[D]).$$

Proof. By Corollary 2.6.5, we have

$$\Omega^\bullet(\log D)(\mathcal{M}) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{M}).$$

Moreover, we have

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{M}) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathbb{D}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}), \mathbb{D}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{O}_{\mathbb{C}^n}))$$

$$\cong \boldsymbol{R}\mathcal{H}om_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})}(\mathcal{M}^*, \mathcal{O}_{\mathbb{C}^n}) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}^*, \mathcal{O}_{\mathbb{C}^n})$$

$$\cong \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathbb{D}_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{O}_{\mathbb{C}^n}), \mathbb{D}_{\mathbb{C}^n}(\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}^*))$$

$$\cong \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{M}[D]).$$

□

Corollary 2.8.15. ([12], Corollary 3.1.6) *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. The following properties are equivalent*

1. *$\mathcal{M}^*$ is admissible;*
2. *$\mathcal{M}[D]$ is admissible;*
3. *The de Rham logarithmic complex $\Omega^\bullet(\log D)(\mathcal{M})$ is a perverse sheaf.*

Theorem 2.8.16. *Let $D \subset \mathbb{C}^n$ be a Koszul free divisor and let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$. Then the de Rham logarithmic complex $\Omega^\bullet(\log D)(\mathcal{M})$ is a perverse sheaf.*

Proof. By hypothesis D is Koszul and hence by Theorem 2.8.8, $\mathcal{M}[D]$ is admissible. The result is direct consequence of Corollary 2.8.15. □

Corollary 2.8.17. ([12], Corollary 3.1.8) *Let $\mathcal{M}$ be a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module that is locally free of finite rank over $\mathcal{O}_{\mathbb{C}^n}$ such that $\mathcal{M}^*$ is admissible. Then there is a natural* *isomorphism in the derived category*

$$\Omega^\bullet(\log D)(\mathcal{M}) \cong \Omega^\bullet(\log D)(\mathcal{M}^*[-D])^\vee,$$

where $\vee$ *denote the Verdier dual.*

Chapter 3

Deformation theory for free divisors

This chapter contains the core part of this thesis: we introduce the notions of admissible and linearly admissible deformation respectively for germs of free and linear free divisors, and we show how to compute the space of first order infinitesimal (linearly) admissible deformations.

After defining the notions of admissible and linearly admissible deformation, we introduce two functors, one for each notion. Then we show how to use the de Rham logarithmic complex, introduced in Section 2.2, to compute the space of first order infinitesimal admissible deformations and we use the theory developed in Chapter 2 to show that for Koszul free divisors such that we can define an integrable logarithmic connection by modifying the adjoint representation on $\mathrm{Der}_{\mathbb{C}^n}$ and $\mathrm{Der}(-\log D)$, as described in Section 2.4, the space of infinitesimal admissible deformations is finite dimensional. We show how to use the standard Lie algebra cohomology complex (see B.3) to compute the space of first order infinitesimal linearly admissible deformations and, using representation theory for reductive Lie algebras (see B.2), we show that reductive linear free divisors are formally rigid. We describe in more details the case of free divisors defined by a weighted homogeneous equation and show that in this case, the space of infinitesimal admissible deformations is finite dimensional. Finally, we show how to use the theory of deformations for non-isolated singularities introduced in [16] and [18] by T. de Jong and D. van Straten to deform germs of free divisors in another way and we show that the two approaches are equivalent.

3.1 Admissible and linearly admissible deformations

In this section we introduce the notions of admissible and linearly admissible deformation respectively for germs of free and linear free divisors, we define the associated functors and we describe the basic properties of an (linearly) admissible deformation.

Let us fix on $(\mathbb{C}^n, 0)$ a set of coordinates $x_1, \ldots, x_n$.

Definition 3.1.1. *Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a free divisor and let (S, s) be a complex space germ. An* admissible deformation *of $(D, 0)$ over (S, s) consists of a flat morphism $\phi\colon (X, x) \longrightarrow (S, s)$ of complex spaces, where $(X, x) \subset (\mathbb{C}^n \times S, (0, s))$, together with an isomorphism from $(D, 0)$ to the central fibre of $\phi, (D, 0) \longrightarrow (X_s, x) := (\phi^{-1}(s), x)$, such that*

$$\operatorname{Der}(-\log X/S)/\mathfrak{m}_{S,s} \operatorname{Der}(-\log X/S) = \operatorname{Der}(-\log D) \tag{3.1}$$

where $\mathfrak{m}_{S,s}$ is the maximal ideal of $\mathcal{O}_{S,s}$.

Definition 3.1.2. *Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a linear free divisor and let (S, s) be a complex space germ. A* linearly admissible deformation *of $(D, 0)$ over (S, s) is an admissible deformation of $(D, 0)$ over (S, s) such that there exists a set of generators of $\operatorname{Der}(-\log X/S)$ as $\mathcal{O}_{\mathbb{C}^n \times S,(0,s)}$-module consisting of vector fields all of whose coefficients are linear in $x_1, \ldots, x_n$.*

Definition 3.1.3. *In the Definitions 3.1.1 and 3.1.2, (X, x) is called the* total space*, (S, s) the* base space *and $(X_s, x) \cong (D, 0)$ the* special fibre *of the (linearly) admissible deformation.*

Let us now look at some examples of admissible deformations.

Example 3.1.4. *1. Consider $f = xy(x-y)(x+y) \in \mathbb{C}[x, y]$ as defining equation of the germ of a free divisor $(D, 0) \subset (\mathbb{C}^2, 0)$. Then we can consider the Saito matrix*

$$A_0 = \begin{bmatrix} x & 0 \\ y & x^2y - y^3 \end{bmatrix}.$$

We can now consider $F = xy(x-y)(x+(1+t)y) \in \mathbb{C}[x, y, t]$ as defining equation of germ of the hypersurface $(X, 0) \subset (\mathbb{C}^3, 0)$. This is a one parameter admissible deformation for $(D, 0)$, in fact $\operatorname{Der}(-\log X/\mathbb{C})$ is generated by the columns of the following matrix

$$A_t = \begin{bmatrix} x & 0 \\ y & x^2y - y^3 + t(xy^2 - y^3) \end{bmatrix}.$$

Notice that $(X,0) \subset (\mathbb{C}^3,0)$ *is a germ of a free divisor with Saito matrix*

$$\begin{bmatrix} x & 0 & 0 \\ y & 0 & xy+y^2 \\ 0 & yt-x+y & -xt-2x \end{bmatrix}.$$

2. *Consider* $f = x^4 + y^4 \in \mathbb{C}[x,y]$ *as defining equation of the germ of a free divisor* $(D,0) \subset (\mathbb{C}^2,0)$*. Then we can consider the Saito matrix*

$$A_0 = \begin{bmatrix} x & -y^3 \\ y & x^3 \end{bmatrix}.$$

We can now consider $F = x^4 + y^4 + tx^2y^2 \in \mathbb{C}[x,y,t]$ *as defining equation of germ of the hypersurface* $(X,0) \subset (\mathbb{C}^3,0)$*. This is a one parameter admissible deformation for* $(D,0)$*, in fact* $\mathrm{Der}(-\log X/\mathbb{C})$ *is generated by the columns of the following matrix*

$$A_t = \begin{bmatrix} x & -x^2yt-2y^3 \\ y & xy^2t+2x^3 \end{bmatrix}.$$

Notice that $(X,0) \subset (\mathbb{C}^3,0)$ *is not free.*

3. *Consider* $f = xyz(x-y)(x+y)(x-z) \in \mathbb{C}[x,y,z]$ *as defining equation of the germ of a free divisor* $(D,0) \subset (\mathbb{C}^3,0)$*. Then we can consider the Saito matrix*

$$A_0 = \begin{bmatrix} x & 0 & 0 \\ y & 0 & x^2y-y^3 \\ z & xz-z^2 & 0 \end{bmatrix}.$$

We can now consider $F = xyz(x-y)(x+y)(x-(1+t)z) \in \mathbb{C}[x,y,z,t]$ *as defining equation of germ of the hypersurface* $(X,0) \subset (\mathbb{C}^4,0)$*. This is a one parameter admissible deformation for* $(D,0)$*, in fact* $\mathrm{Der}(-\log X/\mathbb{C})$ *is generated by the columns of the following matrix*

$$A_t = \begin{bmatrix} x & 0 & 0 \\ y & 0 & x^2y-y^3 \\ z & xz-(1+t)z^2 & 0 \end{bmatrix}.$$

We can write a (linearly) admissible deformation as a commutative diagram

$$\begin{array}{ccc} (D,0) & \overset{i}{\hookrightarrow} & (X,x) \\ \downarrow & & \downarrow \phi \\ \{*\} & \hookrightarrow & (S,s) \end{array} \tag{3.2}$$

where i is a closed embedding mapping $(D,0)$ isomorphically onto (X_s,x). We denote a (linearly) admissible deformation by

$$(i,\phi)\colon\ (D,0) \overset{i}{\hookrightarrow} (X,x) \overset{\phi}{\longrightarrow} (S,s)\ .$$

Remark 3.1.5. *In the Definitions 3.1.1 and 3.1.2, the requirement that ϕ be a flat morphism implies that (X,x) is a germ of a hypersurface in $(\mathbb{C}^n \times S,(0,s))$.*

Definition 3.1.6. *Given two (linearly) admissible deformations $(i,\phi)\colon (D,0) \hookrightarrow (X,x) \longrightarrow (S,s)$ and $(j,\psi)\colon (D,0) \hookrightarrow (Y,y) \longrightarrow (T,t)$, of $(D,0)$ over (S,s) and (T,t) respectively. A* morphism of (linearly) admissible deformations *from (i,ϕ) to (j,ψ) is a morphism of the diagram* (3.2) *being the identity on $(D,0) \longrightarrow \{*\}$. Hence, it consists of two morphisms (τ,σ) such that the following diagram commutes*

$$\begin{array}{ccccc} & & (D,0) & & \\ & {}^{i}\swarrow & & \searrow^{j} & \\ (X,x) & & \overset{\tau}{\longrightarrow} & & (Y,y) \\ \phi\downarrow & & & & \downarrow\psi \\ (S,s) & & \overset{\sigma}{\longrightarrow} & & (T,t) \end{array}$$

Definition 3.1.7. *Two (linearly) admissible deformations over the same base space (S,s) are* isomorphic *if there exists a morphism (τ,σ) with τ an isomorphism and σ the identity map.*

Proposition 3.1.8. *Let $(i,\phi)\colon (D,0) \hookrightarrow (X,x) \longrightarrow (S,s)$ be a (linearly) admissible deformation and let $\psi\colon (T,t) \longrightarrow (S,s)$ be a morphism of complex space germs.*

Consider the commutative diagram

$$\begin{array}{ccc} & (D,0) & \\ \swarrow^{\psi^* i} & & \searrow^{i} \\ (X,x)\times_{(S,s)}(T,t) & \xrightarrow{\ \tau\ } & (X,x) \\ \downarrow{\scriptstyle \psi^*\phi} & & \downarrow{\scriptstyle \phi} \\ (T,t) & \xrightarrow{\ \psi\ } & (S,s) \end{array}$$

where $(X,x)\times_{(S,s)}(T,t) := \{(x,t)\in(X,x)\times(T,t) \mid \psi(t)=\phi(x)\}$*,* $\psi^*\phi$*, respectively* τ*, are induced by the second, respectively the first, projection and* $\psi^* i = (\tau|_{(\psi^*\phi)^{-1}(t)})^{-1}\circ i$*. Then* $(\psi^* i, \psi^*\phi)\colon (D,0)\hookrightarrow (X,x)\times_{(S,s)}(T,t)\longrightarrow (T,t)$ *is a (linearly) admissible deformation of* $(D,0)$ *over* $(T,0)$*.*

Proof. Because $(X,x)\subset(\mathbb{C}^n\times S,(0,s))$, then $(Y,0):=(X,x)\times_{(S,s)}(T,t)\subset(\mathbb{C}^n\times S,(0,s))\times_{(S,s)}(T,t)=(\mathbb{C}^n\times T,(0,t))$. By Remark 3.1.5, (X,x) is an hypersurface and hence it exists a $F\in\mathcal{O}_{\mathbb{C}^n\times S,(0,s)}$ such that $X=V(F)$, then, by construction, $Y=V(G)$, where $G=F\circ(id_{(\mathbb{C}^n,0)}\times\psi)$. This implies that if $\sigma\in\mathrm{Der}(-\log Y/T)$, then $\sigma=\sum_i\alpha\sigma_i^*$, where $\alpha_i\in\mathcal{O}_{\mathbb{C}^n\times T,(0,t)}$ and $\sigma_i^*=\sum_{j=1}^n\nu_{i,j}^*\partial/\partial x_j$ for $\nu_{i,j}^*=\nu_{i,j}\circ(id_{(\mathbb{C}^n,0)}\times\psi)$, where $\sum_{j=1}^n\nu_{i,j}\partial/\partial x_j\in\mathrm{Der}(-\log X/S)$. This implies that $(\psi^* i,\psi^*\phi)\colon(D,0)\hookrightarrow(X,x)\times_{(S,s)}(T,t)\longrightarrow(T,t)$ is a (linearly) admissible deformation. □

Definition 3.1.9. *The (linearly) admissible deformation of the previous Proposition is called the* induced (linearly) admissible deformation by ψ from (i,ϕ)*, or just the* pull-back deformation.

In the following, we denote by **Art** the category of local Artin rings with residue field k and by **Set** the category of pointed sets with distinguished element $*$. We can now introduce the functors relative to the two types of deformation that we are interested in.

Definition 3.1.10. *Let* $(D,0)\subset(\mathbb{C}^n,0)$ *be a germ of a free divisor. Define the functor* $\mathbf{FD}_D\colon\mathbf{Art}\longrightarrow\mathbf{Set}$ *by setting*

$$\mathbf{FD}_D(A) := \left\{\begin{array}{c}\textit{Isomorphism classes of admissible}\\ \textit{deformations of } (D,0) \textit{ over } \operatorname{Spec} A\end{array}\right\}.$$

If $(D,0)\subset(\mathbb{C}^n,0)$ *is a germ of a linear free divisor, we define similarly the functor*

$\mathbf{LFD}_D\colon \mathbf{Art} \longrightarrow \mathbf{Set}$ *by setting*

$$\mathbf{LFD}_D(A) := \left\{ \begin{array}{c} \textit{Isomorphism classes of linearly} \\ \textit{admissible deformations of } (D,0) \textit{ over } \operatorname{Spec} A \end{array} \right\}.$$

In order to fit into the general pattern as described in Appendix A, we need to check some technical properties of the functors $\mathbf{FD}_D$ and $\mathbf{LFD}_D$.

Theorem 3.1.11. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a free divisor. Then the functor $\mathbf{FD}_D$ satisfies Schlessinger's conditions (H1) and (H2) from Definition A.1.13. Moreover, if $(D,0)$ is linear, then also the functor $\mathbf{LFD}_D$ satisfies conditions (H1) and (H2).*

Proof. Let $A' \longrightarrow A$ and $A'' \longrightarrow A$ be maps in **Art** such that the latter is a small extension, see Definition A.1.2. Consider now $X \in \mathbf{FD}_D(A), X' \in \mathbf{FD}_D(A')$ and $X'' \in \mathbf{FD}_D(A'')$. Define $Y := (D, \mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''})$, by Lemma A.1.18, it is flat over $A' \times_A A''$ and it is an element of $\mathbf{FD}_D(A' \times_A A'')$. Hence the map $\tau_{A',A'',A}$ of $(H1)$ is surjective.

We want to show now that $\tau_{A',A'',A}$ is a bijection in the case $A'' = k[\epsilon]$ and $A = k$. Let $W \in \mathbf{FD}_D(A' \times_A A'')$ restrict to X' and X''. Then we can choose immersions $q'\colon X' \hookrightarrow W$ and $q''\colon X'' \hookrightarrow W$. Since these maps are all compatible with the immersions from D, they agree with the chosen maps $u'\colon X \hookrightarrow X'$ and $u''\colon X \hookrightarrow X''$, since in this case $X = D$. Now by the universal property of fibered product of rings, there is a map $Y \longrightarrow W$ compatible with the above maps. Since Y and W are both flat over $A' \times_A A''$, and the map becomes an isomorphism when restricted to D, we find that, by Lemma A.1.20, Y is isomorphic to W and hence they are equal as elements of $\mathbf{FD}_D(A' \times_A A'')$.

The previous proof works similarly also for the functor $\mathbf{LFD}_D$. ☐

Notice that the above proof of (H2) works also for any small extension A'' of k.

We now translate the requirement (3.1) of Definition 3.1.1 in a more geometrical way and to describe its consequences.

Proposition 3.1.12. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a (linear) free divisor. Then a deformation of $(D,0)$ is a (linearly) admissible deformation if and only if the singular locus of $(D,0)$ is deformed in a flat way.*

Proof. Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be a defining equation for $(D,0)$ and let $\phi\colon (X,x) \longrightarrow (S,s)$ be a (linearly) admissible deformation of $(D,0)$. Any element of $\operatorname{Der}(-\log D)$ can be seen as a relation among $f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n$ and similarly, any element of

$\mathrm{Der}(-\log X/S)$ can be seen as a relation among $F, \partial F/\partial x_1, \dots, \partial F/\partial x_n$, where $F \in \mathcal{O}_{\mathbb{C}^n \times S,(0,s)}$ is a defining equation for (X,x). The requirement (3.1) of Definition 3.1.1 holds true if and only if any relation among $f, \partial f/\partial x_1, \dots, \partial f/\partial x_n$ lifts to a relation among $F, \partial F/\partial x_1, \dots, \partial F/\partial x_n$ and this is equivalent to deform the singular locus of $(D,0)$ in a flat way, see [30], Chapter I, Proposition 1.91. □

Notice that by the previous Proposition and Theorem 1.1.21, $\mathcal{O}_{\mathbb{C}^n \times S,(0,s)}/J(X)_{rel}$ is a Cohen-Macaulay $\mathcal{O}_{\mathbb{C}^n \times S,(0,s)}$-module, where $J(X)_{rel} = (F, \partial F/\partial x_1, \dots, \partial F/\partial x_n)$ and $F \in \mathcal{O}_{\mathbb{C}^n \times S,(0,s)}$ is a defining equation for (X,x).

One might think that the requirement (3.1) forces also (X,x) to be a free divisor, but by Example 3.1.4, we see that the total space of an admissible deformation might not be free. However, we have the following:

Proposition 3.1.13. *In the situation of Definition 3.1.1, the requirement* (3.1) *implies that* $\mathrm{Der}(-\log X/S)$ *is a locally free* $\mathcal{O}_{\mathbb{C}^n \times S,(0,s)}$*-module of rank n.*

Proof. By Proposition 3.1.12, the singular locus of $(D,0)$ is deformed flatly and so $\mathcal{O}_{\mathbb{C}^n \times S,(0,s)}/J(X)_{rel}$ is a flat $\mathcal{O}_{S,s}$-module and represents a deformation of $\mathcal{O}_{\mathbb{C}^n,0}/J(D)$, where $J(X)_{rel} = (F, \partial F/\partial x_1, \dots, \partial F/\partial x_n)$ and $J(D) = (f, \partial f/\partial x_1, \dots, \partial f/\partial x_n)$. Hence, a free resolution of $\mathcal{O}_{\mathbb{C}^n,0}/J(D)$ lifts to a free resolution of $\mathcal{O}_{\mathbb{C}^n \times S,(0,s)}/J(X)_{rel}$. Because $(D,0)$ is free, then a free resolution of $\mathcal{O}_{\mathbb{C}^n \times S,(0,s)}/J(X)_{rel}$ looks like

$$0 \longrightarrow \mathcal{O}^n_{\mathbb{C}^n \times S,(0,s)} \longrightarrow \mathcal{O}^{n+1}_{\mathbb{C}^n \times S,(0,s)} \xrightarrow{(F,\partial F/\partial x_1,\dots,\partial F/\partial x_n)} \mathcal{O}_{\mathbb{C}^n \times S,(0,s)} \longrightarrow$$

$$\longrightarrow \mathcal{O}_{\mathbb{C}^n \times S,(0,s)}/J(X)_{rel} \longrightarrow 0$$

But as explained in Proposition 3.1.12, we can identify $\mathrm{Der}(-\log X/S)$ with the syzygy module of $(F, \partial F/\partial x_1, \dots, \partial F/\partial x_n)$, and hence, it is locally free of rank n. □

The "naive" idea of (linearly) admissible deformations is to construct families where each fiber is a (linear) free divisor. However, as seen in the previous two Propositions, the requirement (3.1) is a much stronger property. In fact we have the following:

Remark 3.1.14. *Let* $(D,0) \subset (\mathbb{C}^n,0)$ *be a singular free divisor with a quasi-homogeneous defining equation* f*. Then we can consider* $(X,0) = (V(f-t),0) \subset (\mathbb{C}^n \times \mathbb{C},0)$ *and* ϕ *the projection on* $(\mathbb{C},0)$*. In this case each fiber is a free divisor but this is not an admissible deformation of* $(D,0)$

Proof. Because f is quasi-homogeneous, we can take $\chi, \sigma_1, \ldots, \sigma_{n-1}$ as a basis of $\mathrm{Der}(-\log D)$, where $\chi = \sum_{i=1}^n \alpha_i x_i \partial/\partial x_i$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ is the Euler vector field and $\sigma_1, \ldots, \sigma_{n-1}$ annihilate f. Hence $\chi(f) = \sum_{i=1}^n \alpha_i x_i \partial f/\partial x_i = f$. Notice that because $(X, 0)$ is non-singular, it is a free divisor in $(\mathbb{C}^n \times \mathbb{C}, 0)$ and so we can take as Saito matrix for $(X, 0)$, the matrix

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \partial f/\partial x_1 & \partial f/\partial x_2 & \cdots & \partial f/\partial x_n & f - t \end{bmatrix}$$

Let λ_i be the vector field represented by the i-th column of A. Consider now the vector fields $\sigma_i^* = \sigma_i$ seen as a vector field in $\mathbb{C}^n \times \mathbb{C}$ and $\tau_i = t\lambda_i + \partial f/\partial x_i \lambda_{n+1} - \partial f/\partial x_i \sum_{j=1}^n \alpha_j x_j \lambda_j$. Clearly, $\sigma_i^*(f - t) = \sigma_i(f) = 0$ and so $\sigma_i^* \in \mathrm{Der}(-\log X/\mathbb{C})$. Similarly, $\tau_i \in \mathrm{Der}(-\log X/\mathbb{C})$ because $\tau_i \in \mathrm{Der}(-\log X)$ and its coefficient of $\partial/\partial t$ is equal to $t\partial f/\partial x_i + \partial f/\partial x_i(f - t) - \partial f/\partial x_i \sum_{j=1}^n \alpha_j x_j \partial f/\partial x_j = \partial f/\partial x_i(f - \chi(f)) = 0$. This implies that we have an inclusion $\langle \sigma_1^*, \ldots, \sigma_{n-1}^*, \tau_1, \ldots, \tau_n \rangle \subset \mathrm{Der}(-\log X/\mathbb{C})$. However, because $\sigma_1, \ldots, \sigma_{n-1}$ are the generators of $\mathrm{Ann}(D)$, then any element of $\mathrm{Der}(-\log X/\mathbb{C})$ that is a linear combination of $\lambda_1, \ldots, \lambda_n$ is a linear combinations of $\sigma_1^*, \ldots, \sigma_{n-1}^*$. Consider now an element of $\mathrm{Der}(-\log X/\mathbb{C})$ that can be written as a linear combination of the λ_i involving λ_{n+1}. Because it is independent of $\partial/\partial t$, then the coefficient of λ_{n+1} is forced to be in the Jacobian ideal of f. Because t appear only in λ_{n+1}, this implies that, modulo the σ_i^*, it is a linear combination of $\tau_1, \ldots, \tau_n$. Hence $\sigma_1^*, \ldots, \sigma_{n-1}^*, \tau_1, \ldots, \tau_n$ generate $\mathrm{Der}(-\log X/\mathbb{C})$.

Because f is singular, $\partial f/\partial x_i \in (x_1, \ldots, x_n)$ for all $i = 1, \ldots, n$ and so each τ_i has weight bigger than zero, i.e. $\deg(\partial f/\partial x_i \alpha_j x_j) - \deg(x_j) > 0$. This tells us that the Euler vector field $\chi \notin \mathrm{Der}(-\log X/\mathbb{C})/\mathfrak{m}_{\mathbb{C},0}\,\mathrm{Der}(-\log X/\mathbb{C})$ because χ has weight zero and is not a linear combination of $\sigma_1, \ldots, \sigma_{n-1}$. □

In the previous Remark, if the defining equation f is smooth, then the deformation defined is an admissible deformation. In fact, we can suppose $f = x_1$ and we can take as Saito matrix

$$\begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

With a similar argument as the proof of the previous Remark, we have that $\mathrm{Der}(-\log X/\mathbb{C})$ is generated by the columns of the matrix

$$\begin{bmatrix} x_1 - t & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and hence the requirement (3.1) of the Definition 3.1.1 is fulfilled.

Definition 3.1.15. *A (linearly) admissible deformation* $(i, \phi)\colon (D, 0) \hookrightarrow (X, x) \longrightarrow (S, s)$ *is called* trivial *if it is isomorphic to the* product deformation

$$(D, 0) \overset{j}{\hookrightarrow} (D \times S, (0, s)) \xrightarrow{p} (S, s)$$

with j *the canonical inclusion and* p *the second projection.*

Remark 3.1.16. *Let* $(i, \phi)\colon (D, 0) \hookrightarrow (X, x) \longrightarrow (S, s)$ *be a (linearly) admissible deformation. Then it is a trivial (linearly) admissible deformation if and only if it is trivial as deformation of* $(D, 0)$ *as complex space germ.*

Definition 3.1.17. *A (linear) free divisor* $(D, 0) \subset (\mathbb{C}^n, 0)$ *is called* rigid *if all its (linearly) admissible deformations are trivial and it is called* formally rigid *if all its (linearly) admissible deformations are trivial but the isomorphisms involved are only formal.*

Definition 3.1.18. *The complex space germ* T_ϵ *consists of one point with local ring* $\mathbb{C}[\epsilon] = \mathbb{C} + \epsilon \cdot \mathbb{C}, \epsilon^2 = 0$*, that is* $\mathbb{C}[\epsilon] = \mathbb{C}[t]/(t^2)$*, where* t *is an indeterminate. Thus* $T_\epsilon = \mathrm{Spec}(\mathbb{C}[t]/(t^2))$.

Definition 3.1.19. *An* infinitesimal (linearly) admissible deformation *of a (linear) free divisor* $(D, 0) \subset (\mathbb{C}^n, 0)$ *is a (linearly) admissible deformation of* $(D, 0)$ *over* T_ϵ.

Definition 3.1.20. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a free divisor. Then $\mathcal{FT}^1(D) := \mathbf{FD}_D(\mathbb{C}[t]/(t^2))$. If $(D,0)$ is linear, then $\mathcal{LFT}^1(D) := \mathbf{LFD}_D(\mathbb{C}[t]/(t^2))$.*

3.2 The shape of (linearly) admissible deformations

In this section, we show how to characterise a (linearly) admissible deformation from the equation but also just looking at the logarithmic vector fields.

Definition 3.2.1. *Consider $f \in \mathbb{C}[[x_1,\dots,x_n]]$ such that $f(0)=0$. Then an* unfolding *of f is a power series $F \in \mathbb{C}[[x_1,\dots,x_n,t_1,\dots,t_r]]$ with $F(x,0)=f(x)$, that is*

$$F(x,t) = f(x) + \sum_{\substack{i\in\mathbb{N}^r \\ i\neq(0,\dots,0)}} g_i(x)t^i$$

for some $g_i \in \mathbb{C}[[x_1,\dots,x_n]]$.

Proposition 3.2.2. ([30], Chapter II, Proposition 1.5) *Consider a commutative diagram of complex space germs*

$$\begin{array}{ccc} (X_0,x) & \hookrightarrow & (X,x) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ (S_0,s) & \hookrightarrow & (S,s) \end{array}$$

where the horizontal maps are closed embeddings. Assume that f_0 factors as

$$(X_0,x) \xhookrightarrow{i_0} (\mathbb{C}^n,0)\times(S_0,s) \xrightarrow{p_0} (S_0,s),$$

with i_0 a closed embedding and p_0 the second projection. Then there exists a commutative diagram

$$\begin{array}{ccccc} & (X_0,x) & \hookrightarrow & (X,x) & \\ & {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle i} & \\ f_0 & (\mathbb{C}^n,0)\times(S_0,s) & \hookrightarrow & (\mathbb{C}^n,0)\times(S,s) & f \\ & {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} & \\ & (S_0,s) & \hookrightarrow & (S,s) & \end{array}$$

with i a closed embedding and p the second projection. That is, the embedding of f_0 over (S_0,s) extends to an embedding of f over (S,s).

The previous Proposition allows us to describe the defining equation of the total space of an (linearly) admissible deformation.

Corollary 3.2.3. *Any (linearly) admissible deformation of* $(D,0) = (V(f),0) \subset (\mathbb{C}^n,0)$ *over a complex space germ* (S,s) *is of the form* $(X,(0,s)) = (V(F),(0,s)) \subset (\mathbb{C}^n \times S,(0,s))$*, for some unfolding* F *of* f *with* ϕ *just the projection on* (S,s)*.*

Similarly as the classic deformation theory of singularities, see [30], we have the following:

Remark 3.2.4. *Let* $f \in \mathcal{O}_{\mathbb{C}^n,0}$ *be a defining equation for a germ of a (linear) free divisor* $(D,0) \subset (\mathbb{C}^n,0)$*. Any infinitesimal (linearly) admissible deformation of* $(D,0)$ *is of the form* $(X,0) = (V(f+\epsilon\cdot f'),0) \subset (\mathbb{C}^n \times T_\epsilon,0)$*, for some* $f' \in \mathcal{O}_{\mathbb{C}^n,0}$ *where* ϕ *is just the projection on* T_ϵ*.*

By Remark 3.1.16 and Chapter II, 1.4 from [30], we have the following:

Remark 3.2.5. *An infinitesimal (linearly) admissible deformation* $(X,0) = (V(f+\epsilon\cdot f'),0) \longrightarrow T_\epsilon$ *is trivial if and only if there is an isomorphism*

$$\mathcal{O}_{\mathbb{C}^n\times T_\epsilon,0}/(f) \cong \mathcal{O}_{\mathbb{C}^n\times T_\epsilon,0}/(f+\epsilon\cdot f')$$

which is the identity modulo ϵ *and which is compatible with the inclusion of* $\mathcal{O}_{T_\epsilon}$ *in* $\mathcal{O}_{\mathbb{C}^n\times T_\epsilon,0}$*. Such an isomorphism is induced by an automorphism* φ *of* $\mathcal{O}_{\mathbb{C}^n\times T_\epsilon,0}$*, mapping* $x_j \mapsto x_j + \epsilon\sigma_j(x)$ *and* $\epsilon \mapsto \epsilon$ *such that*

$$(\varphi^* f) = (f(x+\epsilon\cdot\sigma(x))) = (f+\epsilon\cdot f'),$$

where $x = (x_1,\dots,x_n)$ *and* $\sigma = \sum_{j=1}^n \sigma_j \partial/\partial x_j$*.*

We are now going to prove a relative Saito's Lemma in order to be able to characterise an (linearly) admissible deformation by logarithmic vector fields.

Lemma 3.2.6. *Let* (S,s) *be a complex space germ with an embedding* $(S,s) \subset (\mathbb{C}^r,0)$ *and let* $t = (t_1,\dots,t_r)$ *be coordinates on the ambient space* $(\mathbb{C}^r,0)$*. Let* $(X,x) \subset (\mathbb{C}^n \times S,(0,s))$ *be a (linearly) admissible deformation of a germ of a (linear) free divisor* $(D,0) \subset (\mathbb{C}^n,0)$ *and let* $h_p = 0$ *be a reduced equation for* (X,x)*, locally at* $p = (x_0,t_0) \in (\mathbb{C}^n \times S,(0,s))$*. Suppose* $\delta_i' = \sum_{j=1}^n a_i^j(x,t)\partial/\partial x_j \in \mathrm{Der}_p(-\log X/S)$*,* $\forall\, i = 1,\dots,n$*. Then* $\det(a_i^j) \in (h_p)\mathcal{O}_{\mathbb{C}^n\times S,p}$*.*

Proof. Suppose that $\det(a_i^j)$ does not vanish at p. Hence it does not vanish in a small neighbourhood U of p. This implies that $\delta_1,\dots,\delta_n$ are linearly independent in U. Consider now the fibre X_{t_0}. We have that $\widetilde{\delta_i} = \sum_{j=1}^n a_i^j(x,t_0)\partial/\partial x_j \in$

$\mathrm{Der}(-\log X_{t_0})$ and they are linearly independent, but this implies that X_{t_0} is n-dimensional, contradicting the fact that (X, x) is a flat (linearly) admissible deformation of $(D, 0)$, that is $(n-1)$-dimensional. $\square$

Proposition 3.2.7. *Let (S, s) be a complex space germ with an embedding $(S, s) \subset (\mathbb{C}^r, 0)$ and let $t = (t_1, \ldots, t_r)$ be coordinates on the ambient space $(\mathbb{C}^r, 0)$. Let $(X, x) \subset (\mathbb{C}^n \times S, (0, s))$ be a (linearly) admissible deformation of a germ of a (linear) free divisor $(D, 0) \subset (\mathbb{C}^n, 0)$ and let $h_p = 0$ be a reduced equation for (X, x), locally at $p = (x_0, t_0) \in (\mathbb{C}^n \times S, (0, s))$. Then there exist $\delta'_1, \ldots, \delta'_n \in \mathrm{Der}_p(-\log X/S)$ with $\delta'_i = \sum_{j=1}^n a_i^j(x, t)\partial/\partial x_j$, such that $\det(a_i^j)$ is a multiple unit of h_p.*

Proof. By Proposition 3.1.13, $\mathrm{Der}_p(-\log X/S)$ is a free $\mathcal{O}_{\mathbb{C}^n \times S, p}$-module of rank n. Since $\mathrm{Der}(-\log X/S)$ is coherent, then there exists a neighbourhood U of p such that $\mathrm{Der}(-\log X/S)|_U$ is free. Let $\delta'_1, \ldots, \delta'_n$ be a basis of $\mathrm{Der}(-\log X/S)|_U$ with $\delta'_i = \sum_{j=1}^n a_i^j(x, t)\partial/\partial x_j$. By Lemma 3.2.6, $\det(a_i^j) = g h_p$, where g is a holomorphic function on U. Since $\partial/\partial x_1, \ldots, \partial/\partial x_n$ is a basis for $p \in U \setminus X$, then g does not vanish on $U \setminus X$. At a smooth point $p \in X$, we can suppose $X = V(x_1)$ and hence, we may choose as a basis of $\mathrm{Der}(-\log X/S)$ on $\mathrm{Reg}(X) \cap U$ the vector fields $x_1\partial/\partial x_1, \ldots, \partial/\partial x_n$. Thus g does not vanish anywhere on $U \setminus (U \cap \mathrm{Sing}(X))$, but because $\mathrm{codim}_U(U \cap \mathrm{Sing}(X)) > 1$, then g does not vanish anywhere on U and so it is a unit. $\square$

Lemma 3.2.8. *Let R be a commutative ring, let A and B be two $n \times n$ matrices with coefficients in R and let $a_1, \ldots, a_n$ be the columns of A. Then*

$$\sum_{i=1}^n \det[a_1, \ldots, a_{i-1}, Ba_i, a_{i+1}, \ldots, a_n] = \mathrm{trace}(B)\det(A).$$

Proof. It is know that if we consider a $n \times n$ matrix C with columns $c_1, \ldots, c_n$, then

$$d_A \det(C) = \sum_{i=1}^n \det[a_1, \ldots, a_{i-1}, c_i, a_{i+1}, \ldots, a_n],$$

where d is the tangent map. Then we have the following equalities

$$\sum_{i=1}^n \det[a_1, \ldots, a_{i-1}, Ba_i, a_{i+1}, \ldots, a_n] = d_A \det(BA)$$

$$= \frac{d}{dt}(\det(A + tBA))|_{t=0} = \det(A)\frac{d}{dt}(\det(I + tB))|_{t=0}$$
$$= \det(A) d_I \det(B) = \det(A)\,\mathrm{trace}(B).$$

□

Lemma 3.2.9. *Let (S,s) be a complex space germ with an embedding $(S,s) \subset (\mathbb{C}^r,0)$ and let $t=(t_1,\dots,t_r)$ be coordinates on the ambient space $(\mathbb{C}^r,0)$. Consider $(D,0)=(V(f),0)\subset(\mathbb{C}^n,0)$ a germ of a (linear) free divisor such that $\mathrm{Der}_{x_0}(-\log D)$ is generated by $\delta_1,\dots,\delta_n$. Let $\delta'_i=\sum_{j=1}^n a_i^j(x,t)\partial/\partial x_j$, $i=1,\dots,n$, be a system of holomorphic vector fields at $(x_0,s)\in(\mathbb{C}^n\times S,(0,s))$ such that*

1. $\delta'_i|_{\mathbb{C}^n,x_0}=\delta_i$ *for all* $i=1,\dots,n$;
2. $[\delta'_i,\delta'_j]\in\sum_{k=1}^n \mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}\delta'_k$ *for* $i,j=1,\dots,n$;
3. $\det(a_i^j)=h$ *defines a reduced hypersurface* X.

Then for $X=\{h(x,t)=0\}$, $\delta'_1,\dots,\delta'_n$ belongs to $\mathrm{Der}_{(x_0,s)}(-\log X/S)$, $\{\delta'_1,\dots,\delta'_n\}$ is a free basis of $\mathrm{Der}_{(x_0,s)}(-\log X/S)$ and X is a (linearly) admissible deformation of $(D,0)$ over (S,s).

Proof. First of all we need to show that each $\delta'_k\in\mathrm{Der}_{(x_0,s)}(-\log X/S)$. Because $[\delta'_k,\delta'_j]=\delta'_k(\delta'_j)-\delta'_j(\delta'_k)$, where $\delta'_k(\delta'_j)=\sum_{l=1}^n\delta'_k(a_l^j)\partial/\partial x_l$, then we have the following equalities

$$\delta'_k(h)=\delta'_k(\det[\delta'_1,\dots,\delta'_n])=\sum_{j=1}^n\det[\delta'_1\dots,\delta'_{j-1},\delta'_k(\delta'_j),\delta'_{j+1},\dots,\delta'_n]$$

$$=\sum_{j=1}^n\det[\delta'_1\dots,\delta'_{j-1},[\delta'_k,\delta'_j]+\delta'_j(\delta'_k),\delta'_{j+1},\dots,\delta'_n]$$

$$=\sum_{j=1}^n\det[\delta'_1\dots,\delta'_{j-1},[\delta'_k,\delta'_j],\delta'_{j+1},\dots,\delta'_n]+\sum_{j=1}^n\det[\delta'_1\dots,\delta'_{j-1},\delta'_j(\delta'_k),\delta'_{j+1},\dots,\delta'_n].$$

By 2 and Lemma 3.2.6, $\det[\delta'_1\dots,\delta'_{j-1},[\delta'_k,\delta'_j],\delta'_{j+1},\dots,\delta'_n]\in(h)\mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}$ for all $j=1,\dots,n$, and so the first part of the last equality is in $(h)\mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}$. Furthermore, if we consider the matrices $A=[\delta'_1,\dots,\delta'_n]$ and $B=(\partial a_k^i/\partial x_j)_{i,j=1,\dots,n}$, we can apply Lemma 3.2.8 and obtain

$$\sum_{j=1}^n\det[\delta'_1\dots,\delta'_{j-1},\delta'_j(\delta'_k),\delta'_{j+1},\dots,\delta'_n]=\sum_{i=1}^n\frac{\partial a_k^i}{\partial x_i}h\in(h)\mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}.$$

This shows that $\delta'_k(h)\in(h)\mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}$ and so $\delta'_k\in\mathrm{Der}_{(x_0,s)}(-\log X/S)$ for all $k=1,\dots,n$.

Notice now that by 1 and 3, $h|_{\mathbb{C}^n,x_0} = f$. Moreover, by 1

$$\mathrm{Der}_{x_0}(-\log D) \subset \mathrm{Der}_{(x_0,s)}(-\log X/S)/\mathfrak{m}_{S,s}\,\mathrm{Der}_{(x_0,s)}(-\log X/S).$$

Consider $\sigma \in \mathrm{Der}_{(x_0,s)}(-\log X/S)$ such that $\sigma|_{\mathbb{C}^n,x_0} \notin \mathrm{Der}_{x_0}(-\log D)$. But $\sigma(h) = \alpha h$ for some $\alpha \in \mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}$. Hence $(\sigma(h))|_{\mathbb{C}^n,x_0} = \sigma|_{\mathbb{C}^n,x_0}(f) = \alpha|_{\mathbb{C}^n,x_0} f$ and so $\sigma|_{\mathbb{C}^n,x_0} \in \mathrm{Der}_{x_0}(-\log D)$, but this is a contradiction. Hence $\mathrm{Der}_{x_0}(-\log D) = \mathrm{Der}_{(x_0,s)}(-\log X/S)/\mathfrak{m}_{S,s}\,\mathrm{Der}_{(x_0,s)}(-\log X/S)$ and so X is a (linearly) admissible deformation of $(D,0)$ over (S,s).

Consider $\sigma \in \mathrm{Der}_{(x_0,s)}(-\log X/S)$. We want to prove that $\sigma \in \sum_{i=1}^n \mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}\delta'_i$. By Cramer's rule, $h\partial/\partial x_j \in \sum_{i=1}^n \mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}\delta'_i$ for all $j = 1,\dots,n$. Hence we can consider $h\sigma = \sum_{i=1}^n f_i\delta'_i$, for some $f_i \in \mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}$. By Lemma 3.2.6, we have that $\det[\delta'_1,\dots,\delta'_{i-1},\sigma,\delta'_{i+1},\dots,\delta'_n] \in (h)\mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}$. Thus

$$h\det[\delta'_1,\dots,\delta'_{i-1},\sigma,\delta'_{i+1},\dots,\delta'_n]$$

$$= \det[\delta'_1,\dots,\delta'_{i-1},h\sigma,\delta'_{i+1},\dots,\delta'_n]$$

$$= \det[\delta'_1,\dots,\delta'_{i-1},f_i\delta'_i,\delta'_{i+1},\dots,\delta'_n]$$

$$= f_i\det[\delta'_1,\dots,\delta'_n] = f_i h \in (h^2)\mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}.$$

Thus $f_i \in (h)\mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}$ for all i. This show that we can write $\sigma = \sum_{i=1}^n (f_i/h)\delta'_i \in \sum_{i=1}^n \mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}\delta'_i$. □

Notice that if we consider S to be a reduced point, then the previous Lemma is the same statement of Lemma 1.1.18, but the proof is different from the one given in [48].

We can now state and prove the main result of this section:

Theorem 3.2.10. *Let $(D,0) = (V(f),0) \subset (\mathbb{C}^n,0)$ be a germ of a free divisor and let $\delta_1,\dots,\delta_n$ be a set of generators for $\mathrm{Der}(-\log D)$. Any infinitesimal admissible deformation of $(D,0)$ can be represented by n classes of vector fields $\tilde{\delta}_1,\dots,\tilde{\delta}_n \in \mathrm{Der}_{\mathbb{C}^n}/\mathrm{Der}(-\log D)$ such that the $\mathcal{O}_{\mathbb{C}^n\times T_\epsilon,0}$-module generated by $\delta'_1 = \delta_1 + \epsilon\cdot\tilde{\delta}_1,\dots,\delta'_n = \delta_n + \epsilon\cdot\tilde{\delta}_n$ is closed under Lie brackets. Moreover, if the deformation is linearly admissible, then the coefficients of all $\tilde{\delta}_i$ need to be linear functions too.*

Proof. Let $(X,x) \subset (\mathbb{C}^n \times T_\epsilon, 0)$ be a infinitesimal (linearly) admissible deformation of $(D,0)$. By Remark 3.2.4, it is of the form $(X,0) = (V(f+\epsilon\cdot f'),0) \subset (\mathbb{C}^n \times T_\epsilon, 0)$. By Proposition 3.2.7, the fact that $(X,0)$ is the total space of an infinitesimal (linearly) admissible deformation of $(D,0)$ implies that exists a $n\times n$ matrix $A(\epsilon)$ with coefficients in $\mathbb{C}[x_1,\ldots,x_n,\epsilon]$ such that $\det A(\epsilon)=(f+\epsilon\cdot f')$, but $\epsilon^2=0$ implies that we can write $A(\epsilon)=B+\epsilon\cdot C$, where B and C are $n\times n$ matrices with coefficients in $\mathbb{C}[x_1,\ldots,x_n]$. Hence $f=\det A(0)=\det B$ and so B is a Saito matrix for $(D,0)$. We can then take δ_i as the column of B and $\tilde{\delta}_i$ as the column of C and this prove that the Lie algebra $\operatorname{Der}(-\log X/T_\epsilon)$ is generated by $\delta_1+\epsilon\cdot\tilde{\delta}_1,\ldots,\delta_n+\epsilon\cdot\tilde{\delta}_n$ as required. Because $\operatorname{Der}(-\log X/T_\epsilon)$ is a Lie algebra, then $[\delta'_i,\delta'_j]\in\operatorname{Der}(-\log X/T_\epsilon)$ for all $i,j=1,\ldots,n$, but then $[\delta'_i,\delta'_j]\in\sum_{k=1}^n\mathcal{O}_{\mathbb{C}^n\times S,(x_0,s)}\delta'_k$ for $i,j=1,\ldots,n$.

We consider the classes of $\tilde{\delta}_1,\ldots,\tilde{\delta}_n$ modulo $\operatorname{Der}(-\log D)$, because if $\tilde{\delta}_1,\ldots,\tilde{\delta}_n\in\operatorname{Der}(-\log D)$, then $f'\in(f)\mathcal{O}_{\mathbb{C}^n,0}$ and hence, by [30], Chapter II, 1.4, the admissible deformation is trivial.

On the other hand, let $\tilde{\delta}_1,\ldots,\tilde{\delta}_n\in\operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D)$ be n classes of vector fields such that the $\mathcal{O}_{\mathbb{C}^n\times T_\epsilon,0}$-module generated by $\delta_1+\epsilon\cdot\tilde{\delta}_1,\ldots,\delta_n+\epsilon\cdot\tilde{\delta}_n$ is closed under Lie brackets. The determinant of the matrix of coefficients $[\delta_1+\epsilon\cdot\tilde{\delta}_1,\ldots,\delta_n+\epsilon\cdot\tilde{\delta}_n]$ is equal to $f+\epsilon\cdot f'$ and so by Lemma 3.2.9 it is enough to show that this determinant is reduced. First, noticed that for $\epsilon=0$ the determinant is equal to f and hence, it is reduced. Now, reducedness is an open property and so the result holds.

The last part of the statement is trivial. □

3.3 The complexes $\mathcal{C}^\bullet$ and $\mathcal{C}_0^\bullet$

As seen in the Example 2.2.4, we can consider the $\mathcal{O}_{\mathbb{C}^n}$-module $\operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D)$ and introduce on it the integrable logarithmic connection defined by $\nabla_\delta=[\delta,-]$. This fact allows us to treat $\operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D)$ as a $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module and hence we can use the complex of Definition 2.2.9:

Definition 3.3.1. *Let $\mathcal{C}^\bullet$ be the complex with modules*

$$\mathcal{C}^p:=\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\bigwedge^p\operatorname{Der}(-\log D),\operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D))$$

and differentials

$$(d^p(\psi))(\delta_1\wedge\cdots\wedge\delta_{p+1}):=\sum_{i=1}^{p+1}(-1)^i[\delta_i,\psi(\delta_1\wedge\cdots\wedge\widehat{\delta_i}\wedge\cdots\wedge\delta_{p+1})]+$$

$$+\sum_{1\leq i<j\leq p+1}(-1)^{i+j-1}\psi([\delta_i,\delta_j]\wedge\delta_1\wedge\cdots\wedge\widehat{\delta_i}\wedge\cdots\wedge\widehat{\delta_j}\wedge\cdots\wedge\delta_{p+1}).$$

Notice that

$$\mathcal{C}^0 = \operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D)$$

and the map d^0 is defined by

$$d^0 \colon \mathcal{C}^0 \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\operatorname{Der}(-\log D), \operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))$$

$$\sigma \mapsto (\delta \mapsto [\delta, \sigma]).$$

Furthermore, by Corollary 2.6.4, we have that the complex $\mathcal{C}^\bullet$ is isomorphic to $\Omega^\bullet(\log D)(\operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))$.

Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be the germ of a linear free divisor. Then we denote by $(\operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))_0$ the weight zero part of $\operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D)$. Notice that it is a finite dimensional complex Lie algebra. Because the Lie bracket of two weight zero vector fields is a weight zero vector field, we can consider $(\operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))_0$ as a $\operatorname{Der}(-\log D)_0$-module via the representation

$$\varrho\colon \operatorname{Der}(-\log D)_0 \longrightarrow (\operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))_0$$

defined by

$$\varrho(\delta) := [\delta, -].$$

See B.2 for the general theory. This fact allows us to use the Lie algebra cohomology complex. See B.3 for the general theory.

Definition 3.3.2. *Let $\mathcal{C}_0^\bullet$ be the complex defined by*

$$\mathcal{C}_0^p := Hom_{\mathbb{C}}(\bigwedge^p \operatorname{Der}(-\log D)_0, (\operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))_0)$$

and the differentials

$$(d_0^p(\psi))(\delta_1 \wedge \cdots \wedge \delta_{p+1}) := \sum_{i=1}^{p+1} (-1)^i [\delta_i, \psi(\delta_1 \wedge \cdots \wedge \hat{\delta_i} \wedge \cdots \wedge \delta_{p+1})] +$$

$$+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j-1} \psi([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \wedge \hat{\delta_i} \wedge \cdots \wedge \hat{\delta_j} \wedge \cdots \wedge \delta_{p+1}).$$

3.4 Infinitesimal admissible deformations

In this section we show how to use the complex $\mathcal{C}^\bullet$, introduced in the previous section, to compute the space of first order infinitesimal admissible deformations of a germ of a free divisor. Finally, we show that if $D \subset \mathbb{C}^n$ is a Koszul free divisor such that we can put a logarithmic connection on $\operatorname{Der}_{\mathbb{C}^n}$ and $\operatorname{Der}(-\log D)$, as explained in Section 2.4, then the space of infinitesimal admissible deformations is finite dimensional.

Theorem 3.4.1. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a free divisor. Then the germ at the origin of the first cohomology sheaf of the complex $\mathcal{C}^\bullet$ is isomorphic to $\mathcal{FT}^1(D)$, i.e. $\mathcal{H}^1(\mathcal{C}^\bullet)_0 \cong \mathcal{FT}^1(D)$.*

Proof. To prove that we can identify $\mathcal{H}^1(\mathcal{C}^\bullet)_0$ with $\mathcal{FT}^1(D)$, two things have to be checked: we must first identify the elements of $\ker(d^1\colon \mathcal{C}^1 \longrightarrow \mathcal{C}^2)$ with the admissible deformations of $(D,0)$, and then we have to show that the image of $d^0\colon \mathcal{C}^0 \longrightarrow \mathcal{C}^1$ is the collection of trivial admissible deformations of $(D,0)$.

By Proposition 3.2.10, we are looking for n classes of vector fields $\tilde{\delta}_1,\dots,\tilde{\delta}_n \in \operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D)$ such that the $\mathcal{O}_{\mathbb{C}^n\times T_\epsilon,0}$-module generated by $\delta_1+\epsilon\cdot\tilde{\delta}_1,\dots,\delta_n+\epsilon\cdot\tilde{\delta}_n$ is closed under Lie brackets.

Take an element $\psi \in \ker(d^1)$, which means that

$$\psi([\delta,\nu]) - [\delta,\psi(\nu)] + [\nu,\psi(\delta)] = 0 \text{ in } \operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D)$$

for all $\delta,\nu \in \operatorname{Der}(-\log D)$. Then ψ corresponds to the admissible deformation given by the $\mathcal{O}_{\mathbb{C}^n\times T_\epsilon,0}$-module $\mathcal{L}$ generated by

$$\delta_1+\epsilon\cdot\psi(\delta_1),\dots,\delta_n+\epsilon\cdot\psi(\delta_n).$$

By $\mathbb{C}$-linearity of the Lie brackets, $\mathcal{L}$ is closed under Lie brackets if and only if for any two elements $\delta+\epsilon\cdot\psi(\delta), \nu+\epsilon\cdot\psi(\nu) \in \mathcal{L}$ we have $[\delta+\epsilon\cdot\psi(\delta),\nu+\epsilon\cdot\psi(\nu)] \in \mathcal{L}$, which is equivalent to

$$F := [\delta,\nu] + \epsilon\cdot([\delta,\psi(\nu)] - [\nu,\psi(\delta)]) \in \mathcal{L}.$$

Consider $G := [\delta,\nu]+\epsilon\cdot\psi([\delta,\nu])$ which is an element of $\mathcal{L}$, so the condition $F \in \mathcal{L}$ is equivalent to $G - F \in \mathcal{L}$, that is

$$\psi([\delta,\nu]) - [\delta,\psi(\nu)] + [\nu,\psi(\delta)] \in \operatorname{Der}(-\log D).$$

This means exactly that $\psi \in \ker(d^1)$.

Let us consider now an infinitesimal admissible deformation $(X,0) = (V(f + \epsilon\cdot f'),0)$. Then by the previous part, we have that $\operatorname{Der}(-\log X/T_\epsilon) = \langle \delta_1+\epsilon\cdot\psi(\delta_1),\dots,\delta_n+\epsilon\cdot\psi(\delta_n)\rangle$ for some $\psi \in \ker(d^1)$. By Remark 3.2.5, we have that

$f+\epsilon\cdot f'$ is trivial if and only if $(\varphi^* f) = (f(x+\epsilon\cdot\sigma(x))) = (f+\epsilon\cdot f')$, for some $\varphi\in\mathrm{Aut}(\mathbb{C}^n\times T_\epsilon)$. On the level of logarithmic vector fields this is equivalent to say that

$$\langle D_{\varphi^{-1}(x)}\varphi(\delta_1(\varphi^{-1}(x))),\dots,D_{\varphi^{-1}(x)}\varphi(\delta_n(\varphi^{-1}(x)))\rangle = \langle\delta_1+\epsilon\cdot\psi(\delta_1),\dots,\delta_n+\epsilon\cdot\psi(\delta_n)\rangle,$$

where if $h\colon X\longrightarrow Y$, then $D_x h\colon T_xX\longrightarrow T_{h(x)}Y$ is the holomorphic derivative. But we have the following equalities

$$D_{\varphi^{-1}(x)}\varphi(\delta_i(\varphi^{-1}(x))) = D_{x-\epsilon\cdot\sigma(x)}\varphi(\delta_i(x-\epsilon\cdot\sigma(x)))$$

$$=\delta_i(x-\epsilon\cdot\sigma(x))+\epsilon\cdot D_{x-\epsilon\cdot\sigma(x)}\sigma(\delta_i(x-\epsilon\cdot\sigma(x)))$$

$$=\delta_i(x)-\epsilon\cdot(D_x\delta_i(\sigma(x))-D_{x-\epsilon\cdot\sigma(x)}\sigma(\delta_i(x)))$$

$$=\delta_i(x)+\epsilon\cdot(D_x\sigma(\delta_i(x))-D_x\delta_i(\sigma(x)))=\delta_i(x)+\epsilon\cdot[\sigma,\delta_i](x)$$

and that tells us that $\psi(\delta_i)=[\sigma,\delta_i]$, i.e. $\psi\in\mathrm{image}(d^0)$. □

Lemma 3.4.2. *Let $D\subset\mathbb{C}^n$ be a free divisor. Then* $\mathrm{Der}(-\log D)$ *is a self-normalising Lie subalgebra of* $\mathrm{Der}_{\mathbb{C}^n}$*. That is, if we consider $\chi\in\mathrm{Der}_{\mathbb{C}^n}$ such that $[\chi,\delta]\in$* $\mathrm{Der}(-\log D)$ *for all $\delta\in$* $\mathrm{Der}(-\log D)$*, then $\chi\in$* $\mathrm{Der}(-\log D)$*.*

Proof. By the definition of $\mathrm{Der}(-\log D)$, it is enough to show that if we consider $p\in D$ a smooth point, then $\chi(p)\in T_pD$. Without loss of generality, we can suppose that at p the divisor D is defined by the equation $x_1=0$, that its Saito matrix is

$$[\delta_1,\cdots,\delta_n]=\begin{bmatrix} x_1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

and that $\chi(p)=\sum_{i=1}^n a_i\partial/\partial x_i$ with $a_i\in\mathcal{O}_{\mathbb{C}^n,p}$. In this way, we have reduced the problem to prove that $a_1\in(x_1)\mathcal{O}_{\mathbb{C}^n,p}$.

By hypothesis, $[\chi,\delta]\in\mathrm{Der}_p(-\log D)$ for all $\delta\in\mathrm{Der}_p(-\log D)$, and in particular $[\chi,\delta_1]=a_1\partial/\partial x_1-\sum_{i=1}^n x_1\partial a_i/\partial x_1\partial/\partial x_i=(a_1-x_1\partial a_1/\partial x_1)\partial/\partial x_1-\sum_{i=2}^n x_1\partial a_i/\partial x_1\partial/\partial x_i\in\mathrm{Der}_p(-\log D)$. Hence, $(a_1-x_1\partial a_1/\partial x_1)\in(x_1)\mathcal{O}_{\mathbb{C}^n,p}$ and so $a_1\in(x_1)\mathcal{O}_{\mathbb{C}^n,p}$ as required. □

An immediate consequence of the previous Lemma is the following:

Lemma 3.4.3. *Let $D \subset \mathbb{C}^n$ be a linear free divisor. Then* $\mathrm{Der}(-\log D)_0$ *is a self-normalising Lie subalgebra of* $(\mathrm{Der}_{\mathbb{C}^n})_0$*.*

Proposition 3.4.4. $\mathcal{H}^0(\mathcal{C}^\bullet) = 0$*.*

Proof. Consider $\sigma \in \mathcal{H}^0(\mathcal{C}^\bullet) = \ker(d^0)$. Hence, $[-, \sigma]$ is the zero map, i.e. for all $\delta \in \mathrm{Der}(-\log D)$ we have that $[\delta, \sigma] \in \mathrm{Der}(-\log D)$. Then by Lemma 3.4.2, $\sigma \in \mathrm{Der}(-\log D)$. □

Proposition 3.4.5. *Let $(D, 0)$ be a germ of a smooth divisor, then $\mathcal{FT}^1(D) = 0$.*

Proof. We can suppose $f = x_1$ and we can take as Saito matrix

$$S = [\delta_1, \dots, \delta_n] = \begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

In addition, we can represent an element of $\mathcal{C}^1$ as the column of the $n \times n$ matrix T, where T is the matrix

$$T = [\tilde{\delta}_1, \dots, \tilde{\delta}_n] = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $g_i = g_i(x_2, \dots, x_n) \in \mathcal{O}_{\mathbb{C}^n,0}$.

Because $[\delta_i, \delta_j] = 0$ for every $i, j = 1, \dots, n$, then the element represented by T is in the kernel of d^1 if and only if $g_i = -\partial g_1/\partial x_i$ for all $i = 2, \dots, n$. To show that this element is zero in cohomology, it is enough to find $\sigma \in \mathcal{C}^0 = \mathrm{Der}_{\mathbb{C}^n} / \mathrm{Der}(-\log D)$ such that $[\sigma, \delta_i] = \tilde{\delta}_i$ for all $i = 1, \dots, n$, i.e. T is in the image of d^0. Consider then $\sigma = g_1 \partial/\partial x_1$. □

Proposition 3.4.6. *Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be a germ of the normal crossing divisor. Then $\mathcal{FT}^1(D) = 0$.*

Proof. Let $f = x_1 \cdots x_n$ be a defining equation for $(D, 0)$. We can take as Saito matrix

$$S = [\delta_1, \ldots, \delta_n] = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}.$$

Moreover, we can represent an element of $\mathcal{C}^1$ as the column of the $n \times n$ matrix T, where T is the matrix

$$T = [\tilde{\delta}_1, \ldots, \tilde{\delta}_n] = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ \vdots & \vdots & & \vdots \\ g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{bmatrix}$$

and $g_{i,j} = g_{i,j}(x_1, \ldots, \hat{x}_i, \ldots, x_n) \in \mathcal{O}_{\mathbb{C}^n}$.

Because $[\delta_i, \delta_j] = 0$ for every $i, j = 1, \ldots, n$, then the element represented by T is in the kernel of d^1 if and only if $A_{i,j} = -[\delta_i, \tilde{\delta}_j] + [\delta_j, \tilde{\delta}_i] \in \operatorname{Der}(-\log D)$ for all $i, j = 1, \ldots, n$. Let us suppose that $i < j$, then we have that

$$A_{i,j} = \begin{bmatrix} -x_i \partial g_{1,j}/\partial x_i \\ \vdots \\ g_{i,j} \\ \vdots \\ -x_i \partial g_{j,j}/\partial x_i \\ \vdots \\ -x_i \partial g_{n,j}/\partial x_i \end{bmatrix} + \begin{bmatrix} x_j \partial g_{1,i}/\partial x_j \\ \vdots \\ x_j \partial g_{i,i}/\partial x_j \\ \vdots \\ -g_{j,i} \\ \vdots \\ x_j \partial g_{n,i}/\partial x_j \end{bmatrix}.$$

Now, we can write $A_{i,j} = \sum_{k=1}^n A_{ijk} \partial/\partial x_k$, where $A_{ijk} = A_{ijk}(x_1, \ldots, \hat{x}_k, \ldots, x_n) \in \mathcal{O}_{\mathbb{C}^n}$. Then $A_{i,j} \in \operatorname{Der}(-\log D)$ for all $i, j = 1, \ldots, n$ if and only if $A_{i,j} = 0$ if and only if

$$T = \begin{bmatrix} g_{1,1} & -x_2 \partial g_{1,1}/\partial x_2 & \cdots & -x_n \partial g_{1,1}/\partial x_n \\ -x_1 \partial g_{2,2}/\partial x_1 & g_{2,2} & \cdots & -x_n \partial g_{2,2}/\partial x_n \\ \vdots & \vdots & & \vdots \\ -x_1 \partial g_{n,n}/\partial x_1 & -x_2 \partial g_{n,n}/\partial x_2 & \cdots & g_{n,n} \end{bmatrix}.$$

To show that this element is zero in cohomology, it is enough to find $\sigma \in \mathcal{C}^0 = \operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D)$ such that $[\sigma, \delta_i] = \tilde{\delta}_i$ for all $i = 1, \ldots, n$, i.e. T is in the image of d^0. Consider then σ the vector field represented by the column vector

$$\begin{bmatrix} g_{1,1} \\ \vdots \\ g_{n,n} \end{bmatrix}.$$

□

Remark 3.4.7. *In general $\mathcal{FT}^1(D) \neq 0$.*

Proof. Consider $f = xy(x-y)(x+y) \in \mathbb{C}[x,y]$ as defining equation of the germ of a free divisor $(D,0) \subset (\mathbb{C}^2,0)$. Then we can consider the Saito matrix

$$\begin{bmatrix} x & 0 \\ y & x^2y - y^3 \end{bmatrix}.$$

To find an infinitesimal admissible deformation for $(D,0)$ we have to find a non-zero element $\alpha \in \mathcal{H}^1(\mathcal{C}^\bullet)_0 = \mathcal{FT}^1(D)$. Let α be defined by the columns of the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & xy^2 - y^3 \end{bmatrix}.$$

This is an element of $\mathcal{H}^1(\mathcal{C}^\bullet)_0$ that describes the infinitesimal admissible deformation $X = V(xy(x-y)(x+(1+\epsilon)y)) = V(f + \epsilon(x^2y^2 - xy^3)) \subset \mathbb{C}^2 \times T_\epsilon$. This infinitesimal admissible deformation is non-trivial because it is a non-trivial deformation of f as a germ of function, in fact $x^2y^2 - xy^3 \notin J(D)$. □

With the notation of Section 2.4, we have that:

Theorem 3.4.8. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a Koszul free divisor such that $\sum_{k=1}^n a_{kl}\partial(b^i_{jk})/\partial x_r = 0$ and $\sum_{k=1}^n a_{lk}\partial(b^i_{jr})/\partial x_k = 0$, for $i,j,l,r = 1,\dots,n$. Then all $\mathcal{H}^i(\mathcal{C}^\bullet)$ are constructible sheaves of finite dimensional complex vector spaces.*

Proof. Let us denote $\mathcal{E}_0 = \mathrm{Der}_{\mathbb{C}^n} / \mathrm{Der}(-\log D)$, $\mathcal{E}_1 = \mathrm{Der}_{\mathbb{C}^n}$ and $\mathcal{E}_2 = \mathrm{Der}(-\log D)$. By Proposition 2.4.2 and 2.4.4, we can consider the short exact sequence

$$0 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

as a resolution of the $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-module $\mathcal{E}_0$. By twisting with $\mathcal{O}_{\mathbb{C}^n}[D]$, we find another $\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})$-resolution

$$0 \longrightarrow \mathcal{E}_2[D] \longrightarrow \mathcal{E}_1[D] \longrightarrow \mathcal{E}_0[D] \longrightarrow 0.$$

By Theorem 2.8.8, the complexes $\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{E}_i[D]$, for $i = 1, 2$ are concentrated in degree zero. Hence, we can compute the complex $\mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{E}_0[D]$ through the above resolution as

$$\mathcal{D}_{\mathbb{C}^n} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{E}_2[D] \longrightarrow \mathcal{D}_{\mathbb{C}^n} \otimes_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{E}_1[D].$$

By Theorem 2.8.8, the above complex is holonomic in each degree and we deduce that $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{E}_0[D])$ is constructible. By Corollary 2.6.4 and 2.8.14, we have the following isomorphisms

$$\mathcal{C}^\bullet \cong \Omega^\bullet(\log D)(\mathcal{E}_0) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{D}_{\mathbb{C}^n} \overset{L}{\otimes}_{\mathcal{V}_0(\mathcal{D}_{\mathbb{C}^n})} \mathcal{E}_0[D])$$

and hence, we can conclude. □

Corollary 3.4.9. *Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a Koszul free divisor such that $\sum_{k=1}^n a_{kl}\partial(b^i_{jk})/\partial x_r = 0$ and $\sum_{k=1}^n a_{lk}\partial(b^i_{jr})/\partial x_k = 0$, for $i, j, l, r = 1, \ldots, n$. Then $\mathbf{FD}_D$ has a hull.*

Proof. By Theorem 3.4.8, condition (H3) from Theorem A.1.17 is satisfied. Then the result follows from Theorem 3.1.11 and Theorem A.1.17. □

Corollary 3.4.10. *Let $(D, 0) \subset (\mathbb{C}^2, 0)$ be a germ of a free divisor defined by a weighted homogeneous equation. Then $\mathbf{FD}_D$ has a hull.*

Proof. By Proposition 2.7.5, $(D, 0)$ is Koszul. Because $(D, 0)$ is defined by f a weighted homogenous equation, then we can choose χ, δ as a basis of $\mathrm{Der}(-\log D)$, where χ is an Euler vector field and $\delta(f) = 0$. Then $[\chi, \delta] = \alpha\delta$, where $\alpha \in \mathbb{C}$ and so all the $b^i_{jk} \in \mathbb{C}$. Hence all the hypothesis of previous Corollary are fulfilled. □

Corollary 3.4.11. *Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a Koszul linear free divisor. Then all $\mathcal{H}^i(\mathcal{C}^\bullet)$ are constructible sheaves of finite dimensional complex vector spaces.*

Proof. This follows from Theorem 3.4.8 and the fact that if $(D, 0)$ is linear then $b^i_{jk} \in \mathbb{C}$. □

The author does not know whether there exists a subclass of the Koszul free divisors that satisfy the assumptions of Theorem 3.4.8. However, we know that not all Koszul free divisors satisfy them. A direct computation shows that the Koszul free divisor $D = V(2^8z^3 - 2^7x^2z^2 + 2^4x^4z + 2^43^2xy^2z - 2^2x^3y^2 - 3^3y^4) \subset \mathbb{C}^3$, described in Example 2.7.3, does not fulfil them.

3.5 Infinitesimal linearly admissible deformations

In this section we show how to use the Lie algebra cohomology complex (see B.3) to compute the space of first order infinitesimal linearly admissible deformations of a germ of a linear free divisor. In addition, using representation theory for reductive Lie algebras (see B.2), we show that if $(D,0) \subset (\mathbb{C}^n,0)$ is a germ of a reductive linear free divisor then the space of first order infinitesimal linearly admissible deformations is zero, and hence $(D,0)$ is formally rigid.

Theorem 3.5.1. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a linear free divisor. Then the germ at the origin of the first cohomology sheaf of the complex $\mathcal{C}_0^\bullet$ is isomorphic to $\mathcal{LFT}^1(D)$, i.e. $H^1(\mathcal{C}_0^\bullet)_0 \cong \mathcal{LFT}^1(D)$.*

Proof. This is a consequence of Theorem 3.4.1 and the second part of Theorem 3.2.10. □

Corollary 3.5.2. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a linear free divisor. Then the functor $\mathbf{LFD}_D$ satisfies Schlessinger's condition (H3) from Definition A.1.13.*

Proof. This is a consequence of the previous Theorem and of the fact that the cohomology with respect to a finite dimensional representation of a finite dimensional Lie algebra is finite dimensional. □

From Theorem 3.1.11 and the previous Corollary, we know that the functor $\mathbf{LFD}_D$ satisfies conditions (H1), (H2) and (H3). This allows us to deduce immediately an important result:

Corollary 3.5.3. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a linear free divisor. Then $\mathbf{LFD}_D$ has a hull.*

Proof. This is a consequence of Theorem A.1.17, Theorem 3.1.11 and the previous Corollary. □

Proposition 3.5.4. $H^0(\mathcal{C}_0^\bullet) = 0$.

Proof. Like the proof of Proposition 3.4.4 but using Lemma 3.4.3. □

Notice that when we compute $\mathcal{LFT}^1(D)$, we are just computing the cohomology of the Lie algebra $\operatorname{Der}(-\log D)_0$ with coefficients in the non-trivial representation $(\operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))_0$. See B.3.

Lemma 3.5.5. *Let $D \subset \mathbb{C}^n$ be a reductive linear free divisor. Then all the elements in the centre of $\operatorname{Der}(-\log D)_0$ are diagonalizable.*

Proof. By definition, $\mathfrak{g}_D$ is a reductive Lie algebra and hence by Lemmas 1.2.6 and 1.2.9, G_D° is a reductive Lie group. Hence by definition, the centre $Z_{G_D^\circ}$ of G_D° is composed of semisimple transformations. Moreover, the Lie algebra of the identity component of $Z_{G_D^\circ}$ coincides with $Z(\mathfrak{g}_D)$ the centre of $\mathfrak{g}_D$ and hence it is composed of diagonalizable elements. □

Proposition 3.5.6. *Let $D \subset \mathbb{C}^n$ be a reductive linear free divisor. Then the representation of* $\operatorname{Der}(-\log D)_0$ *in* $(\operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))_0$ *is semisimple.*

Proof. This is a consequence of Proposition B.2.12 and Lemma 3.5.5. □

We now state and prove the main result of the section:

Theorem 3.5.7. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a reductive linear free divisor. Then $\mathcal{LFT}^1(D) = 0$.*

Proof. By Lemma 3.4.3, $(\operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))_0^{\operatorname{Der}(-\log D)_0} = 0$ and hence by Theorem B.3.16, $\mathcal{LFT}^1(D) = 0$. □

Notice that if $\mathcal{LFT}^1(D) = 0$, then $(D,0) \longrightarrow (\operatorname{Spec}(\mathbb{C}),0)$ is a miniversal linearly admissible deformation. This implies that any linearly admissible deformation is trivial and so $(D,0)$ is formally rigid. The same argument applies in the case that $\mathcal{FT}^1(D) = 0$. See [50] and [51] for the general case.

Corollary 3.5.8. *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of a reductive linear free divisor. Then it is formally rigid.*

The statement of Theorem 3.5.7 is false if we consider non-reductive germs of linear free divisors. In fact Brian Pike suggested me to look at the following:

Example 3.5.9. *Consider $f = x_5(x_4^4 - 2x_5x_4^2x_3 + x_5^2x_3^2 + 2x_5^2x_4x_2 - 2x_5^3x_1) \in \mathbb{C}[x_1,\dots,x_5]$ as a defining equation of the germ of a linear free divisor $(D,0) \subset (\mathbb{C}^5,0)$. Then it has Saito matrix*

$$\begin{bmatrix} x_4 & x_3 & x_2 & x_1 & 0 \\ x_5 & x_4 & 0 & 0 & x_2 \\ 0 & x_5 & 2x_4 & -x_3 & 2x_3 \\ 0 & 0 & x_5 & -2x_4 & 3x_4 \\ 0 & 0 & 0 & -3x_5 & 4x_5 \end{bmatrix}$$

Consider $\sigma = 16x_1\partial/\partial x_1 + 11x_2\partial/\partial x_2 + 6x_3\partial/\partial x_3 + x_4\partial/\partial x_4 - 4x_5\partial/\partial x_5$. Then $\sigma \in \operatorname{Ann}(D)$ and $\operatorname{trace}(\sigma) = 30$. Hence by Lemma 1.2.13, $(D,0)$ is the germ of a non-reductive linear free divisor.

To find an infinitesimal linearly admissible deformation for $(D,0)$ we have to find a non-zero element $\alpha \in \mathcal{H}^1(\mathcal{C}_0^\bullet)_0 = \mathcal{LFT}^1(D)$. Let α be defined by the columns of the following matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x_3 & 0 & 0 \\ 0 & 0 & -2x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is an element of $\mathcal{H}^1(\mathcal{C}_0^\bullet)_0$ that describes the infinitesimal linearly admissible deformation $X = V(x_5(x_4^4(1-\epsilon) - 2x_5x_4^2x_3 + x_5^2x_3^2 + 2x_5^2x_4x_2 - 2x_5^3x_1)) = V(f - \epsilon(x_4^4x_5)) \subset \mathbb{C}^5 \times T_\epsilon$. This infinitesimal linearly admissible deformation is non-trivial because it is a non-trivial deformation of f as a germ of function, in fact $x_4^4x_5 \notin J(D)$. Moreover, one can check, via a long Macaulay 2 computation, that $\mathcal{LFT}^1(D)$ is 3-dimensional and this element is one of its generators. See Appendix C.1 for more details.

This example is of particular interest also because if we consider $f + t(x_4^4x_5)$, where t is a complex parameter, then this defines a family of liner free divisors with an exceptional value. In fact, if we fix t, then $D = V(f + t(x_4^4x_5)) \subset \mathbb{C}^5$ is a linear free divisor, unless $t = -1$.

3.6 Propagation of deformations

Throughout this section we suppose that $(D,0) \subset (\mathbb{C}^n,0)$ is a germ of a free divisor such that there exists a germ of a free divisor $(D',0) \subset (\mathbb{C}^{n-1},0)$ such that $(D,0) = (D' \times \mathbb{C},0)$, i.e. there exists a defining equation for $(D,0)$ in $\mathbb{C}[[x_1,\dots,x_{n-1}]]$.

The section is devoted to prove a result which highlights the difference between the theory of admissible deformations and the classical deformation theory of singularities. Observe that in the ordinary deformation theory of singularities, if $(D,0) = (D' \times \mathbb{C},0)$ and $T^1_{(D',0)}$ is non-zero then $T^1_{(D,0)}$ is infinite dimensional, see [30], Chapter II, 1.4 for more details. However, for admissible deformations, if $(D,0) = (D' \times \mathbb{C},0)$, like in the previous assumptions, then $\mathcal{FT}^1(D')$ and $\mathcal{FT}^1(D)$ are isomorphic. See Corollary 3.6.10.

Because $(D,0)$ is a product, then $\mathrm{Der}(-\log D)$ also decomposes. In fact we have the following:

Lemma 3.6.1. *In this situation*

$$\mathrm{Der}(-\log D) = (\mathrm{Der}(-\log D') \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \mathcal{O}_{\mathbb{C}^n,0}) \oplus \mathcal{O}_{\mathbb{C}^n,0}\frac{\partial}{\partial x_n}$$

and

$$\operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D) = \operatorname{Der}_{\mathbb{C}^{n-1}}/\operatorname{Der}(-\log D') \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \mathcal{O}_{\mathbb{C}^n,0}.$$

Hence, if $\delta \in \operatorname{Der}(-\log D)$*, it can be written as* $\delta = (\delta', h\partial/\partial x_n)$*, where* $\delta' \in \operatorname{Der}(-\log D') \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \mathcal{O}_{\mathbb{C}^n,0}$ *and* $h \in \mathcal{O}_{\mathbb{C}^n,0}$*.*

In general, we have the following result that will be useful in what follows:

Lemma 3.6.2. *Let* R *be a commutative ring and let* A *and* B *be two* R*-modules. Then*

$$\bigwedge^p (A \oplus B) = \bigoplus_{i+j=p} (\bigwedge^i A \otimes \bigwedge^j B).$$

To distinguish between the complexes for $(D,0)$ and for $(D',0)$ we denote them respectively by $(\mathcal{C}_D^\bullet, d_D^\bullet)$ and $(\mathcal{C}_{D'}^\bullet, d_{D'}^\bullet)$.

Proposition 3.6.3. *We have the following isomorphism*

$$\varrho\colon \bigwedge^p \operatorname{Der}(-\log D) \longrightarrow (\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \bigwedge^p \operatorname{Der}(-\log D'))$$

$$\oplus (\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \bigwedge^{p-1} \operatorname{Der}(-\log D'))$$

$$(\delta_1 \wedge \cdots \wedge \delta_p) = (\delta_1', h_1 \frac{\partial}{\partial x_n}) \wedge \cdots \wedge (\delta_p', h_p \frac{\partial}{\partial x_n}) \mapsto$$

$$(\delta_1' \wedge \cdots \wedge \delta_p', \sum_{k=1}^p (-1)^{p-k} h_k \delta_1' \wedge \cdots \wedge \widehat{\delta_k'} \wedge \cdots \wedge \delta_p').$$

Proof. That follows by applying Lemmas 3.6.1 and 3.6.2 and using the fact that $\bigwedge^p \mathcal{O}_{\mathbb{C}^n,0} = 0$ for all $p \geq 2$. □

Also the complex $\mathcal{C}_D^\bullet$ decomposes in this situation. In fact we have the following:

Corollary 3.6.4. *As a consequence*

$$\mathcal{C}_D^p = \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n,0}}(\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \bigwedge^p \operatorname{Der}(-\log D'), \operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D))$$

$$\oplus \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n}}(\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \bigwedge^{p-1} \operatorname{Der}(-\log D'), \operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D)) =$$

$$= \mathcal{C}_{D'}^p \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \mathcal{O}_{\mathbb{C}^n,0} \oplus \mathcal{C}_{D'}^{p-1} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \mathcal{O}_{\mathbb{C}^n,0}.$$

Lemma 3.6.5. *It is possible to write an element* $\Gamma \in \mathcal{C}^p_D$ *for* $p > 0$ *as*

$$\Gamma = (\psi, \phi) = (\sum_{i\geq 0} x_n^i \psi_i, \sum_{i\geq 0} x_n^i \phi_i) = \sum_{i\geq 0} x_n^i (\psi_i, \phi_i)$$

with $\psi_i \in \mathcal{C}^p_{D'}$ *and* $\phi_i \in \mathcal{C}^{p-1}_{D'}$.

Notice that by Lemma 3.6.5, we can describe the differential

$$\tilde{d}^p \colon \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n,0}}(\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \bigwedge^p \operatorname{Der}(-\log D'), \operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D)) \longrightarrow$$

$$\longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n,0}}(\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}} \bigwedge^{p+1} \operatorname{Der}(-\log D'), \operatorname{Der}_{\mathbb{C}^n} / \operatorname{Der}(-\log D))$$

as

$$(\tilde{d}^p(\psi))(\sigma_1 \wedge \cdots \wedge \sigma_{p+1}) = \sum_{i\geq 0} x_n^i (d^p_{D'}(\psi_i))(\sigma_1 \wedge \cdots \wedge \sigma_{p+1}).$$

Proposition 3.6.6. *The differential on* $\mathcal{C}^\bullet_D$ *becomes*

$$d^p_D \colon \mathcal{C}^p_D \longrightarrow \mathcal{C}^{p+1}_D$$

$$(\psi, \phi) \mapsto (\tilde{d}^p(\psi), \tilde{d}^{p-1}(\phi) + (-1)^{p+1}[\frac{\partial}{\partial x_n}, \psi(-)]).$$

Proof. Consider $\Gamma = (\psi, \phi) \in \mathcal{C}^p_D$. We want now to compute $d^p_D(\Gamma) \in \mathcal{C}^{p+1}_D$:

$$(d^p_D(\Gamma))(\sigma_1 \wedge \cdots \wedge \sigma_{p+1}, \nu_1 \wedge \cdots \wedge \nu_p)$$

$$= (d^p_D(\Gamma))(\sigma_1 \wedge \cdots \wedge \sigma_{p+1}) + (d^p_D(\Gamma))(\nu_1 \wedge \cdots \wedge \nu_p) \tag{3.3}$$

where $\sigma_i, \nu_j \in \operatorname{Der}(-\log D')$, this is possible by Remark 3.6.5.

Let us now look at the first part of the right hand side of the previous equality

$$(d^p_D(\Gamma))(\sigma_1 \wedge \cdots \wedge \sigma_{p+1})$$

$$= \sum_{i=1}^{p+1} (-1)^i [\sigma_i, \Gamma(\sigma_1 \wedge \cdots \wedge \widehat{\sigma_i} \wedge \cdots \wedge \sigma_{p+1})] +$$

$$+ \sum_{1\leq i<j\leq p+1} (-1)^{i+j-1} \Gamma([\sigma_i, \sigma_j'] \wedge \sigma_1 \wedge \cdots \wedge \widehat{\sigma_i} \wedge \cdots \wedge \widehat{\sigma_j} \wedge \cdots \wedge \sigma_{p+1})$$

$$= (\tilde{d}^p(\psi))(\sigma_1 \wedge \cdots \wedge \sigma_{p+1}).$$

Now we consider the second part of the equation (3.3) from above. Let us put

$\nu_{p+1} := \partial/\partial x_n$. Then

$$\varrho((\nu_1, 0 \cdot \frac{\partial}{\partial x_n}) \wedge \cdots \wedge (\nu_p, 0 \cdot \frac{\partial}{\partial x_n}) \wedge (0, 1 \cdot \frac{\partial}{\partial x_n})) = (0, \nu_1 \wedge \cdots \wedge \nu_p)$$

and hence

$$(d_D^p(\Gamma))(\nu_1 \wedge \cdots \wedge \nu_p) = (d_D^p(\Gamma))(\nu_1 \wedge \cdots \wedge \nu_{p+1})$$

$$= \sum_{i=1}^{p+1} (-1)^i [\nu_i, \Gamma(\nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \nu_{p+1})] +$$

$$+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j-1} \Gamma([\nu_i, \nu_j] \wedge \nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \widehat{\nu_j} \wedge \cdots \wedge \nu_{p+1}).$$

Let us now look at the first part of the right hand side of the previous equality

$$= \sum_{i=1}^{p+1} (-1)^i [\nu_i, \Gamma(\nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \nu_{p+1})]$$

$$= (-1)^{p+1} [\frac{\partial}{\partial x_n}, \Gamma(\nu_1 \wedge \cdots \wedge \nu_p)] + \sum_{i=1}^{p} (-1)^i [\nu_i, \Gamma(\nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \nu_p \wedge \frac{\partial}{\partial x_n})]$$

$$= (-1)^{p+1} [\frac{\partial}{\partial x_n}, \psi(\nu_1 \wedge \cdots \wedge \nu_p)] + \sum_{i=1}^{p} (-1)^i [\nu_i, \phi(\nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \nu_p)].$$

Moreover, we have

$$\sum_{1 \leq i < j \leq p+1} (-1)^{i+j-1} \Gamma([\nu_i, \nu_j] \wedge \nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \widehat{\nu_j} \wedge \cdots \wedge \nu_{p+1})$$

$$= \sum_{i=1}^{p} (-1)^{i+p} \Gamma([\nu_i, \frac{\partial}{\partial x_n}], \wedge \nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \nu_p) +$$

$$+ \sum_{1 \leq i < j \leq p} (-1)^{i+j-1} \Gamma([\nu_i, \nu_j] \wedge \nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \widehat{\nu_j} \wedge \cdots \wedge \nu_p \wedge \frac{\partial}{\partial x_n})$$

$$= \sum_{1 \leq i < j \leq p} (-1)^{i+j-1} \phi([\nu_i, \nu_j] \wedge \nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \widehat{\nu_j} \wedge \cdots \wedge \nu_p).$$

Here the important point is that $[\nu_i, \partial/\partial x_n] = 0$ because $\nu_i \in \mathrm{Der}(-\log D')$. Hence, we obtain

$$(d_D^p(\Gamma))(\nu_1 \wedge \cdots \wedge \nu_p)$$

$$= (-1)^{p+1} [\frac{\partial}{\partial x_n}, \psi(\nu_1 \wedge \cdots \wedge \nu_p)] + \sum_{i=1}^{p} (-1)^i [\nu_i, \phi(\nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \nu_p)] +$$

$$+\sum_{1\le i<j\le p}(-1)^{i+j-1}\phi([\nu_i,\nu_j]\wedge\nu_1\wedge\cdots\wedge\widehat{\nu_i}\wedge\cdots\wedge\widehat{\nu_j}\wedge\cdots\wedge\nu_p)$$

$$=(-1)^{p+1}[\frac{\partial}{\partial x_n},\psi(\nu_1\wedge\cdots\wedge\nu_p)]+(\tilde{d}^{p-1}(\phi))(\nu_1\wedge\cdots\wedge\nu_p).$$

□

Proposition 3.6.7. *We can rewrite the differential as*

$$d^p_D\colon \mathcal{C}^p_D\longrightarrow\mathcal{C}^{p+1}_D$$

$$(\psi,\phi)\mapsto\sum_{i\ge 0}x_n^i(d^p_{D'}(\psi_i),d^{p-1}_{D'}(\phi_i)+(-1)^{p+1}(i+1)\psi_{i+1}).$$

Definition 3.6.8. *We define the morphism J to be the inclusion*

$$J\colon \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^{n-1},0}}(\bigwedge^p\operatorname{Der}(-\log D'),\operatorname{Der}_{\mathbb{C}^{n-1}}/\operatorname{Der}(-\log D'))=\mathcal{C}^p_{D'}\hookrightarrow$$

$$\mathcal{C}^p_D=\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n,0}}(\mathcal{O}_{\mathbb{C}^n,0}\otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}}\bigwedge^p\operatorname{Der}(-\log D'),\operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D))\oplus$$

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^n,0}}(\mathcal{O}_{\mathbb{C}^n,0}\otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}}\bigwedge^{p-1}\operatorname{Der}(-\log D'),\operatorname{Der}_{\mathbb{C}^n}/\operatorname{Der}(-\log D))=$$

$$=\mathcal{C}^p_{D'}\otimes_{\mathcal{O}_{\mathbb{C}^{n-1}}}\mathcal{O}_{\mathbb{C}^n,0}\oplus\mathcal{C}^{p-1}_{D'}\otimes_{\mathcal{O}_{\mathbb{C}^{n-1},0}}\mathcal{O}_{\mathbb{C}^n,0}$$

$$\psi\mapsto x_n^0(\psi,0).$$

All the previous work is devoted to prove that in order to compute the cohomology of D is enough to compute the one of D':

Theorem 3.6.9. *The morphism J is a quasi-isomorphism.*

Proof. It is enough to show that the cokernel of J is acyclic. Consider

$$\Gamma:=\sum_{i\ge 1}x_n^i(\psi_i,\phi_i)+(0,\phi_0)\in\operatorname{coker}(J)$$

and suppose that $\Gamma\in\ker(d^p_D)$. Then we have $d^p_{D'}(\psi_i)=0$ and $d^p_{D'}(\phi_i)=(-1)^p(i+1)\psi_{i+1}$ for all $i\ge 0$. Define now

$$\Lambda:=\sum_{i\ge 1}x_n^i(\frac{(-1)^p\phi_{i-1}}{i},0)\in\mathcal{C}^{p-1}_D.$$

Then we have that

$$d_D^{p-1}(\Lambda) = \sum_{i\geq 1} x_n^i(d_{D'}^{p-1}\Lambda_i, (-1)^p(i+1)\Lambda_{i+1}) = \Gamma.$$

Hence, Γ vanishes in cohomology. □

Corollary 3.6.10. *There is an isomorphism of sheaves*

$$\pi^{-1}\mathcal{H}^i(\mathcal{C}^\bullet_{D'}) \cong \mathcal{H}^i(\mathcal{C}^\bullet_D),$$

where $\pi\colon (D,0) \longrightarrow (D',0)$ *is the projection. In particular, for* $i=1$*:*

$$\pi^{-1}\mathcal{FT}^1(D') \cong \mathcal{FT}^1(D)$$

Proof. This follows because π^{-1} is an exact functor. □

3.7 The weighted homogenous case

The aim of this section is to describe more in detail the case of germs of free divisors defined by a weighted homogeneous equation. We show that in this case, the space of infinitesimal admissible deformations is finite dimensional and that, in the case of curves, its dimension depends only on the weights and the degree of the defining equation.

Proposition 3.7.1. *Let* $(D,0) \subset (\mathbb{C}^n,0)$ *be a germ of a free divisor defined by a weighted homogeneous polynomial of degree* k*. Then an element of* $\mathcal{FT}^1(D)$ *can be represented by* $f' \in \mathbb{C}[x_1,\ldots,x_n]_k$*, where* $\mathbb{C}[x_1,\ldots,x_n]_k$ *is the space of polynomials of weighted degree* k*.*

Proof. Let f be a defining equation for $(D,0)$. Because f is weighted homogeneous, then there exists $\chi \in \mathrm{Der}(-\log D)$ such that $\chi(f) = f$.

Consider (X,x) an infinitesimal admissible deformation of $(D,0)$. By Remark 3.2.4, we can supposed it is defined by the equation $f+\epsilon\cdot f'$, where $f' \in \mathcal{O}_{\mathbb{C}^n,0}$. Let us suppose that f' is weighted homogeneous of degree β. Because (X,x) is admissible, then χ lifts. This means that there exists $\chi' \in \mathrm{Der}_{\mathbb{C}^n}$ such that $(\chi+\epsilon\cdot\chi')(f+\epsilon\cdot f') = (1+\epsilon\cdot\alpha)(f+\epsilon\cdot f')$ and so $\chi'(f)+\chi(f') = \alpha f + f'$, where $\alpha \in \mathcal{O}_{\mathbb{C}^n,0}$. Because f' is weighted homogeneous of degree β, then $\chi(f') = \beta f'$. Hence, the previous expression becomes $(\chi'-\alpha)f = (1-\beta)f'$. Because f is weighted homogenous, $(\chi'-\alpha)f$ is a combination of the partial derivative of f and so $(1-\beta)f'$ is in the

Jacobian ideal of D. If f' is in the Jacobian ideal, then the admissible deformation is trivial, by [30], Chapter II, 1.4. Otherwise $\beta = 1$ and so f' is of weighted degree k as f.

If f' is not weighted homogeneous, we can apply the previous argument to each of its weighted homogeneous parts. □

Notice that the previous result highlights another difference between the theory of admissible deformations and the classical deformation theory of singularities. In the latter, the weighted homogeneity is not preserved under deformations like in the case of admissible deformations.

Corollary 3.7.2. *Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a free divisor defined by a weighted homogeneous polynomial of degree k. Then*

$$\dim_{\mathbb{C}} \mathcal{FT}^1(D) \leq \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]_k / J(D) \cap \mathbb{C}[x_1, \ldots, x_n]_k,$$

where $J(D)$ is the Jacobian ideal of D.

Proof. Because $(D, 0)$ is defined by a weighted homogeneous polynomial, then a basis of $\mathcal{FT}^1(D)$ can be chosen to be made of monomials. Then it is a consequence of Proposition 3.7.1 and that $J(D)$ defines only trivial deformations. □

Corollary 3.7.3. *Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a free divisor defined by a weighted homogeneous polynomial. Then $\mathbf{FD}_D$ has a hull.*

Proof. By Corollary 3.7.2, condition (H3) from Definition A.1.13 is satisfied. Then the result follows from Theorem 3.1.11 and Theorem A.1.17. □

In Corollary 3.5.3, we have seen that for a germ of a linear free divisor $\mathbf{LFD}_D$ has a hull. Because each germ of a linear free divisor $(D, 0) \subset (\mathbb{C}^n, 0)$ is defined by a homogeneous equation of degree n, then by the previous Corollary, we also have that

Corollary 3.7.4. *Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a linear free divisor. Then $\mathbf{FD}_D$ has a hull.*

In the case of plane curves, we can be more precise. In fact we have the following:

Theorem 3.7.5. *Let $(D, 0) \subset (\mathbb{C}^2, 0)$ be a germ of a free divisor defined by a weighted homogeneous polynomial of degree k. Then $\mathcal{FT}^1(D) \cong \mathbb{C}[x, y]_k / J(D) \cap \mathbb{C}[x, y]_k$.*

Proof. Let f be a defining equation for $(D, 0)$. Because f is weighted homogeneous, then there exists $\chi \in \mathrm{Der}_{\mathbb{C}^n}$ such that $\chi(f) = f$. Consider $\delta = \partial f/\partial x \partial/\partial y - \partial f/\partial y \partial/\partial x$. Because $(D, 0)$ has an isolated singularity, then δ, χ form a basis of $\mathrm{Der}(-\log D)$.

By Proposition 3.7.1, we know that if (X, x) is an infinitesimal admissible deformation of $(D, 0)$ defined by $f + \epsilon \cdot f'$, then $f' \in \mathbb{C}[x, y]_k$.

On the other hand, if $f' \in \mathbb{C}[x, y]_k$ and we consider $(X, 0)$ defined by $f + \epsilon \cdot f' = F$, then it is an infinitesimal admissible deformation because both δ and χ lift. In fact, we can consider $\delta' = \partial F/\partial x \partial/\partial y - \partial F/\partial y \partial/\partial x$ and χ as elements of $\mathrm{Der}(-\log X/T_\epsilon)$.

We have to go modulo $J(D) \cap \mathbb{C}[x, y]_k$ to avoid trivial admissible deformations. □

Remark 3.7.6. *The previous Theorem is false in higher dimensions.*

Proof. Consider $f = 4x^3y^2 - 16x^4z + 27y^4 - 144xy^2z + 128x^2z^2 - 256z^3 \in \mathbb{C}[x, y, z]$. It is weighted homogeneous of degree 12 with weights $(2, 3, 4)$ and it defines a germ of a free divisor $(D, 0) \subset (\mathbb{C}^3, 0)$. A Macaulay 2 computation shows that $\dim_{\mathbb{C}} \mathbb{C}[x, y, z]_{12}/J(D) \cap \mathbb{C}[x, y, z]_{12} = 3$ but $\mathcal{FT}^1(D) = 0$. □

Corollary 3.7.7. *Let $(D, 0) \subset (\mathbb{C}^2, 0)$ be a germ of a free divisor defined by a homogeneous polynomial of degree k. Then $\dim_{\mathbb{C}} \mathcal{FT}^1(D) = k - 3$ if $k \geq 3$, and is zero otherwise.*

Proof. If $k = 1$, then $J(D) = \mathbb{C}[x, y]$ and if $k = 2$, then $J(D) = (x, y)$ and so, by Theorem 3.7.5, in both cases $\mathcal{FT}^1(D) = 0$.

Let us suppose now that $k \geq 3$. We have that $\dim_{\mathbb{C}} \mathbb{C}[x, y]_k = k + 1$ and that $J(D) \cap \mathbb{C}[x, y]_k$ gives us 4 relations: $x\partial f/\partial x, x\partial f/\partial y, y\partial f/\partial x, y\partial f/\partial y$. Because $(D, 0)$ is an isolated singularity, then $\partial f/\partial x, \partial f/\partial y$ form a regular sequence and so the Koszul relation generates the relations between the partial derivative of f. Because the Koszul relation is of degree $k-1 > 1$, then $x\partial f/\partial x, x\partial f/\partial y, y\partial f/\partial x, y\partial f/\partial y$ are linearly independent. Hence, $\dim_{\mathbb{C}} \mathbb{C}[x, y]_k/J(D) \cap \mathbb{C}[x, y]_k = k + 1 - 4 = k - 3$. We conclude by Theorem 3.7.5. □

Example 3.7.8.

1. *Consider $f = xy(x - y)(x + y) \in \mathbb{C}[x, y]$ as defining equation of the germ of a free divisor $(D, 0) \subset (\mathbb{C}^2, 0)$. Then $\mathcal{FT}^1(D)$ is 1-dimensional and it is generated by x^2y^2. See Appendix C.2 for more details.*

2. *Consider $f = x^5 + y^4 \in \mathbb{C}[x, y]$ and the germ of a free divisor $(D, 0) = (V(f), 0) \subset (\mathbb{C}^2, 0)$. A direct computation shows that $\mathcal{FT}^1(D) = 0$ and so $(D, 0)$ is formally rigid.*

Notice that the last Example is a particular case of the following:

Proposition 3.7.9. *Let* $(D,0) = (V(f),0) \subset (\mathbb{C}^2,0)$ *be a germ of an irreducible binomial curve, i.e.* $f = x^a + y^b$ *for some coprime positive integers* $a, b \in \mathbb{N}$*. Then* $\mathcal{FT}^1(D) = 0$ *and so* $(D,0)$ *is formally rigid.*

Proof. Because a, b are coprime, f is weighted homogeneous of degree ab with respect to weights (b, a). Now $\mathbb{C}[x,y]_{ab}$ has basis $\{x^a, y^b\}$. However, both elements belong to $J(D) \cap \mathbb{C}[x,y]_{ab}$. We conclude by Theorem 3.7.5. □

Remark 3.7.10. *Let* $(D,0) \subset (\mathbb{C}^n,0)$ *be a germ of a free divisor defined by a weighted homogeneous equation. Then we can compute the cohomology of* $\mathcal{C}^\bullet$ *degree by degree, because each module and map involved is degree preserving.*

Theorem 3.7.11. *Let* $(D,0) \subset (\mathbb{C}^n,0)$ *be a germ of a free divisor defined by a weighted homogeneous polynomial. Then* $\mathcal{FT}^1(D) \cong (\mathcal{H}^1(\mathcal{C}^\bullet)_0)_0$*, where* $(\mathcal{H}^1(\mathcal{C}^\bullet)_0)_0$ *is the weight zero part of* $\mathcal{H}^1(\mathcal{C}^\bullet)_0$*.*

Proof. Let f be a defining equation for $(D,0)$ weighted homogeneous of degree k and let (X,x) be an infinitesimal admissible deformation of $(D,0)$. By Proposition 3.7.1, we can suppose (X,x) has defining equation $f + \epsilon \cdot f'$, with f' weighted homogeneous of degree k.

By Proposition 1.1.24, we can take $\delta_1, \ldots, \delta_n \in \mathrm{Der}(-\log D)$ a weighted homogeneous basis. By Proposition 3.2.7, $\mathrm{Der}(\log X/T_\epsilon)$ is generated by $\delta_1 + \epsilon \cdot \tilde{\delta}_1, \ldots, \delta_n + \epsilon \cdot \tilde{\delta}_n$ such that the determinant of their coefficients is $f + \epsilon \cdot f'$. Because f and f' are both weighted homogenous of the same degree, then each $\tilde{\delta}_i$ is weighted homogeneous of the same degree as δ_i, for all $i = 1, \ldots, n$.

As seen in the proof of Theorem 3.4.1, there exists $\psi \in \mathcal{C}^1$ such that $\psi(\delta_i) = \tilde{\delta}_i$. So by the previous argument ψ is a weight preserving map and so it represents an element of $(\mathcal{H}^1(\mathcal{C}^\bullet)_0)_0$. □

Corollary 3.7.12. *Let* $(D,0) \subset (\mathbb{C}^n,0)$ *be a germ of a linear free divisor. Then* $\mathcal{FT}^1(D) \cong \mathcal{LFT}^1(D)$*.*

Proof. It is clear that $(\mathcal{H}^1(\mathcal{C}^\bullet)_0)_0 = \mathcal{H}^1(\mathcal{C}_0^\bullet)_0$. Then we can conclude by Theorems 3.5.1 and 3.7.11. □

Corollary 3.7.13. *Let* $(D,0) \subset (\mathbb{C}^n,0)$ *be a germ of a reductive linear free divisor. Then* $\mathcal{FT}^1(D) = 0$ *and hence, it is formally rigid also as free divisor.*

Proof. This follows from Theorem 3.5.7 and Corollary 3.7.12. □

3.8 Another possible approach

In [16] and [18] T. de Jong and D. van Straten developed a deformation theory for non-isolated singularities in which not only the singularity but also the singular locus are deformed in a flat way. By Theorem 1.1.21, free divisors are a particular class of non-isolated singularities, hence we can apply their theory in our case. This gives us another way to deform a germ of a free divisor in such a way that each fiber of the deformation is a free divisor. In fact if the singular locus of the special fiber is Cohen-Macaulay of codimension 2, i.e. it is a free divisor, then the same is true for each fiber and hence, each fiber is free. It turns out that the two methods give the same answer.

Their idea is to consider $(X, 0)$ a germ of an analytic space and $(\Sigma, 0) \subset (\mathrm{Sing}(X), 0)$ the subspace of the singular locus of $(X, 0)$ that we want to deform in a flat way and require that the deformed $(\Sigma, 0)$ is still contained in the relative singular locus of the deformed $(X, 0)$. This idea leads to a well defined deformation functor $\mathrm{Def}(\Sigma, X)$. On the other hand, in this way $\mathrm{Def}(\Sigma, X)$ becomes a subfunctor of the deformations of the diagram $(\Sigma, 0) \hookrightarrow (X, 0)$ and not of the deformation functor $\mathrm{Def}(X)$ of flat deformations of $(X, 0)$. However, under some geometrically reasonable circumstances, that are always satisfied in the case of free divisors, $\mathrm{Def}(\Sigma, X)$ is a subfunctor of $\mathrm{Def}(X)$.

Definition 3.8.1. *Let $(X, x) \longrightarrow (Y, y)$ be flat map of relative codimension n between germs of analytic spaces. Let $J(X/Y) := F_n(\Omega^1_{X/Y})$ be the n-th Fitting ideal of the module of relative Kähler one forms. We call $J(X/Y)$ the* Jacobian ideal *of $(X, x) \longrightarrow (Y, y)$.*

Definition 3.8.2. *The* critical locus *$(\mathcal{C}, x) := (\mathcal{C}_{X/Y}, x)$ of $(X, x) \longrightarrow (Y, y)$ is the locus defined by $J(X/Y)$ and the* critical space *is $(\mathcal{C}, x)$ together with the structural sheaf $\mathcal{O}_{\mathcal{C},x} = \mathcal{O}_{X,x}/J(X/Y)$.*

Notice that if Y is a point then the critical locus is $\mathrm{Sing}(X)$, and if $(X, x) \subset (V \times Y, (v, y))$ and the map is the second projection then the critical locus is $\mathrm{Sing}(X)_{rel}$.

Definition 3.8.3. *Let (S, s) be a complex space germ. A* diagram *over (S, s) is a triple (Σ_S, X_S, i), where (Σ_S, x) and (X_S, x) are germs of analytic spaces with a map to (S, s) and $i\colon (\Sigma_S, x) \longrightarrow (X_S, x)$ is a map such that the following diagram commutes*

$$\begin{array}{ccc} (\Sigma_S, x) & \xrightarrow{\;i\;} & (X_S, x) \\ \downarrow & & \downarrow \\ (S, s) & = & (S, s) \end{array}$$

Remark 3.8.4. *Usually we will say that* $(\Sigma_S, x) \longrightarrow (X_S, x)$ *is a diagram over* (S, s)*, without even mentioning the map* i.

Definition 3.8.5. *A diagram* $(\Sigma_S, x) \longrightarrow (X_S, x)$ *over* (S, s) *is said to be* admissible*, if the map* $i\colon (\Sigma_S, x) \longrightarrow (X_S, x)$ *factorizes over the inclusion map* $(\mathcal{C}_{X_S/S}, x) \hookrightarrow (X_S, x)$.

Definition 3.8.6. *A morphism between admissible diagrams over* (S, s) *is just a morphism of the underlying diagrams over* (S, s).

Definition 3.8.7. *Let* (T, t) *be a complex space germ, let* $(\Sigma_T, x) \longrightarrow (X_T, x)$ *be a diagram over* (T, t) *and let* $(T, t) \longrightarrow (S, s)$ *be a map. A diagram* $(\Sigma_S, x) \longrightarrow (X_S, x)$ *over* (S, s) *is said to be a* deformation *of the diagram* $(\Sigma_T, x) \longrightarrow (X_T, x)$ *if and only if*

1. (Σ_S, x) *and* (X_S, x) *are flat over* (S, s)*;*
2. $((\Sigma_T, x) \longrightarrow (X_T, x)) \cong ((\Sigma_S, x) \longrightarrow (X_S, x)) \times_{(S,s)} (T, t)$.

Definition 3.8.8. *A deformation* $(\Sigma_S, x) \longrightarrow (X_S, x)$ *of* $(\Sigma_T, x) \longrightarrow (X_T, x)$ *is said to be an* admissible deformation *if the diagram* $(\Sigma_S, x) \longrightarrow (X_S, x)$ *is admissible.*

As usual, we are mainly interested in the case of T a point.

Definition 3.8.9. ([16], Definition 1.4) *Let* $\mathbf{C}$ *be the category of local analytic complex algebras. Let* $(\Sigma, 0) \longrightarrow (X, 0)$ *be an admissible diagram over a point. The functor*

$$\mathbf{C} \longrightarrow \mathbf{Set}$$

$$S \mapsto \left\{ \begin{array}{c} \textit{Isomorphism classes of admissible} \\ \textit{deformations of } (\Sigma_S, x) \longrightarrow (X_S, x) \textit{ over } S \end{array} \right\}$$

is called the functor of admissible deformations *and it is denoted by* $\mathrm{Def}(\Sigma, X)$.

Proposition 3.8.10. ([16], Corollary 1.8) *If* $T^1(\Sigma, X) := \mathrm{Def}(\Sigma, X)(\mathbb{C}[t]/(t^2))$ *is a finite dimensional vector space, then there exists a formal miniversal deformation for* $\mathrm{Def}(\Sigma, X)$.

Theorem 3.8.11. ([16], Theorem 1.11) *Let* $(\Sigma, 0) \longrightarrow (X, 0)$ *be an admissible diagram and let* I *be the ideal of* $(\Sigma, 0)$ *in* $\mathcal{O}_{X,0}$*. Assume that*

1. $(X, 0)$ *is Cohen-Macaulay;*
2. $(\Sigma, 0)$ *is Cohen-Macaulay of codimension* c *in* $(X, 0)$*;*

3. $\dim \operatorname{Supp}(I/J(X)) < \dim(\Sigma)$.

Then the natural forgetful transformation $\operatorname{Def}(\Sigma, X) \longrightarrow \operatorname{Def}(X)$ *is injective.*

In the case in which we are interested, we will consider $(X, 0)$ a germ of a free divisor and $(\Sigma, 0)$ its singular locus, i.e. $I = J(X)$, and hence the previous hypothesis are always satisfied.

We now specialize to the case of a germ of a free divisor $(D, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$. In this case the singular locus $(\mathcal{C}_D, 0)$ is described by the ideal $J(D) = (f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ and we consider $(\Sigma, 0) = (\mathcal{C}_D, 0)$.

Definition 3.8.12. ([16], Definition 2.5, Variation 2.7) *The complex* $\mathbf{D}(\Sigma, D)$ *is defined by*

$$0 \longrightarrow \operatorname{Der}_{\mathbb{C}^n,0} \xrightarrow{\partial_0} \mathcal{O}_{D,0} \oplus Hom(\mathcal{O}^{n+1}_{\mathbb{C}^n,0}, \mathcal{O}_{\Sigma,0}) \xrightarrow{\partial_1} P^1 \otimes \mathcal{O}_{\Sigma,0} \oplus Hom(\mathcal{R}, \mathcal{O}_{\Sigma,0}) \longrightarrow 0$$

with maps

$$\partial_0(\theta) := (\theta(f), \theta(f)e_1^* + \sum_{i>1} \theta(\frac{\partial f}{\partial x_{i-1}})e_i^*)$$

$$\partial_1((g, \sum n_i e_i^*)) := (g - n_1, dg - \sum_{i>1} n_i e_{i-1}, (\sum r_i e_i \mapsto \sum r_i n_i)),$$

where $P^1 := \mathcal{O}_{(\mathbb{C}^n,0)} \oplus \Omega^1$ *is the space of* 1*-jets of functions and* $\mathcal{R}$ *is the module of relations between the elements* $f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n$ *modulo the Koszul relations.*

Definition 3.8.13. *The cohomology group* $\mathbf{D}(\Sigma, D)$ *are denoted by* $T^i(\Sigma, D)$ *for* $i = 0, 1, 2$.

Theorem 3.8.14. ([16], Proposition 2.6) *The complex* $\mathbf{D}(\Sigma, D)$ *describes the infinitesimal admissible deformations*

1. $T^0(\Sigma, D) = \{\theta \in \operatorname{Der}(-\log D) \mid \theta(J(D)) \subset J(D)\}$;
2. $T^1(\Sigma, D) = \operatorname{Def}(\Sigma, D)(\mathbb{C}[t]/(t^2))$;
3. $T^2(\Sigma, D)$ *is the obstruction space.*

Remark 3.8.15. *From the definition of the complex* $\mathbf{D}(\Sigma, D)$ *and the previous Theorem, we see that an infinitesimal admissible deformation of* $(\Sigma, 0) \longrightarrow (D, 0)$ *is given by an element* $(\Sigma_\epsilon, 0) \longrightarrow (D_\epsilon, 0)$ *where* $(D_\epsilon, 0)$ *has defining equation* $f + \epsilon \cdot g$, Σ_ϵ *is described by the ideal* $(f + \epsilon \cdot g, \partial f/\partial x_1 + \epsilon \cdot \partial g/\partial x_1, \ldots, \partial f/\partial x_n + \epsilon \cdot \partial g/\partial x_n)$ *and* $g \in \mathcal{O}_{D,0}$, *such that all relations between* $f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n$ *lift.*

Definition 3.8.16. *Let* $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ *be an ideal. Then the* primitive ideal *of* I *is* $\int I := \{g \in \mathcal{O}_{\mathbb{C}^n,0} \mid (g, \partial g/\partial x_1, \dots, \partial g/\partial x_n) \subset I\}$.

Proposition 3.8.17. ([16], Remark 2.9) *We have the following exact sequence*

$$0 \longrightarrow \int J(D)/(f, J_\Sigma(f)) \longrightarrow T^1(\Sigma, D) \longrightarrow \ker(w_f) \longrightarrow 0$$

where $J_\Sigma(f) := \{\theta(f) \mid \theta(J(D)) \subset J(D)\}$ *and the map* $w_f \colon T^1(\Sigma) \longrightarrow \Omega^1_{\Sigma,0}$ *is defined by* $w_f(\sum n_i e_i^*) := dn_1 - \sum_{i>1} n_i e_{i-1}$.

Proposition 3.8.18. ([16], Remark 2.9) $T^2(\Sigma, D) \cong \operatorname{coker}(w_f)$.

Notice that in general the groups $T^1(\Sigma, D)$ and $T^2(\Sigma, D)$ do not have a natural structure of $\mathcal{O}_{\mathbb{C}^n}$-module and the map w_f is not $\mathcal{O}_{\mathbb{C}^n}$-linear.

We are now able to show that, in the case of free divisors, this type of deformations is equivalent to the notion of admissible deformations introduced in Definition 3.1.1.

Theorem 3.8.19. *Let* $(D, 0) \subset (\mathbb{C}^n, 0)$ *be a germ of a free divisor and let* $(\Sigma, 0)$ *be its singular locus. Then we can identify* $\mathcal{FT}^1(D)$ *with* $T^1(\Sigma, D)$.

Proof. By Definition 3.1.1 and Remark 3.2.4, an admissible deformation of $(D, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$ is $(X, 0) = (V(f + \epsilon \cdot g), 0) \longrightarrow T_\epsilon$ such that the condition (3.1) is satisfied. By Proposition 3.1.12, each relations between the elements $f, \partial f/\partial x_1, \dots, \partial f/\partial x_n$ lift. Hence by Remark 3.8.15, to have an admissible deformation of $(\Sigma, 0) \longrightarrow (D, 0)$ it is enough to consider $(\mathrm{Sing}(X)_{rel}, 0) \longrightarrow (X, 0)$.

Similarly, each admissible deformation of the diagram $(\Sigma, 0) \longrightarrow (D, 0)$ gives rise to an admissible deformation of $(D, 0)$. In fact by Remark 3.8.15, each relation between $f, \partial f/\partial x_1, \dots, \partial f/\partial x_n$ lift and so by Proposition 3.1.12, the condition (3.1) is satisfied.

It is clear that in both settings there is the same notion of triviality. □

Appendix A

Formal deformation theory

The aim of this appendix is to give a review of abstract deformation theory as developed by M. Schlessinger, M. Artin and others. All facts presented in this appendix are well known.

A.1 Deformation functors

The classical reference for the theory of functor on Artin rings is [50], where conditions for a functor to have a formally miniversal deformation is given. In [50] the term hull is used in place of formally miniversal deformation. Moreover, M. Schlessinger, in [50], introduced for a functor F the vector space T^1_F called the tangent space of the functor F and the most important of the above conditions is that its dimension is finite.

Definition A.1.1. *Denote by* **Art** *the category of local Artin rings with residue field k, by* $\widehat{\mathbf{Art}}$ *the category of complete local (noetherian) rings with residue field k, by* **Set** *the category of pointed sets with distinguished element $*$ and by* **Fun** *the category of functors F from* **Art** *to* **Set** *such that $F(k) = *$ together with natural transformations.*

Definition A.1.2. *Consider the short exact sequence*

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

in **Art**. *Such a sequence is called a* small extension *of A by M if and only if $\mathfrak{m}_B M = 0$. Small extensions with 1-dimensional kernel, that is, sequences of the form*

$$0 \longrightarrow k \longrightarrow B \longrightarrow A \longrightarrow 0$$

are called principal small extensions.

Definition A.1.3. *Let $\tau\colon F \longrightarrow G$ be a natural transformation of functors, i.e. a morphism in* **Fun***. Then we will call τ*

- smooth *if and only if, for any surjection $A' \longrightarrow A$, the canonical map*

$$F(A') \longrightarrow G(A') \times_{G(A)} F(A)$$

  is surjective. A functor $F \in$ **Fun** *is called smooth if the morphism $F \longrightarrow \{*\}$ to the constant functor is smooth.*

- unramified *if and only if the induced morphism on the tangent spaces*

$$T^1_F := F(k[\epsilon]) \longrightarrow T^1_G := G(k[\epsilon])$$

  is injective.

- étale *if and only if it is smooth and unramified.*

Definition A.1.4. *A functor $F \in$* **Fun** *is called* pro-representable *if and only if there exists $R \in \widehat{\mathbf{Art}}$ such that F is isomorphic to the functor $h_R\colon \mathbf{Art} \longrightarrow \mathbf{Set}$ defined by $h_R(A) := Hom(R, A)$.*

Consider now $R \in \widehat{\mathbf{Art}}$ with maximal ideal $\mathfrak{m}$ and $F \in$ **Fun**. Let

$$\tau\colon h_R \longrightarrow F$$

be a natural transformation of functors. Then for each n this will give a map

$$\tau_n\colon Hom(R, R/\mathfrak{m}^n) \longrightarrow F(R/\mathfrak{m}^n)$$

and the image of the quotient map of R to $R/\mathfrak{m}^n$ gives an element $\xi_n \in F(R/\mathfrak{m}^n)$. These elements ξ_n are compatible, in the sense that the natural map $R/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^n$ induces a map of sets $F(R/\mathfrak{m}^{n+1}) \longrightarrow F(R/\mathfrak{m}^n)$ that sends ξ_{n+1} to ξ_n. Thus the collection $\{\xi_n\}$ defines an element $\xi \in \varprojlim F(R/\mathfrak{m}^n)$.

Definition A.1.5. *We will call such a collection $\xi = \{\xi_n\}$ a* formal family *of F over the ring R.*

The category $\widehat{\mathbf{Art}}$ contains the category **Art** and we can extend any functor $F \in$ **Fun** to a functor

$$\widehat{F}\colon \widehat{\mathbf{Art}} \longrightarrow \mathbf{Set}$$

by defining

$$\widehat{F}(R) := \varprojlim F(R/\mathfrak{m}^n)$$

for any $R \in \widehat{\mathbf{Art}}$. In this notation, $\widehat{F}(R)$ is the set of formal families of F over R.

Moreover, if we consider $\xi = \{\xi_n\} \in \widehat{F}(R)$ a formal family, then it defines a natural transformation of functors

$$\tau\colon h_R \longrightarrow F.$$

For any $A \in \mathbf{Art}$ and any homomorphism $f\colon R \longrightarrow A$, since A is artinian, it factors through $R/\mathfrak{m}^n$ for some n, say $f = g \circ \pi$, where $\pi\colon R \longrightarrow R/\mathfrak{m}^n$ and $g\colon R/\mathfrak{m}^n \longrightarrow A$. Then let $\tau(f)$ be the image of ξ_n under the map $F(g)\colon F(R/\mathfrak{m}^n) \longrightarrow F(A)$.

Proposition A.1.6. ([31], Proposition 15.1) *If $F \in$ **Fun** and $R \in \widehat{\mathbf{Art}}$, then there is a natural bijection between the set $\widehat{F}(R)$ of formal families $\{\xi_n \mid \xi_n \in F(R/\mathfrak{m}^n)\}$ and the set of transformations of functors h_R to F.*

Notice that if F is pro-representable, then there is an isomorphism $\xi\colon h_R \longrightarrow F$ for some R, and we can think of ξ as an element of $\widehat{F}(R)$.

Definition A.1.7. *We say that the pair (R, ξ)* pro-represents *the functor F.*

Remark A.1.8. *One can verify that if F is pro-representable, the pair (R, ξ) is unique up to unique isomorphism.*

Definition A.1.9. *Let $F \in$ **Fun**. A pair (R, ξ) with $R \in \widehat{\mathbf{Art}}$ and $\xi \in \widehat{F}(R)$ is a* versal family *for F if the associated map $\varphi\colon h_R \longrightarrow F$ is a smooth transformation of functors. In our case this means that given a surjection $B \longrightarrow A$ in* **Art**, *a map $R \longrightarrow A$ inducing an element $\eta \in F(A)$ and given $\theta \in F(B)$ mapping to η, one can lift the map $R \longrightarrow A$ to a map $R \longrightarrow B$ inducing θ.*

Definition A.1.10. *If in addition, the transformation $\varphi\colon h_R \longrightarrow F$ is étale, we say that the pair (R, ξ) is a* miniversal family, *or that the functor F has a* hull.

Definition A.1.11. *We say that (R, ξ) is a* universal family *if it pro-represents the functor F.*

Proposition A.1.12. ([31], Proposition 15.2) *Let (R, ξ) be a formal family of the functor F. Then*

1. *if (R, ξ) is a versal family, then for any other formal family (S, η), there is a ring homomorphism $f\colon R \longrightarrow S$ such that the induced map $\widehat{F}(R) \longrightarrow \widehat{F}(S)$ sends ξ to η;*

2. *if* (R, ξ) *is miniversal, then for any* (S, η) *the map* $f\colon R \longrightarrow S$ *of 1. induces a unique homomorphism* $R/\mathfrak{m}_R^2 \longrightarrow S/\mathfrak{m}_S^2$*;*

3. *if* (R, ξ) *is a universal family, then for any* (S, η)*, the corresponding map* $f\colon R \longrightarrow S$ *is unique.*

Definition A.1.13. (cf. [50], Theorem 2.11) *Let* $F \in \mathbf{Fun}$ *and* $A' \longrightarrow A$ *and* $A'' \longrightarrow A$ *be morphisms in* **Art***, the latter being a small extension. Consider the canonical map*

$$\tau_{A',A'',A}\colon F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A'').$$

Then we have the following conditions for the functor F

(H1) *the map* $\tau_{A',A'',A}$ *is surjective for all small extensions* $A'' \longrightarrow A$*;*

(H2) *the map* $\tau_{A',A'',A}$ *is bijective for* $A = k$ *and* $A'' = k[\epsilon]$*;*

(H2') *the map* $\tau_{A',A'',A}$ *is bijective for* $A = k$ *and any* A''*;*

(H3) *the tangent space* $T_F^1 = F(k[\epsilon])$ *of* F *is finite dimensional over* k*. (Note that (H2) guarantees that* T_F^1 *is a vector space.)*

(H4) *the map* $\tau_{A',A'',A}$ *is bijective for every small extension* $A'' \longrightarrow A$*.*

Definition A.1.14. *A functor* $F \in \mathbf{Fun}$ *satisfying (H1) and (H2) is called* deformation functor*.*

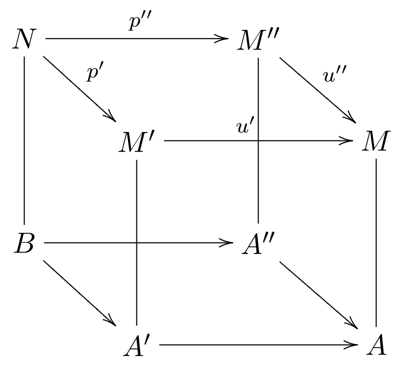
Definition A.1.15. *A functor* $F \in \mathbf{Fun}$ *satisfying (H1) and (H2') is called* deformation functor with obstruction theory*.*

Definition A.1.16. *A functor* $F \in \mathbf{Fun}$ *satisfying (H4) is called* homogeneous*.*

We now state the main result of the section:

Theorem A.1.17. ([53], Theorem A.8) *Let* $F \in \mathbf{Fun}$ *be a deformation functor with finite dimensional tangent space, i.e. (H1), (H2) and (H3) are satisfied. Then there is a miniversal family* (R, ξ)*. If in addition, (H4) holds, then* (R, ξ) *pro-represents* F*.*

Lemma A.1.18. ([50], Lemma 3.4) *Consider the commutative diagram*

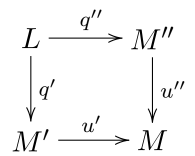


of compatible ring and module homomorphisms, where $B = A' \times_A A'', N = M' \times_M M''$ *and* M' *(resp.* M''*) is a flat* A' *(resp.* A''*) module. Suppose that*

1. $A''/J \longrightarrow A$ *is an isomorphism, where* J *is a nilpotent ideal in* A''*;*
2. u' *(resp.* u''*) induces an isomorphism* $M' \otimes_{A'} A \longrightarrow M$ *(resp.* $M'' \otimes_{A''} A \longrightarrow M$*).*

Then N *is flat over* B *and* p' *(resp.* p''*) induces an isomorphism* $N \otimes_B A' \longrightarrow M'$ *(resp.* $N \otimes_B A'' \longrightarrow M''$*).*

Corollary A.1.19. ([50], Corollary 3.6) *With the notation as above, let* L *be a* B*-module which may be inserted in a commutative diagram*



where q' *induces an isomorphism* $L \otimes_B A' \longrightarrow M'$*. Then the canonical morphism* $q' \times q'' \colon L \longrightarrow N = M' \times_M M''$ *is an isomorphism.*

Lemma A.1.20. ([31], Exercise 4.2) *Let* $A \in$ **Art***, let* X_1 *and* X_2 *be schemes of finite type flat over* A *and let* $f \colon X_1 \longrightarrow X_2$ *be an* A*-morphism that induces an isomorphism of closed fibres* $f \otimes_A k \colon X_1 \times_A k \longrightarrow X_2 \times_A k$*. Then* f *is an isomorphism too.*

A.2 Obstruction theory

From Theorem A.1.17 we know that if a functor $F \in$ **Fun** satisfies Schlessinger's conditions (H1), (H2) and (H3), then it admits a hull R. However, we do not have any informations on the structure of the space Spec(R). In particular, we do not know whether it is smooth or not. Obstruction theory is concerned with this question. More precisely, one asks whether for a given small extension

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

the induced map $F(B) \longrightarrow F(A)$ is surjective, i.e. we are studying if F is smooth in the sense of Definition A.1.3.

This section is mainly taken from [42] and [23].

Definition A.2.1. *Let $F \in$ **Fun**. Then an* obstruction theory *of F, denoted by (V, v_F), consists of the following data*

- *a k-vector space V;*
- *a map $v_F(e)\colon F(A) \longrightarrow V \otimes_k M$ associated to any small extension $e\colon 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$*

such that the following properties are satisfied

1. *Let $\nu \in F(A)$ be given such that ν lies in the image of the map $F(B) \longrightarrow F(A)$. Then $v_F(e)(\nu) = 0$;*
2. *Let $\alpha\colon e_1 \longrightarrow e_2$ be a morphism of small extension*

$$\begin{array}{lccccccccc} e_1\colon & 0 & \longrightarrow & M_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_A} & & \\ e_2\colon & 0 & \longrightarrow & M_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

and $\nu \in F(A_1)$. Then $v_F(e_2)(F(\alpha_A)(\nu)) = (Id_V \otimes \alpha_M)(v_F(e_1)(\nu))$.

Definition A.2.2. *An obstruction theory for which the converse of 1. holds is called* complete.

Definition A.2.3. *A morphism of obstruction theories is defined as a map $\psi\colon V \longrightarrow V'$ such that $v'_F(e) = \psi \circ v_F(e)$.*

Definition A.2.4. *An obstruction theory (O_F, v_F) is called* universal *if and only if it is the smaller one possible, i.e. if, for any given obstruction theory (V, w_F), there is an unique morphism $(O, v_F) \longrightarrow (V, w_F)$.*

Theorem A.2.5. ([23], Theorem 3.2) *Let $F \in$ **Fun** be a functor satisfying conditions (H1) and (H2'). Then it has a unique universal obstruction theory (O_F, v_F)*

which is complete and every element of the vector space O_F is of the form $v_F(\chi)$ for some principal extension $e\colon 0 \longrightarrow k \longrightarrow B \longrightarrow A \longrightarrow 0$ and some $\chi \in F(A)$.

Definition A.2.6. *Let $\tau\colon F \longrightarrow G$ be a morphism of functors and let (V, v_F) and (W, v_G) be obstruction theories respectively for F and G. A linear map $v_\tau\colon V \longrightarrow W$ is called* compatible *if and only if for each small extension $0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$ and each $\nu \in F(A)$, we have that $(v_G \circ \tau)(\nu) = (v_\tau \otimes Id_M) \circ v_F(\nu)$.*

Theorem A.2.7. *Let $\tau\colon F \longrightarrow G$ be a morphism of functors, let (V, v_F) and (W, v_G) be obstruction theories respectively for F and G and let $v_\tau\colon V \longrightarrow W$ be a compatible map. If (V, v_F) is complete, v_τ is injective and $T^1_F \longrightarrow T^1_G$ is surjective, then τ is smooth.*

Proof. First we prove the following preliminary fact: for any functor $F \in \mathbf{Fun}$ and any small extension $0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$, there is a natural transitive action of $T^1_F \otimes M$ on the fibers of $F(B) \longrightarrow F(A)$. For this, one first needs to identify $F(k \otimes M)$, where $k \otimes M$ is the trivial extension of M by k, with $T^1_F \otimes M$ which is easily done by induction on the length of B. Then consider $C = B \times_A B$. We have that $C \cong B \times_k (k \oplus M)$ so, by (H2') we obtain

$$F(C) = F(B) \times (T^1_F \otimes M).$$

From the natural morphism $\alpha\colon F(C) \longrightarrow F(B) \times_{F(A)} F(B)$ we obtain a map

$$F(B) \times (T^1_F \otimes M) \longrightarrow F(B) \times_{F(A)} F(B)$$

which by construction is the identity on the first factor. Composing with the second projection, we get finally a map $F(B) \times (T^1_F \otimes M) \longrightarrow F(B)$ which induces the group action we are looking for. Transitivity follows immediately from subjectivity of α which comes from condition (H1).

Let an element $(a, b') \in F(A) \times_{G(A)} G(B)$ be given. Our task is to find $b \in F(B)$ which projects to $a \in F(A)$ and $b' \in G(B)$. Denote by $a' \in G(A)$ the common image of a and b' in $G(A)$. As b' is a lift of a' to $G(B)$, we have that $v_G(a') = 0 \in W \otimes M$. But (V, v_F) is complete, so we can find $\hat{b} \in F(B)$ lifting $a \in F(A)$. It is not true that the image $\hat{b}' = \tau(\hat{b})$ is equal to b'. But as $(\hat{b}'.b')$ is in $G(B) \times_{G(A)} G(B)$, we find $t' \in T^1_G \otimes M$ which sends $\hat{b}'$ to b'. By subjectivity of $T^1_F \longrightarrow T^1_G$ there is $t \in T^1_F \otimes M$ which can be used to find an element b lying in the same fiber of $F(B) \longrightarrow F(A)$ as $\hat{b}$ and having the desired properties. □

Remark A.2.8. *For any morphism of functor $\tau\colon F \longrightarrow G$ and any obstruction theory (W, v_G) of G, the composition $(W, v_G \circ \tau)$ is an obstruction theory for F.*

Notice that if, in the previous Remark, we take $W = O_G$ and we use the universality of O_F, we obtain a linear map $O_F \longrightarrow O_G$.

Corollary A.2.9. *Let $\tau\colon F \longrightarrow G$ be a morphism of functors and consider the universal obstruction theories O_F and O_G. Then*

1. *τ is smooth if and only if $T^1_F \longrightarrow T^1_G$ is surjective and $O_F \longrightarrow O_G$ is injective;*
2. *F is smooth if and only if $O_F = 0$.*

Proof. By Theorem A.2.7, it remains to prove that for a smooth morphism τ, the map $o_\tau\colon O_F \longrightarrow O_G$ is injective. Suppose that there is an element $y \in O_F$ such that $o_\tau(y) = 0$. By universality, there is a small extension $B \twoheadrightarrow A$ and $\nu \in F(A)$ such that $v_F(\nu) = y$. As O_G is complete, we can lift $\tau(\nu) \in G(A)$ to $G(B)$. But then by smoothness of τ, there is a lift of ν to $F(B)$ which in turn implies that $v_F(\nu) = y$ vanishes. □

Definition A.2.10. *Let $e\colon 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$ be a small extension and let $\psi\colon A' \longrightarrow A$ be a morphism. Then the* pullback *$\psi^* e$ of e is defined by*

$$0 \longrightarrow M \longrightarrow A' \times_A B \longrightarrow A' \longrightarrow 0.$$

Definition A.2.11. *Let $e\colon 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$ be a small extension and let $\phi\colon M \longrightarrow M'$ be a morphism. Then the* pushforward *$\phi_* e$ of e is defined by*

$$0 \longrightarrow M' \longrightarrow B' \longrightarrow A \longrightarrow 0,$$

where $B' := (B \otimes M')/(\{(m, \phi(m)) \mid m \in M\})$.

The universal obstruction theory of a pro-representable functor can be explicitly described.

Theorem A.2.12. *Let $R = P/I$, where $P = k[[x_1, \ldots, x_n]]$ and $I \subset \mathfrak{m}_P^2$. Then we have the small extension*

$$u_R\colon 0 \longrightarrow I/\mathfrak{m}_P I \longrightarrow P/\mathfrak{m}_P I \longrightarrow R \longrightarrow 0$$

and the universal obstruction space of the functor pro-represented by R is $O_R := (I/\mathfrak{m}_P I)^*$

Proof. Define the obstruction map v_R as follows. Let

$$e\colon 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

be any small extension and $\nu \in h_R(A) = Hom(R, A)$. This induces a morphism $\nu\colon P \longrightarrow A$. Chose any lift to a morphism $\nu'\colon P \longrightarrow B$. Obviously, $\nu'(I) \subset M$ and ν' maps $\mathfrak{m}_P$ to $\mathfrak{m}_B$. Therefore, $\nu(\mathfrak{m}_P I) = 0 \in B$ and we obtain a map $P/\mathfrak{m}_P \longrightarrow B$ which in turn induces the map

$$\lambda_\nu\colon I/\mathfrak{m}_P I \longrightarrow M.$$

Then define $v_R(\nu) = \lambda_\nu \in (I/\mathfrak{m}_P I)^* \otimes M$. We see that λ_ν is zero if and only if $\nu(I) = 0 \in B$. This means that there is a lift of ν to B showing that we have a well defined obstruction theory. That it is indeed universal is proved in [23].

We note that using the above definitions of pullback and pushforward, we could have defined λ_ν as the element of $(I/\mathfrak{m}_P I)^* \otimes M = Hom(I/\mathfrak{m}_P I, M)$ such that $\nu^* e = \lambda_{\nu_*} u_R$. □

Appendix B

Lie algebras, representations and cohomology

The aim of this appendix is to review the theory of Lie algebras, representation theory and Lie algebra cohomology.

The material of this appendix is mainly taken from: [35] and [55].

B.1 Lie algebras

In this section we recall the notion of Lie algebras and their properties.

Definition B.1.1. *A vector space* $\mathfrak{g}$ *over a field* K*, with an operation* $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$*, denoted by* $(a, b) \mapsto [a, b]$ *and called* bracket *or* commutator *of* a *and* b*, is called* Lie algebra *over* K *if the following axioms are satisfied*

1. *the bracket operation is bilinear;*
2. $[a, a] = 0$ *for all* $a \in \mathfrak{g}$*;*
3. (Jacobi identity) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ *for all* $a, b, c \in \mathfrak{g}$*.*

Notice that conditions 1 and 2 imply that $[a, b] = -[b, a]$, for all $a, b \in \mathfrak{g}$. Conversely, if $\mathrm{Char}\, K \neq 2$, the previous condition implies 2.

Example B.1.2. *If* V *is a finite dimensional vector space over* K*, we denote by* $\mathrm{End}(V)$ *the set of linear transformations* $V \longrightarrow V$*.* $\mathrm{End}(V)$ *is a ring relative to the usual product operation and we can endow it with a structure of Lie algebra by* $[a, b] = ab - ba$*, for all* $a, b \in \mathrm{End}(V)$*. We will call it* general linear algebra *and we will denote it by* $\mathfrak{gl}(V)$*.*

Definition B.1.3. *Let V be a finite dimensional vector space over K and $x \in \mathfrak{gl}(V)$. Then x is called* semisimple *if the roots of its minimal polynomial over K are all distinct.*

Remark B.1.4. *If K is algebraically closed, x is semisimple if and only if x is diagonalizable.*

Definition B.1.5. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras over K. A* homomorphism *between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is a linear transformation $\psi\colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ satisfying $\psi([a,b]) = [\psi(a), \psi(b)]$ for all $a, b \in \mathfrak{g}_1$. If ψ is surjective and injective we will say that $\mathfrak{g}_1$ are $\mathfrak{g}_2$ are* isomorphic *and we will write $\mathfrak{g}_1 \cong \mathfrak{g}_2$.*

Definition B.1.6. *A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a* Lie subalgebra *if $[a,b] \in \mathfrak{h}$ whenever $a, b \in \mathfrak{h}$.*

Definition B.1.7. *A Lie algebra $\mathfrak{g}$ is called* abelian *if $[a,b] = 0$ for all $a, b \in \mathfrak{g}$.*

Definition B.1.8. *A subspace I of a Lie algebra $\mathfrak{g}$ is called an* ideal *if $[a,b] \in I$ for all $a \in \mathfrak{g}$ and $b \in I$.*

Remark B.1.9. *Every ideal of $\mathfrak{g}$ is also a Lie subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{g}$ itself and 0 are always ideals of $\mathfrak{g}$.*

Definition B.1.10. *Let $\mathfrak{g}$ be a Lie algebra and I an ideal of $\mathfrak{g}$. The* quotient Lie algebra *$\mathfrak{g}/I$ is defined by considering the vector space $\mathfrak{g}/I$ with bracket $[a+I, b+I] := [a,b] + I$ for all $a, b \in \mathfrak{g}$.*

Definition B.1.11. *The* center *of a Lie algebra $\mathfrak{g}$ is $Z(\mathfrak{g}) := \{a \in \mathfrak{g} \mid [a,b] = 0 \ \forall b \in \mathfrak{g}\}$. It is an ideal of $\mathfrak{g}$.*

Definition B.1.12. *The* derived algebra *of a Lie algebra $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}] := \{[a,b] \mid a, b \in \mathfrak{g}\}$. It is an ideal of $\mathfrak{g}$.*

Definition B.1.13. *The* abelianization *of a Lie algebra $\mathfrak{g}$ is $\mathfrak{g}^{ab} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. It is clear that it is an abelian Lie algebra.*

Definition B.1.14. *A Lie algebra $\mathfrak{g}$ is called* simple *if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and $\mathfrak{g}$ has no ideal except itself and 0.*

Definition B.1.15. *The* normaliser *of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is $N_{\mathfrak{g}}(\mathfrak{h}) := \{a \in \mathfrak{g} \mid [a,b] \in \mathfrak{h} \ \forall b \in \mathfrak{h}\}$. If there are no possible confusions, we denote it just by $N(\mathfrak{h})$. By Jacobi identity, it is a Lie subalgebra of $\mathfrak{g}$.*

Definition B.1.16. *If $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, then we call $\mathfrak{h}$* self-normalising.

Definition B.1.17. *The* centraliser *of a subset Y of $\mathfrak{g}$ is $C_{\mathfrak{g}}(Y) := \{a \in \mathfrak{g} \mid [a, b] = 0\ \forall b \in Y\}$. By Jacobi identity, it is a Lie subalgebra of $\mathfrak{g}$.*

Remark B.1.18. $C_{\mathfrak{g}}(\mathfrak{g}) = Z(\mathfrak{g})$.

Definition B.1.19. *Given a Lie algebra $\mathfrak{g}$, we define the* derived series *of $\mathfrak{g}$ to be the following descending sequence of ideals*

$$\mathfrak{g} \supset \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{g}'' = (\mathfrak{g}')' = [\mathfrak{g}', \mathfrak{g}'] \supset \cdots \supset \mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \supset \cdots .$$

We say that $\mathfrak{g}$ is a solvable *Lie algebra if $\mathfrak{g}^{(n)} = 0$ for some n.*

Proposition B.1.20. ([35], Proposition 3.1) *Let $\mathfrak{g}$ be a Lie algebra. Then*

1. *If $\mathfrak{g}$ is solvable, then so are all Lie subalgebras and homomorphic images of $\mathfrak{g}$;*
2. *If I is a solvable ideal of $\mathfrak{g}$ such that $\mathfrak{g}/I$ is solvable, then $\mathfrak{g}$ itself is solvable;*
3. *If I, J are solvable ideals of $\mathfrak{g}$, then so is $I + J$.*

Corollary B.1.21. *Let $\mathfrak{g}$ be a Lie algebra. Then there exists a unique maximal solvable ideal, i.e. included in no larger solvable ideal of $\mathfrak{g}$.*

Definition B.1.22. *Let $\mathfrak{g}$ be a Lie algebra. Then we denote the unique maximal solvable ideal of $\mathfrak{g}$ by* $\operatorname{Rad}(\mathfrak{g})$ *and we will call it the* radical *of $\mathfrak{g}$.*

Definition B.1.23. *A Lie algebra $\mathfrak{g}$ is called* semisimple *if $\mathfrak{g} \neq 0$ and $\operatorname{Rad}(\mathfrak{g}) = 0$.*

Notice that any simple Lie algebra $\mathfrak{g}$ is semisimple, because $\mathfrak{g}$ has no ideal except itself and 0 and $\mathfrak{g}$ is non-solvable.

In addition, if we consider $\mathfrak{g}$ a non-solvable Lie algebra, i.e. $\mathfrak{g} \neq \operatorname{Rad}(\mathfrak{g})$, then $\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ is semisimple.

Theorem B.1.24. ([55], Theorem 7.8.5) *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field of characteristic zero. Then $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}$ is a finite product of simple Lie algebras. In particular, every ideal of a semisimple Lie algebra is semisimple.*

Definition B.1.25. *Given a Lie algebra $\mathfrak{g}$, we define the* lower central series *of $\mathfrak{g}$ to be the following descending sequence of ideals*

$$\mathfrak{g} \supset \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{g}^3 = [\mathfrak{g}^2, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}] \supset \cdots .$$

We say that $\mathfrak{g}$ is a nilpotent *Lie algebra if $\mathfrak{g}^n = 0$ for some n.*

Proposition B.1.26. ([35], Proposition 3.2) *Let $\mathfrak{g}$ be a Lie algebra. Then*

1. *If $\mathfrak{g}$ is nilpotent, then so are all Lie subalgebras and homomorphic images of $\mathfrak{g}$;*
2. *If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ itself is nilpotent;*
3. *If $\mathfrak{g}$ is nilpotent, then $Z(\mathfrak{g}) \neq 0$.*

Proposition B.1.27. *Every nilpotent Lie algebra $\mathfrak{g}$ is solvable.*

Proof. It is suffices to show that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$, because then by induction this implies that $\mathfrak{g}^{(n)} \subset \mathfrak{g}^n$. To see this, we proceed by induction on j. The case $j = 1$ is trivially true by definition of $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}]$. Inductively, we compute:

$$[\mathfrak{g}^i, \mathfrak{g}^{j+1}] = [\mathfrak{g}^i, [\mathfrak{g}^j, \mathfrak{g}]] \subset [[\mathfrak{g}^i, \mathfrak{g}], \mathfrak{g}^j] + [[\mathfrak{g}^i, \mathfrak{g}^j], \mathfrak{g}] \subset [\mathfrak{g}^{i+1}, \mathfrak{g}^j] + [\mathfrak{g}^{i+j}, \mathfrak{g}] = \mathfrak{g}^{i+j+1}.$$

□

Definition B.1.28. *A Lie algebra $\mathfrak{g}$ is called* reductive *if $\mathfrak{g} \cong Z(\mathfrak{g}) \times [\mathfrak{g}, \mathfrak{g}]$.*

B.2 Representations of Lie algebras

In this section, we recall the notions of representation of a Lie algebra and of module over a Lie algebra and show that the two are equivalent. Moreover, we describe properties of representations in the case of reductive Lie algebras.

Definition B.2.1. *Let V be a vector space and let $\mathfrak{g}$ be a Lie algebra. A* representation of $\mathfrak{g}$ in V *is a homomorphism of Lie algebras $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. The dimension of V is called the* dimension *of the representation.*

Example B.2.2. *Let $\mathfrak{g}$ be a Lie algebra. Define the* adjoint representation *as*

$$\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

$$a \mapsto (b \mapsto [a, b]).$$

Notice that $\ker(\mathrm{ad}) = Z(\mathfrak{g})$.

It is often convenient to use the language of modules along with the language of representations.

Definition B.2.3. *Let* $\mathfrak{g}$ *be a Lie algebra. A* $\mathfrak{g}$-module *is a vector space* V *endowed with an operation*

$$\mathfrak{g} \times V \longrightarrow V$$
$$(a, v) \mapsto a \cdot v$$

that satisfies the following conditions

1. $(\alpha a + \beta b) \cdot v = \alpha(a \cdot v) + \beta(b \cdot v)$ *for all* $\alpha, \beta \in K$*,* $a, b \in \mathfrak{g}$ *and* $v \in V$*;*

2. $a \cdot (\alpha v + \beta w) = \alpha(a \cdot v) + \beta(a \cdot w)$ *for all* $\alpha, \beta \in K$*,* $a \in \mathfrak{g}$ *and* $v, w \in V$*;*

3. $[a, b] \cdot v = ab \cdot v - ba \cdot v$ *for all* $a, b \in \mathfrak{g}$ *and* $v \in V$*.*

If $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, then V may be seen as a $\mathfrak{g}$-module via the action $a \cdot v := \varrho(a)(v)$. Conversely, given a $\mathfrak{g}$-module V we can define a representation $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ by $\varrho(a)(v) := a \cdot v$.

In what follows, we will refer both to the homomorphism ϱ and to the vector space V as representation of $\mathfrak{g}$.

Definition B.2.4. *A* homomorphism of $\mathfrak{g}$-modules *is a linear map* $\phi\colon V \longrightarrow W$ *that is product preserving, that is* $\phi(a \cdot v) = a \cdot \phi(v)$ *for all* $a \in \mathfrak{g}$ *and* $v \in V$*. We write* $Hom_{\mathfrak{g}}(V, W)$ *for the set of all such homomorphism of* $\mathfrak{g}$*-modules.*

Remark B.2.5. *If* $\phi \in Hom_{\mathfrak{g}}(V, W)$ *and* $\alpha \in K$*, then also* $\alpha\phi \in Hom_{\mathfrak{g}}(V, W)$*, so* $Hom_{\mathfrak{g}}(V, W)$ *is a vector subspace of* $Hom_K(V, W)$*.*

Definition B.2.6. *A* $\mathfrak{g}$*-module* V *is called* trivial *if* $\mathfrak{g}$ *acts as zero on it, i.e.* $a \cdot v = 0$ *for all* $a \in \mathfrak{g}$ *and* $v \in V$*.*

Definition B.2.7. *Let* V *be a* $\mathfrak{g}$*-module. The* invariant submodule *of* V *is* $V^{\mathfrak{g}} := \{v \in V \mid a \cdot v = 0 \ \forall a \in \mathfrak{g}\}$.

Remark B.2.8. *If we consider* K *as a trivial* $\mathfrak{g}$*-module, then* $V^{\mathfrak{g}} \cong Hom_{\mathfrak{g}}(K, V)$*.*

Definition B.2.9. *A* $\mathfrak{g}$*-module* V *is called* irreducible *or* simple *if it has precisely two submodules: itself and* 0*.*

Definition B.2.10. *A* $\mathfrak{g}$*-module* V *is called* completely reducible *or* semisimple *if it is a direct sum of irreducible* $\mathfrak{g}$*-modules.*

Proposition B.2.11. ([55], Exercise 7.8.5) *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra over a field of characteristic zero. Then the following statements are equivalent*

1. $\mathfrak{g}$ *is a reductive Lie algebra;*

2. $\mathrm{Rad}(\mathfrak{g})$ *is abelian and equals the center of* $\mathfrak{g}$*;*

3. $\mathfrak{g}$ *is a completely reducible* $\mathfrak{g}$*-module via the adjoint representation;*

4. $\mathfrak{g} \cong \mathfrak{h} \times \mathfrak{l}$*, where* $\mathfrak{h}$ *is abelian and* $\mathfrak{l}$ *is semisimple.*

Proposition B.2.12. ([19], Corollary 1.6.4) *Let* $\mathfrak{g}$ *be a reductive Lie algebra and let* $\varrho \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ *be a finite dimensional representation of* $\mathfrak{g}$*. Then the following conditions are equivalent*

1. V *is completely reducible* $\mathfrak{g}$*-module;*

2. for all $a \in Z(\mathfrak{g})$*,* $\varrho(a)$ *is semisimple.*

B.3 Lie algebra cohomology

In this last section, we recall the notion of Lie algebra cohomology, we show its interpretations and finally, we show some of its properties in the cases of semisimple and reductive Lie algebras.

Definition B.3.1. *Let* $\mathfrak{g}$ *be a Lie algebra over a field* K *and let* V *be a* $\mathfrak{g}$*-module. We define the space of linear maps*

$$C^p(\mathfrak{g}, V) := Hom_K(\bigwedge^p \mathfrak{g}, V)$$

which we call the space of p-forms on $\mathfrak{g}$ with values in V.

Remark B.3.2. *We can identify* $C^0(\mathfrak{g}, V)$ *with* V*.*

Definition B.3.3. *We define on each* $C^p(\mathfrak{g}, V)$ *the structure of a* $\mathfrak{g}$*-module. By the previous Remark, the structure of* $\mathfrak{g}$*-module is already defined on* $C^0(\mathfrak{g}, V)$*. For* $p > 0$*,* $\psi \in C^p(\mathfrak{g}, V)$*,* $a, b_1, \dots, b_p \in \mathfrak{g}$*, we define the transform* $a \cdot \psi$ *by the formula*

$$(a\cdot\psi)(b_1\wedge\cdots\wedge b_p) := a\cdot(\psi(b_1\wedge\cdots\wedge b_p)) - \sum_{i=1}^{p} \psi(b_1\wedge\cdots\wedge b_{i-1}\wedge[a, b_i]\wedge b_{i+1}\wedge\cdots\wedge b_p).$$

Definition B.3.4. *We define the differentials*

$$d^p \colon C^p(\mathfrak{g}, V) \longrightarrow C^{p+1}(\mathfrak{g}, V)$$

by

$$(d^p(\psi))(a_1 \wedge \cdots \wedge a_{p+1}) := \sum_{i=1}^{p+1} (-1)^i a_i \cdot (\psi(a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_{p+1})) +$$

$$+ \sum_{1 \le i < j \le p+1} (-1)^{i+j-1} \psi([a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge \widehat{a_j} \wedge \cdots \wedge a_{p+1}).$$

Remark B.3.5. *It is clear that $d^{p+1} \circ d^p = 0$. Moreover, notice that*

$$d^0 \colon C^0(\mathfrak{g}, V) = V \longrightarrow C^1(\mathfrak{g}, V)$$

is defined by

$$v \mapsto (a \mapsto a \cdot v).$$

Definition B.3.6. *The cohomology of the complex $(C^\bullet(\mathfrak{g}, V), d^\bullet)$ is denoted by $H^\bullet(\mathfrak{g}, V)$.*

Definition B.3.7. *Let $\mathfrak{g}$ be a Lie algebra and let V be a $\mathfrak{g}$-module. Then a* derivation *from $\mathfrak{g}$ to V is a K-linear map $D \colon \mathfrak{g} \longrightarrow V$ such that the Leibnitz formula holds*

$$D([a, b]) = a \cdot D(b) - b \cdot D(a),$$

for all $a, b \in \mathfrak{g}$. The set of all derivations from $\mathfrak{g}$ to V is denoted by $\mathrm{Der}(\mathfrak{g}, V)$.

Definition B.3.8. *$D \in \mathrm{Der}(\mathfrak{g}, V)$ is called an* inner derivation *if there exists $v \in V$ such that $D(a) = a \cdot v$ for all $a \in \mathfrak{g}$. The set of inner derivations is denoted by* $\mathrm{Der}_{Inn}(\mathfrak{g}, V)$.

Theorem B.3.9. ([55], Theorem 7.4.7 and Corollary 7.4.8) *Let $\mathfrak{g}$ be a Lie algebra and let V be a $\mathfrak{g}$-module. Then $H^1(\mathfrak{g}, V) \cong \mathrm{Der}(\mathfrak{g}, V) / \mathrm{Der}_{Inn}(\mathfrak{g}, V)$. Moreover if V is trivial, then $H^1(\mathfrak{g}, V) \cong Hom_k(\mathfrak{g}^{ab}, V)$.*

Definition B.3.10. *Let $\mathfrak{g}$ be a Lie algebra. An* extension of Lie algebras of $\mathfrak{g}$ by $\mathfrak{h}$ *is a short exact sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

where $\mathfrak{h}$ is an abelian Lie algebra.

Definition B.3.11. *The set of equivalence classes of extensions of $\mathfrak{g}$ by $\mathfrak{h}$ is denoted by* $\mathrm{Ext}(\mathfrak{g}, \mathfrak{h})$.

Remark B.3.12. *An extension of Lie algebras makes $\mathfrak{h}$ into a $\mathfrak{g}$-module. If $a \in \mathfrak{g}$ and $b \in \mathfrak{h}$, then define $a \cdot b$ to be the product $[\tilde{a}, b] \in \mathfrak{e}$, where $\pi(\tilde{a}) = a$. Since $\mathfrak{h}$ is abelian, $a \cdot b$ is independent of the choice of $\tilde{a}$.*

Theorem B.3.13. ([55], Theorem 7.6.3) *Let V be a $\mathfrak{g}$-module. The set* $\mathrm{Ext}(\mathfrak{g}, V)$ *is in 1-1 correspondence with $H^2(\mathfrak{g}, V)$.*

Theorem B.3.14. ([55], Theorem 7.8.9) *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero. If V is a simple $\mathfrak{g}$-module and $V \neq K$, then $H^i(\mathfrak{g}, V) = 0$ for all i.*

Proposition B.3.15. ([55], Corollary 7.8.10 and 7.8.12) *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero and let V be a finite dimensional $\mathfrak{g}$-module. Then $H^1(\mathfrak{g}, V) = 0 = H^2(\mathfrak{g}, V)$.*

Theorem B.3.16. ([33], Theorem 10) *Let $\mathfrak{g}$ be a reductive Lie algebra of finite dimension over $\mathbb{C}$ and let V be a finite dimensional semisimple $\mathfrak{g}$-module such that $V^{\mathfrak{g}} = (0)$. Then $H^i(\mathfrak{g}, V) = 0$ for all $i \geq 0$.*

Appendix C

Examples of procedure to compute $\mathcal{LFT}^1(D)$ and $\mathcal{FT}^1(D)$

The aim of this last appendix is to show two examples: in the first one we compute $\mathcal{LFT}^1(D)$ for a germ of a non-reductive linear free divisor in $(\mathbb{C}^5, 0)$ and in the second one we compute $\mathcal{FT}^1(D)$ for a germ of a weighted homogeneous free divisor in $(\mathbb{C}^2, 0)$.

C.1 Example 3.5.9

In this section we describe a Macaulay 2 procedure to compute $\mathcal{LFT}^1(D)$ for the germ of the linear free divisor $(D, 0) \subset (\mathbb{C}^5, 0)$ defined by $f = x_5(x_4^4 - 2x_5x_4^2x_3 + x_5^2x_3^2 + 2x_5^2x_4x_2 - 2x_5^3x_1) \in \mathbb{C}[x_1, \ldots, x_5]$ of Example 3.5.9.

First of all, notice that we represent each vector field of $\mathrm{Der}_{\mathbb{C}^5}$ with a column matrix.

Before starting with the procedure, we need to introduce a function that computes the Lie bracket of vector fields:

```
LB = method()
LB(Matrix,Matrix):=Matrix=>(v,h)->transpose((transpose(v)*jacobian(transpose(h)))+
-(transpose(h)*jacobian(transpose(v))));
```

We are now ready to compute the kernel of $d^1 \colon \mathcal{C}_0^1 \longrightarrow \mathcal{C}_0^2$. We represent each element with a 5×5 matrix.

```
S:=QQ[a_1..a_25,b_1..b_25,c_1..c_25,d_1..d_25,e_1..e_25];
R:=S[x_1..x_5];
f:=x_5*(x_4^4-2*x_5*x_4^2*x_3+x_5^2*x_3^2+2*x_5^2*x_4*x_2-2*x_5^3*x_1);
%We compute a Saito matrix for (D,0)
D:=modulo(gens(ideal(jacobian(ideal(f)))),matrix{{f}});
```

```
%We compute the Lie brackets of the generators of Der(- log D)
v12:=LB(D_{0},D_{1});
v13:=LB(D_{0},D_{2});
v14:=LB(D_{0},D_{3});
v15:=LB(D_{0},D_{4});
v23:=LB(D_{1},D_{2});
v24:=LB(D_{1},D_{3});
v25:=LB(D_{1},D_{4});
v34:=LB(D_{2},D_{3});
v35:=LB(D_{2},D_{4});
v45:=LB(D_{3},D_{4});
%We compute the coefficients to write the vij as a linear combination of the basis of Der(-log D)
d12:=v12//D;
d13:=v13//D;
d14:=v14//D;
d15:=v15//D;
d23:=v23//D;
d24:=v24//D;
d25:=v25//D;
d34:=v34//D;
d35:=v35//D;
d45:=v45//D;
%A is the matrix of possible element in the kernel of d^1
A:=matrix{{a_1*x_1+b_1*x_2+c_1*x_3+d_1*x_4+e_1*x_5,a_6*x_1+b_6*x_2+c_6*x_3+d_6*x_4+e_6*x_5,
a_11*x_1+b_11*x_2+c_11*x_3+d_11*x_4+e_11*x_5,a_16*x_1+b_16*x_2+c_16*x_3+d_16*x_4+e_16*x_5,
a_21*x_1+b_21*x_2+c_21*x_3+d_21*x_4+e_21*x_5},
{a_2*x_1+b_2*x_2+c_2*x_3+d_2*x_4+e_2*x_5,a_7*x_1+b_7*x_2+c_7*x_3+d_7*x_4+e_7*x_5,
a_12*x_1+b_12*x_2+c_12*x_3+d_12*x_4+e_12*x_5,a_17*x_1+b_17*x_2+c_17*x_3+d_17*x_4+e_17*x_5,
a_22*x_1+b_22*x_2+c_22*x_3+d_22*x_4+e_22*x_5},
{a_3*x_1+b_3*x_2+c_3*x_3+d_3*x_4+e_3*x_5,a_8*x_1+b_8*x_2+c_8*x_3+d_8*x_4+e_8*x_5,
a_13*x_1+b_13*x_2+c_13*x_3+d_13*x_4+e_13*x_5,a_18*x_1+b_18*x_2+c_18*x_3+d_18*x_4+e_18*x_5,
a_23*x_1+b_23*x_2+c_23*x_3+d_23*x_4+e_23*x_5},
{a_4*x_1+b_4*x_2+c_4*x_3+d_4*x_4+e_4*x_5,a_9*x_1+b_9*x_2+c_9*x_3+d_9*x_4+e_9*x_5,
a_14*x_1+b_14*x_2+c_14*x_3+d_14*x_4+e_14*x_5,a_19*x_1+b_19*x_2+c_19*x_3+d_19*x_4+e_19*x_5,
a_24*x_1+b_24*x_2+c_24*x_3+d_24*x_4+e_24*x_5},
{a_5*x_1+b_5*x_2+c_5*x_3+d_5*x_4+e_5*x_5,a_10*x_1+b_10*x_2+c_10*x_3+d_10*x_4+e_10*x_5,
a_15*x_1+b_15*x_2+c_15*x_3+d_15*x_4+e_15*x_5,a_20*x_1+b_20*x_2+c_20*x_3+d_20*x_4+e_20*x_5,
a_25*x_1+b_25*x_2+c_25*x_3+d_25*x_4+e_25*x_5}}
%Bij represents the image of vij via the element defined by A
B12:=A*(d12);
B13:=A*(d13);
B14:=A*(d14);
B15:=A*(d15);
B23:=A*(d23);
B24:=A*(d24);
B25:=A*(d25);
B34:=A*(d34);
B35:=A*(d35);
B45:=A*(d45);
%We now compute the map d^1 on our basis of Der(-log D)
C12:=-LB(D_{0},A_{1})+LB(D_{1},A_{0})+B12;
C13:=-LB(D_{0},A_{2})+LB(D_{2},A_{0})+B13;
C14:=-LB(D_{0},A_{3})+LB(D_{3},A_{0})+B14;
```

```
C15:=-LB(D_{0},A_{4})+LB(D_{4},A_{0})+B15;
C23:=-LB(D_{1},A_{2})+LB(D_{2},A_{1})+B23;
C24:=-LB(D_{1},A_{3})+LB(D_{3},A_{1})+B24;
C25:=-LB(D_{1},A_{4})+LB(D_{4},A_{1})+B25;
C34:=-LB(D_{2},A_{3})+LB(D_{3},A_{2})+B34;
C35:=-LB(D_{2},A_{4})+LB(D_{4},A_{2})+B35;
C45:=-LB(D_{3},A_{4})+LB(D_{4},A_{3})+B45;
%We now compute for which coefficients each Cij is in Der(-log D)
T:=R[p_1..p_10,q_1..q_10,r_1..r_10,u_1..u_10,t_1..t_10];
DT:=sub(D,T);
%Each si is a generic element of Der(-log D)
s1:=p_1*sub(DT_{0},T)+q_1*sub(DT_{1},T)+r_1*sub(DT_{2},T)+u_1*sub(DT_{3},T)+t_1*sub(DT_{4},T);
s2:=p_2*sub(DT_{0},T)+q_2*sub(DT_{1},T)+r_2*sub(DT_{2},T)+u_2*sub(DT_{3},T)+t_2*sub(DT_{4},T);
s3:=p_3*sub(DT_{0},T)+q_3*sub(DT_{1},T)+r_3*sub(DT_{2},T)+u_3*sub(DT_{3},T)+t_3*sub(DT_{4},T);
s4:=p_4*sub(DT_{0},T)+q_4*sub(DT_{1},T)+r_4*sub(DT_{2},T)+u_4*sub(DT_{3},T)+t_4*sub(DT_{4},T);
s5:=p_5*sub(DT_{0},T)+q_5*sub(DT_{1},T)+r_5*sub(DT_{2},T)+u_5*sub(DT_{3},T)+t_5*sub(DT_{4},T);
s6:=p_6*sub(DT_{0},T)+q_6*sub(DT_{1},T)+r_6*sub(DT_{2},T)+u_6*sub(DT_{3},T)+t_6*sub(DT_{4},T);
s7:=p_7*sub(DT_{0},T)+q_7*sub(DT_{1},T)+r_7*sub(DT_{2},T)+u_7*sub(DT_{3},T)+t_7*sub(DT_{4},T);
s8:=p_8*sub(DT_{0},T)+q_8*sub(DT_{1},T)+r_8*sub(DT_{2},T)+u_8*sub(DT_{3},T)+t_8*sub(DT_{4},T);
s9:=p_9*sub(DT_{0},T)+q_9*sub(DT_{1},T)+r_9*sub(DT_{2},T)+u_9*sub(DT_{3},T)+t_9*sub(DT_{4},T);
s10:=p_10*sub(DT_{0},T)+q_10*sub(DT_{1},T)+r_10*sub(DT_{2},T)+u_10*sub(DT_{3},T)+t_10*sub(DT_{4},T);
v1=sub(C12,T);
v2=sub(C13,T);
v3=sub(C14,T);
v4=sub(C15,T);
v5=sub(C23,T);
v6=sub(C24,T);
v7=sub(C25,T);
v8=sub(C34,T);
v9=sub(C35,T);
v10=sub(C45,T);
w1:=sub(v1-s1,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w2:=sub(v1-s1,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
w3:=sub(v1-s1,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w4:=sub(v1-s1,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w5:=sub(v1-s1,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
w6:=sub(v2-s2,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w7:=sub(v2-s2,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
w8:=sub(v2-s2,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w9:=sub(v2-s2,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w10:=sub(v2-s2,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
w11:=sub(v3-s3,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w12:=sub(v3-s3,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
w13:=sub(v3-s3,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w14:=sub(v3-s3,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w15:=sub(v3-s3,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
w16:=sub(v4-s4,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w17:=sub(v4-s4,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
w18:=sub(v4-s4,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w19:=sub(v4-s4,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w20:=sub(v4-s4,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
w21:=sub(v5-s5,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w22:=sub(v5-s5,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
```

```
w23:=sub(v5-s5,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w24:=sub(v5-s5,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w25:=sub(v5-s5,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
w26:=sub(v6-s6,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w27:=sub(v6-s6,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
w28:=sub(v6-s6,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w29:=sub(v6-s6,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w30:=sub(v6-s6,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
w31:=sub(v7-s7,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w32:=sub(v7-s7,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
w33:=sub(v7-s7,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w34:=sub(v7-s7,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w35:=sub(v7-s7,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
w36:=sub(v8-s8,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w37:=sub(v8-s8,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
w38:=sub(v8-s8,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w39:=sub(v8-s8,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w40:=sub(v8-s8,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
w41:=sub(v9-s9,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w42:=sub(v9-s9,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
w43:=sub(v9-s9,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w44:=sub(v9-s9,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w45:=sub(v9-s9,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
w46:=sub(v10-s10,{x_1=>1,x_2=>0,x_3=>0,x_4=>0,x_5=>0});
w47:=sub(v10-s10,{x_1=>0,x_2=>1,x_3=>0,x_4=>0,x_5=>0});
w48:=sub(v10-s10,{x_1=>0,x_2=>0,x_3=>1,x_4=>0,x_5=>0});
w49:=sub(v10-s10,{x_1=>0,x_2=>0,x_3=>0,x_4=>1,x_5=>0});
w50:=sub(v10-s10,{x_1=>0,x_2=>0,x_3=>0,x_4=>0,x_5=>1});
%I is the ideal of relations of the parameters, if they are all zero then all Cij are in Der(-log D)
I:=ideal(w1..w50);
S:=QQ[p_1..p_10,q_1..q_10,r_1..r_10,u_1..u_10,t_1..t_10,a_1..a_25,b_1..b_25,c_1..c_25,d_1..d_25,
e_1..e_25, MonomialOrder=>Eliminate 50];
J:=sub(I,S);
%We now eliminate the additional parameters p_i, r_i, q_i, u_i and t_i
G:=selectInSubring(1,gens gb J);
R;
F:=sub(G,R);
H:=ideal(F);
%We put the relations in R
P:=R/H;
%We finally compute the element of the kernel of d^1
E:=sub(A,P);
%We write now the outcome of the computation
%First column of E
|-5x_1c_9 +4x_1d_4+3x_1d_10-3x_1e_5+x_2b_24+x_2e_4+3x_3c_24-x_3d_8+x_3d_25+x_3e_3+2x_3e_9+x_4d_1+x_ 5e_1|
| -x_1d_5-3x_2c_9+3x_2d_4+x_2d_10-2x_2e_5-1/2x_3d_15+x_4d_2+x_5e_2                                  |
| -2x_3c_9+2x_3d_4+x_3d_10-x_3e_5+x_4d_3+x_5e_3                                                     |
| x_4d_4+x_5e_4                                                                                     |
| x_4d_5+x_5e_5                                                                                     |
%Second column of E
|3x_1b_24-x_1d_3+4x_1d_9+3/2x_1d_15+2x_1e_4-3x_1e_10+3x_2c_24-x_2d_8+x_2d_25+3x_2e_9-1/2x_3c_12
   +x_3d_1+x_3d_7-1/2x_3d_13-x_3e_2+x_3e_14-1/2x_4d_12-3/2x_4d_23+x_4e_1+x_4e_7+1/2x_4e_13+3x_4e_24+x_ 5e_6|
|-x_1c_9+2x_2b_24+3x_2d_9+x_2d_15-2x_2e_10-x_3c_24+x_3d_2+x_3d_8-2x_3d_25-x_3e_3-2x_3e_9+x_4d_7 +x_5e_7    |
```

```
|-x_1d_5-x_2d_10+x_3d_3+2x_3d_9-2x_3e_4-x_3e_10+x_4d_8+x_5e_8                                                |
|x_3c_9+x_4d_9+x_5e_9                                                                                        |
|x_3d_5+x_4d_10+x_5e_10                                                                                      |
%Third column of E
|10x_1c_24-x_1d_2-x_1d_8+4x_1d_14-5x_1d_25+x_1e_3+2x_1e_9-3x_1e_15+x_2d_1-x_2e_2+x_2e_14-1/2x_3d_12
             -3/2x_3d_23+3/2x_3e_13+3x_3e_24-x_4d_22+2x_4e_6+x_4e_12+x_4e_23+x_5e_11      |
|4x_2c_24+x_2d_2+x_2d_8+3x_2d_14-7x_2d_25-x_2e_3-2x_2e_9-2x_2e_15+x_3c_12+x_4d_12+x_5e_12 |
|-2x_1c_9+x_2d_3-2x_2e_4+6x_3c_24+2x_3d_14-3x_3d_25-x_3e_15+x_4d_13+x_5e_13               |
|-x_1d_5+x_2c_9-x_2d_10-1/2x_3d_15+x_4d_14+x_5e_14                                        |
|x_2d_5+x_4d_15+x_5e_15                                                                   |
 %Fourth column of E
|4x_1d_19-3x_1e_20-x_2e_1+x_2e_19-2x_3e_6+x_3e_18-3x_4e_11+x_4e_17-x_5e_21                                   |
|-2x_1c_24+x_1d_2+x_1d_8-2x_1d_25-x_1e_3-2x_1e_9+3x_2d_19-2x_2e_20-1/2x_3d_12-1/2x_3d_23+1/2x_3e_13
                    +x_3e_24-x_4d_22+x_4e_18+x_4e_23+x_5e_17                                                 |
|-2x_1b_24+2x_1d_3-4x_1e_4-2x_2c_24+x_2d_8-x_2d_25-2x_2e_9+2x_3d_19-x_3e_20-x_4d_23+2x_4e_19+2x_4e_24+x_5e_18|
|3x_1c_9 -x_2b_24-x_3c_24+x_4d_19+x_5e_19                                                                    |
|4x_1d_5+3x_2d_10+x_3d_15-x_4d_25+x_5e_20                                                                    |
 %Fifth column of E
|4x_1d_24-3x_1e_25+x_2e_1+x_2e_24+2x_3e_6+x_3e_23+3x_4e_11+x_4e_22+x_5e_21                  |
|2x_1c_24-x_1d_2-x_1d_8+2x_1d_25+x_1e_3+2x_1e_9+3x_2d_24-2x_2e_25+1/2x_3d_12+1/2x_3d_23
                                    -1/2x_3e_13 -x_3e_24+x_4d_22+x_5e_22                    |
|2x_1b_24-2x_1d_3+4x_1e_4+2x_2c_24-x_2d_8+x_2d_25+2x_2e_9+2x_3d_24-x_3e_25+x_4d_23+x_5e_23  |
|-3x_1c_9+x_2b_24+x_3c_24+x_4d_24+x_5e_24                                                   |
|-4x_1d_5-3x_2d_10-x_3d_15+x_4d_25+x_5e_25                                                  |
%Note that in E we have 47 free parameters and each of them gives us an element of the kernel of d^1.
```

We are now ready to compute the image of $d^0\colon \mathcal{C}_0^0 \longrightarrow \mathcal{C}_0^1$. Also here, we represent each element with a 5×5 matrix.

```
R:=QQ[x_1..x_5];
f:=x_5*(x_4^4-2*x_5*x_4^2*x_3+x_5^2*x_3^2+2*x_5^2*x_4*x_2-2*x_5^3*x_1);
D:=modulo(gens(ideal(jacobian(ideal(f)))),matrix{{f}});
M:=matrix{{x_1,0,0,0,0,x_2,0,0,0,0,x_3,0,0,0,0,x_4,0,0,0,0,x_5,0,0,0,0},
{0,x_1,0,0,0,0,x_2,0,0,0,0,x_3,0,0,0,0,x_4,0,0,0,0,x_5,0,0,0},
{0,0,x_1,0,0 ,0,0,x_2,0,0,0,0,x_3,0,0,0,0,x_4,0,0,0,0,x_5,0,0},
{0,0,0,x_1,0,0,0,0,x_2,0,0,0,0,x_3,0,0,0,0,x_4,0,0,0,0,x_5,0},
{0,0,0,0,x_1,0,0,0,0,x_2,0,0,0,0,x_3,0,0,0,0,x_4,0,0,0,0,x_5}};
l:={};
for i from 0 to 24 do(
l=l|{matrix{{LB(D_{0},M_{i}),LB(D_{1},M_{i}),LB(D_{2},M_{i}),LB(D_{3},M_{i}),LB(D_{4},M_{i})}}});
%The 25 elements of the list l are 5x5 matrices that span the image of d^0
```

Notice that if in the matrix E we substitute the value one to the parameter c_{12} and zero to all the other parameters, then we obtain the element

$$\begin{bmatrix} 0 & -1/2x_3 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that differs from the one described in Example 3.5.9 by an element of the image of d^0 and so they describe the same admissible deformation.

To conclude we have now to check if the elements obtained from the matrix E by substituting successively the value 1 to one parameter and 0 to the others, can be written as linear combination of the matrices in the list l. We also have to eliminate the elements that have all columns in $\mathrm{Der}(-\log D)$.

After this long checking, we notice that only the three elements relative to the parameters e_4, c_{12} and d_7 and are not zero in cohomology and so $\mathcal{LFT}^1(D)$ is three dimensional and is generated by these elements. This are the elements:

$$\begin{bmatrix} x_2 & 2x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2x_2 & -4x_1 & 4x_1 \\ x_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1/2x_3 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & x_3 & 0 & 0 & 0 \\ 0 & x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

C.2 Example 3.7.8, 1)

In this section we describe a Macaulay 2 procedure to compute $\mathcal{FT}^1(D)$ for the germ of the free divisor $(D,0) \subset (\mathbb{C}^2,0)$ defined by $f = xy(x-y)(x+y) \in \mathbb{C}[x,y]$ of Example 3.7.8 and Remark 3.4.7. Notice that the procedure is similar to the one of the previous Section.

Similarly to the previous section, we represent each vector field of $\mathrm{Der}_{\mathbb{C}^2}$ with a column matrix and each element of the kernel of $d^1 \colon \mathcal{C}^1 \longrightarrow \mathcal{C}^2$ with a 2×2 matrix.

```
S:=QQ[a_1..a_12];
R:=S[x,y];
f:=x*y*(x-y)*(x+y);
D:=modulo(gens(ideal(jacobian(ideal(f)))),matrix{{f}});
v:=LB(D_{0},D_{1});
A:=matrix{{a_1*x+a_2*y,a_5*x^3+a_6*x^2*y+a_7*x*y^2+a_8*y^3},
{a_3*x+a_4*y,a_9*x^3+a_10*x^2*y+a_11*x*y^2+a_12*y^3}};
B:=2*A_{1};
%We now compute the map d^1 on our basis of Der(-log D)
```

```
C:=-LB(D_{0},A_{1})+LB(D_{1},A_{0})+B;
%We now want to find the value of the coefficient such that C is in Der(-log D)
%Notice that C has degree 3:
| x2ya_2-y3a_2                           |
| -x3a_3-2x2ya_1-2xy2a_2+3xy2a_3+2y3a_4 |
T:=R[p,q,r,s];
Ct=sub(C,T);
Dt=sub(D,T);
dt=p*x^2*Dt_{0}+q*x*y*Dt_{0}+r*y^2*Dt_{0}+s*Dt_{1};
Et=Ct-dt;
| -px3-qx2y-rxy2+x2ya_2-y3a_2                                  |
| -px2y-qxy2-ry3-sx2y+sy3-x3a_3-2x2ya_1-2xy2a_2+3xy2a_3+2y3a_4 |
%We now impose that all the coefficients of Et are zero.
I:=ideal(-p,-q+a_2,-r,a_2,a_3,-p-s-2*a_1,-q-2*a_2+3*a_3,-r+s+2*a_4);
S:=QQ[p,q,r,s,a_1..a_12, MonomialOrder=>Eliminate 4];
J:=sub(I,S);
G:=selectInSubring(1,gens gb J);
R;
F:=sub(G,R);
H:=ideal(F);
P:=R/H;
E:=sub(A,P);
| xa_4 x3a_5+x2ya_6+xy2a_7+y3a_8    |
| ya_4 x3a_9+x2ya_10+xy2a_11+y3a_12 |
```

Notice that if in the matrix E we substitute the value 1 to the parameter a_{11} and 0 to all the other parameters, then we obtain the element E_{11}, such that $\det(D + \epsilon \cdot E_{11}) = f + \epsilon \cdot x^2y^2$ as described in Example 3.7.8.

$$E_{11} = \begin{bmatrix} 0 & 0 \\ 0 & xy^2 \end{bmatrix}.$$

We are now ready to compute the image of $d^0 \colon \mathcal{C}^0 \longrightarrow \mathcal{C}^1$. We represent each element with a 2×2 matrix.

```
R:=QQ[x,y];
f:=x*y*(x-y)*(x+y);
D:=modulo(gens(ideal(jacobian(ideal(f)))),matrix{{f}});
M:=matrix{{x,0,y,0},{0,x,0,y}};
l:={};
for i from 0 to 3 do(
l= l|{matrix{{LB(D_{0},M_{i}),LB(D_{1},M_{i})}}});
{| 0 0     |, | 0 0        |, | 0 x2y-y3 |, | 0 0   |}
 | 0 -2x2y |  | 0 -x3+3xy2 |  | 0 -2xy2  |  | 0 2y3 |
```

As in the previous example, to conclude we have now to check if the elements obtained from the matrix E by substituting successively the value 1 to one parameter and 0 to the others, can be written as linear combination of the matrices in the list l. We also have to eliminate the elements that have all columns in $\mathrm{Der}(-\log D)$.

After this long checking, we notice that only the element relative to the parameter a_{11} is not zero in cohomology and so $\mathcal{FT}^1(D)$ is one dimensional and is generated by this element.

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