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Abstract

This short paper shows in an example of strategic market game that the Cournot-Nash equilibrium converges to the Walras equilibrium, even in the case of an exchange economy with infinitely many commodities.

1 Introduction

Economies with infinitely many commodities allow to analyse issues as commodity differentiation, infinite time horizon and uncertainty. In these economies the standard proofs of existence of a Walras equilibrium, as Debreu (1959), are no longer true. Bewley (1972) made one of the first attempts to prove its existence in such economies. In the context of cooperative game theory, Debreu and Scarf (1963) proved that if an economy with a finite number of commodities is replicated infinitely many times, then the set of core allocations shrinks to the set of Walras equilibrium allocations. This result was extended by Peleg and Yaari (1970) to the case of a countable infinity of commodities and by Gabszewicz (1991) to the case of a continuum of commodities.

In the context of non cooperative game theory, as far as I know, there are no contributions on strategic market games with an infinite number of commodities. In this paper, it is shown an example of a strategic market game with infinitely many commodities. The example is developed using the strategic market game introduced by Shubik (1973) and Shapley (1976). This is a strategic market game in which there is a trading post for each commodity and in each of them a commodity is exchanged for commodity money.

This kind of strategic market game was analysed in details by Dubey and Shubik (1978) for the case of finite commodities. In their Theorem 2, they proved that if traders are replicated infinitely many times, then the price vector and the allocation, at the Cournot-Nash equilibrium, converge to the Walras equilibrium of the exchange economy associated to the strategic market game.

In the main proposition of this paper, it is shown that the same convergence result is obtained in the example, even if there are infinitely many commodities. This result

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does not rely on Dubey and Shubik's Theorem 2 because they considered a strategic market game with a finite number of commodities and in particular the proof is based on Kuhn-Tucker Theorem that holds only if the commodity space is a subset of Euclidean space.

This problem is circumvented extending a proposition in Mas-Colell, Whinston, and Green (1995), on exchange economy with infinitely many commodities, to the strategic market game. In Proposition 20.D.2, they showed that if an utility function is defined on the space of bounded sequences and has an additive form with a discount factor, then an allocation is optimal when it solves the first order conditions of the Lagrangian function and satisfies the budget constraint. In Proposition 2, the analogous result is established for the strategic market game.

2 An example of strategic market game with infinitely many commodities

Consider an exchange economy with two types of trader 1 and 2. Let T_1 and T_2 be countable set, with cardinality k such that $k \geq 2$, and let $T = T_1 \cup T_2$. Elements of T_1 are the traders of type 1 and elements of T_2 are the traders of type 2. Let X be the consumption set such that $X \subseteq \ell^\infty_+$, with $\ell^\infty_+ = \{x \in \mathbb{R}^\infty_+ : \sup_j \|x_j\| < \infty\}$, that is the space of non-negative bounded sequences. Elements of X are called commodity bundles and are denoted by $x = (x_0, x_1, \dots)$, where x_j denotes the quantity of commodity j. A list of commodity bundles is an allocation and is denoted by $\mathbf{x}$. The price vector is denoted by $p = (p_0, p_1, \dots)$. Each trader is characterized by an utility function and an initial endowment. In this example, the utility function is $u^t(x^t) = \sum_{j=0}^\infty \delta^j \ln x^t_j$, with $\delta \in (0,1)$, for all $t \in T$, and the initial endowments are $w^t = (M, E, 0, E, 0, \dots)$, for all $t \in T_1$, and $w^t = (M, 0, E, 0, E, \dots)$, for all $t \in T_2$. Walras equilibrium can be defined as follows.

Definition 1. *A pair $(p^*, \boldsymbol{x}^*)$ is a Walras equilibrium for the exchange economy, if the commodity bundle x^{t*} solves the trader optimization problem at p^*,*

$$\begin{aligned} &\max_{x^t \in X} && \sum_{j=0}^\infty \delta^j \ln x^t_j \\ &\textit{subject to} && \sum_{j=0}^\infty p^*_j x^t_j \leq \sum_{j=0}^\infty p^*_j w^t_j, \end{aligned} \tag{1}$$

for all $t \in T$, and all markets clear

$$\begin{aligned} &\sum_{t \in T} x^{t*}_0 = 2kM, \\ &\sum_{t \in T} x^{t*}_j = kE, \textit{ for } j = 1, 2, \dots. \end{aligned} \tag{2}$$

As there are infinitely many commodities, the standard method of constrained maximization, namely Kuhn-Tucker Theorem, cannot be applied to find the maximum of the utility function. By using the following proposition from Mas-Colell et al. (1995), it is possible to circumvent this problem.

Proposition 1. *If the commodity bundle $x^t \in X$ satisfies the constraint $\sum_{j=0}^{\infty} p_j x_j^t = \sum_{j=0}^{\infty} p_j w_j^t < \infty$ and, for some $\lambda^t > 0$,*

$$\delta^j \frac{1}{x_j^t} - \lambda^t p_j = 0, \text{ for } j = 0, 1, \dots.$$

then it is the solution of the maximization problem in (1).

Proof. See Proposition 20.D.2 in Mas Colell et al. (1995). □

By Proposition 1 and market clearing conditions (2), the Walras equilibrium is

$$\begin{aligned}
(p_0^*, p_1^*, p_2^*, \dots) &= \left(1, 2\delta^1 \frac{M}{E}, 2\delta^2 \frac{M}{E}, \dots\right), \\
(x_0^{t*}, x_1^{t*}, x_2^{t*}, \dots) &= \left(\frac{1+2\delta-\delta^2}{1+\delta} M, \frac{1+2\delta-\delta^2}{1+\delta} \frac{E}{2}, \frac{1+2\delta-\delta^2}{1+\delta} \frac{E}{2}, \dots\right), \text{ for all } t \in T_1, \\
(x_0^{t*}, x_1^{t*}, x_2^{t*}, \dots) &= \left(\frac{1+\delta^2}{1+\delta} M, \frac{1+\delta^2}{1+\delta} \frac{E}{2}, \frac{1+\delta^2}{1+\delta} \frac{E}{2}, \dots\right), \text{ for all } t \in T_2.
\end{aligned} \tag{3}$$

The Walras equilibrium does not depend on k.

Consider now the following strategic market game, ${}^k\Gamma$, associated with the exchange economy. For each commodity $j = 1, 2, \dots$, there is a trading post where commodity j is exchanged for commodity money 0. Let W_t be the set of commodities, different from 0, that trader t holds and $\overline{W}_t$ be the set of commodities, different from 0, that trader t does not hold, for all $t \in T$. Assume that each trader t can only bid on commodities that he does not own and can only send to the market commodities that he owns. Hence, each trader has only one strategy for each commodity. The strategy sets are

$$S^t = \{(q_1^t, b_2^t, q_3^t, b_4^t, \dots) \in \ell_+^\infty : q_j^t \leq E, \text{ for all } j \in W_t, \text{ and } \sum_{j \in \overline{W}_t} b_j^t \leq M\}, \text{ for all } t \in T_1,$$

$$S^t = \{(b_1^t, q_2^t, b_3^t, q_4^t, \dots) \in \ell_+^\infty : q_j^t \leq E, \text{ for all } j \in W_t, \text{ and } \sum_{j \in \overline{W}_t} b_j^t \leq M\}, \text{ for all } t \in T_2,$$

where q_j^t denotes the quantity of commodity j that trader t offers to sale, with $j \in W_t$, and b_j^t denotes the amount of commodity money that trader t bids on commodity j, with $j \in \overline{W}_t$. Let $S = S^1 \times \dots \times S^{2k}$ and $S^{-t} = S^1 \times \dots \times S^{t-1} \times S^{t+1} \times \dots \times S^{2k}$. Let s, s^t and s^{-t} be elements of S, S^t and S^{-t} respectively. For all $s \in S$, $p(s)$ is a price vector determined according the following rule

$$p_j(s) = \begin{cases} \frac{\overline{b}_j}{\overline{q}_j} & \text{if } \overline{q}_j \neq 0 \\ 0 & \text{if } \overline{q}_j = 0 \end{cases} \tag{4}$$

for $j = 1, 2, \dots$. If j is odd, $\overline{b}_j = \sum_{t \in T_2} b_j^t$ and $\overline{q}_j = \sum_{t \in T_1} q_j^t$. If j is even, $\overline{b}_j = \sum_{t \in T_1} b_j^t$ and $\overline{q}_j = \sum_{t \in T_2} q_j^t$. For all $s \in S$, trader t's commodity bundle is a vector $x^t(s) \in X$

such that

$$
\begin{aligned}
x_0^t(s) &= M - \sum_{j\in\overline{W}_t} b_j^t + \sum_{j\in W_t} q_j^t p_j, \\
x_j^t(s) &= E - q_j^t, \text{ for all } j \in W_t, \\
x_j^t(s) &= \begin{cases} \frac{b_j^t}{p_j} & \text{if } p_j \neq 0 \\ 0 & \text{if } p_j = 0 \end{cases}, \text{ for all } j \in \overline{W}_t.
\end{aligned}
\tag{5}
$$

The payoff function of trader t is

$$
\pi^t(s) = \ln\left(M - \sum_{j\in\overline{W}_t} b_j^t + \sum_{j\in W_t} q_j^t \frac{\overline{b}_j}{q_j^t + \overline{q}_j^t}\right) + \sum_{j\in W_t} \delta^j \ln(E - q_j^t) + \sum_{j\in\overline{W}_t} \delta^j \ln b_j^t \frac{\overline{q}_j}{b_j^t + \overline{b}_j^t},
$$

where $\overline{q}_j^t = \overline{q}_j - q_j^t$ and $\overline{b}_j^t = \overline{b}_j - b_j^t$. The notion of interior type symmetric Cournot-Nash equilibrium is now introduced.

Definition 2. *An $\hat{s} \in S$ is an interior type symmetric Cournot-Nash equilibrium*[1] *of the strategic market game ${}^k\Gamma$, if all traders of the same type play the same strategies, and if $\hat{s}^t$ solve the following maximization problem at $\hat{s}^{-t}$,*

$$
\begin{aligned}
\max_{s^t\in S^t} \quad & \ln\left(M - \sum_{j\in\overline{W}_t} b_j^t + \sum_{j\in W_t} q_j^t \frac{\overline{b}_j}{q_j^t + \overline{q}_j^t}\right) + \sum_{j\in W_t} \delta^j \ln(E - q_j^t) + \sum_{j\in\overline{W}_t} \delta^j \ln b_j^t \frac{\overline{q}_j}{b_j^t + \overline{b}_j^t} \\
\text{subject to} \quad & \sum_{j\in\overline{W}_t} b_j^t < M \\
& q_j^t < E, \text{ for all } j \in W_t,
\end{aligned}
\tag{6}
$$

for all $t \in T$.

As before, since each trader has an infinite number of strategies, the standard method of constrained maximization cannot be applied to find the maximum of the payoff function. By using the following proposition, that is the analogous of Proposition 1 for the strategic market game, it is possible to circumvent this problem.

Proposition 2. *If the strategy $s^t \in S^t$ satisfies the constraints $\sum_{j\in\overline{W}_t} b_j^t < M$ and $q_j^t < E$, for all $j \in W_t$, and*

$$
\begin{aligned}
& -\frac{1}{M - \sum_{j\in\overline{W}_t} b_j^t + \sum_{j\in W_t} q_j^t \frac{\overline{b}_j}{q_j^t + \overline{q}_j^t}} + \frac{\delta^j}{b_j^t}\left(1 - \frac{b_j^t}{b_j^t + \overline{b}_j^t}\right) = 0, \text{ for all } j \in \overline{W}_t, \\
& \frac{1}{M - \sum_{j\in\overline{W}_t} b_j^t + \sum_{j\in W_t} q_j^t \frac{\overline{b}_j}{q_j^t + \overline{q}_j^t}} \frac{\overline{b}_j}{q_j^t + \overline{q}_j^t}\left(1 - \frac{q_j^t}{q_j^t + \overline{q}_j^t}\right) - \frac{\delta^j}{E - q_j^t} = 0, \text{ for all } j \in W_t,
\end{aligned}
\tag{7}
$$

then it is the solution of the maximization problem in (6).

Proof. First, it is impossible to improve the payoff by changing a finite number of strategies. Indeed, (7) implies that the first order sufficient conditions for any such

[1] Henceforth, for simplicity, only "Cournot-Nash equilibrium"

constrained payoff maximization problem are satisfied. Suppose that there exists a strategy $s'^t \in S^t$ yielding an higher payoff than s^t. Then for a sufficiently large h, consider the strategy s''^t, with $s''^t_j = s'^t_j$ for $j \leq h$ and $s''^t_j = s^t_j$ for $j > h$. Because $\delta < 1$, if h is large enough there is a payoff improvement in going from s^t to s''^t. Moreover, the strategy s''^t is feasible. All strategies q''^t_j are feasible because $q'^t_j < E$ and $q^t_j < E$, for all $j \in W_t$. If $\sum_{j>h}(b^t_j - b'^t_j) \leq 0$ then s''^t is feasible. Suppose not and that $\sum_{j>h}(b^t_j - b'^t_j) > 0$. Since s_t and s'_t are feasible, then $\sum_j b^t_j < M$ and $\sum_j b'^t_j < M$. But then, the sequences of strategies $\{b^t_j\}$ and $\{b'^t_j\}$ converge to 0 and $\sum_{j>h}(b^t_j - b'^t_j)$ can be made arbitrary small. Since $\sum_j b'^t_j < M$, there exists an $\epsilon > 0$ such that $\sum_j b'^t_j + \epsilon < M$. Hence, for large h, $\sum_{j>h}(b^t_j - b'^t_j) \leq \epsilon$ and then s''^t is a feasible strategy. But then, s''^t is feasible and gives an higher payoff of s^t but we have altered only a finite number of strategies in the process, a contradiction. □

In order to obtain a type symmetric Cournot-Nash equilibrium, for all $t \in T_1$, (7) becomes

$$\begin{aligned} &\frac{1}{M - \sum_{j\in\overline{W}_t} b^t_j + \sum_{j\in W_t} b^s_j} - \frac{\delta^j}{b^t_j}\left(1 - \frac{1}{k}\right) = 0, \text{ for all } j \in \overline{W}_t, \\ &\frac{1}{M - \sum_{j\in\overline{W}_t} b^t_j + \sum_{j\in W_t} b^s_j}\frac{b^s_j}{q^t_j}\left(1 - \frac{1}{k}\right) - \frac{\delta^j}{E - q^t_j} = 0, \text{ for all } j \in W_t, \end{aligned} \tag{8}$$

with $s \in T_2$. Similarly, for all $s \in T_2$, (7) becomes

$$\begin{aligned} &\frac{1}{M - \sum_{j\in\overline{W}_s} b^s_j + \sum_{j\in W_s} b^t_j} - \frac{\delta^j}{b^s_j}\left(1 - \frac{1}{k}\right) = 0, \text{ for all } j \in \overline{W}_s, \\ &\frac{1}{M - \sum_{j\in\overline{W}_s} b^s_j + \sum_{j\in W_s} b^t_j}\frac{b^t_j}{q^s_j}\left(1 - \frac{1}{k}\right) - \frac{\delta^j}{E - q^s_j} = 0, \text{ for all } j \in W_s, \end{aligned} \tag{9}$$

with $t \in T_1$. Let $G = \left(1 - \frac{1}{k}\right)$. Combining the equations in (8) and in (9), for all commodities $j = 1, 2, \ldots$, the Cournot-Nash equilibrium proves to be

$$\begin{aligned} \hat{b}^t_j &= \delta^j G \frac{1 + 2G\delta - \delta^2}{1 + G\delta - \delta^2 + G\delta^2} M, \text{ for all } j \in \overline{W}_t, \\ \hat{q}^t_j &= G^2 \frac{1 - \delta^2 + 2G\delta^2}{1 + G^2 + 2G\delta - \delta^2 - G^2\delta^2 + 2G^3\delta^2} E, \text{ for all } j \in W_t, \end{aligned} \tag{10}$$

for all $t \in T_1$,

$$\begin{aligned} \hat{b}^s_j &= \delta^j G \frac{1 - \delta^2 + 2G\delta^2}{1 + G\delta - \delta^2 + G\delta^2} M, \text{ for all } j \in \overline{W}_t, \\ \hat{q}^s_j &= G^2 \frac{1 + 2G\delta - \delta^2}{1 + G^2 + 2G^3\delta - \delta^2 + 2G\delta^2 - G^2\delta^2} E, \text{ for all } j \in W_t, \end{aligned} \tag{11}$$

for all $s \in T_2$. In the next proposition, it is shown that if traders are replicated infinitely many times, then the price vector and the allocation, at the Cournot-Nash equilibrium, converge to the Walras equilibrium of the exchange economy associated to the strategic market game.

Proposition 3. *Consider the sequence of Cournot-Nash equilibria, $\{^k\hat{s}\}$, associated to the sequence of strategic market games, $\{^k\Gamma\}$, converging to Γ. The sequence of Cournot-Nash equilibria converges to $\tilde{s}$ and the pair $(p(\tilde{s}), \boldsymbol{x}(\tilde{s}))$ is a Walras equilibrium of the exchange economy associated to Γ.*

Proof. To find the limit of the sequence of Cournot-Nash equilibria, consider the case of $k \to \infty$. Then, $\lim_{k\to\infty} G = 1$ and by substituting it in equations (10) and (11), the limit of the sequence of Cournot-Nash equilibria proves to be

$$(\tilde{q}_1^t, \tilde{b}_2^t, \tilde{q}_3^t, \tilde{b}_4^t, \dots) = \left(\frac{1+\delta^2}{1+\delta}\frac{E}{2}, \delta^2 \frac{1+2\delta-\delta^2}{1+\delta} M, \frac{1+\delta^2}{1+\delta}\frac{E}{2}, \delta^4 \frac{1+2\delta-\delta^2}{1+\delta} M, \dots \right),$$

for all $t \in T_1$,

$$(\tilde{b}_1^t, \tilde{q}_2^t, \tilde{b}_3^t, \tilde{q}_4^t \dots) = \left(\delta \frac{1+\delta^2}{1+\delta} M, \frac{1+2\delta-\delta^2}{1+\delta}\frac{E}{2}, \delta^3 \frac{1+\delta^2}{1+\delta} M, \frac{1+2\delta-\delta^2}{1+\delta}\frac{E}{2}, \dots \right),$$

for all $t \in T_2$. By substituting these strategies in (4) and (5), $p(\tilde{s})$ and $\mathbf{x}(\tilde{s})$ are obtained. It is immediate to see that these are the same price vector and allocation obtained in the Walras equilibrium (3) of the exchange economy, i.e. $p(\tilde{s}) = p^*$ and $\mathbf{x}(\tilde{s}) = \mathbf{x}^*$. □

References

[1] Bewley T. (1972), "Existence of equilibria in economies with infinitely many commodities," *Journal of Economic Theory* **4**, 514-40.

[2] Debreu G. (1959), *Theory of Value*, Yale University Press, New Haven.

[3] Debreu G., Scarf H. (1963), "A Limit Theorem on the Core of an Economy," *International Economic Review* **4**, 235-246.

[4] Dubey P., Shubik, M. (1978), "The noncooperative equilibria of a closed trading economy with market supply and bidding strategies," *Journal of Economic Theory* **17**, 1-20.

[5] Gabszewicz J.J. (1991), "A limit theorem on the core of an economy with a continuum of commodities," in: Khan M.A., Yannelis N.C. (eds.), Equilibrium theory with infinitely many commodities. Springer-Verlag, New York.

[6] Mas-Colell A., Whinston M.D. and Green J. R. (1995), *Microeconomic Theory*, Oxford University Press, New York.

[7] Peleg B., Yaari M.E. (1970), "Markets with Countably Many Commodities," *International Economic Review* **11**, 369-377.

[8] Shapley L.S. (1976), "Noncooperative general exchange", in: Lin S.A.Y. (Ed.), Theory of Measurement of Economic Externalities. Academic Press, New York.

[9] Shubik M. (1973), "Commodity, money, oligopoly, credit and bankruptcy in a general equilibrium model," *Western Economic Journal* **11**, 24-38.