Topics in orbifold geometry and Gorenstein homogeneous spaces

Umar Hayat

A thesis submitted in fulfilment of the requirements

for the degree of Doctor of Philosophy

Mathematics Department, University of Warwick

November 2011

THE UNIVERSITY OF WARWICK

Contents

List of Tables

Acknowledgements

It is a pleasure to express my deep gratitude to my supervisor Prof. Miles Reid for his continuous advice and encouragement throughout my PhD. His enthusiasm, dedication and way of explaining mathematics have been a constant source of inspiration.

I am thankful to Alvaro Nolla de Celis, Stephen Coughlan, Gavin Brown, Timothy Logvinenko, Diane Maclagan, Stéphane Lamy, Stavros Papadakis and Shengtian Zhou for many conversations and explanations. There are many other algebraic geometers and mathematicians who have helped me and I regret that I cannot name them all here.

I am grateful to many people at Warwick for their help and friendship, especially Sarah Davis, Eduardo Dias, Sohail Iqbal, Taro Sano, Michael Selig and Seung-Jo Jung.

I would like to thank the Higher Education Commission, Pakistan for providing financial support during my research at Warwick. Part of this work was carried out during a trip to Sogang University. This trip was funded by the Korean government WCU Grant R33- 2008-000-10101-0. I thank Prof. Yongnam Lee for making this trip possible and everyone at Sogang for their excellent hospitality.

Finally, I wish to thank my family for their support. I especially thank my wife Saima, whose love, support and encouragement has kept me going.

Declaration

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated, cited, or commonly known.

The material in this thesis is submitted for the degree of PhD to the University of Warwick only, and has not been submitted to any other university.

Abstract

I study two problems from different domains. The first problem is related to orbifold geometry and the second to Gorenstein homogeneous spaces. Though two different topics, they share a common theme: the Gorenstein property.

The first half of the thesis is related to the McKay correspondence. In particular we study a relation between the McKay correspondence in dimensions two and three. The primary purpose is to prove a theorem that generalises a conjecture given by Barth, proved by Boissière and Sarti.

The second half of the thesis is mainly about Gorenstein homogeneous spaces. We prove a theorem that gives a necessary and sufficient condition for the canonical divisor to vanish on a quasi-homogeneous affine algebraic variety.

Chapter 1

Introduction

An orbifold is locally of the form M/G where $M = \mathbb{C}^n$ and a finite group $G \subset \mathrm{GL}(n, \mathbb{C})$, acting in the standard way, and with no quasireflections. A quasireflection is an element $g \in \mathrm{GL}(n, \mathbb{C})$ of finite order such that $g - I_n$ has rank 1. A finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ is small if it contains no quasi-reflections. There might be some fixed points in M, which do not transform freely under the action of G. These points of M/G are singular and therefore the orbifold will usually not be a manifold. Although we start with a fairly simple manifold M, the orbifold M/G may have a quite complicated structure.

We mainly study orbifolds, orbifold singularities and their resolution. In particular we are interested in Gorenstein quotient singularities, their resolutions and the relations between them. It is well known that when a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$ acts on $\mathbb{C}^n$ then the quotient $\mathbb{C}^n/G$ is Gorenstein. For $n = 2$, these quotients are known as Kleinian singularities, studied by Felix Klein [Kle93] circa 1870. These singularities can be resolved by surfaces containing set of -2 curves, and the resolution has the same configuration graph as an ADE diagram. In dimension three many mathematicians such as Reid, [Rei97, Rei02, IR96], Ito, [Ito03, Ito02, Ito95, Ito94, IN96], Yau and Yu, [YY93], etc., studied these quo-

tients in different contexts.

John McKay [McK80] made the remarkable observation in the 1980s that there is a one-to-one correspondence between the ADE resolution graph of the Kleinian singularities and the nontrivial irreducible representations of the abstract group G, called the McKay correspondence. Reid's conjecture [IR96] generalised the McKay correspondence in higher dimensions. Many people have explored the relation between the equivariant geometry of M and the geometry of a crepant resolution of the quotient $\mathbb{C}^n/G$.

In chapter 2 we recall the basic facts of the McKay correspondence. We also give a brief introduction to some well known results about the cyclic quotient singularities. In the second half of the chapter we provide some elementary material on algebraic groups, homogeneous spaces and basic classes of homogeneous spaces.

In chapter 3 we study the relation between the McKay correspondence in dimensions two and three. We prove a theorem that generalises a conjecture of Barth proved by Boissière and Sarti [BS07].

A homogeneous space for an algebraic group G is a space M with a transitive action of G on M. Equivalently, it is a space of the form G/H, where G is an algebraic group and H a closed subgroup of G. Homogeneous spaces play a vital role in the representation theory of the algebraic group because these are often realised as the space of sections of vector bundles over homogeneous spaces.

Reid and Corti [RC02] studied weighted analogs of the homogeneous spaces such as Grassmannian $\mathrm{Gr}(2,5)$ and the Orthogonal Grassmanian $\mathrm{OGr}(5,10)$ and how to use these as weighted projective constructions.

In chapter 4 we study quasi-homogeneous affine algebraic varieties as the closure of an open orbit in a suitable representation with the aim of giving a criterion

for when these varieties are Gorenstein. In particular, we explicitly calculate the tangent bundle and the canonical class of the open orbit and write these in terms of group representations.

In chapter 5 we start with an example of a quasi-homogeneous algebraic variety that we analyse completely and show that it is Gorenstein. Then we make a natural substantial generalisation to a large class of varieties. We also discuss one interesting case that has links with the Orthogonal Grassmannian $\mathrm{OGr}(5, 10)$.

In the final chapter we formulate a conjecture related to Gorenstein quasi-homogeneous affine algebraic varieties and verify that it holds in certain cases including the varieties studied in chapters 4 and 5.

Chapter 2

Preliminaries

This chapter contains some well known results. In section 2.1 we explain the Du Val singularities (see [Reia]) and section 2.2 focuses on the McKay correspondence in dimension two. We review the McKay correspondence in dimension three in section 2.3. In the second half of this chapter we recall basic facts about algebraic groups, root systems, Weyl groups and homogeneous spaces.

2.1 Du Val singularities

Finite subgroups $G \subset \mathrm{SL}(2, \mathbb{C})$ were known to Felix Klein [Kle93] in the late nineteenth century. Up to conjugation, there are five different types of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$. These are:

- the cyclic groups $\mathbb{Z}_n$ of order n with generator

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix},$$

  where ε is a primitive root of unity of order n, throughout this thesis we denote it by $\frac{1}{n}(1, -1)$;

- the binary dihedral groups BD_{4n} of order $4n$ is generated by

$$\alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

  where ε is a primitive root of unity of order $2n$;

- the binary tetrahedral group BT_{24} has order 24;

- the binary octahedral group BO_{48} has order 48;

- the binary icosahedral group BI_{120} has order 120.

If we consider the action of G on $M = \mathbb{C}^2$ then the quotient $X := M/G$ is known as Du Val singularity, rational double point or simple singularity and has been studied in many different contexts. The quotients $\mathbb{C}^2/G$ are the Klein quotient points. We view them as germs of a singular point, given by the equations in Table 2.1. The quotient can be embedded as a hypersurface $X \subset \mathbb{C}^3_{x,y,z}$ with an isolated singularity at the origin, where x, y and z are generators for the ring of invariant polynomials under the action of G. The defining equation for the singularity is determined by the conjugacy class of the group G.

These singularities have minimal resolution $f\colon \mathcal{X} \longrightarrow X$, where minimal means that the resolution has no -1-curves. The exceptional set E consists of smooth rational curves of self-intersection -2, intersecting transversely. The dual graph of the exceptional set is one of the Dynkin diagrams as shown in Table 2.2. There are many characterisations of the Du Val singularities. See [Dur79, Reib] for more detail. The resolution is crepant, meaning $K_{\mathcal{X}} = f^* K_X$.

Name	Equation	Group
A_n	$x^2 + y^2 + z^{n+1}$	cyclic $\mathbb{Z}_{n+1}$
D_n	$x^2 + y^3 + z^{n-1}$	binary dihedral $\mathrm{BD}_{4(n-2)}$
E_6	$x^2 + y^3 + z^4$	binary tetrahedral BT_{24}
E_7	$x^2 + y^3 + yz^3$	binary octahedral BO_{48}
E_8	$x^2 + y^3 + z^5$	binary icosahedral BI_{120}

Table 2.1: The Du Val singularities

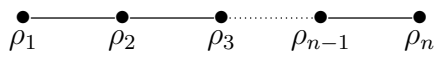
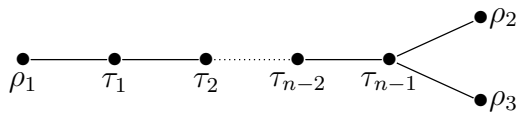
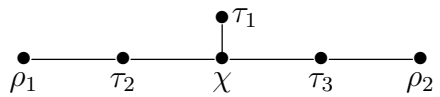
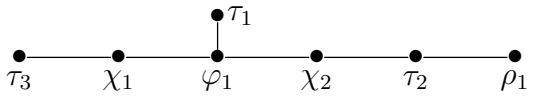
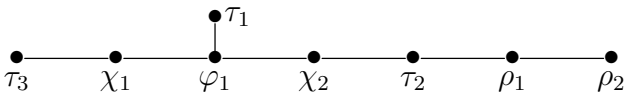
A_n	
D_n	
E_6	
E_7	
E_8	

Table 2.2: The Dynkin diagrams ADE

2.2 McKay quiver

This section is on the paper of Boissière and Sarti [BS07]. If G is a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$, its action on $\mathbb{C}^n$ defines a n-dimensional representation Q of G, called the natural representation. Let $\mathrm{Irr}(G) = \{\rho_0, \ldots, \rho_k\}$ be the set of irreducible finite dimensional complex representations of G, of which ρ_0 is the trivial one. For each ρ_i, we can decompose the tensor product

$$\rho_i \otimes Q \cong \sum_{\rho_j \in \mathrm{Irr}(G)} a_{i,j} \rho_j \tag{2.1}$$

for some positive integers $a_{i,j}$. If the character of the representation Q is real-valued, then $a_{i,j} = a_{j,i}$ for all i, j. The McKay quiver of $G \subset \mathrm{SL}(n, \mathbb{C})$ is the unoriented graph with vertices ρ_i belonging to $\mathrm{Irr}(G)$ and $a_{i,j}$ arrows between the vertices ρ_i and ρ_j, where $a_{i,j}$ is as in (2.1). Table 2.2 illustrates the graphs underlying the McKay quiver without the vertex corresponding to the trivial representation ρ_0. It is a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8, and has no loops or multiple edges. The McKay quiver is obtained from this by adding the vertex ρ_0 to form the extended Dynkin diagram, and then doubling every edge with one arrow in each direction.

2.2.1 McKay correspondence in dimension two

In 1980, McKay [McK80] made the remarkable observation that the graph of ADE type associated to a Du Val singularity $\mathbb{C}^2/G$ can be reconstructed in terms of the representation theory of the finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$.

McKay [McK80] showed that the reduced McKay quiver is the Dynkin diagram of the quotient singularity $\mathbb{C}^2/G$ as shown in Table 2.2. In other words McKay's observation gives a one-to-one correspondence between the components of the

crepant resolution $f\colon \mathcal{X} \longrightarrow X$ and the isomorphism classes of G. This is called the classical McKay correspondence for a finite $G \subset \mathrm{SL}(2, \mathbb{C})$.

2.3 McKay correspondence in dimension three

Ito and Nakamura [IN96] introduced the notion of Hilbert schemes G-Hilb$(\mathbb{C}^2)$ of G-clusters. Here a G-cluster $\mathcal{Z}$ is a G-invariant 0-dimensional subscheme of $\mathbb{C}^2$ such that $\mathcal{O}_{\mathcal{Z}}$ is the regular representation of G. They proved that for $G \subset \mathrm{SL}(2, \mathbb{C})$, G-Hilb$(\mathbb{C}^2)$ is the minimal resolution of the Du Val singularities.

Bridgeland, King and Reid [BKR01] establish that G-Hilb$(\mathbb{C}^3) \longrightarrow \mathbb{C}^3/G$ is a crepant resolution for a finite subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$. Nakamura [Nak01] calculated A-Hilb$(\mathbb{C}^3)$ for an Abelian subgroup A in $\mathrm{SL}(3, \mathbb{C})$, and Craw and Reid [CR02] gave an explicit method of computing it.

McKay quiver for finite subgroups in $\mathrm{SL}(3, \mathbb{C})$

For each finite subgroup in $\mathrm{SL}(3, \mathbb{C})$, we draw the reduced McKay quiver with our conventions. In chapter 3, we fix a normal subgroup $H \subset G$ then we denote by white vertices for the H-trivial representations and use black vertices for those representations of G on which H does not act trivially.

2.4 Semi-invariant polynomials

A group $G \subset \mathrm{SL}(2, \mathbb{C})$ acts on the complex plane $\mathbb{C}^2_{u,v}$. It also acts on the polynomial ring $\mathbb{C}[u, v]$ breaking it up into different eigenspaces. We say that a polynomial $f \in \mathbb{C}[u, v]$ belongs to a one-dimensional representation ρ (or f is ρ-invariant

or f is semi-invariant with respect to ρ), if

$$f(g \cdot P) = \rho(g) f(P) \qquad \text{for all} \quad g \in G,\ P \in \mathbb{C}^2_{u,v}.$$

Similarly a pair of polynomials (f, h) belongs to the 2-dimensional representation ρ_2 if and only if

$$(f(g_1 \cdot P), h(g_1 \cdot P)) = \rho_2(g_1)(f(P), h(P))$$
$$(f(g_2 \cdot P), h(g_2 \cdot P)) = \rho_2(g_2)(f(P), h(P)),$$

where g_1 and g_2 are the generators of G. See [IN96] for further details.

2.5 Cyclic quotient singularities

In this section we recall the cyclic quotient singularities and their resolutions [Reia], which we will use in the following chapters.

Let G be a finite cyclic subgroup of $\mathrm{SL}(2, \mathbb{C})$. If we consider the action of $G = \mathbb{Z}_n$ on $M = \mathbb{C}^2$ then the quotient $X = \mathbb{C}^2/G$ is an isolated quotient singularity at the origin, denoted by $\frac{1}{n}(1, -1)$, with toric structure. The resolution of this cyclic singularity is given by the Hirzebruch-Jung continued fraction $\frac{n}{n-1} = [2, \ldots, 2]$, (see [Reia]), here 2s are repeated $(n-1)$ times.

The Newton polygon of the lattice $L = \mathbb{Z}^2 + \mathbb{Z}.\frac{1}{n}(1, -1) \subset \mathbb{R}^2$ is defined as the convex hull in $\mathbb{R}^2$ of all nonzero lattice points in the positive quadrant. We write

$$e_0 = (0, 1), \qquad e_1 = \frac{1}{n}(1, -1), \qquad e_2, \ldots, e_{n-1} \qquad e_n = (1, 0).$$

Then any two consecutive lattice points, e_i, e_{i+1}, for $i = 0, \ldots, n$, form an oriented basis of L and for $i = 1, \ldots, n$, any three consecutive points satisfy the relation

$$e_{i+1} + e_{i-1} = 2e_i.$$

We construct the resolution of singularities $\mathcal{X} \longrightarrow X$ as follows: for each $i = 0, \ldots, n$, let λ_i, μ_i be the monomials forming the dual basis, i.e.,

$$e_i(\lambda_i) = 1, \quad e_i(\mu_i) = 0, \qquad \text{and} \qquad e_{i+1}(\lambda_i) = 0, \quad e_{i+1}(\mu_i) = 1.$$

Then $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1 \cdots \cup \mathcal{X}_n$ where each $\mathcal{X}_i$ is a copy of $\mathbb{C}^2$ with coordinates λ_i, μ_i and the gluing

$$\mathcal{X}_i \setminus (\mu_i = 0) \to \mathcal{X}_{i+1} \setminus (\lambda_{i+1} = 0)$$

is the isomorphism defined by

$$\lambda_{i+1} = \mu_i^{-1} \text{ and } \mu_{i+1} = \lambda_i \mu_i^2.$$

2.6 Algebraic groups

This material is mainly taken from Humphreys' book [Hum75]. An algebraic group G is a group equipped with the structure of an algebraic variety such that the multiplication $\mu\colon G \times G \longrightarrow G$ and the inversion $i\colon G \longrightarrow G$ are morphisms of varieties. An algebraic group is said to be a linear algebraic group if the underlying variety is affine. An algebraic group is called connected if its algebraic variety is connected. We call an algebraic group simple if it has no nontrivial proper closed connected normal subgroup. An algebraic group is called reductive if the unipotent radical is identity.

2.6.1 Root system and Weyl group

Let G be a reductive algebraic group and $T \subset G$ a maximal torus. Let $X(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$ be the set of all group homomorphisms from T to $\mathbb{C}^\times$. We can give $X(T)$ an additive group structure by letting

$$(\chi_1 + \chi_2)(t) = \chi_1(t)\chi_2(t) \text{ for all } \chi_1, \chi_2 \in X(T) \text{ and } t \in T.$$

If $T = \mathbb{C}^\times$ as an algebraic group, then $X(T) \cong \mathbb{Z}$. In general, as T is isomorphic to n copies of $\mathbb{C}^\times$, we have

$$X(T) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}.$$

We call $X(T)$ the character group of T or lattice of weights with respect to T. The individual characters $\chi \colon T \to \mathbb{C}^\times$ are *weights*. Now suppose $Y(T) = \mathrm{Hom}(\mathbb{C}^\times, T)$ and $Y(T)$ is also isomorphic to the free Abelian group of rank n. We call $Y(T)$ the cocharacter group of T or dual lattice of 1- parameter subgroups. If we compose $f \in X(T)$ and $g \in Y(T)$, we have a canonical perfect pairing

$$\langle\ ,\ \rangle \colon X(T) \times Y(T) \longrightarrow \mathbb{Z}.$$

We define the normaliser of T to be the subgroup

$$N_G(T) = \{g \in G : gT = Tg\} = \left\{g \in G : gTg^{-1} = T\right\}$$

of G. T is a normal subgroup of $N_G(T)$ and the quotient group

$$W(G, T) = N_G(T)/T$$

is called the Weyl group of G with respect to T. The Weyl group acts on T by conjugation and this action induces an action of the Weyl group on the character and cocharacter groups of T.

Let V be a vector space and $\rho\colon T \longrightarrow \mathrm{GL}(V)$ a representation of T. Now, V is a direct sum of one-dimensional subrepresentations. On each of these subspaces an element of $t \in T$ acts by $\chi(t)$, where χ is a character of T. The characters so obtained are the weights of T in V. The nonzero subspace

$$V_\chi = \{v \in V \mid \rho(t)v = \chi(t)v \text{ for all } t \in T\},$$

is called the weight space associated to a character χ. The roots of G are defined as the weights of T appearing due to the adjoint action of T on the Lie algebra $\mathfrak{g}$. A set of roots $\{\alpha_1, \ldots, \alpha_n\}$ is called simple if it is a vector space basis for $\mathfrak{h}^*$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ and every root α can be written in the form

$$\alpha = k_1\alpha_1 + \cdots + k_n\alpha_n,$$

where each k_i is an integer such that they are either all nonnegative or all nonpositive. A root α is said to be positive with respect to the given simple roots if all the k_i are nonnegative; otherwise α is negative.

We fix once and for all a set of simple roots. This defines the following partial order on weights: $\lambda \geq \mu$. Let λ and μ be two weights, we say that λ is higher than μ if

$$\lambda - \mu = k_1\alpha_1 + \cdots + k_n\alpha_n$$

with $k_i \in \mathbb{Z}_{\geq 0}$. A weight λ for a representation is called highest if no other weight is higher than λ.

2.6.2 Homogeneous spaces

Let G be an algebraic group and H its closed subgroup, we may construct the quotient space $M = G/H$ of right cosets. The group G acts on M transitively. The space M together with this action is called a homogeneous space.

A Borel subgroup of G is a closed, connected, solvable subgroup of G, which is maximal for these properties. A closed subgroup of G is parabolic if and only if it contains a Borel subgroup. If G is an algebraic group and P is a parabolic subgroup of G, then the quotient G/P is a projective variety.

Example 2.6.1. The group $G = \mathrm{GL}(3, \mathbb{C})$ acts on affine space $\mathbb{A}^3$. Under this action we have two orbits. One is $\{(0,0,0)\}$, and the other is $\mathbb{A}^3 - \{(0,0,0)\}$. One orbit is closed and the other is open. In this example G does not act transitively on $\mathbb{A}^3$ because we have more than one orbit. If $v = (1,0,0)$, then the stabiliser of v is the set of matrices of the form

$$H = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

which is a closed subgroup of G since it is the zero set of $a_{11} - 1 = a_{21} = a_{31} = 0$.

The group $G = \mathrm{GL}(3, \mathbb{C})$ acts on the projective plane $\mathbb{P}^2$ transitively. Let $v = (1:0:0)$, then the stabiliser of v is

$$P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

This is also a closed subgroup of G since it is given by the zeros of the equations

$a_{21} = a_{31} = 0$. There is a one-to-one correspondence between the G/P and projective space $\mathbb{P}^2$ via $gP \leftrightarrow g \cdot v$. We can transfer the variety structure of the projective plane to the set G/P via the above bijection.

2.7 Vanishing theorems

This section is based on Esnault and Viehweg's book "lectures on vanishing theorems" [EV92]. Let $X = G/H$ be a projective manifold defined over an algebraically closed field k and let $\mathfrak{L}$ be an invertible sheaf on X. We state the vanishing theorems that we need in chapter 6.

Theorem 2.7.1. *If $\mathfrak{L}$ is ample and $\mathfrak{F}$ a coherent sheaf, then there is some $v_0 \in \mathbb{N}$ such that*

$$H^b(X, \mathfrak{F} \otimes \mathfrak{L}^v) = 0 \text{ for } b > 0 \text{ and } v \geq v_0$$

In particular, for $\mathfrak{F} = \mathcal{O}_X$, one obtains the vanishing of $H^b(X, \mathfrak{L}^v)$ for $b > 0$ and v sufficiently large.

If char(k)= 0, then v sufficiently large can be made more precise. For example, it is enough to choose v such that $A = \mathfrak{L}^v \otimes \omega_X^{-1}$ is ample, where $\omega_X = \Omega_X^n$ is the canonical sheaf of X, and to use:

Theorem 2.7.2. Let X be a complex projective manifold and let A be an ample invertible sheaf. Then

(1) $H^b(X, \omega_X \otimes A) = 0$ for $b > 0$

(2) $H^{b'}(X, A^{-1}) = 0$ for $b' < n =$ dim X

Further information on vanishing theorems can be found in [Ser55, Kod53, AN54].

2.8 Cohen-Macaulay and Gorenstein rings

A local Noetherian ring R is called Cohen-Macaulay if the depth of R as an R-module is equal to the dimension of R. More generally, a Noetherian ring R is called Cohen-Macaulay, if for every maximal ideal m of R the localisation R_m is Cohen-Macaulay.

A local Noetherian ring R is called Gorenstein if it is Cohen-Macaulay, the dualising module ω_R exists, and ω_R is isomorphic to R as R-modules. More generally, a Noetherian ring R is called Gorenstein, if for every maximal ideal m of R the localisation R_m is Gorenstein. There are many equivalent characterisations of Gorenstein rings, see for example [Mat89]. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a small subgroup. Then by a result of Watanabe [Wat74] $\mathbb{C}^n/G$ is Gorenstein if and only if $G \subset \mathrm{SL}(n, \mathbb{C})$.

A projective variety is said to be projectively Gorenstein if its homogeneous coordinate ring is a Gorenstein ring.

Chapter 3

Resolution of quotients by $\mathrm{SL}(2, \mathbb{C})$ groups and their subgroups

In this chapter, we study the relation between the two-dimensional and three-dimensional McKay correspondence. Suppose $G \subset \mathrm{SL}(2, \mathbb{C})$ is a finite subgroup. If we consider the action of G on $M = \mathbb{C}^2$ then the quotient M/G is a surface singularity that has an isolated singular point at the origin. The resolution is a configuration of -2 curves with an intersection graph that is an ADE Dynkin diagram. The classical McKay correspondence [McK80] relates the nontrivial irreducible representations of G to the nodes of the graph. Let H be a normal subgroup in G and $T := G/H$ the quotient group. Instead of taking the action of G on M, we first take the action of H on M and then act by T on the intermediate quotient $I := M/H$. Clearly $I/T = M/G$. Note that the intermediate quotient $I := M/H$ is a hypersurface singularity and its inclusion $I = M/H \hookrightarrow \mathbb{C}^3_{x,y,z}$ is defined by a set of generators x, y and z for the ring of invariant functions, and

is therefore T-equivariant. Diagrammatically this can be shown as follows

$$\begin{tikzcd}
\mathcal{X} \arrow[dr] \arrow[ddd] & & \mathcal{Y} \arrow[ddd] \\
& \mathcal{P} \arrow[ur, hook] \arrow[d] & \\
& & \\
X = \mathbb{C}^2/G \arrow[r, "\sim"] & P = I/T \arrow[r, hook] & \mathbb{C}^3/T = Y
\end{tikzcd}$$

where $\mathcal{X}$=G-Hilb$(\mathbb{C}^2)$, $\mathcal{Y} = T$-Hilb$(\mathbb{C}^3)$ and $\mathcal{P}$ the partial resolution induced[1] by $\mathcal{Y} \to Y$.

Boissière and Sarti [BS07] consider only the case $H = \{\pm 1\}$. In this case T always acts on $\mathbb{C}^3$ as a subgroup in $\mathrm{SL}(3, \mathbb{C})$. They constructed the map between $\mathcal{X} \to \mathcal{Y}$ mapping isomorphically the exceptional curves corresponding to the representations on which H acts trivially and contracting others.

For H any normal subgroup in G, except for two cases, $T = G/H$ acts on $\mathbb{C}^3$ as a subgroup of $\mathrm{SL}(3, \mathbb{C})$. The first case when $T \cong \mathbb{Z}_4$ acts on $\mathbb{C}^3$ by $\frac{1}{4}(1, 3, 2; 2)$ corresponds to Mori's special class of terminal singularities, see [Rei87, Mor85] for details. The second case when T is noncyclic is listed in Table 3.1.

A crepant resolution of singularities $\mathbb{C}^3/T$ exists when T is a finite subgroup of $\mathrm{SL}(3, \mathbb{C})$ [BKR01]. Our main aim is to prove the following theorem assuming that crepant resolutions of singularities of $\mathbb{C}^3/T$ exist .

Theorem 3.0.1. *The crepant resolution of singularities $\mathcal{Y} \longrightarrow Y$ of $Y = \mathbb{C}^3/T$ induces a birational transformation $\mathcal{P} \longrightarrow P = I/T \cong X$ together with a resolution morphism $\mathcal{X} \longrightarrow \mathcal{P}$. This has the property that the exceptional curves on $\mathcal{X}$*

[1]The partial quotient P is in Y. We have the resolution of singularities morphism $\mathcal{Y} \to Y$; this induces a partial resolution of singularities $\mathcal{P} \to P$, obtained as the birational transform (proper transform) of P. Specifically, we take the inverse image of P in $\mathcal{Y}$, and consider in it the component $\mathcal{P}$ that maps birationally to P. We can also think it as by first restricting to the locus $\mathcal{Y}^0 \to Y^0$ where $\mathcal{Y} \to Y$ is an isomorphism, taking the inverse image of P^0 there, then define $\mathcal{P}$ to be closure in $\mathcal{Y}$.

that correspond under the McKay correspondence to representations on which H acts trivially are mapped isomorphically and the remainder are contracted.

Proof. The list of all possible triples (G, H, T) is given in Table 3.1. All these cases will be addressed later in the chapter. The α and β are the matrices defined on page 13.

G	H	$T \cong$
$\frac{1}{rs}(1,-1)$	$\frac{1}{r}(1,-1)$	$\frac{1}{s}(1,-1,0)$
BD_{4n}	$\frac{1}{2n}(1,-1)$	$\frac{1}{2}(1,1,0)$
$\mathrm{BD}_{4n},\ n$ odd	$\frac{1}{n}(1,-1)$	$\frac{1}{4}(1,3,2;2) \subset \mathrm{GL}(2,\mathbb{C})$
$\mathrm{BD}_{4n},\ n$ even	$\langle \alpha^2, \beta \rangle$	$\frac{1}{2}(1,1,0)$
$\mathrm{BD}_{4n},\ n$ even	$\langle \alpha^2, \alpha\beta \rangle$	$\frac{1}{2}(1,1,0)$
BD_{4n}	$\langle \alpha^a \rangle, a$ divides both $2n$ and n	$D_{2a} \subset \mathrm{SL}(2,\mathbb{C})$
BD_{4n}	$\langle \alpha^a \rangle, a$ divides $2n$ but not n	$D_{2a} \subset \mathrm{GL}(2,\mathbb{C})$
BT_{24}	BD_8	$\frac{1}{3}(0,1,2)$
BO_{48}	BT_{24}	$\frac{1}{3}(1,0,1)$
BO_{48}	BD_8	S_3

Table 3.1: List of all possible triples (G, H, T)

□

3.1 Contraction versus non-contraction

It is well known from the time of Klein [Kle93] that when a finite nontrivial subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{C}^2$, the ring of invariants is always generated by three generators. Let H be a normal subgroup of G which acts on $\mathbb{C}^2$, and let x, y and z be the generators of the ring of invariants of H.

Let $\rho_0, \rho_1, \ldots, \rho_n$ be the set of distinct irreducible representations of G. There are certain polynomials which belong to each of these irreducible representations. We will observe it experimentally; see for example section 8 for the binary dihedral case. Theorem 3.0.1 asserts that if the polynomials belonging to a particular representation, say ρ_i, can be written in terms of H-invariant polynomials then the curve that corresponds to the representation ρ_i under the McKay correspondence will survive, otherwise it will be contracted.

We argue below that the curves on $\mathcal{X}$ marked by representations that are not H-invariant are necessarily contracted by $\mathcal{X} \to \mathcal{P}$ and others survive; for further details see [IN96, BS07, GSV83]. Ito and Nakamura [IN96] showed that C_ρ is parametrised by functions $(f_1, f_2) \in \rho$. Let C_ρ denotes the curve marked by the representation ρ.

Proposition 3.1.1. *If ρ is not H-trivial then the corresponding curve C_ρ is contracted.*

Proof. This proves the only if part of Theorem 3.0.1 without any case by case argument. Let $\rho\colon G \to \mathrm{GL}(V)$ be an irreducible representation of G. For ρ to descend to a representation of T, we require $\rho(g) = \rho(gh)$, where $g \in G$ and $h \in H$. This implies that $\rho(g) = \rho(g)\rho(h)$, which implies that $\rho(h) = 1$ for all h, that is, H acts trivially. Therefore the only rational functions involved in the 3-fold resolution $Y \to \mathcal{Y}$ are ratios of H-invariant polynomials, so the only way

that a curve of $\mathcal{X}$ can survive in $\mathcal{P}$ is if it is mapped nontrivially by H-invariant rational functions.

In other words the only functions on $\mathcal{Y} = T\text{-Hilb}(\mathbb{C}^3)$ are H-invariant ratios. Therefore the polynomials which do not belong to H-invariant representations will not appear on $\mathcal{P} \subset \mathcal{Y}$. Hence all curves correspond to representations which are not H-trivial will be contracted.

□

In what follows we prove if the representation ρ is H-trivial then C_ρ is not contracted. We observe case by case what happens in $\mathbb{C}^3/T$.

3.2 Cyclic subgroups $\frac{1}{rs}(1,-1)$ of $\mathrm{SL}\,(2,\mathbb{C})$

Let $G \cong \mathbb{Z}_{rs}$ be the cyclic subgroup of $\mathrm{SL}\,(2,\mathbb{C})$ with generator:

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix},$$

where ε is a primitive root of unity of order rs.

Let $\mathbb{Z}_r \cong H$ be a cyclic subgroup of $G \cong \mathbb{Z}_{rs}$ and $\mathbb{Z}_s \cong T := G/H$. The subgroup H acts on $M = \mathbb{C}^2_{u,v}$ by $\frac{1}{r}(1,-1)$ and $x = u^r$, $y = v^r$ and $z = uv$ are invariants under the action of $\mathbb{Z}_r$ and these are the generators of the ring of invariants. Now $T \cong \mathbb{Z}_s$ acts on $\mathbb{C}^3_{x,y,z}$ with generator

$$\begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ξ is a primitive root of unity of order s, by $\frac{1}{s}(1, s-1, 0)$.

To observe the contraction of fibres in dimensions 2 and 3, it is enough to check the irreducible representations of G, on which H acts trivially. Since G is a cyclic group of order rs, we have in total rs one-dimensional irreducible representations (including the trivial one) given by the matrices (ε^i), $i = 0, \ldots, (rs-1)$. The subgroup H acts trivially on those representations for which i is a multiple of r.

In other words we want to check which coordinates on the resolution graph $\mathcal{X}$ can be written in terms of H-invariant polynomials. We know that a monomial $u^i v^j$ belongs to ρ_k if and only if $\alpha(u^i v^j) = \varepsilon^k u^i v^j$. So

$$\{1\}, \{u, v^{rs-1}\}, \ldots, \{u^i, v^{rs-i}\}, \ldots, \{u^{rs-1}, v\}$$

belong to $\rho_0, \rho_1, ..., \rho_{rs-1}$ respectively. The only possibility when we can write u^i, v^{rs-i} in terms of H-invariant polynomials is when i is a multiple of r.

For example, if $G \cong \mathbb{Z}_6$ then we have two possible normal subgroups, namely, $H_1 \cong \mathbb{Z}_2$ and $H_2 \cong \mathbb{Z}_3$. So

$$\{1\}, \{u, v^5\}, \{u^2, v^4\}, \{u^3, v^3\}, \{u^4, v^2\}, \{u^5, v\}$$

belong to $\rho_0, \rho_1, \ldots, \rho_5$ respectively. In the case $H_1 \cong \mathbb{Z}_2$, the representations on which H acts trivially are $\rho = (\varepsilon^i)$, $i = 0, 2, 4$, because we can write $\{u^2, v^4\}$ and $\{u^4, v^2\}$ in terms of H_2-invariant polynomials. Similarly we can check that H_2=$\mathbb{Z}_3$ acts trivially on $\rho = (\varepsilon^i)$, $i = 0, 3$, because we can write $\{u^3, v^3\}$ in terms of H-invariant polynomials. See Table 3.2 for the reduced McKay quiver.

	Dimension 2	Dimension 3	
$\mathbb{Z}_6$	$\bullet\!\!-\!\!\circ\!\!-\!\!\bullet\!\!-\!\!\circ\!\!-\!\!\bullet$ ρ_1 ρ_2 ρ_3 ρ_4 ρ_5	$\circ\!\!-\!\!\circ$ ρ_2 ρ_4	$\mathbb{Z}_2$
$\mathbb{Z}_6$	$\bullet\!\!-\!\!\bullet\!\!-\!\!\circ\!\!-\!\!\bullet\!\!-\!\!\bullet$ ρ_1 ρ_2 ρ_3 ρ_4 ρ_5	$\circ$ ρ_3	$\mathbb{Z}_3$

Table 3.2: Reduced McKay quiver ($G \cong \mathbb{Z}_6,\ T \cong \mathbb{Z}_2,\ \mathbb{Z}_3$)

3.3 Binary dihedral subgroup of $\mathrm{SL}(2,\mathbb{C})$

Throughout this section $G = \mathrm{BD}_{4n} \subset \mathrm{SL}(2,\mathbb{C})$ is the binary dihedral group of order $4n$ is generated by

$$\alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where ε is a primitive root of unity of order $2n$. The group has the presentation

$$\left\langle \alpha, \beta \mid \alpha^{2n} = \beta^4 = 1, \alpha^n = \beta^2, \alpha\beta = \beta\alpha^{-1} \right\rangle.$$

BD_{4n} has four one-dimensional and $(n-1)$ two-dimensional irreducible representations. The one-dimensional representations are given by

$$\rho_0(\alpha) = 1,\ \rho_0(\beta) = 1, \qquad \rho_1(\alpha) = 1,\ \rho_1(\beta) = -1,$$

$$\rho_2(\alpha) = -1,\ \rho_2(\beta) = i^n, \qquad \rho_3(\alpha) = -1,\ \rho_3(\beta) = -i^n.$$

For $j = 1, ..., (n-1)$ we define a two-dimensional irreducible representation τ_j as

follows

$$\tau_j(\alpha) = \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix}, \quad \tau_j(\beta) = \begin{pmatrix} 0 & 1 \\ (-1)^j & 0 \end{pmatrix}.$$

To use in the next sections, we write the polynomials in Table 3.2 that belong to the above representations, Table 3.3 lists the polynomials and pairs of polynomials that belong to the representations ρ_i and τ_j, see [IN96, p.217].

ρ_0	$1,\ u^2v^2,\ u^{2n}+v^{2n}$
ρ_1	$uv,\ u^{2n}-v^{2n}$
ρ_2	$u^n - i^n v^n,\ uv(u^n + i^n v^n)$
ρ_3	$u^n + i^n v^n,\ uv(u^n - i^n v^n)$
τ_j	$(u^j,\ v^j),\ (u^{j+1}v,\ -uv^{j+1}),\ (v^{2n-j},\ (-1)^j u^{2n-j})$

Table 3.3: Representations for binary dihedral groups

3.3.1 $H = \langle \alpha \rangle$, $T \cong \mathbb{Z}_2$

Let H be a normal subgroup of order $2n$ generated by α. If we consider the action of H on M then the ring of invariants is generated by $x = u^{2n}$, $y = v^{2n}$ and $z = uv$ and $T \cong \mathbb{Z}_2$ acts on $\mathbb{C}^3_{z,x-y,x+y}$ by $\frac{1}{2}(1,-1,0)$ with new eigenbasis z, $x-y$ and $x+y$. We have a singular surface inside this quotient, see [CR02] for resolution of this quotient. The group $\mathbb{Z}_2$ has only two irreducible representations and one of those is the trivial representation. The only nontrivial irreducible representation on which H acts trivially is given by $\rho(\alpha) = 1$ and $\rho(\beta) = -1$. In terms of coordinates, we can see that the polynomials which belong to the representations ρ_0 and ρ_1 are the only polynomials which can be written in terms of H-invariant monomials, see Table 3.4 for contraction of fibres.

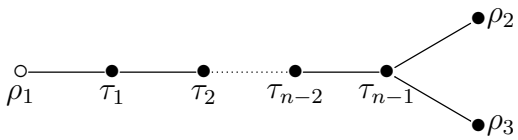
Dimension 2		Dimension 3	
BD_{4n}		ρ_1	$\mathbb{Z}_2$

Table 3.4: Reduced McKay quiver ($G = \mathrm{BD}_{4n},\ T \cong \mathbb{Z}_2$)

3.3.2 $H = \langle \alpha^2 \rangle$, $T \cong \mathbb{Z}_4$, n odd

The normal subgroup $H = \langle \alpha^2 \rangle$ acts on M and $x = u^n$, $y = v^n$ and $z = uv$ are invariants under this action. Now β sends $x \mapsto y$, $y \mapsto -x$ and $xy \mapsto -xy$. In other words β acts on $\mathbb{C}^3_{x,y,z}$ with generator

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

But if we diagonalise the above matrix then β acts on $\mathbb{C}^3$ by $\frac{1}{4}(1,3,2) \nsubseteq \mathrm{SL}(3,\mathbb{C})$ with a new eigenbasis, $x - iy$, $x + iy$ and z.

As we can see from Table 3.3, we can write the polynomials which belong to representations ρ_0, ρ_1, ρ_2 and ρ_3 in terms of H-invariant polynomials. Since $\mathbb{Z}_4$ has four linear representations, all curves corresponding to the two-dimensional representations are contracted and only one-dimensional representations survive, as shown in Table 3.5.

This case is not covered by the assumptions of Theorem 3.0.1, and in this case theorem 3.0.1 does not give any conclusion.

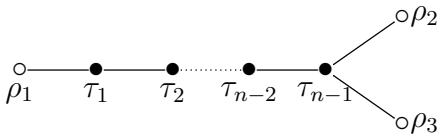
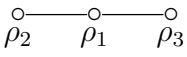
Dimension 2		Dimension 3	
BD_{4n}			$\mathbb{Z}_4$

Table 3.5: Reduced McKay quiver ($G = \mathrm{BD}_{4n},\ T \cong \mathbb{Z}_4$)

3.3.3 $H = \mathrm{BD}_{2n}$, $T \cong \mathbb{Z}_2$, n even

Consider the subgroups $H_1 = \langle \alpha^2, \beta \rangle$ and $H_2 = \langle \alpha^2, \alpha\beta \rangle$. Both H_1 and H_2 are themselves binary dihedral groups of index 2.

If we consider the action of H_1 on M then the ring of invariants is generated by $x = u^n + v^n$, $y = uv(u^n - v^n)$ and $z = u^2v^2$ and the group $T = \mathrm{BD}_{4n}/H$ is isomorphic to the cyclic group of order 2 and acts on $\mathbb{C}^3_{x,y,z}$ by $\frac{1}{2}(1,1,0)$. On the other hand, if we first take the action of H_2 on M then $x = u^n - v^n$, $y = uv(u^n + v^n)$ and $z = u^2v^2$ generates the ring of invariants and T acts on $\mathbb{C}^3_{x,y,z}$ by $\frac{1}{2}(1,1,0)$.

Since T is isomorphic to the cyclic group of order two we have only one non-trivial irreducible representation. Therefore we need to check on which linear representation of BD_{4n} the group H acts trivially.

As we can see from Table 3.3, we can write those polynomials which belong to ρ_2 and ρ_3 in terms of H_1 and H_2-invariant polynomials. So one of ρ_2 or ρ_3 will survive depending on the choice of H_1 and H_2.

For both of these cases the reduced McKay quiver is explained in Table 3.6.

3.3.4 $H = \langle \alpha^a \rangle$, T is non-cyclic

Let $H = \langle \alpha^a \rangle$ be a subgroup of the binary dihedral group. In this section, we explain two cases depending upon whether a divides $2n$ but not n or a divides both n and $2n$.

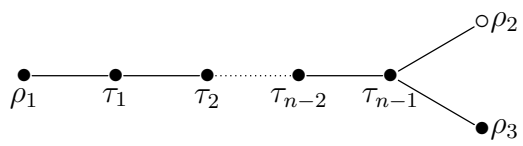
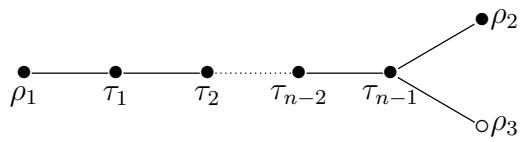
Dimension 2		Dimension 3	
BD_{4n}		ρ_2	$\mathbb{Z}_2$
BD_{4n}		ρ_3	$\mathbb{Z}_2$

Table 3.6: Reduced McKay quivers ($G = \mathrm{BD}_{4n},\ T \cong \mathbb{Z}_2$)

Case 1: a divides both n and $2n$

We first take the action of $H = \langle \alpha^a \rangle$ on M then $x = u^{2n/a}$, $y = v^{2n/a}$ and $z = uv$ are invariants under this action and the quotient group $T := \mathrm{BD}\,/H$ acts on $\mathbb{C}^3_{x,y,z}$ with generators

$$A = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where ξ is a primitive root of unity of order a. T is isomorphic to the dihedral group D_{2a}.

We need to take care as to whether a is odd or even. If a is odd then the polynomials which belong to representations ρ_1 and τ_j, where j is a multiple of a and for even a polynomials belonging to ρ_1, ρ_2, ρ_3 and τ_j, again j is a multiple of a, can be written in terms of H-invariant polynomials.

Hence for odd a, representations ρ_1 and τ_j, where j is a multiple of a and for even a, all one-dimensional representations and τ_j, where j is a multiple of a, are mapped bijectively and the remainders are contracted.

Case 2: a divides $2n$ but not n

In this case, when the subgroup $H = \langle \alpha^a \rangle$ acts on M, it leaves $x = u^{2n/a}$, $y = v^{2n/a}$ and $z = uv$ as invariants and since $\frac{2n}{a}$ is odd, β sends $y \mapsto -y$. In other words the quotient group T acts on $\mathbb{C}^3_{x,y,z}$ with generators

$$A = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where ξ is a primitive root of unity of order a. We observe that T does not act on $\mathbb{C}^3$ as a subgroup of $\mathrm{SL}(3, \mathbb{C})$, see Table 3.1.

3.4 Binary tetrahedral subgroup of $\mathrm{SL}\,(2, \mathbb{C})$

The binary tetrahedral group $\mathrm{BT}_{24} \subset \mathrm{SL}\,(2, \mathbb{C})$ is generated by

$$\alpha = \begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \quad \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix},$$

where ε is a primitive root of unity of order 8.

3.4.1 $H = \mathrm{BD}_8$, $T \cong \mathbb{Z}_3$

The group generated by α and β is the binary dihedral group of order 8 and is normal in BT_{24}. We denote it by H. If we take the action of H on M then the ring of invariants is generated by $x = uv(u^4 - v^4)$, $y = u^4 + v^4$ and $z = u^2v^2$. Then $T = \mathrm{BT}_{24}/H \cong \mathbb{Z}_3$, $\mathbb{Z}_3$ acts on $\mathbb{C}^3$ by $\frac{1}{3}(0, 1, 2)$ with new eigenbasis x, $y + (\frac{2\omega+1}{6})z$ and $y + (\frac{2\omega^2+1}{6})z$, where ω is a cube root of unity. The binary tetrahedral group

has six nontrivial irreducible representations of which two are linear, three are of degree two and one is of degree of three, but $\mathbb{Z}_3$ has only two nontrivial one-dimensional representations.

Irreducible representations on which H acts trivially are given by $\rho(\alpha) = \rho(\beta) = 1$, $\rho(\mu) = \omega$, $\rho(\alpha) = \rho(\beta) = 1$ and $\rho(\mu) = \omega^2$, where ω is a cube root of unity. The polynomials φ, ψ^2 and ψ, φ^2 , where

$$\varphi = (u^2 + v^2)^2 + 4\omega(uv)^2,$$

$$\psi = (u^2 + v^2)^2 + 4\omega^2(uv)^2,$$

belong to nontrivial irreducible one-dimensional representations and they are the only ones that can be written in terms of H-invariant polynomials, see [IN96, p.228]. Therefore the exceptional curves corresponding to the one-dimensional representations are in one-to-one correspondence while all others are contracted. The contraction of fibres is shown in Table 3.7, where the ρ_i, τ_i and χ denotes one, two and three-dimensional representations of BT_{24} receptively.

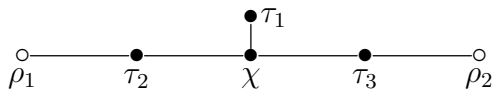
	Dimension 2	Dimension 3
BT_{24}		 $\mathbb{Z}_3$

Table 3.7: Reduced McKay quiver ($G = \mathrm{BT}_{24},\ T \cong \mathbb{Z}_3$)

3.5 Binary octahedral subgroup of $\mathrm{SL}\,(2,\mathbb{C})$

The binary octahedral group $\mathrm{BO}_{48} \subset \mathrm{SL}\,(2,\mathbb{C})$ is generated by

$$\alpha = \begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix} \text{ and } \kappa = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}$$

where $\varepsilon = \exp \dfrac{2\pi i}{8}$. So BO_{48} is generated by BT_{24} and κ (note $\kappa^2 = \alpha$). We know that BO_{48} has two normal subgroups other than $\{\pm 1\}$ namely, BT_{24} and BD_8. The case $\{\pm 1\}$ has been already considered by Boissière and Sarti [BS07]. So we consider two cases in the following sections.

3.5.1 $H = \mathrm{BT}_{24}$, $T \cong \mathbb{Z}_2$

The binary tetrahedral group $H = \mathrm{BT}_{24}$ is a maximal normal subgroup in $G = \mathrm{BO}_{48}$ which acts on M. It is well known [Kle93] that the ring of invariants is generated by homogeneous polynomials of degree 6, 8 and 12. We will follow the notation used in [IN96]. The homogeneous polynomials are denoted by S, W and φ^3, which are given below.

$$\begin{aligned} S &= (u^2 - v^2)(u^2 + v^2)(uv), \\ \varphi &= (u^2 + v^2)^2 + 4\omega(uv)^2, \\ \psi &= (u^2 + v^2)^2 + 4\omega^2(uv)^2, \\ W &= \varphi\psi. \end{aligned}$$

The ring of invariants is generated by S, W and φ^3 (or by ψ^3) [Kle93, p.51].

We observe that $T = G/H \cong \mathbb{Z}_2$ acts on $\mathbb{C}^3_{S,W,\varphi^3}$ by $\frac{1}{2}(1,0,1)$ and this quotient has a resolution similar to $\frac{1}{2}(1,1)$, with third axes fixed in the resolution graph,

see [CR02] for more details. The only nontrivial irreducible representation on which H acts trivially is given by

$$\rho(\alpha) = \rho(\kappa) = -1, \rho(\beta) = \rho(\delta) = 1.$$

One can observe [IN96] that the polynomials $uv(u^4 - v^4)$ and $u^{12} - 33u^8v^4 - 33u^4v^8 + v^{12}$ can be written in terms of H-invariant polynomials and both these polynomials belong to ρ_1. That means that in dimension 3, we will get one exceptional curve that corresponds to ρ_1 on which H acts trivially and all others are contracted to ordinary nodes as shown in Table 3.8.

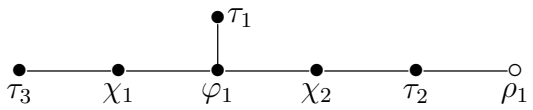
Dimension 2		Dimension 3	
BO_{48}			$\mathbb{Z}_2$

Table 3.8: Reduced McKay quiver ($G = \mathrm{BO}_{48},\ T \cong \mathbb{Z}_2$)

In the graph, τ_i, χ_i and φ_1 are 2, 3 and 4-dimensional representations of BO_{48} respectively.

3.5.2 $H = \mathrm{BD}_8$, $T \cong \mathrm{S}_3$

The binary dihedral subgroup of order 8 acts on M and $x = u^4+v^4$, $y = uv(u^4-v^4)$ and $z = u^2v^2$ are invariants under this action. The quotient group $T = \mathrm{BO}_{24}/H \cong$

S_3 acts on $\mathbb{C}^3_{x,y,z}$ by

$$A = \begin{pmatrix} \xi^{-1} & 0 & 0 \\ 0 & \xi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1/2 & 0 & -3 \\ 0 & 1 & 0 \\ 1/4 & 0 & -1/2 \end{pmatrix},$$

where $\xi^2 = 1$. If we diagonalise B, then it acts on $\mathbb{C}^3$ with a new eigenbasis by generator

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ω is a primitive cube root of unity.

As we know BO_{48} has only one nontrivial one-dimensional representation and there are three two-dimensional irreducible representations so we need to check which one among these three survives, because S_3 has only one two-dimensional representation. The following polynomials

$$\psi^2, \ -\varphi^2, \ u^5v\psi - uv^5\varphi, \ -u^5v\varphi + uv^5\psi$$

and

$$uv(u^4 - v^4), \ u^{12} - 33u^8v^4 - 33u^4v^8 + v^{12}$$

belong by to τ_1 and ρ_1 respectively [IN96] and can be written in terms of H-invariant polynomials, therefore τ_1 and ρ_1 will survive and all others will be contracted as shown in Table 3.9.

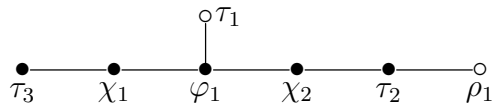
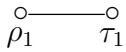
Dimension 2		Dimension 3
BO_{48}		 S_3

Table 3.9: Reduced McKay quiver ($G = \mathrm{BO}_{48},\ T \cong \mathrm{S}_3$)

3.6 Binary icosahedral subgroup of $\mathrm{SL}\,(2,\mathbb{C})$

The binary icosahedral subgroup of $\mathrm{SL}(2,\mathbb{C})$ is a group of order 120 and has only one proper normal subgroup namely, $\{\pm 1\}$, that has been already considered by Boissière and Sarti [BS07] .

Chapter 4

Gorenstein quasi-homogeneous affine varieties

In this chapter we study quasi-homogeneous affine algebraic varieties. In particular we calculate their tangent bundle and canonical class, with the aim of characterising the cases that are Gorenstein. In the first part of this chapter, we work out the tangent bundle and canonical class to the open orbit of these varieties. In the second part, we write the tangent bundle and canonical class in terms of group representations.

4.1 The variety in equations

Let M and X be $r \times (r+1)$ and $(r+1) \times 1$ matrices as given below

$$M_{r,r+1} = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,r+1} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,r+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{r,1} & m_{r,2} & \cdots & m_{r,r+1} \end{pmatrix} \quad \text{and} \quad X_{r+1,1} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r+1} \end{pmatrix}.$$

We define a variety as follows:

$$V = \left\{ (M, X, \omega) : MX = 0 \text{ and } \bigwedge^r M = \omega X \right\},$$

where $\bigwedge^r M$ denotes the $r \times r$ minors of M, $\omega \in \mathbb{C}$ and $(-1)^i \omega x_i = M_{\hat{i}}$ where $M_{\hat{i}}$ is the complementary minor of M obtained by deleting the ith column of M. We call these varieties Cramer varieties and denote them by $\mathrm{Cr}(r, r+1, 1)$. The case $r = 3$ is the original codimension 4 example in the first paper of Kustin and Miller [KM80]. Understanding this case led them to the more general notion of Gorenstein unprojection. The original motivation for my problem was to give a more geometric interpretation of the Gorenstein property.

Now if we consider

$$V_0 = \left\{ (M, X, \omega) \in V : \text{rank of } M = r \right\},$$

then V_0 has codimension $r+1$ in $\mathbb{C}^{(r\times(r+1))+(r+1)+1}$ and $V = \overline{V_0}$. The V_0 is a

homogeneous space, the orbit of the vector

$$M_{r,r+1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad X_{r+1,1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{and } \omega = 1,$$

under the action of $G = \mathrm{GL}(r) \times \mathrm{GL}(r+1)$ as explained in section 4.2. It has codimension $(r+1)$ in $\mathbb{C}^{r\times(r+1)+(r+1)+1}$ and $V = \overline{V_0}$ is its closure. When M is a matrix of maximal rank then $MX = 0$ forces X to live in one-dimensional vector space and ω must be nonzero.

Let $S = V \setminus V_0$ be the complement of V_0 in V. For $(M, X, \omega) \in S$, $\bigwedge^r M = 0$. We have three possibilities for the elements of S:

1. either $X \neq 0$ and $\omega = 0$;
2. or $X = 0$ and $\omega \neq 0$;
3. or $X = \omega = 0$.

In case 1, if $V_i = \{x_i \neq 0\}$ then M must be of rank less than or equal to $(r-1)$. In this case by counting conditions we can show that V_i in V is an irreducible divisor. For example, if r is 2 then it is two conditions for M to have rank ≤ 1, and one condition comes from $MX = 0$, which leaves with six free variable. In general a typical element of this is given below,

$$M_{r,r+1} = \begin{pmatrix} I_{(r-1)\times(r-1)} & 0_{(r-1)\times 2} \\ 0_{1\times(r-1)} & 0_{1\times 2} \end{pmatrix} \quad \text{and} \quad X_{r+1,1} = \begin{pmatrix} 0_{r\times 1} \\ 1 \end{pmatrix} \text{ and } \omega = 0.$$

In all other cases where the rank $M \leqslant (r-2)$ and in both the remaining cases

(2 and 3) the codimension of this part is greater than or equal to 2 so we are not worried about this part of V.

These varieties in terms of equations are unprojections and have been studied by Papadakis and Reid, see [Pap04, PR04]. In fact the variety is a single unprojection because all $x_i = 0$ is a codimension $(r+1)$ complete intersection D and all $\sum m_{ij}x_j = 0$ is a codimension r complete intersection X containing D. So Kustin-Miller unprojection applies to give ω as an unprojection variable with the second set of equations as unprojection equations. Also Hochster [Hoc75] studied these examples in relation to the variety of complexes.

4.1.1 The canonical class of V

In this section we calculate the canonical class of the variety. We cover V in such a way that we are not missing any divisor.

If x_1 is nonzero then we can use that to write the first column of M and ω in terms of the remaining entries of M and the x_i. Similarly if x_2 is nonzero then we can solve for the second column of M and ω in terms of the remaining entries of M and the x_i. In the same way if we assume that entry x_i of X is nonzero then we can use that to solve for the ith column of M and ω where the coordinates will be the remaining entries of M and the x_i.

Let V_1 be a chart for V with coordinates $\xi_1, \ldots, \xi_{r^2}, \xi_{r^2+1}, \ldots, \xi_{r^2+r+1}$ where

$$\begin{pmatrix} \xi_1 & \cdots & \xi_r \\ \xi_{r+1} & \cdots & \xi_{2r} \\ \vdots & \ddots & \vdots \\ \xi_{2r+1} & \cdots & \xi_{r^2} \end{pmatrix} = \begin{pmatrix} m_{1,2} & \cdots & m_{1,r+1} \\ m_{2,2} & \cdots & m_{2,r+1} \\ \vdots & \ddots & \vdots \\ m_{r,2} & \cdots & m_{r,r+1} \end{pmatrix}, \quad \begin{pmatrix} \xi_{r^2+1} \\ \vdots \\ \xi_{r^2+r+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{r+1} \end{pmatrix}. \tag{4.1}$$

Similarly, let V_2 be a chart with coordinates $\eta_1, \ldots, \eta_{r^2}, \eta_{r^2+1}, \ldots, \eta_{r^2+r+1}$ where

$$\begin{pmatrix} \eta_1 & \cdots & \eta_r \\ \eta_{r+1} & \cdots & \eta_{2r} \\ \vdots & \ddots & \vdots \\ \eta_{2r+1} & \cdots & \eta_{r^2} \end{pmatrix} = \begin{pmatrix} m_{1,1} & \cdots & m_{1,r+1} \\ m_{2,1} & \cdots & m_{2,r+1} \\ \vdots & \ddots & \vdots \\ m_{r,1} & \cdots & m_{r,r+1} \end{pmatrix}, \quad \begin{pmatrix} \eta_{r^2+1} \\ \vdots \\ \eta_{r^2+r+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{r+1} \end{pmatrix}. \tag{4.2}$$

and the second column of

$$\begin{pmatrix} m_{1,1} & \cdots & m_{1,r+1} \\ m_{2,1} & \cdots & m_{2,r+1} \\ \vdots & \ddots & \vdots \\ m_{r,1} & \cdots & m_{r,r+1} \end{pmatrix}$$

is omitted.

The change of coordinates from one chart to the other is given by the Jacobian

$$\mathrm{Jac}\left(\frac{\xi_1, \ldots, \xi_{r^2}, \xi_{r^2+1}, \ldots, \xi_{r^2+r+1}}{\eta_1, \ldots, \eta_{r^2}, \eta_{r^2+1}, \ldots, \eta_{r^2+r+1}}\right)$$

and the determinant of this Jacobian matrix is given by $(-1)^r \left(\frac{x_1}{x_2}\right)^r$.

We know that

$$\mathcal{O}(K_V) = \bigwedge^{r^2+r+1} \Omega_V^1 \text{ and } \mathcal{O}(K_V)\,|_{V_1} = \mathcal{O}_{V_1} \cdot \sigma_1$$

where σ_1 is given by

$$\sigma_1 = \frac{d\xi_1 \wedge \cdots \wedge d\xi_{r^2} \wedge d\xi_{r^2+1} \wedge \cdots \wedge d\xi_{r^2+r+1}}{(x_1)^r} \tag{4.3}$$

and similarly σ_2 is given by

$$\sigma_2 = \frac{d\eta_1 \wedge \cdots \wedge d\eta_{r^2} \wedge d\eta_{r^2+1} \wedge \cdots \wedge d\eta_{r^2+r+1}}{(x_2)^r}. \tag{4.4}$$

The above calculation of the Jacobian determinant shows that $\sigma_1 = \pm\sigma_2$. The function x_1 is nonzero on V_1 and putting $(x_1)^r$ in the denominator is a convenient trick to cancel out the Jacobian determinant. By repeating the same calculations we notice that $\sigma = \sigma_i$ is independent of i. Since σ_i is the basis of Ω^{r^2+r+1} on V_i (no zeros or poles, because x_i is invertible on V_i), we have

$$K_V = \operatorname{div}(\sigma) = 0.$$

Proposition 4.1.1. *The canonical divisor of the Cramer variety* $\mathrm{Cr}(r, r+1, 1)$ *is Cartier.*

Proof. A Cartier divisor on a variety is an open cover $\{(U_i)\}$ and rational functions $f_i \in k(U_i)^*$ such that for all i, j, $f_i f_j^{-1} \in \mathcal{O}^*(U_i \cap U_j)$. By definition, a scheme is called Gorenstein if it is Cohen-Macaulay and the canonical divisor K is a Cartier divisor. We have an open cover $\{(V_i)\}$ for the variety V, with $f_i = \dfrac{1}{x_i} \in k(V_i)^*$ and $f_i f_j^{-1} = \dfrac{x_j}{x_i} \in \mathcal{O}^*(V_1 \cap V_j) = \mathcal{O}^*(V_{1,2})$. Therefore we have proved that the canonical divisor of $\mathrm{Cr}(r, r+1, 1)$ is a Cartier divisor.

□

We conjecture that they are Cohen-Macaulay on the basis of computer algebra calculations for small values of r; if this conjecture can be established, they are Gorenstein. Magma output is available at the end of thesis in appendix.

4.2 The variety V as quasi-homogeneous space

We study $\mathrm{Cr}(r, r+1, 1)$ as a closure of the orbit of a special vector. Let $G = \mathrm{GL}(r) \times \mathrm{GL}(r+1)$ which is a reductive algebraic group. Let U and W be the given r and $(r+1)$-dimensional representations of $\mathrm{GL}(r)$ and $\mathrm{GL}(r+1)$.

Now for $A \in \mathrm{GL}(r)$, $B \in \mathrm{GL}(r+1)$, $M \in \mathrm{Hom}(U, W)$, $X \in \mathrm{Hom}(W, \mathbb{C})$ and $\omega \in \mathbb{C}$, we define the action of $G = \mathrm{GL}(r) \times \mathrm{GL}(r+1)$ on the representation $R = \mathrm{Hom}(U, W) \oplus \mathrm{Hom}(W, \mathbb{C}) \oplus \mathbb{C}$ as follows:

$$\begin{aligned} M &\longmapsto AMB^{-1} \\ X &\longmapsto BX \\ \omega &\longmapsto \lambda\omega, \text{ where } \lambda = \frac{\det(A)}{\det(B)}. \end{aligned}$$

Let M_0 be the matrix of maximal rank written below and X_0 be the highest weight vector for the representation $\mathrm{Hom}(W, \mathbb{C})$, so by using row and column operations we can write M_0 as

$$M_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad X_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The stabiliser of a vector $v = (M_0, X_0, \omega = 1)$ is

$$H = \mathrm{Stab}(v) = \left\{ (A, B) \;\middle|\; B = \begin{pmatrix} A & 0_{1\times r} \\ * & 1 \end{pmatrix} \right\}.$$

In this part we consider the open orbit

$$V_0 = G/H \simeq G \cdot v \hookrightarrow R.$$

The variety V is a quasi-homogeneous space with the natural action of G and $V = \overline{G \cdot v}$.

4.2.1 The Weyl group $W(G)$

We study the algebraic group G and its representations with the help of the Weyl group $W(G) \cong \mathrm{S}_r \times \mathrm{S}_{r+1}$, which acts as a permutation group. We know from section 4.2 that $R = \mathrm{Hom}(U, W) \oplus \mathrm{Hom}(W, \mathbb{C}) \oplus \mathbb{C}$ is a representation of G. We describe here how the Weyl group $W(G) \cong \mathrm{S}_r \times \mathrm{S}_{r+1}$ acts on R, where S_r and S_{r+1} are symmetric groups of order $r!$ and $(r+1)!$ respectively. The group S_r acts on any $M \in \mathrm{Hom}(U, W)$ from the left and permutes the rows while S_{r+1} acts on the right and permutes the columns. Also S_{r+1} acts on $X \in \mathrm{Hom}(W, \mathbb{C})$ by permuting the coordinates.

4.2.2 Torus action and Weyl group

If $\mathbb{T} \subset G$ is the maximal torus given by

$$\left\{ T_A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_r \end{pmatrix}, \quad T_B = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{r+1} \end{pmatrix} \right\},$$

then it acts on

$$\left(M_{11} = \begin{pmatrix} m_{11} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_{r+1} \end{pmatrix} \text{ and } \quad \omega \right),$$

as explained in section 4.2 and under this action M_{11}, X_0 and ω are the weight vectors with weight $\frac{a_1}{b_1}$, b_{r+1} and $\det(T_A) \times ((\det(T_B))^{-1}$.

4.2.3 One parameter subgroup and elements of V_i

Let $P(t) = (T_A(t), T_B)$ be the one parameter subgroup given by

$$\left\{ T_A(t) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & t \end{pmatrix} \quad \text{and} \quad T_B = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\}.$$

For $p_i \in V_i$, we want to show that $p_i \in \overline{G \cdot v}$. For this we show that there exists one-parameter subgroup $P(t)$ such that $p_i \in \overline{P(t) \cdot v}$.

In fact the one parameter subgroup $P(t) = (T_A(t), T_B)$ acts on

$$p_{r+1} = \left(\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega = 1 \right)$$

and we get

$$P(t) \cdot p_{r+1} = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega = t \right\}.$$

Therefore we have shown that there exists a one parameter subgroup $P(t)$ such that $V_i \in \overline{P(t) \cdot v}$. That is for $t \to 0$ along a one parameter subgroup we get the typical vector of V_i.

4.2.4 Relationship between the canonical class of V and representations of G

In this section we observe the relationship between the tangent bundle and the canonical class of the open orbit and the representations of G. We know from section 4.1.1 that the tangent bundle T_V to V is a vector bundle of rank $r^2 + r + 1$

and the top wedge of $T_V^\vee$ is equal to

$$\bigwedge^{r^2+r+1} T_V^\vee = \bigwedge^{r^2} \mathrm{Hom}(U, W_{r-1} \subset W) \otimes \bigwedge^{r+1} (\mathrm{Hom}(W, \mathbb{C})),$$

where U and W are the given r and $(r+1)$-dimensional $GL(r)$- and $GL(r+1)$-representations and W_{r-1} is an $(r-1)$-dimensional subspace of W.

4.2.5 Gorenstein criterion in terms of weights

The aim of this section is to observe whether we can assign a weight to the canonical differential which is invariant under the Weyl group. We will give a criterion whether the variety is Gorenstein or not depending upon the invariant weight of the canonical differential. We observe that the σ_1 given in equation 4.3 is the weight vector for the maximal torus $\mathbb{T} \subset G$ with weight $\dfrac{(\det(T_A))^r}{(\det(T_B))^{r-1}}$ and similarly all the σ_i are weight vectors with weight $\dfrac{(\det(T_A))^r}{(\det(T_B))^{r-1}}$. But the maximal torus $\mathbb{T} \subset G$ does not normalise the stabiliser H of the special vector. We take the restricted torus $\mathbb{T}_H = \mathbb{T} \cap N_H$ to overcome this problem, where N_H denotes the normaliser of the H. The restricted torus $\mathbb{T}_H$ given by

$$\left\{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_r \end{pmatrix}, \begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_r & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\}.$$

Proposition 4.2.1. *The weight of the canonical differential $K_{G/H}$ is the determinant of the restricted torus.*

Proof. The restricted torus $\mathbb{T}_H$ acts on $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras

of G and H respectively. The weights of G minus the weights of H are given by

$$\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \ddots & & \vdots & \vdots \\ * & * & \cdots & * & * \\ \hline & & & & * \end{array}\right).$$

There are in total $(r^2 + r + 1)$ weight vectors. When we multiply the weights corresponding to the weight vectors in the top left $(r \times r)$ block we get 1. Also the weight vector in bottom (1×1) block has weight 1. So the only contribution comes from the first r weights in the last column. When we multiply the weights corresponding to these r weight vectors we get $(a_1 \cdots a_r)$, which is the determinant of the restricted torus. $\square$

We have shown that the restricted torus $\mathbb{T}_H$ acts on the canonical differential and its weight is the determinant of the $\mathbb{T}_H$ which is invariant under the action of the Weyl group..

In fact we can take the action of the group $G = \mathrm{GL}(r) \times \mathrm{GL}(r+1) \times \mathbb{C}^\times$ on the representation $R = \mathrm{Hom}(U, W) \oplus \mathrm{Hom}(W, \mathbb{C}) \oplus \mathbb{C}$ as follows:

$$\begin{aligned} M &\longmapsto AMB^{-1} \\ X &\longmapsto BXC \\ \omega &\longmapsto \lambda\omega, \text{ where } \lambda = \frac{\det(A)}{c(\det(B))}, \end{aligned}$$

where $A \in \mathrm{GL}(r)$, $B \in \mathrm{GL}(r+1)$ and $c \in \mathbb{C}^\times$. The stabiliser of $v = (M_0, X_0, \omega = 1)$

is

$$H = \mathrm{Stab}(v) = \left\{ (A, B, C) \;\middle|\; B = \begin{pmatrix} A & 0_{1\times r} \\ * & C \end{pmatrix} \right\}.$$

Chapter 5

The Cramer varieties $\mathrm{Cr}(r, r+s, s)$

In this chapter we study a key example of a quasi-homogeneous affine algebraic variety, in particular we calculate its canonical class. We want to give criteria as to why this variety is Gorenstein. In the first part of this chapter, we write the canonical class of the open orbit. In the second part, we write the canonical class in terms of group representations. At the end of this chapter we discuss a generalisation of this example and cases studied in chapter 4. This example is based on examples of Hochster [Hoc75] and is related to the variety of complexes.

5.1 The variety $\mathrm{Cr}(2,5,3)$ in equations

Let M and N be the 2×5 and 5×3 matrices as given below:

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{15} \\ m_{21} & m_{22} & \cdots & m_{25} \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ \vdots & \vdots & \vdots \\ n_{51} & n_{52} & n_{53} \end{pmatrix}.$$

We define a variety $V \subset \mathbb{C}^{(2\times 5)+(5\times 3)+1}$ by the two sets of equations

$$MN = 0 \qquad \text{and} \qquad \omega \bigwedge^{2} M = \bigwedge^{3} N,$$

where $\omega \in \mathbb{C}$ and $\bigwedge^2 M$, $\bigwedge^3 N$ denote the 2×2 and 3×3 minors of M and N respectively. To equate the 2×2 minors of M with 3×3 minors of N we use the method given in [Hoc75]. If M_{ij} is the 2×2 minor of M formed by the ith and jth column, and $N_{\widehat{ij}}$ is the complementary 3×3 minor given by deleting the ith and jth row then we have

$$\omega(-1)^{i+j} M_{ij} = N_{\widehat{ij}}.$$

We denote this Cramer variety by $\mathrm{Cr}(2,5,3)$.

Now if we consider

$$V_0 = \Big\{ (M,N,\omega) : \text{rank of } M = 2, \text{rank of } N = 3 \text{ and } \omega \neq 0 \Big\},$$

then V_0 has codimension 7 in $\mathbb{C}^{(2\times 5)+(5\times 3)+1}$ and $V = \overline{V_0}$. The V_0 is a homogeneous space, the orbit of the vector

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad N_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \omega = 1$$

under the action of $G = \mathrm{GL}(2)\times \mathrm{GL}(5)\times \mathrm{GL}(3)$ as explained in section 5.2.

When M and N are of maximal rank then we can assume the first minor M_{12} of M is nonzero. We can use that to solve the top two rows of N and ω in terms

of remaining entries of N and M.

Let $S = V \setminus V_0$ be the complement of V_0 in V. If $(M, N, \omega) \in S$ then for $\bigwedge^3 N = 0$ we have three possibilities for the elements of S:

1. either the rank of M is full and $\omega = 0$;
2. or the rank of M is strictly less than 2 and $\omega \neq 0$;
3. or the rank of M is strictly less than 2 and $\omega = 0$.

In case 1, when M is of maximal rank then one of the minors M_{ij} is nonzero. For N of rank 2, we get a codimension one irreducible variety, say V_1. A typical element of a divisor V_1 looks like

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad N_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \omega = 0.$$

In all other cases where rank $N \leq 1$ or rank $M \leq 1$ the codimension is greater than or equal to two so we are not worried about that part of S: these subvarieties are not divisorial so do not appear in the canonical class.

5.1.1 The canonical class of $\mathrm{Cr}(2,5,3)$

Suppose that the first minor M_{12} of M is nonzero. Then we can write n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23} and ω in terms of remaining entries of M and N. Similarly if we assume that the minor M_{13} of M is nonzero then we can solve for n_{11}, n_{12}, n_{13}, n_{31},n_{32}, n_{33} and ω. In the same way if we assume that the minor M_{ij} of M is

nonzero then we can use that to solve for the ith and jth rows of N and ω where the coordinates will be remaining entries of M and N.

Let $V_{M_{12}\neq 0}$ and $V_{M_{13}\neq 0}$ be the two charts for V with coordinates $\xi_1, \ldots, \xi_{19}$ and $\eta_1, \ldots, \eta_{19}$ where

$$\begin{pmatrix} \xi_1 & \xi_6 \\ \vdots & \vdots \\ \xi_5 & \xi_{10} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{21} \\ \vdots & \vdots \\ m_{15} & m_{25} \end{pmatrix}, \quad \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{14} & \xi_{15} & \xi_{16} \\ \xi_{17} & \xi_{18} & \xi_{19} \end{pmatrix} = \begin{pmatrix} n_{31} & n_{32} & n_{33} \\ n_{41} & n_{42} & n_{43} \\ n_{51} & n_{52} & n_{53} \end{pmatrix} \tag{5.1}$$

and

$$\begin{pmatrix} \eta_1 & \eta_6 \\ \vdots & \vdots \\ \eta_5 & \eta_{10} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{21} \\ \vdots & \vdots \\ m_{15} & m_{25} \end{pmatrix}, \quad \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{14} & \eta_{15} & \eta_{16} \\ \eta_{17} & \eta_{18} & \eta_{19} \end{pmatrix} = \begin{pmatrix} n_{21} & n_{22} & n_{23} \\ n_{41} & n_{42} & n_{43} \\ n_{51} & n_{52} & n_{53} \end{pmatrix}. \tag{5.2}$$

There are sixteen coordinates common to both charts. The change of coordinates from one chart to other is given by the 19×19 Jacobian matrix J whose first 16×16 block is $I_{16\times 16}$:

$$J = \begin{pmatrix} I_{16\times 16} & 0_{16\times 3} \\ 0_{3\times 16} & C \end{pmatrix} \text{ where } C = \begin{pmatrix} \frac{M_{13}}{M_{12}} & 0 & 0 \\ 0 & \frac{M_{13}}{M_{12}} & 0 \\ 0 & 0 & \frac{M_{13}}{M_{12}} \end{pmatrix}$$

and the determinant of J is $\left(\dfrac{M_{13}}{M_{12}}\right)^3$.

We know that

$$\mathcal{O}(K_V) = \bigwedge^{19} \Omega^1_V \text{ and } \mathcal{O}(K_V)\,|_{U_{M_{12}\neq 0}} = \mathcal{O}_{U_{M_{12}\neq 0}} \cdot \sigma_{12}$$

where

$$\sigma_{12} = \frac{d\xi_1 \wedge \cdots \wedge d\xi_{19}}{(M_{12})^3} \text{ and similarly } \sigma_{13} = \frac{d\eta_1 \wedge \cdots \wedge d\eta_{19}}{(M_{13})^3}. \tag{5.3}$$

The minor M_{12} is invertible on $V_{M_{12}}$ and putting $(M_{12})^3$ in the denominator is a convenient trick to cancel out the Jacobian matrix, which will appear again later. The above calculation of the Jacobian determinant shows that $\sigma_{12} = \sigma_{13}$ and repeating the same calculation defines $\sigma = \sigma_{ij}$ independently of ij. Since σ_{ij} is a basis of $\bigwedge^{19} \Omega^1_V$ and has no zeros or poles, exactly because of the M_{ij} in the denominator, we have

$$K_V = \mathrm{div}(\sigma) = 0.$$

Proposition 5.1.1. *The Cramer variety* $\mathrm{Cr}(2,5,3)$ *is Gorenstein.*

Proof. A Cartier divisor on a variety is an open cover $\{(U_i)\}$ and rational functions $f_i \in k(U_i)^*$ such that for all i, j, $f_i f_j^{-1} \in \mathcal{O}^*(U_i \cap U_j)$. By definition, a scheme is called Gorenstein if it is Cohen-Macaulay and the canonical divisor K is Cartier. We have an open cover $\{(V_{M_{ij}})\}$ for $\mathrm{Cr}(2,5,3)$, with transition functions $\dfrac{1}{M_{ij}} \in k(V_{M_{ij}})^*$. Note that $\dfrac{M_{12}^3}{M_{13}^3} \in \mathcal{O}^*(V_{M_{12}} \cap V_{M_{13}}) = \mathcal{O}^*(V_{M_{12},M_{13}})$. Hence K is Cartier for $\mathrm{Cr}(2,5,3)$.

We have checked that $\mathrm{Cr}(2,5,3)$ is Cohen-Macaulay with computer algebra calculations. Magma output is available at the end of thesis in appendix. Therefore $\mathrm{Cr}(2,5,3)$ is Gorenstein.

□

5.2 The variety $\mathrm{Cr}(2,5,3)$ as a quasi-homogeneous space

Our aim is to study the variety V as the closure of the orbit of a special vector. Let $G = \mathrm{GL}(2) \times \mathrm{GL}(5) \times \mathrm{GL}(3)$ which is a reductive algebraic group. Let W_2, W_5 and W_3 be the given 2, 5 and 3-dimensional representations of GL(2), GL(5) and GL(3) respectively.

We want to define an action of $G = \mathrm{GL}(2) \times \mathrm{GL}(5) \times \mathrm{GL}(3)$ on the representation $R = \mathrm{Hom}(W_2, W_5) \oplus \mathrm{Hom}(W_5, W_3) \oplus \mathbb{C}$ such that the variety V is invariant under this action. In coordinate-free terms, $M \in \mathrm{Hom}(W_2, W_5)$, $N \in \mathrm{Hom}(W_5, W_3)$ and $\omega \in \mathbb{C}$ and the action of $(A, B, C) \in G$ with $A \in \mathrm{GL}(2)$, $B \in \mathrm{GL}(5)$, $C \in \mathrm{GL}(3)$ is defined as follows,

$$\begin{aligned} M &\longmapsto AMB^{-1} \\ N &\longmapsto BNC^{-1} \\ \omega &\longmapsto \lambda\omega, \text{ where } \lambda = \frac{\det(B)}{\det(A) \times \det(C)}. \end{aligned}$$

Let M and N be matrices of maximal rank. By using row and column operations we can write M and N as follows

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The stabiliser of $v = (M_0,\ N_0,\ \omega = 1)$ is

$$H = \mathrm{Stab}(v) = \left\{ (A, B, C) \ \middle| B = \begin{pmatrix} A & 0 \\ * & C \end{pmatrix} \right\}.$$

In this part we consider the open orbit

$$V_0 = G/H \simeq G \cdot v \hookrightarrow R.$$

Cramer variety $\mathrm{Cr}(2,5,3)$ is a quasi-homogeneous space with the natural action of G and $V = \overline{G \cdot v}$.

5.2.1 The Weyl group $W(G)$

To study the algebraic group $G = \mathrm{GL}(2) \times \mathrm{GL}(5) \times \mathrm{GL}(3)$ and its representations we use the Weyl group $W(G) \cong \mathrm{S}_2 \times \mathrm{S}_5 \times \mathrm{S}_3$ which acts as a permutation group. We know from section 5.2 that $R = \mathrm{Hom}(W_2, W_5) \oplus \mathrm{Hom}(W_5, W_3) \oplus \mathbb{C}$ is a representation of G. We describe here how the Weyl group acts on R. The group S_2 acts on any $M \in \mathrm{Hom}(W_2, W_5)$ from the left and permutes the rows while S_5 acts on the right and permutes the columns. Similarly, S_5 acts on $N \in \mathrm{Hom}(W_5, W_3)$ from the left and permutes the rows and S_3 acts on right and permutes the columns.

5.2.2 The torus action and Weyl group

If $\mathbb{T} \subset G$ is the maximal torus given by

$$\left\{ T_A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \; T_B = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{55} \end{pmatrix}, \; T_C = \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \right\},$$

then it acts on

$$M_{11} = \begin{pmatrix} m_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad N_{11} = \begin{pmatrix} n_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and } \omega$$

as explained in section 5.2. Under this action M_{11}, N_{11} and ω are the weight vectors with weights $\frac{a_{11}}{b_{11}}$, $\frac{b_{11}}{c_{11}}$ and $\det(T_B) \times (\det T_A)^{-1} \times (\det T_C)^{-1}$ respectively.

5.2.3 One parameter subgroup and elements of $V \setminus V_0$

Let $P(t) = (T_A(t), T_B(t) \text{ and } T_C(t))$ be the one parameter subgroup given by

$$\left\{ T_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; T_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; T_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

In section 5.1 we showed that we have only one irreducible divisor $V \backslash V_0$, say V_1. We want to show that $V_1 \subset \overline{G \cdot v}$. For this we show that there exists a one parameter subgroup $P(t)$ such that $V_1 \subset \overline{P(t) \cdot v}$.

In fact the one parameter subgroup $P(t) = (T_A(t), T_B(t)$ and $T_C(t))$ acts on

$$p_{r+1} = \left(M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, N_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \omega = 1 \right)$$

and we get

$$P(t).p_{r+1} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega = t \right\}.$$

Therefore we have shown that there exists one parameter subgroup $P(t)$ such that $V_1 \subset \overline{P(t) \cdot v}$. That is for $t \to 0$ along the one parameter subgroup we get the typical vector of V_1.

5.2.4 The relationship between the canonical class of $\mathrm{Cr}(2,5,3)$ and representations of G

In this section we observe the relationship between the tangent bundle and the canonical class to the open orbit and the representations of G. We know from

section 5.1.1 that the tangent bundle T_V to V is a vector bundle of rank 19 and top wedge of its dual is equal to

$$\bigwedge^{19} T_V^\vee = \bigwedge^{10} \mathrm{Hom}(W_2, W_5) \otimes \bigwedge^{9} \mathrm{Hom}(W_3' \subset W_5, W_3)$$

where W_2, W_3, W_3' and W_5 are the given 2, 3, 3 and 5-dimensional GL(2)-, GL(3)-, GL(3)- and GL(5)- representations.

5.2.5 Gorenstein criterion in terms of weights

The canonical differential σ_{12} in equation (5.3) is a weight vector for the maximal $\mathbb{T} \subset G$ with weight $\dfrac{(\det(A))^2 \times \det(B)}{(\det(C))^3}$ and similarly all the σ_{ij} are weight vectors with weight $\dfrac{(\det(A))^2 \times \det(B)}{(\det(C))^3}$. The only problem is $\mathbb{T} \subset G$ does not normalise the stabiliser H. If we take the restricted torus $\mathbb{T}_H = \mathbb{T} \cap N_H$, where N_H is the normaliser of H then $\mathbb{T}_H$

$$\left\{ T_A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix},\ T_B = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & 0 & c_{11} \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{33} \end{pmatrix},\ T_C = \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \right\}$$

acts on the canonical differential σ_{12} in equation 5.3 and σ_{12} is a weight vector with weight $\dfrac{(\det \mathbb{T}_A)^3}{(\det \mathbb{T}_C)^2}$.

Let $\mathfrak{h}$ and $\mathfrak{g}$ be the Lie algebras of H and G respectively. The tangent bundle $T_{G/H}$ to G/H comes from the representation $\mathfrak{g}/\mathfrak{h}$: $\mathfrak{g}/\mathfrak{h}$ is the tangent space to G/H at the identity H, and the tangent space to any other $gH \in G/H$ is given

by $\mathfrak{g}/g\mathfrak{h}g^{-1}$. The canonical class of the variety G/H is

$$K_{G/H} = \mathrm{div}(\bigwedge^{19} T^{\vee}_{G/H}).$$

The canonical differential $K_{G/H}$ is a weight vector for $T_{\mathbb{H}}$ and its weight is exactly the product of those weights of G which are not weights of H.

Proposition 5.2.1. *The weight of the canonical differential $K_{G/H}$ is the determinant of the restricted torus.*

Proof. In this present example there are nineteen weights that are weights of G but not of H and the product of those weight is $\dfrac{(\det \mathbb{T}_A)^3}{(\det \mathbb{T}_C)^2}$.

In the matrix below, each asterisk represents a weight space of G that is not a weight space for H,

$$\left(\begin{array}{cc|ccc} * & * & * & * & * \\ * & * & * & * & * \\ \hline & & * & * & * \\ & & * & * & * \\ & & * & * & * \end{array}\right).$$

When we multiply these weights then the only contribution comes from the top right 2×3 block because the product of all the weights in the 2×2 and 3×3 square blocks is 1.

This shows that the canonical differential is a multiple of the determinant $\dfrac{(\det \mathbb{T}_A)^3}{(\det \mathbb{T}_C)^2}$ under the action of $\mathbb{T}_H$ on $\mathfrak{g}/\mathfrak{h}$. $\square$

5.3 The general case $\mathrm{Cr}(r, r+s, s)$

In this section we discuss the general case. Let M and N be $r \times t$ and $t \times s$ matrices with $t = r + s$, $r \leq s$. We define a variety $V \subset \mathbb{C}^{(r\times s)+(s\times t)+1}$ by the two sets of equations

$$MN = 0 \qquad \text{and} \qquad \omega \bigwedge^{r} M = \bigwedge^{s} N,$$

where $\omega \in \mathbb{C}$. We follow the method given in [Hoc75] to equate ω times the $r \times r$ minors of M with the $s \times s$ minors of N. Let $T = \{1, \ldots, t\}$ be a set of t elements and T_r be a subset of any r elements of T. Let $M_{1,\ldots,r}$ be the minor of M obtained from any r columns of M and let $N_{\widehat{1,\ldots,r}}$ be the minor of N obtained by deleting the r rows of N. We equate these minors as follows

$$(-1)^{\Sigma}\omega M_{1,\ldots,r} = N_{\widehat{1,\ldots,r}},$$

where $\Sigma = \sum t_i$. We call these varieties Cramer varieties and denote them by $\mathrm{Cr}(r, r+s, s)$.

We can do the general case of $\mathrm{Cr}(r, r+s, s)$ by the same method as explained in chapter 4 and in the key example and can prove that the canonical divisor K is Cartier for the Cramer varieties. We conjecture that these varieties are Cohen-Macaulay on the basis of computer algebra calculations for small values of r, s; if this conjecture can be established, they are Gorenstein.

We can also study V as the closure of the orbit of a vector

$$\left(M_0 = \begin{pmatrix} I_{r\times r} & 0_{r\times s} \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0_{r\times s} \\ I_{s\times s} \end{pmatrix} \quad \text{and } \omega = 1 \right).$$

The stabiliser of the vector $v = (M_0,\ N_0,\ \omega = 1)$ is

$$H = \mathrm{Stab}(v) = \left\{ (A, B, C) \;\middle|\; B = \begin{pmatrix} A & 0 \\ * & C \end{pmatrix} \right\}.$$

The restricted torus $\mathbb{T}_H$ acts on the canonical differential and under this action it has weight $\dfrac{(\det \mathbb{T}_A)^s}{(\det \mathbb{T}_C)^r}$. This shows that the canonical differential is a multiple of trivial determinant under the action of restricted torus $\mathbb{T}_H$. This weight is a multiple of all those weights which are only weights of G but not of H.

5.3.1 Special case $\mathrm{Cr}(2, 4, 2) = \mathrm{OGr}(5, 10)$

Let M and N be 2×4 and 4×2 matrices respectively given by

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \\ n_{41} & n_{42} \end{pmatrix}.$$

We define a variety $V \subset \mathbb{C}^{(2\times 4)+(4\times 2)}$ by the two sets of equations

$$MN = 0 \qquad \text{and} \qquad \bigwedge^2 M = \bigwedge^2 N.$$

If M is not of maximal rank then it follows from the second equation that N must be of rank less than 2. This is locus of codimension greater than or equal to 2 so we are not worried about this locus when calculating the canonical class of the variety.

We can assume the first entry m_{11} of M is nonzero. We can use that to solve the top row of N and m_{22}, m_{23} and m_{24} in terms of remaining entries of M and

N. Similarly if we assume the entry m_{21} of M is nonzero then we can use that to eliminate the first row of N and m_{12}, m_{13} and m_{14} in terms of remaining entries of M and N.

Let $V_{m_{11}}$ and $V_{m_{21}}$ be the two charts for V with coordinates given above. These two charts differ by three coordinates and the Jacobian determinant is given by $\left(\frac{m_{11}}{m_{21}}\right)^3$.

We know that

$$\mathcal{O}(K_V) = \bigwedge^{11} \Omega^1_V \text{ and } \mathcal{O}(K_V)\mid_{U_{m_{11}\neq 0}} = \mathcal{O}_{U_{m_{11}\neq 0}} \cdot \sigma_{11}$$

where $\sigma_{11} = \big(dm_{11}\wedge\cdots\wedge dm_{14}\wedge dm_{21}\wedge dn_{21}\wedge dn_{22}\wedge dn_{31}\wedge dn_{32}\wedge dn_{41}\wedge dn_{42}\big)/(m_{11})^3$. Similarly $\sigma_{12} = \big(dm_{21} \wedge \cdots \wedge dm_{24} \wedge dm_{11} \wedge dn_{21} \wedge dn_{22} \wedge dn_{31} \wedge dn_{32} \wedge dn_{41} \wedge dn_{42}\big)/(m_{21})^3$, with $\sigma_{11} = \sigma_{12}$ and repeating the same calculation gives that $\sigma = \sigma_{ij}$ is independent of ij. Since σ_{ij} is a basis for Ω^{11} on $V_{m_{ij}}$ (no zeros or poles, exactly because of the m_{ij} in the denominator), we have

$$K_V = \mathrm{div}(\sigma) = 0.$$

There are 16 variables and 10 equations, each of them of 4 terms. That makes it similar to $\mathrm{OGr}(5, 10)$. One checks that the two sets of equations and the two varieties are identical, although $\mathrm{Cr}(2, 4, 2)$ is only quasi-homogeneous under $G = \mathrm{GL}(2) \times \mathrm{GL}(4) \times \mathrm{GL}(2)$ with an open orbit with complement V_0 of codimension 2. The relation between $\mathrm{OGr}(5, 10)$ and $\mathrm{Cr}(2, 4, 2)$ seems to be an intriguing sporadic phenomenon that has possibly not been noticed before.

Chapter 6

On the canonical divisor of quasi-homogeneous affine algebraic varieties

The aim of this chapter is to prove a theorem that gives a necessary and sufficient condition for the canonical divisor to vanish on a quasi-homogeneous affine algebraic variety. To start, we explain the terminology that we need to state and prove the Theorem 6.3.5 in the general setting. The varieties studied in chapters 4 and 5 are also good relevant examples and we revisit them here too.

6.1 General set up

Let G be a reductive algebraic group and V its complex representation. Let $H \subset G$ be the stabiliser of a vector $v \in V$. The variety $\Omega_0 := G/H = G \cdot v \subset \overline{G/H} = \Omega \subset V$ is a homogeneous space with a natural left G-action. We assume throughout that Ω_0 in Ω has complement of codimension ≥ 2, or equivalently that the orbit Ω_0 intersects every divisor of Ω. Under this assumption, we say Ω_0 is a

big open orbit.

In important cases Ω_0 has an open cover $\bigcup U_i$ with each open U_i parametrised by an open set of $\mathfrak{g}/\mathfrak{h}$ and the transition functions are given by certain elements of the Weyl group. For example in case of Cramer $\mathrm{Cr}(2,3,1)$ the coordinates on two different open pieces V_3 and V_1 are given by

$$\mathfrak{g}/\mathfrak{H}_3 = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right) \text{ and } \mathfrak{g}/\mathfrak{H}_1 = \left(\begin{array}{c|cc} * & 0 & 0 \\ \hline * & * & * \\ * & * & * \end{array}\right).$$

We can go from $\mathfrak{g}/\mathfrak{H}_3$ to $\mathfrak{g}/\mathfrak{H}_1$ by using an element $(13) \in W(G)$ of the Weyl group as transition function. We need to pre and post multiply $\mathfrak{g}/\mathfrak{H}_3$ by the permutation (13) to go to the chart $\mathfrak{g}/\mathfrak{H}_1$.

If $K_{\Omega_0} = 0$ then a global generator $\sigma \in \mathcal{O}(K)$ can be written on each U_i as

$$\sigma = (d\xi_1 \wedge \cdots \wedge d\xi_n)/f_i$$

where ξ_i are the coordinates and f_i is an invertible function on U_i. We use this representation in calculating examples, although it does not play any role in the general statement and proof of Theorem 6.3.5.

Choosing a set-theoretic splitting of $N(T) \to W$, say $w \mapsto \tilde{w}$ gives a linear map $\tilde{w}\colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. In the case of $\mathrm{GL}(n,\mathbb{C})$ we have a natural splitting and $\widetilde{wy} = \tilde{w}\tilde{y}$ for all w, $y \in W$. In the general case, this is no longer the case.

6.2 Vector bundles on G/H

We can think of the quotient space G/H as the base space of a bundle. The group G is the total space for this bundle with a standard projection map $\pi\colon G \longrightarrow G/H$ that takes g to the coset $\overline{g} = gH$. Each coset $gH \in G/H$ is fixed by gHg^{-1} under the transitive action of G on G/H. For each $\overline{g} = gH \in G/H$, we denote $\pi^{-1}(\overline{g})$ as the fibre over $\overline{g}$, which is a copy of H. We can think of the total space of the bundle as a family of copies of H, parametrised by points of the base space. This bundle is called a principal H-bundle. See [FH91, Hum75, Dei00] for the construction of vector bundles and line bundles on G/H and properties of the Lie algebras.

Let ρ be any representation of H on a vector space E. We construct a vector bundle over G/H with fibre E as follows. We quotient out the product space $G \times E$ by the following action of H:

$$(g, e) \in (G, E) \to (gh^{-1}, \rho(h)e).$$

We denote this quotient space by $G \times_H E$. It is an example of a vector bundle because the fibre over each point $\overline{g} \in G/H$ is a copy of the vector space E. In general we can think of a vector bundle as a family of vector spaces of same dimension, parametrised by the base space. In particular each one-dimensional representation of H gives us a line bundle. The tangent bundle $T_{G/H}$ to G/H comes from the representation $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}/\mathfrak{h}$ is the tangent space to G/H at the identity H and the tangent space to any other $gH \in G/H$ is given by $\mathfrak{g}/g\mathfrak{h}g^{-1}$. The top wedge of the tangent bundle $\bigwedge^{\text{top}} T_{G/H}$ gives rise to the anticanonical line bundle. It can be expressed as the product of those weights of G which are not weights of H. The canonical class $K_{G/H}$ of the variety G/H comes from the dual

$\bigwedge^{\text{top}} T^{\vee}_{G/H}$.

Now we are ready to state our main result on quasi-homogeneous spaces.

6.3 Statement of the main result

Let G be a connected reductive algebraic group, T its maximal torus, H its closed connected subgroup. Let $T_H = T \cap H$ be the maximal torus of H for some choice of T.

Let $\mathcal{C}$ be the category of G-equivariant vector bundles on $X = G/H$ and $\mathcal{D}$ the category of representations of H. If $F\colon \mathcal{C} \to \mathcal{D}$ is the functor then for an equivariant vector bundle E over X we get a representation H on the fibre $[H]$, where $[H]$ is the identity coset. On the other hand if $G\colon \mathcal{D} \to \mathcal{C}$ is the functor then for a representation ρ of H on a finite dimensional vector space V we let H act from the right on $G \times V$ by $(g, v)h = (gh, \rho(h)^{-1}v)$ and define $E_\rho = (G \times V)/H$. This is the vector bundle over X.

Remark 6.3.1. The tangent space to G/H at $[H]$ is $\mathfrak{g}/\mathfrak{h}$. This corresponds to H-module $\mathfrak{g}/\mathfrak{h}$ where H acts on $\mathfrak{g}/\mathfrak{h}$ by adjoint action. The line bundle corresponds to $M = (\bigwedge^{\text{top}} \mathfrak{g}/\mathfrak{h})^*$ on which H acts as a character $\kappa\colon H \to \mathbb{C}^*$. We want the line bundle K_X as a G-equivariant sheaf that corresponds to κ.

Now we are ready to state and prove our main result.

Theorem 6.3.2. The canonical class $K_{G/H}$ of the homogeneous space G/H is trivial if and only if the action of the restricted torus T_H on $\mathfrak{g}/\mathfrak{h}$ has trivial determinant.

Proof. The first observation is that the category $\mathcal{C}$ of G-equivariant vector bundles on $X = G/H$ and the category $\mathcal{D}$ of representation of H are equivalent; see for example [Dei00], page 1.

The tangent space to G/H at $[H]$ is $\mathfrak{g}/\mathfrak{h}$. This corresponds to H-module $\mathfrak{g}/\mathfrak{h}$ where H acts on $\mathfrak{g}/\mathfrak{h}$ by adjoint action. The line bundle corresponds to $M = (\bigwedge^{\text{top}} \mathfrak{g}/\mathfrak{h})^*$ on which H acts as a character $\kappa\colon H \to \mathbb{C}^*$.

Since M has rank 1, it is trivial on the commutator $[H, H]$ and actually a representation of the abelianisation $H_a = H/[H, H]$. We use fact from [Sha95] about connected abelian algebraic groups that every connected abelian algebraic group is isomorphic to the direct product of $G_u \times G_s$ where G_u is a connected group whose elements are all unipotent and G_s is a connected group whose elements are all semisimple. Furthermore G_s is isomorphic to a direct product $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$ of groups isomorphic to the multiplicative group $\mathbb{C}^*$. In this case G_u is isomorphic to a product of $\mathbb{C}^+ \times \cdots \times \mathbb{C}^+$ of groups isomorphic to the additive group $\mathbb{C}^+$. Therefore abelian group H_a is isomorphic to a product $V \times T_1$ of a vector space and a torus T_1, where $V \cong \mathbb{C}^+ \times \cdots \times \mathbb{C}^+$ and $T_1 = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$.

If $T_H = T \cap H$ is the maximal torus of H for some choices of T then the natural map $T_H \to T_1$ is surjective; see [Hum75, p.136] and result follows.

□

The equivalence of categories of G-equivariant vector bundles on $X = G/H$ and the category of representation of H has the following consequences.

Corollary 6.3.3. *The G-equivariant line bundles on G/H are in one-to-one correspondence with characters of H, or* $\operatorname{Pic}^G(G/H) = \operatorname{Hom}(H, \mathbb{C}^\times)$.

Let L be a line bundle on G/H and $\pi\colon G \to G/H$ be the canonical map.

Corollary 6.3.4. $\operatorname{Pic}(G/H) = \operatorname{Hom}(H, \mathbb{C}^\times)/\operatorname{Hom}(G, \mathbb{C}^\times)$*; that is, line bundles on G/H correspond to characters of H modulo those that extend to the whole of G.*

Sketch proof: $\pi^* L$ is a line bundle on G, and so is trivial; see for example [KKLV89]. Since $\pi^* L =$ trivial bundle $\mathbb{C} \times G$, L must be obtained by dividing this by an action of H that is a character of H times the given action of G. This is then an equivariant G bundle.

Theorem 6.3.5. *The canonical class $K_{G/H}$ of the homogeneous space G/H is trivial if and only if the determinant of the action of the restricted torus T_H on $\mathfrak{g}/\mathfrak{h}$ extends to a character of G.*

Sketch proof: Now suppose that L on G/H is a G-equivariant line bundle. If it is the trivial line bundle then the G-action on it is given by a character of G that extends the action of H (given by the equivalence of categories we have been arguing about in the preceding sections). And conversely, it is given by a character of H that extends to G, then we can twist back by this character of G to make the bundle correspond to the trivial character of H, so the bundle is trivial.

6.3.1 Secant variety of the Grassmannian $\mathrm{Gr}(2,5)$

We denote by $\Omega = \mathrm{Sec}^2(\mathrm{Gr}(2,n))$ the second secant variety of the Grassmannian $\mathrm{Gr}(2,5)$ in its Plücker embedding, where $n \geq 5$. The affine cone $\Omega_0 \subset \bigwedge^2 \mathbb{C}^n$ is the orbit of $v = e_1 \wedge f_1 + e_2 \wedge f_2$ under the action of $G = \mathrm{GL}(n, \mathbb{C})$. The subgroup H of G that stabilises the special vector is given by

$$H = \begin{pmatrix} A_{4\times 4} & *_{4\times n-4} \\ 0_{(n-4)\times 4} & *_{(n-4)\times(n-4)} \end{pmatrix},$$

where $A \in \mathrm{Sp}(4, \mathbb{C})$ is $\mathrm{Sp}(4, \mathbb{C}$ of $v = e_1 \wedge f_1 + e_2 \wedge f_2$. We know that the Lie algebras of G and $\mathrm{Sp}(4, \mathbb{C})$ have dimension n^2 and 10, respectively, therefore $\mathfrak{g}/\mathfrak{h}$ has dimension $4n - 10$. The restricted torus $\mathbb{T}_H = \mathbb{T} \cap N_H$, where N_H is the normaliser of the H, is given by

$$\mathbb{T}_H = \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix}$$

with $t_1 t_2 t_3 t_4 = 1$.

Now when we take the action of the restricted torus on $\mathfrak{g}/\mathfrak{h}$ then there are a total of $4n - 10$ weight spaces for $\mathbb{T}_H$. There are six weight spaces with weight 1. The product of the remaining weights is given by

$$\frac{(t_5 \dots t_n)^4}{(t_1 \dots t_4)^{n-4}}.$$

We know that in case of the restricted torus $t_1 t_2 t_3 t_4 = 1$, hence the canonical class $K_{G/H}$ of the second secant variety is a multiple of $(t_5 \dots t_n)^4$ which is the determinant of the restricted torus $\mathbb{T}_H$.

6.3.2 Segre threefold

Suppose $G = \mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(3, \mathbb{C})$ and let $e_1 \otimes f_1$ be the highest weight vector for the representation $V_\chi = V^2 \otimes V^3$, where V^2 and V^3 are 2- and 3-dimensional GL(2) and GL(3) modules. The stabiliser of the highest weight vector $v = e_1 \otimes f_1$

is given by

$$H = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \quad \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix} \text{ with } a_{11}b_{11} = 1 \right\}.$$

In fact H is the kernel of the the $\chi : P \longrightarrow \mathbb{C}^{\times}$ where P is the parabolic subgroup of G,

$$P = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \quad \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix} \right\}.$$

For any $p \in P$ and $t \in \mathbb{T}$, we have $\chi(tpt^{-1}) = \chi(t)\chi(p)\chi(t^{-1}) = \chi(p)$ where $\mathbb{T}$ is the maximal torus in G

$$\mathbb{T} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \begin{pmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix} \right\}.$$

The variety $G/P = \mathbb{P}(G \cdot v) \subset \mathbb{P}(V^2 \otimes V^3)$ is not Gorenstein because the product of the weights of G minus the weights of H is not a multiple of the weight of the highest weight vector v. We also observe that $G/H = G \cdot v \subset (V^2 \otimes V^3)$ is a 4-dimensional affine variety whose canonical class is not a multiple of the trivial determinant of the torus $\mathbb{T}$. In other words the torus $\mathbb{T}$ does not act on $\mathfrak{g}/\mathfrak{h}$ as a trivial determinant.

6.3.3 Rational normal curve

Suppose $G = \mathrm{GL}(2, \mathbb{C})$ acts on $\mathbb{C}^2$. We are interested in the orbit of the highest weight vector e_1. In this case $\mathbb{P}^1 = G/P = \mathbb{P}(G \cdot e_1)$, where

$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Let $\chi_k : P \to \mathbb{C}^\times$ be the character for P given by a_{11}^k and the kernel that corresponds to this character is given by

$$H_k = \begin{pmatrix} \mu_k & * \\ 0 & * \end{pmatrix},$$

where μ_k is the primitive kth root of unity. We can give coordinates on one affine piece by $\mathfrak{g}/\mathfrak{h}_k$. We want to calculate the product of the weights that are only weights for G but not of H_k under the action of the torus $\mathbb{T} \subset G$ on $\mathfrak{g}/\mathfrak{h}_k$, where $\mathfrak{g}$ and $\mathfrak{h}_k$ denote the Lie algebras of G and H_k respectively.

The maximal torus

$$\mathbb{T} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

acts on

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

by

$$\mathbb{T} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbb{T}^{-1}$$

and we get

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

as weight vector with weight $\frac{t_2}{t_1}$. As we know that χ_k is the character for P, when we restrict this to the maximal torus then this character is given by t_1^k. We can only get a symmetric weight when we multiply t_2/t_1 by $\chi_2 = t_1^2$.

We have shown here that the canonical differential is a multiple of the trivial determinant of the $\mathbb{T}$ exactly when $k = 1$ or 2.

6.4 Relation to the varieties in chapter 4 and 5

We study quasi-homogeneous affine algebraic varieties in chapters 4 and 5 defined by the following two sets of equations

$$MN = 0 \qquad \text{and} \qquad \omega \bigwedge^r M = \bigwedge^s N.$$

In fact chapter 4 is the special case with N a column vector. In chapter 5 section 5.3 we have shown that the weight of the canonical differential of these varieties is a multiple of the trivial determinant of the restricted torus $\mathbb{T}_H$. Also we have shown, in both chapters, that the canonical class $K_{G/H}$ of the open orbit is trivial.

6.5 Conjecture on quasi-homogeneous affine algebraic varieties

Conjecture 6.5.1. *The quasi-homogeneous affine algebraic varieties are Cohen-Macaulay.*

Here is the sketch why quasi-homogeneous affine algebraic varieties are Cohen-Macaulay. We consider two separate cases for H depending upon whether it is a subgroup of a parabolic subgroup or not.

If H is a subgroup of a parabolic group P. Then the homogeneous space $V_0 = G/H$ has the structure of a fibre bundle $G/H \to G/P$, with the G/P projective homogeneous space. The G/P are Fano varieties in interesting cases, we can use Kodaira vanishing to show that certain cohomology groups vanish and the Cohen-Macaulay part of the Gorenstein property for our variety follows.

On the other hand, $V = \overline{V_0}$ is affine, because we assume that V_0 in R is the closed orbit $G \cdot v$ of a vector v.

Each fibre of $G/H \to G/P$ is the coset space gPg^{-1}/H. We can build a coherent sheaf $\mathfrak{F}$ on G/P by taking the pushdown of linear forms on V restricted to gPg^{-1}; this is locally free (by homogeneity), and is generated by its global sections (since by construction, $V \subset H^0(G/P, \mathfrak{F})$.

Therefore by Kodaira vanishing, all the higher cohomology of $\mathfrak{F}$ vanishes.

This should imply that V is Cohen-Macaulay by an argument similar to the usual argument for G/P.

Appendix A

Magma outputs

Magma output for $\mathrm{Cr}(2,4,2)$ to check whether it is Cohen-Macaulay or not

```
> RR<a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p> := PolynomialRing(Rationals(),16);
>AA := Spec(RR);
>V:=Scheme(AA,[
a*i + b*j + c*k + d*l,a*m + b*n + c*o + d*p,e*i + f*j + g*k + h*l,
e*m + f*n + g*o + h*p,c*h - d*g - i*n + j*m,b*h - d*f + i*o - k*m,
 b*g - c*f - i*p + l*m,a*h - d*e - j*o + k*n,a*g - c*e + j*p - l*n,
a*f - b*e - k*p + l*o ]);
>Dimension(V);
11
>C := FreeResolution(GradedModule(Ideal(V)));
>C;
>Chain complex with terms of degree 6 down to -1
>Dimensions of terms: 0 1 10 16 16 10 1 0
```

This reports that the minimal free resolution of I_V: $0 \leftarrow R/I_V \leftarrow R \leftarrow R^{10} \leftarrow$

$\cdots \leftarrow R \leftarrow 0$. The codimension is 5, the free resolution has length 5, so V is Cohen-Macaulay.

Magma output for $\mathrm{Cr}(2,5,3)$ to check whether it is Cohen-Macaulay or not

```
> RR<m1,m2,m3,m4,m5,l1,l2,l3,l4,l5,n1,k1,p1,n2,k2,p2,n3,k3,p3,n4,k4,p4,
n5,k5,p5,la>
>   := PolynomialRing(Rationals(),26);
> M := Matrix(5,[m1,m2,m3,m4,m5,l1,l2,l3,l4,l5]);
> N := Matrix(3,[n1,k1,p1,n2,k2,p2,n3,k3,p3,n4,k4,p4,n5,k5,p5]);
> AA := Spec(RR);
>  V:=Scheme(AA,[
>  m1*n1 + m2*n2 + m3*n3 + m4*n4 + m5*n5,
>  m1*k1 + m2*k2 + m3*k3 + m4*k4 + m5*k5,
>  m1*p1 + m2*p2 + m3*p3 + m4*p4 + m5*p5,
>  l1*n1 + l2*n2 + l3*n3 + l4*n4 + l5*n5,
>  l1*k1 + l2*k2 + l3*k3 + l4*k4 + l5*k5,
>  l1*p1 + l2*p2 + l3*p3 + l4*p4 + l5*p5,
> -m1*l2*la + m2*l1*la - n3*k4*p5 + n3*p4*k5 + k3*n4*p5 -
 k3*p4*n5 - p3*n4*k5 + p3*k4*n5,
> m1*l3*la - m3*l1*la - n2*k4*p5 + n2*p4*k5 + k2*n4*p5 -
 k2*p4*n5 - p2*n4*k5 + p2*k4*n5,
> -m1*l4*la + m4*l1*la - n2*k3*p5 + n2*p3*k5 + k2*n3*p5 -
k2*p3*n5 - p2*n3*k5 + p2*k3*n5,
> m1*l5*la - m5*l1*la - n2*k3*p4 + n2*p3*k4 + k2*n3*p4 -
k2*p3*n4 - p2*n3*k4 + p2*k3*n4,
```

```
> -m2*l3*la + m3*l2*la - n1*k4*p5 + n1*p4*k5 + k1*n4*p5 -
k1*p4*n5 - p1*n4*k5 + p1*k4*n5,
> m2*l4*la - m4*l2*la - n1*k3*p5 + n1*p3*k5 + k1*n3*p5 -
k1*p3*n5 - p1*n3*k5 + p1*k3*n5,
> -m2*l5*la + m5*l2*la - n1*k3*p4 + n1*p3*k4 + k1*n3*p4 -
k1*p3*n4 - p1*n3*k4 + p1*k3*n4,
> -m3*l4*la + m4*l3*la - n1*k2*p5 + n1*p2*k5 + k1*n2*p5 -
k1*p2*n5 - p1*n2*k5 + p1*k2*n5,
> m3*l5*la - m5*l3*la - n1*k2*p4 + n1*p2*k4 + k1*n2*p4 -
k1*p2*n4 - p1*n2*k4 + p1*k2*n4,
> -m4*l5*la + m5*l4*la - n1*k2*p3 + n1*p2*k3 + k1*n2*p3 -
k1*p2*n3 - p1*n2*k3 + p1*k2*n3
>  ]);
> Dimension(V);
19
> C := FreeResolution(GradedModule(Ideal([
>  m1*n1 + m2*n2 + m3*n3 + m4*n4 + m5*n5,
>  m1*k1 + m2*k2 + m3*k3 + m4*k4 + m5*k5,
>  m1*p1 + m2*p2 + m3*p3 + m4*p4 + m5*p5,
>  l1*n1 + l2*n2 + l3*n3 + l4*n4 + l5*n5,
>  l1*k1 + l2*k2 + l3*k3 + l4*k4 + l5*k5,
>  l1*p1 + l2*p2 + l3*p3 + l4*p4 + l5*p5,
> -m1*l2*la + m2*l1*la - n3*k4*p5 + n3*p4*k5 + k3*n4*p5 -
k3*p4*n5 - p3*n4*k5 + p3*k4*n5,
> m1*l3*la - m3*l1*la - n2*k4*p5 + n2*p4*k5 + k2*n4*p5 -
k2*p4*n5 - p2*n4*k5 + p2*k4*n5,
```

```
> -m1*l4*la + m4*l1*la - n2*k3*p5 + n2*p3*k5 + k2*n3*p5 -
k2*p3*n5 - p2*n3*k5 + p2*k3*n5,
> m1*l5*la - m5*l1*la - n2*k3*p4 + n2*p3*k4 + k2*n3*p4 -
k2*p3*n4 - p2*n3*k4 + p2*k3*n4,
> -m2*l3*la + m3*l2*la - n1*k4*p5 + n1*p4*k5 + k1*n4*p5 -
 k1*p4*n5 - p1*n4*k5 + p1*k4*n5,
> m2*l4*la - m4*l2*la - n1*k3*p5 + n1*p3*k5 + k1*n3*p5 -
k1*p3*n5 - p1*n3*k5 + p1*k3*n5,
> -m2*l5*la + m5*l2*la - n1*k3*p4 + n1*p3*k4 + k1*n3*p4 -
k1*p3*n4 - p1*n3*k4 + p1*k3*n4,
> -m3*l4*la + m4*l3*la - n1*k2*p5 + n1*p2*k5 + k1*n2*p5 -
k1*p2*n5 - p1*n2*k5 + p1*k2*n5,
> m3*l5*la - m5*l3*la - n1*k2*p4 + n1*p2*k4 + k1*n2*p4 -
k1*p2*n4 - p1*n2*k4 + p1*k2*n4,
> -m4*l5*la + m5*l4*la - n1*k2*p3 + n1*p2*k3 + k1*n2*p3 -
k1*p2*n3 - p1*n2*k3 + p1*k2*n3
>      ])));
> C;
Chain complex with terms of degree 8 down to -1
Dimensions of terms: 0 1 16 50 97 97 50 16 1 0
```

This reports that the minimal free resolution of I_V: $0 \leftarrow R/I_V \leftarrow R \leftarrow R^{16} \leftarrow \cdots \leftarrow R \leftarrow 0$. The codimension is 7, the free resolution has length 7, so V is Cohen-Macaulay. In particular in both examples V is Gorenstein since the rank of the end modules in the resolution is 1.

Bibliography

[AN54] Yasuo Akizuki and Shigeo Nakano. Note on Kodaira-Spencer's proof of Lefschetz theorems. *Proc. Japan Acad.*, 30:266–272, 1954.

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14(3):535–554 (electronic), 2001.

[BS07] Samuel Boissière and Alessandra Sarti. Contraction of excess fibres between the McKay correspondences in dimensions two and three. *Ann. Inst. Fourier (Grenoble)*, 57(6):1839–1861, 2007.

[CR02] Alastair Craw and Miles Reid. How to calculate A-Hilb $\mathbb{C}^3$. In *Geometry of toric varieties*, volume 6 of *Sémin. Congr.*, pages 129–154. Soc. Math. France, Paris, 2002.

[Dei00] Anton Deitmar. Differential operators on equivariant vector bundles over symmetric spaces. *Electron. J. Differential Equations*, pages No. 59, 8 pp. (electronic), 2000.

[Dur79] Alan H. Durfee. Fifteen characterizations of rational double points and simple critical points. *Enseign. Math. (2)*, 25(1-2):131–163, 1979.

[EV92] Hélène Esnault and Eckart Viehweg. *Lectures on vanishing theorems*, volume 20 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1992.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[GSV83] G. Gonzalez-Sprinberg and J.-L. Verdier. Construction géométrique de la correspondance de McKay. *Ann. Sci. École Norm. Sup. (4)*, 16(3):409–449 (1984), 1983.

[Hoc75] Melvin Hochster. *Topics in the homological theory of modules over commutative rings*. Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I., 1975. Expository lectures from the CBMS Regional Conference held at the University of Nebraska, Lincoln, Neb., June 24–28, 1974, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 24.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[IN96] Yukari Ito and Iku Nakamura. McKay correspondence and Hilbert schemes. *Proc. Japan Acad. Ser. A Math. Sci.*, 72(7):135–138, 1996.

[IR96] Yukari Ito and Miles Reid. The McKay correspondence for finite subgroups of $\mathrm{SL}(3, \mathbf{C})$. In *Higher-dimensional complex varieties (Trento, 1994)*, pages 221–240. de Gruyter, Berlin, 1996.

[Ito94] Yukari Ito. Crepant resolution of trihedral singularities. *Proc. Japan Acad. Ser. A Math. Sci.*, 70(5):131–136, 1994.

[Ito95] Yukari Ito. Crepant resolution of trihedral singularities and the orbifold Euler characteristic. *Internat. J. Math.*, 6(1):33–43, 1995.

[Ito02] Yukari Ito. Special McKay correspondence. In *Geometry of toric varieties*, volume 6 of *Sémin. Congr.*, pages 213–225. Soc. Math. France, Paris, 2002.

[Ito03] Yukari Ito. The McKay correspondence—a bridge from algebra to geometry. In *European women in mathematics (Malta, 2001)*, pages 127–147. World Sci. Publ., River Edge, NJ, 2003.

[KKLV89] Friedrich Knop, Hanspeter Kraft, Domingo Luna, and Thierry Vust. Local properties of algebraic group actions. In *Algebraische Transformationsgruppen und Invariantentheorie*, volume 13 of *DMV Sem.*, pages 63–75. Birkhäuser, Basel, 1989.

[Kle93] Felix Klein. *Vorlesungenüber das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. Birkhäuser Verlag, Basel, 1993. Reprint of the 1884 original, Edited, with an introduction and commentary by Peter Slodowy.

[KM80] Andrew R. Kustin and Matthew Miller. Algebra structures on minimal resolutions of Gorenstein rings of embedding codimension four. *Math. Z.*, 173(2):171–184, 1980.

[Kod53] K. Kodaira. On a differential-geometric method in the theory of analytic stacks. *Proc. Nat. Acad. Sci. U. S. A.*, 39:1268–1273, 1953.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[McK80] John McKay. Graphs, singularities, and finite groups. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 183–186. Amer. Math. Soc., Providence, R.I., 1980.

[Mor85] Shigefumi Mori. On 3-dimensional terminal singularities. *Nagoya Math. J.*, 98:43–66, 1985.

[Nak01] Iku Nakamura. Hilbert schemes of abelian group orbits. *J. Algebraic Geom.*, 10(4):757–779, 2001.

[Pap04] Stavros Argyrios Papadakis. Kustin-Miller unprojection with complexes. *J. Algebraic Geom.*, 13(2):249–268, 2004.

[PR04] Stavros Argyrios Papadakis and Miles Reid. Kustin-Miller unprojection without complexes. *J. Algebraic Geom.*, 13(3):563–577, 2004.

[RC02] Miles Reid and Alessio Corti. Weighted Grassmannians. In *Algebraic geometry*, pages 141–163. de Gruyter, Berlin, 2002.

[Reia] Miles Reid. Cyclic surface quotient singularities and Hirzebruch-Jung resolutions. `http://www.maths.warwick.ac.uk/~miles/surf/more/cyclic.pdf`.

[Reib] Miles Reid. The Du Val singularities. `http://www.maths.warwick.ac.uk/~miles/surf/more/DuVal.pdf`.

[Rei87] Miles Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[Rei97] Miles Reid. McKay correspondence. In T. Katsura, editor, *Proc. of algebraic geometry symposium (Kinosaki, Nov 1996)*, pages 14–41, 1997.

[Rei02] Miles Reid. La correspondance de McKay. *Astérisque*, (276):53–72, 2002. Séminaire Bourbaki, Vol. 1999/2000.

[Ser55] Jean-Pierre Serre. Faisceaux algébriques cohérents. *Ann. of Math. (2)*, 61:197–278, 1955.

[Sha95] I. R. Shafarevich, editor. *Algebra. IX*, volume 77 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1995. Finite groups of Lie type. Finite-dimensional division algebras, A translation of ıt Algebra. 9 (Russian), Vsesoyuz. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow, 1992 [MR1170352 (93a:00017)], Translation edited by A. I. Kostrikin and I. R. Shafarevich.

[Wat74] Keiichi Watanabe. Certain invariant subrings are Gorenstein. I, II. *Osaka J. Math.*, 11:1–8; ibid. 11 (1974), 379–388, 1974.

[YY93] Stephen S.-T. Yau and Yung Yu. Gorenstein quotient singularities in dimension three. *Mem. Amer. Math. Soc.*, 105(505):viii+88, 1993.